UNIVERSITY OF CAPE TOWN

DOCTORAL THESIS

Point Symmetry Methods for Itô Stochastic Differential Equations (SDE) with a Finite Jump Process

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Declaration of Authorship

I, Aminu Ma'aruf Nass, declare that this thesis titled, "Point Symmetry Methods for Itô Stochastic Differential Equations (SDE) with a Finite Jump Process" is my own unaided work, both in concept and execution, and that apart from the normal guidance from my supervisor, I have received no assistance. And I confirm that neither the substance nor any part of the above thesis has been in the past, or is being, or is to be submitted for a degree at this University, or any other university.

The following chapters are based on the listed publications:

- **Chapter Two :** Aminu M. Nass and E. Fredericks : Lie Symmetry of Itô Stochastic Differential Equation Driven by Poisson Process, American Review of Mathematics and Statistics, 4(1) (2016).
- **Chapter Three :** Aminu M. Nass and E. Fredericks : N-Symmetry of Itô Stochastic Differential Equation Driven by Poisson Process, International Journal of Pure and Applied Mathematics, 110(1), (2016).
- **Chapter Four :** Aminu M. Nass and E. Fredericks : Symmetry of Jump-Diffusion Stochastic Differential Equations, Global and Stochastic Analysis, 3(1), (2016).
- **Chapter Five :** Aminu M. Nass and E. Fredericks : W-symmetries of Jump-diffusion Itô Stochastic Differential Equations, **(Submitted)**.

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"We must have perseverance and above all confidence in ourselves. We must believe that we are gifted for something and that this thing must be attained."

Marie Curie

UNIVERSITY OF CAPE TOWN

Abstract

Faculty of Science
Department of Mathematics and Applied Mathematics

Doctor of Philosophy

Point Symmetry Methods for Itô Stochastic Differential Equations (SDE) with a Finite Jump Process

by Aminu Ma'aruf Nass

The mixture of Wiener and a Poisson processes are the primary tools used in creating jump-diffusion process which is very popular in mathematical modeling. In financial mathematics, they are used to describe the change of stock rates and bonanzas, and they are often used in mathematical biology modeling and population dynamics.

In this thesis, we extended the Lie point symmetry theory of deterministic differential equations to the class of jump-diffusion stochastic differential equations, i.e., a stochastic process driven by both Wiener and Poisson processes. The Poisson process generates the jumps whereas the Brownian motion path is continuous. The determining equations for a stochastic differential equation with finite jump are successfully derived in an Itô calculus context and are found to be deterministic, even though they represent a stochastic process.

This work leads to an understanding of the random time change formulae for Poisson driven process in the context of Lie point symmetries without having to consult much of the intense Itô calculus theory needed to formally derive it. We apply the invariance methodology of Lie point transformation together with the more generalized Itô formulae, without enforcing any conditions to the moments of the stochastic processes to derive the determining equations and apply it to few models.

In the first part of the thesis, point symmetry of Poisson-driven stochastic differential equations is discussed, by considering the infinitesimals of not only spatial and temporal variables but also infinitesimals of the Poisson process variable. This was later extended, in the second part, to define the symmetry of jump-diffusion stochastic differential equations (i.e., stochastic differential equations driven by both Wiener and Poisson processes).

Acknowledgements

All praise is for Allah. We praise Him and seek His help and forgiveness. We seek refuge in Allah from evils of ourselves and the wickedness of our own deeds. Whomever Allah guides, cannot be lead astray and whoever Allah misguides, none can guide him. I bear witness that none has right to be worshipped except Allah, and I bear witness that Muhammad is his slave, and messenger. May peace of Allah be upon him.

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I dedicated this work to my beloved parents
Late. Alh. Ma'aruf Musa Nass and Hajia Binta Aliyu,
who have been giving me never-ending love and
absolute support through life.

Chapter 1

Introduction and Preliminaries

1.1 Introduction

The method of studying differential equations using their symmetries was introduced by the Norwegian mathematician Sophus Lie, who also founded the theory of infinitesimal transformations and Lie groups. Lie's classical approach is based on finding a symmetry group associated with a differential equation. This is a local Lie group of point transformations that maps the solution of differential equations (DEs) to that of the same DE. The classical method of Lie allows computing the symmetry group associated with a given differential equation. This symmetry group can further be used for many important applications in the context of differential equations. For instance, for determination of group-invariant solutions, solving the first order differential equation, reducing the order of higher ODE, reducing the number of variables of partial differential equations and finding conservation laws. Lie symmetry theory is one of the most powerful method in analyzing deterministic differential equations and is widely used in literature [36] - [50] and [56] - [64].

In contrast to the deterministic differential equation, only a few attempts have been made to extend Lie group theory to the stochastic differential equations. It is worth noticing that the theory is still developing. Gaeta and Quintero [10] made the first approach to extend Lie symmetry of differential equations to Itô stochastic ordinary differential equations by which they consider a small class of transformations, i.e., fiber preserving transformations

$$\bar{x} = \theta_1(t, x, \epsilon), \qquad \bar{t} = \theta_2(t, \epsilon).$$

The method has been used to study the relationship between symmetries of stochastic systems to the symmetries of their corresponding Fokker-Planck equation. This is a restricted transformation that can only work to a fiber-preserving class of transformations which is a small sub-class of all possible transformations.

The second attempt [7, 8, 9, 13, 15, 20] succeed in applying symmetry transformations that include all the dependent variables in the transformation.

$$\bar{x} = \theta_1(t, x, \epsilon), \qquad \bar{t} = \theta_2(t, x, \epsilon).$$

This approach has been used to study the symmetry of scalar stochastic ordinary differential equations of the first order [8] which reconciled the works of S. V. Meleshko , B. S. Srihirun, E. Schultz [13] and C. Wafo Soh, F. M. Mahomed [15]. Furthermore, the formal method for finding Lie Point symmetries of scalar Itô stochastic differential equations of the first order driven by the Wiener process was also discussed by E. Fredericks and F. M. Mohamed [7] with intention to correct and reconcile the findings of Srihirun and Schultz [13].

In [16, 14] G. Gaeta introduced "W-symmetries" by considering symmetries that involve both the spatial, temporal variables and the vector Wiener process $w(t)$. However, G. Gaeta [16, 14] enforced conditions that transformed the Wiener process to be consistent with the original process in terms of its momenta, i.e., the instantaneous mean and variance of the transformed process are forced to be exactly the instantaneous mean and variance of the original Wiener process.

To the best of our knowledge in literature, all the methods above were applied only to the Itô stochastic differential equations driven by Wiener processes [7, 8, 9, 10, 11, 13, 14, 15, 16, 20, 21, 26, 27, 28, 30, 31, 32, 33, 35, 52, 53].

In this thesis, we extend the Lie symmetry methods to the class of Itô stochastic differential equations driven by both Wiener and Poisson processes i.e., the jump-diffusion process by implementing a more generalized Itô formula. The primary tools for finding admitted Lie point symmetry transformations for stochastic differential equations (SDEs) are the Itô formula and the random change of time.

We are going to follow the methodology of E. Fredericks and F. M. Mohamed [7, 29] and G. Gaeta [10, 16] in this regard:

- apply infinitesimal transformations on the spatial, temporal variables as well as Wiener and jump diffusion processes
- apply infinitesimal transformations on the drift and diffusion coefficients of the stochastic differential equation.
- apply infinitesimal transformations on the moments of the differential stochastic processes.
- induce an invariance transformation argument on the transformed stochastic differential equation in differential form.

The thesis is organised as follows. In chapter two, we discuss Lie point symmetry of Poisson driven stochastic differential equations (SDE) by considering infinitesimals of spatial x and temporal t variables [1]. In chapter three, this is extended to include not only spatial and temporal variables but also the Poisson process variable $N(t)$ in the transformation, which subsequently leads to the derivation of generalised random time change for the Poisson process [2].

In chapter four, we define Lie symmetry of stochastic differential equations driven by the Wiener and the Poisson process, known as the jump-diffusion stochastic process, by considering infinitesimals of spatial x and temporal t variables [3]. This was achieved by using the random time change formula of the Poisson process derived in chapter three and the random time change transformation for the Brownian motion process (Wiener) [8, 9, 13, 20, 28, 32].

Furthermore, the symmetry of jump-diffusion stochastic differential equations is extended in chapter five and chapter six to include not only infinitesimals of spatial x and temporal t but also those of Wiener $W(t)$ and Poisson $N(t)$ processes respectively. This leads to the derivation of the random time change transformation for the Brownian motion process.

In both cases, the determining equations are derived and found to be deterministic even though they represent a stochastic process. This is accomplished by ensuring the finite jump stochastic differential equation, as well as the moments of the processes, remain invariant under one-parameter Lie groups of transformation. Finally, applications to some stochastic differential equations are presented and later showed the Lie bracket relations between the admitted infinitesimals generators.

1.2 Preliminaries

In this sections, we give the basic definitions and results obtained in the literature regarding the Lie symmetry method for differential equations and stochastic processes that will provide the necessary background for the research work carried out in this thesis. The short review consists of presentation of the notion of Lie symmetries of differential equations [38, 40, 43, 45, 46, 47, 50], the necessary properties and theorems of Brownian motion [4, 22, 23, 34], as well as Poisson and jump-diffusion processes for Itô stochastic differential equations [5, 25, 24].

1.2.1 Lie Point Symmetry of Differential Equations

This section focus on basic ideas of Lie symmetry method which serves as the basis for our research, the main objective is to give a short review of the standard background in Lie symmetry method. A comprehensive account of the subject of Lie symmetry of differential equations is contained in many standard books on the topic cf [21, 27, 37, 40, 43, 45].

Definition 1.2.1 *Any transformation mapping a differential equation into an equivalent equation of the same form is called a symmetry of the differential equation.*

Definition 1.2.2 *The set of invertible point transformation G in (t,x) plane,*

$$\overline{x} = \theta_1(t,x,\epsilon), \qquad \overline{t} = \theta_2(t,x,\epsilon) \tag{1.1}$$

depending on a parameter ϵ is called a Lie group of transformation if it contains the identical transformation $I = T_0$ and include the inverse T_ϵ^{-1} as well as composition $T_{\epsilon_2}T_{\epsilon_1}$ for all its element $T_{\epsilon_1}, T_{\epsilon_2} \in G$, where θ_1 and θ_2 are sufficiently smooth functions and $\theta_1\Big|_{\epsilon=0} = x$ and $\theta_2\Big|_{\epsilon=0} = t$.

Definition 1.2.3 *Infinitesimal Generators:*
Consider a one parameter Lie group of transformation

$$\overline{x} = x(\epsilon) = \theta_1(t,x,\epsilon), \qquad \overline{t} = t(\epsilon) = \theta_2(t,x,\epsilon) \tag{1.2}$$

using Taylor's series expansion in the parameter ϵ near $\epsilon = 0$ (1.2) gives

$$\overline{x} = x + \xi(t,x)\epsilon + O(\epsilon^2), \qquad \overline{t} = t + \tau(t,x)\epsilon + O(\epsilon^2), \tag{1.3}$$

with

$$\xi(t,x) = \frac{d\overline{x}}{d\epsilon}\Big|_{\epsilon=0} \qquad and \qquad \tau(t,x) = \frac{d\overline{t}}{d\epsilon}\Big|_{\epsilon=0}. \tag{1.4}$$

This geometrically means the infinitesimals transformation (1.3) defined a tangent vector $\Big(\tau(x,y),\xi(x,y)\Big)$ and can be written as first order linear differential operator

$$H = \tau(t,x)\frac{\partial}{\partial t} + \xi(t,x)\frac{\partial}{\partial x}. \tag{1.5}$$

Equation (1.5) is called the infinitesimal generator of (1.3), while $\Big(\tau(x,y),\xi(x,y)\Big)$ are called the infinitesimals of the transformation.

Definition 1.2.4 *The Lie Point transformation (1.3) or the correspondent infinitesimal generator (1.5) is called a fiber-preserving transformation or projectile transformation if* $\frac{\partial \tau(t,x)}{\partial x} = 0$ *i.e.,*

$$H = \tau(t)\frac{\partial}{\partial t} + \xi(t,x)\frac{\partial}{\partial x}. \tag{1.6}$$

1.2.2 Lie Algebra [61]

Let H_1 and H_2 be first order differential operators of the form

$$H_1 = \tau(t)\frac{\partial}{\partial t} + \xi(t,x)\frac{\partial}{\partial x}, \tag{1.7}$$

$$H_2 = \dot{\tau}(t)\frac{\partial}{\partial t} + \dot{\xi}(t,x)\frac{\partial}{\partial x}. \tag{1.8}$$

Their commutator $\left[H_1, H_2\right]$ is define by

$$\left[H_1, H_2\right] = H_1 H_2 - H_2 H_1. \tag{1.9}$$

Definition 1.2.5 *The vector space* ℓ *of operator*

$$H_1 = \tau(t)\frac{\partial}{\partial t} + \xi(t,x)\frac{\partial}{\partial x}, \quad H_2 = \dot{\tau}(t)\frac{\partial}{\partial t} + \dot{\xi}(t,x)\frac{\partial}{\partial x} \tag{1.10}$$

is called a Lie algebra if it is closed under the Lie bracket i.e.,

$$\left[H_1, H_2\right] \in \ell \tag{1.11}$$

for any H_1, $H_2 \in \ell$.

Definition 1.2.6 *For any finite Lie algebra* ℓ_r *spanned by* $H_1, H_2, H_3 ... H_r$. *A subspace* K_s $(s < r)$ *of a vector space* ℓ_r *spanned by linearly independent operators* $X_1, X_2, X_3 ... X_s \in \ell$ *is called a subalgebra of* ℓ_r *if*

$$\left[X_1, X_2\right] \in K_s \tag{1.12}$$

for any X_1, $X_2 \in K_s$.

Example 1.2.7 *Consider the two-dimensional Lie algbra with the basis*

$$H_1 = \frac{\partial}{\partial x} \qquad H_2 = x\frac{\partial}{\partial x} + \frac{3t}{4}\frac{\partial}{\partial t}. \tag{1.13}$$

Therefore,

$$\begin{aligned} \Big[H_1, H_2\Big] &= \Big(H_1(x) - X_2(1)\Big)\frac{\partial}{\partial x} + \Big(H_1(\frac{3t}{4}) - X_2(0)\Big)\frac{\partial}{\partial y} \\ &= \frac{\partial}{\partial x} \\ &= H_1. \end{aligned} \tag{1.14}$$

Therefore, from the anti-commutator property we have

$$\Big[H_1, H_2\Big] = -\Big[H_2, H_1\Big] = -H_1$$

and

$$\Big[H_1, H_1\Big] = \Big[H_2, H_2\Big] = \Big[H_3, H_3\Big] = 0.$$

1.2.3 Stochastic Processes

In this section we introduce basic properties and important theorems of Wiener processes and simple Poisson jump processes in differential form, which are the primary tools used in creating jump-diffusion process models.

Definition 1.2.8 *Stochastic process is the collection of parametrised random variables*

$$X = \{X(t) : t \in T\} \tag{1.15}$$

defined on a probability space $\Big(\Omega, \mathbb{F}, \mathbb{P}\Big) \in \mathbb{R}^n$*. Where* Ω *is a sample space,* $\mathbb{F}$ *is a* σ*-algebra and* $\mathbb{P}$ *is a probability measure.*

1.2.3.1 Stochastic Differential Equations for Wiener Processes

Definition 1.2.9 *The standard Wiener process*

$$W(t) : \mathbb{R}^+ \times \Omega \to \mathbb{R} \tag{1.16}$$

over a probability space $(\Omega, \mathbb{F}, \mathbb{P})$ *has the following properties*

- *Standard Wiener or Brownian motion starts at zero with probability one i.e.,*

$$\mathbb{P}\Big(W(0)=0\Big)=1.$$

- *The covariance of two Wiener processes at different time is equal to the minimum between the two different times i.e.,*

$$Cov\Big(W(t),W(s)\Big)=min(t,s). \tag{1.17}$$

- *Standard Brownian motion has independents increments i.e, for* $\{t_i\}\subset\mathbb{R}^+$*, the Wiener process* $W(t_{i+1})-W(t_i)$ *are independents for all*

$$0\leq t_1<t_2<...<t_i.$$

Lemma 1.2.10 *Differential Products* $dtdW(t)$ *and* $dW(t)dW(t)$ *[4]*
Given a standard Wiener $W(t)$ *process, we have*

$$\int_0^t dsdW(s)=0 \tag{1.18}$$

and

$$\int_0^t dW(s)dW(s)=t. \tag{1.19}$$

Equations (4.66) and (4.68) can be rewritten in symbolic form respectively as

$$dtdW(t)=0 \qquad and \qquad dW(t)dW(t)=dt. \tag{1.20}$$

For any two independent Wiener processes $dW_l(t)$ and $dW_m(t)$, $l,m=1,2,3...,n$ the differential product of the Brownian motion is giving in *Table 1.1* below

TABLE 1.1: Itô Multiplication Table of Wiener Processes

	$dW_l(t)$	$dW_m(t)$	dt
$dW_l(t)$	dt	0	0
$dW_m(t)$	0	dt	0
dt	0	0	0

Definition 1.2.11 *The infinitesimal moments of the standard differential Wiener process are*

(1.21)
$$\mu\Big(dW(t)\Big) = E\Big[dW(t)\Big|W = w\Big] = 0 \quad and \quad Var\Big[dW(t)\Big|W = w\Big] = E\Big[dW(t)dW(t)\Big|W = w\Big] = dt.$$

Theorem 1.2.12 *(Itô Formula)*

Let $X_i(t)$ be a solution of the stochastic differential equation

$$dX_i(t) = f_i(t, X(t))dt + G_{il}(t, X(t))dW_l(t) \qquad with \qquad 0 \le t \le T \tag{1.22}$$

with initial value $X(0) = x_0$, where $f_i(t, X(t))$ and $G_{il}(t, X(t))$ are bounded and integrable respectively. Assume $F(t, X(t))$ is differentiable partially up to order two (C^2) with respect to spatial variable x and once differentiable with respect to time t, then

(1.23)
$$dF_j(t, X(t)) = \left(\frac{\partial F_j}{\partial t}dt + f_i\frac{\partial F_j}{\partial x_i} + \frac{1}{2}\sum_{k=1}^{M} G_{ik}(t, X(t))G_{mk}(t, X(t))\frac{\partial^2 F_j}{\partial x_i \partial x_m}\right)dt + G_{il}(t, X(t))\frac{\partial F_j}{\partial x_i}dW_l.$$

The Einstein summation convention is assumed through out this work.

Theorem 1.2.13 *(Existence and Uniqueness)*

There exists a unique solution of the Itô stochastic differential equation (1.22) if the coefficients satisfy the following conditions

1. *The drift and diffusion coefficients are uniformly Lipschitz in temporal t and locally at spatial x variables i.e., for any constant K depending on M and T such that $|x|, |y| \le M$ and $0 \le t \le T$ then*

$$\Big|f(t, X_1(t)) - f(t, X_2(t))\Big| + \Big|G(t, X_1(t)) - G(t, X_2(t))\Big| < K\Big|X_1(t) - X_2(t)\Big|. \tag{1.24}$$

2. *The drift and Wiener diffusion coefficients satisfy the linear growth condition i.e.,*

$$\left|f(t,X(t))\right| + \left|G(t,X(t))\right| < K\Big(1+|x|\Big). \tag{1.25}$$

3. *The initial condition* $X(0) = x_0$ *is independent of the Wiener process* $W(t)$, $0 \leq t \leq T$ *and*

$$EX^2(0) < \infty. \tag{1.26}$$

1.2.3.2 Stochastic Differential Equations for Poisson Processes

Definition 1.2.14 *[5] Stochastic process* $N(t)$ *for* $t \geqslant 0$ *is called a Poisson Process with intensity* $\lambda > 0$ *if*

- $N(t)$ *takes values* $0,1,2,...$ *with* $N_0 = 0$
- *Poisson increments i.e., for* $s \leq t$, *the* $N(t)-N(s)$ *is Poisson distribution with intensity* $\lambda(t-s)$

$$\mathbb{P}\Big(N(t)-N(s)=k\Big) = \frac{(\lambda(t-s))^k}{k!}, \qquad k = 0,1,2,... \tag{1.27}$$

- *Independence of increments i.e.,* $N(t)-N(s)$ *is independent of the past* $N(u)$ *with* $u \leq s$.

Definition 1.2.15 *[4] Assume the process* $Y(t) = F(t,X(t))$ *is a continuous transformation of the process* $X(t)$ *with a jump function* $[X](t)$ *at* t. *Then the jump function* $F(t,X(t))$ *is defined as*

$$\Big[Y\Big](t) = \Big[F\Big](t,X(t)) = F\Big(t,X(t)+[X](t)\Big) - F(t,X(t)). \tag{1.28}$$

Lemma 1.2.16 *[4]*

Let the process $Y(t) = F(t,X(t))$ *be a continuous transformation of the process* $X(t)$ *with a jump function*

$$\Big[X\Big](t) = J(t,X(t))dN(t) \tag{1.29}$$

at t, *then*

$$\Big[Y\Big](t) = \Big[F\Big](t,X(t)) = \big(F(t,X(t)+J(t,X(t))) - F(t,X(t))\big)dN(t). \tag{1.30}$$

The Lemma 1.2.16 can easily be prove by using definition 1.2.15 and by the use of zero-one jump law [4].

Definition 1.2.17 *Chain rule for $\mathscr{H}(N(t),t)$*

Let $\mathscr{H}(N(t),t)$ be once continuously differential in t and continuous N.

$$d\mathscr{H}(N(t),t) = d_{(Cont)}\mathscr{H}(N(t),t) + d_{(Jump)}\mathscr{H}(N(t),t) \tag{1.31}$$

where

$$d_{(Cont)}\mathscr{H}(N(t),t) = \mathscr{H}_t(N(t),t), \quad d_{(Jump)}\mathscr{H}(N(t),t) = \Big[\mathscr{H}\Big](N(t),t).$$

Lemma 1.2.18 *Differential Products $dtdN(t)$ and $dN(t)dW(t)$ [4]*

Given an independent random variables of the Wiener $W(t)$ and Poisson processes $N(t)$ respectively, we have

$$\int_0^t dsdN(s) = 0 \tag{1.32}$$

and

$$\int_0^t dN(s)dW(s) = 0. \tag{1.33}$$

Equation (1.32) and (1.33) can be rewritten in symbolic form respectively as

$$dtdN(t) = 0 \qquad and \qquad dN(t)dW(t) = 0. \tag{1.34}$$

Definition 1.2.19 *The infinitesimal moments of the standard differential scalar Poisson process with jump rate $\lambda > 0$ is*

$$\mu\Big(dN(t)\Big) = E\Big[dN(t)\Big] = \lambda dt \quad and \quad Var\Big[dN(t)\Big] = E\Big(N_t - \mu\Big(dN(t)\Big)\Big)^2 = \lambda dt. \tag{1.35}$$

consequently, from (1.33) we also have

$$E\Big[dN(t)dW(t)\Big|W = w\Big] = E\Big[dW(t)\Big|W = w\Big]E\Big[dN(t)\Big] = 0. \tag{1.36}$$

Theorem 1.2.20 *(Mean Square Limit Form Of The Zero-one Law)*

Let m be a non-negative integer and

$$\mu\Big(dN(t)\Big) = E\Big[dN(t)\Big] = \lambda dt \qquad then \tag{1.37}$$

$$\int_0^t (dN)^m(s) = N(t) \tag{1.38}$$

symbolically $(dN)^m(t) = dN(t), \quad m = 1,2,3...,n.$

Theorem 1.2.21 *(Itô Lemma for Poisson diffusion process) [4, 5]*

Let $Y(t) = F_j(t,X(t))$*, such that the function* $F_j(t,X(t))$ *is once continuously differentiable with respect to x and t. Let the* $X_i(t)$ *process satisfy the Itô stochastic differential equation with finite jump of the form,*

$$dX_i(t) = f_i(t,X(t))dt + J_i(t,X(t))dN(t) \tag{1.39}$$

$X(0) = x_0$*, where* $f_i(t,X(t))$ *and* $J_i(t,X_i(t))$ *are bounded and integrable respectively. Then*

$$dF_j(t,X(t)) = \Big(\frac{\partial F_j}{\partial t} + f\frac{\partial F_j}{\partial x}\Big)dt + \Big(F_j\Big(t,X(t)+J_j(t,X(t))\Big) - F_j(t,X(t))\Big)dN(t). \tag{1.40}$$

Proof : *Formally, using the increment form of the differential,*

$$\begin{aligned} dY(t) =& Y(t+dt) - Y(t) \\ &= F(X(t+dt), t+dt) - F(X(t),t) \\ &= F(X(t)+dX(t), t+dt) - F(X(t),t). \end{aligned} \tag{1.41}$$

Using definition 1.2.17 the instantaneous jump changes (terms in $dN(t)$ *only, such that* $\Big[X\Big](t) = J(X(t),t)dN(t)$*) are treated separately from the continuous and smooth deterministic changes (terms in* dt *only, such that* $dX^{(det)}(t)) = f(X(t),t)dt$*).*

Using the mean square approximation [4] we have

$$\begin{aligned} dF_j(t,X(t)) =& \frac{\partial F_j}{\partial t}dt + f\frac{\partial F_j}{\partial x}dt + \Big(F_j\Big(t,X(t)+[X]\Big) - F_j(t,X(t))\Big) \\ &= \Big(\frac{\partial F_j}{\partial t} + f\frac{\partial F_j}{\partial x}\Big)dt + \Big(F_j\Big(t,X(t)+J_j(t,X(t)dN(t)\Big) - F_j(t,X(t))\Big) \\ &= \Big(\frac{\partial F_j}{\partial t} + f\frac{\partial F_j}{\partial x}\Big)dt + \Big(F_j\Big(t,X(t)+J_j(t,X(t))\Big) - F_j(t,X(t))\Big)dN(t). \end{aligned} \tag{1.42}$$

where the zero-one jump law [4] has been used to take the $dN(t)$ *out of the argument of F and let it multiply the jump change in F in the last line of the above equation. Note that the jump change has been defined, so that if there is no Poisson jump, then the jump function is zero.*

1.2.3.3 Stochastic Differential Equations with Jump-diffusion Processes

Jump-diffusion stochastic differential equations appear in a process combining both Wiener and Poisson processes. The Wiener term provides the diffusion while the Poisson term provides the jump. i.e.,

$$dX_i(t) = f_i(t, X(t))dt + G_{il}(t, X(t))dW_l(t) + J_i(t, X(t))dN(t). \tag{1.43}$$

Where $f_i(t, X(t))$ and $J_i(t, X(t))$ are $n \times 1$ dimensional drift vector and jump diffusion coefficients respectively, while $G_{il}(t, X(t))$ is the Wiener diffusion matrix coefficient of $n \times M$ dimension. $dW(t)$ is called the infinitesimal increment of the Wiener process while $dN(t)$ is called the infinitesimal increment of the Poisson process and is used to determine the existence of the jump.

The stochastic jump-diffusion equation (1.43) can be decomposed into the continuous part

$$dX_i^c(t) = f_i(t, X(t))dt + G_{il}(t, X(t))dW_l(t) \tag{1.44}$$

and the jump part

$$dX_i^J(t) = \Big[X\Big](t) = J_i(t, X(t))dN(t). \tag{1.45}$$

Now, the jump-diffusion equation (1.43) can be rewritten as

$$dX_i(t) = dX_i^c(t) + dX_i^J(t). \tag{1.46}$$

Definition 1.2.22 *For any two independent Brownian motion processes $dW_l(t)$ and $dW_m(t)$ and a scalar Poisson process $dN(t)$ where $l, m = 1, 2, 3..., n$. The differential product of the jump-diffusion process can be summarised viz Table 1.2 below*

TABLE 1.2: Itô Multiplication Properties Table of Jump-diffusion

	dt	dW_l	dW_m	dN
dt	0	0	0	0
dW_l	0	dt	0	0
dW_m	0	0	dt	0
dN	0	0	0	dN

Theorem 1.2.23 *(Itô Lemma For Jump-diffusion Processes)*
Let $X_i(t)$ be a solution of jump-diffusion stochastic differential equation

$$dX_i(t) = f_i(t, X(t))dt + G_{il}(t, X(t))dW_l(t) + J_i(t, X(t))dN(t). \tag{1.47}$$

Where $f_i(t,X(t))$ and $J_i(t,X(t))$ are $n \times 1$ dimensional drift vector and jump diffusion coefficients respectively and $G_{il}(t,X(t))$ is the Wiener diffusion matrix coefficient of $n \times M$ dimension. with initial value $X(0) = x_0$. Assume $F(t,X(t))$ is differentiable partially up to order two (C^2) with respect to spatial variable x and once differentiable with respect to time t, then

$$\begin{aligned} dF_j(t,X(t)) = \frac{\partial F_j}{\partial t}dt + \frac{\partial F_j}{\partial x_i}dX_i^c(t) + \frac{1}{2}\sum_{k=1}^{M} \frac{\partial^2 F_j}{\partial x_i \partial x_m}\Big(dX_i^c(t)dX_m^c(t)\Big) \\ + \Big(F_j\Big(t, X_i(t) + J_j(t, X(t))\Big) - F_j(t, X(t))\Big)dN(t). \end{aligned} \tag{1.48}$$

The solution of the jump-diffusion stochastic differential equation (1.47) exists and is unique if the drift vector and jump diffusion as well as the Wiener diffusion matrix coefficients satisfy the following conditions [5, 25, 34, 51, 66]

1. The drift and diffusion coefficients are uniformly Lipschitz in temporal t and locally at spatial variable x i.e., for any constant K depending on M and T such that $|x|, |y| \le M$ and $0 \le t \le T$ then
(1.49)
$$\Big|f(t,X_1(t)) - f(t,X_2(t))\Big|^2 + \Big|G(t,X_1(t)) - G(t,X_2(t))\Big|^2 + \Big|J(t,X_1(t)) - J(t,X_2(t))\Big|^2 < K\Big|X_1(t) - X_2(t)\Big|^2.$$

2. The drift and Wiener diffusion coefficients satisfied linear growth condition i.e.,

$$\left|f(t,X(t))\right|^2 + \left|G(t,X(t)))\right|^2 + \left|J(t,X(t))\right|^2 < K\left(1+|x|^2\right). \tag{1.50}$$

3. The initial condition $X(0) = x_0$ is independent of the Wiener process $W(t)$, $0 \le t \le T$ and

$$EX^2(0) < \infty. \tag{1.51}$$

Proof : See [25].

Chapter 2

Lie Symmetry of Itô Stochastic Differential Equations Driven by the Poisson Processes

In this chapter, we defined a Lie point symmetry transformation of a class of Poisson driven Itô stochastic differential equations [1] which extended the earlier theory of Lie symmetry on Brownian motion driven stochastic differential equations [7, 8, 9, 10, 15, 20]. This was achieved by utilising Itô lemma [4, 5] for Poisson stochastic Processes and by considering the infinitesimals of temporal and spatial variables.

2.1 Introduction

We consider Lie point symmetries of Itô stochastic differential equations driven by Poisson processes of the form,

$$dX_i(t) = f_i(t, X(t))dt + J_i(t, X(t))dN(t) \tag{2.1}$$

with initial condition $X(0) = x_0$. So, equation (2.1) can be written in integral form as

$$X(t) = x_0 + \int_o^t f(s, X(s))ds + \int_o^t J(s, X(s))dN(s), \tag{2.2}$$

where $f_i(t, X(t))$ and $J_i(t, X(t))$ are $n \times 1$ dimensional drift vector coefficients and Poisson diffusion coefficient respectively, which are assumed to satisfy Ikeda and Watanabe [51, 66] conditions for the uniqueness and existence of the solution of (2.1). $dN(t)$ is the infinitesimal increment of the Poisson Process [17, 18, 19].

Symmetries of (2.1) are analysed by considering an infinitesimal generator

$$H = \tau(t,x)\frac{\partial}{\partial t} + \xi_i(t,x)\frac{\partial}{\partial x_i}. \tag{2.3}$$

The determining equations for Itô stochastic differential equations (SDE) driven by Poisson processes (2.1) are derived using Itô calculus and are found to be non-stochastic.

Starting with an arbitrary function $F(t,X(t))$ which is once differential with respect to the spatial coordinate x and differentiable once with respect to temporal variable t, the Itô Poisson diffusion process for $F(t,X(t))$ of (2.1) exists [4, 5] and is

$$dF_j(t,X(t)) = \left(\frac{\partial F_j}{\partial t} + f_i\frac{\partial F_j}{\partial x_i}\right)dt + \left(F_j\Big(t,X_i + J_j(t,X_i)\Big) - F_j(t,X_i)\right)dN(t). \tag{2.4}$$

For simplicity let

$$\Gamma_{(F)_j} = \frac{\partial F_j}{\partial t} + f_i\frac{\partial F_j}{\partial x_i} \tag{2.5}$$

and

$$\Gamma^*_{(F)_j} = F_j(t,X_i(t) + J_j(t,X(t))) - F_j(t,X(t)). \tag{2.6}$$

Therefore (4.68) can be rewritten as;

$$dF_j(t,X(t)) = \Gamma_{(F)_j}dt + \Gamma^*_{(F)_j}dN(t). \tag{2.7}$$

Using the Itô multiplication properties in *Table 1.2* [4, 5] and application of infinitesimal transformations the determining equations for (SDE) with Poisson processes are derived and are non-stochastic. The main result can be summarised as

Theorem 2.1.1 *The Itô stochastic differential equation driven by Poisson processes*

$$dX_i(t) = f_i(t,X_i(t))dt + J_i(t,X_i(t))dN(t) \tag{2.8}$$

where $f_i(t,X(t))$ and $J_i(t,X(t))$ are the n-dimensional drift vector coefficient and the Poisson diffusion coefficient, with infinitesimal generator

$$H = \tau(t,x)\frac{\partial}{\partial t} + \xi_i(t,x)\frac{\partial}{\partial x_i}, \tag{2.9}$$

has admitted the following determining equations;

$$\left(f_j\Gamma_{(\tau)} + \frac{\lambda J_j}{2}\Gamma_{(\tau)} + H(f_j) - \Gamma_{(\xi)_j}\right)(t,X(t)) = 0 \tag{2.10}$$

$$\left(\frac{J_j}{2}\Gamma_{(\tau)} + H(J_j) - \Gamma^*_{(\xi)_j}\right)(t,X(t)) = 0 \tag{2.11}$$

with additional conditions,

$$\Gamma^*_{(\tau)}(t,X(t)) = 0, \qquad and \qquad \Gamma_{(\tau)}(t,X(t)) = c_1. \tag{2.12}$$

The operators $\Gamma(t,X(t))$ and $\Gamma^*(t,X(t))$ are defined as in (2.5) and (2.6), and $\lambda > 0$ is called the intensity of the jump process or jump rate.

2.2 Lie Group Transformations

Consider a one parameter group of transformations of the time index t and the spatial variable x respectively,

$$\bar{t} = \theta_1(x,t,\epsilon), \qquad \bar{x} = \theta_2(x,t,\epsilon)$$

with the infinitesimals

$$\frac{\partial\theta_1}{\partial\epsilon} = \tau(t,x), \qquad \frac{\partial\theta_2}{\partial\epsilon} = \xi(t,x)$$

satisfying the following initial conditions at $\epsilon = 0$

$$\bar{t}\Big|_{\epsilon=0} = t, \qquad \bar{X}(\bar{t})\Big|_{\epsilon=0} = X(t).$$

A one parameter Lie group of infinitesimal transformations is therefore

$$\bar{t} = t + \epsilon\tau(t,x) + O(\epsilon) \tag{2.13}$$

and

$$\overline{X_j}(\bar{t}) = X_j(t) + \epsilon \xi_j(t,x) + O(\epsilon), \tag{2.14}$$

where ϵ is the parameter of the group, with the corresponding generator of the Lie algebra of the form

$$H = \tau(t,x)\frac{\partial}{\partial t} + \xi_i(t,x)\frac{\partial}{\partial x_i}.$$

The differential point transformations of the spatial, temporal and the Poisson process variables are as follows

$$d\bar{t} = dt + \epsilon d\tau + O(\epsilon), \tag{2.15}$$

$$d\overline{X_j}(\bar{t}) = dX_j(t) + \epsilon d\xi_j + O(\epsilon) \tag{2.16}$$

and

$$d\overline{N}(\bar{t}) = dN(t) + \frac{\epsilon}{2}\frac{d\tau}{dt}\Big(\lambda dt + dN(t)\Big) + O(\epsilon). \tag{2.17}$$

Using the Itô formula (2.7), we have the spatial and temporal infinitesimals in Itô forms as

$$d\xi_j = \Gamma_{(\xi)_j}(t,X(t))dt + \Gamma^*_{(\xi)_j}(t,X(t))dN(t) \tag{2.18}$$

and

$$d(\tau) = \Gamma_{(\tau)}(t,X(t))dt + \Gamma^*_{(\tau)}(t,X(t))dN(t), \tag{2.19}$$

where $\Gamma_{(\xi)_j}(t,X(t))$, $\Gamma^*_{(\xi)_j}(t,X(t))$, $\Gamma_{(\tau)}(t,X(t))$ and $\Gamma^*_{(\tau)}(t,X(t))$ are the drift and diffusion coefficients of the spatial and temporal infinitesimals, respectively defined using the operators (2.5) and (2.6).

By substitution of the infinitesimal of spatial (2.18) and temporal variables (2.19) in (2.15), (2.16) and (2.17), and also using the Itô multiplication properties *table 1.2* we proceed to get the group transformations of temporal, spatial and jump variables in Itô forms

$$d\bar{t} = dt + \epsilon\Big(\Gamma_{(\tau)}(t,X(t))dt + \Gamma^*_{(\tau)}(t,X(t))\Big)dN(t) + O(\epsilon), \tag{2.20}$$

$$d\overline{X}_j(\bar{t}) = dX_j(t) + \epsilon\Big(\Gamma_{(\xi)_j}(t,X(t))dt + \Gamma^*_{(\xi)_j}(t,X(t)\Big)dN(t) + O(\epsilon) \tag{2.21}$$

and

$$d\overline{N}(\bar{t}) = dN(t) + \frac{\epsilon}{2}\frac{\Gamma_{(\tau)}dt + \Gamma^*_{(\tau)}dN(t)}{dt}\Big(\lambda dt + dN(t)\Big)(t,X(t)) + O(\epsilon). \tag{2.22}$$

Expending the Itô infinitesimal of the jump variable (2.22) by utilising the Poisson process differential multiplication properties we get

$$d\overline{N}(\bar{t}) = dN(t) + \tfrac{\epsilon}{2}\left(\lambda(\Gamma_{(\tau)}(t,X(t))dt + \Gamma^*_{(\tau)}(t,X(t)))dN(t)) + \Gamma_{(\tau)}(t,X(t))dN(t) + \tfrac{\Gamma^*_{(\tau)}(t,X(t))dN(t)}{dt}\right) + O(\epsilon). \tag{2.23}$$

2.2.1 Invariance Form of the Spatial Process

To ensure the recovery of the finite transformations from the infinitesimal transformation, we need to transform $dX_j(t)$ into

$$d\overline{X}_j(\bar{t}) = \overline{f_j}(\bar{t},\overline{X}(\bar{t}))d\bar{t} + \overline{J_j}(\bar{t},\overline{X}(\bar{t}))d\overline{N}(\bar{t}), \tag{2.24}$$

where the transformed drift component $f_j(t,X(t))$ and jump component $J_j(t,X(t))$ using the infinitesimal generator

$$H = \tau(t,x)\frac{\partial}{\partial t} + \xi_i(t,x)\frac{\partial}{\partial x_i},$$

respectively are

$$\begin{aligned}\overline{f_j}(\bar{t},\overline{X}(\bar{t})) &= \Big(f_j + \epsilon H(f_j)\Big)(t,X(t))\\ &= f_j(t,X(t)) + \epsilon\left(\tau\frac{\partial f_j}{\partial t} + \xi_i\frac{\partial f_j}{\partial x_i}\right)(t,X(t))\end{aligned} \tag{2.25}$$

and

$$\begin{aligned}\overline{J_j}(\bar{t},\overline{X}(\bar{t})) &= \Big(J_j + \epsilon H(J_j)\Big)(t,X(t))\\ &= J_j + \epsilon\left(\tau\frac{\partial J_j}{\partial t} + \xi_i\frac{\partial J_j}{\partial x_i}\right)(t,X(t)).\end{aligned} \tag{2.26}$$

2.2.2 Poisson Invariance Properties

We apply the invariance to the moments of the Poisson process to ensure it remains invariant under the group transformations, *viz* the instantaneous mean and variance of the Poisson process which are:

$$E_Q\Big[dN(t)\Big] = \lambda dt \tag{2.27}$$

$$E_Q\Big[dN(t)dN(t)\Big] = \lambda dt. \tag{2.28}$$

The invariance of the instantaneous mean of the transformed Poisson process under new measure $\overline{Q}$ is

$$E_{\overline{Q}}\Big[d\overline{N}(\bar{t})\Big] = \lambda d\bar{t}. \tag{2.29}$$

Expanding (2.29) using the Itô forms of temporal (2.20) and jump group transformations (2.23) we get

$$\Gamma^*_{(\tau)}(t, X(t)) = 0. \tag{2.30}$$

Next, we apply the invariance form to instantaneous variance of the transformed Poisson process measure (2.28) from which using (2.23) we have

$$E_{\overline{Q}}\Big[d\overline{N}(\bar{t})d\overline{N}(\bar{t})\Big] = \lambda d\bar{t}. \tag{2.31}$$

Thus, using (2.30) and the Itô temporal group transformation (2.20) we have derived the following generalised random time change formula

$$\bar{t} = \int^t \Gamma_{(\tau)}(s)ds \tag{2.32}$$

with

$$\Gamma_{(\tau)} = constant = c_1 \tag{2.33}$$

using the probabilistic invariance property of the transformed time index differential, i.e.

$$E_{\overline{Q}}\Big[d\bar{t}(t, N)\Big] = d\bar{t}. \tag{2.34}$$

Finally, we can conclude from (2.30) the temporal infinitesimal $\tau(t,x)$ does not depend on x, therefore $\tau(t,x) = \tau(t)$.

Definition 2.2.1 *The infinitesimal transformations (2.15) and (2.16) i.e.,*

$$\bar{t} = t + \epsilon\tau(t,x) + O(\epsilon), \qquad \overline{X_j}(\bar{t}) = X_j(t) + \epsilon\xi_j(t,x) + O(\epsilon) \tag{2.35}$$

are called Lie symmetry transformations of (2.1) if they leave the Itô stochastic differential equation (2.1)

$$dX(t) = f(t,X(t))dt + J(t,X(t))dN(t) \tag{2.36}$$

and the infinitesimal moments for the differential Poisson process i.e.,

$$E_Q\Big[dN(t,N)\Big] = \lambda dt, \qquad E_Q\Big[dN(t,N)dN(t,N)\Big] = \lambda dt \tag{2.37}$$

and $E_Q\Big[dt\Big] = dt$ invariant. Where $\lambda > 0$ and ϵ are the jump intensity and group parameter respectively.

2.3 Derivation of the Determining Equations

In this section, we will derive the determining equations for the admitted symmetries of (2.1).

The intention is to transform $dX_j(t)$ into

$$d\overline{X_j}(\bar{t}) = \overline{f_j}(\bar{t},\overline{X}(\bar{t}))d\bar{t} + \overline{J_j}(\bar{t},\overline{X}(\bar{t}))d\overline{N}(\bar{t}). \tag{2.38}$$

Substituting the transformed drift coefficient (2.25), Poisson vector coefficients (2.26), Itô forms of temporal (2.20) and Poisson group transformation (2.23) into (2.38) we get

$$\begin{aligned} d\overline{X_j}(\bar{t}) = dX_j(t) + \epsilon&\left(f_j\Gamma_{(\tau)}(t,X(t)) + \frac{\lambda J_j}{2}\Gamma_{(\tau)}(t,X(t)) + H(f_j)\right)dt \\ &+ \left(\Gamma^*_{(\tau)}(t,X(t)) + \frac{J_j}{2}\Gamma_{(\tau)}(t,X(t)) + H(J_j)\right)dN(t). \end{aligned} \tag{2.39}$$

Therefore, by comparing transformed stochastic differential equation (2.39) and the Itô form of the spatial group transformation (2.21) we have the following determining equations

$$\left(f_j\Gamma_{(\tau)} + \frac{\lambda J_j}{2}\Gamma_{(\tau)} + H(f_j) - \Gamma_{(\xi_j)}\right)(t,X(t)) = 0 \tag{2.40}$$

and

$$\left(\frac{J_j}{2}\Gamma_{(\tau)} + H(J_j) - \Gamma^*_{(\xi_j)}\right)(t, X(t)) = 0. \tag{2.41}$$

The invariance of the instantaneous mean of the transformed differential Poisson process (2.29) gives additional conditions i.e., from (2.30) we get

$$\Gamma^*_{(\tau)}(t, X(t)) = 0. \tag{2.42}$$

Equation (2.40) can be interpreted using the definition of first prolongation of an infinitesimal generator for non-stochastic ordinary differential equations as follows;

$$H^{[1]} = H + \eta_i^{[1]}\frac{\partial}{\partial \dot{x}_i}, \tag{2.43}$$

where

$$\dot{x}_i = \frac{dx_i}{dt} = D_t x_i \tag{2.44}$$

and

$$\begin{aligned} \eta_i^{[1]} &= D_t(\xi_i) - \dot{x}_i D_t(\tau) \\ &= \frac{\partial \xi_i}{\partial t} + \dot{x}_i\frac{\partial \xi_i}{\partial x} - \dot{x}_i\left(\frac{\partial \tau}{\partial t} + \dot{x}_i\frac{\partial \tau}{\partial x}\right) \end{aligned} \tag{2.45}$$

with total time derivative D_t defined as

$$D_t = \frac{\partial}{\partial t} + \dot{x}_i\frac{\partial}{\partial x} + \ddot{x}_i\frac{\partial}{\partial \dot{x}_i} + \dots \tag{2.46}$$

using the definition of first prolongation on $(\dot{x}_i - f_i)$ at $\dot{x}_i = f_i$, can be expressed as

$$H^{[1]}(\dot{x}_i - f_i)|_{\dot{x}_i = f_i} = \eta_i^{[1]} - H(f_i). \tag{2.47}$$

Using (2.45) and (2.47) equation (2.41) can be written as

$$H^{[1]}(\dot{x}_i - f_i)|_{\dot{x}_i = f_i} - \frac{\lambda J_i}{2}\left(\frac{\partial \tau}{\partial t} + f_i\frac{\partial \tau}{\partial x}\right) = 0, \tag{2.48}$$

where the operators $\Gamma_{(\tau)}(t, X(t))$, $\Gamma^*_{(\tau)}(t, X(t))$ are defined in (2.5), (2.6) respectively.

Remark 2.3.1 *The extra condition obtained from the invariance of the instantaneous mean of the transformed differential Poisson process (2.29) forces the temporal infinitesimal $\tau(t,x)$ to be a function of the time variable only. This implies that, we are now dealing with a fiber-preserving infinitesimal generator i.e.,*

$$H = \tau(t)\frac{\partial}{\partial t} + \xi_i(t,x)\frac{\partial}{\partial x_i}. \tag{2.49}$$

2.3.1 Unal's Extra Condition

Unal's G. in [9] commented that the Itô multiplication properties for the transformed processes has to be satisfied i.e.,

$$d\overline{N}(\bar{t})d\overline{N}(\bar{t}) = d\overline{N}(\bar{t}), \quad d\overline{W}_l(\bar{t})d\overline{N}(t), \quad d\overline{N}(\bar{t})d\bar{t} = 0, \tag{2.50}$$

This led to an additional condition

$$\Gamma_{(\tau)}(t,X(t)) = 0. \tag{2.51}$$

2.4 Applications

In this section, we are going to apply the derived determining equations of Poisson Itô stochastic differential equations obtained in the previous section to some Poisson process models to show how the determining equations can be used to find the admitted Lie point symmetries of each model.

Example 2.4.1 *Consider the Poisson SDE, linear in the state process $X(t)$, with constant coefficients,*

$$dX(t) = X(t)\Big(u_0(t)dt + v_0(t)dN(t)\Big) \tag{2.52}$$

with initial condition $X(t_0) = x_0 > 0$, $u_0(t) = 2$ is the drift or deterministic coefficient and $v_0(t) = 1$ is the jump amplitude coefficient of the jump term, with jump rate $\lambda = \lambda_0$.

Using the determining equations (2.40) and (2.41) respectively we have

$$\left(2x\Gamma_{\tau(t)} + \lambda_0 x\frac{\Gamma_{\tau(t)}}{2} + 2\xi(t,x) - \Gamma_{(\xi)}\right)(t,X(t)) = 0 \tag{2.53}$$

$$2x\frac{\partial\tau(t)}{\partial t}+\frac{\lambda_0 x}{2}\frac{\partial\tau(t)}{\partial t}+2\xi(t,x)-\frac{\partial\xi(t,x)}{\partial t}-2x\frac{\partial\xi(t,x)}{\partial x}=0 \tag{2.54}$$

and

$$\left(x\frac{\Gamma_{(\tau)}}{2}+\xi(t,x)-\Gamma^{*}_{(\xi_j)}\right)(t,X(t))=0 \tag{2.55}$$

$$\frac{x}{2}\frac{\partial\tau(t)}{\partial t}+\xi(t,x)-\xi(t,x+x)+\xi(t,x)=0. \tag{2.56}$$

Using (2.30) and (2.33) we get the temporal infinitesimal as

$$\tau(t)=c_1 t+c_2. \tag{2.57}$$

Substituting temporal infinitesimal (2.57) in (2.54) and (2.56) respectively gives

$$\frac{c_1 x(4+\lambda_0)}{2}+2\xi(t,x)-\frac{\partial\xi(t,x)}{\partial t}-2x\frac{\partial\xi(t,x)}{\partial x}=0 \tag{2.58}$$

and

$$\frac{c_1 x}{2}+2\xi(t,x)-\xi(t,2x)=0. \tag{2.59}$$

Differentiating (2.58) with respect to x gives

$$\frac{c_1(4+\lambda_0)}{2}+2\frac{\partial\xi(t,x)}{\partial x}-\frac{\partial^2\xi(t,x)}{\partial x\partial t}-2\frac{\partial\xi(t,x)}{\partial x}-2x\frac{\partial^2\xi(t,x)}{\partial x^2}=0. \tag{2.60}$$

Differentiating (2.59) with respect to x gives

$$\frac{c_1}{2}+2\frac{\partial\xi(t,x)}{\partial x}-2\frac{\partial\xi(t,2x)}{\partial x}=0. \tag{2.61}$$

Differentiating (2.61) with respect to t gives

$$\frac{\partial^2\xi(t,x)}{\partial x\partial t}=\frac{\partial^2\xi(t,2x)}{\partial x\partial t}. \tag{2.62}$$

Equation (2.62) implies

$$\frac{\partial^2\xi(t,x)}{\partial x\partial t}=\frac{\partial^2\xi(t)}{\partial x\partial t}=\frac{df(t)}{dt}. \tag{2.63}$$

Solving the differential equation (2.63) we get

$$\xi(t,x) = f(t)x + g(x). \tag{2.64}$$

By substituting (2.64) into (2.60) we get

$$\frac{c_1(4+\lambda_0)}{2} = \frac{df(t)}{dt} + 2x\frac{d^2g(x)}{dx^2}. \tag{2.65}$$

When differentiating (2.65) with respect to time we get

$$\frac{d^2f(t)}{dt^2} = 0. \tag{2.66}$$

Solving the ordinary differential equation (2.66) implies $f(t)$ is linear in t i.e.,

$$f(t) = c_3t + c_4. \tag{2.67}$$

After substituting (2.67) into (2.64) we arrive at spatial infinitesimal

$$\xi(t,x) = (c_3t + c_4)x + g(x). \tag{2.68}$$

Substituting (2.68) into (2.65) results in

$$\frac{c_1(4+\lambda_0)}{2} = c_3 + 2x\frac{d^2g(x)}{dx^2}, \tag{2.69}$$

which implies that

$$\frac{d^2g(x)}{dx^2} = \frac{\frac{c_1(4+\lambda_0)}{2} - c_3}{2x}. \tag{2.70}$$

Solving the differential equation (2.70) for $g(x)$ finally gives

$$g(x) = \frac{\frac{c_1(4+\lambda_0)}{2} - c_3}{2}\Big(x\ln|x| - x\Big) + c_5x + c_6, \tag{2.71}$$

therefore, using (2.71) the spatial infinitesimal (2.68) can be written as

$$\xi(t,x) = (c_3t + c_4)x + \frac{\frac{c_1(4+\lambda_0)}{2} - c_3}{2}\big(x\ln|x| - x\big) + c_5x + c_6. \tag{2.72}$$

However, substituting (2.72) in (2.59) we have

$$\frac{c_1 x}{2} + 2\left((c_3 t + c_4)x + \frac{\frac{c_1(4+\lambda_0)}{2} - c_3}{2}\left(x\ln|x| - x\right) + c_5 x + c_6 \right) = 2(c_3 t + c_4)x + \frac{\frac{c_1(4+\lambda_0)}{2} - c_3}{2}\left(2x\ln|2x| - 2x\right) + 2c_5 x + c_6. \tag{2.73}$$

which can be simplified to get

$$\frac{c_1 x}{2} + c_6 = \frac{\frac{c_1(4+\lambda_0)}{2} - c_3}{2}\left(x\ln|4|\right). \tag{2.74}$$

Further comparison of the coefficients of powers of x in (2.74), gives

- $x : c_3 = c_1\left(\frac{(4+\lambda_0)}{2} - \frac{1}{\ln|4|}\right)$ *and*
- $x^0 : c_6 = 0.$

Thus, the spatial infinitesimal (2.72) finally becomes

$$\xi(t,x) = c_1\left(\left(\frac{\ln|4|(4+\lambda_0) - 2}{\ln|16|}\right)tx + \frac{(x\ln|x| - x)}{\ln|16|}\right) + c_4 x + c_5 x. \tag{2.75}$$

So we have three symmetry generators corresponding to the infinitesimals

$$H_1 = t\frac{\partial}{\partial t} + \left(\left(\frac{\ln|4|(4+\lambda_0) - 2}{\ln|16|}\right)tx + \frac{(x\ln|x| - x)}{\ln|16|}\right)\frac{\partial}{\partial x}, \tag{2.76}$$

$$H_2 = \frac{\partial}{\partial t}, \qquad H_3 = 2x\frac{\partial}{\partial x}. \tag{2.77}$$

After finding the generators (2.76) and (2.77) they give the following Lie bracket relations in Table 2.1.

TABLE 2.1: Commutator Table for the Lie Algebra Generators (2.76) and (2.77)

$[H_i, H_j]$	H_1	H_2	H_3
H_1	0	$-H_4$	$-\frac{H_3}{\ln\lvert 16\rvert}$
H_2	H_4	0	0
H_3	$\frac{H_3}{\ln\lvert 16\rvert}$	0	0

The commutator Table 2.1 shows that the infinitesimals generator (2.76) and (2.77) is closed under Lie bracket relations and hence is Lie algebra, where H_4 is linear combination of H_3 and H_2 given as $H_4 = \alpha H_3 + H_2$ with $\alpha = \frac{\ln|16|-1+\ln|2|\lambda_0}{\ln|16|}$.

Applying the extra condition (2.51) obtain by making sure the transformed Poisson variables satisfy Watanabe characterisation of Poisson, we get from (2.33)

$$\Gamma_{(\tau)} = c_1 = 0. \tag{2.78}$$

Which reduced the symmetry infinitesimals to

$$H_2 = \frac{\partial}{\partial t}, \qquad H_3 = 2x\frac{\partial}{\partial x}. \tag{2.79}$$

Example 2.4.2 *Consider a Poisson driven stochastic differential equation*

$$dX = -kt^2dt + bdN(t) \qquad with \qquad b \neq 0 \tag{2.80}$$

and initial condition $X(0) = x_0$.

Using the determining equations (2.40) and (2.41) we get

$$-kt^2\left(\frac{\partial\tau}{\partial t} - kt^2\frac{\partial\tau}{\partial x}\right) + \frac{b\lambda}{2}\left(\frac{\partial\tau}{\partial t} - kt^2\frac{\partial\tau}{\partial x}\right) - 2kt\tau = \frac{\partial\xi}{\partial t} - kt^2\frac{\partial\xi}{\partial x} \tag{2.81}$$

$$\frac{b}{2}\left(\frac{\partial\tau}{\partial t} - kt^2\frac{\partial\tau}{\partial x}\right) = \xi(t, x+b) - \xi(t, x). \tag{2.82}$$

Using equation (2.30) and (2.33) we get the temporal infinitesimal as

$$\tau(t) = c_1 t + c_2. \tag{2.83}$$

Using temporal infinitesimal (2.83) in (2.81) and (2.82) we respectively have

$$c_1\left(\frac{b\lambda}{2} - kt^2\right) - 2kt(c_1 t + c_2) = \frac{\partial \xi(t,x)}{\partial t} - kt^2 \frac{\partial \xi(t,x)}{\partial x} \tag{2.84}$$

and

$$\frac{c_1 b}{2} = \xi(t, x+b) - \xi(t,x). \tag{2.85}$$

Differentiating (2.84) and (2.85) with respect to x respectively gives

$$\frac{\partial^2 \xi(t,x)}{\partial x \partial t} - kt^2 \frac{\partial^2 \xi(t,x)}{\partial x^2} = 0 \tag{2.86}$$

and

$$\frac{\partial \xi(t, x+b)}{\partial x} = \frac{\partial \xi(t,x)}{\partial x}. \tag{2.87}$$

Equation (2.87) implies

$$\frac{\partial \xi(t,x)}{\partial x} = \frac{\partial \xi(t)}{\partial x} = f(t). \tag{2.88}$$

Differentiating (2.88) with respect to x gives

$$\frac{\partial^2 \xi(t,x)}{\partial x^2} = 0, \tag{2.89}$$

solving the differential equation (2.89) we have

$$\xi(x,t) = f(t)x + g(t). \tag{2.90}$$

Substituting (2.90) into (2.86)

$$\frac{df(t)}{dt} = 0. \tag{2.91}$$

Equation (2.91) implies $f(t)$ is constant i.e.,

$$f(t) = c_3, \tag{2.92}$$

therefore, using (2.92) and (2.90) we have

$$\xi(x,t) = c_3 x + g(t). \tag{2.93}$$

Substituting (2.93) in (2.85) gives this relation

$$c_1 = 2c_3. \tag{2.94}$$

Using (2.93) and (2.94), equation (2.84) gives

$$c_1\left(\frac{b\lambda}{2} - 3kt^2\right) - 2ktc_2 = \frac{dg(t)}{dt} - \frac{kt^2 c_1}{2}. \tag{2.95}$$

Solving the differential equation (2.95) gives

$$g(t) = c_1\left(\frac{b\lambda t}{2} - \frac{5kt^3}{6}\right) - kt^2 c_2 + c_4. \tag{2.96}$$

Therefore, substituting (2.96) into (2.93) the spatial infinitesimal finally becomes

$$\xi(x,t) = c_1\left(\frac{b\lambda t}{2} - \frac{5kt^3}{6} + \frac{x}{2}\right) - kt^2 c_2 + c_4. \tag{2.97}$$

Finally the jump-diffusion model admitted three dimensional symmetry infinitesimal generators;

$$H_1 = t\frac{\partial}{\partial t} + \left(\frac{b\lambda t}{2} - \frac{5kt^3}{6} + \frac{x}{2}\right)\frac{\partial}{\partial x}, \qquad H_2 = \frac{\partial}{\partial t} - kt^2\frac{\partial}{\partial x}, \qquad H_3 = \frac{\partial}{\partial x}. \tag{2.98}$$

With corresponding Lie bracket relations of the generators (2.98) given in Table 2.2 as

TABLE 2.2: Commutator Table for the Lie Algebra Generators (2.98)

$[H_i, H_j]$	H_1	H_2	H_3
H_1	0	$-H_4$	$-\frac{H_3}{2}$
H_2	H_4	0	0
H_3	$\frac{H_3}{2}$	0	0

The Lie bracket relations in Table 2.2 above show that the infinitesimal generator (2.98) satisfied Lie commutative relation properties and hence forms a Lie algebras, where $H_4 = H_2 - \frac{b\lambda}{2}H_3$ *is the linear combination of* H_2 *and* H_3.

Applying the extra condition (2.51) obtain by making sure the transformed Poisson variables satisfy Watanabe characterisation of Poisson, we get from (2.33)

$$\Gamma_{(\tau)} = c_1 = 0. \tag{2.99}$$

Which reduced the symmetry infinitesimals to

$$H_2 = \frac{\partial}{\partial t} - kt^2\frac{\partial}{\partial x}, \qquad H_3 = \frac{\partial}{\partial x}. \tag{2.100}$$

Example 2.4.3 *Consider a Poisson stochastic differential equation*

$$dX(t) = a\,dt + dN(t), \qquad a \in \Re \neq 0 \tag{2.101}$$

given initial condition $X(0) = x_0$.

Using the determining equations (2.40) and (2.41) respectively we have

$$\left(a\Gamma_{\tau(t)} + \lambda\frac{\Gamma_{\tau(t)}}{2} - \Gamma_{(\xi)}\right)(t, X(t)) = 0 \tag{2.102}$$

$$a\frac{\partial\tau(t)}{\partial t} + \frac{\lambda}{2}\frac{\partial\tau(t)}{\partial t} - \frac{\partial\xi(t,x)}{\partial t} - a\frac{\partial\xi(t,x)}{\partial x} = 0 \tag{2.103}$$

and

$$\left(\frac{\Gamma_{(\tau)}}{2} - \Gamma^*_{(\xi_j)}\right)(t, X(t)) = 0 \tag{2.104}$$

$$\frac{\partial\tau(t)}{\partial t} - \xi(t, x+1) + \xi(t,x) = 0. \tag{2.105}$$

Equation (2.30) and (2.33) gives the temporal infinitesimal

$$\tau(t) = c_1 t + c_2. \tag{2.106}$$

Substituting temporal infinitesimal (2.106) into equations (2.103) and (2.105) respectively gives

$$c_1\left(a+\frac{\lambda}{2}\right)=\frac{\partial\xi(t,x)}{\partial t}+a\frac{\partial\xi(t,x)}{\partial x} \tag{2.107}$$

and

$$c_1=\xi(t,x+1)-\xi(t,x). \tag{2.108}$$

Differentiating (2.107) and (2.108) with respect to x respectively gives

$$\frac{\partial^2\xi(t,x)}{\partial t\partial x}+a\frac{\partial^2\xi(t,x)}{\partial x^2}=0 \tag{2.109}$$

and

$$\frac{\partial\xi(t,x+1)}{\partial t}=\frac{\partial\xi(t,x)}{\partial x}=\frac{\partial\xi(t)}{\partial x}. \tag{2.110}$$

From (2.110)

$$\frac{\partial\xi(t,x)}{\partial x}=f(t). \tag{2.111}$$

Differentiating (2.111) with respect to x gives

$$\frac{\partial^2\xi(t,x)}{\partial x^2}=0. \tag{2.112}$$

Equation (2.111) and (2.112) implies

$$\xi(t,x)=f(t)x+g(t). \tag{2.113}$$

Substituting (2.113) into (2.109)

$$\frac{df(t)}{dt}=0. \tag{2.114}$$

Equation (2.114) implies $f(t)$ is constant i.e.,

$$f(t)=c_3. \tag{2.115}$$

Therefore, equation (2.113) using (2.115) becomes

$$\xi(t,x)=c_3x+g(t). \tag{2.116}$$

Using (2.116) and (2.108) reveals the relation

$$c_1 = c_3. \tag{2.117}$$

Equation (2.116) and (2.117) implies

$$\xi(t,x) = c_1 x + g(t). \tag{2.118}$$

Substituting (2.118) into (2.103) gives

$$\frac{dg(t)}{dt} = c_1 \frac{\lambda}{2}. \tag{2.119}$$

Solving the differential equation (2.119) and substituting the result in (2.118) gives the required spatial infinitesimal as

$$\xi(t,x) = c_1 \left(x + \frac{\lambda t}{2} \right) + c_4. \tag{2.120}$$

So we have three symmetry generators corresponding to the infinitesimals as

$$H_1 = t\frac{\partial}{\partial t} + \left(x + \frac{\lambda t}{2} \right) \frac{\partial}{\partial x} \qquad H_2 = \frac{\partial}{\partial t} \qquad H_3 = \frac{\partial}{\partial x}. \tag{2.121}$$

The commutation relations of the generators (2.121) given in Table 2.3 as

TABLE 2.3: Commutator Table for the Lie Algebra Generators (2.121)

$[H_i, H_j]$	H_1	H_2	H_3
H_1	0	$-(H_2 + \frac{\lambda}{2} H_3)$	$-H_3$
H_2	$H_2 + \frac{\lambda}{2} H_3$	0	0
H_3	H_3	0	0

The commutator Table 2.3 shows that the infinitesimals generator (2.121) is closed under Lie bracket relations and hence is a Lie algebra.

Applying the extra condition (2.51) obtain by making sure the transformed Poisson variables satisfy Watanabe characterisation of Poisson, we get from (2.33)

$$\Gamma_{(\tau)} = c_1 = 0. \tag{2.122}$$

Which reduced the symmetry to

$$H_2 = \frac{\partial}{\partial t} \qquad H_3 = \frac{\partial}{\partial x}. \tag{2.123}$$

Example 2.4.4 *Consider a Poisson noise stochastic differential equation*

$$dX(t) = \alpha dN(t), \qquad \alpha \in \Re \neq 0 \tag{2.124}$$

with initial condition $X(0) = x_0$.

Using the determining equations (2.40) and (2.41) respectively we have

$$\left(\lambda \frac{\Gamma_{\tau(t)}}{2} - \Gamma_{(\xi)}\right)(t, X(t)) = 0, \tag{2.125}$$

$$\frac{\lambda}{2}\frac{\partial \tau(t)}{\partial t} - \frac{\partial \xi(t,x)}{\partial t} = 0 \tag{2.126}$$

and

$$\left(\frac{\Gamma_{(\tau)}}{2} - \Gamma^{*}_{(\xi_j)}\right)(t, X(t)) = 0 \tag{2.127}$$

$$\frac{\partial \tau(t)}{\partial t} - \xi(t, x+\alpha) + \xi(t,x) = 0. \tag{2.128}$$

Similarly from (2.83), we get the temporal infinitesimal as

$$\tau(t) = c_1 t + c_2. \tag{2.129}$$

Substituting temporal infinitesimal (2.129) into (2.126) and (2.128) respectively gives

$$\frac{\lambda}{2} c_1 = \frac{\partial \xi(t,x)}{\partial t} \tag{2.130}$$

and

$$c_1 = \xi(t, x+\alpha) - \xi(t, x). \tag{2.131}$$

Differentiating (2.130) and (2.131) with respect to x respectively gives

$$\frac{\partial^2 \xi(t,x)}{\partial x \partial t} = 0 \tag{2.132}$$

and

$$\frac{\partial \xi(t, x+\alpha)}{\partial x} = \frac{\partial \xi(t,x)}{\partial x} = \frac{\partial \xi(t)}{\partial x}. \tag{2.133}$$

Equation (2.133) implies

$$\frac{\partial \xi(t,x)}{\partial x} = f(t). \tag{2.134}$$

Differentiating (2.134) with respect to x gives

$$\frac{\partial^2 \xi(t,x)}{\partial x^2} = 0. \tag{2.135}$$

Solving the differential equation (2.135) with (2.134) in mind we have

$$\xi(t,x) = f(t)x + g(t). \tag{2.136}$$

Substituting (2.136) into (2.132) gives

$$f(t) = c_3. \tag{2.137}$$

Equation (2.137) into (2.136) gives

$$\xi(t,x) = c_3 x + g(t). \tag{2.138}$$

Using (2.138) in (2.131) gives the following relation

$$c_1 = \alpha c_3. \tag{2.139}$$

Substituting (2.138) in (2.130) using (2.139) gives

$$g(t) = \frac{c_1 \lambda t}{2} + c_4. \tag{2.140}$$

Finally, using (2.140), (2.138) and (2.139) gives the spatial infinitesimal

$$\xi(t,x) = c_1\left(\frac{x}{\alpha} + \frac{\lambda t}{2}\right) + c_4, \qquad \alpha \neq 0. \tag{2.141}$$

So we have three dimensional symmetry generators as

$$H_1 = t\frac{\partial}{\partial t} + \left(\frac{x}{\alpha} + \frac{\lambda t}{2}\right)\frac{\partial}{\partial x} \qquad H_2 = \frac{\partial}{\partial t} \qquad H_3 = \frac{\partial}{\partial x} \qquad \alpha \neq 0. \tag{2.142}$$

With corresponding Lie bracket relations given in Table 2.4.

TABLE 2.4: Commutator Table for the Lie Algebra Generators (2.142)

$[H_i,H_j]$	H_1	H_2	H_3
H_1	0	$-(H_2 + \frac{\lambda}{2}H_3)$	$-\frac{H_3}{\alpha}$
H_2	$H_2 + \frac{\lambda}{2}H_3$	0	0
H_3	$\frac{H_3}{\alpha}$	0	0

The commutator Table 2.4 shows that the infinitesimals generators (2.142) is closed under Lie bracket relations and hence is a Lie algebra.

Applying the extra condition (2.51) obtain by making sure the transformed Poisson variables satisfy Watanabe characterisation of Poisson, we get from (2.33)

$$\Gamma_{(\tau)} = c_1 = 0. \tag{2.143}$$

Which reduced the symmetry to only

$$H_2 = \frac{\partial}{\partial t} \qquad H_3 = \frac{\partial}{\partial x} \qquad \alpha \neq 0. \tag{2.144}$$

2.5 Conclusion

Lie symmetry analysis for Itô Stochastic differential equations driven by the Poisson process was carried out, infinitesimals of the Poisson process $dN(t)$ where derived using the moments invariance properties of the process. Determining equations where derived and found to be deterministic even thought they describe stochastic differential equation. Examples are given to show how the determining equations can be used to find the symmetries, symmetries admitted by (2.1) are found to be fiber preserving symmetries.

Fredericks E. and Mahomed F. M. [8] proved that in the case of Wiener driven stochastic differential equation, the invariance of the moments of the process is sufficient with no recourse to the Itô multiplication properties of the transformed variables. However, it was proved in this chapter, is not the case for Poisson driven stochastic equations i.e., the Itô multiplication properties of the transformed variables must be satisfied. This lead to the reduction of the symmetry infinitesimals by at least one dimension.

Finally, the Lie bracket relations was obtained which shows that all the infinitesimals generators found are closed under the Lie bracket and hence they form a Lie algebra. Classification of the given examples is presented in *Table 2.5*.

TABLE 2.5: Lie Group Classification Chapter 2

Group Dimension	Basis Operators	Equations
2	$H_2 = \frac{\partial}{\partial t}, \qquad H_3 = 2x\frac{\partial}{\partial x}$	$dX(t) = X(t)(2dt + dN(t))$
2	$H_2 = \frac{\partial}{\partial t} - kt^2\frac{\partial}{\partial x}, \qquad H_3 = \frac{\partial}{\partial x}$	$dX = -kt^2dt + bdN(t)$
2	$H_2 = \frac{\partial}{\partial t} \qquad H_3 = \frac{\partial}{\partial x}$	$dX(t) = adt + dN(t) \qquad a \neq 0$
3	$H_2 = \frac{\partial}{\partial t}, \qquad H_3 = \frac{\partial}{\partial x}$	$dX(t) = \alpha dN(t) \qquad \alpha \neq 0$

Chapter 3

N- symmetry of Itô Stochastic Differential Equations Driven by the Poisson Process

Lie point symmetry transformation of the class of Itô stochastic differential equation driven by Poisson Processes was successfully carried out [2]. We consider symmetries involving not only spatial and time variables (t,x), but also the Poisson process term $N(t)$. The result was achieved by following the invariance methodology of Lie point transformation and the use of Itô formula for Poisson stochastic differential equation without enforcing any conditions to the momenta of the stochastic process.

3.1 Introduction

Lie symmetry methods for the class of Itô stochastic differential equations driven by Poisson processes of the form

$$dX_i(t) = f_i(t,X(t))dt + J_i(t,X(t))dN(t), \tag{3.1}$$

were discussed by extending the symmetry generator to include the infinitesimal transformations of Poisson processes $N(t)$. We now consider infinitesimal transformations involving not only the spatial and time variables (t,x), but also the Poisson diffusion processes N(t). We named this symmetry generator "N(t)-symmetries" of (3.1), where $f_i(t,x)$ and $J_i(t,x)$ are the $n \times 1$ dimensional drift vector coefficient and Poisson diffusion coefficient respectively. Here $dN(t)$ is the infinitesimal increment of the Poisson process. We assume that the coefficient functions $f(t,x)$ and $J(t,x)$ satisfy the Ikeda Watanabe conditions for the existence and uniqueness of the solution of (3.1) [4, 9].

Symmetries of (3.1) are analysed by considering the infinitesimal generator

$$H = \tau(t,x,N)\frac{\partial}{\partial t} + \xi_i(t,x)\frac{\partial}{\partial x_i} + \phi_i(t,x,N)\frac{\partial}{\partial N}. \tag{3.2}$$

The determining equations for Itô stochastic differential equations driven by Poisson processes (3.1) are successfully derived in an Itô calculus context, and they are found to be deterministic even though they represent a stochastic process.

Finally, using Itô multiplication properties of stochastic differential equations driven by Poisson diffusion processes in *Table 1.2* [4, 5] and application of infinitesimal transformations the determining equations for the Poisson process stochastic differential equation (3.1) are derived. The following result was obtained.

Theorem 3.1.1 *The Itô stochastic differential equation driven by Poisson processes*

$$dX_i(t) = f_i(t,X_i(t))dt + J_i(t,X_i(t))dN(t), \tag{3.3}$$

where the coefficient functions $f_i(t,X(t))$ and $J_i(t,X(t))$ are $n\times 1$ dimensional drift vector coefficient and Poisson diffusion coefficients, with infinitesimal generator

$$H = \tau(t,x,N)\frac{\partial}{\partial t} + \xi_i(t,x)\frac{\partial}{\partial x_i} + \phi_i(t,x,N)\frac{\partial}{\partial N}. \tag{3.4}$$

has admitted the following determining equations;

$$\Big(f_j\Gamma(\tau) + \lambda J_j[\Gamma(\tau) - \Gamma^*(\phi)_j] + H(f_j) - \Gamma(\xi)_j\Big)(t,X(t)) = 0, \tag{3.5}$$

$$\Big(J_j\Gamma^*(\phi)_j + H(J_j) - \Gamma^*(\xi)_j\Big)(t,X(t)) = 0 \tag{3.6}$$

and

$$\Gamma(\phi)_j + \lambda\Gamma^*(\phi)_j = \lambda\Gamma(\tau) \tag{3.7}$$

$$\Gamma^*(\phi)_j = \frac{\Gamma(\tau)}{2} \tag{3.8}$$

with additional conditions,

$$\Gamma^*(\tau) = 0 \quad and \quad \Gamma(\tau) = c_1. \tag{3.9}$$

The operators $\Gamma(t, X(t,N))$ and $\Gamma^*(t, X(t,N))$ are defined as in (2.5) and (2.6). The infinitesimals $\tau(t,x,N)$, $\xi(t,x,N)$ and $\phi(t,x,N)$ are called the admitted symmetries of (3.1) if and only if they satisfied the determining equations (3.5)-(3.9).

3.2 Lie Group Transformations

Consider a one parameter group of transformation with temporal variable t, the spatial variable x and Poisson process variable N respectively,

$$\bar{t} = \theta_1(t,x,N\epsilon) \qquad \bar{x} = \theta_2(t,x,\epsilon) \qquad \overline{N} = \theta_3(t,x,N,\epsilon) \tag{3.10}$$

with the infinitesimals

$$\frac{\partial \theta_1}{\partial \epsilon} = \tau(\theta_1,\theta_2,\theta_3), \qquad \frac{\partial \theta_2}{\partial \epsilon} = \xi(\theta_1,\theta_2,\theta_3), \qquad \frac{\partial \theta_3}{\partial \epsilon} = \phi(\theta_1,\theta_2,\theta_3)$$

satisfying the following initial conditions at $\epsilon = 0$

$$\bar{t}\Big|_{\epsilon=0} = t \qquad \overline{X}(\bar{t})\Big|_{\epsilon=0} = X(t) \qquad \overline{N}(\bar{t})\Big|_{\epsilon=0} = N(t).$$

Where ϵ is the parameter of the group, hence the corresponding generator of the Lie algebra is of the form of

$$H = \tau(t,x,N)\frac{\partial}{\partial t} + \xi_i(t,x)\frac{\partial}{\partial x_i} + \phi_i(t,x,N)\frac{\partial}{\partial N}. \tag{3.11}$$

The group transformations can be expressed in term of of the symmetry operator (3.11) as

$$\bar{t} = e^{\epsilon H}(t) \tag{3.12}$$

$$\bar{x} = e^{\epsilon H}(x) \tag{3.13}$$

and

$$\overline{N} = e^{\epsilon H}(N). \tag{3.14}$$

The Itô stochastic differential equation with Poisson related to the group of transformations are

$$d\bar{t} = \Gamma e^{\epsilon H}(t)dt + \Gamma^* e^{\epsilon H}(t)dN(t), \tag{3.15}$$

$$d\overline{N}(\bar{t}) = \Gamma e^{\epsilon H}(N)dt + \Gamma^* e^{\epsilon H}(N)dN(t), \tag{3.16}$$

and

$$d\overline{X}(\bar{t}) = \Gamma e^{\epsilon H}(x)dt + \Gamma^* e^{\epsilon H}(x)dN(t). \tag{3.17}$$

Using the Itô formula (2.7), we can also write the spatial, temporal and Poisson infinitesimals as Itô processes respectively as,

$$d\tau = \Gamma(\tau)dt + \Gamma^*(\tau)dN(t), \tag{3.18}$$

$$d\xi_j = \Gamma(\xi)_j dt + \Gamma^*(\xi)_j dN(t) \tag{3.19}$$

and

$$d\phi = \Gamma(\phi)_j dt + \Gamma^*(\phi)_j dN(t). \tag{3.20}$$

The operators $\Gamma(t,x)$ and $\Gamma^*(t,x)$ were been defined in (2.5) and (2.6) respectively, these operators are in fact instantaneous and standard deviation of the temporal, spatial and Poisson infinitesimals τ, ξ and ϕ respectively.

3.2.1 Invariance Form of the Spatial Process

To ensure the recovery of the finite transformations from the infinitesimal transformation, we need to transform $dX(t)$ into

$$d\overline{X_j}(\bar{t}) = \overline{f_j}(\bar{t},\overline{X}(\bar{t}))d\bar{t} + \overline{J_j}(\bar{t},\overline{X}(\bar{t}))d\overline{N}(\bar{t}) \tag{3.21}$$

where the transformed drift component using our generator (3.11) is

$$\overline{f_j}(\bar{t},\overline{X}(\bar{t},\overline{N})) = e^{\epsilon H}(f_j) \tag{3.22}$$

and the transformed Poisson amplitude is

$$\overline{J_j}(\bar{t},\overline{X}(\bar{t},\overline{N})) = e^{\epsilon H}(J_j). \tag{3.23}$$

Expanding the drift component (3.22) we have

$$\overline{f_j}(\bar{t},\overline{X}(\bar{t}))d\bar{t} = (f_j(t,X) + \epsilon\Big(\Gamma H(t) + H\Big)f_j(t,X(t))dt. \tag{3.24}$$

The following condition is necessary to ensure the recovery of the finite transformations from the infinitesimal transformation

$$e^{\epsilon H(t)}(t,X) = \Gamma(e^{\epsilon H(t)}(t,X)). \tag{3.25}$$

Condition (3.25) is to ensure that the higher order terms depend on the first term associated with $O(\epsilon)$. All the ordered terms contribute in the construction of the finite transformations, the zeroth and the first order terms contribute towards the construction of the infinitesimal transformations. This also forces the instantaneous drift coefficient of the temporal infinitesimal to be a constant. i.e.,

$$\Gamma(\tau) = c_1. \tag{3.26}$$

Expanding the Poisson amplitude (3.23) gives

$$J_j(\bar{t},\overline{X}(\bar{t},\overline{N}))d\overline{N}(\bar{t}) = J_j f_j(t,X)dN + \lambda\epsilon\Big(J_j\Gamma H(N) - J_j\Gamma^* H(N)\Big)dt + \epsilon\Big(J_j\Gamma^* H(t) + H(J_j)\Big)dN(t). \tag{3.27}$$

3.2.2 Poisson Invariance Properties

Before deriving the determining equations, we apply the invariance to the moments of the Poisson process to make sure it remain invariant under the group transformations, *viz* the instantaneous mean and variance of the Poisson processes which are:

$$E_Q\Big[dN(t)\Big] = \lambda dt \tag{3.28}$$

and

$$E_Q\Big[dN(t)dN(t)\Big] = \lambda dt. \tag{3.29}$$

The invariance of the instantaneous mean of the transformed Poisson process under new measure $\overline{Q}$ is

$$E_Q\left[d\overline{N}(\bar{t})\right] = \lambda d\bar{t}, \tag{3.30}$$

using (3.15) and (3.16) equation (3.30) gives

$$E_Q\left[\Gamma e^{\epsilon H}(N)dt + \Gamma^* e^{\epsilon H}(N)dN(t)\right] = \lambda\left(\Gamma e^{\epsilon H}(t)dt + \Gamma^* e^{\epsilon H}(t)dN(t)\right). \tag{3.31}$$

Now, expending (3.31) using (3.28) gives

$$\left(\Gamma e^{\epsilon H}(N) + \lambda\Gamma^* e^{\epsilon H}(N) - \lambda(\Gamma e^{\epsilon H}(t))\right)dt = \Gamma^* e^{\epsilon H}(t)\lambda dN(t). \tag{3.32}$$

Next, we apply the invariant form to instantaneous variance of the transformed Poisson process measure i.e.,

$$E_{\overline{Q}}\left[d\overline{N}(\bar{t})d\overline{N}(\bar{t})\right] = \lambda d\bar{t} \tag{3.33}$$

which using (3.16) gives

$$E_{\overline{Q}}\left[\Gamma^* e^{\epsilon H}(N)\Gamma^* e^{\epsilon H}(N)dN(t)dN(t)\right] = \lambda d\bar{t}, \tag{3.34}$$

finally, from (3.15) and (3.16), equation (3.34) gives the following differential relation

$$\Gamma^* e^{\epsilon H}(N)\Gamma^* e^{\epsilon H}(N)dt = \Gamma e^{\epsilon H}(t)dt + \Gamma^* e^{\epsilon H}(t)dN(t). \tag{3.35}$$

Comparing the jump and Riemann integrals from (3.35) we have the following relations

$$\Gamma^* e^{\epsilon H}(t) = 0 \tag{3.36}$$

and

$$\Gamma^* e^{\epsilon H}(N)\Gamma^* e^{\epsilon H}(N) = \Gamma e^{\epsilon H}(t). \tag{3.37}$$

Using (3.36), (3.32) reduced to

$$\Gamma e^{\epsilon H}(N) + \lambda\Gamma^* e^{\epsilon H}(N) = \lambda\Gamma e^{\epsilon H}(t). \tag{3.38}$$

Thus, we have successfully derived the generalised random time change formula

$$\bar{t} = \int^{t} \Gamma e^{\epsilon H}(s)ds, \tag{3.39}$$

with

$$\Gamma e^{\epsilon H}(t) = constant = c_1 \tag{3.40}$$

using the probabilistic invariance property of the transformed time index differential, i.e.

$$E_{\overline{Q}}\Big[d\bar{t}\Big] = d\bar{t}. \tag{3.41}$$

Therefore, the generalised infinitesimal jump process is

$$d\overline{N} = \lambda\Big(\Gamma e^{\epsilon H}(t) - \Gamma^* e^{\epsilon H}(N)\Big)dt + \Gamma^* e^{\epsilon H}(N)dN(t) \tag{3.42}$$

with

$$\Gamma^* e^{\epsilon H}(N)\Gamma^* e^{\epsilon H}(N) = \lambda\Gamma e^{\epsilon H}(t). \tag{3.43}$$

This is a generalized random time change formula for the Poisson process.

Definition 3.2.1 *The infinitesimal transformations*

$$\bar{t} = e^{\epsilon H}(t), \quad \bar{x} = e^{\epsilon H}(x) \quad and \quad \overline{N} = e^{\epsilon H}(N) \tag{3.44}$$

are called admitted Lie symmetry transformations of stochastic differential equations driven by Poisson process

$$dX_i(t,N) = f_i(t,X(t,N))dt + J_i(t,X(t,N))dN(t), \tag{3.45}$$

if they leave (3.45) and infinitesimal moments of the Poisson process i.e.,

$$E_Q\Big[dN(t)\Big] = \lambda dt, \quad E_Q\Big[dN(t)dN(t)\Big] = \lambda dt \quad and \quad E_Q\Big[dt\Big] = dt \tag{3.46}$$

invariant, where $\lambda > 0$ is the Poisson jump intensity.

3.3 Derivation of the Determining Equations

This section is devoted to finding the determining equations of the admitted Lie symmetries of the stochastic differential equations driven by Poisson processes (3.1).

The intention is to transform given stochastic differential equation (3.1) into

$$d\overline{X}_j(\bar{t}) = \overline{f}_j(\bar{t},\overline{X}(\bar{t}))d\bar{t} + \overline{J}_j(\bar{t},\overline{X}(\bar{t}))d\overline{N}(\bar{t}). \tag{3.47}$$

This can be achieved by substituting (3.24) and (3.27) into (3.47) which gives

$$d\overline{X}(\bar{t}) = dX_j(t) + \epsilon\Big(f_j\Gamma H(t) + H(f_j) + \lambda J_j\Gamma H(t) - \lambda J_j\Gamma^* H(N)\Big)dt + \Big(J_j\Gamma^* H(N) + H(J_j)\Big)dN(t). \tag{3.48}$$

Therefore, by comparing equation (3.48) and (3.17) we have the following determining equations

$$\Big(f_j\Gamma(\tau) + \lambda J_j\Big(\Gamma(\tau) - \Gamma^*(\phi)_j\Big) + H(f_j) - \Gamma(\xi)_j\Big)(t,X(t)) = 0 \tag{3.49}$$

and

$$\Big(J_j\Gamma^*(\phi)_j + H(J_j) - \Gamma^*(\xi)_j\Big)(t,X(t)) = 0. \tag{3.50}$$

Equations (3.37) and (3.38) give

$$\Gamma^*(\phi_j) + \Gamma^*(\phi_j) = \Gamma(\tau) \tag{3.51}$$

and

$$\Gamma(\phi_j) + \lambda\Gamma^*(\phi_j) = \lambda\Gamma(\tau). \tag{3.52}$$

From (3.51), we have

$$\Gamma^*(\phi_j) = \frac{\Gamma(\tau)}{2} \tag{3.53}$$

Equations (3.52) and (3.53) give

$$\Gamma(\phi_j) = \lambda\frac{\Gamma(\tau)}{2}. \tag{3.54}$$

To show the relationship between (3.49) and the first prolongation of ordinary differential equations we proceed by; using the definition of first prolongation of an infinitesimal generator for non-stochastic ordinary differential equations

$$H^{[1]} = H + \eta_i^{[1]} \frac{\partial}{\partial \dot{x}_i}, \tag{3.55}$$

for

$$\dot{x_i} = \frac{dx_i}{dt} = D_t x_i \tag{3.56}$$

and

$$\begin{aligned} \eta_i^{[1]} &= D_t(\xi_i) - \dot{x}_i D_t(\tau) \\ &= \frac{\partial \xi_i}{\partial t} + \dot{x}_i \frac{\partial \xi_i}{\partial x} - \dot{x}_i \left(\frac{\partial \tau}{\partial t} + \dot{x}_i \frac{\partial \tau}{\partial x} \right) \end{aligned} \tag{3.57}$$

with total time derivative D_t defined as

$$D_t = \frac{\partial}{\partial t} + \dot{x}_i \frac{\partial}{\partial x} + \ddot{x}_i \frac{\partial}{\partial \dot{x}_i} + \ldots \tag{3.58}$$

using the definition of first prolongation on $(\dot{x}_i - f_i)$ at $\dot{x}_i = f_i$, can be expressed as

$$H^{[1]}(\dot{x}_i - f_i)|_{\dot{x}_i = f_i} = \eta_i^{[1]} - H(f_i). \tag{3.59}$$

Using (3.59) and (3.57) equation (3.49) can be written as

$$H^{[1]}(\dot{x}_j - f_j)|_{\dot{x}=f} - \lambda J_j \left(\Gamma(\tau) - \Gamma(\phi)\right) = 0, \tag{3.60}$$

where $\Gamma(t, X(t,N))$ and $\Gamma^*(t, X(t,N))$ are defined using the operators (2.5) and (2.6) respectively.

3.3.1 Unal's Extra Condition

Unal's G. in [9] commented that the Itô multiplication properties for the transformed processes has to be satisfied i.e.,

$$d\overline{N}(\bar{t})d\overline{N}(\bar{t}) = d\overline{N}(\bar{t}), \quad d\overline{W}_l(\bar{t})d\overline{N}(t), \quad d\overline{N}(\bar{t})d\bar{t} = 0, \tag{3.61}$$

the properties led to an additional condition

$$\Gamma(\tau) = 0. \tag{3.62}$$

3.4 Applications

In this section, we are going to apply the derived determining equations obtained in the previous section to some Poisson stochastic differential equations to show how they can be used to find the admitted Lie point symmetries.

Example 3.4.1 *Consider a stochastic differential equation*

$$dX(t,N) = -kt^2 dt + b dN(t) \tag{3.63}$$

with $b \neq 0$ and initial condition $X(0) = x_0$.

Using the determining equations (3.49), (3.50), (3.53) and (3.54) we get respectively

$$-kt^2\left(\frac{\partial \tau}{\partial t} - kt^2\frac{\partial \tau}{\partial x}\right) + \frac{b\lambda}{2}\left(\frac{\partial \tau}{\partial t} - kt^2\frac{\partial \tau}{\partial x}\right) - 2kt\tau = \frac{\partial \xi}{\partial t} - kt^2\frac{\partial \xi}{\partial x}, \tag{3.64}$$

$$b\Big(\phi(t,x+b,N) - \phi(t,x,N)\Big) = \xi(t,x+b) - \xi(t,x), \tag{3.65}$$

$$\phi(t,x+b,N) - \phi(t,x,N) = \frac{1}{2}\left(\frac{\partial \tau}{\partial t} - kt^2\frac{\partial \tau}{\partial x}\right) \tag{3.66}$$

and

$$\left(\frac{\partial \phi}{\partial t} - kt^2\frac{\partial \phi}{\partial x}\right) = \frac{\lambda}{2}\left(\frac{\partial \tau}{\partial t} - kt^2\frac{\partial \tau}{\partial x}\right). \tag{3.67}$$

From (3.36) and (3.62) we have

$$\frac{\partial \tau(t,x,N)}{\partial x} = 0, \quad \frac{\partial \tau(t,x,N)}{\partial N} = 0. \tag{3.68}$$

Therefore, using (3.68) and (3.36) we have the temporal infinitesimal as

$$\tau(t,N) = c_1 t + c_0. \tag{3.69}$$

Using (3.69) equations (3.64), (3.66) and (3.67) can be rewritten as

$$c_1\left(\frac{b\lambda}{2} - 3kt^2\right) - 2ktc_0 = \frac{\partial \xi(t,x)}{\partial t} - kt^2\frac{\partial \xi(t,x)}{\partial x}, \tag{3.70}$$

(3.71) $$\frac{\partial \phi(t,x,N)}{\partial t} - kt^2 \frac{\partial \phi(t,x,N)}{\partial x} = \frac{\lambda c_1}{2}$$

and

(3.72) $$\phi(t,x+b,N) - \phi(t,x,N) = \frac{c_1}{2}.$$

Differentiating (3.70) and (3.71) respectively with respect to x gives

(3.73) $$\frac{\partial^2 \xi(t,x)}{\partial t \partial x} - kt^2 \frac{\partial^2 \xi(t,x)}{\partial x^2} = 0,$$

(3.74) $$\frac{\partial^2 \phi(t,x,N)}{\partial t \partial x} - kt^2 \frac{\partial^2 \phi(t,x,N)}{\partial x^2} = 0.$$

Differentiate (3.72) with respect to x gives

(3.75) $$\frac{\partial \phi(t,x+b,N)}{\partial x} = \frac{\partial \phi(t,x,N)}{\partial x} = g(t,N).$$

Differentiating (3.65) with respect to x and using (3.75) gives

(3.76) $$\frac{\partial \xi(t,x+b)}{\partial x} = \frac{\partial \xi(t,x)}{\partial x} = h(t).$$

From (3.75) and (3.76) we have

(3.77) $$\frac{\partial^2 \phi(t,x,N)}{\partial x^2} = 0 \qquad and \qquad \frac{\partial^2 \xi(t,x)}{\partial x^2} = 0,$$

which implies from (3.74) and (3.76)

(3.78) $$\frac{\partial^2 \phi(t,x,N)}{\partial x \partial t} = 0$$

and

(3.79) $$\xi(t,x) = h(t)x + g(t).$$

Substituting (3.79) into (3.73) we get

(3.80) $$f(t) = c_3,$$

therefore, using (3.80) and (3.79) we have

$$\xi(x,t) = c_3 x + g(t). \tag{3.81}$$

Substituting (3.81) in (3.65) using (3.72) gives the relation

$$c_1 = 2c_3. \tag{3.82}$$

Using (3.81) and (3.82), equation (3.70) gives

$$c_1\left(\frac{b\lambda}{2} - 3kt^2\right) - 2ktc_0 = \frac{dg(t)}{dt} - \frac{kt^2 c_1}{2}. \tag{3.83}$$

Solving the differential equation (3.83) gives

$$g(t) = c_1\left(\frac{b\lambda t}{2} - \frac{5kt^3}{6}\right) - kt^2 c_0 + c_4. \tag{3.84}$$

Therefore, substituting (3.84) into (3.81) the spatial infinitesimal finally becomes

$$\xi(x,t) = c_1\left(\frac{b\lambda t}{2} - \frac{5kt^3}{6} + \frac{x}{2}\right) - kt^2 c_0 + c_4. \tag{3.85}$$

Therefore from (3.75) and (3.78) we get

$$g(t,N) = g(N). \tag{3.86}$$

Which implies from (3.75) and (3.78) respectively

$$\phi(t,x,N) = g(N)x + g_1(t,N), \tag{3.87}$$

and

$$\xi(t,t) = h(t)x + g(t). \tag{3.88}$$

Substituting (3.87) into (3.71) gives

$$g_1(t,N) = \frac{kt^3 g(N)}{3} + \frac{\lambda c_1 t}{2} + g_2(N), \tag{3.89}$$

therefore using (3.89) in (3.87) gives

$$\phi(t,x,N) = g(N)x + \frac{kt^3 g(N)}{3} + \frac{\lambda c_1 t}{2} + g_2(N). \tag{3.90}$$

Substituting (3.90) into (3.72) gives the following relation

$$g(N) = \frac{c_1}{2b}. \tag{3.91}$$

Substituting (3.91) in (3.90) yields the infinitesimal of the Poisson process as

$$\phi(t,x,N) = c_1\left(\frac{x}{2b} + \frac{kt^3}{6b} + \frac{\lambda t}{2}\right) + g_2(N). \tag{3.92}$$

Equations (3.92) and (3.85) satisfy (3.65) automatically. So we have infinite symmetry generators as

$$H_1 = t\frac{\partial}{\partial t} + \left(\frac{x}{2} - \frac{5kt^3}{6} + \frac{b\lambda t}{2}\right)\frac{\partial}{\partial x} + \left(\frac{x}{2b} + \frac{kt^3}{6b} + \frac{\lambda t}{2}\right)\frac{\partial}{\partial N} \tag{3.93}$$

$$H_2 = \frac{\partial}{\partial t} - kt^2\frac{\partial}{\partial x} \qquad H_3 = \frac{\partial}{\partial x} \qquad H_4 = g_2(N)\frac{\partial}{\partial N}. \tag{3.94}$$

with corresponding Lie bracket relations for a simplest $g_2(N) = c$, given by

TABLE 3.1: Commutator Table for the Lie Algebra Generators (3.93) and (3.94)

$[H_i, H_j]$	H_1	H_2	H_3	H_4
H_1	0	H_5	H_6	0
H_2	$-H_5$	0	0	0
H_3	$-H_6$	0	0	0
H_4	0	0	0	0

where H_5 is a linear combination of H_2, H_3 and H_4 given as

$$H_5 = -\left(H_2 + \frac{\lambda}{2}\left(bH_3 + H_4\right)\right)$$

and H_6 is the linear combination of H_3 and H_4 given as

$$H_6 = -\frac{1}{2}\left(H_3 - \frac{H_4}{b}\right), \quad b \neq 0.$$

The commutative Table 3.1 shows that the infinitesimals generators (3.93) and (3.94) are closed under Lie bracket relations and hence form a Lie algebra.

Applying the extra condition (3.62) obtain by making sure the transformed Poisson variables satisfy Watanabe characterisation of Poisson, we get from (3.40)

$$\Gamma_{(\tau)} = c_1 = 0. \tag{3.95}$$

Which reduced the symmetry infinitesimals to only

$$H_2 = \frac{\partial}{\partial t} - kt^2\frac{\partial}{\partial x} \qquad H_3 = \frac{\partial}{\partial x} \qquad H_4 = \frac{\partial}{\partial N}. \tag{3.96}$$

Example 3.4.2 *Consider a model*

$$dX(t,N) = -k^2 t dt + \alpha t dN(t), \qquad \alpha \neq 0 \tag{3.97}$$

with initial condition $X(0) = x_0$.

Using the determining equations (3.49), (3.50), (3.53) and (3.54) we get respectively

$$-k^2 t\left(\frac{\partial \tau}{\partial t} - k^2 t\frac{\partial \tau}{\partial x}\right) + \frac{\alpha\lambda t}{2}\left(\frac{\partial \tau}{\partial t} - k^2 t\frac{\partial \tau}{\partial x}\right) - k^2\tau = \left(\frac{\partial \xi}{\partial t} - k^2 t\frac{\partial \xi}{\partial x}\right), \tag{3.98}$$

$$\alpha t\Big(\phi(t,x+\alpha t,N) - \phi(t,x,N)\Big) + \alpha\tau = \xi(t,x+\alpha t,N) - \xi(t,x,N), \tag{3.99}$$

$$\phi(t,x+\alpha t,N) - \phi(t,x,N) = \frac{1}{2}\left(\frac{\partial \tau}{\partial t} - k^2 t\frac{\partial \tau}{\partial x}\right) \tag{3.100}$$

and

$$\left(\frac{\partial \phi}{\partial t} - k^2 t\frac{\partial \phi}{\partial x}\right) = \frac{\lambda}{2}\left(\frac{\partial \tau}{\partial t} - k^2 t\frac{\partial \tau}{\partial x}\right). \tag{3.101}$$

From (3.36) we have

$$\frac{\partial \tau(t,x,N)}{\partial x} = 0. \tag{3.102}$$

Therefore, using (3.102) and (3.36) the infinitesimal of the temporal variable becomes

$$\tau(t,N) = c_1 t + c_0. \tag{3.103}$$

Using (3.103) equation (3.98), (3.100) and (3.101) can be rewritten as

$$\frac{\partial \xi}{\partial t} - k^2 t \frac{\partial \xi}{\partial x} = \left(\frac{\alpha \lambda t}{2} - 2k^2 t\right) c_1 - c_0 k^2, \tag{3.104}$$

$$\alpha t\Big(\phi(t,x+\alpha t,N) - \phi(t,x,N)\Big) + \alpha\Big(c_1 t + c_0\Big) = \xi(t,x+\alpha t) - \xi(t,x), \tag{3.105}$$

$$\phi(t,x+\alpha t,N) - \phi(t,x,N) = \frac{c_1}{2}, \tag{3.106}$$

and

$$\frac{\partial \phi}{\partial t} - k^2 t \frac{\partial \phi}{\partial x} = \frac{c_1 \lambda}{2}. \tag{3.107}$$

Differentiating (3.106) and (3.105) with respect to x gives

$$\frac{\partial \phi(t,x+\alpha t,N)}{\partial x} = \frac{\partial \phi(t,x,N)}{\partial x} = g(t,N) \tag{3.108}$$

and

$$\alpha t\left(\frac{\partial \phi(t,x+\alpha t,N)}{\partial x} - \frac{\partial \phi(t,x,N)}{\partial x}\right) = \frac{\partial \xi(t,x+\alpha t)}{\partial x} - \frac{\partial \xi(t,x)}{\partial x}. \tag{3.109}$$

Using (3.108) in (3.109) yields

$$\frac{\partial \xi(t,x+\alpha t)}{\partial x} = \frac{\partial \xi(t,x)}{\partial x} = h(t). \tag{3.110}$$

Similarly using (3.108) into (3.107) gives

$$\frac{\partial^2 \phi(t,x,N)}{\partial t \partial x} = 0 \tag{3.111}$$

which implies

$$\phi(t,x,N) = g_1(N)x + g_2(t,N). \tag{3.112}$$

Substituting (3.112) in (3.106) we get

$$g_1(N) = c_1 = 0 \tag{3.113}$$

which implies

$$\phi(t,x,N) = g_2(t,N). \tag{3.114}$$

Similarly, substituting (3.114) in (3.107) gives

$$\phi(t,x,N) = g_3(N). \tag{3.115}$$

solving (3.110) yields

$$\xi(t,x) = h(t)x + h_1(t). \tag{3.116}$$

Substituting (3.116) in (3.105) using (3.113) and (3.115) gives

$$h(t) = \frac{c_0}{t}. \tag{3.117}$$

Therefore substituting (3.117) into (3.116) we have

$$\xi(t,x) = \frac{c_0}{t}x + h_1(t). \tag{3.118}$$

Substituting (3.118) into (3.104) using (3.113) and gives

$$h_1(t) = c_3 \quad and \quad c_0 = 0. \tag{3.119}$$

Using (3.113) and (3.119) the infinitesimals of the temporal and spatial variables (3.103) and (3.118) respectively become

$$\tau(t,x,N) = 0 \qquad and \qquad \xi(t,x) = c_3. \tag{3.120}$$

Therefore we have infinite symmetry infinitesimal generators

$$H_1 = \frac{\partial}{\partial x}, \qquad H_g = g_3(N)\frac{\partial}{\partial N}. \tag{3.121}$$

The infinitesimals generators (3.121) forms an abelian group with the Lie bracket relations for a simplest form of $g_3(N) = c$, given as

$$[H_1, H_{g=c}] = [H_{g=c}, H_1] = 0.$$

The infinitesimals in this case automatically satisfy Unal extra condition (3.62)

$$\Gamma_{(\tau)} = c_1 = 0. \tag{3.122}$$

Example 3.4.3 *Consider a Poisson stochastic differential model*

$$dX(t,N) = -k^2 x dt + \sqrt{2k^2} dN(t), \qquad k \neq 0 \tag{3.123}$$

where k is a non-negative real number and initial condition $X(0) = x_0$.

Using the determining equations (3.49), (3.50), (3.53) and (3.54) we respectively yield

$$-k^2 x\left(\frac{\partial \tau}{\partial t} - k^2 x \frac{\partial \tau}{\partial x}\right) + \frac{\sqrt{2k^2}\lambda}{2}\left(\frac{\partial \tau}{\partial t} - k^2 x \frac{\partial \tau}{\partial x}\right) - k^2 \xi = \left(\frac{\partial \xi}{\partial t} - k^2 x \frac{\partial \xi}{\partial x}\right), \tag{3.124}$$

$$\sqrt{2k^2}\Big(\phi(t, x + \sqrt{2k^2}, N) - \phi(t,x,N)\Big) = \xi(t, x + \sqrt{2k^2}) - \xi(t,x), \tag{3.125}$$

$$\phi(t, x + \sqrt{2k^2}, N) - \phi(t,x,N) = \frac{1}{2}\left(\frac{\partial \tau}{\partial t} - k^2 x \frac{\partial \tau}{\partial x}\right) \tag{3.126}$$

and

$$\left(\frac{\partial \phi}{\partial t} - k^2 x \frac{\partial \phi}{\partial x}\right) = \frac{\lambda}{2}\left(\frac{\partial \tau}{\partial t} - k^2 x \frac{\partial \tau}{\partial x}\right). \tag{3.127}$$

From (3.36) we have

$$\frac{\partial \tau(t,x,N)}{\partial x} = 0. \tag{3.128}$$

Therefore using (3.128) and (3.36) we have the temporal infinitesimal as

$$\tau(t) = c_1 t + c_0. \tag{3.129}$$

Using (3.129) equation (3.124), (3.126) and (3.127) can be rewritten respectively as

$$\Big(-k^2 x + \frac{\sqrt{2k^2}\lambda}{2}\Big)c_1 - k^2 \xi = \left(\frac{\partial \xi}{\partial t} - k^2 x \frac{\partial \xi}{\partial x}\right), \tag{3.130}$$

$$\phi(t,x+\sqrt{2k^2},N)-\phi(t,x,N)=\frac{c_1}{2} \tag{3.131}$$

and

$$\left(\frac{\partial\phi}{\partial t}-k^2x\frac{\partial\phi}{\partial x}\right)=\frac{c_1\lambda}{2}. \tag{3.132}$$

Differentiate (3.125) and (3.131) with respect to x respectively gives

$$\sqrt{2k^2}\Big(\frac{\partial\phi(t,x+\sqrt{2k^2},N)}{\partial x}-\frac{\partial\phi(t,x,N)}{\partial x}\Big)=\frac{\partial\xi(t,x+\sqrt{2k^2})}{\partial x}-\frac{\partial\xi(t,x)}{\partial x} \tag{3.133}$$

and

$$\frac{\partial\phi(t,x+\sqrt{2k^2},N)}{\partial x}=\frac{\partial\phi(t,x,N)}{\partial x}=g(t,N). \tag{3.134}$$

Substituting (3.134) into (3.133) we get

$$\frac{\partial\xi(t,x+\sqrt{2k^2})}{\partial x}=\frac{\partial\xi(t,x)}{\partial x}=h(t). \tag{3.135}$$

Differentiating (3.130) and (3.132) with respect to x respectively gives

$$\frac{\partial^2\xi}{\partial t\partial x}-xk^2\frac{\partial^2\xi}{\partial x^2}=-k^2c_1, \tag{3.136}$$

and

$$\frac{\partial^2\phi}{\partial t\partial x}-xk^2\frac{\partial^2\phi}{\partial x^2}-k^2\frac{\partial\phi}{\partial x}=0. \tag{3.137}$$

From (3.134) and (3.135) we have

$$\frac{\partial^2\phi}{\partial x^2}=0 \qquad and \qquad \frac{\partial^2\xi}{\partial x^2}=0. \tag{3.138}$$

Substituting (3.138) in (3.137) and (3.136) respectively yields

$$\frac{\partial^2\phi}{\partial t\partial x}-k^2\frac{\partial\phi}{\partial x}=0. \tag{3.139}$$

and

$$\frac{\partial^2\xi}{\partial t\partial x}+k^2c_1=0. \tag{3.140}$$

Solving (3.139) and (3.140) respectively using (3.134) and (3.135) yields

$$\phi = g_1(N)e^{k^2 t}x + g_2(t,N) \tag{3.141}$$

and

$$\xi(t,x) = -k^2 x t c_1 + h_1(t) + c_3 x. \tag{3.142}$$

Substituting (3.141) in (3.132) gives

$$g_2(t,N) = \frac{c_1 t\lambda}{2} + g_3(N). \tag{3.143}$$

Similarly substituting (3.141) in (3.131) using (3.143) gives

$$g_1(N) = \frac{c_1 e^{-k^2 t}\sqrt{2k^2}}{4k^2}, \tag{3.144}$$

therefore, using (3.144) and (3.141) the infinitesimal of the Poisson process is reduced to

$$\phi = \left(\frac{x\sqrt{2k^2}}{4k^2} + \frac{t\lambda}{2}\right)c_1 + g_3(N). \tag{3.145}$$

Using (3.145), equation (3.133) implies

$$\frac{\partial\xi(t,x+\sqrt{2k^2})}{\partial x} - \frac{\partial\xi(t,x)}{\partial x} = 0. \tag{3.146}$$

Substituting (3.142) into (3.125) using (3.131) yields

$$c_1 = c_3 = 0. \tag{3.147}$$

Therefore using (3.144), (3.147) and (3.145) gives

$$\xi = h_1(t), \quad \phi = g_3(N). \tag{3.148}$$

Finally, substituting (3.148) into (3.130) using (3.144) and (3.145) gives the spatial infinitesimal as

$$\xi(t) = c_4 e^{-k^2 t}. \tag{3.149}$$

So we have infinite symmetry infinitesimal generators as

$$H_1 = \frac{\partial}{\partial t}, \qquad H_2 = e^{-k^2 t}\frac{\partial}{\partial x}, \qquad H_3 = g_3(N)\frac{\partial}{\partial N}, \tag{3.150}$$

with corresponding Lie bracket relations of the generators (3.150) for a simplest form of $g_3(N) = c$ given in Table 3.2 as

TABLE 3.2: Commutator Table for the Lie Algebra Generators (3.150)

$[H_i, H_j]$	H_1	H_2	H_3
H_1	0	$-k^2 H_2$	0
H_2	$k^2 H_2$	0	0
H_3	0	0	0

The Lie bracket relations in Table 3.2 show that the infinitesimal generator (3.150) satisfied Lie commutative relation properties and hence forms a Lie algebras.

The infinitesimals in this case automatically satisfy Unal extra condition (3.62)

$$\Gamma_{(\tau)} = c_1 = 0. \tag{3.151}$$

3.5 Conclusion

Lie point symmetry transformations for the class of Itô stochastic differential equations driven by Poisson Processes was successfully carried out. We considered symmetries involving not only spatial x and time variables t, but also included the Poisson process term $N(t)$. This was achieved by following the methodology of Lie point transformations [3, 6] and the use of Itô formula for Poisson stochastic differential equations [1] without enforcing any conditions at the momenta of the stochastic processes.

We ensured the instantaneous mean and variance of the Poisson stochastic processes remained invariant under the transformation (3.10).

Finally, the determining equations for the Itô stochastic differential equation driven by Poisson processes

$$dX_i(t,N) = f_i(t,X(t,N))dt + J_i(t,X(t,N))dN(t) \tag{3.152}$$

were successfully derived and they were found to be non-stochastic even though they represent stochastic processes. Determining equations found were later applied to a few examples which, while simple, are non-trivial to find their correspondent admitted Lie symmetries. Classification of the given examples are presented in *Table 3.1* below. The simplest form of $g_2(N) = g_3(N) = c$ is used throughout to calculate the commutator tables.

TABLE 3.3: Lie Group Classification Chapter 3

Group Dimension	Basis Operators	Equations
$infinite$	$H_2 = \frac{\partial}{\partial t} - kt^2\frac{\partial}{\partial x},\quad H_3 = \frac{\partial}{\partial x}\quad H_4 = g(N)\frac{\partial}{\partial N}.$	$dX(t,N) = -kt^2dt + bdN(t)$
$infinite$	$H_1 = \frac{\partial}{\partial x},\qquad H_2 = g(N)\frac{\partial}{\partial N}.$	$dX(t,N) = t(-k^2tdt + \alpha dN(t))$
$infinite$	$H_1 = \frac{\partial}{\partial t},\qquad H_2 = e^{-k^2t}\frac{\partial}{\partial x},\qquad H_3 = g(N)\frac{\partial}{\partial N}$	$dX(t,N) = -k^2xdt + \sqrt{2k^2}dN(t)$

Chapter 4

Lie Symmetry of Jump-Diffusion Itô Stochastic Differential Equations

In this chapter, we define the Lie symmetry of jump-diffusion stochastic differential equations by considering infinitesimals of the spatial and temporal variables [3]. This was achieved by using the random time formula for standard Brownian motion to transform the Wiener process term [33, 53] as well as the random time formula for Poisson processes to transform the Poisson term which was derived in chapter 3.

4.1 Introduction

Lie symmetries of Wiener process stochastic differential equation were discussed in [7, 10, 13, 14, 16, 20, 28] which is based on the standard method of the random time change of Brownian motion [33, 53]. That is, the Wiener process is transformed as

$$d\overline{W}(\bar{t}) = \sqrt{\frac{d\bar{t}(t)}{dt}}\, dW(t). \tag{4.1}$$

In chapter 3, we derived a similar random time change formula for Poisson processes in the context of Lie point symmetries by ensuring the instantaneous mean and variance of the Poisson process remained invariant under Lie group transformations [2]. i.e.,

$$d\overline{N}(\bar{t}) = dN(t) + \frac{\epsilon}{2}\frac{d\tau}{dt}\Big(\lambda dt + dN(t)\Big) + O(\epsilon). \tag{4.2}$$

In this chapter, we considered the random time change of Wiener processes (4.1) and that of Poisson processes (4.2) and discussed Lie point symmetries for Itô stochastic differential equations (SDE) driven by both Wiener process and Poisson processes (jump-diffusion);

$$dX_i(t) = f_i(t,X(t))dt + G_{ik}(t,X(t))dW_k(t) + J_i(t,X(t))dN(t) \tag{4.3}$$

where $f_i(t,X(t))$ and $J_i(t,X(t))$ are $n\times 1$ dimensional drift vector coefficients and jump diffusion coefficients respectively. While $G_{ik}(t,X(t))$ is the Wiener diffusion matrix coefficient of $n\times M$ dimensions, $dW(t)$ is called the infinitesimal increment of the Wiener process while $dN(t)$ is called the infinitesimal increment of the Poisson process.

To ensure the existence and uniqueness of the solution of (4.3), the instantaneous drift coefficient $f_i(t,X(t))$, Wiener diffusion coefficient $G_{ik}(t,X(t))$ and the jump diffusion coefficient $J_i(t,X(t))$ are assumed to comply with Ikeda-Watanabe conditions [66].

The Lie point symmetries of (4.3) are discussed by considering infinitesimals involving the spatial variable x and time variable t, using the infinitesimal generating operator

$$H = \tau(t,x)\frac{\partial}{\partial t} + \xi_i(t,x)\frac{\partial}{\partial x_i}. \tag{4.4}$$

The determining equations for Ito stochastic differential equations (SDE) with finite jumps (4.3) are derived in an Ito calculus context and were found to be non-stochastic though they represent a stochastic process.

For an arbitrary function $F(t,X(t))$ which is twice contentiously differential with respect to spatial coordinates x and differentiable once with respect to time t, then by the Itô lemma for jump diffusion process, the Itô process $F(t,X(t))$ of (4.3) exists and is

$$\begin{aligned} dF_j(t,X(t)) &= \left(\frac{\partial F_j}{\partial t} + f_i\frac{\partial F_j}{\partial x_i} + \frac{1}{2}\sum_{k=1}^{M} G_{ik}(t,X(t))G_{mk}(t,X(t))\frac{\partial^2 F_j}{\partial x_i \partial x_m}\right)dt \\ &+ G_{il}(t,X(t))\frac{\partial F_j}{\partial x_i}dW(t) + \Big(F_j\big(t,X_i(t)+J(t,X_i(t))\big) - F_j(t,X_i(t))\Big)dN(t). \end{aligned} \tag{4.5}$$

The Einstein summation convention is assumed through out, for the matter of convenience let introduce the following operators;

$$\Gamma_{(F)_j} = \frac{\partial F_j}{\partial t}dt + f_i\frac{\partial F_j}{\partial x_i} + \frac{1}{2}\sum_{k=1}^{M} G_{ik}(t,X(t))G_{mk}(t,X(t))\frac{\partial^2 F_j}{\partial x_i \partial x_m}, \tag{4.6}$$

$$\Gamma^{*}_{(F)_j} = G_{il}(t,X(t))\frac{\partial F_j}{\partial x_i} \tag{4.7}$$

and

$$\Gamma^{**}_{(F)_j} = F_j(t,X_i(t)+J(t,X_i(t))) - F_j(t,X_i(t)). \tag{4.8}$$

Therefore, the Itô process (4.5)

$$\begin{aligned} dF_j(t,X(t)) &= \left(\frac{\partial F_j}{\partial t} + f_i\frac{\partial F_j}{\partial x_i} + \frac{1}{2}\sum_{k=1}^{M} G_{ik}(t,X(t))G_{mk}(t,X(t))\frac{\partial^2 F_j}{\partial x_i \partial x_m}\right)dt \\ &+G_{il}(t,X(t))\frac{\partial F_j}{\partial x_i}dW(t) + \left(F_j\left(t,X_i(t)+J(t,X_i(t))\right) - F_j(t,X_i(t))\right)dN(t) \end{aligned} \tag{4.9}$$

can be rewritten as;

$$dF_j(t,X(t)) = \Gamma_{(F)_j}dt + \Gamma^{*}_{(F)_j}dW(t) + \Gamma^{**}_{(F)_j}dN(t). \tag{4.10}$$

By using the Itô multiplication properties for stochastic differential equations driven by both Wiener and Poisson processes in *Table 1.3* and application of infinitesimal transformations the determining equations for stochastic differential equations with finite jump processes (SDEJ) are derived and found to be deterministic. The main result can be summarised as

Theorem 4.1.1 *The Itô stochastic differential equation driven by both Wiener and Poisson process*

$$dX(t) = f_i(t,X(t))dt + \sum_{k=1}^{M} G_{ik}(t,X(t))dW_k(t) + J_i(t,X(t))dN(t)$$

i.e., stochastic differential equation with a finite jump, where $f(t,X(t))$ and $J(t,X(t))$ are the $n\times 1$ dimensional drift vector coefficient and jump diffusion coefficient respectively and $G_{ik}(t,X(t))$ is the Wiener diffusion matrix coefficient of $n\times M$ dimension, with infinitesimal generator

$$H = \tau(t,x)\frac{\partial}{\partial t} + \xi_i(t,x)\frac{\partial}{\partial x_i}, \tag{4.11}$$

has admitted the following determining equations;

$$\left(f_j\Gamma_{(\tau)} + \frac{\lambda J_j}{2}\Gamma_{(\tau)} + H(f_j) - \Gamma_{(\xi)_j}\right)(t,X(t)) = 0, \tag{4.12}$$

$$\left(\frac{G_{jk}}{2}\Gamma_{(\tau)} + H(G_{jk}) - \Gamma^{*}_{(\xi)_j}\right)(t, X(t)) = 0 \tag{4.13}$$

and

$$\left(\frac{J_j}{2}\Gamma_{(\tau)} + H(J_j) - \Gamma^{**}_{(\xi_j)}\right)(t, X(t)) = 0. \tag{4.14}$$

With additional conditions

$$\Gamma^{*}_{(\tau)}(t, X(t)) = 0 \tag{4.15}$$

and

$$\Gamma^{**}_{(\tau)}(t, X(t)) = 0, \tag{4.16}$$

where the operators $\Gamma(t,x)$, $\Gamma^{*}(t,x)$ and $\Gamma^{**}(t,x)$ are defined as in (4.6), (4.7) and (4.8). $\lambda > 0$ is called the intensity of the jump process or simply the jump rate, and infinitesimals $\xi(t,x)$ and $\tau(t,x)$ are called the admitted symmetries of (4.3) if and only if they satisfied the determining equations (4.12) - (4.16).

4.2 Lie Group Transformations

Consider a one parameter group of transformations of time index t and the spatial variable x respectively,

$$\bar{t} = \theta_1(x, t, \epsilon), \qquad \bar{x} = \theta_2(x, t, \epsilon)$$

with the infinitesimals

$$\frac{\partial \theta_1}{\partial \epsilon} = \tau(t, x), \qquad \frac{\partial \theta_2}{\partial \epsilon} = \xi(t, x).$$

The following initial conditions are satisfied at $\epsilon = 0$

$$\bar{t}\Big|_{\epsilon=0} = t, \qquad \overline{X}(\bar{t})\Big|_{\epsilon=0} = X(t).$$

Therefore, considering a one parameter Lie group of infinitesimal transformations

$$\bar{t} = t + \epsilon\tau(t,x), \tag{4.17}$$

$$\overline{X_j}(\bar{t}) = X_j(t) + \epsilon\xi_j(t,x), \tag{4.18}$$

where ϵ is the parameter of the group, with the corresponding infinitesimal generator of the Lie algebra

$$H = \tau(t,X(t))\frac{\partial}{\partial t} + \xi_i(t,X_i(t))\frac{\partial}{\partial x_i}.$$

The differential group transformation of the spatial x, temporal t, Wiener process $W(t)$ and the jump process $N(t)$ variables respectively are

$$d\bar{t} = dt + \epsilon d\tau + O(\epsilon), \tag{4.19}$$

$$\overline{X_j}(\bar{t}) = dX_j(t) + \epsilon d\xi_j + O(\epsilon), \tag{4.20}$$

$$d\overline{W_j}(\bar{t}) = dW_j(t) + \frac{\epsilon}{2}\frac{d\tau}{dt}dW_j(t) + O(\epsilon), \tag{4.21}$$

and

$$d\overline{N}(\bar{t}) = dN(t) + \frac{\epsilon}{2}\frac{d\tau}{dt}\Big(\lambda dt + dN(t)\Big) + O(\epsilon). \tag{4.22}$$

While using the Itô formula (4.10), the spatial and temporal infinitesimals in Itô forms can be written respectively as

$$d\xi_j = \Gamma_{(\xi)_j}(t,X(t))dt + \Gamma^*_{(\xi)_j}(t,X(t))dW(t) + \Gamma^{**}_{(\xi)_j}(t,X(t))dN(t) \tag{4.23}$$

and

$$d\tau = \Gamma_{(\tau)}(t,X(t))dt + \Gamma^*_{(\tau)}(t,X(t))dW(t) + \Gamma^{**}_{(\tau)}(t,X(t))dN(t). \tag{4.24}$$

Substituting the Itô forms of the spatial infinitesimal (4.23) and the temporal infinitesimal (4.24) into equations (4.19), (4.20) and (4.21) the point group transformation of the spatial, temporal and Wiener processes respectively can be written in Itô form as

$$d\bar{t} = dt + \epsilon\Big(\Gamma_{(\tau)}(t,X(t))dt + \Gamma^*_{(\tau)}(t,X(t))dW(t) + \Gamma^{**}_{(\tau)}(t,X(t))dN(t)\Big) + O(\epsilon), \tag{4.25}$$

(4.26)
$$d\overline{X_j}(\bar{t}) = dX_j(t) + \epsilon\Big(\Gamma_{(\xi)_j}(t,X(t))dt + \Gamma^{*}_{(\xi)_j}(t,X(t))dW_j(t) + \Gamma^{**}_{(\xi)_j}(t,X(t))\Big)dN(t) + O(\epsilon)$$

and

(4.27)
$$d\overline{W_j}(\bar{t}) = dW_j(t) + \frac{\epsilon}{2}\frac{\Gamma_{(\tau)}(t,X(t))dt + \Gamma^{*}_{(\tau)}(t,X(t))dW_j(t) + \Gamma^{**}_{(\tau)}(t,X(t))dN(t)}{dt}dW_j(t) + O(\epsilon).$$

The Itô form of Wiener group transformation (4.27) can be further simplified using the Itô multiplication properties of Wiener process *viz Table 1.1* to get

$$d\overline{W_j}(\bar{t}) = dW_j(t) + \frac{\epsilon}{2}\Big(\Gamma_{(\tau)}(t,X(t))dW_j(t) + \Gamma^{*}_{(\tau)}(t,X(t))\Big) + O(\epsilon). \tag{4.28}$$

Similarly, substituting the Itô temporal infinitesimal of the (4.24) into (4.22) the jump process variables group transformation can be written in Itô form as

(4.29)
$$d\overline{N}(\bar{t}) = dN(t) + \frac{\epsilon}{2}\frac{\Gamma_{(\tau)}(t,X(t))dt + \Gamma^{*}_{(\tau)}(t,X(t))dW_j(t) + \Gamma^{**}_{(\tau)}(t,X(t))dN(t)}{dt}\Big(\lambda dt + dN(t)\Big) + O(\epsilon).$$

Expanding the Itô group transformation of jump variable (4.29) and by the use of Itô multiplication properties of Poisson processes *viz Table 1.2* we get

(4.30)
$$\begin{aligned} d\overline{N}(\bar{t}) = dN(t) &+ \frac{\epsilon}{2}\Big(\lambda(\Gamma_{(\tau)}(t,X(t))dt + \Gamma^{*}_{(\tau)}(t,X(t))dW_j(t) + \Gamma^{**}_{(\tau)}(t,X(t))dN(t))\Big) \\ &+ \frac{\epsilon}{2}\left(\Gamma_{(\tau)}(t,X(t))dN(t) + \frac{\Gamma^{**}_{(\tau)}(t,X(t))dN(t)}{dt}\right) + O(\epsilon). \end{aligned}$$

After transforming the temporal, spatial, Wiener process and Poisson process infinitesimals using one parameter group of transformation in Itô context, we are now going to proceed next to find the transformed drift vector, Wiener diffusion and Poisson process coefficients using our infinitesimal generator in the subsequent section.

4.2.1 Invariance Form of the Spatial Process

To ensure the recovery of the finite transformations from the infinitesimal transformation, we need to transform (4.3) into

$$d\overline{X_j}(\bar{t}) = f_j(\bar{t},\overline{X}(\bar{t}))d\bar{t} + G_{jk}(\bar{t},\overline{X}(\bar{t}))d\overline{W(t)} + J_j(\bar{t},\overline{X}(\bar{t}))d\overline{N}(\bar{t}). \tag{4.31}$$

In order to accomplish this, we need to transform the drift vector coefficient $f_j(t,X(t))$, the Wiener diffusion coefficient $G_{jk}(t,X(t))$ as well as the jump diffusion coefficient $J_j(t,X(t))$ using the infinitesimal generator (4.4) i.e.,

$$H = \tau(t,x)\frac{\partial}{\partial t} + \xi_i(t,x)\frac{\partial}{\partial x_i}.$$

The drift vector coefficient, the Wiener diffusion coefficient, and the jump diffusion coefficient can be transformed respectively as follows

$$\begin{aligned} \overline{f_j}(\bar{t},\overline{X}(\bar{t})) &= \Big(f_j + \epsilon H(f_j)\Big)(t,X(t)) \\ &= f_j(t,X(t)) + \epsilon\left(\tau\frac{\partial f_j}{\partial t} + \xi_i\frac{\partial f_j}{\partial x_i}\right)(t,X(t)), \end{aligned} \tag{4.32}$$

$$\begin{aligned} \overline{G^l_j}(\bar{t},\overline{X}(\bar{t})) &= \Big(G_{jk} + \epsilon H(G_{jk})\Big)(t,X(t)) \\ &= G_{jk}(t,X(t)) + \epsilon\left(\tau\frac{\partial G_{jk}}{\partial t} + \xi_i\frac{\partial G_{jk}}{\partial x_i}\right)(t,X(t)). \end{aligned} \tag{4.33}$$

and

$$\begin{aligned} \overline{J_j}(\bar{t},\overline{X}(\bar{t})) &= \Big(J_j + \epsilon H(J_j)\Big)(t,X(t)) \\ &= J_j(t,X(t)) + \epsilon\left(\tau\frac{\partial J_j}{\partial t} + \xi_i\frac{\partial J_j}{\partial x_i}\right)(t,X(t)). \end{aligned} \tag{4.34}$$

In the next sections, we ensure sure the properties of both Wiener and Poisson process moments remains invariant under Lie group transformation. This will help us to obtain extra conditions for the determining equations of jump-diffusion stochastic differential equation (4.3) [7, 9], and ensure (4.3) remains unchanged under the Lie group of transformations.

4.2.2 Wiener Invariance Properties

We apply the invariance to the moments of the Wiener process to make sure it remains invariant under the group transformations, *viz* the instantaneous mean and variance of the Wiener process which are:

$$E_Q\Big[dW(t,w)|W = w\Big] = 0 \tag{4.35}$$

and

$$E_Q\left[dW_l(t,w)dW_m(t,w)|W=w\right]=\delta^l_m dt. \tag{4.36}$$

The invariance of the instantaneous mean of the transformed Wiener process under new measure $\overline{Q}$ is

$$E_{\overline{Q}}\left[d\overline{W}(t,w)|W=w\right]=0. \tag{4.37}$$

Expanding (4.37) using (4.28) gives

$$E_{\overline{Q}}\left[dW+\frac{\epsilon}{2}\left(\Gamma_{(\tau)}(t,X(t))dW(t)+\Gamma^*_{(\tau)}(t,X(t))\right)+O(\epsilon)\right]=0, \tag{4.38}$$

simplifying (4.38) and the use of instantaneous mean property (4.35) yields

$$\Gamma^*_{(\tau)}(t,X(t))=0. \tag{4.39}$$

Next we apply the invariance form to instantaneous variance of the transformed Wiener process measure i.e., using (4.28) and (4.39) we get

$$\begin{aligned}E_{\overline{Q}}\left[d\overline{W}_l(t,w)d\overline{W}_m(t,w)|W=w\right]&=E_{\overline{Q}}\left[dW_l(t,w)dW_m(t,w)|W=w\right]\\ +\epsilon E_{\overline{Q}}\left[\left(\frac{\Gamma_{(\tau)}(t,X(t))dt+\Gamma^*_{(\tau)}(t,X(t))dW(t)+\Gamma^{**}_{(\tau)}(t,X(t))dN}{dt}\right)\right.&\left.d\overline{W}_l(t,w)d\overline{W}_m(t,w)|W=w\right].\end{aligned} \tag{4.40}$$

Expanding (4.40) gives

$$E_{\overline{Q}}\left[d\overline{W}_l(t,w)d\overline{W}_m(t,w)|W=w\right]=\delta^l_m d\bar{t}. \tag{4.41}$$

Equation (4.40) implies that instantaneous variance remain invariant under the Lie group of transformations.

Remark 4.2.1 *We have seen that applying the invariance transformation to the mean and variance of the Wiener process lead to additional conditions for the determining equations. This is the same extra condition obtained by [9, 8, 20, 32, 33] when studying the symmetry of stochastic equations driven by Wiener processes.*

4.2.3 Poisson Invariance Properties

Similarly, we apply the invariance to moments of the Poisson process to make sure it remains invariant under the Lie group transformations, *viz* the instantaneous mean and variance of the Poisson process which are;

$$E_Q\left[dN(t)\right] = \lambda dt \tag{4.42}$$

and

$$E_Q\left[dN(t)dN(t)\right] = \lambda dt. \tag{4.43}$$

The invariance of the instantaneous mean of the transformed Poisson process under new measure $\overline{Q}$ is

$$E_{\overline{Q}}\left[d\overline{N}(\bar{t})\right] = \lambda d\bar{t}. \tag{4.44}$$

Expanding (4.44) using (4.30) and (4.25) gives

$$\Gamma^{**}_{(\tau)}(t, X(t)) = 0. \tag{4.45}$$

Next, we apply the invariant form to instantaneous variance of the transformed Poisson process measure (4.43) from which using (4.30) and (4.45) yields

$$E_{\overline{Q}}\left[d\overline{N}(\bar{t})d\overline{N}(\bar{t})\right] = \lambda d\bar{t}. \tag{4.46}$$

Thus using (4.39) and (4.45) we have derived the following generalised random time change formula

$$\bar{t} = \int^{t} \Gamma_{(\tau)}(s)ds, \tag{4.47}$$

with

$$\Gamma_{(\tau)} = constant = c_1 \tag{4.48}$$

obtained using the probabilistic invariance property of the transformed time index differential i.e.,

$$E_{\overline{Q}}\left[d\bar{t}(t,w)|W = w\right] = d\bar{t}. \tag{4.49}$$

Remark 4.2.2 *In [10, 13, 14, 28] while discussing symmetries of stochastic differential equations driven by the Brownian motion they restrict their work such that the temporal infinitesimal $\tau(t,x)$ depends only on t not x in the beginning i.e., fiber-preserving transformations*

$$H = \tau(t)\frac{\partial}{\partial t} + \xi_i(t,x)\frac{\partial}{\partial x_i}. \tag{4.50}$$

Invariance of the instantaneous mean of the Poisson process under the group transformation yields the same result in this case. That is from (4.45) we can conclude the temporal infinitesimal $\tau(t,x)$ does not depend on x, therefore $\tau(x,t) = \tau(t)$.

Definition 4.2.3 *A symmetry of jump-diffusion stochastic differential equation (4.3) i.e.,*

$$dX_i(t) = f_i(t,X(t))dt + G_{ik}(t,X(t))dW_k(t) + J_i(t,X(t))dN(t) \tag{4.51}$$

is a one parameter group of transformations that leaves (4.51) and infinitesimal moments of the Wiener and the Poisson processes invariant.

4.3 Derivation of the Determining Equations

In this section, we concentrate on finding determining equations for the admitted Lie group symmetries of (4.3).

The intention is to transform (4.3) into

$$d\overline{X_j}(\bar{t}) = \overline{f_j}(\bar{t},\overline{X}(\bar{t}))d\bar{t} + \overline{G_{jk}}(\bar{t},\overline{X}(\bar{t}))d\overline{W}(\bar{t}) + \overline{J_j}(\bar{t},\overline{X}(\bar{t}))d\overline{N}(\bar{t}). \tag{4.52}$$

Substituting (4.25), (4.28), (4.30), (4.32), (4.33) and (4.34) into (4.52) with consideration of equations (4.39) and (4.45), gives

$$\begin{aligned} d\overline{X_j}(\bar{t}) = dX_j + \epsilon\left(f_j\Gamma_{(\tau)}(t,X(t)) + \frac{\lambda J_j}{2}\Gamma_{(\tau)}(t,X(t)) + H(f_j)\right)dt \\ + \left(\frac{G_{jk}}{2}\Gamma_{(\tau)}(t,X(t)) + H(G_{jk})\right)dW(t) + \left(\frac{J_j}{2}\Gamma_{(\tau)}(t,X(t)) + H(J_j)\right)dN(t). \end{aligned} \tag{4.53}$$

Therefore, by comparing (4.26) and (4.53) we successfully obtain the following determining equations

$$\left(f_j\Gamma_{(\tau)} + \frac{\lambda J_j}{2}\Gamma_{(\tau)} + H(f_j) - \Gamma_{(\xi)_j}\right)(t, X(t)) = 0, \tag{4.54}$$

$$\left(\frac{G_{jk}}{2}\Gamma_{(\tau)} + H(G_{jk}) - \Gamma^{*}_{(\xi)_j}\right)(t, X(t)) = 0 \tag{4.55}$$

and

$$\left(\frac{J_j}{2}\Gamma_{(\tau)} + H(J_j) - \Gamma^{**}_{(\xi)_j}\right)(t, X(t)) = 0. \tag{4.56}$$

With additional conditions obtained from the invariance of both Wiener (4.39) and Poisson momenta (4.45) respectively as;

$$\Gamma^{*}_{(\tau)}(t, X(t)) = 0 \tag{4.57}$$

and

$$\Gamma^{**}_{(\tau)}(t, X(t)) = 0. \tag{4.58}$$

Equation (4.54) can be related with the first prolongation of the ordinary differential equations as follows.

Using the definition of first prolongation of an infinitesimal generator for non-stochastic ordinary differential equations

$$H^{[1]} = H + \eta_i^{[1]}\frac{\partial}{\partial \dot{x}_i}, \tag{4.59}$$

where

$$\dot{x}_i = \frac{dx_i}{dt} = D_t x_i \tag{4.60}$$

and

$$\eta_i^{[1]} = D_t(\xi_i) - \dot{x}_i D_t(\tau) \tag{4.61}$$

$$= \frac{\partial \xi_i}{\partial t} + \dot{x}_i\frac{\partial \xi_i}{\partial x} - \dot{x}_i\left(\frac{\partial \tau}{\partial t} + \dot{x}_i\frac{\partial \tau}{\partial x}\right). \tag{4.62}$$

With total time derivative D_t defined as

$$D_t = \frac{\partial}{\partial t} + \dot{x}_i \frac{\partial}{\partial x} + \ddot{x}_i \frac{\partial}{\partial \dot{x}_i} + \ldots, \tag{4.63}$$

using the definition of first prolongation on $(\dot{x}_i - f_i)$ at $\dot{x}_i = f_i$, can be expressed as

$$H^{[1]}(\dot{x}_i - f_i)|_{\dot{x}_i = f_i} = \eta_i^{[1]} - H(f_i). \tag{4.64}$$

Using (4.60) and (4.62), equation (4.54) can be rewritten as

$$H^{[1]}(\dot{x}_i - f_i)|_{\dot{x}=f} - \frac{\lambda J_i}{2}\left(\frac{\partial \tau}{\partial t} + f_i \frac{\partial \tau}{\partial x}\right) + \frac{1}{2}\sum_{k=1}^{M} G_{ik}(t, X(t)) G_{mk}(t, X(t)) \left(\frac{\partial^2 \xi_j}{\partial x_i \partial x_m} - f_j \frac{\partial^2 \tau}{\partial x_i \partial x_m}\right) = 0. \tag{4.65}$$

The operators $\Gamma(t,x)$, $\Gamma^*(t,x)$ and $\Gamma^{**}(t,x)$ are defined as in (4.6), (4.7) and (4.8), the temporal infinitesimals $\xi(t,x)$ and $\tau(t,x)$ are called the admitted symmetries of (4.3) if and only if they satisfied the determining equations (4.54) - (4.58).

Remark 4.3.1 *Note that by removing the jump diffusion term i.e., substituting $J(t,x) = 0$ in (4.3), the determining equations was partially covered in [7, 10] when considering stochastic differential equations driven by Wiener processes using the so called fiber preserving transformation and the ignoring extra condition found in (4.57). Similarly, for jump diffusion term $J(t,x) = 0$ determining equations (4.54), (4.55) and (4.57) are derived in [8, 9] while studying Wiener stochastic differential equations.*

4.3.1 Unal's Extra Condition

Unal's G. in [9] commented that the Itô multiplication properties for the transformed processes has to be satisfied i.e.,

$$d\overline{N}(\bar{t})d\overline{N}(\bar{t}) = d\overline{N}(\bar{t}), \quad d\overline{W}_l(\bar{t})d\overline{N}(t), \quad d\overline{N}(\bar{t})d\bar{t} = 0 \tag{4.66}$$

and

$$dW_l(t)dW_m(t) = \delta_m^l dt, \quad dW(t)dt = 0, \quad dtdt = 0. \tag{4.67}$$

Fredericks E. and Mahomed F. M. [8] proof that in the case of Wiener driven stochastic differential equation, the invariance of the moments of the process is sufficient with no recourse to the Itô multiplication properties of the transformed variables. However, we prove in chapter 2 and 3 that, is not the case for Poisson driven stochastic equations, in which equation (4.66) led to extra condition

$$\Gamma(\tau)(t, X(t)) = 0. \tag{4.68}$$

4.4 Applications

In this section, we apply the determining equations obtained for some jump-diffusion models to find their respective infinitesimals.

Example 4.4.1 *Consider a stochastic model driven by both Wiener and Poisson processes*

$$dX(t) = -kt^2 dt + \sqrt{D} dW + b dN(t) \tag{4.69}$$

with initial condition $X(0) = x_0$, where D is non-negative constant and $b \neq 0$.

From the jump-diffusion model (4.69) we have the drift vector, Wiener diffusion and jump coefficients respectively as

$$f(t,x) = -kt^2, \quad G(x,t) = \sqrt{D}, \quad D > 0 \quad and \quad J(t,x) = b \quad b \neq 0. \tag{4.70}$$

Using the determining equations (4.54), (4.55) and (4.56) respectively, we get

$$-kt^2\left(\frac{\partial \tau}{\partial t} - kt^2 \frac{\partial \tau}{\partial x} + \frac{D}{2}\frac{\partial^2 \tau}{\partial x^2}\right) + \frac{b\lambda}{2}\left(\frac{\partial \tau}{\partial t} - kt^2 \frac{\partial \tau}{\partial x} + \frac{D}{2}\frac{\partial^2 \tau}{\partial x^2}\right) - 2kt\tau = \frac{\partial \xi}{\partial t} - kt^2 \frac{\partial \xi}{\partial x} + \frac{D}{2}\frac{\partial^2 \xi}{\partial x^2}, \tag{4.71}$$

$$\frac{\sqrt{D}}{2}\left(\frac{\partial \tau}{\partial t} - kt^2 \frac{\partial \tau}{\partial x} + \frac{D}{2}\frac{\partial^2 \tau}{\partial x^2}\right) = \sqrt{D}\frac{\partial \xi}{\partial x}, \tag{4.72}$$

and

$$\frac{b}{2}\left(\frac{\partial \tau}{\partial t} - kt^2 \frac{\partial \tau}{\partial x} + \frac{D}{2}\frac{\partial^2 \tau}{\partial x^2}\right) = \xi(t, x+b) - \xi(t,x). \tag{4.73}$$

Using (4.48) and (4.58) we get the temporal infinitesimal as

$$\tau(t) = c_1 t + c_2. \tag{4.74}$$

Substituting the temporal infinitesimal (4.74) into (4.71), (4.72) and (4.73) we respectively get

$$\left(-kt^2+\frac{b\lambda}{2}-2kt^2\right)c_1-2ktc_2=\frac{\partial\xi}{\partial t}-kt^2\frac{\partial\xi}{\partial x}+\frac{D}{2}\frac{\partial^2\xi}{\partial x^2}, \tag{4.75}$$

$$\frac{\partial\xi}{\partial x}=\frac{c_1}{2} \tag{4.76}$$

and

$$\xi(t,x+b)-\xi(t,x)=\frac{bc_1}{2}. \tag{4.77}$$

Solving the differential equation in (4.76) yields the spatial infinitesimal

$$\xi(t,x)=\frac{c_1x}{2}+f(t). \tag{4.78}$$

Substituting the spatial infinitesimal (4.78) in (4.75) gives

$$\left(-kt^2+\frac{b\lambda}{2}-2kt^2\right)c_1-2ktc_2=\frac{df(t)}{dt}-\frac{kt^2c_1}{2}. \tag{4.79}$$

Solving (4.79) for $f(t)$ yields

$$f(t)=\left(\frac{tb\lambda}{2}-\frac{5kt^3}{6}\right)c_1-kt^2c_2+c_3. \tag{4.80}$$

Substituting (4.80) into (4.78) and by using (4.77) the spatial infinitesimal becomes

$$\xi(t,x)=\left(\frac{x}{2}+\frac{tb\lambda}{2}-\frac{5kt^3}{6}\right)c_1-kt^2c_2+c_3. \tag{4.81}$$

Finally, using the extra condition (4.68) the spatial and temporal infinitesimals reduce to

$$\xi(t,x)=-kt^2c_2+c_3, \quad and \quad \tau(t,x)=c_2. \tag{4.82}$$

Therefore, jump-diffusion stochastic differential equation (4.69) admitted two dimensional Lie symmetry infinitesimal generators

$$H_1=\frac{\partial}{\partial t}-kt^2\frac{\partial}{\partial x}, \qquad H_2=\frac{\partial}{\partial x}. \tag{4.83}$$

Example 4.4.2 *Consider the system of stochastic differential equations studied by Giuseppe Gaeta [16] with additional jump term*

$$\begin{aligned} dX_1(t) &= X_2 dt \\ dX_2(t) &= -k^2 X_2 dt + \sqrt{2k^2} dW + \alpha t dN(t) \end{aligned} \tag{4.84}$$

with k^2 a positive constant and $X(0) = x_0$.

Therefore, the drift, jump vector and Wiener diffusion matrix are respectively

$$f_j(t,x) = \begin{pmatrix} X_2 \\ -k^2 X_2 \end{pmatrix}, \quad J_j(t,x) = \begin{pmatrix} 0 \\ \alpha t \end{pmatrix}, \quad G_{ij}(t,x) = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{2k^2} \end{pmatrix}. \tag{4.85}$$

Using determining equation (4.54) for $j=1$ and $j=2$ gives

$$x_2\left(\frac{\partial \tau}{\partial t} + x_2 \frac{\partial \tau}{\partial x_1} - k^2 x_2 \frac{\partial \tau}{\partial x_2}\right) + \xi_2(t,x_1,x_2) = \frac{\partial \xi_1}{\partial t} + x_2 \frac{\partial \xi_1}{\partial x_1} - k^2 x_2 \frac{\partial \xi_1}{\partial x_2} + k^2 \frac{\partial^2 \xi_1}{\partial x_2^2} \tag{4.86}$$

and

$$\begin{aligned} -k^2 x_2\left(\frac{\partial \tau}{\partial t} + x_2 \frac{\partial \tau}{\partial x_1} - k^2 x_2 \frac{\partial \tau}{\partial x_2}\right) - k^2 \xi_2(t,x_1,x_2) + \frac{\alpha \lambda t}{2}\left(\frac{\partial \tau}{\partial t} + x_2 \frac{\partial \tau}{\partial x_1} - k^2 x_2 \frac{\partial \tau}{\partial x_2}\right) = \frac{\partial \xi_2}{\partial t} \\ + x_2 \frac{\partial \xi_2}{\partial x_1} - k^2 x_2 \frac{\partial \xi_2}{\partial x_2} + k^2 \frac{\partial^2 \xi_2}{\partial x_2^2}. \end{aligned} \tag{4.87}$$

While equation (4.55) for $j=1$ and $j=2$ gives

$$\frac{\partial \xi_1}{\partial x_2} = \frac{1}{2}\left(\frac{\partial \tau}{\partial t} + x_2 \frac{\partial \tau}{\partial x_1} - k^2 x_2 \frac{\partial \tau}{\partial x_2}\right) \tag{4.88}$$

and

$$\frac{\partial \xi_2}{\partial x_2} = \frac{1}{2}\left(\frac{\partial \tau}{\partial t} + x_2 \frac{\partial \tau}{\partial x_1} - k^2 x_2 \frac{\partial \tau}{\partial x_2}\right). \tag{4.89}$$

Using (4.48) and (4.58) we get the temporal infinitesimal

$$\tau(t) = c_1 t + c_2. \tag{4.90}$$

Substituting temporal infinitesimal (4.90) into (4.86) and (4.87) respectively gives

$$x_2 c_1 + \xi_2 = \frac{\partial \xi_1}{\partial t} + x_2 \frac{\partial \xi_1}{\partial x_1} - k^2 x_2 \frac{\partial \xi_1}{\partial x_2} + k^2 \frac{\partial^2 \xi_1}{\partial x_2^2}, \tag{4.91}$$

$$\left(\frac{\alpha\lambda t}{2} - k^2 x_2\right) c_1 - k^2 \xi_2 = \frac{\partial \xi_2}{\partial t} + x_2 \frac{\partial \xi_2}{\partial x_1} - k^2 x_2 \frac{\partial \xi_2}{\partial x_2} + k^2 \frac{\partial^2 \xi_2}{\partial x_2^2}. \tag{4.92}$$

Similarly, substituting (4.90) into (4.88) and (4.89) yields

$$\frac{\partial \xi_1}{\partial x_2} = \frac{c_1}{2} \tag{4.93}$$

and

$$\frac{\partial \xi_2}{\partial x_2} = \frac{c_1}{2}. \tag{4.94}$$

Solving (4.93) and (4.94) respectively gives

$$\xi_1 = \frac{c_1 x_2}{2} + f(t, x_1) \tag{4.95}$$

and

$$\xi_2 = \frac{c_1 x_2}{2} + g(t, x_1). \tag{4.96}$$

Substituting (4.95) and (4.96) into (4.91) and (4.92) respectively gives

$$x_2 c_1 + \frac{c_1 x_2}{2} + g(t, x_1) = \frac{\partial f(t, x_1)}{\partial t} + x_2 \frac{\partial f(t, x_1)}{\partial x_1} \tag{4.97}$$

and

$$\left(\frac{\alpha\lambda t}{2} - k^2 x_2\right) c_1 - \frac{c_1 k^2 x_2}{2} - k^2 g(t, x_1) = \frac{\partial g(t, x_1)}{\partial t} + x_2 \frac{\partial g(t, x_1)}{\partial x_1}. \tag{4.98}$$

It is clear to see that, since $f(t,x_1)$ and $g(t,x_1)$ does not depend on x_2 we have from (4.97) and (4.98)

$$c_1 = \frac{\partial f(t, x_1)}{\partial x_1} = \frac{\partial g(t, x_1)}{\partial x_1} = 0. \tag{4.99}$$

Substituting from (4.99) into (4.97) and (4.98) we respectively get

$$\frac{\partial f(t, x_1)}{\partial t} - g(t, x_1) = 0 \tag{4.100}$$

and

$$\frac{\partial g(t, x_1)}{\partial t} + k^2 g(t, x_1) = 0. \tag{4.101}$$

Solving (4.101) gives

$$g(t,x_1) = c_3 e^{-k^2 t}. \tag{4.102}$$

Substituting (4.102) into (4.100) leads to

$$f(t,x_1) = -\frac{c_3 e^{-k^2 t}}{k^2} + c_4. \tag{4.103}$$

Therefore, by using (4.99) in (4.90), (4.95) and (4.96) respectively gives the following infinitesimals

$$\tau(t) = c_2, \qquad \xi_1 = \frac{-c_3 e^{-k^2 t}}{k^2} + c_4 \qquad and \qquad \xi_2 = c_3 e^{-k^2 t}. \tag{4.104}$$

We can clearly see ξ_1 *satisfied (4.56) automatically for* $j = 1$. *For* $j = 2$, *substituting* $\xi_2 = c_3 e^{-k^2 t}$ *and* $\tau(t) = c_2$ *into (4.56) we get*

$$c_2 \alpha = 0, \tag{4.105}$$

this implies $c_2 = 0$, *since* $\alpha \neq 0$.

Therefore, substituting $c_2 = 0$ *into (4.104) we have the infinitesimals reduced to*

$$\tau(t) = 0, \qquad \xi_1 = \frac{-c_3 e^{-k^2 t}}{k^2} + c_4 \qquad and \qquad \xi_2 = c_3 e^{-k^2 t}. \tag{4.106}$$

Therefore the symmetries of the infinitesimal generators are two dimensional given as:

$$H_1 = \frac{-e^{-k^2 t}}{k^2}\frac{\partial}{\partial x_1} + e^{-k^2 t}\frac{\partial}{\partial x_2}, \qquad H_2 = \frac{\partial}{\partial x_1} \tag{4.107}$$

with corresponding Lie bracket relations $[H_1, H_2] = [H_2, H_1] = 0$. *Which shows that the symmetries generator (4.107) forms an abelian group.*

Remark 4.4.3 *After considering the model that involves both Wiener and Poisson diffusion we recover two of the three symmetries obtained by Giuseppe Gaeta [16]. This is to be expected since the jump term adds uncertainty to the model.*

Example 4.4.4 *Consider the jump SDE, linear in the state process* $X(t)$, *with constant coefficients;*

$$dX(t) = X(t)\Big(u_0(t)dt + \alpha_0(t)dW + v_0(t)dN(t)\Big) \tag{4.108}$$

with initial condition $X(t_0) = x_0 > 0$. The coefficient $u_0(t)$ is called the drift or deterministic coefficient, $v_0(t)$ is called the jump amplitude coefficient of the jump term and $\alpha_0(t)$ is called Wiener diffusion coefficient, with jump intensity $\lambda = \lambda_0$.

Therefore, we have the drift, Brownian motion diffusion and jump coefficients as

$$f(t,x) = u_0 x, \qquad g(t,x) = \alpha_0 x \qquad \text{and} \qquad J(t,x) = v_0 x. \tag{4.109}$$

with u_0, α_0 v_0 non-zero.

Using the determining equations (4.54), (4.55) and (4.56) we respectively get

$$u_0 x\left(\frac{\partial\tau}{\partial t} + u_0 x\frac{\partial\tau}{\partial x} + \frac{\alpha_0^2 x^2}{2}\frac{\partial^2\tau}{\partial x^2}\right) + \frac{v_0 x\lambda_0}{2}\left(\frac{\partial\tau}{\partial t} + u_0 x\frac{\partial\tau}{\partial x} + \frac{\alpha_0^2 x^2}{2}\frac{\partial^2\tau}{\partial x^2}\right) + u_0\xi(t,x) = \frac{\partial\xi}{\partial t} + u_0 x\frac{\partial\xi}{\partial x} + \frac{\alpha_0^2 x^2}{2}\frac{\partial^2\xi}{\partial x^2}, \tag{4.110}$$

$$\frac{\alpha_0 x}{2}\left(\frac{\partial\tau}{\partial t} + u_0 x\frac{\partial\tau}{\partial x} + \frac{\alpha_0^2 x^2}{2}\frac{\partial^2\tau}{\partial x^2}\right) + \alpha_0\xi(t,x) = \alpha_0 x\frac{\partial\xi}{\partial x} \tag{4.111}$$

and

$$\frac{v_0 x}{2}\left(\frac{\partial\tau}{\partial t} + u_0 x\frac{\partial\tau}{\partial x} + \frac{\alpha_0^2 x^2}{2}\frac{\partial^2\tau}{\partial x^2}\right) + v_0\xi = \xi(t, x+v_0 x) - \xi(t,x). \tag{4.112}$$

From (4.48) and (4.58) we have the temporal infinitesimal as

$$\tau(t) = c_1 t + c_2. \tag{4.113}$$

Substituting temporal infinitesimal (4.113) into (4.110), (4.111) and (4.112) we respectively have

$$\left(u_0 x + \frac{v_0 x\lambda_0}{2}\right)c_1 + u_0\xi(t,x) = \frac{\partial\xi}{\partial t} + u_0 x\frac{\partial\xi}{\partial x} + \frac{\alpha_0^2 x^2}{2}\frac{\partial^2\xi}{\partial x^2}, \tag{4.114}$$

$$\frac{c_1 x}{2} + \xi(t,x) = x\frac{\partial\xi}{\partial x}, \tag{4.115}$$

and

$$\frac{v_0 c_1 x}{2} + v_0\xi = \xi(t, x+v_0 x) - \xi(t,x). \tag{4.116}$$

Solving (4.115) we get the spatial infinitesimal as

$$\xi(t,x) = \frac{xc_1 \ln|x|}{2} + f(t)x, \quad for \quad x > 0. \tag{4.117}$$

By substituting spatial infinitesimal (4.117) into (4.116), we finally get

$$c_1 = 0. \tag{4.118}$$

Therefore, substituting (4.118) into (4.117) and (4.113) the temporal and spatial infinitesimals respectively reduce to

$$\tau(t) = c_2 \tag{4.119}$$

and

$$\xi(t,x) = f(t)x. \tag{4.120}$$

Substituting (4.120) in (4.114) gives

$$\frac{df(t)}{dt} = 0. \tag{4.121}$$

Which implies

$$\xi(t,x) = c_3 x. \tag{4.122}$$

Therefore, the symmetries algebra is two dimensional given as

$$H_1 = \frac{\partial}{\partial t}, \qquad H_2 = x\frac{\partial}{\partial x}. \tag{4.123}$$

The Lie bracket relation of the generator (4.123) is $[H_1, H_2] = [H_2, H_1] = 0$, *which shows that the symmetries algebra is also an abelian group .*

Remark 4.4.5 *Symmetry algebra of geometric Brownian motion driven stochastic differential equation was discussed by Ebrahim and Mahomed F. M. [8] and the generators are found to be generated by three-dimensional algebra. In this example, we see that adding Poisson diffusion to the model reduces the symmetry by one dimension. Interestingly, the two generators found are the only ones that leave a stochastic differential invariant [8].*

4.5 Conclusion

Lie symmetry of jump-diffusion stochastic differential equations was discussed, by considering infinitesimals of the spatial and temporal variables. This was achieved by utilising the random time formula for standard Brownian motion used in [13, 14, 20, 28, 33] to study symmetry of Wiener process stochastic differential equations i.e.,

$$d\overline{W}(\bar{t}) = \sqrt{\frac{d\bar{t}(t)}{dt}}dW(t) \tag{4.124}$$

as well as the random time formula for Poisson processes derived in Chapter 3 i.e.,

$$d\overline{N}(\bar{t}) = dN(t) + \frac{\epsilon}{2}\frac{d\tau}{dt}\Big(\lambda dt + dN(t)\Big) + O(\epsilon). \tag{4.125}$$

The determining equations of the jump-diffusion stochastic differential equation

$$dX(t) = f_i(t, X(t))dt + G_{ik}(t, X(t))dW_k(t) + J_i(t, X(t))dN(t) \tag{4.126}$$

were derived and are found to be deterministic after applying the invariance methodology to the moments of both Wiener and Poisson processes. The determining equations found are similar to the one used in [8, 9, 13, 20, 28, 32] if the Poisson terms are removed. We apply the determining equations to several jump-diffusion models to show how they can be used to find the admitted Lie infinitesimals transformation of the respective models. Finally, a Lie bracket relation was found to show the relationship between the infinitesimals generators, the which show that the infinitesimals generators are closed under Lie relations and hence form a Lie algebra. The Lie group classification of the given examples is given in *Table 4.2* below.

TABLE 4.1: Lie Group Classification Chapter 4

Group Dimension	Basis Operators	Equations
2	$H_1 = \frac{\partial}{\partial t} - kt^2\frac{\partial}{\partial x}, \quad H_2 = \frac{\partial}{\partial x}.$	$dX = -kt^2dt + \sqrt{D}dW + bdN(t)$
2	$H_1 = \frac{-e^{-k^2t}}{k^2}\frac{\partial}{\partial x_1} + e^{-k^2t}\frac{\partial}{\partial x_2}, \quad H_2 = \frac{\partial}{\partial x_1}.$	$dX_1 = X_2dt$ $dX_2 = -k^2X_2dt + \sqrt{2k^2}dW + \alpha tdN(t)$
2	$H_1 = \frac{\partial}{\partial t}, \quad H_2 = x\frac{\partial}{\partial x}.$	$dX(t) = X(t)(u_0(t)dt + \alpha_0(t)dW + v_0(t)dN(t))$

Chapter 5

W- symmetry of Itô Stochastic Differential Equations with a Finite Jump Process

In this Chapter, we discuss Lie point symmetry of stochastic differential equations driven by Wiener and Poisson processes. The symmetry is obtained by considering infinitesimals involving not only spatial and temporal variables but also that of vector Wiener process variable $W(t)$. This work leads to the derivation of the random time change formula of Itô Brownian motion [35] in Lie transformation context.

5.1 Introduction

In Chapter 4, we consider symmetries of Itô stochastic differential equations (SDE) with a finite jump process involving the infinitesimals of temporal t and spatial variables x. In this chapter, we extend Chapter 4 by including the infinitesimal of the Wiener process $W(t)$ i.e., we now consider infinitesimals involving not only the spatial and time variables, but also the Wiener diffusion process.

The Lie point symmetries of Itô stochastic differential equations (SDE) driven by both Wiener and Poisson processes

$$dX_i(t) = f_i(t, X(t,W))dt + G_{il}(t, X(t,W))dW_l(t,W) + J_i(t, X(t,W))dN(t) \tag{5.1}$$

are considered. Where $f_i(t, X(t,W))$ and $J_i(t, X(t,W))$ are $n \times 1$ dimensional drift vector coefficient and jump diffusion coefficient respectively and $G_{ik}(t, X(t,W))$ is the $n \times M$ dimensional Wiener diffusion matrix coefficient, with jump rate $\lambda > 0$.

The determining equations for Itô stochastic differential equations (SDE) with finite jump (5.1) are derived in an Itô calculus context and find to be non-stochastic.

In Chapter 4, we start with an arbitrary function $F(t,x)$ which is at least twice differentiable with respect to spatial x and temporal t variables respectively, which satisfy the finite jump stochastic differential equation

(5.2) $$dF_j(t,X(t,W)) = \Gamma_{(F)_j}dt + \Gamma^*_{(F)_j}dW(t) + \Gamma^{**}_{(F)_j}dN(t)$$

i.e., by the Itô lemma of the jump diffusion process. Where

(5.3) $$\Gamma_{(F)_j} = \frac{\partial F_j}{\partial t}dt + f_i\frac{\partial F_j}{\partial x_i} + \frac{1}{2}G_{ik}(t,X(t,W))G_{mk}(t,X(t,W))\frac{\partial^2 F_j}{\partial x_i \partial x_m},$$

(5.4) $$\Gamma^*_{(F)_j} = G_{il}(t,X(t,W))\frac{\partial F_j}{\partial x_i}$$

and

(5.5) $$\Gamma^{**}_{(F)_j} = F_j\Big(t, X_i(t,W) + J(t,X_i(t,W))\Big) - F_j(t,X_i(t,W))$$

In [29] E. Fredericks, successfully extended the operators (5.3) and (5.4) using the same arbitrary function $F(t,x)$ so that they include the Wiener process i.e.,

(5.6) $$dF_j(t,X(t,W)) = \Gamma_{(w)}(F)_j dt + \Gamma^*_{(w)}(F)_j dW(t,w) + \Gamma^{**}_{(w)}(F)_j dN(t,w),$$

with

(5.7) $$\Gamma_{(w)}(F)_j = \frac{\partial F_j}{\partial t}dt + f_i\frac{\partial F_j}{\partial x_i} + \frac{1}{2}\frac{\partial^2 F_j}{\partial W_i \partial W_m} + \frac{1}{2}G_{ik}(t,X(t,W))G_{mk}(t,X(t,W))\frac{\partial^2 F_j}{\partial x_i \partial x_m}$$

(5.8) $$\Gamma^*_{(w)}(F)_j = \frac{\partial F_j}{\partial W} + G_{il}(t,X(t,W))\frac{\partial F_j}{\partial x_i}$$

and additional jump term

(5.9) $$\Gamma^{**}_{(w)}(F)_j = F_j\Big(t, X_i(t,W) + J(t,X_i(t,W))\Big) - F_j(t,X_i(t,W)).$$

The operators (5.7), (5.8) and (5.9) still satisfy (5.6) since it is not a function of the Wiener variable w. Thus operators were used to confirm the finding of Gaeta [16].

The main result obtained in this chapter is summarised in the theorem below.

Theorem 5.1.1 *The Itô stochastic differential equation with a finite jump*

$$dX_i(t) = f_i(t, X(t,w))dt + G_{il}(t, X(t,w))dW(t) + J_i(t, X(t,w))dN(t)$$

where $f_i(t, X(t,w))$ and $J_i(t, X(t,w))$ are $n \times 1$ dimensional drift vector coefficient and jump diffusion coefficient respectively and $G_{ik}(t, X(t,w))$ is the Wiener diffusion matrix coefficient of $n \times M$ dimension, with infinitesimal generator

$$H = \tau(t,x,w)\frac{\partial}{\partial t} + \xi_i(t,x)\frac{\partial}{\partial x_i} + \phi_i(t,x,w)\frac{\partial}{\partial W}, \tag{5.10}$$

has the following determining equations

$$\left(f_j\Gamma_{(w)}(\tau) + \frac{J_j\lambda\Gamma_{(w)}(\tau)}{2} + H(f_j) - \Gamma_{(w)}(\xi)_j\right)(t, X(t,w)) = 0, \tag{5.11}$$

$$\left(H(G_{jk}) + G_{jk}\frac{\Gamma_{(w)}(\tau)}{2} - \Gamma^*_{(w)}(\xi)_j\right)(t, X(t,w)) = 0 \tag{5.12}$$

and

$$\left(H(J_j) + \frac{J_j\Gamma(\tau)}{2} - \Gamma^{**}_{(w)}(\xi)_j\right)(t, X(t,w)) = 0 \tag{5.13}$$

with the following additional conditions;

$$\Gamma^*_{(w)}(\tau)(t, X(t,w)) = 0, \tag{5.14}$$

$$\Gamma^{**}_{(w)}(\tau)(t, X(t)) = 0, \tag{5.15}$$

$$\Gamma^{**}_{(w)}(\phi_l)(t, X(t,w)) = 0, \tag{5.16}$$

$$\Gamma^*_{(w)}(\phi_l)(t, X(t,w)) = 0, \tag{5.17}$$

(5.18) $$\Gamma_{(\tau)}(t, X(t,w)) = 0$$

and

(5.19) $$\Gamma_{(w)}(\phi_l)(t, X(t,w)) = 0.$$

The operators $\Gamma_{(w)}(t,x)$, $\Gamma^{*}_{(w)}(t,x)$ and $\Gamma^{**}_{(w)}(t,x)$ are defined in (5.7), (5.8) and (5.9) respectively, then the infinitesimals $\xi(t,x)$, $\tau(t,x,w)$ and $\phi(t,x,w)$ are called the admitted symmetries of (5.1) if and only if they satisfied the determining equations (5.11) - (5.19). The proof of the theorem will follow in the subsequent sections.

5.2 Lie Group Transformations

Consider a one-parameter group of transformations of time index t, the spatial variable x and jump variable J respectively,

(5.20) $$\bar{t} = \theta_1(t,x,w,\epsilon), \qquad \bar{x} = \theta_2(t,x,\epsilon), \qquad \overline{W} = \theta_3(t,x,w,\epsilon)$$

with the infinitesimals of the temporal, spatial and jump variables respectively as

(5.21) $$\frac{\partial \theta_1}{\partial \epsilon} = \tau(\theta_1,\theta_2,\theta_3), \qquad \frac{\partial \theta_2}{\partial \epsilon} = \xi(\theta_1,\theta_2,\theta_3), \qquad \frac{\partial \theta_3}{\partial \epsilon} = \phi(\theta_1,\theta_2,\theta_3)$$

satisfying the following initial conditions at $\epsilon = 0$

$$\bar{t}\Big|_{\epsilon=0} = t, \qquad \overline{X}\Big|_{\epsilon=0} = X, \qquad \overline{W}\Big|_{\epsilon=0} = W.$$

The group transformations can be expressed as

(5.22) $$\bar{t} = t + \epsilon\tau(t,x,w),$$

(5.23) $$\overline{X_j} = X_j + \epsilon\xi_j(t,x)$$

and

(5.24) $$\overline{W_l} = W_l + \epsilon\phi_l(t,x,w).$$

Where ϵ is the parameter of the group, hence the corresponding generator of the Lie algebra is of the form

(5.25) $$H = \tau(t,x,w)\frac{\partial}{\partial t} + \xi_i(t,x)\frac{\partial}{\partial x_i} + \phi_l(t,x,w)\frac{\partial}{\partial w_l}.$$

The Point differential transformation of the temporal and spatial variables are respectively

(5.26) $$d\bar{t} = dt + \epsilon d\tau + O(\epsilon)$$

and

(5.27) $$d\overline{X_j} = dX_j + \epsilon d\xi_j + O(\epsilon).$$

Similarly, the point differential transformations of Wiener and Jump variables are respectively

(5.28) $$d\overline{W_l}(\bar{t}) = dW_l(t) + \epsilon d\phi_l + O(\epsilon)$$

and

(5.29) $$d\overline{N}(\bar{t}) = dN(t) + \frac{\epsilon}{2}\frac{d\tau}{dt}\Big(\lambda dt + dN(t)\Big) + O(\epsilon).$$

The infinitesimals of spatial, temporal and jump variables using Itô formula (5.6) can be written in Itô forms of the stochastic differential equations of jump form as

(5.30)
$$d\xi_j = \Gamma_{(w)}(\xi)_j(t,X(t,w))dt + \Gamma^*_{(w)}(\xi)_j(t,X(t,w))dW(t) + \Gamma^{**}_{(w)}(\xi)_j(t,X(t,w))dN(t),$$

(5.31) $$d\tau = \Gamma_{(w)}(\tau)(t,X(t,w))dt + \Gamma^*_{(w)}(\tau)(t,X(t,w))dW(t) + \Gamma^{**}_{(w)}(\tau)(t,X(t,w))dN(t)$$

and

(5.32)
$$d\phi_l = \Gamma_{(w)}(\phi_l)(t,X(t,w))dt + \Gamma^*_{(w)}(\phi)_l(t,X(t,w))dW(t) + \Gamma^{**}_{(w)}(\phi_l)(t,X(t,w))dN(t).$$

Substituting the Itô form of the spatial infinitesimal (5.30) and temporal infinitesimal (5.31) in equations (5.26) and (5.27), spatial and temporal differential transformations respectively can be written in Itô form as

$$d\bar{t} = dt + \epsilon\Big(\Gamma_{(w)}(\tau)dt + \Gamma^{*}_{(w)}(\tau)dW(t) + \Gamma^{**}_{(w)}(\tau)dN(t)\Big)(t, X(t,w)) + O(\epsilon) \tag{5.33}$$

and

$$d\overline{X_j}(\bar{t}) = dX_j(t) + \epsilon\Big(\Gamma_{(w)}(\xi)_j dt + \Gamma^{*}_{(w)}(\xi)_j dW(t) + \Gamma^{**}_{(w)}(\xi)_j dN(t)\Big)(t, X(t,w)) + O(\epsilon). \tag{5.34}$$

Similarly, substituting the Itô form of the jump infinitesimal (5.32) in (5.28) and the temporal infinitesimal (5.31) in (5.29), the Wiener process and the jump process group transformations in Itô form is respectively;

$$d\overline{W_l}(\bar{t}) = dW_l(t) + \epsilon\Big(\Gamma_{(w)}(\phi_l)dt + \Gamma^{*}_{(w)}(\phi_l)dW(t) + \Gamma^{**}_{(w)}(\phi_l)dN(t)\Big)(t, X(t,w)) + O(\epsilon) \tag{5.35}$$

and

$$d\overline{N}(\bar{t}) = dN(t) + \frac{\epsilon}{2}\frac{\Gamma_{(w)}(\tau)dt + \Gamma^{*}_{(w)}(\tau)dW(t) + \Gamma^{**}_{(w)}(\tau)dN(t)}{dt}\Big(\lambda dt + dN(t)\Big)(t, X(t,w)) + O(\epsilon). \tag{5.36}$$

Equation (5.36) can be expand using the Itô multiplication properties *viz Table 1.2* to

$$\begin{aligned} d\overline{N}(\bar{t}) = dN(t) + \frac{\epsilon}{2}\bigg(\lambda\Big(\Gamma_{(w)}(\tau)dt + \Gamma^{*}_{(w)}(\tau)dW(t) + \Gamma^{**}_{(w)}(\tau)dN(t)\Big)(t, X(t,w))\bigg) \\ + \frac{\epsilon}{2}\bigg(\Gamma_{(w)}(\tau)dN(t) + \frac{\Gamma^{**}_{(w)}(\tau)dN(t)}{dt}\bigg)(t, X(t,w)) + O(\epsilon). \end{aligned} \tag{5.37}$$

Setting the differential group transformations of the temporal, spatial, Wiener and jump processes in Itô form, we are now ready to apply them and transform the jump-diffusion coefficients as well as the moments of the processes i.e., to make sure they remain invariant under the Lie group of transformations.

5.2.1 Invariance Form of the Spatial Process

To ensure the recovery of the finite transformations from the infinitesimal transformation, we need to transform (5.1) into

$$d\overline{X_j}(\bar{t}, \bar{w}) = f_j(\bar{t}, \overline{X}(\bar{t}, \bar{w}))d\bar{t} + G_{jl}(\bar{t}, \overline{X}(\bar{t}, \bar{w}))d\overline{W}(\bar{t}) + J_j(\bar{t}, \overline{X}(\bar{t}, \bar{w}))d\overline{N}(\bar{t}). \tag{5.38}$$

This will be achieved by transforming the drift vector $f_j(t,X(t,w))$, Wiener diffusion $G_{jk}(t,X(t,w))$ and jump coefficients $J_j(t,X(t,w))$ using the infinitesimal generator of the Lie group of transformations

$$H=\tau(t,x,w)\frac{\partial}{\partial t}+\xi_i(t,x)\frac{\partial}{\partial x_i}+\phi_l(t,x,w)\frac{\partial}{\partial w_l}.$$

The drift vector component, Wiener and jump diffusion coefficients can be respectively transformed to;

$$\begin{aligned}\overline{f_j}(\bar{t},\overline{X}(\bar{t},\overline{w}))&=\left(f_j+\epsilon H(f_j)\right)(t,X(t,w))\\&=f_j(t,X(t,w))+\epsilon\left(\tau\frac{\partial f_j}{\partial t}+\xi_i\frac{\partial f_j}{\partial x_i}+\phi_l\frac{\partial f_j}{\partial w_l}\right)(t,X(t,w)),\end{aligned}\tag{5.39}$$

$$\begin{aligned}G^l_{jl}(\bar{t},\overline{X}(\bar{t}))&=\left(G_{jl}+\epsilon H(G_{jl})\right)(t,X(t,w))\\&=G_{jl}(t,X(t,w))+\epsilon\left(\tau\frac{\partial G_{jl}}{\partial t}+\xi_i\frac{\partial G_{jl}}{\partial x_i}+\phi_l\frac{\partial G_{jl}}{\partial w_l}\right)(t,X(t,w))\end{aligned}\tag{5.40}$$

and

$$\begin{aligned}J_j(\bar{t},\overline{X}(\bar{t}))&=\left(J_j+\epsilon H(J_j)\right)(t,X(t,w))\\&=J_j(t,X(t,w))+\epsilon\left(\tau\frac{\partial J_j}{\partial t}+\xi_i\frac{\partial J_j}{\partial x_i}+\phi_l\frac{\partial J_j}{\partial w_l}\right)(t,X(t,w)).\end{aligned}\tag{5.41}$$

In the next sections, we are going to make sure the transformed Wiener and Poisson processes together with there moments are still Wiener and Poisson processes, that is, to ensure they remain invariant under the Lie group transformation and later move to derive the determining equations.

5.2.2 Wiener Invariance Properties

We first verify that the transformed Wiener process is still Wiener by ensuring the Itô multiplication properties of the Brownian motion [9, 25] are satisfied i.e.,

$$dW_l(t)dW_m(t)=\delta^l_m dt,\quad dW(t)dt=0,\quad dtdt=0.\tag{5.42}$$

Using the differential group transformation of Wiener process (5.35), we find

(5.43)
$$d\overline{W}_l(\bar{t})d\overline{W}_m(\bar{t})=dW_l(t)dW_m(t)+\epsilon(\Gamma^*_{(w)}(\phi_l)+\Gamma^*_{(w)}(\phi_m))dW_l(t)dW_m(t)=\delta^l_m d\bar{t}.$$

Substituting the temporal group transformation (5.33) in (5.43) and by comparing the Wiener, jump and Riemann integrals we have the following relations

$$\Gamma^{*}_{(w)}(\phi_l)+\Gamma^{*}_{(w)}(\phi_m)=\Gamma_{(w)}(\tau)(t,X(t)), \tag{5.44}$$

$$\Gamma^{*}_{(w)}(\tau)(t,X(t))=0 \tag{5.45}$$

and

$$\Gamma^{**}_{(w)}(\tau)(t,X(t))=0. \tag{5.46}$$

Similarly, using the differential group transformation of Wiener process (5.35) and temporal variable (5.33) we have

$$d\overline{W}(\bar{t})d\bar{t}=\Gamma^{*}_{(w)}(\tau)(t,X(t))=0, \tag{5.47}$$

its easy the see $d\bar{t}d\bar{t}=0$.

Next, we apply the invariance to the moments of the Wiener process to make sure it remains invariant under the group transformations, *viz* the instantaneous mean and variance of the Wiener process which are:

$$E_Q\left[dW_l(t)|W=w\right]=0 \tag{5.48}$$

and

$$E_Q\left[dW_l(t)dW_m(t)|W=w\right]=\delta^l_m dt. \tag{5.49}$$

The invariance of the instantaneous mean of the transformed Wiener process under new measure $\overline{Q}$ is

$$E_{\overline{Q}}\left[d\overline{W}_l(t)|W=w\right]=0. \tag{5.50}$$

Substituting the differential group transformation of the Wiener process (5.35) into (5.50) we get

(5.51)
$$E_{\overline{Q}}\left[dW(t)+\epsilon\Big(\Gamma_{(w)}(\phi_l)dt+\Gamma^{*}_{(w)}(\phi_l)dW(t)+\Gamma^{**}_{(w)}(\phi_l)dN(t)\Big)(t,X(t,w))+O(\epsilon)|W=w\right]=0.$$

Expanding (5.51) using Itô multiplication properties of the Wiener process *viz Table 1.1* gives

(5.52)
$$\Big(\Gamma_{(w)}(\phi_l) + \lambda\Gamma^{**}_{(w)}(\phi_l)\Big)(t, X(t,w)) = 0.$$

Next, we apply the invariance form to instantaneous variant of the transformed Wiener process measure (5.49)

(5.53)
$$E_{\overline{Q}}\Big[d\overline{W}_l(\bar{t})d\overline{W}_m(\bar{t})|W = w\Big] = \delta^l_m d\bar{t}.$$

From which using differential group transformation of the Wiener process (5.35) gives

(5.54)
$$E_{\overline{Q}}\Big[dW_l(t)dW_m(t) + \epsilon\Big(\Gamma^{*}_{(w)}(\phi_l) + \Gamma^{*}_{(w)}(\phi_m)\Big)dW_l(t)dW_m(t)\Big] = \delta^l_m d\bar{t}.$$

Finally, substituting the Itô form of the group transformation of the index time variable (5.33) into (5.54) gives the following differential relation

(5.55)
$$\Big(\Gamma^{*}_{(w)}(\phi_l) + \Gamma^{*}_{(w)}(\phi_m)\Big)dt = \Gamma_{(w)}(\tau)(t, X(t,w))dt + \Gamma^{*}_{(w)}(\tau)(t, X(t,w))dW(t) + \Gamma^{**}_{(w)}(\tau)(t, X(t,w))dN(t).$$

Comparing the Wiener, jump and Riemann integrals we have the following relations

(5.56)
$$\Gamma^{*}_{(w)}(\phi_l) + \Gamma^{*}_{(w)}(\phi_m) = \delta^l_m \Gamma_{(w)}(\tau)(t, X(t,w)),$$

(5.57)
$$\Gamma^{*}_{(w)}(\tau)(t, X(t,w)) = 0$$

and

(5.58)
$$\Gamma^{**}_{(w)}(\tau)(t, X(t,w)) = 0.$$

5.2.3 Poisson Invariance Properties

Before deriving the determining equations, we need to apply the invariance to the transformed Poisson process and its moments to make sure it remains invariant under the group transformations.

To ensure the transformed Poisson process is still Poisson the following identities has to be satisfy [4, 5]

$$d\overline{N}(\bar{t})d\overline{N}(\bar{t}) = d\overline{N}(\bar{t}), \quad d\overline{W}_l(\bar{t})d\overline{N}(t), \quad d\overline{N}(\bar{t})d\bar{t} = 0. \tag{5.59}$$

From which using the Itô form of the group transformation of the jump variable (5.36), (5.59) satisfied for

$$\Gamma_{(w)}(\tau)(t, X(t)) = 0. \tag{5.60}$$

While using the Itô form of the group transformation of the Poisson and Wiener processes we have

$$d\overline{W}_l(\bar{t})d\overline{N}(\bar{t}) = \Gamma^{**}_{(w)}(\phi_l)(t, X(t)) = 0 \tag{5.61}$$

and finally $d\overline{N}(\bar{t})d\bar{t} = 0$.

Similarly, we verify that the instantaneous mean and variance of the Poisson process

$$E_Q\left[dN(t)|W = w\right] = \lambda dt \tag{5.62}$$

$$E_Q\left[dN(t)dN(t)|W = w\right] = \lambda dt \tag{5.63}$$

as well as the expectation of the differential product of Wiener and Poisson process group transformations remains invariant i.e.,

$$E_Q\left[dW_l(t)dN(t)\Big|W = w\right] = 0. \tag{5.64}$$

The invariance of the instantaneous mean of the transformed jump process under new measure $\overline{Q}$ is

$$E_{\overline{Q}}\left[d\overline{N}(\bar{t})|W = w\right] = \lambda d\bar{t}. \tag{5.65}$$

Expanding (5.65) using the Itô form of the group transformation of the jump variable (5.36) and that of the time variable (5.33), lead to

$$\Gamma^{**}_{(w)}(\tau)(t, X(t, w)) = 0. \tag{5.66}$$

Next, we apply the invariance form to instantaneous variance of the transformed jump process measure (5.63) from which using the Itô form of the group transformation of the jump variable (5.36) yields

$$E_{\overline{Q}}\left[d\overline{N}(\bar{t})d\overline{N}(\bar{t})\middle|W=w\right]=\lambda d\bar{t}. \tag{5.67}$$

Finally, equation (5.64) under new measure $\overline{Q}$ can be transformed to

$$E_{\overline{Q}}\left[d\overline{W}_l(\bar{t})d\overline{N}(\bar{t})\middle|W=w\right]=0 \tag{5.68}$$

Using the Itô form of the group transformation of Brownian motion (5.35) and that of jump variable (5.36) in (5.68) gives

$$E_{\overline{Q}}\left[\Gamma^{**}_{(w)}(\phi_l)dN(t)\right]=0 \tag{5.69}$$

which finally implies

$$\Gamma^{**}_{(w)}(\phi_l)(t,X(t,w))=0. \tag{5.70}$$

Thus, by using (5.57) and (5.58) we have derived the following generalised random time change formula

$$\bar{t}=\int^{t}\Gamma_{(w)}(\tau)(s)ds. \tag{5.71}$$

We have from (5.52)

$$\left(\Gamma_{(w)}(\phi_l)+\lambda\Gamma^{**}_{(w)}(\phi_l)\right)(t,X(t,w))=0, \tag{5.72}$$

substituting (5.70) into (5.72) we get

$$\Gamma_{(w)}(\phi_l)(t,X(t,w))=0. \tag{5.73}$$

From (5.56) and (5.60), we have

$$\Gamma^{*}_{(w)}(\phi)(t,X(t,w))=0. \tag{5.74}$$

Remark 5.2.1 *From equation (5.66) and (5.70), we can conclude that the infinitesimals of the temporal and Wiener processes are independent of the spatial variable*

x. This result was assumed by Gaeta to achieve his result on W-symmetries of Wiener stochastic differential equations [16].

Remark 5.2.2 *Note that by using the Lie group transformation to ensure the mean and variance of both Wiener and Poisson processes remain invariant under the transformations, we have obtained the same random time change formula*

$$d\overline{W}(\bar{t}) = \sqrt{\frac{d\bar{t}(t)}{dt}}\, dW(t), \tag{5.75}$$

used in [8, 9, 13, 20, 28, 32] to transform the Wiener process term. Even though, it was not derived formally using Lie group of transformations.

Definition 5.2.3 *The infinitesimals group transformations*

$$\bar{t} = t + \epsilon\tau(t,x,w), \quad \overline{X} = X_i + \epsilon\xi_i(t,x) \quad and \quad \overline{W} = W + \epsilon\phi(t,x,w) \tag{5.76}$$

are called symmetry transformations of the jump-diffusion stochastic differential equation (SDEJ) i.e.,

$$dX_i(t) = f_i(t,X(t))dt + G_{il}(t,X(t))dW(t) + J_i(t,X(t))dN(t) \tag{5.77}$$

if they leave (5.77). (5.59) and (5.42) invariant.

5.3 Derivation of the Determining Equations

In the previous sections, we transformed the drift, Wiener and jump diffusion components using the Lie group of transformations. We also ensured that the transform Wiener and Poisson process together with there mean and variance properties remain invariant under the Lie group of transformations, which gives us additional necessary and sufficient conditions in order for the stochastic equation (5.1) to remain invariant. The intention is to transform the finite jump stochastic differential equation (5.1) into a similar one using the transformation (5.76) i.e.,

$$d\overline{X_j}(\bar{t}) = f_j(\bar{t},\overline{X}(\bar{t}))d\bar{t} + G_{jl}(\bar{t},\overline{X}(\bar{t}))d\overline{W}(\bar{t}) + J_j(\bar{t},\overline{X}(\bar{t}))d\overline{N}(\bar{t}). \tag{5.78}$$

This can be achieved by substituting the Itô forms of the temporal, Wiener and jump group transformations as well as the transformed drift vector, Wiener diffusion and jump coefficients i.e., (5.33), (5.35), (5.36), (5.39), (5.40) and (5.41) into

(5.78) respectively

$$d\overline{X_j}(\bar{t}) = dX_j(t) + \epsilon\left(f_j\Gamma_{(w)}(\tau) + \frac{J_j\Gamma_{(w)}(\tau)}{2} + H(f_j) + G_{jk}\Gamma_{(w)}(\phi_l)\right)dt$$
$$+\epsilon\left(\frac{J_j\lambda}{2}\Gamma^*_{(w)}(\tau) + H(G_{jl}) + f_j\Gamma^*_{(w)}(\tau) + G_{jl}\Gamma^*_{(w)}(\phi)\right)dW(t) \tag{5.79}$$
$$+\epsilon\left(G_{jl}\Gamma^{**}_{(w)}(\phi_l) + H(J_j) + f_j\Gamma^{**}_{(w)}(\tau) + \frac{J_j\Gamma(\tau)}{2} + \frac{J_j\lambda\Gamma^{**}_{(w)}(\tau)}{2}\right)dN(t).$$

Using (5.57), (5.58), (5.70) and (5.73) equation (5.79) can be reduced to

$$d\overline{X_j}(\bar{t}) = dX_j(t) + \epsilon\left(f_j\Gamma_{(w)}(\tau) + \frac{J_j\Gamma_{(w)}(\tau)}{2} + H(f_j) + G_{jl}\Gamma_{(w)}(\phi_l)\right)dt$$
$$+\epsilon\left(H(G_{jl}) + G_{jl}\Gamma^*_{(w)}(\phi_l)\right)dW(t) + \epsilon\left(H(J_j) + \frac{J_j\Gamma_{(w)}(\tau)}{2}\right)dN(t). \tag{5.80}$$

Therefore, by comparing the Itô form of spatial group transformation (5.34) and (5.80) we have the following determining equations

$$\left(f_j\Gamma_{(w)}(\tau) + \frac{J_j\lambda\Gamma_{(w)}(\tau)}{2} + H(f) - \Gamma_{(w)}(\xi)_j\right)(t, X(t,w)) = 0, \tag{5.81}$$

$$\left(H(G_{jl}) + G_{jl}\Gamma^*_{(w)}(\phi_l) - \Gamma^*_{(w)}(\xi)_j\right)(t, X(t,w)) = 0 \tag{5.82}$$

and

$$\left(H(J_j) + \frac{J_j\Gamma_{(w)}(\tau)}{2} - \Gamma^{**}_{(w)}(\xi)_j\right)(t, X(t,w)) = 0. \tag{5.83}$$

With the following additional conditions *viz* (5.57), (5.58) and (5.70) respectively;

$$\Gamma^*_{(w)}(\tau)(t, X(t,w)) = 0, \tag{5.84}$$

$$\Gamma^{**}_{(\tau)}(t, X(t,w)) = 0, \tag{5.85}$$

and

$$\Gamma^{**}_{(w)}(\phi_l)(t, X(t)) = 0. \tag{5.86}$$

Similarly, we respectively have from (5.73), (5.60) and (5.74)

$$\Gamma_{(w)}(\phi_l)(t,X(t))=0, \tag{5.87}$$

$$\Gamma_{(w)}(\tau)(t,X(t))=0 \tag{5.88}$$

and

$$\Gamma^{*}_{(w)}(\phi_l)(t,X(t))=0. \tag{5.89}$$

Those additional conditions (5.84-5.89) are necessary for making the jump-diffusion stochastic differential equation invariant and are obtained from the invariance forms of the moments of the Wiener and Poisson processes.

Where the operators $\Gamma_{(w)}(t,x)$, $\Gamma^{*}_{(w)}(t,x)$ and $\Gamma^{**}_{(w)}(t,x)$ are defined as in (5.7), (5.8) and (5.9); while $\lambda>0$ is called the intensity of the jump process, then the infinitesimals $\xi(t,x)$, $\tau(t,x,w)$ and $\phi(t,x,w)$ are called the admitted symmetries of (5.1) if and only if they satisfied the determining equations (5.81) - (5.89).

Remark 5.3.1 *After removing the infinitesimal of the Wiener process, we obtained the same determining equations as the one we obtained in Chapter 4. Also, by removing the jump term the same result as Gaeta [16] was derived, even though Gaeta makes use of restricted transformation i.e., fiber preserving transformation from the beginning.*

5.4 Applications

This section, is devoted to finding admitted Lie point symmetries of some few jump-diffusion stochastic differential equations by utilising the determining equations derived in the previous section.

Example 5.4.1 *Consider the jump SDE, linear in the state process $X(t)$, with constant coefficients;*

$$dX(t)=u_0X(t)dt+\alpha_0X(t)dW(t)+v_0X(t)dN(t) \tag{5.90}$$

with initial condition $X(t_0)=x_0>0$. The coefficient u_0 is called the drift or deterministic coefficient, v_0 is called the jump amplitude coefficient of the jump term and α_0 is called Wiener diffusion coefficient, with jump intensity $\lambda=\lambda_0$.

Therefore, we have the drift, Brownian motion diffusion and jump coefficients as

$$f(t,x) = u_0 x, \qquad g(t,x) = \alpha_0 x \qquad and \qquad J(t,x) = v_0 x. \tag{5.91}$$

with u_0, α_0 and v_0 non-zero.

Using the determining equations (5.81), (5.82), (5.83) and (5.89) for drift $f(t,x) = u_0 x$, Wiener diffusion $G(t,x) = \alpha_0 x$ and jump $J(t,x) = v_0 x$ coefficients we respectively have

$$\begin{aligned}\frac{\partial\tau}{\partial t} + u_0 x\frac{\partial\tau}{\partial x} + \frac{1}{2}\frac{\partial^2\tau}{\partial w^2} + \frac{\alpha_0^2 x^2}{2}\frac{\partial^2\tau}{\partial x^2} + \frac{v_0 x\lambda}{2}\left(\frac{\partial\tau}{\partial t} + u_0 x\frac{\partial\tau}{\partial x} + \frac{1}{2}\frac{\partial^2\tau}{\partial w^2} + \frac{\alpha_0^2 x^2}{2}\frac{\partial^2\tau}{\partial x^2}\right)\\ + u_0\xi(t,x) = \frac{\partial\xi}{\partial t} + u_0 x\frac{\partial\xi}{\partial x} + \frac{\alpha_0^2 x^2}{2}\frac{\partial^2\xi}{\partial x^2},\end{aligned} \tag{5.92}$$

$$x\frac{\partial\xi}{\partial x} - \xi(t,x) = 0 \tag{5.93}$$

and

$$\frac{x}{2}\left(\frac{\partial\tau}{\partial t} + u_0 x\frac{\partial\tau}{\partial x} + \frac{1}{2}\frac{\partial^2\tau}{\partial w^2} + \frac{\alpha_0^2 x^2}{2}\frac{\partial^2\tau}{\partial x^2}\right) + v_0\xi(t,x) = \xi(t,x+v_0 x) - \xi(t,x). \tag{5.94}$$

From (5.84), (5.89) and (5.87) we respectively get

$$\left(\frac{\partial\tau}{\partial w} + \alpha_0 x\frac{\partial\tau}{\partial x}\right)(t,x) = 0, \tag{5.95}$$

$$\left(\frac{\partial\phi}{\partial w} + \alpha_0 x\frac{\partial\phi}{\partial x}\right)(t,x) = 0 \tag{5.96}$$

and

$$\frac{\partial\phi}{\partial t} + u_0 x\frac{\partial\phi}{\partial x} + \frac{\partial^2\phi}{2\partial w^2} + \frac{\alpha_0^2 x^2\partial^2\phi}{2\partial x^2} = 0. \tag{5.97}$$

From equation (5.85) and (5.86) we obtain the following

$$\tau(t,x,w) = \tau(t,w) \tag{5.98}$$

and

$$\phi(t,x,w) = \phi(t,w). \tag{5.99}$$

Substituting (5.98) into (5.95) gives

$$\frac{\partial \tau(t,w)}{\partial w} = 0. \tag{5.100}$$

By using (5.98), (5.100) and (5.88) we obtain the time infinitesimal

$$\tau(t,x,w) = c_1. \tag{5.101}$$

Substituting the temporal infinitesimal (5.101) into (5.96) and (5.97) using (5.99) we obtain

$$\frac{\partial \phi}{\partial w} = 0 \tag{5.102}$$

and

$$\frac{\partial \phi}{\partial t} + \frac{\partial^2 \phi}{2\partial w^2} = 0. \tag{5.103}$$

Using (5.102), (5.99) and (5.103) we finally obtain the jump infinitesimal

$$\phi(t,w) = c_2. \tag{5.104}$$

Substituting the temporal infinitesimal (5.101) into (5.92), (5.93) and (5.94) we respectively obtain

$$u_0\xi(t,x) = \frac{\partial \xi(t,x)}{\partial t} + u_0 x \frac{\partial \xi(t,x)}{\partial x} + \frac{\alpha_0^2 x^2}{2}\frac{\partial^2 \xi(t,x)}{\partial x^2}, \tag{5.105}$$

$$x\frac{\partial \xi(t,x)}{\partial x} - \xi(t,x) = 0 \tag{5.106}$$

and

$$\xi(t,x+v_0 x) - \xi(t,x) = v_0\xi(t,x). \tag{5.107}$$

Differentiating (5.107) once with respect to both x and t gives

(5.108) $$\frac{\partial^2 \xi(t,2x)}{\partial x \partial t} = \frac{\partial^2 \xi(t,x)}{\partial x \partial t} = \frac{dg(t)}{dt}.$$

Therefore, solving (5.108) we obtain

(5.109) $$\xi(t,x) = g(t)x + h(x).$$

Substituting (5.109) in (5.106) we gives

(5.110) $$x\frac{dh(x)}{dx} - h(x) = 0.$$

Solving (5.110) for $h(x)$ gives

(5.111) $$h(x) = c_5 x.$$

Substituting (5.111) into (5.109) gives

(5.112) $$\xi(t,x) = g(t)x + c_5 x.$$

Substituting (5.112) into (5.105) gives

(5.113) $$g(t) = c_6$$

Finally, using (5.112), (5.113) and (5.107) we have the spatial infinitesimal

(5.114) $$\xi(t,x) = c_6 x + c_5 x.$$

Which implies from (5.114), (5.101) and (5.104), respectively

(5.115) $$\xi(t,x) = 2c_7 x, \qquad \tau(t,x,w) = c_1 \qquad and \qquad \phi(t,x,w) = c_2.$$

Therefore (5.90) admit three dimensional symmetry algebras, given as

(5.116) $$H_1 = \frac{\partial}{\partial t}, \qquad H_2 = 2x\frac{\partial}{\partial x}, \qquad H_3 = \frac{\partial}{\partial w}.$$

Example 5.4.2 *Consider a stochastic model driven by both Wiener and Poisson processes*

(5.117) $$dX(t) = -kt^2 dt + \sqrt{D}dW + bdN(t)$$

with initial condition $X(0) = x_0$, where D is non-negative constant and $b \neq 0$

From the jump-diffusion model (5.117) we have the drift vector, Wiener diffusion and jump coefficients respectively as

$$f(t,x) = -kt^2, \quad G(x,t) = \sqrt{D}, \quad D > 0 \quad and \quad J(t,x) = b, \quad b \neq 0. \tag{5.118}$$

From (5.84), (5.85), (5.88) (5.86), (5.87) and (5.89) the temporal and Poisson term infinitesimals are

$$\tau(t,x,w) = c_1, \quad \phi(t,x,w) = c_2. \tag{5.119}$$

Using the determining equations (5.81), (5.82), (5.83), (5.89) and (5.119) respectively, we get

$$\frac{\partial \xi}{\partial t} - kt^2 \frac{\partial \xi}{\partial x} + \frac{D}{2}\frac{\partial^2 \xi}{\partial x^2} = -2ktc_1, \tag{5.120}$$

$$\frac{\partial \xi(t,x)}{\partial x} = 0 \tag{5.121}$$

and

$$\xi(t,x+b) = \xi(t,x). \tag{5.122}$$

Solving the differential equation (4.75) using (4.76) yields the spatial infinitesimal

$$\xi(t,x) = -kt^2 c_1 + c_3. \tag{5.123}$$

Therefore, jump-diffusion stochastic differential equation (5.117) admitted three dimensional Lie symmetry infinitesimal generators

$$H_1 = \frac{\partial}{\partial t} - kt^2 \frac{\partial}{\partial x}, \quad H_2 = \frac{\partial}{\partial w} \qquad H_3 = \frac{\partial}{\partial x}. \tag{5.124}$$

Example 5.4.3 *Consider a jump stochastic differential equations*

$$dX(t) = -k^2 X(t)dt + \sqrt{2k^2}dW(t) + \alpha t dN(t) \tag{5.125}$$

with k^2 a positive constant and $X(0) = x_0$. From the jump-diffusion model (4.69) we have the drift vector, Wiener diffusion and jump coefficients respectively *as*

$$f(t,x) = -k^2 x, \quad G(x,t) = \sqrt{2k^2}, \quad k > 0 \quad and \quad J(t,x) = \alpha t, \quad \alpha \neq 0. \tag{5.126}$$

Using the determining equations (5.81), (5.82), (5.83), (5.89) and (5.130) respectively, we get

$$-k^2 \xi(t,x) = \frac{\partial \xi(t,x)}{\partial t} - k^2 x \frac{\partial \xi(t,x)}{\partial x} + k^2 \frac{\partial^2 \xi(t,x)}{\partial x^2}, \tag{5.127}$$

$$\frac{\partial \xi(t,x)}{\partial x} = 0, \tag{5.128}$$

and

$$\xi(t, x + \alpha t) - \xi(t,x) = \alpha \tau(t,x,w). \tag{5.129}$$

It follow from (5.84), (5.85), (5.88) (5.86), (5.87) and (5.89) the temporal and Poisson term infinitesimals are respectively

$$\tau(t,x,w) = c_1, \quad \phi(t,x,w) = c_2. \tag{5.130}$$

Solving (5.127) and (5.128) simultaneously we have the spatial infinitesimal as

$$\xi(t,x) = c_3 e^{k^2 t}. \tag{5.131}$$

Substituting (5.131) in (5.129) using (5.130) implies

$$\tau(t,x,w) = c_1 = 0. \tag{5.132}$$

Therefore the symmetries of the infinitesimal generators are two dimensional given as:

$$H_1 = e^{k^2 t} \frac{\partial}{\partial x}, \qquad H_2 = \frac{\partial}{\partial w} \tag{5.133}$$

with corresponding Lie bracket relations $[H_1, H_2] = [H_2, H_1] = 0$. *Which shows that the symmetries generator (5.133) forms an abelian group.*

Remark 5.4.4 *Its worth noticing that extending the Lie transformations to include not only the spatial and temporal variables but also the Wiener process variable gives at least one additional infinitesimal generator, this is similar to the* *finding of F. G. Geata [14, 16] and Ebrahim [29] while discussing "W"-symmetry of Wiener stochastic differential equations.*

Lie classification of the jump-diffusion stochastic differential equation discusses in this chapter is given in Table 5.1 below.

TABLE 5.1: Lie Group Classification chapter 5

Group Dimension	Basis Operators	Equations
2	$H_1 = e^{k^2 t}\frac{\partial}{\partial x}, \qquad H_2 = \frac{\partial}{\partial w}$	$dX(t) = -k^2 X(t)dt + \sqrt{2k^2}dW(t) + \alpha t dN(t)$
3	$H_1 = \frac{\partial}{\partial t}, \qquad H_2 = 2x\frac{\partial}{\partial x}, \qquad H_3 = \frac{\partial}{\partial w}.$	$dX(t) = u_0 X(t)dt + \alpha_0 X(t)dW(t) + v_0 X(t)dN(t)$
3	$H_1 = \frac{\partial}{\partial t} - kt^2\frac{\partial}{\partial x}, \qquad H_2 = \frac{\partial}{\partial x}, \qquad H_3 = \frac{\partial}{\partial w}.$	$dX(t) = -kt^2 dt + \sqrt{D}dW(t) + bdN(t), D > 0, b \neq 0$

Chapter 6

N-symmetry of Itô Stochastic Differential Equations with a Finite Jump Process

Lie point symmetry of jump-diffusion stochastic differential equations is defined, by considering infinitesimals of spatial, temporal and vector jump process variables $N(t)$. We utilised the standard random time change formula to transform the Brownian motion [35]. The determining equations for the admitted Lie symmetries are derived in Itô context and happen to be deterministic. This was later applied to a number jump-diffusion models to obtain the corresponding infinitesimal generators.

6.1 Introduction

In this chapter, we discuss Lie point symmetries of Itô stochastic differential equations driven by both Wiener processes and Poisson processes i.e., finite jump processes (SDEJ) given as;

$$dX_i(t) = f_i(t,X(t,N))dt + G_{il}(t,X(t,N))dW(t) + J_i(t,X(t,N))dN(t). \tag{6.1}$$

Where $f_i(t,X(t,N))$ and $J_i(t,X(t,N))$ are $n \times 1$ dimensional drift vector coefficient and jump diffusion coefficient respectively and $G_{il}(t,X(t,N))$ is the $n \times M$ dimensional Wiener diffusion matrix coefficient. The coefficients are assumed to satisfy the standard theorem of existence and uniqueness of the solution of (6.1).

The Lie Symmetries of (6.1) are analysed by extending the symmetry generator used in chapter three to include the infinitesimal transformations of the Poisson process. That is, we now consider infinitesimals involving not only the spatial and time variables (t,x), but also the Poisson diffusion processes $N(t)$, while keeping the standard random change formula for Brownian motion unaltered.

The determining equations for Itô stochastic differential equations (SDE) with finite jump (6.1) are derived in an Itô calculus context and found to be non-stochastic.

For an arbitrary differentiable function $F(t,X(t,N))$ we determine by the Itô lemma of the jump diffusion process, the Itô process $F(t,X(t))$ of (6.1) exists and is

$$dF_j(t,X(t,W)) = \Gamma_{(N)}(F)_j dt + \Gamma^*_{(N)}(F)_j dW(t) + \Gamma^{**}_{(N)}(F)_j dN(t). \tag{6.2}$$

with

$$\Gamma_{(N)}(F)_j = \frac{\partial F_j}{\partial t}dt + f_i\frac{\partial F_j}{\partial x_i} + \frac{1}{2}\frac{\partial^2 F_j}{\partial N \partial N} + \frac{1}{2}G_{ik}(t,X(t,N))G_{mk}(t,X(t,N))\frac{\partial^2 F_j}{\partial x_i \partial x_m}, \tag{6.3}$$

$$\Gamma^*_{(N)}(F)_j = \frac{\partial F_j}{\partial N} + G_{il}(t,X(t,N))\frac{\partial F_j}{\partial x_i} \tag{6.4}$$

and

$$\Gamma^{**}_{(N)}(F)_j = F_j\Big(t, X_i(t) + J(t,X_i(t,N))\Big) - F_j(t,X_i(t)). \tag{6.5}$$

Using the Itô multiplication properties of both Brownian motion and Poisson processes and application of infinitesimal transformations the determining equations for (SDEJ) are derived and are non-stochastic. We obtained the following result;

Theorem 6.1.1 *The Itô stochastic differential equation with a finite jump*

$$dX_i(t) = f_i(t,X(t,N))dt + G_{il}(t,X(t,N))dW_l(t) + J_i(t,X(t,N))dN(t) \tag{6.6}$$

where $f_i(t,X(t,N))$ and $J_i(t,X(t))$ are $n\times 1$ dimensional drift vector coefficient and jump diffusion coefficient respectively and $G_{il}(t,X(t))$ is the Wiener diffusion matrix coefficient of $n \times M$ dimension, with infinitesimal generator

$$H = \tau(t,X,N)\frac{\partial}{\partial t} + \xi_i(t,X_i)\frac{\partial}{\partial x_i} + \gamma_i(t,X_i,N)\frac{\partial}{\partial N}, \tag{6.7}$$

has the following determining equations;

$$\Big(f_j\Gamma_{(N)}(\tau) + J_j\Gamma_{(N)}(\gamma_j) + H(f_j) - \Gamma_{(N)}(\xi)_j\Big)(t,X(t,N)) = 0, \tag{6.8}$$

$$\left(\frac{G_{jl}}{2}\Gamma_{(N)}(\tau) + H(G_{jl}) - \Gamma^{*}_{(N)}(\xi)_j\right)(t, X(t,N)) = 0 \tag{6.9}$$

and

$$\left(J_j\Gamma^{**}_{(N)}(\gamma_j) + H(J_j) - \Gamma^{**}_{(N)}(\xi)_j\right)(t, X(t,N)) = 0 \tag{6.10}$$

with the following additional conditions;

$$\Gamma^{*}_{(N)}(\tau)(t, X(t,N)) = 0, \tag{6.11}$$

$$\Gamma^{**}_{(N)}(\tau)(t, X(t,N)) = 0, \tag{6.12}$$

$$\Gamma^{*}_{(N)}(\gamma_j)(t, X(t,N)) = 0, \tag{6.13}$$

$$\Gamma^{**}_{(N)}(\gamma_j) = 0, \tag{6.14}$$

$$\Gamma_{(N)}(\tau)(t, X(t,N)) = 0 \tag{6.15}$$

and

$$\Gamma_{(N)}(\gamma_j)(t, X(t,N)) = 0. \tag{6.16}$$

The operators $\Gamma_{(N)}(t,X(t))$, $\Gamma^{*}_{(N)}(t,X(t))$ and $\Gamma^{**}_{(N)}(t,X(t))$ are defined as in (6.3), (6.4) and (6.5), the infinitesimals $\xi(t,x)$, $\tau(t,x,N)$ and $\phi(t,x,N)$ are called the admitted symmetries of (6.1) if and only if they satisfied the determining equations (6.8) - (6.16).

6.2 Lie Group Transformations

Consider a one-parameter group of transformation of time index t, the spatial x and jump variables J respectively,

$$\bar{t} = \theta_1(t,x,N,\epsilon), \qquad \bar{x} = \theta_2(t,x,\epsilon), \qquad \overline{N} = \theta_3(t,x,N,\epsilon) \tag{6.17}$$

with the infinitesimals

$$\frac{\partial\theta_1}{\partial\epsilon} = \tau(\theta_1,\theta_2,\theta_3), \qquad \frac{\partial\theta_2}{\partial\epsilon} = \xi(\theta_1,\theta_2,\theta_3), \qquad \frac{\partial\theta_3}{\partial\epsilon} = \gamma(\theta_1,\theta_2,\theta_3) \tag{6.18}$$

satisfying the following initial conditions at $\epsilon = 0$

$$\bar{t}\Big|_{\epsilon=0} = t \qquad \overline{X}\Big|_{\epsilon=0} = X \qquad \overline{N}\Big|_{\epsilon=0} = N. \tag{6.19}$$

Where ϵ is the parameter of the group, hence the corresponding generator of the Lie transformation is of the form

$$H = \tau(t,x,N)\frac{\partial}{\partial t} + \xi_i(t,X_i)\frac{\partial}{\partial x_i} + \gamma_i(t,X_i,N)\frac{\partial}{\partial N}. \tag{6.20}$$

The differential Point transformation of the temporal t and spatial x variables respectively are

$$d\bar{t} = dt + \epsilon d\tau + O(\epsilon) \tag{6.21}$$

and

$$d\overline{X_j}(\bar{t}) = dX_j(t) + \epsilon d\xi_j + O(\epsilon). \tag{6.22}$$

Similarly, the differential Point transformation of Wiener process W and the jump process J variables respectively are

$$d\overline{W_l}(\bar{t}) = dW_l(t) + \frac{\epsilon}{2}\frac{d\tau}{dt} + O(\epsilon) \tag{6.23}$$

and

$$d\overline{N}(\bar{t}) = dN(t) + \epsilon d\gamma_j + O(\epsilon). \tag{6.24}$$

Using Itô formula (6.2), the spatial, temporal and jump infinitesimals can be written in Itô forms respectively as

$$d\xi_j = \Gamma_{(N)}(\xi)_j(t,X(t))dt + \Gamma^*_{(N)}(\xi)_j(t,X(t))dW(t) + \Gamma^{**}_{(N)}(\xi)_j(t,X(t))dN(t), \tag{6.25}$$

$$d\tau = \Gamma_{(N)}(\tau)(t,X(t))dt + \Gamma^*_{(N)}(\tau)(t,X(t))dW(t) + \Gamma^{**}_{(N)}(\tau)(t,X(t))dN(t) \tag{6.26}$$

and

$$d\gamma_j = \Gamma_{(N)}(\gamma_j)(t,X(t))dt + \Gamma^*_{(N)}(\gamma_j)(t,X(t))dW(t) + \Gamma^{**}_{(N)}(\gamma_j)(t,X(t))dN(t). \tag{6.27}$$

Substituting the Itô form of the temporal (6.26) and spatial infinitesimals (6.25) into (6.21) and (6.22) the spatial and temporal group transformations can be written in Itô forms respectively as

$$d\bar{t} = dt + \epsilon\Big(\Gamma_{(\tau)}(t,X(t))dt + \Gamma^*_{(\tau)}(t,X(t))dW(t) + \Gamma^{**}_{(\tau)}(t,X(t))dN(t)\Big) + O(\epsilon) \tag{6.28}$$

and

$$d\overline{X_j}(\bar{t}) = dX_j(t) + \epsilon\Big(\Gamma_{(\xi)_j}(t,X(t))dt + \Gamma^*_{(\xi)_j}(t,X(t))dW(t) + \Gamma^{**}_{(\xi)_j}(t,X(t))dN(t)\Big) + O(\epsilon). \tag{6.29}$$

Similarly, substituting the Itô form of the temporal (6.26) and jump infinitesimals (6.27) into (6.23) and (6.24) the Wiener process and the jump process group transformation respectively can be written in Itô form as

$$d\overline{W_l}(\bar{t}) = dW_l(t) + \frac{\epsilon}{2}\,\frac{\Gamma_{(\tau)}(t,X(t))dt + \Gamma^*_{(\tau)}(t,X(t))dW(t) + \Gamma^{**}_{(\tau)}(t,X(t))dN(t)}{dt}\,dW(t) + O(\epsilon) \tag{6.30}$$

and

$$d\overline{N}(\bar{t}) = dN(t) + \epsilon\Big(\Gamma_{(\gamma_j)}(t,X(t))dt + \Gamma^*_{(\gamma_j)}(t,X(t))dW(t) + \Gamma^{**}_{(\gamma_j)}(t,X(t))dN(t)\Big) + O(\epsilon). \tag{6.31}$$

Simplifying the Itô Wiener group transformation (6.30) using the Itô multiplication properties of Brownian motion in *Table 1.1*, the group transformation (6.30) can be reduced to

$$d\overline{W}(\bar{t}) = dW(t) + \frac{\epsilon}{2}\Big(\Gamma_{(\tau)}(t,X(t))dW(t) + \Gamma^*_{(\tau)}(t,X(t))\Big) + O(\epsilon). \tag{6.32}$$

We are now in a position to proceed to use the transformed Itô forms of the group transformation of spatial, temporal and jump process variables as well as the Itô form of the infinitesimals of temporal, spatial and jump variables to transforms the properties of both Brownian motion and Poisson process.

6.2.1 Wiener Invariance Properties

We apply the invariance to the moments of the Wiener process to ensure it remains invariant under the group transformations, *viz* the instantaneous mean and variance of the Wiener process which are:

$$E_Q\left[dW(t)\middle|W=w\right]=0 \tag{6.33}$$

and

$$E_Q\left[dW_l(t)dW_m(t)\middle|W=w\right]=\delta^l_m dt. \tag{6.34}$$

The invariance of the instantaneous mean of the transformed Wiener process under new measure $\overline{Q}$ is

$$E_{\overline{Q}}\left[d\overline{W}(t)\middle|W=w\right]=0. \tag{6.35}$$

Substituting the Itô forms of the group transformation of Wiener process variable (6.32) into (6.35) gives

$$E_{\overline{Q}}\left[dW(t)+\frac{\epsilon}{2}\left(\Gamma_{(\tau)}(t,X(t))dW(t)+\Gamma^*_{(\tau)}(t,X(t))\right)+O(\epsilon)\middle|W=w\right]=0 \tag{6.36}$$

Expanding (6.36) using (6.33) gives

$$\Gamma^*_{(\tau)}(t,X(t))=0. \tag{6.37}$$

Next, we apply the invariant form to instantaneous variance of the transformed Wiener process measure (6.34) from which using the Itô forms of the group transformation of Wiener process variable (6.32) we obtain

$$E_{\overline{Q}}\left[d\overline{W}_l(t)d\overline{W}_m(t)\middle|W=w\right]=\delta^l_m d\bar{t}. \tag{6.38}$$

The Levy characterization of Brownian motions [9, 25] are follow automatically using (6.37) i.e.,

$$d\overline{W}_l(t)d\overline{W}_m(t) = \delta^l_m d\bar{t}, \quad d\overline{W}_l(t)d\bar{t} = 0, \quad d\bar{t}d\bar{t} = 0. \tag{6.39}$$

6.2.2 Poisson Invariance Properties

Before deriving the determining equations, we apply the invariance to the moments of the jump process to make sure it remains invariant under the group transformations, *viz* the instantaneous mean and variance of the jump process which are:

$$E_Q\left[dN(t)\right] = \lambda dt, \tag{6.40}$$

$$E_Q\left[dN(t)dN(t)|N = n\right] = \lambda dt \tag{6.41}$$

and the differential product of the Itô forms of the group transformation of Wiener and Poisson process variables i.e.,

$$E_Q\left[dW_l(t)dN(t)|W = w, N = n\right] = 0. \tag{6.42}$$

The invariance of the instantaneous mean of the transformed jump process under new measure $\overline{Q}$ is

$$E_Q\left[d\overline{N}(\bar{t})|N = n\right] = \lambda d\bar{t}. \tag{6.43}$$

Using the Itô forms of the temporal (6.28) and Poisson process group transformation (6.31) gives

$$\begin{aligned} E_Q\left[\Gamma_{(\gamma_j)}dt + \Gamma^{*}_{(\gamma_j)}dW(t) + \Gamma^{**}_{(\gamma_j)}(t, X(t))dN(t)\Big|W = w\right] = \lambda\Big(\Gamma_{(\tau)}(t, X(t))dt \\ + \Gamma^{*}_{(\tau)}(t, X(t))dW(t) + \Gamma^{**}_{(\tau)}(t, X(t))dN(t)\Big). \end{aligned} \tag{6.44}$$

Expanding (6.44) using the properties of the moments of the processes i.e., (6.33), (6.34) and (6.40), equation (6.44) can be simplified to

$$\left(\Gamma_{(\gamma_j)} + \lambda\Gamma^{**}_{(\gamma_j)} - \lambda\Gamma_{(\tau)}\right)dt = \Gamma^{**}_{(\tau)}dN(t). \tag{6.45}$$

Next, we apply the invariant form to instantaneous variance of the transformed jump process measure i.e.,

$$E_{\overline{Q}}\left[d\overline{N}(t)d\overline{N}(t)|W=w\right]=\lambda d\bar{t}. \tag{6.46}$$

Expanding (6.46) using the Itô form of the Poisson process variable group transformation (6.31) we obtain

$$E_{\overline{Q}}\left[(\Gamma^{**}_{(N)}(\gamma_j)+\Gamma^{**}_{(N)}(\gamma_m))dN(t)dN(t)\Big|W=w\right]=\lambda d\bar{t}. \tag{6.47}$$

Finally, using the Itô temporal group transformation (6.28), equation (6.47) gives the following differential relation

$$(\Gamma^{**}_{(N)}(\gamma_j)+\Gamma^{**}_{(N)}(\gamma_m))dt=\Gamma_{(N)}(\tau)dt+\Gamma^{**}_{(N)}(\tau)dN(t,N). \tag{6.48}$$

Comparing the jump and Riemann integrals in (6.48) we have the following relations

$$\Gamma^{**}_{(N)}(\tau)(t,X(t))=0 \tag{6.49}$$

and

$$\Gamma_{(N)}(\gamma_j)+\lambda\Gamma^{**}_{(N)}(\gamma_j)=\lambda\Gamma_{(N)}(\tau). \tag{6.50}$$

Using (6.49), equation (6.48) can be reduced to

$$\Gamma^{**}_{(N)}(\gamma_j)+\Gamma^{**}_{(N)}(\gamma_j)=\Gamma_{(N)}(\tau). \tag{6.51}$$

Finally equation (6.42) under new measure $\overline{Q}$ can be transformed to

$$E_{\overline{Q}}\left[d\overline{W}_l(t)d\overline{N}(t)\Big|W=w\right]=0. \tag{6.52}$$

Substituting the Itô group transformation of Wiener (6.32) and Poisson processes (6.31) into (6.52) we get

$$E_{\overline{Q}}\left[\epsilon\Gamma^{*}_{(N)}(\gamma_j)dt+\frac{\epsilon}{2}\Gamma^{*}_{(N)}(\tau)dN|W=w\right]=0, \tag{6.53}$$

taking the expectation in (6.53) and by using (6.40) we have

$$\Gamma^{*}_{(N)}(\gamma_j)+\frac{\lambda}{2}\Gamma^{*}_{(N)}(\tau)=0. \tag{6.54}$$

Finally, using equation (6.37) and (6.54) we have the following relation

$$\Gamma^{*}_{(N)}(\gamma_j)(t,X(t))=0. \tag{6.55}$$

Thus, we have successfully derived a generalised random time change formula for Poisson processes

$$\bar{t}=\int^{t}\Gamma_{(N)}(\tau(s))(t,X(t))ds \tag{6.56}$$

with

$$\Gamma_{(N)}(\tau)(t,X(t))=constant=c_1 \tag{6.57}$$

using the probabilistic invariance property of the transformed time index differential, i.e.,

$$E_{\overline{Q}}\left[d\bar{t}(t)\right]=d\bar{t}. \tag{6.58}$$

Therefore, the generalised infinitesimal of the jump process variable is

$$d\overline{N}(\bar{t})=dN(t)+\lambda\left(\Gamma_{(\tau)}-\Gamma^{**}_{(\gamma)}\right)dt+\Gamma^{**}_{(\gamma)}dN(t). \tag{6.59}$$

This is a generalised random time change formula for jump process variables that transform the Poisson process variable.

However, the invariant of the Watanabe characterisation of Poisson processes

$$d\overline{N}(\bar{t})d\overline{N}(\bar{t})=d\overline{N}(\bar{t}),\quad d\overline{W}_l(\bar{t})d\overline{N}(t),\quad d\overline{N}(\bar{t})d\bar{t}=0 \tag{6.60}$$

led to

$$\Gamma_{(N)}(\gamma_j)(t,X(t))=0,\quad and\quad \Gamma^{**}_{(N)}(\gamma_j)(t,X(t))=0. \tag{6.61}$$

From which using (6.61) in (6.50), (6.51) and (6.57) we get

$$\Gamma_{(N)}(\tau)(t,X(t))=0. \tag{6.62}$$

Remark 6.2.1 *Note that the random time change formula derived here confirmed our finding in Chapter 3 even though different operators were used.*

6.2.3 Invariance Form of the Spatial Process

To ensure the recovery of the finite transformations from the infinitesimal transformation, we need to transform $d\overline{X}(\bar{t},\overline{N})$ into

(6.63)
$$d\overline{X_j}(\bar{t}) = f_j(\bar{t},\overline{X}(\bar{t})d\bar{t} + G_{jl}(\bar{t},\overline{X}(\bar{t})d\overline{W}(\bar{t}) + J_j(\bar{t},\overline{X}(\bar{t})d\overline{N}(\bar{t})$$

where the transformed drift component using the infinitesimal generator (6.20) is

(6.64)
$$\begin{aligned} f_j(\bar{t},\overline{X}(\bar{t}) &= \Big(f_j + \epsilon H(f_j)\Big)(t,X(t)) \\ &= f_j(t,X(t)) + \epsilon\left(\tau(t,X,N)\frac{\partial f_j}{\partial t} + \xi_i(t,X_i)\frac{\partial f_j}{\partial x_i} + \gamma_i(t,X_i,N)\frac{\partial f_j}{\partial N}\right) \end{aligned}$$

while the transformed Wiener diffusion component gives

(6.65)
$$\begin{aligned} G_{jl}(\bar{t},\overline{X}(\bar{t},\overline{N})) &= \Big(G_{jl} + \epsilon H(G_{jl})\Big)(t,X(t,N)) \\ &= G_{jl}(t,X(t,N)) + \epsilon\left(\tau(t,X,N)\frac{\partial G_{jl}}{\partial t} + \xi_i(t,X_i)\frac{\partial G_{jl}}{\partial x_i} + \gamma_i(t,X_i,N)\frac{\partial G_{jl}}{\partial N}\right). \end{aligned}$$

Finally, the transformed jump diffusion component using the infinitesimal generator

(6.66)
$$\begin{aligned} J_j(\bar{t},\overline{X}(\bar{t},\overline{N})) &= \Big(J_j + \epsilon H(J_j)\Big)(t,X(t,N)) \\ &= J_j(t,X(t,N)) + \epsilon\left(\tau(t,X,N)\frac{\partial J_j}{\partial t} + \xi_i(t,X_i)\frac{\partial J_j}{\partial x_i} + \gamma_i(t,X_i,N)\frac{\partial J_j}{\partial N}\right). \end{aligned}$$

By now, all the necessary tools are available to derive the determining equations for the admitted Lie symmetries of stochastic differential equations driven by both Wiener and Poisson processes.

Definition 6.2.2 *The infinitesimals group transformations*

(6.67)
$$\bar{t} = t + \epsilon\tau(t,x,N), \quad \overline{X_i}(\bar{t}) = X_i(t) + \epsilon\xi_i(t,x), \quad \overline{N_i}(\bar{t}) = N_i(t) + \epsilon\gamma_i(t,x_i,N)$$

are called symmetry transformations of the jump-diffusion stochastic differential equation (SDEJ) i.e.,

(6.68)
$$dX_i(t) = f_i(t,X(t))dt + G_{il}(t,X(t))dW(t) + J_i(t,X(t))dN(t)$$

if they leave (6.68) and the properties of the Brownian motion and Poisson processes invariant.

6.3 Derivation of the Determining Equations

This section is devoted to obtaining the determining equations for the admitted Lie point symmetry of (6.1). This can be achieved by transforming (6.1) into

$$d\overline{X}_j(\bar{t}) = f_j(\bar{t},\overline{X}(\bar{t})d\bar{t} + G_{jl}(\bar{t},\overline{X}(\bar{t})d\overline{W}(\bar{t}) + J_j(\bar{t},\overline{X}(\bar{t}))d\overline{N}(\bar{t}) \tag{6.69}$$

through utilising the results obtained in the previous sections as follows.

Substituting the Itô temporal (6.28), jump (6.31), Wiener variables group transformations (6.32) as well as the transformed drift (6.64), Wiener diffusion (6.65) and jump components (6.66) into (6.69) gives

$$\begin{aligned} d\overline{X}_j(\bar{t}) =& dX_j(t) + \epsilon\left(f_j\Gamma_{(N)}(\tau) + J_j\Gamma_{(N)}(\gamma_j) + H(f_j)\right)dt \\ &+\epsilon\left(\frac{G_{jl}}{2}\Gamma_{(N)}(\tau) + H(G_{jl}) + f_j\Gamma^*_{(N)}(\tau) + J_j\Gamma^*_{(N)}(\gamma_j)\right)dW(t) \\ &+\epsilon\left(J_j\Gamma^{**}_{(N)}(\gamma_j) + H(J_j) + f_j\Gamma^{**}_{(N)}(\tau)\right)dN(t) + \frac{\epsilon G_{jl}}{2}\Gamma^*_{(N)}(\tau). \end{aligned} \tag{6.70}$$

Substituting (6.37), (6.49) and (6.55) into (6.70), equation (6.70) can be reduced to

$$\begin{aligned} d\overline{X}_j(\bar{t}) = dX_j(t) + \epsilon\left(f_j\Gamma_{(N)}(\tau) + J_j\Gamma_{(N)}(\gamma_j) + H(f_j)\right)dt \\ +\epsilon\left(\frac{G_{jl}}{2}\Gamma_{(N)}(\tau) + H(G_{jl})\right)dW(t) + \epsilon\left(J_j\Gamma^{**}_{(N)}(\gamma_j) + H(J_j)\right)dN(t). \end{aligned} \tag{6.71}$$

Therefore, by comparing (6.29) and the Itô form of the spatial variable group transformation (6.71) we obtain the following determining equations

$$\left(f_j\Gamma_{(N)}(\tau) + J_j\Gamma_{(N)}(\gamma_j) + H(f_j) - \Gamma_{(N)}(\xi)_j\right)(t,X(t)) = 0, \tag{6.72}$$

$$\left(\frac{G_{jl}}{2}\Gamma_{(N)}(\tau) + H(G_{jl}) - \Gamma^*_{(N)}(\xi)_j\right)(t,X(t)) = 0 \tag{6.73}$$

and

$$\left(J_j\Gamma^{**}_{(N)}(\gamma_j)+H(J_j)-\Gamma^{**}_{(N)}(\xi)_j\right)(t,X(t))=0. \tag{6.74}$$

With the following additional conditions from the invariant forms of the moments properties of the Wiener and Poisson processes i.e., (6.37), (6.49) and (6.55) respectively

$$\Gamma^{*}_{(N)}(\tau)(t,X(t))=0, \tag{6.75}$$

$$\Gamma^{**}_{(N)}(\tau)(t,X(t))=0, \tag{6.76}$$

and

$$\Gamma^{*}_{(N)}(\gamma_j)(t,X(t))=0. \tag{6.77}$$

Equation (6.61) gives

$$\Gamma^{**}_{(N)}(\gamma_j)(t,X(t))=0 \tag{6.78}$$

and

$$\Gamma_{(N)}(\gamma_j)(t,X(t))=0 \tag{6.79}$$

Finally, from (6.62), we can obtain

$$\Gamma_{(N)}(\tau)(t,X(t))=0. \tag{6.80}$$

Where the operators $\Gamma_{(N)}(t,x)$, $\Gamma^{*}_{(N)}(t,x)$ and $\Gamma^{**}_{(N)}(t,x)$ are defined as in (6.3), (6.4) and (6.5), while $\lambda>0$ is called the intensity of the jump process, then the infinitesimals $\xi(t,x)$, $\tau(t,x,N)$ and $\phi(t,x,N)$ are called the admitted symmetries of (6.1) if and only if they satisfied the determining equations (6.72) - (6.80).

6.4 Applications

In this section, we are going to apply the derived determining equations of Poisson Itô stochastic differential equations obtained in the previous section to some stochastic models to show how the determining equations can be used to find the admitted Lie point symmetries of each model.

Example 6.4.1 *Consider a jump-diffusion stochastic differential equation with constant Wiener and jump coefficients*

$$dX = -kt^2dt + \sqrt{D}dW(t) + bdN(t) \tag{6.81}$$

$X(0) = x_0$, *while* D *and* b *are positive constants with* $b \neq 0$.

Therefore, the drift, Wiener and jump process coefficients are

$$f(t,x) = -kt^2, \qquad G(x,t) = \sqrt{D} \qquad and \qquad J(t,x) = b \quad b \neq 0 \tag{6.82}$$

Using the determining equations (6.72), (6.73) and (6.74) we respectively get

$$\begin{aligned} -kt^2\left(\frac{\partial\tau}{\partial t} - kt^2\frac{\partial\tau}{\partial x} + \frac{D}{2}\frac{\partial^2\tau}{\partial x^2} + \frac{1}{2}\frac{\partial^2\tau}{\partial N^2}\right) + \frac{b\lambda}{2}\left(\frac{\partial\tau}{\partial t} - kt^2\frac{\partial\tau}{\partial x} + \frac{D}{2}\frac{\partial^2\tau}{\partial x^2} + \frac{1}{2}\frac{\partial^2\tau}{\partial N^2}\right) \\ -2kt\tau = \frac{\partial\xi}{dt} - kt^2\frac{\partial\xi}{\partial x} + \frac{D}{2}\frac{\partial^2\xi}{\partial x^2}, \end{aligned} \tag{6.83}$$

$$\frac{\sqrt{D}}{2}\left(\frac{\partial\tau}{\partial t} - kt^2\frac{\partial\tau}{\partial x} + \frac{D}{2}\frac{\partial^2\tau}{\partial x^2} + \frac{1}{2}\frac{\partial^2\tau}{\partial N^2}\right) = \sqrt{D}\frac{\partial\xi}{\partial x} \tag{6.84}$$

and

$$\frac{b}{2}\left(\frac{\partial\tau}{\partial t} - kt^2\frac{\partial\tau}{\partial x} + \frac{D}{2}\frac{\partial^2\tau}{\partial x^2} + \frac{1}{2}\frac{\partial^2\tau}{\partial N^2}\right) = \xi(t,x+b) - \xi(t,x). \tag{6.85}$$

Similarly, using the additional conditions (6.75), (6.79), (6.77) and (6.78) we respectively obtain

$$\frac{\partial\tau}{\partial N} + \sqrt{D}\frac{\partial\tau}{\partial x} = 0, \tag{6.86}$$

$$\frac{\partial\gamma}{\partial t} - kt^2\frac{\partial\gamma}{\partial x} + \frac{D}{2}\frac{\partial^2\gamma}{\partial x^2} + \frac{1}{2}\frac{\partial^2\gamma}{\partial N^2} = 0, \tag{6.87}$$

$$\frac{\partial\gamma}{\partial N} + \sqrt{D}\frac{\partial\gamma}{\partial x} = 0 \tag{6.88}$$

and

$$\gamma(t,x+b,N)-\gamma(t,x,N)=0. \tag{6.89}$$

From equation (6.76), the temporal infinitesimal is independent of variable x i.e,

$$\tau(t,x,N)=\tau(t,N). \tag{6.90}$$

Substituting (6.90) into (6.86) implies temporal infinitesimal is the function of the time variable t i.e.,

$$\frac{\partial\tau}{\partial N}=0. \tag{6.91}$$

Using temporal infinitesimals (6.90) and (6.91) in (6.57) gives the temporal infinitesimal as

$$\tau(t,x,N)=c_2. \tag{6.92}$$

Substituting (6.92) in (6.83), (6.84) and (6.85) we respectively obtain

$$\frac{\partial\xi(t,x)}{\partial t}-kt^2\frac{\partial\xi(t,x)}{\partial x}+\frac{D\partial^2\xi(t,x)}{2\partial x^2}=-2ktc_2 \tag{6.93}$$

$$\frac{\partial\xi(t,x)}{\partial x}=0 \tag{6.94}$$

and

$$\xi(t,x+b)=\xi(t,x). \tag{6.95}$$

Solving (6.94) and (6.93) simultaneously, gives the spatial infinitesimal

$$\xi(t,x)=-kt^2c_2+c_3. \tag{6.96}$$

Using (6.89) and (6.88) respectively gives

$$\frac{\partial\gamma}{\partial x}=0 \quad \textit{and} \quad \frac{\partial\gamma}{\partial N}=0. \tag{6.97}$$

Finally, from (6.97) and (6.87) we have a constant jump infinitesimal i.e.,

$$\gamma(t,x,N) = c_4. \tag{6.98}$$

Therefore, (6.81) admit three dimensional symmetry infinitesimal generators;

$$H_1 = \frac{\partial}{\partial t} - kt^2\frac{\partial}{\partial x}, \qquad H_2 = \frac{\partial}{\partial x}, \qquad H_3 = \frac{\partial}{\partial N}. \tag{6.99}$$

Example 6.4.2 *Consider the jump SDE, linear in the state process $X(t)$, with constant coefficients,*

$$dX(t) = X(t)(u_0(t)dt + \alpha_0(t)dW(t) + v_0(t)dN(t)) \tag{6.100}$$

with initial condition $X(t_0) = x_0 > 0$, $u_0(t)$ called the drift or deterministic coefficient, $v_0(t)$ is called the jump amplitude coefficient of the jump term and $\alpha_0(t)$ is called the Wiener diffusion coefficient, with jump intensity $\lambda = \lambda_0 > 0$. Therefore, the drift, Wiener and jump coefficients are;

$$f(t,x) = u_0 x, \qquad g(t,x) = \alpha_0 x \qquad and \qquad J(t,x) = v_0 x. \tag{6.101}$$

Using the determining equations (6.72), (6.73) and (6.74) we respectively get

$$\begin{aligned} u_0 x\left(\frac{\partial \tau}{\partial t} + u_0 x\frac{\partial \tau}{\partial x} + \frac{\alpha_0^2 x^2}{2}\frac{\partial^2 \tau}{\partial x^2}\right) + \frac{v_0 x\lambda_0}{2}\left(\frac{\partial \tau}{\partial t} + u_0 x\frac{\partial \tau}{\partial x} + \frac{\alpha_0^2 x^2}{2}\frac{\partial^2 \tau}{\partial x^2}\right) \\ + u_0\xi(t,x) = \frac{\partial \xi}{\partial t} + u_0 x\frac{\partial \xi}{\partial x} + \frac{\alpha_0^2 x^2}{2}\frac{\partial^2 \xi}{\partial x^2}, \end{aligned} \tag{6.102}$$

$$\frac{\alpha_0 x}{2}\left(\frac{\partial \tau}{\partial t} + u_0 x\frac{\partial \tau}{\partial x} + \frac{\alpha_0^2 x^2}{2}\frac{\partial^2 \tau}{\partial x^2}\right) + \alpha_0\xi(t,x) = \alpha_0 x\frac{\partial \xi}{\partial x} \tag{6.103}$$

and

$$\frac{v_0 x}{2}\left(\frac{\partial \tau}{\partial t} + u_0 x\frac{\partial \tau}{\partial x} + \frac{\alpha_0^2 x^2}{2}\frac{\partial^2 \tau}{\partial x^2}\right) + v_0\xi = \xi(t,x+v_0 x) - \xi(t,x). \tag{6.104}$$

Similarly, using the additional conditions (6.75), (6.79), (6.77) and (6.78) we respectively have

$$\frac{\partial \tau}{\partial N} + \alpha_0 x\frac{\partial \tau}{\partial x} = 0, \tag{6.105}$$

(6.106)
$$\frac{\partial \gamma}{\partial t}+u_0x\frac{\partial \gamma}{\partial x}+\frac{\alpha_0^2x^2}{2}\frac{\partial^2\gamma}{\partial x^2}+\frac{1}{2}\frac{\partial^2\gamma}{\partial N^2}=0,$$

(6.107)
$$\frac{\partial \gamma}{\partial N}+\alpha_0x\frac{\partial \gamma}{\partial x}=0$$

and

(6.108)
$$\gamma(t,x+v_0x,N)-\gamma(t,x,N)=0.$$

From equation (6.76), (6.75),(6.62) and(6.57) we have the infinitesimal of the temporal variable as;

(6.109)
$$\tau(t,x,N)=c_2.$$

Substituting the infinitesimal of the temporal variable (6.109) into (6.102), (6.103) and (6.104) respectively gives

(6.110)
$$u_0\xi(t,x)=\frac{\partial \xi}{\partial t}+u_0x\frac{\partial \xi}{\partial x}+\frac{\alpha_0^2x^2}{2}\frac{\partial^2\xi}{\partial x^2},$$

(6.111)
$$\xi(t,x)-x\frac{\partial \xi}{\partial x}=0$$

and

(6.112)
$$v_0\xi=\xi(t,x+v_0x)-\xi(t,x).$$

Solving (6.111) gives the spatial infinitesimal as

(6.113)
$$\xi(t,x)=f(t)x.$$

Substituting spatial infinitesimal (6.113) into (6.110) gives

(6.114)
$$\frac{df(t)}{dt}=0.$$

Using (6.114) and (6.113) the spatial infinitesimal becomes

$$\xi(t,x) = c_3 x. \tag{6.115}$$

From (6.108), we show that the infinitesimal of the jump variable does not depend on x i.e.,

$$\frac{\partial \gamma}{\partial x} = 0 \tag{6.116}$$

Similarly, using equation (6.116) and (6.105) we can conclude that the infinitesimal of the jump variable is independent of N i.e.,

$$\frac{\partial \gamma}{\partial N} = 0, \tag{6.117}$$

while substituting (6.116) and (6.117) into (6.106) gives

$$\frac{\partial \gamma}{\partial t} = 0. \tag{6.118}$$

This implies from (6.116), (6.117) and (6.118) that the infinitesimal of the jump variable is constant i.e.,

$$\gamma(t,x,N) = c_4. \tag{6.119}$$

Therefore, the symmetry algebra corresponding to the infinitesimals is three dimensional and given as

$$H_1 = \frac{\partial}{\partial t}, \qquad H_2 = x\frac{\partial}{\partial x}, \qquad H_3 = \frac{\partial}{\partial N}. \tag{6.120}$$

Example 6.4.3 *Consider the system of jump-diffusion Stochastic differential equations*

$$\begin{aligned} dX_1 &= X_2 dt \\ dX_2 &= -k^2 X_2 dt + \sqrt{2k^2} dW(t) + \alpha t dN(t), \end{aligned} \tag{6.121}$$

where k^2 positive is a constant and $\alpha \neq 0$ with initial condition $X_j(0) = x_0$. Therefore, the drift, jump vector and Wiener diffusion matrix are respectively

$$f_j = \begin{pmatrix} X_2 \\ -k^2 X_2 \end{pmatrix}, \quad J_j = \begin{pmatrix} 0 \\ \alpha t \end{pmatrix}, \quad G_{ij} = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{2k^2} \end{pmatrix}. \tag{6.122}$$

From the determining equation (6.72) for $j = 1$ and $j = 2$ we respectively obtain

$$x_2\left(\frac{\partial \tau}{dt} + x_2\frac{\partial \tau}{\partial x_1} - k^2 x_2\frac{\partial \tau}{\partial x_2}\right) + \xi_2(t,x_1,x_2) = \frac{\partial \xi_1}{\partial t} + x_2\frac{\partial \xi_1}{\partial x_1} - k^2 x_2\frac{\partial \xi_1}{\partial x_2} + k^2\frac{\partial^2 \xi_1}{\partial x_2^2} \tag{6.123}$$

and

(6.124)
$$-k^2 x_2\left(\frac{\partial \tau}{\partial t} + x_2\frac{\partial \tau}{\partial x_1} - k^2 x_2\frac{\partial \tau}{\partial x_2}\right) - k^2\xi_2(t,x_1,x_2) + \frac{\alpha\lambda t}{2}\left(\frac{\partial \tau}{\partial t} + x_2\frac{\partial \tau}{\partial x_1} - k^2 x_2\frac{\partial \tau}{\partial x_2}\right) = \frac{\partial \xi_2}{\partial t} + x_2\frac{\partial \xi_2}{\partial x_1} - k^2 x_2\frac{\partial \xi_2}{\partial x_2} + k^2\frac{\partial^2 \xi_2}{\partial x_2^2}.$$

While using equation (6.73) for $j = 1$ and $j = 2$ gives

$$\frac{\partial \xi_1}{\partial x_2} = \frac{1}{2}\left(\frac{\partial \tau}{\partial t} + x_2\frac{\partial \tau}{\partial x_1} - k^2 x_2\frac{\partial \tau}{\partial x_2}\right) \tag{6.125}$$

and

$$\frac{\partial \xi_2}{\partial x_2} = \frac{1}{2}\left(\frac{\partial \tau}{\partial t} + x_2\frac{\partial \tau}{\partial x_1} - k^2 x_2\frac{\partial \tau}{\partial x_2}\right). \tag{6.126}$$

From (6.75), (6.79), (6.77) and (6.78) we respectively obtain

$$\frac{\partial \tau}{\partial N} + \sqrt{2k^2}\frac{\partial \tau}{\partial x} = 0, \tag{6.127}$$

$$\frac{\partial \gamma_j}{\partial t} + f_j\frac{\partial \gamma_j}{\partial x} + \frac{G_{ij}^2}{2}\frac{\partial^2 \gamma_j}{\partial x^2} + \frac{1}{2}\frac{\partial^2 \gamma_j}{\partial N^2} = 0, \tag{6.128}$$

$$\frac{\partial \gamma_j}{\partial N} + G_{ij}\frac{\partial \gamma_j}{\partial x} = 0 \tag{6.129}$$

and

$$\gamma_j(t, x + J_j, N) - \gamma_j(t,x,N) = 0. \tag{6.130}$$

Using equation (6.76), (6.75) and (6.57) gives the temporal infinitesimal

$$\tau(t,x,N) = c_1 t + c_2. \tag{6.131}$$

Substituting temporal infinitesimal (6.131) into (6.123), (6.124), (6.125) and (6.126) respectively gives

$$x_2c_1+\xi_2=\frac{\partial\xi_1}{\partial t}+x_2\frac{\partial\xi_1}{\partial x_1}-k^2x_2\frac{\partial\xi_1}{\partial x_2}+k^2\frac{\partial^2\xi_1}{\partial x_2^2}, \tag{6.132}$$

$$\left(\frac{\alpha\lambda_0 t}{2}-k^2x_2\right)c_1-k^2\xi_2=\frac{\partial\xi_2}{\partial t}+x_2\frac{\partial\xi_2}{\partial x_1}-k^2x_2\frac{\partial\xi_2}{\partial x_2}+k^2\frac{\partial^2\xi_2}{\partial x_2^2}, \tag{6.133}$$

$$\frac{\partial\xi_1}{\partial x_2}=\frac{c_1}{2}, \tag{6.134}$$

and

$$\frac{\partial\xi_2}{\partial x_2}=\frac{c_1}{2}. \tag{6.135}$$

Solving (6.134) and (6.135) respectively gives

$$\xi_1=\frac{c_1x_2}{2}+f(t,x_1) \tag{6.136}$$

and

$$\xi_2=\frac{c_1x_2}{2}+g(t,x_1). \tag{6.137}$$

Substituting (6.136) and (6.137) into (6.132) and (6.133) respectively gives

$$x_2c_1+\frac{c_1x_2}{2}+g(t,x_1)=\frac{\partial f(t,x_1)}{\partial t}+x_2\frac{\partial f(t,x_1)}{\partial x_1} \tag{6.138}$$

and

$$\left(\frac{\alpha\lambda t}{2}-k^2x_2\right)c_1-\frac{c_1k^2x_2}{2}-k^2g(t,x_1)=\frac{\partial g(t,x_1)}{\partial t}+x_2\frac{\partial g(t,x_1)}{\partial x_1}. \tag{6.139}$$

We can conclude the following from (6.138) and (6.139)

$$c_1=0,\qquad f_{x_1}(t,x_1)=0\qquad \mathit{and}\qquad g_{x_1}(t,x_1)=0. \tag{6.140}$$

Substituting (6.140) into (6.138) and (6.139) gives

$$\frac{df(t)}{dt}=g(t) \tag{6.141}$$

and

$$\frac{dg(t)}{dt} = k^2 g(t). \tag{6.142}$$

Solving the differential equation (6.142) gives

$$g(t) = c_3 e^{-k^2 t}, \tag{6.143}$$

substituting (6.143) in (6.141), and solving for $f(t)$ we get

$$f(t) = -\frac{c_3 e^{-k^2 t}}{k^2} + c_4. \tag{6.144}$$

Finally, substituting (6.144), (6.143) in (6.136) and (6.137) using (6.140) respectively gives

$$\xi_1 = \frac{-c_3 e^{-k^2 t}}{k^2} + c_4 \tag{6.145}$$

and

$$\xi_2 = c_3 e^{-k^2 t}. \tag{6.146}$$

Similarly, using (6.140) in (6.131) and (6.80) reduces the temporal infinitesimal to

$$\tau(t) = c_2. \tag{6.147}$$

We can clearly see that for $j = 1$ (6.145) satisfied (6.74), while for $j = 2$ equation (6.146) and (6.74) led to

$$c_2 \alpha = 0 \tag{6.148}$$

which implies

$$c_2 = 0 \qquad since \qquad \alpha \neq 0. \tag{6.149}$$

To find the infinitesimal of the jump process we proceed as follows, using (6.130) implies the infinitesimal is independent of x i.e.,

$$\frac{\partial \gamma_j}{\partial x} = 0. \tag{6.150}$$

Using equation (6.150), (6.128) and (6.129) shows that the infinitesimal of the jump process variable is constant i.e.,

$$\gamma_j(t,x,N) = c_5 \qquad for \qquad j = 1,2. \tag{6.151}$$

Finally, we have the following infinitesimals

$$\tau(t) = 0, \qquad \xi_1 = \frac{-c_3 e^{-k^2 t}}{k^2} + c_4 \qquad and \qquad \xi_2 = c_3 e^{-k^2 t} \tag{6.152}$$

and

$$\gamma_j(t,x,N) = c_5 \qquad for \qquad j = 1,2. \tag{6.153}$$

Therefore, the symmetry infinitesimal generators corresponding to (6.152) and (6.153) is three dimensional given as;

$$H_1 = \frac{-e^{-k^2 t}}{k^2}\frac{\partial}{\partial x_1} + e^{-k^2 t}\frac{\partial}{\partial x_2}, \qquad H_2 = \frac{\partial}{\partial x_1}, \qquad H_3 = \frac{\partial}{\partial N} \tag{6.154}$$

Remark 6.4.4 *Note that if $c_1 = 0$ the determining equations (6.77), (6.78) and (6.79) will automatically give the infinitesimal of the jump process $\gamma(t,x,N) =$ constant otherwise we obtain a non-constant infinitesimal. Also extending the symmetry generator to include the infinitesimal transformations of the Poisson process $N(t)$ gives at least one extra symmetry generator. The Lie group classification is given in Table 6.1 below.*

TABLE 6.1: Lie Group Classification Chapter 6

Group Dimension	Basis Operators	Equations
3	$H_2 = \frac{\partial}{\partial t} - kt^2\frac{\partial}{\partial x}, \quad H_3 = \frac{\partial}{\partial x}, \quad H_4 = \frac{\partial}{\partial N}.$	$dX = -kt^2 dt + \sqrt{D}dW + bdN$
3	$H_1 = \frac{-e^{-k^2 t}}{k^2}\frac{\partial}{\partial x_1} + e^{-k^2 t}\frac{\partial}{\partial x_2}, \quad H_2 = \frac{\partial}{\partial x_1}, \quad H_3 = \frac{\partial}{\partial N}.$	$dX_1 = X_2(t)dt$ $dX_2 = -k^2 X_2 dt + \sqrt{2k^2}dW + \alpha t dN$
3	$H_1 = \frac{\partial}{\partial t}, \quad H_2 = x\frac{\partial}{\partial x} \quad H_3 = \frac{\partial}{\partial N}.$	$dX(t) = X(t)(u_0(t)dt + \alpha_0(t)dW + v_0(t)dN(t))$

Chapter 7

Conclusions

The mixture of Wiener and a Poisson processes are the primary tools used in creating jump-diffusion process which is very popular in mathematical modeling. In financial mathematics, they are used to describe the change of stock rates and bonanzas, and they are often used in mathematical biology modeling and population dynamics.

In this thesis, we have successfully extended the Lie symmetry methods to the class of jump-diffusion stochastic differential equations, i.e., a stochastic process driven by both Wiener and Poisson processes. The main goals are first to construct the determining equations for the jump-diffusion stochastic differential equations by applying the invariance methodology of Lie point transformation together with Jump-diffusion Itô lemma, without enforcing any conditions to the moments of the stochastic processes. And applied the developed theory to some stochastic models with physical significance to shown that the determining equations found can be solved to obtained the required infinitesimals.

First, we defined a Lie point symmetry transformation of a class of Poisson driven stochastic differential equations (7.1), which extended the earlier theory of the Lie symmetry on Wiener driven stochastic differential equations [7, 8, 9, 10, 15, 20].

$$dX_i(t) = f_i(t, X(t))dt + J_i(t, X(t))dN(t). \tag{7.1}$$

The determining equations are obtained by considering the infinitesimals of temporal and spatial variables i.e.,

$$H = \tau(t,x)\frac{\partial}{\partial t} + \xi_i(t,x)\frac{\partial}{\partial x_i} \tag{7.2}$$

and they were given as;

$$\left(f_j\Gamma_{(\tau)} + \frac{\lambda J_j}{2}\Gamma_{(\tau)} + H(f_j) - \Gamma_{(\xi)_j}\right)(t, X(t)) = 0, \tag{7.3}$$

$$\left(\frac{J_j}{2}\Gamma_{(\tau)} + H(J_j) - \Gamma^*_{(\xi)_j}\right)(t, X(t)) = 0 \tag{7.4}$$

and

$$\Gamma^*_{(\tau)}(t, X(t)) = 0, \qquad \Gamma_{(\tau)}(t, X(t)) = 0. \tag{7.5}$$

We have successfully applied the theory to some stochastic models and they both admitted two-dimensional Abelian Lie algebras $L_2^I = [H_1, H_2] = 0$, the Canonical structures are given in *Table 7.1* below

TABLE 7.1: Canonical Forms of Poisson Driven Stochastic Equations

Algebra	Basis Operators	Representing Equations
L_2^I	$H_1 = \frac{\partial}{\partial t}, \qquad H_2 = 2x\frac{\partial}{\partial x}$	$dX(t) = X(t)(2dt + dN(t))$
L_2^I	$H_1 = \frac{\partial}{\partial t} - kt^2\frac{\partial}{\partial x}, \qquad H_2 = \frac{\partial}{\partial x}$	$dX = -kt^2dt + bdN(t)$
L_2^I	$H_1 = \frac{\partial}{\partial t} \qquad H_2 = \frac{\partial}{\partial x}$	$dX(t) = adt + dN(t) \qquad a \neq 0$
L_2^I	$H_1 = \frac{\partial}{\partial t}, \qquad H_2 = \frac{\partial}{\partial x}$	$dX(t) = \alpha dN(t) \qquad \alpha \neq 0.$

Extending the symmetry generator to include the infinitesimal transformations of Poisson processes $N(t)$ the determining equations $(7.3) - (7.5)$ has extended to include;

$$\Gamma(\phi)_j = 0, \quad and \quad \Gamma^*(\phi)_j = 0. \tag{7.6}$$

The stochastic models discussed in this case both admitted infinite-dimensional Lie algebras. An examples for a trivial form of the Poisson infinitesimal generator are given as follows;

- $dX(t,N) = -kt^2dt + bdN(t)$ admitted three-dimensional Abelian Lie algebras L_3^I
$$H_1 = \frac{\partial}{\partial t} - kt^2\frac{\partial}{\partial x}, \quad H_2 = \frac{\partial}{\partial x}, \quad H_3 = \frac{\partial}{\partial N}$$
with $L_3^I = [H_1,H_2] = [H_1,H_3] = [H_2,H_3] = 0$.

- $dX(t,N) = t(-k^2tdt + \alpha dN(t))$ admitted two-dimensional Abelian Lie algebras $L_2^I = [H_1,H_2] = 0$,
$$H_1 = \frac{\partial}{\partial x}, \qquad H_2 = \frac{\partial}{\partial N}.$$

- $dX(t,N) = -k^2xdt + \sqrt{2k^2}dN(t)$ admitted three-dimensional Lie algebras
$$H_1 = \frac{\partial}{\partial t}, \quad H_2 = e^{-k^2t}\frac{\partial}{\partial x}, \quad H_3 = \frac{\partial}{\partial N}.$$
Simple computation shows that
$$[H_1,H_2] = -k^2H_2, \quad [H_1,H_3] = 0, \qquad [H_2,H_3] = 0.$$
Thus the symmetries span a Lie algebras.

The theory was later extended to define the Lie point symmetries for jump-diffusion stochastic differential equations (i.e., stochastic differential equations driven by both Wiener and Poisson processes).

$$dX_i(t) = f_i(t,X(t))dt + G_{ik}(t,X(t))dW_k(t) + J_i(t,X(t))dN(t). \tag{7.7}$$

The determining equations are derived in an Itô calculus context and were found to be non-stochastic though they represent a stochastic process.

The Lie point symmetries of (4.3) are discussed by considering infinitesimals involving the spatial variable x and time variable t

$$H = \tau(t,x)\frac{\partial}{\partial t} + \xi_i(t,x)\frac{\partial}{\partial x_i}, \tag{7.8}$$

and has admitted the following determining equations;

$$\left(f_j\Gamma_{(\tau)} + \frac{\lambda J_j}{2}\Gamma_{(\tau)} + H(f_j) - \Gamma_{(\xi)_j}\right)(t,X(t)) = 0, \tag{7.9}$$

$$\left(\frac{G_{jk}}{2}\Gamma_{(\tau)} + H(G_{jk}) - \Gamma^{*}_{(\xi)_j}\right)(t, X(t)) = 0, \tag{7.10}$$

$$\left(\frac{J_j}{2}\Gamma_{(\tau)} + H(J_j) - \Gamma^{**}_{(\xi_j)}\right)(t, X(t)) = 0, \tag{7.11}$$

$$\Gamma^{*}_{(\tau)}(t, X(t)) = 0, \quad \Gamma_{(\tau)}(t, X(t)) = 0 \tag{7.12}$$

and

$$\Gamma^{**}_{(\tau)}(t, X(t)) = 0, \tag{7.13}$$

where the operators $\Gamma(t,x)$, $\Gamma^{*}(t,x)$ and $\Gamma^{**}(t,x)$ are defined in (4.6), (4.7) and (4.8), $\lambda > 0$ is the jump rate.

The stochastic models consider admitted two-dimensional Abelian Lie algebras $L_2^I = [H_1, H_2] = 0$, the canonical forms of Poisson driven stochastic equations admitting two-dimensional symmetry Lie algebras are given in *Table 7.2* below

TABLE 7.2: Canonical Forms

Algebra	Basis Operators	Representing Equations
L_2^I	$H_1 = \frac{\partial}{\partial t} - kt^2\frac{\partial}{\partial x}, \qquad H_2 = \frac{\partial}{\partial x}.$	$dX = -kt^2 dt + \sqrt{D}dW + bdN(t)$
L_2^I	$H_1 = \frac{-e^{-k^2 t}}{k^2}\frac{\partial}{\partial x_1} + e^{-k^2 t}\frac{\partial}{\partial x_2}, \qquad H_2 = \frac{\partial}{\partial x_1}.$	$dX_1 = X_2 dt$ $dX_2 = -k^2 X_2 dt + \sqrt{2k^2}dW + \alpha t dN(t)$
L_2^I	$H_1 = \frac{\partial}{\partial t}, \qquad H_2 = x\frac{\partial}{\partial x}.$	$dX(t) = X(t)(u_0(t)dt + \alpha_0(t)dW + v_0(t)dN(t)).$

We extended the Lie transformation theory of the jump-diffusion stochastic equations by including dependency of the infinitesimal of the Wiener $W(t)$ and the Poisson $N(t)$ processes in the Lie generator. This means the determining equations (7.9) – (7.13) now include

$$\Gamma^{**}_{(w)}(\phi_l)(t, X(t,w)) = 0, \quad \Gamma^{*}_{(w)}(\phi_l)(t, X(t,w)) = 0, \quad \Gamma_{(w)}(\phi_l)(t, X(t,w)) = 0 \tag{7.14}$$

and

$$\Gamma^{*}_{(N)}(\gamma_j)(t,X(t,N)) = 0, \quad \Gamma^{**}_{(N)}(\gamma_j) = 0, \quad \Gamma_{(N)}(\gamma_j)(t,X(t,N)) = 0, \tag{7.15}$$

where the operators $\Gamma_{(w)}(t,X(t,w))$, $\Gamma^{*}_{(w)}(t,X(t,w)) = 0$, $\Gamma^{**}_{(w)}(t,X(t,w))$, $\Gamma_{(N)}(t,X(t))$, $\Gamma^{*}_{(N)}(t,X(t))$ and $\Gamma^{**}_{(N)}(t,X(t))$ are defined as in (6.3), (6.4), (6.5), (5.7), (5.8) and (5.9) .

It is worth noticing that extending the Lie transformations to include the dependency of the Wiener and Poisson process infinitesimals rise to the larger algebra. The Canonical forms in each case are given in *Table 7.3* and *Table 7.4* respectively

TABLE 7.3: Canonical Forms of Jump-diffusion SDE with Wiener Infinitesimal Dependency

Algebra	Basis Operators	Representing Equations
L_2^I	$H_1 = e^{k^2 t}\frac{\partial}{\partial x}$, $H_2 = \frac{\partial}{\partial w}$	$dX(t) = -k^2 X(t)dt + \sqrt{2k^2}dW(t) + \alpha t dN(t)$
L_3^I	$H_1 = \frac{\partial}{\partial t}$, $H_2 = 2x\frac{\partial}{\partial x}$, $H_3 = \frac{\partial}{\partial w}$.	$dX(t) = u_0 X(t)dt + \alpha_0 X(t)dW(t) + v_0 X(t)dN(t)$
L_3^I	$H_1 = \frac{\partial}{\partial t} - kt^2\frac{\partial}{\partial x}$, $H_2 = \frac{\partial}{\partial x}$, $H_3 = \frac{\partial}{\partial w}$.	$dX(t) = -kt^2 dt + \sqrt{D}dW(t) + b dN(t), D > 0, b \neq 0$

TABLE 7.4: Canonical Forms of Jump-diffusion SDE with Poisson Infinitesimal Dependency

Algebra	Basis Operators	Representing Equations
L_3^I	$H_2 = \frac{\partial}{\partial t} - kt^2\frac{\partial}{\partial x}$, $H_3 = \frac{\partial}{\partial x}$, $H_4 = \frac{\partial}{\partial N}$.	$dX = -kt^2 dt + \sqrt{D}dW + b dN$
L_3^I	$H_1 = \frac{-e^{-k^2 t}}{k^2}\frac{\partial}{\partial x_1} + e^{-k^2 t}\frac{\partial}{\partial x_2}$, $H_2 = \frac{\partial}{\partial x_1}$, $H_3 = \frac{\partial}{\partial N}$.	$dX_1 = X_2(t)dt$ $dX_2 = -k^2 X_2 dt + \sqrt{2k^2}dW + \alpha t dN$
L_3^I	$H_1 = \frac{\partial}{\partial t}$, $H_2 = x\frac{\partial}{\partial x}$ $H_3 = \frac{\partial}{\partial N}$.	$dX(t) = X(t)(u_0(t)dt + \alpha_0(t)dW + v_0(t)dN(t))$.

The following problems remain unsolved;

- Extending Lie transformation theory to a class of stochastic differential equations driven by fractional Brownian motions.
- Is the Lie transformation theory of stochastic differential equations as applicable as that of deterministic differential equations?

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