

A Framework for Homotopy Theory and
Differential Geometry in the Category of
Frölicher Spaces

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Chapter 1

Introduction

Frölicher spaces are a modern invention originating in the mid 1980's, but they are quite classical in spirit. They allow one to define a notion of smoothness of functions between sets by reducing all questions of smoothness to a problem in the calculus of real-valued functions of a single real variable.

The notion of a Frölicher space is based on work by A. Frölicher and A. Kriegl [47], and L. Lawvere, S. Schanuel and W. R. Zame [81]. Frölicher and Kriegl showed in [47] that Frölicher spaces form a category, which we denote **FRL**, with useful properties; the most useful one from our point of view being Cartesian closedness.

Cherenack began investigating homotopy theory in the category of Frölicher spaces in [22]. In this paper he showed that some of the basic objects needed for homotopy theory existed in **FRL** in a natural way. In particular he showed how to construct the homotopy groups of a Frölicher space, and then in [24] Cherenack gave an indication of how one might construct the left Puppe sequence in the category **FRL**.

In his masters' thesis [36] this author showed that under the assumption that a certain type of modified product structure could be placed on certain products in **FRL**, the right Puppe sequence exists in **FRL** and is right exact.

In their book [78] Kriegl and Michor elaborated on the uses of Frölicher spaces in the context of functional analysis, and incidentally began calling the spaces 'Frölicher spaces'. Up until this point they had generally been referred to as 'smooth spaces' which was a somewhat unsatisfactory name due to the fact that any subset of \mathbb{R}^n can be given a 'smooth structure' in a natural way, but the resulting structure need not be smooth in terms of our intuitive idea of smoothness.

At about the same time Cherenack independantly began calling these spaces Frölicher spaces in his series of papers relating Frölicher spaces to cosmology and differential spaces. See [23], [27] and [25].

This thesis divides naturally into two parts. The aim of the first part is to lay a solid foundation for the study of homotopy theory in **FRL**. We do this by defining suitable notions of cofibration and fibration in **FRL**, and then showing

that Baues' cofibration and fibration axioms are satisfied by **FRL** with these definitions.

Of the two dual sets of axioms, it is the cofibration axioms that are the more difficult to deal with, due to problems with smooth extensions of smooth functions. We verify the cofibration axioms in Chapter 6, using tools developed in the earlier part of the thesis.

The fibration axioms can be shown to be satisfied in **FRL** in a way that is quite close to the verification of the axioms for topological spaces. We do this in Appendix A.

In addition to this we show in Appendix B that the 'cube axiom' of Doeraene [32] is largely satisfied by **FRL**. The 'cube axiom' together with the fibration and cofibration axioms implies that all of Doeraene's work on L.S.-category holds in **FRL**, see [32].

The second part of the thesis is a short look at differential geometry in **FRL**. Some of the basic notions of differential geometry, such as tangent spaces, have been defined for Frölicher spaces by Cherenack in [23]. Because Frölicher spaces are much more general than smooth manifolds (they may have singularities), one cannot expect usually that the corresponding results hold. In particular we give an example to show that the inverse function theorem does not hold for general Frölicher spaces.

For this reason we will restrict our attention to quotients of smooth manifolds where the quotient is taken by a compact submanifold. These are not in general manifolds, but have a natural Frölicher structure. We use this Frölicher structure to show that one has an inverse function theorem for Frölicher spaces arising from quotients by a finite set on a smooth manifold. For quotients by a submanifold of a smooth manifold we obtain an explicit description of the 'tangent cone' at the identified point. The 'tangent cone' is a notion specific to Frölicher spaces that is similar to the notion of 'tangent space'. For a smooth manifold the two notions coincide.

1.1 How Frölicher spaces arise in functional analysis

The book by Frölicher and Kriegl [47] uses Frölicher spaces in their theory of the calculus of convenient vector spaces. Their idea is to observe that for a classical smooth manifold or a Banach space, the family of smooth curves and the family of smooth real-valued functions determine each other in the sense that a map is a *smooth curve* if and only if its composite with every smooth real-valued function is a smooth real mapping, and a map is a *smooth real-valued function* if and only if its composite with every smooth curve is a smooth real mapping. Frölicher and Kriegl show that one can enlarge the category of Banach spaces to a category of 'convenient vector spaces'. Manifolds modelled on convenient vector spaces are shown by Frölicher and Kriegl to be a class of spaces in which functional analysis plays a useful role.

1.2 Frölicher spaces and their relation to smooth manifolds

One can look at Frölicher spaces from the point of view of differential geometry. They can then be thought of as a generalisation of the concept of smooth manifold. In particular, a Frölicher space abstracts only the notion of ‘smooth mapping’, and so ignores all of the other structure of a smooth manifold, such as the local Euclidean nature of a smooth manifold.

The structure curves and structure functions of a smooth manifold, M , are exactly the smooth curves $c : \mathbb{R} \rightarrow M$ and smooth real-valued functions $f : M \rightarrow \mathbb{R}$, respectively. One can define the notions of ‘tangent vector’, ‘tangent space’ and ‘tangent bundle’ for a Frölicher space in such a way that these notions agree with the usual definitions in the case that the Frölicher space is a smooth manifold. Of course a Frölicher space does not have nearly as rigid a structure as a smooth manifold, and so these constructs of differential geometry can take an unexpected form when the Frölicher space is not a smooth manifold. See Cherenack [23], [27].

1.3 Homotopy in the category FRL

This thesis is concerned primarily with homotopy theory in the category of Frölicher spaces. The structure of a Frölicher space determines a natural topology on the space, and so one can study the homotopy theory of Frölicher spaces from a purely topological standpoint. However, it is often desirable and instructive to work within the category when studying the various homotopy constructs. A starting point is to look at which maps between two Frölicher spaces may be deformed into one another in a smooth way. The papers by Cherenack [22], [24] begin with this idea, and make adjustments to the usual topological methods, arriving at the ‘left Puppe sequence’. This approach is somewhat problematical when one tries to construct a ‘right Puppe sequence’, see [36]. In this thesis we begin with the notions of ‘smooth homotopy equivalence’ and ‘smooth cofibration’, and then show that Baues’ cofibration axioms [7] are satisfied by the category of Frölicher spaces with these two classes of mappings.

This approach puts much of the homotopy theory of Frölicher spaces on a secure foundation, allowing all the results that are derived from the axioms to be utilised.

1.4 Breakdown of chapters

Chapter one is this introduction; the second chapter consists of the preliminary results that are used throughout the thesis. These results are mainly concerned with the definition of a Frölicher space, and the categorical constructions associated with the spaces, such as product spaces, coproduct spaces and subspaces.

Quotient spaces are also discussed, and the notion of a smooth braking function is introduced, following Hirsch [58]. Some useful results concerning braking functions are proved.

Chapter 3 introduces the new notion of a ‘pre-metrizable’ Frölicher space. The notion of a pre-metrizable space is similar to the notion of a metrizable topological space. The topological procedure of embedding a topological space into a ‘Hilbert cube’ is imitated in a smooth setting, enabling us to show that certain Frölicher spaces are pre-metrizable. The Stone-Weierstrass theorem is applied to pre-metrizable Frölicher spaces enabling us to determine the underlying topology of certain Frölicher spaces. This chapter also provides smooth variants of the ‘Urysohn Lemma’ and a weak ‘Tietze Extension Theorem’.

The purpose of Chapter 4 is to introduce the basic homotopy constructs in a smooth setting. Some constructs are variants of work by Cherenack [22], others are original. In particular we define the notion of smooth cofibration which is important for our later work in the homotopy theory of Frölicher spaces. We also use the work of the previous chapter to investigate the underlying topologies of certain homotopy objects in the category of Frölicher spaces.

Chapter 5 gives a solution to the problem of how to define a reasonable notion of ‘neighbourhood deformation retract’ (NDR) for a Frölicher space. For topological spaces NDR’s are essentially equivalent to closed cofibrations. We prove a similar equivalence for Frölicher spaces. We then show how a right exact ‘Puppe sequence’ may be constructed in the category of Frölicher spaces.

The sixth chapter provides a general setting for homotopy theory in the category of Frölicher spaces by verifying that Baues’ cofibration axioms are satisfied by this category. The process of verifying these axioms give a good indication where homotopy in the category of Frölicher spaces differs from that of topological spaces. Our work on smooth NDR pairs is used in the verification of one of the axioms.

Chapter 7 provides the basic definitions needed for the study of differential geometry of Frölicher spaces. These definitions are adapted from similar definitions for smooth manifolds, and some are due to Cherenack [23].

Chapters 8 and 9 are concerned with quotients on smooth manifolds. We identify a submanifold of a manifold to a point, and give the resulting space the quotient Frölicher structure. Our aim is to show that for such spaces one has an inverse function theorem similar to the usual inverse function theorem for smooth manifolds. Chapter 8 introduces the ideas by concentrating on manifolds where a finite set of points is identified to a point, and derives an inverse function theorem for such Frölicher spaces, while Chapter 9 is concerned with the more general situation where the quotient is taken by a submanifold, and concludes with an explicit description of the ‘tangent cone’ of such a Frölicher space at the identified point. We expect that an inverse function theorem will hold for these Frölicher spaces, and our description of the ‘tangent cone’ could serve as a starting point for further study.

Appendix A is a verification of Baues’ fibration axioms [7]. These axioms are dual to the cofibration axioms. As with the cofibration axioms, certain definitions and constructions are more elaborate than in the topological situation.

Appendix B is a partial verification of the J-axioms for the category of Frölicher spaces. These axioms give a framework in which Lusternick-Schnirelmann theory can be studied in a category and are due to Doeraene [32]. For a complete verification of Doeraene's axioms one still needs to show that the pullback of a cofibration by fibrations is a cofibration.

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Chapter 2

Preliminaries

In this chapter we describe the category of Frölicher spaces, and define the basic ideas that are used throughout this thesis. We also prove some technical results that are needed later.

2.1 Frölicher spaces

We begin by recalling some basic notions regarding Frölicher spaces. For more detail, see Frölicher and Kriegl [47], Cherenack [24], [22] or Dugmore[36].

The category of Frölicher spaces is defined as follows.

2.1.1. Definition. *An object in the category **FRL** is a triple (X, C_X, F_X) where X is a set, called the **underlying set**, C_X is a collection of maps of the form $c: \mathbb{R} \rightarrow X$, called the **structure curves**, and F_X is a collection of maps of the form $f: X \rightarrow \mathbb{R}$, called the **structure functions**, which satisfy the following conditions:*

1. $\Phi C_X = \{f: X \rightarrow \mathbb{R} \mid f \circ c \text{ smooth for all } c \in C_X\} = F_X,$
2. $\Gamma F_X = \{c: \mathbb{R} \rightarrow X \mid f \circ c \text{ smooth for all } f \in F_X\} = C_X.$

*A morphism $g: (X, C_X, F_X) \rightarrow (Y, C_Y, F_Y)$ in **FRL** is a map $g: X \rightarrow Y$, such that $g \circ C_X \subset C_Y$ or equivalently $F_Y \circ g \subset F_X$. The objects of **FRL** are called **Frölicher spaces**, and the pair (C_X, F_X) is called the **smooth structure on X** , or the **Frölicher structure on X** . We will usually write X for a Frölicher space (X, C_X, F_X) .*

The following result is of fundamental importance.

2.1.2. Proposition. *The category **FRL** has initial and final structures with respect to the forgetful functor to **Sets**.*

Proposition 2.1.2 is essentially a paraphrase of Proposition 1.1.4 in Frölicher and Kriegl [47].

Explicitly the initial and final structures are defined as follows. Let X_j be a family of Frölicher spaces, and let $g_j: A \rightarrow X_j$ be a family of maps from a set A to the Frölicher space X_j . The **initial structure** on A generated by the family of maps g_j is given by

$$F_A = \Phi\{c: \mathbb{R} \rightarrow A \mid g_j \circ c \in C_{X_j}, \text{ for all } j\},$$

$$C_A = \Gamma F_A.$$

Given a collection $g_j: X_j \rightarrow A$ of maps from the Frölicher space X_j to a set A , the **final structure** on A generated by the family of maps g_j is given by

$$C_A = \Gamma\{f: A \rightarrow \mathbb{R} \mid f \circ g_j \in F_{X_j}, \text{ for all } j\},$$

$$F_A = \Phi C_A.$$

As we shall see, many Frölicher structures such as products, coproducts, quotients and subspaces are defined as initial or final structures generated by a given collection of maps.

An important special case of a final structure is the following. If we start with an arbitrary collection F of functions $f: X \rightarrow \mathbb{R}$, we can define $C_X = \Gamma F$, and $F_X = \Phi C_X$. We call this smooth structure the **smooth structure generated by the set of functions F** .

Let X and Y be Frölicher spaces with the same underlying set. We say that the Frölicher structure on X is **coarser** than the Frölicher structure on Y if $F_X \subseteq F_Y$. Conversely, if $F_Y \subseteq F_X$, then we say that the Frölicher structure on X is **finer** than the Frölicher structure on Y .

2.2 Subspaces of a Frölicher space

Let $A \subseteq X$, and let $i: A \rightarrow X$ denote the inclusion map. Then (A, C_A, F_A) is called a (Frölicher) subspace of the Frölicher space (X, C_X, F_X) , provided C_A and F_A are defined by the initial smooth structure generated by the set of functions $\{f \circ i: A \rightarrow \mathbb{R} \mid f \in F_X\}$ (see Frölicher and Kriegel [47]). Thus, one can show that

1. the structure curves on A are the elements of the set

$$C_A = \{c: \mathbb{R} \rightarrow A \mid i \circ c \in C_X\}, \text{ and}$$

2. the structure functions on A are elements of the set $F_A = \Phi C_A$.

In the theory of differential spaces (see Sikorski [111]) one defines structure functions on a subspace A of a differential space X to be locally the restrictions of structure functions on X .

For a Frölicher space observe that the structure functions on a subspace A of X are generated by the restrictions of structure functions on X , as described above.

2.3 Products and coproducts

Suppose we are given a collection of Frölicher spaces

$$\{(X_i, C_{X_i}, F_{X_i})\}, (1 \leq i \leq n), n \in \mathbb{N}.$$

Then the Cartesian product $\prod_{i=1}^n X_i$ (in **Sets**) is given the initial structure generated by the set of projection maps

$$p_j: \prod_{i=1}^n X_i \rightarrow X_j, \text{ for } j = 1, \dots, n.$$

The coproduct $\sqcup_{i=1}^n X_i$ (in **Sets**) is given the final structure generated by the inclusions $j_i: X_i \rightarrow \sqcup_{i=1}^n X_i$. We shall often have occasion to deal with maps of the form $h: X \sqcup Y \rightarrow Z$, for Frölicher spaces X, Y and Z . If h has components f and g , we shall often write $h = [f, g]$.

One extends these definitions to infinite products and coproducts.

2.4 The underlying topology of a Frölicher space

There is a functor

$$\mathcal{T}: \mathbf{FRL} \rightarrow \mathbf{TOP},$$

which replaces the Frölicher structure on a set X with a topology given by the Frölicher structure, called the underlying topology, defined as follows.

2.4.1. Definition. *The underlying topology of a Frölicher space X is the topology with subbasis*

$$\mathcal{U} = \{f^{-1}(0, 1)\}_{f \in F_X}.$$

Frölicher and Kriegl define the underlying topology as in Definition 2.4.1, but for us there is another basis that is also useful:

2.4.2. Lemma. *The collection of subsets of X ,*

$$\mathcal{B} = \{f^{-1}(0, \infty)\}_{f \in F_X}$$

forms a basis for the underlying topology. Similarly, \mathcal{U} is a basis.

Proof. We show that \mathcal{B} is a basis. The proof of the fact that \mathcal{B} generates the underlying topology is left to the reader. (*Hint:* For a subbasic open set $U = f^{-1}(0, 1)$, compose f^{-1} with $\phi(t) = e^{-\frac{1}{t}}$ for $t > 0$ to get $U = f^{-1} \circ \phi(0, \infty)$.)

To show that \mathcal{B} is a basis we need to show that \mathcal{B} is closed under finite intersections.

Let $\alpha: \mathbb{R} \rightarrow \mathbb{R}$ be the smooth function with the properties that

1. $\alpha(t) = 0$ for $t \leq 0$,
2. $\alpha(t) = e^{-\frac{1}{t}}$ for $t > 0$.

See Section 2.8.

To see that \mathcal{B} is closed under finite intersections, let $U_1, U_2 \in \mathcal{B}$. Suppose that $U_1 = f_1^{-1}(0, \infty)$ and $U_2 = f_2^{-1}(0, \infty)$, where $f_1, f_2 \in F_X$. Then let $f_3 = \alpha(f_1)\alpha(f_2)$. Let $x \in U_1 \cap U_2$. Clearly $f_3(x) \in (0, \infty)$, and so $U_1 \cap U_2 \subseteq f_3^{-1}(0, \infty)$.

On the other hand, suppose that $f_3(x) \in (0, \infty)$. Then $\alpha(f_1(x)) \in (0, \infty)$ and $\alpha(f_2(x)) \in (0, \infty)$. From the definition of α , we deduce that $f_1(x) \in (0, \infty)$ and $f_2(x) \in (0, \infty)$. This implies that $x \in U_1 \cap U_2$. \square

The functor \mathcal{T} leaves morphisms unchanged on the level of sets. The open intervals in \mathbb{R} form a basis for the Euclidean topology, and each open interval is diffeomorphic to $(0, 1)$. Thus for any smooth map $f: X \rightarrow Y$, the map $\mathcal{T}(f)$ is continuous in the underlying topology, and so is a morphism in the category **TOP**.

We shall usually suppress the functor notation, and talk about X as a Frölicher space and a topological space interchangeably, when we really mean that X is the Frölicher space, and that $\mathcal{T}(X)$ is the associated topological space.

For topological spaces one has the useful property that a continuous injective mapping on a compact space is a homeomorphism onto its image. It is unfortunately not true that a smooth injective mapping on a compact Frölicher space has to be a diffeomorphism. There is the following counterexample.

2.4.3. Example. Let I denote the unit interval with the Frölicher structure it inherits as a subspace of \mathbb{R} . Then a curve $c: \mathbb{R} \rightarrow I$ is a structure curve if and only if when composed with the inclusion of I into \mathbb{R} it is a structure curve on \mathbb{R} . Let I_0 be the Frölicher space having the unit interval as its underlying set with the Frölicher structure on I_0 the structure generated by the two functions $f_0, f_1: I_0 \rightarrow \mathbb{R}$, given by

$$f_0(t) = \begin{cases} \frac{1}{2} - t & \text{if } t \leq \frac{1}{2} \\ 2t - 1 & \text{if } t > \frac{1}{2}, \end{cases}$$

and

$$f_1(t) = \begin{cases} \frac{1}{2} - t & \text{if } t \leq \frac{1}{2} \\ 4t - 2 & \text{if } t > \frac{1}{2}. \end{cases}$$

One can show without much difficulty that any $c \in C_{I_0}$ with $c(t_0) = \frac{1}{2}$, for some $t_0 \in \mathbb{R}$, is a structure curve on I with the property that $c'(t_0) = 0$. (*Hint:* Look at left and right derivatives).

Now define $g: I_0 \rightarrow I$ by $g(t) = t$. This is clearly a bijection. It is smooth since any curve on I_0 is a curve on I , and so the composite of g with any element of C_{I_0} is an element of C_I . One now observes that the mapping $g^{-1}: I \rightarrow I_0$ is not smooth, since not every curve on I is a curve on I_0 .

In fact, one can show that $\mathcal{T}(I_0)$ and $\mathcal{T}(I)$ in the above example are homeomorphic at the topological level, (one follows the same procedure that is used in Section 4.3 to show that the ‘flattened unit interval’ is homeomorphic to the usual unit interval), and then g is an example of a map which is not a diffeomorphism, but is a homeomorphism in the underlying topology. This is an indication that the underlying topology of a Frölicher space does not carry all the information regarding the Frölicher structure on the Frölicher space, which is not surprising.

An important point worth noticing right away is contained in the following lemma.

2.4.4. Lemma. *If a subset A of a Frölicher space X is given the Frölicher subspace structure, then the topology underlying the Frölicher subspace structure on A is finer than the subspace topology that the set A inherits as a topological subspace of X .*

Proof. Let $i : A \rightarrow X$ denote the inclusion map, let \mathcal{T} be the topology underlying the Frölicher subspace structure on A , and let \mathcal{T}^* be the subspace topology that A inherits as a topological subspace of X . We wish to show that $\mathcal{T}^* \subseteq \mathcal{T}$. Let $U \in \mathcal{T}^*$. Then there is an open set $V \subseteq X$ in the underlying topology on X , such that $U = i^{-1}(V)$.

From Lemma 2.4.2, we may write the open set V as

$$V = \bigcup_{\lambda \in \Lambda} f_\lambda^{-1}(0, \infty),$$

where each f_λ is a structure function on X , and Λ is some indexing set. Then

$$\begin{aligned} U &= i^{-1}(V) \\ &= i^{-1}\left(\bigcup_{\lambda \in \Lambda} f_\lambda^{-1}(0, \infty)\right) \\ &= \bigcup_{\lambda \in \Lambda} (f_\lambda \circ i)^{-1}(0, \infty), \end{aligned}$$

and since each $f_\lambda \circ i$ is a structure function on A , the last line represents an open set in the underlying topology on A . This shows that $U \in \mathcal{T}$. \square

In other words the Frölicher subspace structure can induce a strictly finer topology than the underlying topology. An interesting example of this is the inclusion of the **Koch snowflake** K into \mathbb{R}^2 . (See Dugmore [36]). In this case K is homeomorphic to the unit circle in the subspace topology, but the topology underlying the Frölicher subspace structure is discrete. See Falconer [40] for the construction of the Koch snowflake, and a discussion of other aspects of this fractal curve.

We now prove a result concerning the countable product of Frölicher spaces. We show that the underlying topology on a countable Frölicher product is finer than the usual product topology. In fact, it is not difficult to prove that this is

the case, even for an arbitrary product of Frölicher spaces (see Cherenack [22]), but we shall only be concerned with countable products, and so we only prove the result in this case.

2.4.5. Lemma. *Let $\{X_n | n \in \mathbb{N}\}$ be a countable collection of Frölicher spaces. Then the underlying topology on*

$$X = \prod_{n=1}^{\infty} X_n$$

is finer than the usual product topology.

Proof. Let \mathcal{T}_X^* denote the usual product topology on X , and \mathcal{T}_X denote the topology induced on the product by the Frölicher structure. We wish to show that $\mathcal{T}_X^* \subseteq \mathcal{T}_X$. A set of the form

$$U = \prod_{n=1}^{\infty} U_n$$

is an open set in \mathcal{T}_X^* if each of the U_n is open set in X_n , and all but a finite number of the $U_n = X_n$. We show that $U \in \mathcal{T}_X$.

Suppose that each U_n is a basic open set in X_n , unless $U_n = X_n$. There are smooth functions such that $U_{n_i} = f_{n_i}^{-1}(0, \infty)$ for each $n_i \in \mathbb{N}$ for which $U_{n_i} \neq X_{n_i}$. Define $g_{n_i} : X \rightarrow \mathbb{R}$ by $g_{n_i}(x) = f_{n_i} \circ p_{n_i}(x)$, where $p_{n_i} : X \rightarrow X_{n_i}$ is the projection map. Then each g_{n_i} is clearly smooth, and each $g_{n_i}^{-1}(0, \infty) \in \mathcal{T}_X$. But

$$U = \bigcap_{i=1}^k g_{n_i}^{-1}(0, \infty) \in \mathcal{T}_X.$$

□

In particular, the result holds for finite products.

2.5 Function spaces

It has been shown by Frölicher and Kriegl [47] that the category **FRL** of Frölicher spaces is Cartesian closed (See also Kriegl and Michor [78]). This useful fact implies that function spaces of smooth mappings have a natural Frölicher structure. This structure is generated in the following way.

Let X and Y be Frölicher spaces. Let **FRL**(X, Y) denote the set of smooth maps from X to Y . Define a set of curves on **FRL**(X, Y) by

$$C = \{e : \mathbb{R} \rightarrow \mathbf{FRL}(X, Y) | \hat{e} : \mathbb{R} \times X \rightarrow Y \text{ is smooth}\}$$

where \hat{e} is the evaluation map given by

$$\hat{e}(t, x) = e(t)(x).$$

We then define the structure functions on $\mathbf{FRL}(X, Y)$ to be

$$F_{\mathbf{FRL}(X, Y)} = \Phi C,$$

and the structure curves can then be shown to be

$$C_{\mathbf{FRL}(X, Y)} = \Gamma \Phi C = C.$$

We always give the function space $\mathbf{FRL}(X, Y)$ the structure defined above. It is shown by Frölicher and Kriegl [47] that this function space structure behaves functorially, and that with the given structure on $\mathbf{FRL}(X, Y)$, the category \mathbf{FRL} is Cartesian closed.

Since the category \mathbf{FRL} is Cartesian closed we have the exponential law

$$\mathbf{FRL}(X \times Y, Z) \approx \mathbf{FRL}(X, \mathbf{FRL}(Y, Z))$$

for any Frölicher spaces X , Y and Z .

2.6 Quotient Frölicher spaces

2.6.1. Definition. Let \sim be an equivalence relation on a Frölicher space X . Let $q: X \rightarrow X/\sim$ be the set quotient map. We define the structure curves $C_{X/\sim}$ and the structure functions $F_{X/\sim}$ on X/\sim as follows.

- $F_{X/\sim} = \{f: X/\sim \rightarrow \mathbb{R} \mid f \circ q: X \rightarrow \mathbb{R} \text{ is smooth on } X\}$.
- $C_{X/\sim} = \Gamma(F_{X/\sim})$.

It is not difficult to see that $\Phi \Gamma(F_{X/\sim}) = F_{X/\sim}$. We will be dealing mostly with a special case of the above construction; namely the identification of a closed subset of a Frölicher space to a point. If A is a Frölicher subspace of the Frölicher space X , and $q: X \rightarrow X/A$ is the identification map, then we consider the following sequence of maps,

$$A \xrightarrow{i} X \xrightarrow{q} X/A \xrightarrow{f} \mathbb{R}.$$

The map $f: X/A \rightarrow \mathbb{R}$ is a structure function if and only if the composite $f \circ q: X \rightarrow \mathbb{R}$ is a structure function on X . Thus every structure function on X/A gives rise to a structure function on X which is constant over A .

Conversely, let $\tilde{f}: X \rightarrow \mathbb{R}$ be a structure function on X which is constant over A . Then the map $f: X/A \rightarrow \mathbb{R}$, given by

$$f([x]) = \tilde{f}(x), \text{ for } [x] \in X/A$$

is a structure function on X/A . So there is a one-to-one correspondence between structure functions on X/A , and structure functions on X which are constant over A . This is a point to which we shall refer in later work.

In fact we have the following simple but important lemma.

2.6.2. **Lemma.** *Given the commutative diagram of solid arrows below,*

$$\begin{array}{ccccc} A & \xrightarrow{i} & X & \xrightarrow{q} & X/A \\ & \searrow y_0 & \downarrow g & \swarrow \tilde{g} & \\ & & Y & & \end{array}$$

where $y_0 \in Y$ and q is the quotient map, we can fill in an arrow \tilde{g} such that the diagram commutes. Thus, q is the quotient map in **FRL**.

Proof. Define $\tilde{g}: X/A \rightarrow Y$ by

$$\begin{aligned} \tilde{g}([x]) &= \tilde{g}(q(x)) \\ &= g(x). \end{aligned}$$

The fact that $g \circ i(a) = y_0$ for all $a \in A$ ensures that \tilde{g} is well-defined.

To see that \tilde{g} is smooth, let $f: Y \rightarrow \mathbb{R}$ be a structure function on Y . Then we have

$$\begin{aligned} f \circ \tilde{g}([x]) &= f \circ \tilde{g}(q(x)) \\ &= f \circ g(x). \end{aligned}$$

The map $f \circ g$ is smooth since g is, and the result is proved. \square

Notice that the structure curves into X/A can be described in the following way. We have the diagram,

$$\mathbb{R} \xrightarrow{c} X \xrightarrow{q} X/A \xrightarrow{f} \mathbb{R}.$$

Obviously, if c is a structure curve on X , then $q \circ c$ is a structure curve on X/A . In addition to curves on X/A arising in this way, if the composite $g \circ c$ is a smooth real mapping, where c is not necessarily a structure curve on X , for all structure functions, f , on X that are constant over A , then $q \circ c$ is a structure curve on X/A .

2.7 Wedge products

2.7.1. **Definition.** *The wedge product $\mathbb{R}^m \vee \mathbb{R}^n$, ($m, n \in \mathbb{N}$), of \mathbb{R}^n and \mathbb{R}^m , is defined to be*

$$(\mathbb{R}^m \sqcup \mathbb{R}^n) / \{0^m, 0^n\},$$

the quotient in **FRL** with the origins $0^m, 0^n$ in \mathbb{R}^m and \mathbb{R}^n , respectively, identified to a point. We denote this identified point by 0 .

The above construction is a special case of a more general construction. (See end of Section 2.9). We extend the above definition to a finite wedge product using the notation

$$\dot{X} = \bigvee_{i=1}^k \mathbb{R}_i^{m_i}, \quad m_i \in \mathbb{N}.$$

In fact, given a collection of k topological (or Frölicher) spaces with basepoint,

$$\{(X_i, x_i)\}, \quad 1 \leq i \leq k,$$

one can define the wedge product

$$X = \bigvee_{i=1}^k X_i = \left(\bigsqcup_{i=1}^k X_i \right) / \{x_1, x_2, \dots, x_k\},$$

in FRL or TOP. We shall use the same definition in a slightly different context. Let V_1, V_2, \dots, V_k be a collection of k vector spaces over \mathbb{R} . Let 0_i denote the zero element of the vector space V_i , for $1 \leq i \leq k$. Define

$$\bigvee_{i=1}^k V_i = \bigsqcup_{i=1}^k V_i / \{0_1, 0_2, \dots, 0_k\},$$

in the category \mathbf{Sets}_* , where the zero element 0_i forms the basepoint of the underlying set of V_i , for each $1 \leq i \leq k$.

Of course $\bigvee_{i=1}^k V_i$ is not a vector space itself, but each subset $V_j \subseteq \bigvee_{i=1}^k V_i$ has the structure of a vector space for each $1 \leq j \leq k$.

2.8 Smooth braking functions

In this section we will show how to construct smooth braking functions, following the method of Hirsch [58]. Smooth braking functions are tools used throughout this thesis. These functions are useful because they allow us to convert piecewise smooth functions to smooth functions, by ‘smoothing’ across the joins of the smooth pieces.

At the end of this section we introduce some notation for a special class of smooth braking functions, and we prove a technical result that is of use in Chapter 6.

It is well known that the function $\phi: \mathbb{R} \rightarrow \mathbb{R}$, given by

$$\phi(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ e^{-\frac{1}{x^2}} & \text{if } x > 0. \end{cases}$$

is smooth (See, for example, Binmore [10]).

Now let us construct a smooth function $\alpha: \mathbb{R} \rightarrow \mathbb{R}$ with the following properties.

Let $0 \leq a < b$. We want $\alpha(t)$ to satisfy:

- $\alpha(t) = 0$ for $t \leq a$,
- $0 < \alpha(t) < 1$ for $a < t < b$,
- α strictly increasing for $a < t < b$,
- $\alpha(t) = 1$ for $t \geq b$.

To define α , start with the smooth function $\phi: \mathbb{R} \rightarrow \mathbb{R}$, defined above. Next define a new function $\gamma: \mathbb{R} \rightarrow \mathbb{R}$ by

$$\gamma(x) = \phi(x - a)\phi(b - x).$$

Then $\alpha: \mathbb{R} \rightarrow I$ is given by

$$\alpha(t) = \frac{\int_a^t \gamma(x) dx}{\int_a^b \gamma(x) dx}.$$

Of course the above construction need not be the only way of constructing a smooth braking function. We shall call a smooth braking function defined in the above way a **standard smooth braking function**, or a **standard braking function**, it being understood that all braking functions considered in this thesis are assumed to be smooth.

Furthermore, the lowercase Greek letter α always refers to some smooth braking function, although not necessarily a standard one.

More specifically, for $0 < \epsilon < \frac{1}{2}$, the notation α_ϵ always refers to a smooth braking function with the following properties.

- $\alpha_\epsilon(t) = 0$ for $t \leq \epsilon$,
- $0 < \alpha_\epsilon(t) < 1$ for $\epsilon < t < 1 - \epsilon$,
- α strictly increasing for $\epsilon < t < 1 - \epsilon$,
- $\alpha_\epsilon(t) = 1$ for $1 - \epsilon \leq t$.

We now prove a series of lemmas that give us the following useful property. For a standard braking function α_ϵ , and $0 < \epsilon < \frac{1}{2}$ there is a constant $0 < k_\epsilon < \frac{1}{2}$ such that $0 \leq t - \alpha_\epsilon(t) < k_\epsilon$ for $0 \leq t \leq \frac{1}{2}$.

2.8.1. Lemma. *Let $0 < \epsilon < \frac{1}{2}$. Then $\alpha_\epsilon(\frac{1}{2}) = \frac{1}{2}$ if α_ϵ is a standard braking function.*

Proof. We have

$$\alpha_\epsilon(t) = \frac{\int_\epsilon^t \gamma_\epsilon(x) dx}{\int_\epsilon^{1-\epsilon} \gamma_\epsilon(x) dx},$$

where $\gamma_\epsilon(x) = \phi(x - \epsilon)\phi(1 - \epsilon - x)$. Thus

$$\begin{aligned}
 1 &= \frac{\int_\epsilon^{1-\epsilon} \gamma_\epsilon(x) dx}{\int_\epsilon^{1-\epsilon} \gamma_\epsilon(x) dx} \\
 &= \frac{\int_\epsilon^{\frac{1}{2}} \gamma_\epsilon(x) dx}{\int_\epsilon^{1-\epsilon} \gamma_\epsilon(x) dx} + \frac{\int_{\frac{1}{2}}^{1-\epsilon} \gamma_\epsilon(x) dx}{\int_\epsilon^{1-\epsilon} \gamma_\epsilon(x) dx} \\
 &= \frac{\int_\epsilon^{\frac{1}{2}} \phi(x - \epsilon)\phi(1 - \epsilon - x) dx}{\int_\epsilon^{1-\epsilon} \phi(x - \epsilon)\phi(1 - \epsilon - x) dx} + \frac{\int_{\frac{1}{2}}^{1-\epsilon} \phi(x - \epsilon)\phi(1 - \epsilon - x) dx}{\int_\epsilon^{1-\epsilon} \phi(x - \epsilon)\phi(1 - \epsilon - x) dx} \\
 &= 2 \frac{\int_\epsilon^{\frac{1}{2}} \phi(x - \epsilon)\phi(1 - \epsilon - x) dx}{\int_\epsilon^{1-\epsilon} \phi(x - \epsilon)\phi(1 - \epsilon - x) dx} \\
 &= 2 \frac{\int_\epsilon^{\frac{1}{2}} \gamma_\epsilon(x) dx}{\int_\epsilon^{1-\epsilon} \gamma_\epsilon(x) dx} \\
 &= 2\alpha_\epsilon\left(\frac{1}{2}\right)
 \end{aligned}$$

Thus $\alpha_\epsilon\left(\frac{1}{2}\right) = \frac{1}{2}$. □

2.8.2. Lemma. *Let $0 < \epsilon < \frac{1}{2}$. Then the function $f(t) = t - \alpha_\epsilon(t)$ is concave down on the interval $(\epsilon, \frac{1}{2})$, if α_ϵ is a standard braking function.*

Proof. We have

$$\begin{aligned}
 f'(t) &= 1 - \alpha'_\epsilon(t) \\
 &= 1 - A_\epsilon \gamma_\epsilon(t), \text{ by the fundamental theorem of calculus,}
 \end{aligned}$$

where $\gamma_\epsilon(t) = \phi(t - \epsilon)\phi(1 - \epsilon - t)$, and

$$A_\epsilon = \left(\int_\epsilon^{1-\epsilon} \gamma_\epsilon(x) dx \right)^{-1} \geq 0.$$

From this we compute

$$\begin{aligned}
 f''(t) &= -\frac{d}{dt} A_\epsilon \gamma_\epsilon(t) \\
 &= -A_\epsilon \frac{d}{dt} (\phi(t - \epsilon)\phi(1 - \epsilon - t)) \\
 &= -A_\epsilon \phi'(t - \epsilon)\phi(1 - \epsilon - t) + A_\epsilon \phi(t - \epsilon)\phi'(1 - \epsilon - t) \\
 &= -A_\epsilon \left[\underbrace{\frac{1}{(t - \epsilon)^3}}_{\geq 0, \epsilon < t < \frac{1}{2}} + \underbrace{\frac{1}{(1 - \epsilon - t)^3}}_{\geq 0, \epsilon < t < \frac{1}{2}} \right] \underbrace{e^{-\frac{1}{(t-\epsilon)^2}} e^{-\frac{1}{(1-\epsilon-t)^2}}}_{\geq 0} \\
 &\leq 0.
 \end{aligned}$$

□

2.8.3. Lemma. *Let $0 < \epsilon < \frac{1}{2}$ then there exists $0 < k_\epsilon < \frac{1}{2}$ such that $0 \leq t - \alpha_\epsilon(t) < k_\epsilon$ for $0 \leq t < \frac{1}{2}$, if α_ϵ is a standard braking function.*

Proof. Define $f(t) = t - \alpha_\epsilon(t)$. We have $f(t) = t$ for $0 \leq t \leq \epsilon$ and $f(0) = 0 = f(\frac{1}{2})$. From Lemma 2.8.2 we know that f is concave down on the interval $(\epsilon, \frac{1}{2})$. Thus we may use elementary calculus to deduce that $f(t) \geq 0$ for $\epsilon < t < \frac{1}{2}$. This gives us the inequality $0 \leq f(t) < \frac{1}{2}$ for $0 \leq t \leq \frac{1}{2}$. Thus there is some number $0 < k < \frac{1}{2}$ such that $0 \leq f(t) \leq k < \frac{1}{2}$ for $0 \leq t \leq \frac{1}{2}$. Let $k_\epsilon = \frac{k + \frac{1}{2}}{2}$. Then we have $0 \leq f(t) \leq k < k_\epsilon < \frac{1}{2}$ for $0 \leq t \leq \frac{1}{2}$. □

2.9 The category of pointed Frölicher spaces

We denote the category of pointed Frölicher spaces and basepoint preserving maps by \mathbf{FRL}_* :

2.9.1. Definition. *The objects in \mathbf{FRL}_* are Frölicher spaces with a distinguished basepoint. The morphisms are basepoint preserving maps in \mathbf{FRL} . We will represent the basepoint of a pointed Frölicher space X by $*_X$ or $*$ if there is no chance of confusion.*

The basepoint of the Cartesian product $\prod_{i=1}^n X_i$ of Frölicher spaces is the point $(*_X, *_X, \dots, *_X)$. Thus projection maps are basepoint preserving smooth maps.

The coproduct or wedge product of two pointed Frölicher spaces X_1 and X_2 is $X_1 \sqcup_* X_2 = (X_1 \sqcup X_2) / \sim$ where \sim identifies the basepoints of X_1 and X_2 .

The unit interval I is given the basepoint 0.

Note that the wedge product defined in Section 2.7 is just the categorical sum of the two pointed Frölicher spaces, \mathbb{R}^n and \mathbb{R}^m .

2.10 Notation

2.10.1. Definition. *Let (A, d) be a metric space. Let $\epsilon > 0$. We define*

$$B_\epsilon(a) = \{x \in A \mid d(x, a) < \epsilon\},$$

called the open ball centred at a , with radius ϵ .

Chapter 3

Pre-metrizable Frölicher spaces

We begin by defining the notion of “pre-metrizability” for a Frölicher space, and show that if a Frölicher space X is pre-metrizable and compact in the underlying topology, an application of the Stone-Weierstrass Theorem allows us to deduce that X is metrizable (by a metric whose square is a smooth function).

As an aside we apply the Stone-Weierstrass theorem to show that Frölicher spaces that are compact in the underlying topology have useful properties, such as a “Urysohn lemma”, a weak form of the “Tietze extension theorem”, existence of smooth partitions of unity subordinate to an open cover, etc.

It has been shown by Cherenack [25] (see also Kriegl and Michor [78]) that any Frölicher space X can be embedded in a function space of the form \mathbb{R}^F where $F \subseteq F_X$ is a generating set of functions, provided F has a sufficiently “rich” structure. We prove a similar result, and use it to derive properties of certain Frölicher spaces that admit a countable generating set of smooth functions.

In particular we show that a Frölicher space X is pre-metrizable if the Frölicher structure on X is generated by a countable collection of functions which “separate points in X ”. If the Frölicher structure on X is generated by a finite set of functions F , with this “point separation property”, then this fact is equivalent to the fact that X can be embedded in some \mathbb{R}^n .

3.1 Pre-metrizable Frölicher spaces

We define a new notion for Frölicher spaces called “pre-metrizability”, which will turn out to be very useful in later work.

3.1.1. Definition. *Let X be a Frölicher space.*

1. *A structure function $f: X \rightarrow \mathbb{R}$ is said to generate the underlying topology on X if the set*

$$U = \{f^{-1}(V) : V \text{ open in } \mathbb{R}\}$$

forms a basis for the underlying topology on X .

2. A Frölicher space X is called **pre-metrizable** if there exists a smooth function

$$d: X \times X \rightarrow \mathbb{R}^+,$$

where $\mathbb{R}^+ = \{t \in \mathbb{R} \mid t \geq 0\}$ with the Frölicher subspace structure, and where \sqrt{d} satisfies the usual metric space axioms. That is, for each $x, y, z \in X$ the following must hold.

- (a) $0 \leq \sqrt{d(x, y)} < \infty$,
- (b) $\sqrt{d(x, y)} = 0$ if and only if $x = y$,
- (c) $\sqrt{d(x, y)} = \sqrt{d(y, x)}$,
- (d) $\sqrt{d(x, y)} + \sqrt{d(y, z)} \geq \sqrt{d(x, z)}$.

We call the function d a **pre-metric** on X . If there exists a pre-metric d that generates the underlying topology on X , then we say that X is **metrizable**, and call \sqrt{d} a **metric**.

It is important to notice that the definition of a pre-metrizable Frölicher space X only postulates the *existence* of a smooth pre-metric on X , but does not *specify* any particular one.

3.1.2. Lemma. Let $d: X \times X \rightarrow \mathbb{R}^+$ be a pre-metric on X . Then d and \sqrt{d} generate the same topology on X .

Proof. The result follows from the fact that an open ball $B_\epsilon(x_0)$ of radius ϵ ($\epsilon > 0$) and centred at x_0 can be written in either of the following ways:

$$\begin{aligned} B_\epsilon(x_0) &= \{x \in X \mid \sqrt{d(x, x_0)} < \epsilon\} \\ &= \{x \in X \mid d(x, x_0) < \epsilon^2\}. \end{aligned}$$

□

3.1.3. Example. Let $d: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^+$ be the function $d(x, y) = (x - y)^2$. Then d generates the Euclidean topology on \mathbb{R} , which is also the underlying topology of \mathbb{R} when treated as a Frölicher space. Thus d is a pre-metric, and $\sqrt{d(x, y)} = \sqrt{(x - y)^2} = |x - y|$ defines a metric on \mathbb{R} .

If a Frölicher space X is pre-metrizable, then the underlying topology on X is always finer than the topology induced on X by any pre-metric d . To see this, notice that for every $x \in X$, the map

$$d(x, \cdot): X \rightarrow \mathbb{R}$$

is smooth, and so each open ball is given by

$$B_\epsilon(x) = d(x, \cdot)^{-1}([0, \epsilon^2]).$$

Since $d(x, \cdot)$ is smooth on X (and hence continuous), the ball defined above must be an open set in the underlying topology.

Sometimes we will need to specify that a subset A of a pre-metrizable Frölicher space X is open or closed in a given pre-metric d . To simplify our notation, we shall often say simply that A is d -open or d -closed. Of course a d -open set is always open in the underlying topology on X , for any pre-metric d . A subset of X that is bounded for a pre-metric d will be called d -bounded, etc. If we wish to refer to a property P associated with a pre-metric d on X , we shall always refer to the d - P property.

The underlying topology on X is by definition the coarsest topology in which every structure function is continuous. Since any d -topology is even coarser, in general, for a pre-metric d , there can be structure functions which are not continuous in a given d -topology. Clearly if there is a pre-metric for which all structure functions are continuous, then the Frölicher space is metrizable, and this pre-metric is a metric.

To show that a Frölicher space is pre-metrizable, all we need to do is embed the space in another pre-metrizable Frölicher space, as the following lemma indicates.

3.1.4. Lemma. *Let A be a Frölicher subspace of the Frölicher space X , and let i be the inclusion map. If X is pre-metrizable with pre-metric*

$$d: X \times X \rightarrow \mathbb{R}^+,$$

then A is pre-metrizable with pre-metric

$$d \circ (i \times i): A \times A \rightarrow \mathbb{R}^+.$$

Proof. The map $d \circ (i \times i)$ is obviously smooth, and it is a triviality to verify the metric space axioms for $\sqrt{d \circ (i \times i)}$. \square

3.2 The Stone-Weierstrass theorem applied to the underlying topology and d -topologies of a pre-metrizable Frölicher space

Applying the Stone-Weierstrass theorem to a pre-metrizable Frölicher space that is compact in the underlying topology, we arrive at the main result of this section, which states that a pre-metrizable Frölicher space that is compact in the underlying topology is metrizable and, furthermore, any pre-metric defines a metric generating the underlying topology.

Before we prove this result we introduce some notation. For the rest of this chapter we shall be concerned mainly with Frölicher spaces that are compact in the underlying topology. For the sake of brevity we make the following definition.

3.2.1. Definition. *A Frölicher space whose underlying topology is compact is called a compact Frölicher space.*

3.2.2. Definition. Let X be a compact Frölicher space. Let $C(X)$ denote the algebra of all real-valued functions on X that are continuous in the underlying topology on X . We give $C(X)$ the norm

$$\|f\| = \sup \{|f(x)| : x \in X\}.$$

If X is pre-metrizable, then for a given pre-metric d , we let $C_d(X)$ denote the subalgebra of $C(X)$ of all real-valued functions on X that are continuous in the d -topology. We give $C_d(X)$ the norm that it inherits from $C(X)$.

3.2.3. Definition. A generating set F of functions for a Frölicher space X is said to separate points in X if for every $x, y \in X$ there exists an $f \in F$ such that $f(x) \neq f(y)$.

We now state the Stone-Weierstrass Theorem exactly as it appears in Kelley [75]. See [75] for an outline of the proof, or Engelking [38] for a full proof. Naber [94] gives a proof of the theorem for Euclidean spaces, and the book by Arkhangel'skiĭ [2] discusses the space $C(X)$ in detail. See also Arkhangel'skiĭ [3].

3.2.4. Theorem. Let X be a compact topological space. Then a subalgebra R of $C(X)$ is dense in $C(X)$ if it has the property that for distinct points x and $y \in X$ and real numbers a and b , there is a function f in R such that $f(x) = a$ and $f(y) = b$. This is called the two-point property.

Equivalently, R is dense in $C(X)$ if it separates points (for $x, y \in X$ there exists an $f \in R$ with $f(x) \neq f(y)$).

This result will enable us to approximate continuous maps (in the underlying topology) arbitrarily closely by smooth maps, and also to use d -continuous maps to approximate smooth or continuous (in the underlying topology) maps as closely as necessary.

3.2.5. Lemma. If a compact Frölicher space X admits a generating set that separates points, then F_X is dense in $C(X)$.

Proof. The constant functions are always smooth, so F_X is a subalgebra of $C(X)$ that satisfies the conditions of Theorem 3.2.4. \square

3.2.6. Lemma. If X is a compact pre-metrizable Frölicher space with pre-metric d , then $C_d(X)$ is dense in $C(X)$.

Proof. The constant maps are always d -continuous, and since the d -topology is Hausdorff (it is metrizable) there is a d -continuous function that separates points in X . Given distinct $x, y \in X$, one can choose the d -continuous function $d(x, -)$ to separate x and y . Apply Theorem 3.2.4. \square

3.2.7. Corollary. Let X be a compact Frölicher space that is pre-metrizable with a pre-metric d . Consider a structure function $f \in F_X$. Then for any $\epsilon > 0$, there exists $h \in C_d(X)$ with $|f(x) - h(x)| < \epsilon$ for every $x \in X$.

Another application of Lemma 3.2.6 is that for a compact pre-metrizable Frölicher space any d -continuous real-valued function can be approximated arbitrarily closely by a d' -continuous real-valued function, where d and d' are different pre-metrics on X .

We now use these results to compare the underlying topology of a compact pre-metrizable Frölicher space with the d -topology determined by a pre-metric d .

3.2.8. Lemma. *Let U be a basic open set in the underlying topology of a compact pre-metrizable Frölicher space X . Then there exists a sequence of d -continuous functions $\{g_n\}_{n \in \mathbb{N}}$ such that*

$$U = \bigcup_{n=1}^{\infty} g_n^{-1}\left(\frac{1}{n}, 1 - \frac{1}{n}\right).$$

Proof. Since U is a basic open set in the underlying topology, there is a smooth function $f \in F_X$ such that

$$U = f^{-1}(0, 1).$$

For each $n \in \mathbb{N}$, choose a d -continuous function $g_n: X \rightarrow \mathbb{R}$ such that

$$\|f - g_n\| < \frac{1}{n}.$$

Notice that for each $x \in X$,

$$g_n(x) - \frac{1}{n} < f(x) < g_n(x) + \frac{1}{n}.$$

Choose $x \in g_n^{-1}\left(\frac{1}{n}, 1 - \frac{1}{n}\right)$. Then $\frac{1}{n} < g_n(x) < 1 - \frac{1}{n}$, and so the previous inequality becomes

$$0 = \frac{1}{n} - \frac{1}{n} < f(x) < 1 - \frac{1}{n} + \frac{1}{n} = 1.$$

This gives the inclusion

$$\bigcup_{n=1}^{\infty} g_n^{-1}\left(\frac{1}{n}, 1 - \frac{1}{n}\right) \subseteq U,$$

since each $g_n^{-1}\left(\frac{1}{n}, 1 - \frac{1}{n}\right) \subseteq U$.

We now prove the reverse inclusion. Let $x \in U$. Then $f(x) \in (0, 1)$. Let

$$\delta = \min\{f(x), 1 - f(x)\}.$$

Let $m \in \mathbb{N}$ be large enough that $\frac{1}{m} < \frac{\delta}{2}$. Then we have

$$|g_m(x) - f(x)| < \frac{1}{m} < \frac{\delta}{2}.$$

Notice that

$$\frac{\delta}{2} \leq f(x) - \frac{\delta}{2}$$

and

$$f(x) + \frac{\delta}{2} \leq 1 - \frac{\delta}{2}.$$

This implies that

$$\frac{1}{m} < \frac{\delta}{2} \leq f(x) - \frac{\delta}{2} < g_m(x) < f(x) + \frac{\delta}{2} \leq 1 - \frac{\delta}{2} < 1 - \frac{1}{m}.$$

Thus $x \in g_m^{-1}(\frac{1}{m}, 1 - \frac{1}{m})$, giving us the inclusion

$$U \subseteq \bigcup_{n=1}^{\infty} g_n^{-1}(0, 1).$$

□

We have the following corollary.

3.2.9. Corollary. *A compact pre-metrizable Frölicher space is metrizable and all pre-metrics yield equivalent metrics.*

3.3 Smooth variants of the Urysohn lemma and the Tietze Extension Theorem

This section is a short diversion where we use the Stone-Weierstrass theorem to prove a smooth version of the classical “Urysohn lemma”, and a weakened form of the classical “Tietze extension theorem” for compact, Hausdorff Frölicher spaces.

Smooth extensions of functions are somewhat problematic, due partly to the additional difficulties of extending *smoothly*, but also due in a large part to the fact that a Frölicher subspace does not induce the subspace topology as the underlying topology. Hence the “extension theorem” below is weaker than Tietze’s Theorem for topological spaces, but is strong enough for our uses.

This section does not depend on the notion of pre-metrizability.

3.3.1. Lemma. (Urysohn Lemma) *Let X be a compact Frölicher space that admits a generating set of functions that separate points in X . Let $A \subset X$ and $B \subset X$ be disjoint closed sets in the underlying topology. Then there is a smooth function $f: X \rightarrow [0, 1]$ with the property that $f(A) \equiv 0$ and $f(B) \equiv 1$.*

Proof. It is known that there is a continuous function $g: X \rightarrow [0, 1]$ with $g(A) \equiv 0$ and $g(B) \equiv 1$. This is the classical Urysohn Lemma for topological spaces. See Jänich [73], for example. Choose half-open intervals $[0, \epsilon_1)$ and $(\epsilon_2, 1]$, $\epsilon_1,$

$\epsilon_2 > 0$, with the property that $g^{-1}([0, \epsilon_1]) \cap g^{-1}((\epsilon_2, 1]) = \emptyset$. Apply Lemma 3.2.5 to find a smooth function $\tilde{g}: X \rightarrow \mathbb{R}$ with the property that

$$\|g - \tilde{g}\| < \min\left(\frac{\epsilon_1}{2}, \frac{\epsilon_2}{2}\right).$$

Now choose a smooth braking function $\alpha: \mathbb{R} \rightarrow [0, 1]$ with the following properties.

1. $\alpha(t) = 0$ for $0 \leq t < \epsilon_1$.
2. $\alpha(t) = 1$ for $1 - \epsilon_2 < t \leq 1$.
3. $0 \leq \alpha(t) \leq 1$ for $t \in \mathbb{R}$.

The required smooth function is then $f = \alpha \circ \tilde{g}: X \rightarrow [0, 1]$. \square

For our purposes the following weaker form of Tietze's extension theorem will be sufficient.

3.3.2. Lemma. (Extension Theorem) *Let X be a compact Frölicher space which admits a generating set of functions that separates points in X . Suppose we are given a smooth function*

$$f: A \rightarrow [0, 1]$$

defined on a closed subset $A \subseteq X$. Then for any closed set $B \subset \text{int}(A)$, the smooth map $f|_B$ can be extended to smooth function

$$\tilde{f}: X \rightarrow [0, 1].$$

Proof. Consider the closed set $B \subset \text{int}(A)$. Now choose a smooth function $g: X \rightarrow [0, 1]$ such that $g(B) \equiv 1$ and $g(X - A) \equiv 0$. Then define

$$\tilde{f}(x) = \begin{cases} \alpha_\epsilon(g(x))f(x) & \text{if } x \in A \\ 0 & \text{if } x \notin A, \end{cases}$$

where α_ϵ is a smooth braking function as described in Section 2.8, with $0 < \epsilon < \frac{1}{2}$. \square

3.4 Smooth partitions of unity subordinate to an open cover

In this section we take another diversion to show that if X is a compact Frölicher space whose underlying topology admits a generating set of functions that separate points in X , then one can always find a smooth partition of unity subordinate to an open cover of X .

This section is also not dependent on the notion of pre-metrizability.

3.4.1. Definition. Let X be a Frölicher space.

1. A collection of smooth functions

$$\{f_\lambda\}_{\lambda \in \Lambda}, \text{ where } f_\lambda: X \rightarrow [0, 1],$$

is called a **smooth partition of unity** if the following properties hold.

(a) Each $x \in X$ has a neighbourhood in which all but a finite number of the f_λ vanish.

(b) For each $x \in X$,

$$\sum_{\lambda \in \Lambda} f_\lambda(x) = 1.$$

2. The **support** of each f_λ is defined as

$$\text{supp } f_\lambda = \overline{\{x \in X : f_\lambda(x) \neq 0\}}.$$

3. A smooth partition of unity is called **subordinate to an open cover** \mathcal{U} of X if for each λ ,

$$\text{supp } f_\lambda \subseteq U$$

for some $U \in \mathcal{U}$.

3.4.2. Lemma. Let $\mathcal{U} = \{U\}_{\lambda \in \Lambda}$ be an open cover of a compact Frölicher space X that admits a generating set of functions that separates points in X . Then there is a smooth partition of unity subordinate to \mathcal{U} .

Proof. We may assume that \mathcal{U} is a locally finite open cover of X , since the underlying topology is paracompact. Now find an open cover of X , called $\mathcal{V} = \{V_\lambda\}_{\lambda \in \Lambda}$ with the property that $\overline{V_\lambda} \subset U_\lambda$. Again, the paracompactness of the underlying topology allows this. (See Jänich, [73]).

We now apply the smooth Urysohn Lemma (Lemma 3.3.1) to get smooth functions $f_\lambda: X \rightarrow [0, 1]$ with the property that that

$$f_\lambda|_{\overline{V_\lambda}} \equiv 1$$

and

$$f_\lambda|_{X - U_\lambda} \equiv 0.$$

The sum

$$\phi = \sum_{\lambda \in \Lambda} f_\lambda$$

is positive everywhere. It is smooth, since given a structure curve $c: \mathbb{R} \rightarrow X$, the composite

$$\phi \circ c(t) = \sum_{\lambda \in \Lambda} f_\lambda \circ c(t)$$

is a finite sum of smooth real functions in a neighbourhood of every $t \in \mathbb{R}$.

Set $g_\lambda = \frac{f_\lambda}{\phi}$, for $\lambda \in \Lambda$ to get the desired partition of unity. \square

3.5 The smooth Hilbert cube is pre-metrizable

We now show that there is a ‘cube’ which is pre-metrizable, and later we shall show that certain Frölicher spaces can be embedded into this cube, allowing us to deduce that these Frölicher spaces themselves are pre-metrizable. We are essentially imitating a topological procedure whereby certain topological spaces are embedded into a metrizable ‘Hilbert cube’, allowing one to deduce that the spaces themselves are metrizable.

The metric space $([0, 1]^{\mathbb{N}}, d')$, where

$$d'(\underline{x}, \underline{y}) = \sum_{n=1}^{\infty} 2^{-n} |x_n - y_n|, \quad (\underline{x}, \underline{y}) \in [0, 1]^{\mathbb{N}} \times [0, 1]^{\mathbb{N}},$$

is known as the **Hilbert cube**. See van Mill [127] for a discussion of the Hilbert cube and other infinite dimensional topological spaces, or Kelley [75] for a more general discussion of cubes.

The metric d' is not smooth, due to the presence of the absolute-value function, but we shall show that there is a smooth pre-metric

$$d: [0, 1]^{\mathbb{N}} \times [0, 1]^{\mathbb{N}} \rightarrow \mathbb{R}^+$$

on the Frölicher space $[0, 1]^{\mathbb{N}}$, (which we shall call the **smooth Hilbert cube**), and that \sqrt{d} is equivalent to the usual metric d' given above. This implies that the smooth Hilbert cube is pre-metrizable, and thus any Frölicher subspace of the smooth Hilbert cube is also pre-metrizable.

We proceed as follows: For each $n \in \mathbb{N}$, let $\alpha_n: \mathbb{R} \rightarrow \mathbb{R}$ be a smooth braking function with the following properties.

1. $\alpha_n(t) = 0$ for $t \leq n - \frac{3}{4}$,
2. $\alpha_n(t) = 1$ for $t \geq n - \frac{1}{4}$,
3. $0 \leq \alpha_n(t) \leq 1$ for $t \in \mathbb{R}$.

Now define

$$\phi: [0, 1]^{\mathbb{N}} \rightarrow [0, 1]^{\mathbb{R}}$$

as follows.

$$\phi(\underline{x})(t) = \begin{cases} x_{n-1} & \text{for } t < n-1 \\ x_{n-1} + (x_n - x_{n-1})\alpha_n(t), & \text{for } n-1 \leq t \leq n, \\ x_n & \text{for } t > n \end{cases} = f_{\underline{x}}(t), \text{ say.}$$

It is easy to see that $f_{\underline{x}}$ is smooth, since it is constant in a neighbourhood of each n , and smooth on each $(n-1, n)$, for $n \in \mathbb{N}$. We wish to verify that $\phi: [0, 1]^{\mathbb{N}} \rightarrow [0, 1]^{\mathbb{R}}$ is smooth. By the Cartesian closedness of **FRL**, we need only show that the evaluation map

$$\hat{\phi}: [0, 1]^{\mathbb{N}} \times \mathbb{R} \rightarrow [0, 1]$$

is smooth.

To this end, let

$$c: \mathbb{R} \rightarrow [0, 1]^{\mathbb{N}} \times \mathbb{R}$$

be a structure curve, given by

$$\begin{aligned} c(s) &= (\underline{x}(s), t(s)) \\ &= \left((x_1(s), x_2(s), \dots), t(s) \right). \end{aligned}$$

Then

$$\hat{\phi} \circ c: \mathbb{R} \rightarrow \mathbb{R}$$

is given by

$$\begin{aligned} \hat{\phi} \circ c(s) &= \hat{\phi}(\underline{x}(s), t(s)) \\ &= x_{n-1}(s) + (x_n(s) - x_{n-1}(s))\alpha_n(t(s)), \text{ for } n-1 \leq t(s) \leq n, \end{aligned}$$

which is clearly smooth.

We may now define

$$d: [0, 1]^{\mathbb{N}} \times [0, 1]^{\mathbb{N}} \rightarrow \mathbb{R}$$

by

$$d(\underline{x}, \underline{y}) = \int_1^{\infty} 2^{-2t} (f_{\underline{x}}(t) - f_{\underline{y}}(t))^2 dt.$$

The function d is smooth since the improper integral converges by the comparison test, and the integrand is smooth.

3.5.1. Lemma. *The map \sqrt{d} satisfies the metric space axioms.*

Proof. The axiom of symmetry is immediate. For the other axioms, let $\underline{x}, \underline{y}, \underline{z} \in [0, 1]^{\mathbb{N}}$.

To see that $d(\underline{x}, \underline{y}) = 0$ if and only if $\underline{x} = \underline{y}$ one needs only observe that, by the definition of $f_{\underline{x}}$, the integrand in the definition of $d(\underline{x}, \underline{y})$ is zero if and only if $\underline{x} = \underline{y}$.

To see that the triangle inequality holds, define an inner product

$$\underline{x} \cdot \underline{y} = \int_1^M 2^{-2t} f_{\underline{x}}(t) f_{\underline{y}}(t) dt$$

for $0 < M < \infty$, and $\|\underline{x}\| = \sqrt{\underline{x} \cdot \underline{x}}$. We then have the usual triangle inequality for a norm. Allowing $M \rightarrow \infty$, we have $\sqrt{d(\underline{x}, \underline{y})} = \lim_{M \rightarrow \infty} \|\underline{x} - \underline{y}\|$, and the result follows. \square

We shall now show that the underlying topological space on the smooth Hilbert cube is the usual (topological) Hilbert cube.

3.5.2. Lemma. Let $([0, 1]^{\mathbb{N}}, d')$ be the topological Hilbert cube. Suppose that $d^* : [0, 1]^{\mathbb{N}} \times [0, 1]^{\mathbb{N}} \rightarrow \mathbb{R}$ is another metric defined on $[0, 1]^{\mathbb{N}}$. If there exists a constant $K > 0$ such that

$$d^*(\underline{x}, \underline{y}) \leq K d'(\underline{x}, \underline{y}), \text{ for all } \underline{x}, \underline{y} \in [0, 1]^{\mathbb{N}},$$

then the map

$$\phi : ([0, 1]^{\mathbb{N}}, d') \rightarrow ([0, 1]^{\mathbb{N}}, d^*),$$

given by $\phi(\underline{x}) = \underline{x}$ is a homeomorphism.

Proof. Notice that K is a Lipschitz constant for ϕ , and Lipschitz mappings are continuous. Hence ϕ is a continuous bijection on a compact space, and is thus a homeomorphism. \square

We now prove the main result.

3.5.3. Theorem. The metric spaces $([0, 1]^{\mathbb{N}}, d')$ and $([0, 1]^{\mathbb{N}}, \sqrt{d})$ are homeomorphic.

Proof. Let $\underline{x}, \underline{y} \in [0, 1]^{\mathbb{N}}$. Notice that for $n \leq t \leq n+1$, we either have the inequality

$$(f_{\underline{x}}(t) - f_{\underline{y}}(t))^2 \leq (x_n - y_n)^2$$

or the inequality

$$(f_{\underline{x}}(t) - f_{\underline{y}}(t))^2 \leq (x_{n+1} - y_{n+1})^2.$$

Thus we certainly have the inequality

$$(f_{\underline{x}}(t) - f_{\underline{y}}(t))^2 \leq (x_n - y_n)^2 + (x_{n+1} - y_{n+1})^2.$$

Then

$$\begin{aligned}
\sqrt{d(\underline{x}, \underline{y})} &= \sqrt{\int_1^\infty 2^{-2t} (f_{\underline{x}}(t) - f_{\underline{y}}(t))^2 dt} \\
&= \sqrt{\sum_{n=1}^\infty \int_n^{n+1} 2^{-2t} (f_{\underline{x}}(t) - f_{\underline{y}}(t))^2 dt} \\
&\leq \sum_{n=1}^\infty \sqrt{\int_n^{n+1} 2^{-2t} (f_{\underline{x}}(t) - f_{\underline{y}}(t))^2 dt} \\
&\leq \sum_{n=1}^\infty \sqrt{\int_n^{n+1} 2^{-2n} (f_{\underline{x}}(t) - f_{\underline{y}}(t))^2 dt} \\
&\leq \sum_{n=1}^\infty 2^{-n} \sqrt{\left(\int_n^{n+1} (x_n - y_n)^2 dt + \int_n^{n+1} (x_{n+1} - y_{n+1})^2 dt \right)} \\
&= \sum_{n=1}^\infty 2^{-n} \sqrt{\int_n^{n+1} (x_n - y_n)^2 dt} + \sum_{n=1}^\infty 2^{-n} \sqrt{\int_n^{n+1} (x_{n+1} - y_{n+1})^2 dt} \\
&= \sum_{n=1}^\infty 2^{-n} \sqrt{(x_n - y_n)^2} + \sum_{n=1}^\infty 2^{-n} \sqrt{(x_{n+1} - y_{n+1})^2} \\
&= \sum_{n=1}^\infty 2^{-n} |x_n - y_n| + \sum_{n=1}^\infty 2^{-n} |x_{n+1} - y_{n+1}| \\
&\leq d'(\underline{x}, \underline{y}) + \frac{1}{2} \sum_{n=2}^\infty 2^{-n} |x_n - y_n| \\
&\leq d'(\underline{x}, \underline{y}) + \frac{1}{2} d'(\underline{x}, \underline{y}) \\
&= \frac{3}{2} d'(\underline{x}, \underline{y}).
\end{aligned}$$

Hence $\sqrt{d(\underline{x}, \underline{y})} \leq \frac{3}{2} d'(\underline{x}, \underline{y})$, and Lemma 3.5.2 now shows that the metrics \sqrt{d} and d' define the same topology, so we are done. \square

3.6 Countably generated Frölicher spaces

In this section we define what is meant by a “countably generated” and “finitely generated” Frölicher space. We then derive some embedding results for these spaces.

3.6.1. Definition. A Frölicher space (X, C_X, F_X) is called *countably generated* if there exists a countable subset $F \subset F_X$ which generates the Frölicher

structure on X . In other words, $C_X = \Gamma F$. We call F the **generating functions** for X . In particular, if F is finite, we say that X is **finitely generated**.

The following result allows us to identify many countably and finitely generated Frölicher spaces.

3.6.2. Lemma. *Suppose the set X has been given the initial Frölicher structure defined by a countable (respectively finite) collection of maps*

$$\{g_j: X \rightarrow Y_j | j = 1, \dots, m\},$$

where each Y_j is countably (finitely) generated. Then X is countably (finitely) generated.

Proof. We prove the finitely generated case. The proof for the countable situation is almost identical. Let

$$\{f_j^i | 1 \leq i \leq n_j\}$$

be the generating functions for Y_j . Then

$$\bigcup_{j=1}^m \{f_j^i g_j | 1 \leq i \leq n_j\}$$

is a generating set of functions for X , since, for any map $c: \mathbb{R} \rightarrow X$, we have

$$\begin{aligned} c \in C_X & \text{ if and only if } g_j \circ c \in C_{Y_j} \text{ for each } j \\ & \text{ if and only if } f_j^i \circ g_j \circ c \in c^\infty(\mathbb{R}, \mathbb{R}) \text{ for each } i \text{ and } j. \end{aligned}$$

□

This lemma has a number of immediate consequences.

3.6.3. Corollary. *A countable (finite) product of countably (finitely) generated spaces is countably (finitely) generated.*

Proof. Products are given the initial structure defined by the projection maps. □

3.6.4. Corollary. *A subspace of a countably (finitely) generated Frölicher space is countably (finitely) generated.*

Proof. A subspace is given the initial structure defined by the inclusion map. □

Recall that the structure functions on a disjoint union $X \sqcup Y$ of Frölicher spaces are mappings $f: X \sqcup Y \rightarrow \mathbb{R}$ which restrict to structure functions on each of the components of the disjoint union. Thus if each of X and Y are countably (finitely) generated, so is the disjoint union $X \sqcup Y$. To see this, suppose that $\{f_n\}, \{g_n\}, n \in \mathbb{N}$ are generating sets of functions for X and Y respectively.

Choose $x_0 \in X$ and $y_0 \in Y$. For each $f_n: X \rightarrow \mathbb{R}$ define $f_n^*: X \sqcup Y \rightarrow \mathbb{R}$ as follows

$$f_n^*(z) = \begin{cases} \exp(f_n(z)) & \text{if } z \in X \\ -1 & \text{if } z \in Y. \end{cases}$$

For each $g_n: Y \rightarrow \mathbb{R}$ define $g_n^*: X \sqcup Y \rightarrow \mathbb{R}$ as follows

$$g_n^*(z) = \begin{cases} -1 & \text{if } z \in X \\ \exp(g_n(z)) & \text{if } z \in Y. \end{cases}$$

It is not difficult to see that the countable collection $\{f_n^*\} \cup \{g_n^*\}$ is a generating set of functions for $X \sqcup Y$.

In fact one observes that a disjoint union of countably (finitely) many countably (finitely) generated Frölicher spaces is countably (finitely) generated.

3.7 Examples of countably generated spaces

3.7.1. Example. 1. The real line \mathbb{R} is finitely generated, since $1_{\mathbb{R}}: \mathbb{R} \rightarrow \mathbb{R}$ is clearly a generating function.

2. Euclidean n -space is finitely generated, since it is the n -fold product of \mathbb{R} , which is finitely generated.

3. Any subspace of Euclidean space is finitely generated if it is given the Frölicher subspace smooth structure.

3.7.2. Example. A compact finite-dimensional smooth manifold is finitely generated since it can be embedded in \mathbb{R}^n for some $n \in \mathbb{N}$, by the well-known Whitney embedding theorem. It can be shown that the Frölicher subspace structure in this case coincides with the usual manifold structure.

3.7.3. Example. We have seen in Example 3.7.1 (3) that any Frölicher space that can be embedded in Euclidean space is finitely generated. The converse is not true. Take

$$X = \bigsqcup_{\lambda \in \Lambda} \mathbb{R},$$

where the cardinality of Λ is $|\Lambda| > c$, $c = |\mathbb{R}|$. The space X is finitely generated by the single function

$$[1_{\mathbb{R}}, 1_{\mathbb{R}}, \dots]: X \rightarrow \mathbb{R},$$

but X can not be embedded in any \mathbb{R}^n , since $|X| > |\mathbb{R}^n|$, for any $n \in \mathbb{N}$.

3.8 Embedding a countably generated space in the smooth Hilbert cube

It is a well-known topological fact that a Tychonoff space is characterized by the fact that it is homeomorphic to a subspace of a cube. By "cube" we mean

a topological space $[0, 1]^\Lambda$, where Λ is a set. The product is given the topology of pointwise convergence. See for example Kelley [75].

Cherenack [25] has shown that every Frölicher space X which admits a generating set F of functions that separates points in X can be embedded in the Frölicher space \mathbb{R}^F . His method is analogous to that in Kelley [75] for topological spaces. We shall use this method to embed X in a cube $[0, 1]^F$, and then derive some consequences of this result.

Note that the map $\nu: \mathbb{R} \rightarrow (0, 1)$ given by

$$\nu(t) = 1 - \frac{1}{e^t + 1}$$

is an isomorphism of Frölicher spaces. In fact it is not difficult to verify that F is a generating set for a Frölicher space X if, and only if X has the initial structure generated by the set $\nu \circ F$. Moreover F separates points in X if, and only if for each pair $(x, y) \in X \times X$, there exists an $f \in F$ such that $\nu \circ f(x) \neq \nu \circ f(y)$.

3.8.1. Lemma. *Let X be a Frölicher space with a generating set F that separates points in X . Then X can be embedded in the cube $[0, 1]^F$.*

Proof. Define $\alpha: X \rightarrow [0, 1]^F$ as follows. Let

$$\alpha(x): F \rightarrow X$$

be given by

$$\alpha(x)(f) = \nu \circ f(x).$$

Now α is injective, since if

$$\alpha(x) = \alpha(y): F \rightarrow X,$$

then $x = y$, because otherwise there would be a function $f \in F$ for which $\nu \circ f(x) \neq \nu \circ f(y)$, contradicting the assertion that $\alpha(x) = \alpha(y)$.

To see that $\alpha: X \rightarrow [0, 1]^F$ is smooth, let $c: \mathbb{R} \rightarrow X$ be a structure curve. Then

$$\alpha \circ c: \mathbb{R} \rightarrow [0, 1]^F$$

is smooth if, and only if

$$\widehat{\alpha \circ c}: \mathbb{R} \times F \rightarrow [0, 1]$$

is, where this map is given by

$$\begin{aligned} \widehat{\alpha \circ c}(t, f) &= \alpha(c(t))(f) \\ &= f(c(t)). \end{aligned}$$

But $f \circ c$ is smooth by the hypothesis that c is a structure curve on X .

Finally, to see that α is an isomorphism onto its image, we need to show that if $c: \mathbb{R} \rightarrow \alpha(X) \subseteq \mathbb{R}^F$ is a structure curve, where $\alpha(X)$ has the subspace structure, then $\alpha^{-1} \circ c$ is a structure curve on X .

To see this, assume that c is such a structure curve. Recall that by the Cartesian closedness of **FRL**, $c: \mathbb{R} \rightarrow \mathbb{R}^F$ is smooth if and only if

$$\tilde{c}: \mathbb{R} \times F \rightarrow \mathbb{R},$$

given by $\tilde{c}(t, f) = c(t)(f)$ is smooth. But

$$\begin{aligned} c(t)(f) &= ((\alpha \circ \alpha^{-1})(c(t)))(f) \\ &= \alpha(\alpha^{-1}(c(t)))(f) \\ &= \nu \circ f(\alpha^{-1}(c(t))) \\ &= \nu(f \circ \alpha^{-1}(c(t))). \end{aligned}$$

Since ν is a smooth isomorphism, $f \circ \alpha^{-1} \circ c$ and hence $\alpha^{-1} \circ c$ is smooth. \square

A Frölicher space X that is finitely generated by a finite set of n functions F that separates points in X is characterized by the fact that it can be embedded in \mathbb{R}^n .

3.8.2. Corollary. *A Frölicher space X is finitely generated by a set F of n functions that separate points in X if and only if X can be embedded in \mathbb{R}^n .*

Proof. By Lemma 3.8.1, if X has a finite generating set F of n functions, then X can be embedded in the finite product $[0, 1]^F$.

Conversely, if X can be embedded in \mathbb{R}^n , then as we have seen, X is finitely generated by the n projection maps

$$p_i: \mathbb{R}^n \rightarrow \mathbb{R},$$

which obviously separate points in X . \square

3.8.3. Example. The function $f(t) = t$ is a function that generates the Frölicher structure on \mathbb{R} . The function f separates points in \mathbb{R} , and so we may embed \mathbb{R} in $[0, 1]$.

3.8.4. Example. Let Λ be a set with cardinality $|\Lambda| > c$, where $c = |\mathbb{R}|$ is the cardinality of the real line. Then the function $f: \Lambda \rightarrow \mathbb{R}$ given by $f(x) = 0$ for each $x \in \Lambda$ generates the indiscrete topology on Λ . We cannot embed Λ in \mathbb{R} , since $|\Lambda| > |\mathbb{R}|$. This shows that the separating property of the generating set of functions is necessary.

We end off with another application of the cube embedding.

3.8.5. Lemma. *Let X be a Frölicher space with a structure generated by a countable collection, F , of real-valued functions, which separate points in X . Then X is pre-metrizable.*

Proof. There is a smooth embedding

$$\alpha: X \rightarrow [0, 1]^F,$$

and $[0, 1]^F$ is pre-metrizable if F is countable. \square

Chapter 4

The basic constructions of homotopy theory in **FRL**

In this chapter we define the fundamental notions of homotopy theory in the category **FRL**, such as the homotopy relation and the mapping cylinder. We begin with an overview of our approach to homotopy in **FRL**, and then discuss alternate Frölicher structures on the unit interval which are used in this and later chapters.

4.1 An overview of our approach to homotopy theory in **FRL**

One might begin investigating homotopy theory in **FRL** by simply following the homotopy theory of topological spaces, replacing continuous functions with smooth ones.

One can certainly define the notion of a homotopy $H: I \times X \rightarrow Y$ between the smooth maps $H(0, -)$ and $H(1, -)$ in this way (which we do). One can even get as far as the left Puppe sequence (see Cherenack [24]), but eventually difficulties begin to arise.

Extending functions defined on a subspace of a Frölicher space tends to be a little tricky, and so the definition of a cofibration in **FRL** is one that needs careful consideration. A fairly direct approach to the construction of the right Puppe sequence in **FRL** was attempted in [36], but a full proof of the existence of this sequence was not obtained.

We shall construct the right Puppe sequence in Chapter 6. To do this we define a slightly weaker notion of cofibration than the notion obtained from topological spaces. In addition we define the mapping cylinder of a smooth map $f: X \rightarrow Y$ using not the unit interval, but a modified version called the **weakly flattened unit interval**, denoted \mathbb{I} , which we show is topologically homeomorphic to the unit interval.

This modified structure on the unit interval allows us to show that the inclusion of a space X into the mapping cylinder of $f: X \rightarrow Y$ is a cofibration (in our weaker sense).

The weakly flattened unit interval is useful, but it also has its drawbacks. It would be ideal to have a single structure on the unit interval that can be used throughout our homotopy theory, but the weakly flattened unit interval is not suitable, because it has the rather restrictive property that a smooth map $f: I \rightarrow I$ on the usual unit interval often does not define a smooth map $f: \mathbb{I} \rightarrow \mathbb{I}$ unless the endpoints of the interval are mapped to the endpoints. This restrictive property means that we only use the flattened unit intervals where they are absolutely necessary.

We later show that with our modified notions of cofibration and mapping cylinder, Baues' cofibration axioms are satisfied.

In appendix A we define the notion of a **fibration** in **FRL**, which is slightly different to the corresponding topological notion. We also define the notion of **mapping fibre** of a smooth map $f: X \rightarrow Y$ in terms of the weakly flattened unit interval. With these modified notions we show that Baues' fibration axioms are satisfied.

4.2 Flattened structures on the unit interval

As mentioned earlier it is convenient to define various modified Frölicher structures on the unit interval, I . We shall define two main Frölicher structures which we call the **flattened unit interval** and the **weakly flattened unit interval**. Using some of our results on pre-metrizable Frölicher spaces we shall show that each of these alternate Frölicher structures generates the usual topology on the underlying set.

The flattened unit interval, \mathbf{I} is generated by structure functions which are locally constant at 0 and at 1. The weakly flattened unit interval, \mathbb{I} , is generated by structure functions having zero n -th derivatives at the endpoints of the interval, for all n .

4.2.1. Definition. *The Frölicher space, $(\mathbf{I}, C_{\mathbf{I}}, F_{\mathbf{I}})$, with the structure defined below is called the **flattened unit interval**. The underlying set of the flattened unit interval is the set $\{t \in \mathbb{R} \mid 0 \leq t \leq 1\}$. Let F denote the following collection of functions on the underlying set.*

$$F = \{f \in F_I \mid \text{there exists } 0 < \epsilon < \frac{1}{4} \text{ with } f(t) = f(0) \text{ for } t \in [0, \epsilon) \text{ and} \\ f(t) = f(1) \text{ for } t \in (1 - \epsilon, 1]\}.$$

We define the structure on \mathbf{I} , namely $(C_{\mathbf{I}}, F_{\mathbf{I}})$, to be the structure generated by the set of functions F .

Thus $\mathbf{I} = I$ in **Sets**. The functions in F which are used to generate the smooth structure on \mathbf{I} are all constant near each endpoint of the unit interval.

4.2.2. Note. It is easy to see that any continuous map $c: \mathbb{R} \rightarrow I$ which is smooth at every point $s \in \mathbb{R}$, where $c(s) \in (0, 1)$, defines a structure curve on I , since given any $f \in F$, $f \circ c: \mathbb{R} \rightarrow \mathbb{R}$ is smooth on $\{s \in \mathbb{R} | c(s) \in (0, 1)\}$ and constant on a neighbourhood of each point in $\mathbb{R} \setminus \{s \in \mathbb{R} | c(s) \in (0, 1)\}$. In fact, one can show that the converse is also true.

We define the **left (respectively right) flattened unit interval**, denoted by I^- (respectively I^+), to be the Frölicher structure on the set $\{t \in \mathbb{R} : 0 \leq t \leq 1\}$ generated by the structure functions on I which are constant near 0 (respectively 1).

4.2.3. Note. One can see that a continuous map $c: \mathbb{R} \rightarrow I$ that is smooth on points mapping into $(0, 1]$ (respectively $[0, 1)$) defines a structure curve on I^- (respectively I^+).

We define the weakly flattened unit interval as follows.

4.2.4. Definition. The Frölicher space, $(\mathbb{I}, C_{\mathbb{I}}, F_{\mathbb{I}})$, with the structure defined below is called the **weakly flattened unit interval**. The underlying set of the weakly flattened unit interval is the unit interval. In other words $\mathbb{I} = I$ in the category Sets. Let a set of functions F on the unit interval be given by

$$F = \{f \in F_I | \lim_{t \rightarrow 0^+} \frac{d^n}{dt^n} f(t) = 0, \lim_{t \rightarrow 1^-} \frac{d^n}{dt^n} f(t) = 0, n \geq 1\}.$$

We define the structure on \mathbb{I} , $(C_{\mathbb{I}}, F_{\mathbb{I}})$, to be the structure generated by the set of functions F . We call the property of f ,

$$\lim_{t \rightarrow 0^+} \frac{d^n}{dt^n} f(t) = 0, \lim_{t \rightarrow 1^-} \frac{d^n}{dt^n} f(t) = 0,$$

the **zero derivative property of f** .

In other words, $C_{\mathbb{I}} = \Phi F$ and $F_{\mathbb{I}} = \Gamma \Phi F$. We shall prove that all structure functions on \mathbb{I} have the zero derivative property, in other words, that $F_{\mathbb{I}} = F$, where F is the collection of functions described in Definition 4.2.4. We prove the result via the following lemma.

4.2.5. Lemma. Let $c: \mathbb{R} \rightarrow \mathbb{R}$ be a smooth real function at $t = t_0$, and let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a smooth real function at $t = c(t_0)$. Then

$$\begin{aligned} \frac{d^n}{dt^n} (f \circ c)(t_0) &= f^{(n)}(c(t_0)) (c'(t_0))^n \\ &+ \text{terms of the form } a f^{(k)}(c(t_0)) (c'(t_0))^{m_1} (c''(t_0))^{m_2} \dots (c^{(n-1)}(t_0))^{m_{n-1}}, \end{aligned}$$

where $k < n$ and $a \in \mathbb{R}$. In addition, if $a \neq 0$ then at least one of m_2, m_3, \dots, m_{n-1} is also non-zero.

Proof. We proceed by induction. For the sake of brevity, we call the term $f^{(n)}(c(t_0)) (c'(t_0))^n$ the **primary term**, and the terms of the form

$$a f^{(k)}(c(t_0)) (c'(t_0))^{m_1} (c''(t_0))^{m_2} \dots (c^{(n-1)}(t_0))^{m_{n-1}}$$

the lower order terms for each n . The statement is true for $n = 1$ and for $n = 2$, since

$$\frac{d}{dt}(f \circ c)(t_0) = f'(c(t_0)) \cdot c'(t_0),$$

and

$$\frac{d^2}{dt^2}(f \circ c)(t_0) = f''(c(t_0)) \cdot (c'(t_0))^2 + f'(c(t_0)) \cdot c''(t_0).$$

Suppose the result is true for $n = k$. Then

$$\begin{aligned} \frac{d^{k+1}}{dt^{k+1}}(f \circ c)(t_0) &= \frac{d}{dt} \left(f^{(k)}(c(t_0)) (c'(t_0))^k \right) \\ &+ \text{terms of the form} \\ &\frac{d}{dt} \left(a f^{(j)}(c(t_0)) (c'(t_0))^{m_1} (c''(t_0))^{m_2} \dots (c^{(k-1)}(t_0))^{m_{k-1}} \right), \end{aligned}$$

which satisfy the requirements of the lemma.

Now

$$\begin{aligned} \frac{d}{dt} \left(f^{(k)}(c(t_0)) (c'(t_0))^k \right) &= f^{(k+1)}(c(t_0)) (c'(t_0))^{k+1} \\ &+ k f^{(k)}(c(t_0)) (c'(t_0))^{k-1} c''(t_0). \end{aligned}$$

The first term after the equal sign is the primary term for $n = k + 1$, and the second term after the equal sign is of the form of a lower order term for $n = k + 1$. Thus we need only show that

$$\frac{d}{dt} \left(a f^{(j)}(c(t_0)) (c'(t_0))^{m_1} (c''(t_0))^{m_2} \dots (c^{(k-1)}(t_0))^{m_{k-1}} \right)$$

gives rise to lower order terms for $n = k + 1$. Now,

$$\begin{aligned} &\frac{d}{dt} \left(a f^{(j)}(c(t_0)) (c'(t_0))^{m_1} (c''(t_0))^{m_2} \dots (c^{(k-1)}(t_0))^{m_{k-1}} \right) \\ &= a f^{(j+1)}(c(t_0)) (c'(t_0))^{m_1+1} (c''(t_0))^{m_2} \dots (c^{(k-1)}(t_0))^{m_{k-1}} \\ &+ a f^{(j)}(c(t_0)) \frac{d}{dt} \left((c'(t_0))^{m_1} (c''(t_0))^{m_2} \dots (c^{(k-1)}(t_0))^{m_{k-1}} \right) \end{aligned}$$

$$\begin{aligned}
&= af^{(j+1)}(c(t_0))(c'(t_0))^{m_1+1}(c''(t_0))^{m_2} \dots (c^{(k-1)}(t_0))^{m_{k-1}} \\
&\quad + (af^{(j)}(c(t_0))) \left((c'(t_0))^{m_1} \dots \frac{d}{dt} \left[(c^{(i)}(t_0))^{m_i} \right] \dots (c^{(k-1)}(t_0))^{m_{k-1}} \right. \\
&\quad \left. + (c^{(i)}(t_0))^{m_i} \frac{d}{dt} (c'(t_0))^{m_1} \dots (c^{(i-1)}(t_0))^{m_{i-1}} (c^{(i+1)}(t_0))^{m_{i+1}} \right. \\
&\quad \quad \quad \left. \dots (c^{(k-1)}(t_0))^{m_{k-1}} \right)
\end{aligned}$$

where $i > 1$ and $m_i \neq 0$

$$\begin{aligned}
&= af^{(j+1)}(c(t_0))(c'(t_0))^{m_1+1}(c''(t_0))^{m_2} \dots (c^{(k-1)}(t_0))^{m_{k-1}} \\
&\quad + (am_i f^{(j)}(c(t_0)))(c'(t_0))^{m_1} \dots (c^{(i)}(t_0))^{m_i-1} (c^{(i+1)}(t_0))^{m_{i+1}+1} \dots (c^{(k-1)}(t_0))^{m_{k-1}} \\
&\quad + (c^{(i)}(t_0))^{m_i} \frac{d}{dt} \left((c'(t_0))^{m_1} \dots (c^{(i-1)}(t_0))^{m_{i-1}} (c^{(i+1)}(t_0))^{m_{i+1}} \right. \\
&\quad \quad \quad \left. \dots (c^{(k-1)}(t_0))^{m_{k-1}} \right)
\end{aligned}$$

We observe, using induction, that each of the three terms after the last equal sign is of the form of a lower order term for $n = k+1$. This completes the proof, since $j < k$ implies that $j+1 < k+1$. \square

4.2.6. Theorem. $F_{\mathbb{I}} = \{f \in F_I \mid \lim_{t \rightarrow 0^+} \frac{d^n}{dt^n} f(t) = 0 = \lim_{t \rightarrow 1^-} \frac{d^n}{dt^n} f(t)\} = F$.

Proof. We know that $F \subseteq F_{\mathbb{I}}$. We must show the reverse inequality. Let $0 \leq \epsilon \leq \frac{1}{2}$, and $M \in (-1, 1)$. Consider the function $c_M: \mathbb{R} \rightarrow \mathbb{R}$, given by

$$c_M(t) = [1 - \alpha_\epsilon(|t|)]\beta_M(t) + \alpha_\epsilon(|t|)$$

where $\alpha_\epsilon: \mathbb{R} \rightarrow \mathbb{R}$ is a smooth braking function as defined in Section 2.8 and $\beta_M: \mathbb{R} \rightarrow \mathbb{R}$ is given by

$$\beta_M(t) = \begin{cases} -Mt & \text{if } t \leq 0 \\ t & \text{if } t > 0. \end{cases}$$

Since $\alpha_\epsilon(|t|) = 0$ for $t \in (-\epsilon, \epsilon)$, we observe that $\alpha_\epsilon(|t|)$ is smooth on \mathbb{R} . It is clear that c_M is continuous on \mathbb{R} , and smooth on the subset $\mathbb{R} \setminus \{0\}$ of \mathbb{R} . Also note that $c_M(\mathbb{R}) \subseteq [0, 1]$, and that for $0 \leq t < \epsilon$, $c_M(t) = \beta_M(t)$, since $\alpha_\epsilon(t) = 0$ here. Thus $c_M(0) = 0$. Now,

$$\frac{d}{dt} c_M(t) = \frac{d}{dt} \beta_M(t) = -M, \text{ for } t \in (-\epsilon, 0),$$

and

$$\frac{d}{dt} c_M(t) = \frac{d}{dt} \beta_M(t) = 1, \text{ for } t \in (0, \epsilon).$$

For $n > 1$, we have

$$\frac{d^n}{dt^n} c_M(t) = \frac{d^n}{dt^n} \beta_M(t) = 0, \text{ for } t \in (-\epsilon, 0) \cup (0, \epsilon).$$

We now show that for $M \neq \pm 1$, $c_M \in \Phi(F)$. To this end, let $f \in F$. To show that $f \circ c_M: \mathbb{R} \rightarrow \mathbb{R}$ is smooth, we need only concentrate on the point $t = 0$, since $f \circ c_M$ is the composite of smooth functions for $t \neq 0$, and is hence smooth there.

For $t \neq 0$, and $n \in \mathbb{N}$, Lemma 4.2.5 tells us that

$$\begin{aligned} \frac{d^n}{dt^n} (f \circ c_M)(t) &= f^{(n)}(c_M(t)) (c'_M(t))^n \\ &+ \text{ terms of the form } a f^{(k)}(c_M(t)) (c'_M(t))^{m_1} (c''_M(t))^{m_2} \dots (c_M^{(n-1)}(t))^{m_{n-1}}, \end{aligned}$$

where $k < n$, $a \in \mathbb{R}$ and if $a \neq 0$, then at least one of m_2, m_3, \dots, m_{n-1} is also non-zero.

But, as $t \rightarrow 0$, $c_M(t) \rightarrow 0^+$, and so, letting $s = c_M(t)$, we have

$$\lim_{t \rightarrow 0} f^{(j)}(c_M(t)) = \lim_{s \rightarrow 0^+} f^{(j)}(s) = 0,$$

for all $j \in \mathbb{N}$, by the zero derivative property of f .

Thus as t approaches the value 0, the primary term and all the lower order terms of $\frac{d^n}{dt^n} (f \circ c_M)(t)$ fall away, and we have shown that $f \circ c_M$ is smooth at $t = 0$. This implies that $f \circ c_M \in C^\infty(\mathbb{R}, \mathbb{R})$ for all $f \in F$. We deduce that $c_M \in \Phi F$.

We are now ready to show that $F_{\mathbb{1}} \subseteq F$. To this end, suppose now that we are given a structure function $f \in F_{\mathbb{1}}$. We shall show that this f has the zero derivative property, and is thus an element of F .

Since $f \in F_{\mathbb{1}}$, we know that $f \circ c$ is a smooth real function for every $c \in \Phi F$. In particular, $f \circ c_M$ is smooth for all $M \in (-1, 1)$. Thus, for any $n \in \mathbb{N}$,

$$\lim_{t \rightarrow 0^-} \frac{d^n}{dt^n} (f \circ c_M)(t) = \lim_{t \rightarrow 0^+} \frac{d^n}{dt^n} (f \circ c_M)(t).$$

Lemma 4.2.5 tells us that, for $t \neq 0$,

$$\begin{aligned} \frac{d^n}{dt^n} (f \circ c_M)(t) &= f^{(n)}(c_M(t)) (c'_M(t))^n \\ &+ \text{ terms of the form } a f^{(k)}(c_M(t)) (c'_M(t))^{m_1} (c''_M(t))^{m_2} \dots (c_M^{(n-1)}(t))^{m_{n-1}}, \end{aligned}$$

where $k < n$, $a \in \mathbb{R}$ and if $a \neq 0$ then at least one of m_2, m_3, \dots, m_{n-1} is also non-zero.

As $t \rightarrow 0^-$, $c_M(t) \rightarrow 0^+$. Let us consider the lower order terms for n first. Each term of the form

$$a f^{(k)}(c_M(t)) (c'_M(t))^{m_1} (c''_M(t))^{m_2} \dots (c_M^{(n-1)}(t))^{m_{n-1}}$$

has some term $(c_M^{(i)}(t))^{m_i}$, for some $i > 1$, with $m_i \neq 0$. But $\lim_{t \rightarrow 0^-} c_M^{(i)}(t) = 0$, if $i > 1$, and so

$$\lim_{t \rightarrow 0^-} a f^{(k)}(c_M(t)) (c_M'(t))^{m_1} (c_M''(t))^{m_2} \dots (c_M^{(n-1)}(t))^{m_{n-1}} = 0.$$

So all the lower order terms fall away, leaving us with

$$\begin{aligned} \lim_{t \rightarrow 0^-} \frac{d^n}{dt^n} (f \circ c_M)(t) &= \lim_{t \rightarrow 0^-} f^{(n)}(c_M(t)) [c_M'(t)]^n \\ &= \lim_{t \rightarrow 0^-} f^{(n)}(c_M(t)) (-M)^n \\ &= \lim_{s \rightarrow 0^+} f^{(n)}(s) (-M)^n \end{aligned}$$

where $s = c_M(t)$. In a similar way, we can calculate that

$$\begin{aligned} \lim_{t \rightarrow 0^+} \frac{d^n}{dt^n} (f \circ c_M)(t) &= \lim_{s \rightarrow 0^+} f^{(n)}(s) (1)^n \\ &= \lim_{s \rightarrow 0^+} f^{(n)}(s) \end{aligned}$$

But $f \circ c_M$ is a smooth real function, so all its derivatives are continuous. Thus

$$\begin{aligned} \lim_{s \rightarrow 0^+} f^{(n)}(s) (-M)^n &= \lim_{t \rightarrow 0^-} \frac{d^n}{dt^n} (f \circ c_M)(t) \\ &= \lim_{t \rightarrow 0^+} \frac{d^n}{dt^n} (f \circ c_M)(t) \\ &= \lim_{s \rightarrow 0^+} f^{(n)}(s) \end{aligned}$$

This implies that $\lim_{s \rightarrow 0^+} f^{(n)}(s) = 0$.

We have shown that the zero derivative property of f holds for the left endpoint of the unit interval. To show that the zero derivative property of f holds for the right endpoint of f , notice that $d_M(t) = 1 - c_M(t): \mathbb{R} \rightarrow \mathbb{R}$ is a smooth real function with $d(0) = 1$, and $d_M(\mathbb{R}) \subseteq [0, 1]$. One can follow a similar procedure to the above, using d_M instead of c_M to show that $\lim_{s \rightarrow 1^-} f^{(n)}(s) = 0$. \square

4.3 The underlying topology of the flattened unit intervals

We shall now show that the underlying topology on the flattened unit interval, weakly flattened unit interval and their variants is the usual Euclidean topology on the unit interval. Specifically, we shall answer the following two questions:

1. Are there structure functions on I that are not locally constant near the endpoints of I ?

2. Are \mathbf{I} , \mathbb{I} and their variants homeomorphic to the usual unit interval I at the topological level?

We proceed as follows. Firstly note that the underlying topology on \mathbf{I} is coarser than the usual topology on I , since there are fewer functions on \mathbf{I} .

Next, for each integer $n \geq 2$, define a smooth increasing braking function $\alpha_n: [0, 1] \rightarrow \mathbb{R}$ with the following properties:

$$\alpha_n(t) = \begin{cases} 0 & \text{if } t < \frac{1}{4n} \\ t & \text{if } \frac{3}{4n} < t < 1 - \frac{3}{4n} \\ 1 & \text{if } 1 - \frac{1}{4n} < t. \end{cases}$$

We have the following lemma:

4.3.1. Lemma. *The collection of functions $\{\alpha_n\}_{n \in \mathbb{N}}$ is a generating set of functions for the Frölicher structure on \mathbf{I} which separates points in \mathbf{I} .*

Proof. Firstly note that the set of functions $\{\alpha_n\}_{n \in \mathbb{N}}$ obviously separates points in \mathbf{I} .

As we have seen, a continuous map $c: \mathbb{R} \rightarrow \mathbf{I}$ is a structure curve if and only if c is smooth as a real function at each $s \in \mathbb{R}$ for which $c(s) \in (0, 1)$.

Suppose that for some map $c: \mathbb{R} \rightarrow \mathbf{I}$ we have $\alpha_n \circ c: \mathbb{R} \rightarrow \mathbb{R}$ is smooth for each $n \in \mathbb{N}$. Let $s \in \mathbb{R}$ be a point for which $c(s) \in (0, 1)$. Choose n large enough that α_n is given by $\alpha_n(t) = t$ near $c(s)$. Then $\alpha_n \circ c(s) = c(s)$ is smooth at s .

It is also clear that if $c(a) = 0$ or $c(a) = 1$, then c is continuous at a . Hence, as in 4.2.2, c is a structure curve on \mathbf{I} . \square

This lemma implies that \mathbf{I} is countably generated by a collection of functions that separates points, and is thus pre-metrizable with a pre-metric that we call d . This pre-metric is induced by the pre-metric on the smooth Hilbert cube.

Now note that the smooth pre-metric d generates the Euclidean topology as the d -topology on I since we have shown that the smooth pre-metric d is equivalent to the metric d' on the Hilbert cube which generates the Euclidean topology on the unit interval (Kelley [75]). See Section 3.5. For properties of the topological Hilbert cube, see Van Mill [127] or Kelley [75].

Finally, because the underlying topology on any pre-metrizable Frölicher space is finer than the d -topology, we combine this fact with our first observation (that the underlying topology is coarser than the usual topology) to conclude that the underlying topology and usual topology coincide. This conclusion obviously answers our second question in the affirmative. The answer to the first question must also be “yes”, since at every point t_0 of \mathbf{I} , the metric squared defines a smooth function $d(t_0, -)$ on \mathbf{I} that is not locally constant at t_0 . Taking $t_0 = 0$ or $t_0 = 1$, we see that the functions $d(0, -)$ and $d(1, -)$ are smooth functions on \mathbf{I} that are not locally constant at 0 and at 1 respectively.

To conclude, observe that \mathbb{I} has fewer structure functions than I , but more structure functions than \mathbf{I} . Thus the underlying topology on \mathbb{I} is coarser than the underlying topology on I , but finer than the underlying topology on \mathbf{I} . But the underlying topologies on I and \mathbf{I} are both the Euclidean topology, and so the underlying topology on \mathbb{I} is also the Euclidean topology.

4.4 Some properties of smooth functions between the flattened unit intervals

One has to be careful when dealing with the various flattened unit intervals. A smooth function $g: I \rightarrow I$ from the usual unit interval to itself need not define a smooth function $g: \mathbf{I} \rightarrow \mathbf{I}$, for example.

Conversely, not every smooth function $h: \mathbf{I} \rightarrow \mathbf{I}$ defines a smooth function $h: I \rightarrow I$.

In particular we need to be aware of the fact that the addition and multiplication of functions when defined between the various flattened unit intervals does not preserve smoothness, as is the case with the usual unit interval.

4.4.1. Example. The function $g: I \rightarrow I$ given by $g(t) = \frac{1}{2}t$ is clearly smooth, but the corresponding function $g: \mathbf{I} \rightarrow \mathbf{I}$ given by the same formula is not smooth. To see this, let $\alpha: \mathbb{R} \rightarrow \mathbb{R}$ be a smooth braking function with the properties that

- $\alpha(t) = -1$ for $t < -\frac{3}{4}$,
- $\alpha(t) = t$ for $-\frac{1}{4} < t < \frac{1}{4}$ and
- $\alpha(t) = 1$ for $t > \frac{3}{4}$.

Define $c: \mathbb{R} \rightarrow \mathbf{I}$ by $c(t) = 1 - |\alpha(t)|$. The curve c is smooth everywhere except at $t = 0$, where $c(0) = 1$. However, every generating function f on \mathbf{I} is constant near 1, and so the composite $f \circ c(t)$ is a smooth real function. Thus c is a structure curve on \mathbf{I} .

Now $g \circ c: \mathbb{R} \rightarrow \mathbf{I}$ is given by $g \circ c(t) = \frac{1}{2}(1 - |\alpha(t)|)$. Let $f_1: \mathbf{I} \rightarrow \mathbb{R}$ be a structure function with the properties that

- $f_1(s) = 0$ for $s < \frac{1}{8}$,
- $f_1(s) = s$ for $\frac{1}{4} < s < \frac{3}{4}$ and
- $f_1(s) = 1$ for $s > \frac{7}{8}$.

Then $f_1 \circ g \circ c$ is given by $f_1 \circ g \circ c(t) = \frac{1}{2}(1 - |\alpha(t)|)$ for t near 0, and this composite is not a smooth real function at $t = 0$.

Thus g does not define a smooth function from \mathbf{I} to \mathbf{I} .

4.4.2. Example. The function $g: \mathbf{I} \rightarrow \mathbf{I}$ given by $g(t) = \sqrt{t}$ is smooth, but the corresponding function $g: I \rightarrow I$ given by the same formula is not smooth.

To see this, notice that g is a smooth real function on the interval $(0, 1)$, and any generating function f on \mathbf{I} is constant near 0 and 1. Thus $f \circ g$ is a smooth function on \mathbf{I} .

It is clear that g is not a smooth function from I to I , because if $c: \mathbb{R} \rightarrow I$ is a structure curve with $c(t) = t^2$ near $t = 0$, then $g \circ c(t) = |t|$ near $t = 0$ which is not smooth on I at $t = 0$.

4.4.3. Example. The functions $g_1, g_2: \mathbf{I}^- \rightarrow \mathbf{I}^-$ given by $g_1(t) = \frac{1}{2}\sqrt{t}$ and $g_2(t) = \frac{1}{4}$ are both smooth, but the sum $g_1(t) + g_2(t) = \frac{1}{2}\sqrt{t} + \frac{1}{4}$ is not smooth.

The following result follows from the definition of the Frölicher structures on the various flattened unit intervals.

4.4.4. Lemma. *Let $g: I \rightarrow I$ be a smooth function with the properties that $g(0) = 0$ and $g(1) = 1$. Then the corresponding maps*

1. $g: I \rightarrow \mathbf{I}^\pm$,
2. $g: I \rightarrow \mathbb{I}$,
3. $g: \mathbf{I}^\pm \rightarrow \mathbf{I}$,
4. $g: \mathbb{I} \rightarrow \mathbf{I}$,
5. $g: I \rightarrow \mathbf{I}$

are all smooth.

Proof. The proof is immediate from the definitions of the various flattened unit intervals. \square

The function defined in the following example is important for the work in Chapter 5.

4.4.5. Example. Let $H: \mathbf{I} \times \mathbf{I}^- \rightarrow \mathbf{I}^-$ be given by $H(t, s) = (1 - \alpha(t))s$, where $\alpha: \mathbb{R} \rightarrow \mathbb{R}$ is a smooth braking function with the properties that

- $\alpha(t) = 0$ for $t < \frac{1}{4}$,
- $0 \leq \alpha(t) \leq 1$ for all t and
- $\alpha(t) = 1$ for $t > \frac{3}{4}$.

We show that H is smooth. To see this, let $f: \mathbf{I}^- \rightarrow \mathbb{R}$ be a generating function on \mathbf{I}^- . Then f is constant near 0.

Now let $c: \mathbb{R} \rightarrow \mathbf{I} \times \mathbf{I}^-$ be a structure curve, given by $c(v) = (t(v), s(v))$. The curve t is a structure curve on \mathbf{I} , and so is a smooth real function for all v , except possibly when $t(v) = 0$ or $t(v) = 1$.

The curve s is a structure curve on \mathbf{I}^- and so is a smooth real function for all v except possibly when $s(v) = 0$.

Now consider the composite $H \circ c: \mathbb{R} \rightarrow \mathbf{I}^-$. We know that $\alpha(t(v))$ is a smooth real function for all v , since the only possible points on non-smoothness occur when $t(v) = 0$ or $t(v) = 1$, and $\alpha(t(v))$ is locally constant near these points.

Thus $H \circ c$ is a smooth everywhere except possibly when $s(v) = 0$. In this case $H \circ c(v) = 0$.

When we now take the composite $f \circ H \circ c: \mathbb{R} \rightarrow \mathbb{R}$, the only potential point of non-smoothness is when $s(v) = 0$, giving us $H \circ c(v) = 0$. But f is a generating function on \mathbf{I}^- , and so is locally constant near 0. This shows that the composite $f \circ H \circ c$ is a smooth real function for all $v \in \mathbb{R}$, and we deduce that H is smooth.

4.5 Homotopy in FRL and objects in FRL relating to homotopy

This section is devoted to the definition of smooth homotopy, and the definitions of some basic homotopy objects in **FRL**.

We define the notion of smooth homotopy (that is, homotopy within the category **FRL**) as follows.

- 4.5.1. Definition.** 1. If X is a Frölicher space, and $x_0, x_1 \in X$, then we say that x_0 is smoothly path-connected to x_1 if there is a smooth path $c: I \rightarrow X$ such that $c(0) = x_0$ and $c(1) = x_1$. We write $x_0 \simeq x_1$. The relation \simeq is called **smooth homotopy** when it is applied to hom-sets.
2. Suppose we have a smooth map $f: X \rightarrow Y$. If there exists a smooth map $g: Y \rightarrow X$ such that $fg \simeq 1_Y$ and $gf \simeq 1_X$, then we call f a **smooth homotopy equivalence**.
3. We denote the class of smooth cofibrations (defined in the following section) by **cof** and the class of smooth homotopy equivalences by **we**. We also call the elements of these classes the **cofibrations** and **weak equivalences** respectively, in keeping with Baues' [7] terminology.

One can show that smooth homotopy is a congruence on **FRL**.

In practice we usually say that smooth maps $f, g: X \rightarrow Y$ are smoothly homotopic if there exists a smooth map $H: I \times X \rightarrow Y$ with $H(0, -) = f$ and $H(1, -) = g$. If $A \subseteq X$ is a Frölicher subspace of X then we say that H is a **smooth homotopy (rel A)** if the map H has the property that $H(t, a) = a$ for each $t \in I$ and $a \in A$. See Cherenack [22] and Dugmore [36] for more detail regarding smooth homotopy.

The notion of **deformation retract** is fundamental to topological homotopy theory. The following definitions are adapted for smooth homotopy, and will be needed at a later stage.

- 4.5.2. Definition.** Let $A \subseteq X$ be a Frölicher subspace of a Frölicher space X , and let $i: A \hookrightarrow X$ denote the inclusion map. Then

1. We say that A is a **retract** of X if there exists a smooth map $r: X \rightarrow A$ such that $ri = 1_A$. We call r a **retraction**.
2. We call A a **weak deformation retract** of X if the inclusion i is a smooth homotopy equivalence.
3. The subspace A is called a **deformation retract** of X if there exists a retraction $r: X \rightarrow A$ such that $ir \simeq 1_X$.
4. The subspace A is called a **strong deformation retract** of X if there exists a retraction $r: X \rightarrow A$ such that $ir \simeq 1_X(\text{rel } A)$.

4.5.3. Definition. The mapping cylinder I_f of $f: X \rightarrow Y$ is defined by the following pushout

$$\begin{array}{ccc} I \times X & \longrightarrow & I_f \\ \uparrow i_1 & & \uparrow \\ X & \xrightarrow{f} & Y \end{array}$$

where $i_1: X \rightarrow I \times X$ is given by $i_1(x) = (1, x)$, for any $x \in X$. We denote the elements of I_f by $[t, x]$ or $[y]$, where $(t, x) \in I \times X$ or $y \in Y$.

Replacing $I \times X$ in the above pushout diagram by $\mathbf{I} \times X$ or $\mathbb{I} \times X$, we obtain the **flattened mapping cylinder I_f** and **weakly flattened mapping cylinder \mathbb{I}_f** of f respectively. We use the same notation for elements of these flattened mapping cylinders as described above for the mapping cylinder.

There is also a map $i_0: X \rightarrow I \times X$ defined by $i_0(x) = (0, x)$ for $x \in X$. This induces an inclusion map $i'_0: X \rightarrow I_f$, which identifies X with the Frölicher subspace $i'_0(X)$ of I_f .

An inclusion is induced in a similar way for the flattened mapping cylinders.

If one identifies $0 \times X$ to a point in the mapping cylinder I_f of a map $f: X \rightarrow Y$, then one obtains the **mapping cone T_f** of the map f . The map defined by $x \mapsto [\frac{1}{2}, x]$ is an inclusion.

In a similar fashion we define the **flattened mapping cone T_f** and **weakly flattened mapping cone \mathbb{T}_f** of a smooth map $f: X \rightarrow Y$. As above we have inclusions given by $x \mapsto [\frac{1}{2}, x]$.

When working in the category \mathbf{FRL}_* of pointed Frölicher spaces and basepoint preserving maps we identify the set $\{(t, *X) \mid \text{all } t\}$ to a point in each of the various mapping cylinders and mapping cones defined above. The identified set is taken to be the basepoint in each case.

We shall need the **flattened double mapping cylinder $I_{f,g}$** later:

4.5.4. Definition. Given two smooth maps $f: X \rightarrow Y$ and $g: X \rightarrow Z$, we define the **flattened double mapping cylinder $I_{f,g}$** of f and g by the pushout

$$\begin{array}{ccc} \mathbf{I}_f & \longrightarrow & \mathbf{I}_{f,g} \\ \uparrow i'_0 & & \uparrow \\ X & \xrightarrow{g} & Z, \end{array}$$

where i is the inclusion into the flattened mapping cylinder.

Of course one can similarly define a **double mapping cylinder** and **weakly flattened double mapping cylinder**, but we shall not need these objects in this thesis.

4.6 Cofibrations in FRL

The notion of a **cofibration**, and the dual notion of a **fibration** are central to homotopy theory. In this thesis we concentrate on the role of cofibrations in the homotopy theory of **FRL**. For some aspects of the role that fibrations play in **FRL**, see Appendix A.

A cofibration is a map $i: A \rightarrow B$ for which the problem of extending functions from $i(A)$ to B is a homotopy problem. In other words, if a map $f: i(A) \rightarrow Z$ can be extended to a map $f^*: B \rightarrow Z$, then so can any map homotopic to f .

In fact, for topological spaces, the usual definition is phrased in a slightly more restrictive way. The extension of a map $g \simeq_H f$, for some homotopy $H: I \times i(A) \rightarrow Z$, is required to exist at every level of the homotopy simultaneously. In other words, one requires each $H(t, -)$ to be extendable in such a way that the resulting homotopy $H^*: I \times B \rightarrow Z$ is continuous.

We weaken this definition somewhat, to enable smooth homotopy extensions to be more easily constructed using a flattening at the endpoints of the homotopy. This enables us to characterise smooth cofibrations in terms of a flattened unit interval (see Lemma 4.6.3), and then later to relate smooth cofibrations to smooth neighbourhood deformation retracts, which is the subject of Chapter 5. This gives the following definition.

4.6.1. Definition. A smooth map

$$A \xrightarrow{i} B$$

is called a **smooth cofibration** if, corresponding to every commutative diagram of the form

$$\begin{array}{ccc} A & \xrightarrow{i} & B \xrightarrow{f} Z, \\ \downarrow (0, 1_A) & \nearrow G & \\ I \times A & & \end{array}$$

there exists a commutative diagram in **FRL** of the form

$$\begin{array}{ccc} B & \xrightarrow{f} & Z \\ \downarrow (0, 1_B) & \nearrow F & \uparrow G' \\ I \times B & \xleftarrow{1_I \times i} & I \times A \end{array}$$

where $G' : I \times A \rightarrow Z$ is given by $G'(t, a) = G(\alpha_\epsilon(t), a)$ for some $0 < \epsilon < \frac{1}{2}$, and each $t \in I, a \in A$.

The problem of extending a map smoothly from a subspace of a Frölicher space to the entire space is a more difficult problem than simply extending a map continuously. It is mainly for this reason that the definition of smooth cofibration differs somewhat from the corresponding definition of a topological cofibration.

All topological cofibrations are inclusions, and this result is true for smooth cofibrations too. The proof of the following lemma is essentially the same as the proof given by James [71] for the topological result, although James' proof is in some sense dual to ours, using path spaces in place of Cartesian products and the adjoint versions of our homotopies.

4.6.2. Lemma. A cofibration

$$B \xrightarrow{i} A$$

is a smooth inclusion.

Proof. Let \mathbb{I}_i be the mapping cylinder of i and let $j : A \rightarrow \mathbb{I}_i$ be the standard inclusion map. See Definition 4.5.3, where $j(a) = [a]$.

We have the following commutative diagram

$$\begin{array}{ccc} B & \xrightarrow{i} & A & \xrightarrow{j} & \mathbb{I}_i \\ & & \searrow G & & \uparrow \\ & & I \times B & & \end{array}$$

(0,1_B) ↓

where $G(t, b) = [1 - t, b]$. Notice that the map G is smooth, since a structure function on the weakly flattened unit interval \mathbb{I} also defines a structure function on the usual unit interval I . Since i is a cofibration we have the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{j} & \mathbb{I}_i \\ & \searrow F & \uparrow G' \\ I \times A & \xleftarrow{1_I \times i} & I \times B \end{array}$$

(0,1_A) ↓

where $G'(t, b) = G(\alpha_\epsilon(t), b)$ for some $0 < \epsilon < \frac{1}{2}$.

Define $U : A \rightarrow \mathbb{I}_i$ by $U(a) = F(1, a)$. We have $U \circ i = G'(1, -)$. The map $G'(1, -)$ is given by $G'(1, b) = [0, b]$. Thus the assignment $b \mapsto G'(1, b)$ defines the usual inclusion of B into the mapping cylinder. From this we deduce that $U \circ i$ is an inclusion and hence i is an inclusion. \square

There is an equivalent formulation of definition 4.6.1 given in the following lemma.

4.6.3. Lemma. A smooth map

$$A \xrightarrow{i} B$$

is a cofibration if and only if for every smooth map

$$h: (0 \times B) \cup (\mathbf{I}^- \times i(A)) \rightarrow Z$$

the following diagram exists in **FRL**,

$$\begin{array}{ccc} (0 \times B) \cup (\mathbf{I}^- \times i(A)) & \xrightarrow{h} & Z \\ j \downarrow & \nearrow G & \\ \mathbf{I}^- \times B & & \end{array}$$

where j is the evident inclusion.

Proof. Suppose that the inclusion

$$A \xrightarrow{i} B$$

is a smooth cofibration. We use the notation $A \subseteq B$ or $i(A) \subseteq B$ if we wish to emphasise the inclusion map.

Suppose that

$$h: (0 \times B) \cup (\mathbf{I}^- \times i(A)) \rightarrow Z$$

is a smooth map. We have the diagram

$$\begin{array}{ccc} (0 \times B) \cup (\mathbf{I}^- \times i(A)) & \xrightarrow{h} & Z \\ j \downarrow & & \\ \mathbf{I}^- \times B & & \end{array}$$

We need to fill in a smooth map $G: \mathbf{I}^- \times X \rightarrow Y$ which makes the resulting diagram commute.

To do this, notice that $h|_{\mathbf{I}^- \times i(A)}$ is smooth and thus the corresponding map $h|_{I \times i(A)}$ using the usual unit interval is also smooth. We have the following diagram,

$$\begin{array}{ccc} A & \xrightarrow{i} & B \xrightarrow{h|_{0 \times B}} & Z \\ (0, 1_A) \downarrow & & \nearrow h|_{\mathbf{I}^- \times A} & \\ I \times A & & & \end{array}$$

and thus the fact that i is a smooth cofibration gives the commutative diagram in **FRL**

$$\begin{array}{ccc} B & \xrightarrow{h|_{0 \times B}} & Z \\ (0, 1_B) \downarrow & \nearrow F & \uparrow (h|_{\mathbf{I}^- \times A})' \\ I \times B & \xleftarrow{1_I \times i} & I \times A \end{array}$$

where $(h|_{\mathbb{I}^- \times A})'(t, a) = h|_{\mathbb{I}^- \times A}(\alpha_\epsilon(t), a)$, for some $0 < \epsilon < \frac{1}{2}$.

Now choose a smooth braking function $\alpha: \mathbb{R} \rightarrow \mathbb{R}$ with the following properties.

- $\alpha(t) = 0$ for $t < \frac{\epsilon}{2}$,
- $\alpha(t) = t$ for $\epsilon < t$.

One can verify that F is a map which makes the required diagram commute, but F may not be smooth on $\mathbb{I}^- \times A$ due to the flattening requirements of the left flattened unit interval. To correct this, set $G(t, a) = F(\alpha(t), a)$. Notice that the insertion of this braking function does not affect the commutativity conditions of G , since the only adjustments to F occur in the first coordinate where the map $(h|_{\mathbb{I}^- \times B})'$ is constant.

Conversely, suppose that corresponding to every smooth map

$$h: (0 \times B) \cup (\mathbb{I}^- \times i(A)) \rightarrow Z$$

there exists a commutative diagram

$$\begin{array}{ccc} (0 \times B) \cup (\mathbb{I}^- \times i(A)) & \xrightarrow{h} & Z \\ j \downarrow & \nearrow H & \\ \mathbb{I}^- \times B & & \end{array}$$

We wish to show that the inclusion $i: A \rightarrow B$ is a cofibration, so assume we have the following diagram

$$\begin{array}{ccccc} A & \xrightarrow{i} & B & \xrightarrow{j} & Z \\ (0, 1_A) \downarrow & & & \nearrow G & \\ I \times A & & & & \end{array}$$

There exists the diagram

$$\begin{array}{ccccc} A & \xrightarrow{i} & B & \xrightarrow{j} & Z \\ (0, 1_A) \downarrow & & & \nearrow G' & \\ I \times A & & & & \end{array}$$

where $G'(t, a) = G(\alpha_\epsilon(t), a)$. Our hypothesis allows us to construct the diagram

$$\begin{array}{ccc} (0 \times B) \cup (\mathbb{I}^- \times i(A)) & \xrightarrow{j \cup G'} & Z \\ j \downarrow & \nearrow H & \\ \mathbb{I}^- \times B & & \end{array}$$

Note that $f \cup G'$ is smooth since $\alpha_\epsilon(t)$ is constant near $t = 0$.

Since H is smooth on $I^- \times B$ it defines a smooth map on $I \times B$. One can verify then that the diagram

$$\begin{array}{ccc}
 B & \xrightarrow{f} & Z \\
 (0,1x) \downarrow & \nearrow H & \uparrow G' \\
 I \times B & \xleftarrow{1_I \times i} & I \times A
 \end{array}$$

commutes as required. □

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Chapter 5

Smooth neighbourhood deformation retracts

This chapter is concerned with the formulation of a suitable notion of **smooth neighbourhood deformation retract**.

For topological spaces the statement that a closed subspace A of X is a neighbourhood deformation retract of X is equivalent to the statement that the inclusion $i: A \hookrightarrow X$ is a closed cofibration.

We show that in the category of Frölicher spaces there is a notion of ‘smooth neighbourhood deformation retract’ that gives rise to an analogous result that a closed Frölicher subspace A of the Frölicher space X is a smooth neighbourhood deformation retract of X if and only if the inclusion $i: A \hookrightarrow X$ comes from a certain subclass of cofibrations.

As an application we construct the right Puppe sequence.

5.1 SNDR pairs and SDR pairs

In this section we shall define the notion of ‘smooth neighbourhood deformation retract pair’. The definition is similar to the definition of ‘R-SNDR pair’ suggested in [36], but we have modified the definition in order to retain only the essential aspects of ‘first coordinate independence’ defined in [36].

We begin by defining the ‘first coordinate independence property’ of a function on a product of a Frölicher space with \mathbf{I} (or \mathbf{I}^- , \mathbf{I}^+).

5.1.1. Definition. Let $i: A \rightarrow X$ be a smooth map. Let $c: \mathbb{R} \rightarrow X$ be a structure curve on X . Define

$$\Lambda(c, i) = \{t_* \in c^{-1}(i(A)) \mid \exists \text{ a sequence } \{t_n\} \text{ of real numbers} \\ \text{with } \lim_{n \rightarrow \infty} t_n = t_* \text{ and each } t_n \in c^{-1}(X \setminus i(A))\}.$$

The points in $\Lambda(c, i)$ are those values in \mathbb{R} where the curve c ‘enters’ $i(A)$ from $X \setminus i(A)$, or ‘touches’ a point in $i(A)$ whilst remaining in $X \setminus i(A)$ nearby.

Now we are ready to define the 'first coordinate independence property' for a structure function on a product.

5.1.2. Definition. Let $i: A \rightarrow X$ be a smooth map and suppose $f: \mathbf{I} \times X \rightarrow \mathbb{R}$ is a structure function on $\mathbf{I} \times X$. Let $c: \mathbb{R} \rightarrow \mathbf{I} \times X$, given by $c(s) = (t(s), x(s))$ have the following properties.

1. The map $x(s)$ is a structure curve on X .
2. For all $\epsilon > 0$, $t(s)$ is a smooth real function on $\mathbb{R} \setminus \bigcup_{s_* \in \Lambda(x, i)} [s_* - \epsilon, s_* + \epsilon]$.

If, for every such map c , the composite $f \circ c$ is a smooth real function, then we say that $f: \mathbf{I} \times X \rightarrow \mathbb{R}$ has the **first coordinate independence property (FCIP)** with respect to i .

Extending the definition, we say that a map $g: \mathbf{I} \times X \rightarrow Y$ has the **FCIP** with respect to i if the composite $f \circ g: \mathbf{I} \times X \rightarrow \mathbb{R}$ has the FCIP with respect to i for every $f \in F_Y$.

In the previous definition we are considering curves into the product $\mathbf{I} \times X$ whose first coordinate may become singular as the second coordinate enters the subspace $i(A)$ from $X \setminus i(A)$. A function $f: \mathbf{I} \times X \rightarrow \mathbb{R}$ has the FCIP if it can be composed with such curves to yield a smooth real function.

Notice that we can formulate a similar definition of the FCIP if we replace \mathbf{I} throughout by \mathbf{I}^- or \mathbf{I}^+ , leaving the rest of the definition unchanged. We will have occasion to use this type of first coordinate independence in the later part of this work.

5.1.3. Note. Let $i: A \rightarrow X$ and suppose that we are given a map $g: \mathbf{I} \times X \rightarrow Y$. Let $f: Y \rightarrow \mathbb{R}$ be a structure function on Y , and suppose that $f \circ g: \mathbf{I} \times X \rightarrow \mathbb{R}$ has the FCIP with respect to i for any such f . Then, given a smooth map $h: Y \rightarrow Z$, the composite $f' \circ h \circ g: \mathbf{I} \times X \rightarrow \mathbb{R}$ has the FCIP with respect to i for any structure function f' on Z . This is immediately obvious, since the composite $f' \circ h: Y \rightarrow \mathbb{R}$ is a structure function on Y .

Again, observe that the above note applies equally well if $g: \mathbf{I}^- \times X \rightarrow Y$ or $g: \mathbf{I}^+ \times X \rightarrow Y$ has the FCIP with respect to i when composed with a smooth function h on Y .

5.1.4. Example. 1. For any $A \xrightarrow{i} X$, the projection onto the second coordinate $p_2: \mathbf{I} \times X \rightarrow X$ has the FCIP.

2. Let $\alpha: \mathbb{R} \rightarrow \mathbb{R}$ be a smooth braking function with the properties that

- $\alpha(t) = 0$ if $t < \frac{1}{4}$,
- $0 < \alpha(t) < 1$ if $\frac{1}{4} \leq t \leq \frac{3}{4}$,
- $\alpha(t) = 1$ if $\frac{3}{4} < t$.

Consider $0 \hookrightarrow \mathbf{I}^-$. Let $H: \mathbf{I} \times \mathbf{I}^- \rightarrow \mathbf{I}^-$ be given by $H(t, s) = (1 - \alpha(t))s$. Then $f \circ H: \mathbf{I} \times \mathbf{I}^- \rightarrow \mathbb{R}$ has the FCIP with respect to the inclusion $0 \hookrightarrow \mathbf{I}^-$, for any $f \in F_{\mathbf{I}^-}$. See Lemma 5.1.7 for a proof.

5.1.5. Definition. Consider a smooth inclusion $i: A \hookrightarrow X$. Suppose that there exists a smooth map $u: X \rightarrow \mathbf{I}$, with $u^{-1}(0) = i(A)$. If there exists a smooth map

$$H: \mathbf{I} \times X \rightarrow X,$$

that satisfies the following properties, then we call the pair (X, A) a **smooth neighbourhood deformation retract pair**, or **SNDR pair** for short.

1. H has the FCIP with respect to i .
2. $H(0, x) = x$ for all $x \in X$.
3. $H(t, x) = x$ for all $(t, x) \in \mathbf{I} \times i(A)$.
4. $H(1, x) \in i(A)$ for all $x \in X$ with $u(x) < 1$.

If, in addition, we may choose H so that $H(1 \times X) \subseteq i(A)$, then we call the pair (X, A) a **smooth deformation retract pair** or an **SDR pair** for short.

We say that A is a **smooth neighbourhood deformation retract** or **smooth deformation retract** of X if (X, A) is an SNDR pair or SDR pair, respectively.

We call (u, H) a **representation** for the SNDR (or SDR) pair.

- 5.1.6. Example.**
1. The pair (X, \emptyset) is an SNDR pair. A representation is $u(x) = 1$, $H(t, x) = x$, for each $t \in \mathbf{I}$ and $x \in X$.
 2. The pair (X, X) is an SNDR pair. A representation is $u(x) = 0$, $H(t, x) = x$, for each $t \in \mathbf{I}$ and $x \in X$.

In the next lemma we present our first example of a non-trivial SNDR pair. We give a detailed proof of the fact that $0 \hookrightarrow \mathbf{I}^-$ is an SNDR pair. This example forms the basis of many important constructions, such as the mapping cylinder.

5.1.7. Lemma. *The pair $(\mathbf{I}^-, 0)$ is an SDR pair.*

Proof. Let $\alpha: \mathbb{R} \rightarrow \mathbb{R}$ be a smooth braking function, such as the one in Example 5.1.4. A representation for $(\mathbf{I}^-, 0)$ as an SDR pair is (u, H) , where $u: \mathbf{I}^- \rightarrow \mathbf{I}$ and $H: \mathbf{I} \times \mathbf{I}^- \rightarrow \mathbf{I}^-$ are given by $u(s) = s$, and $H(t, s) = (1 - \alpha(t))s$.

Since the structure on \mathbf{I}^- is finer than the structure on \mathbf{I} , the identity $u: \mathbf{I}^- \rightarrow \mathbf{I}$ is smooth.

The map H is smooth, as shown in Example 4.4.5, and H clearly has the FCIP with respect to the inclusion, since whenever v approaches a value for which $s(v) = 0$, the composite

$$g\left(\left(1 - \alpha(t'(v))\right)s(v)\right) = g(0)$$

for v in a neighbourhood of this value. □

5.1.8. Example. The pair $(\mathbf{I}, \{0, 1\})$ is an SNDR pair.

Proof. A representation (u, H) for the SNDR pair can be given as follows. Define $u: \mathbf{I} \rightarrow \mathbf{I}$ to be a bump function with the following properties

- $u(t) = 0$ for $t = 0$ or $t = 1$,
- $u(t) = 1$ for $t \in [\frac{1}{4}, \frac{3}{4}]$ and
- $0 < u(t) < 1$ otherwise,

Let $\alpha_{\frac{1}{4}}: \mathbf{I} \rightarrow \mathbf{I}$ be a braking function with the properties

- $\alpha_{\frac{1}{4}}(s) = 0$ for $0 \leq s \leq \frac{1}{4}$,
- $\alpha_{\frac{1}{4}}(s) = 1$ for $\frac{3}{4} \leq s \leq 1$.

Let $0 < \epsilon < \frac{1}{4}$, and define $H: \mathbf{I} \times \mathbf{I} \rightarrow \mathbf{I}$ by $H(t, s) = (1 - \alpha_{\epsilon}(t))s + \alpha_{\epsilon}(t)\alpha_{\frac{1}{4}}(s)$.

Notice that $H(0, s) = s$, $H(t, 0) = 0$ and $H(t, 1) = 1$. Suppose that $u(s) < 1$. Then $s \in [0, \frac{1}{4}) \cup (\frac{3}{4}, 1]$. This implies that $\alpha_{\frac{1}{4}}(s) = 0$ or $\alpha_{\frac{1}{4}}(s) = 1$. We then have $H(1, s) = \alpha_{\frac{1}{4}}(s) = 0$ or $H(1, s) = \alpha_{\frac{1}{4}}(s) = 1$, which means that $H(1, s) \in \{0, 1\}$ if $u(s) < 1$.

To see that H is smooth, let $f: \mathbf{I} \rightarrow \mathbb{R}$ be a generating function for the flattened unit interval. Then f is locally constant at 0 and 1. Since each of the variables t and s comes from a flattened unit interval, a structure curve into either the t or the s component will only have points of non-smoothness when $t = 0, 1$ and $s = 0, 1$. The braking function α_{ϵ} ensures that H is locally constant in the t variable whenever t is near 0 or 1, so no problems arise from the t component. When s is near $s = 0$, we have $H(t, s)$ near 0, and so the generating function f is locally constant. Similarly, when s is near $s = 1$, we have $H(t, s)$ near 1, and the generating function f is again locally constant. \square

We now show that the 'product' of SNDR pairs is again an SNDR pair.

5.1.9. Theorem. *Let $i: A \hookrightarrow X$ and $j: B \hookrightarrow Y$ be inclusion mappings. If (X, A) and (Y, B) are SNDR pairs, then so is*

$$\left(X \times Y, (X \times B) \cup (A \times Y) \right).$$

If one of (X, A) or (Y, B) is an SDR pair then so is the pair

$$\left(X \times Y, (X \times B) \cup (A \times Y) \right).$$

Proof. Let $\alpha: \mathbb{R} \rightarrow \mathbb{R}$ be a smooth braking function with the properties

- $\alpha(t) = 0$ for $t \leq \frac{1}{4}$,
- $\alpha(t) = 1$ for $t \geq \frac{3}{4}$,

and let $\beta: \mathbb{R} \rightarrow \mathbb{R}$ be a smooth increasing braking function with the properties

- $\beta(t) = t$ for $t \leq \frac{1}{4}$,
- $\beta(t) = 1$ for $t \geq \frac{3}{4}$.

Suppose that (u, H) and (v, J) are representations for the SNDR pairs (X, A) and (Y, B) , respectively.

Let $\bar{u}: X \rightarrow \mathbf{I}$ and $\bar{v}: Y \rightarrow \mathbf{I}$ be given by $\bar{u}(x) = \beta(u(x))$ and $\bar{v}(y) = \beta(v(y))$ respectively.

Define

$$w: X \times Y \rightarrow \mathbf{I}$$

by $w(x, y) = \bar{u}(x)\bar{v}(y)$. The braking function β ensures smoothness when $\bar{u}(x) \neq 1$ and $\bar{v}(y) = 1$, or when $\bar{u}(x) = 1$ and $\bar{v}(y) \neq 1$. We need to take this into account to ensure that when w is composed with a structure curve on $X \times Y$ we have a structure curve on \mathbf{I} .

The map w is clearly smooth elsewhere. We have $w^{-1}(0) = (X \times B) \cup (A \times Y)$, as required.

Define $Q: \mathbf{I} \times X \times Y \rightarrow X \times Y$ as follows.

$$Q(t, x, y) = \begin{cases} \left(H(\alpha(t), x), J(\alpha(t), y) \right) & \text{if } u(x) = v(y) = 0, \\ \left(H(\alpha(t), x), J\left(\alpha\left(\frac{\bar{u}(x)}{\bar{v}(y)}\right)\alpha(t), y\right) \right) & \text{if } \bar{v}(y) \geq \bar{u}(x), \bar{v}(y) > 0, \\ \left(H\left(\alpha\left(\frac{\bar{v}(y)}{\bar{u}(x)}\right)\alpha(t), x\right), J(\alpha(t), y) \right) & \text{if } \bar{u}(x) \geq \bar{v}(y), \bar{u}(x) > 0. \end{cases}$$

We must show that Q is a smooth map, with the first coordinate independence property with respect to the inclusion $X \times B \cup A \times Y \hookrightarrow X \times Y$.

We first consider each part of the definition of Q separately. The first part is clearly smooth. Let us verify that Q is smooth on the second part of its definition; the third part is similar.

Restricting our attention then to the second line in the definition of Q , we observe that the first component $H(\alpha(t), x)$ is smooth, so we focus our attention on the second component, namely $J\left(\alpha\left(\frac{\bar{u}(x)}{\bar{v}(y)}\right)\alpha(t), y\right)$. Each function making up $J\left(\alpha\left(\frac{\bar{u}(x)}{\bar{v}(y)}\right)\alpha(t), y\right)$ is smooth individually, so we need only pay extra attention to those parts that involve flattened unit intervals, remembering that the addition and multiplication on the flattened unit interval need not preserve smoothness, as is the case for the usual unit interval.

The braking function α is always smooth, so let us consider the composite of α with the quotient $\frac{\bar{u}(x)}{\bar{v}(y)}$, namely $\alpha\left(\frac{\bar{u}(x)}{\bar{v}(y)}\right)$. This is always smooth, except possibly when $\bar{u}(x)$ or $\bar{v}(y)$ each approach 0 or 1, since it is here that structure curves on the flattened unit interval need not be smooth in the usual sense.

If $u(x)$ approaches 0 and $v(y)$ does not approach 0, then the braking function α ensures that $\frac{\bar{u}(x)}{\bar{v}(y)} = 0$ 'near' such points. If $\bar{v}(y)$ approaches 0 then $\bar{u}(x)$ must also approach 0. This situation is dealt with later.

Thus, Q in part two of the definition is smooth, and one can show similarly that Q in the third part of the definition is smooth.

Let us now consider the overlaps of the three parts of the definition of Q . Observe that if $\bar{u}(x)$ is in a sufficiently small neighbourhood of $\bar{v}(y)$, with $\bar{u}(x) \neq 0$ and $\bar{v}(y) \neq 0$, then we have $\alpha(\frac{\bar{u}(x)}{\bar{v}(y)}) = \alpha(\frac{\bar{v}(y)}{\bar{u}(x)}) = 1$, and so the second and third parts of the definition of Q coincide here. Thus it remains only to show that Q is smooth as $\bar{u}(x)$ and $\bar{v}(y)$ both approach 0.

If Q is smooth in each of its coordinates then it is smooth, so consider the coordinate involving the map J . Let $c: \mathbb{R} \rightarrow \mathbb{I} \times X \times Y$ be a structure curve that is given by $c(s) = (t(s), x(s), y(s))$. Then the map $c_1: \mathbb{R} \rightarrow \mathbb{I} \times Y$, given by

$$c_1(s) = \begin{cases} \left(\alpha(t(s)), y(s) \right) & \text{if } \bar{u}(x(s)) = \bar{v}(y(s)) = 0, \\ \left(\alpha\left(\frac{\bar{u}(x(s))}{\bar{v}(y(s))}\right)\alpha(t(s)), y(s) \right) & \text{if } \bar{v}(y(s)) \geq \bar{u}(x(s)), \bar{v}(y(s)) > 0, \\ \left(\alpha(t(s)), y(s) \right) & \text{if } \bar{u}(x(s)) \geq \bar{v}(y(s)), \bar{u}(x(s)) > 0. \end{cases}$$

is a map satisfying the conditions of Definition 5.1.2, since its second coordinate is smooth, but its first coordinate may become singular as $\bar{v}(y(s))$ (and hence $\bar{u}(x(s))$) approaches zero. Since J has the first coordinate independence property, the map

$$(J \circ c_1)(s) = \begin{cases} J(\alpha(t(s)), y(s)) & \text{if } \bar{u}(x(s)) = \bar{v}(y(s)) = 0, \\ J\left(\alpha\left(\frac{\bar{u}(x(s))}{\bar{v}(y(s))}\right)\alpha(t(s)), y(s)\right) & \text{if } \bar{v}(y(s)) \geq \bar{u}(x(s)), \bar{v}(y(s)) > 0, \\ J(\alpha(t(s)), y(s)) & \text{if } \bar{u}(x(s)) \geq \bar{v}(y(s)), \bar{u}(x(s)) > 0. \end{cases}$$

is smooth. Thus $Q \circ c$ is smooth, and since c is arbitrary, Q is smooth. In a similar way, the coordinate of Q involving H can be shown to be smooth.

We now verify that Q satisfies the required boundary conditions. When $t = 0$, all three lines defining Q reduce to $\left(H(0, x), J(0, y) \right)$. If $x \in A$ and $y \notin B$ then Q is given by the second part of its definition, which reduces to $\left(H(\alpha(t), x), J(0, y) \right)$. The case when $x \notin A$ and $y \in B$ is similar. If $t = 1$ and $0 < w(x, y) < 1$ then either $0 < \bar{u}(x) < 1$ or $0 < \bar{v}(y) < 1$. Suppose that $0 < \bar{u}(x) < 1$. Then either $\bar{u}(x) \leq \bar{v}(y)$ or $\bar{v}(y) < \bar{u}(x)$. If $\bar{u}(x) \leq \bar{v}(y)$ then Q is given by the second part of its definition, which reduces to $\left(H(1, x), J\left(\alpha\left(\frac{\bar{u}(x)}{\bar{v}(y)}\right), y\right) \right) \in i(A) \times Y$. If $\bar{v}(y) < \bar{u}(x)$ then the third part of the definition of Q applies and Q reduces to $\left(H\left(\alpha\left(\frac{\bar{v}(y)}{\bar{u}(x)}\right), x\right), J(1, y) \right) \in X \times j(B)$.

Finally, we must show that for any $f \in F_{X \times Y}$, $f \circ Q$ has the first coordinate independence property with respect to the inclusion $(X \times B) \cup (A \times Y) \hookrightarrow X \times Y$. To this end consider a map $c: \mathbb{R} \rightarrow \mathbb{I} \times X \times Y$, given by $c(s) = (t(s), x(s), y(s))$, and a structure function $f \in F_{X \times Y}$. Let $\{s_n\}$ be a sequence of real numbers converging to s_* with $c(s_n) \in (X \times Y) \setminus \left((A \times Y) \cup (X \times B) \right)$, and $c(s_*) \in (A \times Y) \cup (X \times B)$. There are three cases to consider.

1. Suppose that $c(s_*) \in A \times B$. Then $x(s_*) \in A$ and $y(s_*) \in B$. The fact that H has the first coordinate independence property with respect to i and J has the first coordinate independence property with respect to j means that each coordinate of Q is smooth, and so Q is smooth.
2. Suppose that $c(s_*) \in A \times Y$ and that $y(s_*) \notin B$. Then at each of the points $c(s_n)$, $Q \circ c(s_n)$ is given by the second part of the definition of Q , for n large enough. Since $x(s_*) \in A$, the component of Q involving H is smooth, since H has the first coordinate independence property. For s in a neighbourhood of s_* , $\alpha\left(\frac{\bar{u}(x(s))}{\bar{v}(y(s))}\right) = 0$, since $\frac{\bar{u}(x(s_*))}{\bar{v}(y(s_*))} = 0$. Thus the component of Q involving J is constant for s in a neighbourhood of s_* , and so is smooth there.
3. The case with $c(s_*) \in X \times B$, and $x(s_*) \notin A$ is similar to the second case above.

For the last part of the theorem, suppose that (u, H) represent (X, A) as an SDR pair. If we replace u by $u' = \frac{1}{2}u$, then (u', H) also represent (X, A) as an SDR pair. Making the above constructions now with u' in place of u , it follows that $w(x, y) < 1$ for all (x, y) and so $Q(1, x, y) \in (X \times B) \cup (A \times Y)$.

This completes the proof. \square

The proof of the corresponding topological result is fairly delicate (see Steenrod [119]); so it is not surprising that our proof requires some care to ensure that all the required boundary conditions are satisfied, whilst keeping the map Q smooth.

5.2 Cofibrations

In this section we show that for a subspace $A \subseteq X$ that is closed in the underlying topology, the inclusion $i: A \rightarrow X$ is a cofibration (where any homotopy extension can be chosen with the FCIP) if and only if (X, A) is an SNDR pair.

5.2.1. Definition. Let $i: A \rightarrow B$ be a cofibration. We call i a **cofibration with the FCIP** if any homotopy extension can be chosen to have the FCIP with respect to i .

Using the equivalent formulation of the notion of cofibration given by Lemma 4.6.3, we may state Definition 5.2.1 as follows.

A cofibration $i: A \rightarrow X$ is a cofibration with the FCIP if and only if the map G that we may fill in to complete the commutative diagram

$$\begin{array}{ccc} (0 \times X) \cup (\mathbf{I}^- \times A) & \xrightarrow{h} & Y \\ j \downarrow & \nearrow G & \\ \mathbf{I}^- \times X & & \end{array}$$

may be chosen to have the FCIP with respect to the inclusion i .

We have the following result, which corresponds to a similar topological result.

5.2.2. Lemma. *A smooth map $i: A \rightarrow X$ is a cofibration (with the FCIP) if and only if*

$$(0 \times X) \cup (\mathbf{I}^- \times A)$$

is a retract of $\mathbf{I}^- \times X$, (where the retraction

$$r: \mathbf{I}^- \times X \rightarrow (0 \times X) \cup (\mathbf{I}^- \times A)$$

has the FCIP).

Proof. In the one direction, suppose that $(0 \times X) \cup (\mathbf{I}^- \times A)$ is a retract of $\mathbf{I}^- \times X$. We wish to complete the following diagram.

$$\begin{array}{ccc} (0 \times X) \cup (\mathbf{I}^- \times A) & \xrightarrow{h} & Y \\ j \downarrow & \nearrow G & \\ \mathbf{I}^- \times X & & \end{array}$$

By hypothesis, there exists $r: \mathbf{I}^- \times X \rightarrow (0 \times X) \cup (\mathbf{I}^- \times i(A))$ such that $r \circ j = 1$. Define $G = h \circ r$. If r has the FCIP, then so does $h \circ r$.

Conversely, suppose that $i: A \rightarrow X$ is a cofibration (with the FCIP). We may find a map r such that the diagram

$$\begin{array}{ccc} (0 \times X) \cup (\mathbf{I}^- \times A) & \xrightarrow{1} & (0 \times X) \cup (\mathbf{I}^- \times i(A)) \\ j \downarrow & \nearrow r & \\ \mathbf{I}^- \times X & & \end{array}$$

commutes. Thus $r \circ j = 1$. If i is a cofibration with the FCIP with respect to i , then r can be chosen to have the FCIP. \square

The next theorem shows the relationship between cofibrations, retracts and SNDR pairs.

5.2.3. Theorem. *Let $i: A \hookrightarrow X$ be an inclusion. Suppose that A is closed in the underlying topology on X . Then the following are equivalent.*

1. The pair (X, A) is an SDR pair.
2. There is a smooth retraction $r: \mathbf{I}^- \times X \rightarrow (0 \times X) \cup (\mathbf{I}^- \times A)$ with the FCIP.
3. The map $i: A \rightarrow X$ is a cofibration with the FCIP.

Proof. To show that 1 and 2 are equivalent, notice that the pair

$$\left(\mathbf{I}^- \times X, (0 \times X) \cup (\mathbf{I}^- \times A) \right)$$

is an SDR pair, as a consequence of Lemma 5.1.7 and Theorem 5.1.9.

Let (w, Q) be a representation for the pair

$$\left(\mathbf{I}^- \times X, (0 \times X) \cup (\mathbf{I}^- \times A) \right)$$

as an SDR pair, and let Q be constructed as in Theorem 5.1.9. Define

$$r: \mathbf{I}^- \times X \rightarrow (0 \times X) \cup (\mathbf{I}^- \times A)$$

by $r(t, x) = Q(1, t, x)$, where $(t, x) \in \mathbf{I}^- \times X$. We observe that r has the FCIP, since Q has this property, and Q has this property since each of its components has this property.

The equivalence of 2 and 3 is Lemma 5.2.2.

We need now only show 2 implies 1. Let $r: \mathbf{I}^- \times X \rightarrow (0 \times X) \cup (\mathbf{I}^- \times i(A))$ be a retraction with the FCIP with respect to i . Define $H: \mathbf{I} \times X \rightarrow X$ by $H(t, x) = p \circ r(\alpha(t), x)$, where p is the projection onto the second coordinate, and $\alpha: \mathbb{R} \rightarrow \mathbb{R}$ is a braking function with the following properties:

- $\alpha(t) = 0$ for $t \leq 0$,
- $\alpha(t) = 1$ for $t \geq \frac{3}{4}$,
- $0 < \alpha(t) < 1$ for $0 < t < \frac{3}{4}$.

This braking function is necessary to ensure smoothness at the right endpoint of the flattened unit interval \mathbf{I} . Smoothness at the left endpoint is already taken care of by the fact that r is defined in terms of the left flattened unit interval.

Then H satisfies the following properties:

1. The map H has the FCIP since r has this property.
2. We have $H(0, x) = (p \circ r)(0, x) = p(0, x) = x$, for $x \in X$.
3. We have $H(t, x) = (p \circ r)(\alpha(t), x) = p(\alpha(t), x) = x$, for $x \in A$.

We now construct $u: X \rightarrow \mathbf{I}$. Let $p_1: \mathbf{I} \times X \rightarrow \mathbf{I}$ denote the projection onto the first coordinate of $\mathbf{I} \times X$. Define a function $\beta: \mathbb{R} \rightarrow \mathbb{R}$ by

$$\beta(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ e^{-\frac{1}{t^2}} & \text{if } t > 0. \end{cases}$$

It is well known that β is a smooth real function.

Now define $u: X \rightarrow \mathbf{I}$ by

$$u(x) = \frac{\int_0^1 \beta(\alpha(t) - (p_1 r)(1, x)(p_1 r)(\alpha(t), x)) dt}{\int_0^1 \beta(\alpha(t)) dt}.$$

It is clear that u is a smooth mapping.

We now verify that (u, H) represents (X, A) as an SNDR pair.

1. If $x \in A$, then $p_1 r(1, x) = 1$ and $p_1 r(\alpha(t), x) = \alpha(t)$ and so $p_1 r(1, x)p_1 r(\alpha(t), x) = \alpha(t)$.

Thus $\int_0^1 \beta(\alpha(t) - p_1 r(1, x)p_1 r(\alpha(t), x)) dt = 0$. This implies that $u(x) = 0$.

2. Suppose that $x \in X \setminus A$. Since $0 \times (X \setminus A)$ is open in the underlying topology on $(0 \times X) \cup (\mathbf{I}^- \times A)$, we may choose a neighbourhood $W \subseteq 0 \times (X \setminus A)$ of $(0, x)$. Since r is continuous, there is a neighbourhood $V \subseteq \mathbf{I}^- \times X$ such that $rV \subseteq W \subseteq 0 \times (X \setminus A)$.

Now consider the mapping $q_x: \mathbf{I} \rightarrow \mathbf{I} \times X$, given by $q_x(t) = (\alpha(t), x)$, for each $x \in X$. This is clearly smooth, and hence continuous in the underlying topologies. Thus there exists a neighbourhood $U \subseteq \mathbf{I}^-$ such that $q_x(U) \subseteq V$. In other words, $U \times x \subseteq V$. So we have $p_1 r(\alpha(t), x) = 0$, for all $t \in U$. Thus we have

$$\begin{aligned} u(x) &= \frac{\int_0^1 \beta(\alpha(t) - p_1 r(1, x)p_1 r(\alpha(t), x)) dt}{\int_0^1 \beta(\alpha(t)) dt} \\ &= \frac{\int_{\mathbf{I} \setminus U} \beta(\alpha(t) - p_1 r(1, x)p_1 r(\alpha(t), x)) dt + \int_U \beta(\alpha(t)) dt}{\int_0^1 \beta(\alpha(t)) dt} \\ &> 0, \end{aligned}$$

since $\int_U \beta(\alpha(t)) dt \neq 0$. Thus, combining this with part 1, we have $u^{-1}(0) = A$.

3. Suppose that x is such that $u(x) < 1$. Then, since we have

$$u(x) = \frac{\int_0^1 \beta(\alpha(t) - p_1 r(1, x) p_1 r(\alpha(t), x)) dt}{\int_0^1 \beta(\alpha(t)) dt}$$

there must be a neighbourhood U of \mathbf{I} such that $p_1 r(1, x) p_1 r(\alpha(t), x) > 0$, for $t \in U$. Thus $p_1 r(1, x) > 0$. But this implies that $r(1, x) \in \mathbf{I} \times A$ and $H(1, x) \in A$.

This completes the proof. \square

5.3 The Mapping cylinder

In this section we show that the inclusion of X into the flattened mapping cylinder \mathbf{I}_f of a map $f: X \rightarrow Y$ is a cofibration with the FCIP.

5.3.1. Theorem. *Let $f: X \rightarrow Y$ be a smooth map. Then the pair (\mathbf{I}_f, X) is an SNDR pair.*

Proof. Let $\alpha: \mathbf{I} \rightarrow \mathbb{R}$ be a smooth braking function with the following properties.

- $\alpha(t) = 0$ if $0 \leq t \leq \frac{1}{4}$,
- $\alpha(t) = 1$ if $\frac{3}{4} \leq t \leq 1$,
- $0 < \alpha(t) < 1$ otherwise.

Define two more smooth braking functions as follows. Let $\alpha_1: \mathbf{I} \rightarrow \mathbb{R}$ satisfy:

- $\alpha_1(t) = 0$ if $t = 0$,
- $0 < \alpha_1(t) < 1$ if $0 < t < \frac{3}{4}$,
- $\alpha_1(t) = 1$ if $\frac{3}{4} \leq t \leq 1$.

Let $\alpha_2: \mathbf{I} \rightarrow \mathbb{R}$ satisfy:

- $\alpha_2(t) = 0$ if $0 \leq t \leq \frac{3}{4}$,
- $\alpha_2(t) = 1$ if $\frac{7}{8} \leq t \leq 1$.

Now define $u: \mathbf{I}_f \rightarrow \mathbf{I}$ by $u([t, x]) = \alpha_1(t)$ and $u([y]) = 1$, for $(t, x) \in \mathbf{I} \times X$ and $y \in Y$. Define $H: \mathbf{I} \times \mathbf{I}_f \rightarrow \mathbf{I}_f$ by

- $H(s, [t, x]) = [(1 - \alpha(s))t + \alpha(s)\alpha_2(t), x]$ if $(t, x) \in \mathbf{I} \times X$,
- $H(s, [y]) = [y]$ if $y \in Y$.

To see that u is a smooth map, notice that u is smooth when restricted to each component of the coproduct $\mathbf{I} \times X \sqcup Y$, and is thus smooth on the quotient \mathbf{I}_f .

To see that the map $H: \mathbf{I} \times \mathbf{I}_f \rightarrow \mathbf{I}_f$ is smooth, note that since we are working in a cartesian closed category, products commute with quotients, i.e. if q is a quotient, then so is $1 \times q$, where 1 is an identity map. Thus we may think of H as being defined on the space

$$((\mathbf{I} \times \mathbf{I} \times X) \sqcup (\mathbf{I} \times Y)) / \sim$$

where \sim is the identification $(t, 1, x) = (t, f(x))$ for $t \in \mathbf{I}$, and $x \in X$. Since H is smooth when restricted to each component of the coproduct

$$(\mathbf{I} \times \mathbf{I} \times X) \sqcup (\mathbf{I} \times Y),$$

H is smooth on the quotient $\mathbf{I} \times \mathbf{I}_f$.

Let us verify that (u, H) is a representation for (\mathbf{I}_f, X) as an SNDR pair.

1. $u^{-1}(0) = [0, x] = i_0'(X)$.
2. $H(0, [t, x]) = [t, x]$ and $H(0, [y]) = [y]$.
3. $H(s, [0, x]) = [0, x]$.
4. If $u[t, x] < 1$ then $t < \frac{3}{4}$ and so $\alpha_2(t) = 0$. Thus $H(1, [t, x]) = [\alpha_2(t), x] = [0, x]$.

This completes the proof. □

Finally, we have the following important corollary.

5.3.2. Corollary. *Given any smooth map $f: X \rightarrow Y$, the inclusion $X \hookrightarrow \mathbf{I}_f$ is a cofibration with the FCIP.*

5.4 The Exact sequence of a cofibration

Our aim in this section is to show how one can use SNDR pairs to prove the existence of the right exact Puppe sequence. We state the result in Theorem 5.4.1, and break the proof of the result up into a number of lemmas. We follow the method used by Whitehead [130] for the topological case.

Throughout this section we work in the category \mathbf{FRL}_* of Frölicher spaces with basepoint, and basepoint preserving smooth maps.

5.4.1. Theorem. *Let W be an object in \mathbf{FRL}_* , and suppose that $i: A \hookrightarrow X$ is a cofibration in \mathbf{FRL} . For any basepoint $x_0 \in A \subseteq X$ there is a sequence*

$$\begin{aligned} \dots \longrightarrow [\Sigma^{n+1}A, W] &\xrightarrow{(\Sigma^n k)^*} [\Sigma^n \mathbf{T}_i, W] \xrightarrow{(\Sigma^n j)^*} [\Sigma^n X, W] \xrightarrow{(\Sigma^n i)^*} [\Sigma^n A, W] \longrightarrow \\ \dots \longrightarrow [\Sigma A, W] &\xrightarrow{k^*} [\mathbf{T}_i, W] \xrightarrow{j^*} [X, W] \xrightarrow{i^*} [A, W]. \end{aligned}$$

which is an exact sequence in \mathbf{Sets}_* , where $j: X \rightarrow \mathbf{T}_i$ is the inclusion discussed in Section 4.5 and $k: \mathbf{T}_i \rightarrow \Sigma A$ is the quotient map defined below.

It is, in fact, possible to prove that the sequence above is an exact sequence of groups as far as $[\Sigma A, W]$ and that the morphisms to this point are group homomorphisms, but we shall not do so here.

The **reduced (flattened) suspension** of a pointed Frölicher space X is defined as

$$\Sigma X = (I/\{0, 1\}) \wedge X,$$

where the reduced join is defined as for topological spaces with the identified set taken as basepoint, and with 0 the basepoint of I .

In this chapter, whenever we refer to the suspension of a space, we mean the reduced flattened suspension defined above. In the next chapter we shall define the reduced suspension of a space in terms of the weakly flattened unit interval, and show that they are of the same smooth homotopy type.

We break the proof of Theorem 5.4.1 into a number of lemmas, giving an indication how the proof of each should proceed.

5.4.2. Lemma. *If (X, A) is an SNDR pair and $p: X \rightarrow X/A$ is the quotient map, then the sequence*

$$A \xrightarrow{i} X \xrightarrow{p} X/A$$

is right exact.

Proof. To show that the given sequence is right exact we must show that for any Frölicher space W the following sequence is exact in \mathbf{Sets}_* :

$$[X/A, W] \xrightarrow{p^*} [X, W] \xrightarrow{i^*} [A, W].$$

It is easy to see that $\text{im } p^* \subseteq \text{ker } i^*$. To see the reverse inclusion, let $g: X \rightarrow W$ be an element of $[X, W]$, with $g|_A \simeq w_0$ (rel w_0). Since $A \xrightarrow{i} X$ is an SNDR pair, the map $i: A \rightarrow X$ is a cofibration, and so we may extend w_0 to a smooth map $g': X \rightarrow W$ such that $g' \simeq g$. But g' is constant on A and so there exists a smooth map $g_1: X/A \rightarrow W$ such that $g_1 \circ p = g'$. See Section 2.6. This shows that $\text{ker } i^* \subseteq \text{im } p^*$. \square

5.4.3. Lemma. *For any smooth map $f: X \rightarrow Y$, the sequence*

$$X \xrightarrow{f} Y \xrightarrow{l} \mathbf{T}_f$$

is right exact, where l is the usual inclusion of Y into the mapping cone; i.e. $y \mapsto [y] \in \mathbf{T}_f$.

Proof. One can show that there is a homotopy commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow i & \downarrow j \\ & & I_f \\ & & \xrightarrow{p} \mathbf{T}_f \end{array}$$

where i , j , and l are the usual inclusions, and p is the quotient map. Since (\mathbf{I}_f, X) is an SNDR pair, by Lemma 5.4.2 the sequence

$$X \xrightarrow{i} \mathbf{I}_f \xrightarrow{p} \mathbf{T}_f$$

is right exact. It is fairly easy to show that $j : Y \rightarrow \mathbf{I}_f$ is a homotopy equivalence. This is done explicitly for the weakly flattened mapping cylinder in 6.3.2 to 6.3.5. This fact allows us to deduce that the sequence

$$X \xrightarrow{j} Y \xrightarrow{l} \mathbf{T}_f$$

is right exact. □

5.4.4. Lemma. *For any smooth map $i : A \rightarrow X$, there is an infinite right exact sequence*

$$A \xrightarrow{i} X \xrightarrow{i^1} \mathbf{T}_i \xrightarrow{i^2} \dots \xrightarrow{i^{n-1}} \mathbf{T}_{i^{n-2}} \xrightarrow{i^n} \mathbf{T}_{i^{n-1}} \xrightarrow{i^{n+1}} \dots$$

where i^n , $n \geq 1$ are inclusion maps.

Proof. The pair (\mathbf{T}_i, X) is an SNDR pair. The representation for the pair (\mathbf{I}_f, X) in Theorem 5.3.1 can be adapted to show this. One now iterates the procedure of Lemma 5.4.2 and Lemma 5.4.3. □

One can easily see that there is an isomorphism between \mathbf{T}_i/X and ΣA . Define $q : \mathbf{T}_i \rightarrow \Sigma A$ to be the map which identifies $X \subset \mathbf{T}_i$ to a point, followed by the isomorphism $\mathbf{T}_i/X \rightarrow \Sigma A$.

5.4.5. Lemma. *The sequence*

$$X \xrightarrow{i^1} \mathbf{T}_i \xrightarrow{q} \Sigma A$$

is right exact.

Proof. As noted above the pair (\mathbf{T}_i, X) is an SNDR pair. We have the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{i^1} & \mathbf{T}_i & \xrightarrow{p} & \mathbf{T}_i/X \\ & & & \searrow q & \downarrow q_0 \\ & & & & \Sigma A \end{array}$$

where $p : \mathbf{T}_i \rightarrow \mathbf{T}_i/X$ is the identification map, and $q_0 : \mathbf{T}_i/X \rightarrow \Sigma A$ is an isomorphism. The top line of the diagram is right exact, by Lemma 5.4.2, and so the sequence

$$X \xrightarrow{i^1} \mathbf{T}_i \xrightarrow{q} \Sigma A$$

is right exact. □

There is a commutative diagram

$$\begin{array}{ccccc}
 X & \xrightarrow{i^1} & \mathbf{T}_i & \xrightarrow{i^2} & \mathbf{T}_{i^1} \\
 & & & \searrow q & \downarrow q_1 \\
 & & & & \Sigma A
 \end{array}$$

where q_1 is a homotopy equivalence. (See Whitehead [130] for more details of this map.) Using commutative diagrams of this form, one can now proceed almost exactly as one does in the topological situation, as in Whitehead [130] for example, to get the following infinite right exact sequence:

$$\begin{array}{ccccccccccc}
 A & \xrightarrow{i} & X & \xrightarrow{i^1} & \mathbf{T}_i & \xrightarrow{q} & \Sigma A & \xrightarrow{\Sigma i} & \Sigma X & \xrightarrow{\Sigma i^1} & \Sigma \mathbf{T}_i & \xrightarrow{\Sigma q} & \dots \\
 & & & & & & & & & & & & \\
 & & & & & & \dots & \longrightarrow & \Sigma^n A & \xrightarrow{\Sigma^n i} & \Sigma^n X & \xrightarrow{\Sigma^n i^1} & \dots
 \end{array}$$

The definition of right exactness now gives us the exact sequence of Theorem 5.4.1.

Chapter 6

The cofibration axioms

Our aim in this chapter is to show that the category **FRL** satisfies Baues' [7] four cofibration axioms.

In addition we follow Baues [7] construction of the right Puppe sequence for a 'cofibration category', i.e. one that satisfies his four 'cofibration axioms', obtaining this sequence for Frölicher spaces. The construction of the right Puppe sequence for Frölicher spaces was partially obtained in [36], but certain assumptions needed to be made regarding the Frölicher product structure. We show that the sequence obtained by using the cofibration axioms is the same as the sequence discussed in [36], that is the 'usual' Puppe sequence that one would expect in **FRL**.

Baues' cofibration axioms may be formulated in a dual way, giving axioms for a fibration category. We show that Baues' fibration axioms are satisfied by the category **FRL** in Appendix A.

Because **FRL** is topological over sets (see Frölicher and Kriegl [47]), one can work in **FRL** largely as one would work with topological spaces.

We now investigate the four axioms for a cofibration category with respect to the category **FRL**. So that the exposition is self-contained, we restate each axiom (exactly as it appears in Baues' book) before considering it. See Baues [7] for a more complete discussion of the axioms.

The symbol **C** denotes a cofibration category.

6.1 Axiom (C1): Composition axiom

The isomorphisms in **C** are weak equivalences and cofibrations. (A map which is both a cofibration and weak equivalence is sometimes called a **trivial cofibration**). For two maps

$$A \xrightarrow{f} B \xrightarrow{g} C,$$

if any two of f , g , gf are weak equivalences, then so is the third. The composite of cofibrations is a cofibration.

Let us now verify this axiom. It is obvious that isomorphisms in the category **FRL** are both cofibrations and weak equivalences, as required.

Suppose we have two smooth maps

$$A \xrightarrow{f} B \xrightarrow{g} C.$$

If f and g are smooth homotopy equivalences, then so is gf .

Suppose that f and gf above are smooth homotopy equivalences. Then there exists an $\bar{f}: B \rightarrow A$, and $k: C \rightarrow A$ such that $f\bar{f} \simeq 1_B$, $\bar{f}f \simeq 1_A$, $gfk \simeq 1_C$ and $kgf \simeq 1_A$.

Define $\bar{g} = fk$. Then we have $g\bar{g} = gfk \simeq 1_C$ and $\bar{g}g = fkg \simeq fkgf\bar{f} \simeq f\bar{f} \simeq 1_B$.

Now suppose that g and gf are smooth homotopy equivalences. Then there exist $\bar{g}: C \rightarrow B$ and $k: C \rightarrow A$ such that $g\bar{g} \simeq 1_C$, $\bar{g}g \simeq 1_B$, $gfk \simeq 1_C$ and $kgf \simeq 1_A$.

Define $\bar{f} = kg$. Then $f\bar{f} = fkg \simeq \bar{g}gfk \simeq \bar{g}g \simeq 1_C$, and $\bar{f}f = kgf \simeq 1_A$. The above are explicit computations in the category **FRL**.

Of course, alternatively, one could just use the fact that homotopy equivalences are isomorphisms in the homotopy category to arrive at the same results.

Finally, we need to verify that the composite of two cofibrations is again a cofibration. To this end, suppose that we have two cofibrations

$$X \xrightarrow{f} Y \xrightarrow{h} W$$

and suppose that we are given the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{hf} & W \xrightarrow{k} Z \\ \downarrow (0,1_k) & \nearrow K & \\ I \times X & & \end{array}$$

We need to construct a smooth $J: I \times W \rightarrow Z$ making the following diagram commute,

$$\begin{array}{ccc} W & \xrightarrow{k} & Z \\ (0,1_w) \downarrow & \nearrow J & \uparrow K' \\ I \times W & \xleftarrow{1_I \times hf} & I \times X \end{array} \quad (*)$$

where $K'(t, x) = K(\alpha_\epsilon(t), x)$ for some $0 < \epsilon < \frac{1}{2}$ and each $t \in I$, $x \in X$. But f is a smooth cofibration, so given the following diagram, which clearly commutes

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \xrightarrow{kh} Z \\ (0,1_x) \downarrow & \nearrow K & \\ I \times X & & \end{array}$$

we may construct the following commutative diagram,

$$\begin{array}{ccc}
 Y & \xrightarrow{kh} & Z \\
 (0,1_Y) \downarrow & \nearrow F & \uparrow K'' \\
 I \times Y & \xleftarrow{1_I \times f} & I \times X
 \end{array}$$

where $K''(t, x) = K(\alpha_{\epsilon_1}^*(t), x)$ for some $0 < \epsilon_1 < \frac{1}{2}$ and each $t \in I$, and $x \in X$. Also, h is a smooth cofibration, so given the following commutative diagram

$$\begin{array}{ccccc}
 Y & \xrightarrow{h} & W & \xrightarrow{k} & Z \\
 (0,1_Y) \downarrow & & & \nearrow F & \\
 I \times Y & & & &
 \end{array}$$

we have the following commutative diagram.

$$\begin{array}{ccccc}
 W & \xrightarrow{k} & Z & & \\
 (0,1_W) \downarrow & \nearrow H & \uparrow F' & & (**) \\
 I \times W & \xleftarrow{1_I \times h} & I \times Y & &
 \end{array}$$

where $F'(t, y) = F(\alpha_{\epsilon_2}(t), y)$ for some $0 < \epsilon_2 < \frac{1}{2}$, and each $t \in I$ and $y \in Y$. Now we define $J(t, w) := H(t, w)$. Then the left triangle of (*) commutes, since the left triangle of (**) commutes. The right triangle of (*) commutes:

$$\begin{aligned}
 J(t, hfx) &= H(t, hfx) \\
 &= F'(t, fx) \\
 &= F(\alpha_{\epsilon_2}(t), fx) \\
 &= K''(\alpha_{\epsilon_2}(t), x) \\
 &= K(\alpha_{\epsilon_1}(\alpha_{\epsilon_2}(t)), x), x \in X, \text{ and } t \in I.
 \end{aligned} \tag{6.1}$$

Thus if we set $\epsilon = \alpha_{\epsilon_2}^{-1}(\epsilon_1)$, then diagram (*) commutes for the smooth braking function $\alpha_\epsilon: \mathbb{R} \rightarrow \mathbb{R}$, with $\alpha_\epsilon(t) = \alpha_{\epsilon_1}(\alpha_{\epsilon_2}(t))$. Notice that this smooth braking function may not be a standard smooth braking function. Thus we have verified axiom (C1).

6.2 Axiom (C2): Pushout axiom

For a cofibration

$$B \xrightarrow{i} A$$

and a map $f: B \rightarrow Y$ there exists the pushout in \mathbf{C}

$$\begin{array}{ccc} A & \xrightarrow{\bar{f}} & A \sqcup_B Y \\ \uparrow i & & \uparrow \bar{i} \\ B & \xrightarrow{f} & Y \end{array}$$

and \bar{i} is a cofibration. Moreover:

- (a) if f is a weak equivalence, so is \bar{f} ,
- (b) if i is a weak equivalence, so is \bar{i} .

If \mathbf{C} has an initial object, denoted \emptyset , then we call an object X in \mathbf{C} \emptyset -cofibrant if $\emptyset \rightarrow X$ is a cofibration. The empty set, \emptyset , is an initial object in **FRL**, and it is obvious that all objects are \emptyset -cofibrant.

It is shown in Baues [7], Lemma 1.4, that in this situation axiom (C2)(b) is redundant. Alternatively, if all objects in \mathbf{C} are cofibrant, then Baues shows that (C2)(b), (C1) and (C3) (discussed below) imply (C2)(a). Thus we only verify (C2)(b), because all objects are \emptyset -cofibrant in **FRL**.

We now verify axiom (C2). Suppose that we are given a cofibration

$$B \xrightarrow{i} A$$

and a smooth map

$$B \xrightarrow{f} Y,$$

we define the adjunction space as usual,

$$A \sqcup_B Y := (A \sqcup Y) / \sim$$

where \sim is the identification defined by setting $i(b) \sim f(b)$, for $b \in B$, and quotients are taken in **FRL**.

Define the smooth maps

$$\bar{f}: A \rightarrow A \sqcup_B Y \text{ as } \bar{f}(a) = [a], a \in A,$$

and

$$\bar{i}: Y \rightarrow A \sqcup_B Y \text{ as } \bar{i}(y) = [y], y \in Y,$$

where $[a]$ denotes the equivalence class under the above identification.

We can then verify that we have the following pushout diagram in **FRL**.

$$\begin{array}{ccc} A & \xrightarrow{\bar{f}} & A \sqcup_B Y \\ \uparrow i & & \uparrow \bar{i} \\ B & \xrightarrow{f} & Y \end{array}$$

To verify that \bar{i} is a smooth cofibration, suppose that we have a commutative diagram

$$\begin{array}{ccccc} Y & \xrightarrow{\bar{i}} & A \sqcup_B Y & \xrightarrow{g} & Z \\ (0,1_Y) \downarrow & & & \nearrow G & \\ I \times Y & & & & \end{array}$$

We wish to construct a commutative diagram as follows,

$$\begin{array}{ccccc} A \sqcup_B Y & \xrightarrow{g} & Z & & \\ (0,1_{A \sqcup_B Y}) \downarrow & \nearrow F & \nearrow G' & & \\ I \times A \sqcup_B Y & \xleftarrow{1_I \times \bar{i}} & I \times Y & & \end{array} \quad (***)$$

where $G'(t, y) = G(\alpha_\epsilon(t), y)$ for some $0 < \epsilon < \frac{1}{2}$, and each $t \in I, y \in Y$. But i is a smooth cofibration, so since we have the following commutative diagram

$$\begin{array}{ccccc} B & \xrightarrow{i} & A & \xrightarrow{g\bar{f}} & Z \\ (0,1_B) \downarrow & & & \nearrow G(t, f(-)) & \\ I \times B & & & & \end{array}$$

we may construct the following commutative diagram,

$$\begin{array}{ccccc} A & \xrightarrow{g\bar{f}} & Z & & \\ (0,1_A) \downarrow & \nearrow H & \nearrow G(\alpha_{\epsilon_1}(t), f(-)) & & \\ I \times A & \xleftarrow{1_I \times i} & I \times B & & \end{array}$$

for some $0 < \epsilon_1 < \frac{1}{2}$. We define $F: I \times (A \sqcup_B Y) \rightarrow Z$ as follows

$$F(t, \bar{f}(a)) = H(t, a), \quad a \in A$$

and

$$F(t, \bar{i}(y)) = G(\alpha_{\epsilon_1}(t), y), \quad y \in Y.$$

(Note that an element of $A \sqcup_B Y$ is either of the form $\bar{f}(a)$, for $a \in A$, or $\bar{i}(y)$ for $y \in Y$, and we may distribute the Cartesian product over the adjunction, by the Cartesian closedness of **FRL**).

We firstly verify that F is well defined.

Suppose that $i(b) = f(b)$ for some $b \in B$. Then

$$F(t, \bar{i}f(b)) = G(\alpha_{\epsilon_1}(t), f(b)) = H(t, i(b)),$$

and

$$F(t, \bar{f}i(b)) = H(t, i(b)),$$

by definition.

Next we verify that the diagram (***) commutes:

$$F(0, \bar{f}(a)) = H(0, a) = g\bar{f}(a), \text{ for } a \in A,$$

and

$$F(0, \bar{i}(y)) = G(0, y) = g\bar{i}(y).$$

Thus the left triangle of (***) commutes. The right triangle of (***) commutes by definition of F , with $\epsilon = \epsilon_1$. Finally, note that F is smooth since Cartesian closedness allows us to distribute the Cartesian product over the adjunction, and F is smooth on each component of the adjunction.

Finally, we need to show that if i is a weak equivalence, then so is \bar{i} . We prove this via the following sequence of lemmas.

6.2.1. Lemma. *If*

$$B \xrightarrow{i} A$$

is a trivial cofibration, then B is a smooth deformation retract of A . It then follows that $B \cong i(B)$ in FRL under the map i .

Proof. Let $j: A \rightarrow B$ be a homotopy inverse to i . Let $F: I \times B \rightarrow B$ be the homotopy giving $ji \simeq_F 1_B$. We have the commutative diagram

$$\begin{array}{ccc} B & \xrightarrow{i} & A & \xrightarrow{j} & B \\ & & \searrow & \nearrow & \\ & & I \times B & & \end{array}$$

(0,1_B) ↓
F

Since i is a cofibration, there exists the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{j} & B \\ & \searrow & \nearrow \\ & & I \times B \\ & \swarrow & \downarrow \\ I \times A & \xleftarrow{1_I \times i} & I \times B \end{array}$$

(0,1_A) ↓
G ↑
F'

where $F'(t, b) = F(\alpha_\epsilon(t), b)$ for some $0 < \epsilon < \frac{1}{2}$. Define $k := G(1, -): A \rightarrow B$. Notice that $ki = 1_B$ and that $ik \simeq ij \simeq 1_A$. Hence B is a retract of A . It now follows that $B \cong i(B)$ under i .

Let $K: I \times A \rightarrow A$ be the smooth homotopy giving $1_A \simeq ik$. Then K is a deformation retraction of A onto $i(B)$. To see this we need only observe that $K(0, -) = 1_A$ and $K(1, -) = ik \in i(B)$. Also $K(1, i(b)) = iki(b) = i(b)$ gives B as a retract of A . \square

6.2.2. Lemma. *If*

$$B \xrightarrow{i} A$$

is a trivial cofibration with the FCIP, then B is a strong deformation retract of A .

Proof. The previous lemma shows that we can identify B with $i(B)$.

Let $j: A \rightarrow B$ be a smooth homotopy inverse of i . We have homotopies $F: I \times B \rightarrow B$ and $G: I \times A \rightarrow A$ giving $1_B \simeq_F j \circ i$ and $1_A \simeq_G i \circ j$, respectively.

Since i is a trivial cofibration, j is homotopic to a retraction, by Lemma 6.2.1, and so we might as well assume from the start that j is a retraction.

Let $P = I \times A$ and $Q = (\{0, 1\} \times A) \cup (I \times B)$. Since $\{0, 1\} \hookrightarrow I$ and $B \xrightarrow{i} A$ are SNDR pairs, (see Example 5.1.8 and Theorem 5.2.3, respectively), so is their product pair $Q \hookrightarrow P$, by Theorem 5.1.9.

Define $H_*: 0 \times P \cup I \times Q \rightarrow A$ by

- $H_*(s, 0, a) = a$,
- $H_*(s, 1, a) = G(\alpha_\epsilon(1-s), ij(a))$,
- $H_*(s, t, b) = G\left(\alpha_\epsilon((1-s)\alpha_\epsilon(t), i(b))\right)$, and
- $H_*(0, t, a) = G(\alpha_\epsilon(t), a)$,

for some $0 < \epsilon < \frac{1}{2}$.

A routine verification shows that H_* is well defined, and the smooth braking function α_ϵ ensures smoothness across the overlaps in the definition of H_* .

Since $Q \hookrightarrow P$ is an SNDR pair, it is a cofibration with the FCIP. Thus H_* admits an extension $H: I \times P \rightarrow A$. The end value of H at $s = 1$ is the required strong retracting deformation. \square

6.2.3. Corollary. *For a trivial cofibration*

$$B \xrightarrow{i} A$$

the subspace $B \subseteq I_i$ of the mapping cylinder I_i is a strong deformation retract of I_i .

Proof. We know that $B \hookrightarrow I_i$ is an SNDR pair, see Corollary 5.3.2, and hence a cofibration with the FCIP. We also have the diagram

$$B \begin{array}{c} \xrightarrow{h} \\ \xrightarrow{p} \\ \xrightarrow{r} \end{array} I_i \xrightarrow{p} A$$

where h is the inclusion into the mapping cylinder, r is the retraction given by Lemma 6.2.1, and p is the weak equivalence mentioned in Section 4.5. Since p and r are weak equivalences, so is $B \hookrightarrow I_i$ by axiom (C1). Apply Lemma 6.2.2. \square

6.2.4. Note. Before we continue it is useful to examine the construction of the strong retracting deformation $H(1, -, -): I \times I_i \rightarrow I_i$ given by Lemma 6.2.2.

Define $j: I_i \rightarrow B$ by $j = rp$, where $p: I_i \rightarrow A$ is the map defined by

- $p([a]) = a$
- $p([t, b]) = i(b)$,

which is obviously well defined, and $r: A \rightarrow B$ is the retraction given by Lemma 6.2.1. Then with the definitions of Lemma 6.2.2 we have

$$\begin{aligned}
 H(1, 1, [1, b]) &= G(0, hj[1, b]) \\
 &= G(0, hji(b)) \\
 &= G(0, h(b)) \\
 &= G(0, [0, b]) \\
 &= [0, b] \\
 &= h(b).
 \end{aligned}$$

This observation will be of use later.

6.2.5. Lemma. *Given the pushout diagram*

$$\begin{array}{ccc}
 A & \xrightarrow{\bar{j}} & A \sqcup_B Y \\
 \uparrow i & & \uparrow \bar{i} \\
 B & \xrightarrow{f} & Y
 \end{array}$$

if B is a strong deformation retract of A , then \bar{i} is a weak equivalence.

Proof. Let $F: I \times A \rightarrow A$ be the strong retracting deformation, with $F(0, a) = a$ and $F(1, a) \in i(B)$. Define a map $k: A \rightarrow B$ by $k(a) = i^{-1}F(1, a)$. This is smooth since i is an isomorphism of B onto $i(B)$ by Lemma 6.2.1. Define $\bar{j}: A \sqcup_B Y \rightarrow Y$ by

1. $\bar{j}([a]) = fk(a)$ and
2. $\bar{j}([y]) = y$.

We show firstly that \bar{j} is well-defined. Notice that for $[y_1], [y_2] \in A \sqcup_B Y$, we have $[y_1] = [y_2]$ if and only if there exist $b_1, b_2 \in B$ with $y_1 = f(b_1)$ and $y_2 = f(b_2)$ and $i(b_1) = i(b_2)$. Since i is injective, we deduce that $[y_1] = [y_2]$ if and only if $y_1 = y_2$. Thus part 2 of the definition of \bar{j} makes sense.

For part 1, suppose that $[a_1] = [a_2]$. Then there exist $b_1, b_2 \in B$ with $a_1 = i(b_1)$ and $a_2 = i(b_2)$, and $f(b_1) = f(b_2)$. Then we have

$$\begin{aligned}
 \bar{j}([a_1]) &= fki(b_1) \\
 &= fi^{-1}F(1, i(b_1))
 \end{aligned}$$

$$\begin{aligned}
 &= f(b_1) \\
 &= f(b_2) \\
 &= fi^{-1}F(1, i(b_2)) \\
 &= fki(b_2) \\
 &= \bar{j}([a_2]).
 \end{aligned}$$

Thus the first part of the definition of \bar{j} makes sense. To see that \bar{j} is well-defined on the adjunction, notice that

$$\bar{j}([i(b)]) = fki(b) = f(b) = \bar{j}([f(b)]).$$

Finally, we need to verify that \bar{j} is indeed a homotopy inverse to \bar{i} . We have

$$\bar{j}\bar{i}(y) = \bar{j}([y]) = y.$$

On the other hand,

$$\bar{i}\bar{j}([y]) = \bar{i}(y) = [y].$$

Also, notice that $1_A \simeq_{F^{-1}} ik = ii^{-1}F(1, -)$, giving us

$$\bar{i}\bar{j}|_A = \bar{i}fk = \bar{f}ik \simeq \bar{f}.$$

That is, $\bar{i}\bar{j}([a]) \simeq [a]$ for $[a] \in A \sqcup_B Y$. Notice that this smooth homotopy is well-defined on the adjunction because F is a strong retracting deformation.

This completes the proof. \square

6.2.6. Theorem. *Given the pushout diagram*

$$\begin{array}{ccc}
 A & \xrightarrow{\bar{j}} & A \sqcup_B Y \\
 \uparrow i & & \uparrow \bar{i} \\
 B & \xrightarrow{f} & Y
 \end{array}$$

where i is a cofibration, if i is a weak equivalence, then so is \bar{i} .

Proof. This proof deals with three different quotient spaces, namely the adjunction space $A \sqcup_B Y$, the mapping cylinder of i , I_i and the double mapping cylinder of i and f , $I_{i,f}$. See Definition 4.5.4. To make the proof easier to follow we denote the elements of these spaces as follows. A point in the adjunction $A \sqcup_B Y$ is denoted by $[a]$ or $[y]$, where $[i(b)] = [f(b)]$. A point in the mapping cylinder I_i is denoted by $\langle a \rangle$ or $\langle t, b \rangle$, where $\langle 1, b \rangle = \langle i(b) \rangle$. A point in the double mapping cylinder $I_{i,f}$ is denoted by $[a]$ or $[t, b]$ or $[b]$, where we have $[i(b)] = [1, b]$ and $[f(b)] = [0, b]$.

For convenience we shall sometimes abuse this notation in the following way. For example, suppose that $x \in I_i$. We shall sometimes say that $[x] \in I_{i,f}$ when

we really mean that $x = \langle t, b \rangle$ or $x = \langle a \rangle$ and that $[t, b]$ or $[a]$, respectively, is an element in $\mathbf{I}_{i,j}$. We abuse notation with regard to the adjunction space similarly.

Of course when using this notational convenience we are especially careful to ensure that the equivalence relations are compatible in such a way that the notation still makes sense.

Consider the inclusion $h: B \hookrightarrow \mathbf{I}_i$. This is a trivial cofibration with the FCIP by Corollary 6.2.3, and thus $h(B)$ is a strong deformation retract of \mathbf{I}_i . We then have a homotopy $F: I \times \mathbf{I}_i \rightarrow \mathbf{I}_i$ ($\text{rel } h(B)$), with the properties that

- $F(0, \langle x \rangle) = \langle x \rangle$, and
- $F(1, \langle x \rangle) \in h(B)$, where $\langle x \rangle \in \mathbf{I}_i$.

Note 6.2.4 shows that we may choose F so that

$$\begin{aligned} F(1, \langle i(b) \rangle) &= F(1, \langle 1, b \rangle) \\ &= \langle 0, b \rangle \quad (*) \\ &= h(b). \end{aligned}$$

We have the following diagram

$$\begin{array}{ccccc} \mathbf{I}_i & \xrightarrow{\bar{J}} & \mathbf{I}_{i,j} & \xrightarrow{k} & A \sqcup_B Y \\ \uparrow h \sim & & \uparrow \bar{h} \sim & \nearrow \bar{i} & \\ B & \xrightarrow{J} & Y & & \end{array}$$

push

where \bar{h} is the weak equivalence given by Lemma 6.2.5 and k is defined below. We shall show that k is a weak equivalence, and that the right triangle is commutative with our definition of k . This will imply by axiom (C1) that \bar{i} is a weak equivalence.

Define $P: I \times \mathbf{I}_{i,j} \rightarrow \mathbf{I}_{i,j}$ by

- $P(s, [a]) = [F(s, \langle a \rangle)]$,
- $P(s, [t, b]) = [F(s, \langle t, b \rangle)]$ and
- $P(s, [y]) = [y]$.

We need to check that P is well-defined. We have

$$\begin{aligned} P(s, [i(b)]) &= [F(s, \langle i(b) \rangle)] \\ &= [F(s, \langle 1, b \rangle)] \text{ since } F \text{ is well-defined on } \mathbf{I}_i. \end{aligned}$$

Also,

$$\begin{aligned} P(s, [0, b]) &= [F(s, \langle 0, b \rangle)] \\ &= [0, b] \text{ since } F \text{ is a homotopy rel } h(B), \\ &= [f(b)]. \end{aligned}$$

Now define $\bar{k}: A \sqcup_B Y \rightarrow \mathbf{I}_{i,f}$ by

- $\bar{k}([a]) = [F(1, \langle a \rangle)]$ and
- $\bar{k}([y]) = [y]$.

Notice that \bar{k} is well-defined, since

$$\begin{aligned} \bar{k}([i(b)]) &= [F(1, \langle i(b) \rangle)] \\ &= [h(b)] \text{ as seen in } (*), \\ &= [f(b)] \\ &= \bar{k}([f(b)]). \end{aligned}$$

Define $k: \mathbf{I}_{i,f} \rightarrow A \sqcup_B Y$ by $k(x) = [P(1, x)]$, for $x \in \mathbf{I}_{i,f}$. We now observe that \bar{k} is a homotopy inverse to $k := [P(1, -)]$.

It is clear that $P(0, -) = 1_{\mathbf{I}_{i,f}}$. Notice that

$$P(1, [a]) = [F(1, \langle a \rangle)]$$

where $F(1, \langle a \rangle) \in \bar{f}(\bar{h}(B)) \subset \mathbf{I}_{i,f}$, and hence $\bar{k}k([a]) = k([a]) = P(1, [a])$.

As above, we have $P(1, [t, b]) = [F(1, \langle t, b \rangle)]$, and so $\bar{k}k([t, b]) = P(1, [t, b])$. and $P(1, [y]) = [y] = \bar{k}k([y])$. Thus $1_{\mathbf{I}_{i,f}} \simeq_P \bar{k}k$.

For the other direction, first define $q: \mathbf{I}_{i,f} \rightarrow A \sqcup_B Y$ by

- $q([a]) = [a]$,
- $q([t, b]) = [f(b)] = [i(b)]$, and
- $q([y]) = [y]$.

Observe that q is well defined, for $q([1, b]) = [i(b)] = q([i(b)])$, and $q([0, b]) = [f(b)] = q([f(b)])$.

Notice that we have

$$[a] = qP(0, [a]),$$

and

$$[y] = qP(0, [y]).$$

Thus $1_{A \sqcup_B Y} = qP(0, [-])$.

At the other endpoint we have

$$qP(1, [a]) = [P(1, [a])] = k\bar{k}([a]),$$

and

$$qP(1, [y]) = [P(1, [y])] = k([y]) = k\bar{k}([y]).$$

Thus $1_{A \sqcup_B Y} \simeq_{qP} k\bar{k}$. Clearly, $k \circ \bar{h} = \bar{i}$.

We have shown that k is a weak equivalence. Since \bar{h} is a weak equivalence we deduce by axiom (C1) that \bar{i} is a weak equivalence. \square

6.3 Axiom (C3): C-Factorization axiom

For any morphism $f: B \rightarrow Y$ in \mathbf{C} , there exists a commutative diagram

$$\begin{array}{ccc} B & \xrightarrow{f} & Y \\ & \searrow i & \nearrow g \\ & & A \end{array}$$

\sim

where i is a cofibration and g is a weak equivalence. We follow Doeraene [32] and call this the C-factorization axiom, to distinguish it from the F-factorization axiom discussed in Appendix A. We verify this axiom.

The following definition is paraphrased from Section 4.5.

6.3.1. Definition. Let $f: B \rightarrow Y$ be a smooth map. The smooth (weakly flattened) mapping cylinder \mathbb{I}_f of f is defined by:

$$\mathbb{I}_f = ((\mathbb{I} \times B) \sqcup Y) / \sim,$$

where \sim is the identification defined by setting $(1, b) \sim f(b)$. We use $[t, b]$ to represent the image of (t, b) under the identification above, where $t \in \mathbb{I}$ and $b \in B$. We use $[y]$ to represent the image of $y \in Y$ under the identification above.

We use the notation $B \subset \mathbb{I}_f$, $0 \times B \subset \mathbb{I}_f$ interchangeably, and the notation $y \in \mathbb{I}_f$ or $[y] \in \mathbb{I}_f$ interchangeably, obtaining inclusions, $i: B \hookrightarrow \mathbb{I}_f$, and $j: Y \hookrightarrow \mathbb{I}_f$. One can quite easily show that $i: B \hookrightarrow \mathbb{I}_f$ and $j: Y \hookrightarrow \mathbb{I}_f$ are smooth isomorphisms.

6.3.2. Definition. For a smooth map $f: B \rightarrow Y$, define a map $\tilde{f}: \mathbb{I}_f \rightarrow Y$ by:

$$\tilde{f}([t, b]) = f(b), \text{ and } \tilde{f}([y]) = y.$$

6.3.3. Lemma. For a smooth map $f: B \rightarrow Y$, the map $\tilde{f}: \mathbb{I}_f \rightarrow Y$ is smooth.

Proof. Let $q : (\mathbb{I} \times B) \sqcup Y \rightarrow \mathbb{I}_f$ be the identification map that identifies $(1, b)$ and $f(b)$. Then $\tilde{f} \circ q : (\mathbb{I} \times B) \sqcup Y \rightarrow Y$ is given by $\tilde{f} \circ q(t, b) = f(b)$ and $\tilde{f} \circ q(y) = y$. Since $\tilde{f} \circ q$ is smooth on each component of the coproduct, it is smooth, and this implies that \tilde{f} is smooth, by definition of the quotient structure. See Section 2.6. \square

6.3.4. Lemma. *The map $\tilde{f} : \mathbb{I}_f \rightarrow Y$ is a retraction of \mathbb{I}_f onto $Y \subset \mathbb{I}_f$.*

Proof. One has,

$$\tilde{f} \circ j(y) = \tilde{f}([y]) = y.$$

Thus \tilde{f} is a retraction of \mathbb{I}_f onto Y . \square

6.3.5. Lemma. *For a smooth map $f : B \rightarrow Y$,*

$$1_{\mathbb{I}_f} \simeq j \circ \tilde{f} : \mathbb{I}_f \rightarrow \mathbb{I}_f, \text{ (rel } Y\text{)}.$$

Proof. Let $0 < \epsilon < \frac{1}{2}$. Define $k : \mathbb{I}_f \rightarrow \mathbb{I}_f$ by

- $k([t, b]) = [\alpha_\epsilon(t), b]$, for $[t, b] \in \mathbb{I}_f$ and
- $k([y]) = y$ for $y \in Y$.

Now k is smooth, since it is the composite of smooth maps, and

- $\alpha_\epsilon(t) = 0$ for $t \leq \epsilon$, and
- $\alpha_\epsilon(t) = 1$ for $t \geq 1 - \epsilon$,

so k has the required zero derivative property, since k is independent of t for t near 0 and t near 1.

First we show $k \simeq j \circ \tilde{f}$. Define a smooth homotopy $H : I \times \mathbb{I}_f \rightarrow \mathbb{I}_f$ as follows:

- $H(s, [t, b]) = [(1-s)\alpha_\epsilon(t) + s, b]$, for $s \in I$ and $[t, b] \in \mathbb{I}_f$, and
- $H(s, [y]) = [y]$.

Note that H is independent of t for t near 0 and t near 1. Thus H has the zero derivative property required of a function on $I \times \mathbb{I}$.

We now verify smoothness of H . Consider the following diagram,

$$\mathbb{R} \xrightarrow{c} I \times (\mathbb{I} \times B) \sqcup I \times Y \xrightarrow{\tilde{q}} I \times \mathbb{I}_f \xrightarrow{H} \mathbb{I}_f \xrightarrow{g} \mathbb{R}$$

where $c(\lambda)$ is a structure curve given by either

- $c(\lambda) = (s(\lambda), t(\lambda), x(\lambda))$ or,
- $c(\lambda) = (s(\lambda), y(\lambda))$.

Also

$$(1 - s(\lambda))\alpha_\epsilon(t(\lambda)) + s(\lambda)t(\lambda)$$

is smooth for $t(\lambda) \neq 0$.

We have $J(0, [t, b]) = [\alpha_\epsilon(t), b]$, and $J(1, [t, b]) = [t, b]$, and so

$$1_{\mathbb{I}_f} \simeq_{J^{-1}} k \simeq_H j \circ \tilde{f},$$

which is the required result, where $J^{-1}(s, -) = J(1 - s, -)$. □

This gives us the commutative diagram

$$\begin{array}{ccc} B & \xrightarrow{f} & Y \\ \downarrow i & \nearrow \tilde{f} & \\ \mathbb{I}_f & & \end{array}$$

\sim

The next result is adapted from the proof given by Spanier [114] for the topological situation.

6.3.6. Lemma. *Let B, Y be Frölicher spaces and $f : B \rightarrow Y$ be a smooth map. Then the inclusion $i : B \hookrightarrow \mathbb{I}_f$ is a smooth cofibration.*

Proof. Suppose we are given the following commutative diagram

$$\begin{array}{ccccc} B & \xrightarrow{i} & \mathbb{I}_f & \xrightarrow{g} & W \\ \downarrow (0, 1_B) & & \searrow G & & \uparrow \\ I \times B & & & & \end{array}$$

where W is a Frölicher space, and G and g are smooth maps.

We must show the existence of the dotted arrow in the following commutative diagram

$$\begin{array}{ccccc} \mathbb{I}_f & \xrightarrow{g} & W & & \\ \downarrow (0, 1_{\mathbb{I}_f}) & \nearrow H & \uparrow G' & & \\ I \times \mathbb{I}_f & \xleftarrow{1_I \times i} & I \times B & & \end{array}$$

where $G'(t, b) = G(\alpha_\epsilon(t), b)$ for some smooth braking function α_ϵ and each $t \in I$ and $b \in B$.

We construct the homotopy $H : I \times \mathbb{I}_f \rightarrow W$ as follows. Let $0 < \epsilon < \frac{1}{2}$. Define H by

$$H(s, [y]) = g([y]), \text{ for } y \in Y, s \in I$$

$$H(s, [t, b]) = \begin{cases} g([\frac{2t-s}{2-s}, b]) & \text{if } 0 \leq s \leq 2t \leq 2, b \in B \\ G(\alpha_{\epsilon}(s - \alpha_\epsilon(2t)), b) & \text{if } 0 \leq 2t \leq s \leq 1, b \in B, \end{cases}$$

where α_ϵ is a standard smooth braking function, and k_ϵ is the constant whose existence is shown in Lemma 2.8.3.

We now prove that H is a smooth map. Firstly observe that for $0 \leq t \leq 1$ and $0 \leq s \leq 1$, H is smooth on each of the two pieces defining it, since it is the composite of smooth functions there. We need to verify smoothness of H as $s \rightarrow 2t$, and we need to verify the zero derivative property of H with respect to the flattened mapping cylinder involved.

We first show smoothness as $s \rightarrow 2t$. To this end, let

$$c: \mathbb{R} \rightarrow I \times \left((\mathbb{I} \times B) \sqcup Y \right)$$

be a structure curve. We have the following diagram.

$$\mathbb{R} \xrightarrow{c} I \times \left((\mathbb{I} \times B) \sqcup Y \right) \xrightarrow{q} I \times \mathbb{I}_f \xrightarrow{H} W$$

We need to show that $H \circ q \circ c: \mathbb{R} \rightarrow W$ is a structure curve on W . Now, the structure curve c is of the form

$$\begin{aligned} c(\lambda) &= (s(\lambda), t(\lambda), b(\lambda)) \text{ or,} \\ c(\lambda) &= (s(\lambda), y(\lambda)). \end{aligned}$$

For a structure curve of the second type, smoothness of $H \circ q \circ c$ is clear, so suppose we have a structure curve of the first type. In addition, assume that for some $a \in \mathbb{R}$ and $0 < \eta$, we have

$$\begin{aligned} s(\lambda) &\leq 2t(\lambda) \text{ if } a - \eta < \lambda \leq a \\ s(\lambda) &\geq 2t(\lambda) \text{ if } a \leq \lambda < a + \eta. \end{aligned}$$

(In general, one must divide $(a - \eta, a + \eta)$ into subsets where each of these conditions hold. The proof then proceeds similarly). We will also assume that η is sufficiently small so that $s(\lambda) - \alpha_\epsilon(2t(\lambda)) < k_\epsilon$ for $a \leq \lambda < a + \eta$. We can do this by Lemma 2.8.3. Then

$$H \circ q \circ c(\lambda) = \begin{cases} g\left(\left[\frac{2t(\lambda) - s(\lambda)}{2 - s(\lambda)}, b(\lambda)\right]\right) & \text{if } a - \eta < \lambda \leq a \\ G\left(\alpha_{k_\epsilon}(s(\lambda) - \alpha_\epsilon(2t(\lambda))), b(\lambda)\right) & \text{if } a \leq \lambda < a + \eta. \end{cases}$$

Now, observe that the map $c_1: \mathbb{R} \rightarrow I \times \mathbb{I} \times B$ given by

$$c_1(\lambda) = \begin{cases} (s(\lambda), t(\lambda), b(\lambda)) & \text{if } \lambda \leq a \\ (s(\lambda), \frac{s(\lambda)}{2}, b(\lambda)) & \text{if } \lambda \geq a. \end{cases}$$

is also a structure curve, since $t(a) = \frac{s(a)}{2}$, and so c_1 is continuous and piecewise smooth, with the only possible point of non-smoothness occurring at $\lambda = a$.

Note that for $a \leq \lambda < a + \eta$,

$$\begin{aligned} G\left(\alpha_{k_\epsilon}(s(\lambda) - \alpha_\epsilon(s(\lambda))), b(\lambda)\right) \\ = G(0, b(\lambda)) \\ = g([0, b(\lambda)]), \end{aligned}$$

and so

$$H \circ c(\lambda) = H \circ [c_1(\lambda)] = \begin{cases} g\left(\left[\frac{2t(\lambda) - s(\lambda)}{2 - s(\lambda)}, b(\lambda)\right]\right) & \text{if } a - \eta < \lambda \leq a \\ g([0, b(\lambda)]) & \text{if } a \leq \lambda < a + \eta. \end{cases}$$

Thus, since c_1 is a structure curve, and g is a structure function on \mathbb{I}_f , H is smooth across the join where $s = 2t$, since it always the composite of smooth functions in a neighbourhood of the join.

We need to show that H has the zero derivative property as $t \rightarrow 0$ and $t \rightarrow 1$. Suppose that H is given by

$$g\left(\left[\frac{2t - s}{2 - s}, b\right]\right)$$

as $t \rightarrow 0$. Then, since $0 \leq s \leq 2t$, we have $s \rightarrow 0$ also. Thus, $\frac{2t-s}{2-s} \rightarrow 0$ as $t \rightarrow 0$. Since g has the zero derivative property, so does H in this case. If H is given by

$$G(\alpha_{k_\epsilon}(s - \alpha_\epsilon(2t)), b)$$

as $t \rightarrow 0$, then $\alpha_\epsilon(2t) = 0$ for t near 0, and so H is independent of t for t near 0, and so H has the zero derivative property in this case too.

Suppose $t \rightarrow 1$. Then eventually H is given by

$$g\left(\left[\frac{2t - s}{2 - s}, b\right]\right).$$

But $\frac{2t-s}{2-s} \rightarrow 1$ as $t \rightarrow 1$. Since g has the zero derivative property here, so does H .

This completes the verification that H is smooth. It is easy to verify that H is the required smooth homotopy, and we have $G'(s, b) = G(\alpha_{k_\epsilon}(s), b)$. \square

Putting the above results together, we have the following commutative diagram,

$$\begin{array}{ccc} B & \xrightarrow{J} & Y \\ & \searrow i & \nearrow \tilde{f} \\ & & \mathbb{I}_f \end{array}$$

where i is a cofibration and \tilde{f} is a weak equivalence. Thus axiom (C3) is verified.

6.4 Axiom (C4): Axiom on fibrant models

Before stating this axiom, we need the following definition from Baues' book [7].

6.4.1. Definition. *An object R in a cofibration category \mathbf{C} is called a **fibrant model** or **fibrant** if every trivial cofibration $i: R \rightarrow Q$ admits a retraction $r: Q \rightarrow R$, $ri = 1_R$.*

The axiom on fibrant models is then the following. For each object X in \mathbf{C} there is a trivial cofibration

$$X \xrightarrow{\sim} RX$$

where RX is fibrant. We call RX a **fibrant model** of X .

Lemma 6.2.1 shows that every object in **FRL** is fibrant, and thus every object is its own fibrant model.

Hence axiom (C4) has been verified, since each object in **FRL** is its own fibrant model, with the identity map the trivial cofibration.

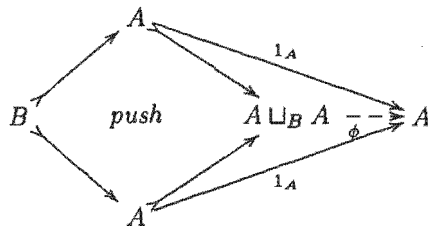
6.5 Homotopy in FRL via the Cofibration Axioms

We now follow Baues [7], and construct axiomatically the notion of homotopy in **FRL**, via the axioms above. We then show that this notion of homotopy coincides with the usual notion of smooth homotopy defined in Chapter 4. See also Cherenack [22] and Dugmore [36].

6.5.1. Definition. *Suppose that*

$$B \xrightarrow{\quad} A$$

is a smooth cofibration. By (C2) we have the pushout diagram



By (C3) we may factorize $\phi = [1_A, 1_A]_B$ (called the folding map) as follows

$$A \sqcup_B A \xrightarrow{i} \mathbb{I}_\phi \xrightarrow[p]{\sim} A$$

We also denote \mathbb{I}_ϕ by $I_B A$, in keeping with Baues' notation, and call this object a relative cylinder on

$$B \xrightarrow{j} A.$$

In our situation, the relative cylinders are always the mapping cylinders \mathbb{I}_ϕ . We will sometimes write i_ϕ or p_ϕ to accentuate the particular ϕ on which i or p depend.

6.5.2. Definition. If

$$B \xrightarrow{j} A$$

is a smooth cofibration, then two maps $\alpha, \beta: A \rightarrow X$ are called **homotopic relative B** and we write $\alpha \simeq \beta \text{ rel } B$ if there is a commutative diagram

$$\begin{array}{ccc} A \sqcup_B A & \xrightarrow{i=[i_0, i_1]_B} & \mathbb{I}_\phi \\ & \searrow [\alpha, \beta]_B & \swarrow H \\ & & X \end{array}$$

where \mathbb{I}_ϕ is the relative cylinder of Definition 6.5.1. We call H a **homotopy from α to β (rel B)**.

6.5.3. Proposition. The notion of homotopy in Definition 6.5.2 coincides with the usual notion of relative homotopy in **FRL** discussed in Chapter 4.

Proof. Let $\alpha: \mathbb{R} \rightarrow \mathbb{R}$ be a smooth braking function with the properties

$$\alpha(t) = \begin{cases} 0 & \text{if } t \leq \frac{1}{4} \\ 1 & \text{if } t \geq \frac{3}{4} \end{cases}$$

Suppose we have a map $H: I \times A \rightarrow X$ such that

1. $H(0, a) = f(a)$, $a \in A$,
2. $H(1, a) = g(a)$, $a \in A$ and
3. $H(t, j(b)) = fj(b) = gj(b)$, $b \in B$,

where

$$B \xrightarrow{j} A$$

is a smooth cofibration.

Note that

$$\mathbb{I}_\phi \cong \underbrace{[(\mathbb{I} \times A)]}_P \sqcup_{(\mathbb{I} \times B)} \underbrace{[(\mathbb{I} \times A)]}_Q / \sim,$$

where P denotes the first copy of $\mathbb{I} \times A$ and Q denotes the second copy of $\mathbb{I} \times A$, and \sim is the following quotient. For $(1, a_1) \in P$ and $(1, a_2) \in Q$, we have $(1, a_1) \sim (1, a_2)$ if $a_1 = a_2$.

Construct $F: \mathbb{I}_\phi \rightarrow X$ as follows

$$F([t, a]) = \begin{cases} H(\alpha(\frac{t}{2}), a) & \text{if } (t, a) \in P \\ H(\alpha(\frac{1}{2} + \frac{t}{2}), a) & \text{if } (t, a) \in Q. \end{cases}$$

One can easily see that F is well-defined, since $F([t, j(b)]) = f(j(b))$ irrespective of to which copy, P or Q , $[t, b]$ belongs.

The map F is smooth, since it is smooth on each component of the coproduct, and the braking function gives it the zero derivative property required of the flattened unit interval in the definition of \mathbb{I}_ϕ .

Conversely, suppose that we have a commutative diagram

$$\begin{array}{ccc} A \sqcup_B A & \xrightarrow{i=[i_0, i_1]_B} & \mathbb{I}_\phi \\ & \searrow f \cup_B g & \swarrow F \\ & & X \end{array}$$

We may define $H: I \times A \rightarrow X$ by

$$H(t, a) = \begin{cases} F(\alpha(2t), i_0(a)) & \text{if } 0 \leq t \leq \frac{1}{2} \\ F(\alpha(2-2t), i_1(a)) & \text{if } \frac{1}{2} \leq t \leq 1, \end{cases}$$

where the braking function α defined earlier ensures that H is smooth. \square

6.5.4. Note. For an \emptyset -cofibrant object A , Baues [7] defines the sum of A and an object Y , denoted by $A + Y$ as the pushout $A \sqcup_\emptyset Y$. In the category **FRL** all objects are \emptyset -cofibrant, and so we always have the following diagram

$$\begin{array}{ccc} A & \longrightarrow & A \sqcup Y \\ \uparrow & & \uparrow \\ \emptyset & \longrightarrow & Y \end{array}$$

where A and Y are any two Frölicher spaces. Thus whenever we quote a result of Baues that involves a sum $A + Y$, we replace this sum with the coproduct $A \sqcup Y$ (a disjoint union) in **FRL** without further comment.

6.5.5. Definition. The axiomatic mapping cylinder Z_f of a smooth map $f: A \rightarrow Y$ is defined by the factorization of $[1_Y, f]: Y \sqcup A \rightarrow Y$

$$[1_Y, f]: Y \sqcup A \longrightarrow \mathbb{I}_{[1_Y, f]} := Z_f \xrightarrow{q} Y.$$

Note that in general the smooth mapping cylinder \mathbb{I}_f of a map f is not isomorphic to Z_f . Thus we adopt the name **axiomatic** mapping cylinder for Z_f to distinguish between these two cylinders. However, as the next lemma indicates, \mathbb{I}_f and Z_f are of the same smooth homotopy type.

6.5.6. Lemma. *Let $f: A \rightarrow Y$ be a smooth map. Then \mathbb{I}_f and Z_f are of the same smooth homotopy type.*

Proof. Note firstly, that

$$\mathbb{I}_f = (\mathbb{I} \times A) \sqcup_f Y,$$

and

$$Z_f = (\mathbb{I} \times (A \sqcup Y)) \sqcup_{[1_Y, f]} Y.$$

Define $\gamma: \mathbb{I}_f \rightarrow Z_f$ by

1. $\gamma[(t, a)] = [(t, a)]$ for $t \in I, a \in A$,
2. $\gamma([y]) = [y]$ for $y \in Y$.

If $y = f(a)$, then $\gamma[(1, a)] = [(1, a)] = [y] = \gamma([y])$, so γ is well defined.

Next define $\xi: Z_f \rightarrow \mathbb{I}_f$ by

1. $\xi[(t, a)] = [(t, a)]$ for $t \in I, a \in A$,
2. $\xi[(t, y)] = [y]$ for $t \in I, y \in Y$,
3. $\xi([y]) = [y]$ for $y \in Y$.

Note that $\xi[(1, y)] = [y] = \xi([y])$ and $\xi[(1, a)] = [(1, a)] = [y] = \xi([y])$ if $f(a) = y$, so ξ is well defined.

Clearly $1_{\mathbb{I}_f} = \xi\gamma: \mathbb{I}_f \rightarrow \mathbb{I}_f$. Consider $H: I \times Z_f \rightarrow Z_f$ defined as follows:

1. $H(s, [(t, a)]) = [(t, a)]$ for $t \in I, a \in A$,
2. $H(s, [(t, y)]) = [(s + (1 - s)\alpha_\epsilon(t), y)]$ for $y \in Y$,
3. $H(s, [y]) = [y]$ for $y \in Y$.

Note that $H(s, [(1, y)]) = [(1, y)] = [y] = H(s, [y])$ and $H(s, [(1, a)]) = [(1, a)] = [y] = H(s, [y])$, so H is well defined.

One can easily verify that $H(0, -) \simeq 1_{Z_f}$ and $H(1, -) = \gamma\xi$, so $\mathbb{I}_f \simeq Z_f$. Notice that one only has $H(0, -) \simeq 1_{Z_f}$ because the braking function $\alpha_\epsilon(t)$ in part 2 of the definition of H needs to be deformed into the identity function. This is easily done. \square

6.6 The Cofibration Axioms in \mathbf{FRL}_*

To construct the smooth homotopy groups using the axioms, and to construct the right Puppe sequence, we need to work in a pointed category. We will work in the category \mathbf{FRL}_* of Frölicher spaces with basepoint, and smooth basepoint preserving maps. We must define cofibrations and weak equivalences in this category and verify the necessary cofibration axioms before we can continue.

6.6.1. Definition. A pointed smooth map $H : I \times X \rightarrow Y$ is called a smooth homotopy relative to $*_X$ if $H(t, *_X) = *_Y$ for every $t \in I$. If $f = H(0, -) : X \rightarrow Y$ and $g = H(1, -) : X \rightarrow Y$ then we write $f \simeq_H g \text{ (rel } *)$.

6.6.2. Definition. A pointed smooth map

$$A \xrightarrow{i} B$$

is called a **smooth cofibration** if, corresponding to every commutative diagram in \mathbf{FRL}_* of the form

$$\begin{array}{ccccc} A & \xrightarrow{i} & B & \xrightarrow{j} & Z \\ \downarrow (0, 1_A) & & & \nearrow G & \\ I \times A & & & & \end{array}$$

where G is a homotopy relative to $*_A$, there exists a commutative diagram in \mathbf{FRL}_* of the form

$$\begin{array}{ccccc} B & \xrightarrow{j} & Z & & \\ \downarrow (0, 1_B) & & \nearrow F & & \\ I \times B & \xleftarrow{1_I \times i} & I \times A & \xrightarrow{G'} & Z \end{array}$$

where $G' : I \times A \rightarrow Z$ is given by $G'(t, a) = G(\alpha_\epsilon(t), a)$ for some $0 < \epsilon < \frac{1}{2}$, and each $t \in I$, $a \in A$.

Notice that the commutativity of the last diagram implies that F is a homotopy relative to $*_B$.

6.6.3. Definition. A pointed smooth map $p : A \rightarrow B$ is called a **weak equivalence** if there exists a pointed smooth map $\bar{p} : B \rightarrow A$ such that

1. There exists a smooth homotopy $H : I \times A \rightarrow A \text{ (rel } *)$, such that

- $H(0, a) = \bar{p} \circ p$, for $a \in A$,
- $H(1, a) = 1_A$ for $a \in A$.

2. There exists a smooth map $G : I \times B \rightarrow B \text{ (rel } *)$, such that

- $G(0, b) = p \circ \bar{p}$ for $b \in B$,
- $G(1, b) = 1_B$ for $b \in B$,

One can readily verify the following useful fact.

6.6.4. Lemma. If $f_0, f_1 : A \rightarrow B$ and $g_0, g_1 : B \rightarrow C$ with $f_0 \simeq f_1 \text{ (rel } *)$ and $g_0 \simeq g_1 \text{ (rel } *)$, then $g_0 \circ f_0 \simeq g_1 \circ f_1 \text{ (rel } *)$.

We now define the pointed version of the smooth mapping cylinder, and prove a few results that we will need when verifying the necessary cofibration axioms.

6.6.5. Definition. Given a pointed smooth map $f: B \rightarrow Y$, the **reduced (weakly flattened) mapping cylinder of f** is

$$\mathbb{I}_f = (\mathbb{I} \times B \sqcup Y) / \sim,$$

where \sim is the relation defined by setting $(1, b) \sim f(b)$ and $(t, *_B) \sim (0, *_B)$ for every $t \in I$. We denote points in \mathbb{I}_f by $[t, b]$ or $[y]$, and the point $[(0, *_B)] = [*_Y]$ is taken as the basepoint of \mathbb{I}_f .

Although we use the same notation for the (weakly flattened) mapping cylinder and the reduced (weakly flattened) mapping cylinder, there should be no possibility for confusion, since they are objects in different categories, and these categories will be treated separately.

For ease of discussion, in this section we shall often omit the words ‘weakly flattened’, since all mapping cylinders are the weakly flattened mapping cylinders.

6.6.6. Definition. Given a map $f: B \rightarrow Y$, we define the following maps.

- The map $i: B \hookrightarrow \mathbb{I}_f$ is given by $i(b) = [0, b]$, for $b \in B$.
- The map $j: Y \hookrightarrow \mathbb{I}_f$ is given by $j(y) = [y]$, for $y \in Y$.
- The map $\tilde{f}: \mathbb{I}_f \rightarrow Y$ is given by $\tilde{f}[t, b] = f(b)$ for $b \in B$ and $\tilde{f}[y] = y$ for $y \in Y$.

It is easy to show that i and j are smooth isomorphisms onto their images in \mathbb{I}_f , and that \tilde{f} is a well-defined pointed smooth map. We may prove the following properties of the maps i , j and \tilde{f} .

6.6.7. Lemma. 1. $\tilde{f} \circ i = f$.

2. $1_{\mathbb{I}_f} \simeq j \circ \tilde{f}$ (rel Y).

Proof. Part 1 is obvious. For part 2, we can use the same smooth homotopy that we used when proving the non-pointed version of this lemma (ie Lemma 6.3.5). \square

6.6.8. Lemma. The map $i: B \hookrightarrow \mathbb{I}_f$ is a pointed smooth cofibration.

Proof. If one looks through the proof of Lemma 6.3.6, one can verify that all the maps involved are basepoint preserving maps if they are treated as maps in \mathbf{FRL}_* . Thus the proof of this result is identical to the proof of Lemma 6.3.6. \square

(C1) Composition Axiom

The proof that this axiom holds in \mathbf{FRL}_* is identical to the proof for \mathbf{FRL} , except that we require all smooth maps to be basepoint preserving.

(C2) Pushout Axiom

For this axiom one can simply observe that all the constructions of Section 6.2 may be transferred verbatim to the category \mathbf{FRL}_* , and all the maps used become basepoint preserving maps. As $*_X$ is the initial object in \mathbf{FRL}_* , we need the fact that $*_X \hookrightarrow X$ is a cofibration, which we prove below.

6.6.9. Lemma. *For a pointed Frölicher space X the inclusion $*_X \hookrightarrow X$ is a cofibration.*

Proof. Suppose we have a commutative diagram

$$\begin{array}{ccc} *_X & \longrightarrow & X \xrightarrow{g} Z \\ \downarrow 0 \times *_X & \searrow G & \\ I \times *_X & & \end{array}$$

where G is a smooth homotopy (rel $*$). Then, we may construct the commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & Z \\ \downarrow (0, 1_X) & \searrow F & \uparrow G' \\ I \times X & \xleftarrow{1_I \times *_X} & I \times *_X \end{array}$$

where $G'(t, *_X) = G(\alpha_\epsilon(t), *_X)$, some $0 < \epsilon < \frac{1}{2}$, and F is given by $F(t, x) = g(x)$. \square

(C3) C-Factorization Axiom

This axiom follows from Lemma 6.6.7 and Lemma 6.6.8.

(C4) Axiom on Fibrant Models

As before we need to show that there is a trivial cofibration from every object in \mathbf{FRL}_* to a fibrant object, called a fibrant model. We show that every object in \mathbf{FRL}_* is itself a fibrant model.

Suppose that

$$A \xrightarrow[\sim]{i} B$$

is a trivial cofibration. Then there exists $\bar{i}: B \rightarrow A$ such that $\bar{i}i \simeq_G 1_A$ (rel $*$).

We then have the following commutative diagram

$$\begin{array}{ccc} A & \xrightarrow[\sim]{i} B \xrightarrow{\bar{i}} A \\ \downarrow (0, 1_A) & \searrow G & \\ I \times A & & \end{array}$$

where G is a homotopy relative to $*_A$. But i is a smooth cofibration, so we may construct the following commutative diagram

$$\begin{array}{ccc} I \times A & \xrightarrow{(1_I, i)} & I \times B \\ & \searrow G' & \downarrow F \\ & & A \end{array}$$

where $G'(t, -) = G(\alpha_\epsilon(t), -)$, for $0 < \epsilon < \frac{1}{2}$. Note that $*_A = G'(t, *_A) = F(t, i(*_A)) = F(t, *_B)$ for every $t \in I$, and so F is a smooth homotopy relative to $*_B$. Define $r: B \rightarrow A$ by $r(b) = F(1, b)$ for $b \in B$. Then r is a pointed smooth map, since F is a homotopy relative to $*_B$, and

$$ri(a) = F(1, i(a)) = G(1, a) = a.$$

Thus A is a fibrant model.

6.7 Homotopy Groups

In this section we define the homotopy groups of an object in \mathbf{FRL}_* axiomatically, following Baues [7]. In section 6.8 we verify that these groups are isomorphic to those defined by Cherenack [22] and Dugmore [36].

6.7.1. Definition. We define the (weakly flattened) torus $\Sigma_Y X$ on an inclusion $Y \subset X$ by the pushout diagram

$$\begin{array}{ccc} \mathbb{I}_\phi & \longrightarrow & \Sigma_Y X \\ \uparrow i_\phi & & \uparrow i \\ X \sqcup_Y X & \xrightarrow{\phi} & X \end{array}$$

where $\phi = [1_X, 1_X]_Y$ is the folding map, and the cofibration

$$X \sqcup_Y X \twoheadrightarrow \mathbb{I}_\phi$$

is obtained from the C -factorization of the folding map, as in Definition 6.5.1.

6.7.2. Definition. For an object A in \mathbf{FRL}_* , the (weakly flattened) suspension ΣA of A is given by the pushout diagram

$$\begin{array}{ccccc} \mathbb{I}_\phi & \longrightarrow & \Sigma_* A & \longrightarrow & \Sigma A \\ \uparrow i_\phi & & \uparrow & & \uparrow \\ A \sqcup_* A & \xrightarrow{\phi} & A & \xrightarrow{*} & * \end{array}$$

where $\phi = [1_A, 1_A]_*$ is the folding map and the cofibration

$$A \sqcup_* A \twoheadrightarrow \mathbb{I}_\phi$$

is obtained from the C -factorization of the folding map, as in definition 6.5.1.

We define inductively $\Sigma^n A = \Sigma(\Sigma^{n-1}A)$ for $n \geq 1$, $\Sigma^0 A = A$.

6.7.3. **Definition.** For $n \geq 1$, we define the homotopy groups

$$\pi_n^A(U) = [\Sigma^n A, U]$$

It is shown by Baues [7], using the cofibration axioms, that the $\pi_n^A(U)$ are groups.

6.7.4. **Note.** The usual n -th homotopy group in \mathbf{FRL}_* is given by

$$\pi_n(X) = \pi_n^{S^0}(X) = [\Sigma^n S^0, X].$$

6.8 The Puppe Sequence

We now investigate the Puppe sequence that is constructed axiomatically by Baues [7] using the cofibration axioms, and show that it is the same as the 'usual' Puppe sequence in \mathbf{FRL}_* found in Section 5.4.

6.8.1. **Definition.** Let A be a pointed smooth space. We define the (weakly flattened) cone CA by the pushout

$$\begin{array}{ccc} \mathbb{I}\phi & \longrightarrow & CA \\ \uparrow i_\phi & & \uparrow j_A \\ A \sqcup_* A & \xrightarrow{[* , 1_A]_*} & A \end{array}$$

Here ϕ is the same ϕ as in Definition 6.7.2.

6.8.2. **Definition.** We now define the (weakly flattened) (axiomatic) mapping cone C_f of a map $f: B \rightarrow A$ by the pushout diagram

$$\begin{array}{ccc} CB & \xrightarrow{\pi_f} & C_f \\ \uparrow j_B & & \uparrow i \\ B & \xrightarrow{f} & A \end{array}$$

Note that we call this the **axiomatic** mapping cone to distinguish it from the mapping cone \mathbb{T}_f defined in Section 4.5. Baues shows in his book [7] that one has the following right exact sequence in a cofibration category,

$$B \xrightarrow{f} A \xrightarrow{i} C_f \xrightarrow{q} \Sigma B \xrightarrow{\Sigma f} \dots$$

where $q: C_f \rightarrow \Sigma B$ is the quotient map that identifies all points $[b] \in C_f$, $b \in B$ to a point.

We then have the induced sequence in \mathbf{hFRL}_* , the homotopy category of \mathbf{FRL}_* :

$$\cdots \longrightarrow [\Sigma A, U] \xrightarrow{q^*} [C_f, U] \xrightarrow{i^*} [A, U] \xrightarrow{f^*} [B, U]$$

We wish to show that this is the same as the sequence discussed in Section 5.4. To do this it suffices to show that the suspension, and mapping cone defined in 5.4 are of the same homotopy type as those defined above.

Let X, Y be Frölicher spaces, with basepoints x_0, y_0 , respectively, and let $f: X \rightarrow Y$ be a smooth mapping. We denote the flattened suspension of X as defined in Section 5.4 by $\bar{\Sigma}X$. Recall that the flattened mapping cone of f , defined in Section 4.5, is denoted \mathbf{T}_f .

6.8.3. Lemma. *For a Frölicher space A , and a smooth map f , we have:*

1. $\Sigma A \simeq \bar{\Sigma}A$, and
2. $C_f \simeq \mathbf{T}_f$.

Proof. We prove 1. Part 2 is similar.

One constructs the suspension ΣA via two pushouts. If one follows this construction step by step, one can verify that

$$\Sigma A \cong \underbrace{(\mathbb{I} \times A)}_P \sqcup \underbrace{(\mathbb{I} \times A)}_Q / \sim,$$

where \sim represents a quotient. As indicated above, we denote the first copy of $\mathbb{I} \times A$ by P and the second copy of $\mathbb{I} \times A$ by Q . Thus any point in ΣX can be written as $[t_1, a_1]$ or $[t_2, a_2]$, where $(t_1, a_1) \in P$ or $(t_2, a_2) \in Q$ and $[t, a]$ is the equivalence class of (t, a) under the quotient \sim .

One can verify that

$$\bar{\Sigma}A \cong \mathbb{I} \times A / \approx,$$

where \approx represents the relation that identifies the set $(\{0, 1\} \times A) \cup (\mathbb{I} \times *_A)$ to a point. We represent a point in $\bar{\Sigma}A$ by $[t, a]$, where $[t, a]$ represents the equivalence class of (t, a) under the quotient \approx . It should be clear in the following whether a point $[t, a]$ belongs to ΣA or $\bar{\Sigma}A$.

Define a braking function $\alpha: \mathbb{I} \rightarrow \mathbb{R}$ with the properties:

1. $\alpha(t) = 0$ for $0 \leq t \leq \frac{1}{4}$,
2. $\alpha(t) = 1$ for $\frac{3}{4} \leq t \leq 1$,
3. $0 \leq \alpha(t) \leq 1$.

Define $\gamma: \Sigma A \rightarrow \bar{\Sigma}A$ by

$$\gamma[t, a] = \begin{cases} [\frac{\alpha(t)}{2}, a] & \text{if } [t, a] \in P \\ [1 - \frac{\alpha(t)}{2}, a] & \text{if } [t, a] \in Q. \end{cases}$$

One can easily verify that this mapping is well-defined.

We now define $\xi: \bar{\Sigma}A \rightarrow \Sigma A$ by

$$\xi[t, a] = \begin{cases} [\alpha(2t), a] \in P & \text{if } 0 \leq t \leq \frac{1}{2} \\ [\alpha(2-2t), a] \in Q & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

We must show $\gamma\xi \simeq 1_{\bar{\Sigma}A}$ and $\xi\gamma \simeq 1_{\Sigma A}$. Note that

$$\gamma\xi[t, a] = \begin{cases} [\frac{\alpha(\alpha(2t))}{2}, a] & \text{if } 0 \leq t \leq \frac{1}{2} \\ [1 - \frac{\alpha(\alpha(2-2t))}{2}, a] & \text{if } \frac{1}{2} \leq t \leq 1, \end{cases}$$

and

$$\xi\gamma[t, a] = \begin{cases} [\alpha(\alpha(t)), a] \in P & \text{if } [t, a] \in P \\ [\alpha(\alpha(t)), a] \in Q & \text{if } [t, a] \in Q. \end{cases}$$

Construct $H: I \times \bar{\Sigma}A \rightarrow \bar{\Sigma}A$ as follows:

$$H(s, [t, a]) = \begin{cases} [st + (1-s)\frac{\alpha(\alpha(2t))}{2}, a] & \text{if } 0 \leq t \leq \frac{1}{2} \\ [st + (1-s)\{1 - \frac{\alpha(\alpha(2-2t))}{2}\}, a] & \text{if } \frac{1}{2} \leq t \leq 1, \end{cases}$$

It is easy to verify that H is the required homotopy giving $\gamma\xi \simeq 1_{\bar{\Sigma}A}$.

The case involving $\xi\gamma$ is similar. □

Chapter 7

The basic constructions of differential geometry in FRL

This short chapter defines the basic notions needed for the study of differential geometry in the category **FRL**. In the two subsequent chapters we restrict our attention to the differential geometry of Frölicher spaces that arise from quotients on smooth manifolds.

7.1 The tangent cone of a Frölicher space

The tangent cone of a Frölicher space at a point was defined by Cherenack. See [22], [24]. His definition is given below as Definition 7.1.1.

7.1.1. Definition. Let X be a Frölicher space and let $x \in X$. The **tangent cone** of X at the point x is defined to be the set

$$\mathbf{TC}_x X = \{v_c: F_X \rightarrow \mathbb{R} : c \in C_X, c(0) = x\},$$

where, for a given $c \in C_X$ with $c(0) = x$, we define

$$v_c(f) = \lim_{t \rightarrow 0} \frac{f \circ c(t) - f \circ c(0)}{t},$$

for each $f \in F_X$. We call $v_c \in \mathbf{TC}_x X$ a **tangent vector** even though $\mathbf{TC}_x X$ is not, in general, a vector space.

Notice that two different curves c_1 and c_2 could define the same tangent vector, $v_{c_1} = v_{c_2}$.

7.1.2. Definition. Let (X, C_X, F_X) be a Frölicher space. A map $v: F_X \rightarrow \mathbb{R}$ is called a **derivation at $x \in X$** if it is a linear map that satisfies the property

$$v(fg) = f(x)v(g) + v(f)g(x).$$

The set of all derivations at a point $x \in X$ is called the tangent space at x of the Frölicher space X , and is denoted $\mathbf{T}_x X$.

The tangent space at $x \in X$ is always a vector space. Notice that for an n -dimensional smooth manifold M , the tangent space at $x \in M$ is an n -dimensional vector space which coincides with the tangent cone $\mathbf{TC}_x M$.

The tangent cone $\mathbf{TC}_x X$ at x of a Frölicher space X is contained in the vector space $\mathbf{T}_x X$. However, the tangent cone need not be a vector space, since it is not necessarily closed under addition (although it is easy to see that the tangent cone is always closed under scalar multiplication).

If we define

$$D = \mathbf{FRL}(F_X, \mathbb{R}),$$

we then see that the tangent cone $\mathbf{TC}_x X \subseteq D$ and the tangent space $\mathbf{T}_x X \subseteq D$ each have a natural Frölicher structure which they inherit from the function space D .

We shall often refer to the ' m -dimensional tangent cone $\mathbf{TC}_x X$ '. For our purposes it is sufficient to define this dimension to be the topological dimension of the underlying topological space. (See Definition 2.4.1 for the definition of the underlying topology of a Frölicher space). Although we make this definition, we shall only talk about the dimension of the tangent cone when the tangent cone is a finite dimensional vector space.

7.1.3. Definition. Let X be a Frölicher space. Let

$$\mathbf{TC}X = \{(x, v) | x \in X, v \in \mathbf{TC}_x X\}.$$

We call $\mathbf{TC}X$ the tangent cone bundle of X .

As $\mathbf{TC}X \subset X \times D$, it has a natural Frölicher structure.

7.2 Manifolds

Whenever we refer to a **manifold**, we mean a Hausdorff c^∞ -manifold. A **submanifold** M of a Frölicher space X is a Frölicher subspace of X which is a Hausdorff c^∞ -manifold under the inherited Frölicher structure. Thus for us, the statement ' A is a submanifold of B ' does not imply that B is itself a manifold.

It has been shown that the category of smooth manifolds is a subcategory of the category of Frölicher spaces. (See [22]). Briefly, the structure curves of a smooth manifold M are exactly those curves $c: \mathbb{R} \rightarrow M$ that are smooth in the usual sense. Similarly, the structure functions are exactly those functions $f: M \rightarrow \mathbb{R}$ that are smooth in the usual sense.

If M is an m -dimensional manifold, and hence a Frölicher space, then at every point $x \in M$ there is, as we saw above, an associated m -dimensional vector space, called the **tangent space at x** . We denoted the tangent space of M at x by $\mathbf{T}_x M$. We denote a basis for $\mathbf{T}_x M$ by $\mathbf{B}_x M$.

Our definition of the tangent cone $\text{TC}_x X$ of a Frölicher space X at a point x is modelled on a common method of defining the tangent space $\text{T}_a M$ of a manifold M at a point a . It follows that for manifolds $\text{TC}_a M = \text{T}_a M$.

The following is a useful result concerning manifolds.

7.2.1. Theorem. *Let $V \subseteq M$ be an open subset of the manifold M . Suppose $f: V \rightarrow \mathbb{R}$ is a smooth map defined on V . Let $U \subset V$ be an open subset of V with $\bar{U} \subset V$. Then there exists a smooth function $\tilde{f}: M \rightarrow \mathbb{R}$ such that $\tilde{f}(x) = f(x)$ for each $x \in U$.*

Proof. If $V = M$ then we are done, so assume that $V \neq M$. Let W be an open subset of V with $\bar{W} \subset V$ and $\bar{U} \subset W$. Suppose that $f: V \rightarrow \mathbb{R}$ is a smooth map. It is known (see, for example, Golubitsky and Guillemin [48], Chapter 1, Corollary 4.7) that there exists a smooth function $g: M \rightarrow \mathbb{R}$ such that

$$g(x) = \begin{cases} 1 & \text{if } x \in U \\ 0 & \text{if } x \notin W \\ 0 \leq f(x) \leq 1 & \text{otherwise.} \end{cases}$$

Define $\tilde{f}: M \rightarrow \mathbb{R}$ by

$$\tilde{f}(x) = \begin{cases} f(x)g(x) & \text{if } x \in W \\ 0 & \text{if otherwise.} \end{cases}$$

□

Let $A \subseteq M$. Any smooth function $f: M \rightarrow \mathbb{R}$ can be restricted to define a smooth function $f|_A: A \rightarrow \mathbb{R}$. Conversely, the above result tells us that any smooth function $f: U \rightarrow \mathbb{R}$ defined on an open neighbourhood of the closed set A (in M) can be extended to $\tilde{f}: M \rightarrow \mathbb{R}$ so that f and \tilde{f} coincide on A .

7.3 Some standard definitions

For completeness we now include some well known definitions.

7.3.1. Definition. *Let E and X be manifolds, and let $\pi: E \rightarrow X$ be a submersion. Let $E_U = \pi^{-1}(U)$ for any subset $U \subseteq X$. We call E a family of vector spaces over X of dimension k if, for every $x \in X$, E_x is equipped with the structure of a topological vector space over \mathbb{R} , in the subspace topology that E_x inherits from E .*

7.3.2. Definition. *Given a submersion $E \xrightarrow{\pi} X$, a section of E is a smooth mapping $s: X \rightarrow E$ such that $\pi \circ s = 1_X$.*

7.3.3. Definition. *If $E \xrightarrow{\pi_1} X$ and $F \xrightarrow{\pi_2} X$ are each families of vector spaces over X , then a smooth map $\phi: E \rightarrow F$ is a homomorphism if $\pi_2 \circ \phi = \pi_1$ and $\phi: E_x \rightarrow F_x$ is a vector space homomorphism for each $x \in X$.*

If ϕ is a diffeomorphism and a homomorphism, then we call ϕ an isomorphism of these families.

7.3.4. Definition. Let V be a vector space, and suppose that $X \times V \xrightarrow{p_1} X$ is a family of vector spaces over X , where p_1 is the projection onto the first coordinate. We call $X \times V \xrightarrow{p_1} X$ a **product family**. A family of vector spaces is called **trivial** if it is isomorphic to some product family.

7.3.5. Definition. Let $E \xrightarrow{\pi} X$ be a family of vector spaces over X . If each $x \in X$ has an open neighbourhood U_x so that $E_{U_x} \xrightarrow{\pi} U_x$ is a trivial family of vector spaces, then we call $E \xrightarrow{\pi} X$ a **vector bundle** over X .

In other words, a vector bundle is a *locally* trivial family of vector spaces. If a vector bundle is isomorphic to a product family, we call the vector bundle **trivial**.

7.3.6. Definition. Let M be a manifold. Then we call

$$\mathbf{TM} = \bigcup_{x \in M} \{x\} \times \mathbf{T}_x M$$

the **tangent bundle** to M . We denote the projection onto the first coordinate by $\pi: \mathbf{TM} \rightarrow M$.

One can show that \mathbf{TM} can be given a manifold structure and then π defines \mathbf{TM} to be a vector bundle over M . One can show that the Frölicher space structure on \mathbf{TM} as a Frölicher subspace of $M \times D$ is isomorphic in FRL to the Frölicher space structure on \mathbf{TM} viewed as a manifold.

We define the analogous concept of a **vector bundle** over a Frölicher space X in the obvious way.

Notice that the tangent cone bundle, \mathbf{TCX} of a Frölicher space X is not, in general, a vector bundle over X .

Chapter 8

Finite-Set Quotients on a Manifold

In this chapter we prove some results about the critical points of a function defined on a Frölicher space which arises as the quotient space M/A of a manifold M by a finite subset $A \subseteq M$.

8.1 Some motivating examples

We present two examples. The first is an example of a Frölicher space with a tangent cone at a point which is not a vector space. The second example is of a Frölicher space which has a tangent cone which is a vector space, but where the Frölicher space is not locally a manifold.

8.1.1. Example. Let \mathbb{R}_0 and \mathbb{R}_1 each be a copy of the real line \mathbb{R} , with the usual Frölicher structures. Let $\mathbb{R}_0 \vee \mathbb{R}_1$ denote the wedge product of \mathbb{R}_0 and \mathbb{R}_1 obtained by identifying the zeroes of each copy of \mathbb{R} , as described previously. Let

$$q: \mathbb{R}_0 \sqcup \mathbb{R}_1 \rightarrow \mathbb{R}_0 \vee \mathbb{R}_1.$$

denote the identification map. Then, as we have seen before, f is smooth if, and only if, $f \circ q$ is smooth.

One can easily show that there are Frölicher space isomorphisms

$$\mathbb{R}_0 \cong q(\mathbb{R}_0) \text{ and } \mathbb{R}_1 \cong q(\mathbb{R}_1)$$

induced from q .

Thus $f: \mathbb{R}_0 \vee \mathbb{R}_1 \rightarrow \mathbb{R}$ is smooth if, and only if, $f|_{q(\mathbb{R}_0)}$ and $f|_{q(\mathbb{R}_1)}$ are each smooth real functions.

For any $x \in \mathbb{R}_0 \vee \mathbb{R}_1$ with $x \neq 0$, the Frölicher space $\mathbb{R}_0 \vee \mathbb{R}_1$ has the structure of a 1-dimensional manifold locally at x , so we are interested in the Frölicher structure of $\mathbb{R}_0 \vee \mathbb{R}_1$ at $0 \in \mathbb{R}_0 \vee \mathbb{R}_1$.

We have characterized the functions on $\mathbb{R}_0 \vee \mathbb{R}_1$, so we now characterize the curves on $\mathbb{R}_0 \vee \mathbb{R}_1$.

Suppose that $c: \mathbb{R} \rightarrow \mathbb{R}_0 \vee \mathbb{R}_1$ is a structure curve on $\mathbb{R}_0 \vee \mathbb{R}_1$, with $c(t_0) = 0$, for some $t_0 \in \mathbb{R}$. Then we have the diagram

$$\mathbb{R} \xrightarrow{c} \mathbb{R}_0 \vee \mathbb{R}_1 \xrightarrow{f} \mathbb{R},$$

where $f \circ c: \mathbb{R} \rightarrow \mathbb{R}$ is a smooth real function.

Either we have $c(\mathbb{R}) \subset q(\mathbb{R}_i)$, for $i = 0, 1$, in which case c can be any smooth curve, or we have $c(\mathbb{R}) \not\subset q(\mathbb{R}_0)$ and $c(\mathbb{R}) \not\subset q(\mathbb{R}_1)$. If there exists a neighbourhood $U \subseteq \mathbb{R}$ of t_0 such that $c(U) \subseteq q(\mathbb{R}_0)$ or $c(U) \subseteq q(\mathbb{R}_1)$, then we are effectively in the first situation.

So suppose that there is no neighbourhood of t_0 whose image under c is contained entirely in $q(\mathbb{R}_0)$ or in $q(\mathbb{R}_1)$. Then any neighbourhood $U \subseteq \mathbb{R}$ of t_0 can be partitioned into three subsets

$$U = U^0 \cup U^* \cup U^1,$$

where

$$c(t) \in \begin{cases} q(\mathbb{R}_0) - \{0\}, & \text{if } t \in U^0 \\ \{0\}, & \text{if } t \in U^* \\ q(\mathbb{R}_1) - \{0\}, & \text{if } t \in U^1. \end{cases}$$

Clearly U^* is closed, and U^0, U^1 are each open. Since the image of U under c is not contained in $q(\mathbb{R}_0)$ or $q(\mathbb{R}_1)$, each of U^0 and U^1 is non-empty.

Now, since $f \circ c$ is a smooth real function, the derivatives $\frac{d(f \circ c)(t_0)}{dt}$ exist, and

$$\begin{aligned} \frac{d(f \circ c)(t_0)}{dt} &= \lim_{t \rightarrow t_0} \frac{(f \circ c)(t) - (f \circ c)(t_0)}{t - t_0} \\ &= \lim_{t \rightarrow t_0, t \in U^0} \frac{(f \circ c)(t) - (f \circ c)(t_0)}{t - t_0} \\ &= \lim_{t \rightarrow t_0, t \in U^1} \frac{(f \circ c)(t) - (f \circ c)(t_0)}{t - t_0} \\ &= \lim_{t \rightarrow t_0, t \in U^1} \frac{(f_1 \circ c)(t) - (f_1 \circ c)(t_0)}{t - t_0}. \end{aligned}$$

In the last line above, we may replace the function f by a different smooth function $f_1: \mathbb{R}_0 \vee \mathbb{R}_1 \rightarrow \mathbb{R}$, with $f_1(c(t)) = f(c(t))$, ($t \in U^1$) since any function $g: \mathbb{R}_0 \vee \mathbb{R}_1 \rightarrow \mathbb{R}$ is smooth if and only if it is smooth on each of $q(\mathbb{R}_0)$ and $q(\mathbb{R}_1)$ separately. Thus in the last line above, the values of $(f \circ c)(t)$ for $t \in U^1$ can be taken constant irrespective of the values of $(f \circ c)(t)$ for $t \in U^0$.

It is not difficult to see that

$$\lim_{t \rightarrow t_0, t \in U^0} \frac{c(t) - c(t_0)}{t - t_0} = 0,$$

and that

$$\lim_{t \rightarrow t_0, t \in U^1} \frac{c(t) - c(t_0)}{t - t_0} = 0.$$

Thus one can take f to be $f = 1_{\mathbb{R}} \vee k$, firstly, where $k = 0$, and then, secondly, take $f = k \vee 1_{\mathbb{R}}$ to get

$$\lim_{t \rightarrow t_0, t \in U^0} \frac{d}{dt} c(t_0) = 0$$

and

$$\lim_{t \rightarrow t_0, t \in U^1} \frac{d}{dt} c(t_0) = 0.$$

We conclude that $\lim_{t \rightarrow t_0} \frac{d}{dt} c(t_0) = 0$. It follows from the chain rule that, for every $f \in F_{\mathbb{R}_0 \vee \mathbb{R}_1}$, $f \circ c: \mathbb{R} \rightarrow \mathbb{R}$ is a smooth real function with the property that

$$\frac{d(f \circ c)(t_0)}{dt} = 0.$$

To summarize, a structure curve of $\mathbb{R}_0 \vee \mathbb{R}_1$ at 0 is locally a smooth curve from a subset of \mathbb{R} to

$$q(\mathbb{R}_0) \cong \mathbb{R},$$

or

$$q(\mathbb{R}_1) \cong \mathbb{R},$$

if c is locally contained in either $q(\mathbb{R}_0)$ or $q(\mathbb{R}_1)$, respectively. If $c(t_0) = 0$ and c is not locally contained in either $q(\mathbb{R}_0)$ or $q(\mathbb{R}_1)$, then we have $\frac{d(f \circ c)(t_0)}{dt} = 0$. In fact, it can easily be shown that all higher order derivatives of $f \circ c$ must be zero at t_0 too. We do not use this fact here.

Now that we know what the structure curves on $\mathbb{R}_0 \vee \mathbb{R}_1$ look like, we are ready to investigate the structure of the tangent cone at 0, $\mathbf{TC}_0(\mathbb{R}_0 \vee \mathbb{R}_1)$. Define a subset of the structure curves on $\mathbb{R}_0 \vee \mathbb{R}_1$, $C \subseteq C_{\mathbb{R}_0 \vee \mathbb{R}_1}$, by

$$C = \{c \in C_{\mathbb{R}_0 \vee \mathbb{R}_1} \mid \text{there exists } t_0 \in \mathbb{R} \text{ with } c(t_0) = 0\}.$$

We partition C into three subsets,

$$C^0 = \{c \in C : c \text{ maps locally into } q(\mathbb{R}_0) \text{ at } t_0\},$$

$$C^1 = \{c \in C : c \text{ maps locally into } q(\mathbb{R}_1) \text{ at } t_0\}, \text{ and}$$

$$C^* = C - (C^0 \cup C^1).$$

For any $c \in C$ we have a derivation

$$v_c: F_{\mathbb{R}_0 \vee \mathbb{R}_1} \rightarrow \mathbb{R},$$

given by

$$v_c(f) = \lim_{t \rightarrow t_0} \frac{(f \circ c)(t) - (f \circ c)(t_0)}{t - t_0},$$

for each $f \in F$.

In part 2 above we showed that for any $c \in C^*$, we have

$$\begin{aligned} 0 &= \lim_{t \rightarrow t_0} \frac{(f \circ c)(t) - (f \circ c)(t_0)}{t - t_0} \\ &= v_c(f). \end{aligned}$$

Thus all curves contained in C^* define the zero derivation.

Let $c \in C^0$, with $v_c(f) \neq 0$, for some $f \in F_{\mathbb{R}_0 \vee \mathbb{R}_1}$. We have the diagram

$$\mathbb{R} \xrightarrow{c} q(\mathbb{R}_0) \cong \mathbb{R} \xrightarrow{f|_{q(\mathbb{R}_0)}} \mathbb{R}.$$

Since $q(\mathbb{R}_0) \cong \mathbb{R}$, we may treat c as a real function of one variable locally at t_0 , and we have already observed that $f|_{q(\mathbb{R}_0)}$ is a real function of one variable, so we get

$$\begin{aligned} v_c(f) &= \lim_{t \rightarrow t_0} \frac{(f \circ c)(t) - (f \circ c)(t_0)}{t - t_0} \\ &= \frac{df(c(t_0))}{dx} \frac{dc(t_0)}{dt}. \end{aligned}$$

Since we are assuming that $v_c(f) \neq 0$ for some $f \in F_{\mathbb{R}_0 \vee \mathbb{R}_1}$, we conclude that

$$\frac{dc(t_0)}{dt} \neq 0.$$

Let $c_1 \in C^0$, with $c_1(t_1) = 0$. Then

$$\begin{aligned} v_{c_1}(f) &= \lim_{t \rightarrow t_1} \frac{(f \circ c_1)(t) - (f \circ c_1)(t_1)}{t - t_1} \\ &= \frac{df(c_1(t_1))}{dx} \frac{dc_1(t_1)}{dt} \\ &= \frac{df(c(t_0))}{dx} \frac{dc_1(t_1)}{dt} \\ &= \frac{df(c(t_0))}{dx} \frac{dc(t_0)}{dt} \left(\frac{dc_1(t_1)}{dt} \Big/ \frac{dc(t_0)}{dt} \right) \\ &= v_c(f) \left(\frac{dc_1(t_1)}{dt} \Big/ \frac{dc(t_0)}{dt} \right). \end{aligned}$$

Thus the vector v_c generates a 1-dimensional vector space. In a similar way, any non-zero vector defined by a curve in C^1 generates a different 1-dimensional vector space. It is easy to verify that this vector space is different from the one defined by curves in C^0 . All one needs to observe is that a structure function

can be chosen which assigns a constant value to points in $q(\mathbb{R}_1)$ irrespective of values on $q(\mathbb{R}_0)$.

Thus the tangent cone can be expressed as

$$\mathbf{TC}_0(\mathbb{R}_0 \vee \mathbb{R}_1) = V^- \vee V^+,$$

where V^- and V^+ are each distinct 1-dimensional vector spaces.

8.1.2. Example. Let \mathbb{R}' be the set \mathbb{R} with the Frölicher structure generated by the set containing the single function $F = \{f\}$ where $f(x) = |x|$. This set generates structure curves

$$C_{\mathbb{R}'} = \Gamma F = \{c: \mathbb{R} \rightarrow \mathbb{R}' \mid f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})\}.$$

Since

$$\lim_{x \rightarrow 0^-} \frac{f(x) - f(0)}{x} = -1, \text{ and } \lim_{x \rightarrow 0^+} \frac{f(x) - f(0)}{x} = 1,$$

each $c \in \Gamma F$ has the property that $c'(t_0) = 0$ if $c(t_0) = 0$. Thus $\mathbf{TC}_0\mathbb{R}' = 0$.

Now let $X = \mathbb{R}' \vee \mathbb{R}$, the wedge product obtained from \mathbb{R}' and \mathbb{R} by identifying the zeroes of each space. Following the method of Example 8.1.1, we can verify that \mathbf{TC}_0X is a 1-dimensional vector space, but X is not Euclidean near 0.

One can construct higher dimensional counter-examples in a similar way. We have shown that the tangent cone of a Frölicher space need not be a vector space, and even if it is a vector space, the Frölicher space need not look like a manifold near that point.

8.2 \mathbb{R}^m -invertible pairs

8.2.1. Definition. Let X be a Frölicher space, and let $x \in X$. An \mathbb{R}^m -invertible pair at x , which we denote $(c, f)_{t_0}$, is a pair (c, f) , where

$$c: \mathbb{R}^m \rightarrow X \text{ is smooth, with } c(t_0) = x, t_0 \in \mathbb{R}^m,$$

and $f: X \rightarrow \mathbb{R}^m$ is smooth and such that the composite $f \circ c: \mathbb{R}^m \rightarrow \mathbb{R}^m$ is a smooth, locally invertible function at t_0 .

8.2.2. Theorem. Let X be a Frölicher space and let $x \in X$. Suppose that $\mathbf{TC}_x X$ contains an m -dimensional vector subspace of $\mathbf{T}_x X$, and suppose that there exists a smooth map $c: \mathbb{R}^m \rightarrow X$, with $c(a) = x$, ($a \in \mathbb{R}^m$), which induces an injective map on the tangent cones. Then X admits an \mathbb{R}^m -invertible pair at x .

Proof. The set

$$\left\{ \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_m} \right\} \subseteq \mathbf{T}_a \mathbb{R}^m$$

forms a basis for the tangent cone (=tangent space) at $a \in \mathbb{R}^m$, and so the image of this set under the injection $c_*: \mathbf{T}_a \mathbb{R}^m \rightarrow \mathbf{TC}_x X$

$$\left\{ c_* \frac{\partial}{\partial x_1}, c_* \frac{\partial}{\partial x_2}, \dots, c_* \frac{\partial}{\partial x_m} \right\} \subseteq \mathbf{TC}_x X$$

is a linearly independent subset of $\mathbf{TC}_x X$.

We claim that there exists a set of m structure functions on X ,

$$\{f_i \in F_X : i = 1, \dots, m\},$$

with the property that

$$|J(f_1, f_2, \dots, f_m)| = \begin{vmatrix} \frac{\partial}{\partial x_1}(f_1 c)(a) & \frac{\partial}{\partial x_2}(f_1 c)(a) & \dots & \frac{\partial}{\partial x_m}(f_1 c)(a) \\ \frac{\partial}{\partial x_1}(f_2 c)(a) & \frac{\partial}{\partial x_2}(f_2 c)(a) & \dots & \frac{\partial}{\partial x_m}(f_2 c)(a) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial}{\partial x_1}(f_m c)(a) & \frac{\partial}{\partial x_2}(f_m c)(a) & \dots & \frac{\partial}{\partial x_m}(f_m c)(a) \end{vmatrix} \neq 0.$$

To see this, suppose the converse. Then for any set $\{f_i \in F_X : i = 1, \dots, m\}$, of m structure functions on X , we have $|J(f_1, f_2, \dots, f_m)| = 0$. Expanding the determinant by cofactors along the first row, we get

$$\begin{aligned} |J(f_1, f_2, \dots, f_m)| &= \frac{\partial}{\partial x_1}(f_1 c)(a)C_{11} + \frac{\partial}{\partial x_2}(f_1 c)(a)C_{12} + \dots + \frac{\partial}{\partial x_m}(f_1 c)(a)C_{1m} \\ &= 0, \end{aligned}$$

where $\{C_{11}, C_{12}, \dots, C_{1m}\}$ are the relevant cofactors in the expansion of the determinant.

In particular, for every $f_1 \in F_X$,

$$\frac{\partial}{\partial x_1}(f_1 c)(a)C_{11} + \frac{\partial}{\partial x_2}(f_1 c)(a)C_{12} + \dots + \frac{\partial}{\partial x_m}(f_1 c)(a)C_{1m} = 0. \quad (*)$$

Note that each of the cofactors $C_{11}, C_{12}, \dots, C_{1m}$ depends only on the $m-1$ functions f_2, f_3, \dots, f_m , and so we may think of each of $C_{11}, C_{12}, \dots, C_{1m}$ as being constant with respect to f_1 . Since the set

$$\left\{ c_* \frac{\partial}{\partial x_1}, c_* \frac{\partial}{\partial x_2}, \dots, c_* \frac{\partial}{\partial x_m} \right\} \subseteq \mathbf{TC}_x X$$

is linearly independent, and (*) implies that $C_{11} = 0$. Let us consider this cofactor. We have

$$C_{11} = \begin{vmatrix} \frac{\partial}{\partial x_2}(f_2 c)(a) & \frac{\partial}{\partial x_3}(f_2 c)(a) & \dots & \frac{\partial}{\partial x_m}(f_2 c)(a) \\ \frac{\partial}{\partial x_2}(f_3 c)(a) & \frac{\partial}{\partial x_3}(f_3 c)(a) & \dots & \frac{\partial}{\partial x_m}(f_3 c)(a) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial}{\partial x_2}(f_m c)(a) & \frac{\partial}{\partial x_3}(f_m c)(a) & \dots & \frac{\partial}{\partial x_m}(f_m c)(a) \end{vmatrix} = 0.$$

As before, we expand the determinant by cofactors along the first row to get

$$C_{11} = \frac{\partial}{\partial x_2}(f_2 c)(a)C_{22}^1 + \frac{\partial}{\partial x_3}(f_2 c)(a)C_{23}^1 + \cdots + \frac{\partial}{\partial x_m}(f_2 c)(a)C_{2m}^1 = 0,$$

for all $\{f_2, f_3, \dots, f_m\} \subset F_X$, where

$$C_{22}^1, C_{23}^1, \dots, C_{2m}^1$$

are the relevant cofactors in the expansion of the determinant C_{11} . Again, note that each of the cofactors $C_{22}^1, C_{23}^1, \dots, C_{2m}^1$ depends only on f_3, f_4, \dots, f_m , and so we may think of each of $C_{22}^1, C_{23}^1, \dots, C_{2m}^1$ as being constant with respect to f_2 . Since the set

$$\left\{ c_* \frac{\partial}{\partial x_2}, c_* \frac{\partial}{\partial x_3}, \dots, c_* \frac{\partial}{\partial x_m} \right\} \subseteq \mathbf{TC}_x X$$

is linearly independent, and $\frac{\partial}{\partial x_2}(f_2 c)(a) \neq 0$, for some $f_2 \in F_X$, we must have $C_{22}^1 = 0$.

Expanding C_{22}^1 and continuing in this way, we find that

$$\begin{aligned} C_{m-1, m-1}^{m-2} &= \begin{vmatrix} \frac{\partial}{\partial x_{m-1}}(f_{m-1}c)(a) & \frac{\partial}{\partial x_m}(f_{m-1}c)(a) \\ \frac{\partial}{\partial x_{m-1}}(f_m c)(a) & \frac{\partial}{\partial x_m}(f_m c)(a) \end{vmatrix} \\ &= \frac{\partial}{\partial x_{m-1}}(f_{m-1}c)(a) \cdot \frac{\partial}{\partial x_m}(f_m c)(a) - \frac{\partial}{\partial x_m}(f_{m-1}c)(a) \cdot \frac{\partial}{\partial x_{m-1}}(f_m c)(a) \\ &= 0 \end{aligned}$$

for all $f_{m-1}, f_m \in F_X$. But the set

$$\left\{ c_* \frac{\partial}{\partial x_{m-1}}, c_* \frac{\partial}{\partial x_m} \right\} \subseteq \mathbf{TC}_x X$$

is linearly independent, and $\frac{\partial}{\partial x_{m-1}}(f_{m-1}c)(a) \neq 0$, for some $f_{m-1} \in F_X$. This implies that $\frac{\partial}{\partial x_m}(f_m c)(a) = 0$ for all $f_m \in F_X$. This is a contradiction, since $c_* \frac{\partial}{\partial x_m} \neq 0$.

Now define $f: X \rightarrow \mathbb{R}^m$ by $f(x) = (f_1(x), f_2(x), \dots, f_m(x))$. We then have the following sequence of maps.

$$\mathbb{R}^m \xrightarrow{c} X \xrightarrow{f} \mathbb{R}^m.$$

Since $|J(f_1, f_2, \dots, f_m)| \neq 0$ is the determinant of the transformation

$$(f \circ c): \mathbb{R}^m \rightarrow \mathbb{R}^m,$$

we may apply the inverse function theorem for \mathbb{R}^m to $f \circ c$. This implies that there exists an open set $V \subseteq \mathbb{R}^m$ such that $f \circ c: V \rightarrow f(V) = W$ is a Frölicher space isomorphism with $f(V)$ open in \mathbb{R}^n . Hence $(c, f)_a$ is an \mathbb{R}^m -invertible pair at x . \square

Notice that if X admits an \mathbb{R}^m -invertible pair $(c, f)_{t_0}$ at $x \in X$, and there exists an open neighbourhood of x , $U \subseteq X$, onto which c maps surjectively, then we may invert c locally at x . This implies that X is locally Euclidean at x . In other words, X looks like an m -dimensional manifold at x .

Equivalently, if f is injective on an open neighbourhood of x , then we may invert f locally at $f(x)$, and again we deduce that X looks like an m -dimensional manifold at x .

8.2.3. Proposition. *Let X be a Frölicher space with a non-zero tangent cone at $x \in X$. Then X admits an \mathbb{R} -invertible pair at x .*

Proof. Since $\mathbf{TC}_x X \neq 0$, there exists some $v_c \in \mathbf{TC}_x X$ such that $v_c \neq 0$, where $c \in C_X$ with $c(t_0) = x$, for some $t_0 \in \mathbb{R}$. This curve c induces an injective map $c_*: \mathbf{T}_{t_0} \mathbb{R} \rightarrow \mathbf{TC}_x X$. Apply Theorem 8.2.2 with $m = 1$. \square

8.2.4. Definition. *A smooth map $g: X \rightarrow Y$ is called*

1. **locally surjective at $y \in Y$** if there exists an open set $V \subseteq Y$ containing y , in the underlying topology such that $g(g^{-1}(V)) = V$, and
2. **locally injective at $x \in X$** if there exists an open set $U \subseteq X$ containing x , in the underlying topology such that $g|_U$ is injective.
3. **locally open at $x \in X$** if there exists an open neighbourhood W of x in the underlying topology on X such that $g|_W$ is an open mapping.

8.2.5. Definition. *Let X be a Frölicher space, and let $(c, f)_{t_0}$ be an \mathbb{R}^m -invertible pair ($m \in \mathbb{N}$) at $x \in X$.*

1. If $m \in \mathbb{N}$ is the largest integer for which an \mathbb{R}^m -invertible pair exists at x , then we say that the pair $(c, f)_{t_0}$ is **maximal at x** .
2. If c is locally an open mapping at t_0 , then we say that the pair $(c, f)_{t_0}$ is **open at x** .

8.2.6. Proposition. *Let X be a Frölicher space, and let $(c, f)_{t_0}$ be an \mathbb{R}^m -invertible pair ($m \in \mathbb{N}$) at $x \in X$. The following are equivalent.*

1. The pair $(c, f)_{t_0}$ is open at x .
2. The mapping c is locally open at $t_0 \in \mathbb{R}^m$.
3. The mapping c is locally surjective at x .
4. The mapping f is locally injective at x .
5. The space X is locally m -dimensional Euclidean at x .

Proof. The equivalence of 1 and 2 is by definition.

It is clear from the definition that if c is locally open at t_0 , then c is locally surjective at x , so 2 implies 3.

Since the composite $f \circ c$ is locally an isomorphism (in the category **FRL**) on an open set containing t_0 , it follows from purely set-theoretic considerations that 3 and 4 are equivalent, and that in either case both c and f are, in fact, local isomorphisms at t_0 and x respectively.

It now follows that 3 implies 5, since if c is locally surjective at x , then c is a local isomorphism at t_0 , by a set-theoretic argument. In other words X is locally isomorphic to \mathbb{R}^m at x .

Finally, 5 implies 2, since if X is locally Euclidean at x , then the map c , which induces a bijective map of tangent cones at x , can be locally inverted by the inverse function theorem for \mathbb{R}^m . \square

8.2.7. Proposition. *If X admits an \mathbb{R}^m -invertible pair at $x \in X$, then there exists a Frölicher subspace $A \subseteq X$, with $x \in A$, which inherits an m -dimensional Euclidean structure from X .*

Proof. Let $(c, f)_{t_0}$ be the \mathbb{R}^m -invertible pair at x , with $c(t_0) = x$. Then we have the following diagram.

$$\mathbb{R}^m \xrightarrow{c} X \xrightarrow{f} \mathbb{R}^m.$$

There exists an open set $V \subseteq \mathbb{R}^m$ such that $f \circ c$ is invertible on V . Let $U = (f \circ c)^{-1}(V)$. This is an open set in \mathbb{R}^m which contains t_0 . Let $c(U) \subseteq X$ have the inherited Frölicher structure. The map c is locally surjective at $x \in c(U)$ in this inherited Frölicher structure, so Proposition 8.2.6 tells us that $c(U)$ is locally m -dimensional Euclidean at x . \square

8.2.8. Proposition. *Let X be a locally m -dimensional Euclidean Frölicher space at some $x_0 \in X$. There exists an \mathbb{R}^k -invertible pair, $(c_k, f_k)_{t_0}$ at $x_0 \in X$, for each integer $1 \leq k \leq m$. The \mathbb{R}^m -invertible pair is maximal.*

Proof. Since X is locally m -dimensional Euclidean at x_0 , there exists an isomorphism $\phi: U \rightarrow V$, where U is an open neighbourhood of \mathbb{R}^m , and V is an open neighbourhood of x_0 . Suppose that $\phi(t_0) = x_0$, some $t_0 \in U$. Let $B_\epsilon(t_0) \subseteq U$ be an open ball with $\overline{B_\epsilon(t_0)} \subseteq U$. Define a smooth braking function $\alpha: \mathbb{R} \rightarrow \mathbb{R}$ with the properties

- $\alpha(t) = 1$ for $t \leq \frac{\epsilon}{2}$,
- $0 < \alpha(t) < 1$ for $\frac{\epsilon}{2} < t < \epsilon$,
- $\alpha(t) = 0$ for $t \geq \epsilon$.

We can use this to define a smooth mapping

$$c: \mathbb{R}^m \rightarrow X,$$

given by

$$c(t) = \begin{cases} \phi\left(\alpha(|t-t_0|)t + (1-\alpha(|t-t_0|))t_0\right) & \text{if } t \in B_\varepsilon(t_0) \\ \phi(t_0) & \text{otherwise.} \end{cases}$$

The map c is locally an isomorphism on $B_{\frac{\varepsilon}{2}}(t_0)$. An isomorphism maps tangent cones bijectively, so we may apply Theorem 8.2.2, allowing us to construct an \mathbb{R}^m -invertible pair $(c, f)_{x_0}$, where $f: X \rightarrow \mathbb{R}^m$.

Define

$$c_k: \mathbb{R}^k \rightarrow \mathbb{R}^m$$

by

$$c_k(t_1, t_2, \dots, t_k) = (t_1, t_2, \dots, t_k, 0, 0, \dots, 0).$$

Define

$$f_k: \mathbb{R}^m \rightarrow \mathbb{R}^k$$

by

$$f_k(t_1, t_2, \dots, t_k, t_{k+1}, \dots, t_m) = (t_1, t_2, \dots, t_k).$$

Then $(c \circ c_k, f_k \circ f)_{t_0}$ is an \mathbb{R}^k -invertible pair.

The pair $(c, f)_{t_0}$ is maximal, since if there exists an \mathbb{R}^{m+1} -invertible pair $(c^{m+1}, f^{m+1})_{t_0}$, then the induced map of tangent cones

$$c_*^{m+1}: \mathbf{T}_{t_0}\mathbb{R}^{m+1} \rightarrow \mathbf{T}_{x_0}X$$

is injective. This is a contradiction, since $\mathbf{T}_{t_0}\mathbb{R}^{m+1}$ is an $(m+1)$ -dimensional vector space, while $\mathbf{T}_{x_0}X$ is an m -dimensional vector space. \square

The space $X = \mathbb{R}' \vee \mathbb{R}$ in Example 8.1.2 above admits an \mathbb{R} -invertible pair at 0 since it has a non-zero tangent cone there. We observed that X was not locally Euclidean at 0, and so this shows that the mere existence of the \mathbb{R}^k -invertible pairs in Proposition 8.2.8 is not sufficient to ensure that the space is locally Euclidean. Thus the implication of Proposition 8.2.8 cannot be reversed.

8.2.9. Theorem. *Let M be a manifold, and let $g: M \rightarrow X$ be a smooth Frölicher map that induces an injective map $g_*: \mathbf{T}_a M \rightarrow \mathbf{TC}_{g(a)}X$. Then there is an open neighbourhood U of a , such that $g(U)$ is m -dimensional Euclidean.*

Proof. There exists a local isomorphism

$$\gamma: U \rightarrow M,$$

where U is an open subset of \mathbb{R}^m . Then $g \circ \gamma: U \rightarrow g\gamma(U)$ is a locally surjective mapping which induces an injective map of the tangent cones. By Theorem 8.2.2 there exists $f: g\gamma(U) \rightarrow \mathbb{R}^m$ such that $(g\gamma, f)_{t_0}$ is an \mathbb{R}^m -invertible pair at a . Since the composite $g \circ \gamma$ is locally surjective, we conclude by Proposition 8.2.6 that $g\gamma(U)$ is locally Euclidean at $a \in M$. \square

8.3 The Inverse function theorem

The obvious inverse function theorem that one might expect to hold for Frölicher spaces would be something like the following.

If a smooth map $g: X \rightarrow Y$ induces a bijective map $g_: \mathbf{TC}_x X \rightarrow \mathbf{TC}_{g(x)} Y$ at a point $x \in X$, then there exists an open neighbourhood of $g(x)$ on which the map g may be inverted.*

Such a theorem does not exist, as the following counter-example illustrates.

8.3.1. Example. Let $X = \mathbb{R}' \vee \mathbb{R}$ be the Frölicher space defined in Example 8.1.2. The map

$$g: \mathbb{R} \rightarrow \mathbb{R}' \vee \mathbb{R}$$

given by $g(t) = t \in \mathbb{R}$ induces a bijective map of tangent cones at 0, but is not locally invertible there, since the space X is not locally Euclidean at 0.

8.3.2. Proposition. Let $X = \mathbb{R}_0 \vee \mathbb{R}_1$. Suppose $g: X \rightarrow X$ induces a bijection $g_*: \mathbf{TC}_x X \rightarrow \mathbf{TC}_{g(x)} X$ for some $x \in X$. Then g is locally a Frölicher isomorphism at $x \in X$.

Proof. We know from Example 8.1.1 that $\mathbf{TC}_x X$ is a 1-dimensional vector space at all points $x \in X$, except at $x = 0$ where $\mathbf{TC}_x X = V^- \cup V^+$, the union of two 1-dimensional vector spaces. So we may apply the usual inverse function theorem for manifolds at each $x \in X$, except when $x = 0$. We describe this case below.

If $x = 0$, then $g(x) = 0$, since otherwise $\mathbf{TC}_{g(x)} X \neq V^- \cup V^+$. Let

$$i_0: \mathbb{R}_0 \rightarrow \mathbb{R}_0 \sqcup \mathbb{R}_1$$

and

$$i_1: \mathbb{R}_1 \rightarrow \mathbb{R}_0 \sqcup \mathbb{R}_1$$

denote the inclusions into the coproduct. We summarise the situation in the following diagram

$$\begin{array}{ccccc}
 \mathbb{R}_0 & & & & \\
 & \searrow^{i_0} & & & \\
 & & \mathbb{R}_0 \sqcup \mathbb{R}_1 & \xrightarrow{q} & \mathbb{R}_0 \vee \mathbb{R}_1 & \xrightarrow{g} & \mathbb{R}_0 \vee \mathbb{R}_1 \\
 & & \nearrow_{i_1} & & & & \\
 & & \mathbb{R}_1 & & & &
 \end{array}$$

Each of the inclusions i_0, i_1 induces a bijective map on the tangent spaces. The quotient map q induces an injective map of tangent cones in this case and g induces a bijective map of tangent cones, by hypothesis. Thus each composite, $g \circ q \circ i_0$ and $g \circ q \circ i_1$ induces an injective map on the tangent cones.

By Theorem 8.2.9, there exist open sets $U_0 \subseteq \mathbb{R}_0$ and $U_1 \subseteq \mathbb{R}_1$, with $0 \in U_0$ and $0 \in U_1$ such that $gq_i_0(U_0)$ and $gq_i_1(U_1)$ are 1-dimensional Euclidean spaces.

We must have $gq_i_0(U_0) \cap gq_i_1(U_1) = \{0\}$, since otherwise g would not induce a bijective map of tangent cones. Thus we have either

$$gq_i_0(U_0) \subseteq \mathbb{R}_0 \text{ and } gq_i_1(U_1) \subseteq \mathbb{R}_1,$$

or

$$gq_i_0(U_0) \subseteq \mathbb{R}_1 \text{ and } gq_i_1(U_1) \subseteq \mathbb{R}_0.$$

The two cases are similar, so we assume the first. We may apply the inverse function theorem for real functions of one variable to each of gq_i_0 and gq_i_1 at 0, to get open sets $V_0 \subseteq gq_i_0(U_0)$ of \mathbb{R}_0 and $V_1 \subseteq gq_i_1(U_1)$ of \mathbb{R}_1 each containing 0, on which we may invert gq_i_0 and gq_i_1 . This gives us the following diagram.

$$\begin{array}{ccccc}
 U_0 & & & & V_0 \subseteq \mathbb{R}_0 \\
 & \searrow^{i_0} & & & \swarrow^{q_{i_0}} \\
 & & \mathbb{R}_0 \sqcup \mathbb{R}_1 & \xrightarrow{g} & \mathbb{R}_0 \vee \mathbb{R}_1 & \xrightarrow{g} & \mathbb{R}_0 \vee \mathbb{R}_1 \\
 & \swarrow_{i_1} & & & \nwarrow_{q_{i_1}} \\
 U_1 & & & & V_1 \subseteq \mathbb{R}_1 \\
 & & & & \xleftarrow{(gq_{i_1})^{-1}}
 \end{array}$$

The set $q_{i_0}(V_0) \cup q_{i_1}(V_1)$ is open in $\mathbb{R}_0 \vee \mathbb{R}_1$. Define

$$g^{-1}: q_{i_0}(V_0) \cup q_{i_1}(V_1) \rightarrow \mathbb{R}_0 \vee \mathbb{R}_1$$

by

$$g^{-1}(x) = \begin{cases} q_{i_0}(gq_{i_0})^{-1}(t) & \text{if } x = q_{i_0}(t) \\ q_{i_1}(gq_{i_1})^{-1}(t) & \text{if } x = q_{i_1}(t) \end{cases}$$

□

We may generalise the above theorem, to get the following.

8.3.3. Proposition. *Let $X = M \vee N$, (M, N smooth manifolds of dimension m and n , respectively), be the space defined by identifying a point in M with a point in N . Suppose $g: X \rightarrow X$ induces a bijection*

$$g_*: \text{TC}_x X \rightarrow \text{TC}_{g(x)} X$$

for some $x \in X$. Then g is locally a Frölicher isomorphism at $x \in X$.

8.4 Critical points of a smooth mapping

8.4.1. Definition. Let $g: X \rightarrow Y$ be a smooth mapping between Frölicher spaces. We say that g has a critical point at $x \in X$ if the induced map

$$g_*: \mathbf{TC}_x X \rightarrow \mathbf{TC}_{g(x)} Y$$

is not surjective. For such x , we call $g(x)$ a critical value of g .

8.4.2. Lemma. Let A be a finite subset of a manifold M . Let

$$q: M \rightarrow M/A$$

be the quotient map. If, for $a \in A$, a smooth map

$$g: M/A \rightarrow Y$$

has a critical point at $[a] \in M/A$ then $g \circ q$ has a critical point at each $x \in q^{-1}([a])$.

Proof. The composition of smooth maps

$$M \xrightarrow{q} M/A \xrightarrow{g} Y$$

induces a composite of maps on the tangent cones,

$$\mathbf{T}_a M \xrightarrow{q_*} \mathbf{TC}_{[a]} M/A \xrightarrow{g_*} \mathbf{TC}_{g([a])} Y$$

for each $a \in A$. Since g_* is not surjective, the composite

$$(g \circ q)_* = g_* \circ q_*$$

is not surjective for any $a \in A$. □

8.4.3. Proposition. Let A be a finite subset of the m -dimensional manifold M . Let

$$q: M \rightarrow M/A$$

be the quotient map. A smooth real-valued function

$$g: M/A \rightarrow \mathbb{R}$$

has a critical point at $[a] \in M/A$ if, and only if,

$$g \circ q: M \rightarrow \mathbb{R}$$

has a critical point at each $a \in A$.

Proof. Suppose that $f: M/A \rightarrow \mathbb{R}$ has a critical point at $[a]$. Applying Proposition 8.4.2, we deduce that

$$g \circ q: M \rightarrow \mathbb{R}$$

has a critical point at each $a \in A$.

Conversely, suppose that $g \circ q: M \rightarrow \mathbb{R}$ has a critical point at each $a \in A$. Let $v_c \in \text{TC}_{[a]}M/A$. One sees readily that (see Example 8.1.1)

$$\text{TC}_{[a]}M/A = \bigvee_{i=1}^k V^i$$

where each V^i is an m -dimensional vector space. Hence there exists $1 \leq i \leq k$ such that $v_c \in V^i$. Thus, for some $a \in A$, there exists some $v'_c \in \mathbb{T}_a M$ such that $v_c = q_*(v'_c)$, and so

$$\begin{aligned} f_*(v_c) &= f_*(q_*(v'_c)) \\ &= (f \circ q)_*(v'_c) \\ &= 0. \end{aligned}$$

□

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Chapter 9

Submanifold Quotients on a Manifold

In this chapter we investigate the structure of Frölicher spaces which arise from a quotient of the form M/A , where $A \subseteq M$ is a submanifold of a manifold M . Throughout the chapter A denotes a compact k -dimensional submanifold of an m -dimensional manifold M , and $q: M \rightarrow M/A$ denotes the set quotient map with M/A equipped with the quotient Frölicher structure. Of course, M/A is not a manifold, in general.

9.1 Submanifolds

In this section we prove a few technical results that will be used in subsequent sections of this chapter.

9.1.1. Proposition. *Let A be a submanifold of a smooth manifold M . Given $a \in A$, there exists an open neighbourhood U of a in M , and a diffeomorphism*

$$\phi = (\phi_1, \phi_2, \dots, \phi_m): U \rightarrow \mathbb{R}^m$$

such that $\phi(a) = \underline{0}$, and

$$\phi(x) = (\phi_1(x), \phi_2(x), \dots, \phi_k(x), \underbrace{0, \dots, 0}_{m-k \text{ times}}),$$

if and only if $x \in U \cap A$.

Proof. Let $E \xrightarrow{\pi} A$ be a tubular neighbourhood of A in M , and let $\xi: E \rightarrow M$ be the associated diffeomorphism between E and an open submanifold of M . A tubular neighbourhood can be shown to be isomorphic to the normal bundle to A in M . See Hirsch [58] for definitions and further references. Choose $a \in A$. Each fibre of the normal bundle has dimension $m - k$, and since a

tubular neighbourhood is a locally trivial family of vector spaces, there is an open neighbourhood W of a contained in $\xi(E)$ such that $\xi^{-1}(W) = V_1 \times V_2$, where V_1 is an open neighbourhood in A and V_2 is an open neighbourhood in \mathbb{R}^{m-k} . Let $\psi: Z \rightarrow \mathbb{R}^m$ be a coordinate patch containing a , small enough that $A \cap Z$ is diffeomorphic to an open neighbourhood of $\psi(a)$ in \mathbb{R}^k , and shifted if necessary, so that a is identified with the origin of \mathbb{R}^k . Let $\chi: A \cap Z \rightarrow V_3$, be this diffeomorphism where V_3 is an open set in \mathbb{R}^k .

Let $U = \xi\left(\xi^{-1}(W) \cap (\xi^{-1}(A \cap Z) \times V_2)\right)$ and define

$$\phi: U \rightarrow \mathbb{R}^m$$

by

$$\phi(x) = (\chi, 1_{\mathbb{R}^{m-k}}) \circ \xi^{-1}(x), \text{ for each } x \in U.$$

If $\phi = (\phi_1, \phi_2, \dots, \phi_m)$, then the fact that ξ identifies A with the zero section of the bundle $E \xrightarrow{\pi} A$ gives us the property that

$$\phi(a) = (\phi_1(a), \phi_2(a), \dots, \phi_k(a), \underbrace{0, \dots, 0}_{m-k \text{ times}}),$$

for each $a \in U \cap A$. □

9.1.2. Proposition. *The map*

$$q|_{M-A}: (M - A) \rightarrow (M/A - \{[a]\})$$

is a diffeomorphism of manifolds.

Proof. By definition q is a smooth bijection on $M - A$. It remains only to prove that

$$q^{-1}|_{M/A - \{[a]\}}: (M/A - \{[a]\}) \rightarrow (M - A)$$

is smooth.

Let $c: \mathbb{R} \rightarrow M/A - \{[a]\}$ be a structure curve on $M/A - \{[a]\}$, given by $c(t) = [x(t)]$. Then $q^{-1} \circ c(t) = q^{-1}[x(t)] = x(t)$. We must show that $x(t)$ is a structure curve on $M - A$.

To this end, let $f: M - A \rightarrow \mathbb{R}$ be a structure function. Then $\bar{f}: M/A - \{[a]\} \rightarrow \mathbb{R}$, given by $\bar{f}([m]) = f(m)$ is smooth, since $\bar{f} \circ q = f$. Thus,

$$\begin{aligned} f \circ q^{-1} \circ c(t) &= f(x(t)) \\ &= \bar{f}([x(t)]) \\ &= \bar{f} \circ c(t). \end{aligned}$$

But \bar{f} is a structure function and c is a structure curve on $M - A$, and so the composite is a smooth real function. □

9.2 The structure of the tangent cone

In this section we investigate the structure of the tangent cone of M/A at the identified point $[a]$, $a \in A$.

9.2.1. Lemma. *Let $\mathbf{B}_a A$ be a basis for $\mathbf{T}_a A$ contained in a basis $\mathbf{B}_a M$ of $\mathbf{T}_a M$. For each $a \in A$,*

$$q_*(\mathbf{T}_a M) = \text{span}(q_*(\mathbf{B}_a M - \mathbf{B}_a A)),$$

where $q_*: \mathbf{T}_a M \rightarrow \mathbf{TC}_{[a]} M/A$ is the induced map of tangent cones.

Proof. Let $a \in A$. It is clear that

$$\text{span}(q_*(\mathbf{B}_a M - \mathbf{B}_a A)) \subseteq q_*(\mathbf{T}_a M).$$

We show the reverse inclusion. Abusing notation somewhat, suppose that

$$\mathbf{B}_a A = \{v_{c_1}, v_{c_2}, \dots, v_{c_k}\},$$

and that

$$\mathbf{B}_a M = \{v_{c_1}, v_{c_2}, \dots, v_{c_k}, v_{c_{k+1}}, \dots, v_{c_m}\},$$

where $c_i: \mathbb{R} \rightarrow M$ are structure curves with the property that $c_i(\mathbb{R}) \subseteq A$ for $1 \leq i \leq k$. These bases can be chosen in this way, because the Frölicher structure on the submanifold A is generated by the structure curves on M that lie entirely within A . Here $\mathbf{T}_a A = \mathbf{TC}_a A$ and $\mathbf{T}_a M = \mathbf{TC}_a A$.

Let $v_c \in \mathbf{T}_a M$. Then there exist constants $\alpha_1, \alpha_2, \dots, \alpha_m \in \mathbb{R}$ such that

$$v_c = \alpha_1 v_{c_1} + \alpha_2 v_{c_2} + \dots + \alpha_m v_{c_m}.$$

Let $f \in F_{M/A}$. For each basic vector v_{c_i} , we have

$$q_*(v_{c_i})(f) = \lim_{t \rightarrow 0} \frac{(f \circ q \circ c_i)(t) - (f \circ q \circ c_i)(0)}{t}.$$

If $c_i(\mathbb{R}) \subseteq A$, then $q \circ c_i(t) = [a]$ (constant), and so

$$\lim_{t \rightarrow 0} \frac{(f \circ q \circ c_i)(t) - (f \circ q \circ c_i)(0)}{t} = 0.$$

Thus, using the linearity of q_* ,

$$q_*(v_c) = \alpha_{k+1} q_*(v_{c_{k+1}}) + \alpha_{k+2} q_*(v_{c_{k+2}}) + \dots + \alpha_m q_*(v_{c_m}).$$

□

9.2.2. Lemma. *Let $v_c \in \ker(q_*)$. Then $v_c(f) = 0$ for every $f \in F_M$ that is constant on A .*

Proof. Let $g \in F_{M/A}$. Then

$$\begin{aligned} 0 &= q_*(v_c)(g) \\ &= \lim_{t \rightarrow 0} \frac{(g \circ q \circ c)(t) - g \circ q \circ c(0)}{t} \\ &= v_c(g \circ q). \end{aligned}$$

This shows that $v_c(f) = 0$ if f is of the form $g \circ q$ for some $g \in F_{M/A}$. In section 2.6 it is shown that every function on M that is constant over A can be factored through the quotient map in this way. \square

9.2.3. Lemma. For each $a \in A$,

$$\ker(q_*) \subseteq \mathbf{T}_a A.$$

Proof. Let $a \in A$ and let $v_c \in \ker(q_*)$. This means that $v_c \in \mathbf{T}_a M$, and that $q_*(v_c) = 0$. We wish to show that $v_c \in \mathbf{T}_a A$.

Applying Lemma 9.1.1, choose a neighbourhood U of a in M , and a diffeomorphism

$$\phi = (\phi_1, \phi_2, \dots, \phi_m): U \rightarrow \mathbb{R}^m$$

such that

$$\phi(x) = (\phi_1(x), \phi_2(x), \dots, \phi_k(x), \underbrace{0, \dots, 0}_{m-k \text{ times}}),$$

for each $x \in U \cap A$, with $\phi(a) = \underline{0}$.

Now,

$$\left\{ \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_m} \right\}$$

is a basis for the tangent space $\mathbf{T}_{\phi(a)}\mathbb{R}^m$. Define

$$v_i = \phi_*^{-1}\left(\frac{\partial}{\partial x_i}\right), \quad i = 1, \dots, m.$$

Then $\{v_1, v_2, \dots, v_m\}$ is a basis for the tangent space $\mathbf{T}_a M$, and $\{v_1, v_2, \dots, v_k\}$ is a basis for the tangent space $\mathbf{T}_a A$, a vector subspace of $\mathbf{T}_a M$.

Let $\epsilon > 0$ be small enough that $B_\epsilon(\underline{0}) \subset \phi(U)$. Define a smooth braking function $\alpha: \mathbb{R} \rightarrow \mathbb{R}$ by

$$\alpha(t) = \begin{cases} 1 & \text{if } t < \frac{\epsilon^2}{4} \\ 0 & \text{if } t > \frac{3\epsilon^2}{4} \\ 0 \leq \alpha(t) \leq 1 & \text{otherwise.} \end{cases}$$

Let $D: \mathbb{R}^m \rightarrow \mathbb{R}$ be given by

$$D(t) = t_1^2 + t_2^2 + \dots + t_m^2,$$

where $\underline{t} = (t_1, t_2, \dots, t_m)$. Next, define $T: \mathbb{R}^m \rightarrow \mathbb{R}^m$ by

$$T(\underline{t}) = \left(\alpha(D(\underline{t}))t_1, \alpha(D(\underline{t}))t_2, \dots, \alpha(D(\underline{t}))t_m \right).$$

Notice that $T(\underline{t}) \in \phi(U)$ for each $\underline{t} \in \mathbb{R}^m$, and for \underline{t} sufficiently close to the origin, the mapping T is the identity.

Suppose that

$$v_c = \sum_{i=1}^m \alpha_i v_i.$$

Define $\hat{c}: \mathbb{R} \rightarrow \phi(U)$ by

$$\hat{c}(t) = T(\underbrace{0, 0, \dots, 0}_{k \text{ times}}, \alpha_{k+1}t, \dots, \alpha_m t).$$

It is clear that \hat{c} is smooth, and notice that for t sufficiently close to zero,

$$\hat{c}(t) = (\underbrace{0, 0, \dots, 0}_{k \text{ times}}, \alpha_{k+1}t, \dots, \alpha_m t),$$

since T is the identity mapping on \mathbb{R}^m near $\underline{0}$. Clearly

$$v_{\hat{c}} = \sum_{i=1}^m \alpha_i \frac{\partial}{\partial x_i}.$$

We may deduce that

$$(\phi^{-1})_*(v_{\hat{c}}) = \sum_{i=1}^m \alpha_i v_i = v_c.$$

Now, let $f \in F_M$. Define $\tilde{f}: B_\epsilon(\underline{0}) \rightarrow \mathbb{R}$ by

$$\tilde{f}(\underline{t}) = f \circ \phi^{-1} \circ T(\underbrace{0, 0, \dots, 0}_{k \text{ times}}, t_{k+1}, \dots, t_m).$$

Notice that for $D(\underline{t}) < \frac{\epsilon^2}{4}$, we have

$$\tilde{f}(t_1, t_2, \dots, t_m) = f \circ \phi^{-1}(\underbrace{0, 0, \dots, 0}_{k \text{ times}}, t_{k+1}, \dots, t_m).$$

Observe that for $i = k+1, \dots, m$, we have

$$\begin{aligned} \frac{\partial \tilde{f}(\underline{0})}{\partial x_i} &= \lim_{t \rightarrow 0} \frac{\tilde{f}(\underbrace{0, 0, \dots, 0}_{i-1 \text{ times}}, t, 0, 0, \dots, 0) - \tilde{f}(\underline{0})}{t} \\ &= \lim_{t \rightarrow 0} \frac{(f \circ \phi^{-1})(\underbrace{0, 0, \dots, 0}_{i-1 \text{ times}}, t, 0, 0, \dots, 0) - (f \circ \phi^{-1})(\underline{0})}{t} \\ &= \frac{\partial (f \circ \phi^{-1})(\underline{0})}{\partial x_i} \end{aligned}$$

Using this fact, we have

$$\begin{aligned}
 v_{\hat{c}}(\tilde{f}) &= \frac{d(\tilde{f} \circ \hat{c})(0)}{dt} \\
 &= \sum_{i=1}^m \frac{\partial \tilde{f}(\hat{c}(0))}{\partial x_i} \frac{\partial x_i(0)}{\partial t} \\
 &= \sum_{i=k+1}^m \alpha_i \frac{\partial \tilde{f}(\hat{c}(0))}{\partial x_i} \\
 &= \sum_{i=k+1}^m \alpha_i \frac{\partial (f \circ \phi^{-1})(\hat{c}(0))}{\partial x_i} \\
 &= \frac{d(f \circ \phi^{-1} \circ \hat{c})(0)}{dt} \\
 &= v_{\hat{c}}(f \circ \phi^{-1}).
 \end{aligned}$$

Also, for $D(t) > \frac{3\epsilon^2}{4}$, we have \tilde{f} constant, and in addition \tilde{f} is constant on $\phi(A \cap U) \cap B_{\epsilon}(0)$.

Thus we may extend \tilde{f} to the whole of $\phi(U)$ by defining

$$\tilde{f}(x) = \begin{cases} \tilde{f}(x) & \text{if } x \in B_{\epsilon}(0) \\ \tilde{f}(0) & \text{otherwise.} \end{cases}$$

This mapping is smooth. Now we construct a new mapping $f_1 \in F_M$ by defining

$$f_1(x) = \begin{cases} \tilde{f} \circ \phi(x) & \text{if } x \in U \\ \tilde{f}(a) & \text{otherwise.} \end{cases}$$

Again, it is easy to see that this map is smooth. Now, since f_1 is a function on M that is constant on A , we have

$$\begin{aligned}
 0 &= v_c(f_1) \\
 &= \sum_{i=1}^m \alpha_i v_i(f_1) \\
 &= \sum_{i=k+1}^m \alpha_i v_i(f_1) \quad (f_1 \text{ constant on } A), \\
 &= (\phi^{-1})_*(v_{\hat{c}})(f_1) \\
 &= \lim_{t \rightarrow 0} \frac{(f_1 \circ \phi^{-1} \circ \hat{c})(t) - (f_1 \circ \phi^{-1} \circ \hat{c})(0)}{t} \\
 &= \lim_{t \rightarrow 0} \frac{(\tilde{f} \circ \phi \circ \phi^{-1} \circ \hat{c})(t) - (\tilde{f} \circ \phi \circ \phi^{-1} \circ \hat{c})(0)}{t} \\
 &= v_{\hat{c}}(\tilde{f})
 \end{aligned}$$

$$\begin{aligned}
&= v_{\hat{c}}(f \circ \phi^{-1}) \\
&= (\phi^{-1})_*(v_{\hat{c}})(f).
\end{aligned}$$

Since $(\phi^{-1})_*(v_{\hat{c}})(f) = 0$ for each $f \in F_M$, and

$$(\phi^{-1})_*(v_{\hat{c}}) = \sum_{i=k+1}^m \alpha_i v_i,$$

by definition of \hat{c} , we must have $\alpha_{k+1} = \cdots = \alpha_m = 0$. Thus

$$v_c = \sum_{i=1}^k \alpha_i v_i \in \mathbf{T}_a A.$$

□

9.2.4. Lemma. For each $a \in A$, the set of tangent vectors $q_*(\mathbf{B}_a M - \mathbf{B}_a A)$ is linearly independent.

Proof. Suppose that

$$\mathbf{B}_a M = \{v_{c_1}, v_{c_2}, \dots, v_{c_k}, v_{c_{k+1}}, \dots, v_{c_m}\},$$

where $c_i: \mathbb{R} \rightarrow M$ are structure curves with the property that $c_i(\mathbb{R}) \subseteq A$ for $1 \leq i \leq k$, and the vectors defined by these curves form a basis for $\mathbf{T}_a A$. Then, as we observed in the proof of Lemma 9.2.1,

$$q_*(v_{c_i}) = 0, \text{ for } 1 \leq i \leq k.$$

Suppose that there exist constants $\beta_{k+1}, \beta_{k+2}, \dots, \beta_m \in \mathbb{R}$ such that

$$\beta_{k+1} q_*(v_{c_{k+1}}) + \beta_{k+2} q_*(v_{c_{k+2}}) + \cdots + \beta_m q_*(v_{c_m}) = 0.$$

Then we have

$$\begin{aligned}
0 &= q_*(\beta_{k+1} v_{c_{k+1}} + \beta_{k+2} v_{c_{k+2}} + \cdots + \beta_m v_{c_m}) \\
&= q_*(v_c), \text{ say.}
\end{aligned}$$

By Lemma 9.2.3, $v_c \in \mathbf{T}_a A$, and so $v_c = 0$, since none of the vectors

$$\{v_{c_j} | j = k+1, \dots, m\}$$

is a basis element for $\mathbf{T}_a A$. This implies that each

$$\beta_j = 0, (j = k+1, \dots, m),$$

since the

$$v_{c_j}, (j = 1, \dots, m)$$

are elements of the basis for $\mathbf{T}_a M$.

□

9.2.5. **Corollary.** For each $a \in A$,

$$q_*(\mathbf{T}_a M) = \text{span} (q_*(\mathbf{B}_a M - \mathbf{B}_a A)),$$

is an $m - k$ -dimensional vector space.

Proof. The result follows from Lemma 9.2.1 and Lemma 9.2.4. \square

9.2.6. **Corollary.**

$$\bigcup_{a \in A} \text{span} (q_*(\mathbf{B}_a M) - q_*(\mathbf{B}_a A)) \subseteq \mathbf{TC}_{[a]} M/A.$$

Proof. For each $a \in A$, the induced map q_* is injective, and

$$q_*(\mathbf{T}_a M) \subseteq \mathbf{TC}_{[a]} M/A.$$

From Lemma 9.2.1 and Corollary 9.2.5, we know that

$$\begin{aligned} q_*(\mathbf{T}_a M) &= \text{span} (q_*(\mathbf{B}_a M - \mathbf{B}_a A)) \\ &= \text{span} (q_*(\mathbf{B}_a M) - q_*(\mathbf{B}_a A)). \end{aligned}$$

Thus

$$\begin{aligned} \bigcup_{a \in A} \text{span} (q_*(\mathbf{B}_a M) - q_*(\mathbf{B}_a A)) &= \bigcup_{a \in A} q_*(\mathbf{T}_a M) \\ &\subseteq \mathbf{TC}_{[a]} M/A. \end{aligned}$$

\square

9.2.7. **Lemma.**

$$\mathbf{TC}_{[a]} M/A \subseteq \bigcup_{a \in A} \text{span} (q_*(\mathbf{B}_a M) - q_*(\mathbf{B}_a A)).$$

Proof. Let $v_c \in \mathbf{TC}_{[a]} M/A$. If $v_c = 0$, then we are done, so assume that $v_c \neq 0$.

We have

$$v_c(f) = \lim_{t \rightarrow 0} \frac{(f \circ c)(t) - (f \circ c)(0)}{t}$$

for every $f \in F_{M/A}$. Since we are assuming that $v_c \neq 0$, there must be an open interval

$$U = (-\epsilon, \epsilon), (\epsilon > 0),$$

such that

$$c(U - \{0\}) = c((-\epsilon, 0) \cup (0, \epsilon)) \subseteq M/A - \{[a]\}.$$

The quotient map $q: M \rightarrow M/A$ is a diffeomorphism on $M - A$, and so

$$(q^{-1} \circ c)(-\epsilon, 0) \text{ and } (q^{-1} \circ c)(0, \epsilon)$$

each represent parts of a smooth curve on M .

Let $\{t_n\}$, ($n \in \mathbb{N}$, $t_n \neq 0$) be a sequence of real numbers converging to 0. Notice that since $c(0) = [a]$, every open neighbourhood of $[a]$ contains

$$c(-\eta, \eta) \text{ some } 0 < \eta < \epsilon.$$

This implies that every open neighbourhood of A contains

$$(q^{-1} \circ c)(-\eta, 0) \cup (q^{-1} \circ c)(0, \eta) \text{ some } 0 < \eta < \epsilon.$$

Thus every open neighbourhood of A contains the tail of the sequence $\{(q^{-1} \circ c)(t_n)\}$.

Cover the compact manifold A by finitely many coordinate patches of M

$$\{W_p | 1 \leq p \leq P, P \in \mathbb{N}\}.$$

The union

$$\bigcup_{p=1}^P W_p \supseteq A$$

is an open neighbourhood of A , and thus contains the tail of the sequence $\{(q^{-1} \circ c)(t_n)\}$. Since there are only a finite number of neighbourhoods in this union, there is a W_p which contains infinitely many of the elements of the sequence $\{(q^{-1} \circ c)(t_n)\}$. But the closure, $\overline{W_p}$, of this W_p is a compact Hausdorff set, and is thus sequentially compact. (See Kelley [75]). This means that a subsequence of $\{(q^{-1} \circ c)(t_n)\}$ converges to a point in $\overline{W_p}$.

Denote this convergent subsequence by

$$\{(q^{-1} \circ c)(t_{n_j})\}, j \in \mathbb{N}.$$

We have

$$\lim_{j \rightarrow \infty} c(t_{n_j}) = \lim_{n \rightarrow \infty} c(t_n) = c(0) = [a].$$

This implies that

$$\lim_{j \rightarrow \infty} (q^{-1} \circ c)(t_{n_j}) \in A.$$

Define

$$a_0 = \lim_{j \rightarrow \infty} (q^{-1} \circ c)(t_{n_j}).$$

Use Proposition 9.1.1 to find a neighbourhood U of a_0 in M , and a diffeomorphism

$$\phi = (\phi_1, \phi_2, \dots, \phi_m): U \rightarrow \mathbb{R}^m$$

such that $\phi(a_0) = \underline{0}$, and

$$\phi(x) = (\phi_1(x), \phi_2(x), \dots, \phi_k(x), \underbrace{0, \dots, 0}_{m-k \text{ times}}),$$

for each $x \in U \cap A$.

Let $f \in F_{M/A}$. Then, for $t \neq 0$,

$$\begin{aligned} \frac{d(f \circ c)(t)}{dt} &= \frac{d(f \circ q \circ q^{-1} \circ c)(t)}{dt} \\ &= \frac{d(f \circ q \circ \phi^{-1} \circ \phi \circ q^{-1} \circ c)(t)}{dt} \\ &= \sum_{i=1}^m \frac{\partial(f \circ q \circ \phi^{-1})((\phi \circ q^{-1} \circ c)(t))}{\partial \phi_i} \frac{d(\phi_i \circ q^{-1} \circ c)(t)}{dt} \end{aligned}$$

Since f and c are smooth, the limit

$$\lim_{t \rightarrow 0} \frac{d(f \circ c)(t)}{dt} = \lim_{j \rightarrow \infty} \frac{d(f \circ c)(t_{n_j})}{dt}$$

exists. Since ϕ_i is smooth and hence continuous, we deduce that

$$L_i = \lim_{j \rightarrow \infty} \frac{d(\phi_i \circ q^{-1} \circ c)(t_{n_j})}{dt}$$

exists for each $i = 1, \dots, m$.

Now define the straight line $l: \mathbb{R} \rightarrow \mathbb{R}^m$ by

$$l(t) = (L_1 t, L_2 t, \dots, L_m t).$$

Use a suitable braking function (a function like $T: \mathbb{R}^m \rightarrow \mathbb{R}^m$ of Lemma 9.2.3 will do) to truncate the ends of the line l so that the resulting curve remains in $\phi(U)$, but so that l is unchanged near $t = 0$. Suppose that this truncated line, $l^*: \mathbb{R} \rightarrow \mathbb{R}^m$, is given by

$$l^*(t) = (x_1^*(t), x_2^*(t), \dots, x_m^*(t)).$$

Then $\phi^{-1} \circ l: \mathbb{R} \rightarrow M$ is a structure curve in M , and

$$\begin{aligned} q_*(v_{\phi^{-1} \circ l})(f) &= \lim_{t \rightarrow 0} \frac{(f \circ q \circ \phi^{-1} \circ l)(t) - (f \circ q \circ \phi^{-1} \circ l)(0)}{t} \\ &= \sum_{i=1}^m \lim_{j \rightarrow \infty} \frac{\partial(f \circ q \circ \phi^{-1})(l(t_{n_j}))}{\partial x_i^*} \frac{d(x_i^*(t_{n_j}))}{dt} \\ &= \sum_{i=1}^m \lim_{j \rightarrow \infty} \frac{\partial(f \circ q \circ \phi^{-1})(l(t_{n_j}))}{\partial x_i^*} \cdot L_j \\ &= \frac{d(f \circ c)(0)}{dt} \\ &= v_c(f). \end{aligned}$$

But $v_c = q_*(v_{\phi^{-1} \circ l}) \in \text{span}(q_*(\mathbf{B}_a M) - q_*(\mathbf{B}_a A))$ by Lemma 9.2.1. \square

Combining these results, we have the following theorem.

9.2.8. Theorem.

$$TC_{[a]}M/A = \bigcup_{a \in A} \text{span} (q_*(\mathbf{B}_a M) - q_*(\mathbf{B}_a A)).$$

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Appendix A

The fibration axioms in FRL

In this chapter we state and verify Baues' (see [7]) axioms for a **fibration category**. Before we investigate the axioms we define a few concepts in the next section that we shall need when verifying the fibration axioms.

A.1 Fibrations and fibre homotopy in FRL

First, we define the notion of **smooth fibration**.

A.1.1. Definition. *A smooth map*

$$E \xrightarrow{p} B$$

is called a smooth fibration if, given commutative diagram of the form

$$\begin{array}{ccc} Z & \xrightarrow{J} & E \\ (0,1_x) \downarrow & & \downarrow p \\ I \times Z & \xrightarrow{F} & B \end{array}$$

there exists a commutative diagram of the form

$$\begin{array}{ccc} Z & \xrightarrow{J} & E \\ (0,1_x) \downarrow & \nearrow G & \downarrow p \\ I \times Z & \xrightarrow{F'} & B \end{array}$$

where $F'(t, z) = F(\alpha_\epsilon(t), z)$, for some $0 < \epsilon < \frac{1}{2}$ and some smooth braking function $\alpha_\epsilon: \mathbb{R} \rightarrow \mathbb{R}$.

A.1.2. Definition. Let $p: E \rightarrow B$ be a smooth fibration. A smooth homotopy $H: I \times X \rightarrow E$ is called a **fibre homotopy over B** if $pH(t, x) = pH(0, x)$ for all $t \in I$.

A.1.3. Definition. A **fibration category** is a category \mathbf{C} together with a class of morphisms, denoted fib , called the **fibrations**, and a class of morphisms, denoted we , called the **weak equivalences** which satisfy axioms (F1), (F2), (F3) and (F4) described below. We define **trivial fibrations** to be morphisms which are both fibrations and weak equivalences.

We will verify that **FRL**, with the class of smooth fibrations and smooth homotopy equivalences defined above, is a fibration category.

A.2 Axiom (F1): Composition axiom

Isomorphisms are trivial fibrations. For any two maps

$$A \xrightarrow{f} B \xrightarrow{g} C,$$

if any two of f, g, gf are weak equivalences, so is the third. The composite of fibrations is a fibration.

We now verify this axiom for **FRL**. Note that when we verified axiom (C1) for a cofibration category, we verified that if any two of f, g and gf are weak equivalences, then so is the third.

Smooth isomorphisms are clearly weak equivalences. They are also smooth fibrations, for if $p: E \rightarrow B$ is a smooth isomorphism, and we are given a commutative diagram of the form

$$\begin{array}{ccc} Z & \xrightarrow{f} & E \\ (0,1z) \downarrow & & \downarrow p \\ I \times Z & \xrightarrow{F} & B \end{array}$$

then we may define $G: I \times Z \rightarrow E$ by $G(t, z) = p^{-1} \circ F(\alpha_\epsilon(t), z)$, for any $0 < \epsilon < \frac{1}{2}$, some smooth braking function $\alpha_\epsilon: \mathbb{R} \rightarrow \mathbb{R}$, and each $t \in I, z \in Z$. Then the following diagram clearly commutes

$$\begin{array}{ccc} Z & \xrightarrow{f} & E \\ (0,1z) \downarrow & \nearrow G & \downarrow p \\ I \times Z & \xrightarrow{F'} & B \end{array}$$

where $F'(t, z) = F(\alpha_\epsilon(t), z)$.

Finally, suppose that

$$E \xrightarrow{p_0} B \quad \text{and} \quad B \xrightarrow{p} C$$

are smooth fibrations. Suppose we are given the commutative diagram below.

$$\begin{array}{ccc}
 Z & \xrightarrow{f} & E \\
 \downarrow (0,1z) & & \downarrow p_0 \\
 & & B \\
 & & \downarrow p \\
 I \times Z & \xrightarrow{F} & C
 \end{array}$$

Since p is a smooth fibration, we have the commutative diagram,

$$\begin{array}{ccc}
 Z & \xrightarrow{f} & E \\
 \downarrow (0,1z) & & \downarrow p_0 \\
 & \nearrow G & B \\
 & & \downarrow p \\
 I \times Z & \xrightarrow{F'} & C
 \end{array}$$

where $F'(t, z) = F(\alpha_{\epsilon_1}(t), z)$ for some $0 < \epsilon_1 < \frac{1}{2}$, and some smooth braking function $\alpha_{\epsilon_1}: \mathbb{R} \rightarrow \mathbb{R}$. Since p_0 is a smooth fibration, we have the commutative diagram

$$\begin{array}{ccc}
 Z & \xrightarrow{f} & E \\
 \downarrow (0,1z) & \nearrow H & \downarrow p_0 \\
 I \times Z & \xrightarrow{G'} & B
 \end{array}$$

where $G'(t, z) = G(\alpha_{\epsilon_2}(t), z)$ for some $0 < \epsilon_2 < \frac{1}{2}$, and some smooth braking function α_{ϵ_2} .

We have

$$\begin{aligned}
 p \circ p_0 \circ H(t, z) &= p \circ G'(t, z) \\
 &= p \circ G(\alpha_{\epsilon_2}(t), z) \\
 &= F'(\alpha_{\epsilon_2}(t), z) \\
 &= F(\alpha_{\epsilon_1}(\alpha_{\epsilon_2}(t)), t) \\
 &= F''(t, z),
 \end{aligned}$$

where $F''(t, z) = F(\alpha_{\epsilon_3}(t), z)$, and $\epsilon_3 = \min(\epsilon_1, \epsilon_2)$, for the smooth braking

function $\alpha_{\epsilon_3}(t) = \alpha_{\epsilon_1}(\alpha_{\epsilon_2}(t))$. This gives us the commutative diagram

$$\begin{array}{ccc}
 Z & \xrightarrow{I} & E \\
 \downarrow (0,1z) & \nearrow H & \downarrow p_0 \\
 & & B \\
 & & \downarrow p \\
 I \times Z & \xrightarrow{F''} & C
 \end{array}$$

and so $p \circ p_0$ is a smooth fibration.

A.3 Axiom (F2): Pullback axiom

For any fibration

$$E \xrightarrow{p} B$$

and map $f: B' \rightarrow B$ there exists the pullback in \mathbf{C}

$$\begin{array}{ccc}
 B' \times_B E & \xrightarrow{I'} & E \\
 \downarrow p' & & \downarrow p \\
 B' & \xrightarrow{f} & B
 \end{array}$$

and p' is a fibration (which we sometimes call the fibration induced from p by f). Moreover:

- (a) if f is a weak equivalence, so is f' ,
- (b) if p is a weak equivalence, so is p' .

If \mathbf{C} has a final object, denoted e , then we call an object X in \mathbf{C} e -fibrant if the unique map $X \rightarrow e$ is a fibration. The axioms (F1), (F2)(b), and (F3) imply axiom (F2)(a) if all objects are e -fibrant in \mathbf{C} . (See Baues [7]). We show below that all objects in **FRL** are indeed e -fibrant, and so we do not verify part (a) of this axiom.

We now verify the pullback axiom for **FRL**. The one point space $*$ is a final object in **FRL**, and it is easy to see that all objects in **FRL** are $*$ -fibrant. Given a commutative diagram,

$$\begin{array}{ccc}
 Z & \xrightarrow{I} & E \\
 \downarrow (0,1z) & & \downarrow \\
 I \times Z & \longrightarrow & *
 \end{array}$$

define $G : I \times Z \rightarrow E$ by $G(t, z) = f(z)$ for $z \in Z$. Then G is a smooth homotopy, and the diagram

$$\begin{array}{ccc} Z & \xrightarrow{f} & E \\ (0, 1_z) \downarrow & \nearrow G & \downarrow \\ I \times Z & \longrightarrow & * \end{array}$$

commutes.

One can easily verify that pullbacks exist in **FRL**, and that the pullback $B' \times_B E$ above is a subset of $B' \times E$:

$$B' \times_B E = \{(b', e) | f(b') = p(e)\},$$

with the Frölicher subspace structure. The maps p' and f' are the usual projections onto each component of the product.

Now suppose we are given a pullback diagram like the one above. We need to show that $p_1 : B' \times_B E \rightarrow B'$, the projection onto the first component of the product, is a smooth fibration if $p : E \rightarrow B$ is. To this end, suppose we have the following commutative diagram.

$$\begin{array}{ccccc} Z & \xrightarrow{g} & B' \times_B E & \xrightarrow{p_2=f'} & E \\ (0, 1_z) \downarrow & & \downarrow p_1=p' & & \downarrow p \\ I \times Z & \xrightarrow{G} & B' & \xrightarrow{f} & B \end{array}$$

Since p is a smooth fibration, we may fill in the smooth homotopy $H : I \times Z \rightarrow E$, such that the diagram

$$\begin{array}{ccccc} Z & \xrightarrow{g} & B' \times_B E & \xrightarrow{p_2=f'} & E \\ (0, 1_z) \downarrow & & \downarrow p_1=p' & \nearrow H & \downarrow p \\ I \times Z & \xrightarrow{G'} & B' & \xrightarrow{f} & B \end{array}$$

commutes, where $G'(t, z) = G(\alpha_\epsilon(t), z)$ for some $0 < \epsilon < \frac{1}{2}$, and some smooth braking function $\alpha_\epsilon : \mathbb{R} \rightarrow \mathbb{R}$. But the right hand square of solid arrows is a pullback square, so we may factor the maps $H : I \times Z \rightarrow E$ and $G' : I \times Z \rightarrow B'$ through the product, enabling us to fill in the arrow $H' : I \times Z \rightarrow B' \times_B E$, in

the following commutative diagram.

$$\begin{array}{ccccc}
 Z & \xrightarrow{g} & B' \times_B E & \xrightarrow{p_2=f'} & E \\
 \downarrow (0,1_z) & \nearrow H' & \downarrow p_1=p' & \nearrow H & \downarrow p \\
 I \times Z & \xrightarrow{G'} & B' & \xrightarrow{f} & B
 \end{array}$$

Explicitly, H' is given by $H'(t, z) = (G'(t, z), H(t, z))$. Notice that $p_1 \circ H' = G'$, since p_1 is the projection onto the first component of the product. Also, we have

$$H'(0, z) = (G'(0, z), H(0, z)) = (p_1 \circ g(z), p_2 \circ g(z)) = g(z).$$

This shows that $p_1 : B' \times_B E \rightarrow B'$ is a smooth fibration. Thus we have verified axiom (F2).

As shown above, all objects in **FRL** are e -fibrant, thus we need now only verify part (b) of axiom (F2). We do this via the following sequence of lemmas. The proof is basically the same as the proof for topological spaces, with a few adjustments to give smooth maps, and of course the slightly modified definition of fibration. Compare James [71], for example.

A.3.1. Lemma. *Suppose we are given the commutative diagram*

$$\begin{array}{ccc}
 E & \xrightarrow{g} & E \\
 & \searrow p & \downarrow p \\
 & & B
 \end{array}$$

where p is a fibration in **FRL** and g is a smooth map homotopic to 1_E . Then there exists a smooth map $g' : E \rightarrow E$ such that gg' is fibre homotopic to 1_E .

Proof. Let $F : I \times E \rightarrow E$ be the smooth homotopy giving $1_E \simeq_F g$. Then $pF : I \times E \rightarrow B$ gives the homotopy $p \simeq_{pF} pg$. We have the commutative diagram

$$\begin{array}{ccccc}
 E & \xrightarrow{1_E} & E & \xrightarrow{p} & B \\
 \downarrow (0,1_E) & & \nearrow pF & & \\
 I \times E & & & &
 \end{array}$$

Since p is a fibration there exists the diagram

$$\begin{array}{ccccc}
 E & \xrightarrow{1_E} & E & & \\
 \downarrow (0,1_E) & \nearrow G & \downarrow p & & \\
 I \times E & \xrightarrow{pF'} & B & & (*)
 \end{array}$$

where $F'(t, -) = F(\alpha_\epsilon(t), -)$, for some $0 < \epsilon < \frac{1}{2}$.

Define $g'(e) := G(1, e)$. Then $pg'(e) = pG(1, e) = pF'(1, e) = pg(e)$. Also G is a homotopy giving $1_E \simeq_G g'$.

Now consider the homotopy $g \circ G: I \times E \rightarrow E$. We have $gG(0, -) = g$ and $gG(1, -) = gg'$. Define $H: I \times E \rightarrow E$ by

$$H(t, e) = \begin{cases} g \circ G(1 - \alpha_{\epsilon_1}(2t), e) & \text{if } 0 \leq t \leq \frac{1}{2} \\ F'(2 - \alpha'(2t), e) & \text{if } \frac{1}{2} \leq t \leq 1, \end{cases}$$

for some $0 < \epsilon_1 < \frac{1}{2}$, and where α' is a smooth braking function with the properties

- $\alpha'(t) = 1$ for $1 \leq t \leq \frac{9}{8}$,
- $\alpha'(t) = t$ for $\frac{5}{4} \leq t \leq 2$ and
- $1 \leq \alpha'(t) \leq 2$ for $1 \leq t \leq 2$.

It is clear that H is smooth. We have $H(0, -) = gg'$ and $H(1, -) = 1_E$.

Now define a smooth homotopy $K: I \times I \times E \rightarrow B$ by

$$K(t, s, e) = \begin{cases} p \circ g \circ G((1 - \alpha_{\epsilon_1}(2s))(1 - t), e) & \text{if } 0 \leq s \leq \frac{1}{2} \\ p \circ F'(1 + (1 - \alpha'(2s))(1 - t), e) & \text{if } \frac{1}{2} \leq s \leq 1. \end{cases}$$

To see that K is smooth we need only observe that for s near $\frac{1}{2}$ we have K given by either $p \circ g \circ G(0, e) = p \circ g$ or $p \circ F'(1, e) = p \circ g$.

Notice that $K(0, s, e) = p \circ H(s, e)$. We have the diagram

$$\begin{array}{ccccc} I \times E & \xrightarrow{H} & E & \xrightarrow{p} & B \\ (0, 1_{I \times E}) \downarrow & & \nearrow K & & \\ I \times I \times E & & & & \end{array}$$

Since p is a fibration, we have the diagram

$$\begin{array}{ccccc} I \times E & \xrightarrow{H} & E & & \\ (0, 1_{I \times E}) \downarrow & & \nearrow L & & \downarrow p \\ I \times I \times E & \xrightarrow{K'} & B, & & (**) \end{array}$$

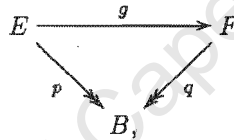
where $K'(t, s, e) = K(\alpha_{\epsilon_2}(t), s, e)$, some $0 < \alpha_{\epsilon_2} < \frac{1}{2}$.

We have

$$\begin{aligned}
 gg' &= H(0, -) \\
 &= L(0, 0, -) \\
 &\simeq L(1, 0, -) \\
 &\simeq L(1, 1, -) \\
 &\simeq L(0, 1, -) \\
 &= H(1, -) \\
 &= 1_E.
 \end{aligned}$$

Notice that from (**), and the definition of K' , all the above homotopies are fibre homotopies over B . For example, to see that $L(0, 0, -) \simeq L(1, 0, -)$ is a fibre homotopy over B , notice that from (**), $pL(t, 0, -) = K'(t, 0, -)$. But from the definition of K' we see that $K'(t, 0, -)$ does not depend on t . \square

A.3.2. Lemma. *Suppose that we are given the commutative diagram*



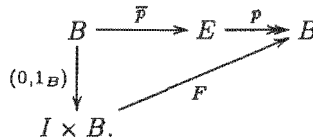
where p and q are fibrations. If g is a homotopy equivalence then g is a fibre homotopy equivalence over B .

Proof. Let $g': E \rightarrow E$ be a homotopy inverse for g . Then $gg' \simeq 1_E$, and so by Lemma A.3.1 there exists a smooth map $g'': E \rightarrow E$ such that $gg'g''$ is fibre homotopic to 1_E . Thus $g'g''$ is right fibre homotopy inverse to g .

The composite $g'g''$ is a homotopy equivalence, since g is. Apply the above procedure to $g'g''$ to get a right fibre homotopy inverse to $g'g''$. But $g'g''$ also has a left fibre homotopy inverse, namely g . Thus $g'g''$ is a fibre homotopy equivalence. Hence g is itself a fibre homotopy equivalence. \square

A.3.3. Lemma. *If $p: E \rightarrow B$ is a trivial fibration, then p is a fibre homotopy equivalence.*

Proof. Let $\bar{p}: B \rightarrow E$ be a homotopy inverse for p . Let $F: I \times B \rightarrow B$ be the homotopy giving $p\bar{p} \simeq_F 1_B$. We have the diagram



Since p is a fibration there exists the diagram

$$\begin{array}{ccc} B & \xrightarrow{\bar{p}} & E \\ (0, 1_B) \downarrow & \nearrow G & \downarrow p \\ I \times B & \xrightarrow{F'} & B. \end{array}$$

Let $s := G(1, -): B \rightarrow E$. Notice that s is a section of p , since $ps(b) = pG(1, b) = F'(1, b) = b$.

We have the diagram

$$\begin{array}{ccc} B & \xrightarrow{s} & E \\ & \searrow & \downarrow p \\ & & B. \end{array}$$

Since s is a homotopy equivalence, it is a fibre homotopy equivalence by Lemma A.3.2. To complete the proof we need only observe that the fibre homotopy inverse of s is p . We have $ps = 1_B$. Let $k: E \rightarrow B$ be the fibre homotopy inverse of s . Then we have a fibre homotopy $sk \simeq 1_E$. Thus $psk \simeq p$, and so $k \simeq p$, both homotopies being fibre homotopies. \square

A.3.4. Theorem. *Given the pullback diagram*

$$\begin{array}{ccc} B' \times_B E & \xrightarrow{f'} & E \\ p' \downarrow & & \downarrow p \\ B' & \xrightarrow{f} & B, \end{array}$$

where p is a fibration, if p is a weak equivalence then p' is a weak equivalence.

Proof. Let s be the section of p that is given by Lemma A.3.3. Define $s': B' \rightarrow B' \times_B E$ by $s'(b') = (b', sf(b'))$. Notice that s' is well-defined, since $p(sf(b')) = f(b')$ as required.

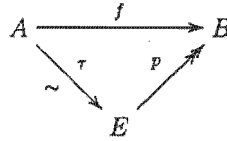
We have $p's'(b') = p'(b', sf(b')) = b'$. On the other hand

$$\begin{aligned} s'p'(b', e) &= s'(b') \\ &= (b', sf(b')) \\ &= (b', sp(e)) \\ &\simeq (b', e). \end{aligned}$$

The homotopy above is defined by deforming sp into 1_E . This homotopy is well-defined because by Lemma A.3.3 we may choose the homotopy to be a fibre homotopy over B . \square

A.4 Axiom (F3): F-Factorization axiom

For any morphism $f: A \rightarrow B$ in \mathbf{C} , there exists a commutative diagram



where τ is a weak equivalence and p is a fibration.

We verify this axiom, using Spanier's [114] notation, and following his methods for the topological situation.

A.4.1. Definition. Given a smooth map $f: A \rightarrow B$, define

1. $\bar{B} = [(I \times A) \sqcup (\{0\} \times B)] / \sim$ where \sim is the relation $(0, a) \sim (0, f(a))$. We will use $[t, a]$ and $[0, b]$ to denote points of \bar{B} corresponding to $(t, a) \in I \times A$ and $(0, b) \in \{0\} \times B$ respectively.

2. Define $\bar{i}: \bar{B} \rightarrow I \times B$ by

$$\begin{aligned}
 \bar{i}[t, a] &= (t, f(a)), \text{ for } a \in A, t \in I, \\
 \bar{i}[0, b] &= (0, b), \text{ for } b \in B.
 \end{aligned}$$

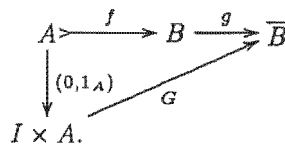
3. For $0 < \epsilon < \frac{1}{2}$, an α_ϵ -weak retracting function for f is a map $\rho: I \times B \rightarrow \bar{B}$ with the properties

$$\begin{aligned}
 \rho(0, b) &= [0, b] \text{ for } b \in B \\
 \rho(t, f(a)) &= [\alpha_\epsilon(t), a] \text{ for } t \in I, a \in A
 \end{aligned}$$

Note that if f is an inclusion map, so is \bar{i} , and an α_ϵ -weak retracting function for f is a weak retraction of $I \times B$ to a subspace which is the image of $I \times A \cup \{0\} \times B$ in \bar{B} .

A.4.2. Lemma. A map $f: A \rightarrow B$ is a smooth cofibration if and only if there exists an α_ϵ -weak retracting function for f , for some $0 < \epsilon < \frac{1}{2}$, and some braking function $\alpha_\epsilon: \mathbb{R} \rightarrow \mathbb{R}$.

Proof. Suppose f is a smooth cofibration. Let $g: B \rightarrow \bar{B}$ and $G: I \times A \rightarrow \bar{B}$ be given by $g(b) = [0, b]$ and $G(t, a) = [t, a]$. Since $G(0, a) = [0, a] = [0, f(a)] = gf(a)$, we have the following commutative diagram



Thus there exists a commutative diagram as follows

$$\begin{array}{ccc}
 B & \xrightarrow{g} & \bar{B} \\
 (0, 1_B) \downarrow & \nearrow \rho & \uparrow G' \\
 I \times B & \xleftarrow{1_f \times f} & I \times A,
 \end{array}$$

where $G'(t, a) = G(\alpha_\epsilon(t), a)$ for some braking function $\alpha_\epsilon: \mathbb{R} \rightarrow \mathbb{R}$, and some $0 < \epsilon < \frac{1}{2}$. Here $\rho(0, b) = g(b)$ and

$$\begin{aligned}
 \rho(t, f(a)) &= G'(t, a) \\
 &= G(\alpha_\epsilon(t), a) \\
 &= [\alpha_\epsilon(t), a].
 \end{aligned}$$

Conversely, given maps $g: B \rightarrow Z$ and $G: I \times A \rightarrow Z$ such that

$$\begin{array}{ccc}
 A & \xrightarrow{f} B & \xrightarrow{g} Z \\
 (0, 1_A) \downarrow & \nearrow G & \\
 I \times A & &
 \end{array}$$

commutes, define $\bar{G}: \bar{B} \rightarrow Z$ by

$$\begin{aligned}
 \bar{G}[t, a] &= G(t, a), \\
 \bar{G}[0, b] &= g(b).
 \end{aligned}$$

If, for some braking function $\alpha_\epsilon: \mathbb{R} \rightarrow \mathbb{R}$, and some $0 < \epsilon < \frac{1}{2}$, $\rho: I \times B \rightarrow \bar{B}$ is an α_ϵ -weak retracting function for f , then the map

$$F = \bar{G} \circ \rho: I \times B \rightarrow Z$$

makes the diagram

$$\begin{array}{ccc}
 B & \xrightarrow{g} & Z \\
 (0, 1_B) \downarrow & \nearrow F & \uparrow G' \\
 I \times B & \xleftarrow{1_f \times f} & I \times A
 \end{array}$$

commute, where $G'(t, a) = G(\alpha_\epsilon(t), a)$, since

$$\begin{aligned}
 F(t, f(a)) &= \bar{G} \circ \rho(t, f(a)) \\
 &= \bar{G}[\alpha_\epsilon(t), a] \\
 &= G(\alpha_\epsilon(t), a).
 \end{aligned}$$

□

A.4.3. Theorem. *Let $f: A \rightarrow B$ be a smooth cofibration, and let Y be any Frölicher space. The map $p: Y^B \rightarrow Y^A$ defined by $p(g) = g \circ f$ is a smooth fibration.*

Proof. Let $\rho: I \times B \rightarrow \bar{B}$ be an α_ϵ -weak retracting function for f (which exists by Lemma A.4.2). Then, ρ defines a map

$$\rho': Y^{\bar{B}} \rightarrow Y^{I \times B}$$

by $\rho'(g) = g \circ \rho$ for $g: \bar{B} \rightarrow Y$. But **FRL** is Cartesian closed, so we have $Y^{I \times B} \cong (Y^B)^I$. Also,

$$Y^{\bar{B}} \cong \{(g, G) \in Y^B \times Y^{I \times A} \mid g \circ f = G(0, -)\}.$$

In other words, a map $h: \bar{B} \rightarrow Y$ can be broken up into a map $g: B \rightarrow Y$ and a map $G: I \times A \rightarrow Y$ such that $g \circ f(a) = G(0, a)$.

Note that for $(g, G) \in Y^{\bar{B}}$,

$$\begin{aligned} [\rho'(g, G)](0, b) &= g(b) \\ [\rho'(g, G)](t, f(a)) &= [(g, G)]\rho(t, f(a)) \\ &= [(g, G)]\alpha_\epsilon(t, a) \\ &= G(\alpha_\epsilon(t), a). \end{aligned}$$

To show $p: Y^B \rightarrow Y^A$ a smooth fibration, suppose that we have the commutative diagram

$$\begin{array}{ccc} Z & \xrightarrow{h} & Y^B \\ (0, 1_Z) \downarrow & & \downarrow p \\ I \times Z & \xrightarrow{F} & Y^A \end{array}$$

Define $s: Z \rightarrow Y^{I \times A}$ by $s(z)(t, a) = F(t, z)(a)$.

Now define $G: I \times Z \rightarrow Y^B$ by

$$G(t, z)(b) = \left(\underbrace{\rho'(h(z))}_{\in Y^B}, \underbrace{s(z)}_{\in Y^{I \times A}} \right)(t, b).$$

One shows readily that $h(z)(f(a)) = s(z)(0, a)$. The map G is a composite of smooth maps, followed by an evaluation, so it is smooth. Note that from above,

$$\begin{aligned} G(0, z)(b) &= \left(\rho'(h(z), s(z)) \right)(0, b) \\ &= h(z)(b), \end{aligned}$$

and

$$\begin{aligned}
 (p \circ G(t, z))(a) &= (G(t, z))(f(a)) \\
 &= (\rho'(h(z), s(z)))(t, f(a)) \\
 &= (h(z), s(z))(\alpha_\epsilon(t), a) \\
 &= s(z)(\alpha_\epsilon(t), a) \\
 &= F(\alpha_\epsilon(t), z)(a) \\
 &= (F(\alpha_\epsilon(t), z))(a)
 \end{aligned}$$

Thus we have the following commutative diagram,

$$\begin{array}{ccc}
 Z & \xrightarrow{h} & Y^B \\
 (0, 1z) \downarrow & \nearrow G & \downarrow p \\
 I \times Z & \xrightarrow{F'} & Y^A
 \end{array}$$

where $F'(t, z) = F(\alpha_\epsilon(t), z)$ for the smooth braking function $\alpha_\epsilon: \mathbb{R} \rightarrow \mathbb{R}$ determined by the α_ϵ -weak retracting function $\rho: I \times B \rightarrow \bar{B}$. Thus we have shown that p is a smooth fibration. \square

A.4.4. Corollary. For any space Y , let $p: Y^I \rightarrow Y$ be the map $p(\omega) = \omega(0)$, for $\omega: I \rightarrow Y$. Then p is a smooth fibration.

Proof. Since the inclusion map $0 \hookrightarrow I$ is a smooth cofibration, the result follows from Lemma A.4.3 with the observation that $Y^{\{0\}} \cong Y$. \square

A.4.5. Definition. Let $f: X \rightarrow Y$. Define the **mapping fibre** of f by

$$P_f = \{(x, \omega) \in X \times Y^I \mid \omega(0) = f(x)\}.$$

A.4.6. Definition. Let $p': Y^I \rightarrow Y$ be the smooth fibration defined by $p'(\omega) = \omega(0)$ (see Corollary A.4.4), and let $p: P_f \rightarrow X$ be the smooth fibration induced from p' by f . (In other words, $p(x, \omega) = x$. We call p the **mapping path fibration**.

There is a smooth section $s: X \rightarrow P_f$ of p defined by $s(x) = (x, \omega_{f(x)})$, where $\omega_{f(x)}$ is the constant path in Y at $f(x)$.

There is a smooth map $p'': P_f \rightarrow Y$ defined by setting $p''(x, \omega) = \omega(1)$.

A.4.7. Theorem. Given a map $f: X \rightarrow Y$, there is a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow^s & \nearrow_{p''} \\ & & P_f \end{array}$$

such that

1. there is a fibre homotopy $1_{P_f} \simeq s \circ p$ over X ,
2. p'' is a smooth fibration.

Proof. The triangle commutes by definition of the maps involved.

1. We prove this part in two steps. Define $F_1: I \times P_f \rightarrow P_f$ by

$$F_1(s, (x, \omega)) = (x, \omega_{1-s}),$$

where $\omega_{1-s}(t) = \omega((1-s)\alpha_\epsilon(t))$, for some $0 < \epsilon < \frac{1}{2}$, and some smooth braking function $\alpha_\epsilon: \mathbb{R} \rightarrow \mathbb{R}$. Note that $\omega_{1-s}(t)$ is independent of t for $t \in [0, \epsilon]$ and for $t \in (1-\epsilon, 1]$. Then F_1 is a smooth fibre homotopy from $F_1(0, -): P_f \rightarrow P_f$, given by

$$F_1(0, (x, \omega)) = (x, \omega(\alpha_\epsilon(-))),$$

to $s \circ p$.

Next, define $F_2: I \times P_f \rightarrow P_f$ by

$$F_2(s, (x, \omega)) = (x, \omega'_s),$$

where $\omega'_s(t) = \omega((1-s)t + s\alpha_\epsilon(t))$.

Note that F_2 is a smooth fibre homotopy from 1_{P_f} to $F_2(1, -): P_f \rightarrow P_f$ given by

$$F_2(1, (x, \omega)) = (x, \omega(\alpha_\epsilon(-))).$$

Note that as $t \rightarrow 0$, then $(1-s)t + s\alpha_\epsilon(t) \rightarrow 0$, and as $t \rightarrow 1$, then $(1-s)t + s\alpha_\epsilon(t) \rightarrow 1$, and so ω'_t is a smooth function on the flattened unit interval.

2. To show that p'' is a smooth fibration, suppose there exists a commutative diagram.

$$\begin{array}{ccc} W & \xrightarrow{g} & P_f \\ (0, 1_\omega) \downarrow & & \downarrow p'' \\ I \times W & \xrightarrow{G} & Y \end{array}$$

Then $g: W \rightarrow P_f \subseteq X \times Y^I$ may be decomposed into two maps $g': W \rightarrow X$ and $g'': W \rightarrow Y^I$ such that $g''(w)(0) = fg'(w)$. In other words $g(w) = (g'(w), g''(w))$.

Define $H: I \times W \rightarrow P_f$ by

$$H(t, w) = (g'(w), \bar{g}(t, w)),$$

where $\bar{g}(t, w) \in Y^I$ is defined by

$$\bar{g}(t, w)(s) = \begin{cases} g''(w)\left(\frac{2s}{2-t}\right) & \text{if } 0 \leq 2s \leq 2-t \leq 2, w \in W \\ G\left(\alpha_{k_\epsilon}\left(2\left(\alpha_\epsilon(s) + \frac{t}{2} - 1\right)\right), w\right) & \text{if } 1 \leq 2-t \leq 2s \leq 2, w \in W, \end{cases}$$

and k_ϵ is the constant given by Lemma 2.8.3. Note that $s \in I$. Observe now that \bar{g} is clearly smooth at $s = 0$, since $g''(\omega) \in Y^I$, and at $s = 1$ because of the presence of the braking function α_ϵ . One can now prove that $\bar{g}(t, w)$ is smooth in much the same way as one proves the map $H: I \times I_f \rightarrow W$ smooth in Lemma 6.3.6.

We now have the commutative diagram,

$$\begin{array}{ccc} W & \xrightarrow{g} & P_f \\ (0, 1_w) \downarrow & \nearrow H & \downarrow p'' \\ I \times W & \xrightarrow{G'} & Y \end{array}$$

where $G'(t, w) = G(\alpha_\epsilon(t), w)$ and α_ϵ is a smooth braking function defined by the construction of H above. This completes the proof. □

This completes the verification of the F-Factorization axiom.

A.5 Axiom (F4): Axiom on cofibrant models

For every object X in \mathbf{C} , there exists a cofibrant model F with a trivial fibration

$$F \xrightarrow{\sim} X.$$

We show that every object, X , in \mathbf{FRL} is cofibrant. Let

$$E \xrightarrow{\sim} X$$

be a trivial fibration. Then there exists a smooth map $\bar{p}: X \rightarrow E$ such that $p \circ \bar{p} \simeq_G 1_X$, where $G: I \times X \rightarrow X$ is a smooth homotopy with $G(0, x) = p \circ \bar{p}(x)$ and $G(1, x) = 1_X$.

We have the following commutative diagram.

$$\begin{array}{ccc} X & \xrightarrow{\bar{p}} & E \\ (0, 1_x) \downarrow & & \sim \downarrow p \\ I \times X & \xrightarrow{G} & X \end{array}$$

But p is a smooth fibration, so we may construct a smooth homotopy $H: I \times X \rightarrow E$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\bar{p}} & E \\ (0,1x) \downarrow & \nearrow H & \downarrow \sim p \\ I \times X & \xrightarrow{G'} & X \end{array}$$

commutes, where $G'(t, x) = G(\alpha_\epsilon(t), x)$ for some $0 < \epsilon < \frac{1}{2}$, and some smooth braking function $\alpha_\epsilon: \mathbb{R} \rightarrow \mathbb{R}$. Define $s: X \rightarrow E$ by $s(x) = H(1, x)$. Then

$$\begin{aligned} ps(x) &= pH(1, x) \\ &= G'(1, x) \\ &= G(\alpha_\epsilon(1), x) \\ &= G(1, x) \\ &= x. \end{aligned}$$

Thus every object in **FRL** is its own cofibrant model.

This concludes the verification of the axioms for a fibration category.

Appendix B

The Axioms for a J-category

In this appendix we investigate the **cube axiom**, as defined by Doeraene [32]. A category with an object $*$ that is both initial and final and that satisfies the cube axiom and the cofibration and fibration axioms is called a **J-category** by Doeraene [32].

A J-category has sufficient structure for a reasonable notion of Lusterick-Schnirelmann category to be defined from the axioms. The paper by Doeraene [32] derives many results in this setting that correspond to known topological results. Thus, if one knows that **FRL** satisfies the necessary axioms for a J-category, one automatically has access to all of Doeraene's results. We provide a proof of only a special case of the cube axiom. We believe that a full proof exists, but the proof would require the result that the pullback by fibrations of a cofibration is a cofibration. This is a fairly subtle point that is beyond the scope of this appendix.

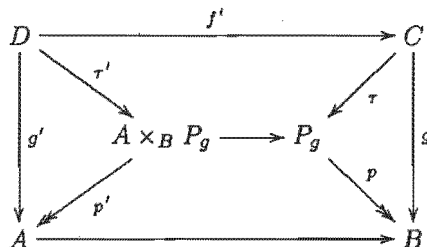
First we need to define the notions of **homotopy pullbacks** and **homotopy pushouts**. See also Grandis [49]. The following definition is taken from the paper by Doeraene [32].

B.0.1. Definition. 1. *A commutative diagram*

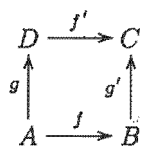
$$\begin{array}{ccc} D & \xrightarrow{f'} & C \\ \downarrow g' & & \downarrow g \\ A & \xrightarrow{f} & B \end{array}$$

is called a homotopy pullback if, for some F-factorization $g = p\tau$, the

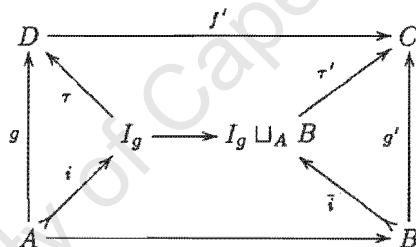
induced map τ' below is a weak equivalence.



2. Dually, a commutative diagram



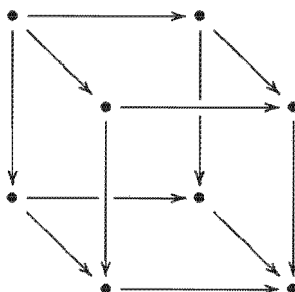
is called a homotopy pushout if, for some C -factorization $g = \tau i$, the induced map τ' below is a weak equivalence.



See also Grandis [49] for work on homotopy pullbacks and related constructions in an axiomatic setting.

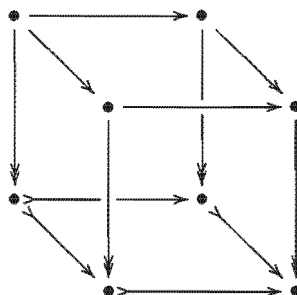
B.1 The Cube Axiom

The cube axiom, as given by Doeraene [32] is as follows. Suppose we are given a commutative cube in \mathcal{C} ,



where the bottom face is a homotopy pushout, the vertical faces are homotopy pullbacks. Then the top face is a homotopy pushout.

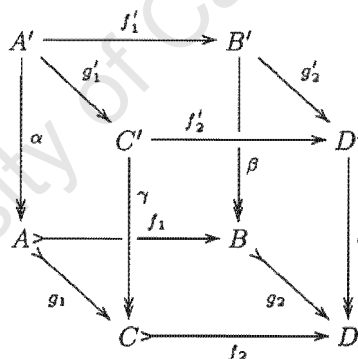
It is proved by Doeraene [32], (Theorem A.1), that if a category \mathbf{C} is a fibration category and a cofibration category, then \mathbf{C} satisfies the cube axiom if and only if for any commutative cube in \mathbf{C} of the following form,



if the bottom face is the pushout of two cofibrations, the vertical maps are fibrations, and the vertical faces are pullbacks, then the top face is a homotopy pushout.

We prove the following special case of the above result.

B.1.1. Theorem. *Suppose we are given the following commutative diagram in FRL.*



*Suppose the bottom face is the pushout of two cofibrations, the vertical maps are fibrations, and the vertical faces are pullbacks. If $D = *$, the terminal object, then the top face is the pushout of two cofibrations.*

Proof. We proceed as far as possible without the assumption that $D = *$. This proof relies on Lemma 4.6.2. In other words we assume implicitly that f_1, f_2, g_1, g_2 are all inclusions.

Since the bottom face is pushout, $D = C \sqcup_A B$, and $f_2 : C \rightarrow C \sqcup_A B$ is given by $f_2(c) = [c]$, for $c \in C$, where the square bracket denotes the equivalence class under the identification defining the adjunction. Similarly, $g_2 : B \rightarrow C \sqcup_A B$ is given by $g_2(b) = [b]$ for $b \in B$.

The front face is a pullback, and so $C' = C \times_D D'$, and $\gamma: C \times_D D' \rightarrow C$ and $f'_2: C \times_D D' \rightarrow D'$ are the projections onto the first and second coordinates of the product, respectively.

The right face is a pullback, and so $B' = B \times_D D'$, and $\beta: B \times_D D' \rightarrow B$ and $g'_2: B \times_D D' \rightarrow D'$ are the projections onto the first and second coordinates of the product, respectively.

The left face is a pullback, so $A' = A \times_C C'$. By the pasting properties of pullbacks,

$$\begin{aligned} A \times_C C' &= A \times_D D' \\ &= \{(a, d') \in A \times D' : f_2 g_1(a) = \delta(d')\} \\ &= \{(a, d') \in A \times D' : g_2 f_1(a) = \delta(d')\} \end{aligned}$$

The map $\alpha: A' \rightarrow A$ is given by $\alpha(a, c, d') = a$, for $(a, c, d') \in A'$. The map $g'_1: A' \rightarrow C'$ is given by $g'_1(a, c, d') = (c, d')$, for $(a, c, d') \in A'$.

The rear face is a pullback, so $A' = A \times_B B'$. Then

$$\begin{aligned} A \times_B B' &= \{(a, b') \in A \times B' : f_1(a) = \beta(b')\} \\ &= \{(a, b, d') \in A \times B \times D' : f_1(a) = \beta(b') = b, g_2(b) = \delta(d')\} \end{aligned}$$

The map $\alpha: A' \rightarrow A$ is given by $\alpha(a, b, d') = a$, for $(a, b, d') \in A'$. The map $f'_1: A' \rightarrow B'$ is given by $f'_1(a, b, d') = (b, d')$, for $(a, b, d') \in A'$.

Let us now consider the top face of the cube. We may form the pushout of the maps f'_1 and g'_1 , obtaining the following commutative diagram,

$$\begin{array}{ccc} A' & \xrightarrow{f'_1} & B' \\ \downarrow g'_1 & & \downarrow g'_2 \\ C' & \xrightarrow{f'_2} & D' \end{array} \quad \begin{array}{c} \nearrow i \\ C' \sqcup_{A'} B' \\ \searrow \tau \end{array}$$

where $\tau: C' \sqcup_{A'} B' \rightarrow D'$ is the unique map induced by the pushout. We shall show that τ is an isomorphism of Frölicher spaces, by defining a smooth inverse to τ .

Before we construct this inverse, let us consider the following spaces. Note that

$$D \times D' = (C \sqcup_A B) \times D' = (C \times D') \sqcup_{A \times D'} (B \times D'),$$

where we form the identification $(c, d') = (b, d')$ if and only if $c = g_1(a)$ and $b = f_1(a)$, in the adjunction on the right. The fact that the spaces above are isomorphic follows from the fact that **FRL** is a Cartesian closed category, and so the product of a quotient and identity map is a quotient map. Note also that

$$C' \sqcup_{A'} B' = (C \times_D D') \sqcup_{A'} (B \times_D D'). \quad (*)$$

We assume from here on that $D = \{*\}$. Before we define the map $\bar{\tau}: D' \rightarrow C' \sqcup_{A'} B'$, define

$$\rho: (C \times D') \sqcup_{A \times D'} (B \times D') \rightarrow (C \times_D D') \sqcup_{A'} (B \times_D D')$$

by $\rho([c, d']) = i(c, d')$ for $(c, d') \in C \times D'$ or $\rho([b, d']) = j(b, d')$ for $(b, d') \in B \times D'$. One can easily verify that ρ is well defined, and consistent with the various identifications that are made.

To see that ρ is smooth, let $f: (C \times_D D') \sqcup_{A'} (B \times_D D') \rightarrow \mathbb{R}$ be a structure function on $(C \times_D D') \sqcup_{A'} (B \times_D D')$. Consider the following composite map.

$$(C \times D') \sqcup_{A \times D'} (B \times D') \xrightarrow{\rho} (C \times_D D') \sqcup_{A'} (B \times_D D') \xrightarrow{f} \mathbb{R}.$$

To see that $f \circ \rho: (C \times D') \sqcup_{A \times D'} (B \times D') \rightarrow \mathbb{R}$ is smooth, note that for $(c, d') \in C \times D'$,

$$f(\rho([c, d'])) = fi(c, d'),$$

and $f \circ i$ is smooth. For $(b, d') \in B \times D'$,

$$f(\rho([b, d'])) = fj(b, d'),$$

and $f \circ j$ is smooth. Thus $f \circ \rho$ is smooth on each component of the coproduct defining $(C \times D') \sqcup_{A \times D'} (B \times D')$ and is therefore smooth.

We now define (using $(*)$)

$$\bar{\tau}: D' \rightarrow C' \sqcup_{A'} B'$$

by $\bar{\tau} = \rho \circ (\delta \times 1)$, where $\delta \times 1: D' \rightarrow D \times D'$. The map $\bar{\tau}$ is smooth, as it is the composite of smooth maps. We need only verify that it is, in fact, an inverse for τ . To this end, let $d' \in D'$. Then

$$\begin{aligned} (\tau \circ \bar{\tau})(d') &= \tau(\rho \circ (\delta \times 1)(d')) \\ &= \tau(\rho([\delta(d'), d'])) \end{aligned}$$

Now, $\delta(d') \in D = C \sqcup_A B$ is of the form either $f_2(c)$ or $g_2(b)$ for some $c \in C$ or $b \in B$. Consider the first possibility (the other case is similar). We then have

$$\begin{aligned} \tau(\rho([\delta(d'), d'])) &= \tau(\rho([f_2(c), d'])) \\ &= \tau(i(c, d')) \\ &= d' \end{aligned}$$

On the other hand, consider an element of $(C \times_D D') \sqcup_{A'} (B \times_D D')$ of the form $i(c, d')$. The other case is similar. Then

$$\begin{aligned} (\bar{\tau} \circ \tau)(i(c, d')) &= \bar{\tau}(d') \\ &= i(c, d'), \end{aligned}$$

as we saw in the above calculation. This shows that $\bar{\tau}$ is an inverse for τ .

We have shown that the top face of the cube is a pushout square. To complete the proof of this theorem, we need to show that the top face is a pushout of two cofibrations, and then that will imply that the top face is a homotopy pushout, as required by our special case of the cube axiom. We will show that $g'_1: A' \rightarrow C'$ is a cofibration. The proof that $f'_1: A' \rightarrow B'$ is a cofibration is similar.

Recall that $A' = \{(a, c, d') \in A \times C \times D' : g_1(a) = c, f_2(c) = \delta(d')\}$. Since we always have $c = g_1(a)$, we can think of A' as

$$A' = \{(a, d') \in A \times D' : f_2(g_1(a)) = \delta(d')\}.$$

If we think of A' in this way, then the map $g'_1: A' \rightarrow C'$ is given by

$$g'_1(a, d') = (g_1(a), d') \text{ for all } (a, d') \in A'.$$

In other words, we are thinking of A' as the pullback given by

$$\begin{array}{ccc} A' & \xrightarrow{f'_2 \circ g'_1} & D' \\ \downarrow \alpha & & \downarrow \delta \\ A & \xrightarrow{f_2 \circ g_1} & D \end{array}$$

(or use the fact that pullbacks of pullbacks are pullbacks). Thus $A' = A \times_D D'$.

Before we continue, recall that, since **FRL** is a Cartesian closed category, given two Frölicher spaces W and Y , the function space W^Y has a natural smooth structure, and a map $\xi: X \rightarrow W^Y$, where X is some Frölicher space, is smooth if and only if the induced map $\tilde{\xi}: X \times Y \rightarrow W$ is smooth, where $\tilde{\xi}(x, y) = [\xi(x)](y)$. We will use this fact in proving that g'_1 is a cofibration.

Suppose that we are given the following commutative diagram, (where the fibred products such as $A \times_D D'$ are equal to the usual products such as $A \times D'$ because $D = *$)

$$\begin{array}{ccc} A \times D' & \xrightarrow{g'_1} & C \times D' \xrightarrow{f} Z \\ \downarrow (0, 1_{A \times D'}) & & \nearrow G \\ I \times (A \times D') & & \end{array}$$

We must first show the existence of a commutative diagram of the form

$$\begin{array}{ccc} C \times D' & \xrightarrow{f} & Z \\ \downarrow (0, 1_{C \times D'}) & & \nearrow F \\ I \times (C \times D') & \xleftarrow{1_I \times g'_1} & I \times (A \times D') \end{array} \quad \begin{array}{c} \uparrow G' \\ Z \end{array}$$

where $G'(t, a, d') = G(\alpha_\epsilon(t), a, d')$ for $(t, a, d') \in I \times (A \times D')$.

We know that $g_1: A \rightarrow C$ is a cofibration, and by the Cartesian closedness of **FRL** we have the following diagram,

$$\begin{array}{ccc} A & \xrightarrow{g_1} & C & \xrightarrow{\bar{f}} & Z^{D'} \\ (0,1_A) \downarrow & & \nearrow \bar{G} & & \\ I \times A & & & & \end{array}$$

where, for any $c \in C$, $\bar{f}(c): D' \rightarrow Z$ is given by

$$(\bar{f}(c))(d') = f(c, d'), \text{ for } d' \in D',$$

and for any $(t, a) \in I \times A$, $\bar{G}(t, a): D' \rightarrow Z$ is given by

$$(\bar{G}(t, a))(d') = G(t, a, d'), \text{ for } d' \in D'.$$

Thus there exists the following commutative diagram in **FRL**.

$$\begin{array}{ccc} C & \xrightarrow{\bar{f}} & Z^{D'} \\ (0,1_C) \downarrow & \nearrow \bar{F} & \uparrow \bar{G}' \\ I \times C & \xleftarrow{1_I \times g_1} & I \times A \end{array}$$

where $\bar{G}'(t, a) = \bar{G}(\alpha_\epsilon(t), a)$, for $(t, a) \in I \times A$, and some braking function α_ϵ , $0 < \epsilon < \frac{1}{2}$.

We now define $F: I \times C \times D' \rightarrow Z$ by

$$F(t, c, d') = (\bar{F}(t, c))(d'), \text{ for } (t, c, d') \in I \times (C \times_D D').$$

We have

$$\begin{aligned} F(0, c, d') &= (\bar{F}(0, c))(d') \\ &= (\bar{f}(c))(d') \\ &= f(c, d'), \end{aligned}$$

for $(c, d') \in C \times D'$, and

$$\begin{aligned} F(t, g_1'(a, d')) &= (\bar{F}(t, g_1(a)))(d') \\ &= (\bar{G}'(t, a))(d') \\ &= (\bar{G}(\alpha_\epsilon(t), a))(d') \\ &= G(\alpha_\epsilon(t), a, d'), \end{aligned}$$

for $t \in I$, and $(a, d') \in A \times D'$. This shows that the required commutativity properties of F are satisfied, and so the proof is complete. \square

\square

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