

TOPICS IN NONABELIAN TENSOR PRODUCTS OF TOPOLOGICAL GROUPS

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ABSTRACT. The well-known notion of tensor product is used to describe multilinear relations between objects and enjoys many applications in pure and applied mathematics. The tensor product has been studied extensively in linear algebra with generalisations to abstract abelian group theory and modules. In this MSc thesis we study further generalisations of tensor products to non-abelian groups as well as topological groups. We encounter a rich existing theory of compact topological groups, which we are going to investigate. Finally we consider some recent problems in the theory of nonabelian tensor products of topological groups, showing a series of relevant connections between algebraic topology, topological group theory, and homological algebra.

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INTRODUCTION

The nonabelian tensor product of abstract groups was introduced by R. Brown and J.-L. Loday in [4] and studied by several authors and in different contexts in the last twenty years, as mentioned in [5, 6]. Generalisations of nonabelian tensor products to categories of topological groups exist, and serve as the main motivation for the present MSc thesis. It is a relatively recent topic of investigation, mentioned for instance in [26, 28].

Section 1 is dedicated to preliminaries. Here we lay out some of the necessary notions and results from general topology and topological groups. We emphasise on compact topological groups and discover that compact groups occur frequently, and we give some classical examples via constructions of direct limits or projective limits. We then discuss categorically these constructions thanks to the notions of pullbacks and pushouts in an arbitrary category. Always in Section 1, we introduce characters of abelian groups and mention some classical results through the Pontryagin duality for locally compact groups. Roughly speaking, this is a parallel theory to the well-known reflexivity of vector spaces. We also exhibit interactions with category theory by interpreting some classical notions of topology in a categorical language: it turns out to be useful in the final chapters.

Section 2 is dedicated to the tensor product of abstract abelian groups. Here we study free abstract groups and discuss tensor products of abstract abelian groups, showing that they may be realised as appropriate quotients of free abstract abelian groups. This is an idea which will play a prominent role in the rest of the present MSc thesis. We also show interactions between tensor products and homology.

In Section 3 we study free topological groups, introducing briefly some classical results on free topological groups in general. Here we encounter the notions of the Graev free topological group and Markov free topological group, which have recently been generalised to paratopological groups. On the other hand, we just mention these notions, since we mainly study the well established theory of free compact groups in this section of the MSc thesis. At the end, we study the tensor product in the category of compact abelian groups and show that it can be also obtained as an appropriate quotient of a free compact abelian group.

Section 4 considers the notion of the nonabelian tensor product of abstract groups, as originally presented in [3, 4]. We look at this notion firstly when it coincides with the usual tensor product of abstract abelian groups, and then discuss examples when it does not coincide, in order to show that the nonabelian tensor product of abstract groups really generalizes the classical notion of abstract tensor product. We need to discuss some interactions with homological group theory (referring to [27]), because one of the conditions, when these two notions coincide, happens for groups with trivial Schur multipliers, or for groups with small complex cohomology. We conclude the thesis with Section 5 where we consider the nonabelian tensor product of topological groups and see recent results, which motivate to study new problems for further research.

The material of the present thesis relies heavily on the references [12, 13, 14, 27, 33], which have been mentioned appropriately in order to make clear what is our contribution to the literature.

1. PRELIMINARIES

In this section we collect some preliminary notions of topological groups.

1.1. Background on Groups and Topological Groups. Given a group G with a binary operation $\cdot : G \times G \rightarrow G$, we may look at the binary operation as a multiplication on G , so that

$$(1.1) \quad \cdot : (x, y) \in G \times G \mapsto \cdot(x, y) = x \cdot y \in G.$$

The first axiom of a group with respect to \cdot may be given as $\cdot(\cdot(x, y), z) = \cdot(x, \cdot(y, z))$, which is a compatibility relation between \cdot and the images of pairs (x, y) . The second axiom may be given by looking at the map

$$(1.2) \quad ()^{-1} : x \in G \mapsto (x)^{-1} = x^{-1} \in G,$$

and saying that it is well defined and bijective. Finally, the final axiom may be interpreted as the presence of a unique fixed point for the map $()^{-1}$. At this point we may wish to speak about the continuity of these maps, and so we endow G with additional structure of a topology.

Definition 1.1 (See [25], Definition 1.3.1). A topology on a nonempty set X is a collection \mathcal{T} of subsets of X satisfying

- (1) \emptyset and X are in \mathcal{T} .
- (2) Finite intersections of elements of \mathcal{T} are in \mathcal{T} .
- (3) Arbitrary unions of elements of \mathcal{T} are in \mathcal{T} .

Every set X always has at least two topologies: the so-called indiscrete topology on X consisting of only the empty set \emptyset and the set X itself, as well as the so-called discrete topology on X consisting of every subset of X . We call a pair (X, \mathcal{T}) where \mathcal{T} is a topology on X a *topological space*, and simply write X . The elements U in \mathcal{T} are called *open sets* and their complements $X \setminus U$ are called *closed sets*. By a *clopen* set we mean a subset of X that is simultaneously open and closed. For any $A \subseteq X$, the *closure* \bar{A} of A refers to the smallest closed set containing A . A *neighbourhood* of a point x in X is a subset of X containing an open set containing x .

We may wish to specify a topology \mathcal{T} on a set X in a minimal way without directly mentioning every set in \mathcal{T} . This can be achieved by giving a subcollection \mathcal{B} of \mathcal{T} with sufficient open sets such that \mathcal{T} may be recovered by taking arbitrary unions of elements of \mathcal{B} . This prompts the following definition.

Definition 1.2 (See [25], Definition 1.4.1). Let (X, \mathcal{T}) be a topological space. A *basis* for \mathcal{T} is a subset $\mathcal{B} \subseteq \mathcal{T}$ such that every element of \mathcal{T} is a union of elements of \mathcal{B} . The elements of such a \mathcal{B} are called *basic open sets* and \mathcal{T} is said to be *generated* by \mathcal{B} .

We say \mathcal{B} is a basis for a topological space X to mean that it is a basis for some specified topology possessed by X . At a local level, that is, at the level of the points of a topological space, we have the following corresponding notion.

Definition 1.3 (See [25], Definition 1.4.13). Let X be a topological space, and x a point in X . A *local basis* at x is a collection \mathcal{U}_x of neighbourhoods of x such that every neighbourhood of x contains a neighbourhood of x from \mathcal{U}_x .

The following condition, which describes how points in a topological space may be separated, is of great importance to us, and almost every topological space we consider is assumed to satisfy it.

Definition 1.4 (See [25], Definition 7.1.1). A topological space X is *Hausdorff* if for every two distinct points x and y in X , there are disjoint open sets U and V such that x is in U and y is in V .

A map $f : X \rightarrow Y$ between topological spaces is *continuous* if for any open subset V of Y the preimage $f^{-1}(V)$ is open in X . A *homeomorphism* of topological spaces X and Y is a bijective continuous function $f : X \rightarrow Y$ such that its inverse $f^{-1} : Y \rightarrow X$ is continuous.

Given a surjective map $f : X \rightarrow Y$, where (X, \mathcal{T}) is a topological space, we can endow Y with a topology \mathcal{T}_f called the *quotient topology* with respect to f , given by

$$(1.3) \quad \mathcal{T}_f = \{U \subseteq Y \mid f^{-1}(U) \in \mathcal{T}\}$$

It is easy to show that \mathcal{T}_f is indeed a topology on Y (see [17, Definition 5.1]). Moreover, \mathcal{T}_f is the largest topology making f continuous (see [17, Exercise 5.3(a)]), and indeed this property sufficiently describes the quotient topology. The notion of *induced topology* is dual to the notion of quotient topology (see [17, Definition 4.1, Exercises 45.(c)]), and given an injective function $g : X \rightarrow Y$ into a topological space Y , the induced topology on X with respect to g is defined as the smallest topology making g continuous. When $X \subset Y$ and $g : X \rightarrow Y$ is the inclusion, the induced topology is sometimes called the *relative* or *subspace topology* of X with respect to Y .

The last construction we need at this point is a particular topology on the cartesian product

$$(1.4) \quad \prod_{j \in J} X_j = \left\{ f : J \rightarrow \bigcup_{j \in J} X_j \mid f(j) \in X_j, \text{ for each } j \in J \right\}$$

of a family X_j of topological spaces indexed by a set J .

Definition 1.5 (See [33], Definition 8.3). Let X_j be a family of topological spaces indexed by some set J . The *product topology* on $\prod_{j \in J} X_j$ is the topology generated by the basis consisting of sets of the form $\prod_{j \in J} U_j$ where $U_j \subseteq X_j$ is open in X_j and $U_j \neq X_j$ for at most finitely many j in J .

We can endow a group G with a topology and combine the algebraic structure with the geometric one.

Definition 1.6 (See [13], Definition 1.1). A group G with a topology such that multiplication

$$(1.5) \quad \cdot : (x, y) \in G \times G \mapsto x \cdot y \in G,$$

and inversion

$$(1.6) \quad ()^{-1} : x \in G \mapsto x^{-1} \in G$$

are both continuous is called a *topological group*.

A homomorphism $\varphi : G \rightarrow H$ of topological groups is a homomorphism with respect to their group structures, which we call an *abstract homomorphism*, that is also continuous with respect to their topologies. We say that two topological

groups are isomorphic as topological groups if there is a homeomorphism between them that is also an abstract group homomorphism.

Example 1.7 (See [13], Examples 1.3).

- (1) Every finite group is a topological group with the discrete topology.
- (2) The set of real numbers \mathbb{R} with addition has the usual topology in which each open set is a union of open intervals of the form $(a - \varepsilon, a + \varepsilon)$ for $a, \varepsilon \in \mathbb{R}$ and $\varepsilon > 0$. In fact, for all $n \geq 1$, \mathbb{R}^n with its usual topology and component-wise addition is a topological group, replacing the 1-dimensional ball $B_1^\circ(a, \varepsilon)$ with the n -dimensional ball $B_n^\circ(a, \varepsilon)$.
- (3) The set of integers \mathbb{Z} with addition and the induced topology from the usual topology on \mathbb{R} is a topological group. Also, the set of rational numbers \mathbb{Q} with addition and the induced topology from the usual topology on \mathbb{R} is a topological group.
- (4) The multiplicative group \mathbb{C}^\times of non-zero complex numbers is a topological group.

Example 1.8. Take \mathbb{R} with the usual topology. The quotient map

$$(1.7) \quad p : t \in \mathbb{R} \mapsto t + \mathbb{Z} \in \mathbb{R}/\mathbb{Z}$$

is a surjective abstract homomorphism of groups. The torus group is usually denoted as a quotient of the additive group of the reals:

$$(1.8) \quad \mathbb{T} = \mathbb{R}/\mathbb{Z}$$

and it turns out to be a compact topological group, if it is endowed with the quotient topology with respect to p . The map

$$(1.9) \quad \exp : t \in \mathbb{R} \mapsto \exp(2\pi it) \in \mathbb{S}^1 \subseteq \mathbb{C}^\times$$

is another abstract homomorphism of groups since for all $t_1, t_2 \in \mathbb{R}$

$$(1.10) \quad \begin{aligned} \exp(t_1 + t_2) &= \exp(2\pi i(t_1 + t_2)) \\ &= \exp(2\pi it_1) \cdot \exp(2\pi it_2) = \exp(t_1) \cdot \exp(t_2). \end{aligned}$$

Note that the kernel of (1.9) is equal to \mathbb{Z} and (1.9) is surjective. From the First Homomorphism Theorem (see [13, pp. 7]), there is a commutative diagram

$$(1.11) \quad \begin{array}{ccc} & \mathbb{T} & \\ p \nearrow & & \searrow h \\ \mathbb{R} & \xrightarrow{\exp} & \mathbb{S}^1 \end{array}$$

where $h : t + \mathbb{Z} \in \mathbb{T} \mapsto h(t) = \exp(2\pi it) \in \mathbb{S}^1$ is an isomorphism such that $h \circ p = \exp$. Since both p and \exp are continuous, h is continuous and in fact h is also a homeomorphism, so we have that both algebraically and topologically the torus and the circle are isomorphic. The peculiarity of having a sphere which has simultaneously an algebraic structure and a topological structure does not happen for all \mathbb{S}^n . In fact $n = 1$ is a very special situation, see The Sphere Theorem for Connected Compact Lie Groups in [13, Theorem 6.95].

General results on topological groups, such as separation in topological groups, basis and local basis of topological groups, metrizable, uniformizability, and so on, can be found in [15] and [13, Appendix 4]. We shall mention those concepts

as far as we need them here. For instance, we have already seen in Example 1.8 that the adjective “compact” for a topological group refers directly to the topology which is possessed by the group. In fact, compactness is defined as follows.

Definition 1.9 (See [25], Definition). A topological space X is compact if for any union $\bigcup_{i \in I} U_i \supseteq X$ where U_i is open in X for all $i \in I$, there is a finite subset $F \subseteq I$ such that $\bigcup_{i \in F} U_i \supseteq X$.

In an equivalent way, Definition 1.9 may be rephrased, saying that X is compact if it is possible to extract from any open cover $\bigcup_{i \in I} U_i$ of X a finite subcover $\bigcup_{i \in F} U_i$ of X . Weaker versions are possible, such as the following important localised version of compactness.

Definition 1.10 (See [33], Definition 18.11). A topological space X is *locally compact* if each point x in X has a local basis consisting of compact sets.

We mostly consider topological groups whose topologies are compact and Hausdorff. We refer to these simply as compact groups, and locally compact groups when the topology is locally compact and Hausdorff. Some compact groups that we have encountered so far are \mathbb{T} in Example 1.8, and all finite groups with discrete topology in Example 1.7(1). These are also locally compact groups because compactness implies local compactness. Examples of locally compact groups that are not compact are \mathbb{R} and \mathbb{Z} in Example 1.7(2,3). The group \mathbb{Q} as in Example 1.7(3) is a Hausdorff topological group that is neither compact nor locally compact.

Consider \mathbb{K} to be the real field \mathbb{R} or the complex field \mathbb{C} . Recall that a *Banach algebra* over a field \mathbb{K} is a normed vector space A over \mathbb{K} which is also a ring such that $a(vw) = (av)w = v(aw)$ and $\|vw\| \leq \|v\|\|w\|$ for all $a \in \mathbb{K}, v, w \in A$ (see [13, pp. 2-3]). Topological groups are well-known in functional analysis due to the fact that the group of invertible elements A^{-1} of a Banach algebra A is a topological group with respect to the subspace topology induced by the norm topology (see [13, Proposition 1.4]).

Notably, if $A = \mathbb{K}^n$ is an n - dimensional \mathbb{K} vector space, then the general linear group $\text{Gl}(n, \mathbb{K})$ is a topological group. This offers an example of a nonabelian compact group when $n \geq 2$. It may be useful to note that the topology on $\text{Gl}(n, \mathbb{K})$ is the topology induced by the operator norm $\|T\| = \sup_{\|x\| \leq 1} \|Tx\|$, where $x \in \mathbb{K}^n$

and the norm on \mathbb{K}^n is the one induced by the standard inner product on \mathbb{K}^n . On the other hand, it is interesting to note that all vector space norms on a finite-dimensional \mathbb{K} vector space \mathbb{K}^n are equivalent. In particular, since $\text{Gl}(n, \mathbb{K})$ is the n^2 - dimensional vector space \mathbb{K}^{n^2} , one may wish consider the norm on $\text{Gl}(n, \mathbb{K})$ that is induced by the determinant.

Example 1.11 (See [13], Examples 1.6).

- (1) The special linear group

$$(1.12) \quad \text{Sl}(n, \mathbb{K}) = \{A \in \text{Gl}(n, \mathbb{K}) \mid \det(A) = 1\}$$

is a topological group.

- (2) If $(\cdot \mid \cdot)$ is the usual scalar product on \mathbb{K}^n , then the unitary group

$$(1.13) \quad \text{U}(n) = \{A \in \text{Gl}(n, \mathbb{K}) \mid (Ax \mid Ax) = (x \mid x) \text{ for all } x \in \mathbb{K}^n\}$$

on \mathbb{K}^n is a compact group. When $\mathbb{K} = \mathbb{R}$, $U(n)$ is usually written $O(n)$ and it is called the orthogonal group on \mathbb{R}^n .

We can form new compact groups from existing ones. In particular, arbitrary products of compact groups are compact groups, by [13, Proposition 1.14]. Closed subgroups of compact groups are compact groups, and quotients of compact groups by closed normal subgroups are compact groups, by [13, Proposition 1.10(ii)]. Amongst those topological groups which reside in the group of units of Banach algebras is a very important class called compact Lie Groups. To define these groups we need the following notions.

Definition 1.12 (See [13], Definition 2.37). A topological group G has *no small subgroups*, respectively, *no small normal subgroups* if there is an identity neighbourhood U such that if H is a subgroup, respectively, a normal subgroup of G and $H \subseteq U$, then $H = \{1\}$.

This means that anytime we get close to the identity element with a subgroup contained in a neighbourhood of the identity, then such a subgroup collapses. In other words, subgroups have a condition of homogeneity of the algebraic structure and the topological structure when we are close to the identity.

For instance, in \mathbb{R} of Example 1.7(2) the identity neighbourhood $U = (-1, 1)$ does not contain a nontrivial subgroup of \mathbb{R} . Similarly, the torus group \mathbb{T} of Example 1.8 has no small subgroups.

Groups with no small subgroups are frequent due to the fact that every subgroup of the multiplicative group of any Banach algebra has no small subgroups (see [13, Lemma 2.38]). Moreover, compact groups with no small subgroups admit several characterisations, and one of these is relevant in Lie theory.

Theorem 1.13 (See [13], Corollary 2.40). *If G is a compact group, then the following properties are equivalent:*

- (1) G has no small subgroups.
- (2) G has no small normal subgroups.
- (3) G is isomorphic as a compact group to a group of orthogonal or unitary matrices.
- (4) G is isomorphic as a compact group to a closed subgroup of the multiplicative group of some Banach algebra.

Thanks to Theorem 1.13, it is possible to avoid the Lie correspondence (see [13, Definition 5.32]), and focus on a more efficient definition of a Lie group, depending on the context.

Definition 1.14 (See [13], Definition 2.41). A compact group G is called a *compact Lie group* if it satisfies any of the statements in Theorem 1.13.

Some examples are here.

Remark 1.15 (See [13], Exercise E.28).

- (1) Any finite group is a compact Lie group.
- (2) Finite products of compact Lie groups are compact Lie groups. In particular, $\mathbb{T}^n \times E$ where E is a discrete finite abelian group, is a compact Lie group.
- (3) Every closed subgroup of a compact Lie group is a compact Lie group.

Recall that an n -dimensional manifold, or n -manifold for short, is a compact connected topological space such that each of its points has a neighbourhood that is homeomorphic to the Euclidean space \mathbb{R}^n . The following proposition says that abelian compact Lie groups may be regarded as appropriate n -manifolds.

Proposition 1.16 (See [13], Proposition 2.42). *Every abelian compact Lie group G is of the form $G = \mathbb{T}^n \times E$ for some $n \in \mathbb{N}$ and finite abelian group E . If G is connected, then $G \cong \mathbb{T}^n$ for some $n \in \mathbb{N}$.*

Remark 1.17. If G is a connected abelian compact Lie group, then G is an n -manifold. This follows from Proposition 1.16 and the definition of n -manifold.

Compact Lie groups are important in that they form building blocks of compact groups. To make this precise, we need another construction of compact groups from existing ones.

Firstly, we recall that a set J is *directed* if it is endowed with a reflexive and transitive relation \leq such that every finite subset of J is bounded from above.

Definition 1.18 (See [13], Definition 1.25). Let J be a directed set.

- (1) A *projective* (or *inverse*) system of topological groups is a family of topological groups $G_{j \in J}$ and continuous group homomorphisms $f_{jk} : G_k \rightarrow G_j$ for $j \leq k$, such that $f_{jj} : G_j \rightarrow G_j$ is the identity on G_j for all $j \in J$, and $i \leq j \leq k$ implies that $f_{ik} = f_{ij} \circ f_{jk}$.
- (2) For a projective system $f_{ij} : G_j \rightarrow G_i$ of topological groups, the subgroup consisting of all (x_j) such that $j \leq k$ implies that $f_{jk}(x_k) = x_j$, is called the *projective limit* of the system, and it is written $\lim_{j \in J} G_j$.
- (3) The maps $f_{jk} : G_k \rightarrow G_j$ are called *bonding maps*. Let $\text{pr}_i : \prod_{j \in J} G_j \rightarrow G_i$ be the projection $\text{pr}_i((g_j)_{j \in J}) = g_i$ onto the i^{th} coordinate, and $\text{incl} : \lim_{j \in J} G_j \rightarrow \prod_{j \in J} G_j$ be the inclusion. Then the maps $f_i = \text{pr}_i \circ \text{incl} : \lim_{j \in J} G_j \rightarrow G_i$ are called *limit maps*.

The commutative diagram

$$\begin{array}{ccccccc}
 & & & & & & f_{ik} \\
 & & & & & \swarrow & \\
 & & & & & f_{ij} & \swarrow \\
 & & & & & \leftarrow & \leftarrow \\
 & & & & & G_j & \leftarrow & G_k \dots \\
 & & & & & \uparrow & \nearrow & \\
 & & & & & f_j & & f_k \\
 & & & & & \lim_{j \in J} G_j & & \\
 & & & & & \swarrow & \searrow & \\
 & & & & & \dots & G_i & \leftarrow \dots & G_l & \leftarrow \dots & G_m & \leftarrow \dots
 \end{array}$$

(1.14)

illustrates the setup of definition 1.18

The projective limit $\lim_{j \in J} G_j$ of a projective system of topological groups is a closed subgroup of the product $\prod_{j \in J} G_j$ (see [13, Lemma 1.26]). When $\{G_j\}_{j \in J}$ is a family of compact groups, then $\lim_{j \in J} G_j$ being a closed subgroup of the compact space $\prod_{j \in J} G_j$ is a compact group (see [13, Lemma 1.26]).

When the limit maps and the bonding maps are surjective, then the projective system is called a *strict projective system* and the projective limit is called a *strict projective limit*. In the case of compact groups, surjectivity of bonding maps is equivalent to surjectivity of limit maps (see [13, Proposition 1.31]). A classical construction is reported below.

Example 1.19. For any prime p and any natural number n , the subgroup $\frac{1}{p^n}\mathbb{Z}/\mathbb{Z}$ of \mathbb{T} is the cyclic group of order p^n . For each $n \in \mathbb{N}$, $\frac{1}{p^n}\mathbb{Z}/\mathbb{Z}$ with the discrete topology is a compact group. Moreover, for each $n \geq 2$, the map

$$(1.15) \quad \varphi_n : \frac{m}{p^n} + \mathbb{Z} \in \frac{1}{p^n}\mathbb{Z}/\mathbb{Z} \mapsto \frac{m}{p^{n-1}} + \mathbb{Z} \in \frac{1}{p^{n-1}}\mathbb{Z}/\mathbb{Z}$$

is continuous. For all $j, k \in \mathbb{N}$ such that $j \leq k$, define $f_{jk} : \frac{1}{p^k}\mathbb{Z}/\mathbb{Z} \rightarrow \frac{1}{p^j}\mathbb{Z}/\mathbb{Z}$ by (1.15) as follows:

$$(1.16) \quad f_{jk} = \varphi_{j+1} \circ \varphi_{j+2} \circ \dots \circ \varphi_k,$$

where f_{kk} is the identity map of $\frac{1}{p^k}\mathbb{Z}/\mathbb{Z}$. Clearly, the system of compact groups $\{\frac{1}{p^n}\mathbb{Z}/\mathbb{Z} \mid n \in \mathbb{N}\}$ and continuous homomorphisms $\{f_{jk} \mid j, k \in \mathbb{N}\}$ is a strict projective system of compact groups, visualised in the diagram

$$(1.17) \quad 0 \longleftarrow \frac{1}{p}\mathbb{Z}/\mathbb{Z} \xleftarrow{f_{12}} \frac{1}{p^2}\mathbb{Z}/\mathbb{Z} \xleftarrow{f_{23}} \frac{1}{p^3}\mathbb{Z}/\mathbb{Z} \xleftarrow{f_{34}} \dots$$

The strict projective limit $\lim_{n \in \mathbb{N}} \frac{1}{p^n}\mathbb{Z}/\mathbb{Z}$ is called the group of *p -adic integers* and it is written as \mathbb{Z}_p . Its limit maps are $f_j = \text{pr}_j \circ \text{incl}$, for all $j \in \mathbb{N}$, and the commutative diagram

$$(1.18) \quad \begin{array}{ccc} \mathbb{Z}_p & \xrightarrow{\text{incl}} & \prod_{n \in \mathbb{N}} \frac{1}{p^n}\mathbb{Z}/\mathbb{Z} \\ \swarrow f_j & \downarrow f_k & \swarrow \text{pr}_k \\ \frac{1}{p^j}\mathbb{Z}/\mathbb{Z} & \xleftarrow{f_{jk}} & \frac{1}{p^k}\mathbb{Z}/\mathbb{Z} \end{array}$$

illustrates the setup.

We need to recall some well known notions of connected and disconnected topological spaces in order to appreciate the different topologies of \mathbb{Z}_p and \mathbb{T} .

Definition 1.20 (See [33], Definition 26.1). A topological space X is said to be *disconnected* if there are disjoint nonempty open subsets K and H of X such that $K \cup H = X$. If X has no such subsets, then X is *connected*.

The largest connected subset C_x of X containing the point x in X is called the *connected component* of x . The collection of all connected components of X form a partition of X , and each of them is closed in X (see [33, Theorem 26.12]). In the extreme case where the connected components are singletons we have a special terminology.

Definition 1.21 (See [33], Definition 26.1). A topological space X is said to be *totally disconnected* if the only connected subsets of X are singletons.

Let X/conn denote that set of all connected components of a topological space X . The function

$$(1.19) \quad x \in X \mapsto C_x \in X/\text{conn}$$

which sends each point of X to its connected component is surjective, and thus X/conn may be given the quotient topology with respect to (1.19). With this topology X/conn becomes a totally disconnected topological space, and it plays a role in describing the structure of free compact abelian groups.

Of course, \mathbb{Z}_p , being a projective limit of compact groups, is a compact group. Moreover, using the fact that the continuous image of a connected space is connected, it is easy to show that \mathbb{Z}_p is totally disconnected. Note that \mathbb{Z}_p is torsion-free, meaning that it has no elements of finite order. The subgroups of \mathbb{Z}_p are precisely those of the form $p^n\mathbb{Z}_p$ for $n \in \mathbb{N} \cup \{0\}$. In fact,

$$(1.20) \quad \{p^n\mathbb{Z}_p : n \in \mathbb{N} \cup \{0\}\}$$

is a local basis of the identity consisting of clopen subgroups. For any integer $z \in \mathbb{Z}$, write

$$(1.21) \quad \psi : z \in \mathbb{Z} \mapsto \psi(z) = \left(\frac{z}{p^n} + \mathbb{Z} \right)_{n \in \mathbb{N}} \in \mathbb{Z}_p.$$

Now, for each $g = (z_n + \mathbb{Z})_{n \in \mathbb{N}} \in \mathbb{Z}_p$, the sequence $(\psi(p^n z_n))_{n \in \mathbb{N}}$ converges to g . Therefore, every $g \in \mathbb{Z}_p$ may be approximated by an infinite sequence of the form $(\psi(z_n))_{n \in \mathbb{N}}$, where $z_n \in \mathbb{Z}$. This is called the p -adic expansion of g , and so every element of \mathbb{Z}_p admits a p -adic expansion.

The notion of a projective limit is a more general categorical construction. To see this, we firstly recall some elementary notions of category theory.

Definition 1.22 (See [13], Definition A3.1). A category \mathcal{C} is a collection consisting of a class of objects, denoted $\text{ob}(\mathcal{C})$, such that

- (1) for each ordered pair (A, B) of objects A and B in $\text{ob}(\mathcal{C})$, there is a set $\mathcal{C}(A, B)$ consisting of elements called *arrows* (or *morphisms*) from A to B . We write $f : A \rightarrow B$ (or $A \xrightarrow{f} B$) for f in $\mathcal{C}(A, B)$.
- (2) for any X in $\text{ob}(\mathcal{C})$, there is an arrow 1_A in $\mathcal{C}(A, A)$, called the identity arrow or identity morphism.
- (3) for each triple of objects (A, B, C) there is a function

$$(1.22) \quad \circ : (f, g) \in \mathcal{C}(B, C) \times \mathcal{C}(A, B) \mapsto \circ(f, g) = f \circ g \in \mathcal{C}(A, C)$$

called composition. Moreover, composition satisfies the following:

- (i) $1_B \circ f = f$, and $f \circ 1_A = f$ for all arrows $f : A \rightarrow B$.
- (ii) $(f \circ g) \circ h = f \circ (g \circ h)$ for all $f : C \rightarrow D, g : B \rightarrow C, h : A \rightarrow B$.

We briefly describe some categories that we are mainly interested in.

- (1) Recall that a group A is abelian if $ab = ba$ for all $a, b \in A$. We denote by $\mathbb{A}\mathbb{B}$ the category of abstract abelian groups, which has abstract abelian groups for objects, and (abstract) homomorphisms between abstract abelian groups for arrows.
- (2) The category $\mathbb{C}\mathbb{A}\mathbb{B}$ of compact abelian groups, where the objects are compact abelian groups, and the arrows are continuous homomorphisms between compact abelian groups.

- (3) The category TOP_0 of pointed Hausdorff topological spaces. Here, the objects are ordered pairs (X, x_0) where X is a Hausdorff topological space, and $x_0 \in X$ is an identified basepoint. The arrows are basepoint preserving continuous functions, that is, continuous functions $f : (X, x_0) \rightarrow (Y, y_0)$ such that $f(x_0) = y_0$.
- (4) The category KTOPG of compact groups. Where the objects are compact groups and the arrows are continuous homomorphisms between compact groups.

Definition 1.23 (See [13], Definition A3.17). A functor $F : \mathcal{A} \rightarrow \mathcal{B}$ of a category \mathcal{A} to a category \mathcal{B} is a function which assigns to an object A of \mathcal{A} an object FA of \mathcal{B} , and maps an arrow $A \rightarrow B$ of \mathcal{A} to an arrow $FA \rightarrow FB$ of \mathcal{B} such that:

- (1) $F(1_A) = 1_{FA}$ for all $A \in \text{ob}(\mathcal{A})$.
- (2) $F(f \circ g) = F(f) \circ F(g)$ whenever $f \circ g$ is defined.

A category \mathcal{C} is said to be *small* if both its collections of objects and arrows are sets. If \mathcal{A} is a small category, then F is called a *diagram*.

Definition 1.24 (See [13], [31] pp.7). Let $F : \mathcal{A} \rightarrow \mathcal{B}$ be a functor between categories.

- (1) F is a *contravariant functor* if it reverses the direction of arrows, i.e., $F(f) : F(B) \rightarrow F(A)$ for all $f : A \rightarrow B$.
- (2) F is a *covariant functor* if it preserves the direction of arrows, i.e., $F(f) : F(A) \rightarrow F(B)$ for all $f : A \rightarrow B$.

Let $\mathbb{A}\mathbb{B}_R$ be the category of left R -modules, where R is a ring with unity 1, and $\mathbb{A}\mathbb{B}$ the category of abstract abelian groups. For any R -module A in $\mathbb{A}\mathbb{B}_R$, the maps

$$(1.23) \quad B \in \mathbb{A}\mathbb{B}_R \mapsto \text{Hom}(A, B) \in \mathbb{A}\mathbb{B}$$

and

$$(1.24) \quad f \in \text{Hom}(B, C) \mapsto \text{Hom}(A, f) \in \text{Hom}(\text{Hom}(A, B), \text{Hom}(A, C)),$$

given by

$$(1.25) \quad \text{Hom}(A, f)(g) = f \circ g$$

for all $g : A \rightarrow B$ define a covariant functor. This is called the covariant Hom functor of A , written $\text{Hom}(A, -) : \mathbb{A}\mathbb{B}_R \rightarrow \mathbb{A}\mathbb{B}$.

The maps

$$(1.26) \quad B \in \mathbb{A}\mathbb{B}_R \mapsto \text{Hom}(B, A) \in \mathbb{A}\mathbb{B}$$

and

$$(1.27) \quad f \in \text{Hom}(B, C) \mapsto \text{Hom}(f, A) \in \text{Hom}(\text{Hom}(C, A), \text{Hom}(B, A)),$$

given by

$$(1.28) \quad \text{Hom}(f, A)(g) = g \circ f$$

for all $g : C \rightarrow A$ defines a contravariant functor called the contravariant Hom functor of A , and it is often written $\text{Hom}(-, A) : \mathbb{A}\mathbb{B}_R \rightarrow \mathbb{A}\mathbb{B}$. See [13, Example A3.26] for details.

We can compare two functors $S, T : \mathcal{A} \rightarrow \mathcal{B}$ through the notion of a natural transformation, which is defined as follows.

Definition 1.25 (See [13], Definition A3.31). A *natural transformation* between two functors $S, T : \mathcal{A} \rightarrow \mathcal{B}$ is a function $\alpha : S \rightarrow T$ which maps an object $A \in \mathcal{A}$ to an arrow $\alpha_A : SA \rightarrow TA$ in \mathcal{B} such that for all arrows $f : A_1 \rightarrow A_2$ in \mathcal{A} the diagram

$$(1.29) \quad \begin{array}{ccc} SA_1 & \xrightarrow{Sf} & SA_2 \\ \alpha_{A_1} \downarrow & & \downarrow \alpha_{A_2} \\ TA_1 & \xrightarrow{Tf} & TA_2 \end{array}$$

is commutative, i.e., $Sf \circ \alpha_{A_2} = Tf \circ \alpha_{A_1}$.

The following notion is intimately tied to the notion of a Universal Property (see [13, Theorem A3.28]) which we exploit frequently.

Definition 1.26 (See [13], Definition A3.29). Functors $F : \mathcal{A} \rightarrow \mathcal{B}$ and $U : \mathcal{B} \rightarrow \mathcal{A}$ are said to be *adjoint* if for every object A in $\text{ob } \mathcal{A}$, there is an arrow $\eta_A : A \rightarrow UF(A)$ in \mathcal{A} such that for all arrows $f : A \rightarrow UB$ in \mathcal{A} , there is a unique arrow $f' : FA \rightarrow B \in \mathcal{B}$ such that the diagram

$$(1.30) \quad \begin{array}{ccc} A & \xrightarrow{\eta_A} & UF(A) \\ f \downarrow & & \downarrow Uf' \\ UB & \xrightarrow{1_{UB}} & UB \end{array}$$

is commutative, i.e., $f = (Uf') \circ \eta_A$.

Note that we can compose the functors $F : \mathcal{A} \rightarrow \mathcal{B}$ and $U : \mathcal{B} \rightarrow \mathcal{A}$ in the obvious way to obtain the functor $UF : \mathcal{A} \rightarrow \mathcal{A}$. We can then compare UF to the identity functor $\text{id}_{\mathcal{A}}$ using, once again, natural transformations. In fact, one sees that Definition 1.26 is saying that $\eta_A : \text{id}_{\mathcal{A}} \rightarrow UF$ is a natural transformation. See [13, Proposition A3.33] for more details.

Recall that a *set-based category* is a category whose objects are sets with additional structure, and arrows are set functions that preserve that structure (see [13], Definition A3.4).

Definition 1.27 (See [13], Example A3.19). If \mathcal{C} is a set-based category, then a functor $U : \mathcal{C} \rightarrow \mathbb{S}$ to the category of sets which maps each object X to its underlying set UX , and maps each arrow $f : X \rightarrow Y$ to its underlying set function $Uf : UX \rightarrow UY$, is called a *grounding functor* or a *forgetful functor*.

In general, a forgetful functor may not necessarily forget the entire structure of objects. For example, a forgetful functor on the category of compact groups may forget the group structure and preserve the pointed topological structure. So, a functor $U : \mathbb{K}\text{TOPG} \rightarrow \text{TOP}_0$ which sends every compact topological group to the underlying Hausdorff pointed topological space (with identity element as basepoints) is also called a forgetful functor.

Let A be an object in a category \mathcal{C} , and J a small category. Denote by $\text{const}(A) : J \rightarrow \mathcal{C}$ the constant functor at A which sends every object of J to A , and every morphism in J to the identity morphism 1_A on A in \mathcal{C} . Let $D : J \rightarrow \mathcal{C}$ be a

functor, or rather a diagram since J is a small category. Then a *cone* with *vertex* A and *base* D is a natural transformation

$$(1.31) \quad \alpha : \text{const}(A) \rightarrow D,$$

and it may be visualised through the following cone-shaped commutative diagram:

$$(1.32) \quad \begin{array}{ccccc} & & A & & \\ & \swarrow \alpha_i & \downarrow \alpha_j & \searrow \alpha_k & \\ D(i) & \xleftarrow{D(i,j)} & D(j) & \xleftarrow{D(j,k)} & D(k) \end{array}$$

which justifies the terminology of *vertex* for A and *base* for D .

Fixing a diagram $D : J \rightarrow \mathcal{C}$, we can form cones of arbitrary objects in \mathcal{C} . Furthermore, the collection of all these cones forms a category where the morphisms $\lambda : \text{cone}(B) \rightarrow \text{cone}(A)$ are morphisms $\lambda : B \rightarrow A$ such that $\alpha'_i = \lambda \circ \alpha_i$ for all $i \in J$, where $\alpha' : \text{const}(B) \rightarrow D$ is a cone with vertex B . A *terminal object* in this category, that is, an object with a unique arrow to it from every other object, is called a *limit* of the diagram D , more precisely we have the following definition.

Definition 1.28 (See [13], Definition A3.41). A diagram $D : J \rightarrow \mathcal{C}$ is said to have a *limit* $\lim D$ in $\text{ob}(\mathcal{C})$ if there is a unique cone $\lambda : \text{const}(\lim D) \rightarrow D$ such that for any cone $\alpha : \text{const}(A) \rightarrow D$ there is a unique morphism $\alpha' : A \rightarrow \lim D$ such that $\alpha = \lambda \circ \text{const}(\alpha')$.

If J is a set partially ordered by \leq , then J is a category with morphisms $j \rightarrow k$ whenever $j \leq k$. Composition of morphisms is given by transitivity of \leq , and for each $j \in J$ the identity morphism 1_j is made available by the reflexivity of \leq . Now the p -adic integers \mathbb{Z}_p can be seen as the limit of the contravariant diagram $D : \mathbb{N} \cup \{0\} \rightarrow \mathbb{C}\text{AB}$ from $\mathbb{N} \cup \{0\}$ with its natural order, seen as a category, to the category $\mathbb{C}\text{AB}$ of compact abelian groups, which maps each natural number n to the compact abelian group $\frac{1}{p^n}\mathbb{Z}/\mathbb{Z}$, and every arrow $j \leq k$ to the continuous homomorphism $f_{jk} : \frac{1}{p^k}\mathbb{Z}/\mathbb{Z} \rightarrow \frac{1}{p^j}\mathbb{Z}/\mathbb{Z}$ of Example 1.19.

Notice that in Definition 1.28 of a limit, the diagram $J : D \rightarrow \mathcal{C}$ reverses the direction of arrows, that is, an arrow

$$(1.33) \quad j \rightarrow k$$

in J is mapped to the arrow

$$(1.34) \quad D(k) \rightarrow D(j)$$

in \mathcal{C} . In other words, the diagram D is a contravariant functor. Instead, by taking $D : J \rightarrow \mathcal{C}$ to be a covariant functor, so that the direction of arrows is preserved and an arrow

$$(1.35) \quad j \rightarrow k$$

in J is mapped to the arrow

$$(1.36) \quad D(j) \rightarrow D(k)$$

in \mathcal{C} , we obtain the notions of a *cocone* and a *colimit*, which are dual to the notions a cone and a limit, respectively. Consequently, dual to the notions of a projective system and a projective limit of topological groups are the notions of a *direct system* and a *direct limit* of topological groups, respectively. More precisely, we have the following definition.

Definition 1.29 (See [2], pp. 108-109). Let $D : J \rightarrow \mathcal{C}$ be a diagram. For any object A in $\text{obj}(\mathcal{C})$, a *cocone* with vertex A and base D is a natural transformation $\alpha : D \rightarrow \text{const}(A)$, where $\text{const}(A) : J \rightarrow \mathcal{C}$ is the constant functor at A .

A cocone with vertex A and base D may be visualised by the following commutative diagram:

$$(1.37) \quad \begin{array}{ccc} & A & \\ \alpha_j \nearrow & & \nwarrow \alpha_j \\ D(j) & \xrightarrow{D(jk)} & D(k) \end{array}$$

Just as cones with natural transformations form a category, cocones with natural transformations also form a category. An *initial object* in this category, that is, an object having a unique arrow from it to every other object, is called a colimit. More precisely, we have the following definition.

Definition 1.30 (See [2], pp. 109). Given a diagram $D : J \rightarrow \mathcal{C}$. A colimit is a cocone $\alpha : D \rightarrow \text{const}(\text{colim}D)$ such that for any cocone $\alpha' : D \rightarrow \text{const}(A)$ with vertex A there is a morphism $\lambda : \text{colim}D \rightarrow A$ such that $\alpha'_j = \lambda \circ \alpha_j$ for all $j \in J$.

By varying the category J , and the diagram $D : J \rightarrow \mathcal{C}$, various limits may be obtained. For instance, let J be a discrete category, i.e., a category where the only morphisms are the identity morphisms. Then $\lim D$ satisfies the universal property of products, therefore $\lim D = \prod_{j \in J} D(j)$. On the other hand, $\text{colim}D$ satisfies the

universal property of coproducts, therefore $\text{colim}D = \coprod_{j \in J} D(j)$. More limits such

as pullbacks, and pushouts can be obtained similarly. See [13, pp. 779 - 780] for details.

Example 1.31 (See [13], pp.726). Let p be a prime number, and consider the discrete compact group $\frac{1}{p^n}\mathbb{Z}/\mathbb{Z}$ as in Example 1.19. Note that for any $m \in \mathbb{Z}$ we have $\frac{m}{p} + \mathbb{Z} = \frac{pm}{p^{n+1}} + \mathbb{Z}$, therefore $\frac{1}{p^n}\mathbb{Z}/\mathbb{Z} \subseteq \frac{1}{p^{n+1}}\mathbb{Z}/\mathbb{Z}$, and hence the inclusion

$$(1.38) \quad \varphi_n : \frac{m}{p^n} + \mathbb{Z} \in \frac{1}{p^n}\mathbb{Z}/\mathbb{Z} \mapsto \frac{m}{p^n} + \mathbb{Z} \in \frac{1}{p^{n+1}}\mathbb{Z}/\mathbb{Z}$$

is a well-defined homomorphism of compact groups. For any $i, j \in \mathbb{N}$ with $i \leq j$, define a homomorphism $f_{ij} : \frac{1}{p^i}\mathbb{Z}/\mathbb{Z} \rightarrow \frac{1}{p^j}\mathbb{Z}/\mathbb{Z}$ of compact groups by

$$(1.39) \quad f_{ij} = \varphi_{j-1} \circ \varphi_{j-2} \circ \dots \circ \varphi_{i+1} \circ \varphi_i.$$

Then the groups $\frac{1}{p^j}\mathbb{Z}/\mathbb{Z}$, together with the homomorphisms f_{ij} , $i, j \in \mathbb{N}, i \leq j$, form a direct system of groups, visualised in the diagram

$$(1.40) \quad 0 \longrightarrow \frac{1}{p}\mathbb{Z}/\mathbb{Z} \xrightarrow{f_{12}} \frac{1}{p^2}\mathbb{Z}/\mathbb{Z} \xrightarrow{f_{23}} \frac{1}{p^3}\mathbb{Z}/\mathbb{Z} \xrightarrow{f_{34}} \dots$$

Note that we can consider this directed system as a covariant diagram from the category $\mathbb{N} \cup \{0\}$ to the category $\mathbb{C}\mathbb{A}\mathbb{B}$, and its colimit or direct limit is denoted $\mathbb{Z}(p^\infty)$ and it is called the *Prüfer group*.

We end with some relevant examples of limits and topological groups.

Definition 1.32 (See [25], Definition 12.2.2). Let X be a nonempty set. A collection \mathcal{F} of subsets of X is called a *filter* if

- (1) $B \supseteq A \in \mathcal{F}$ implies that $B \in \mathcal{F}$.
- (2) $A_1, \dots, A_n \in \mathcal{F}$ implies that $A_1 \cap \dots \cap A_n \in \mathcal{F}$.

Example 1.33. In a topological group G , for any $g \in G$, a local basis of g is a filter.

The importance of the projective limit construction for compact groups is understood through [13, Proposition 1.33], where we see that a compact group G has a filter $\mathcal{U}(G)$ of compact normal subgroups that converges to $\{1\}$, and this allows an approximation of G via quotients of G , that is, G is isomorphic as a compact group to $\lim_{N \in \mathcal{U}(G)} G/N$. In fact, the following filters of compact groups are of special interest.

$$(1.41) \quad \mathcal{P}(G) = \{N = \overline{N} \trianglelefteq G \mid G/N \text{ is a finite } p\text{-group}\}$$

$$(1.42) \quad \mathcal{F}(G) = \{N = \overline{N} \trianglelefteq G \mid G/N \text{ is finite}\}$$

$$(1.43) \quad \mathcal{N}(G) = \{N = \overline{N} \trianglelefteq G \mid G/N \text{ is a compact Lie group}\}$$

Note that $N = \overline{N}$ abbreviates the fact that N is a closed subgroup, while $N \trianglelefteq G$ abbreviates the fact that N is normal in G . Note also that G/N is a finite p -group means that its cardinality $|G/K| = p^{k_N}$ for some $k_N \in \mathbb{N}$, depending on N . Notice that $\mathcal{P}(G) \subseteq \mathcal{F}(G) \subseteq \mathcal{N}(G)$. With the above notations, a compact group G is called a *pro- p -group* if $G \cong \lim_{N \in \mathcal{P}(G)} G/N$, it is called a *profinite group* if $G \cong \lim_{N \in \mathcal{F}(G)} G/N$, see [13, p.23] for details.

1.2. Background on Characters and Duality. It is a well-known fact of linear algebra that every finite dimensional vector space is isomorphic to its algebraic dual. It is also isomorphic to its double dual through an isomorphism that does not depend on any basis (see [11, pp.25]). We say that a finite dimensional vector space is naturally isomorphic to its double dual; this property is known as *reflexivity*. Reflexivity is also exhibited by abelian groups, where a strong interplay between abelian groups and compact abelian groups is observed.

Definition 1.34 (See [13], Definition 1.15). Let G be an abelian group. With respect to pointwise operations and the product topology, the set

$$(1.44) \quad \text{Hom}(G, \mathbb{T}) = \{\chi : G \rightarrow \mathbb{T} \mid \chi \text{ is an abstract homomorphism}\} \subseteq \mathbb{T}^G$$

is an abelian topological group called the *character group* of G . It is written \widehat{G} and its elements are called *characters* of G .

A few examples of character groups are here.

Example 1.35 (See [13], pp. 11).

- (1) $\widehat{\mathbb{Z}} \cong \mathbb{T}$.
- (2) $\widehat{\mathbb{Z}(n)} \cong \mathbb{Z}(n)$.

We can calculate character groups of larger spaces from smaller ones. In particular, for a family $G_j, j \in J$, of compact abelian groups, $\widehat{\bigoplus_{j \in J} G_j} \cong \prod_{j \in J} \widehat{G_j}$, by [13,

Proposition 1.17].

Characters of compact abelian groups are defined in the obvious way as follows.

Definition 1.36 (See [13], Definition 1.22). Let G be a compact abelian group. The space

$$(1.45) \quad \text{Hom}(G, \mathbb{T}) = \{\chi : G \rightarrow \mathbb{T} \mid \chi \text{ is a continuous homomorphism}\} \subseteq \mathbb{T}^G$$

of continuous homomorphisms of G to \mathbb{T} is an abstract abelian group with respect to pointwise operations. We call this abelian group the *character group* of G and its elements are called *characters* of G .

Definition 1.37 (See [13], Lemma 1.23(i),(ii)). Let A be an abelian group. The map $\eta_A : A \rightarrow \widehat{\widehat{A}}$ which maps a in A to $\eta_A(a) : \widehat{A} \rightarrow \mathbb{T}$, which is in turn given by

$$(1.46) \quad \chi \in \widehat{\widehat{A}} \mapsto \eta_A(a)(\chi) = \chi(a) \in \mathbb{T}$$

for all $\chi \in \widehat{\widehat{A}}$ is called the evaluation map.

For an abelian group G , $\eta_G : G \rightarrow \widehat{\widehat{G}}$ is an injective homomorphism of groups since

$$(1.47) \quad \ker \eta_G = \{\chi \in \widehat{\widehat{G}} \mid \chi(g) = 0 \text{ for all } g \in G\} = \{0\}$$

When G is compact, η_G is an injective homomorphism of compact groups (see [13, Lemma 1.23]). If in addition, η_G is an isomorphism (either of abstract groups or compact groups) we say that G has *duality*. In fact, any abstract abelian group has duality, by [13, Theorem 1.37], and any compact abelian group has duality, by [13, Theorem 2.32], although the latter is much harder to prove. This duality of abelian groups, and compact abelian groups is known as *Pontryagin duality*, and it holds even when G is locally compact as illustrated below.

Theorem 1.38 (Pontryagin-Van Kampen Duality, See [13], Theorem 7.63). *Every locally compact abelian group is reflexive, i.e., for any locally compact abelian group G the evaluation map $\eta_G : G \rightarrow \widehat{\widehat{G}}$ is an isomorphism of topological groups.*

Duality of compact groups can be used to calculate character groups of strict projective limits of compact abelian groups, as illustrated by the following proposition.

Proposition 1.39 (See [13], Proposition 1.36, Exercise E.14). *If $G = \lim_{j \in J} G_j$ is a strict projective limit of compact abelian groups G_j , then $\widehat{G} = \bigcup_{j \in J} \widehat{G_j}$. Moreover, if each of the G_j 's has duality then G has duality.*

As an application of Proposition 1.39 consider the following example.

Example 1.40 (See [13], Example 1.38). The group \mathbb{Z}_p of p -adic integers is a strict projective limit of compact groups $\frac{1}{p^n}\mathbb{Z}/\mathbb{Z}$, each of which having duality. Therefore, \mathbb{Z}_p has duality, by Proposition 1.39. Furthermore, $\mathbb{Z}_p = \lim_{n \in \mathbb{N}} \frac{1}{p^n}\mathbb{Z}/\mathbb{Z}$, so Proposition

1.39 implies that $\widehat{\mathbb{Z}_p} = \bigcup_{n \in \mathbb{N}} \widehat{\frac{1}{p^n}\mathbb{Z}/\mathbb{Z}}$. Now, $\frac{1}{p^n}\mathbb{Z}/\mathbb{Z}$ has duality for all $n \in \mathbb{N}$, therefore

$$\bigcup_{n \in \mathbb{N}} \widehat{\frac{1}{p^n}\mathbb{Z}/\mathbb{Z}} = \bigcup_{n \in \mathbb{N}} \frac{1}{p^n}\mathbb{Z}/\mathbb{Z} = \mathbb{Z}(p^\infty).$$

Roughly, another way of seeing that $\widehat{\mathbb{Z}_p} = \mathbb{Z}(p^\infty)$ is to observe that the strict projective system

$$(1.48) \quad 0 \longleftarrow \frac{1}{p}\mathbb{Z}/\mathbb{Z} \xleftarrow{f_{12}} \frac{1}{p^2}\mathbb{Z}/\mathbb{Z} \xleftarrow{f_{23}} \frac{1}{p^3}\mathbb{Z}/\mathbb{Z} \xleftarrow{f_{34}} \dots$$

dualises to become a direct system

$$(1.49) \quad 0 \longrightarrow \frac{1}{p}\mathbb{Z}/\mathbb{Z} \xrightarrow{f'_{12}} \frac{1}{p^2}\mathbb{Z}/\mathbb{Z} \xrightarrow{f'_{23}} \frac{1}{p^3}\mathbb{Z}/\mathbb{Z} \xrightarrow{f'_{34}} \dots,$$

where the map $f'_{nn+1} : \frac{1}{p^n}\mathbb{Z}/\mathbb{Z} \rightarrow \frac{1}{p^{n+1}}\mathbb{Z}/\mathbb{Z}$ is the inclusion. So the colimit of this directed system is the Prüfer group $\mathbb{Z}(p^\infty)$, defined in Example 1.31. Note also that duality preserves compactness in this case. However, from $\widehat{\mathbb{Z}} = \mathbb{T}$ and $\widehat{\mathbb{T}} = \mathbb{Z}$ we easily understand that this is not always true.

Example 1.41 (See [13], Exercise 1.38). Denote by $\frac{1}{p^\infty}\mathbb{Z}$, the subgroup of \mathbb{Q} consisting of all rational numbers of the form $\frac{m}{p^k}$, for some $m \in \mathbb{Z}, k \in \mathbb{N}$. Define

$$(1.50) \quad \varphi_n : \frac{m}{p^k} + p^{n+1}\mathbb{Z} \in \frac{1}{p^\infty}\mathbb{Z} \mapsto \frac{m}{p^k} + p^n\mathbb{Z} \in \frac{1}{p^\infty}\mathbb{Z}.$$

This defines a projective system of topological groups, visualised in the following diagram

$$(1.51) \quad 0 \longleftarrow \frac{1}{p\mathbb{Z}} \xleftarrow{\frac{1}{p^\infty}\mathbb{Z}} \frac{1}{p^2\mathbb{Z}} \xleftarrow{\frac{1}{p^\infty}\mathbb{Z}} \frac{1}{p^3\mathbb{Z}} \xleftarrow{\dots},$$

whose limit $\lim_{n \in \mathbb{N}} \frac{1}{p^\infty}\mathbb{Z} = \mathbb{Q}_p$ is called the group of p -adic rationals. It is a torsion-free locally compact abelian group containing \mathbb{Z}_p . Here $\widehat{\mathbb{Q}_p} = \mathbb{Q}_p$, as it happens for $\widehat{\mathbb{R}} = \mathbb{R}$ and $\widehat{\mathbb{Q}} = \mathbb{Q}$.

We now mention precisely how compact Lie groups are building blocks of compact groups.

Theorem 1.42 (See [13], Corollary 2.43). *Every compact group is a strict projective limit of compact Lie groups.*

In other words, with the notation established in (1.43), we have $G = \lim_{N \in \mathcal{N}(G)} G/N$ for any compact group G . Theorem 1.42 gives the topological structure of a compact group. However, it says very little about its group theoretic structure. For this we turn to another structure theorem of compact groups. To that end, let us first

recall some preliminary notions. Recall that the centre of a compact group G is the subgroup

$$(1.52) \quad Z(G) = \{g \in G : gx = xg \text{ for all } x \in G\}$$

which is always a closed normal subgroup, by [15, Proposition 9]. The commutator subgroup of G is the subgroup

$$(1.53) \quad G' = \langle ghg^{-1}h^{-1} \mid g, h \in G \rangle = \langle [g, h] \mid g, h \in G \rangle$$

which is not always a closed subgroup of G , so we take $\overline{G'}$ when we deal with compact groups, where $\overline{G'}$ denotes, as usual, the closure of G' in G . Note that if G is a compact Lie group, then G' is closed, by [13, Theorem 6.11]. The connected component of the identity of G is denoted by G_0 . According to [15, Proposition], G_0 is a closed normal subgroup of G .

For a compact group G , let $G_A = G/\overline{G'}$, and let $\pi : Z(G) \rightarrow G_A$ be the restriction of the quotient morphism to the center $Z(G)$, and $\theta : g\overline{G'} \in G_A \mapsto gZ(G)\overline{G'} \in G/(Z(G)\overline{G'})$. Then we have the following theorem due to Lie-Kolchin and Mal'cev.

Theorem 1.43 (See [13], Theorem 9.23). *For a compact group G :*

- (1) $G_0 \subseteq Z(G)\overline{G'}$, and $Z(G) \cap \overline{G'}$ is totally disconnected.
- (2) There are injective homomorphism δ and surjective homomorphism μ of compact groups such that

$$(1.54) \quad 1 \longrightarrow Z_0(G) \cap (\overline{G'})_0 \xrightarrow{\delta} Z_0(G) \times (\overline{G'})_0 \xrightarrow{\mu} G_0 \longrightarrow 1$$

satisfies $\ker \mu \subseteq \text{Im } \delta$, that is, (1.54) is a short exact sequence.

- (3) We have $(G_A)_0 \cong G_0\overline{G'}/\overline{G'} = Z_0(G)\overline{G'}/\overline{G'} \cong Z_0(G)/(Z_0(G) \cap \overline{G'})$, and there are exact sequences:

$$(1.55) \quad \begin{array}{ccccccc} 0 & \longrightarrow & Z_0(G) \cap \overline{G'} & \xrightarrow{\text{incl}} & Z_0(G) & \xrightarrow{\pi|_{Z_0(G)}} & G_A & \longrightarrow & G_A / (G_A)_0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \parallel & & \downarrow & & \\ 0 & \longrightarrow & Z(G) \cap \overline{G'} & \xrightarrow{\text{incl}} & Z(G) & \xrightarrow{\pi} & G_A & \longrightarrow & G / (Z(G)\overline{G'})_0 & \longrightarrow & 0 \end{array}$$

2. THE TENSOR PRODUCT OF ABELIAN GROUPS

By an abstract abelian group we mean an abelian group without a topology. In this section we shall primarily focus on abstract abelian groups, which together with abstract homomorphisms of abelian groups form the category $\mathbb{A}\mathbb{B}$.

Definition 2.1 (See [13] p. 695). Let A, B, C in $\mathbb{A}\mathbb{B}$ be abstract abelian groups. A map $f : A \times B \rightarrow C$ is a *bilinear map* if

$$(2.1) \quad f(a_1 + a_2, b) = f(a_1, b) + f(a_2, b)$$

and

$$(2.2) \quad f(a, b_1 + b_2) = f(a, b_1) + f(a, b_2),$$

for all $a, a_1, a_2 \in A, b, b_1, b_2 \in B$.

For abstract abelian groups A and B it can be useful to consider, if it exists, an abelian group $A \otimes B$ with the property that every bilinear map from $A \times B$ to C extends uniquely to a linear map from $A \otimes B$ to C . This section presents information about such a group, called the tensor product of *abstract* abelian groups. We are primarily concerned with the existence of the group $A \otimes B$ as well as some of its properties. To that end, some basic notions and results regarding *abstract* abelian groups are useful.

2.1. Free Abstract Abelian Groups. To construct $A \otimes B$ we need the notion of a free abelian group over an arbitrary set X , which in turn requires the notion of a direct sum of abelian groups defined below.

Definition 2.2 (See [13], Definition A1.3). Let $A_{j \in J}$ in $\mathbb{A}\mathbb{B}$ be a family of abstract abelian groups over an index set J .

(1) The *direct product*

$$(2.3) \quad \prod_{j \in J} A_j = \left\{ f : J \rightarrow \bigcup_{j \in J} A_j \mid f(j) \in A_j \right\}$$

is an abelian group under pointwise operations.

(2) The subgroup

$$(2.4) \quad \bigoplus_{j \in J} A_j = \left\{ f \in \prod_{j \in J} A_j \mid f(j) = 0 \text{ for all but finitely many } j \in J \right\} \subseteq \prod_{j \in J} A_j$$

is called the *direct sum* of $A_{j \in J}$. Each A_j is called a *direct summand*.

When $A_j = A$ for all $j \in J$, $\prod_{j \in J} A_j$ coincides with A^J , the set of all functions from J to A . In this case, $\bigoplus_{j \in J} A_j$ is written $A^{(J)}$.

When A_j is a finite family of abelian groups, that is, when the cardinality $|J|$ of J is finite, then $\prod_{j \in J} A_j = \bigoplus_{j \in J} A_j$. On the other hand, when J is infinite, we have,

for instance, according to [13, pp. 474], that the group $\mathbb{Z}^{\mathbb{N}}/\mathbb{Z}^{(\mathbb{N})}$ always contains a set of cardinality at least $|\mathbb{R}|$. In particular, it contains a copy of \mathbb{Z}_p as a direct summand, for each prime p . Note that it is possible to look at Definition 2.2(1),(2) as special cases of Definition 1.28, i.e., as limits of some appropriate diagram.

Example 2.3 (See [13], pp. 709).

- (1) For any set X , $\mathbb{Z}^{(X)}$ and \mathbb{Z}^X are abelian groups.
- (2) Denote by $\mathbb{Z}(n) = \langle x \mid n \cdot x = 0 \rangle$ the cyclic group of order n , then $\mathbb{Z}(n) = \bigoplus_{i=1}^k \mathbb{Z}(p_i^{n_i})$ where $n = p_1^{n_1} \dots p_k^{n_k}$ is the unique prime factorisation of n (Structure of Finite Abelian Groups, see [13, Exercise EA1.5]).

Definition 2.4 (See [13] A1.4). Let X denote a set and $F(X)$ in $\mathbb{A}\mathbb{B}$ an abstract abelian group together with a function $j : X \rightarrow F(X)$. We say that $F(X)$ is a *free abstract abelian group* over the set X if for every function $f : X \rightarrow A$ into an abstract abelian group A in $\mathbb{A}\mathbb{B}$ there is a unique abstract homomorphism $f' : F(X) \rightarrow A$ such that $f = f' \circ j$.

The condition of Definition 2.4 is summarised by the commutativity of the following diagram in $\mathbb{A}\mathbb{B}$

$$(2.5) \quad \begin{array}{ccc} X & \xrightarrow{j} & F(X) \\ f \downarrow & \swarrow f' & \\ A & & \end{array}$$

By observing that the category $\mathbb{A}\mathbb{B}$ of abstract abelian groups is simply the category $\mathbb{A}\mathbb{B}_{\mathbb{Z}}$ of left \mathbb{Z} -modules, Definition 2.4 is simply the Universal Property of Free Functors in [13, Proposition A3.21]. The universal property of free abelian groups implies that up to isomorphism there is only one free abelian group over a set X . Moreover, [13, Proposition A3.21] suggests a construction of the free abstract abelian group over any set. In fact, we have the following proposition.

Proposition 2.5 (See [13], Proposition A1.6). *The group $\mathbb{Z}^{(X)}$ is the free abstract abelian group over a given set X .*

The importance of free abstract abelian groups is clear through the following classical result.

Proposition 2.6 (See [27], 2.3.6). *Every abstract abelian group is a quotient of a free abstract abelian group over some set.*

Subgroups of free abstract abelian groups are themselves free, by [13, Theorem A1.9], but their quotients may not be free as shown by $\mathbb{Z}(n) = \mathbb{Z}/n\mathbb{Z}$. In fact, by Proposition 2.5 we readily see that no nontrivial finite abstract abelian group is free. Put another way; the category of finite abstract abelian groups does not have all free objects. With this we encounter perhaps the simplest example of a category without all free objects.

Free abstract abelian groups are a special case of a more general notion of free groups in a variety of abstract groups, where free objects always exist. Firstly, recall that for a set X , a word w in X is a finite sequence of symbols $x_1^{l_1} \dots x_r^{l_r}$, where $l_i \in \mathbb{Z}$. If G is an abstract group, not necessarily abelian, then for any word $w = x_1^{l_1} \dots x_r^{l_r}$ in X , and any sequence g_1, \dots, g_r of elements in G , the *value* of w at g_1, \dots, g_r is the element $w(g_1, \dots, g_r) = g_1^{l_1} \dots g_r^{l_r}$. If W is a set of words in X , then the subgroup

$$(2.6) \quad W(G) = \langle w(g_1^{l_1} \dots g_r^{l_r}) \mid g_i \in G, w = x_1^{l_1} \dots x_r^{l_r} \in W \rangle$$

of G is called the *verbal* subgroup of G determined by W .

Example 2.7. For any abstract group G , the derived subgroup $G' = \langle [g, h] \mid g, h \in G \rangle$ is the verbal subgroup of $W = \{xyx^{-1}y^{-1}\}$.

According to [27, pp. 58], if W is a set of words in x_1, x_2, \dots , then the class $\mathfrak{B}(W)$ of all groups G such that $W(G) = 1$ is called a *variety* determined by W . Abstract abelian groups are precisely the variety $\mathfrak{B}(W)$ determined by $W = \{xyx^{-1}y^{-1}\}$ since an abstract group G is abelian if and only if $[g, h] = 1$ for all $g, h \in G$.

Definition 2.8 (See [27], pp.60). Let X be a set and F an abstract group in a variety \mathfrak{B} , and $j : X \rightarrow F$ a function. Then F is said to be \mathfrak{B} -free on X if for each function $f : X \rightarrow G$ to a \mathfrak{B} group G , there exists a homomorphism $f' : F \rightarrow G$, unique in making the diagram

$$(2.7) \quad \begin{array}{ccc} X & \xrightarrow{j} & F \\ f \downarrow & \swarrow f' & \\ G & & \end{array}$$

is commutative, i.e. $f = f' \circ j$.

When \mathfrak{B} is the variety of abstract abelian groups, then the \mathfrak{B} -free groups are exactly the free abelian groups.

The notion of variety of groups was first studied by S.A. Morris in [20, 21, 22] for topological groups and turns out to be slightly different from the notion of varieties of abstract groups that is primarily due to H. Neumann. Even the problem of the existence of free topological groups is not exactly the same when we work in topological group theory.

Now we formalise what we have seen already in Theorem 1.43, but in $\mathbb{A}\mathbb{B}$.

Definition 2.9 (See [31], pp. 22). Given $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$, where A, B, C are in $\mathbb{A}\mathbb{B}$ and α, β are homomorphisms in $\mathbb{A}\mathbb{B}$. We say

- (1) $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ is *exact* at B if $\text{Im } \alpha = \ker \beta$.
- (2) a sequence $0 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 0$ of abstract abelian groups and abstract homomorphisms is a *short exact sequence* if α is injective, β is surjective, and $\text{Im } \alpha = \ker \beta$.

Of course Definition 2.9 may be reformulated in $\mathbb{K}\text{TOPG}$ assuming the additional condition that the abstract homomorphisms which occur in Definition 2.9 are continuous.

Example 2.10 (See [31], pp.23).

- (1) For any natural number n , $0 \rightarrow n\mathbb{Z} \xrightarrow{j} \mathbb{Z} \xrightarrow{p} \mathbb{Z}(n) \rightarrow 0$, where j is the inclusion and p is the quotient morphism, is a short exact sequence.
- (2) For any natural number n , $0 \rightarrow n\mathbb{Z} \xrightarrow{j} n\mathbb{Z} \oplus \mathbb{Z}(n) \xrightarrow{p} \mathbb{Z}(n) \rightarrow 0$, where j is the coprojection, and p is the projection, is a short exact sequence.
- (3) See Theorem 1.43 for an example of a short exact sequence in $\mathbb{K}\text{TOPG}$.

In light of Definition 2.9, Proposition 2.6 says that for any abstract abelian group A there is a short exact sequence

$$(2.8) \quad 0 \rightarrow R \rightarrow F(A) \rightarrow A \rightarrow 0,$$

where R is such that $F(A)/R \cong A$. Such a short exact sequence is called a *presentation* of A in $\mathbb{A}\mathbb{B}$. Again, we may discuss topological presentations which are not algebraic presentations but we do not mention this theory here. See [1, 8] for more for details on topological presentations of locally compact groups and in particular [23, 24, 34] for the structure of finitely generated profinite groups.

Definition 2.11. We say that a sequence $0 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 0$ of groups A, B, C in $\mathbb{A}\mathbb{B}$ and homomorphisms α, β in $\mathbb{A}\mathbb{B}$ is an *extension* of A by C if it is a short exact sequence. In this case we also say that B is an extension of A by C .

If $0 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 0$ is an extension, then $B/A \cong C$. Therefore, an extension B of A by C tells us that an abstract group B may be constructed from abstract groups A and C . From Example 2.10 we see two nonisomorphic extensions, \mathbb{Z} and $n\mathbb{Z} \oplus \mathbb{Z}(n)$, of the same groups $n\mathbb{Z}$ and $\mathbb{Z}(n)$. Therefore, extensions constructed from the same groups may not be the same.

Two extensions

$$(2.9) \quad 0 \longrightarrow A \xrightarrow{j_1} B_1 \xrightarrow{p_1} C \longrightarrow 0$$

and

$$(2.10) \quad 0 \longrightarrow A \xrightarrow{j_2} B_2 \xrightarrow{p_2} C \longrightarrow 0$$

in $\mathbb{A}\mathbb{B}$ are said to be *equivalent* if there exists an isomorphism $f : B_1 \rightarrow B_2$ of abelian groups such that the diagram

$$(2.11) \quad \begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{j_1} & B_1 & \xrightarrow{p_1} & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ & & & & 1_A & & f & & 1_C \\ 0 & \longrightarrow & A & \xrightarrow{j_2} & B_2 & \xrightarrow{p_2} & C & \longrightarrow & 0 \end{array}$$

is commutative.

This defines an equivalence relation on the collection of all extensions of A by C . For details see [13, Page 737]. The set $\text{Ext}(C, A)$ of equivalence classes of extensions of A by C can be made into an abelian group by defining the sum of two extensions (2.9) and (2.10) to be the extension

$$(2.12) \quad 0 \longrightarrow A \xrightarrow{j} B \xrightarrow{p} C \longrightarrow 0,$$

where

$$(2.13) \quad B = \{(b_1, b_2) + K \mid p_1(b_1) = p_2(b_2)\}, b_1 \in B_1, b_2 \in B_2.$$

and

$$(2.14) \quad K = \{(j_1(a), -j_2(a)) \mid a \in A\},$$

and $j : A \rightarrow B$ is given by

$$(2.15) \quad j(a) = (j_1(a), 0) + K,$$

while $p : B \rightarrow C$ is given by

$$(2.16) \quad p((b_1, b_2) + K) = p_1(b_1).$$

This is the well-known ‘‘Baer sum’’ in [18, Chapter III]. More details can be found in [13, pp.737] and [18, p.68 - 72]. In particular, we note that with the Baer sum, the set $\text{Ext}(C, A)$ becomes an abelian group (see [13, Lemma A1.50]).

Ext is indeed a functor (see [13, Lemma A1.52]) and it plays a fundamental role in homology theory, but we do not enter the details here. We just mention that Ext can be used to characterise freeness of abstract abelian groups through the following theorem due to Pontryagin.

Theorem 2.12 (Pontryagin, See [13], Theorem A1.62). *A countable abstract abelian group G is free if and only if $\text{Ext}(G, \mathbb{Z}) = 0$*

See [13, Theorem A1.62] for a proof of Theorem 2.12. It is interesting to consider whether Ext can characterise freeness of all abstract abelian groups, that is, even those that are not countable, and we have a proper terminology for this.

Definition 2.13 (See [13], Definition A1.64). An abstract abelian group G in $\mathbb{A}\mathbb{B}$ is called a *Whitehead* group if $\text{Ext}(G, \mathbb{Z}) = 0$.

Theorem 2.14 (See [13]). *If G in $\mathbb{A}\mathbb{B}$ is an abstract abelian group, then $\text{Ext}(G, \mathbb{Z}) = 0$ implies that G is free*

The details to demonstrate this fact are interesting but highly nontrivial and involve many advanced notions of axiomatic set theory, as well as ordinal and cardinal arithmetic, which we do not enter into here. Instead, we refer the reader to [13, pp.747-760] for a thorough treatment of this.

According to [27, pp. 311 - 312], every extension

$$(2.17) \quad 0 \longrightarrow A \xrightarrow{j} B \xrightarrow{p} C \longrightarrow 0$$

of abstract abelian groups induces an *outer automorphism* $\chi : C \rightarrow \text{Out}A = \text{Aut}A/\text{Inn}A$, called a *coupling*, where $\text{Aut}A$ is the group of automorphisms of A , and $\text{Inn}A$ the subgroup of $\text{Aut}A$ consisting of *inner automorphisms*, that is, those automorphisms of A that are induced by conjugation $a \in A \mapsto g^{-1}ag \in A$ for all $a \in A$ and some fixed $g \in A$. Since A is abelian, we have $\text{Inn}A = 0$, and so each coupling is in fact an *action* $\chi : C \rightarrow \text{Aut}A$.

Definition 2.15 (See [27], pp. 312). Given two abstract groups B and N , a *coupling* of B to N is an abstract homomorphism $\chi : B \rightarrow \text{Out}N$.

According to [27, 11.1.1], equivalent extensions (2.9) and (2.10) induce the same coupling. Next we explore some interactions between the theory of extensions and cohomology via the notion of coupling.

Definition 2.16 (See [13], Definition A1.13). An abstract abelian group P in $\mathbb{A}\mathbb{B}$ is *projective* if for every surjective abstract homomorphism $f : A \rightarrow B$ of abstract abelian groups A and B in $\mathbb{A}\mathbb{B}$ and every abstract homomorphism $p : P \rightarrow B$, there is an abstract homomorphism $h : P \rightarrow A$ such the diagram

$$(2.18) \quad \begin{array}{ccc} P & & \\ \downarrow h & \searrow p & \\ A & \xrightarrow{f} & B \end{array}$$

is commutative, i.e., $p = f \circ h$.

Freeness is related to projectivity in $\mathbb{A}\mathbb{B}$. In fact, we have the following proposition.

Proposition 2.17 (See [13], Proposition A1.14). *An abstract abelian group A in $\mathbb{A}\mathbb{B}$ is projective if and only if it is a free abstract abelian group.*

In other words, the projective abstract abelian groups are of the form $\mathbb{Z}^{(X)}$ for some set X . We also define projective objects in $\mathbb{C}\mathbb{A}\mathbb{B}$ in exactly the same way as in Definition 2.16, making obvious adjustments. It turns out that the projective compact abelian groups are exactly those that are torsion-free (see [13, Theorem 8.78]). So projectivity does not fully characterise freeness in $\mathbb{C}\mathbb{A}\mathbb{B}$, but it does characterise it to some extent (see [13, Proposition 8.79]).

Definition 2.18 (See [27], pp. 326). A *complex* \mathbf{C} in $\mathbb{A}\mathbb{B}$ is a sequence

$$(2.19) \quad \dots \longrightarrow C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \dots$$

$n \in \mathbb{Z}$ such that $\text{Im } d_{n+1} \subseteq \ker d_n$. If $C_n = 0$ for all $n < 0$, then \mathbf{C} is called a positive complex, or a group exact sequence.

For a complex \mathbf{C} , the abstract abelian groups

$$(2.20) \quad H_n \mathbf{C} = \ker d_n / \text{Im } d_{n+1}$$

are called *homology* groups. Each of them measures exactness at C_n in the complex. Clearly, a complex is exact if and only if each of its homology groups is trivial.

Definition 2.19 (See [27], pp.329). A positive complex \mathbf{C} in $\mathbb{A}\mathbb{B}$, together with a surjective homomorphism of abstract abelian groups $d_0 : C_0 \rightarrow A$, such that

$$(2.21) \quad \dots \longrightarrow C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} A$$

is exact is called a *resolution* of A . It is called a *projective resolution* of A if C_n is projective for all $n \in \mathbb{N} \cup \{0\}$. We write $\mathbf{C} \rightarrow A$ for a resolution of A .

By recursively extending a presentation $0 \rightarrow R \rightarrow F \rightarrow A \rightarrow 0$ in $\mathbb{A}\mathbb{B}$, where F is a free abstract abelian group such that $F/R \cong A$, we can show the following proposition.

Proposition 2.20 (See [27], 11.2.3). *Every abstract abelian group A in $\mathbb{A}\mathbb{B}$ has a projective resolution.*

For any abstract abelian group M in \mathbf{C} , and a projective resolution $\mathbf{C} \rightarrow A$ in $\mathbb{A}\mathbb{B}$, take $\text{Hom}(\mathbf{C}, M)$, that is,

$$(2.22) \quad \dots \longrightarrow \text{Hom}(C_{n-1}, M) \xrightarrow{d'_n} \text{Hom}(C_n, M) \xrightarrow{d'_{n+1}} \text{Hom}(C_{n+1}, M) \xrightarrow{d'_{n+2}} \dots,$$

where $d'_n : \text{Hom}(C_{n-1}, M) \rightarrow \text{Hom}(C_n, M)$ is given by $d'_n(f) = f \circ d_n$ for all $n \in \mathbb{N}$. The abstract abelian group

$$(2.23) \quad H^n(A, M) = H_n(\text{Hom}(\mathbf{C}, M)) = \frac{\ker d'_{n+1}}{\text{Im } d'_n}$$

is called the n^{th} -*cohomology* group of A with coefficients in M . When $M = \mathbb{C}^\times$, the group

$$(2.24) \quad H^2(A, \mathbb{C}^\times)$$

is called the *Schur multiplier* of A . The Schur multiplier arises in the theory of projective representations which was initiated by Issai Schur, and has since been studied in greater generality yielding various ways of defining it. It is related to the theory of extensions through the following theorem.

Theorem 2.21 (See [27], 11.4.10). *Let $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be an extension with coupling $\chi : C \rightarrow \text{Aut}A$, where A, B, C in $\mathbb{A}\mathbb{B}$ are abstract abelian groups. Then there is a bijective correspondence between the equivalence classes of extensions of A by C with coupling χ and $H^2(C, A)$.*

In particular, if G is a finite group, we mention a classical result:

Theorem 2.22 (Schur-Zassenhaus, see [27]). *If a finite group G possesses N normal subgroup such that $|N|$ and $|G : N|$ are coprime, then G contains a subgroup H of order equal to $|G : N|$ such that $H \cap N = 1$ and $G = NH$.*

One of the main steps of the proof of the above results is to check that in the assumptions of the theorem G has trivial $H^2(G, \mathbb{C}^\times)$. Therefore information on the size of the Schur multiplier in a finite group yields information on the structure of the group.

2.2. The Tensor Product of Abstract Abelian Groups. In search of the abelian group $A \otimes B$ postulated at the beginning of this section, Proposition 2.6 is suggestive of how it can be constructed. Specifically, we would like to realise $A \otimes B$ as an appropriate quotient of some free abstract abelian group. Right away, we have the following proposition.

Proposition 2.23 (See [13], Proposition A1.44). *Let A, B in $\mathbb{A}\mathbb{B}$ be abstract abelian groups. Then*

- (1) *there exists an abstract abelian group $A \otimes B$ in $\mathbb{A}\mathbb{B}$, unique up to isomorphism, with a bilinear map $\varepsilon : (a, b) \in A \times B \mapsto a \otimes b \in A \otimes B$ such that for every bilinear map $f : A \times B \rightarrow C$ to an abstract abelian group C in $\mathbb{A}\mathbb{B}$ there exists an abstract homomorphism $f' : A \otimes B \rightarrow C$, unique in making the diagram*

$$(2.25) \quad \begin{array}{ccc} A \times B & \xrightarrow{\varepsilon} & A \otimes B \\ f \downarrow & \swarrow f' & \\ C & & \end{array}$$

commutative, i.e., $f'(a \otimes b) = f(a, b)$ for all $a \in A, b \in B$.

- (2) *The groups $\text{Hom}(A \otimes B, C)$, $\text{Hom}(A, \text{Hom}(B, C))$, and $\text{Hom}(B, \text{Hom}(A, C))$ are naturally isomorphic in $\mathbb{A}\mathbb{B}$ and are isomorphic to the group of all bilinear maps from $A \times B$ to C .*

Variation on the theme of the proof of Proposition A1.44 of [13]. (1) Define $e : A \times B \rightarrow \mathbb{Z}^{(A \times B)}$ by

$$(2.26) \quad e(a, b)(x, y) = \begin{cases} 1, & \text{if } (x, y) = (a, b) \\ 0, & \text{otherwise} \end{cases}$$

Let W be the subgroup of $\mathbb{Z}^{(A \times B)}$ that is algebraically generated by

$$(2.27) \quad e(a_1 + a_2, b) - e(a_1, b) - e(a_2, b), e(a, b_1 + b_2) - e(a, b_1) - e(a, b_2)$$

for all $a, a_1, a_2 \in A, b, b_1, b_2 \in B$. Since $\mathbb{Z}^{(A \times B)}$ is the free abstract abelian group over the set $A \times B$, there exists an abstract homomorphism $g : \mathbb{Z}^{(A \times B)} \rightarrow C$ unique in making the diagram

$$(2.28) \quad \begin{array}{ccc} A \times B & \xrightarrow{e} & \mathbb{Z}^{(A \times B)} \\ \downarrow f & \swarrow g & \\ C & & \end{array}$$

commutative, i.e., $f = g \circ e$. Since f is bilinear,

$$(2.29) \quad \begin{aligned} g(e(a_1 + a_2, b) - e(a_1, b) - e(a_2, b)) &= g(e(a_1 + a_2, b)) - g(e(a_1, b)) - g(e(a_2, b)) \\ &= f(a_1 + a_2, b) - f(a_1, b) - f(a_2, b) = 0 \end{aligned}$$

Similarly,

$$(2.30) \quad g(e(a, b_1 + b_2) - e(a, b_1) - e(a, b_2)) = 0$$

Hence g vanishes on W . Let $p : \mathbb{Z}^{(A \times B)} \rightarrow \mathbb{Z}^{(A \times B)}/W$ be the quotient homomorphism given by

$$(2.31) \quad p(\varphi) = \varphi + W$$

for all $\varphi \in \mathbb{Z}^{(A \times B)}$. Then there exists an abstract homomorphism $f' : \mathbb{Z}^{(A \times B)}/W \rightarrow C$ unique in making the diagram

$$(2.32) \quad \begin{array}{ccc} \mathbb{Z}^{(A \times B)} & \xrightarrow{p} & \mathbb{Z}^{(A \times B)}/W \\ \downarrow g & \swarrow f' & \\ C & & \end{array}$$

commutative, i.e., $g = p \circ f'$. Putting $\mathbb{Z}^{(A \times B)}/W = A \otimes B$ and $a \otimes b = p(e(a, b))$, it remains to show that f' is unique in making the diagram

$$(2.33) \quad \begin{array}{ccccc} A \times B & \xrightarrow{e} & \mathbb{Z}^{(A \times B)} & \xrightarrow{p} & \mathbb{Z}^{(A \times B)}/W \\ \downarrow f & & & \searrow f' & \\ C & & & & \end{array}$$

commutative. To that end, let $h : \mathbb{Z}^{(A \times B)}/W \rightarrow C$ be a homomorphism of abstract abelian groups such that $h \circ p \circ e = f = f' \circ p \circ e$. By the uniqueness of g in making 2.28 commutative, it follows that $h \circ p = g = f' \circ p$. Moreover, since p is surjective it follows that $h = f'$, as required. The uniqueness of $A \otimes B$ follows from applying the universal property of $A \otimes B$. In particular, suppose T is another abstract abelian group together with a bilinear map $j : A \times B \rightarrow T$ such that any bilinear map

$f : A \times B \rightarrow C$ induces an abstract homomorphism $f' : T \rightarrow C$ unique with respect to satisfying $f' \circ j = f$. Then apply the universal property of $A \otimes B$ to the bilinear map $j : A \times B \rightarrow T$, to obtain an abstract homomorphism $\beta_1 : A \otimes B \rightarrow T$ such that $\beta_1 \circ e = j$. Similarly, we obtain an abstract homomorphism $\beta_2 : T \rightarrow A \otimes B$ such that $\beta_2 \circ j = e$. Then $\beta_1 \circ \beta_2 \circ e = e$. However, the identity $\text{id}_{A \otimes B} : A \otimes B \rightarrow A \otimes B$ is unique in making the diagram

$$(2.34) \quad \begin{array}{ccc} A \times B & \xrightarrow{e} & A \otimes B \\ \downarrow e & \swarrow \text{id}_{A \otimes B} & \\ A \otimes B & & \end{array}$$

commutative. Therefore, $\beta_2 \circ \beta_1 = 1_{A \otimes B}$. Similarly, $\beta_1 \circ \beta_2 = 1_T$, and hence $A \otimes B \cong T$ as required.

(2) Notice that the universal property of tensor products can be rephrased to the statement that $\text{Hom}(A \otimes B, C)$ is isomorphic to the group of all bilinear maps from $A \times B$ to C . Furthermore, for each bilinear map $f : A \times B \rightarrow C$, and for each $a \in A$, the map $f_a : B \rightarrow C$ given by $f_a(b) = f(a, b)$ is in $\text{Hom}(B, C)$. Specifically, the map $a \in A \rightarrow f_a \in \text{Hom}(B, C)$ is a homomorphism. On the other hand, for each $\varphi \in \text{Hom}(A, \text{Hom}(B, C))$, the map $(a, b) \in A \times B \mapsto \varphi(a, b) \in C$ is bilinear, hence the result follows. \square

In practice, we seldom use the construction of the abelian group $A \otimes B$. What is essential is its defining property. In fact, proving some basic properties of the tensor product reduces to an exercise in using the properties (1) and (2) of Proposition 2.23. We record some of these properties without proof in the following proposition.

Proposition 2.24 (See [13], Proposition A1.47, [31] Lemma 2). *Let A, B, C in $\mathbb{A}\mathbb{B}$ be abstract abelian groups. Then*

- (1) $A \otimes \mathbb{Z} \cong A$ (*Absorption*)
- (2) $A \otimes B \cong B \otimes A$ (*Commutativity*)
- (3) $(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$ (*Associativity*)
- (4) $(A \oplus B) \otimes C \cong (A \otimes C) \oplus (B \otimes C)$ (*Distributivity*)

The method of proving the statements in Proposition 2.24 is an exercise in using the universal property of tensor products in the sense that one finds an appropriate bilinear map which then extends to the desired isomorphism.

Note that Proposition 2.24(4) can be generalised to

$$(2.35) \quad A \otimes \bigoplus_{j \in J} B_j \cong \bigoplus_{j \in J} (A \otimes B_j)$$

for any family B_j of abstract abelian groups. Also, we note in passing that Proposition 2.24 is essentially the statement that the category $\mathbb{A}\mathbb{B}$ of abstract abelian groups with tensor product \otimes is a commutative monoidal category (see [13, Definition A3.62]).

Recalling the notion of a covariant functor in Definition 1.23 we show the following example.

Example 2.25 (See [31], pp. 18-19). The tensor product of abstract abelian groups defines a covariant self-functor on the category $\mathbb{A}\mathbb{B}$ of abstract abelian groups.

To see this, fix an abstract abelian group $A \in \mathbb{A}\mathbb{B}$, and consider the maps

$$(2.36) \quad B \in \mathbb{A}\mathbb{B} \mapsto A \otimes B \in \mathbb{A}\mathbb{B},$$

and

$$(2.37) \quad f \in \text{Hom}(B, C) \mapsto f' \in \text{Hom}(A \otimes B, A \otimes C)$$

where f' is the unique homomorphism of abelian groups such that $f'(a \otimes b) = a \otimes f(b)$ for all $a \in A, b \in B$, whose existence is ensured by the universal property of tensor products (see Proposition 2.23). In fact, the functor $A \otimes |$ preserves coproducts. Recall the following notions of exactness of functors.

Definition 2.26 (See, [31], pp.24). Let $F : \mathbb{A}\mathbb{B}_R \rightarrow \mathbb{A}\mathbb{B}$ be a covariant functor, $0 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 0$ an exact sequence of R -modules and homomorphisms. We say that F

- (1) is *half exact* if $F(A) \xrightarrow{F(\alpha)} F(B) \xrightarrow{F(\beta)} F(C)$ is exact.
- (2) is *left exact* if $0 \rightarrow F(A) \xrightarrow{F(\alpha)} F(B) \xrightarrow{F(\beta)} F(C)$ is exact.
- (3) is *right exact* if $F(A) \xrightarrow{F(\alpha)} F(B) \xrightarrow{F(\beta)} F(C) \rightarrow 0$ is exact.
- (4) is *exact* if $0 \rightarrow F(A) \xrightarrow{F(\alpha)} F(B) \xrightarrow{F(\beta)} F(C) \rightarrow 0$ is exact.

Clearly a functor F is exact if and only if it is both left exact and right exact. More details on exactness of functors may be found in [31, 18]. In particular, from [31] we recall the following well-known statements.

Proposition 2.27 (See [31], Proposition 2.6, Proposition 2.7). *For any abstract abelian group G in $\mathbb{A}\mathbb{B}$*

- (1) *The tensor product of abstract abelian groups is right exact. That is, if*

$$(2.38) \quad 0 \longrightarrow A \xrightarrow{j} B \xrightarrow{p} C \longrightarrow 0$$

is a short exact sequence of abstract abelian groups A, B, C in $\mathbb{A}\mathbb{B}$, then

$$(2.39) \quad A \otimes G \xrightarrow{j \otimes 1} B \otimes G \xrightarrow{p \otimes 1} C \otimes G \longrightarrow 0$$

is an exact sequence.

- (2) *The covariant Hom functor of abstract abelian groups is left exact. Equivalently, if*

$$(2.40) \quad 0 \longrightarrow A \xrightarrow{j} B \xrightarrow{p} C \longrightarrow 0$$

is a short exact sequence of abstract abelian groups A, B, C in $\mathbb{A}\mathbb{B}$, then

$$(2.41) \quad 0 \longrightarrow \text{Hom}(A, G) \longrightarrow \text{Hom}(B, G) \longrightarrow \text{Hom}(C, G)$$

is an exact sequence.

Proposition 2.27(1), often referred to as the *right exactness property of tensor products* is important in homology for the computation of the integral homology groups. In particular, the right exactness property of tensor products is used in the proof of [27, 11.2.7], which is in turn used to compute the integral homology groups.

It is possible to measure how far the tensor product and covariant Hom functors may be from being exact. It turns out that the Ext functor we encounter in the theory of extensions may be used to measure how far the Hom functor is from being exact (see [13, Theorem A1.55]). For the tensor product functor, we show in the next section that the Tor functor which we study next does the job.

2.3. Torsion Subgroups and Tor Functor in Abstract Abelian Groups. In this section, we shall briefly describe the torsion subgroup, the Tor functor and depart the section with a look at the Universal Coefficient Theorem for homology. We begin by defining the torsion subgroup of an abstract abelian group as follows.

Definition 2.28 (See [13], Definition A1.16). If A is an abstract abelian group in $\mathbb{A}\mathbb{B}$, then

$$(2.42) \quad \text{tor}(A) = \{a \in A \mid n \cdot a = 0 \text{ for some } n \in \mathbb{N}\}$$

is called the *torsion subgroup* of A .

An abelian group A in $\mathbb{A}\mathbb{B}$ is *torsion-free* if $\text{tor}(A) = \{0\}$. It is called *torsion* if $\text{tor}(A) = A$. The torsion subgroup of an abstract abelian group is the subgroup consisting of all elements of finite order. Some torsion abstract abelian groups that we have encountered before in Section 1 are the Prüfer group $\mathbb{Z}(p^\infty)$ and all finite cyclic groups $\mathbb{Z}(n)$, $n \in \mathbb{N}$, and indeed all finite groups. The torsion-free abstract abelian groups that we have encountered include $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{Z}_p$. We record some obvious properties of $\text{tor}(A)$ in the following proposition.

Proposition 2.29 (See [13], Remark A1.7). *Let A be an abstract abelian group in $\mathbb{A}\mathbb{B}$, then*

- (1) *the quotient $A/\text{tor}(A)$ is torsion-free.*
- (2) *if B is a torsion-free abelian group and $f : A \rightarrow B$ is a homomorphism of abelian groups, then*
 - (i) $\text{tor}(A) \subseteq \ker f$.
 - (ii) *there is a unique abstract homomorphism $F : A/\text{tor}(A) \rightarrow B$ such that $F(a + \text{tor}(A)) = f(a)$*

The theory of torsion subgroups in $\mathbb{C}\mathbb{A}\mathbb{B}$ is not different from that of $\mathbb{A}\mathbb{B}$. In fact, the torsion subgroups of some relevant compact abelian groups are captured below.

Compact abelian group	Torsion subgroup
\mathbb{Z}_p	$\{0\}$
\mathbb{T}	\mathbb{Q}/\mathbb{Z}
$\mathbb{Z}(p^\infty)$	$\mathbb{Z}(p^\infty)$
$\mathbb{Z}(n)$	$\mathbb{Z}(n)$

TABLE 1. Table of torsion subgroups of some groups in $\mathbb{C}\mathbb{A}\mathbb{B}$.

Definition 2.30 (See [13], Definition A1.18). Let A be an abstract abelian group. For a prime number p , the subgroup

$$(2.43) \quad A_p = \{a \in A \mid p^n \cdot a = 0 \text{ for some } n \in \mathbb{N}\}$$

is called the *p -primary component* of A , or the *p -Sylow subgroup* of A .

The torsion subgroup of an abstract abelian group has a decomposition in terms of its p -primary components. In particular, we have that

$$(2.44) \quad \text{tor}(A) = \bigoplus_{p \text{ prime}} A_p$$

for any abstract abelian group A (see [13, Theorem A1.19]). The p -Sylow subgroups of an abstract abelian group are examples of abstract groups with all their elements having prime power order for some prime p , these are known as p -groups.

Remark 2.31. The notions of a p -group and a p -Sylow subgroup exist for locally compact abelian groups. They are defined through characters and are different from that of abstract abelian groups (see [13, Definition 8.7], or more generally [12] for locally compact abelian groups). We do not enter into the details here, we only mention as an illustration, that for $G = \mathbb{Z}_p$, considered as a compact abelian group, we have $\text{tor}(G) = 0$, but its p -Sylow subgroup $G_p = G$, which shows that there are torsion-free p -Sylow subgroups in $\mathbb{C}\mathbb{A}\mathbb{B}$, in contrast to $\mathbb{A}\mathbb{B}$ where this cannot happen.

Definition 2.32 (See [13], Definition A1.22). A subgroup B of an abstract abelian group A in $\mathbb{A}\mathbb{B}$ is pure if for all $n \in \mathbb{N}$, and for all $a \in A$, if $n \cdot a \in B$ then there is $b \in B$ such that $n \cdot b = n \cdot a$.

Pure subgroups of an abstract abelian group A are direct summands, by [13, Page 719]. Further, both the torsion subgroup $\text{tor}(A)$ and the p -Sylow subgroups A_p are pure in A , by [13, Lemma A1.23], and hence are direct summands. For each prime p , the torsion subgroup of an abstract abelian group A has an important subgroup:

$$(2.45) \quad S_p(A) = \{a \in A \mid p \cdot a = 0\}$$

called the p -socle of A . The p -socle of an abstract abelian group leads to a fundamental definition which requires some elementary notions from linear algebra. So, recall that a finite subset $\{x_1, x_2, \dots, x_n\}$ of a vector space V over a field \mathbb{F} is *linearly independent* if

$$(2.46) \quad a_1 \cdot x_1 + a_2 \cdot x_2 + \dots + a_n \cdot x_n = 0$$

implies that $a_i = 0$ for all $i = 1, 2, \dots, n$, $a_i \in \mathbb{F}$. Also recall that a *basis* X is a linearly independent set of vectors in V such that every vector in V is a linear combination of elements in X . A vector space is a *finite dimensional* vector space if it has a finite basis. It turns out that every basis of a finite dimensional vector space has the same number of vectors (see [11, Theorem 8.1]), allowing for the following definition.

Definition 2.33 (See [11], pp.14). The *dimension* of a finite dimensional vector space V over a field \mathbb{F} is the cardinality of any one of its bases, written $\dim_{\mathbb{F}} V$.

By [11, Theorem 8.2], any basis of a finite dimensional vector space is maximal with respect to being linearly independent, meaning that any other linearly independent subset containing a basis is necessarily equal to it.

This linear algebra is relevant because the p -socle of an abstract abelian group A is a finite-dimensional vector space over the finite field $\text{GF}(p)$ of order p (see [13, Remark A1.21]). And so we make the following definition.

Definition 2.34 (See [13], Appendix 1). For a prime number p and an abstract abelian group A in $\mathbb{A}\mathbb{B}$, the dimension $\dim_{\text{GF}(p)} S_p(A)$ of $S_p(A)$ over $\text{GF}(p)$ is called p -rank of A . This number is denoted by $\text{rank}_p(A)$.

The following table contains the p -ranks of some abelian groups.

Abelian group in $\mathbb{A}\mathbb{B}$	p -socle in $\mathbb{A}\mathbb{B}$	p -rank in $\mathbb{A}\mathbb{B}$
\mathbb{T}	$\frac{1}{p}\mathbb{Z}/\mathbb{Z} \cong \mathbb{Z}(p)$	1
\mathbb{Z}_p	$\{0\}$	0
$\mathbb{Z}_p \oplus \mathbb{Z}_p$	$\{0\}$	0
$\mathbb{Z}(p) \oplus \mathbb{Z}(p)$	$\mathbb{Z}(p) \oplus \mathbb{Z}(p)$	2
\mathbb{Q}	$\{0\}$	0
\mathbb{R}	$\{0\}$	0
\mathbb{Z}	$\{0\}$	0
$\mathbb{Z}(p^\infty)^n$	$\left(\frac{1}{p}\mathbb{Z}/\mathbb{Z}\right)^n$	n

TABLE 2. Table of p -ranks of some abelian groups.

Note that the notion of p -rank for compact abelian group in $\mathbb{C}\mathbb{A}\mathbb{B}$ may differ from the notion of p -rank just given in $\mathbb{A}\mathbb{B}$, see [12, Definition 3.6]. In fact the topological generation is different from the algebraic generation, as the example of \mathbb{Z}_p shows: it is topologically one generated, meaning that there is $g \in \mathbb{Z}_p$ such that $\langle g \rangle = \mathbb{Z}_p$, but it is algebraically infinitely generated. We do not use the p -rank of a compact abelian group, referring to the topological generation, so Definition 2.34 will be sufficient for what we discuss in the present thesis.

Since for any homomorphism $f : A \rightarrow B$ of abstract abelian groups $f(\text{tor}(A)) \subseteq \text{tor}(B)$, taking torsion subgroups induces a homomorphism $\text{tor}(f) : \text{tor}(A) \rightarrow \text{tor}(B)$. Therefore, $\text{tor} : \mathbb{A}\mathbb{B} \rightarrow \mathbb{A}\mathbb{B}$ is a covariant functor (see [13, Example A3.22]). The tor functor is actually a special case of a more general functor Tor as we illustrate below.

Recall that the tensor product of abstract abelian groups can be viewed as a functor from the category $\mathbb{A}\mathbb{B}$ to itself. Moreover, it is right exact in that if

$$(2.47) \quad 0 \longrightarrow A \xrightarrow{j} B \xrightarrow{p} C \longrightarrow 0$$

is a short exact sequence of abstract abelian groups A, B, C in $\mathbb{A}\mathbb{B}$, then for any abstract abelian group G in $\mathbb{A}\mathbb{B}$ the sequence

$$(2.48) \quad A \otimes G \xrightarrow{j \otimes 1} B \otimes G \xrightarrow{p \otimes 1} C \otimes G \longrightarrow 0$$

is always exact. However, the sequence

$$(2.49) \quad 0 \longrightarrow A \otimes G \xrightarrow{j \otimes 1} B \otimes G \xrightarrow{p \otimes 1} C \otimes G \longrightarrow 0$$

may fail to be exact.

To measure how far \otimes is from being exact we introduce the Tor functor which generalises tor . Firstly, we recall a few definitions and results.

For any projective resolution $\mathbf{C} \xrightarrow{d_0} A$ of an abstract abelian group A in $\mathbb{A}\mathbb{B}$, and any abelian group G in $\mathbb{A}\mathbb{B}$ we can form the complex

$$(2.50) \quad \dots \longrightarrow C_2 \otimes G \xrightarrow{d_2 \otimes 1} C_1 \otimes G \xrightarrow{d_1 \otimes 1} C_0 \otimes G \xrightarrow{d_0 \otimes 1} A \otimes G$$

from which we replace the term $A \otimes G$ by the trivial group to obtain the positive complex

$$(2.51) \quad \dots \longrightarrow C_2 \otimes G \xrightarrow{d_2 \otimes 1} C_1 \otimes G \xrightarrow{d_1 \otimes 1} C_0 \otimes G \longrightarrow 0$$

whose homology groups are

$$(2.52) \quad \text{Tor}_n(G, A) = \ker(d_{n+1} \otimes 1) / \text{Im}(d_n \otimes 1)$$

When $A = \mathbb{Z}$, we write

$$(2.53) \quad \text{Tor}_n(G, \mathbb{Z}) = H_n(G, \mathbb{Z}) = H_n(G),$$

and call $H_n(G)$ the n^{th} homology of G with coefficients in \mathbb{Z} , or the *integral homology* of G . With this terminology $\text{Tor}_n(G, A)$ is the n^{th} homology of G with coefficients in A . This notion leads to a different duality due to Poincaré (see [10]).

Theorem 2.35 (See [27], Exercise 11.4.7). *For a finite group G , there is an isomorphism between the abelian groups $H_2(G, \mathbb{Z})$ and $H^2(G, \mathbb{C}^\times)$. This isomorphism is known as Poincaré duality.*

Note that the Poincaré duality does not happen in general for infinite groups. We go ahead and illustrate the relationship between tor and Tor through the following example.

Example 2.36 (See [31], Example 10). Let A in $\mathbb{A}\mathbb{B}$ be an abstract abelian group. For any prime number p , consider the projective resolution

$$(2.54) \quad \dots \longrightarrow 0 \longrightarrow \mathbb{Z} \xrightarrow{p} \mathbb{Z} \xrightarrow{d_0} \mathbb{Z}(p)$$

where the map $p : \mathbb{Z} \rightarrow \mathbb{Z}$ is the multiplication by p , and $d_0 : \mathbb{Z} \rightarrow \mathbb{Z}(p)$ is the quotient map $n \in \mathbb{Z} \mapsto n + p\mathbb{Z} \in \mathbb{Z}(p)$. We obtain the complex

$$(2.55) \quad \dots \longrightarrow 0 \longrightarrow \mathbb{Z} \otimes A \xrightarrow{p \otimes 1} \mathbb{Z} \otimes A \longrightarrow 0$$

after tensoring (1) by A and discarding the term $\mathbb{Z}(p) \otimes A$. Since $\mathbb{Z} \otimes A \cong A$, (2) becomes

$$(2.56) \quad \dots \longrightarrow 0 \longrightarrow A \xrightarrow{p} A \longrightarrow 0$$

Therefore

$$(2.57) \quad \text{Tor}_0(A, \mathbb{Z}(p)) = \frac{\ker 0}{\text{Im } p} = A/pA$$

and

$$(2.58) \quad \text{Tor}_1(A, \mathbb{Z}(p)) = \frac{\ker p}{\text{Im } 0} \cong \ker p = \{a \in A \mid p \cdot a = 0\} = \text{tor}(A)$$

while $\text{Tor}_n(A, \mathbb{Z}(p)) = 0$ for all $n \geq 2$.

From Example 2.36, it is worth emphasising the equation $\text{Tor}_1(A, \mathbb{Z}(p)) = \text{tor}(A)$. We make precise how Tor measures how far \otimes is from being exact through the following proposition.

Proposition 2.37 (See [31], Theorem 3.4(a)). *If $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is a short exact sequence of abstract abelian groups A, B, C in $\mathbb{A}\mathbb{B}$, then there is a long exact sequence*

$$\begin{array}{ccccccc}
 & & & & \dots & \longrightarrow & \text{Tor}_{n+1}(C, G) \\
 & & & & & & \downarrow \\
 & & \text{Tor}_n(C, G) & \longleftarrow & \text{Tor}_n(B, G) & \longleftarrow & \text{Tor}_n(A, G) \\
 & & \downarrow & & & & \\
 & & \vdots & & & & \\
 & & \downarrow & & & & \\
 & & \text{Tor}_1(A, G) & \longrightarrow & \text{Tor}_1(B, G) & \longrightarrow & \text{Tor}_1(A, G) \\
 & & & & & & \downarrow \\
 0 & \longleftarrow & C \otimes G & \longleftarrow & B \otimes G & \longleftarrow & A \otimes G
 \end{array}$$

Clearly, \otimes is exact when $\text{Tor}_1(A, G) = 0$, and indeed when $\text{Tor}_n(A, G) = 0$ for all $n \geq 1$. This is the sense in which Tor measures how far \otimes is from being exact. This formalism of completing a right (or left) exact sequence obtained from a functor, to a long exact sequence through another functor, consequently obtaining a measure of the exactness of the original functor, is a subject of the theory of derived functors which can be found in [18, Chapter 12].

We finish the section with the so-called Universal Coefficient Theorem for Homology, which relates the n^{th} homology of any abstract abelian group G in $\mathbb{A}\mathbb{B}$ with coefficients in any abstract abelian group A in $\mathbb{A}\mathbb{B}$ to the n^{th} integral homology of G .

Theorem 2.38 (Universal Coefficient Theorem for Homology, See [18], Theorem 11.1). *If $\mathbf{C} \rightarrow A$ is a projective resolution of A in $\mathbb{A}\mathbb{B}$, then there is a short exact sequence*

$$0 \longrightarrow H_n(\mathbf{C}) \otimes G \longrightarrow \text{Tor}_n(G, A) \longrightarrow \text{Tor}_1(H_{n-1}(\mathbf{C}), G) \longrightarrow 0$$

Again, one can formalise Theorem 2.38 in the classical way once $\mathbb{C}\mathbb{A}\mathbb{B}$ is considered and this can be found in [13, Chapter 8].

2.4. Divisibility in Abstract Abelian Groups. Any element x of an abstract abelian group A can be multiplied by a natural number. Say, for example $n \cdot x = a$, where $n \in \mathbb{N}, a \in A$. We call x a root of a . Clearly, x may not be unique. On the other hand, given an arbitrary element $b \in A$, we can ask if b has a root. The answer is generally no. However, in some situations, like in the additive group \mathbb{Q} of rational numbers, every element has a root, one for each natural number $n \in \mathbb{N}$; we are interested in abstract abelian groups with this property. This makes way for the following definition.

Definition 2.39 (See [13], Definition A1.29). Let A in $\mathbb{A}\mathbb{B}$ be an abstract abelian group. An element a in A is divisible if for each natural number n there is an x in A such that $n \cdot x = a$. The group A is divisible if each of its elements is divisible.

Fix an abstract abelian group A in $\mathbb{A}\mathbb{B}$. Since $n \cdot x = a$ and $n \cdot y = b$ implies that $a + b = n(x + y)$, where $n \in \mathbb{N}$ and $a, b, x, y \in A$, the set $\text{Div}(A)$ of all divisible elements of A is a subgroup of A . Clearly, the group A is divisible if it coincides with $\text{Div}(A)$. We say that it is *reduced* if $\text{Div}(A) = \{0\}$.

If m and n are relatively prime integers and $m \cdot a = 0$ for some a in A then there is an x in A such that $n \cdot x = a$ (see [13, Exercise EA1.11(iii)]). In particular, every element of finite order n is divisible by any integer coprime to n .

According to [13, Exercise EA1.11(iii)], divisibility has a good behavior with respect to direct products and cartesian products, that is, if $\{A_j \mid j \in J\}$ is a family of divisible abstract abelian groups, then both $\bigoplus_{j \in J} A_j$ and $\prod_{j \in J} A_j$ are divisible. It also has a good behavior with respect to quotients, that is, every quotient of a divisible abstract abelian group is divisible; more generally, every homomorphic image of an abstract abelian group is divisible (see [13, Exercise EA1.11(vi)]).

Divisibility does not have a good behavior with respect to formation of subgroups. For example, the subgroup \mathbb{Z} of the additive group \mathbb{Q} is not divisible, while \mathbb{Q} is a divisible group. In fact, the additive group of any rational vector space is divisible. No cyclic group is divisible and therefore no finite group is divisible. Note, however, that taking pure subgroups preserves divisibility (see [13, Exercise EA1.11(vii)]).

According to [13, Definition A1.30], the Prüfer group may be realised as $\frac{1}{p^\infty}\mathbb{Z}/\mathbb{Z} = \mathbb{Z}(p^\infty)$. To see that $\mathbb{Z}(p^\infty)$ is divisible, take $a \in \mathbb{Z}(p^\infty)$ and observe that since every element of $\mathbb{Z}(p^\infty)$ has order a power of p , for any natural number n coprime to p there is $x \in \mathbb{Z}(p^\infty)$ such that $n \cdot x = a$. Further, for any power p^n of p , there is $y \in \mathbb{Z}(p^\infty)$ such that $p^n \cdot y = a$. Putting these facts together, it follows that $\mathbb{Z}(p^\infty)$ is divisible.

An analogue to the fact that every abstract abelian group is a quotient of a free abstract abelian group is that every abstract abelian group G is a subgroup of a divisible abstract abelian group D (see [13, Proposition A1.33]). Moreover, D can be chosen in such a way that every nontrivial subgroup of D meets G nontrivially. This prompts the following definition.

Definition 2.40 (See [13], pp.727). For an abstract abelian group A , a divisible group $D \supseteq A$ such that every nontrivial subgroup of D meets A nontrivially is called a divisible hull of A .

The divisible hull of an abstract abelian group is unique up to isomorphism, and $S_p(A) = S_p(D)$ (see [13, Cororally A1.36]).

Example 2.41. \mathbb{Z} is a subgroup of the divisible group \mathbb{Q} . For any nontrivial rational number $\frac{a}{b}$, $b\frac{a}{b} = a \in \mathbb{Z}$, so that every nontrivial subgroup of \mathbb{Q} meets \mathbb{Z} nontrivially. So \mathbb{Q} is the divisible hull of \mathbb{Z} .

According to [13, Proposition A1.38], in a divisible abstract abelian group A , the torsion subgroup $\text{tor}(A)$ being a pure subgroup is a divisible direct summand with a torsion-free complement. Since torsion subgroups are direct sums of p -groups, the study of divisible abstract abelian groups is reduced to the study of those divisible abstract abelian groups that are p -groups and those that are torsion-free. It turns out that the divisible torsion-free abstract abelian groups are rational vector spaces by [13, Proposition A1.39], and the divisible abstract abelian p -groups are direct sums of $\mathbb{Z}(p^\infty)$ by [13, Proposition A1.42]. In order to write down the precise structure of divisible abstract abelian groups we need the notion of rank defined below.

Definition 2.42 (See [13], pp. 692). The rank of an abelian group A in $\mathbb{A}\mathbb{B}$ is the dimension $\dim_{\mathbb{Q}} D$ of its divisible hull D when D is viewed as a \mathbb{Q} -vector space. We write $\text{rank}(A)$ for the rank of an abelian group A in $\mathbb{A}\mathbb{B}$.

Theorem 2.43 (See [13], Theorem A1.42). *Let A in $\mathbb{A}\mathbb{B}$ be a divisible abstract abelian group. Then*

$$(2.59) \quad A = \mathbb{Q}^{(\text{rank}(A))} \oplus \bigoplus_{p \text{ prime}} \mathbb{Z}(p^\infty)^{(\text{rank}_p(A))}$$

We say that a subgroup of an abstract abelian group is divisible if it is a divisible group. According to [13, Proposition A1.31], every abstract abelian group A contains a largest divisible group, denoted $\text{div}(A)$, which does not always coincide with $\text{Div}(A)$.

Remark 2.44. We can consider $\text{div}(G)$ and $\text{Div}(G)$ for G in $\mathbb{C}\mathbb{A}\mathbb{B}$. It turns out that if G is a compact abelian group, or if it is a discrete torsion-free abelian group, then $\text{div}(G) = \text{Div}(G)$ (see [13, Proposition 8.2]). Moreover, there is a connection between divisibility and connectedness in the context of compact abelian groups. More precisely, a compact abelian group is connected if and only if it is divisible (see [13, Corollary 8.5]).

An example of an abstract abelian group having $\text{div}(A) \neq \text{Div}(A)$ is the group

$$(2.60) \quad \nabla = \mathbb{Z}^{(\mathbb{N})}/W$$

where W is the subgroup of $\mathbb{Z}^{(\mathbb{N})}$ defined as follows: for each $n \in \mathbb{N}$, define

$$(2.61) \quad e_n : m \in \mathbb{N} \mapsto e_n(m) = \begin{cases} 1, & \text{if } m = n \\ 0, & \text{otherwise} \end{cases} \in \mathbb{Z}.$$

Then put

$$(2.62) \quad W = \langle e_1 - n \cdot e_n \mid n \in \mathbb{N} \rangle$$

In order to illustrate its properties we recall that an element b of an abstract abelian group B is a *root* of $b_1 \in B$ if $n \cdot b = b_1$ for some $n \in \mathbb{N}$. An element b_1 in B is said to have a *consecutive set of roots* $\{b_n \in B \mid n \in \mathbb{N}\}$, if for any two natural numbers m and n , the relation $mb_{mn} = b_n$ holds. Finally, returning to the group ∇ in (2.60), put

$$(2.63) \quad g_n = e_n + W \in \mathbb{Z}^{(\mathbb{N})}/W$$

and consider the following theorem.

Theorem 2.45 (See [13], Theorem A1.32). *Let B in $\mathbb{A}\mathbb{B}$ be an abstract abelian group with a divisible element b_1 in B having a consecutive set of roots $\{b_n \in B \mid n \in \mathbb{N}\}$. Then the following statements about ∇ hold.*

- (1) *There is a unique abstract homomorphism $d : g_n \in \nabla \mapsto b_n \in B$. In particular, taking $B = \mathbb{Q}$ and $b_1 = 1 \in \mathbb{Q}$, there are a surjective homomorphism*

$\pi : g_n \in \nabla \mapsto \frac{1}{n} \in \mathbb{Q}$ and a homomorphism $q : \frac{n}{m} \in \mathbb{Q} \mapsto m \cdot b_n \in B$ which are unique in making the diagram

$$\begin{array}{ccc} & & \mathbb{Q} \\ & \nearrow \pi & \downarrow q \\ \nabla & \xrightarrow{d} & B \end{array}$$

commutative.

- (2) For each $n \in \mathbb{N}$, g_n generates a torsion-free infinite cyclic group $\langle g_n \rangle \cong \mathbb{Z}$, and each g_n is a root of $g_1 = n \cdot g_n$.
- (3) We have $\ker \pi = \text{tor}(\nabla)$.
- (4) The quotient $\nabla / \langle g_1 \rangle \cong \bigoplus_{n=2}^{\infty} \mathbb{Z}(n)$.
- (5) If G is any subgroup of ∇ such that $G \cap \langle g_1 \rangle = \{0\}$, then $G \subseteq \text{tor}(\nabla)$.
- (6) $\text{div}(\nabla) = \{0\}$ and $\text{Div}(\nabla) = \langle g_1 \rangle$.
- (7) The torsion subgroup $\text{tor}(\nabla)$ is not a direct summand of ∇ .
- (8) The group $S \stackrel{\text{def}}{=} \bigoplus_{n=2}^{\infty} \mathbb{Z}(n)$ contains a subgroup $K \cong \text{tor}(\nabla)$ such that $S/K \cong \mathbb{Q}/\mathbb{Z}$. The subgroup K of S is not a direct summand.
- (9) Let \mathbb{P} denote the set of all prime numbers. We have

$$S = \bigoplus_{p \in \mathbb{P}} \left(\bigoplus_{n \in \mathbb{N}} \mathbb{Z}(p^n) \right),$$

and if K_p is the p -primary component of K , there is an exact sequence

$$0 \rightarrow K_p \xrightarrow{\text{incl}} \bigoplus_{n \in \mathbb{N}} \mathbb{Z}(p^n) \xrightarrow{\gamma_p} \mathbb{Z}(p^\infty) \rightarrow 0.$$

In particular, the p -rank of K_p , and thus the p -rank of ∇ , are infinite.

We end this section with some interactions between divisibility of abstract abelian groups and their tensor products. We have the following theorem.

Proposition 2.46 (See [13], Proposition A1.45). *If A in $\mathbb{A}\mathbb{B}$ is an abstract abelian group, then there is an abstract homomorphism:*

$$\alpha : a \in A \mapsto 1 \otimes a \in A \otimes \mathbb{Q}.$$

Moreover, the abstract abelian group $\mathbb{Q} \otimes A$ is a rational vector space and the following properties hold:

- (1) If $x \in \mathbb{Q} \otimes A$, then $x = q \otimes a$ for some $q \in \mathbb{Q}$ and $a \in A$.
- (2) The quotient group $(\mathbb{Q} \otimes A) / (1 \otimes A)$ is torsion.
- (3) The group $\mathbb{Q} \otimes A$, being a rational vector space, is a torsion-free divisible group, and every nontrivial subgroup of $\mathbb{Q} \otimes A$ meets $1 \otimes A$ nontrivially.
- (4) We have $\ker \alpha = \text{tor}(A)$. Therefore, $A/\text{tor}(A) \cong \text{Im } \alpha \subseteq \mathbb{Q} \otimes A$. Hence the quotient morphism $p : A/\text{tor}(A) \rightarrow \mathbb{Q} \otimes A$ is an embedding.
- (5) The embedding $p : A/\text{tor}(A) \rightarrow \mathbb{Q} \otimes A$ is the unique embedding of a torsion-free abstract abelian group A into a smallest divisible group. Moreover, $\mathbb{Q} \otimes (A/\text{tor}(A)) \cong \mathbb{Q} \otimes A$.

(6) If A is a subgroup of an abstract abelian group B in $\mathbb{A}\mathbb{B}$, then the exact sequence:

$$0 \longrightarrow A \xrightarrow{j} B \xrightarrow{p} B/A \longrightarrow 0$$

induces an exact sequence:

$$0 \longrightarrow \mathbb{Q} \otimes A \xrightarrow{\text{id}_{\mathbb{Q}} \otimes j} \mathbb{Q} \otimes B \xrightarrow{\text{id}_{\mathbb{Q}} \otimes p} \mathbb{Q} \otimes (B/A) \longrightarrow 0$$

Looking at Proposition 2.46, one sees an alternative construction of the divisible hull of an abstract abelian group via tensor products. Consequently, the rank of an abstract abelian A group may alternatively be defined via

$$(2.64) \quad \text{rank}(A) = \dim_{\mathbb{Q}}(\mathbb{Q} \otimes A)$$

For instance, note that by Proposition 2.46(2) the group $\mathbb{Q} \otimes \mathbb{Z} \cong \mathbb{Q}$ is smallest divisible group containing \mathbb{Z} , and that this is consistent with with Example 2.41 where \mathbb{Q} was seen to be the divisible hull of \mathbb{Z} . Then $\text{rank}(\mathbb{Z}) = \dim_{\mathbb{Q}}(\mathbb{Q}) = 1$.

3. FREE TOPOLOGICAL GROUPS

Recall that in section 2.1 we present the tensor product of two abstract abelian groups as a quotient of the free abstract abelian group over their cartesian product. This was preceded by the study of free abstract abelian groups which gave a strong clue on how this tensor product may be obtained. In this section we consider free topological groups with the aim of perhaps studying tensor products of topological groups as quotients of free topological groups. We are mainly interested in free compact and free compact abelian groups.

3.1. Free Topological and Paratopological Groups.

Definition 3.1 (See [9], Definition 1.4.3). The *Markov free topological group* over a topological space X is a topological group $F(X)$ together with a continuous map $\varepsilon : X \rightarrow F(X)$, such that for any continuous map $f : X \rightarrow G$ to a topological group G there is a continuous group homomorphism $f' : F(X) \rightarrow G$, unique in making the diagram

$$(3.1) \quad \begin{array}{ccc} X & \xrightarrow{\varepsilon} & F(X) \\ f \downarrow & \swarrow f' & \\ G & & \end{array}$$

commutative, i.e., $f' \circ \varepsilon = f$. The *Markov free abelian topological group* over X is defined similarly, but requiring that G and $F(X)$ be abelian topological groups.

It is not evident that X can be embedded as a topological space in the Markov free topological group $F(X)$ like it is the case for abstract free abelian groups over sets. In fact, the proof that the map $\varepsilon : X \rightarrow F(X)$ is injective in the case of abstract free abelian groups over a set X depends on being able to find, for distinct $x, y \in X$, a set function $f : X \rightarrow G$ to a nontrivial group such that $f(x) \neq f(y)$. Unfortunately, this fails for topological groups as f cannot, in general, be guaranteed to be continuous, an impediment that is absent when continuity is not imposed. This can be remedied by imposing certain separation axioms on X . So we recall the following separation axioms.

Definition 3.2 (See [33], Definition 14.8). A topological space X is *completely regular* if for any closed set K in X and any point x in $X \setminus K$, there is a continuous function $f : X \rightarrow \mathbb{R}$ such that $f(x) = a$ and $f(K) = b \neq a$.

The condition of being Hausdorff can be weakened to the following separation axiom.

Definition 3.3 (See [33], Definition 13.3). A topological space X is called a T_1 *space* if for any two distinct points $x, y \in X$ there is an open set that contains x and does not contain y , and an open set that contains y and does not contain x .

Clearly, every Hausdorff topological space is a T_1 space. A topological space that is T_1 and completely regular is called a *Tychonoff space* (see [33, Definition 14.8]). According to [9, p.11], when X is completely regular, then the map $\varepsilon : X \rightarrow F(X)$ in Definition 3.1 is an embedding. In this case, X may be considered a subspace of $F(X)$ so that the induced function $f' : F(X) \rightarrow G$ is the unique extension of f to $F(X)$.

The notion of a free topological group exists for pointed topological spaces. It is defined as follows.

Definition 3.4 (See [9], Definition 1.4.4). The *Graev free topological group* over a pointed topological space (X, e) is a topological group $F_G(X, e)$ together with a continuous basepoint preserving map $\varepsilon : (X, e) \rightarrow F_G(X, e)$, such that for any continuous map $f : X \rightarrow G$ to a topological group G satisfying $f(e) = 1_G$, there is a continuous group homomorphism $f' : F_G(X, e) \rightarrow G$, unique in making the diagram

$$(3.2) \quad \begin{array}{ccc} (X, e) & \xrightarrow{\varepsilon} & F_G(X, e) \\ f \downarrow & \swarrow f' & \\ (G, 1_G) & & \end{array}$$

commutative, i.e., $f' \circ \varepsilon = f$. The *Graev free abelian topological group* over X is defined similarly, but requiring that (G, e_G) and $F_G(X, e)$ be abelian topological groups.

Again, according to [9, pp. 11], (X, e) is embedded in the Graev free topological group via $\varepsilon : X \rightarrow F_G(X, e)$ if and only if X is completely regular.

Some categorical remarks are due here: the map $X \in \text{ob TOP} \mapsto F(X) \in \text{ob TOPG}$, where $F(X)$ is the Markov free topological group on X , is a functor from the category TOP of topological spaces to the category TOPG of topological groups. Moreover, if $U : \text{TOPG} \rightarrow \text{TOP}$ is the forgetful functor, which sends a topological group to its underlying topological space and each continuous homomorphism to the corresponding continuous function of topological spaces, then F and U are adjoint functors.

Similarly, the Graev free topological group defines a functor $F_G : \text{TOP}_0 \rightarrow \text{TOPG}$ that is adjoint to the forgetful functor $U : \text{TOPG} \rightarrow \text{TOP}_0$. Pointing out the adjoint situation in this case is not necessarily superfluous since the question of the existence of free topological groups may be settled by the Adjoint Functor Existence Theorem (see [13, Theorem A3.60]).

The notion of a topological group can be weakened by not requiring that the inversion map $x \in G \mapsto x^{-1} \in G$ of a group G be continuous (see (1.2) in §1.1). Consequently, we obtain the following notion.

Definition 3.5 (See [9], pp. 6). A *paratopological group* G is a group with a topology such that multiplication $(x, y) \mapsto x \cdot y : G \times G \rightarrow G$ is continuous.

Clearly, every topological group is a paratopological group. However, the converse is not true. A standard example of a paratopological group which fails to be a topological group is the additive group \mathbb{R} with the Sorgenfrey topology (see [33]), showing that the notion of a paratopological group is not superfluous. It is interesting to consider when a paratopological group G is a topological group. This happens, for instance, when G is compact (see [9, pp. 7]).

The notions of free Graev and free Markov topological groups over a space have paratopological group analogues that are defined in a similar way.

Definition 3.6 (See [9], Remark 2.2.9). The *Markov free paratopological group* over a topological space X is a paratopological group $FP(X)$ together with continuous map $\varepsilon : X \rightarrow FP(X)$ such that for any continuous map $f : X \rightarrow G$ to a paratopological group G , there is a continuous homomorphism $f' : FP(X) \rightarrow G$ of paratopological groups, unique in making the diagram

$$(3.3) \quad \begin{array}{ccc} X & \xrightarrow{\varepsilon} & FP(X) \\ f \downarrow & \swarrow f' & \\ G & & \end{array}$$

commutative, i.e., $f' \circ \varepsilon = f$.

The *Markov free abelian paratopological group* is defined similarly, but requiring that G and $FP(X)$ are abelian paratopological groups in Definition 3.6.

Unlike in the case of the free Markov topological group over X , the map $\varepsilon : X \rightarrow FP(X)$ in Definition 3.6 is an embedding for any topological space (see [9, p.44]). So we may assume that X is a subspace of the Markov free paratopological group $FP(X)$ and give an alternative definition of $FP(X)$ seen in [9, Definition 2.2.1], where X is assumed to be a subspace of $FP(X)$. With this, we encounter the first difference between the theory of free topological groups and that of free paratopological groups.

There is also the notion of the free Graev paratopological group.

Definition 3.7 (See [9], Remark 2.2.8). The *Graev free paratopological group* over a pointed topological space (X, e) is a paratopological group $FP_G(X, e)$ together with a continuous map $\varepsilon : (X, e) \rightarrow FP_G(X, e)$ such that for any continuous map $f : X \rightarrow G$ to a paratopological group G satisfying $f(e) = 1_G$, there is a continuous homomorphism $f' : FP_G(X, e) \rightarrow G$ of paratopological groups, unique in making the diagram

$$(3.4) \quad \begin{array}{ccc} X & \xrightarrow{\varepsilon} & FP_G(X, e) \\ f \downarrow & \swarrow f' & \\ (G, 1_G) & & \end{array}$$

commutative, i.e., $f' \circ \varepsilon = f$.

The *Graev free abelian paratopological group* is defined similarly, but requiring that (G, e_G) and $FP_G(X, e)$ are abelian paratopological groups in Definition 3.7.

Again, according to [9, pp. 44], the map $\varepsilon : (X, e) \rightarrow FP_G(X, e)$ is an embedding for any topological space, leading to an alternative definition where X is assumed to be a subspace of $FP_G(X, e)$ (see [9, Definition 2.1.1]).

The notion of Graev free topological group generalises the notion of Markov free topological group in that every free Markov topological group is a Graev topological group. The same for the Graev free and Markov free paratopological group. Take note that the reason these notions do not generally coincide is that the notion of Graev fixes a basepoint while that of Markov does not even require a function $f : X \rightarrow G$ to have the identity 1_G of G in its image. However, one can force f to

have 1_G in its image by taking the topological sum $X \oplus \{e\}$ whose characteristic property allows one to extend $f : X \rightarrow G$ to a continuous function that necessarily maps some point of X to 1_G and also agrees with f . This new function then induces a continuous homomorphism from $F_G(X \oplus \{e\})$ to G (see [9], pp. 44). In this way, we obtain the following connection between the notions of Graev and Markov: $F(X) \cong F_G(X \oplus \{e\})$ as topological groups, and $FP(X) \cong FP_G(X \oplus \{e\})$ as paratopological groups, where $e \in X$.

3.2. Free Compact Abelian Groups. In this section we study free objects in the category $\mathbb{C}\mathbb{A}\mathbb{B}$ of compact abelian groups.

Let X be a pointed topological space, and \mathbb{T} the torus as in Example 1.8. The set $C_0(X, \mathbb{T})$ of basepoint preserving continuous functions from X to \mathbb{T} is an abelian group with respect to pointwise operations. With the discrete topology, it becomes a topological group denoted by $C_0(X, \mathbb{T})_d$. Its character group is $\widehat{C_0(X, \mathbb{T})_d}$.

Remark 3.8. Whenever we discuss pointed topological spaces and basepoint preserving maps, we assume that the basepoint of a topological group is its identity and we omit this information in practice.

Definition 3.9 (See [13], Definition 8.51). A free compact abelian group over a pointed topological space (X, x_0) is a compact group $F(X)$ in $\mathbb{C}\mathbb{A}\mathbb{B}$ together with a basepoint preserving continuous map $\varepsilon : X \rightarrow F(X)$ such that for any basepoint preserving continuous map $f : X \rightarrow G$ to a compact abelian group G in $\mathbb{C}\mathbb{A}\mathbb{B}$, there exists a continuous homomorphism $f' : F(X) \rightarrow G$, unique in making the diagram

$$(3.5) \quad \begin{array}{ccc} X & \xrightarrow{\varepsilon_X} & F(X) \\ \downarrow G & \swarrow f' & \\ G & & \end{array}$$

is commutative, i.e., $f = f' \circ \varepsilon$.

Just like the free abstract abelian group, the free compact abelian group $F(X)$ over a topological space X is unique up to isomorphism of compact abelian groups, by [13, Proposition 8.52]. Moreover, the group $\langle \varepsilon_X(X) \rangle$ is the abstract free abelian group over $\varepsilon_X(X)$, by [13, Proposition 8.52].

Theorem 3.10 (See [13], Theorem 8.53). *For a pointed topological space X , the compact group $F(X) = C_0(\widehat{X, \mathbb{T}})_d$ is the free compact abelian group over X with the continuous map $x \in X \mapsto \varepsilon(x) \in C_0(\widehat{X, \mathbb{T}})_d$, where $\varepsilon(x)$ is given by $\varepsilon(x)(f) = f(x)$ for all $f \in C_0(X, \mathbb{T})_d$.*

Example 3.11. The free compact abelian group $F(X)$ of a singleton space X is the trivial group along with the zero map $X \rightarrow F(X) = \{0\}$. This can be shown by verifying that the trivial group satisfies Definition 3.9.

Example 3.12. Let \mathbb{T}_d denote the torus with the discrete topology, instead of \mathbb{T} which denotes the compact torus of Example 1.8. Let $X = \{x_0, x\}$ be a two-point space with discrete topology and basepoint x_0 . Since any $f \in C_0(X, \mathbb{T})_d$ necessarily maps x_0 to $0 \in \mathbb{T}$, any such f is completely determined by the image $f(x)$ of x . This defines an injective continuous homomorphism

$$(3.6) \quad f \in C_0(X, \mathbb{T})_d \mapsto f(x) \in \mathbb{T}_d.$$

On the other hand, for any $t \in \mathbb{T}_d$, the map sending $x_0 \in X \mapsto 0 \in \mathbb{T}$ and $x \in X \mapsto t \in \mathbb{T}$ determines a basepoint preserving continuous function $f_t \in C_0(X, \mathbb{T})_d$ given by $f_t(x_0) = 0$ and $f_t(x) = t$. Therefore (3.6) is a continuous bijective homomorphism, and $C_0(X, \mathbb{T}) \cong \mathbb{T}_d$. Applying Theorem 3.10, we obtain

$$(3.7) \quad F(X) = C_0(\widehat{X}, \widehat{\mathbb{T}})_d = \widehat{\mathbb{T}}_d$$

getting to the character group of \mathbb{T}_d .

The arguments in Example 3.11 and Example 3.12 are laborious and may not work for a large class of topological spaces. Therefore, we need more general structure theorems for free compact abelian groups which may make calculations of free compact abelian groups over even more sophisticated topological spaces easier. To that end, recall that a family of sets can be well-ordered by cardinality, and hence has a minimal element. In particular, we have the following definition.

Definition 3.13 (See [13], Definition A4.7). For a topological space X , the *weight* $w(X)$ of X is the minimal element of $\{|\mathcal{B}| : \mathcal{B} \text{ is a basis for } X\}$. The *reduced weight* of X is defined to be $w_0(X) = w(X) - 1$.

Clearly, the weight of a discrete space is the same as its cardinality.

Theorem 3.14 (See [13], Theorem A4.9). *If X is a compact Hausdorff topological space, then $w(X)^{\aleph_0} = \text{card } C_0(X, \mathbb{R}) = \text{card } C_0(X, \mathbb{T})$.*

Consider the short exact sequence:

$$(3.8) \quad 0 \rightarrow \mathbb{Z} \xrightarrow{j} \mathbb{R} \xrightarrow{q} \mathbb{T} \rightarrow 0$$

of topological groups and morphisms of topological groups. By [13, Proposition 8.50], (3.8) induces an exact sequence

$$(3.9) \quad 0 \longrightarrow C_0(X, \mathbb{Z}) \xrightarrow{C_0(X, j)} C_0(X, \mathbb{R}) \xrightarrow{C_0(X, q)} C_0(X, \mathbb{T}) \xrightarrow{\text{quot}} [X, \mathbb{T}] \longrightarrow 0$$

where

$$(3.10) \quad [X, \mathbb{T}] = C_0(X, \mathbb{T}) / C_0(X, \mathbb{T})_0.$$

Further, $C_0(X, \mathbb{T}) = C_0(X, \mathbb{T})_0 \oplus H$ where H is a discrete and torsion-free subgroup isomorphic to $[X, \mathbb{T}]$. Moreover, this splitting is topological since $C_0(X, \mathbb{T})_0$ is open, being the identity component of a Lie group, by [13, Exercise E8.9]. Applying this to the free compact abelian group $F(X)$ of X yields:

$$(3.11) \quad F(X) = C_0(\widehat{X}, \widehat{\mathbb{T}})_d \cong C_0(\widehat{X}, \widehat{\mathbb{T}})_0 \times \widehat{[X, \mathbb{T}]} \cong \left(\frac{C_0(X, \mathbb{R})_d}{C_0(X, \mathbb{Z})} \right) \times \widehat{[X, \mathbb{T}]}.$$

In particular, when X is a connected space, then $FX \cong C_0(\widehat{X}, \widehat{\mathbb{R}}) \times \widehat{[X, \mathbb{T}]}$.

We proceed to give a more precise structure of compact abelian groups. Recall from Theorem 2.43 that any divisible abelian group D has the structure

$$(3.12) \quad D = \mathbb{Q}^{(\text{rank } D)} \oplus \bigoplus_{p \text{ prime}} \mathbb{Z}(p^\infty)^{(\text{rank}_p(D))}.$$

In particular, $D = \frac{C_0(X, \mathbb{R})_d}{C_0(X, \mathbb{Z})_d}$ is divisible as a quotient of a rational vector space, it admits a decomposition as in (3.12), where $\text{rank}(D) = w(X)^{\aleph_0}$ and $\text{rank}_p(D) = w_0(X/\text{conn})$. In this situation one can see from [13, Proposition 8.50] that

$$(3.13) \quad C_0(X, \mathbb{T})_d = \mathbb{Q}^{(w(X)^{\aleph_0})} \oplus \bigoplus_{p \text{ prime}} \mathbb{Z}(p^\infty)^{(w_0(X/\text{conn}))} \oplus [X, \mathbb{T}]$$

which we dualise getting to a structural theorem for free compact abelian groups.

Theorem 3.15 (See [13], Lemma 8.65). *For a nonsingleton compact topological space X , we have*

$$(3.14) \quad F(X) = \widehat{\mathbb{Q}}^{w(X)^{\aleph_0}} \times \prod_{p \text{ prime}} \mathbb{Z}_p^{w_0(X/\text{conn})} \times \widehat{[X, \mathbb{T}]}$$

As an application of Theorem 3.15, consider the following example.

Example 3.16. Let X be a discrete compact space with a least two points. Then X is necessarily finite, and totally disconnected. Then $w(X) = \text{card}X = n$ for some $n \in \mathbb{N}, n \geq 2$. Hence

$$(3.15) \quad F(X) = \widehat{\mathbb{Q}}^{w(X)^{\aleph_0}} \times \prod_{p \text{ prime}} \mathbb{Z}_p^{w_0(X/\text{conn})} \times \widehat{[X, \mathbb{T}]} = \widehat{\mathbb{Q}}^{n^{\aleph_0}} \times \prod_{p \text{ prime}} \mathbb{Z}_p^{n-1}.$$

In (3.15) we used the fact that $\widehat{[X, \mathbb{T}]} = 0$ whenever X is a compact totally disconnected space (see [13, Exercise E8.14]). Now, when $n = 2$ we have:

$$(3.16) \quad F(X) = \widehat{\mathbb{Q}}^{2^{\aleph_0}} \times \prod_{p \text{ prime}} \mathbb{Z}_p = \widehat{\mathbb{T}}_d$$

which agrees with Example 3.12.

It is well known that every abstract abelian group is a quotient of a free one in $\mathbb{A}\mathbb{B}$; here we have the corresponding result in $\mathbb{C}\mathbb{A}\mathbb{B}$. We show a short proof, illustrating the methods which are direct consequences of the formalization which we made until here and of the appropriate definitions in the corresponding categories.

Proposition 3.17. *Every compact abelian group is a quotient of a free compact abelian group.*

Proof. For a compact abelian group G , the identity map $\text{id}_G : G \rightarrow G$ induces a surjective continuous homomorphism $f : F(G) \rightarrow G$ such that the diagram

$$(3.17) \quad \begin{array}{ccc} G & \xrightarrow{\varepsilon_G} & F(G) \\ \text{id}_G \downarrow & & \swarrow f \\ G & & \end{array}$$

commutes. Therefore $G \cong F(G) / \ker f$ as compact abelian groups. \square

We finish the section on free compact abelian groups by defining, constructing, and proving the uniqueness of the tensor product of compact abelian groups. The formalism is modelled after Proposition 2.23, but here bilinear maps are replaced by continuous bilinear maps.

Definition 3.18. Let G, H, K in \mathbb{CAB} be compact abelian groups. A continuous bilinear map is a continuous map $f : G \times H \rightarrow K$ satisfying

$$(3.18) \quad f(g_1 + g_2, h) = f(g_1, h) + f(g_2, h),$$

$$(3.19) \quad f(g, h_1 + h_2) = f(g, h_1) + f(g, h_2)$$

for all $g, g_i \in G$, and $h, h_i \in H, i = 1, 2$.

Then we give the corresponding notion of tensor product in \mathbb{CAB} .

Definition 3.19 (Tensor Product of Compact Groups). Let G and H in \mathbb{CAB} be compact abelian groups. A compact abelian group $G \otimes H$ in \mathbb{CAB} , together with a continuous bilinear map $G \times H \rightarrow G \otimes H$, is called a tensor product of G and H if for all continuous bilinear maps $f : G \times H \rightarrow A$ to a compact abelian group A in \mathbb{CAB} there exists a continuous homomorphism $f' : G \otimes H \rightarrow A$, unique in making the diagram

$$(3.20) \quad \begin{array}{ccc} G \times H & \xrightarrow{(g,h) \mapsto g \otimes h} & G \otimes H \\ f \downarrow & \swarrow f' & \\ A & & \end{array}$$

commutative, i.e., $f'(g \otimes h) = f(g, h)$.

Next, we show that the tensor product of compact abelian groups exists, offering an argument of proof which adapt to \mathbb{CAB} what is known for \mathbb{AB} .

Theorem 3.20 (Existence of Tensor Products of Compact Abelian Groups). *Let G and H in \mathbb{CAB} be compact abelian groups. There exists a compact abelian group $G \otimes H$ in \mathbb{CAB} together with a continuous bilinear map $(g, h) \in G \times H \mapsto g \otimes h \in G \otimes H$, such that for any continuous bilinear map $f : G \times H \rightarrow A$ into a compact abelian group A in \mathbb{CAB} there is a continuous homomorphism $f' : G \otimes H \rightarrow A$, unique in making the diagram*

$$(3.21) \quad \begin{array}{ccc} G \times H & \xrightarrow{(g,h) \mapsto g \otimes h} & G \otimes H \\ f \downarrow & \swarrow f' & \\ A & & \end{array}$$

is commutative, that is, $f'(g \otimes h) = f(g, h)$.

Proof. Note that $G \times H$ is a compact group and so the free compact abelian group $F(G \times H)$ over the underlying topological space of $G \times H$ exists by 3.10. By the universal property of free compact abelian groups (see Theorem 3.9), the continuous function $f : G \times H \rightarrow A$ induces a homomorphism of compact abelian groups $g : F(G \times H) \rightarrow A$, unique in making the diagram

$$(3.22) \quad \begin{array}{ccc} G \times H & \xrightarrow{\varepsilon} & F(G \times H) \\ f \downarrow & \swarrow g & \\ A & & \end{array}$$

commutative. Let K be the (topological) closure of the subgroup of $F(G \times H)$ that is algebraically generated by the elements

$$(3.23) \quad \varepsilon(g_1 + g_2, h) - \varepsilon(g_1, h) - \varepsilon(g_2, h), \varepsilon(g, h_1 + h_2) - \varepsilon(g, h_1) - \varepsilon(g, h_2)$$

for all $g, g_1, g_2 \in G, h, h_1, h_2 \in H$. Moreover

$$(3.24) \quad \begin{aligned} g(\varepsilon(g_1 + g_2, h) - \varepsilon(g_1, h) - \varepsilon(g_2, h)) &= g(\varepsilon(g_1 + g_2, h)) - g(\varepsilon(g_1, h)) - g(\varepsilon(g_2, h)) \\ &= f(g_1 + g_2, h) - f(g_1, h) - f(g_2, h) = 0 \end{aligned}$$

for all $g_1, g_2 \in G, h \in H$. Similarly,

$$(3.25) \quad g(\varepsilon(g, h_1 + h_2) - \varepsilon(g, h_1) - \varepsilon(g, h_2)) = 0$$

for all $g \in H, h_1, h_2 \in H$. Therefore g vanishes on K and hence, letting

$$(3.26) \quad \pi : \phi \in F(G \times H) \mapsto \phi + K \in F(G \times H)/K$$

to be the quotient morphism of compact groups, there is a continuous homomorphism $f' : F(G \times H)/K \rightarrow A$, unique in making the diagram

$$(3.27) \quad \begin{array}{ccc} F(G \times H) & \xrightarrow{\pi} & F(G \times H)/K \\ \downarrow g & \swarrow f' & \\ A & & \end{array}$$

commutative. Setting $g \otimes h = \pi(\varepsilon_{G \times H}(g, h))$ and $G \otimes H = F(G \times H)/K$, we have $f' : G \otimes H \rightarrow A$ as desired. It remains to show that f' is unique in making the diagram

$$(3.28) \quad \begin{array}{ccccc} G \times H & \xrightarrow{\varepsilon} & F(G \times H) & \xrightarrow{\pi} & F(G \times H)/K \\ \downarrow f & & & \nearrow f' & \\ A & & & & \end{array}$$

commutative. To that end, let $h : G \otimes H \rightarrow A$ be another continuous homomorphism such that $h(g \otimes h) = f(g, h)$ for all $g \in G, h \in H$. Then $h \circ \pi \circ \varepsilon = f' \circ \pi \circ \varepsilon$, and hence $h \circ \pi = f' \circ \pi$ the uniqueness of f' in making (3.22) commutative. Since π is surjective, it follows that $h = f'$, thus establishing the uniqueness of f' . \square

Next we show that the tensor product of compact abelian groups is unique. The nature of the proof adapts a well-known argument.

Theorem 3.21 (Uniqueness of the Tensor Product of Compact Abelian Groups). *Let G, H in $\mathbb{C}\mathbb{A}\mathbb{B}$ be compact abelian groups. Let T in $\mathbb{C}\mathbb{A}\mathbb{B}$ be a compact abelian group together with a continuous bilinear map $\alpha : G \times H \rightarrow T$ such that for any continuous bilinear map $f : G \times H \rightarrow A$ to a compact abelian group A in $\mathbb{C}\mathbb{A}\mathbb{B}$ there is a continuous homomorphism $f' : T \rightarrow A$ such that $f = f' \circ \alpha$. Then $T \cong G \otimes H$.*

Proof. Let $j : (g, h) \in G \times H \mapsto g \otimes h \in G \otimes H$ be the universal map of $G \otimes H$ as in Theorem 3.20. Apply Theorem 3.20 to the continuous bilinear map $\alpha : G \times H \rightarrow T$

to obtain a morphism $\beta_1 : G \otimes H \rightarrow T$ of compact abelian groups, unique in making the diagram

$$(3.29) \quad \begin{array}{ccc} G \times H & \xrightarrow{j} & G \otimes H \\ \alpha \downarrow & & \swarrow \beta_1 \\ T & & \end{array}$$

commutative, i.e., $\beta_1 \circ j = \alpha$. Similarly, applying the hypothesis of this theorem to the bilinear map $j : G \times H \rightarrow G \otimes H$, we obtain a morphism $\beta_2 : T \rightarrow G \otimes H$ of compact abelian groups, unique in making

$$(3.30) \quad \begin{array}{ccc} G \times H & \xrightarrow{\beta_2} & T \\ j \downarrow & & \swarrow \beta_2 \\ G \otimes H & & \end{array}$$

and so the diagram

$$(3.31) \quad \begin{array}{ccc} G \times H & \xrightarrow{\varepsilon} & G \otimes H \\ j \downarrow & & \swarrow \beta_2 \circ \beta_1 \\ G \otimes H & & \end{array}$$

is commutative. However, the identity $\text{id}_{G \otimes H}$ on $G \otimes H$ is unique in making the diagram

$$(3.32) \quad \begin{array}{ccc} G \times H & \xrightarrow{j} & G \otimes H \\ j \downarrow & & \swarrow \text{id}_{G \otimes H} \\ G \otimes H & & \end{array}$$

commutative. So $\beta_2 \circ \beta_1 = \text{id}_{G \otimes H}$. Similarly, $\beta_1 \circ \beta_2 = 1_T$, and hence $T \cong A \otimes B$ as required. \square

3.3. Free Compact Groups. We have seen in the category of abstract abelian groups and the category of compact abelian groups that objects are quotients of free objects. This fact is used to realise tensor products as certain quotients of free objects. With this formalism in mind, in this section we study free compact groups that may not necessarily be abelian.

Again, the free compact group over a topological space X is defined by a universal property as follows.

Definition 3.22 (See [13] Definition 11.1). A free compact group over a pointed topological space (X, x_0) is a compact group $F(X)$ in $\mathbb{K}\text{TOPG}$ together with a basepoint preserving continuous map $\varepsilon : X \rightarrow F(X)$ such that for every basepoint preserving continuous function $f : X \rightarrow G$ to a compact group G in $\mathbb{K}\text{TOPG}$ there

is a continuous homomorphism $f' : F(X) \rightarrow G$, unique in making the diagram

$$(3.33) \quad \begin{array}{ccc} X & \xrightarrow{\varepsilon} & F(X) \\ \downarrow f & \searrow f' & \\ G & & \end{array}$$

commutative, i.e., $f' \circ \varepsilon = f$.

Recall that there is another way of forming a compact Hausdorff space from any topological space, called a compactification. More precisely, recall the following definition.

Definition 3.23 (See, [33] Definition 19.1). Let X be a topological space. A compact Hausdorff space K together with an embedding $\alpha : X \rightarrow K$ of X into K such that $\overline{\alpha(X)} = K$ is called a *compactification* of X . We write (K, α) for brevity.

There are several ways of compactifying a topological space.

If (Y, f) and (Z, g) are compactifications of X , then we can consider the relation $(Y, f) \geq (Z, g)$ if there is a continuous function $h : Y \rightarrow Z$ such that the diagram

$$(3.34) \quad \begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow g & \downarrow h \\ & & Z \end{array}$$

is commutative; i.e., $g = h \circ f$.

In fact, according to [16, Theorem 22] the collection of all compactifications of X is partially ordered by \geq , where two compactifications (Y, f) and (Z, g) are equivalent if $h : Y \rightarrow Z$ in (3.34) is a homeomorphism.

Note that a compact Hausdorff space is a Tychonoff space, that is, a completely regular Hausdorff space (see [25, Corollary 7.2.7]), and this property is hereditary. Therefore a space X with a compactification is necessarily a Tychonoff space. With respect to this ordering, any Tychonoff space X has a largest compactification $(\beta X, e)$ called the *Stone-Ćech compactification* (see [16, Theorem 24]), which is constructed by embedding X into a cube as briefly illustrated below. This is a well known construction in topology and can be found in [10, 16, 17, 25, 33].

Remark 3.24. Take $[0, 1] \subseteq \mathbb{R}$, the Stone-Ćech compactification of a topological space X is realised as a pair $(e, \beta(X))$, where $e : X \rightarrow [0, 1]^{C(X, [0, 1])}$ is given by the evaluation map $e(x)(f) = f(x)$ for all $x \in X, f \in C(X, [0, 1])$. And $\beta(X)$ is the closure of $e(X)$ in the cube $[0, 1]^{C(X, [0, 1])}$. When X is a Tychonoff space, then e turns out to be an embedding, see for instance [16, Theorem 24].

The fact that the Stone-Ćech compactification of a topological space X is the largest with respect to the aforementioned partial order translates to a universal property which characterises the Stone-Ćech compactification of a topological space. In particular, we have the following proposition.

Proposition 3.25 (See [16], Theorem 24). *Let X be a completely regular topological space. If $f : X \rightarrow Y$ is a continuous map of X to a compact Hausdorff space Y , then there is a continuous map $f' : \beta X \rightarrow Y$ of f such that $f' \circ e = f$.*

$$(3.35) \quad \begin{array}{ccc} X & \xrightarrow{e} & \beta X \\ & \searrow f & \downarrow f' \\ & & Y \end{array}$$

In the language of categories, the Stone-Ćech compactification of X is adjoint to the inclusion functor $\mathbf{KTOP} \rightarrow \mathbf{TOP}$ from the category of compact Hausdorff spaces to the category of Hausdorff topological spaces (see [13, Lemma 11.2]).

From Definition 3.22 we have a functor $F : \mathbf{TOP}_0 \rightarrow \mathbf{KTOPG}$ where \mathbf{KTOPG} is the category of compact groups. There is a relation between β and F which essentially says that the free compact group over a topological space X can be obtained up to isomorphism by taking the Stone-Ćech compactification of X and then taking the free compact group of the compactification. Therefore, since every topological space X has a Stone-Ćech compactification, it is enough to study free compact groups taken over topological spaces that are already compact (see [13, Lemma 11.3]).

This also holds for free compact abelian groups. In fact, by [13, Lemma 11.4(i)], the group

$$(3.36) \quad F_{\text{ab}}X = F(X) / \overline{F(X)'} ,$$

where $F(X)'$ is the commutator subgroup of $F(X)$ is naturally isomorphic to the free compact abelian group over X (see Definition 3.9). Other types of free compact groups can be obtained similarly, that is, as quotients of free compact groups.

For instance, by [15, Proposition 17] and [13, Theorem 1.34] the group $F_z X \stackrel{\text{def}}{=} F(X) / (F(X))_0$ is profinite. It is a free object in the category of profinite groups (see [13, Lemma 11.4(ii)]). In particular, it is the *free profinite group* over X .

The universal property of free compact groups can be refined towards a more concrete statement. To that end, recall that by [13, Corollary 2.29] any compact group is isomorphic to some closed subgroup of some product of unitary groups. The idea of how to refine the universal property of free compact groups is straightforward; roughly, if every continuous map $f : X \rightarrow U(n_j)$ from a compact space to a unitary group extends to a morphism $f' : F(X) \rightarrow U(n_j)$ of compact groups, then, since every compact group is isomorphic to a subgroup of a product of unitary groups $U(n_j)$, any continuous map into a compact group can be extended to $F(X)$ by passing to the appropriate subgroup of the product. The details can be found in [13, Proposition 11.6] which we state without proof in the following proposition.

Proposition 3.26 (See [13], Proposition 11.6(i)). *Let X be a compact pointed topological space. If G is a compact group, $e : X \rightarrow G$ a continuous basepoint preserving function such that any basepoint preserving function $f : X \rightarrow U(n)$ to a unitary group extends to a continuous homomorphism of compact groups $f' : G \rightarrow U(n_j)$, then $\overline{\langle e(X) \rangle}$ with the corestriction of e to $\overline{\langle e(X) \rangle}$ is the free compact group on X .*

Proposition 3.26 allows for the construction of the free compact group over a topological space X . In particular, we have the following proposition.

Proposition 3.27 (See [13], Proposition 11.6(ii)). *Let X be a compact pointed topological space, $C_0(X, U(n))$ the space of continuous base point preserving maps*

from X into $U(n)$. Then the group $G = \prod_{n=1}^{\infty} U(n)^{C_0(X, U(n))}$ with $e : X \rightarrow G$ given by $e(x) = (e_n(x))_{n \in \mathbb{N}}$, where $e_n(x) : C_0(X, U(n)) \rightarrow U(n)$ is the evaluation $e_n(x)(f) = f(x)$, satisfies the hypothesis of Proposition 3.26.

Details of Proposition 3.27 are not very technical and can be found in [13, Proposition 11.6]. We only remark that part of the information obtained from Proposition 3.27 is not entirely surprising. In particular, we already know that any compact group and hence any free compact group is a subgroup of a product of unitary groups. Nonetheless, Proposition 3.27 gives us more information. Notably, it tells us which product we may consider as containing the free compact group on X as a closed subgroup, and gives at the very least an upper bound for the free compact group on X .

Remark 3.28. We shall now focus on the structure of free compact groups. Here we will find connections with the structure of finite groups, mentioned in Theorem 2.22, that is, we will find groups G possessing closed normal subgroups N such that $G = NH$ for H closed subgroup of G such that $H \cap N = 1$.

Firstly, recall the notion of a semidirect product defined as follows.

Definition 3.29 (See [27], pp.27). If N is a normal subgroup of an abstract group G , H a subgroup of G such that $G = HN$ and $H \cap N = \{1\}$, then G is said to be the (*internal*) *semidirect product* of N and H , written $G = N \rtimes H$ or $G = H \ltimes N$. N is called a *semidirect factor* of G , and H is a *complement* of N .

The semidirect product is described in [27, pp. 27 - 28].

Example 3.30. Let N in $\mathbb{A}\mathbb{B}$ be an abelian group, and $\mathbb{Z}(2)$ in $\mathbb{A}\mathbb{B}$ be the cyclic group $\mathbb{Z}(2) = \{1, -1\}$ of order 2. There is a homomorphism of groups $\beta : \mathbb{Z}(2) \rightarrow \text{Aut}N$ given by $\beta(1) = \text{id}_N$ and $\beta(-1) \in \text{Aut}N$ is the inversion map of the group N which is given by $n \in N \mapsto -n = \beta(-1)(n) \in N$. The semidirect product $N \rtimes_{\beta} \mathbb{Z}(2)$ is called the *generalised dihedral group* with respect to N , written $D(N)$. Notable examples are:

- (1) $D(\mathbb{Z}(n)) = \mathbb{Z}(n) \rtimes_{\beta} \mathbb{Z}(2)$ finite dihedral group pertaining to the symmetries of a regular polygon with n edges;
- (2) $D(\mathbb{Z}) = \mathbb{Z} \rtimes_{\beta} \mathbb{Z}(2)$ infinite dihedral group;
- (3) $D(\mathbb{T}) = \mathbb{T} \rtimes_{\beta} \mathbb{Z}(2)$ group of the isometries of the circle.

If, in Definition 3.29, all the groups considered are topological groups, and N is closed, then G it is a *topological semidirect factor*. The references [13, 14] have many examples of groups which are constructed in this way.

We return to the identity component of a free compact group. The following theorem says that the identity component of a free compact group is a topological semidirect factor.

Theorem 3.31. *Let X be a compact pointed topological space, and F_0X be the identity component of the free compact group $F(X)$ on X . Then there is a closed subgroup $H \cong F_z X$ of $F(X)$ such that $F(X) = F_0X \rtimes H$, i.e., F_0X is a topological semidirect factor of $F(X)$.*

We now pay attention to the center $Z(F(X))$ of the free compact group $F(X)$ over a compact pointed topological space X .

Theorem 3.32 (See [13], Theorem 11.14). *If X is a pointed compact topological space with $\text{card}X \geq 3$, then $Z(F(X)) \subseteq F_0X$. In particular, if $\text{card}X = 2$, then $Z(F(X)) = F(X)$.*

From Theorem 3.32 we observe that when X is a compact pointed space with two points, then $F(X)$ is a compact abelian group, and it coincides with the free compact abelian group $F_{\text{ab}}X$ over X . Further, if $F(X)$ is abelian, then $F(X) = Z(F(X)) \subseteq F_0X$. Therefore, a free compact group that is abelian is necessarily connected.

Lastly, we look at the commutator subgroup $F'X$ of the free compact group $F(X)$ over X . We see that under certain conditions, the commutator subgroup is a direct product factor of $F(X)$, according to the following theorem.

Theorem 3.33 (See [13], Theorem 11.19). *For a compact connected pointed topological space X , $F(X) \cong F_{\text{ab}}X \times F(X)'$ as compact groups if and only if the group $[X, \mathbb{T}]$ in (3.10) is divisible.*

4. NONABELIAN TENSOR PRODUCTS OF GROUPS

In this section we discuss the nonabelian tensor product of abstract groups, from which the tensor product of abstract abelian groups may be obtained as a special case. This notion can be found in [3, 4, 5, 28].

The nonabelian tensor product of abstract groups takes into account the actions of groups. So we first discuss the notion of a group action for abstract groups, and in anticipation of the tensor product of topological groups we also briefly discuss actions of topological groups as well.

Definition 4.1 (See [13], pp.5-6). An abstract group G is said to *act* on a set X if there is a function $(g, x) \in G \times X \mapsto g \cdot x \in X$ such that for all $x \in X$

- (1) $1_G \cdot x = x$,
- (2) $(gh) \cdot x = g(h \cdot x)$ for all $g, h \in G$.

An action of G on a set X defines a homomorphism $\pi : g \in G \mapsto \pi(g) \in \text{Sym}(X)$ given by $\pi(g)(x) = g \cdot x$ in the group of permutations $\text{Sym}(X)$ on X .

Of course, a proper definition is designed for our context.

Definition 4.2 (See [13], Definition 1.9). A topological group G acts on a topological space X if there is a continuous function $(g, x) \in G \times X \mapsto g \cdot x \in X$ that implements an action, in the sense of Definition 4.1, of the underlying abstract group of G on the underlying set of the topological space X .

We exclusively consider actions where G and X are both abstract groups or both topological groups, and are simultaneously acting on each other. Note that an abstract group G always acts on itself via conjugation $(g, x) \in G \times G \mapsto gxg^{-1} \in G$. If G is a topological group, Definitions 4.1 and 4.2 give an action of G onto itself putting $X = G$.

Strictly speaking, the definition of an action that we have given is called a *left action*. The notion of a *right action* is defined in the same way but the function that implements the right action of G on X is written $(x, g) \in X \times G \mapsto x \cdot g \in X$.

Remark 4.3. We write, without loss of generality, all actions as left actions, and, since we will deal only with actions via conjugation of a topological group G onto itself, the symbol x^g will denote until the rest of the present thesis always the conjugate of x by g in $G = X$, that is, we will have always the following situation: $(g, x) \in G \times G \mapsto g \cdot x = x^g = g^{-1}xg \in G$. Finally, we say that G acts on itself *trivially by conjugation*, if $x^g = gx = x$ for all $g \in G$.

The nonabelian tensor product of abstract groups was introduced in [4].

Definition 4.4 (See [3] pp.178). Two abstract groups, G and H , which act on themselves by conjugation are said to act *compatibly* on each other if

$$(4.1) \quad g^{(h^z)} = g^{z^{-1}hz} \quad \text{and} \quad h^{(g^t)} = h^{t^{-1}gt}$$

for all $g, z \in G, h, t \in H$.

The nonabelian tensor product of abstract groups is defined in [3] via generators and relations. However, in keeping with the spirit with which we defined the tensor product of abstract abelian groups and the tensor product of compact abelian groups, we shall define it using a universal property and then prove its existence which will implicitly yield the generators and relations anyway. Firstly, we define

a new type of map that will play the role that is played by bilinear maps in the abelian tensor product.

Definition 4.5 (See [3], Remark 3). Let A be an abstract group, and G, H be abstract groups acting on themselves via conjugation and on each other compatibly. A function $f : G \times H \rightarrow A$ is called a *crossed pairing* if for all $g, z \in G, h, t \in H$,

$$(4.2) \quad f(gz, h) = f(z^g, h^g) f(g, h) \quad \text{and} \quad f(g, ht) = f(g, h) f(g^h, t^h)$$

When G and H are abelian groups acting on each other trivially, then we may specialize to

$$(4.3) \quad f(gz, h) = f(g, h) f(z, h) \quad \text{and} \quad f(g, ht) = f(g, h) f(g, t).$$

respectively, and so in this case crossed pairings specialise to bilinear maps encountered before in Definition 2.1.

The following universal property defines the nonabelian tensor product of abstract groups.

Definition 4.6 (Nonabelian Tensor Product of Abstract Groups, See [3], Remark 3). Let A be any abstract group, and G and H be abstract groups acting on themselves by conjugation and on each other compatibly. A tensor product of G and H is an abstract group $G \otimes H$ together with a crossed pairing $(g, h) \in G \times H \mapsto g \otimes h \in G \otimes H$ such that for any crossed pairing $G \times H \rightarrow A$ there is an abstract homomorphism $f' : G \otimes H \rightarrow A$, unique in making the diagram

$$(4.4) \quad \begin{array}{ccc} G \times H & \xrightarrow{(g,h) \mapsto g \otimes h} & G \otimes H \\ f \downarrow & \swarrow f' & \\ A & & \end{array}$$

commutative, i.e., $f'(g \otimes h) = f(g, h)$.

The above notion is a generalization of the notion of abelian tensor product in $\mathbb{A}\mathbb{B}$ via the notion of bilinear map: we have largely discussed the details of the notions in previous chapters. Here we are extending the notions out from $\mathbb{A}\mathbb{B}$, but still no topology is present on the objects which we consider. The following result shows that nonabelian tensor products have the universal property, exactly as it happened in the abelian case in $\mathbb{A}\mathbb{B}$, and, having this property is a condition to ensure their existence in a category of abstract groups.

Theorem 4.7 (Existence of the Nonabelian Abstract Tensor Product). *Let G and H be abstract groups that act on themselves by conjugation and on each other compatibly. Then there is an abstract group $G \otimes H$ and a crossed pairing $(g, h) \in G \times H \mapsto g \otimes h \in G \otimes H$ such that for any crossed pairing $f : G \times H \rightarrow A$ to an abstract group, there is an abstract homomorphism $f' : G \otimes H \rightarrow A$, unique in making the diagram*

$$(4.5) \quad \begin{array}{ccc} G \times H & \xrightarrow{(g,h) \mapsto g \otimes h} & G \otimes H \\ f \downarrow & \swarrow f' & \\ A & & \end{array}$$

commutative, i.e., $f'(g \otimes h) = f(g, h)$.

Proof. Let $F(G \times H)$ be the free abstract group over $G \times H$, with universal map $\sigma : (g, h) \in G \times H \mapsto [(g, h)] \in F(G \times H)$, where $[(g, h)]$ is the equivalence class containing the word (g, h) . By the universal property of abstract free groups (see [27, pp.44 - 45]) there is an abstract homomorphism $p : F(G \times H) \rightarrow A$, unique in making the diagram

$$(4.6) \quad \begin{array}{ccc} G \times H & \xrightarrow{\sigma} & F(G \times H) \\ f \downarrow & \swarrow p & \\ A & & \end{array}$$

commutative. Let K be the subgroup of $F(G \times H)$ that is generated by the elements

$$(4.7) \quad \sigma(gz, h)\sigma(g, h)^{-1}\sigma(z^g, h^g)^{-1} \text{ and } \sigma(g, ht)\sigma(g^t, h^t)^{-1}\sigma(g, h)^{-1}$$

for all $g, z \in G, h, t \in H$. Then

$$(4.8) \quad \begin{aligned} p(\sigma(gz, h)\sigma(g, h)^{-1}\sigma(z^g, h^g)^{-1}) &= p(\sigma(gz, h))p(\sigma(g, h)^{-1})p(\sigma(z^g, h^g)^{-1}) \\ &= f(gz, h)f(g, h)^{-1}f(z^g, h^g)^{-1} \\ &= f(z^g, h^g)f(g, h)f(g, h)^{-1}f(z^g, h^g)^{-1} = 1 \end{aligned}$$

Similarly, we have

$$(4.9) \quad p(\sigma(g, ht)\sigma(g^t, h^t)^{-1}\sigma(g, h)^{-1}) = 1,$$

therefore, p vanishes on K and hence, taking $\pi : (g, h) \in G \times H \mapsto (g, h)K \in F(G \times H)/K$ to be the quotient homomorphism, there is an abstract homomorphism $f' : F(G \times H)/K \rightarrow A$, unique in making the diagram

$$(4.10) \quad \begin{array}{ccc} F(G \times H) & \xrightarrow{\pi} & F(G \times H)/K \\ p \downarrow & \swarrow f' & \\ A & & \end{array}$$

commutative. Setting $G \otimes H = F(G \times H)/K$, and $g \otimes h = \pi(\sigma(g, h))$, it remains to show that f' is unique in making the diagram

$$(4.11) \quad \begin{array}{ccccc} G \times H & \xrightarrow{\sigma} & F(G \times H) & \xrightarrow{\pi} & F(G \times H)/K \\ f \downarrow & & & \nearrow f' & \\ A & & & & \end{array}$$

commutative. To that end, let $f'' : F(G \times H)/K$ be another abstract homomorphism such that $f'' \circ \pi \circ \sigma = f' \circ \pi \circ \sigma$. Then $f'' \circ \pi = p$ by the uniqueness of p in making (4.6) commutative. Therefore, $f'' \circ \pi = f' \circ \pi$ and hence $f'' = f'$ since π is surjective. \square

The nonabelian tensor product $G \otimes H$ of abstract groups G and H is unique up to isomorphism, and a proof of this works the same as the proof of Theorem 3.21. When G and H are abelian groups, then conjugation is a trivial action on both of them. Still, the nonabelian tensor product may not coincide with the abelian tensor product of G and H since the actions of these groups on each other must be taken

into account. However, if G and H act trivially on each other, then their nonabelian tensor product is exactly their tensor product as abstract abelian groups.

Now we specialise to the case where $G = H$. The group $G \otimes G$ is called the *tensor square* of G . The *exterior square* of G is the group

$$(4.12) \quad G \wedge G = G \otimes G / \nabla(G),$$

where

$$(4.13) \quad \nabla(G) = \langle g \otimes g \mid g \in G \rangle$$

turns out to be a normal subgroup of $G \otimes G$. The commutator map $(g_1, g_2) \in G \times G \mapsto [g_1, g_2] \in G$ is a crossed pairing, and hence induces a homomorphism $\kappa : G \otimes G \rightarrow G$ such that

$$(4.14) \quad \kappa(g_1 \otimes g_2) = [g_1, g_2]$$

for all $g_1, g_2 \in G$. Moreover, the kernel

$$(4.15) \quad J_2(G) = \ker \kappa$$

of κ is central in $G \otimes G$ by [3, Proposition 4], and induces the following central extension

$$(4.16) \quad 1 \longrightarrow \nabla(G) \longrightarrow J_2(G) \longrightarrow H_2(G, \mathbb{C}^\times) \longrightarrow 1,$$

where $H_2(G, \mathbb{C}^\times)$ is the Schur multiplier of G . In addition, the following diagram is commutative with central rows:

$$(4.17) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \nabla(G) & \longrightarrow & G \otimes G & \longrightarrow & [G, G] \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H_2(G, \mathbb{C}^\times) & \longrightarrow & G \wedge G & \longrightarrow & [G, G] \longrightarrow 0. \end{array}$$

These commutative diagrams can be used to make computations and to deduce properties of the nonabelian exterior square. For example, we see in [3, Proposition 5] that $G \otimes G$ is finite whenever G is finite, and it is a p -group whenever G is a p -group.

Definition 4.8 (See [3], pp.182). The universal covering group of a group G is a central extension

$$(4.18) \quad 1 \longrightarrow H_2(G, \mathbb{C}^\times) \xrightarrow{j} \widehat{G} \longrightarrow G \longrightarrow 1$$

of the Schur multiplier $H_2(G, \mathbb{C}^\times)$ by G where $\text{Im } j \subseteq \widehat{G}'$.

When G is perfect, $G \otimes G$ is the unique covering group of G , by [3, Corollary 1]. Furthermore, when $H_2(G, \mathbb{C}^\times)$ is finitely generated, then $G \wedge G \cong [\widehat{G}, \widehat{G}]$, by [3, Corollary 2].

As an application consider the following example.

Example 4.9 (Tensor Products of the Alternating Groups, See [3], pp. 196-197).

- (1) Since the alternating groups A_n are simple for $n > 4$, they are perfect and so $A_n \otimes A_n$ is the unique covering group of A_n , by [3, Corollary 1]. So determining $A_n \otimes A_n$ is tantamount to determining the covering group of A_n .
- (2) $A_4 \otimes A_4 = \mathbb{Z}(3) \times Q_2$, where Q_2 is the quaternionic group of order 8 defined in [3, p.187]. See [3, p.183] for details.

5. CONCLUSIONS

We illustrate that it is possible to look at the nonabelian tensor product of topological groups as a topological generalisation of the nonabelian tensor product of abstract groups. Firstly we briefly describe how to adapt some notions of abstract groups to a topological setting.

We say that two topological groups G and H act on each other *compatibly with continuity* if their actions on each other are continuous and compatible. Further, we say that a function $f : G \times H \rightarrow A$ is a *continuous crossed pairing*, where A is also a topological group, if it is a crossed pairing of the underlying abstract groups and a continuous function of the underlying topological spaces.

Definition 5.1 (See [28]). Let G and H be topological groups which act on themselves via conjugation and which act on each other compatibly with continuity. The nonabelian tensor product of G and H is defined to be a topological group $G \otimes H$ together with a continuous cross pairing $j : G \times H \rightarrow G \otimes H$ such that for any continuous crossed pairing $f : G \times H \rightarrow A$ to a topological group A , there exists a homomorphism of topological groups $f' : G \otimes H \rightarrow A$, unique in making the diagram

$$(5.1) \quad \begin{array}{ccc} G \times H & \xrightarrow{j} & G \otimes H \\ \downarrow f & \swarrow f' & \\ A & & \end{array}$$

commutative, i.e., $f' \circ j = f$.

The notion of nonabelian tensor product of topological group has been recently discussed in [28, 29, 30] and is subject to a categorical approach, as shown in [5]. If we restrict ourselves to the category of compact groups, profinite groups, and pro- p -groups, by replacing each occurrence of topological group(s) by compact group(s), profinite group(s), and pro- p -group(s), respectively, we obtain the notion of a nonabelian tensor product of compact groups, profinite groups, and pro- p -groups, respectively.

Of course, if we focus on a specific category, then we can make an appropriate study of the corresponding notion of nonabelian tensor product in it, and this has been done in [28, 29, 30]. On the other hand, we noted that cohomological methods are necessary when we introduce the nonabelian tensor products, since central extensions and corresponding commutative diagrams occur. Here is useful to mention the reference [19], which illustrate the continuous cohomology of the locally compact groups, in fact a cohomological theory exists for several classes of topological groups, especially for the locally compact groups.

Recall the filters of compact normal subgroups $\mathcal{F}(G)$, $\mathcal{P}(G)$, and $\mathcal{N}(G)$ of a compact group G , which we introduced earlier in (1.41), (1.42), (1.43).

Theorem 5.2 (See [28], Theorem 1.1). *Let $G = \varinjlim_{N \in \mathcal{F}(G)} G/N$ and $H = \varinjlim_{M \in \mathcal{F}(H)} H/M$ be profinite groups. Then there exists a natural isomorphism of profinite groups such that*

$$(5.2) \quad \varinjlim_{(N,M) \in \mathcal{F}(G) \times \mathcal{F}(H)} (G/N \otimes H/M) \cong \varinjlim_{N \in \mathcal{F}(G)} G/N \otimes \varinjlim_{M \in \mathcal{F}(H)} H/M.$$

In particular, if G and H are pro- p -groups, then

$$(5.3) \quad \lim_{(N,M) \in \mathcal{P}(G) \times \mathcal{P}(H)} (G/N \otimes H/M) \cong \lim_{N \in \mathcal{P}(G)} G/N \otimes \lim_{M \in \mathcal{P}(H)} H/M.$$

This result contains a series of important information for the nonabelian tensor products of topological groups.

The first information is given by the fact that (5.2) and (5.3) provide approximations of the nonabelian tensor product of certain topological groups, or, in other words, properties of invariance of the nonabelian tensor product under projective limits of profinite groups in the category of compact groups.

This is relevant, because it allows us to reconstruct the nonabelian tensor product of profinite groups from the usual nonabelian tensor product of finite groups.

The second information is given by the fact that one could use the isomorphism in the proof of Theorem 5.2, in order to introduce from the beginning the notion of nonabelian tensor product of profinite groups.

Note that the notion of topological generator, as we saw, is completely different from that of abstract generator, and in fact a theory for this is present for locally compact groups (see [7, 8], for instance).

Finally, one could find more interactions between the existing theory of cohomology of locally compact groups and the classical theory of cohomology of finite groups via (5.2) and (5.3). We saw that (4.17) connects the notion of nonabelian tensor square with the notion of Schur multiplier.

We end with a few considerations and questions, which we noted in the course of our work for the present MSc thesis.

Repeatedly, we have seen that the existence of free groups allows for tensor products to be realised as quotients of certain free groups by appropriately chosen subgroups. Further, we have seen from Theorem 5.2 that the topology of the nonabelian tensor product of profinite groups may be described as an appropriate projective limit of nonabelian tensor products of certain quotients.

Following these facts, it is reasonable to think that $F(X) = F_0X \rtimes K$ as in Theorem 3.31 implies:

$$(5.4) \quad \lim_{(N,M) \in \mathcal{F}(G) \times \mathcal{F}(H)} (G/N \otimes H/M) \simeq \frac{F_0X \rtimes K}{\ker f}$$

where f are appropriate continuous cross pairings.

This turns out to be an open problem in the literature so far as we have seen. On the other hand, [32] provides evidence that for nonabelian tensor products of abstract groups the answer is positive.

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