THREE-FIELD MIXED FINITE ELEMENT APPROXIMATIONS FOR PROBLEMS IN ELASTICITY

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Declaration

The work described in this dissertation was carried out in the Centre for Research in Computational and Applied Mechanics (CERECAM), University of Cape Town (UCT).

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Abstract

This thesis is concerned with three-field mixed methods for elasticity (often referred to as Hu-Washizu formulations) in which the variables are, for small-strain problems, the displacement, stress and strain. For problems in nonlinear elasticity the corresponding variables are the displacement, first Piola-Kirchhoff stress, and deformation gradient. Of particular interest is the design and analysis of mixed formulations that are uniformly stable in the incompressible limit. The first part of the thesis deals with problems in linear elasticity. Lamichhane, Reddy and Wohlmuth (Numer. Math., 104 (2006)) have shown that the conditions for stability and uniform convergence include an ellipticity condition and, secondly, a condition that the displacement together with a discrete pressure, suitably defined, constitute a stable Stokes pair. The latter condition implies that the inf-sup condition for the three-field formulation is satisfied. In the thesis, families of new stable mixed elements are generated by the following approach. First, a stable Stokes pair is chosen. Then, the space of discrete stresses is defined such that the associated discrete pressure corresponds to that of the Stokes pressure. The space of strains is defined such that it forms a superset of the space of stresses. The final task is that of showing that the spaces chosen in this way satisfy the discrete ellipticity condition. A number of new families of mixed elements are designed and analyzed in this way, and numerical examples in two and three space dimensions are presented to illustrate the theory. The second part of the thesis comprises a short chapter in which the displacement-dilatation-pressure formulation of Taylor *(Int. J. Numer. Meth. Engng,* $47\,(2000)$*)* is shown to be a special case of the general three-field formulation, and is then shown to be uniformly convergent. The final part of the thesis is concerned with the extension of the earlier approach to problems of nonlinear elasticity. The problem considered is the incremental or linearized version, of the kind that forms part of a Newton-Raphson process in numerical implementations, with the unknown variables being the increments in displacement, first Piola-Kirchhoff stress, and deformation gradient. In the discrete formulation the elasticity tensor (that is, the second derivative of the strain energy with respect to deformation gradient) is approximated by its mean value on each element. Conditions are established for the resulting incremental formulation to be stable and uniformly convergent, assuming that the continuous problem is stable. The analysis is illustrated through selected numerical examples.

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CHAPTER 1

INTRODUCTION

1.1 Background and aim of the project

In the present work we are interested in problems that originate in solid mechanics, and which are governed by systems of PDEs for problems in linear and nonlinear elasticity. For such problems the finite element method has been a major tool for obtaining approximate solutions. In fact, much of the early development of the finite element method took place in the context of problems in solid mechanics (see for example [49]). There remain many challenges, though, and in most cases it is important that these challenges be addressed by a combination of computational studies with detailed mathematical analysis. For problems in elasticity a problem that has received attention for some time is that associated with obtaining robust solutions to problems in the incompressible limit. A similar challenge applies in the case of problems involving incompressible fluids. For problems in elasticity, the use of low-order elements in standard displacement formulations leads to poor performance: either locking in the incompressible limit, or poor approximation, or both. An important approach aimed at remedying these shortcomings is to work with mixed finite element approximations, in which additional variables such as the pressure, stress and strain are retained as unknown variables. These methods have the advantage not only of circumventing locking-related problems referred to, provided that they are properly designed, but they also give directly approximations for the other variables such as stress, which are of physical importance. For a review of various relevant mixed and related formulations, see [10, 48].

A popular mixed approach is a three-field formulation, often referred to as the Hu-Washizu formulation [34, 47], though it had been proposed earlier by Fraeijs de Veubeke [30, 31]. In this approach the independent variables are, for problems in linear elasticity, the displacement, stress and strain. While a host of methods have been proposed as special cases of the three-field formulation, results on stability and convergence have been piecemeal, and in many cases absent. However, Lamichhane, Reddy and Wohlmuth [38] have carried out a detailed analysis of finite element approximations based on the three-field formulation. Conditions for uniform convergence have been established for a modified form of the problem, which under certain weak conditions is equivalent to the standard formulation.

The general results include as special cases a number of established mixed and enhanced formulations (such as those proposed, for example, in [37, 42])). The relationships between various established and new elements is set out in [26]. The two-field Hellinger-Reissner formulation, with variables displacement and stress, can also be recovered as a special case of the three-field formulation, as shown in [26]. While this may seem to suggest that the two-field approach is to be preferred, it can be advantageous to use the three-field formulation in certain applications. The motivation in such cases stems from the fact that the displacement is a primary variable, so that the method is well suited to displacement- controlled algorithms of the kind that occur in plasticity. In contrast, in the Hellinger-Reissner formulation the stress is the primary variable while the displacement is the Lagrange multiplier. In such cases the highly effective return mapping algorithm has to be carried out in a less efficient way at element level rather than locally, to be applied at integration points [43].

More recent related work includes that by Cervera, Chiumenti and Codina [19], in which mixed methods are stabilized via a subgrid scale approach. The issue of uniform convergence in the incompressible limit is not addressed.

Also worth mentioning is the work by Mahnken and Caylak [39], who adopt an approach based on a combination of incompatible modes together with mixed enhanced strains, coupled with the use of surface bubble functions, to develop models based on the use of tetrahedral elements in three dimensions. Numerical results indicate the good performance of the elements in the incompressible limit, though no corresponding analysis is carried out.

The formulation introduced by Lamichhane *et al.* [38] and expanded on further by Djoko *et al.* [26] has been analyzed in detail, for a general selection of elements. The motivation in these works has however been the construction and analysis of elements based on the four-noded quadrilateral, and applications are confined to this case. One of the objectives of the thesis is use the framework in [38] as the basis for construction of families of stable elements for the three-field formulation. The strategy relies on the fact that satisfaction of the discrete inf-sup condition is shown to be equivalent to verifying that a displacement-discrete pressure pair satisfies the inf-sup condition for the corresponding Stokes problem. The pressure is constructed via a discrete trace operator which ensures that the corresponding discrete spherical part of the stress belongs to the discrete space of stresses . This condition is not necessarily satisfied by the conventional spherical part.

Thus, starting from such a discrete displacement pressure pair, spaces of discrete stresses and strains may be constructed, after which it remains only to verify that the ellipticity condition is satisfied. In the approach adopted here, this last step is often the most challenging.

The method is used to construct families of stable elements for the three-field formulation, in two and three dimensions, which are based on well-known stable velocity-pressure pairs: specifically, the MINI element [3], the conforming Crouzeix-Raviart elements [24], and the generalized Hood-Taylor elements [33, 35] (see for example [10] for the structure and properties of these elements). The general three-field formulation is also exploited to analyze a mixed formulation due to Kasper and Taylor [36]. This formulation, which involves as unknown variables the displacement, pressure and dilatation, has been shown to perform well in computations, but without an accompanying general analysis. Such an analysis is carried out in this thesis.

The third and final objective of this work is the extension of the linear approach to nonlinear problems. Construction of stable approximations for these problems have been of interest in recent years, and involve a variety of mixed and enhanced methods [6, 22, 36, 41]. Also worth mentioning is the stabilization technique applied to mixed tetrahedral element by Caylak and Mahnken [18]. A different stabilization approach using a perturbation term and bubble function is also addressed in [4]. Two-dimensional formulations have been studied in [5, 6] to determine the ability of numerical formulations to predict physical stability ranges.

The approach taken in this thesis is closer to that of Ten Eyck and Lew [45], who propose a stabilized enhanced strain method based on modifying the fourth-order elasticity tensor that appears in the linearized problem. Like these authors, the linearized problem is considered: this is an important special case as it is essentially the problem that has to be solved in iterative procedures such as the Newton-Raphson method. The continuous problem is assumed to be stable in the sense that the elasticity tensor satisfies a coercivity condition. The novelty of the present approach lies in considering the linearization of the three-field formulation, and making use of the techniques applied to the linear problem, to determine conditions under which the linearized problem is stable and convergent.

The structure of the rest of this thesis is as follows. For the sake of completeness, in Chapter 2 we present a formal description of the stability and convergence of approximations of the Stokes problem.

Chapter 3 deals with the stability and convergence of the three-field formulation in linear elasticity. We describe in detail the assumptions under which the formulation is well posed.

In Chapter 4 we discuss important issues related to new finite element spaces for stable and convergent three-field formulations. We show that our new spaces satisfy the assumptions discussed in Chapter 3. To illustrate the performance of the new elements, some numerical examples are presented in the last part of the chapter.

In Chapter 5 we present an application of the theory developed in Chapter 3. The main concern addressed here is to analyse another three-field mixed variational formulation, introduced by Kasper and Taylor [37]. We have constructed conditions under which this formulation is a special case of the standard three-field formulation. We then prove the stability of the formulation in [37] within the framework of the standard three-field formulation.

In Chapter 6 similar techniques to those discussed in the context of the linear problems are applied to the analysis of nonlinear problems. Starting with a nonlinear three-field, we use a linearization technique to derive the linear three-field formulation. We establish the well-posedness of the continuous weak formulation and conditions for the convergence and stability of its discrete counterpart. Numerical examples in two dimensions spaces are also presented.

1.2 Some definitions and notation

In this section some basic definitions and notation encountered in the remaining part of this work are given. Other definitions and notation will be presented as required in various Chapters. In what follows we deal with a sufficiently regular spatial domain. That is, we always assume that $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, is a non-empty bounded domain. We denote by $\partial\Omega$ the boundary of the domain and assume $\partial\Omega$ to be made up of a finite number of smooth curves, or surfaces.

We denote by $L^2(\Omega)$ the space of (equivalence classes of) Lebesgue-square-integrable functions; that is,

$$L^2(\Omega) = \{\, f \,:\, \Omega \longrightarrow \mathbb{R} \quad \text{such that} \int_\Omega f^2(x)\,\mathrm{d}x \;<\; \infty \,\}\,. \tag{1.1}$$

The $L^2(\Omega)$-inner product is defined by

$$(f\,,\,g)_0 = \int_\Omega f(x)g(x)\,\mathrm{d}x\,, \quad \forall\, f,\; g \in L^2(\Omega)\,,$$

with corresponding norm

$$\|f\|_0 = \|f\|_{L^2(\Omega)} = \sqrt{(f\,,\,f)_0}\,.$$

With this inner product, $L^2(\Omega)$ is a Hilbert space. Other familiar spaces of interest are

- $C(\Omega)$: the space of continuous functions in Ω
- $C^m(\Omega)$: space of m times continuously differentiable functions in Ω, $0 \le m \le \infty$
- $C_0^m(\Omega)$: subspace of $C^m(\Omega)$ containing functions with compact support in Ω

Definition Let f and g be two functions in $L^2(\Omega)$ and $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n$. Furthermore denote by

$$|\alpha| = \alpha_1 + \cdots + \alpha_n \quad \text{and} \quad D^\alpha \varphi = \frac{\partial^{|\alpha|}\varphi}{\partial x_1^{\alpha_1} \cdots x_n^{\alpha_n}} \quad \forall \varphi \in L^2(\Omega).$$

The function g is said to be the weak derivative, of order $|\alpha|$, of f if

$$(\varphi\,,\,g)_0 = (-1)^{|\alpha|}(\partial^\alpha\varphi\,,\,f)_0\,,\quad \forall\varphi\in C_0^\infty(\Omega)\,.$$

See [11, 15].

If the weak derivative of f exists it is unique and is often denoted by $\partial^\alpha f$.

Definition Let $m \geq 0$, be a positive integer. The Hilbert space $H^m(\Omega)$ is the set of all functions whose weak derivatives of order $|\alpha| \leq m$ are square-integrable.

$H^m(\Omega)$ is equipped with the scalar product

$$(f\,,\,g)_m = \sum_{|\alpha|\leq m}(\partial^\alpha f\,,\,\partial^\alpha g)_0\,,\quad \forall\, f,\ g\in L^2(\Omega)\,,$$

with norm and semi-norm defined by

$$\|f\|_m = \|f\|_{H^m(\Omega)} = \sqrt{(f\,,\,f)_m}$$

and

$$|f|_m = |f|_{H^m(\Omega)} = \sqrt{\sum_{|\alpha|=m}(\partial^\alpha f\,,\,\partial^\alpha f)_0}$$

respectively.

1.3 Finite element approximations

We are concerned with finite element approximations based on quasi-uniform, shape-regular meshes $\mathcal{T}_h$, on polygonal domains in $\mathbb{R}^d$, $d = 2, 3$, with simplicial or tensor-product elements $K \in \mathcal{T}_h$ being generated from reference elements, $\hat{K}$, by affine maps.

Here and henceforth we make use of the following notations:

$$
\begin{aligned}
&P_k(K) = \text{set of polynomials on } K \text{ of degree} \le k\\
&Q_k(K) = \text{set of polynomials on } K \text{ of degree} \le k \text{ in each variable}\\
&\mathcal{P}_k = \text{functions } v \text{ defined on } \Omega \text{ such that } v \in C(\Omega) \text{ and } v|_K \in P_k(K)\\
&\mathcal{Q}_k = \text{functions } v \text{ defined on } \Omega \text{ such that } v \in C(\Omega) \text{ and } v|_K \in Q_k(K)\\
&\mathcal{P}'_k = \text{functions } v \text{ defined on } \Omega \text{ such that } v \in L^2(\Omega) \text{ and } v|_K \in P_k(K)\\
&\mathcal{Q}'_k = \text{functions } v \text{ defined on } \Omega \text{ such that } v \in L^2(\Omega) \text{ and } v|_K \in Q_k(K)\\
&\mathcal{Q} = \text{functions } v \text{ defined on } \Omega \text{ such that } v \in L^2(\Omega) \text{ and } \int_\Omega v \,\mathrm{d}x = 0
\end{aligned}
$$

Thus $\mathcal{P}_k$ and $\mathcal{Q}_k$ comprise spaces of continuous piecewise-polynomial functions and $\mathcal{P}'_k$ and $\mathcal{Q}'_k$ are their piecewise-discontinuous counterparts.

Conforming finite element approximations are considered in Chapters 2 to 6, with discrete spaces $V_h\,, Q_h\,, S_h$ and D_h satisfying

$$
\begin{aligned}
V_h \subset V &= \left(H^1_0(\Omega)\right)^d = \{\boldsymbol{v} \in \left(H^1(\Omega)\right)^d \,:\, \boldsymbol{v} = \boldsymbol{0} \text{ in } \partial\Omega\}\,,\\
Q_h \subset Q &= L^2_0(\Omega) = \{q \in L^2_0(\Omega) \,:\, \int_\Omega q \,\mathrm{d}x = 0\,\}\,,\\
S_h \subset S_0 &= \{\boldsymbol{\tau} \in \left(L^2_0(\Omega)\right)^{d\times d} \,:\, \tau_{ji} = \tau_{ij} \text{ and } (\boldsymbol{\tau}\,,\,\boldsymbol{1}) = 0\,\}\,,\\
D_h \subset D &= \{\boldsymbol{d} \in \left(L^2_0(\Omega)\right)^{d\times d} \,:\, d_{ji} = d_{ij}\}\,.
\end{aligned}
$$

We also need the follwoing spaces of piecewise polynomials, continuous or discontinuous:

$$
\begin{aligned}
\boldsymbol{\varepsilon}(V_h) \subset \boldsymbol{\varepsilon}(V) &= \{\boldsymbol{\varepsilon}(\boldsymbol{v}) = \frac{1}{2}\left(\nabla\boldsymbol{v} + (\nabla\boldsymbol{v})^T\right) \,:\, \boldsymbol{v} \in V\}\,,\\
\nabla V_h \subset \nabla V &= \{\nabla\boldsymbol{v} \,:\, \boldsymbol{v} \in V\}\,.
\end{aligned}
$$

Notation. Vector- and tensor- or matrix-valued functions will be written in boldface form. The scalar product of two tensors or matrices $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ will be denoted by $\boldsymbol{\sigma} : \boldsymbol{\tau}$, and is given by $\boldsymbol{\sigma} : \boldsymbol{\tau} = \sigma_{ij}\tau_{ij}$, the summation convention on repeated indices being invoked. The inner product in $\left(L^2_0(\Omega)\right)^{d\times d}$ can then be defined by

$$
(\boldsymbol{\sigma}\,,\,\boldsymbol{\tau})_0 = \sum_{ij} (\sigma_{ij}\,,\,\tau_{ij})_0\,, \quad \text{for } \boldsymbol{\sigma}\,,\ \boldsymbol{\tau} \in \left(L^2_0(\Omega)\right)^{d\times d}
$$

with associated norm $\|\cdot\|_0 = (\cdot,\cdot)_0$. Similarly we define the norm and inner product in V by

$$
(\boldsymbol{u}\,,\,\boldsymbol{v})_V = (\boldsymbol{u}\,,\,\boldsymbol{v})_1 = \sum_i (u_i\,,\,v_i)_1 \quad \text{and} \quad \|\boldsymbol{u}\|_V = \|\boldsymbol{u}\|_1 = \sqrt{(\boldsymbol{u}\,,\,\boldsymbol{u})_1}\,, \quad \forall\,(\boldsymbol{u}\,,\,\boldsymbol{v}) \in V^2\,.
$$

The semi-norm in V is defined by

$$|\boldsymbol{v}|_m = \sum_{|\alpha|=m} \int_\Omega |D^\alpha \boldsymbol{v}| \, \mathrm{d}x \, ,$$

were we have used the notion of Definition 1.2.

CHAPTER 2

STABLE MIXED FINITE ELEMENT APPROXIMATIONS OF THE STOKES PROBLEM

In this chapter we discuss aspects of the Stokes problem that are important for the stability and convergence of a mixed three-field formulation for problems in linear elasticity. It was shown in [38] that such stability is related to one of the stability conditions of the Stokes problem, known as the *inf-sup* or the *Ladyzenskaja-Babuška-Brezzi* (LBB) condition. Therefore in Section 2.1 special attention is given to the introduction of this problem and its well-posedness. In Section 2.2 we present some standard techniques on checking the inf-sup condition. To simplify the difficulties encountered in the analysis of the Stokes equation we introduce, in Section 2.3, a new approach for checking the LBB condition. In Section 2.4 we discuss some standard and new families of stable Stokes pairs.

2.1 Formulation and analysis of the problem

The Stokes problem describes slow flows of an incompressible fluid in the d-dimensional physical domain $\Omega \subset \mathbb{R}^d$, $d = 2, \cdots, 3$ and is governed by the following system of PDE:

$$\begin{aligned} -\Delta \boldsymbol{u} + \nabla p &= \boldsymbol{f} \quad \text{in } \Omega\,, && (2.1a)\\ \operatorname{div} \boldsymbol{u} &= 0 \quad \text{in } \Omega\,, && (2.1b) \end{aligned}$$

where the displacement $\boldsymbol{u} : \Omega \longrightarrow \mathbb{R}^d$ and the pressure $p : \Omega \longrightarrow \mathbb{R}$ are the unknown variables and $\boldsymbol{f}$ represents the external forces per unit mass, see [15]. Without loss of generality we restrict our attention to problems with the homogeneous Dirichlet boundary condition.

The weak or variational formulation of problem (2.1) is the problem of finding $(\boldsymbol{u}\,,\, p) \in V \times Q$ such that

$$\begin{aligned} a(\boldsymbol{u}\,,\,\boldsymbol{v}) + b(p\,,\,\boldsymbol{v}) &= (\boldsymbol{f}\,,\,\boldsymbol{v})_0 \quad \forall\, \boldsymbol{v} \in V, && (2.2a)\\ b(q\,,\,\boldsymbol{u}) &= 0 \qquad\quad\ \forall\, q \in Q, && (2.2b) \end{aligned}$$

where the spaces of displacements and pressures are

$$V = H_0^1(\Omega)^d = \left\{ \boldsymbol{v} \in \left[H^1(\Omega)\right]^d \,:\, \boldsymbol{v} = \boldsymbol{0} \text{ on } \partial\Omega \right\} \quad \text{and} \quad Q = \left\{ q \in L^2(\Omega) \mid \int_\Omega q \,\mathrm{d}x = 0 \right\} \qquad (2.3)$$

respectively and the bilinear forms are given by

$$a(\boldsymbol{u}\,,\,\boldsymbol{v}) = \left(\nabla \boldsymbol{u}\,,\, \nabla \boldsymbol{v}\right)_0 = \int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{v} \,\mathrm{d}x \qquad (2.4a)$$

and

$$b(q\,,\,\boldsymbol{v}) = -\left(q\,,\, \operatorname{div} \boldsymbol{v}\right)_0 = -\int_\Omega q \cdot \operatorname{div} \boldsymbol{v} \,\mathrm{d}x. \qquad (2.4b)$$

The weak formulation (2.2), is consistent in the sense that any of its solutions satisfies also the strong form (2.1) and vice-versa, under certain conditions. The proof of this can be found in [15]. If we denote by P_Q an orthogonal projection onto Q, a simpler approach is to first see that equation (2.2b) is equivalent to

$$P_Q(\operatorname{div} \boldsymbol{u}) = 0 \iff \operatorname{div} \boldsymbol{u} = 0 \quad \text{because} \quad \operatorname{div} V \subset L^2(\Omega).$$

Second, integrating by parts and taking into account the Dirichlet boundary condition, we obtain

$$\begin{aligned} a(\boldsymbol{u}\,,\,\boldsymbol{v}) &= \big(\nabla\boldsymbol{u}\,,\,\nabla\boldsymbol{v}\big)_0 = -\big(\Delta\boldsymbol{u}\,,\,\boldsymbol{v}\big)_0 \\ b(p\,,\,\boldsymbol{v}) &= -\big(p\,,\,\operatorname{div}\boldsymbol{v}\big)_0 = +\big(\nabla p\,,\,\boldsymbol{v}\big)_0, \end{aligned}$$

which if combined with equation (2.2a) implies the following equalities:

$$P_V(-\Delta\boldsymbol{u} + \nabla p) = P_V(\boldsymbol{f}) \iff -\Delta\boldsymbol{u} + \nabla p = \boldsymbol{f}.$$

The problem of linear isotropic elasticity has a close relation to problem (2.2). The elasticity problem has two parameters λ and μ, known as Lamé constants, and is one of finding $(\boldsymbol{u}\,,\,p) \in V \times Q$ that satisfy

$$\mathrm{a}(\boldsymbol{u}\,,\,\boldsymbol{v}) + b(p\,,\,\boldsymbol{v}) = (\boldsymbol{f}\,,\,\boldsymbol{v})_0 \quad \forall\,\boldsymbol{v} \in V\,, \tag{2.5a}$$

$$b(q\,,\,\boldsymbol{u}) - \frac{1}{\lambda}c(p\,,\,q) = 0 \quad \forall\,q \in Q\,, \tag{2.5b}$$

with

$$\mathrm{a}(\boldsymbol{u}\,,\,\boldsymbol{v}) = 2\mu\big(\nabla\boldsymbol{u}\,,\,\nabla\boldsymbol{v}\big)_0,$$

$$c(p\,,\,q) = (p\,,\,q)_0$$

and $b(\cdot\,,\,\cdot)$ is as previously defined. The problem (2.2) is recovered in the incompressible limit; that is, when $\lambda \longrightarrow +\infty$.

We next turn our attention to the well-posedness of problem (2.2) and hence problem (2.5a).

Theorem 2.1.1 *[15] Let Q' be the dual space of Q. The problem (2.2) has a unique solution if and only if*

- *the bilinear form $a(\cdot,\cdot)$ is uniformly continuous:*

$$\exists\,\gamma > 0\,:\,|a(\boldsymbol{w}\,,\,\boldsymbol{v})| \leq \gamma\|\boldsymbol{w}\|_1\|\boldsymbol{v}\|_1 \quad \forall\,(\boldsymbol{w}\,,\,\boldsymbol{v}) \in V \times V \tag{2.6a}$$

  and $V-$elliptic (or coercive):

$$\exists\,C > 0\,:\,|a(\boldsymbol{w}\,,\,\boldsymbol{w})| \geq C\|\boldsymbol{w}\|_1^2 \quad \forall\,\boldsymbol{w} \in V\,; \tag{2.6b}$$

- *the bilinear form* $b(\cdot,\cdot)$ *has a continuous lifting, which means that the image of the transformation*

$$\begin{aligned} B : V &\longrightarrow Q' \\ \boldsymbol{v} &\longmapsto Bv\,, \end{aligned}$$

such that $(B\boldsymbol{v}\,,\,q) = b(v\,,\,q) \quad \forall\, q \in Q'$*, is equal to* Q'*. This is equivalent to the* Ladyzenskaja-Babuška-Brezzi *(LBB) condition or the* inf-sup *condition: that is, there exists* β^* *that satisfies*

$$\inf_{q\in Q}\sup_{\boldsymbol{v}\in V}\frac{(\operatorname{div}\boldsymbol{v}\,,\,q)_0}{\|\boldsymbol{v}\|_1\|q\|_0} \geq \beta^*\,. \tag{2.6c}$$

Proof See [40], [15] and [10]. ∎

From Theorem 2.1.1 the uniqueness of a solution of problem (2.2) can be proven by establishing the uniform continuity of $a(\cdot\,,\,\cdot)$, the ellipticity and the *LBB* conditions.

- **Continuity of** $a(\cdot,\cdot)$: this is a direct implication from the *Cauchy-Schwarz* inequality. Indeed,

$$|a(\boldsymbol{w}\,,\,\boldsymbol{v})| = |(\nabla\boldsymbol{w}\,,\,\nabla\boldsymbol{v})_0| \leq \|\nabla\boldsymbol{w}\|_0\|\nabla\boldsymbol{v}\|_0 \leq \|\boldsymbol{w}\|_1\|\boldsymbol{v}\|_1.$$

- **Ellipticity of** $a(\cdot,\cdot)$: using the *Poincaré-Friedrichs* inequality and the definition of the semi-norm in $H^1(\Omega)$, we can write

$$|a(\boldsymbol{w}\,,\,\boldsymbol{w})| = \|\nabla\boldsymbol{w}\|_0^2 = |\boldsymbol{w}|_1^2 \geq C\|\boldsymbol{w}\|_1^2.$$

- **The inf-sup condition**: we know from [26, 38] that

$$\forall\, q \in Q\ \exists\, \boldsymbol{v} \in V\ :\ \operatorname{div}\boldsymbol{v} = q \quad\Longrightarrow\quad \mathrm{Im}\, B = Q'.$$

Hence B has a continuous lifting which is equivalent to the inf-sup condition.

2.1.1 Mixed finite element approximations [11]

To approximate (2.2) we introduce finite-dimensional subspaces $V_h \subset V$ and $Q_h \subset Q$. Then (2.2) becomes the problem of finding $(\boldsymbol{u}_h\,,\,p_h) \in V_h \times Q_h$ such that

$$a(\boldsymbol{u}_h\,,\,\boldsymbol{v}_h) + b(\boldsymbol{v}_h\,,\,p_h) = \boldsymbol{\ell}(v_h) \qquad \forall\, \boldsymbol{v}_h \in V_h\,, \tag{2.7a}$$

$$b(\boldsymbol{u}_h\,,\,q_h) = 0 \qquad \forall\, q_h \in Q_h\,, \tag{2.7b}$$

where $\ell(\boldsymbol{v}_h) = (\boldsymbol{f} \,,\, \boldsymbol{v}_h)$.

Using the properties of the bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ the solution of the discrete problem (2.7) can be proven to converge to the exact solution of its continuous counterpart (2.2).

It is clear from the equations (2.6a) and (2.6b) of Theorem 2.1.1 that the uniform continuity of the bilinear form $a(\cdot,\cdot)$ and the ellipticity condition in the continuous formulation implies the uniform continuity of $a(\cdot,\cdot)$ and also the ellipticity condition in the discrete formulation (2.7).

In contrast to the ellipticity condition the inequality (2.6c) cannot be always satisfied in the finite-dimensional subspaces V_h and Q_h. This is due to the fact that

$$\inf_{q \in Q} E(q) \leq \inf_{q_h \in Q_h} E(q_h) \quad \text{and} \quad \sup_{\boldsymbol{v} \in V} E(\boldsymbol{v}) \geq \sup_{\boldsymbol{v}_h \in V_h} E(\boldsymbol{v}_h)\,,$$

where $E(\cdot)$ denotes a certain function that could potentially depend on the elements of Q, Q_h, V or V_h, and as a consequence the inequality (2.6c) does not imply that

$$\inf_{q \in Q_h} \frac{(\operatorname{div} \boldsymbol{v}_h \,,\, q_h)_0}{\|\boldsymbol{v}_h\|_1} \geq \beta^* \|q_h\|_0 \quad \text{or that} \quad \sup_{\boldsymbol{v}_h \in V_h} \frac{(\operatorname{div} \boldsymbol{v}_h \,,\, q_h)_0}{\|\boldsymbol{v}_h\|_1} \geq \beta^* \|q_h\|_0\,.$$

Note that the coercivity of $a(\cdot,\cdot)$ gives the existence of the solution $(\boldsymbol{u}_h \,,\, p_h) \in V_h \times Q_h$ and the uniqueness of the displacement $\boldsymbol{u}_h \in V_h$. The inf-sup condition gives the uniqueness of the pressure p. In fact, when it is not satisfied the pressure p_h can be polluted by the presence of spurious pressure modes. That is, there exists $p_h^\star \neq p_h$ such that

$$(p_h^\star \,,\, \operatorname{div} \boldsymbol{v}_h)_0 = 0 \quad \forall \boldsymbol{v}_h \in V_h$$

which implies that

$$b(\boldsymbol{v}_h \,,\, p_h + c p_h^\star) = b(\boldsymbol{v}_h \,,\, p_h)\,,$$

where c is any scalar. Therefore the solution of the discrete problem (2.7) can be obtained only up to a linear combination of the pressure $p_h^\star$ (see the discussion in [40]). However in most cases the task is to prove that the constant $\beta^\star$ is mesh independent.

We next introduce some standard techniques for checking the stability or the inf-sup condition of the problem (2.7).

2.2 Checking the inf-sup condition

Although the inequality (2.6c) seems trivial its analysis in the discrete case is one of the most critical problems for the mixed finite element method. In the first part of this section we review some standard techniques for checking (2.8). Extensive discussion of this is given in [10, 11].

In the second part we present a quite different approach for establishing the inf-sup condition. First, we summarise a classical approach for checking the inf-sup condition.

Definition Let V_h and Q_h be finite-dimensional subspaces of V and Q respectively. The pair $(V_h\,,\,Q_h)$ is said to form a stable Stokes pair if and only if there exists $\beta^\star > 0$, independent of h, such that

$$\sup_{\boldsymbol{v}_h \in V_h} \frac{(\operatorname{div} \boldsymbol{v}_h, q_h)_0}{\|\boldsymbol{v}_h\|_1} \geq \beta^* \|q_h\|_0 \quad \text{for all} \quad q_h \in Q_h\,. \tag{2.8}$$

In this paragraph we introduce a general concept known as the the *Fortin trick*, discussed in [10], for establishing the property (2.8), and then show its application to some finite elements.

Proposition 2.2.1 ([10]) *Assume that the inequality (2.6c) is satisfied and that there exists a linear operator* $\Pi_h : V \longrightarrow V_h$ *satisfying*

$$\big(\operatorname{div}(\boldsymbol{v} - \Pi_h \boldsymbol{v})\,,\, q_h\big)_0 \;=\; 0\,, \quad \forall q_h \in Q_h\,, \tag{2.9a}$$

$$\|\Pi_h \boldsymbol{v}\|_V \leq c\|\boldsymbol{v}\|\,, \tag{2.9b}$$

for $\boldsymbol{v} \in V$, *where* c *is a positive constant independent of the mesh size. Then the discrete inf-sup condition (2.8) is satisfied.*

To show the use of Fortin's trick in establishing Proposition 2.2.1, let Π_1 and Π_2 be two operators such that

$$\sum_K h_K^{2r-2} |\boldsymbol{v} - \Pi_1 \boldsymbol{v}|^2_{r,K} \leq c\|\boldsymbol{v}\|^2_{1,\Omega}\,, \quad r = 0,\,1\,, \quad \forall \boldsymbol{v} \in V\,, \tag{2.10a}$$

and

$$\|\Pi_2 \boldsymbol{v}\|_{1,K} \leq c(h_K^{-1}\|\boldsymbol{v}\|_{0,K} + |\boldsymbol{v}|_{1,K})\,, \quad \forall \boldsymbol{v} \in V\,. \tag{2.10b}$$

Furthermore, if Π_1 and Π_2 satisfy

$$\begin{aligned} \|\Pi_1 \boldsymbol{v}\|_V &\leq c_1 \|\boldsymbol{v}\|_V \quad \forall\, \boldsymbol{v} \in V\,, && (2.11a)\\ \|\Pi_2 (I - \Pi_1)\boldsymbol{v}\|_V &\leq c_2 \|\boldsymbol{v}\|_V \quad \forall\, \boldsymbol{v} \in V\,, && (2.11b)\\ \int_\Omega \operatorname{div}(\boldsymbol{v} - \Pi_2 \boldsymbol{v}) q_h &= 0 \qquad \forall\, \boldsymbol{v} \in V\,,\ \forall\, q_h \in Q_h\,, && (2.11c) \end{aligned}$$

then the operator $\Pi_h \boldsymbol{v} = \Pi_1 \boldsymbol{v} + \Pi_2(\boldsymbol{v} - \Pi_1 \boldsymbol{v})$ satisfies both the equation (2.9a) and the inequality (2.9b).

To prove (2.9a) we observe that

$$\begin{aligned} \big(\operatorname{div}(\boldsymbol{v} - \Pi_h \boldsymbol{v})\,,\, q_h\big)_0 &= \big(\operatorname{div}(\boldsymbol{v} - \Pi_1 \boldsymbol{v} - \Pi_2 \boldsymbol{v} + \Pi_2 \Pi_1 \boldsymbol{v})\,,\, q_h\big)_0 \\ &= \big(\operatorname{div}(\boldsymbol{v} - \Pi_2 \boldsymbol{v})\,,\, q_h\big)_0 - \big(\operatorname{div}(\Pi_1 \boldsymbol{v} - \Pi_2 \Pi_1 \boldsymbol{v})\,,\, q_h\big)_0 \,. \end{aligned} \tag{2.12}$$

From equation (2.11c) and the fact that $\Pi_1 \boldsymbol{v} \in V$ the two last term of the above expression (2.12) vanish, and we therefore have the desired equality (2.9a).

The inequality (2.9b) is a direct consequence of the boundedness of the projection operator.

Note that if V_h is made up of a regular family of affine elements then the operators Π_1 and Π_2 satisfying (2.10a) and (2.10b) exist, where Π_1 can be chosen to be the Clément interpolation operator, see [11].

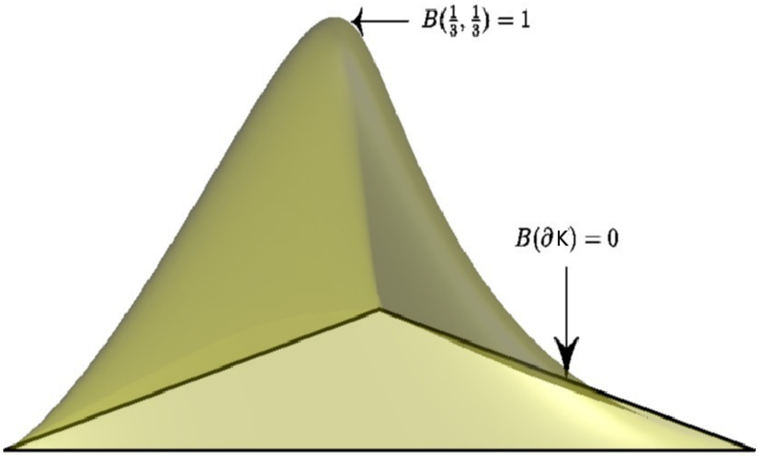


Figure 2.1: Cubic bubble function defined on a triangular element

Definition Let K be an element of the triangulation mesh of the domain $\Omega \subset \mathbb{R}^d$ and $B : \Omega \longrightarrow \mathbb{R}^+$ a continuous piecewise polynomial of minimal degree satisfying the following: B vanishes at the boundary of K and is equal to one at the barycentre of K; that is,

$$B(\boldsymbol{x}) = \begin{cases} 1 & \text{if } \boldsymbol{x} \text{ is the barycentre of } K\,, \\ 0 & \text{if } \boldsymbol{x} \in \partial K\,, \end{cases}$$

then B is said to be a bubble function. A particular case of interest is the space of cubic and quartic bubble functions, which we denote by $\mathcal{B}_{d+1}$, $d = 2, 3$, respectively. A graphical representation of a cubic bubble function is shown in Figure 2.1.

Definition Let E be a face of an element $K \subset \Omega \subset \mathbb{R}^3$ of the triangulation mesh. A continuous piecewise polynomial $A : \Omega \longrightarrow \mathbb{R}^+$, with compact support $E \subset K$, is said to be an area (or surface) bubble function if it vanishes at the boundary of E, is equal to one at its barycentre and has minimal polynomial degree. That is,

$$A(\boldsymbol{x}) = \begin{cases} 1 & \text{if } \boldsymbol{x} \text{ is the barycentre of } E\,, \\ 0 & \text{if } \boldsymbol{x} \in (K \backslash E) \cup \partial E\,. \end{cases}$$

The set of all possible linear combinations of such A generate the space of surface bubble functions $\mathcal{A}_3$. Notice that an element of $\mathcal{A}_3$ has four degrees of freedom on each element.

In the following we give some important conditions for establishing a uniform *inf-sup* condition.

Proposition 2.2.2 (Stability of continuous pressure elements, [10]) *Assume that there exists an operator $\Pi_1 \in \mathcal{L}(V\,,\, V_h)$ satisfying the property (2.10a) of the Clément interpolation operator. If the space of pressures satisfies $Q_h \subset C^0(\Omega)$ and V_h contains the space $\mathcal{B}_{d+1}\big(\mathit{grad}\,(Q_h)\big)$, where*

$$\mathcal{B}_{d+1}\big(\mathit{grad}\,(Q_h)\big) = \{b \nabla q_h \ : \ b \in \mathcal{B}_{d+1} \text{ and } q_h \in Q_h\}\,, \tag{2.13}$$

then $(V_h\,,\, Q_h)$ is a stable Stokes pair.

Proof See [10].

Corollary 2.2.3 ([10]) *Assume that $Q_h \subset Q$ is a space of continuous piecewise-linear functions. If V_h contains $\big(\mathcal{P}_1\big)^2 \oplus \mathcal{B}_{d+1}\big(\mathrm{grad}\,(Q_h)\big)$ then the pair $(V_h\,,\,Q_h)$ satisfies the inf-sup condition (2.8).*

Proof By definition of V_h there exists a Clément operator $\Pi_1 : V \longrightarrow V_h$ satisfying (2.10a). The rest is a consequence of Proposition 2.2.2, see [10, 11] for more details. ∎

Proposition 2.2.4 (Stability of discontinuous pressure elements [10]) *Assume that there exists an operator $\tilde{\Pi}_1$ satisfying*

$$\|\tilde{\Pi}_1 \boldsymbol{v}\|_V \leq C\|\boldsymbol{v}\|_V, \quad \forall \boldsymbol{v} \in V, \tag{2.14a}$$

$$\int_K \mathrm{div}\big(\boldsymbol{v} - \tilde{\Pi}_1 \boldsymbol{v}\big)\, dx = 0 \quad \forall \boldsymbol{v} \in V \quad \forall K \in \mathfrak{T}_h\,. \tag{2.14b}$$

If V_h contains $\mathcal{B}_3\big(\mathrm{grad}\,(Q_h)\big)$ then $(V_h\,,\,Q_h)$ is a stable Stokes pair.

Proof See [10]. ∎

Corollary 2.2.5 *Assume that $Q_h \subset Q$ is a space of piecewise-smooth functions. If V_h contains $\big(\mathcal{P}_2\big)^2 \oplus \mathcal{B}_3\big(\mathrm{grad}\,(Q_h)\big)$ then the pair $(V_h\,,\,Q_h)$ satisfies the inf-sup condition (2.8).*

Proof See [10]. ∎

2.3 A new approach for verifying the inf-sup condition

The technique presented here is based on the properties of the transpose of the discrete kernel of $b(\cdot,\cdot)$, defined by

$$\ker B_h^t = \{q_h \in Q_h \mid b(\boldsymbol{v}_h\,,\,q_h) = 0 \quad \forall\, \boldsymbol{v}_h \in V_h\}\,. \tag{2.15}$$

Indeed, as mentioned in [11] the solution p_h of (2.7) is defined up to an element of $\ker B_h^t$. It is then clear that if $\ker B_h^t = \{0_Q\}$, where 0_Q denote the trivial vector of Q, then the solution of (2.7) is unique. This fact is clarified by the following result.

Proposition 2.3.1 *[11] Assume that the bilinear form* $a(\cdot,\cdot)$ *is* $\ker B_h$*-elliptic, that is, there exists* $C > 0$ *such that*

$$\forall\, \boldsymbol{v}_h \in \ker B_h\,, \quad a(\boldsymbol{v}_h\,,\,\boldsymbol{v}_h) \geq C\|\boldsymbol{v}_h\|_1^2\,, \tag{2.16a}$$

where

$$\ker B_h = \{\boldsymbol{v}_h \in V_h \mid b(\boldsymbol{v}_h\,,\,q_h) = 0 \quad \forall\, q_h \in Q_h\}\,, \tag{2.16b}$$

and C *independent of the dimension of the subspace* V_h*. Then (2.7) has at least one solution. Moreover* $\boldsymbol{u}_h$ *is uniquely determined in* V_h*, and if* $\ker B_h^t = 0_Q$ *then* p_h *is also uniquely determined in* Q_h*.*

Proof The problem (2.7) can equivalently be written as: find $\boldsymbol{u}_h \in \ker B_h$ such that

$$a(\boldsymbol{u}_h\,,\,\boldsymbol{v}_h) = \ell(v_h) \qquad \forall\, \boldsymbol{v}_h \in \ker B_h\,, \tag{2.17a}$$

$$b(\boldsymbol{u}_h\,,\,q_h) = 0 \qquad \forall\, q_h \in Q_h\,. \tag{2.17b}$$

As $a(\cdot,\cdot)$ is $\ker B_h$-elliptic, then using the Lax-Milgram Theorem [15] there exists a unique $\boldsymbol{u}_h \in \ker B_h$ that satisfies the equation (2.17a). And as $\boldsymbol{u}_h$ is in $\ker B_h$ it also satisfies (2.17b). One can observe that any solution in V_h that satisfies equations (2.17a) and (2.17b) it is necessarily in $\ker B_h$, because the equation (2.17b) defines the elements of $\ker B_h$. That is to say that $\boldsymbol{u}_h$ is uniquely defined in V_h. The existence of $p_h \in Q_h$ is also granted; indeed we have

$$b(\boldsymbol{v}_h\,,\,p_h) = 0\,, \quad \forall \boldsymbol{v}_h \in \ker B_h\,,$$

so that any $p_h \in Q_h$ is a candidate solution of the problem (2.7). Now if $\ker B_h^t = \{0_Q\}$, let p_h and p_h' be such that $(\boldsymbol{u}_h\,,\,p_h)$ and $(\boldsymbol{u}_h\,,\,p_h')$ are solutions of (2.7). We then have

$$\begin{cases} a(\boldsymbol{u}_h\,,\,\boldsymbol{v}_h) + b(\boldsymbol{v}_h\,,\,p_h) = \ell(v_h) & \forall\, \boldsymbol{v}_h \in V_h\,, \\ a(\boldsymbol{u}_h\,,\,\boldsymbol{v}_h) + b(\boldsymbol{v}_h\,,\,p_h') = \ell(v_h) & \forall\, \boldsymbol{v}_h \in V_h\,. \end{cases} \tag{2.18}$$

Subtracting the above equations, we obtain

$$b(\boldsymbol{v}_h\,,\,p_h - p_h') = 0 \qquad \forall\, \boldsymbol{v}_h \in V_h\,,$$

which implies that $p_h - p_h' \in \ker B_h^t$, thus $p_h = p_h'$. ∎

We next turn our attention to the new approach for verifying the inf-sup condition.

Proposition 2.3.2 *Assume that* $\ker B_h^t = \ker B^t = \{0_Q\}$ *and that there exists a positive constant* $c > 0$*, independent of the mesh size, such that*

$$\|P_{Q_h} \operatorname{div} \boldsymbol{v}_h\|_0 \geq c\|\boldsymbol{v}_h\|_1 \quad \forall\, \boldsymbol{v}_h \in V_h\,. \tag{2.19}$$

Then the discrete inf-sup condition (2.8) is satisfied.

Proof Clearly the existence of an operator satisfying (2.9a) and (2.9b) is established in [11, page 53, Proposition 2.2].

From the continuous inf-sup condition we have: $\forall\, q_h \in Q_h \subset Q$, there exists $\boldsymbol{v} \in V$ such that

$$\operatorname{div} \boldsymbol{v} = q_h \quad \text{and} \quad \|\boldsymbol{v}\|_1 \leq c\|q_h\|_0\,, \tag{2.20a}$$

which together with equation (2.9a) implies the existence of $\boldsymbol{v}_h \in V_h$ such that

$$b(\boldsymbol{v}, q_h) = \|q_h\|_0^2 = b(\boldsymbol{v}_h, q_h) \quad \text{and} \quad \|\boldsymbol{v}_h\|_1 \leq c\|q_h\|_0\,. \tag{2.20b}$$

The above (2.20b) and (2.20a) imply that

$$q_h = P_{Q_h} \operatorname{div} \boldsymbol{v}_h = \operatorname{div} \boldsymbol{v} \implies \|\operatorname{div} \boldsymbol{v}\|_0 = \|P_{Q_h} \operatorname{div} \boldsymbol{v}_h\|_0$$

and then, using (2.19), we can write

$$\|\boldsymbol{v}\|_1 \geq \|\operatorname{div} \boldsymbol{v}\|_0 = \|P_{Q_h} \operatorname{div} \boldsymbol{v}_h\|_0 \geq c\|\boldsymbol{v}_h\|_1\,. \tag{2.20c}$$

The equalities (2.20c) and (2.20b) give

$$\frac{b(\boldsymbol{v}, q_h)}{\|\boldsymbol{v}\|_1} \leq \frac{b(\boldsymbol{v}_h, q_h)}{c\|\boldsymbol{v}_h\|_1} \tag{2.20d}$$

and using the continuous inf-sup condition we have

$$c\|q_h\|_0 \leq \frac{b(\boldsymbol{v}, q_h)}{\|\boldsymbol{v}\|_1} \leq \frac{b(\boldsymbol{v}_h, q_h)}{c\|\boldsymbol{v}_h\|_1}$$

with c independent of the mesh size. ∎

Lemma 2.3.3 *Consider, the space* $S_h \subset \left(Q\right)^{d\times d}$*, where* d *is the dimension of* Ω*, and* $\boldsymbol{\varepsilon}(V_h)$ *the space of symmetric gradient of velocities* V_h*, or in other words*

$$\boldsymbol{\varepsilon}(V_h) = \left\{ \boldsymbol{\eta}_h \,\middle|\, (\boldsymbol{\eta}_h)_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right),\ i,j = 1,\cdots,d,\ (v_1,\cdots,v_d) \in V_h \right\}.$$

Now assume that

$$(S_h)_{ij} = Q_h \delta_{ij} \text{ and } \ker B_h^t = \ker B^t = \{\mathbf{0}\}\,.$$

If the ellipticity condition

$$\|P_{S_h} \boldsymbol{\varepsilon}(\boldsymbol{v}_h)\|_0 \geq c \|\boldsymbol{v}_h\|_1 \quad \forall\, \boldsymbol{v}_h \in V_h\,, \tag{2.21}$$

is satisfied then the bilinear form $b(\cdot,\cdot)$ satisfies a uniform inf-sup condition.

Proof It is trivial to prove that (2.21) implies (2.19). In fact consider $\boldsymbol{v}_h \in V_h$ and $\boldsymbol{\tau}_h = q_h \mathbf{1} \in S_h$. We then have

$$\begin{aligned}\big(\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\,,\,\boldsymbol{\tau}_h\big) =& \big(\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\,,\,q_h \mathbf{1}\big) = \big(\operatorname{tr} \nabla \boldsymbol{v}_h\,,\,q_h\big)\\ =& \big(\operatorname{div} \boldsymbol{v}_h\,,\,q_h\big) = \big(P_{Q_h} \operatorname{div} \boldsymbol{v}_h\,,\,q_h\big)\\ =& \frac{1}{d}\big(P_{Q_h} \operatorname{div} \boldsymbol{v}_h \mathbf{1}\,,\,\boldsymbol{\tau}_h\big)\,,\end{aligned}$$

and then

$$P_{S_h} \boldsymbol{\varepsilon}(\boldsymbol{v}_h) = \frac{1}{d} P_{Q_h} \operatorname{div}(\boldsymbol{v}_h) \mathbf{1}$$

which implies that

$$\begin{aligned}\|P_{S_h}\big(\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\big)\|_0^2 =& \|P_{Q_h}\big(\operatorname{div} \boldsymbol{v}_h\big)\|_0^2\,,\\ \geq& c\|\boldsymbol{v}_h\|_1\,.\end{aligned}$$

Now using Proposition 2.3.2 the proof follows immediately. ▮

2.4 Families of stable Stokes pairs

In this section we summarize details of elements that are known to satisfy the inf-sup condition (2.8). We present in certain cases the new approach on establishing the stability condition (2.8).

These stable Stokes pair will be used later to establish the stability of new families of mixed finite elements for the linear elasticity problem.

2.4.1 The MINI Element [3]

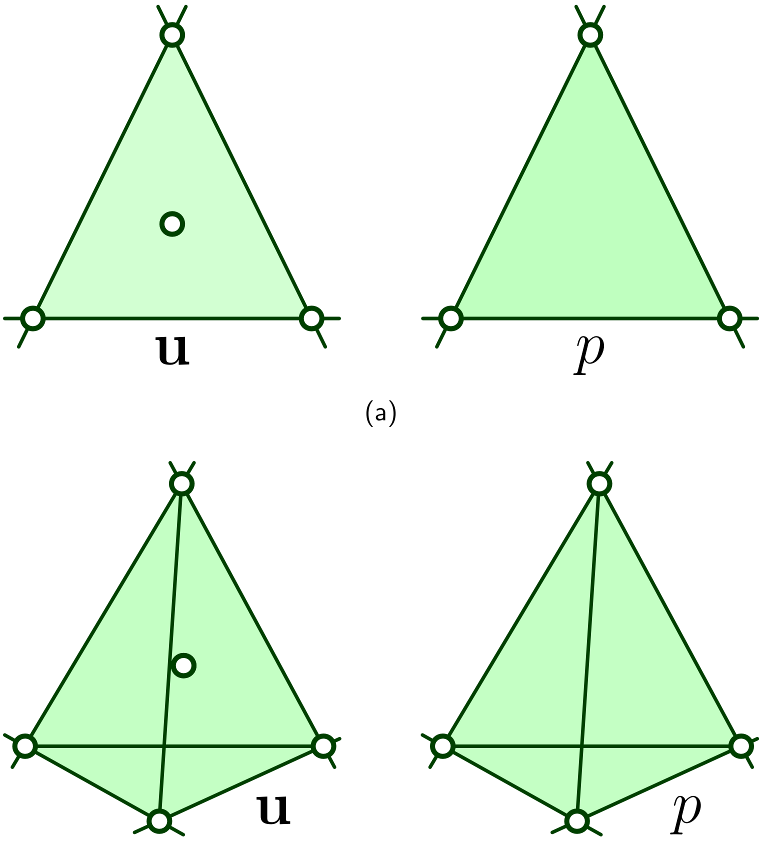
In the conforming case the MINI element is one of the cheapest elements, in the sense that it uses the lowest degree of interpolation polynomials; however this element does not provide well-balanced approximation in that the error estimates are not optimal.

Given a mesh of triangles or tetrahedra , the definition of the spaces is as follows:

$$V_h = (\mathcal{P}_1 \oplus \mathcal{B}_{d+1})^d \cap V \,, \quad d = 2, 3 \tag{2.22a}$$

$$Q_h = \mathcal{P}_1 \cap Q \,. \tag{2.22b}$$

Figures (2.2a) and (2.2b) show the elements with their corresponding degrees of freedom for two-and three-dimensional elements. Note that for $d = 2$, $\mathcal{B}_{d+1}$ is the space of cubic bubble functions, see Figure 2.1, and for $d = 3$, $\mathcal{B}_{d+1}$ is the space of quartic bubble functions.



(a)

(b)

Figure 2.2: (a) Two-dimensional MINI element, (b) Three-dimensional MINI element

Stability based on the Fortin-trick

The stability of this element is a direct consequence of Corollary (2.2.3). In fact, from equations (2.22a) and (2.22b) we have

$$\mathcal{B}_{d+1}\left(\text{grad}\left(Q_h\right)\right) \subset \mathcal{B}_{d+1}^2 \subset \left(\mathcal{P}_1 \oplus \mathcal{B}_{d+1}\right)^2 \subset \mathcal{V}_h, \tag{2.23}$$

which are the necessary conditions of Corollary 2.2.3.

Remark The degrees of freedom prescribed at the vertices of Figures 2.2a and 2.2b indicate the continuity of displacement and pressure variables; that is, for two simplexes K_1 and K_2 such that $\Gamma = K_1 \cap K_2 \neq \emptyset$ we have

$$p_1(\Gamma) = p_2(\Gamma),$$

where $p_i \in \mathcal{P}_1$, $i = 1, 2$. See Figure 2.3.

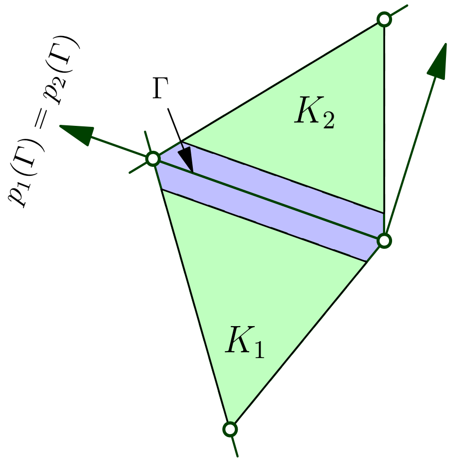


Figure 2.3: Continuity of piecewise polynomials at common vertices

2.4.2 Crouzeix-Raviart Element [24]

This element is an enrichment to the $\mathcal{P}_2 - \mathcal{P}'_0$ element, in order to provide well balanced approximation properties, or equal order estimation error for both velocity and pressure. Two-dimensional approximation spaces are:

$$V_h = \left(\mathcal{P}_2 \oplus \mathcal{B}_3\right)^2 \cap V, \tag{2.24a}$$

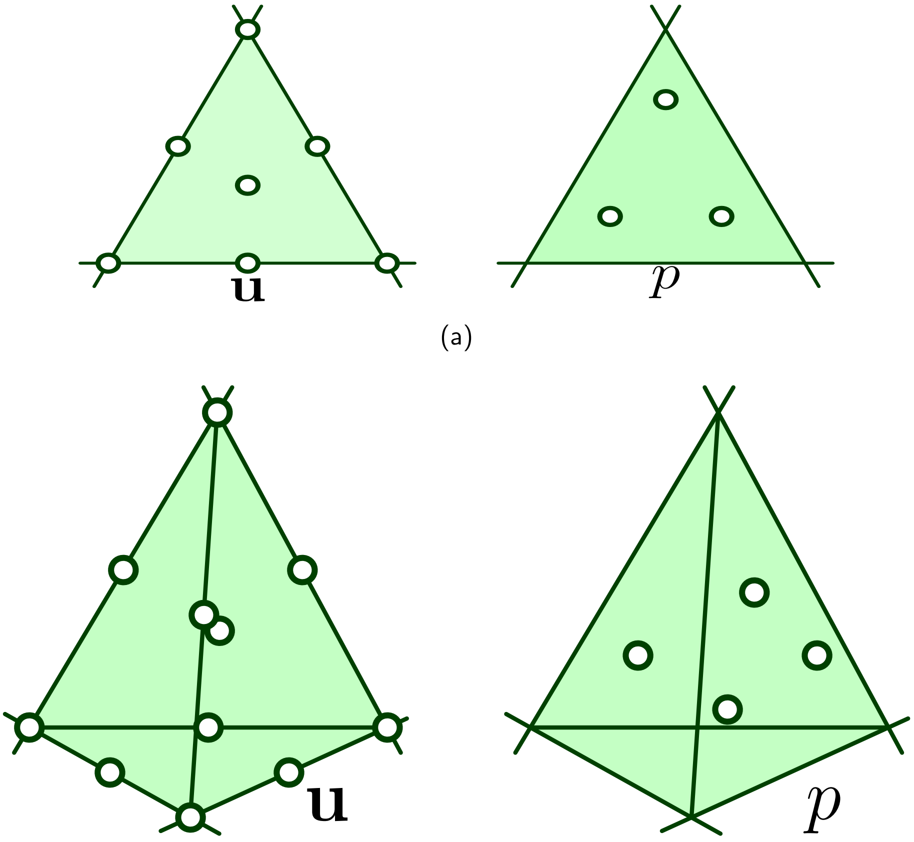
$$Q_h = \mathcal{P}_1^{'} \cap Q\,, \tag{2.24b}$$

and the three-dimensional approximation spaces includes both volumetric and surface bubble functions so that

$$\begin{aligned} V_h &= \left(\mathcal{P}_2 \oplus \mathcal{B}_4 \oplus \mathcal{A}_3\right)^3 \cap V, & \text{(2.25a)} \\ Q_h &= \mathcal{P}_1' \cap Q\,. & \text{(2.25b)} \end{aligned}$$

See Figures (2.4a) and (2.4b) for the degrees of freedom of both the velocity and the pressure variables. We recall that $\mathcal{P}_k'$ is a space of discontinuous piecewise polynomial of degree k.

The stability of the two-dimensional *Crouzeix-Raviart* elements is a consequence of Corollary 2.2.5. For the three-dimensional case see [17].



(a)

(b)

Figure 2.4: (a)Two-dimensional Crouzeix-Raviart element; (b) Three-dimensional Crouzeix-Raviart element; note that the surface bubble is not indicated.

Due to the discontinuity of the pressure we use Corollary 2.2.5 to establish the stability of the *Crouzeix-Raviart* element. A similar relation to expression (2.23) can be derived from (2.24) and is given by

$$\mathcal{B}_{d+1}\left(\text{grad}\left(Q_h\right)\right) \subset \mathcal{B}^2_{d+1} \subset \left(\mathcal{P}_2 \oplus \mathcal{B}_{d+1}\right)^2 \subset \mathcal{V}_h.$$

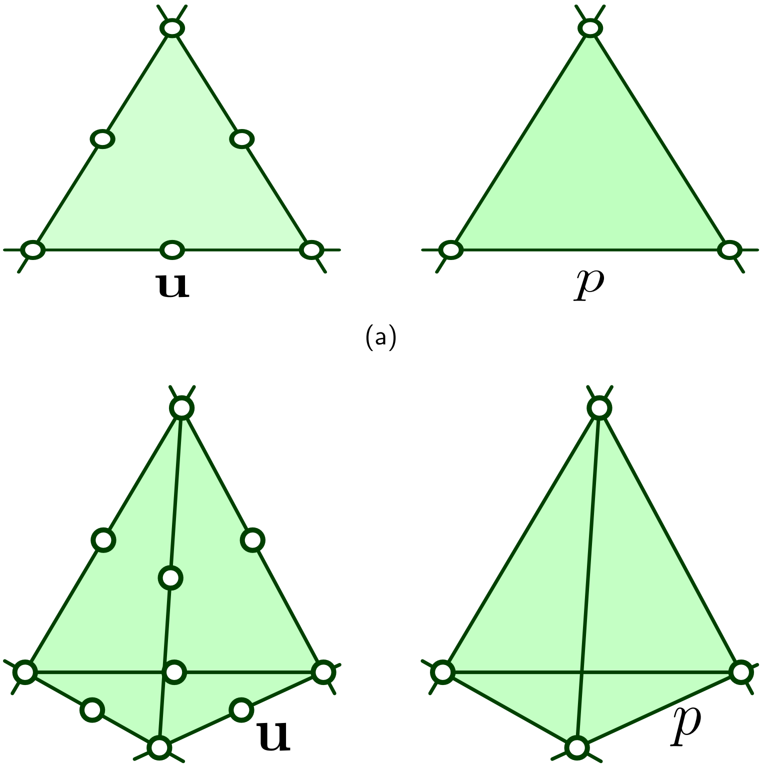
2.4.3 Family of generalised Taylor-Hood Elements ($\mathcal{P}_k/\mathcal{P}_{k-1}$) [33]

For this family spaces of pressures and velocities are continuous piecewise polynomials of degree $k-1$ and k respectively: that is,

$$V_h = \left(\mathcal{P}_k\right)^d \cap V, \tag{2.26a}$$

$$Q_h = \mathcal{P}_{k-1} \cap Q, \tag{2.26b}$$

with $k \geq 2$. Our interest lies in lower-order elements, and we therefore consider only the case $k = 2$. Figures 2.5a and 2.5b show two- and three-dimensional elements of this family. The proof of the inf-sup condition corresponding to these elements can be found in [8, 9].



(a)

(b)

Figure 2.5: (a)Two-dimensional Taylor-Hood element $P_2 - P_1$; (b) Three-dimensional Taylor-Hood element $P_2 - P_1$

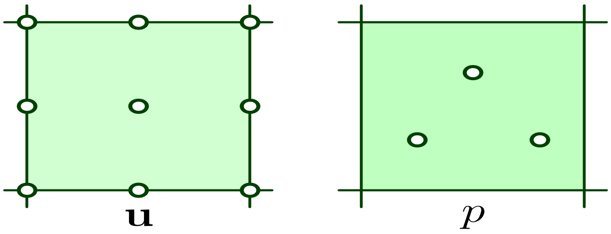
2.4.4 Elements of Taylor-Hood type ($\mathcal{Q}_2/\mathcal{Q}_1$) and ($\mathcal{Q}_2/\mathcal{P}'_1$) [33]

Given a mesh of quadrilaterals in two dimensions and hexahedra in three dimensions the finite element spaces are defined by one of

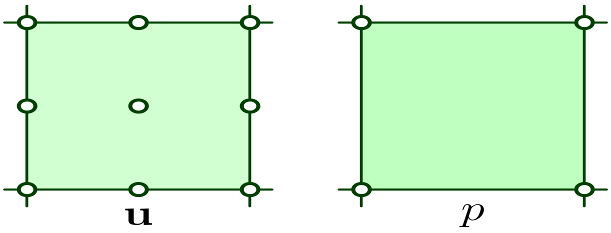
$$V_h = \mathcal{Q}_2^2 \cap V \quad \text{and} \quad Q_h = \mathcal{P'}_1 \cap Q\,, \tag{2.27a}$$

$$V_h = \mathcal{Q}_2^2 \cap V \quad \text{and} \quad Q_h = \mathcal{Q}_1 \cap Q\,. \tag{2.27b}$$

The stability of these elements is a direct consequence of Propositions 2.2.4 and 2.2.2, see the development of its proof in [8, 9, 10]. In fact one can observe that the degrees of freedom in the middle of the elements in Figure 2.7 corresponds to the bubble functions on quadrilaterals and hexahedrons.



(a)



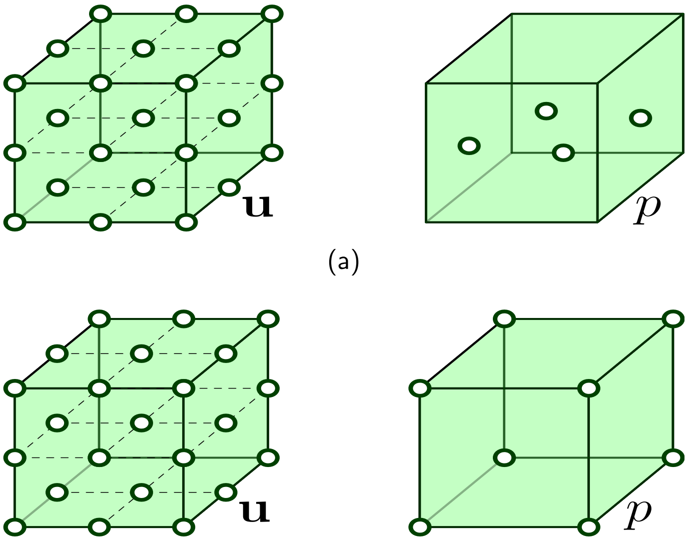
(b)

Figure 2.6: Two-dimensional Taylor-Hood element:(a) $Q_2 - \mathcal{P}'_1$, (b) $Q_2 - Q_1$

2.4.5 Convergence and error estimation

The convergence of finite element approximations depends on the error estimate which is the smallest distance between the discrete space and the exact solution.

Using the properties of polynomial interpolations the errors



(a)

(b)

Figure 2.7: Three-dimensional Taylor-Hood element: (a) $Q_2 - \mathcal{P}'_1$, (b) $Q_2 - Q_1$

$$\boldsymbol{\eta} = \|\boldsymbol{u} - \boldsymbol{u}_h\|_1^2 + \|p - p_h\|_0^2$$

and (3.23) can be bounded by a function of the mesh size.

Theorem 2.4.1 (Ciarlet & Raviart [23]) *Consider $\hat{\Omega}$ and Ω to be two affine-equivalent subset of $\mathbb{R}^d$. And assume that the following conditions hold true:*

$$H^{k+1}(\hat{\Omega}) \subset C(\hat{\Omega}), \quad H^{k+1}(\hat{\Omega}) \subset H^m(\hat{\Omega}), \quad \textit{and} \quad P_k(\hat{\Omega}) \subset H^m(\hat{\Omega}),$$

where m and k are positive integers and $P_k(\hat{\Omega})$ represent the space of polynomials of degree k on $\hat{\Omega}$. Furthermore consider r such that $H^r(\hat{\Omega}) \subset C(\hat{\Omega})$. Then there exists a positive constant C, such that for any $w \in H^r(\Omega_e)$ and w_h its interpolation in $P_k(\Omega_e)$,

$$\|w - w_h\|_{m,\Omega_e} \leq C h_e^{\mu} |w|_{r,\Omega_e}, \tag{2.28a}$$

where $\mu = k + 1 - m$ if $r \geq k + 1$ and $\mu = r - m$ if $r < k + 1$.

However if $w \in H^r(\Omega)$ and w_h its interpolation in $P_k(\Omega)$, then (2.28a) can be rewritten as

$$\|w - w_h\|_{m,\Omega} \leq C h^{\mu} |w|_{r,\Omega}, \quad \textit{for } m = 0 \quad \textit{or } m = 1, \tag{2.28b}$$

where, $h_e = \text{diam}\,(\Omega_e)$, $h = \max\limits_{1 \leq e \leq N} (h_e)$, with N being the total number of nodes.

In problems discussed here the displacements are assumed to belong to $H^1(\Omega)$ and the remaining variables, such as the pressure, the strain and stress are in $H^r(\Omega)$, with $r = 0, 1$, so that using the estimate (2.28b) the error (3.23) can be approximated by

$$\beta \leq Ch^k \,.$$

These results are used to show the theoretical and practical convergence of our finite elements.

2.5 Conclusion

In the first part of this chapter we have given a description of the Stokes problem, discussed its well-posedness accompanied with some standard technique on checking the inf-sup condition. We have presented a different approach for verifying the stability condition and introduced some well known families of stable Stokes pairs. New families, that are not known from the literature, are also presented at the end of the first section. In the last part we have discussed the error estimate of our approximations under the conditions of well-posedness.

CHAPTER 3

STABILITY CONDITIONS FOR THREE FIELD MIXED FINITE ELEMENT APPROXIMATIONS IN ELASTICITY

This chapter is concerned with a class of three-field mixed methods for linear elasticity. The objective of the work is to use a framework in [38] as the basis for construction of families of stable elements for the three-field formulation. The strategy relies on the fact that satisfaction of the discrete inf-sup condition in the elasticity problem is shown to be equivalent to verifying that a displacement-discrete pressure pair satisfies the inf-sup condition for the corresponding Stokes problem. The pressure is constructed via a discrete trace operator which ensures that the corresponding discrete spherical part of the stress belongs to the discrete space of stresses - this condition is not necessarily satisfied by the conventional spherical part.

Thus, starting from such a displacement-discrete pressure pair, spaces of discrete stresses and strains may be constructed, after which it remains only to verify that the ellipticity condition is satisfied. In the approach adopted here, this last step proves to be the most challenging, at least for some cases.

The method is used to construct families of stable elements for the three-field formulation, in two and three dimensions, which are based on well-known stable velocity-pressure pairs: specifically, the MINI element [3], the conforming Crouzeix-Raviart elements [24], and the generalized Hood-Taylor elements [33, 35]. .

The structure of the chapter is as follows. In Section 3.1, we outline the boundary-value problem for linear elasticity, and formulate the weak forms of the standard and modified three- field problems. In Section 3.2 the discrete problems are presented, and sufficient conditions for uniform convergence summarized. Section 3.3 is concerned with the construction and analysis of elements generated from various Stokes-stable pairs.

3.1 The boundary value problem of elasticity

Consider an isotropic linear elastic material body which occupies a bounded domain Ω in $\mathbb{R}^d$ $(d = 2, 3)$ with Lipschitz boundary Γ. For a prescribed body force $\boldsymbol{f}$, the governing equilibrium equation in Ω reads

$$-\operatorname{div} \boldsymbol{\sigma} = \boldsymbol{f} \tag{3.1a}$$

where $\boldsymbol{\sigma}$ is the symmetric Cauchy stress tensor. The infinitesimal strain tensor $\boldsymbol{d}$ is defined as a function of the displacement $\boldsymbol{u}$ by

$$\boldsymbol{d} = \boldsymbol{\varepsilon}(\boldsymbol{u}) := \tfrac{1}{2}(\nabla \boldsymbol{u} + [\nabla \boldsymbol{u}]^t) \ . \tag{3.1b}$$

For convenience a homogeneous Dirichlet boundary condition on the displacement is assumed; that is,

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on} \quad \Gamma. \tag{3.1c}$$

With the fourth-order elasticity tensor denoted by $\mathcal{C}$, the constitutive equation reads

$$\boldsymbol{\sigma} = \mathcal{C}\boldsymbol{d} := \lambda(\operatorname{tr} \boldsymbol{d})\mathbf{1} + 2\mu\, \boldsymbol{d} \ . \tag{3.1d}$$

Here $\mathbf{1}$ is the identity tensor, $\operatorname{tr} \boldsymbol{d}$ denotes the trace of the tensor or matrix $\boldsymbol{d}$, and λ and μ are the Lamé parameters. The parameter μ is the shear modulus.

Of particular interest is the incompressible limit, which corresponds to $\lambda \to \infty$.

Function spaces. For the weak or variational formulations, we will require the space $V := [H_0^1(\Omega)]^d$ of displacements, the space of stresses to be denoted by S, while the space of strains is denoted by D. For the continuous case these spaces are equal, and $D := \{\boldsymbol{e} \mid e_{ji} = e_{ij},\ e_{ij} \in L^2(\Omega)\} =: S$, with norm $\|\cdot\|_0$ generated in the standard way by the L^2-norm. We also introduce the space S_0 defined by

$$S_0 := \{\boldsymbol{\tau} \in S \mid (\boldsymbol{\tau}, \mathbf{1})_0 = 0\} \ ; \tag{3.2}$$

this is a closed subspace of S.

The spherical and deviatoric parts of a member $\boldsymbol{\tau}$ of S are defined by

$$\operatorname{sph} \boldsymbol{\tau} := \tfrac{1}{3}\,(\operatorname{tr} \boldsymbol{\tau})\mathbf{1}, \quad \operatorname{dev} \boldsymbol{\tau} := \boldsymbol{\tau} - \operatorname{sph} \boldsymbol{\tau} \ . \tag{3.3}$$

Then S admits the orthogonal decomposition

$$S = \operatorname{dev} S \oplus \operatorname{sph} S \, .$$

We note that $\operatorname{dev} S$ is a proper subset of S_0.

Three-field formulations. The classical three-field formulation, commonly referred to as the Hu-Washizu formulation [34, 47] - though the contributions by Fraeijs de Veubeke [30, 31] should be acknowledged - is obtained by writing in weak form the equilibrium equation (3.1a), strain-displacement relation (3.1b), and the constitutive relation (3.1d). The resulting problem takes the following form: find $(\boldsymbol{u},\, \boldsymbol{d},\, \boldsymbol{\sigma}) \in V \times D \times S_0$ that satisfy

$$a_1((\boldsymbol{u}, \boldsymbol{d}), (\boldsymbol{v}, \boldsymbol{e})) + b_1((\boldsymbol{v}, \boldsymbol{e}), \boldsymbol{\sigma}) = \ell(\boldsymbol{v}) \ , \qquad (\boldsymbol{v}, \boldsymbol{e}) \in V \times D \ , \tag{3.4a}$$

$$b_1((\boldsymbol{u}, \boldsymbol{d}), \boldsymbol{\tau}) = 0 \ , \qquad \boldsymbol{\tau} \in S_0 \ , \tag{3.4b}$$

where

$$\begin{aligned} a_1((\boldsymbol{u}, \boldsymbol{d}), (\boldsymbol{v}, \boldsymbol{e})) &:= (\mathcal{C}\boldsymbol{d}, \boldsymbol{e})_0 \, , && (3.5a)\\ b_1(\boldsymbol{v}, \boldsymbol{e}, \boldsymbol{\tau}) &:= (\boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{e}, \boldsymbol{\tau})_0 \, , && (3.5b)\\ \ell(\boldsymbol{v}) &:= (\boldsymbol{f}\,,\, \boldsymbol{v}) \, . && (3.5c) \end{aligned}$$

It is readily shown [38] that the problem (3.4) has a unique solution $(\boldsymbol{u}, \boldsymbol{d}, \boldsymbol{\sigma})$ in $V \times D \times S_0$. However, the continuity constant of $a(\cdot,\cdot)$ depends on λ so that the result is not valid in the incompressible limit.

A well-posed formulation, stable in the incompressible limit, may be obtained [38] by extending the formulation (3.4) so that it reads

$$\begin{aligned} a_\alpha((\boldsymbol{u},\boldsymbol{d}),(\boldsymbol{v},\boldsymbol{e})) + b_\alpha((\boldsymbol{v},\boldsymbol{e}),\boldsymbol{\sigma}) &= \ell(\boldsymbol{v}) \ , \quad (\boldsymbol{v},\boldsymbol{e}) \in V \times D \ , && (3.6a) \\ b_\alpha((\boldsymbol{u},\boldsymbol{d}),\boldsymbol{\tau}) - (1-\alpha)(\gamma^2/\lambda)c(\boldsymbol{\sigma},\boldsymbol{\tau}) &= 0 \ , \qquad \boldsymbol{\tau} \in S_0 \ , && (3.6b) \end{aligned}$$

where α is a scalar parameter, and where

$$\gamma = \frac{\lambda}{\kappa} \, , \quad \kappa = 2\mu + d\lambda .$$

Here κ is (d times) the bulk modulus.

The bilinear forms are defined by

$$\begin{aligned} a_\alpha((\boldsymbol{u},\boldsymbol{d}),(\boldsymbol{v},\boldsymbol{e})) &= a_1((\boldsymbol{u},\boldsymbol{d}),(\boldsymbol{v},\boldsymbol{e})) + (\alpha-1)\lambda c(\boldsymbol{d},\boldsymbol{e}) \ , && (3.7a) \\ b_\alpha((\boldsymbol{v},\boldsymbol{e}),\boldsymbol{\tau}) &= b_1((\boldsymbol{v},\boldsymbol{e}),\boldsymbol{\tau}) + (1-\alpha)\gamma c(\boldsymbol{e},\boldsymbol{\tau}) \, , && (3.7b) \\ c(\boldsymbol{e},\boldsymbol{\tau}) &= (\operatorname{tr}\boldsymbol{e}, \operatorname{tr}\boldsymbol{\tau})_0 \, . && (3.7c) \end{aligned}$$

Thus the original three-field formulation is obtained by setting $\alpha = 1$. The motivation behind the modified formulation and its relationship to the original formulation are discussed in [26, 38]. Of importance is the following result.

Lemma 3.1.1 *For $\alpha \neq -\mu/\lambda$, there exists a unique solution $(\boldsymbol{u},\boldsymbol{d},\boldsymbol{\sigma}) \in V \times D \times S_0$ of the modified formulation (3.6).*

Proof This is a repetition of the proof given in [38], with more details. We first show that the solution of problem (3.6) is independent of α. Setting $\boldsymbol{v} = 0$ in equation (3.6a), we have

$$a_\alpha((\boldsymbol{u},\boldsymbol{d}),(0,\boldsymbol{e})) + b_\alpha((0,\boldsymbol{e}),\boldsymbol{\sigma}) = 0 \ , \quad \forall \ \boldsymbol{e} \in D \ . \qquad (3.8)$$

From equation (3.7a) and (3.7b) we have

$$\begin{aligned} a_\alpha((\boldsymbol{u},\boldsymbol{d}),(0,\boldsymbol{e})) &= (\mathcal{C}\boldsymbol{d},\boldsymbol{e})_0 + (\alpha-1)\lambda(\operatorname{tr}\boldsymbol{d},\operatorname{tr}\boldsymbol{e})_0 \\ &= (\mathcal{C}\boldsymbol{d},\boldsymbol{e})_0 + (\kappa\operatorname{tr}\boldsymbol{d},(\alpha-1)\gamma\operatorname{tr}\boldsymbol{e})_0 \\ &= (\mathcal{C}\boldsymbol{d},\boldsymbol{e})_0 + \big(\mathcal{C}\boldsymbol{d},(\alpha-1)\gamma\,(\operatorname{tr}\boldsymbol{e})\mathbf{1}\big)_0 \\ &= \big(\mathcal{C}\boldsymbol{d},\boldsymbol{e} + (\alpha-1)\gamma\,(\operatorname{tr}\boldsymbol{e})\mathbf{1}\big)_0 \, , \end{aligned} \qquad (3.9a)$$

and

$$\begin{aligned} b_\alpha((0,\boldsymbol{e}),\boldsymbol{\sigma}) &= (-\boldsymbol{e},\boldsymbol{\sigma})_0 + (1-\alpha)\gamma(\mathrm{tr}\,\boldsymbol{e},\mathrm{tr}\,\boldsymbol{\sigma})_0 \\ &= (-\boldsymbol{e},\boldsymbol{\sigma})_0 - \big((\alpha-1)\gamma\,(\mathrm{tr}\,\boldsymbol{e})\mathbf{1},\boldsymbol{\sigma}\big)_0 \\ &= -\big(\boldsymbol{\sigma},\boldsymbol{e}+(\alpha-1)\gamma\,(\mathrm{tr}\,\boldsymbol{e})\mathbf{1}\big)_0. \end{aligned} \tag{3.9b}$$

Adding the above equations (3.9a) and (3.9b) and considering the equality (3.8) we obtain

$$\big(\mathcal{C}\boldsymbol{d}-\boldsymbol{\sigma},\boldsymbol{e}+(\alpha-1)\gamma\,(\mathrm{tr}\,\boldsymbol{e})\mathbf{1}\big)_0 = 0 \iff (\mathcal{C}\boldsymbol{d}-\boldsymbol{\sigma},B_\alpha\boldsymbol{e})_0 = 0 \quad \forall\,\boldsymbol{e}\in D, \tag{3.10}$$

where $B_\alpha : \boldsymbol{e} \longmapsto \boldsymbol{e} + (\alpha-1)\gamma\,\mathrm{tr}\,\boldsymbol{e}\mathbf{1}$. The endomorphism B_α is invertible for $\alpha \neq -\dfrac{\mu}{\lambda}$ with inverse defined by

$$B_\alpha^{-1}\boldsymbol{e} = \boldsymbol{e} + \frac{\lambda(1-\alpha)}{2(\mu+\lambda\alpha)}\mathrm{tr}\,\boldsymbol{e}\mathbf{1}.$$

Therefore equation (3.10) implies that

$$\mathcal{C}\boldsymbol{d} = \boldsymbol{\sigma} \implies \mathrm{tr}\,\boldsymbol{\sigma} = \kappa\,\mathrm{tr}\,\boldsymbol{d},$$

which if substituted into equation (3.7b) implies (3.4). As the latter formulation is independent of α, so is the solution of the modified form (3.7).

The existence and uniqueness of a solution of problem (3.6) is carried out by showing that $a_\alpha(\cdot,\cdot)$ is elliptic on (the kernel of B_α)

$$\ker B_\alpha = \{(\boldsymbol{v}\,,\,\boldsymbol{e}) \in V \times D \,:\, b_\alpha((\boldsymbol{v}\,,\,\boldsymbol{e})\,,\,\boldsymbol{\tau}) = 0\,,\ \forall\,\boldsymbol{\tau}\in S_0\}, \tag{3.11}$$

and $b_\alpha(\cdot,\cdot)$ satisfies the *inf-sup* condition.

As the solution of the modified formulation (3.6) is independent of α we now and henceforth fix $\alpha = 0$.

Ellipticity of $a_0(\cdot,\cdot)$

This is shown in [38].

The inf-sup condition

We repeat the proof in [38] with more details. Let

$$\mathsf{M} := \mathrm{tr}\,S_0 = L_0^2(\Omega) = \{q \in L^2(\Omega) \mid \int_\Omega q\,\mathrm{d}x = 0\},$$

then there exists $0 < c < \infty$ such that

$$\forall\, q \in M \;\; \exists\, v_q \in V: \quad \mathsf{dev}\, v_q = q, \quad \|v_q\|_1 \le c\|q\|_0 \,. \tag{3.12}$$

We next consider any $\boldsymbol{\tau} \in S_0$. It is clear that $\mathsf{tr}\,\boldsymbol{\tau} \in M$. Define $\boldsymbol{e}_{\boldsymbol{\mathcal{T}}} = \mathsf{dev}\left(\boldsymbol{\varepsilon}(\boldsymbol{v}_{\mathsf{tr}\,\boldsymbol{\tau}}) - \boldsymbol{\tau}\right) \in D$. Then with the help of equation (3.12) the norm of $\left(\boldsymbol{\varepsilon}(\boldsymbol{v}_{\mathsf{tr}\,\boldsymbol{\tau}}), \boldsymbol{e}_{\boldsymbol{\mathcal{T}}}\right)$ can be bounded as

$$\begin{aligned}
\|\left(\boldsymbol{\varepsilon}(\boldsymbol{v}_{\mathsf{tr}\,\boldsymbol{\tau}}), \boldsymbol{e}_{\boldsymbol{\mathcal{T}}}\right)\|^2_{V\times D} &= \|\boldsymbol{\varepsilon}(\boldsymbol{v}_{\mathsf{tr}\,\boldsymbol{\tau}})\|_1^2 + \|\boldsymbol{e}_{\boldsymbol{\mathcal{T}}}\|_0^2 \\
&\le c\|\mathsf{tr}\,\boldsymbol{\tau}\|_0^2 + \|\mathsf{dev}\,\boldsymbol{\varepsilon}(\boldsymbol{v}_{\mathsf{tr}\,\boldsymbol{\tau}})\|_0^2 + \|\mathsf{dev}\,\boldsymbol{\tau}\|_0^2 \\
&\le C\|\boldsymbol{\tau}\|_0^2 \,.
\end{aligned} \tag{3.13}$$

Noting that $2\mu\mathcal{C}^{-1}\boldsymbol{e}_{\boldsymbol{\mathcal{T}}} = \boldsymbol{e}_{\boldsymbol{\mathcal{T}}}$ and using the equality in (3.12) we have

$$\begin{aligned}
b_0\big((\boldsymbol{v}_{\mathsf{tr}\,\boldsymbol{\tau}}, \boldsymbol{e}_{\boldsymbol{\mathcal{T}}}), \boldsymbol{\tau}\big) &= \Big(\boldsymbol{\varepsilon}(\boldsymbol{v}_{\mathsf{tr}\,\boldsymbol{\tau}}) - \mathsf{dev}\left(\boldsymbol{\varepsilon}(\boldsymbol{v}_{\mathsf{tr}\,\boldsymbol{\tau}}) + \boldsymbol{\tau}\right), \boldsymbol{\tau}\Big) \\
&= \Big(\mathsf{sph}\,\boldsymbol{\varepsilon}(\boldsymbol{v}_{\mathsf{tr}\,\boldsymbol{\tau}}) + \mathsf{dev}\left(\boldsymbol{\tau}\right), \boldsymbol{\tau}\Big) \\
&= \frac{1}{2}\Big(\mathrm{div}\,\boldsymbol{v}_{\mathsf{tr}\,\boldsymbol{\tau}}, \mathsf{tr}\,\boldsymbol{\tau}\Big) + \|\mathsf{dev}\left(\boldsymbol{\tau}\right)\|_0^2 \\
&= \frac{1}{2}\|\mathsf{tr}\,\boldsymbol{\tau}\|_0^2 + \|\mathsf{dev}\left(\boldsymbol{\tau}\right)\|_0^2 \\
&\ge C\|\boldsymbol{\tau}\|_0^2.
\end{aligned} \tag{3.14}$$

The above equation implies that for $\tau \in S_0$ there exists $(\boldsymbol{v}, \boldsymbol{e}) \in V \times D$ such that

$$\frac{b_0\big((\boldsymbol{v}, \boldsymbol{e}), \boldsymbol{\tau}\big)}{\|\big(\boldsymbol{\varepsilon}(\boldsymbol{v}), \boldsymbol{e}\big)\|_{V\times D}}, \ge C\|\boldsymbol{\tau}\|_0$$

from which we deduce that

$$\sup_{(\boldsymbol{v},\boldsymbol{e})\in V\times D} \frac{b_0\big((\boldsymbol{v}, \boldsymbol{e}), \boldsymbol{\tau}\big)}{\|\big(\boldsymbol{\varepsilon}(\boldsymbol{v}), \boldsymbol{e}\big)\|_{V\times D}} \ge C\|\boldsymbol{\tau}\|_0 \quad \forall \boldsymbol{\tau} \in S_0,$$

that is to say that the *inf-sup* condition is satisfied. ∎

Remark Because the bilinear form $c(\cdot,\cdot)$, in (3.7c), is positive semi-definite and uniformly continuous it does not play any role in establishing the stability of the problem (3.6). In fact it has been shown by Braess [14] that for symmetric problems such us (3.6) the existence and uniqueness of the solution depends only on the ellipticity and *inf-sup* condition of the bilinear forms $a_\alpha(\cdot,\cdot)$ and $b_\alpha(\cdot,\cdot)$ respectively. See also [38] and references therein.

3.2 Finite element approximations

3.2.1 Finite element spaces

The discrete form of the standard three-field problem is as follows: find $(\boldsymbol{u}_h, \boldsymbol{d}_h, \boldsymbol{\sigma}_h) \in V_h \times D_h \times S_h$ such that

$$\begin{aligned} a_1((\boldsymbol{u}_h, \boldsymbol{d}_h), (\boldsymbol{v}_h, \boldsymbol{e}_h)) + b_1((\boldsymbol{v}_h, \boldsymbol{e}_h), \boldsymbol{\sigma}_h) &= \ell(\boldsymbol{v}_h), \qquad (\boldsymbol{v}_h, \boldsymbol{e}_h) \in V_h \times D_h \ , \\ b_1((\boldsymbol{u}_h, \boldsymbol{d}_h), \boldsymbol{\tau}_h) &= 0, \qquad \boldsymbol{\tau}_h \in S_h \ . \end{aligned} \tag{3.15}$$

More explicitly, (3.15) reads as follows:

$$\int_\Omega (\mathcal{C}\boldsymbol{d}_h - \boldsymbol{\sigma}_h) : \boldsymbol{e}_h \, dx = 0 \quad \boldsymbol{e}_h \in D_h \ , \tag{3.16a}$$

$$\int_\Omega (\boldsymbol{\varepsilon}(\boldsymbol{u}_h) - \boldsymbol{d}_h) : \boldsymbol{\tau}_h \, dx = 0 \ , \quad \boldsymbol{\tau}_h \in S_h \ , \tag{3.16b}$$

$$\int_\Omega \boldsymbol{\sigma}_h : \boldsymbol{\varepsilon}(\boldsymbol{v}_h) \, dx = \ell(\boldsymbol{v}_h) \ , \quad \boldsymbol{v}_h \in V_h \ . \tag{3.16c}$$

Likewise, the discrete form of the modified problem (3.6) reads as follows: find $(\boldsymbol{u}_h, \boldsymbol{d}_h, \boldsymbol{\sigma}_h) \in V_h \times D_h \times S_h$ such that

$$\begin{aligned} a_\alpha((\boldsymbol{u}_h, \boldsymbol{d}_h), (\boldsymbol{v}_h, \boldsymbol{e}_h)) + b_\alpha((\boldsymbol{v}_h, \boldsymbol{e}_h), \boldsymbol{\sigma}_h) &= \ell(\boldsymbol{v}_h), \qquad (\boldsymbol{v}_h, \boldsymbol{e}_h) \in V_h \times D_h \ , \\ b_\alpha((\boldsymbol{u}_h, \boldsymbol{d}_h), \boldsymbol{\tau}_h) &= 0, \qquad \boldsymbol{\tau}_h \in S_h \ . \end{aligned} \tag{3.17}$$

It is shown in [38] that if the conditions

$$S_h \subset D_h \, , \tag{3.18}$$

$$\mathcal{C} D_h = D_h \, , \tag{3.19}$$

are satisfied, then the solution to the problem (3.15) coincides with that for the discrete modified problem (3.17), for any value of α.

Remark The conditions (3.18) and (3.19) are assumed to hold in the remainder of this chapter. In addition, for convenience we focus attention on the case $\alpha = 0$, for which the standard and modified problems are equivalent.

The assumptions on S_h and D_h result in the equivalence between the three-field and the two-field Hellinger-Reissner problems. The question therefore arises as to whether it might not be more convenient to work with the two-field Hellinger-Reissner formulation. Despite the fact that the three-field formulation involves one more variable, it carries certain advantages. Most particularly, the displacement is one of the primary variables, so that static condensation leads naturally to a displacement-based formulation, which is advantageous in many respects. For example, in problems of elastoplasticity the use of a stress-based Hellinger-Reissner formulation does not permit the highly effective return mapping algorithm to be applied at integration points; instead the computations must be carried out at element level [43].

3.3 Well-posedness and convergence

We present here a summary of the main results on well-posedness and convergence that have been established in [38], and which will be needed in the later analyses. First, it is necessary to define discrete counterparts of the deviatoric and spherical parts of the stress field, as it does not follow that for $\boldsymbol{\sigma}_h \in S_h$,

$$\mathsf{dev}\, \boldsymbol{\sigma}_h \in S_h \quad \text{and} \quad \mathsf{sph}\, \boldsymbol{\sigma}_h \in S_h \,.$$

The discrete deviatoric operator dev_h is defined by

$$\mathsf{dev}_h \boldsymbol{\sigma}_h = P_{S_h} \mathsf{dev}\, \boldsymbol{\sigma}_h \ , \tag{3.20}$$

where P_{S_h} denotes the L^2-orthogonal projection operator onto S_h. Then the discrete spherical operator sph_h is defined in such a way that $\mathsf{sph}_h\, S_h$ is the orthogonal complement of $\mathsf{dev}_h\, S_h$, so that

$$S_h = \mathsf{dev}_h\, S_h \oplus \mathsf{sph}_h\, S_h \,. \tag{3.21}$$

The corresponding space of discrete pressures $\widetilde{M}_h$ is then defined by

$$\widetilde{M}_h := \mathsf{tr}\, \mathsf{sph}_h S_h \ . \tag{3.22}$$

The main result on well-poseness of the discrete problem is as follows, for the case in which the standard and modified formulations are equivalent.

Theorem 3.3.1 *[38] Suppose that the spaces V_h, D_h and S_h satisfy the following conditions:*

(a) $\|P_{S_h} \boldsymbol{\varepsilon}(\boldsymbol{v}_h)\|_0 \geq C_0 \|\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\|_0$ *for all* $\boldsymbol{v}_h \in V_h$,

(b) $(V_h, \widetilde{M}_h)$ *forms a stable Stokes pairing,*

where C_0 *is a positive constant independent of the mesh size* h*. Then the bilinear form* $b_\alpha(\cdot\,,\,\cdot)$ *satisfies a uniform inf-sup condition and the discretization error* $\eta_h^2 := \|\boldsymbol{u} - \boldsymbol{u}_h\|_1^2 + \|\boldsymbol{d} - \boldsymbol{d}_h\|_0^2 + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0^2$ *of problem (3.6) is bounded by the best approximation error*

$$\eta_h^2 \le C \left(\inf_{\boldsymbol{v}_h \in V_h} \|\boldsymbol{u} - \boldsymbol{v}_h\|_1^2 + \inf_{\boldsymbol{e}_h \in D_h} \|\boldsymbol{d} - \boldsymbol{e}_h\|_0^2 + \inf_{\boldsymbol{\tau}_h \in S_h} \|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_0^2 \right) , \tag{3.23}$$

in which the constant $C > 0$ *is independent of* λ*.*

As with the continuum formulation (3.6) the well-posedness of the discrete form (3.17) depends only on the coercivity and the inf-sup condition. We therefore have to prove that based on the conditions of the above Theorem 3.3.1 the bilinear form $a_0(\cdot,\cdot)$ is B_0-elliptic and $b_0(\cdot,\cdot)$ satisfies the *inf-sup* condition, where B_0, as defined in (3.11) with $\alpha = 0$, is the kernel of $b_0(\cdot,\cdot)$.

Lemma 3.3.2 (Coercivity: [38]) *Assume that (a) of Theorem 3.3.1 is satisfied. Then the bilinear form* $a_0(\cdot,\cdot)$ *is uniformly elliptic in the kernel of* $b_0(\cdot,\cdot)$*.*

Proof See [38]. ∎

Lemma 3.3.3 (Inf-sup condition: [38]) *Assume that (3.18), (a) and (b) of Theorem 3.3.1 are satisfied. Then the bilinear form* $b_0(\cdot,\cdot)$ *satisfies a uniform* inf-sup *condition.*

Proof We provide a detailed and simplified proof of [38]. For any $\boldsymbol{\tau}_h \in S_h$ consider its decomposition, as follows

$$\boldsymbol{\tau}_h = \mathsf{dev}_h \boldsymbol{\tau}_h + \mathsf{sph}_h \boldsymbol{\tau}_h = \mathsf{dev}_h \boldsymbol{\tau}_h + q_h \mathbf{1}, \tag{3.24}$$

where $q_h \in \widetilde{M}_h$. As $(V_h, \widetilde{M}_h)$ is a stable Stokes pair for any $q_h \in \widetilde{M}_h$ there exist $\boldsymbol{v}_{q_h} \in V_h$, such that

$$(\operatorname{div} \boldsymbol{v}_{q_h}, q_h) = \|q_h\|_0^2 \quad \text{and} \quad \|\boldsymbol{v}_{q_h}\|_1 \le C\|q_h\|_0 \,.$$

We next define $\boldsymbol{e}_h \in D_h$, such that

$$\bar{e}_h = \mathsf{dev}_h \left(\boldsymbol{\varepsilon}(\boldsymbol{v}_{q_h}) - \boldsymbol{\tau}_h\right)$$

which implies that

$$\begin{aligned}\|\boldsymbol{v}_{q_h}\|_1^2 + \|\bar{\boldsymbol{e}}_h\|_0^2 &\leq C\|q_h\|_0^2 + \|\boldsymbol{\varepsilon}(\boldsymbol{v}_{q_h})\|_0^2 + \|\mathsf{dev}_h\boldsymbol{\tau}_h\|_0^2 \\ &\leq C(\|q_h\|_0^2 + \|\boldsymbol{v}_{q_h}\|_1^2) + \|\mathsf{dev}_h\boldsymbol{\tau}_h\|_0^2 \\ &\leq C\|\boldsymbol{\tau}_h\|_0^2\end{aligned} \tag{3.25}$$

Next we observe that $2\mu\mathcal{C}^{-1}\bar{\boldsymbol{e}}_h = \bar{\boldsymbol{e}}_h$ and then

$$\begin{aligned}b_0\big((\boldsymbol{v}_{q_h}, \bar{\boldsymbol{e}}_h), \boldsymbol{\tau}_h\big) &= \big(\boldsymbol{\varepsilon}(\boldsymbol{v}_{q_h}) - 2\mu\mathcal{C}^{-1}\bar{\boldsymbol{e}}_h, \boldsymbol{\tau}_h\big) \\ &= \Big(\boldsymbol{\varepsilon}(\boldsymbol{v}_{q_h}) - \mathsf{dev}_h\left(\boldsymbol{\varepsilon}(\boldsymbol{v}_{q_h}) - \boldsymbol{\tau}_h\right), \boldsymbol{\tau}_h\Big) \\ &= (\mathsf{sph}_h\, \boldsymbol{\varepsilon}(\boldsymbol{v}_{q_h}), \boldsymbol{\tau}_h) + (\mathsf{dev}\,(\boldsymbol{\tau}_h), \boldsymbol{\tau}_h) \\ &= (\mathrm{div}(\boldsymbol{v}_{q_h}), q_h) + \|\mathsf{dev}\,(\boldsymbol{\tau}_h)\|_0^2 \\ &= \|q_h\|_0^2 + \|\mathsf{dev}\,(\boldsymbol{\tau}_h)\|_0^2 \\ &= \|\boldsymbol{\tau}_h\|_0^2\end{aligned} \tag{3.26}$$

Using (3.25) and (3.26) we can write

$$\frac{b_0\big((\boldsymbol{v}_{q_h}, \bar{\boldsymbol{e}}_h), \boldsymbol{\tau}_h\big)}{\|(\boldsymbol{v}_{q_h}, \bar{\boldsymbol{e}}_h\|_{V_h \times D_h}} \geq C\frac{\|\boldsymbol{\tau}_h\|_0^2}{\|\boldsymbol{\tau}_h\|_0} = C\|\boldsymbol{\tau}_h\|_0,$$

and then

$$\sup_{(\boldsymbol{v}_h, \bar{\boldsymbol{e}}_h)} \frac{b_0\big((\boldsymbol{v}_h, \bar{\boldsymbol{e}}_h), \boldsymbol{\tau}_h\big)}{\|(\boldsymbol{v}_h, \bar{\boldsymbol{e}}_h\|_{V_h \times D_h}} \geq C\|\boldsymbol{\tau}_h\|_0.$$

∎

For the error estimate (3.23) see [38] and [11].

3.4 More general bases

In [26, 38] the key motivation has been to investigate mixed methods in two space dimensions based on the low-order four-noded bilinear quadrilateral element for displacements. For a set of cases studied in those works, the relevant Stokes velocity-pressure pairing is the classical $Q_1 - P_0$ pair of bilinear displacements and piecewise-constant pressures, for which the existence of checkerboard modes is well known. It therefore becomes necessary to define $\widetilde{M}_h$ to be a subset of the piecewise constants, for example by working with macro-elements and extracting from these the checkerboard modes. In any event, it can be shown that the displacement satisfies an a priori error estimate that is uniform in the incompressible limit.

The objective of this chapter is a broader one, viz. that of generating new families of mixed elements based on the following strategy:

(i) start with a known stable Stokes pairing, and on this basis choose the space of discrete displacements;

(ii) since $\widetilde{M}_h$ is given, construct a space S_h of stresses for which $\operatorname{tr}\operatorname{sph}_h S_h = \widetilde{M}_h$, and the ellipticity condition (a) in Theorem 3.3.1 is satisfed;

(iii) select D_h such that assumptions (3.18) and (3.19) are satisfied.

In this chapter a number of examples of mixed formulations constructed in this way will be presented. In all cases the space V_h will comprise continuous functions, while S_h and D_h will comprise piecewise polynomials, continuous or discontinuous, depending on the underlying pressure approximation $\widetilde{M}_h$.

The following section discusses the necessary and sufficient conditions for the finite element approximations to satisfy the assumptions (a) of Theorem 3.3.1.

3.5 The ellipticity condition

As we have discussed in Section 3.3, the stability of the modified formulation (3.6) depends entirely on the assumptions (a) and (b) of Theorem 3.3.1, assuming that (3.18) and (3.19) are trivial.

This section is devoted to necessary conditions for the finite element approximations to satisfy the hypothesis (a) of Theorem 3.3.1. Two different approaches for the satisfaction of such conditions are presented.

In the first approach, which is presented in the following Subsection 3.5.1, we prove that the analysis of the ellipticity condition (a) depends on the properties of the basis of the spaces of stresses. In the second approach, two properties of S_h are proven to be necessary:

a) no non-trivial member of $\boldsymbol{\varepsilon}(V_h)$ is allowed to be orthogonal to S_h;

b) The dimension of S_h must be big enough, compared to the dimension of the symmetric gradient of $\boldsymbol{\varepsilon}(V_h)$.

3.5.1 First approach for satisfaction of the ellipticity condition

There is an extensive set of stable element pairs for the Stokes problem available in the literature (see for example [11, 32]), and no attempt will be made to generate a comprehensive set of corresponding elements for the three-field problem. Rather, the focus will be on the selected elements in Table 3.1, and already discussed in Chapter 2.

Table 3.1: Some Stokes-stable pairs of elements

Element		$V_h - \widetilde{M}_h$
MINI [3]		$(\mathcal{P}_1 \oplus \mathcal{B}_{d+1}) - \mathcal{P}_1$
Crouzeix-Raviart [24]		$(\mathcal{P}_2 \oplus \mathcal{B}_3)/(\mathcal{P}_2 \oplus \mathcal{B}_4 \oplus \mathcal{A}_3) - \mathcal{P}'_1$
Generalized Hood-Taylor [33]	(1)	$\mathcal{P}_2 - \mathcal{P}_1$
	(2)	$\mathcal{Q}_2 - \mathcal{P}'_1$
	(3)	$\mathcal{Q}_2 - \mathcal{Q}_1$

The choice of V_h as in Table 3.1 and a construction of S_h such that (3.22) is satisfied, together ensure that the discrete inf-sup condition is satisfied uniformly. It remains then to verify that the ellipticity condition, condition (a) in Theorem 3.3.1, is satisfied, for the bases chosen. To this end we will require the following results.

Lemma 3.5.1 *Assume that S_h is constructed according to*

$$S_h = (S_h)_{ij} \tag{3.27}$$

in which $(S_h)_{ji} = (S_h)_{ij}$ and each $(S_h)_{ij}$ is a space of piecewise polynomials, continuous or discontinuous. Denote by η_{ij} the components of $\boldsymbol{\eta} \in \boldsymbol{\varepsilon}(V_h)$, and the L^2-orthogonal projection of η_{ij} onto $(S_h)_{ij}$ by $P_{(S_h)_{ij}}\eta_{ij}$. Then if the components η_{ij} satisfy the inequality

$$\|P_{(S_h)_{ij}}\eta_{ij}\|_0 \geq C_0\|\eta_{ij}\|_0 \qquad (i, j = 1, \ldots, d)\,, \tag{3.28}$$

$\boldsymbol{\eta}$ *satisfies the bound*

$$\|P_{S_h}\boldsymbol{\eta}\|_0 \geq C_0\|\boldsymbol{\eta}\|_0\,. \tag{3.29}$$

Proof Note that by definition, $\big(P_{S_h}\boldsymbol{\eta},\boldsymbol{\tau}_h\big) = \big(\boldsymbol{\eta},\boldsymbol{\tau}_h\big) \quad \forall\,\boldsymbol{\tau}_h \in S_h$, which implies that

$$\sum_{i,j}\big((P_{S_h}\boldsymbol{\eta})_{ij},(\boldsymbol{\tau}_h)_{ij}\big)_0 = \sum_{i,j}\big(\eta_{ij},(\boldsymbol{\tau}_h)_{ij}\big)_0.$$

With a proper choice of the components of $\boldsymbol{\eta}$ one can easily show that the above equality also holds for fixed i and j. We therefore have $(P_{S_h}\boldsymbol{\eta})_{ij} = P_{(S_h)_{ij}}\eta_{ij}$. Using the assumption (3.28) we can write

$$\|P_{S_h}\boldsymbol{\eta}\|_0^2 = \sum_{i,j}\|(P_{S_h}\boldsymbol{\eta})_{ij}\|_0^2 = \sum_{i,j}\|P_{(S_h)_{ij}}\eta_{ij}\|_0^2 \geq C_0^2\sum_{i,j}\|\eta_{ij}\|_0^2 = C_0^2\|\boldsymbol{\eta}\|_0^2,$$

which gives the inequality (3.29). ∎

Lemma 3.5.2 *Suppose that* $\{\boldsymbol{\eta}_{h(I)}\}$ *is a basis of* $\varepsilon(V_h)$ *such that*

$$\|P_{S_h}\boldsymbol{\eta}_{h(I)}\|_0 \geq C_0\|\boldsymbol{\eta}_{h(I)}\|_0\,, \qquad 0 < C_0 \leq 1\,. \tag{3.30}$$

If the set $\{P_{S_h}\boldsymbol{\eta}_{h(I)}\}$ *is linearly independent in* S_h*, then the ellipticity condition (a) in Theorem 3.3.1 is satisfied on* $\varepsilon(V_h)$*.*

Lemma 3.5.3 *Assume that the conditions of Lemma 3.5.2 are satisfied. Then the map*

$$\begin{aligned} \|\cdot\|_* :\quad \varepsilon(V_h) &\longrightarrow \mathbb{R}^+ \\ \boldsymbol{\eta}_h &\longmapsto \|\boldsymbol{\eta}_h\|_* = \|P_{S_h}\boldsymbol{\eta}_h\|_0\,, \end{aligned} \tag{3.31}$$

is a continuous in $\varepsilon(V_h)$*.*

Proof The continuity is a consequence of the following inequality

$$\|P_{S_h}\boldsymbol{\eta}_h\|_0 \leq \|\boldsymbol{\eta}_h\|_0 \quad \forall\,\boldsymbol{\eta}_h \in \varepsilon(V_h)\,,$$

which, by the linearity of P_{S_h} implies that

$$\begin{aligned} \Big|\|\boldsymbol{\eta}_h\|_* - \|\boldsymbol{\beta}_h\|_*\Big| &= \Big|\|P_{S_h}\boldsymbol{\eta}_h\|_0 - \|P_{S_h}\boldsymbol{\beta}_h\|_0\Big| \\ &\leq \|P_{S_h}\boldsymbol{\eta}_h - P_{S_h}\boldsymbol{\beta}_h\|_0 \\ &= \|P_{S_h}(\boldsymbol{\eta}_h - \boldsymbol{\beta}_h)\|_0 \\ &\leq \|\boldsymbol{\eta}_h - \boldsymbol{\beta}_h\|_0\,, \end{aligned}$$

$\forall\,\boldsymbol{\eta}_h,\,\boldsymbol{\beta}_h \in \varepsilon(V_h)$. ∎

Proposition 3.5.4 *Consider $\|\cdot\|_*$ of Lemma 3.5.3 and $\|\cdot\|_{0,W_h}$ the restriction of $\|\cdot\|_0$ to the finite-dimensional subspace $W_h = \boldsymbol{\varepsilon}(V_h)$. Then assuming that the conditions of Lemmas 3.5.2 and 3.5.3 are satisfied, there exists a positive constant C, independent of the mesh size, such that*

$$\|\boldsymbol{w}_h\|_* \geq C\|\boldsymbol{w}_h\|_{0,W_h}, \quad \forall\, \boldsymbol{w}_h \in W_h.$$

Proof Consider the unit sphere

$$\Sigma_h = \{\boldsymbol{u}_h \in W_h :\; \|\boldsymbol{u}_h\|_{0,W_h} = 1\}.$$

It is clear that Σ_h is a bounded and closed subset of W_h, thus Σ_h is also compact. Now every continuous function on Σ_h attains its minimum on Σ_h, and as $\|\cdot\|_*$ is a continuous function, there exists $\boldsymbol{u}_h \in \Sigma_h$ that satisfies

$$\|\boldsymbol{w}_h\|_* \geq \|\boldsymbol{u}_h\|_*, \quad \forall\, \boldsymbol{w}_h \in \Sigma_h\,.$$

Next, consider any $\boldsymbol{w}_h \in W_h$ and define $\boldsymbol{v}_h = \dfrac{\boldsymbol{w}_h}{\|\boldsymbol{w}_h\|_{0,W_h}} \in \Sigma_h$; we then obtain

$$\|\boldsymbol{v}_h\|_* = \frac{\|\boldsymbol{w}_h\|_*}{\|\boldsymbol{w}_h\|_{0,W_h}} \geq \|\boldsymbol{u}_h\|_* \quad \Longrightarrow \quad \|\boldsymbol{w}_h\|_* \geq \|\boldsymbol{u}_h\|_*\|\boldsymbol{w}_h\|_{0,W_h}.$$

The existence of C is established by taking it to be equal to $\|\boldsymbol{u}_h\|_*$. It remains to prove its independence of h. Let $c_h = \|\boldsymbol{u}_h\|_*$. If the sequence $\{c_h\}$ can be bounded below by a positive constant independent of h then we have ended the proof. Assume on the contrary that c_h cannot be bounded below; then one can construct a subsequence $\{c_{h_n}\}$ that converges to zero. Thus,

$$\lim_{n\longrightarrow\infty} c_{h_n} = \lim_{n\longrightarrow\infty} \|\boldsymbol{u}_{h_n}\|_* = 0 \;\Longrightarrow\; \lim_{n\longrightarrow\infty} \|P_{S_{h_n}}\boldsymbol{u}_{h_n}\|_0 = 0\,;$$

that is, to say that the sequence $\{P_{S_{h_n}}\boldsymbol{u}_{h_n}\}$ also converges to $\mathbf{0}$. For convenience denote $P_{S_{h_n}}$ by ℓ_n; that is,

$$\begin{aligned} \ell_n \;:\quad \boldsymbol{\varepsilon}(V_{h_n}) &\longrightarrow P_{S_{h_n}}\boldsymbol{\varepsilon}(V_{h_n})\,, \\ \ell\boldsymbol{\eta}_{h_n} &= P_{S_{h_n}}\boldsymbol{\eta}_{h_n}\,. \end{aligned}$$

It is clear that ℓ_n is linear and continuous; indeed the continuity follows from

$$\|P_{S_{h_n}}\boldsymbol{\eta}_{h_n}\|_0 \leq \|\boldsymbol{\eta}_{h_n}\|_0$$

and the linearity is an immediate consequence of its definition. ℓ_n is also a bijection because it maps the basis of $\varepsilon(V_{h_n})$ into a linearly independent family. Thus there exists ℓ_n^{-1}, linear and continuous, the inverse of ℓ_n. We observe that

$$\begin{aligned} \|\ell_n^{-1}\|' &= \sup\{\|\ell_n^{-1}(P_{S_{h_n}}\boldsymbol{\eta}_h)\| \ : \ \boldsymbol{\eta}_h \in \boldsymbol{\varepsilon}(V_{h_n})\,,\ \|P_{S_{h_n}}\boldsymbol{\eta}_{h_n}\|_0 \leq 1\} \\ &\leq \sup\{\|\boldsymbol{\eta}_h\|_0 \ : \ \boldsymbol{\eta}_h \in \boldsymbol{\varepsilon}(V_{h_n})\,,\ \|\boldsymbol{\eta}_{h_n}\|_0 \leq 1\} = 1\,, \end{aligned} \tag{3.32a}$$

where $\|\cdot\|'$ is the dual norm in $\varepsilon(V_{h_n})$. By definition of ℓ_n^{-1} we have

$$\boldsymbol{u}_{h_n} = \ell_n^{-1}(P_{S_{h_n}}\boldsymbol{u}_{h_n})\,,$$

which implies that

$$\|\boldsymbol{u}_{h_n}\|_0 = \|\ell_n^{-1}(P_{S_{h_n}}\boldsymbol{u}_{h_n})\|_0 \leq \|\ell_n^{-1}\|'\|P_{S_{h_n}}\boldsymbol{u}_{h_n}\|_0 \leq \|P_{S_{h_n}}\boldsymbol{u}_{h_n}\|_0 \tag{3.32b}$$

Recalling that the sequence $\{P_{S_{h_n}}\boldsymbol{u}_{h_n}\}$ converges to zero the inequalities (3.32a) and (3.32b) imply that

$$\lim_{n\longrightarrow\infty} \boldsymbol{u}_{h_n} = \lim_{n\longrightarrow\infty} P_{S_{h_n}}\boldsymbol{u}_{h_n} = \boldsymbol{0}\,,$$

which is in contradiction with the fact that $\boldsymbol{u}_{h_n}$ belongs to the unit sphere. Hence the only possibility is the sequence c_{h_n} can be bounded below by a strictly positive constant, independent of the mesh size. ▮

Proof of Lemma 3.5.2. Based on the assumption of this lemma both Lemma 3.5.3 and Proposition 3.5.4 are satisfied. Therefore, from the latter, there exists a positive constant C independent of h such that

$$\|\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\|_* \geq C\|\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\|_0 \quad \forall\, \boldsymbol{v}_h \in V_h\,.$$

Applying the definition of $\|\|_*$ to the above we can write

$$\|P_{S_h}\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\|_0 \geq C\|\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\|_0 \quad \forall\, \boldsymbol{v}_h \in V_h\,,$$

which is the condition (a) in Theorem 3.3.1. ▮

Lemma 3.5.5 *Let T_h be a subspace of S_h. If*

$$\|P_{T_h}\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\|_0 \geq C_0\|\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\|_0 \qquad \textit{for all } \boldsymbol{v}_h \in V_h\,,$$

then condition (a) in Theorem 3.3.1 is satisfied.

Proof Let $T_h^\perp \subset S_h$ be such that $S_h = T_h \oplus T_h^\perp$. Then for any $\boldsymbol{\sigma}_h \in S_h$ one can find $\boldsymbol{\tau}_h \in T_h$ and $\boldsymbol{\tau}_h^\perp \in T_h^\perp$ such that $\boldsymbol{\sigma}_h = \boldsymbol{\tau}_h + \boldsymbol{\tau}_h^\perp$. However as $\boldsymbol{\tau}_h \perp \boldsymbol{\tau}_h^\perp$ we have

$$\|\boldsymbol{\sigma}_h\|_0^2 = \|\boldsymbol{\tau}_h + \boldsymbol{\tau}_h^\perp\|_0^2 = \|\boldsymbol{\tau}_h\|_0^2 + \|\boldsymbol{\tau}_h^\perp\|_0^2.$$

We observe that $P_{S_h}(\boldsymbol{\varepsilon}(\boldsymbol{v}_h)) = P_{T_h}(\boldsymbol{\varepsilon}(\boldsymbol{v}_h)) + P_{T_h^\perp}(\boldsymbol{\varepsilon}(\boldsymbol{v}_h))$, hence

$$\|P_{S_h}(\boldsymbol{\varepsilon}(\boldsymbol{v}_h))\|_0^2 = \|P_{T_h}(\boldsymbol{\varepsilon}(\boldsymbol{v}_h))\|_0^2 + \|P_{T_h^\perp}(\boldsymbol{\varepsilon}(\boldsymbol{v}_h))\|_0^2 \geq \|P_{T_h}(\boldsymbol{\varepsilon}(\boldsymbol{v}_h))\|_0^2 \geq c_0^2\|\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\|_0^2.$$

∎

In our examples the discrete space of displacements comprises continuous, piecewise-polynomial functions. The following lemma gives a sufficient condition for the satisfaction of (3.28) and therefore (3.29).

Lemma 3.5.6 *For $\boldsymbol{\eta} \in \boldsymbol{\varepsilon}(V_h)$, assume that $(P_{S_h}\boldsymbol{\eta})_{ij} \neq 0$ for $\eta_{ij} \neq 0$. Then the inequalities (3.28) and (3.29) are satisfied.*

Proof Using the triangle inequality we can write

$$\|\eta_{ij}\|_0 \leq \|\eta_{ij} - (P_{S_h}\boldsymbol{\eta})_{ij}\|_0 + \|(P_{S_h}\boldsymbol{\eta})_{ij}\|_0. \tag{3.33}$$

There exists a positive constant $0 < a < 1$ such that $\|\eta_{ij} - (P_{S_h}\boldsymbol{\eta})_{ij}\|_0 \leq a\|\eta_{ij}\|_0$. To see clearly that $a < 1$ we first observe that η_{ij} can be decomposed as

$$\eta_{ij} = (P_{S_h}\boldsymbol{\eta})_{ij} \oplus v$$

where v is the orthogonal complement of $(P_{S_h}\boldsymbol{\eta})_{ij}$ in $\big(\boldsymbol{\varepsilon}(V_h)\big)_{ij}$. Second, using the fact that $(P_{S_h}\boldsymbol{\eta})_{ij} \neq 0$ and $(P_{S_h}\boldsymbol{\eta})_{ij} \perp v$ we have

$$\|\eta_{ij}\|_0^2 = \|(P_{S_h}\boldsymbol{\eta})_{ij}\|_0^2 + \|v\|_0^2 > \|v\|_0^2\,.$$

As $v = \eta_{ij} - \big(P_{S_h}\boldsymbol{\eta}\big)_{ij}$, it follows that

$$\|v\|_0^2 = \|\eta_{ij} - \big(P_{S_h}\boldsymbol{\eta}\big)_{ij}\|_0^2 < \|\eta_{ij}\|_0^2\,,$$

which gives rise to the conclusion $a < 1$.

We have therefore shown the existence of a constant $C_0 = 1 - a$ such that the inequalities (3.28) and hence (3.29) are satisfied. ∎

3.5.2 Second approach for satisfaction of the ellipticity condition

In order to establish necessary conditions for the satisfaction of the assumption (a) of Theorem 3.3.1 consider the family $\{\boldsymbol{\eta}_i\}_{1\leq i\leq n}$ and $\{\boldsymbol{\tau}_i\}_{1\leq i\leq m}$ to be the bases of $\varepsilon(V_h)$ and S_h respectively, where n is the dimension of $\varepsilon(V_h)$ and m the dimension of S_h. The orthogonal projection of any $\boldsymbol{\eta}_h \in \varepsilon(V_h)$ onto S_h is defined by the following system of equations:

$$\big(P_{S_h}\boldsymbol{\eta}_h\,,\,\boldsymbol{\tau}_j\big)_0 = \big(\boldsymbol{\eta}_h\,,\,\boldsymbol{\tau}_j\big)_0, \quad \forall\, j = 1,\cdots,m. \tag{3.34a}$$

Let $\boldsymbol{a} = (a_1,\cdots,a_n)^T$ and $\boldsymbol{\alpha} = (\alpha_1,\cdots,\alpha_m)^T$ denote the components of $\boldsymbol{\eta}_h$ and $P_{S_h}\boldsymbol{\eta}_h$ with respect to the relevant bases. Then (3.34a) can explicitly be written as

$$\sum_{i=1}^{m}\alpha_i\big(\boldsymbol{\tau}_i\,,\,\boldsymbol{\tau}_j\big)_0 = \sum_{i=1}^{n}a_i\big(\boldsymbol{\eta}_i\,,\,\boldsymbol{\tau}_j\big)_0, \quad \forall\, j = 1,\cdots,m. \tag{3.34b}$$

The above equation is equivalent to the algebraic formulation

$$\boldsymbol{M}\boldsymbol{\alpha} = \boldsymbol{N}\boldsymbol{a}, \tag{3.34c}$$

where $\boldsymbol{M}$ is the Gram matrix and $\boldsymbol{N}$ the matrix with coefficients $\big(\boldsymbol{\eta}_i\,,\,\boldsymbol{\tau}_j\big)_0$, $i = 1,\cdots,n$ and $j = 1,\cdots,m$.

To prove the condition (a) of Theorem 3.3.1 we state the following lemmas.

Lemma 3.5.7 *If the matrix $\boldsymbol{N}$ in equation (3.34c) is of full rank, $n = \dim\varepsilon(V_h)$, then condition (a) of Theorem 3.3.1 is satisfied.*

Proof Using (3.34c) the norm of the projection of $\boldsymbol{\eta}_h \in \varepsilon(V_h)$ onto S_h is given by

$$\begin{aligned}\|P_{S_h}\boldsymbol{\eta}_h\|_0^2 =& \Big(P_{S_h}\boldsymbol{\eta}_h, P_{S_h}\boldsymbol{\eta}_h\Big) = \sum_{i,j=1}^{m}\alpha_i\alpha_j\big(\boldsymbol{\tau}_i\,,\,\boldsymbol{\tau}_j\big)_0\\ =&\boldsymbol{\alpha}\boldsymbol{M}\boldsymbol{\alpha}^T = \boldsymbol{a}(\boldsymbol{M}^{-1}\boldsymbol{N})^T\boldsymbol{M}(\boldsymbol{M}^{-1}\boldsymbol{N})\boldsymbol{a}^T = \boldsymbol{a}\boldsymbol{N}^T\boldsymbol{M}^{-1}\boldsymbol{N}\boldsymbol{a}^T.\end{aligned} \tag{3.35}$$

By hypothesis the rank of $\boldsymbol{N}$ is equal to n, thus the symmetric matrix $\boldsymbol{N}^T\boldsymbol{M}^{-1}\boldsymbol{N}$ is positive definite of full rank $n = \dim\varepsilon(V_h)$, therefore there exists $\gamma > 0$, the smallest eigenvalue of $\boldsymbol{N}^T\boldsymbol{M}^{-1}\boldsymbol{N}$, such that

$$\boldsymbol{a}\boldsymbol{N}^T\boldsymbol{M}^{-1}\boldsymbol{N}\boldsymbol{a}^T \geq \gamma\boldsymbol{a}\boldsymbol{a}^T = \gamma\|\boldsymbol{\eta}_h\|_{\varepsilon(V_h)}^2\,,$$

where $\| \cdot \|^2_{\varepsilon(V_h)}$ is the Euclidean norm in $\varepsilon(V_h)$; that is, $\|\boldsymbol{\eta}_h\|^2_{\varepsilon(V_h)} = \boldsymbol{a}\boldsymbol{a}^T$ if $\boldsymbol{a}$ is the set of coordinates of $\boldsymbol{\eta}_h$. Without any loss of generality assume that the base $(\boldsymbol{\eta}_j)_{1\leq j\leq n}$ of $\boldsymbol{\varepsilon}(\boldsymbol{v}_h)$ is orthonormal. We then have

$$\|\boldsymbol{\eta}_h\|_0^2 = \|\sum_{j=1}^{n} a_j\boldsymbol{\eta}_j\|_0^2 = \sum_{j=1}^{n} |a_j|^2\|\boldsymbol{\eta}_j\|_0^2 = \sum_{j=1}^{n} |a_j|^2 = \|\boldsymbol{\eta}_h\|^2_{\varepsilon(V_h)}. \tag{3.36}$$

If we consider $c_0 = \min(\sqrt{\gamma}, 1)$ we arrive at the desired inequality

$$\|P_{S_h}\boldsymbol{\eta}_h\|_0 \geq c_0\|\boldsymbol{\eta}_h\|_0.$$

∎

Lemma 3.5.8 *The matrix $\boldsymbol{N}$ given in (3.34c) is of full rank if and only if the conditions*

- *(a) dim $\varepsilon(V_h) \leq$ dim S_h,*
- *(b) zero is the only member of $\varepsilon(V_h)$ that is orthogonal to S_h,*

hold true.

Proof Let $n =$ dim $(\varepsilon(V_h))$ and $m =$ dim (S_h). To prove that the conditions of the lemma are necessary first assume that $n > m$, that is the dimension of $\varepsilon(V_h)$ is greater than the dimension of S_h. Then the order n square matrix $\boldsymbol{N}^T\boldsymbol{M}^{-1}\boldsymbol{N}$ in equation (3.35) is of rank at most $m < n$ (this is because $\boldsymbol{N}$ is of order $n \times m$), which in fact implies that $\boldsymbol{N}^T\boldsymbol{M}^{-1}\boldsymbol{N}$ is semi-definite. Thus there exists a non-zero vector $\underline{a}$ such that $\underline{a}(\boldsymbol{N}^T\boldsymbol{M}^{-1}\boldsymbol{N})\underline{a}^T = 0$. From equality (3.35) we deduce the existence of $\boldsymbol{\varepsilon}(\boldsymbol{v}_h) \in \varepsilon(V_h)$ such that

$$\|P_{S_h}\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\|_0 = 0.$$

As $\|\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\|_0 \neq 0$ the ellipticity condition of (a) of Theorem 3.3.1 is not satisfied.

Second, if there exists a non-zero vector of $\varepsilon(V_h)$ that is orthogonal to S_h obviously the ellipticity condition is also not satisfied.

We next prove that the conditions of the Lemma are sufficient. By contradiction assume that the rank of the matrix $\boldsymbol{N}$, given in (3.34c), is less than $n = \text{dim}\big(\varepsilon(V_h)\big)$. Then the column vectors of $\boldsymbol{N}$ are linearly dependent. That is to say, there exists a constant $c \neq 0$ such that

$$\big(\boldsymbol{\varepsilon}(\boldsymbol{v}_1)\,,\,\boldsymbol{\tau}_i\big) = c\sum_{k=2}^{n}\big(\boldsymbol{\varepsilon}(\boldsymbol{v}_k)\,,\,\boldsymbol{\tau}_i\big), \quad \Longrightarrow \quad \big(\boldsymbol{\varepsilon}(\boldsymbol{v}_1) - c\sum_{k=2}^{n}\boldsymbol{\varepsilon}(\boldsymbol{v}_k)\,,\,\boldsymbol{\tau}_i\big) = 0, \quad i = 1,\cdots,m.$$

We then observe from the above expression that $\boldsymbol{\varepsilon}(\boldsymbol{v}_1) - c\sum_{k=2}^{n}\boldsymbol{\varepsilon}(\boldsymbol{v}_k)$ is orthogonal to S_h and as it is also a non-zero vector of $\varepsilon(V_h)$, we are in contradiction with the second condition of the lemma. ▮

Proposition 3.5.9 *Assume that the conditions of Lemma 3.5.8 are satisfied or equivalently that there exists a constant $C_0 > 0$ such that*

$$\|P_{S_h}\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\|_0 \geq C_0\|\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\|_0, \quad \forall \boldsymbol{v}_h \in V_h\,. \tag{3.37}$$

Then C_0 is independent of the dimension of V_h.

Proof All that needs to be done is to prove that the conditions of Lemma 3.5.2 are satisfied and then apply Proposition 3.5.4 to conclude with the proof.

We first observe that if the inequality (3.37) is satisfied the ellipticity condition (3.30) of Lemma 3.5.2 is also satisfied. It remains to prove that a basis of $\boldsymbol{\varepsilon}(V_h)$ is projected onto S_h to a linearly independent family. Let $\{\boldsymbol{\eta}_i,\, i = 1,\cdots,n\}$, be a basis for $\boldsymbol{\varepsilon}(V_h)$. We assume that its projection onto S_h, defined by $\{P_{S_h}\boldsymbol{\eta}_i,\, i = 1,\cdots,n\}$, is a linearly dependent family. Then clearly there exists $j \in \{1,\cdots,n\}$ such that

$$P_{S_h}\boldsymbol{\eta}_j = \sum_{i\neq j}\alpha_i P_{S_h}\boldsymbol{\eta}_i\,,$$

which using the linearity of orthogonal projection implies that

$$P_{S_h}\boldsymbol{\eta}_j - P_{S_h}\Big(\sum_{i\neq j}\alpha_i\boldsymbol{\eta}_i\Big) = \boldsymbol{0} \quad \text{and then} \quad P_{S_h}\Big(\boldsymbol{\eta}_j - \sum_{i\neq j}\alpha_i\boldsymbol{\eta}_i\Big) = \boldsymbol{0}\,. \tag{3.38}$$

Using the condition (b) of Lemma 3.5.8 and the last equality of the above (3.38) the vector $\boldsymbol{\eta}_h = \boldsymbol{\eta}_j - \sum_{i\neq j}\alpha_i\boldsymbol{\eta}_i$ vanishes. However the family $\{\boldsymbol{\eta}_i\,,\ i = 1,\cdots,n\}$ is a basis of $\boldsymbol{\varepsilon}(V_h)$; that is to say that the coordinates of $\boldsymbol{w}_h$ must vanish, which is not the case because the j^{th} component of $\boldsymbol{\eta}_h$ is equal to one. Therefore if the conditions of Proposition 3.5.9 are met the set $\{P_{S_h}\boldsymbol{\eta}_i,\ i = 1,\cdots,n\}$ is linearly independent. ▮

CHAPTER 4

NEW STABLE MIXED FINITE ELEMENT FOR PROBLEMS IN ELASTICITY

The elements introduced in Table 3.1 all satisfy condition (b) of Theorem 3.3.1. In this chapter, we make simple choices of S_h to ensure satisfaction of (3.22): its components are assumed piecewise-linear, and continuous or discontinuous. Furthermore, it is assumed that $D_h = S_h$. In this way we will also ensure that (3.18) and (3.19) are satsified.

It therefore remains only to verify that the inequality (a) of Theorem 3.3.1 is satisfied. This is shown for each of the elements in Table 3.1. For convenience these are referred to as the extended MINI, Crouzeix-Raviart and generalized Hood-Taylor(i) (i=1,2,3) elements. Here and henceforth we refer to the element as an element of the family P_k/P_{k-1} with $k=2$. See [10, page 75].

4.1 Stability of the new elements in two space dimensions

It will be convenient to use Voigt representation of tensorial or matrix quantities in what follows. Thus, in two dimensions

$$\boldsymbol{d} = [d_{11} \quad d_{22} \quad 2d_{12}]^T \quad \text{and} \quad \boldsymbol{\sigma} = [\sigma_{11} \quad \sigma_{22} \quad \sigma_{12}]^T \,. \tag{4.1}$$

4.1.1 Stability of the extended MINI element

From condition (b) of Theorem 3.3.1 the trace of the discrete spherical part of S_h, which is $\widetilde{M}_h$, must equal the space of pressures corresponding to the MINI element. This could be accomplished by defining the space of stresses such that

$$\operatorname{tr}\operatorname{sph} S_h = \widetilde{M}_h = \mathcal{P}_1 \cap Q\,.$$

We therefore define the direct components of S_h (and of D_h) to be in $\mathcal{P}_1 \cap Q$, and we choose further

$$(S_h)_{12} = \mathcal{P}_1'$$

since there is no need for the shear components to be continuous. The corresponding space of symmetric gradients of displacements is given by

$$\boldsymbol{\varepsilon}(V_h) = \text{span}\left(\mathbf{Id} \oplus \nabla^s(b)\right) = \text{span}\begin{bmatrix} 1 & 0 & 0 & b_\xi & 0 \\ 0 & 1 & 0 & 0 & b_\eta \\ 0 & 0 & 1 & b_\eta & b_\xi \end{bmatrix}\,, \tag{4.2}$$

where $\mathbf{Id}$ is the identity matrix, b is a cubic bubble function, and $\nabla^s(b)$ its symmetric gradient.

Note that in equation (4.2) and also (4.3) the symmetric gradient of V_h is being defined element-wise, from the reference configuration. We next observe that

$$P_{S_h}\left[\boldsymbol{\varepsilon}(V_h)\right] = P_{S_h}[\mathbf{Id} \oplus P_{S_h}\left(\nabla^s(b)\right] = \begin{bmatrix} P_{S_{11}}1 & 0 & 0 & P_{S_{11}}b_\xi & 0 \\ 0 & P_{S_{22}}1 & 0 & 0 & P_{S_{22}}b_\eta \\ 0 & 0 & P_{S_{12}}1 & P_{S_{12}}b_\eta & P_{S_{12}}b_\xi \end{bmatrix}\,. \tag{4.3}$$

Here b_ξ and b_η denote relevant derivatives of the bubble function b. Noting that $P_{(S_h)_{ij}}1$, $P_{(S_h)_{ij}}b_\xi$ and $P_{(S_h)_{ij}}b_\eta$ are non-zero, and also that the derivatives of bubble functions are orthogonal to piecewise-constant functions, the column vectors of the matrix are linearly independent. That is to say, the conditions of Lemmas 3.5.6, 3.5.1 and 3.5.2 are satisfied, and thus the ellipticity condition is also satisfied.

4.1.2 Stability of the extended Crouzeix-Raviart element

For this element we define

$$S_h = \begin{bmatrix} \mathcal{P}'_1 \\ \mathcal{P}'_1 \\ \mathcal{P}'_2 \end{bmatrix} \cap S_0 \,.$$

The higher-order approximation for the shear component is required in order to establish the ellipticity condition. The symmetric gradient of the space of displacements is given by

$$\boldsymbol{\varepsilon}(V_h) = \text{span} \begin{bmatrix} 1 & 0 & 0 & \xi & 0 & 0 & \eta & 0 & 0 & b_\xi & 0 \\ 0 & 1 & 0 & 0 & \xi & 0 & 0 & \eta & 0 & 0 & b_\eta \\ 0 & 0 & 1 & 0 & 0 & \xi & 0 & 0 & \eta & b_\eta & b_\xi \end{bmatrix} . \tag{4.4}$$

Taking into account that $P_{\mathcal{P}'_2}(\nabla b) = \nabla b$, the orthogonal projection of $\boldsymbol{\varepsilon}(V_h)$ onto the space of stresses is given by

$$P_{S_h}\boldsymbol{\varepsilon}(V_h) = \text{span} \begin{bmatrix} 1 & 0 & 0 & \xi & 0 & 0 & \eta & 0 & 0 & P_{S_{11}} b_\xi & 0 \\ 0 & 1 & 0 & 0 & \xi & 0 & 0 & \eta & 0 & 0 & P_{S_{22}} b_\eta \\ 0 & 0 & 1 & 0 & 0 & \xi & 0 & 0 & \eta & b_\eta & b_\xi \end{bmatrix} . \tag{4.5}$$

As the components S_{ij} of S_h corresponding to the extended MINI element are subsets of those for the extended Crouzeix-Raviart element, b_ξ and b_η have non-zero projections onto S_{ij}; this is in fact a consequence of Lemma 3.5.5. Therefore the conditions of Lemma 3.5.1 are satisfied.

Next, in order to obtain the ellipticity condition we require, from the point of view of Lemma 3.5.2, that the column vectors of the matrix in (4.5) be linearly independent. Let $(V_i)_{1\le i\le 11}$ be the column vectors of this matrix. We observe that the sets $A = (V_i)_{1\le i\le 9}$ and $B = (V_i)_{10\le i\le 11}$ are families of linearly independent vectors. Therefore it remains only to prove that V_{10} and V_{11} are linearly independent with respect to $(V_i)_{1\le i\le 9}$. To prove that V_{10} is linearly independent to the family $(V_i)_{1\le i\le 9}$ let $W = \sum_{i=1}^{10} \alpha_i V_i$. Assume that $W = 0$ and let us prove that $\alpha_i = 0$, $i = 1, \cdots, 10$. From (4.5) we observe that the third component of W gives

$$\alpha_3 + \alpha_6 \xi + \alpha_9 \eta + \alpha_{10} b_\eta = 0 \,, \quad \forall (\xi, \eta) \in \mathbb{R}^2 \,.$$

Taking $\xi = \eta = 0$ we obtain $\alpha_3 = 0$. For $\xi = 0$ we obtain $\alpha_9 = 0$. For $\eta = 0$ and $\xi = 1$ we have $\alpha_6 = 0$. We therefore obtain $\alpha_3 = \alpha_6 = \alpha_9 = \alpha_{10} = 0$.

The first and second components of W imply equations of the form

$$\alpha + \beta\xi + \gamma\eta = 0\,, \quad \forall(\xi,\eta) \in \mathbb{R}^2\,,$$

as $\alpha_9 = \alpha_{10} = 0$. Now taking $\xi = \eta = 0$ we have $\alpha = 0$. Next for $\xi = 0$ we obtain $\gamma = 0$ and then taking $\xi = 1$ we have $\beta = 0$. Hence $\alpha_i = 0$, $i = 1, \cdots, 10$, which means that V_i, $i = 1, \cdots, 10$ are linearly independent. Similarly one can prove that V_{11} is linearly independent with respect to the family $\big(V_i\big)_{1\le i\le 9}$.

4.1.3 Extended Generalized Hood-Taylor elements (denoted by EHT(i), $i = 1, 2, 3$)

EHT(1): For this element we set $S_h = D_h = [\mathcal{P}_1\ \mathcal{P}_1\ \mathcal{P}_1]^T \cap S_0$, and note also that $\boldsymbol{\varepsilon}(V_h)$ is as given in (4.4).

A generalized proof, independent of the dimensions of the finite element spaces, of this element is given in the Section 4.2.

EHT(2). Here we choose

$$S_h = D_h = \begin{bmatrix} \mathcal{P}'_1 \\ \mathcal{P}'_1 \\ \mathcal{Q}'_2 \end{bmatrix} \cap S_0\,,$$

and furthermore

$$\boldsymbol{\varepsilon}(V_h) = \mathcal{P}'_1 \cup \text{span}\,\boldsymbol{D} \quad \text{and}$$

where the basis $\boldsymbol{D}$ is given by

$$\boldsymbol{D} = \begin{bmatrix} 2\xi\eta & \eta^2 & \xi\eta^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \xi^2 & 2\xi\eta & \xi^2\eta \\ \xi^2 & 2\xi\eta & \xi^2\eta & 2\xi\eta & \eta^2 & \xi\eta^2 \end{bmatrix}.$$

Ellipticity Condition

Using the second approach developed in subsection 3.5.2, we need to prove that the conditions (a) and (b) of Lemma 3.5.8 are satisfied.

The orthogonal projection of the symmetric gradient of the space of displacements onto the space of stresses is

$$P_{S_h}\boldsymbol{\varepsilon}(V_h) = \mathcal{P}'_1 \cup \operatorname{span} P_{S_h}(\boldsymbol{D}),$$

where

$$P_{S_h}(\boldsymbol{D}) = \begin{bmatrix} 2P_{\mathcal{P}'_1}(\xi\eta) & P_{\mathcal{P}'_1}(\eta^2) & P_{\mathcal{P}'_1}(\xi\eta^2) & 0 & 0 & 0 \\ 0 & 0 & 0 & P_{\mathcal{P}'_1}(\xi^2) & 2P_{\mathcal{P}'_1}(\xi\eta) & P_{\mathcal{P}'_1}(\xi^2\eta) \\ \xi^2 & 2\xi\eta & \xi^2\eta & 2\xi\eta & \eta^2 & \xi\eta^2 \end{bmatrix}.$$

We clearly observe that the column vectors of the above matrix are linearly independent. In fact, either the last components of these vectors appear at different combinations of linearly independent monomials or the vectors formed by their first two components are linearly independent.

We then have

$$\dim \boldsymbol{\varepsilon}(V_h) = \dim P_{S_h}\boldsymbol{\varepsilon}(V_h) \leq \dim S_h\,, \tag{4.6}$$

which is the condition (a) of Lemma 3.5.8. The first equality in the above expression (4.6) also implies the condition (b) of Lemma 3.5.8. In fact if we assume the existence of a non zero vector $\boldsymbol{w}_h \in \boldsymbol{\varepsilon}(V_h)$ whose projection onto S_h vanishes, one can construct a basis of $\boldsymbol{\varepsilon}(V_h)$ containing $\boldsymbol{w}_h$. The projection of such basis onto S_h is linearly dependent, which contradicts the equality in (4.6).

EHT(3): For this case we set

$$S_h = \begin{bmatrix} \mathcal{Q}_1 \\ \mathcal{Q}_1 \\ \mathcal{Q}'_2 \end{bmatrix} \cap S_0\,,$$

and find that

$$\boldsymbol{\varepsilon}(V_h) = \mathcal{P}'_1 \cup \operatorname{span} \boldsymbol{D}\,.$$

Ellipticity Condition

To be able to apply the second approach in subsection 3.5.2 we always need to determine the dimensions of the components of S_h and $\boldsymbol{\varepsilon}(V_h)$. We let $\mathcal{T}_h$ be a quasi-uniform, shape-regular quadrilateral triangulation of the polygonal domain Ω, B and I the number of boundary and internal nodes. Thus the total number of triangles T, edges E and quadrilaterals Q are given by

$$T = B + 2I - 2, \quad E = B + 3I - 3 \quad \text{and} \quad Q = \frac{1}{2}(B + 2I - 2) \tag{4.7}$$

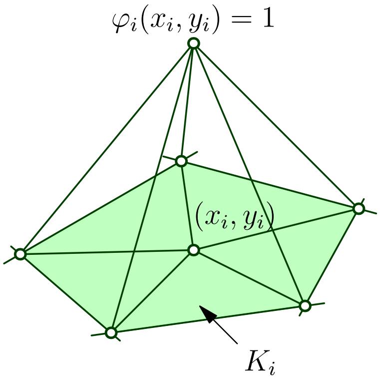


Figure 4.1: Shape function

(see Alfeld [2] and Ewing *et al.* [29]). From the above and [11] the dimension of $\mathcal{Q}_2'$ is

$$\dim \mathcal{Q}_2' = \frac{9}{2}(B + 2I - 2)\,, \tag{4.8}$$

because the elements of $\mathcal{Q}_2'$ have nine degrees of freedom on each quadrilateral and the total number of quadrilaterals is given by the third equation in (4.7).

To obtain the dimension of $\mathcal{Q}_1$ we note that the shape functions, $(\varphi_i)_{i\in\mathbb{N}_n}$, which form the basis of the space of continuous piecewise polynomials, are entirely defined by their compact supports K_i with centre of nodes i, see figure (4.1). Hence the dimension of the space of continuous piecewise polynomial $\mathcal{Q}_1$ is equal to the total number of nodes in the mesh, i.e

$$\dim \mathcal{Q}_1 = B + I. \tag{4.9}$$

As elements of $\mathcal{Q}_2$ have one degree of freedom more than the element of degree one $\mathcal{Q}_1$ on each edge of $\mathcal{T}_h$ it follows that

$$\dim \mathcal{Q}_2 = \dim \mathcal{Q}_1 + E = 2B + 4I - 3\,. \tag{4.10}$$

The equations (4.8) and (4.9) imply that

$$\dim S_h = \frac{17}{2}B + 11I - 9\,. \tag{4.11}$$

The symmetric gradient of V_h is naturally generated by the element of V_h. Thus, using (4.10) we have

$$\dim \boldsymbol{\varepsilon}(V_h) \leq \dim V_h = 2\dim Q_2 = 4B + 8I - 6\,. \tag{4.12}$$

Subtracting equation (4.12) from equation (4.11) we obtain

$$\text{dim}\, S_h - \text{dim}\, \boldsymbol{\varepsilon}(V_h) \geq 4B + 3I - 3 > 0\,.$$

We have thus proved (a) of Lemma 3.5.8.

For the second condition of Lemma 3.5.8 let $\boldsymbol{v}_h = (p, q) \in V_h$ be such that its symmetric gradient $\boldsymbol{\varepsilon}(\boldsymbol{v}_h) = [p_x, q_y, p_y + p_x] \in \boldsymbol{\varepsilon}(V_h)$ has zeros orthogonal projection onto S_h. That is to say that the following is satisfied:

$$(\boldsymbol{\varepsilon}(\boldsymbol{v}_h),\, \boldsymbol{\tau}_h)_0 = 0\,, \quad \forall \boldsymbol{\tau}_h \in S_h\,. \tag{4.13a}$$

With a proper choice of $\boldsymbol{\tau}_h$ the above equation (4.13a) is satisfied component-wise, that is

$$\Big((\boldsymbol{\varepsilon}(\boldsymbol{v}_h))_{ij},\, \tau_h \Big)_0 = 0\,, \quad \forall \tau_h \in (S_h)_{ij}\,. \tag{4.13b}$$

Noting that $(\boldsymbol{\varepsilon}(\boldsymbol{v}_h))_{22} \in (S_h)_{22}$ equation (4.13b) implies that $p_y + q_x = 0$.

From (4.13a), if we define $\boldsymbol{\tau}_h = t\mathbf{1}$, where $\mathbf{1}$ is the identity tensor, we obtain

$$(p_x + q_y, t)_0 = 0 \quad \forall t \in \mathcal{Q}_1 \quad \Longrightarrow \quad (\text{div}\, \boldsymbol{v}_h, t)_0 = 0 \quad \forall t \in \mathcal{Q}_1\,. \tag{4.14}$$

As $(V_h, \mathcal{Q}_1)$ is a stable Stokes pairing the application

$$B_h^t : Q_1 \longrightarrow V_h'$$
$$q \longmapsto B_h^t q = b(\cdot, q)\,,$$

where the bilinear form $b(\cdot,\cdot)$ is given in (2.4b), admits a continuous lifting from V_h' to $\mathcal{Q}_1$ see [11, page 57 and 39], that is

$$\forall f \in V_h'\,,\ \exists q \in \mathcal{Q}_1\ :\quad B_h^t q = f\,. \tag{4.15}$$

Now going back to our $\boldsymbol{v}_h$ the above equation (4.15) implies the existence of $q_h \in \mathcal{Q}_1$ such that

$$B_h^t q_h = b(\boldsymbol{v}_h, \cdot) \quad \text{which implies that} \quad \|B_h^t q_h\|_{V_h'} = \sup_{q_h \in \mathcal{Q}_1} \frac{(\text{div}\, \boldsymbol{v}_h, q_h)}{\|q_h\|_0}\,.$$

By virtue of the *inf-sup* condition and the Banach lifting theorem $\|B_h^t q_h\|_{V_h'}$ is bounded below by a positive constant, that is the exists $\beta^\star > 0$ such that

$$\sup_{q_h \in \mathcal{Q}_1} \frac{(\text{div}\, \boldsymbol{v}_h, q_h)}{\|q_h\|_0} \geq \beta^\star\,, \quad \forall\, \boldsymbol{v}_h \in V_h$$

which is clearly in contradiction with the second equality (4.14).

4.2 Extension to elements in three dimensions

Once again we make use of Voigt notation, so that strains and stresses are represented in vector form according to

$$\boldsymbol{d} = [d_{11} \quad d_{22} \quad d_{33} \quad 2d_{12} \quad 2d_{23} \quad 2d_{13}]^T \quad \text{and} \quad \boldsymbol{\sigma} = [\sigma_{11} \quad \sigma_{22} \quad \sigma_{33} \quad \sigma_{12} \quad \sigma_{23} \quad \sigma_{13}]^T\,. \tag{4.16}$$

We give only the proof of the stability of the three-dimensional Generalized Hood-Taylor elements EHT(1), EHT(2) and EHT(3). For the extended MINI and Crouzeix-Raviart elements, the proof is very similar to their two-dimensional counterparts, so only the details of the elements are summarized.

4.2.1 Extended MINI element

For this element we select

$$(S_h)_{ij} = \begin{cases} \mathcal{P}_1 \cap Q & \text{for } j = i \\ \mathcal{P}_1' & \text{for } j \neq i \end{cases}\,.$$

We recall that the choice of the normal components of S_h are motivated with the fact that $\mathrm{sph}_h(S_h)$ and V_h must form a stable Stokes pair and the shear components of S_h are chosen so that the conditions of Theorem 3.3.1 are also satisfied.

Furthermore,

$$\boldsymbol{\varepsilon}(V_h) = \text{span}\,(\mathbf{Id} \oplus \nabla^s(b))\,;$$

here b is a quartic bubble function.

4.2.2 Extended Crouzeix-Raviart element

In three dimensions the space of displacements for this elements includes both volumetric and surface bubble functions, so that (see Table 3.1)

$$\begin{aligned} V_h &= \mathcal{P}_2 \oplus \mathcal{B}_4 \oplus \mathcal{A}_3\,, \\ Q_h &= \mathcal{P}_1' \cap Q\,. \end{aligned}$$

The corresponding spaces of symmetric gradient of displacements and of stresses are

$$\varepsilon(V_h) = \text{span}\,\big(\nabla^s(\mathcal{P}_2 \oplus \mathcal{B}_4 \oplus \mathcal{A}_3)\big),$$

and

$$(S_h)_{ij} = \begin{cases} \mathcal{P}_1' \cap Q & \text{for } j = i \\ \mathcal{P}_2' & \text{for } j \neq i \end{cases} . \tag{4.17}$$

4.2.3 Extended Generalized Hood-Taylor elements

EHT(1): For this element we have

$$\begin{aligned} V_h &= \big(\mathcal{P}_2 \cap H_0^1(\Omega)\big)^3 , \\ Q_h &= \mathcal{P}_1 \cap Q , \end{aligned}$$

and the corresponding spaces of strains and stresses are given by

$$\varepsilon(V_h) = \text{span}\,\nabla^s(\mathcal{P}_2) \subset \mathcal{P}_1' ,$$

$$(S_h)_{ij} = \begin{cases} \mathcal{P}_1 \cap Q & \text{for } j = i \\ \mathcal{P}_1 & \text{for } j \neq i \end{cases} .$$

respectively.

Lemma 4.2.1 *Let $\{\eta_T\}$ be a basis of $\mathcal{P}_1'$, where η_T is non-zero only on the tetrahedron $T \in \mathcal{T}_h$. Then*

$$P_{\mathcal{P}_1}(\eta_T) \neq 0 . \tag{4.18a}$$

Furthermore, for any $\eta \in \big(\varepsilon(V_h)\big)_{i,j}$,

$$P_{\mathcal{P}_1}(\eta) = 0 \quad \textit{implies} \quad \eta = 0. \tag{4.18b}$$

Lemma 4.2.2 *Let $\big(\nabla V_h\big)_{i,j}$ be the components of the gradient of V_h, which are first-order derivatives of polynomials of degree two in $\mathcal{P}_2$. Then*

$$\mathit{dim}\big(\nabla(V_h)\big)_{i,j} = \mathit{dim}\big(\mathcal{P}_1\big) . \tag{4.19}$$

Proof Let $\mathrm{p}_2 \in \mathcal{P}_2$, such that its restriction, $\mathrm{p}_{2|T}$, on $T \in \mathcal{T}_h$ is given by

$$\mathrm{p}_{2|T} = a_0(T)+a_1(T)x+a_2(T)x^2+b_1(T)y+b_2(T)y^2+c_1(T)z+c_2(T)z^2+d_1(T)xy+d_2(T)xz+d_3(T)yz\,.$$

Du to the continuity of p_2 in x, y and z directions the first degree polynomial p_1, defined by its restriction on $T \in \mathcal{T}_h$ as

$$\mathrm{p}_{1|T} = a_1(T) + a_2(T)x + d_1(T)y + d_2(T)z$$

is also continuous.

We next observe that the partial derivatives of p_2 with respect to x depend only on the degree of freedom of p_1. In fact we have

$$\frac{\partial \mathrm{p}_2}{\partial x} = x\frac{\partial \mathrm{p}_1}{\partial x} + \mathrm{p}_1.$$

Similarly we prove that $\dfrac{\partial \mathrm{p}_2}{\partial y}$ and $\dfrac{\partial \mathrm{p}_2}{\partial z}$ have the same degrees of freedom as some polymials in $\mathcal{P}_1$. Therefore the equality (4.19) is satisfied. ∎

Proof of Lemma 4.2.1 As η_T is linear and continuous on T there exists $\tau \in \mathcal{P}_1$ such that its restriction on T is equal to η_T. If $P_{\mathcal{P}_1}(\eta_T) = 0$ then

$$(\eta_T\,,\,\tau) = (\eta_T, \eta_T) = \|\eta_T\|_0^2 = 0 \quad\Longrightarrow\quad \eta_T = 0,$$

which contradicts the fact that $\{\eta_T \mid T \in \mathcal{T}_h\}$ is a basis member of $\mathcal{P}_1'$. We have therefore proven (4.18a).

To generalize the above assertion on $\big(\nabla(V_h)\big)_{i,j} \subset \mathcal{P}_1'$ consider $\big\{\zeta_T\big\} \subset \big\{\eta_T\big\}$ to be a basis of $\big(\nabla(V_h)\big)_{i,j}$ and $\zeta = \sum_T \alpha_T\zeta_T$ to be any element of $\big(\nabla(V_h)\big)_{i,j}$ with its orthogonal projection $\bar{\zeta} = \sum_T \alpha_T\bar{\zeta}_T$ onto $\mathcal{P}_1$. We want to prove that:

$$\text{if} \quad \bar{\zeta} = 0\,, \quad \text{then} \quad \zeta = 0\,.$$

Observe from equation (4.19) that the family $\big\{\bar{\zeta}_T\big\}$ is linearly independent and hence the map $\|\cdot\|_\star$ defined by

$$\tau = \sum_T \gamma_T\bar{\zeta}_T \longmapsto \|\tau\|_\star = \sum_T \|\gamma_T\bar{\zeta}_T\|_0$$

defines a norm on $P_{\mathcal{P}_1}\left(\left(\nabla(V_h)\right)_{i,j}\right)$. From this we deduce that

$$\|\bar{\zeta}\|_\star = \sum_T \|\alpha_T \bar{\zeta}_T\|_0.$$

Therefore if we assume that $\bar{\zeta}$ vanishes it follows directly, from the above and (4.18a), that

$$\|\bar{\zeta}\|_\star = 0 \implies \sum_T \|\alpha_T \bar{\zeta}_T\|_0 = 0 \implies \|\alpha_T \bar{\zeta}_T\|_0 = 0 \implies \alpha_T = 0.$$

Hence $\eta = 0$, which gives the proof of (4.18b). ▮

Knowing that the components of $\varepsilon(V_h)$ are generated by the components of $\nabla(V_h)$ the proof of the ellipticity condition follows from the above Lemmas 4.2.1 and 3.5.6.

EHT(2): For this element we have

$$V_h = \left(\mathcal{Q}_2 \cap H_0^1(\Omega)\right)^3 , \quad Q_h = \mathcal{P}_1' \cap Q\,,$$

and the space of stresses S_h is defined by

$$(S_h)_{ij} = \begin{cases} \mathcal{P}_1' \cap Q & \text{for } j = i \\ \mathcal{Q}_2' & \text{for } j \neq i \end{cases} . \tag{4.20}$$

The symmetric gradient of V_h is given in Voigt notation by

$$\varepsilon(V_h) = \text{span}\left(\nabla^s \mathcal{Q}_2\right) = \text{span} \begin{bmatrix} i\xi^{i-1}\eta^j\zeta^k & 0 & 0 \\ 0 & j\xi^i\eta^{j-1}\zeta^k & 0 \\ 0 & 0 & k\xi^i\eta^j\zeta^{k-1} \\ j\xi^i\eta^{j-1}\zeta^k & i\xi^{i-1}\eta^j\zeta^k & 0 \\ 0 & k\xi^i\eta^j\zeta^{k-1} & j\xi^i\eta^{j-1}\zeta^k \\ k\xi^i\eta^j\zeta^{k-1} & 0 & i\xi^{i-1}\eta^j\zeta^k \end{bmatrix},$$

in which the indices i, j and k range over 0 to 2.

Thus the orthogonal projection of the symmetric gradient of V_h onto S_h has the form

$$P_{S_h}\big(\boldsymbol{\varepsilon}(V_h)\big) = \mathsf{span}\begin{bmatrix} P_{S_{11}}(i\xi^{i-1}\eta^j\zeta^k) & 0 & 0 \\ 0 & P_{S_{22}}(j\xi^i\eta^{j-1}\zeta^k) & 0 \\ 0 & 0 & P_{S_{33}}(k\xi^i\eta^j\zeta^{k-1}) \\ j\xi^i\eta^{j-1}\zeta^k & i\xi^{i-1}\eta^j\zeta^k & 0 \\ 0 & k\xi^i\eta^j\zeta^{k-1} & j\xi^i\eta^{j-1}\zeta^k \\ k\xi^i\eta^j\zeta^{k-1} & 0 & i\xi^{i-1}\eta^j\zeta^k \end{bmatrix}.$$

To prove the ellipticity condition, first note that the components of each of the column vectors of the above matrix satisfy the properties of Lemma 3.5.1, which in turn implies the condition 3.30 in Lemma 3.5.2 . This can be verified by projecting any monomial of degree less than or equal to 2 onto S_{ij}. For the second condition of Lemma 3.5.2 we observe that the column vectors of $P_{S_h}\big(\boldsymbol{\varepsilon}(V_h)\big)$ appear as different combinations of monomials of different degrees; therefore they are linearly independent. Hence $\big(\boldsymbol{\varepsilon}(V_h)\,,\,S_h\big)$ satisfies the ellipticity condition (a) of Theorem 3.3.1.

EHT(3): Given a mesh of hexahedral elements the stable Stokes pairing (V_h, Q_h) for this element is

$$V_h = \big(\mathcal{Q}_2 \cap H_0^1(\Omega)\big)^3\,, \quad Q_h = \mathcal{Q}_1 \cap Q\,,$$

and S_h is defined by

$$(S_h)_{ij} = \begin{cases} \mathcal{Q}_1 \cap Q & \text{for } j = i\,, \\ \mathcal{Q}_2' & \text{for } j \neq i\,. \end{cases}$$

The proof of ellipticity is similar to that for EHT(2) and is omitted.

Remark A question may arise as to the novelty of the formulations presented in this section: that is, are these formulations distinct from displacement-pressure formulations based on the original elements? It is shown here that the elements in this section do indeed lead to new formulations.

For definiteness consider the displacement-pressure formulation using the MINI element: this is the problem of finding $(\boldsymbol{u}_h\,,\,p_h) \in V_h \times Q_h$ such that

$$\big(2\mu\,\boldsymbol{\varepsilon}(\boldsymbol{u}_h)\,,\,\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\big) - \big(p_h\,,\,\mathrm{div}\,\boldsymbol{v}_h\big) \;=\; (\boldsymbol{f}\,,\,\boldsymbol{v}_h) \quad \forall \boldsymbol{v}_h \in V_h\,, \tag{4.21a}$$

$$-\big(q_h\,,\,\mathrm{div}\,\boldsymbol{u}_h\big) - \frac{1}{\lambda}(p_h\,,\,q_h) \;=\; 0 \quad \forall q_h \in Q_h\,. \tag{4.21b}$$

For convenience we have omitted the subscript 0 to indicate that $(\cdot,\cdot)$ in these equations denotes the standard L^2 inner product. Here V_h is as defined in Table 3.1 for the MINI element and Q_h is the space of continuous piecewise polynomials of degree one.

The corresponding three-field formulation is given by (3.16), with V_h as above and $D_h = S_h$, with the direct components comprising continuous piecewise-linear polynomials and the shear components discontinuous piecewise-linear polynomials.

The strain can be found from the solution of (4.21) in various ways: for example a direct computation gives the strain $\boldsymbol{\varepsilon}(\boldsymbol{u}_h)$ as a piecewise-discontinuous quadratic polynomials on each element. Finally, the stress is found from the relation, using 4.21b),

$$\begin{aligned}\boldsymbol{\sigma}_h &= -p_h\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon}(\boldsymbol{u}_h)\\ &= -\lambda P_{Q_h}(\operatorname{div}\boldsymbol{u}_h)\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon}(\boldsymbol{u}_h)\,,\end{aligned}$$

which is a sum of continuous linear and discontinuous quadratic quantities. On the other hand, the three-field formulation gives the stress and strain as continuous quantities. The two formulations are therefore distinct. Similar considerations show that the other formulations in this section are distinct from the original displacement-pressure elements in Table 3.1 on which they have been based.

4.3 Numerical results

In this section we present a selection of numerical simulations in two and three space dimensions, based on the elements introduced in this work, and comparing the results with those obtained using established elements. We are interested in particular in demonstrating computationally the stability of those schemes that satisfy the conditions for well-posedness set out in Section 4.

We will require the following bases on the reference square $\hat{K}$ to generate spaces of stresses and strains for some elements:

$$\mathbf{Id} := \text{span}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}, \quad \mathbf{A} := \text{span}\begin{bmatrix}\eta&0\\0&\xi\\0&0\end{bmatrix}, \quad \mathbf{C} := \text{span}\begin{bmatrix}0&0\\0&0\\\eta&\xi\end{bmatrix}. \tag{4.22}$$

Unless otherwise stated, all examples are presented for the case of near-incompressibility, and correspond to a value of Poisson's ratio of 0.4999.

4.3.1 Two-dimensional examples

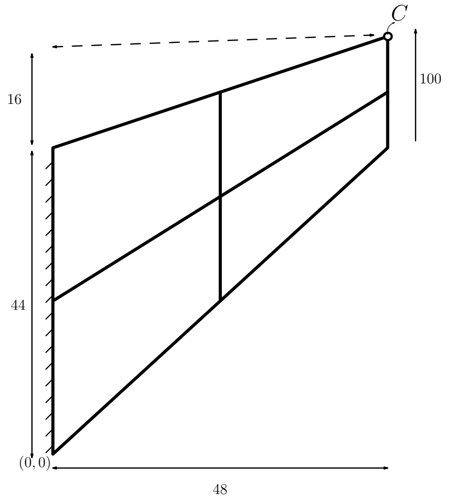
All examples in this section are for problems in plane strain.

The formulations referred to in the examples that follow are:

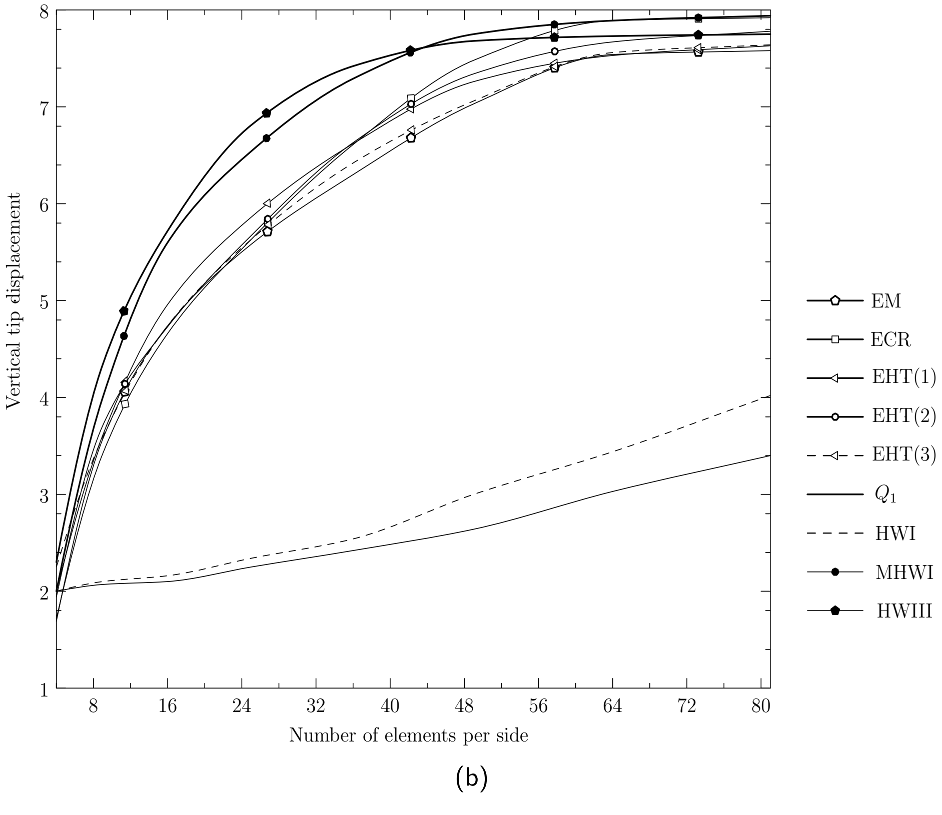
Q1	The standard displacement formulation using the four-noded quadrilateral
HWI	The standard three-field formulation with the spaces of stresses and strains given by $S_h = D_h = \mathbf{Id} \oplus \mathbf{A}$, where the basis $\mathbf{A}$ is given in (4.22) (see [26, 38])
MHWI	The modified three-field formulation with $\alpha = 0$, and with the spaces of stresses and strains as above. This is *not* equivalent to HWI
HWIII	The standard three-field formulation with the spaces of stresses and strains given by $S_h = D_h = \mathbf{Id} \oplus \mathbf{C}$, where the basis $\mathbf{C}$ is given in (4.22) This formulation is independent of α
MES	The mixed enhanced strain formulation with nine enhanced modes (see [37] for details)
EM	The extended MINI element
ECR	The extended Crouzeix-Raviart element
EHT(i)	The extended generalized Hood-Taylor elements (i=1,2,3)
MINI	The standard MINI element of a stable Stokes pair

Example 1: Cook's membrane problem. This benchmark problem, shown in Figure 4.2(a), refers to a tapered panel clamped on one side and subjected to a shearing load at the free end, resulting in deformation that is dominated by a bending response. The material properties are taken to be $E = 250$ and $\nu = 0.4999$, and simulations are carried out for progressive uniform refinements of the mesh.

In Figure 4.2(b) the results obtained from various formulations are compared. The well-known locking response of Q1 is observed, as is the corresponding poor response of HWI, which case is not covered by the well-posedness theory. On the other hand, MHWI shows no such pathologies, and performs well, while HWIII, which corresponds to constant direct stresses and strains, exhibits a response that is most accurate. All of the new elements (EM, ECR, EHT(i)) exhibit good behaviour.



(a)



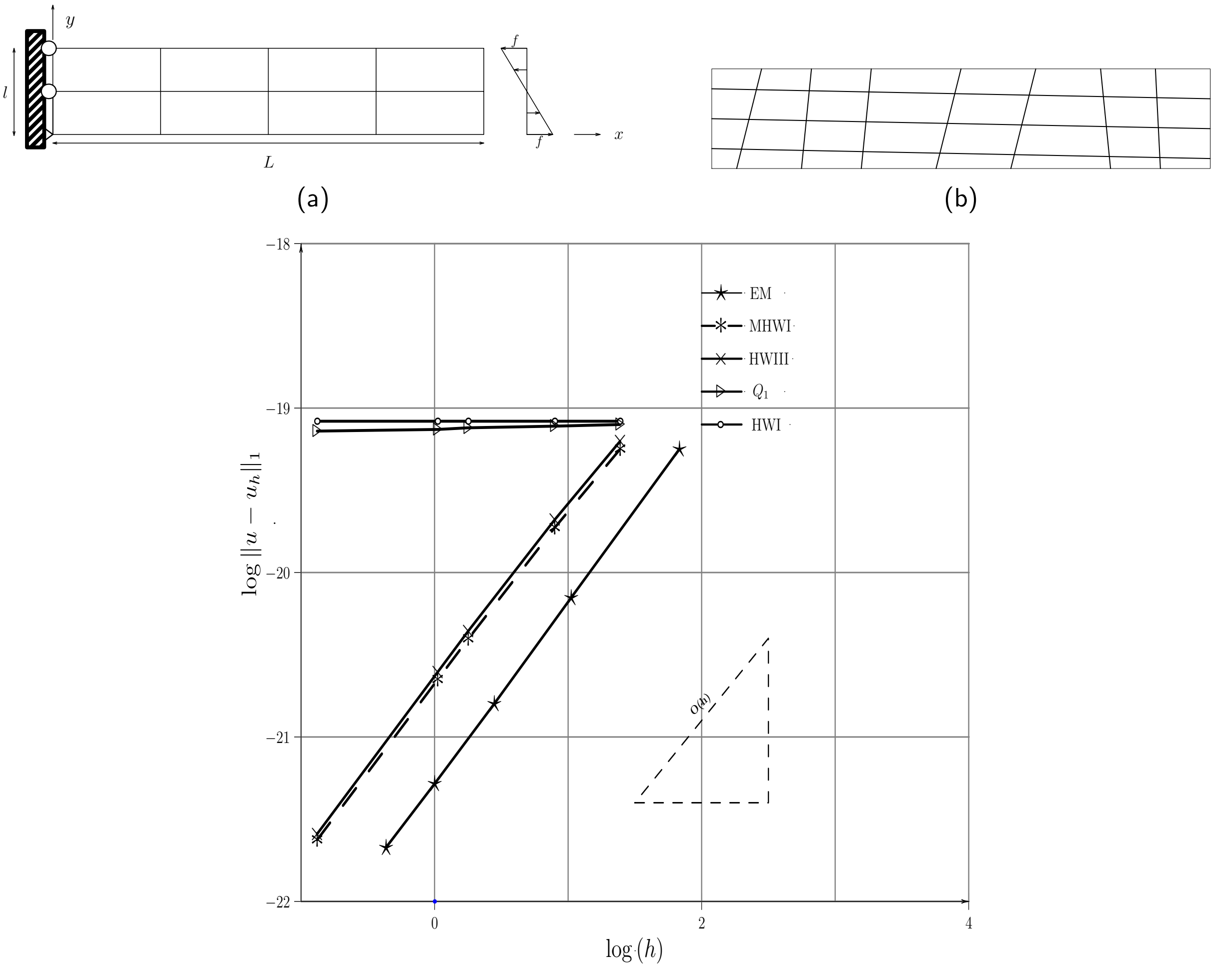
(b)

Figure 4.2: (a) Cook's membrane problem with initial triangulation; (b) Plots of vertical displacement at point C against the number of elements per side

Example 2: Cantilever beam. We consider a beam of unit thickness, subjected to a couple at one end, as shown in Figure 2(a). Along the edge $x = 0$, the horizontal displacement and vertical surface traction are zero. At the point $(0, 0)$, the vertical displacement is also zero. The exact solution is given by

$$u(x,y) = \frac{2f(1-\nu^2)}{El} x \left(\frac{l}{2} - y \right) \text{ , and } \quad v(x,y) = \frac{f(1-\nu^2)}{El} \left[x^2 + \frac{\nu}{1-\nu} y(y-l) \right] .$$

We set $L = 10$, $l = 2$, $E = 1500$, $\nu = 0.4999$, and $f = 3000$. Figure 4.3c shows the behaviour



(a) (b)

(c)

Figure 4.3: (a) Cantilever beam; (b) the distorted mesh; (c) behaviour of displacement error, in the H^1-norm, for both regular and distorted meshes

of the displacement error with mesh refinement, for the case of a series of regular refinements.

Meshes of 2×1, 4×2, 8×4 and 16×8 rectangular and distorted elements are considered.

It is apparent that the convergence rate of that both the standard formulation and the HW1 element are very poor and almost identical, while other formulations exhibit a rate of convergence close to linear.

In Table 3.1 we show the vertical displacement at the point $(0, 2)$ for the new elements, using a mesh of 4×2 elements. The excellent approximation is evident in all cases.

ν	ECR	EHT(1)	EHT(2)	EHT(3)	Exact
0.49	3.0634	3.1028	3.0451	3.0448	3.0396
0.499	3.0048	3.0043	3.0354	3.0050	3.0040
0.4999	3.0009	3.0002	3.0018	3.0009	3.0004
0.49999	3.0000	3.0000	2.9984	3.0005	3.0000

Table 3.1: Vertical displacement at point $(0, 2)$ for the cantilever problem

Example 3: Square with exact solution. We consider a unit square $(0, 1) \times (0, 1)$ with homogeneous Dirichlet boundary condition and with body force given by [16]

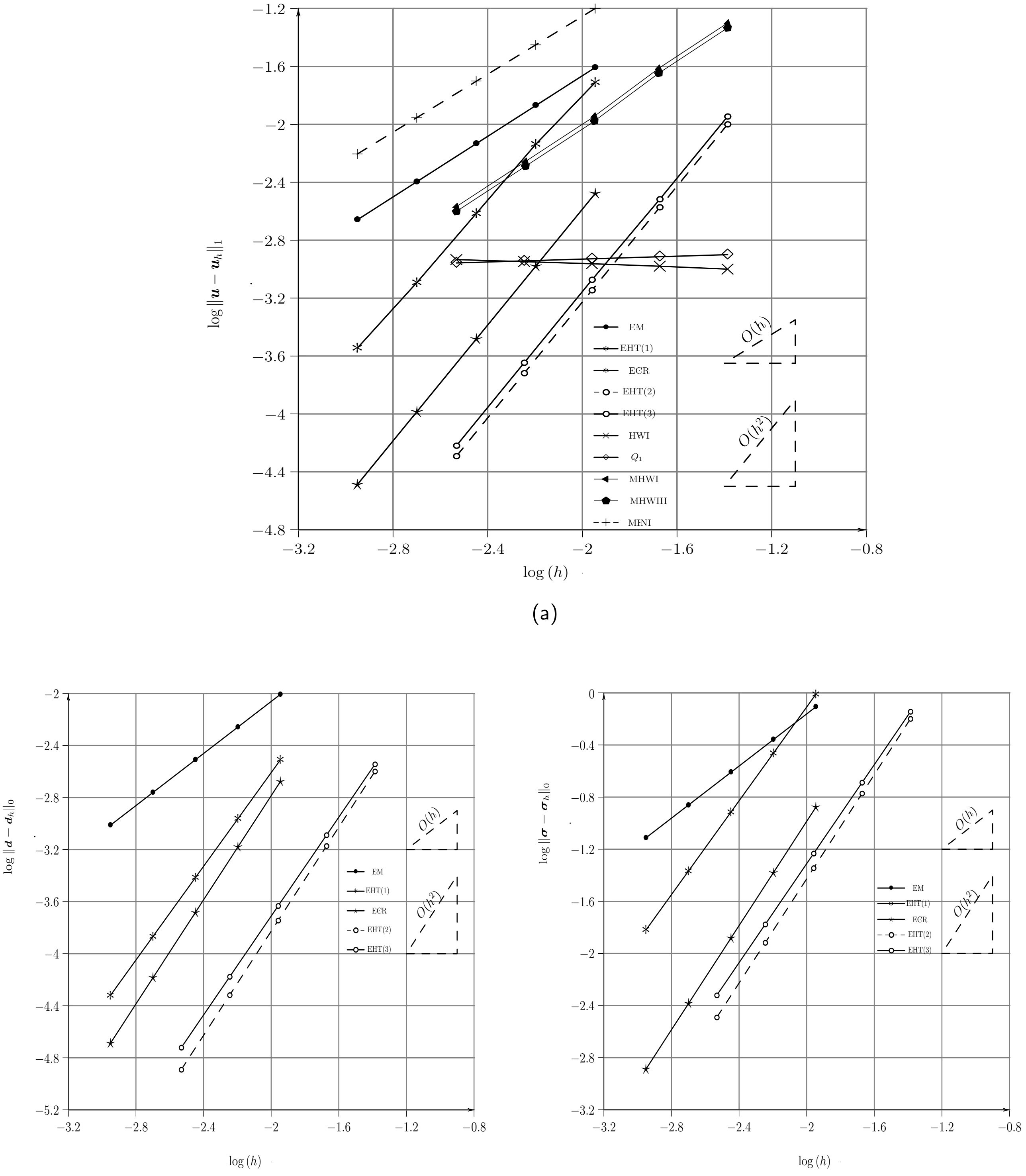
$$\begin{aligned} f_1 &= \pi^2 \left[4 \sin 2\pi y \big(-1 + 2\cos 2\pi x\big) - \cos \pi(x+y) + \frac{2}{1+\lambda} \sin \pi x \sin \pi y \right] , \\ f_2 &= \pi^2 \left[4 \sin 2\pi x \big(-1 + 2\cos 2\pi y\big) - \cos \pi(x+y) + \frac{2}{1+\lambda} \sin \pi x \sin \pi y \right] , \end{aligned}$$

with shear modulus $\mu = 1$. The exact solution, of problem (3.1), with $\mu = 1$, is given by

$$\begin{aligned} u_1 &= \sin 2\pi y \big(-1 + \cos 2\pi x\big) + \frac{1}{1+\lambda} \sin \pi x \sin \pi y \, , \\ u_2 &= \sin 2\pi x \big(1 - \cos 2\pi y\big) + \frac{1}{1+\lambda} \sin \pi x \sin \pi y \, . \end{aligned}$$

Figure 4.4a shows the displacement error as a function of mesh size. The poor convergence of the standard and HWI formulations is apparent, while the other elements all exhibit convergence at the optimal rate. The same figure also demonstrates the optimal-order convergence of the standard MINI element, though the location of the line for this element indicates that the constant in the error estimate is larger for this element than for the various various extended elements.

In Figures 4.4b and 4.4c the errors in stress and strain are plotted for various extended elements, all of which show convergence at the optimal rate.



(a)

(b)

(c)

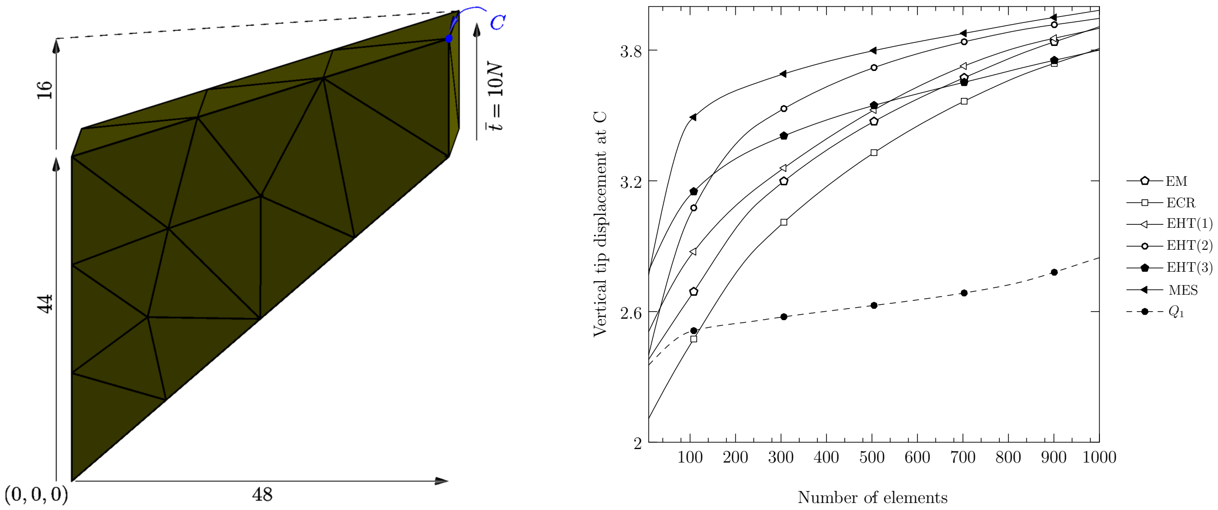
Figure 4.4: Errors for Example 3: (a) displacement; (b) stress; (c) strain

4.3.2 Three-dimensional examples

In this section we report on numerical results for problems in three space dimensions. The element nomenclature is as for the two-dimensional examples

Example 4: Cook's membrane problem. This example is a three-dimensional counterpart of Example 2, with the material properties $E = 1000N/mm^2$, $\nu = 0.4999$ and a uniform load applied at one end in the vertical direction with traction $\bar{t} = 10N/mm^2$.

The beam is fully constrained on the left-hand side, resulting in three-dimensional behaviour (Figure 4.5a). Further details of the problem are given in [39]. As expected, the Q_1 formulation shows locking behaviour while the new elements and the well-known mixed enhanced strain element (MES) all perform well [37], the latter showing the best performance.



(a) (b)

Figure 4.5: (a) 3D Cook's membrane problem; (b) Vertical displacement at point C against the total number of elements

Example 5: Deformation of a unit cube

In this example, which is the three-dimensional version of Example 3, we consider the unit cube $\Omega = (0,1)^3$ with homogeneous Dirichlet boundary conditions and body force $\boldsymbol{f} = (f_1, f_2, f_3)$

given for $\mu = 1$ by

$$\begin{aligned}
f_1 &= \pi^2\Big[12\lambda\cos 2\pi x\sin 2\pi y\sin 2\pi z + 8(3\cos 2\pi x - 1)\sin 2\pi y\sin 2\pi z + \\
&\qquad (1+\frac{3}{\lambda+1})\sin\pi x\sin\pi y\sin\pi z - \cos\pi x\sin\pi(y+z)\Big]\,, \\
f_2 &= \pi^2\Big[12\lambda\sin 2\pi x\cos 2\pi y\sin 2\pi z + 8(3\cos 2\pi y - 1)\sin 2\pi x\sin 2\pi z + \\
&\qquad (1+\frac{3}{\lambda+1})\sin\pi x\sin\pi y\sin\pi z - \cos\pi y\sin\pi(x+z)\Big]\,, \\
f_3 &= \pi^2\Big[12\lambda\sin 2\pi x\sin 2\pi y\cos 2\pi z + 8(3\cos 2\pi z - 1)\sin 2\pi x\sin 2\pi y + \\
&\qquad (1+\frac{3}{\lambda+1})\sin\pi x\sin\pi y\sin\pi z - \cos\pi z\sin\pi(x+y)\Big]\,.
\end{aligned}$$

The exact solution of problem (3.1), with $\mu = 1$, is given by

$$\begin{aligned}
u_1 &= \sin 2\pi y\sin 2\pi z\big(-1+\cos 2\pi x\big) + \frac{1}{1+\lambda}\sin\pi x\sin\pi y\sin\pi z\,, \\
u_2 &= \sin 2\pi x\sin 2\pi z\big(-1+\cos 2\pi y\big) + \frac{1}{1+\lambda}\sin\pi x\sin\pi y\sin\pi z\,, \\
u_3 &= \sin 2\pi x\sin 2\pi y\big(-1+\cos 2\pi z\big) + \frac{1}{1+\lambda}\sin\pi x\sin\pi y\sin\pi z\,.
\end{aligned}$$

Figure 4.6 shows on the one hand the quadratic rate of convergence of the extended Hood-Taylor elements, EHT(i) and the extended Crouzeix-Raviart element ECR, and the linear convergence of the extended MINI element EM and mixed enhanced strain element MES.

Example 6: Torsion of a rectangular bar

Consider a bar having a rectangular cross-section with $\Omega = (0,1)\times(0,3)\times(0,15)$. The bar is subjected a torque $T = 100N$ at the ends $z = 0$ and $z = L$. We also prescribe the Dirichlet boundary condition $\boldsymbol{u} = 0$ along the line $(0,y,0)$. Other surfaces are traction-free. Further details can be found in [28, 46].

The solution to this problem is given by

$$\begin{aligned}
u_1 &= -\alpha yz\,, \\
u_2 &= \alpha xz\,, \\
u_3 &= xy - \frac{32a^2}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^3}\frac{}{\cosh\lambda_n b}\sin\lambda_n x\sinh\lambda_n y\,,
\end{aligned}$$

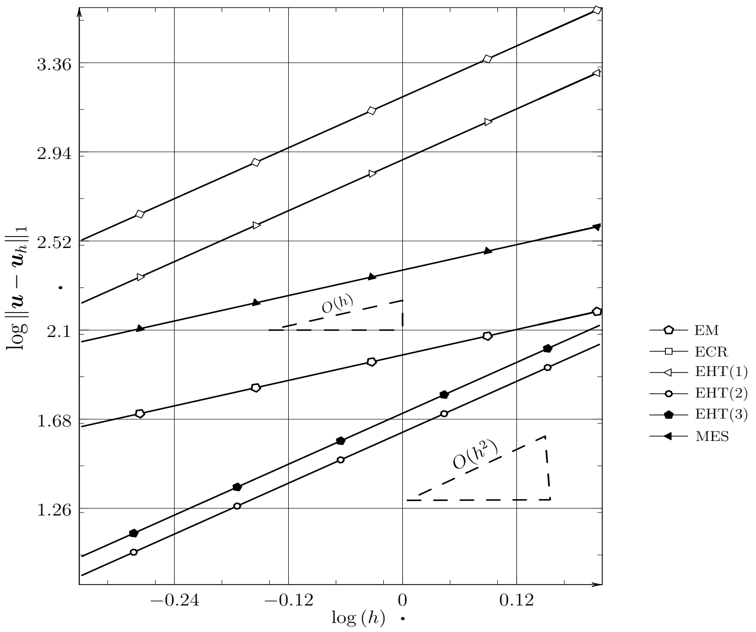
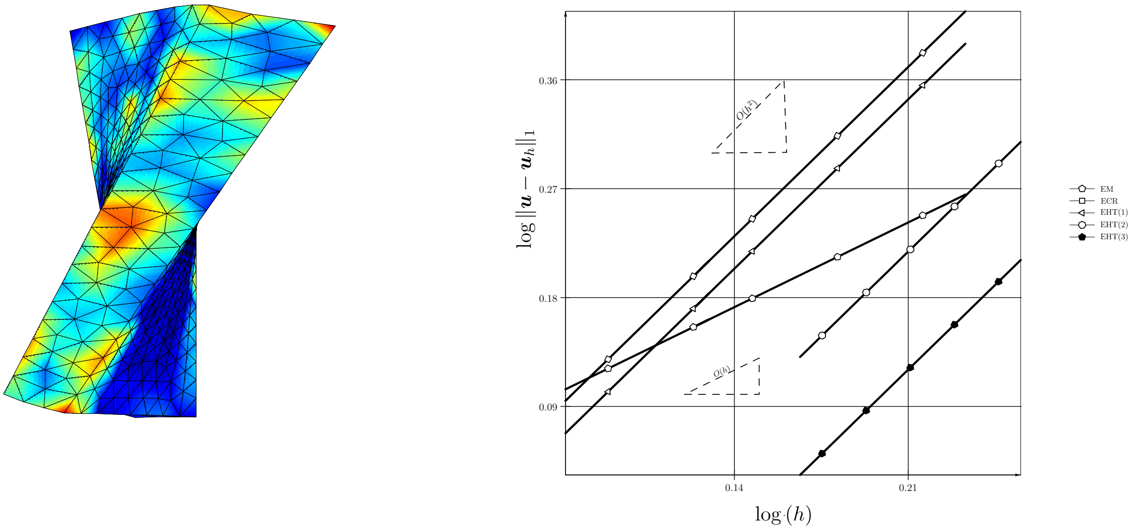


Figure 4.6: Behaviour of displacement error for unit cube



(a)

(b)

Figure 4.7: (a) Stress contours in the twisted bar; (b) displacement error for various elements

where

$$\lambda_n = (2n-1)\frac{\pi}{2}, \quad \alpha = \frac{T}{\mu J}, \quad \text{in our case} \quad J = 12.639\,.$$

Figure 4.7(a) shows a distorted rectangular bar and displays in colour the stress, using the numerical solutions obtained from the extended MINI element.

The results in Figure 4.7(b) once again verify the good convergence properties of the new extended elements, with optimal rates in all cases.

4.4 Concluding remarks

This work is an extension of the theory on the stability and convergence of three-field methods presented in [26, 38]. The developments in that work were centred on the construction of stable elements based on the four-noded quadrilateral in two space dimensions. In this investigation, the general theorems in these earlier works have been used as a basis for constructing new families of elements for three-field formulations, starting with stable Stokes elements. Such an approach has been shown to represent a fertile source of elements: some are characterized by piecewise-discontinuous stresses and strains, as is the case in traditional three-field approaches in solid mechanics, while for others all variables are approximated by piecewise-continuous functions. A further extension of the earlier work has been the application of the general theory in the construction of elements in both two and three space dimensions.

The theory and examples in this study have been confined to consideration of linear problems.The extension to nonlinear problems is an important objective: while this can readily be achieved computationally, the underlying theory presents additional challenges. This is the objective of our work in Chapter 6.

Numerical implementation discussed in the examples above are based on a slight modification of the Matlab code presented in [1].

CHAPTER 5

STABILITY AND CONVERGENCE OF PRESSURE-DILATATION-DISPLACEMENT FORMULATION

Taylor [44] has proposed a three-field formulation with pressure, dilatation and displacement as variables. The approach leads to excellent numerical approximations, but no analysis has been carried out. In this chapter such analysis is done, by showing first that the formulation by Taylor [44] is a special case of the general three-field formulation studied in [38]. Then the framework and set of criteria in the general setting are used to verify the stability and uniform convergence of this approach.

5.1 A pressure-dilatation-displacement mixed formulation

A three-field formulation involving displacement, pressure and volumetric strain has been formulated and studied by Taylor in [44]. The three-dimensional domain Ω, assumed polyhedral, is partitioned into tetrahedral elements, Discrete spaces of displacements, pressure and dilatation are denoted respectively by V_h, Q_h and Θ_h. These spaces are defined as follows:

$$\begin{aligned} Q_h &= \{q_h \in L^2(\Omega) \mid \; q_h|_T \in P_1(T)\} \cap C(\overline{\Omega}) \,, \\ V_h &= \left(Q_h \oplus \mathcal{B}_4\right)^3 \cap (H_0^1(\Omega))^3 \,, \\ \Theta_h &= \{\theta_h \in L^2(\Omega) \mid \theta_h|_T \in P_1(T)\} \,. \end{aligned} \tag{5.1}$$

Here $\mathcal{B}_4$ is the space of quartic bubbles, that is, quartic polynomials on each element with zero values along element boundaries. Then the three-field formulation takes the following form: find $(\boldsymbol{u}_h \,, \theta_h \,, p_h) \in V_h \times \Theta_h \times Q_h$ such that

$$\left(\operatorname{tr} \boldsymbol{\varepsilon}(\boldsymbol{u}_h) - \theta_h \,, q_h\right)_0 = 0 \,, \quad \forall\, q_h \in Q_h \,, \tag{5.2a}$$

$$\left(\tfrac{1}{3} \operatorname{tr} \bar{\boldsymbol{\sigma}}_h - p_h \,, \varphi_h\right)_0 = 0 \,, \quad \forall\, \varphi_h \in \Theta_h \,, \tag{5.2b}$$

$$\left(\operatorname{dev} \bar{\boldsymbol{\sigma}}_h + p_h \mathbf{1} \,, \boldsymbol{\varepsilon}(\boldsymbol{v}_h)\right)_0 = (\boldsymbol{f} \,, \boldsymbol{v}_h)_0 \,, \quad \forall\, \boldsymbol{v}_h \in V_h \,, \tag{5.2c}$$

where $\bar{\boldsymbol{\sigma}}_h = \mathcal{C}\big(\operatorname{dev} \boldsymbol{\varepsilon}(\boldsymbol{u}_h) + \tfrac{1}{3}\theta_h \mathbf{1}\big)$.

We show next that the formulation (5.2a)–(5.2c) is a special case of the general three-field formulation (3.16). To this end, define the spaces of strains and stresses by

$$D_h = \operatorname{dev} \boldsymbol{\varepsilon}(V_h) + \Theta_h \mathbf{1} \,. \tag{5.3a}$$

$$S_h = \operatorname{dev} \boldsymbol{\varepsilon}(V_h) + Q_h \mathbf{1}. \tag{5.3b}$$

From (3.1d) and the definitions (5.1) we note that conditions (3.18) and (3.19) are satisfied.

Starting with (3.16b), set $\boldsymbol{\tau}_h = q_h \mathbf{1}$ for arbitrary $q_h \in Q_h$: then this equation becomes

$$(\operatorname{tr} \boldsymbol{d}_h - \operatorname{tr} \boldsymbol{\varepsilon}(\boldsymbol{u}_h), q_h) = 0 \,. \tag{5.4}$$

Given the structure (5.3a) of D_h, we may set $\operatorname{tr} \boldsymbol{d}_h = \theta_h$ for some $\theta_h \in \Theta_h$. This immediately gives (5.2a).

Next, set $\boldsymbol{\tau}_h = \operatorname{dev} \boldsymbol{\varepsilon}(\boldsymbol{w}_h)$ in (3.16b): this gives

$$(\operatorname{dev}(\boldsymbol{d}_h - \boldsymbol{\varepsilon}(\boldsymbol{u}_h)), \operatorname{dev} \boldsymbol{w}_h) = 0 \,, \tag{5.5}$$

so that

$$\operatorname{dev} \boldsymbol{d}_h = \operatorname{dev} \boldsymbol{\varepsilon}(\boldsymbol{u}_h) \,.$$

It follows that

$$\boldsymbol{d}_h = \operatorname{dev}\boldsymbol{\varepsilon}(\boldsymbol{u}_h) + \tfrac{1}{3}\theta_h\mathbf{1}\,. \tag{5.6}$$

Turning to equation (3.16a), set $\boldsymbol{e}_h = \varphi_h\mathbf{1}$, for some $\varphi_h \in \Theta_h$: then this equation becomes

$$(\operatorname{tr}(\mathcal{C}\boldsymbol{d}_h) - p_h, \varphi_h) = 0\,, \tag{5.7}$$

where

$$p_h := \operatorname{tr}\boldsymbol{\sigma}_h\,. \tag{5.8}$$

Substitution of (5.6) yields (5.2b).

Next, set $\boldsymbol{e}_h = \operatorname{dev}\boldsymbol{\varepsilon}(\boldsymbol{w}_h)$ in (3.16a): then we get

$$(\mathcal{C}(\operatorname{dev}\boldsymbol{\varepsilon}(\boldsymbol{u}_h) + \tfrac{1}{3}\theta_h\mathbf{1}) - \boldsymbol{\sigma}_h, \operatorname{dev}\boldsymbol{\varepsilon}(\boldsymbol{w}_h)) = 0\,, \tag{5.9}$$

which gives

$$\operatorname{dev}\boldsymbol{\sigma}_h = \mathcal{C}\operatorname{dev}\boldsymbol{\varepsilon}(\boldsymbol{u}_h)\,. \tag{5.10}$$

Finally, this identity together with (5.8), when substituted in (3.16c), gives (5.2c).

5.2 Convergence and stability of the pressure-dilatation-displacement formulation

In Chapter 3 a set of conditions are presented that lead to the well-posedness and uniform convergence of general three-field formulations of the type (3.6) as well as its conventional special case obtained by setting $\alpha = 1$. For the purpose of analysing the pressure-dilatation-displacement formulation introduced earlier, it suffices to follow the discussions, on well-posedness, addressed in Chapter 4; that is, to prove that the conditions (a) and (b) of Theorem 3.3.1 are satisfied.

Since it has been shown that the formulation (5.2) is a special case of the general three-field formulation (3.15) it suffices to show that the choice of spaces that define the pressure-based formulation satisfy the conditions of Theorem 3.3.1. Furthermore, it is clear that

$$S_h \subset D_h\,, \quad \operatorname{tr}(D_h)\mathbf{1} \subset D_h\,,$$

and for this choice of spaces

$$M_h = \mathsf{tr}\, S_h = Q_h\,.$$

The pairing (V_h, M_h) is a stable Stokes pair, the MINI element discussed in Chapter 2. Therefore condition (b) of Theorem 3.3.1 is satisfied.

Now it remains to prove that S_h and $\boldsymbol{\varepsilon}(V_h)$ satisfy the ellipticity condition, that is, part (a) of Theorem 3.3.1. This can be done via the conditions of Lemma 3.5.1. To establish (3.29), consider $\boldsymbol{\varepsilon}(\boldsymbol{v}_h) \in \boldsymbol{\varepsilon}(V_h)$. We show that it satisfies the requirement of Lemma 3.5.6, that is,

$$\text{if} \quad P_{S_h}\boldsymbol{\varepsilon}(\boldsymbol{v}_h) = \mathbf{0}, \quad \text{then} \quad \boldsymbol{\varepsilon}(\boldsymbol{v}_h) = \mathbf{0} \quad \forall\, \boldsymbol{v}_h \in V_h\,. \tag{5.11}$$

As $\operatorname{dev}\boldsymbol{\varepsilon}(V_h) \subset S_h$ the above equation (5.11) implies that $\operatorname{dev}\boldsymbol{\varepsilon}(\boldsymbol{v}_h) = \mathbf{0}$.

Therefore equation (5.11) is satisfied only if

$$P_{S_h}\operatorname{tr}\boldsymbol{\varepsilon}(\boldsymbol{v}_h) = \mathbf{0} \Longleftrightarrow \big(\operatorname{div}(\boldsymbol{v}_h),\, q_h\big)_0 = 0 \quad \forall q_h \in M_h\,. \tag{5.12}$$

Let $(\bar{\boldsymbol{v}}_h\,,\, \tilde{\boldsymbol{v}}_h) \in Q_h^2 \times \mathcal{B}_4^2$ be such that $\boldsymbol{v}_h = \bar{\boldsymbol{v}}_h + \tilde{\boldsymbol{v}}_h$. Then $\operatorname{dev}\boldsymbol{\varepsilon}(\boldsymbol{v}_h) = \operatorname{dev}\boldsymbol{\varepsilon}(\bar{\boldsymbol{v}}_h) + \operatorname{dev}\boldsymbol{\varepsilon}(\tilde{\boldsymbol{v}}_h) = \mathbf{0}$. However the components of $\operatorname{dev}\boldsymbol{\varepsilon}(\bar{\boldsymbol{v}}_h)$ and $\operatorname{dev}\boldsymbol{\varepsilon}(\tilde{\boldsymbol{v}}_h)$ are piecewise constant and quadratic polynomials respectively; that is,

$$\operatorname{dev}\boldsymbol{\varepsilon}(\boldsymbol{v}_h) = \mathbf{0} \implies \operatorname{dev}\boldsymbol{\varepsilon}(\bar{\boldsymbol{v}}_h) = \mathbf{0} \quad \text{and} \quad \operatorname{dev}\boldsymbol{\varepsilon}(\tilde{\boldsymbol{v}}_h) = \mathbf{0}\,.$$

Noting that the components of the symmetric gradient of a quartic bubble function has non-zero shear components, for $\operatorname{dev}\boldsymbol{\varepsilon}(\tilde{\boldsymbol{v}}_h) = \mathbf{0}$ we must have $\bar{\boldsymbol{v}}_h = \mathbf{0}$. Then equation (5.12) can equivalently be written as

$$\big(\operatorname{div}(\bar{\boldsymbol{v}}_h),\, q_h\big)_0 = 0 \quad \forall q_h \in M_h$$

which, if we consider q_h to be the standard shape-function with compact support $K \subset \Omega$, gives

$$\big(\operatorname{div}(\bar{\boldsymbol{v}}_h),\, q_h\big)_0 = \operatorname{div}(\bar{\boldsymbol{v}}_h)|_K \mathsf{Vol}(K) = 0\,,$$

where $\mathsf{Vol}(K)$ denotes the volume of K. We therefore have from the above that $\operatorname{div}(\bar{\boldsymbol{v}}_h) = 0$. Thus (5.11) is proven.

To meet all the requirements of Lemma 3.5.1, in order to prove the ellipticity condition (a) of Theorem (3.3.1) it remains to establish that the orthogonal projection of the basis $\boldsymbol{B} = \{\boldsymbol{\eta}_{h(I)}\}_I$ of $\boldsymbol{\varepsilon}(V_h)$ onto S_h forms a linearly independent family of vectors.

Recalling the Voigt notation where a symmetric tensor $\boldsymbol{\sigma} = \left(\sigma_{ij}\right)_{1\leq i,j\leq 3}$ is denoted by

$$\boldsymbol{\sigma} = [\sigma_{11}\,,\,\sigma_{22}\,,\,\sigma_{33}\,,\,\sigma_{12}\,,\,\sigma_{23}\,,\,\sigma_{31}]\,,$$

the basis $\boldsymbol{B}$ is given by

$$\boldsymbol{B} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & b_x & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & b_y & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & b_z \\ 0 & 0 & 0 & 1 & 0 & 0 & b_y & b_x & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & b_z & b_y \\ 0 & 0 & 0 & 0 & 0 & 1 & b_z & 0 & b_x \end{bmatrix}. \tag{5.13}$$

To prove that $P_{S_h}(\boldsymbol{B}) = P_{S_h}\{\boldsymbol{\eta}_{h(I)}\}_I$ is linearly independent let

$$P_{S_h}\boldsymbol{\eta}_h = \sum_I \alpha_I \, P_{S_h}\boldsymbol{\eta}_{h(I)} \quad \text{where} \quad \boldsymbol{\eta}_h = \sum_I \alpha_I \, \boldsymbol{\eta}_{h(I)}. \tag{5.14}$$

We have earlier shown that if the projection onto S_h of an element of $\boldsymbol{\varepsilon}(V_h)$ vanishes then this element also vanishes. Using this fact and the above expression (5.14) we can write

$$P_{S_h}\boldsymbol{\eta}_h = \mathbf{0} \Longrightarrow \boldsymbol{\eta}_h = \mathbf{0} \Longrightarrow \sum_I \alpha_I \, \boldsymbol{\eta}_{h(I)} = \mathbf{0} \Longrightarrow \alpha_I = 0,$$

which means that $P_{S_h}(\boldsymbol{B}) = P_{S_h}\{\boldsymbol{\eta}_{h(I)}\}_I$ is linearly independent. Hence the ellipticity condition (a) of Theorem 3.3.1 is satisfied.

Summing up, we have proved that the mixed formulation (5.2) with the choice (5.1) of spaces converges uniformly at the optimal rate.

CHAPTER 6

THREE-FIELD MIXED FINITE ELEMENT APPROXIMATIONS OF PROBLEMS IN NONLINEAR ELASTICITY

The development of stable methods for non-linear elasticity has been a topic of interest for some time. Prominent among these methods have been enhanced strains: the extension to nonlinear problems was carried out by Simo & Armero [41], with further extensions to avoid numerical instabilities particularly for problems of extreme compression, for example [6, 22, 44]. More recently, the issue of stability has been approached from a different direction, viz. that of adding stabilisation or penalty terms which would also serve to avoid locking in the incompressible limit [5, 45]. Two-dimensional formulations have been studied in Auricchio *et al.* [5, 6] to determine stability ranges. For large loading parameters most of the finite element approximations are unstable, in the sense that they fail to satisfy a coercivity condition. The formulation has also shown numerical instabilities. Issues related to such kinds of instability have also been addressed in Ten Eyck & Lew [45], where a stabilization strategy is used to render the formulation well-posed. Starting with a linearized non-linear problem the method consists of a proper enhancement of the strain variable so that it becomes coercive. The works by Auricchio *et al.* [5, 6] are concerned with the ability of a mixed method to reproduce the stability range. On the other hand, the focus in our case is closer to that of Ten Eyck & Lew [45]: that is assuming that the continuous problem is coercive, so that there are no physical instabilities, we design a finite element scheme that is numerically stable.

The above works all deal with the construction of a numerically stable scheme. The issue of physical instabilities relates to the need to identify bifurcation points, from which two or more solutions are possible, or limit points. The computational challenge is then one of constructing solutions for the primary and secondary paths: see for example Auricchio *et al.* [5] and Duffett & Reddy [27].

In this chapter we take a different approach to the problem of constructing stable formulations for the nonlinear problem, for nearly-incompressible materials. Drawing on the analysis of the linear problem in Chapter 2, we consider the incremental or linearized problem that has to be solved, for example as an iteration in a Newton scheme. It is in fact this problem for which one wants to guarantee solutions that are stable in the incompressible limit. The linearized problem is analysed using the techniques of Chapter 2. This will be shown to lead to conditions for stable solutions that are easily satisfied, provided that the elasticity tensor is such as to rule out physical instabilities.

The structure of this chapter is as follows. In Sections 6.1 to 6.3 we introduce some basic definitions and properties of our primary variables, the velocity, the deformation gradient and the *Piola-Kirchhoff* stress tensor. Section 6.4 is devoted to the weak formulation along with its modified version and the presentation of relevant results on well-posedness. In Section 6.5 we present the analysis of the discrete formulation. And lastly we discuss the error estimate of the approximations.

6.1 Kinematics of Finite Deformations

Consider the motion of a body $\Omega_0 \subset \mathbb{R}^d$ from its initial configuration at time t_0, called the reference configuration, to its position at time t, called the deformed or current configuration. The material is subject to a body force $\boldsymbol{f}$ and surface traction $\bar{\boldsymbol{t}}$, as shown in Figure 6.1, below.

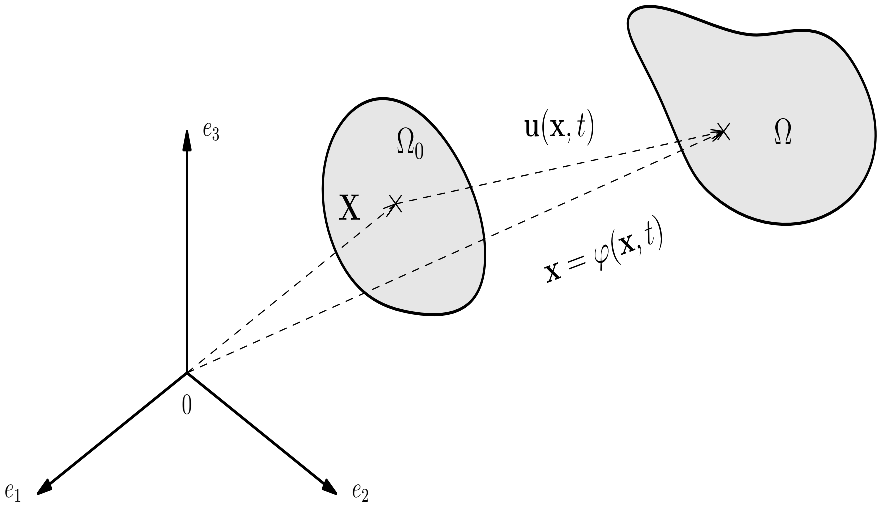


Figure 6.1: Material deformation from the reference to its current configuration

The motion of the body is described by the map

$$\boldsymbol{x} = \boldsymbol{\varphi}(\boldsymbol{X}, t), \tag{6.1a}$$

where $\boldsymbol{X}$ and $\boldsymbol{x}$ are the positions of particles in Ω_0 and Ω, at the reference and current configurations, respectively. It is natural to view the map $\boldsymbol{\varphi}$ as a rigid motion followed by a change in shape of the domain Ω. That is, the equation (6.1a) can be written as

$$\boldsymbol{x} = \boldsymbol{X} + \boldsymbol{u}(\boldsymbol{X}, t), \tag{6.1b}$$

where $\boldsymbol{u}(\boldsymbol{X}, t)$ is the displacement. The deformation of the body is characterized by the transformation

$$\boldsymbol{F} = \frac{\partial \boldsymbol{\varphi}(\boldsymbol{X}, t)}{\partial \boldsymbol{X}} = \nabla \boldsymbol{\varphi}(\boldsymbol{X}, t) = I + \nabla \boldsymbol{u}(\boldsymbol{X}, t), \tag{6.2}$$

called the deformation gradient, where ∇ is the gradient operator in the reference configuration. In what follows the determinant of $\boldsymbol{F}$ is denoted by J.

The left and right Cauchy-Green tensors are defined by:

$$\boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^T \quad \text{and} \quad \boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F}\,, \tag{6.3a}$$

respectively, and the Lagrangian strain tensor is given by

$$\boldsymbol{E} = \tfrac{1}{2}(\boldsymbol{C} - \boldsymbol{I})\ . \tag{6.3b}$$

Of relevance later are the eigenvalues a_i^2 of $\boldsymbol{C}$ or $\boldsymbol{B}$, where a_i are the principal stretches. Using formulas for the deformation gradient (6.2) and the right Cauchy-Green tensor (6.3a) the Lagrangian strain tensor can also be expressed as

$$\boldsymbol{E} = \tfrac{1}{2}\Big(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T + (\nabla\boldsymbol{u})^T\nabla\boldsymbol{u}\Big). \tag{6.4}$$

6.2 Stress and the equilibrium equation

To define the stress consider the surface traction $\boldsymbol{t}(\boldsymbol{n})$, which is the force per unit area acting on a surface with area $\mathrm{d}a$ and unit normal $\boldsymbol{n}$ in the current configuration.

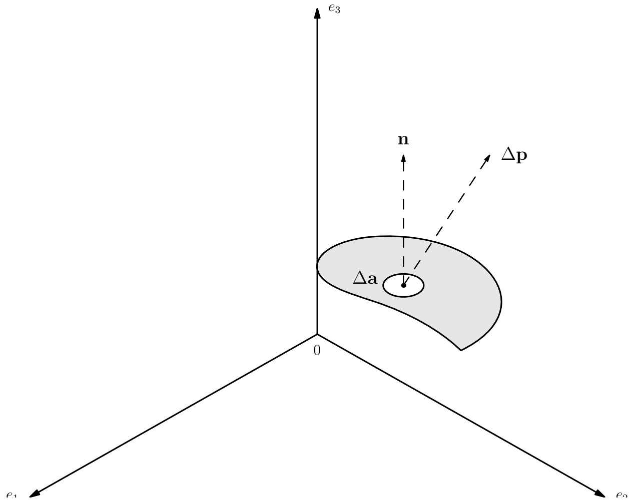


Figure 6.2: Element area, normal vector and resultant force

From the fundamental Theorem of *Cauchy* there exists a second order tensor $\boldsymbol{\sigma}$, called the *Cauchy* stress tensor, which relates the traction $\boldsymbol{t}$ to the normal vector $\boldsymbol{n}$ by

$$\boldsymbol{t}(n) = \boldsymbol{\sigma}\boldsymbol{n}\,. \tag{6.5}$$

Balance of angular momentum gives

$$\boldsymbol{\sigma}^T = \boldsymbol{\sigma}\,.$$

Now consider an element area $\boldsymbol{N}\mathrm{d}A$ that deforms to $\boldsymbol{n}\mathrm{d}a$: it is known that

$$\boldsymbol{n}\mathrm{d}a = J\boldsymbol{F}^{-1}\boldsymbol{N}\mathrm{d}A\,.$$

Thus it is natural from the above to define the first Piola-Kirchhoff stress tensor, denoted by $\boldsymbol{P}$, from the reference configuration as

$$\boldsymbol{P}\boldsymbol{N}\mathrm{d}A = \boldsymbol{\sigma}\boldsymbol{n}\mathrm{d}a \implies \boldsymbol{P} = J\boldsymbol{\sigma}\boldsymbol{F}^{-T}\,. \tag{6.6}$$

If now we further assume the material to be subject to a body force $\boldsymbol{f}$ per unit mass, then the balance of forces for an arbitrary subdomain B vanishes. We then have

$$\int_{\partial B} \boldsymbol{\sigma}\boldsymbol{n} da \; + \int_{B} \rho \boldsymbol{f}\, dv = \boldsymbol{0}, \quad \forall\, B \subset \Omega \tag{6.7a}$$

where ρ is the density of the material. Using the Gauss Theorem the equation (6.7a) can be rewritten as

$$\int_{B} \big(\mathrm{div}\,\boldsymbol{\sigma} \; + \rho \boldsymbol{f}\big) dv = \boldsymbol{0}, \quad \forall\, B \subset \Omega\,, \tag{6.7b}$$

where div is the divergence operator in the current configuration. As the element volume $B \subset \Omega$ in the above equation (6.7b) is arbitrary we have

$$\mathrm{div}\,\boldsymbol{\sigma} \; + \rho \boldsymbol{f} = \boldsymbol{0}. \tag{6.7c}$$

This is the equilibrium equation for a deformable body. Similarly, the equation of equilibrium is

$$\int_{B_0} \big(\mathrm{Div}\,\boldsymbol{P} + \rho_0 \boldsymbol{f}\big) \mathrm{d}v = \boldsymbol{0},$$

so that

$$\mathrm{Div}\,\boldsymbol{P} \; + \rho_0 \boldsymbol{f} = \boldsymbol{0}, \tag{6.8}$$

where Div is the divergence operator at the reference configuration and $\rho_0 = J\rho$.

6.3 Strain energy function

For an elastic material the first Piola-Kirchhoff stress tensor $\boldsymbol{P}$, defined in (6.6), depends only on the deformation gradient $\boldsymbol{F}$ and the material position $\boldsymbol{X}$; that is,

$$\boldsymbol{P} = \boldsymbol{P}(\boldsymbol{F}(\boldsymbol{X}), \boldsymbol{X}).$$

Furthermore, if the material is assumed to be hyperelastic the work done by the stress tensor $\boldsymbol{P}$ from its initial configuration at time t_0 to its current configuration at time t can be expressed as

$$\Psi(\boldsymbol{F}(\boldsymbol{X}), \boldsymbol{X}) = \int_{t_0}^{t} \boldsymbol{P}(\boldsymbol{F}(\boldsymbol{X}), \boldsymbol{X}) : \dot{\boldsymbol{F}} \, \mathrm{d}t \tag{6.9a}$$

which implies that

$$\dot{\Psi}(\boldsymbol{F}) = \frac{\partial \Psi}{\partial F_{ij}} \dot{F}_{ij} \quad \text{and then} \quad \boldsymbol{P} = \frac{\partial \Psi}{\partial \boldsymbol{F}}, \tag{6.9b}$$

see [12] for details, where Ψ is the strain energy. From the principle of material frame indifference, Ψ is in fact a function of $\boldsymbol{C}$: thus

$$\Psi(\boldsymbol{F}) = \Psi(\boldsymbol{C}) .$$

For convenience we denote the functions of $\boldsymbol{F}$ and $\boldsymbol{C}$ by the same symbols. As an example, for a compressible Neo-Hookean material Ψ is given by

$$\Psi = \frac{\mu}{2}\big(\text{tr}(\boldsymbol{C}) - 3\big) - \mu \ln{(J)} + \frac{\lambda}{2}(\ln{(J)})^2, \tag{6.10}$$

where tr$(\boldsymbol{A})$ represents the trace of a tensor $\boldsymbol{A}$ and μ and λ are together the Lamé moduli.

For compressible materials the following generalizations of incompressible relations have been proposed

1. the generalized Neo-Hookean material, with

$$\Psi(\boldsymbol{F}) = \frac{\mu}{2}\big(\frac{\text{tr}(\boldsymbol{C})}{J^{2/3}} - 3\big) + \frac{k_1}{2}(J-1)^2 \,, \tag{6.11a}$$

2. the generalized Mooney-Rivlin material, for which

$$\Psi(\boldsymbol{F}) = \frac{\mu_1}{2}\big(\frac{\text{tr}(\boldsymbol{C})}{J^{2/3}} - 3\big) + \frac{\mu_2}{2}\big(\frac{\text{tr}(\boldsymbol{C} - C^2)}{J^{2/3}} - 3\big) + \frac{k_1}{2}(J-1)^2 \,, \tag{6.11b}$$

where μ and μ_2 and k_1 the bulk modulus. See [13] and [12].

6.4 Mixed variational formulation

The equations (6.8), (6.2) and (6.9b) summarize the strong formulation of the nonlinear elasticity problem:

$$\begin{aligned} \text{Div}\,\boldsymbol{P} + \rho_0 \boldsymbol{f} &= \boldsymbol{0}\,, &(6.12a)\\ \boldsymbol{F} &= \boldsymbol{I} + \nabla \boldsymbol{u}\,, &(6.12b)\\ \boldsymbol{P} &= \frac{\partial \Psi}{\partial \boldsymbol{F}}\,. &(6.12c)\end{aligned}$$

We assume that the motion is subject to the homogeneous Dirichlet boundary condition $\boldsymbol{u} = \boldsymbol{0}$ on $\partial\Omega$.

To formulate the problem in weak form we introduce the following spaces:

the space of displacements $V = \{\boldsymbol{u} : \Omega \longrightarrow \mathbb{R}^d \mid \boldsymbol{u} \in \left[H^1(\Omega)\right]^d \;\text{ with }\; \boldsymbol{u} = \boldsymbol{0} \;\text{ on }\; \partial\Omega\}$, (6.13)

and

$$\mathcal{M} = \{\boldsymbol{F} : \Omega \to \mathbb{R}^{d\times d} \,|\, F_{ij} \in L^2(\Omega) \quad i, j = 1, \cdots, d\}\,, \tag{6.14}$$

for the deformation gradient and the first Piola-Kirchhoff stress tensor.

The three-field functional corresponding to the above strong form, (6.12), is given by

$$W(\boldsymbol{u}, \boldsymbol{F}, \boldsymbol{P}) = \int_{\Omega_0} \Psi(\boldsymbol{F})\,\mathrm{d}V + \int_{\Omega_0} \boldsymbol{P} : (I + \nabla \boldsymbol{u} - \boldsymbol{F})\,\mathrm{d}V - \int_{\Omega_0} \rho_0 \boldsymbol{f} \cdot \boldsymbol{u}\,\mathrm{d}V \tag{6.15}$$

The first variation of this functional in the direction of the virtual displacement, $\boldsymbol{v}$, virtual deformation gradient, $\boldsymbol{G}$, and virtual Piola stress, $\boldsymbol{Q}$ yields the following three-field weak formulation:

$$\begin{aligned} \delta W(\boldsymbol{u}, \boldsymbol{F}, \boldsymbol{P})[\boldsymbol{v}] &= \int_{\Omega_0} \big(\boldsymbol{P} : (\nabla \boldsymbol{v}) - \rho_0 \boldsymbol{f} \cdot \boldsymbol{u}\big)\mathrm{d}V = 0\;; &(6.16a)\\ \delta W(\boldsymbol{u}, \boldsymbol{F}, \boldsymbol{P})[\boldsymbol{G}] &= \int_{\Omega_0} \big(\boldsymbol{P} - \frac{\partial \Psi}{\partial \boldsymbol{F}}\big) : \boldsymbol{G}\,dV = 0\,; &(6.16b)\\ \delta W(\boldsymbol{u}, \boldsymbol{F}, \boldsymbol{P})[\boldsymbol{Q}] &= \int_{\Omega_0} \big(\boldsymbol{F} - (\boldsymbol{I} + \nabla \boldsymbol{u})\big) : \boldsymbol{Q}\,dV = 0\,. &(6.16c)\end{aligned}$$

The system (6.16) is nonlinear, and computational approaches will involve a linearisation of the problem using the Newton method, for example. The problem (6.12), or its finite element approximation, represents a set of nonlinear equations that must be solved by an iterative procedure.

This procedure leads to a linearised problem which is closely related to the rate problem studied in earlier sections.

The weak form of the original discrete problem is

$$(\boldsymbol{P}_h\,,\,\nabla\boldsymbol{v}_h) - (\boldsymbol{f}_h\,,\,\boldsymbol{v}_h) + (\boldsymbol{P}_h - \partial W/\partial\boldsymbol{F}_h\,,\,\boldsymbol{G}_h) + (\boldsymbol{F}_h - (\boldsymbol{I} + \nabla\boldsymbol{v}_h\,,\,\boldsymbol{Q}_h)) = \boldsymbol{0}\,. \tag{6.17}$$

This can be considered to be a set of equations of the form

$$\phi(\mathbf{U}_h) = \boldsymbol{0}\,, \tag{6.18}$$

where $\mathbf{U}_h = (\boldsymbol{u}_h\,,\,\boldsymbol{F}_h\,,\,\boldsymbol{P}_h)$.

Assume that the loading $\boldsymbol{f}$ is divided into steps $\boldsymbol{0} = \boldsymbol{f}_0 < \boldsymbol{f}_1 < \cdots < \boldsymbol{f}_n < \cdots < \boldsymbol{f}_N = \boldsymbol{f}$, and denote the solution corresponding to $\boldsymbol{f}_n$ by $\mathbf{U}_n$. The problem is to find $\mathbf{U}_{n+1}$; that is, solve

$$\phi(\mathbf{U}_{n+1}) = \boldsymbol{0}\,, \tag{6.19}$$

given $\mathbf{U}_{n+1}$. We apply a Newton method to (6.19). Thus,

$$\boldsymbol{0} = \phi(\mathbf{U}_n^{i+1}) = \phi(\mathbf{U}_n^i) + D\phi(\mathbf{U}_n^i)(\mathbf{U}_n^{i+1} - \mathbf{U}_n^i) \tag{6.20}$$

or

$$D\phi(\mathbf{U}_n^i)(\mathbf{U}_n^{i+1} - \mathbf{U}_n^i) = -\phi(\mathbf{U}_n^i) \tag{6.21}$$

where

$$\Delta\mathbf{U}_n = \mathbf{U}_n^{i+1} - \mathbf{U}_n^i\,. \tag{6.22}$$

Now set $\mathbf{U}_{n+1}^1 = \mathbf{U}_n$ to initiate the process.

Application of the Newton process to this nonlinear system leads to the linearised set of equations

$$\begin{aligned}(\mathbb{A}_h^i\Delta\boldsymbol{F}_h - \Delta\boldsymbol{P}_h\,,\,\boldsymbol{G}_h) &= -(\boldsymbol{R}^i\,,\,\boldsymbol{G}_h)\,,\\ (\Delta\boldsymbol{F}_h - \nabla(\Delta\boldsymbol{u}_h)\,,\,\boldsymbol{Q}_h) &= 0\,,\\ (\nabla\boldsymbol{v}_h\,,\,\Delta\boldsymbol{P}_h) &= 0\,,\end{aligned} \tag{6.23}$$

where $\boldsymbol{R}^i$ is the residual.

With this in mind, we consider here the closely related rate problem: given the solution $(\boldsymbol{u}\,,\,\boldsymbol{F}\,,\,\boldsymbol{P})$ of the strong form (6.12) and given the loading rate $\dot{\boldsymbol{f}}$, find $(\dot{\boldsymbol{P}}\,,\,\dot{\boldsymbol{F}}\,,\,\dot{\boldsymbol{u}})$ that satisfy

$$\begin{aligned} \mathrm{Div}\,\dot{\boldsymbol{P}} + \rho_0 \dot{\boldsymbol{f}} &= \boldsymbol{0}\,, && (6.24a)\\ \dot{\boldsymbol{F}} &= \nabla \dot{\boldsymbol{u}}\,, && (6.24b)\\ \dot{\boldsymbol{P}} &= \mathbb{A}\dot{\boldsymbol{F}}\,, && (6.24c) \end{aligned}$$

and $\dot{\boldsymbol{u}} = 0$ on $\partial\Omega$, where

$$\mathbb{A} = \frac{\partial^2 \Psi}{\partial \boldsymbol{F} \partial \boldsymbol{F}} \tag{6.24d}$$

is the first elasticity tensor.

Remark A full treatment of the nonlinear problem would require also an analysis of convergence of the Newton iterations. We focus however on the task of constructing families of elements that are stable for the incremental or rate problem; see also [45].

General properties of the elastic moduli $\mathbb{A}$

It is shown in [20] that $\mathbb{A}$ has the major symmetry

$$\mathbb{A}_{ijkl} = \mathbb{A}_{klij}\,, \tag{6.25a}$$

and is sparse, in the sense that the only non zero entries of $\mathbb{A}$ are

$$\mathbb{A}_{iiii}\,,\ \mathbb{A}_{iijj}\,,\ \mathbb{A}_{ijij}\,,\ \mathbb{A}_{ijji}\,, \quad \text{with } i \neq j\,, \tag{6.25b}$$

where the summation convention is not used.

Assuming (6.24) to be well posed the elastic moduli $\mathbb{A}$ can be assumed to be at least locally invertible. Using the results in [45] $\mathbb{A}$ is also assumed to be $V-$elliptic: that is, there exists $\kappa > 0$ such that

$$\int_{\Omega_0} \nabla \boldsymbol{u} : \mathbb{A} \nabla \boldsymbol{u} \, \mathrm{d}V \geq \kappa \|\boldsymbol{u}\|_1^2\,, \quad \forall\, \boldsymbol{u} \in V\,. \tag{6.26}$$

Since the reference configuration has not been defined explicitly, it may be chosen to coincide with the current configuration. In this case the elasticity tensor is denoted by $\mathbb{B}$, and it is related to $\mathbb{A}$ by [20]

$$\mathbb{B}_{ijkl} = J^{-1} F_{jp} F_{lq} \mathbb{A}_{ipkq}\,. \tag{6.27}$$

Furthermore, when the material is isotropic, then the moduli are related to each other by [21]

$$\mathbb{B}_{ijkl} = J^{-1} a_j a_l \mathbb{A}_{ijkl} \, . \tag{6.28}$$

For example, for the strain energy function given by

$$\Psi(\boldsymbol{F}) = \frac{\mu}{2}\big(a_1^2 + a_2^2 + a_3^2 - 3\big) + f(J) \, , \tag{6.29}$$

the components of $\mathbb{B}$ are

$$\mathbb{B}_{ijkl} = (Jf')'\delta_{ij}\delta_{kl} + \mu J^{-1} a_i^2 \delta_{ik}\delta_{jl} - f'\delta_{il}\delta_{jk} \, . \tag{6.30}$$

For a Neo-Hookean material

$$f = -\mu \ln(J) + \frac{\lambda}{2}\ln^2(J) \Longrightarrow (Jf')' = \frac{\lambda}{J} \quad \text{and} \quad f' = -\frac{\mu}{J} + \frac{\lambda}{J}\ln(J) \, .$$

One can observe from (6.30) that the only non-zero values of $\mathbb{B}$ are

$$B_{iijj} \quad \forall i, j \tag{6.31}$$

$$B_{ijij} \, , \quad B_{ijji} \quad \text{and} \quad B_{jiij} \quad \forall i \neq j \, , \tag{6.32}$$

with no use of the summation convention. The above properties of $\mathbb{B}$ are general for isotropic materials.

6.4.1 Weak formulation

The weak form corresponding to problem (6.24) is as follows: given $\boldsymbol{u}$, $\boldsymbol{F}$ and $\boldsymbol{P}$ along with the load $\dot{\boldsymbol{f}}$ find $(\dot{\boldsymbol{u}} \, , \, \dot{\boldsymbol{F}} \, , \, \dot{\boldsymbol{P}}) \in V \times \mathcal{M} \times \mathcal{M}$ such that

$$a\big((\dot{\boldsymbol{F}} \, , \, \dot{\boldsymbol{u}}), (\boldsymbol{G} \, , \, \boldsymbol{v})\big) + b\big((\boldsymbol{G} \, , \, \boldsymbol{v}), \dot{\boldsymbol{P}}\big) \;=\; \ell(\boldsymbol{v}) \quad \forall \, (\boldsymbol{v} \, , \, \boldsymbol{G}) \in V \times \mathcal{M} \, , \tag{6.33a}$$

$$b\big((\dot{\boldsymbol{F}} \, , \, \dot{\boldsymbol{u}}), \boldsymbol{Q}\big) \;=\; 0 \qquad \forall \, \boldsymbol{Q} \in \mathcal{M} \, , \tag{6.33b}$$

where the bilinear forms $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$ are as follows:

$$a\big((\dot{\boldsymbol{F}} \, , \, \dot{\boldsymbol{u}}), (\boldsymbol{G} \, , \, \boldsymbol{v})\big) \;=\; (\mathbb{A} : \dot{\boldsymbol{F}}, \boldsymbol{G})_0 \, , \tag{6.34a}$$

$$b\big((\boldsymbol{G} \, , \, \boldsymbol{v}), \dot{\boldsymbol{P}}\big) \;=\; \big(\nabla \boldsymbol{v} - \boldsymbol{G}, \dot{\boldsymbol{P}}\big)_0 \, , \tag{6.34b}$$

$$\ell(\boldsymbol{v}) \;=\; (\dot{\boldsymbol{f}}, \boldsymbol{v})_0 \, . \tag{6.34c}$$

Problem (6.33) may also be written in the form

$$\begin{aligned} \left(\mathbb{A}\dot{\boldsymbol{F}} - \dot{\boldsymbol{P}}\,,\, \boldsymbol{G}\right)_0 &= 0 \quad \forall \boldsymbol{G} \in \mathcal{M}\,, && (6.35a)\\ \left(\dot{\boldsymbol{F}} - \nabla\dot{\boldsymbol{u}}\,,\, \boldsymbol{Q}\right)_0 &= 0 \quad \forall \boldsymbol{Q} \in \mathcal{M}\,, && (6.35b)\\ \left(\nabla\boldsymbol{v}\,,\, \dot{\boldsymbol{P}}\right)_0 &= \ell(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in V\,. && (6.35c) \end{aligned}$$

In many situations, depending on the energy functional at hand, the well-posedness of problem (6.33) or (6.35) can not be established due to the presence of the Lamé parameter λ. Indeed in such contexts the uniform continuity constant could depend on λ which does not guarantee the convergence of a finite element approximation in the incompressible limit.

6.4.2 Modified weak formulation

The problem describe here carries its motivation from the work in [38], where one construct an alternative of the formulation (6.33); for the sake of well-posedness.

Assume $\mathbb{A}$ to depend on λ and μ and, further, that it can be decomposed additively as

$$\mathbb{A} = \mu\mathbb{D} + \lambda\mathbb{E}\,, \tag{6.36}$$

where $\mathbb{D}$ and $\mathbb{E}$ are, in many situations, symmetric. The properties of $\mathbb{D}$ and $\mathbb{E}$ can be derived from (6.27) and (6.28) and without loss of generality one can identify the current configuration with the reference configuration and use (6.28) to establish the positive definiteness of $\mathbb{D}$ based on the assumption of strong ellipticity of $\mathbb{A}$. Indeed using (6.30) $\mathbb{D}$ can explicitly be formulated and it is positive-definite at near-incompressible state.

For a Neo-Hookean material we have

$$\mathbb{D} = \frac{\mu}{J}\begin{bmatrix} a_1^2+1 & 0 & 0 & 0\\ 0 & a_2^2+1 & 0 & 0\\ 0 & 0 & a_1^2 & 0\\ 0 & 0 & 0 & a_2^2 \end{bmatrix} \quad \text{and} \quad \mathbb{E} = \frac{\lambda}{J}\begin{bmatrix} 1-\ln J & 1 & 0 & 0\\ 1 & 1-\ln J & 0 & 0\\ 0 & 0 & 0 & -\ln J\\ 0 & 0 & -\ln J & 0 \end{bmatrix}$$

Following the discussions in Chapter 2 we construct a modified form of (6.33) by introducing the bilinear forms

$$\begin{aligned} a_0\big((\dot{\boldsymbol{F}}\,,\,\dot{\boldsymbol{u}})\,,\,(\boldsymbol{G}\,,\,\boldsymbol{v})\big) &= a\big((\dot{\boldsymbol{F}}\,,\,\dot{\boldsymbol{u}}),(\boldsymbol{G}\,,\,\boldsymbol{v})\big) - \lambda\Big(\mathbb{E}\dot{\boldsymbol{F}}\,,\,\boldsymbol{G}\Big)_0 \\ &= \Big(\mathbb{A}\dot{\boldsymbol{F}}\,,\,\boldsymbol{G}\Big)_0 - \Big(\lambda\mathbb{E}\dot{\boldsymbol{F}}\,,\,\boldsymbol{G}\Big)_0\,, \end{aligned} \tag{6.37a}$$

$$\begin{aligned} b_1\big((\boldsymbol{G}\,,\,\boldsymbol{v})\,,\,\dot{\boldsymbol{P}}\big) &= b\big((\boldsymbol{G}\,,\,\boldsymbol{v}),\dot{\boldsymbol{P}}\big) + \Big(\lambda\mathbb{E}\mathbb{A}^{-1}\dot{\boldsymbol{P}}\,,\,\boldsymbol{G}\Big)_0 \\ &= \Big(\nabla\boldsymbol{v}-\boldsymbol{G}\,,\,\dot{\boldsymbol{P}}\Big)_0 + \Big(\lambda\mathbb{E}\mathbb{A}^{-1}\dot{\boldsymbol{P}}\,,\,\boldsymbol{G}\Big)_0 \end{aligned} \tag{6.37b}$$

and

$$b_2\big((\boldsymbol{G}\,,\,\boldsymbol{v})\,,\,\dot{\boldsymbol{P}}\big) = b\big((\boldsymbol{G}\,,\,\boldsymbol{v}),\dot{\boldsymbol{P}}\big)\,. \tag{6.37c}$$

Then the modified problem is that of finding $(\dot{\boldsymbol{u}},\dot{\boldsymbol{F}},\dot{\boldsymbol{P}}) \in V \times \mathcal{M} \times \mathcal{M}$ such that

$$a_0\big((\dot{\boldsymbol{F}}\,,\,\dot{\boldsymbol{u}})\,,\,(\boldsymbol{G}\,,\,\boldsymbol{v})\big) + b_1\big((\boldsymbol{G}\,,\,\boldsymbol{v})\,,\,\dot{\boldsymbol{P}}\big) \;=\; \ell(\boldsymbol{v}) \quad \forall\,(\boldsymbol{v}\,,\,\boldsymbol{G}) \in V \times \mathcal{M}\,, \tag{6.38a}$$

$$b_2\big((\dot{\boldsymbol{F}}\,,\,\dot{\boldsymbol{u}})\,,\,\boldsymbol{Q}\big) \;=\; 0 \qquad \forall\,\boldsymbol{Q} \in \mathcal{M}\,, \tag{6.38b}$$

or equivalently

$$\big(\mu\mathbb{D}(\mathbb{A}^{-1}\dot{\boldsymbol{P}}-\dot{\boldsymbol{F}})\,,\,\boldsymbol{G}\big)_0 \;=\; 0 \quad \forall \boldsymbol{G} \in \mathcal{M}\,, \tag{6.39a}$$

$$\big(\dot{\boldsymbol{F}} - \nabla\dot{\boldsymbol{u}}\,,\,\boldsymbol{Q}\big)_0 \;=\; 0 \quad \forall \boldsymbol{Q} \in \mathcal{M}\,, \tag{6.39b}$$

$$\big(\nabla\boldsymbol{v}\,,\,\dot{\boldsymbol{P}}\big)_0 \;=\; \ell(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in V\,. \tag{6.39c}$$

Now we need to prove that the modified formulation (6.38) is consistent, in that it is equivalent to the standard formulation (6.33), and that it has a unique solution which satisfies the strong form (6.24).

Equivalence between the modified formulation (6.38) and the standard formulation (6.33)

Equations (6.39a) and (6.39c) are obtained from (6.38a) by setting $\boldsymbol{v} = \boldsymbol{0}$ and then $\boldsymbol{G} = \boldsymbol{0}$. Indeed for $\boldsymbol{v} = \boldsymbol{0}$ it follows from (6.38a) that

$$(\mathbb{A}\dot{\boldsymbol{F}}\,,\,\boldsymbol{G})_0 - \lambda(\mathbb{E}\dot{\boldsymbol{F}}\,,\,\boldsymbol{G})_0 - (\boldsymbol{G}\,,\,\dot{\boldsymbol{P}})_0 + \lambda(\mathbb{E}\mathbb{A}^{-1}\dot{\boldsymbol{P}}\,,\,\boldsymbol{G})_0 = 0\,,$$

which if rearranged gives

$$\big((\mathbb{A}-\lambda\mathbb{E})\dot{\boldsymbol{F}}\,,\,\boldsymbol{G}\big)_0-\big((\boldsymbol{I}-\lambda\mathbb{E}\mathbb{A}^{-1})\dot{\boldsymbol{P}}\,,\,\boldsymbol{G})_0=0\,. \tag{6.40}$$

As $\mathbb{A}=\mu\mathbb{D}+\lambda\mathbb{E}$ we have

$$\mathbb{A}-\lambda\mathbb{E}=\mu\mathbb{D}\,,$$

which if multiplied by the inverse of $\mathbb{A}$ gives

$$\boldsymbol{I}-\lambda\mathbb{E}\mathbb{A}^{-1}=\mu\mathbb{D}\mathbb{A}^{-1}\,.$$

Substituting the above two equations into (6.40), we obtain (6.39a). Now taking $\boldsymbol{G}=\boldsymbol{0}$ in (6.38a) we obviously have the equation (6.39c). Equation (6.39b) is clearly identical to (6.38b).

We only need to show the equivalence between (6.35a) and (6.39a). Using the symmetry of elastic modulus $\mathbb{D}$ and $\mathbb{A}$ and their invertibility the equation (6.39a) can be written as

$$\big(\dot{\boldsymbol{P}}-\mathbb{A}\dot{\boldsymbol{F}}\,,\,\mu\mathbb{A}^{-1}\mathbb{D}\boldsymbol{G}\big)_0=0\quad\forall\boldsymbol{G}\in\mathcal{M}\,,$$

which clearly reveals the equivalence between (6.35a) and (6.39a) if we replace $\boldsymbol{G}$ by $(\mu\mathbb{A}^{-1}\mathbb{D})^{-1}\boldsymbol{G}$.

Theorem 6.4.1 *[7, 25] Consider a generalized form of problem (6.38) find $(\dot{\boldsymbol{u}}\,,\,\dot{\boldsymbol{F}}\,,\,\dot{\boldsymbol{P}})\in V\times\mathcal{M}\times\mathcal{M}$ such that*

$$\begin{aligned} a\big((\dot{\boldsymbol{F}}\,,\,\dot{\boldsymbol{u}})\,,\,(\boldsymbol{G}\,,\,\boldsymbol{v})\big)+b_1\big((\boldsymbol{G}\,,\,\boldsymbol{v})\,,\,\dot{\boldsymbol{P}}\big) &= (\dot{\boldsymbol{f}}\,,\,\boldsymbol{v})\quad\forall\,(\boldsymbol{v}\,,\,\boldsymbol{G})\in V\times\mathcal{M}\,, && (6.41a)\\ b_2\big((\dot{\boldsymbol{u}}\,,\,\dot{\boldsymbol{F}})\,,\,\boldsymbol{Q}\big)-c(\dot{\boldsymbol{P}}\,,\,\boldsymbol{Q}) &= (\dot{\boldsymbol{g}}\,,\,\boldsymbol{Q})\qquad\forall\,\boldsymbol{Q}\in\mathcal{M}\,, && (6.41b)\end{aligned}$$

for some $(\dot{\boldsymbol{f}}\,,\,\dot{\boldsymbol{g}})\in V'\times\mathcal{M}'$. Furthermore we define the kernels

$$\begin{aligned} K_i &= \{(\boldsymbol{G}\,,\,\boldsymbol{v})\in\mathcal{M}\times V\,|\,b_i\big((\boldsymbol{G}\,,\,\boldsymbol{v})\,,\,\boldsymbol{Q}\big)=0\quad\forall\boldsymbol{Q}\in\mathcal{M}\}\,,\quad i=1,\cdots,2,\\ \ker c(\cdot,\cdot) &= \{\boldsymbol{G}\in\mathcal{M}\,|\,c\big(\boldsymbol{G}\,,\,\boldsymbol{Q}\big)=0\quad\forall\boldsymbol{Q}\in\mathcal{M}\}\end{aligned}$$

and the corresponding transposes

$$K_i^t=\{\boldsymbol{Q}\in\mathcal{M}\,|\,b_i\big((\boldsymbol{G}\,,\,\boldsymbol{v})\,,\,\boldsymbol{Q}\big)=0\quad\forall(\boldsymbol{G}\,,\,\boldsymbol{v})\in\mathcal{M}\times V\}\,,\quad i=1,\cdots,2\,.$$

Suppose that $a(\cdot,\cdot)$, $b_i(\cdot,\cdot)$, $i=1,\,2$ and $c(\cdot,\cdot)$ satisfy the following:

a) there exists a constant $\beta^*>0$ such that

$$\forall\,(\dot{\boldsymbol{F}}\,,\,\dot{\boldsymbol{u}})\in K_2\,,\quad\sup_{(\boldsymbol{G}\,,\,\boldsymbol{v})\in K_1}\frac{a\big((\dot{\boldsymbol{F}}\,,\,\dot{\boldsymbol{u}})\,,\,(\boldsymbol{G}\,,\,\boldsymbol{v})\big)}{\|(\boldsymbol{G}\,,\,\boldsymbol{v})\|_{\mathcal{M}\times\mathcal{V}}}\geq\beta^*\|(\dot{\boldsymbol{F}}\,,\,\dot{\boldsymbol{u}})\|_{\mathcal{M}\times\mathcal{V}}\,, \tag{6.42a}$$

$$\forall\,(\boldsymbol{G}\,,\,\boldsymbol{v})\in K_1\backslash\boldsymbol{0}\qquad\sup_{(\boldsymbol{F}\,,\,\boldsymbol{u})\in K_2}\;a\big((\dot{\boldsymbol{F}}\,,\,\dot{\boldsymbol{u}})\,,\,(\boldsymbol{G}\,,\,\boldsymbol{v})\big)>0\,; \tag{6.42b}$$

b) the bilinear forms $b_i(\cdot,\cdot)\,,\ i = 1, \cdots, 2$ *satisfy the usual inf-sup condition;*

c) given $m^{\perp} \in (K_1^t)^{\perp}$*, and the continuous bilinear form* $c(\cdot,\cdot)$*, one can find* $m \in K_1^t$ *such that*

$$c(m\,,\,n) = -c(m^{\perp}\,,\,n) \quad \forall n \in K_2^t\,, \tag{6.42c}$$

where $(K_1^t)^{\perp}$ *is the orthogonal complement of* K_1^t *in* $\mathcal{M}$*;*

d) $a(\cdot,\cdot)$*,* $b_1(\cdot,\cdot)$ *and* $b_2(\cdot,\cdot)$ *are uniformly continuous.*

Then for $(\dot{\boldsymbol{f}}\,,\,\dot{\boldsymbol{g}}) \in V' \times \mathit{Im}B_2$ *(*$\mathit{Im}B_2$ *is image of* $B_2(\cdot,\cdot)$*, as defined in (6.43)), the problem (6.41) has a unique solution* $(\dot{\boldsymbol{u}}\,,\,\dot{\boldsymbol{F}}\,,\,\dot{\boldsymbol{P}}) \in V \times \mathcal{M} \times \mathcal{M}/Z_1^t \cap \ker\, c(\cdot,\cdot)$*, which moreover satisfies the bound*

$$\|\dot{\boldsymbol{u}}\|_V + \|\dot{\boldsymbol{F}}\|_{\mathcal{M}} + \|\dot{\boldsymbol{P}}\|_{\mathcal{M}} \le C\big(\|\dot{\boldsymbol{f}}\|_{V'} + \|\dot{\boldsymbol{g}}\|_{\mathcal{M}'}\big)\,, \tag{6.42d}$$

where C *is a strictly positive constant depending on the norm of* $a(\cdot,\cdot)$*,* β^* *and the inf-sup constants of the bilinear forms* $b_i(\cdot,\cdot)$*.*

Note that the image of the bilinear $B_2(\cdot,\cdot)$ *given by*

$$\mathit{Im}\, B_2 = \{b_2\big((\boldsymbol{G}\,,\,\boldsymbol{v})\,,\,\cdot\big) \in \mathcal{M}'\,,\ \forall(\boldsymbol{G}\,,\,\boldsymbol{v}) \in \mathcal{M} \times V\} \tag{6.43}$$

represents the image of the bilinear form $b_2(\cdot,\cdot)$*. And uniform continuity of the bilinear forms* $a(\cdot,\cdot)$ *and* $b_i(\cdot,\cdot)$ *are assumed to be satisfied.*

Proof See [7]. ∎

Well-posedness of problem (6.38)

The existence and uniqueness of a solution of problem (6.38) can be established by verifying the conditions of Theorem 6.4.1. For this problem

$$\begin{aligned} K_1 &= \{(\boldsymbol{G}\,,\,\boldsymbol{v}) \in \mathcal{M} \times V \,|\, b_1\big((\boldsymbol{G}\,,\,\boldsymbol{v})\,,\,\boldsymbol{Q}\big) = 0 \quad \forall \boldsymbol{Q} \in \mathcal{M}\} \\ &= \{(\boldsymbol{G}\,,\,\boldsymbol{v}) \in \mathcal{M} \times V \,|\, \boldsymbol{G} = (\mu\mathbb{D})^{-1}\mathbb{A}\nabla\boldsymbol{v}\} \end{aligned}$$

where the last equation comes from the definition of $b_1(\cdot,\cdot)$ as given in (6.37a), by replacing $\boldsymbol{Q}$ by $\mathbb{A}\boldsymbol{Q}$ and making use of the invertibility of $\mathbb{A}$. If we set

$$\mathbb{M} = (\mu\mathbb{D})^{-1}\mathbb{A}\,, \tag{6.44a}$$

the kernel K_1 can be written as

$$K_1 = \{ \big(\mathbb{M}\nabla \boldsymbol{v} \,,\, \boldsymbol{v} \big) \,, \quad \boldsymbol{v} \in V \} \,. \tag{6.44b}$$

We also note that using the properties of $\mathbb{D}$ and $\mathbb{A}$ $\mathbb{M}$ is strongly elliptic.

We now similarly, obtain

$$\begin{aligned} K_2 &= \{ (\boldsymbol{G} \,,\, \boldsymbol{v}) \in \mathcal{M} \times V \,|\, b_2\big((\boldsymbol{G} \,,\, \boldsymbol{v}) \,,\, \boldsymbol{Q} \big) = 0 \quad \forall \boldsymbol{Q} \in \mathcal{M} \} \\ &= \{ (\nabla \boldsymbol{v} \,,\, \boldsymbol{v}) \,, \quad \boldsymbol{v} \in V \} \,. \end{aligned} \tag{6.44c}$$

The transpose of these kernels can then be derived as

$$K_1^t = \{ \boldsymbol{Q} \in \mathcal{M} \,|\, b_1\big((\boldsymbol{G} \,,\, \boldsymbol{v}) \,,\, \boldsymbol{Q} \big) = 0 \quad \forall (\boldsymbol{G} \,,\, \boldsymbol{v}) \in \mathcal{M} \times V \} = \{ \boldsymbol{0} \} \tag{6.45a}$$

and

$$K_2^t = \{ \boldsymbol{Q} \in \mathcal{M} \,|\, b_2\big((\boldsymbol{G} \,,\, \boldsymbol{v}) \,,\, \boldsymbol{Q} \big) = 0 \quad \forall (\boldsymbol{G} \,,\, \boldsymbol{v}) \in \mathcal{M} \times V \} = \{ \boldsymbol{0} \} \,. \tag{6.45b}$$

Noting that the bilinear form $c(\cdot, \cdot)$ corresponding to problem (6.38) vanishes and using the expressions (6.45a) and (6.45b) the conditions of Theorem 6.4.1 (c) are automatically satisfied. We also observe that $\dot{\boldsymbol{f}} \in V'$ and $\dot{\boldsymbol{g}} = \boldsymbol{0} \in \operatorname{Im} B_2$. It therefore remains only to establish the conditions (a) and (b) of Theorem 6.4.1.

To prove that the inequality (6.42a) is satisfied, let $(\dot{\boldsymbol{F}} \,,\, \dot{\boldsymbol{u}})$ be any element of the kernel K_2: then by definition there exists $\boldsymbol{v} \in V$ such that

$$(\dot{\boldsymbol{F}} \,,\, \dot{\boldsymbol{u}}) = (\nabla \boldsymbol{v} \,,\, \boldsymbol{v}) \,.$$

Similarly, $(\boldsymbol{G} \,,\, \boldsymbol{v}) = (\mathbb{M}\nabla \boldsymbol{v} \,,\, \boldsymbol{v})$ is an element of K_1. We have

$$\begin{aligned} \|(\boldsymbol{G} \,,\, \boldsymbol{v})\|^2_{\mathcal{M}\times\mathcal{V}} &= \|(\mathbb{M}\nabla \boldsymbol{v} \,,\, \boldsymbol{v})\|^2_{\mathcal{M}\times\mathcal{V}} \,, \\ &= \|\mathbb{M}\nabla \boldsymbol{v}\|_0^2 + \|\boldsymbol{v}\|_1^2 \\ &\leq \|\mathbb{M}\|^2_{\mathcal{M}'} \|\nabla \boldsymbol{v}\|_0^2 + \|\boldsymbol{v}\|_1^2 \\ &\leq \|\mathbb{M}\|^2_{\mathcal{M}'} \|\boldsymbol{v}\|_1^2 + \|\boldsymbol{v}\|_1^2 \\ &\leq C_1^2 \|\boldsymbol{v}\|_1^2 \,, \end{aligned}$$

which implies that

$$\|(\boldsymbol{G}\,,\,\boldsymbol{v})\|_{\mathcal{M}\times\mathcal{V}} \leq C_1\|\boldsymbol{v}\|_1$$

where C_1 is a strictly positive constant. Secondly

$$\begin{aligned}\|(\dot{\boldsymbol{F}}\,,\,\dot{\boldsymbol{u}})\|^2_{\mathcal{M}\times\mathcal{V}} &= \|(\nabla\boldsymbol{v}\,,\,\boldsymbol{v})\|^2_{\mathcal{M}\times\mathcal{V}}\,,\\ &= \|\nabla\boldsymbol{v}\|^2_0 + \|\boldsymbol{v}\|^2_1\\ &\leq \|\boldsymbol{v}\|^2_1 + \|\boldsymbol{v}\|^2_1\\ &\leq 2\|\boldsymbol{v}\|^2_1\,,\end{aligned}$$

or equivalently

$$\|(\dot{\boldsymbol{F}}\,,\,\dot{\boldsymbol{u}})\|_{\mathcal{M}\times\mathcal{V}} \leq \sqrt{2}\|\boldsymbol{v}\|_1$$

and

$$\begin{aligned}a\big((\dot{\boldsymbol{F}}\,,\,\dot{\boldsymbol{u}})\,,\,(\boldsymbol{G}\,,\,\boldsymbol{v})\big) &= (\mu\mathbb{D}\dot{\boldsymbol{F}}\,,\,\boldsymbol{G})\\ &= (\mu\mathbb{D}\nabla\boldsymbol{v}\,,\,\mathbb{M}\nabla\boldsymbol{v})_0\,,\\ &\geq C\|\nabla\boldsymbol{v}\|^2_0\,.\end{aligned}$$

Using the Poincaré-Friedrichs inequality the above can be written as

$$a\big((\dot{\boldsymbol{F}}\,,\,\dot{\boldsymbol{u}})\,,\,(\boldsymbol{G}\,,\,\boldsymbol{v})\big) \geq C_2\|\boldsymbol{v}\|^2_1\,,$$

where C_2 is also a strictly positive constant; because $\mathbb{D}$ and $\mathbb{M}$ are both strongly elliptic. It then follows that

$$\begin{aligned}\frac{a\big((\dot{\boldsymbol{F}}\,,\,\dot{\boldsymbol{u}})\,,\,(\boldsymbol{G}\,,\,\boldsymbol{v})\big)}{\|(\boldsymbol{G}\,,\,\boldsymbol{v})\|_{\mathcal{M}\times\mathcal{V}}\|(\dot{\boldsymbol{F}}\,,\,\dot{\boldsymbol{u}})\|_{\mathcal{M}\times\mathcal{V}}} &= \frac{(\dot{\boldsymbol{F}}\,,\,\boldsymbol{G})_0}{\|(\boldsymbol{G}\,,\,\boldsymbol{v})\|_{\mathcal{M}\times\mathcal{V}}\|(\dot{\boldsymbol{F}}\,,\,\dot{\boldsymbol{u}})\|_{\mathcal{M}\times\mathcal{V}}}\,,\\ &= \frac{\mu(\mathbb{D}\mathbb{M}\nabla\boldsymbol{v}\,,\,\nabla\boldsymbol{v})_0}{\|(\nabla\boldsymbol{v}\,,\,\nabla\boldsymbol{v})\|_{\mathcal{M}\times\mathcal{V}}(\mathbb{M}\nabla\boldsymbol{v}\,,\,\nabla\boldsymbol{v})\|_{\mathcal{M}\times\mathcal{V}}}\,,\\ &\geq \frac{C_2}{\sqrt{2}C_1}\,.\end{aligned} \tag{6.46}$$

As the matrix $\mathbb{M}$ depends linearly on the material parameter λ the ratios in the second and last lines of (6.46) are uniformly bounded, in the incompressible limit, by a positive constant, say

β^*. And β^* depends, via $\mathbb{M}$ and $\mathbb{D}$, on the principal stretches (eigenvalues of the Cauchy-Green tensor), which in finite deformation are also bounded away from zero by a positive constant.

To summarize the proof of inequality (6.42a) we have started with a particular element of K_2 and then constructed an element of K_1 to derive the expression (6.46), which will obviously still hold by using the supremum over K_1. That is to say that

$$\forall\,(\dot{\boldsymbol{F}}\,,\,\dot{\boldsymbol{u}}) \in K_2 \qquad \sup_{(\boldsymbol{G}\,,\boldsymbol{v})\in K_1} \frac{a\big((\dot{\boldsymbol{F}}\,,\,\dot{\boldsymbol{u}})\,,\,(\boldsymbol{G}\,,\,\boldsymbol{v})\big)}{\|(\boldsymbol{G}\,,\,\boldsymbol{v})\|_{\mathcal{M}\times\mathcal{V}}\|(\dot{\boldsymbol{F}}\,,\,\dot{\boldsymbol{u}})\|_{\mathcal{M}\times\mathcal{V}}} > \beta^*\,,$$

which is (6.42a).

For the second inequality (6.42b) let $(\boldsymbol{G}\,,\,\boldsymbol{v}) = (\mathbb{M}\nabla\boldsymbol{v}\,,\,\boldsymbol{v})$ be any element of $K_1\backslash\boldsymbol{0}$, for some non-trivial vector $\boldsymbol{v} \in V$, and consider an element of K_2 such that $(\dot{\boldsymbol{F}}\,,\,\dot{\boldsymbol{u}}) = (\nabla\boldsymbol{v}\,,\,\boldsymbol{v})$. We then have, as previously,

$$\|(\boldsymbol{G}\,,\,\boldsymbol{v})\|_{\mathcal{M}\times\mathcal{V}} \leq C_1\|\boldsymbol{v}\|_1\,,$$

$$\|(\dot{\boldsymbol{F}}\,,\,\dot{\boldsymbol{u}})\|_{\mathcal{M}\times\mathcal{V}} \leq C_2\|\boldsymbol{v}\|_1$$

and

$$a\big((\dot{\boldsymbol{F}}\,,\,\dot{\boldsymbol{u}})\,,\,(\dot{\boldsymbol{P}}\,,\,\boldsymbol{v})\big) > 0$$

with C_1, $C_2 > 0$. From these we obtain the inequality (6.42b); that is,

$$\sup_{(\boldsymbol{F}\,,\boldsymbol{u})\in K_2} \frac{a\big((\dot{\boldsymbol{F}}\,,\,\dot{\boldsymbol{u}})\,,\,(\boldsymbol{G}\,,\,\boldsymbol{v})\big)}{\|(\boldsymbol{G}\,,\,\boldsymbol{v})\|_{\mathcal{M}\times\mathcal{V}}\|(\dot{\boldsymbol{F}}\,,\,\dot{\boldsymbol{u}})\|_{\mathcal{M}\times\mathcal{V}}} > \frac{a\big((\nabla\boldsymbol{v}\,,\,\boldsymbol{v})\,,\,(\mathbb{M}\nabla\boldsymbol{v}\,,\,\boldsymbol{v})\big)}{\|(\mathbb{M}\nabla\boldsymbol{v}\,,\,\boldsymbol{v})\|_{\mathcal{M}\times\mathcal{V}}\|(\nabla\boldsymbol{v}\,,\,\boldsymbol{v})\|_{\mathcal{M}\times\mathcal{V}}} > 0\,,$$

which implies that

$$\forall\,(\boldsymbol{G}\,,\,\boldsymbol{v}) \in K_1\backslash\boldsymbol{0} \qquad \sup_{(\boldsymbol{F}\,,\boldsymbol{u})\in K_2} a\big((\dot{\boldsymbol{F}}\,,\,\dot{\boldsymbol{u}})\,,\,(\boldsymbol{G}\,,\,\boldsymbol{v})\big) > 0\,.$$

The last condition (b) of Theorem 6.4.1, to be verified, requires a uniform inf-sup condition for the bilinear forms $b_1(\cdot,\cdot)$ and $b_2(\cdot,\cdot)$ defined in (6.37b) and (6.34b). To do that consider the bilinear form

$$\tilde{b}(\boldsymbol{v}\,,\,\boldsymbol{G}) = (\nabla\boldsymbol{v}\,,\,\boldsymbol{G})\,, \tag{6.47a}$$

from which we also define the linear map

$$\begin{aligned} \tilde{B}\,:V &\longrightarrow \mathcal{M}' \\ \boldsymbol{v} &\longmapsto \tilde{B}\boldsymbol{v} = (\nabla\boldsymbol{v}\,,\,\cdot)_0\,, \end{aligned} \tag{6.47b}$$

where $\mathcal{M}'$ is the dual space of $\mathcal{M}$.

If the image of $\tilde{B}$, denoted as Im $\tilde{B}$, is closed we know [11, page 39] that the bilinear form $\tilde{b}(\cdot,\cdot)$ satisfies a continuous lifting, that is the *inf-sup* condition.

To prove that Im $\tilde{B}$ is closed consider a sequence $(\boldsymbol{v}_k)_{k\in\mathbb{N}} \subset V$ that converges to $\boldsymbol{v} \in V$. We have $\tilde{B}\boldsymbol{v}_k = (\nabla\boldsymbol{v}_k\,,\,\cdot)_0$ and, using the uniform continuity of the $L^2(\Omega)$ inner product, $(\cdot,\cdot)_0$ and of the operator $\nabla(\cdot)$, the sequence $\tilde{B}\boldsymbol{v}_k = (\nabla\boldsymbol{v}_k\,,\,\cdot)_0$ converges to $\tilde{B}\boldsymbol{v} = (\nabla\boldsymbol{v}\,,\,\cdot)_0$. Thus Im $\tilde{B}$ is a closed subset of $\mathcal{M}'$.

Now as we have proven that $\tilde{b}(\cdot,\cdot)$ satisfies an *inf-sup* condition, there exists a constant $\beta^\star > 0$ such that

$$\sup_{\boldsymbol{v}\in V} \frac{\tilde{b}(\boldsymbol{v}\,,\,\boldsymbol{Q})}{\|\boldsymbol{v}\|_V} \geq \beta^\star \|\boldsymbol{Q}\|_{\mathcal{M}\setminus \ker \tilde{B}'} \quad \forall\, \boldsymbol{Q} \in \mathcal{M}. \tag{6.48}$$

From the above inequality (6.48) and the fact that

$$b_i\big((\mathbf{0}\,,\,\boldsymbol{v})\,,\,\boldsymbol{Q}\big) = \tilde{b}(\boldsymbol{v}\,,\,\boldsymbol{Q})\,,\ i=1,\,2, \quad \text{and} \quad \ker \tilde{B}' = \{\mathbf{0}\}\,,$$

we have

$$\sup_{\boldsymbol{v}\in V} \frac{b_i\big((\mathbf{0}\,,\,\boldsymbol{v})\,,\,\boldsymbol{Q}\big)}{\|(\mathbf{0}\,,\,\boldsymbol{v})\|_{\mathcal{M}\times\mathcal{V}}} = \frac{\tilde{b}(\boldsymbol{v}\,,\,\boldsymbol{Q})}{\|\boldsymbol{v}\|_V} \geq \beta^\star \|\boldsymbol{Q}\|_{\mathcal{M}}\,,\ i=1,\,2, \quad \forall\, \boldsymbol{Q} \in \mathcal{M}.$$

Thus

$$\sup_{(\boldsymbol{G}\,,\,\boldsymbol{v})\in\mathcal{M}\times V} \frac{b_i\big((\boldsymbol{G}\,,\,\boldsymbol{v})\,,\,\boldsymbol{Q}\big)}{\|(\boldsymbol{G}\,,\,\boldsymbol{v})\|_{\mathcal{M}\times\mathcal{V}}} \geq \beta^\star \|\boldsymbol{Q}\|_0\,,\ i=1,\,2, \quad \forall\, \boldsymbol{Q} \in \mathcal{M}.$$

The uniform continuity of the bilinear forms

We have only to check their continuity at $\mathbf{0}$. It can easily be checked that

$$\lim_{(\boldsymbol{G}\,,\,\boldsymbol{v})\to\mathbf{0}} a_0\big((\boldsymbol{G}\,,\,\boldsymbol{v})\,,\,(\boldsymbol{G}\,,\,\boldsymbol{v})\big) = 0\,,$$

$$\lim_{(\boldsymbol{G}\,,\,\boldsymbol{v})\,,\,\boldsymbol{Q})\to\mathbf{0}} b_1\big((\boldsymbol{G}\,,\,\boldsymbol{v})\,,\,\boldsymbol{Q}\big) = 0$$

and

$$\lim_{(\boldsymbol{G}\,,\,\boldsymbol{v})\,,\,\boldsymbol{Q})\to\mathbf{0}} b_2\big((\boldsymbol{G}\,,\,\boldsymbol{v})\,,\,\boldsymbol{Q}\big) = 0\,.$$

6.5 Finite element approximations

In this section we develop finite element approximations of problem (6.38) that satisfy the conditions (6.42a), (6.42b), given in Theorem 6.4.1, and the inf-sup conditions for the bilinear forms $b_1(\cdot,\cdot)$ and $b_2(\cdot,\cdot)$. We first introduce the discrete counterpart of the (6.35), (6.38) and Theorem 6.4.1.

6.5.1 Approximation of the standard three-field formulation

Let $V_h \subset V$, $S_h \subset \mathcal{M}$ and $D_h \subset \mathcal{M}$ be finite-dimensional spaces for the rates of displacement, first Piola-Kirchhoff stress and deformation gradient. Then from (6.35) we define the discrete form of the standard formulation as the problem of finding $(\dot{\boldsymbol{u}}_h\,,\,\dot{\boldsymbol{F}}_h\,,\,\dot{\boldsymbol{P}}_h) \in V_h \times D_h \times S_h$ such that

$$\begin{aligned} \big(\mathbb{A}_h\dot{\boldsymbol{F}}_h - \dot{\boldsymbol{P}}_h\,,\,\boldsymbol{G}_h\big)_0 &= 0 \quad \forall \boldsymbol{G}_h \in D_h\,, && (6.49a)\\ \big(\dot{\boldsymbol{F}}_h - \nabla\dot{\boldsymbol{u}}_h\,,\,\boldsymbol{Q}_h\big)_0 &= 0 \quad \forall \boldsymbol{Q}_h \in S_h\,, && (6.49b)\\ \big(\nabla\boldsymbol{v}_h\,,\,\dot{\boldsymbol{P}}_h\big)_0 &= \ell(\boldsymbol{v}_h) \quad \forall \boldsymbol{v}_h \in V_h\,, && (6.49c) \end{aligned}$$

where $\mathbb{A}_h$ is an approximation of $\mathbb{A}$ to be defined later.

6.5.2 Approximation of the modified formulation

Similar to the discussion in Subsection (6.4.2) we construct the discrete modified form of (6.49), by first introducing the bilinear forms

$$\begin{aligned} a_{0,h}\big((\dot{\boldsymbol{F}}_h\,,\,\dot{\boldsymbol{u}}_h)\,,\,(\boldsymbol{G}_h\,,\,\boldsymbol{v}_h)\big) &= \Big(\mathbb{A}_h\dot{\boldsymbol{F}}_h\,,\,\boldsymbol{G}_h\Big)_0 - \Big(\lambda\mathbb{E}_h\dot{\boldsymbol{F}}_h\,,\,\boldsymbol{G}_h\Big)_0\,; && (6.50a)\\ b_{1,h}\big((\boldsymbol{G}_h\,,\,\boldsymbol{v}_h)\,,\,\dot{\boldsymbol{P}}_h\big) &= \Big(\nabla\boldsymbol{v}_h - \boldsymbol{G}_h\,,\,\dot{\boldsymbol{P}}_h\Big)_0 + \Big(\lambda\mathbb{E}_h\mathbb{A}_h^{-1}\dot{\boldsymbol{P}}_h\,,\,\boldsymbol{G}_h\Big)_0\,; && (6.50b)\\ b_{2,h}\big((\boldsymbol{G}_h\,,\,\boldsymbol{v}_h)\,,\,\dot{\boldsymbol{P}}_h\big) &= b\big((\boldsymbol{G}_h\,,\,\boldsymbol{v}_h),\dot{\boldsymbol{P}}_h\big)\,, && (6.50c) \end{aligned}$$

where $\mathbb{A}_h$ is the element wise mean value of $\mathbb{A}$, with the other moduli defined similarly; that is,

$$\mathbb{A}_h|_K = \frac{1}{\mathsf{Vol}(K)}\int_K \mathbb{A}|_K\,\mathrm{d}x\,, \quad \mathbb{D}_h|_K = \frac{1}{\mathsf{Vol}(K)}\int_K \mathbb{D}|_K\,\mathrm{d}x \quad \text{and} \quad \mathbb{E}_h|_K = \frac{1}{\mathsf{Vol}(K)}\int_K \mathbb{E}|_K\,\mathrm{d}x\,,$$

and Vol(K) is the volume or area of K depending on the dimension of the physical domain Ω. The moduli $\mathbb{A}_h$ and $\mathbb{D}_h$ are therefore approximations of $\mathbb{A}$ and $\mathbb{D}$ that are piecewise constant. Because $\mathbb{A}$ and $\mathbb{D}$ are strongly elliptic their approximation $\mathbb{A}_h$ and $\mathbb{D}_h$ are positive definite.

We define the discrete modified formulation as the problem of finding $(\dot{\boldsymbol{u}}_h, \dot{\boldsymbol{F}}_h, \dot{\boldsymbol{P}}_h) \in V_h \times D_h \times S_h$ such that

$$
\begin{aligned}
a_{0,h}\big((\dot{\boldsymbol{F}}_h\,,\,\dot{\boldsymbol{u}}_h)\,,\,(\boldsymbol{G}_h\,,\,\boldsymbol{v}_h)\big) + b_{1,h}\big((\boldsymbol{G}_h\,,\,\boldsymbol{v}_h)\,,\,\dot{\boldsymbol{P}}_h\big) &= \ell(\boldsymbol{v}_h) \quad \forall\,(\boldsymbol{v}_h\,,\,\boldsymbol{G}_h) \in V_h \times D_h\,, && (6.51a)\\
b_{2,h}\big((\dot{\boldsymbol{F}}_h\,,\,\dot{\boldsymbol{u}}_h)\,,\,\boldsymbol{Q}_h\big) &= 0 \qquad\quad \forall\,\boldsymbol{Q}_h \in S_h\,. && (6.51b)
\end{aligned}
$$

Using the definitions in (6.50) and the discrete form of (6.36) the system (6.51) implies that

$$
\begin{aligned}
\big(\mu\mathbb{D}_h(\mathbb{A}_h^{-1}\dot{\boldsymbol{P}}_h - \dot{\boldsymbol{F}}_h)\,,\,\boldsymbol{G}_h\big)_0 &= 0 \quad \forall \boldsymbol{G}_h \in D_h\,, && (6.52a)\\
\big(\dot{\boldsymbol{F}}_h - \nabla\dot{\boldsymbol{u}}_h\,,\,\boldsymbol{Q}_h\big)_0 &= 0 \quad \forall \boldsymbol{Q}_h \in S_h\,, && (6.52b)\\
\big(\nabla\boldsymbol{v}_h\,,\,\dot{\boldsymbol{P}}_h\big)_0 &= \ell(\boldsymbol{v}_h) \quad \forall \boldsymbol{v}_h \in V_h\,. && (6.52c)
\end{aligned}
$$

As before we consider finite element approximations based on shape-regular triangulations of a polygonal or polyhedral domain Ω.

Conditions on the equivalence between the discrete modified and standard three-field formulations

Lemma 6.5.1 *Assume that the spaces S_h and D_h are such that*

$$
\mathbb{A}_h D_h \subset D_h \quad \text{and} \quad \mathbb{M}_h D_h \subset D_h\,, \tag{6.53a}
$$

where $\mathbb{M}_h = (\mu\mathbb{D}_h)^{-1}\mathbb{A}_h$, and

$$
S_h \subset D_h\,. \tag{6.53b}
$$

Then the discrete formulations (6.52) and (6.49) are equivalent.

Proof We only need to show the equivalence between (6.49a) and (6.52a). Using the second condition (6.53b) the equation (6.49a) is strongly satisfied, in fact we have

$$
\mathbb{A}_h\dot{\boldsymbol{F}}_h = \dot{\boldsymbol{P}}_h + (\dot{\boldsymbol{P}}_h)^\perp\,, \quad \text{where} \quad (\dot{\boldsymbol{P}}_h)^\perp \perp S_h \quad \text{and} \quad (\dot{\boldsymbol{P}}_h)^\perp \in D_h\,,
$$

which if substitute into equation (6.49a) implies

$$\big((\dot{\boldsymbol{P}}_h)^{\perp}\,,\,\boldsymbol{G}_h\big)_0 = 0 \quad \forall \boldsymbol{G}_h \in D_h \implies \|(\dot{\boldsymbol{P}}_h)^{\perp}\|_0 = 0 \implies (\dot{\boldsymbol{P}}_h)^{\perp} = \boldsymbol{0}\,.$$

We therefore have that

$$\mathbb{A}_h \dot{\boldsymbol{F}}_h = \dot{\boldsymbol{P}}_h\,,$$

which if multiplied by $\mathbb{A}_h^{-1}$ and $\mu\mathbb{D}_h$ gives the strong form of (6.52a). We now show the converse. The equation (6.52a) can be written as

$$\big(\mathbb{A}_h^{-1}\dot{\boldsymbol{P}}_h\,,\,\mu\mathbb{D}_h\boldsymbol{G}_h\big)_0 - \big(\dot{\boldsymbol{F}}_h\,,\,\mu\mathbb{D}_h\boldsymbol{G}_h\big)_0 = 0 \quad \forall \boldsymbol{G}_h \in D_h$$

and then

$$\big(\dot{\boldsymbol{P}}_h\,,\,\mu\mathbb{A}_h^{-1}\mathbb{D}_h\boldsymbol{G}_h\big)_0 - \big(\dot{\boldsymbol{F}}_h\,,\,\mu\mathbb{D}_h\boldsymbol{G}_h\big)_0 = 0 \quad \forall \boldsymbol{G}_h \in D_h\,.$$

From the second assumption (6.53a) the above equality still holds by replacing $\boldsymbol{G}_h$ with $\mathbb{M}_h\boldsymbol{G}_h$, which gives

$$\big(\dot{\boldsymbol{P}}_h\,,\,\boldsymbol{G}_h\big)_0 - \big(\dot{\boldsymbol{F}}_h\,,\,\mathbb{A}_h\boldsymbol{G}_h\big)_0 = 0 \quad \forall \boldsymbol{G}_h \in D_h\,,$$

or

$$\big(\dot{\boldsymbol{P}}_h - \mathbb{A}_h\dot{\boldsymbol{F}}_h\,,\,\boldsymbol{G}_h\big)_0 = 0 \quad \forall \boldsymbol{G}_h \in D_h\,.$$

Thus (6.52a) implies (6.49a). ∎

We introduce the discrete kernels corresponding to (6.50b) and (6.50c)

$$\begin{aligned} K_{1,h} =& \{(\boldsymbol{G}_h, \boldsymbol{v}_h) \in D_h \times V_h \,|\, b_{1,h}\big((\boldsymbol{G}_h\,,\,\boldsymbol{v}_h)\,,\,\boldsymbol{Q}_h\big) = 0 \quad \forall\, \boldsymbol{Q}_h \in S_h\} \\ =& \{(\boldsymbol{G}_h, \boldsymbol{v}_h) \in D_h \times V_h \,|\, (\nabla\boldsymbol{v}_h - \boldsymbol{G}_h\,,\,\boldsymbol{Q}_h)_0 + (\lambda\mathbb{E}_h\mathbb{A}_h^{-1}\boldsymbol{Q}_h\,,\,\boldsymbol{G}_h)_0 = 0\;\forall\, \boldsymbol{Q}_h \in S_h\} \end{aligned}$$

As the approximate elastic moduli $\mathbb{A}_h$ and $\mathbb{D}_h$ possess the major symmetries properties it follows from the above that

$$\begin{aligned} 0 &= (\nabla\boldsymbol{v}_h - \boldsymbol{G}_h\,,\,\boldsymbol{Q}_h)_0 + (\lambda\mathbb{E}_h\mathbb{A}_h^{-1}\boldsymbol{Q}_h\,,\,\boldsymbol{G}_h)_0 \\ &= (\nabla\boldsymbol{v}_h\,,\,\boldsymbol{Q}_h)_0 - \big((\boldsymbol{I} - \lambda\mathbb{E}_h\mathbb{A}_h^{-1})\boldsymbol{Q}_h\,,\,\boldsymbol{G}_h)_0 \\ &= (\nabla\boldsymbol{v}_h\,,\,\boldsymbol{Q}_h)_0 - \big(\mu\mathbb{D}_h\mathbb{A}_h^{-1}\boldsymbol{Q}_h\,,\,\boldsymbol{G}_h)_0 \\ &= (\nabla\boldsymbol{v}_h\,,\,\boldsymbol{Q}_h)_0 - \big(\boldsymbol{Q}_h\,,\,\mathbb{A}_h^{-1}(\mu\mathbb{D}_h)\boldsymbol{G}_h)_0 \\ &= (\nabla\boldsymbol{v}_h\,,\,\boldsymbol{Q}_h)_0 - \big(\boldsymbol{Q}_h\,,\,\mathbb{M}_h^{-1}\boldsymbol{G}_h)_0. \end{aligned}$$

Hence for $(\boldsymbol{G}_h\,,\,\boldsymbol{v}_h) \in K_{1,h}$ we have

$$(\nabla \boldsymbol{v}_h\,,\,\boldsymbol{Q}_h)_0 = \left(\mathbb{M}_h^{-1}\boldsymbol{G}_h\,,\,\boldsymbol{Q}_h\right)_0$$

or

$$P_{S_h}\left(\mathbb{M}_h^{-1}\boldsymbol{G}_h\right) = P_{S_h}(\nabla \boldsymbol{v}_h)\,, \tag{6.54a}$$

where $\mathbb{M}_h$ is as in (6.53a), and

$$\begin{aligned} K_{2,h} =& \{(\boldsymbol{G}_h, \boldsymbol{v}_h) \in D_h \times V_h \,|\, b_2\big((\boldsymbol{G}_h\,,\,\boldsymbol{v}_h)\,,\,\boldsymbol{Q}_h\big) = 0 \quad \forall\, \boldsymbol{Q}_h \in S_h\} \\ =& \{(\boldsymbol{G}_h, \boldsymbol{v}_h) \in D_h \times V_h \,|\, \big((\nabla \boldsymbol{v}_h - \boldsymbol{G}_h)_0\,,\,\boldsymbol{Q}_h\big)_0 = 0 \quad \forall\, \boldsymbol{Q}_h \in S_h\} \\ =& \{(\boldsymbol{G}_h, \boldsymbol{v}_h) \in D_h \times V_h \,|\, (\nabla \boldsymbol{v}_h\,,\,\boldsymbol{Q}_h\big) = (\boldsymbol{G}_h\,,\,\boldsymbol{Q}_h\big) \quad \forall\, \boldsymbol{Q}_h \in S_h\} \\ =& \{(\boldsymbol{G}_h, \boldsymbol{v}_h) \in D_h \times V_h \,|\, P_{S_h}(\boldsymbol{G}_h) = P_{S_h}(\nabla \boldsymbol{v}_h)\} \end{aligned} \tag{6.54b}$$

respectively. Their respective transposes also follow as

$$K_{1,h}^t = \{\boldsymbol{Q}_h \in S_h \,|\, b_1\big((\boldsymbol{G}_h\,,\,\boldsymbol{v}_h)\,,\,\boldsymbol{Q}_h\big) = 0 \quad \forall (\boldsymbol{G}_h\,,\,\boldsymbol{v}_h) \in D_h \times V_h\} \tag{6.55a}$$

and

$$K_{2,h}^t = \{\boldsymbol{Q}_h \in S_h \,|\, b_2\big((\boldsymbol{G}_h\,,\,\boldsymbol{v}_h)\,,\,\boldsymbol{Q}_h\big) = 0 \quad \forall (\boldsymbol{G}_h\,,\,\boldsymbol{v}_h) \in D_h \times V_h\}\,. \tag{6.55b}$$

The notation P_{S_h} used in the above sets represents the orthogonal projection defined from $D_h \cup \nabla V_h$ onto S_h.

Theorem 6.5.2 *[7, 25] Consider the discrete formulation, which is the problem of finding $(\dot{\boldsymbol{u}}_h\,,\,\dot{\boldsymbol{F}}_h\,,\,\dot{\boldsymbol{P}}_h) \in V_h \times D_h \times S_h$ such that*

$$\begin{aligned} a_{0,h}\big((\dot{\boldsymbol{F}}_h\,,\,\dot{\boldsymbol{u}}_h)\,,\,(\boldsymbol{G}_h\,,\,\boldsymbol{v}_h)\big) + b_{1,h}\big((\boldsymbol{G}_h\,,\,\boldsymbol{v}_h)\,,\,\dot{\boldsymbol{P}}_h\big) &= (\dot{\boldsymbol{f}}\,,\,\boldsymbol{v}_h) \quad \forall\,(\boldsymbol{G}_h\,,\,\boldsymbol{v}_h) \in D_h \times V_h\,, && (6.56a) \\ b_{2,h}\big((\dot{\boldsymbol{F}}_h\,,\,\dot{\boldsymbol{u}}_h)\,,\,\boldsymbol{Q}_h\big) - c(\dot{\boldsymbol{P}}_h\,,\,\boldsymbol{Q}_h) &= (\dot{\boldsymbol{g}}\,,\,\boldsymbol{Q}_h) \qquad \forall\,\boldsymbol{Q}_h \in S_h\,, && (6.56b) \end{aligned}$$

where $(\dot{\boldsymbol{f}}\,,\,\dot{\boldsymbol{g}}) \in V' \times \mathcal{M}'$ and $c(\cdot,\cdot)$ are as in 6.4.1 and $a_{0,h}(\cdot,\cdot)$, $b_{1,h}(\cdot,\cdot)$ and $b_{2,h}(\cdot,\cdot)$ are uniformly continuous bilinear forms.

Suppose that $a_{0,h}(\cdot,\cdot)$, $b_{i,h}(\cdot,\cdot)\,,\ i = 1,\,2$ and $c(\cdot,\cdot)$ satisfy the following:

a) there exists a constant $\beta_{0,h}^* > 0$ such that

$$\forall\,(\dot{\boldsymbol{F}}_h\,,\,\dot{\boldsymbol{u}}_h) \in K_{2,h}\,, \qquad \sup_{(\boldsymbol{G}_h\,,\boldsymbol{v}_h)\in K_{1,h}} \frac{a_{0,h}\big((\dot{\boldsymbol{F}}_h\,,\,\dot{\boldsymbol{u}}_h)\,,\,(\boldsymbol{G}_h\,,\,\boldsymbol{v}_h)\big)}{\|(\boldsymbol{G}_h\,,\,\boldsymbol{v}_h)\|_{\mathcal{M}\times V}} \geq \beta_{0,h}^*\|(\dot{\boldsymbol{F}}_h\,,\,\dot{\boldsymbol{u}}_h)\|_{\mathcal{M}\times V}\,, \tag{6.57a}$$

$$\forall\,(\boldsymbol{G}_h\,,\,\boldsymbol{v}_h) \in K_{1,h}\backslash\boldsymbol{0} \qquad \sup_{(\boldsymbol{F}_h\,,\boldsymbol{u}_h)\in K_{2,h}} a_{0,h}\big((\dot{\boldsymbol{F}}_h\,,\,\dot{\boldsymbol{u}}_h)\,,\,(\boldsymbol{G}_h\,,\,\boldsymbol{v}_h)\big) > 0\,; \tag{6.57b}$$

b) the bilinear forms $b_{i,h}(\cdot,\cdot)\,,\ i=1,\cdots,2$ *satisfy the usual inf-sup condition;*

c) the continuous bilinear form $c(\cdot,\cdot)$ *is positive semi-definite, and there exists a positive constant* γ_h *such that for all* $\boldsymbol{m}_h^{\perp}\in(K_{1,h}^t)^{\perp}$*, one can find* $\boldsymbol{m}_h\in K_{1,h}^t$ *such that*

$$\begin{aligned}\gamma_h\|\boldsymbol{m}_h\|_{\mathcal{M}}&\leq\|\boldsymbol{m}_h^{\perp}\|_{\mathcal{M}}\,,\\ c(\boldsymbol{m}_h\,,\,\boldsymbol{n}_h)&=-c(\boldsymbol{m}_h^{\perp}\,,\,\boldsymbol{n}_h)\quad\forall\boldsymbol{n}_h\in K_{2,h}^t\,.\end{aligned}\tag{6.57c}$$

Then for $(\dot{\boldsymbol{f}}\,,\,\dot{\boldsymbol{g}})\in V'\times$ *Im*$B_{2,h}$ *(image of the bilinear form* $B_{2,h}(\cdot,\cdot)$*), the problem (6.41) has a unique solution* $(\dot{\boldsymbol{u}}_h\,,\,\dot{\boldsymbol{F}}_h\,,\,\dot{\boldsymbol{P}}_h)\in V_h\times D_h\times S_h/K_{1,h}^t\cap\ker\,c(\cdot,\cdot)$*, which moreover satisfies the bound*

$$\|\dot{\boldsymbol{u}}_h\|_V+\|\dot{\boldsymbol{F}}_h\|_{\mathcal{M}}+\|\dot{\boldsymbol{P}}_h\|_{\mathcal{M}}\leq C\big(\|\dot{\boldsymbol{f}}\|_{V'}+\|\dot{\boldsymbol{g}}\|_{\mathcal{M}'}\big)\,,\tag{6.57d}$$

and

$$\|\dot{\boldsymbol{u}}-\dot{\boldsymbol{u}}_h\|_V+\|\dot{\boldsymbol{F}}-\dot{\boldsymbol{F}}_h\|_{\mathcal{M}}+\|\dot{\boldsymbol{P}}-\dot{\boldsymbol{P}}_h\|_{\mathcal{M}}\leq C\big(\|\mathcal{F}\|_{V'}+\|\mathscr{L}\|_{S'}\big)\,,\tag{6.57e}$$

where C *and* $\beta_{0,h}^*$ *are as in Theorem 6.4.1 and* $\mathcal{F}$ *and* $\mathscr{L}$ *are linear functionals defined by*

$$\begin{aligned}&\mathcal{F}:D_h\times V_h\longrightarrow\mathbb{R}\,,\\ &\quad(\boldsymbol{G}_h\,,\,\boldsymbol{v}_h)\mapsto a_{0,h}\big(\dot{\boldsymbol{F}}\,,\,\dot{\boldsymbol{u}})\,,\,(\boldsymbol{G}_h\,,\,\boldsymbol{v}_h)\big)+b_{1,h}\big((\boldsymbol{G}_h\,,\,\boldsymbol{v}_h)\,,\,\dot{\boldsymbol{P}}\big)-(\dot{\boldsymbol{f}}\,,\,\boldsymbol{v}_h)\end{aligned}\tag{6.58a}$$

and

$$\begin{aligned}&\mathscr{L}:S_h\longrightarrow\mathbb{R}\,,\\ &\quad Q_h\mapsto b_{2,h}\big((\dot{\boldsymbol{F}}\,,\,\dot{\boldsymbol{u}})\,,\,\boldsymbol{Q}_h\big)-c(\dot{\boldsymbol{P}}\,,\,\boldsymbol{Q}_h)-(\dot{\boldsymbol{g}}\,,\,\boldsymbol{Q}_h)\,,\end{aligned}\tag{6.58b}$$

Proof For the existence and uniqueness of solution and the boundedness (6.42d) see the original work [7] and also its extension to three-field formulations in [25].

Now to prove the last statement (6.57e) we first observe from (6.51) that: the system

$$a_{0,h}\big((\dot{\boldsymbol{F}}-\dot{\boldsymbol{F}}_h\,,\,\dot{\boldsymbol{u}}-\dot{\boldsymbol{u}}_h)\,,\,(\boldsymbol{G}_h\,,\,\boldsymbol{v}_h)\big)+b_{1,h}\big((\boldsymbol{G}_h\,,\,\boldsymbol{v}_h)\,,\,\dot{\boldsymbol{P}}-\dot{\boldsymbol{P}}_h\big)\ =\ \mathcal{F}(\boldsymbol{G}_h\,,\,\boldsymbol{v}_h)\,,\tag{6.59a}$$

$$b_{2,h}\big((\dot{\boldsymbol{F}}-\dot{\boldsymbol{F}}_h\,,\,\dot{\boldsymbol{u}}-\dot{\boldsymbol{u}}_h)\,,\,\boldsymbol{Q}_h\big)\ =\ \mathscr{L}(\boldsymbol{Q}_h)\,,\tag{6.59b}$$

for any $(\boldsymbol{Q}_h\,,\,\boldsymbol{G}_h\,,\,\boldsymbol{v}_h)\in S_h\times D_h\times V_h$ has a unique solution $(\dot{\boldsymbol{F}}-\dot{\boldsymbol{F}}_h\,,\,\dot{\boldsymbol{u}}-\dot{\boldsymbol{u}}_h\,,\,\dot{\boldsymbol{P}}-\dot{\boldsymbol{P}}_h)$. Applying the boundedness (6.57d) to the problem (6.59) we derive the desired inequality (6.57e). ∎

Corollary 6.5.3 *Assume that the bilinear forms in (6.50) satisfy the condition of Theorem 6.5.2. Then the problem (6.51) has a unique solution and if shape regular-mesh is used it also satisfies the error bound*

$$\|\dot{\boldsymbol{u}}-\dot{\boldsymbol{u}}_h\|_V+\|\dot{\boldsymbol{F}}-\dot{\boldsymbol{F}}_h\|_{\mathcal{M}}+\|\dot{\boldsymbol{P}}-\dot{\boldsymbol{P}}_h\|_{\mathcal{M}}\leq Ch\,.\tag{6.60}$$

We observe that for a well posed problem, the estimate above does not depend on the degree of approximation polynomials used to construct the finite element spaces. The method therefore does not allow for higher-order convergence with the use of polynomials of higher degree. On the other hand we have constructed a method that is stable for lower-order elements.

Proof The existence and uniqueness of solution of problem (6.59) is obtained from Theorem 6.5.2. The inequality (6.57e) also follows from 6.5.2. We therefore only have to prove that

$$\|\mathcal{F}\|_V' + \|\mathscr{L}\|_{S'} \leq Ch\,.$$

Using the expressions in (6.50) and the equalities (6.38) we have

$$\begin{aligned}
\mathscr{L}(\boldsymbol{Q}_h) &= b_{h,0}\big((\dot{\boldsymbol{F}}\,,\,\dot{\boldsymbol{u}})\,,\,\boldsymbol{Q}_h\big) - c(\dot{\boldsymbol{P}}\,,\,\boldsymbol{Q}_h) - (\dot{\boldsymbol{g}}\,,\,\boldsymbol{Q}_h)\\
&= b_2\big((\dot{\boldsymbol{F}}\,,\,\dot{\boldsymbol{u}})\,,\,\boldsymbol{Q}_h\big) - c(\dot{\boldsymbol{P}}\,,\,\boldsymbol{Q}_h) - (\dot{\boldsymbol{g}}\,,\,\boldsymbol{Q}_h)\\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{F}(\boldsymbol{G}_h\,,\,\boldsymbol{v}_h) &= a_{0,h}\big(\dot{\boldsymbol{F}}\,,\,\dot{\boldsymbol{u}})\,,\,(\boldsymbol{G}_h\,,\,\boldsymbol{v}_h)\big) + b_{1,h}\big((\boldsymbol{G}_h\,,\,\boldsymbol{v}_h)\,,\,\dot{\boldsymbol{P}}\big) - (\dot{\boldsymbol{f}}\,,\,\boldsymbol{v}_h)\\
&= a_{0,h}\big(\dot{\boldsymbol{F}}\,,\,\dot{\boldsymbol{u}})\,,\,(\boldsymbol{G}_h\,,\,\boldsymbol{v}_h)\big) + b_{1,h}\big((\boldsymbol{G}_h\,,\,\boldsymbol{v}_h)\,,\,\dot{\boldsymbol{P}}\big)\\
&\quad - a\big(\dot{\boldsymbol{F}}\,,\,\dot{\boldsymbol{u}})\,,\,(\boldsymbol{G}_h\,,\,\boldsymbol{v}_h)\big) - b_1\big((\boldsymbol{G}_h\,,\,\boldsymbol{v}_h)\,,\,\dot{\boldsymbol{P}}\big)\\
&= \big(\mu(\mathbb{D}-\mathbb{D}_h)\dot{\boldsymbol{F}}\,,\,\boldsymbol{G}_h\big) + \big(\mu(\mathbb{D}\mathbb{A}^{-1} - \mathbb{D}_h\mathbb{A}_h^{-1})\dot{\boldsymbol{P}}\,,\,\nabla\boldsymbol{v}_h\big)\,.
\end{aligned}$$

It then follows that

$$\|\mathscr{L}\|_{S'} = 0$$

and

$$\begin{aligned}
\|\mathcal{F}\|_{V'} &= \sup_{(\boldsymbol{G}_h\,,\boldsymbol{v}_h)\in D_h\times V_h} \frac{a_{0,h}\big((\dot{\boldsymbol{F}}\,,\,\dot{\boldsymbol{u}})\,,\,(\boldsymbol{G}_h\,,\,\boldsymbol{v}_h)\big) + b_{1,h}\big((\boldsymbol{G}_h\,,\,\boldsymbol{v}_h)\,,\,\dot{\boldsymbol{P}}\big) - (\dot{\boldsymbol{f}}\,,\,\boldsymbol{v}_h)}{\|(\boldsymbol{G}_h\,,\,\boldsymbol{v}_h)\|_{D_h\times V_h}}\\
&\leq \|\mu\mathbb{D} - \mu\mathbb{D}_h\|_0 \times \|(\dot{\boldsymbol{F}}\,,\,\dot{\boldsymbol{u}})\|_{\mathcal{M}\times V} + \|\lambda\mathbb{E}\mathbb{A}^{-1} - \lambda\mathbb{E}_h\mathbb{A}_h^{-1}\|_0 \times \|\dot{\boldsymbol{P}}\|_{\mathcal{M}}\\
&\leq C\left(\|\mu\mathbb{D} - \mu\mathbb{D}_h\|_0 + \|\lambda\mathbb{E}\mathbb{A}^{-1} - \lambda\mathbb{E}_h\mathbb{A}_h^{-1}\|_0\right)
\end{aligned} \tag{6.61}$$

where C is a positive constant that depend only on $(\dot{\boldsymbol{u}}\,,\,\dot{\boldsymbol{F}}\,,\,\dot{\boldsymbol{P}})$. In the last equality of (6.61) we have considered the norm of any tensor valued function $\mathbb{T}$ to be defined by

$$\|\mathbb{T}\|_0^2 = \int_\Omega \mathbb{T} : \mathbb{T}\,\mathrm{d}x\,.$$

To further bound $\|\mathcal{F}\|_0$ assume $\mathbb{T}$ to have similar properties to $\mu\mathbb{D}$ and $\lambda\mathbb{E}\mathbb{A}^{-1}$; that is, it is uniformly continuous with respect to the physical variable x. Let $K \subset \Omega$ be an element of our triangulation. As K is closed and bounded there exist points $\boldsymbol{a}$ and $\boldsymbol{b}$ of K such that $\mathbb{T}$ satisfies

$$\mathbb{T}(\boldsymbol{a}) = \inf_{\boldsymbol{x}\in K} \mathbb{T}(\boldsymbol{x}) \quad \text{and} \quad \mathbb{T}(\boldsymbol{b}) = \sup_{\boldsymbol{x}\in K} \mathbb{T}(\boldsymbol{x})\,,$$

or equivalently

$$\mathbb{T}(\boldsymbol{a}) \le \mathbb{T}(\boldsymbol{x}) \le \mathbb{T}(\boldsymbol{b})\,, \quad \boldsymbol{x} \in K\,. \tag{6.62a}$$

The integration of this over the domain K gives

$$\mathbb{T}(\boldsymbol{a}) \le \mathbb{T}_h \le \mathbb{T}(\boldsymbol{b})\,, \tag{6.62b}$$

where $\mathbb{T}_h = \dfrac{1}{\text{Vol}(K)} \displaystyle\int_K \mathbb{T}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$. The inequalities (6.62a) and (6.62b) imply that

$$\|\mathbb{T} - \mathbb{T}_h\|_0 \le \mathbb{T}(\boldsymbol{b}) - \mathbb{T}(\boldsymbol{a})\,. \tag{6.62c}$$

The first order Taylor expansion of $\mathbb{T}(\boldsymbol{b})$ gives

$$\mathbb{T}(\boldsymbol{b}) = \mathbb{T}(\boldsymbol{a}) + \mathbf{D}\mathbb{T}(\boldsymbol{a})\cdot(\boldsymbol{b}-\boldsymbol{a}) \quad \Longrightarrow \quad \mathbb{T}(\boldsymbol{b}) - \mathbb{T}(\boldsymbol{a}) \le C\,\text{diam}\,(K) \tag{6.62d}$$

where diam (K) denotes for the diameter of K and $\mathbf{D}\mathbb{T}(\boldsymbol{a})$ the differential of $\mathbb{T}$ evaluated at point $\boldsymbol{a}$. If we denote by h the mesh size of a uniform triangulation then one can always find a positive constant C such that diam $(K) \le Ch$. Therefore from (6.62d) and (6.62c) we have

$$\|\mathbb{T} - \mathbb{T}_h\|_0 \le Ch \quad \Longrightarrow \quad \|\mathbb{T} - \mathbb{T}_h\|_0 \le Ch\,. \tag{6.63}$$

We have stated earlier that $\mathbb{T}$ can be any of the operators $\mu\mathbb{D}$ or $\lambda\mathbb{E}\mathbb{A}^{-1}$. Thus substituting (6.63) into (6.61) we have

$$\|\mathcal{F}\|'_V + \|\mathscr{L}\|_{S'} \le Ch\,. \tag{6.64}$$

∎

Next we introduce a key assumption for the satisfaction of condition (a) of Theorem 6.5.2.

Assumption 6.5.4 *We consider the existence of a constant $C > 0$, such that*

$$\|\boldsymbol{v}_h\|_1 \le C\|P_{S_h}\nabla\boldsymbol{v}_h\|_0\,, \quad \forall\boldsymbol{v}_h \in V_h\,, \tag{6.65}$$

where P_{S_h} is the orthogonal projection operator defined from the gradient of V_h onto S_h

Lemma 6.5.5 *Under the assumption (6.65) above the bilinear form $a(\cdot,\cdot)$ satisfies the inequalities (6.57a) and (6.57b).*

Proof Let $(\dot{\boldsymbol{F}}_h\,,\,\dot{\boldsymbol{u}}_h) = (\boldsymbol{G}'_h\,,\,\boldsymbol{v}'_h)$ be any vector of $K_{2,h}$, which using the definitions (6.54a) and (6.54b) of the discrete kernels implies that the vector $(\boldsymbol{G}_h\,,\,\boldsymbol{v}_h) = (\mathbb{M}_h\boldsymbol{G}'_h\,,\,\boldsymbol{v}'_h)$ is an element of $K_{1,h}$, now using assumption 6.5.4 the norms of these vectors can be bounded as

$$\begin{aligned}
\|(\dot{\boldsymbol{F}}_h\,,\,\dot{\boldsymbol{u}}_h)\|^2_{D_h\times V_h} &= \|\dot{\boldsymbol{F}}_h\|^2_0 + \|\dot{\boldsymbol{u}}_h\|^2_1\,,\\
&= \|\boldsymbol{G}'_h\|^2_0 + \|\boldsymbol{v}'_h\|^2_1\,,\\
&\leq \|\boldsymbol{G}'_h\|^2_0 + C\|P_{S_h}\nabla\boldsymbol{v}'_h\|^2_0\,,\\
&= \|\boldsymbol{G}'_h\|^2_0 + C\|\boldsymbol{G}'_h\|^2_0\,,\\
&= C_1^2\|\boldsymbol{G}'_h\|^2_0
\end{aligned} \tag{6.66a}$$

and

$$\begin{aligned}
\|(\boldsymbol{G}_h\,,\,\boldsymbol{v}_h)\|^2_{D_h\times V_h} &= \|\boldsymbol{G}_h\|^2_0 + \|\boldsymbol{v}_h\|^2_1\,,\\
&= \|\mathbb{M}_h\boldsymbol{G}'_h\|^2_0 + \|\boldsymbol{v}'_h\|^2_1\,,\\
&\leq \|\mathbb{M}_h\boldsymbol{G}'_h\|^2_0 + C\|P_{S_h}\nabla\boldsymbol{v}'_h\|^2_0\,,\\
&= \|\mathbb{M}_h\boldsymbol{G}'_h\|^2_0 + C\|\boldsymbol{G}'_h\|^2_0\,,\\
&= C_2^2\|\boldsymbol{G}'_h\|^2_0\,.
\end{aligned} \tag{6.66b}$$

Next, the bilinear form $a(\cdot,\cdot)$ evaluated at these two pairs gives

$$\begin{aligned}
a((\dot{\boldsymbol{F}}_h\,,\,\dot{\boldsymbol{u}}_h)\,,\,(\boldsymbol{G}_h\,,\,\boldsymbol{v}_h)) &= a((\boldsymbol{G}'_h\,,\,\boldsymbol{v}'_h)\,,\,(\mathbb{M}_h\boldsymbol{G}'_h\,,\,\boldsymbol{v}'_h))\\
&= \mu(\mathbb{D}_h\boldsymbol{G}'_h\,,\,\mathbb{M}_h\boldsymbol{G}'_h)\,,\\
&\geq C_3\|\boldsymbol{G}'_h\|^2_0\,,
\end{aligned} \tag{6.66c}$$

where C_i, $i = 1,\cdots,3$ are strictly positive constants, we observe from the expressions (6.66a), (6.66b) and (6.66c) that

$$\frac{a((\dot{\boldsymbol{F}}_h\,,\,\dot{\boldsymbol{u}}_h)\,,\,(\boldsymbol{G}_h\,,\,\boldsymbol{v}_h))}{\|(\dot{\boldsymbol{F}}_h\,,\,\dot{\boldsymbol{u}}_h)\|_{D_h\times V_h}\times\|(\boldsymbol{G}_h\,,\,\boldsymbol{v}_h)\|_{D_h\times V_h}} \geq \frac{C_3}{C_2C_1} > 0\,,$$

which then implies that

$$\forall\,(\dot{\boldsymbol{F}}_h\,,\,\dot{\boldsymbol{u}}_h)\in K_{2,h} \qquad \sup_{(\boldsymbol{G}_h\,,\boldsymbol{v}_h)\in K_{1,h}} \frac{a((\dot{\boldsymbol{F}}_h\,,\,\dot{\boldsymbol{u}}_h)\,,\,(\boldsymbol{G}_h\,,\,\boldsymbol{v}_h))}{\|(\boldsymbol{G}_h\,,\,\boldsymbol{v}_h)\|_{D_h\times V_h}} \geq C\|(\dot{\boldsymbol{F}}_h\,,\,\dot{\boldsymbol{u}}_h)\|_{D_h\times V_h}\,,$$

where $C = \dfrac{C_3}{C_2 C_1}$

Note that C_2 and C_3 depend on the largest and smallest eigenvalue of $\mathbb{M}_h$, respectively. As the coefficients of $\mathbb{M}_h$ are linearly dependent on λ, the ratio $\dfrac{C_3}{C_2}$ is uniformly bounded below by a constant that could depend only on the eigenvalue of $\mathbb{M}$. Thus C is uniformly bounded and the inequality (6.57a) is satisfied.

To prove the second inequality (6.57b) let $(\boldsymbol{G}_h\,,\,\boldsymbol{v}_h)$ be a non-trivial element of $K_{1,h}$ and define $(\dot{\boldsymbol{F}}_h\,,\,\dot{\boldsymbol{u}}_h) = \left(\mathbb{M}_h^{-1}\boldsymbol{G}_h\,,\,\boldsymbol{v}_h\right)$. It is clear that $(\dot{\boldsymbol{F}}_h\,,\,\dot{\boldsymbol{u}}_h)$ is an element of $K_{2,h}$, and it follows that

$$\begin{aligned} a((\dot{\boldsymbol{F}}_h\,,\,\dot{\boldsymbol{u}}_h)\,,\,(\boldsymbol{G}_h\,,\,\boldsymbol{v}_h)) &= (\mu\mathbb{D}_h\dot{\boldsymbol{F}}_h\,,\,\boldsymbol{G}_h)\,,\\ &= \left(\mu\mathbb{D}_h\mathbb{M}_h^{-1}\boldsymbol{G}_h\,,\,\boldsymbol{G}_h\right)\,. \end{aligned}$$

Observing that $\mu\mathbb{D}_h\mathbb{M}_h^{-1}$ is positive definite we can write

$$\forall\,(\boldsymbol{G}_h\,,\,\boldsymbol{v}_h) \in K_{1,h}\backslash\boldsymbol{0} \qquad \sup_{(\boldsymbol{F}_h\,,\boldsymbol{u}_h)\in K_{2,h}} a((\dot{\boldsymbol{F}}_h\,,\,\dot{\boldsymbol{u}}_h)\,,\,(\boldsymbol{G}_h\,,\,\boldsymbol{v}_h)) > 0\,,$$

hence the inequality (6.57b) is also satisfied.

Now it only remains to introduce sufficient conditions for the bilinear forms $b_1(\cdot,\cdot)$ and $b_2(\cdot,\cdot)$ to satisfy the discrete inf-sup condition.

Lemma 6.5.6 *Assume that S_h and V_h are such that*

$$\inf_{\boldsymbol{Q}_h\in S_h}\sup_{\boldsymbol{v}_h\in V_h}\frac{(\nabla\boldsymbol{v}_h,\,\boldsymbol{Q}_h)_0}{\|\boldsymbol{v}_h\|_1\|\boldsymbol{Q}_h\|_0} \geq \beta_{0,h}\,, \tag{6.67}$$

where $\beta_{0,h}$ is a positive constant independent of the mesh size h. Then the bilinear forms $b_1(\cdot\,,\,\cdot)$ and $b_2(\cdot\,,\,\cdot)$ satisfy a uniform discrete inf-sup condition.

Proof Recall the expression of the bilinear form $b_1(\cdot,\cdot)$, from (6.37b),

$$b_1\big((\boldsymbol{v}_h\,,\,\boldsymbol{G}_h)\,,\,\boldsymbol{Q}_h\big) = \big(\nabla\boldsymbol{v}_h - \boldsymbol{G}_h\,,\,\boldsymbol{Q}_h\big) + \lambda(\mathbb{E}\mathbb{A}^{-1}\boldsymbol{G}_h\,,\,\boldsymbol{Q}_h)\,,$$

for any $\boldsymbol{Q}_h \in S_h$ choose $\boldsymbol{G}_{\boldsymbol{q}_h}$ such that $\boldsymbol{G}_{\boldsymbol{q}_h} = \boldsymbol{0}$, it then follows from the above that

$$\frac{b_1\big((\boldsymbol{v}_h\,,\,\boldsymbol{G}_{\boldsymbol{q}_h})\,,\,\boldsymbol{Q}_h\big)}{\|(\boldsymbol{v}_h\,,\,\boldsymbol{G}_{\boldsymbol{q}_h})\|_{V_h\times D_h}\|\boldsymbol{Q}_h\|_{S_h}} = \frac{(\nabla\boldsymbol{v}_h\,,\,\boldsymbol{Q}_h)_0}{\|\boldsymbol{v}_h\|_1\|\boldsymbol{Q}_h\|_0}\,.$$

Using the assumption of Lemma 6.5.6, that is the inequality (6.67), implies that

$$\sup_{(\boldsymbol{G}_h\,,\,\boldsymbol{v}_h)\in D_h\times V_h} \frac{b_1\big((\boldsymbol{G}_h\,,\,\boldsymbol{v}_h)\,,\,\boldsymbol{Q}_h\big)}{\|(\boldsymbol{G}_h\,,\,\boldsymbol{v}_h)\|_{D_h\times V_h}\|\boldsymbol{Q}_h\|_{S_h}} \geq \sup_{\boldsymbol{v}_h\in S_h} \frac{(\nabla \boldsymbol{v}_h\,,\,\boldsymbol{Q}_h)_0}{\|\boldsymbol{v}_h\|_1\|\boldsymbol{Q}_h\|_0} > \beta \quad \forall \boldsymbol{Q}_h \in S_h\,.$$

Hence $b_1(\cdot,\cdot)$ satisfies a uniform inf-sup condition on $(V_h\,,\,D_h)\times S_h$.

A similar argument holds for the bilinear form $b_2(\cdot,\cdot)$. In fact, observe that for $\boldsymbol{G}_{\boldsymbol{q}_h} = \boldsymbol{0}$ we have

$$b_2\big((\boldsymbol{v}_h\,,\,\boldsymbol{G}_{\boldsymbol{q}_h})\,,\,\boldsymbol{Q}_h\big) = b_1\big((\boldsymbol{v}_h\,,\,\boldsymbol{G}_{\boldsymbol{q}_h})\,,\,\boldsymbol{Q}_h\big)\,.$$

∎

6.5.3 Stable finite element spaces

To illustrate the theory developed in the previous section we focus on the basis corresponding to the mixed enhanced formulation [36].

Corollary 6.5.7 *Let T_h be a quasi-uniform shape-regular triangulation of the domain Ω. Assume that the elements K of T_h are being generated by an isoparametric map F_K from the reference element $\hat{K} = (-1\,,\,1)^2$. Furthermore consider the finite finite element spaces to be defined by*

$$V_h = \{\boldsymbol{v}_h \in V\ :\ (\boldsymbol{v}_h|_K)_i = (\hat{\boldsymbol{v}}_h)_i \circ F_K^{-1}\,,\ (\hat{\boldsymbol{v}}_h)_i \in Q_1(\hat{K})\,,\ K \in T_h\}\,, \tag{6.69a}$$

$$S_h = \{\boldsymbol{Q}_h \in \mathcal{M}\ :\ (\boldsymbol{Q}_h|_K)_{ij} = (\hat{\boldsymbol{Q}}_h)_{ij} \circ F_K^{-1}\,,\ \hat{\boldsymbol{Q}}_h \in \hat{S}(\hat{K})\,,\ K \in T_h\} \tag{6.69b}$$

and

$$D_h = \{\boldsymbol{G}_h \in \mathcal{M}\ :\ (\boldsymbol{G}_h|_K)_{ij} = (\hat{\boldsymbol{G}}_h)_{ij} \circ F_K^{-1}\,,\ \hat{\boldsymbol{G}}_h \in \hat{D}(\hat{K})\,,\ K \in T_h\}\,, \tag{6.69c}$$

where $Q_1(\hat{K})$ is the space of bilinear functions defined on the reference element $\hat{K}$,

$$\hat{S}(\hat{K}) = \mathit{span}\,(\mathcal{I} + \mathcal{A}) \quad \text{and} \quad \hat{D}(\hat{K}) = \mathit{span}\,(\mathcal{I} + \mathcal{A} + \mathcal{B})\ ,$$

with

$$\mathcal{I} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\,, \quad \mathcal{A} = \begin{bmatrix} \xi_2 & 0 \\ 0 & \xi_1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathcal{B} = \begin{bmatrix} \xi_1 & 0 \\ 0 & \xi_2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\,.$$

Then the inequality (6.65) in Assumption 6.5.4 is satisfied. Further more the spaces V_h, $S_h \backslash \ker \tilde{B}_h^t$ satisfy the inequality (6.67) of Lemma 6.5.6. As a consequence, the spaces V_h, $S_h \backslash \ker \tilde{B}_h^t$ and D_h satisfy the well-posedness criteria of problem (6.51).

Proof It was shown in [38] that the spaces V_h and S_h as defined above satisfy the inequality

$$\|P_{S_h}\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\|_0 \geq C\|\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\|_0 \quad \forall \boldsymbol{v}_h \in V_h\,, \tag{6.70a}$$

where C is a positive constant independent of the mesh size. Next consider $\boldsymbol{v}_h = (v_1\,,\, v_2) \in V_h$ and define $\boldsymbol{w}_h = (v_1\,,\, 0) \in V_h$ and $\boldsymbol{w}'_h = (0\,,\, v_2) \in V_h$. We then have that

$$\begin{aligned} \|\nabla \boldsymbol{v}_h\|_0^2 &= \|\partial_x v_1\|_0^2 + \|\partial_y v_1\|_0^2 + \|\partial_x v_2\|_0^2 + \|\partial_y v_2\|_0^2\,, \\ \|\boldsymbol{\varepsilon}(\boldsymbol{w}_h)\|_0^2 &= \|\partial_x v_1\|_0^2 + 2\|\partial_y v_1\|_0^2\,, \\ \|\boldsymbol{\varepsilon}(\boldsymbol{w}'_h)\|_0^2 &= \|\partial_y v_2\|_0^2 + 2\|\partial_x v_2\|_0^2\,, \end{aligned}$$

and thus

$$\|\boldsymbol{\varepsilon}(\boldsymbol{w}_h)\|_0^2 + \|\boldsymbol{\varepsilon}(\boldsymbol{w}'_h)\|_0^2 \geq \|\nabla \boldsymbol{v}_h\|_0^2\,. \tag{6.70b}$$

The inequalities (6.70a) and (6.70b) imply that

$$\|P_{S_h}\boldsymbol{\varepsilon}(\boldsymbol{w}_h)\|_0^2 + \|P_{S_h}\boldsymbol{\varepsilon}(\boldsymbol{w}'_h)\|_0^2 \geq C\|\nabla \boldsymbol{v}_h\|_0^2\,. \tag{6.70c}$$

However, for any $\boldsymbol{u}_h \in V_h$

$$\|P_{S_h}\boldsymbol{\varepsilon}(\boldsymbol{u}_h)\|_0^2 \leq \frac{1}{4}[\|P_{S_h}\nabla \boldsymbol{u}_h\|_0^2 + \|P_{S_h}(\nabla \boldsymbol{u}_h)^T\|_0^2] \leq \frac{1}{2}\|P_{S_h}\nabla \boldsymbol{u}_h\|_0^2\,. \tag{6.70d}$$

From (6.70d) and (6.70c) we have

$$C\|\nabla \boldsymbol{v}_h\|_0^2 \leq 2\Big(\|P_{S_h}\nabla \boldsymbol{w}_h\|_0^2 + \|P_{S_h}\nabla \boldsymbol{w}'_h\|_0^2\Big)\,.$$

Using the linearity of orthogonal projection operator we observe that

$$\left(P_{S_h}\left(\nabla \boldsymbol{u}_h\right)\right)_{ij} = P_{S_{ij}}\left(\frac{\partial u_i}{\partial x_j}\right)\,, \quad \forall \boldsymbol{u}_h \in V_h$$

and then

$$\|P_{S_h}(\nabla \boldsymbol{w}_h)\|_0^2 = \|P_{S_{11}}\partial_x v_1\|_0^2 + \|P_{S_{12}}\partial_y v_1\|_0^2\,,$$

$$\|P_{S_h}(\nabla \boldsymbol{w}'_h)\|_0^2 = \|P_{S_{21}}\partial_x v_2\|_0^2 + \|P_{S_{22}}\partial_y v_2\|_0^2\,,$$

and

$$\begin{aligned} \|P_{S_h}(\nabla \boldsymbol{v}_h)\|_0^2 &= \|P_{S_{11}}\partial_x v_1\|_0^2 + \|P_{S_{12}}\partial_y v_1\|_0^2 + \|P_{S_{21}}\partial_y v_2\|_0^2 + \|P_{S_{22}}\partial_y v_2\|_0^2 \\ &= \|P_{S_h}(\nabla \boldsymbol{w}_h)\|_0^2 + \|P_{S_h}(\nabla \boldsymbol{w}'_h)\|_0^2 \\ &\geq C\|\nabla \boldsymbol{v}_h\|_0^2\,. \end{aligned}$$

To show that the spaces V_h and $S_h \backslash \ker \tilde{B}_h^t$ satisfy the inequality (6.67) of Lemma 6.5.6 let us first prove that $\ker \tilde{B}_h^t = (P_{S_h}\nabla V_h)^{\perp}$, where $(P_{S_h}\nabla V_h)^{\perp}$ represent the orthogonal part of $P_{S_h}\nabla V_h$. To proceed we observe that the following are equivalent

1) $\boldsymbol{Q}_h \in \tilde{B}_h^t$

2) $(\nabla \boldsymbol{v}_h\,,\, \boldsymbol{Q}_h) = 0 \ \ \forall\, \boldsymbol{v}_h \in V_h\,,$

3) $(P_{S_h}\nabla \boldsymbol{v}_h\,,\, \boldsymbol{Q}_h) = 0 \ \ \forall\, \boldsymbol{v}_h \in V_h\,,$

4) $P_{S_h}\nabla \boldsymbol{v}_h \perp \boldsymbol{Q}_h \ \ \forall\, \boldsymbol{v}_h \in V_h\,,$

5) $\boldsymbol{Q}_h \in (P_{S_h}\nabla V_h)^{\perp}$.

The statements 1) and 5) imply that $\ker \tilde{B}_h^t = (P_{S_h}\nabla V_h)^{\perp}$.

Denote by $S_h' = P_{S_h}\nabla V_h$ and consider $\boldsymbol{Q}_h$ to be any element of S_h'. By definition of S_h' there exists $\boldsymbol{v}_{\boldsymbol{Q}_h} \in V_h$ such that $\boldsymbol{Q}_h = P_{S_h}\nabla \boldsymbol{v}_{\boldsymbol{Q}_h}$. We then have

$$(P_{S_h}\nabla \boldsymbol{v}_{\boldsymbol{Q}_h},\, \boldsymbol{Q}_h)_0 = \|P_{S_h}\nabla \boldsymbol{v}_{\boldsymbol{Q}_h}\|_0^2 = \|P_{S_h}\nabla \boldsymbol{v}_{\boldsymbol{Q}_h}\|_0\,.\|\boldsymbol{Q}_h\|_0\,. \tag{6.71}$$

Using (6.65) the above inequality (6.71) implies that

$$(P_{S_h}\nabla \boldsymbol{v}_{\boldsymbol{Q}_h},\, \boldsymbol{Q}_h)_0 = \|P_{S_h}\nabla \boldsymbol{v}_{\boldsymbol{Q}_h}\|_0\|\boldsymbol{Q}_h\|_0 \geq C\|\boldsymbol{v}_{\boldsymbol{Q}_h}\|_1\|\boldsymbol{Q}_h\|_0$$

which, by observing that $(P_{S_h}\nabla \boldsymbol{v}_{\boldsymbol{Q}_h},\, \boldsymbol{Q}_h)_0 = (\nabla \boldsymbol{v}_{\boldsymbol{Q}_h},\, \boldsymbol{Q}_h)_0$, leads to the following

$$\frac{(\nabla \boldsymbol{v}_{\boldsymbol{Q}_h},\, \boldsymbol{Q}_h)_0}{\|\boldsymbol{v}_{\boldsymbol{Q}_h}\|_1} \geq C\|\boldsymbol{Q}_h\|_0\,. \tag{6.72}$$

Nothing that $S_h' = P_{S_h}\nabla V_h = S_h \backslash \ker \tilde{B}_h^t$ the above inequality (6.72) gives

$$\sup_{\boldsymbol{w}_h \in V_h} \frac{(\nabla \boldsymbol{w}_h,\, \boldsymbol{Q}_h)_0}{\|\boldsymbol{w}_h\|_1} \geq C\|\boldsymbol{Q}_h\|_0 \quad \forall\, \boldsymbol{Q}_h \in S_h \backslash \ker \tilde{B}_h^t\,,$$

which in turn implies that

$$\sup_{\boldsymbol{v}_h \in V_h} \frac{\tilde{b}(\boldsymbol{v}_h,\, \boldsymbol{Q}_h)_0}{\|\boldsymbol{v}_h\|_1} \geq C\|\boldsymbol{Q}_h\|_{S\backslash \ker \tilde{B}_h^t}\,.$$

■

Lemma 6.5.8 *Based on the assumption of positive-definiteness of the operators $\mathbb{A}_h$ and $\mathbb{D}_h$ the kernels (6.54a) and (6.54b) of the bilinear forms $b_{i,h}(\cdot,\cdot)$, $i = 1,\, 2$ satisfy*

$$K_{i,h} = \{\mathbf{0}\}\,.$$

Proof Starting with the bilinear form $b_{2,h}(\cdot,\cdot)$ we consider $\boldsymbol{Q}_h \in S_h$ and then define $\boldsymbol{v}_{Q_h} = \mathbf{0}$ and $\boldsymbol{G}_{Q_h} = -\boldsymbol{Q}_h$ to obtain

$$b_{2,h}\left((\boldsymbol{G}_{Q_h},\, \boldsymbol{v}_{Q_h}),\, \boldsymbol{Q}_h\right) = \|\boldsymbol{Q}_h\|_0^2\,. \tag{6.73}$$

Thus if $\boldsymbol{Q}_h \in K_{2,h}$ it follows from the above (6.73) that $\boldsymbol{Q}_h = \mathbf{0}$.

We next prove that $K_{1,h} = \{\mathbf{0}\}$. To proceed let $\boldsymbol{Q}_h \in S_h$ and consider $\boldsymbol{G}_{Q_h} = -\boldsymbol{Q}_h$ and $\boldsymbol{v}_{Q_h} = \mathbf{0}$ it then follows from (6.50b) that

$$\begin{aligned} b_{1,h}\left((\boldsymbol{G}_{Q_h},\, \boldsymbol{v}_{Q_h}),\, \boldsymbol{Q}_h\right) &= -(\boldsymbol{G}_{Q_h},\, \boldsymbol{Q}_h)_0 + (\mathbb{E}_h\mathbb{A}_h^{-1}\boldsymbol{G}_{Q_h},\, \boldsymbol{Q}_h)_0\,, \\ &= (\boldsymbol{Q}_h,\, \mathbb{D}_h\mathbb{A}_h^{-1}\boldsymbol{Q}_h)_0\,. \end{aligned} \tag{6.74}$$

Using the positive definiteness of $\mathbb{A}_h$ and $\mathbb{D}_h$ it follows that $K_{1,h} = \{\mathbf{0}\}$. ∎

Corollary 6.5.9 *Under the conditions of Lemma 6.5.8 the solution of problem (6.51) is uniquely defined in $V_h \times S_h \times D_h$ and satisfies a uniform error estimate.*

Proof From Corollary 6.5.7 we just need to establish the uniqueness and convergence of the pressure variable. The uniqueness comes from the relation $K_{i,h} = \{\mathbf{0}\}$. For the convergence we first recall the equivalence between the three-field formulation (6.49) and the modified formulation (6.51). We then observe that the kernel corresponding to (6.49) is given by $K_{2,h} = \{\mathbf{0}\}$. From the statement in [38], see its original version in [11, Sect. II], this is enough for the uniqueness and a uniform error estimate.

Note that Corollary 6.5.3 together with the results in this section gives the error estimate for the mixed enhanced strain element, see [45]. In what follows we give the estimate by taking into account the effect of the Newton iteration.

6.5.4 Error Estimate and order of convergence

Let $\Delta\mathbf{U}_h = (\Delta\boldsymbol{u}_h\,, \Delta\boldsymbol{F}_h\,, \Delta\boldsymbol{P}_h)$ and $\Delta\mathbf{U} = (\Delta\boldsymbol{u}\,, \Delta\boldsymbol{F}\,, \Delta\boldsymbol{P})$ be the solutions of (6.51) and (6.38) respectively. We also consider $\Delta\mathbf{U}_h^{i+1} = (\Delta\boldsymbol{u}_h^{i+1}\,, \Delta\boldsymbol{F}_h^{i+1}\,, \Delta\boldsymbol{P}_h^{i+1})$ the approximation of $\Delta\mathbf{U}_h$ during Newton iteration.

Now using the triangular inequality we have

$$\|\Delta\boldsymbol{U} - \Delta\boldsymbol{U}_h^{i+1}\|_{V_h\times D_h\times S_h} \leq \|\Delta\boldsymbol{U} - \Delta\boldsymbol{U}_h\|_{V_h\times D_h\times S_h} + \|\Delta\boldsymbol{U}_h - \Delta\boldsymbol{U}_h^{i+1}\|_{V_h\times D_h\times S_h} \qquad (6.75)$$

From Theorem 6.5.2 and Corollary 6.5.3 the first term on the right hand side of (6.75) can be bounded as

$$\|\Delta\boldsymbol{U} - \Delta\boldsymbol{U}_h\|_{V_h\times D_h\times S_h} \leq Ch\,.$$

For the second term $\|\Delta\boldsymbol{U}_h - \Delta\boldsymbol{U}_h^{i+1}\|_{V_h\times D_h\times S_h}$ its order of convergence depends on the tolerance set for the Newton iteration. Assuming it to be proportional to the mesh size, we have

$$\|\Delta\boldsymbol{U} - \Delta\boldsymbol{U}_h^{i+1}\|_{V_h\times D_h\times S_h} \leq Ch\,.$$

6.6 Numerical results

In this section we examine the performance of our finite element approximation in the incompressible limit. We consider the strain energy function corresponding to the compressible Neo-Hookean material, as given in equation (6.9a), in which the Lamé parameter $\nu = 0.4999$ and the Poisson ratio $E = 1N/m^2$.

Example 1: Compression of a cube

This problem has been studied numerically in [45], using enhanced strains. Here the domain considered is initially defined as $\Omega_0 = (0\,,\,1)\times(0\,,\,1)$. A uniform traction $\bar{\boldsymbol{t}} = 0.28/N$, corresponding to the Neumann boundary condition, and normal to the face; that is, in the vertical direction, is imposed on the surface $(0\,,\,0.75)\times\{1\}$. Vertical and horizontal homogeneous Dirichlet boundary conditions, $\boldsymbol{u} = \boldsymbol{0}$, are applied on the faces $\{0\}\times(0\,,\,1)$ and $(0\,,\,1)\times\{0\}$. Figure 6.3 on the left shows the deformed shape, the deformation being similar to that in [45]. On the right side of 6.3 the response with the use of a standard Q_1 element is showing. Locking behavior is evident.

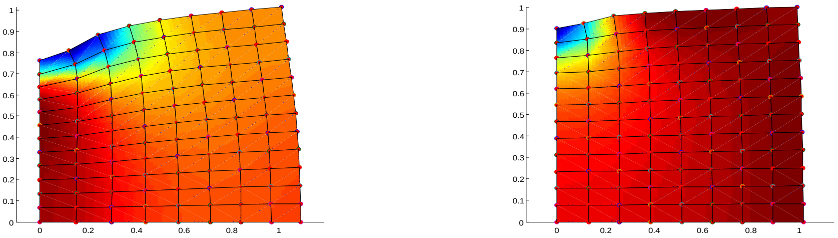


Figure 6.3: Nonlinear elastic deformation in the incompressible limit: new element on the left and standard Q_1 element on the right

Example 2: Shear deformation of rectangular block

We consider the reference configuration $\Omega_0 = (0\,,\,0.25)\times(0\,,\,1)$, on which we impose the Dirichlet boundary condition $\boldsymbol{u} = \mathbf{0}$ on the surface $(0\,,\,0.25)$ and a uniform traction $\bar{\boldsymbol{t}} = 6.25 \times 10^{-3}$ corresponding to the Neumann boundary condition is applied in the horizontal direction of the top face $(0\,,\,0.25) \times \{1\}$. To show the behaviour with respect to degree of compressibility we plot, in Figure 6.5, the horizontal displacement against the parameter $\Xi = \dfrac{\nu}{1-\nu}$. We observe good performance of our implementations compared to standard displacement formulation which shows locking at the incompressible limit.

Figure 6.4 shows the deformed body and displays in colour the magnitude of the stress ($|\boldsymbol{\sigma}|$).

Example 3: The Cook's membrane problem

We reconsider here the Example 4.3.1 as described also in [38] and [22]. Figures 6.6 (a) and 6.6 (b) show the deformed configuration obtained from the numerical simulation of the new element and the standard Q_1 formulation respectively. It can been seen that the new element shows good convergence while the standard displacement formulation gives a poor approximation. In Figure 6.7 we display the convergence of the vertical tip displacement against the number of elements.

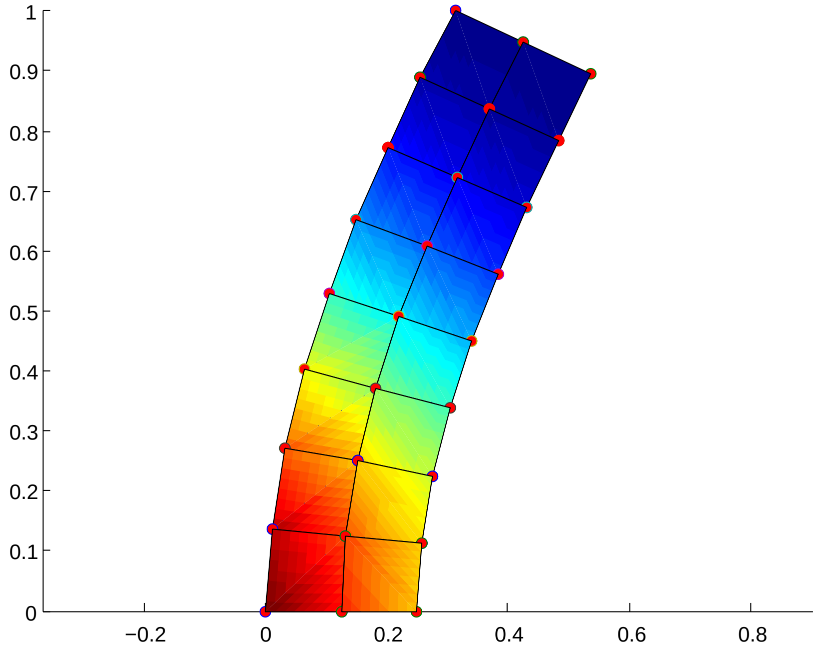


Figure 6.4: Bending of a rectangular domain

6.7 Summary

In this chapter a three-field mixed finite element approximation is applied to the rate or incremental form of the nonlinear elasticity problem. Given the strong form (6.12) we formulate the rate problem (6.24), the weak form of which is the the basis of a finite element approximation. In the finite element formulation the elasticity tensors are approximated by their element-wise mean values. To study the well-posedness of the weak formulation we have adopted a similar approach to [38]; that is, we construct a modified form of the three-field formulation (6.33) and then prove

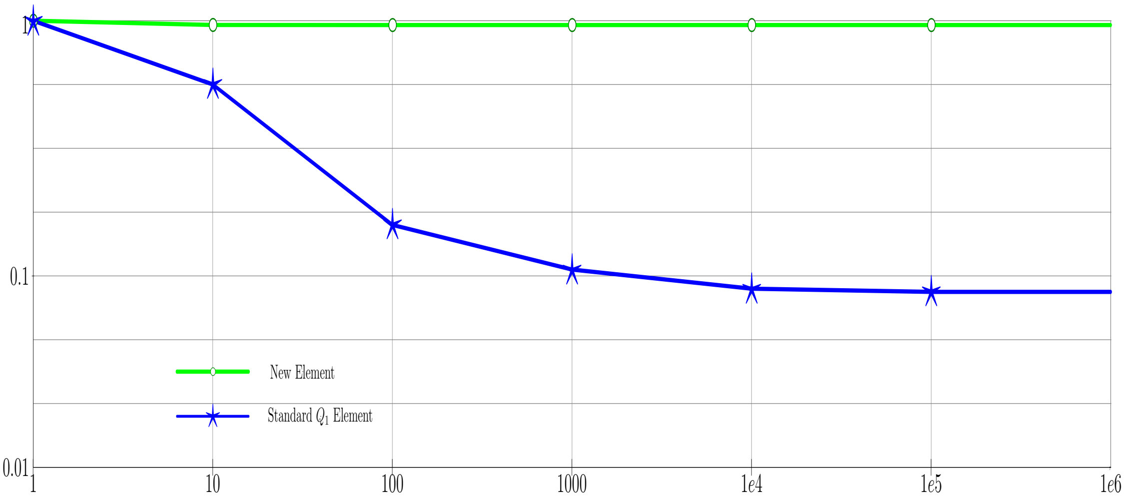


Figure 6.5: Normalized displacement at point $(0, 1)$ against Ξ

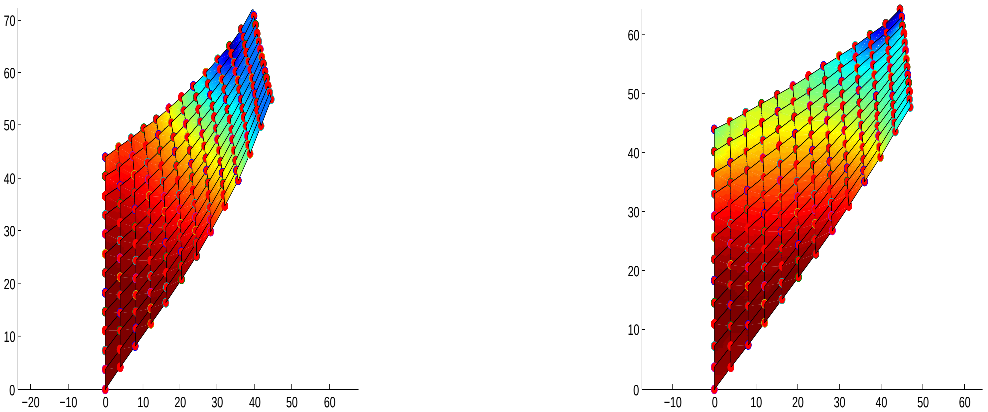


Figure 6.6: Cook' membrane problem: (a) new element; (b) displacement formulation Q_1

that the two formulations are equivalent under certain conditions of the solution-spaces. We have shown that the modified formulation satisfies the inf-sup and the ellipticity conditions, which give rise to the convergence of both formulations. To show the performance of our elements we have presented some examples based a typical choice of lower-order elements. The examples show good performance of our approximations.

(a)

(b)

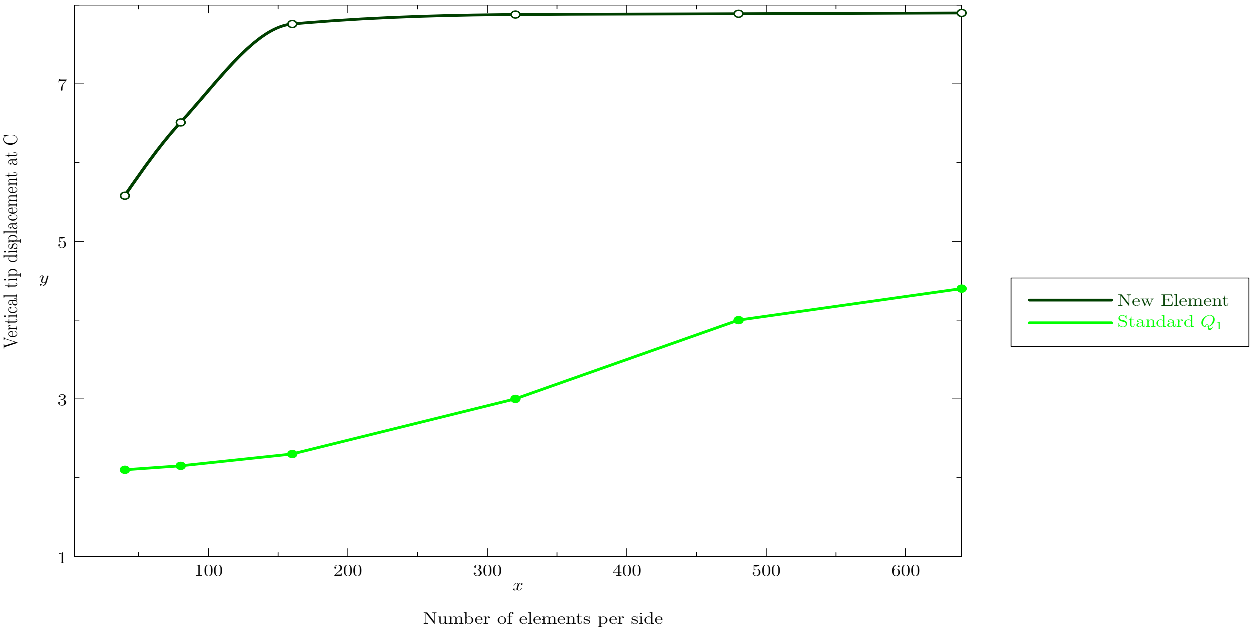


Figure 6.7: Convergence plot: vertical tip displacement against the number of elements

CHAPTER 7

Conclusions

The problems investigated in this thesis have been concerned with the analysis and design of stable and convergent finite elements for problems of near-incompressible elasticity. The basis for the study has been a three-field mixed formulation in which the variables are displacement, stress and strain. Both linear and nonlinear problems have been studied: for the nonlinear case, it is the linearized version that has been investigated, with variables being displacement, first Piola-Kirchhoff stress, and deformation gradient.

A general theory for three-field finite element approximations has been used as the basis for developing new families of stable elements for the linear problem. These are constructed by starting with well-known stable Stokes elements, and then invoking the conditions in the general theory that guarantee stability, to obtain the desired three-field elements. In this way generalizations of the Mini, Crouzeix-Raviart, and Taylor-Hood elements have been obtained. The computational performance of these elements have been demonstrated through a series of numerical examples.

A further contribution has been an analysis of a three-field displacement-pressure-dilatation formulation due to Taylor [44]. This is carried out by embedding the particular formulation within the general three-field approach as a special case.

The linearized version of the nonlinear three-field problem has been formulated and analyzed within the framework of mixed variational formulations. It is assumed that the problem is physically stable in that the elasticity tensor is coercive for the continuous problem. The focus is therefore on determining conditions for the discrete problem to be numerically stable. These conditions are generalizations of those for the linear problem, and essentially take the form of constraints on the choice of the spaces of stresses and strains, involving elementwise-mean values of the elasticity tensor or tensors derived from it. An example in the form of mixed enhanced strains is used to illustrate the numerical performance of an element choice that has been shown to be stable.

Overall, the work reported in this thesis demonstrates the importance and general utility of the three-field approach. At its most general it includes a wide range of popular special cases; and in addition, it serves as a solid basis for designing new families of elements that can be shown to be stable.

There remain various problems worth investigating. Particularly in the domain of nonlinear problems, it would be useful to extend further the investigations in [5] and [6] in order to obtain general guidelines for designing finite element formulations that mimic closely the physical stability ranges of problems. A further area of investigation would be the extension of the approaches developed in this work to more general problems, such as problems of elastoplasticity, contact, and multi-field problems. For all of these extensions the work presented in this thesis serves as a useful point of departure.

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