# Realised Volatility Estimators

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# Declaration

I declare that this dissertation is my own, unaided work. It is being submitted for the Degree of Master of Philosophy in the University of the Cape Town. It has not been submitted before for any degree or examination in any other University.

May 27, 2014

# Abstract

This dissertation is an investigation into realised volatility (RV) estimators. Here, RV is defined as the sum-of-squared-returns (SSR) and is a proxy for integrated volatility (IV), which is unobservable. The study focuses on a subset of the universe of RV estimators. We examine three categories of estimators: historical, high-frequency (HF) and implied. The need to estimate RV is predominantly in the hedging of options and is not concerned with speculation or forecasting. The main research questions are; (1) what is the best RV estimator in a historical study of S&P 500 data? (2) What is the best RV estimator in a Monte Carlo simulation when delta hedging synthetic options? (3) Do our findings support the stylized fact of 'Asymmetry in time scales' (Cont, 2001)? In the answering of these questions, further avenues of investigation are explored. Firstly, the VIX is used as the implied volatility. Secondly, the Monte Carlo simulation generates stock price paths with random components in the stock price and the volatility at each time point. The distribution of the input volatility is varied. The question of asymmetry in time scales is addressed by varying the term and frequency of historical data. The results of the historical and Monte Carlo simulation are compared. The SSR and two of the HF estimators perform best in both cases. Accuracy of estimators using long term data is shown to perform very poorly.

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# Chapter 1

# Introduction

Unlike many other financial variables in the financial industry, latent or integrated volatility (IV) is not observable (McAleer and Medeiros, 2008). Realised volatility (RV) is, however, a consistent estimator of IV (Andersen *et al.*, 2001). The importance of estimating RV and therefore IV is vast. Volatility is a measure of the level of risk (and fear) in the financial markets. Accurately estimating volatility is vital for hedging options. The need to hedge options and manage risk has gained additional attention in the years following the recent global financial crisis. Modelling and forecasting volatility is important in financial applications such as derivative pricing, risk management, asset management and speculative trading (Kulikova and Taylor, 2013). This dissertation does not address these applications, but focuses solely on finding a reliably accurate estimator and applying this estimator to a hedging problem.

Extensive academic research has been conducted on the forecasting, modelling and estimation of RV. The need to estimate volatility was noted as early as 1980 by Parkinson (1980) and Garman and Klass (1980). The Black-Scholes-Merton model has driven this need as it is an input of the model. Several categories of RV estimators exist: historic, high-frequency (HF), implied, stochastic volatility models and (G)ARCH-type models. The latter two categories are not investigated here, as they are more suited for forecasting applications. This dissertation could nonetheless be extended by the inclusion of such categories of estimators. Due to the large number of possible estimators, we have focused on a subset.

Similar comparative research has been conducted with varying research aims. Ahoniemi and Lanne (2013), Bollen and Inder (2002) amongst others, focused on HF estimators and their performance. Christensen and Prabhala (1998) discuss the difference between implied and historical estimators, and Shu and Zhang (2003) compare the performance of these estimators. The purpose of this dissertation is to extend the previously mentioned research. The extensions are: (1) a comparison of the three estimator categories; (2) the use of the VIX as an implied volatility estimator; (3) the implementation of a Monte Carlo delta hedging simulation and (4) the investigation of asymmetry in time scales (Cont, 2001).

The search for a reliably accurate estimator examines the difference between historic and implied estimators. Historic volatility is calculated from past price data, and is "backward looking". The term "backward looking" means that the information used in calculating the estimator is purely obtained from past data. Implied volatility is is derived from derivative market prices, and it is "forward looking". The term "forward looking" means that the information used in calculating the estimator is based on future expectations. This difference between historical and implied estimators/estimates needs to be carefully understood when calculating RV. "Asymmetry of time scales: coarse-grained measures of volatility predict fine-scale volatility better than the other way round" is a "stylized fact" of finance discussed by Cont (2001). This "stylized fact" is interpreted as following: a low-frequency estimator, using say weekly data, is a better estimator for higher frequency RV, say daily, rather than the other way around. This "stylized fact" is investigated by comparing five-minute, weekly and monthly estimators to the daily estimator. The daily estimator is the annualised RV calculated from daily data over 20 trading days. The RV is calculated with the sum-of-squares (SSR) approach. RV using sum-of-squares SSR is commonly used in industry. The SSR approach to estimating RV is also vital in the calculation of the pay-off of volatility swaps. All reference to RV will refer to the most recent 20 trading day SSR.

The general approach to the investigation of RV is intended to be practical. This is the reason for the inclusion of the VIX as an estimator, as it is a widely used proxy for volatility. Initially we compare estimators to RV. Non-overlapping data is used to calculate RV and the estimates. The estimates are calculated form data of a previous time period. RV is calculated from the most recent time period. Thereafter, we analyse the performance of a delta-hedging strategy for at-the-money calls and puts, using each estimator as an input for volatility. The distributions of the profit and loss of the hedged options are then analysed. As an innovation the stock price paths used in the Monte Carlo simulation have randomly generated volatilities at each time step.

The structure of this dissertation will be as following; Section 2 will briefly review some theory regarding Realised Volatility and Integrated Volatility, which will be followed by discussions of the Historical, Intra-day and Implied volatility estimators. Following the theory and description of the estimators, the application of these will be discussed in Section 3. The application includes the data used and the approach to comparing estimators. The results of this dissertation are discussed in Section 4. Lastly conclusions are made in Section 5.

# Chapter 2

# Estimators

The purpose of this section is two fold. Firstly, the background of RV and IV will be explained briefly. Secondly, the RV estimators used in this dissertation will be discussed. The relationship between RV and IV is an important one to grasp, as it brings insight into the purpose of the study. The IV is a measure of the true volatility over a time period. The IV is unobservable and in many cases is the true focus of volatility studies even if not stated explicitly.
Assume the diffusion process:

$$dp(t+\tau) = \mu(t+\tau)d\tau + \sigma(t+\tau)dW((t+\tau),\ 0 \leq \tau \leq 1,\ t = 1, 2, \ldots \tag{2.1}$$

where $p$ is defined as the price of an asset, $\mu$ as the drift component, $\sigma$ as the volatility and $W(t+\tau)$ is defined as standard Brownian motion (McAleer and Medeiros, 2008). The daily returns defined as $r_t = p(t) - p(t-1)$ have been shown to be conditionally normal (Andersen *et al.*, 2001) such that:

$$r_t|\mathcal{F}_t \sim \mathrm{N}\left(\int_0^1 \mu(t+\tau-1)d\tau, \int_0^1 \sigma^2(t+\tau)d\tau\right),$$

where $\mathcal{F}_t$ is the filtration generated by the sample paths of the drift and volatility processes $\mu(t+\tau)$ and $\sigma(t+\tau)$. As explained by McAleer and Medeiros (2008), the integrated variance is defined as the term $\int_0^1 \sigma^2(t+\tau)d\tau$. The IV is the square root of the integrated variance, defined as follows:

$$\mathrm{IV}_t = \sqrt{\int_0^1 \sigma^2(t+\tau)d\tau}. \tag{2.2}$$

Correctly estimating IV is the aim of all hedging activities with a process such as (2.1). Let RV be defined as follows:

$$\mathrm{RV}_t = \sqrt{\sum_{i=1}^{n} r_i^2}. \tag{2.3}$$

It has been shown that $\mathrm{RV}_t$ is a sufficient estimator of IV, and that (Andersen *et al.*, 2001)

$$\mathrm{RV}_t \overset{p}{\sim} \mathrm{IV}_t. \tag{2.4}$$

## 2.1 Historical Estimators

This section will briefly review the historical estimators used in this dissertation. These estimators use high, low, open and close data of the relevant time period to estimate annualised historical volatility. Care needs to be taken when the coarseness and length of data are varied, however the same scaling principles are used throughout the various estimators . The historical volatility is the volatility of the previous time period, and is an estimate of the RV. In order to compare coarseness and length of period, the daily 20-trading-day volatility $\sigma^{(d)}$, annual volatility using monthly data $\sigma^{(m)}$ and the annual volatility using weekly data $\sigma^{(w)}$ are calculated and analysed. All estimates are annualised for ease of comparison.

### 2.1.1 SSR estimator

The SSR equation for RV was shown in (2.3). Lagged SSR is one of the estimators of RV. The same technique will be used to calculate the RV and the historic daily SSR estimate, the difference will merely be the lag of the estimator. The use of SSR was proposed by Andersen *et al.* (2001). Differences in calculating the estimate come in to play when term and frequency vary from that of RV. The annualised volatility (where the annual moving window is taken to be 252 trading days) calculated from $n$ end of day price observations $p_t$ is given by:

$$\sigma_{SSR} = \sqrt{\frac{q}{n}\sum_{t=1}^{n} r_t^2} * 100,$$

where the daily log-return $r_t$ is given by $\ln(p_t) - \ln(p_{t-1})$ (Shu and Zhang, 2003) and $q$ is the annual scaling factor.

For volatility with a term of 20 trading days and a daily observation frequency, $q = 252$

and $n = 20$. For the volatility with weekly observational frequency over a term of a year , $q = 52$ and $n = 52$ and for monthly observed volatility , $q = 12$ and $n = 12$. The values used for $q$ and $n$ will be the same as for the other estimators discussed in this section. The RV estimators are multiplied by 100 for comparative reasons, as the VIX follows this approach. Furthermore, it will amplify small differences, making them easier to observe.

### 2.1.2 Parkinson Estimator

The Parkinson (Park) volatility estimator calculated from the highest and lowest values in a given time period to calculate dispersion (Parkinson, 1980). This estimator squares the differences of the log high and low values for each time period. A scaling factor is required, which can be extracted when deriving the estimator from the probability density function (Parkinson, 1980). The resulting Estimator for a period of $n$ observations is given (Shu and Zhang, 2003):

$$\sigma_{Park} = \sqrt{\frac{q}{n}\sum_{t=1}^{n}\frac{1}{4\ln(2)}(\ln H_t - \ln L_t)^2}$$

where $H_t$ and $L_t$ are the highest and lowest observation in time interval $t$. $H_t$ and $L_t$ may be daily, weekly or monthly highs and lows, respectively, depending on the estimator.

### 2.1.3 Garman-Klass Estimator

The Garman-Klass (GK) estimator for volatility calculated from high and low as well as open and close data, and essentially is a weighted sum of the two previous estimators. The GK estimator was developed in an attempt to create an unbiased estimator with a lower variance than the previous estimators (Garman and Klass, 1980). The resulting GK Estimator for $n$ observations is given as (Shu and Zhang, 2003), (Garman and Klass, 1980):

$$\sigma_{GK} = \sqrt{\frac{q}{n}\sum_{t=1}^{n} 0.5(\ln(H_t/L_t))^2 - 0.39(\ln(p_t/p_{t-1}))^2}.$$

### 2.1.4 Yang-Zhang Estimator

The three estimators discussed so far all assume either that there is no drift in the log returns, or that there are no jumps between close prices and the following opening prices. These assumptions seem to be acceptable for short time intervals, but may cause problems over longer time intervals (Yang and Zhang, 2000). Yang and Zhang (YZ) proposed a drift and jump independent volatility estimator, that is expected to result in a more accurate estimate of volatility while only having a negligible increase in estimator variance. The estimators previously discussed, were shown to behave less favourably when there were drift and jump components in the returns. The YZ estimator is given by (Shu and Zhang, 2003),(Yang and Zhang, 2000):

$$\sigma_{YZ} = \sqrt{q} \times \sqrt{V_o + kV_c + (1-k)V_{RS}}.$$

Where

$$V_o = \frac{1}{n-1}\sum_{t=1}^{n}(o_t - \bar{o})^2$$
$$o_t = \ln(O_t/O_{t-1})$$
$$\bar{o} = 1/n\sum_{t=1}^{n} o_t$$

for $O_t$ which are the opening prices. $V_c$ is defined similarly, where $C_t$ are the closing prices.

$$V_{RS} = \frac{1}{n}\sum_{t=1}^{n}[(\ln H_t - \ln O_t)(\ln H_t - \ln C_t) + (\ln L_t - \ln O_t)(\ln L_t - \ln C_t)]$$

where $H_t$ and $L_t$ are the high and low prices.
The constant $k$ is given by:

$$k = \frac{0.34}{1.34 + \frac{n+1}{n-1}}.$$

## 2.2 High-Frequency Estimators

The estimators discussed thus far, have focused on inter-day data. However, there are several arguments for using high-frequency intra-day price data in estimating volatility. One of the main reasons one would consider intra-day returns in estimating realised volatility, is that it allows the inclusion of overnight returns, which has been shown to be more accurate (Hansen and Lunde, 2005). The reason why overnight returns are relevant is because, even though prices do not change during the night, information still accumulates. This additional overnight information can result in significant price changes, which will consequently affect the volatility (Ahoniemi and Lanne, 2013). In our study of intra-day volatility estimators, five-minute prices over the 20-trading-day day period will be used in the calculation of the realised volatility estimator. The use of a five-minute frequency is justified as it has become common practice. The reason for its widely accepted use is that the frequency is low enough to avoid effects of dependence, yet is high enough to study intra-day effects (Hansen and Lunde, 2005).

Four methods of estimating RV from intra-day returns will be discussed and their performance compared to that of other estimators. The four approaches to using intra-day returns to calculate high frequency (HF) estimators are those suggested by Ahoniemi and Lanne (Ahoniemi and Lanne, 2013). The four approaches are: SSR ignoring overnight effects, equally weighting the open-close difference and daily SSR, scaling SSR estimates and weighting overnight returns and the daily SSR. The four high-frequency estimators will be referred to as $\sigma_{HF1}, \sigma_{HF2}, \sigma_{HF3}, \sigma_{HF4}$, respectively.

The first approach uses the SSR approach and merely uses higher frequency data. The HF1 estimator, however, ignores overnight effects and sums daily SSR over the 20 trading days. This simplistic estimator was used in Wu (2011), Ahoniemi and Lanne (2013) amongst others. Wu applied this approach to data observed every second (Wu, 2011). In our application we will apply the $\sigma_{HF1}$ estimator to five-minute observations. Essentially this estimator treats the price sequence as continuous, and does not account for the longer time period that has elapsed between daily close and open prices. The estimator is given by:

$$\sigma_{HF1} = \sqrt{\frac{k}{n} \sum_{i=1}^{21} \sum_{j=1}^{m} r_{i,j}^2},$$

where $n$ and $k$ are defined as before and $m$ is the number of five minute returns on a given day, $i$. It follows that $r_{i,j}$ is the return on day $i$ at the $j$-th five minute interval. In our case, $k = 252 * m$ and $n = 20 * m$.

The second high-frequency estimator $\sigma_{HF2}$, includes the overnight component by adding the difference between daily open and close prices as another equally weighted squared return observation in the SSR estimator (Ahoniemi and Lanne, 2013). This estimator places equal emphasis on overnight changes as on the intra-day day changes. However, one could argue that it is an improvement on the first estimator, as it recognises the passing of time between close and opening prices. The estimator is calculated as follows:

$$\sigma_{HF2} = \sqrt{\frac{k}{n} \left( \sum_{i=1}^{21} \sum_{j=1}^{m} r_{i,j}^2 + \tilde{r}_{i+1,1}^2 \right)},$$

where $\tilde{r}_{i+1,1}$ is the log return of the close from day $i$ and the open on day $i+1$. It needs to be noted that $n = 20 * (m+1)$, as there is an extra return for each day.

The third high-frequency estimator, $\sigma_{HF3}$, extends the first by still ignoring overnight effects, but "scaling the resulting value upward so that the volatility estimate covers an entire 24-hour day" (Ahoniemi and Lanne, 2013). The estimator is given by:

$$\sigma_{HF3} = \sqrt{\frac{k}{n}\sum_{i=1}^{21}\frac{288}{m}\sum_{j=1}^{m} r_{i,j}^2},$$

which scales the daily SSR to a 24-hour day instead of $m * 5$ minutes. There are 288 five minute intervals in a 24-hour day.

The final high frequency estimator, $\sigma_{HF4}$, suggested by Ahoniemi and Lanne (2013) weights the overnight return and the SSR of the day returns separately (Hansen and Lunde, 2005). This estimator is the most complex of the four, and is given by:

$$\sigma_{HF4} = \sqrt{\frac{k}{n}\sum_{i=1}^{21}\omega_1\sum_{j=1}^{m} r_{i,j}^2 + \omega_2\tilde{r}_{i+1,1}^2},$$

where $\omega_1$ and $\omega_2$ are the weighting factors. The weighting factors are shown by Hansen and Lunde (2005) to be calculated by solving the following optimisation problem:

$$\min_{\omega_1,\omega_2} var\left(\omega_1\sum_{j=1}^{m} r_{i,j}^2 + \omega_2\tilde{r}_{i+1,1}^2\right)$$

such that

$$\omega_1\mu_1 + \omega_2\mu_2 = \mu_0$$

where the parameters are defined as in Hansen and Lunde (2005).

## 2.3 Implied Estimators and the VIX

Option prices are observable in a market, whereas the volatility is not, which means that "volatility" of the underlying can be implied from option prices. This approach to estimating volatility from option prices is unique as there is a one-to-one relationship between option prices and implied volatility. This volatility estimate has been popularised with the development of the Black-Scholes-Merton model as well as others such as the Heston model, which unlike the Black-Scholes-Merton model allows the volatility to be random (Shu and Zhang, 2003). These models price derivatives with volatility as an input, or can imply volatility from market prices. Christensen and Prabhala (1998) showed that implied volatility can be an efficient and accurate estimate if non-overlapping and long time series are used. However, these approaches are very much model dependant, and have their own drawbacks in accurately modelling and estimating volatility. One problem with using volatility implied from option prices is that it is difficult to perform a long term study, as maturities of liquidly traded listed options are short dated, resulting in a need to roll-over the options. Furthermore, options of various strikes need to be analysed. Lastly the behaviour of option prices is erratic close to maturity, which could influence the RV estimate. Due to these practical complications of calculating implied volatilities from individual options, it has been proposed in literature to use volatility indices, such as the VIX, as a proxy for implied volatility (Koopman *et al.*, 2005). In this comparison of volatility estimators, the VIX is used as an implied volatility estimate proxy.

Even though the VIX and other indices are quoted and readily available, it is important to understand how they are calculated. This allows a comparison of the index to historical volatility. In this study, the index of interest is the VIX. The VIX is a volatility index on the Chicago Board of Exchange (CBOE) and is the measure of the markets expectation of the 30-day volatility implied by the S&P 500 (CBOE, 2009). The VIX is a weighted average of implied volatilities of options on the S&P 500 Index with various strikes and maturities. No options that are close to maturity are used, due to the anomalies that occur close to the maturity date. The derivation of the index level formula is based on a technical report by Goldman Sachs regarding the pricing of volatility swaps. Volatility swaps are vanilla swaps with a fixed leg and a floating leg determined by RV per unit Nominal (Demeterfi *et al.*, 2009). As the method of calculating the VIX is fixed, other estimators in this study have been influenced by the VIX estimator. The three main influences that the VIX has had on the choice of the other estimators are: the 30 day RV period (20 trading days), using log returns in the volatility calculations and the use of the SSR for RV and Intra-day estimators Demeterfi *et al.* (2009). The SSR is the used in calculating the floating leg of the volatility swap. The derivation of the formula to calculate the VIX is discussed and explained extensively by Demeterfi *et al.* (2009). The generalised formula for calculating the VIX is:

$$\sigma_{VIX}^2 = \frac{2}{T}\sum_i \frac{\Delta K_i}{K_i^2} e^{\mathrm{rT}} Q(K_i) - \frac{1}{T}\left[\frac{F}{K_0} - 1\right]^2 .$$

where $T$ is the maturity, $r$ is the risk-free rate, $K_i$ are strike prices, $K_0$ is the first strike below the market implied forward price $F$ and $Q(\cdot)$ is the midpoint of the bid-ask spread (CBOE, 2009) (Demeterfi *et al.*, 2009).

# Chapter 3

# Application

The data and methods used to compare estimators are discussed in this section. The S&P 500 spot price data was obtained from Bloomberg. This data will be described and analysed in this section. The data used for the Monte Carlo study of the synthetic delta hedged options is simulated. The method of simulation will be discussed later.

## 3.1 S&P 500 study

### 3.1.1 Data

In this dissertation S&P 500 data is used to calculate the historical, HF and realised volatility. The reason for this choice is that the underlying of the options used in the calculations for the VIX is the S&P 500. Three data sets obtained from Bloomberg will be used. Firstly the daily data of high, low, open and close data, secondly high-frequency open and close price data and lastly the VIX data will be used. It is important to note that the VIX is derived from spot prices, and is itself not a futures Index. The time period in question spans from 3 January 2005 to 3 January 2014, which encompasses 2265 trading days. This time period includes the sub-prime mortgage crisis as well as the subsequent time period which has displayed a decrease in risk appetite and increased regulation. As the results will be investigated across the entire time period as well as over 500 data point intervals, the effects of these events will be noted. The Estimators will be estimated from 2006-2013, and each of the 500 point intervals corresponds to approximately two years. Due to restrictions on high-frequency data, only data from 1 January 2008 until 3 January 2014 was attainable, a total of 121892 data points. As a result the first 502 historical and implied estimates will have no corresponding high frequency estimates. It is important to note that with regards to this HF data, that on some days the NYSE is only open for half a day, such as Christmas day.

### Analysis of Log-Returns

Proper analysis of the data that forms the basis of this investigation is necessary, in order to fully comment on the results. The table below displays the summary statistics of the daily S&P 500 log-returns broken up over smaller intervals. It needs to be noted that the last window is slightly smaller than the others with 492 estimators.

| | **Log-Return Distribution** | | | | |
|---|---|---|---|---|---|
| **data points** | **1-500** | **501-1000** | **1001-1500** | **1500-end** | **1-end** |
| Mean | 0.00034 | -0.00099 | 0.00071 | 0.00052 | 0.00019 |
| Standard Dev. | 0.00640 | 0.01968 | 0.01467 | 0.01042 | 0.01337 |
| Skewness | 0.03142 | -0.17721 | -0.08588 | -0.58452 | -0.31994 |

**Table 3.1:** Distribution of S&P 500 Log-Returns

The mean of log-returns overtime shows that there's a negative mean over the second 500 data points, which is different to all the others. In Addition to this negative mean, this time period also has the highest standard deviation. This time window corresponds to 2008 and 2009, and sensitivity and care is required when analysing the results of the estimators in this period. Another interesting observation that can be made, is that the skewness in the most recent time period, corresponding to 2012-2013 has the most negative skewness.

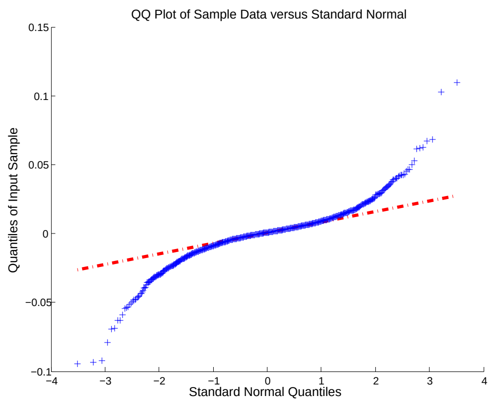


**Figure 3.1:** QQ-plot of S&P 500 log-returns

When analysing the QQ-plot in Figure 3.1, it can be seen that the tails of the log-return distribution are in fact thicker than that of a normal distribution. The thicker tails are to be expected.

#### Analysis of RV

The distribution of the RV estimate, calculated from the data, is summarised in Table 3.2.

| | **RV distrbution** | | | | |
|---|---|---|---|---|---|
| **data points** | **1-500** | **501-1000** | **1001-1500** | **1500-end** | **1-end** |
| Mean | 12.55143 | 29.84629 | 18.81613 | 11.74195 | 18.26177 |
| Standard Dev. | 5.33287 | 18.11196 | 9.02585 | 2.82035 | 12.81167 |
| skewness | 0.85467 | 1.51958 | 1.08069 | 0.46270 | 2.61494 |

**Table 3.2:** Distribution of RV of S&P 500

The RV estimate, similar to the log-returns, has a very high standard deviation in the 2008-2009 time period which is very indicative of the uncertainty in this time period. The mean RV within this time period was double that of the 2006-2007 period, and has decreased significantly since then. The mean of the RV in the last time interval is lower than the pre-crisis levels. The standard deviation of the RV estimate has also decreased significantly. In the most recent time period the standard deviation is less than half of what pre-crisis levels used to be. A possible reason for such low standard deviation of RV in the last two years, is the combined effect of market stabilisation and a decrease in proprietary trading in banks, a decrease in demand of exotic options on the S&P500 and new regulatory requirements. The distribution of the RV will be compared to that of the estimators in order to analyse the accuracy of the estimator. Further comparative measures are discussed below.

### 3.1.2 In-Sample Comparison

The first approach of comparing RV estimators of the S&P 500 study is an in-sample Mean Square Error (MSE). This measure of estimate accuracy squares the difference between the RV and its estimate. A MSE closest to zero is preferred, however due to the estimators being multiplied by 100, relatively high MSE's can be expected. The in-sample MSE will be calculated as in Ahoniemi and Lanne (2013):

$$\mathrm{MSE}_x^{(in)} = \frac{1}{N}\sum_{t=1}^{N}(\sigma_{RV,t} - \sigma_{x,t})^2, \tag{3.1}$$

for the various estimators $x \in \{SSR(d), SSR(w), SSR(m), GK(d), \ldots, HF1, HF2, \ldots, VIX\}$. $N$ is the number of in sample estimates. Here the estimator for realised volatility $\sigma_x$ at time $t$ is compared to $\sigma_{RV}$ at time $t$. The in-sample approach is a direct comparison of the estimates and RV.

### 3.1.3 Out-of-Sample Comparison

The second approach to comparing the RV estimators of the S&P 500 study is a one factor out-of-sample MSE. This approach performs a linear regression on $m$ estimates with the RV as shown in Shu and Zhang (2003):

$$\sigma_{RV,i} = \alpha + \beta\sigma_{x,i} + \epsilon_i, \text{ for } i = 1, 2, \ldots, m.$$

From this regression, estimates $\hat{\alpha}$ and $\hat{\beta}$ over a period $m$ are obtained. The period $m$ may span the entire sample space or just a part thereof. The regression will be performed over the four periods of 500 observations, as well as over the entire data set. Once $\hat{\alpha}$ and $\hat{\beta}$ are obtained, $\hat{\sigma}_x$ is calculated at each time point:

$$\hat{\sigma}_{x,i} = \hat{\alpha} + \hat{\beta}\sigma_{x,i}, \text{ for } i = 1, 2, \ldots, m.$$

The out-of-sample estimate, through regression of past estimates, includes information other than that of the most recent time period. The most recent time period is used to calculate $\sigma_{x,i}$ this can either be data spanning a day, 20 trading days or a year depending on the estimator. On the other hand, $\hat{\alpha}$ and $\hat{\beta}$ are calculated using 500 day samples as well as the entire data set. The out-of-sample estimator therefore minimizes the MSE by regressing the estimator to the RV from a time period greater than that used in calculating $\sigma_x$. The out-of-sample MSE is then calculated similarly to (3.1):

$$\mathrm{MSE}_x^{(out)} = \frac{1}{N}\sum_{t=1}^{N}(\sigma_{RV,t} - \hat{\sigma}_{x,t})^2. \tag{3.2}$$

The out-of-sample MSE can be extended to include a second factor. This approach investigates how well a combination of estimators perform. In order to calculate the out-of-sample MSE, a two factor regression needs to be performed:

$$\sigma_{RV,i} = \alpha + \beta_1 \sigma_{x,i} + \beta_2 \sigma_{y,i} + \epsilon_i, \text{ for } i = 1, 2, \ldots, m,$$

where $y$ is a second estimator from the same set as $x$. Once the estimates $\hat{\alpha}$, $\hat{\beta}_1$ and $\hat{\beta}_2$ are obtained, the new estimator $\hat{\sigma}_{x,y}$ is calculated at each time point $i$:

$$\hat{\sigma}_{x,y,i} = \hat{\alpha} + \hat{\beta}_1 \sigma_{x,i} + \hat{\beta}_2 \sigma_{y,i}, \text{ for } i = 1, 2, \ldots, m.$$

The two factor out-of-sample MSE is then calculated in the same way as (3.2).
The reason for using the out-of-sample estimators, as discussed above, is that any linear relationship between $\sigma_{RV}$ and $\sigma_x$ over a time period can be taken advantage of in order to create a more accurate estimate. This exploitation of a relationship should lead to a more accurate estimate when compared to the in-sample estimate. The out-of-sample approach also lends itself to forecasting future RV by using $\hat{\alpha}$, $\hat{\beta}$ and the previous period's $\sigma_x$. The forecasting however is not implemented in this investigation.

## 3.2 Monte Carlo study

### 3.2.1 Generating Sample Paths

Comparing RV estimators to a chosen method of calculating volatility, in our case the 30-day SSR, has practical applications and has been studied previously by amongst others Shu and Zhang Shu and Zhang (2003). However, this method compares estimators to an estimate of IV. As IV is not truly known, another approach to investigating the estimators is through the dynamic delta hedging in a Monte Carlo approach.
The data for the Monte Carlo Hedge simulation is synthetic. A total of 10 000 paths were simulated, each with 3280 time steps, which corresponds to 41 trading days with 80 intra-day observations. The stock prices were randomly simulated from a log-normal distribution. The parameters used were a risk free rate of 0.05, zero drift and an initial stock price of 100. The input volatility used in simulating the sample paths was also randomly generated at each time step, and is discussed later. The strike price of the Call and Put written on the second 20 days was the forward price of the individual stock

price after 20 days. RV estimates used in the S&P 500 study were calculated on a daily basis and used in the daily delta hedging of the Call and Put. The price path of the Call and Put are calculated using the stock price paths, and the volatility used as an input discussed below. The delta hedge, however, is calculated using the estimator. The two annual estimators (w) and (m) will not be investigated in this instance. The exclusion of the weekly and monthly observed estimators was decided upon ex post, due to their poor performance in the S&P 500 study.

#### Random Volatility

Three different approaches are used to randomly generate the input volatility at each time step: (1) absolute values of a random normal, (2) random log-normal (3) the absolute value of the sum of previous volatility and a random uniform change. For comparative reasons the distributions of the random numbers are chosen such that the first two moments of the SSR estimates are similar. The first approach (Vol 1)generates strictly random normal volatilities with mean 0.11 and standard deviation of 0.02. The strictly positive condition is vital, as volatilities are strictly positive. The log-normal approach addresses this in, and includes skewness in the second approach (Vol 2). The $\mu$ and $\sigma$ of the log-normal distribution from which random numbers are generated are 0.15 and 0.05 respectively. For comparative reasons, the log-normal numbers are scaled down by a factor of 10. The third approach (Vol 3) randomly generates the initial volatility from a Uniform distribution between 0.14 and 0.16, then adds a random value from the Uniform distribution between -0.1 and 0.1 at each time step, ensuring the positive condition. This approach attempts to reflect the behaviour of volatility changing only slightly from the previous time period.

### 3.2.2 Comparing Estimators

As there is no RV calculated in the Monte Carlo study, performance of estimates cannot be compared using MSE. Estimators are compared by analysing the distribution of the final profit and loss of the delta hedged options. An ideal estimator would have a mean of zero, a small as possible standard deviation, and no skew. The reason for this ideal distribution, is that a perfectly delta hedged option has a profit and loss of zero. Distributions of estimators significantly affect the profit and loss, therefore, the estimator distributions need to be analysed together with the distribution of the profit and loss.

# Chapter 4

# Results

## 4.1 S&P 500 Results

This section presents and analyses the results from the RV estimators calculated from the S&P 500 data. Before investigating the MSE of these estimators compared to the RV, insight may be attained from analysis of the distribution of the RV estimates.

| Distribution of RV Estimates | | | |
|---|---|---|---|
| | **Mean** | **Standard Deviation** | **Skewness** |
| $\sigma_{RV}$ | 18.26177 | 12.81167 | 2.61494 |
| $\sigma_{SSR}^{(d)}$ | 18.24853 | 12.83742 | 2.62405 |
| $\sigma_{SSR}^{(w)}$ | 16.28652 | 7.49237 | 1.26547 |
| $\sigma_{SSR}^{(m)}$ | 15.89424 | 8.20906 | 1.42812 |
| $\sigma_{Park}^{(d)}$ | 15.12099 | 9.99167 | 2.75612 |
| $\sigma_{Park}^{(w)}$ | 17.68890 | 8.44050 | 1.26388 |
| $\sigma_{Park}^{(m)}$ | 17.20176 | 8.34555 | 1.26756 |
| $\sigma_{GK}^{(d)}$ | 13.60474 | 8.72558 | 2.83615 |
| $\sigma_{GK}^{(w)}$ | 17.68890 | 8.44050 | 1.26388 |
| $\sigma_{GK}^{(m)}$ | 17.20176 | 8.34555 | 1.26756 |
| $\sigma_{YZ}^{(d)}$ | 20.04582 | 13.51464 | 2.73648 |
| $\sigma_{YZ}^{(w)}$ | 17.76445 | 8.14096 | 1.18836 |
| $\sigma_{YZ}^{(m)}$ | 16.24018 | 7.81271 | 1.24245 |
| $\sigma_{HF1}$ | 19.81243 | 12.48759 | 2.14397 |
| $\sigma_{HF2}$ | 20.30246 | 12.85341 | 2.20531 |
| $\sigma_{HF3}$ | 33.65262 | 21.20889 | 2.14429 |
| $\sigma_{HF4}$ | 19.65561 | 12.39014 | 2.14253 |
| $\sigma_{VIX}$ | 21.75196 | 10.51144 | 2.06711 |

**Table 4.1:** Distribution of RV Estimates

Table 4.1 summarizes the RV distribution over the time period in question, as well as the RV estimators. Comparing the summary statistics of the estimators to the RV gives significant insight into the performance of the estimators at a cursory glance. The first and most notable result, is that $\sigma^{(d)}_{SSR}$ seems to follow a very similar distribution to $\sigma_{RV}$. This is to be expected, as it is in a lagged SSR. However several other trends can be identified. The performance of all $\sigma^{(w)}$ and $\sigma^{(m)}$ is quite poor, as they consistently have a lower standard deviation and skewness. These estimators are affected by smoothing. The smoothing is due to the inclusion of data from the entire year. The lower skewness indicates that these smoothed estimates are not as strongly affected by observations of higher volatility. The $\sigma^{(d)}$ estimates, HF estimates and the VIX however do have skewness similar to $\sigma_{RV}$. The higher frequency of the HF estimators seems not to have effected the skewness or the standard deviation (with the exception of $\sigma_{HF3}$) of the estimate, as it is comparable to $\sigma^{(d)}_{SSR}$. The HF estimates do however have a significantly higher mean. Two comments can be made regarding the VIX, namely that it has the second highest mean, following $\sigma_{HF3}$, and of the estimates with skewness comparable to $\sigma_{RV}$, it has the lowest standard deviation. As the VIX is essentially the market sentiment or expectation of future volatility this indicates that the market over-estimates volatility, and does so consistently. The low standard deviation of the market expectation relative to $\sigma_{RV}$ is quite intuitive, as large jumps in the index, such in the time of crises are not anticipated. When plotting the RV estimates of SSR over the various terms and coarseness the averaging effect of including more information can be seen very clearly.

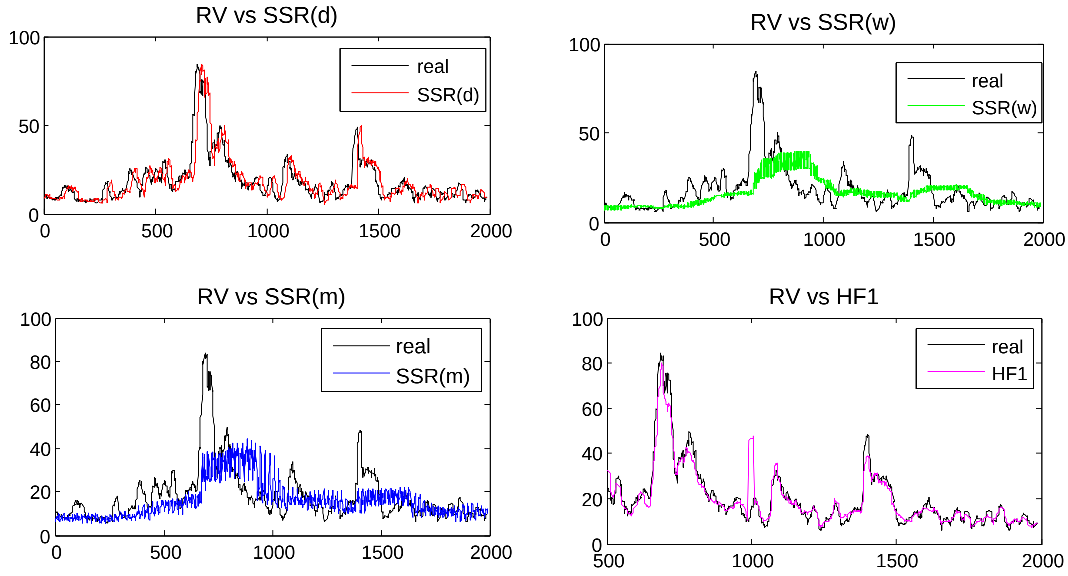


**Figure 4.1:** SSR and HF1 RV Estimates of various terms and coarseness

Figure 4.1 shows the estimates over the various terms. In addition to the smoothing effect of more information a cyclical nature is apparent for more coarse data. This stands to reason as data points will only be included once per cycle, for instance per month, and at the end of the cycle only one new point is included. From a visual inspection, the HF estimator appears to be the best estimator of the four different term and frequency variations.

### 4.1.1 Stylized Fact

One of the main goals of this dissertation was to investigate the stylized fact that longer term, and more coarse data result in more accurate RV estimates Cont (2001). This statements is stated in relative terms, and is as such open to interpretation. An observation that is made, is that estimators with a longer term of input data perform poorly. A further observation that is made is that HF data results in better estimators. Thus far, these observations have been made from visual inspection, and comparing estimator distributions. To quantify estimators performance, the table below shows the in-sample MSE.

| | **In-Sample MSE** | | | | |
|---|---|---|---|---|---|
| **data points** | **1-500** | **501-1000** | **1001-1500** | **1500-end** | **1-end** |
| $\sigma^{(d)}_{SSR}$ | 19.13729 | 149.92764 | 73.47724 | 18.01194 | 65.30405 |
| $\sigma^{(w)}_{SSR}$ | 32.33124 | 397.25052 | 91.61941 | 23.23240 | 136.50485 |
| $\sigma^{(m)}_{SSR}$ | 33.63126 | 353.30046 | 92.77066 | 18.42551 | 124.90465 |
| $\sigma^{(d)}_{Park}$ | 22.28697 | 178.18087 | 72.87483 | 13.42617 | 71.89686 |
| $\sigma^{(w)}_{Park}$ | 26.96430 | 369.58837 | 89.30682 | 28.42153 | 128.92201 |
| $\sigma^{(m)}_{Park}$ | 26.95716 | 335.55691 | 82.37634 | 21.93320 | 117.03877 |
| $\sigma^{(d)}_{GK}$ | 27.57892 | 225.00974 | 83.60954 | 14.70544 | 87.98238 |
| $\sigma^{(w)}_{GK}$ | 26.96430 | 369.58837 | 89.30682 | 28.42153 | 128.92201 |
| $\sigma^{(m)}_{GK}$ | 26.95716 | 335.55691 | 82.37634 | 21.93320 | 117.03877 |
| $\sigma^{(d)}_{YZ}$ | 22.11474 | 187.17703 | 82.32588 | 22.58250 | 78.74661 |
| $\sigma^{(w)}_{YZ}$ | 27.01327 | 388.17163 | 90.25902 | 31.99806 | 134.72002 |
| $\sigma^{(m)}_{YZ}$ | 33.47435 | 404.00289 | 94.03873 | 20.00962 | 138.29540 |
| $\sigma_{HF1}$ | - | 42.24773 | 28.88113 | 2.50340 | 23.93345 |
| $\sigma_{HF2}$ | - | 37.53576 | 28.77541 | 2.52103 | 22.32369 |
| $\sigma_{HF3}$ | - | 467.27790 | 275.24502 | 67.92798 | 270.71504 |
| $\sigma_{HF4}$ | - | 43.99072 | 28.93395 | 2.56465 | 24.55592 |
| $\sigma_{VIX}$ | 21.37075 | 169.08219 | 81.39784 | 31.27052 | 75.93666 |

**Table 4.2:** In-Sample MSE

Table 4.2 showing in-sample MSE values for the different estimators reiterates how poor the $\sigma^{(w)}$ and $\sigma^{(m)}$ estimators are. The MSE of these coarse estimators is materially higher than the $\sigma^{(d)}$. The poor performance is seen across all 500-observation time windows and the entire data set. The HF estimates, however, are with the exception of $\sigma_{HF3}$, the best estimators. From visual inspection, and analysis of the distribution and MSE of the estimators the validity of our interpretation of 'stylized financial fact' is put into question. At the very least, more clarity needs to be given with regard to term and coarseness. The stylized fact states that coarse grained data predicts finer grained estimators better than the other way round Cont (2001) in the findings of this dissertation the opposite was shown.

## 4.2 Best Estimator

Further investigation regarding which of the estimators performed best was undertaken by analysing the Out-of-Sample MSE with both one and two factors. The Out-of-Sample

MSE over the whole period as well as over individual 500-observation periods can be seen in the table below. From this point onwards, the performance of the $\sigma^{(w)}$ and $\sigma^{(m)}$ estimators will not be commented on further, due to their consistently poor performance.

| | **Out-of-Sample one factor MSE** | | | | |
|---|---|---|---|---|---|
| **data points** | **1-500** | **501-1000** | **1001-1500** | **1500-end** | **1-end** |
| $\sigma_{SSR}^{(d)}$ | 16.21729 | 134.06188 | 56.15535 | 7.88660 | 58.69972 |
| $\sigma_{SSR}^{(w)}$ | 17.99425 | 325.81807 | 81.22249 | 7.15638 | 129.89492 |
| $\sigma_{SSR}^{(m)}$ | 18.48163 | 307.57399 | 79.79557 | 7.39907 | 117.40404 |
| $\sigma_{Park}^{(d)}$ | 17.51661 | 149.60283 | 55.85412 | 7.66623 | 62.01977 |
| $\sigma_{Park}^{(w)}$ | 15.19675 | 322.08367 | 81.25478 | 7.20741 | 124.10827 |
| $\sigma_{Park}^{(m)}$ | 13.29018 | 309.98038 | 78.87113 | 7.20684 | 114.25943 |
| $\sigma_{GK}^{(d)}$ | 18.53850 | 160.08831 | 55.46966 | 7.29402 | 64.75258 |
| $\sigma_{GK}^{(w)}$ | 15.19675 | 322.08367 | 81.25478 | 7.20741 | 124.10827 |
| $\sigma_{GK}^{(m)}$ | 13.29018 | 309.98038 | 78.87113 | 7.20684 | 114.25943 |
| $\sigma_{YZ}^{(d)}$ | 17.80904 | 149.00829 | 57.09426 | 7.84050 | 63.44746 |
| $\sigma_{YZ}^{(w)}$ | 16.14601 | 326.05855 | 80.92036 | 7.20727 | 129.40108 |
| $\sigma_{YZ}^{(m)}$ | 16.48371 | 326.67266 | 81.29828 | 7.32055 | 130.22983 |
| $\sigma_{HF1}$ | - | 38.35880 | 27.03060 | 2.47448 | 23.41949 |
| $\sigma_{HF2}$ | - | 36.41623 | 26.34762 | 2.49572 | 22.21493 |
| $\sigma_{HF3}$ | - | 38.24757 | 27.02215 | 2.47671 | 23.38547 |
| $\sigma_{HF4}$ | - | 39.01097 | 27.25134 | 2.49836 | 23.79517 |
| $\sigma_{VIX}$ | 13.88464 | 163.29393 | 57.03174 | 6.88070 | 63.52277 |

**Table 4.3:** Out-of-Sample one factor MSE

Prior to analysing individual performance of the estimators, it is necessary to comment about the estimators over the different time periods . In periods of higher standard deviation of the log index levels, as well as higher standard deviation of the RV, the estimates perform worse than in periods of lower standard deviation. As estimates are by design lagged, they will not include information about the future and will thus perform poorly when there is a sudden fundamental change in the market. The poor performance of the estimators in the 2008-2009 as well as the 2009-2010 time period is observable in the in-sample and out-of-sample MSEs.

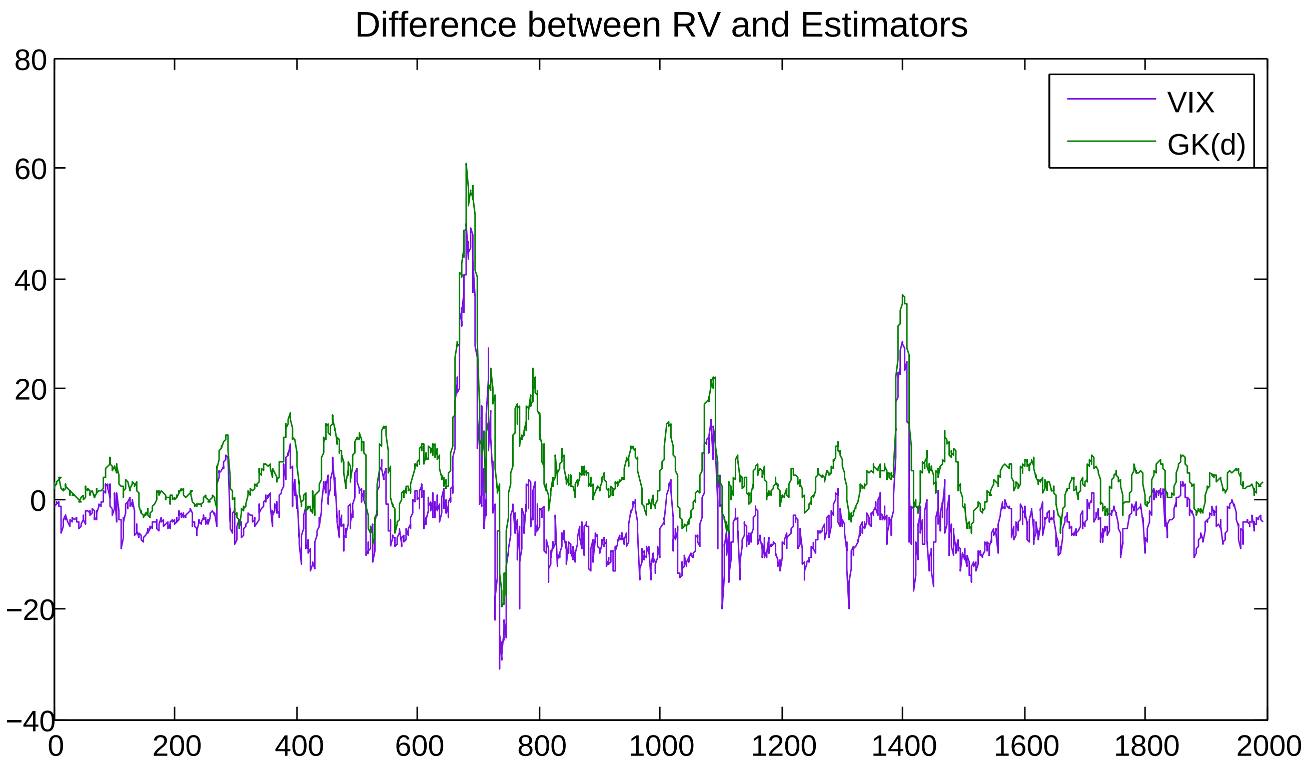


**Figure 4.2:** Exceedance over Estimates by RV

The exceedance ($Exceedance = RV - Estimate$) is shown in Figure 4.2. Showing Exceedances for all the estimators would be difficult to read in one graph. Therefore, Figure 4.2 only shows GK and VIX which consistently have the highest and most negative exceedances. This indicates that the VIX consistently over-estimates, while the daily GK consistently under-estimates RV. A clear spike can be seen between points 600 and 800. It is quite clear that there is a period over which the RV far exceeded both estimates. This spike corresponds to the start of the sub-prime mortgage crisis. As the VIX is an indicator of market expectation and 'fear', this spike shows that the market underestimated the volatility in 2008. Following the big spike around 700, a large overcorrection can be seen. This overcorrection in the VIX shows that the market expected volatility to decline a lot faster than it did in reality. The other $\sigma_{SSR}^{(d)}$ and HF estimates' exceedances mostly lie between those of the VIX and daily GK.

The out-of-sample estimates have performed significantly better than the in-sample estimates, most notably the $\sigma_{HF3}$ estimator. This indicates that the estimate gets scaled back down. Scaling up the $\sigma_{HF1}$ up to cover the whole 24-hours seems to be too high, which is then corrected when performing a regression. The out-of-sample estimates for all the HF estimates are by far the best performing estimators, followed by the $\sigma_{SSR}^{(d)}$ estimator.

The effect of including more than one estimator to estimate RV is analysed by calculating the two factor out-of-sample MSE. As the historic $\sigma^{(d)}$ all contain the same information including two such factors have little to no effect on the out-of-sample MSE. The various $\sigma^{(d)}$ and HF estimators are combined with the VIX, as the VIX reflects a forward looking estimate. The $\sigma^{(w)}$ and $\sigma^{(m)}$ estimates also include more information, namely a long term moving average. They are combined with the $\sigma^{(d)}$ estimators.

| | | **Out-of-Sample two factor MSE** | | | | |
|---|---|---|---|---|---|---|
| $\sigma_X$ | $\sigma_Y$ | **1-500** | **501-1000** | **1001-1500** | **1500-end** | **1-end** |
| $\sigma_{SSR}^{(d)}$ | $\sigma_{VIX}$ | 13.8692 | 134.0465 | 55.0043 | 6.752546 | 56.9994 |
| $\sigma_{Park}^{(d)}$ | $\sigma_{VIX}$ | 13.82162 | 148.3415 | 55.04863 | 6.842564 | 59.41734 |
| $\sigma_{GK}^{(d)}$ | $\sigma_{VIX}$ | 13.78512 | 154.956 | 54.60422 | 6.872332 | 60.52139 |
| $\sigma_{YZ}^{(d)}$ | $\sigma_{VIX}$ | 13.79486 | 147.5631 | 55.82067 | 6.786343 | 59.98602 |
| $\sigma_{HF1}$ | $\sigma_{VIX}$ | - | 36.45404 | 23.5119 | 2.472776 | 21.57446 |
| $\sigma_{HF2}$ | $\sigma_{VIX}$ | - | 34.08712 | 22.92021 | 2.49359 | 20.33982 |
| $\sigma_{HF3}$ | $\sigma_{VIX}$ | - | 36.33162 | 23.50238 | 2.475071 | 21.5356 |
| $\sigma_{HF4}$ | $\sigma_{VIX}$ | - | 37.12892 | 23.69982 | 2.496667 | 21.91774 |
| $\sigma_{SSR}^{(d)}$ | $\sigma_{SSR}^{(w)}$ | 14.85844 | 128.8406 | 55.62367 | 7.147565 | 58.6471 |
| $\sigma_{SSR}^{(d)}$ | $\sigma_{SSR}^{(m)}$ | 14.36813 | 133.9605 | 55.60209 | 7.398234 | 57.51773 |

**Table 4.4:** Out-of-Sample two factor MSE

Refer to Table 4.4 for the analysis of the two factor linear regression. Combining the $\sigma^{(d)}$ and the VIX decreases the MSE, showing that the VIX can improve an estimator and contains more information. Including the $\sigma^{(w)}$ and $\sigma^{(m)}$ estimators also improve the estimators resulting in a lower MSE. The inclusion of the longer term estimators improves the estimators a lot less than the inclusion of the VIX. From the analysis of in-sample, and both out-of-sample estimates it is clear that the HF estimates perform best, followed by the $\sigma_{SSR}^{(d)}$ estimator. The estimator that is most accurate is the $\sigma_{HF2}$ estimate which shows that including overnight volatility improves the estimator, optimal weighting of the daily component and overnight component in $\sigma_{HF4}$ could therefore also improve the estimator, but the choice of weighting needs to be investigated further.

### 4.2.1 Results of Similar Studies

The results found above should be compared to similar studies in order to identify similarities and differences in literature. Ahoniemi and Lanne (2013) in their study of intra-day volatility estimators found that the HF4 estimator performed best, but when compared between stocks the best estimator was not consistent. Their argument that HF estimators perform better than inter-day estimators is reflected in our results. They further state in their conclusion that when deciding on which estimator is best, an in-sample comparison needs to be undertaken to support the modellers decision. The argument for HF estimators is also supported in Bollen and Inder (2002) in their study of estimating daily volatility using intra-day data of S&P 500 Index Futures. The use of HF estimators was not however supported in Ghysels *et al.* (2006) in their study. They similarly estimated realised volatility using different frequency, and in their conclusion wrote: "that the direct use of high-frequency data does not necessarily lead to better volatility forecasts" (Ghysels *et al.*, 2006).

Andersen *et al.* (2001) argue against the use of implied volatility as a RV estimator, due to the distributional assumptions that need to be extrapolated from past data. Our results show that implied volatility is not an accurate estimator. It needs to be noted that the use of the VIX is slightly different to using a Black-Scholes-Merton model. The use of implied volatility as a RV estimator has been criticised in Andersen *et al.* (2001) amongst others, Christensen and Prabhala (1998) acknowledge this popularised view, but show in their study that implied volatility can be a superior estimator if overlapping is not allowed, and if drift is adjusted for. Christensen and Prabhala (1998) state that the non-overlapping sample results in more accurate regression results, their sample results in one implied volatility and one realised volatility per time interval. Adjusting for drift in Christensen and Prabhala (1998) is achieved similarly to the out-of-sample estimator discussed in this investigation of volatility estimation. They argue in their investigation of RV of the S&P 100 that implied volatility makes use of more information than historic estimators (Christensen and Prabhala, 1998). The findings of Christensen and Prabhala (1998) are supported by Shu and Zhang (2003) who compare implied volatility from a Black-Scholes-Merton model and a Heston model to inter-day historic estimators. The findings in Shu and Zhang (2003) and Christensen and Prabhala (1998) are contrary to the results in our study. The performance of estimators in the study in Shu and Zhang (2003) are compared by analysing the R-squared from the regressions they performed.

Such comparison is arguably not as good as the MSE approach.

## 4.3 Monte Carlo Results

The second approach to investigating volatility estimators is a Monte-Carlos simulation of delta hedging vanilla Calls and Puts. In this approach the estimators can be evaluated without the RV. The IV is estimated and a reliably accurate estimator will result in a profit and loss distribution centred around zero with as small as possible a standard deviation. A perfectly hedged option will have a profit and loss of zero. The distribution of the random volatility estimators using the three approaches are displayed below:

| | **Distribution of RV Estimators (Vol 1)** | | | | | |
|---|---|---|---|---|---|---|
| **Estimators** | **Mean** | **Stand Dev.** | **1% VAR** | **5% VAR** | **95% VAR** | **99% VAR** |
| $\sigma^{(d)}_{SSR}$ | 0.11051 | 0.01765 | 0.07204 | 0.08232 | 0.14042 | 0.15353 |
| $\sigma^{(d)}_{Park}$ | 0.10268 | 0.00803 | 0.08554 | 0.08999 | 0.11644 | 0.12254 |
| $\sigma^{(d)}_{GK}$ | 0.09885 | 0.00714 | 0.08249 | 0.08737 | 0.11079 | 0.11615 |
| $\sigma^{(d)}_{YZ}$ | 0.15012 | 0.01541 | 0.11793 | 0.12612 | 0.17692 | 0.18880 |
| $\sigma_{HF1}$ | 0.11108 | 0.00214 | 0.10613 | 0.10758 | 0.11462 | 0.11619 |
| $\sigma_{HF2}$ | 0.11178 | 0.00214 | 0.10683 | 0.10828 | 0.11532 | 0.11688 |
| $\sigma_{HF3}$ | 0.21075 | 0.00407 | 0.20137 | 0.20413 | 0.21748 | 0.22045 |
| $\sigma_{HF4}$ | 0.10598 | 0.00203 | 0.10128 | 0.10266 | 0.10934 | 0.11081 |

**Table 4.5:** Distributions of Vol 1 estimators

| | **Distribution of RV Estimators (Vol 2)** | | | | | |
|---|---|---|---|---|---|---|
| **Estimators** | **Mean** | **Stand Dev.** | **1% VAR** | **5% VAR** | **95% VAR** | **99% VAR** |
| $\sigma^{(d)}_{SSR}$ | 0.11516 | 0.01836 | 0.07536 | 0.08573 | 0.14626 | 0.15952 |
| $\sigma^{(d)}_{Park}$ | 0.10717 | 0.00833 | 0.08931 | 0.09401 | 0.12144 | 0.12777 |
| $\sigma^{(d)}_{GK}$ | 0.10325 | 0.00741 | 0.08623 | 0.09132 | 0.11564 | 0.12119 |
| $\sigma^{(d)}_{YZ}$ | 0.15658 | 0.01599 | 0.12319 | 0.13165 | 0.18437 | 0.19652 |
| $\sigma_{HF1}$ | 0.11573 | 0.00207 | 0.11098 | 0.11235 | 0.11915 | 0.12062 |
| $\sigma_{HF2}$ | 0.11646 | 0.00207 | 0.11169 | 0.11308 | 0.11987 | 0.12134 |
| $\sigma_{HF3}$ | 0.21957 | 0.00392 | 0.21056 | 0.21318 | 0.22607 | 0.22886 |
| $\sigma_{HF4}$ | 0.11041 | 0.00196 | 0.10589 | 0.10721 | 0.11365 | 0.11504 |

**Table 4.6:** Distributions of Vol 2 estimators

| | Distribution of RV Estimators (Vol 3) | | | | | |
|---|---|---|---|---|---|---|
| **Estimators** | **Mean** | **Stand Dev.** | **1% VAR** | **5% VAR** | **95% VAR** | **99% VAR** |
| $\sigma_{SSR}^{(d)}$ | 0.10269 | 0.01647 | 0.06695 | 0.07662 | 0.13061 | 0.14324 |
| $\sigma_{Park}^{(d)}$ | 0.09366 | 0.00761 | 0.07732 | 0.08167 | 0.10659 | 0.11267 |
| $\sigma_{GK}^{(d)}$ | 0.08930 | 0.00685 | 0.07349 | 0.07830 | 0.10064 | 0.10575 |
| $\sigma_{YZ}^{(d)}$ | 0.13798 | 0.01448 | 0.10761 | 0.11550 | 0.16294 | 0.17501 |
| $\sigma_{HF1}$ | 0.10338 | 0.00324 | 0.09613 | 0.09822 | 0.10866 | 0.11097 |
| $\sigma_{HF2}$ | 0.10403 | 0.00324 | 0.09679 | 0.09887 | 0.10930 | 0.11161 |
| $\sigma_{HF3}$ | 0.19614 | 0.00615 | 0.18240 | 0.18635 | 0.20616 | 0.21055 |
| $\sigma_{HF4}$ | 0.09863 | 0.00307 | 0.09177 | 0.09374 | 0.10362 | 0.10582 |

**Table 4.7:** Distributions of Vol 3 estimators

Refer to the Tables 4.5, 4.6 and 4.7. Some consistent trends can be seen in the three random volatility models. Firstly, the GK estimator consistently has the lowest mean, while YZ and HF3 estimators have the two highest means. This observation is consistent with results seen in the S&P 500 study. Furthermore, it is noted that the SSR has the highest standard deviation in all cases. The HF1, HF2 and HF4 have the lowest standard deviation in all three cases.

The summary statistics for the profit and loss of Call options with various random volatilities are displayed in separate tables below.

| | Profit and Loss Distribution of Delta Hedged Call (Vol 1) | | | | | |
|---|---|---|---|---|---|---|
| | **Mean** | **Stand Dev.** | **1% VAR** | **5% VAR** | **95% VAR** | **99% VAR** |
| $\sigma_{SSR}^{(d)}$ | -0.07125 | 0.90376 | -3.60122 | -1.90083 | 0.94572 | 1.19568 |
| $\sigma_{Park}^{(d)}$ | -0.14903 | 0.91061 | -3.77580 | -2.11878 | 0.76962 | 0.93916 |
| $\sigma_{GK}^{(d)}$ | -0.18690 | 0.92784 | -3.86298 | -2.22313 | 0.72950 | 0.90819 |
| $\sigma_{YZ}^{(d)}$ | 0.32431 | 0.84978 | -2.30760 | -1.06406 | 1.40015 | 1.65224 |
| $\sigma_{HF1}$ | -0.06287 | 0.87288 | -3.53387 | -1.87583 | 0.82753 | 0.97649 |
| $\sigma_{HF2}$ | -0.05582 | 0.87045 | -3.51274 | -1.85635 | 0.83534 | 0.98275 |
| $\sigma_{HF3}$ | 0.92788 | 1.11304 | -1.68120 | -1.19136 | 2.33030 | 2.66202 |
| $\sigma_{HF4}$ | -0.11434 | 0.89309 | -3.68230 | -2.02444 | 0.77541 | 0.93933 |

**Table 4.8:** Distribution of Profit and Loss of Call (Vol 1)

| | Profit and Loss Distribution of Delta Hedged Call (Vol 2) | | | | | |
|---|---|---|---|---|---|---|
| | **Mean** | **Stand Dev.** | **1% VAR** | **5% VAR** | **95% VAR** | **99% VAR** |
| $\sigma_{SSR}^{(d)}$ | -0.07535 | 0.93912 | -3.69770 | -1.92126 | 0.99658 | 1.26126 |
| $\sigma_{Park}^{(d)}$ | -0.15579 | 0.94459 | -3.92945 | -2.16509 | 0.80769 | 0.98965 |
| $\sigma_{GK}^{(d)}$ | -0.19502 | 0.96258 | -3.99976 | -2.27760 | 0.76908 | 0.95581 |
| $\sigma_{YZ}^{(d)}$ | 0.33855 | 0.90175 | -2.37781 | -1.12459 | 1.48230 | 1.75475 |
| $\sigma_{HF1}$ | -0.06797 | 0.90802 | -3.66467 | -1.89798 | 0.87002 | 1.03242 |
| $\sigma_{HF2}$ | -0.06062 | 0.90566 | -3.64299 | -1.87800 | 0.87869 | 1.03664 |
| $\sigma_{HF3}$ | 0.96569 | 1.22216 | -1.89761 | -1.34254 | 2.51711 | 2.88724 |
| $\sigma_{HF4}$ | -0.12169 | 0.92819 | -3.82977 | -2.05663 | 0.81755 | 0.98846 |

**Table 4.9:** Distribution of Profit and Loss of Call (Vol 2)

| | Profit and Loss Distribution of Delta Hedged Call (Vol 3) | | | | | |
|---|---|---|---|---|---|---|
| | **Mean** | **Stand Dev.** | **1% VAR** | **5% VAR** | **95% VAR** | **99% VAR** |
| $\sigma_{SSR}^{(d)}$ | -0.07326 | 0.86753 | -3.41603 | -1.79241 | 0.86056 | 1.10534 |
| $\sigma_{Park}^{(d)}$ | -0.16795 | 0.88721 | -3.70795 | -2.05451 | 0.67519 | 0.86458 |
| $\sigma_{GK}^{(d)}$ | -0.21312 | 0.90875 | -3.81798 | -2.17046 | 0.63523 | 0.80687 |
| $\sigma_{YZ}^{(d)}$ | 0.28619 | 0.79269 | -2.37644 | -0.98512 | 1.24606 | 1.47935 |
| $\sigma_{HF1}$ | -0.06922 | 0.83912 | -3.41117 | -1.78611 | 0.75142 | 0.88362 |
| $\sigma_{HF2}$ | -0.06254 | 0.83660 | -3.39332 | -1.76902 | 0.75779 | 0.88838 |
| $\sigma_{HF3}$ | 0.87313 | 0.94976 | -1.36828 | -0.89179 | 2.07599 | 2.36304 |
| $\sigma_{HF4}$ | -0.11802 | 0.85917 | -3.55335 | -1.92374 | 0.70451 | 0.84691 |

**Table 4.10:** Distribution of Profit and Loss of Call (Vol 3)

Refer to the Tables 4.8, 4.9 and 4.10. The estimators with a mean profit and loss closest to zero can be seen to be HF1, HF2 and SSR with HF1 being the best. This trend is consistent in all three cases. The estimators with means furthest from zero are HF3, YZ and GK in decreasing order. The estimator with the lowest standard deviation is consistently the YZ estimator followed by HF1 and HF2. The HF3 estimator has the highest standard deviation.

The summary statistics for the profit and loss of Put options with various random volatilities are displayed in separate tables below.

| | Profit and Loss Distribution of Delta Hedged Put (Vol 1) | | | | | |
|---|---|---|---|---|---|---|
| | **Mean** | **Stand Dev.** | **1% VAR** | **5% VAR** | **95% VAR** | **99% VAR** |
| $\sigma^{(d)}_{SSR}$ | -0.05245 | 2.36259 | -6.20374 | -4.43919 | 2.82597 | 3.49029 |
| $\sigma^{(d)}_{Park}$ | -0.13023 | 2.27938 | -6.09164 | -4.37503 | 2.55720 | 3.07184 |
| $\sigma^{(d)}_{GK}$ | -0.16810 | 2.24430 | -6.07980 | -4.34766 | 2.44861 | 2.95228 |
| $\sigma^{(d)}_{YZ}$ | 0.34311 | 2.77235 | -6.61260 | -4.71065 | 3.86043 | 4.55064 |
| $\sigma_{HF1}$ | -0.04407 | 2.36440 | -6.13387 | -4.42035 | 2.75724 | 3.25682 |
| $\sigma_{HF2}$ | -0.03702 | 2.37143 | -6.14080 | -4.42774 | 2.77558 | 3.27855 |
| $\sigma_{HF3}$ | 0.94668 | 3.40590 | -7.36994 | -5.22702 | 5.57205 | 6.47437 |
| $\sigma_{HF4}$ | -0.09554 | 2.31314 | -6.08285 | -4.38236 | 2.62099 | 3.10904 |

**Table 4.11:** Distribution of Profit and Loss of Put (Vol 1)

| | Profit and Loss Distribution of Delta Hedged Put (Vol 2) | | | | | |
|---|---|---|---|---|---|---|
| | **Mean** | **Stand Dev.** | **1% VAR** | **5% VAR** | **95% VAR** | **99% VAR** |
| $\sigma^{(d)}_{SSR}$ | -0.05644 | 2.49308 | -6.51920 | -4.64857 | 3.00907 | 3.72468 |
| $\sigma^{(d)}_{Park}$ | -0.13688 | 2.40334 | -6.35973 | -4.56640 | 2.71215 | 3.26953 |
| $\sigma^{(d)}_{GK}$ | -0.17611 | 2.36552 | -6.29331 | -4.52893 | 2.60920 | 3.14332 |
| $\sigma^{(d)}_{YZ}$ | 0.35746 | 2.94018 | -6.97361 | -4.99806 | 4.12221 | 4.92546 |
| $\sigma_{HF1}$ | -0.04906 | 2.49430 | -6.44061 | -4.63848 | 2.92430 | 3.45596 |
| $\sigma_{HF2}$ | -0.04172 | 2.50196 | -6.44601 | -4.64273 | 2.94554 | 3.48299 |
| $\sigma_{HF3}$ | 0.98460 | 3.62876 | -7.78117 | -5.52223 | 5.95266 | 6.92741 |
| $\sigma_{HF4}$ | -0.10278 | 2.43847 | -6.38570 | -4.60064 | 2.77846 | 3.29966 |

**Table 4.12:** Distribution of Profit and Loss of Put (Vol 2)

| | Profit and Loss Distribution of Delta Hedged Put (Vol 3) | | | | | |
|---|---|---|---|---|---|---|
| | **Mean** | **Stand Dev.** | **1% VAR** | **5% VAR** | **95% VAR** | **99% VAR** |
| $\sigma^{(d)}_{SSR}$ | -0.01183 | 2.11721 | -5.44472 | -3.94963 | 2.55583 | 3.18372 |
| $\sigma^{(d)}_{Park}$ | -0.10652 | 2.02540 | -5.32856 | -3.91053 | 2.26950 | 2.72425 |
| $\sigma^{(d)}_{GK}$ | -0.15169 | 1.98671 | -5.28702 | -3.89958 | 2.16384 | 2.57331 |
| $\sigma^{(d)}_{YZ}$ | 0.34762 | 2.45603 | -5.73093 | -4.17708 | 3.40776 | 4.09890 |
| $\sigma_{HF1}$ | -0.00779 | 2.11586 | -5.38477 | -3.97176 | 2.49640 | 2.93941 |
| $\sigma_{HF2}$ | -0.00111 | 2.12197 | -5.38894 | -3.97488 | 2.51197 | 2.95495 |
| $\sigma_{HF3}$ | 0.93456 | 3.01370 | -6.30696 | -4.56442 | 5.00421 | 5.87343 |
| $\sigma_{HF4}$ | -0.05659 | 2.07125 | -5.35876 | -3.94510 | 2.37566 | 2.80046 |

**Table 4.13:** Distribution of Profit and Loss of Put (Vol 3)

Refer to the Tables 4.11, 4.12 and 4.13. It is immediately apparent that the standard deviation of the profit and loss is a lot higher for the delta-hedged Puts, when compared to the Calls. When analysing the mean and standard deviation of the profit and loss for various estimators it is clear that the HF3 and YZ estimators have the highest mean and standard deviation across all cases. The HF1, HF2 and SSR estimators persistently have mean profit and loss closest to zero. The profit and loss with the lowest standard deviations are observed when the GK and Park estimators. The performance of estimators regarding dispersion around the mean are summarised by the standard deviation, as well as the percentiles in the tails. The reliability and accuracy of the HF1, HF2 and SSR estimators are clear. The poor performance of the HF3 and YZ estimators are also clear. These findings are consistent with those of the S&P 500 study.

# Chapter 5

# Conclusions

The three areas of investigation set out at the beginning of this investigation of RV estimators revolved around identifying the most reliably accurate estimator, analysing term and frequency of data observations, and comparing the findings from S&P 500 data to a Monte Carlo simulation of delta hedged options. Several historical, implied and high-frequency estimators were discussed and their estimating accuracy analysed. In the study of the S&P 500 data, the best estimators were consistently the SSR estimators with daily observations and the HF estimators, with the exception of the HF3 estimator. By analysing the estimating ability over several non-overlapping intervals, some insights are made about the estimating ability of the various estimators with regard to the standard deviation of the RV. It is clear that in times of higher standard deviation of the RV as well as the log-returns, the estimators perform poorly. The estimating ability of the VIX is not as good as the historical estimators, and it can be shown that the VIX overestimates the RV. The performance of the estimators are compared by examining the in-sample MSE,the one factor out-of-sample MSE and two factor out-of-sample MSE and the distributions of the estimators relative to the RV. Including the VIX as a second factor slightly improves the estimators, while including lower frequency data with daily or five minute data does not improve the estimators significantly.

Through the examinations of the above mentioned comparison approaches it was clear that lower frequency data, namely weekly and monthly data performed very poorly. The daily and five-minute data performed similarly, with the five-minute data performing slightly better. The poor performance of the low-frequency estimators is due to the cyclical and smoothing nature of these estimators. The stylised fact of Asymmetry in time scales discussed by Cont (2010) was questioned, as the results contradicted the stylised fact. It is suggested that the Asymmetry in time scales should be more specific in its mention of time and frequency.

The Monte Carlo simulation of delta hedging forward at the money calls and puts with randomly generated input volatilities showed similar results to the S&P 500 study. The Monte Carlo simulation showed that the HF and SSR estimators perform best. The good performance of the GK and Park estimators with regard to the low standard deviation of profit and loss distribution is noted.

Further research can be done by comparing the performance of the estimators in this dissertation to GARCH type estimators and jump diffusion estimators. A second extension of this investigation is the comparison of absolute returns as a measure of volatility to the squared returns used in this study.

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