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Compactness in asymmetrically normed lattices

by

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Contents

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Dedication

I dedicate this thesis to my late parents Mr Joseph Mabula and Mrs Sarah Radebe and to my late sister Hadio Mabula whom all passed on when I started my first year at the varsity, to my fiancee Ms Nonhlanhla Mbele for patience, support and understanding.

Declaration

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Abstract

If X is a real linear space, a function $p : X \to \mathbb{R}^+$ called an asymmetric norm on X if it is positively homogenous, sub-additive and $p(x) = p(-x) = 0$ if and only if $x = 0$. If $(X, \| \cdot \|)$ is a real normed lattice, an asymmetric norm on X is defined by $p(x) = \|x^+\|$, where $x \in X$ and where $x^+ = x \vee 0$. We call the pair (X, p) an asymmetrically normed lattice. The aim of the thesis is to investigate aspects of the theory of such spaces, concentrating mainly, but not exclusively, on finite dimensional spaces.

One of the main aims of this thesis is to investigate compactness in the setting of asymmetrically normed lattices. In order to do this, it was necessary to study convergence of sequences and left-K-sequential completeness and precompactness of subsets of such spaces. Characterizations of these properties are found in the case of finite dimensional spaces, and some results obtained for sequence spaces. In addition, the Smyth completeness of positive cones of some lattices is also investigated.

We call a subset of an asymmetrically normed space (X, p) strong p-compact if there is a set A_0, compact with respect to the norm p^s defined by $p^s(x) = \max\{p(x), p(-x)\}$, such that such that $A_0 \subseteq A \subseteq A_0 + \{x \in X : p(x) = 0\}$. It is known that strongly p-compact sets are p-compact, but that the converse does not hold in general. We show that if X is finite dimensional and A is p^s-closed, A is p-compact if and only if it is strongly p-compact, but that there are strongly

p-compact sets that are not p^s-closed.

Chapter 0

Introduction

If X is a real linear space, a function $p : X \to \mathbb{R}^+$ will be called an asymmetric norm on X if for all $x, y \in X$ and $\alpha \in \mathbb{R}^+$,

(a) $p(x) = p(-x) = 0$ if and only if $x = 0$,

(b) $p(\alpha x) = \alpha p(x)$,

(c) $p(x + y) \leq p(x) + p(y)$.

The pair (X, p) is called an asymmetrically normed space. The function p^{-1} : $X \to \mathbb{R}^+$ defined by $p^{-1}(x) = p(-x)$ is also an asymmetric norm, and the function $p^s : X \to \mathbb{R}^+$ given by $p^s(x) = \max\{p(x), p(-x)\}$ is a norm on X. An asymmetric norm p on X induces a quasi-metric d_p on X, defined by means of the formula $d_p(x, y) = p(y - x)$ for all $x, y \in X$.

The term asymmetric norm was proposed in 1968, in [16], to study the problem of minimizing the distance from a point of normed linear space to a closed convex subset. In this paper, only asymmetric norms inducing a T_1 topology were considered. Asymmetric norms were also used in the book [6], published in 1986, to study one-parameter semigroups of positive operators on Banach spaces. In this book asymmetric norms are called half norms and the most important example considered is an asymmetric norm on a Banach lattice defined in terms of the positive cone. There has recently been renewed interest in the study of vector spaces equipped with an asymmetric norm. The systematic study of the properties of asymmetrically normed spaces started with the papers ([2, 3, 4, 18, 23]), motivated mainly by the application the theory has in theoretical computer science, specifically complexity theory (see for example [22]). A comprehensive account of various aspects of asymmetric functional analysis can be found in [12].

One of the main differences between asymmetrically normed spaces and normed spaces comes from the fact that the asymmetric norm does not generate a vector topology, since scalar multiplication is not continuous everywhere (see [12]). Regardless of these differences, many results in classical normed spaces have their counterparts in the asymmetrically normed spaces (see [2, 3, 8, 9, 10, 24]).

If $(X, \|\cdot\|)$ is a real normed lattice, then $p(x) = \|x^+\|$ is an asymmetric norm on X ([18]). In this thesis, we study the resulting asymmetrically normed space (X, p), which we call an asymmetrically normed lattice. The aim of the thesis is to investigate aspects of the theory of such spaces, concentrating mainly, but not exclusively, on finite dimensional spaces. The asymmetric norm p induces a T_0-topology which is never T_1. Asymmetric norms on finite dimensional spaces which induce T_1 topology were studied in some detail in [24]. Many results on these spaces are similar to the ones in normed spaces. We show that in the case of asymmetrically normed lattices, the situation is often quite different. For example, the p^s-closed unit ball of the asymmetrically normed lattice ℓ_∞ is p-compact but ℓ_∞ is not finite dimensional.

The main aim of this thesis is to investigate compactness in the setting of asymmetrically normed lattices. In order to do this, it was necessary to study convergence of sequences and left-K-sequential completeness and precompactness of subsets of such spaces, since in the asymmetrically normed space a set is p-compact if and only if it is p-precompact and left-K-sequentially complete. Characterizations of these properties are found in case of finite dimensional spaces, and some results obtained for sequence spaces. In addition, the Smyth completeness of positive cones of some lattices is also investigated.

We call a subset of an asymmetrically normed space (X, p) strongly p-compact if there is a p^s-compact set A_0 such that such that $A_0 \subseteq A \subseteq A_0 + \{x \in X : p(x) = 0\}$. It is known that strongly p-compact sets are p-compact, but that the converse does not hold in general ([2]). A somewhat complicated condition for a p-compact set to be strongly p-compact was given in [2]. We show that if X is finite dimensional a simpler sufficient condition can be given.

We give an outline of the contents of each chapter.

In the first chapter, we recall some known definitions and results which we shall use later in the thesis. We start with sections on quasi-metric spaces and asymmetrically normed spaces. The next section gives an introduction to Riesz spaces (vector lattices). We end this chapter by introducing asymmetrically normed lattices.

In chapter two, we study the convergence of sequences of an asymmetrically normed lattice, since we need convergence to study completeness. We characterize the set of limit points of a convergent sequence in a finite dimensional asymmetrically normed lattice. We investigate the convergent sequences of the asymmetrically normed lattices ℓ_p, where $1 \leq p \leq \infty$, and c_0.

In chapter three, we show that finite dimensional asymmetrically normed lattices are left-K-sequentially complete and characterize the bounded above left-

K-sequentially complete subsets of such spaces. We show that the asymmetrically normed lattices ℓ_p, where $1 \leq p \leq \infty$, c_0 and $C(\Omega)$ are left-K-sequentially complete. The technique we use to study left-K-sequential completeness allows us to study the Smyth completeness of the positive cones of above spaces and we show that the positives cones of $\mathbb{R}^m$, ℓ_p, where $1 \leq p < \infty$, and c_0 are Smyth complete but the positive cones of ℓ_∞ and $C(\Omega)$ are not Smyth complete. Symth completeness was studied in [22] and is used in complexity theory.

In chapter four, we study p-precompactness and p-compactness in asymmetrically normed lattices. We characterize p-precompact subsets of a finite dimensional normed lattice. In asymmetrically normed spaces a distinction has to be made between precompactness and outside precompactness. (In the first the centres of covering balls have to be in the set, in the second this is not required). In a finite dimensional asymmetrically normed lattice, we show that the two notions are equivalent. This is not true for general asymmetrically normed lattices. We show that a subset of a finite dimensional asymmetrically normed lattice is p-precompact if and only if it is p-bounded. We show that if a subset A of a finite dimensional asymmetrically normed lattice $(\mathbb{R}^m, p)$ is p^s-closed, then it is strongly p-compact if and only if it is p-compact. However, there are strongly p-compact sets which are not p^s-closed.

Chapter 1

Preliminaries

In this chapter we recall some definitions and results from the literature that we shall use in the later chapters of the thesis. We use the following [5, 7, 21, 28, 15, 38] and [25, 39, 40] as the references for functional analysis and topology respectively.

We denote by $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{N}$ the set of real numbers, of positive real numbers and of positive integers respectively.

1.1 Quasi-metric spaces

In this section we give some definitions and results in quasi-pseudo metric spaces. Further information on this topic can be found in [19, 29, 30].

Definition 1.1.1. *([12, 32, 37]) A* **quasi-pseudo-metric** *on a set X is a function $\rho : X \times X \to \mathbb{R}^+$ satisfying the following conditions for $x, y, z \in X$:*

(a) $\rho(x, x) = 0$

(b) $\rho(x,z) \le \rho(x,y) + \rho(y,z)$.

If ρ satisfies the following additional condition;

(c) $\rho(x,y) = \rho(y,x) = 0 \Leftrightarrow x = y$,

then ρ is called a **quasi-metric**.

The function $\rho^{-1} : X \times X \to \mathbb{R}^+$ defined by $\rho^{-1}(x,y) = \rho(y,x)$ is also a quasi-(pseudo)metric and called the conjugate of ρ. The function $\rho^s(x,y) = max\{\rho(x,y), \rho^{-1}(y,x)\}$ is a (pseudo)metric on X.
A quasi-metric ρ on X induces a quasi-uniformity $\mathcal{U}_\rho$ on X with basic entourages of the form

$$\{(x,y) \in X \times X : \rho(x,y) < \epsilon\},\ \epsilon > 0.$$

The pair (X,ρ) is called a quasi-(pseudo)metric space. A quasi-metric space (X,ρ) on X is called **bicomplete** if (X,ρ^s) is a complete metric space. Each quasi-metric ρ on X generates a topology τ_ρ, for which the basic open sets are the balls

$$B^\rho_\epsilon(x) = \{y \in X : \rho(x,y) < \epsilon\},\ \epsilon > 0,\ x \in X.$$

Proposition 1.1.2. *([29, p. 242]). Let (X,ρ) be a quasi-pseudo-metric space.*

(a) The topology τ_ρ is T_0 if and only

$$\rho(x,y) = \rho(y,x) = 0 \textit{ implies that } x = y.$$

Hence if ρ is quasi-metric, τ_ρ is a T_0 topology.

(b) The topology τ_ρ is a T_1 if and only if

$$\rho(x,y) = 0 \textit{ implies that } x = y.$$

1.2 Asymmetrically normed spaces

In this section we give some definitions and results in an asymmetrically normed space. We use the following for general reference [2, 12, 24].

Definition 1.2.1. *An* **asymmetric norm** *on a real linear space X is a function $p : X \to \mathbb{R}^+$ satisfying the following conditions, for all $x, y \in X$ and $\alpha \in \mathbb{R}^+$:*

(a) $p(x) = p(-x) = 0 \Leftrightarrow x = 0.$

(b) $p(\alpha x) = \alpha p(x).$

(c) $p(x + y) \leq p(x) + p(y).$

We call the pair (X, p) an asymmetrically normed space. The function $p^{-1} : X \to \mathbb{R}^+$ defined by $p^{-1}(x) = p(-x)$ is also an asymmetric norm. The function $p^s : X \to \mathbb{R}^+$ given by the formula $p^s(x) = \max\{p(x), p(-x)\}$ is a norm on X. An asymmetric norm p induces a quasi-metric d_p by means of the formula

$$d_p(x, y) = p(y - x), \text{ for all } x, y \in X.$$

The sets

$$B^p_\epsilon(0) := \{x \in X : p(x) < \epsilon\}, \ \epsilon > 0,$$

form a fundamental system of neighbourhoods of zero for the topology generated by d_p. In the same way, the translations of these sets,

$$B^p_\epsilon(y) := y + B^p_\epsilon(0),$$

define a fundamental system of neighbourhoods of y for all $y \in X$. We denote by $B^p_\epsilon[0]$, the sets

$$B^p_\epsilon[0] := \{x \in X : p(x) \leq \epsilon\}, \ \epsilon > 0.$$

It is clear from properties (a) and (b) of Definition 1.2.1 that the asymmetric norm is convex function, that is for any $x, y \in X$ we have

$$p(\lambda x + (1-\lambda)y) \leq \lambda p(x) + (1-\lambda)p(y),$$

for all $0 \leq \lambda \leq 1$. This implies that $B_\epsilon^p(0)$ is a convex set. The asymmetrically normed space (X, p) is called a **bi-Banach space** if the associated normed space (X, p^s) is complete ([10]).

Since we will be dealing with the topologies induced by p, p^{-1} and p^s, for the sake of clarity we will refer to a p-closed (respectively p^{-1}-closed, p^s-closed) set to mean a set closed in the topology induced by p (respectively p^{-1}, p^s). A similar convention will be used for other topological properties.

When X has the topology τ_p generated by an asymmetric norm p, addition

$$+ : X \times X \to X : (x, y) \mapsto x + y$$

is a jointly continuous function. But scalar multiplication need not be continuous.

Example 1.2.2. *Let $X = C[0,1]$, the vector space of all continuous real-valued functions on $[0,1]$, and put $p(f) = \sup_{x \in [0,1]} \max\{f(x), 0\}$, for $f \in C([0,1])$. Then p is asymmetric norm on $C[0,1]$. Let*

$$f_n(x) = \begin{cases} nx - 1, & 0 \leq x \leq \frac{1}{n} \\ 0, & \frac{1}{n} < x \leq 1. \end{cases}$$

and $f(x) = 0$ for all $x \in X$. Then

$$p(f_n - f) = 0 \text{ for all } n \in \mathbb{N},$$

therefore $f_n \xrightarrow{p} f$, but

$$p(-f_n - (-f)) = p(-f_n) = 1 \text{ for all } n \in \mathbb{N},$$

so $(-1)f_n \overset{p}{\nrightarrow} (-1)f$. Therefore scalar multiplication is not continuous at $(-1, f)$.

We recall that a normed space is a Hausdorff space. However an asymmetrically normed space need not be a Hausdorff space. Below we recall a characterization of an asymmetrically normed Hausdorff spaces.

Definition 1.2.3. *([23, Definition 2]) Let (X, p) be an asymmetrically normed space. Define the function $\|.\|_p : X \to \mathbb{R}^+$ by*

$$\|x\|_p = \inf_{y \in X} \{p(y) + p(y - x)\}, \; x \in X.$$

Lemma 1.2.4. *([23, Lemma 3]) The function $\|.\|_p$ is a semi-norm. Moreover, it is the supremum of all semi-norms q that satisfy*

$$q(x) \leq p(x), \; x \in X.$$

Theorem 1.2.5. *([23, Theorem 4]) Let (X, p) be an asymmetrically normed space. The following statements are equivalent;*

(a) $\|.\|_p$ is a norm on X.

(b) (X, p) is a Hausdorff space.

(c) (X, p^{-1}) is a Hausdorff space.

Proposition 1.2.6. *([12, Proposition 1.1.6(1)]) Let (X, p) be an asymmetrically normed space, $x \in X$ and $\epsilon > 0$, then*

(a) The ball $B^p_\epsilon(x)$ is p-open and the ball $B^p_\epsilon[x]$ is p^{-1}-closed.

(b) The ball $B^p_\epsilon[x]$ need not be p-closed.

Example 1.2.7. *Let $X = \mathbb{R}$, $q(x) = x^+$ for $x \in \mathbb{R}$. Consider the q-ball $B^q_{\frac{1}{2}}[0]$. Take the sequence $(\frac{1}{2^n})_{n=1}^\infty$ in $B^q_{\frac{1}{2}}[0]$. Then*

$$q\left(\frac{1}{2^n} - 1\right) = \left(\frac{1}{2^n} - 1\right)^+ = 0 \text{ for all } n.$$

Therefore, $\frac{1}{2^n} \xrightarrow{q} 1$ but $1 \notin B^q_{\frac{1}{2}}[0]$. Therefore $B^q_{\frac{1}{2}}[0]$ is not q-closed.

Definition 1.2.8. *([2, Definition 2]) A subset A of an asymmetrically normed space (X, p) is p-***bounded** *if there is a positive constant M such that $p(x) \leq M$ for all $x \in A$.*

Remark 1.2.9. *([2, p. 530]) If a subset A of an asymmetrically normed space (X, p) is p-bounded and p^{-1}-bounded, then it is p^s-bounded.*

Definition 1.2.10. *([24, Definition 1]) Let (X, p) be an asymmetrically normed space and $x \in X$. Define the set θ_x^p by*

$$\theta_x^p = \{y \in X : d_p(x, y) = p(y - x) = 0\}.$$

If $x = 0$, then

$$\theta_0^p = \{y \in X : d_p(0, y) = p(y) = 0\}.$$

The set θ_x^p is the p^{-1}-closure of $\{x\}$ ([24, p. 847]). Hence it is p^{-1}-closed.

Remark 1.2.11. *It will be shown later that the set θ_x^p need not be p-closed.*

Lemma 1.2.12. *([12, Proposition 1.18 (1)]) Let (X, p) be an asymmetrically normed space. Then for any $x \in X$ and every $\epsilon > 0$,*

$$B_\epsilon^{p^s}(x) \subseteq B_\epsilon^p(x)$$

and

$$B_\epsilon^{p^s}[x] + \theta_0^p \subseteq B_\epsilon^p[x].$$

Definition 1.2.13. *([24, Definition 16], [2]) An asymmetrically normed space (X, p) is called* **right-bounded** *if there exists a real constant $r > 0$, such that*

$$rB_1^p(0) \subseteq B_1^{p^s}(0) + \theta_0^p$$

or equivalently

$$rB_1^p[0] \subseteq B_1^{p^s}[0] + \theta_0^p.$$

Definition 1.2.14. *Two asymmetric norms p_1 and p_2 on a linear space X are said to be* **equivalent** *if there exists two constant $K > 0$ and $M > 0$ such that*

$$Kp_1(x) \leq p_2(x) \leq Mp_1(x) \text{ for all } x \in X. \tag{1.1}$$

Remark 1.2.15. *As is the case with norms, two asymmetric norms p_1 and p_2 on X generate the same topology if and only if the inequality* (1.1) *holds.*

1.3 Riesz spaces

In this section, we recall the following definitions and results in Riesz spaces. General references on this topic are [20, 41, 42].

Definition 1.3.1. *A subset A of a partially ordered set X is called* **bounded above** *(respectively* **bounded below***), if there is $y \in X$ such that for all $x \in A$, $x \leq y$ (respectively $y \leq x$).*

Definition 1.3.2. *Let X be a partially ordered set.*

(a) X is called a **lattice** *if every for x, $y \in X$, the supremum $x \vee y$ and the infimum $x \wedge y$ exist.*

(b) A lattice X is **Dedekind complete** *if every non-empty subset of X that is bounded above has a supremum in X.*

Note that if a lattice X is Dedekind complete, every non-empty subset of X which is bounded below has an infimum in X.

Definition 1.3.3. *Let E be a real vector space. If E has a partial ordering such that*

(a) $x \leq y \Leftrightarrow x + z \leq y + z$ for every $x, y, z \in E$ and

(b) $x \geq 0 \Rightarrow \alpha x \geq 0$ *for every* $\alpha \in \mathbb{R}^+$*,*

then E *is called a* **partially ordered vector space***.*

Definition 1.3.4. *If* E *is a partially ordered vector space and a lattice, then* E *is called a* **Riesz space** *(***vector lattice***).*

Definition 1.3.5. *Let* A *be a subset of a partially ordered set* X *and* $x \in X$*, then we write*

(a) $\uparrow A = \{y \in X : y \geq x \text{ for some } x \in A\}$*;*

(b) $\downarrow A = \{y \in X : y \leq x \text{ for some } x \in A\}$*;*

(c) $\uparrow x = \{y \in X : y \geq x\}$*.*

(d) $\downarrow x = \{y \in X : y \leq x\}$*.*

Definition 1.3.6. *A subset* A *of a Riesz space* E *is called* **increasing** *(respectively* **decreasing***) if and only if* $A = \uparrow A$ *(respectively* $A = \downarrow A$*).*

Definition 1.3.7. *Let* E *be a Riesz space. The subset*

$$E^+ = \{x \in E : x \geq 0\}$$

is called the **positive cone** *of* E *and the elements of* E^+ *are called the* **positive elements** *of* E*.*

The positive cone E^+ has the following properties (cone properties):

(a) $x \in E^+$, $y \in E^+$ implies that $x + y \in E^+$.

(b) $x \in E^+$ implies that $\alpha x \in E^+$ for any $\alpha \in \mathbb{R}^+$.

(c) $x \in E^+$, $-x \in E^+$ implies that $x = 0$.

Definition 1.3.8. *Let E be a Riesz space, then for $x, y \in E$ with $x \leq y$ we define the* **order interval** $[x, y]$ *by*

$$[x, y] = \{z \in E : x \leq z \leq y\}.$$

Let E be a Riesz space, then for all $x \in E$ we shall write

$$x^+ = x \vee 0, \ x^- = (-x) \vee 0 \text{ and } |x| = x \vee (-x).$$

Hence x^+, x^{-1} and $|x| \in E^+$.
The elements x^+ and x^- are called the **positive** and **negative parts** of x and the element $|x|$ is called the **absolute value** of x.

Theorem 1.3.9. *Let E be a Riesz space.*

(a) $(-x)^+ = x^-$, $(-x)^- = x^+$ *and* $|-x| = |x|$.

(b) $x = x^+ - x^-$, $x^+ \wedge x^- = 0$ *and* $|x| = x^+ + x^-$.

(c) $0 \leq x^+ \leq |x|$ *and* $0 \leq x^- \leq |x|$.

(d) $x \leq y$ *if and only if* $x^+ \leq y^+$ *and* $x^- \geq y^-$.

Definition 1.3.10. *Let E be a Riesz space. A subset A of E is called*

(a) a **Riesz subspace** *if A is a linear subspace of E and for all $x, y \in A$ we have $x \vee y \in A$ and $x \wedge y \in A$;*

(b) **order bounded** *if there is $x \in E$, $x \geq 0$, such that $|y| \leq x$ for all $y \in A$ (equivalently, A is contained in some order interval);*

(c) **solid** *if $x \in A$ implies $[-|x|, |x|] \subseteq A$;*

(d) an **ideal** *if A is a solid linear subspace.*

Definition 1.3.11. *A linear topology τ on a Riesz space E is* **locally solid***, (and (E, τ) is called a* **locally solid Riesz space***) if τ has a base at zero consisting of solid neighbourhoods.*

We recall that a norm $\|\cdot\|$ on a Riesz space E is called a **lattice norm** whenever

$$|x| \leq |y| \text{ implies } \|x\| \leq \|y\| \text{ for } x, y \in E.$$

This implies that for any $x \in E$,

$$\|x\| = \||x|\|.$$

A **normed Riesz space** is a Riesz space equipped with a lattice norm. The topology of a normed Riesz space is locally solid. If a normed Riesz space is norm complete, then it is called a **Banach lattice**. The mappings

$$(x, y) \mapsto x \vee y,\ (x, y) \mapsto x \wedge y,\ x \mapsto x^+,\ x \mapsto x^- \text{ and } x \mapsto |x|$$

are called **lattice operations** (or **lattice functions**).

Theorem 1.3.12. *([5, Theorem 8.41]) If a linear topology τ on a Riesz space E is locally solid, then the lattice operations are uniformly continuous.*

We recall some special types of Banach lattices which are commonly found in mathematical analysis.

Definition 1.3.13. *Let $(E, \|\cdot\|)$ be a normed Riesz space. Then $(E, \|\cdot\|)$ is called,*

(a) an **L***-normed space if $\|\cdot\|$ satisfies $\|x + y\| = \|x\| + \|y\|$ for all $x, y \in E^+$; an L-normed Banach lattice is called an* **AL***-space;*

(b) an **M***-normed space if $\|\cdot\|$ satisfies $\|x \vee y\| = \|x\| \vee \|y\|$ for all $x, y \in E^+$; an M-normed Banach lattice is called an* **AM***-space.*

Definition 1.3.14. *([18, Definition 1.1]) A normed lattice $(X, \|\cdot\|)$ is said to be an* $\mathbf{E}$*-space, if for x, $y \in X^+$, the following holds*

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

The space $\mathbb{R}^{\mathbb{N}}$ of all real sequences is a Riesz space under pointwise ordering. We mention some subspaces of $\mathbb{R}^{\mathbb{N}}$ which we shall be investigating in this thesis.

(a) $\ell_0 = \{\mathbf{x} \in \mathbb{R}^{\mathbb{N}} : x_i = 0 \text{ except for at most finitely many } i\}$.

(b) $c_0 = \{\mathbf{x} \in \mathbb{R}^{\mathbb{N}} : \lim_{n\to\infty} x_n = 0\}$.

(c) $c = \{\mathbf{x} \in \mathbb{R}^{\mathbb{N}} : \lim_{n\to\infty} x_n \text{ exists}\}$.

(d) $\ell_p = \{\mathbf{x} \in \mathbb{R}^{\mathbb{N}} : \|\mathbf{x}\|_p < \infty\}$, where $\|\mathbf{x}\|_p = (\sum_{n=1}^{\infty} |x_n|^p)^{\frac{1}{p}}$, $0 < p < \infty$.

(e) $\ell_\infty = \{\mathbf{x} \in \mathbb{R}^{\mathbb{N}} : \|\mathbf{x}\|_\infty < \infty\}$, where $\|\mathbf{x}\|_\infty = \sup_n |x_n|$.

These are all Riesz subspaces of $\mathbb{R}^{\mathbb{N}}$, and we have the following inclusions:

$$\ell_0 \subseteq \ell_p \subseteq c_0 \subseteq c \subseteq \ell_\infty \subseteq \mathbb{R}^{\mathbb{N}}.$$

1.4 Asymmetrically normed lattices

In this section we introduce asymmetrically normed lattices and recall some of their properties. Some of these results we have published in [13].
Let $(X, \|.\|)$ be a normed lattice. We consider the asymmetric norm on X defined by

$$p(x) = \|x^+\|, \; x \in X,$$

where $x^+ = 0 \vee x$ (see for example, [4, 18]). We shall call (X, p) an **asymmetrically normed lattice**. In the special case $X = \mathbb{R}$, we write q for the corresponding asymmetric norm, that is

$$q(x) = x^+, \text{ for } x \in \mathbb{R}.$$

Proposition 1.4.1. *([13, Proposition 2.1]) Let* X *be a normed lattice, with asymmetric norm given by* $p(x) = ||x^+||$*. The topology* τ_p *is* T_0 *but is not* T_1*.*

Proof. Since the topology τ_p is induced by a quasi-metric, then by Proposition 1.1.2, it is T_0. We next show that the topology τ_p is not T_1. Since

$$p(x) = ||x^+|| = 0 \Leftrightarrow x^+ = 0 \Leftrightarrow x \leq 0$$

and the topology τ_p is T_1 if and only if $p(x) > 0$ for all $x \in X \setminus \{0\}$ ([24, Lemma 12]), τ_p is not T_1. □

Recall that in a normed lattice $(X, \|.\|)$, for any $x \in X$ $\|x\| = \||x|\|$. However this is not true in general for an asymmetrically normed lattice (X, p). To see this, let $X = \mathbb{R}$, $q(x) = x^+$ for $x \in \mathbb{R}$, then

$$q(-1) = (-1)^+ = 0 \neq 1 = q(|-1|).$$

The following functions were introduced in [18]. Let (X, p) be an asymmetrically normed lattice. For $x \in X$, put

(a)
$$P_L(x) = p(x) + p(-x);$$

(b)
$$P_M(x) = p(x) \vee p(-x);$$

(c)
$$P_E(x) = (p(x)^2 + p(-x)^2)^{\frac{1}{2}}.$$

Proposition 1.4.2. *([18, Theorem 3.1]) Let $(X, \|.\|)$ be a normed lattice, then P_L, P_M and P_E are equivalent to $\|.\|$. Moreover, P_L (respectively P_M, P_E) coincides with $\|.\|$ if X is an L space (respectively an M-space, an E-space).*

Proposition 1.4.3. *([2, Lemma 1]) An asymmetrically normed lattice (X, p) is right-bounded with constant $r = 1$. Hence $B_1^p(0) = B_1^{p^s}(0) + \theta_0^p$.*

Proposition 1.4.4. *([4, Lemma 1]) The subset A of (X, p) is p (respectively p^{-1})-open if and only if it is p^s-open and decreasing (respectively increasing).*

Corollary 1.4.5. *The subset A of (X, p) is p-closed if and only if it is p^s-closed and increasing.*

Proof. This follows from Lemma 1.4.4, since A is p-closed if and only if $X \setminus A$ is p-open, and by ([19, p. 74]) a set A is increasing if and only $X \setminus A$ is decreasing. □

Corollary 1.4.6. *The subset A of (X, p) is p^{-1}-closed if and only if it is p^s-closed and decreasing.*

Proposition 1.4.7. *The positive cone X^+ is p-closed but not p^{-1}-closed.*

Proof. It follows from Corollary 1.4.5 that X^+ is p-closed, since X^+ is p^s-closed and increasing, and X^+ is not p^{-1}-closed, since it is not decreasing. □

In an asymmetrically normed lattice (X, p), for $x \in X$ and $A \subseteq X$:

$$\theta_0^p = \{y \in X : p(y) = 0\} = \{y \in X : y \leq 0\} = -X^+;$$

$$\theta_x^p = \{y \in X : p(y - x) = 0\} = \{y \in X : y \leq x\} = \downarrow x;$$

$$A + \theta_0^p = \{y \in X : y \leq x \text{ for some } x \in A\} = \downarrow A.$$

It follows from Corollary 1.4.5 that the set θ_x^p is not p-closed, since it is not increasing but it is p^{-1}-closed (see [2, p. 529]).

Chapter 2

Convergence of sequences in asymmetrically normed lattices

In this chapter, we study the convergence of sequences in asymmetrically normed lattices. We characterize the set of limits points of a convergent sequence of a finite dimensional asymmetrically normed lattice, and characterize the convergent sequences of ℓ_p, $1 \leq p < \infty$, ℓ_∞ and c_0. Some of the results in this chapter have appeared in [13].

2.1 Some general results on convergence of sequences in asymmetrically normed lattices

In this section we mention some known results about the convergence of sequences and prove some convergence results in an asymmetrically normed lattice. We show that the set of p-limits of a sequence in an asymmetrically normed lattice (X, p) is increasing and that any sequence which is bounded above is p-convergent.

Definition 2.1.1. *([37, Definition 1]) A sequence $(x_n)_{n=1}^{\infty}$ in an asymmetrically normed space (X, p) is said to be p (respectively p^{-1})-**convergent** to $x \in X$ if for every $\epsilon > 0$, there is an $n_\epsilon \in \mathbb{N}$ such that $p(x_n - x) < \epsilon$ (respectively $p^{-1}(x_n - x) < \epsilon$) for all $n \geq n_\epsilon$.*

Proposition 2.1.2. *([12, Proposition 1.1.8(iii)]) Let $(x_n)_{n=1}^{\infty}$ be a sequence in an asymmetrically normed space (X, p), then $x_n \xrightarrow{p^s} x$ if and only if $x_n \xrightarrow{p} x$ and $x_n \xrightarrow{p^{-1}} x$.*

We recall that a sequence $(x_n)_{n=1}^{\infty}$ in an asymmetrically normed space (X, p) is p-bounded if there is an $M \in \mathbb{R}^+$ such that $p(x_n) \leq M$ for all n.

Lemma 2.1.3. *([13, Proposition 2.4]) If a sequence $(x_n)_{n=1}^{\infty}$ in an asymmetrically normed space (X, p) is p-convergent, then it is p-bounded.*

Proof. Suppose $x_n \xrightarrow{p} x$ in X. Take $\epsilon = 1$, then there is $N \in \mathbb{N}^+$ such that for $n \geq N$,

$$p(x_n - x) < 1.$$

Therefore by the triangle inequality, we obtain that for all $n \geq N$

$$\begin{aligned} p(x_n) &\leq p(x_n - x) + p(x) \\ &< 1 + p(x). \end{aligned}$$

Let

$$M = \max\{p(x_1), p(x_2), ..., p(x_{N-1}), 1 + p(x)\}.$$

Then

$$p(x_n) \leq M \text{ for all } n \in \mathbb{N}.$$

Thus, $(x_n)_{n=1}^{\infty}$ is p-bounded. □

Since the topology of an asymmetrically normed lattice (X, p) is not T_1 (see Proposition 1.4.1), the limit of convergent sequence in (X, p) need not be unique.

Let $(x_n)_{n=1}^{\infty}$ be a sequence in X. We write

$$L(x_n) = \{x \in X : x_n \xrightarrow{p} x\}.$$

We recall that a subset A of a Riesz space E is called increasing if $x \in A$, $y \in E$, $x \leq y$ implies that $y \in A$.

Proposition 2.1.4. *([13, Proposition 2.2]) If $(x_n)_{n=1}^{\infty}$ is a p-convergent sequence in an asymmetrically normed lattice (X, p), then $L(x_n)$ is increasing.*

Proof. Let $x \in L(x_n)$. For every $\epsilon > 0$, there is an n_ϵ such that for $n \geq n_\epsilon$, $p(x_n - x) < \epsilon$. Let $y \in X$ be such that $y \geq x$. Then

$$p(x - y) = \|(x - y)^+\| = 0,$$

and

$$p(x_n - y) \leq p(x_n - x) + p(x - y) < \epsilon + 0 = \epsilon \text{ for } n \geq n_\epsilon.$$

Therefore $x_n \xrightarrow{p} y$, and so $y \in L(x_n)$. □

Proposition 2.1.5. *([13, Proposition 2.3]) Let $(x_n)_{n=1}^{\infty}$ be any sequence in an asymmetrically normed lattice (X, p). If $(x_n)_{n=1}^{\infty}$ is bounded above in X, then $x_n \xrightarrow{p} c$ for every upper bound c of $(x_n)_{n=1}^{\infty}$ in X. If $(x_n)_{n=1}^{\infty}$ is bounded below in X, then $x_n \xrightarrow{p^{-1}} d$ for every lower bound d of $(x_n)_{n=1}^{\infty}$ in X.*

Proof. Let $(x_n)_{n=1}^{\infty}$ be a bounded above sequence in (X, p) and $c \in X$ be any upper bound of $(x_n)_{n=1}^{\infty}$, then $x_n \leq c$ for all n. Hence $x_n - c \leq 0$ for all n. Therefore $p(x_n - c) = 0$. Thus $x_n \xrightarrow{p} c$. The proof of the second part is similar. □

Proposition 2.1.6. *Let X be a normed lattice such that every p-bounded sequence in X is bounded above. Let $(x_n)_{n=1}^{\infty}$ be a sequence in X. Then*

(a) $(x_n)_{n=1}^{\infty}$ is p-convergent if and only if it is p-bounded.

(b) $(x_n)_{n=1}^{\infty}$ is p^s-bounded if and only if it is both p- and p^{-1}-convergent.

Proof. (a) Suppose that a sequence $(x_n)_{n=1}^{\infty}$ in X is p-convergent, then it is p-bounded by Lemma 2.1.3.
Conversely, suppose $(x_n)_{n=1}^{\infty}$ is p-bounded, then by assumption, it is bounded above (say by c). Therefore by Proposition 2.1.5, $x_n \xrightarrow{p} c$.
(b) Suppose $(x_n)_{n=1}^{\infty}$ is p^s-bounded, then it is both p and p^{-1}-bounded. Hence by assumption, it is bounded above and below. Therefore by (a), it is both p and p^{-1}-convergent.
Conversely, suppose $(x_n)_{n=1}^{\infty}$ is both p and p^{-1}-convergent in X, then by Proposition 2.1.3, it is both p and p^{-1}-bounded. Hence by Remark 1.2.9, it is p^s-bounded. $\square$

This result is not necessarily true in the normed lattice (X, p^s).

Example 2.1.7. *In $(\mathbb{R}, q^s)$, let (x_n) be the sequence $(1, 2, 1, 2, ...)$. Then $(x_n)_{n=1}^{\infty}$ is q^s-bounded (equivalently bounded) but it does not q^s-converge, since $x_{2n+1} \xrightarrow{q^s} 1$ and $x_{2n} \xrightarrow{q^s} 2$.*

Proposition 2.1.8. *Let $(x_n)_{n=1}^{\infty}$ be a sequence in an asymmetrically normed lattice (X, p) and x, $y \in X$. If $x_n \xrightarrow{p} x$ and $x_n \xrightarrow{p^{-1}} y$, then $y \leq x$.*

Proof. Let $(x_n)_{n=1}^{\infty}$ be a sequence in X such that $x_n \xrightarrow{p} x$ and $x_n \xrightarrow{p^{-1}} y$. Then for every $\epsilon > 0$, there are n_1, $n_2 \in \mathbb{N}$ such

$$p(x_n - x) < \frac{\epsilon}{2} \text{ for } n \geq n_1$$

and

$$p^{-1}(x_n - y) = p(y - x_n) < \frac{\epsilon}{2} \text{ for } n \geq n_2.$$

Therefore, for $n = \max\{n_1, n_2\}$,

$$\begin{aligned} p(y - x) &\leq p(y - x_n) + p(x_n - x) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary,

$$p(y - x) = 0, \text{ so } \|(y - x)^+\| = 0.$$

Therefore $(y - x)^+ = 0$. So $y \leq x$. $\square$

2.2 Convergence of sequences in a finite dimensional space

In this section, we consider convergence of sequences in $\mathbb{R}^m$. Convergence was considered for T_1 finite dimensional asymmetrically normed spaces in [24]. However the topology of an asymmetrically normed lattice is not T_1 (See Proposition 1.4.1).

The following notation will be used:

$$\mathbf{x} = (x_1, x_2, ..., x_n)$$

for an element $\mathbf{x}$ in $\mathbb{R}^m$;

$$\mathbf{x}_n = (x_{n1}, ..., x_{nm})$$

for a term in a sequence $(\mathbf{x}_n)_{n=1}^{\infty}$ in $\mathbb{R}^m$. We use the asymmetric norm derived from the maximum norm on $\mathbb{R}^m$, that is,

$$p(\mathbf{x}) = p((x_1, x_2, ..., x_m)) = \max\{x_1^+, x_2^+, ..., x_m^+\}.$$

There is no loss of generality in doing this as the following result shows.

Proposition 2.2.1. *([13, Proposition 2.5]) Let X be a finite dimensional vector lattice and $\|.\|_1$ and $\|.\|_2$ lattice norms on X. Let $p(x) = \|x^+\|_1$ and $q(x) = \|x^+\|_2$ for $x \in X$. Then p and q are equivalent asymmetric norms.*

Proof. Since X is a finite dimensional space, $\|.\|_1$ and $\|.\|_2$ are equivalent norms (see [28, Theorem 2.4.5]). Therefore there exist positive real numbers m and M such that

$$m\|x\|_1 \leq \|x\|_2 \leq M\|x\|_1 \text{ for all } x \in X.$$

Hence

$$m\|x^+\|_1 \leq \|x^+\|_2 \leq M\|x^+\|_1 \text{ for all } x \in X,$$

from which it follows that p and q are equivalent. $\square$

Proposition 2.2.2. *([13, Proposition 2.7]) Let $(\mathbf{x}_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{R}^m$. Then the following are equivalent:*

(a) $(\mathbf{x}_n)_{n=1}^{\infty}$ *is p-bounded.*

(b) $(\mathbf{x}_n)_{n=1}^{\infty}$ *is bounded above in $\mathbb{R}^m$.*

(c) The coordinate sequences $(x_{ni})_{n=1}^{\infty}$, where $1 \leq i \leq m$ are bounded above in $\mathbb{R}$.

Proof. $(a) \Leftrightarrow (c)$: Suppose $(\mathbf{x}_n)_{n=1}^{\infty}$ is p-bounded, then there is $M \in \mathbb{R}^+$ such that

$$p((x_{n1}, x_{n2}, ..., x_{nm})) = \max\{x_{n1}^+, x_{n2}^+, ..., x_{nm}^+\} \leq M \text{ for all } n.$$

This implies that

$$x_{n1}^+ \leq M,\ x_{n2}^+ \leq M, ..., x_{nm}^+ \leq M \text{ for all } n,$$

which implies that

$$x_{n1} \leq M,\ x_{n2} \leq M, ..., x_{nm} \leq M \text{ for all } n.$$

Therefore $(x_{n1})_{n=1}^{\infty}, (x_{n2})_{n=1}^{\infty}, ..., (x_{nm})_{n=1}^{\infty}$ are bounded above.
Conversely, suppose there are positive constants M_1, M_2,...,M_m such that $x_{ni} \leq$

M_i for all n and $1 \leq i \leq m$. Let $M = \max\{M_1, M_2, ..., M_m\}$, then $M > 0$ and it follows that for all n and $1 \leq i \leq m$

$$x_{ni}^{+} = x_{ni} \vee 0 \leq M_i \vee 0 = M_i, \text{ since } M_i > 0.$$

Hence

$$p(\mathbf{x}_n) = \max\{x_{n_1}^{+}, x_{n_2}^{+}, ..., x_{nm}^{+}\} \leq M \text{ for all } n,$$

and therefore $(\mathbf{x}_n)_{n=1}^{\infty}$ is p-bounded.
$(b) \Leftrightarrow (c)$: Suppose $(\mathbf{x}_n)_{n=1}^{\infty}$ is bounded above, then there is $\mathbf{M} \in \mathbb{R}^m$ such that $\mathbf{x}_n \leq \mathbf{M}$ for all $n \in \mathbb{N}$. Let $\mathbf{M} = (M_1, M_2, ..., M_m)$. Then

$$x_{n1} \leq M_1,\ x_{n2} \leq M_2, ...,x_{nm} \leq M_m \text{ for all } n \in \mathbb{N}.$$

Therefore $(x_{n1})_{n=1}^{\infty}$, $(x_{n2})_{n=1}^{\infty}, ...,$ and $(x_{nm})_{n=1}^{\infty}$ are bounded above in $\mathbb{R}$.
Conversely, suppose that $(x_{n1})_{n=1}^{\infty}, (x_{n2})_{n=1}^{\infty}, ...,$ and $(x_{nm})_{n=1}^{\infty}$ are bounded above in $\mathbb{R}$, then there are M_1, M_2,...,$M_m \in \mathbb{R}$ such that

$$x_{n1} \leq M_1,\ x_{n2} \leq M_2, ,..., x_{nm} \leq M_m \text{ for all } n \in \mathbb{N}.$$

Let $\mathbf{M} = (M_1, M_2, ..., M_m)$, then $\mathbf{x}_n \leq \mathbf{M}$ for all $n \in \mathbb{N}$. Therefore $(\mathbf{x}_n)_{n=1}^{\infty}$ is bounded above in $\mathbb{R}^m$. □

Proposition 2.2.3. *([13, Proposition 2.9 and Corollary 2.10]) Let $(\mathbf{x}_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{R}^m$. Then*

(a) $(\mathbf{x}_n)_{n=1}^{\infty}$ is p-convergent if and only if it is p-bounded.

(b) $(\mathbf{x}_n)_{n=1}^{\infty}$ is p^s-bounded if and only if it is both p- and p^{-1}-convergent.

Proof. The statement follows from Proposition 2.1.6, since by Proposition 2.2.2, a sequence $(\mathbf{x}_n)_{n=1}^{\infty}$ in $\mathbb{R}^m$ is p-bounded if and only if it is bounded above. □

Proposition 2.2.4. *([13, Proposition 2.6]) Let $(\mathbf{x}_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{R}^m$ and $\mathbf{x} \in \mathbb{R}^m$. Then*

$$\mathbf{x}_n \xrightarrow{p} \mathbf{x} \textit{ if and only if } x_{ni} \xrightarrow{q} x_i \textit{ for } i = 1, 2, ...m.$$

Proof. Suppose $x_n \xrightarrow{p} x$. Then for every $\epsilon > 0$, there is $n_\epsilon \in \mathbb{N}$ such that

$$p(\mathbf{x}_n - \mathbf{x}) = \max\{(x_{n1} - x_1)^+, (x_{n2} - x_2)^+, ..., (x_{nm} - x_m)^+\} < \epsilon,$$

for all $n \geq n_\epsilon$. But this implies that for $i = 1, 2, ...m$

$$(x_{ni} - x_i)^+ < \epsilon \text{ for all } n \geq n_\epsilon.$$

Therefore for $i = 1, 2, ...m$

$$x_{ni} \xrightarrow{q} x_i \text{ as } n \to \infty.$$

Suppose that

$$x_{ni} \xrightarrow{q} x_i \text{ for } i = 1, 2, ..., m.$$

Then if $\epsilon > 0$, for $i = 1, 2, ..., m$ there is an $n_\epsilon^i \in \mathbb{N}$ such that for $n \geq n_\epsilon^i$

$$(x_{ni} - x_i)^+ < \epsilon.$$

Let $n_\epsilon = \max_{1 \leq i \leq m}\{n_\epsilon^i\}$, then

$$p(\mathbf{x}_n - \mathbf{x}) = \max\{(x_{n1} - x_1)^+, (x_{n2} - x_2)^+, \ ..., (x_{nm} - x_m)^+\} < \epsilon \text{ for all } n \geq n_\epsilon.$$

That is, $\mathbf{x}_n \xrightarrow{p} \mathbf{x}$. $\square$

Proposition 2.2.5. *([13, Proposition 2.10]) Let $(x_n)_{n=1}^\infty$ be a sequence in $(\mathbb{R}, q)$. If $(x_n)_{n=1}^\infty$ is bounded above, then*

(a) $L(x_n) = \{x \in \mathbb{R} : x \geq \limsup_n x_n\}$ if $\limsup_n x_n$ exists.

(b) $L(x_n) = \mathbb{R}$ if $\limsup_n x_n$ does not exist.

Proof. (a) Suppose $\limsup_n x_n$ exists and

$$x_n \xrightarrow{q} x.$$

We show that $x \geq \limsup_n x_n$. Suppose not, then

$$x < \limsup_n x_n = \inf_n \sup_{k \geq n} x_k.$$

Therefore there is an $\epsilon > 0$ such that

$$x + \epsilon \leq \inf_n \sup_{k \geq n} x_k.$$

So

$$\sup_{k \geq n} x_k \geq x + \epsilon \text{ for all } n \in \mathbb{N}.$$

Therefore for all $n \in \mathbb{N}$, $x + \frac{1}{2}\epsilon$ is not an upper bound of the sequence $(x_k)_{k=n}^{\infty}$. For every n, there is $k_n \in \mathbb{N}$ such that

$$x_{k_n} > x + \frac{1}{2}\epsilon,$$

and without loss of generality we may assume that the sequence (k_n) is strictly increasing. This gives a subsequence (x_{k_n}) of $(x_k)_{k=1}^{\infty}$ such that

$$x_{k_n} - x > \frac{1}{2}\epsilon.$$

Therefore

$$q(x_{k_n} - x) = (x_{k_n} - x)^+ = x_{k_n} - x > \frac{1}{2}\epsilon \text{ for all } n.$$

So $x_{k_n} \stackrel{q}{\nrightarrow} x$, which is a contradiction since $x_n \xrightarrow{q} x$.
Conversely, suppose $(x_n)_{n=1}^{\infty}$ is bounded above and $x \geq \limsup_n x_n$. Let $\epsilon > 0$. Then

$$x + \epsilon > \limsup_n x_n = \inf_n \sup_{k \geq n} x_k.$$

Hence $x + \epsilon$ is not a lower bound for the sequence $(\sup_{k \geq n} x_k)$, and therefore there is an n_0 such that

$$\sup_{k \geq n_0} x_k < x + \epsilon.$$

Then

$$x_k < x + \epsilon \text{ for all } k \geq n_0.$$

For $k \geq n_0$, we have

$$q(x_k - x) = (x_k - x)^+ = (x_k - x) \vee 0 < \epsilon.$$

Hence

$$x_k \xrightarrow{q} x \text{ as } k \to \infty.$$

(b) Suppose $\limsup_n (x_n) = \inf_n \sup_{k\geq n} x_k$ does not exist. Then for any $a \in \mathbb{R}$, a is not a lower bound for $(\sup_{k\geq n} x_k)_{n=1}^{\infty}$. Hence there is an n_0 such that $\sup_{k\geq n_0} x_k < a$. So

$$x_k < a \text{ and } q(x_k - a) = 0 \text{ for all } k \geq n_0.$$

Thus $x_n \xrightarrow{q} a$. $\square$

Proposition 2.2.6. *([13, Proposition 2.11]) Let $(\mathbf{x}_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{R}^m$ which is bounded above, and $L(\mathbf{x}_n)$ be the set of p-limits of $(\mathbf{x}_n)_{n=1}^{\infty}$. Then*

$$L(\mathbf{x}_n) = \{\mathbf{x} \in \mathbb{R}^m : x_i \geq \limsup_n x_{ni} \textit{ for all } i, 1 \leq i \leq m, \textit{ for which } \limsup_n x_{ni} \textit{ exists } \}.$$

Proof. The statement follows from Propositions 2.2.5 and 2.2.4, since

$$\begin{aligned} \mathbf{x} \in L(\mathbf{x}_n) &\Leftrightarrow \mathbf{x}_n \xrightarrow{p} \mathbf{x} \\ &\Leftrightarrow x_{ni} \xrightarrow{q} x_i \text{ for } i = 1, 2, ..., m \\ &\Leftrightarrow x_i \geq \limsup_n x_{ni} \text{ when } \limsup_n x_{ni} \text{ exists }. \end{aligned}$$

$\square$

Corollary 2.2.7. *([13, Corollary 2.12]) A similar result holds for the set of p^{-1}-limits of a sequence which is bounded below, with the $\limsup_n x_{ni}$ replaced by $\liminf_n x_{ni}$ and $\geq$ replaced by $\leq$.*

2.3 Convergence of sequences in sequence spaces

In this section, we consider the convergence of sequences with respect to the asymmetric norm in some classical Banach lattices. We characterize the convergent sequences of some sequence spaces.

Recall that we define ℓ_∞ to be the set of all real bounded sequences $(x_n)_{n=1}^{\infty}$ and c_0 to be the set of all real sequences $(x_n)_{n=1}^{\infty}$ that converge to zero. The norm on ℓ_∞ is defined by $\|\mathbf{x}\|_\infty = \sup_n |x_n|$ and $(\ell_\infty, \|.\|_\infty)$ is a Banach lattice. We consider the supremum norm on ℓ_∞ and define the asymmetric norm p on ℓ_∞ by

$$p(\mathbf{x}) = \sup_n x_n^+, \ \mathbf{x} = (x_1, x_2, ...) \in \ell_\infty.$$

c_0 has the asymmetric norm induced by p.

Proposition 2.3.1. *Let $(\mathbf{x}_n)_{n=1}^{\infty}$ be a sequence in ℓ_∞ and $\mathbf{x} \in \ell_\infty$. If*

$$\mathbf{x}_n \xrightarrow{p} \mathbf{x}, \ \textit{then } x_{ni} \xrightarrow{q} x_i \textit{ for } i = 1, 2, 3, ...$$

Proof. The proof follows as in Proposition 2.2.4. □

Proposition 2.3.2. *Let $(\mathbf{x}_n)_{n=1}^{\infty}$ be a sequence in c_0, and $\mathbf{x} \in c_0$. Then*

$$\mathbf{x}_n \xrightarrow{p} \mathbf{x}$$

if and only if $x_{nk} \xrightarrow{q} x_k$ for all $k \in \mathbb{N}$ and $\sup_{n\in\mathbb{N}} \sup_{k\geq m} x_{nk}^+ \to 0$ as $m \to \infty$.

Proof. Let $(\mathbf{x}_n)_{n=1}^{\infty}$ be a sequence in c_0 and $\mathbf{x} \in c_0$. Suppose $\mathbf{x}_n \xrightarrow{p} \mathbf{x} \in c_0$. Then by Proposition 2.3.1, $x_{ni} \xrightarrow{q} x_i$ for all $i \in \mathbb{N}$. Let $\epsilon > 0$. There is $n_\epsilon \in \mathbb{N}$ such that

$$p(\mathbf{x}_n - \mathbf{x}) = \sup_{i\in\mathbb{N}} (x_{ni} - x_i)^+ < \frac{\epsilon}{2} \text{ for } n \geq n_\epsilon.$$

We show that there is $j_\epsilon \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ and for every $i \geq j_\epsilon$, $x_{ni}^+ < \epsilon$. Since $(x_i) \in c_0$, there is $i_\epsilon \in \mathbb{N}$ such that $|x_i| < \frac{\epsilon}{2}$ for $i \geq i_\epsilon$. Hence $x_i^+ \leq |x_i| < \frac{\epsilon}{2}$

for $i \geq i_\epsilon$. Then if $i \geq i_\epsilon$ and $n \geq n_\epsilon$,

$$\begin{aligned} x_{ni}^+ &\leq (x_{ni} - x_i)^+ + x_i^+ \\ &\leq \sup_{i\in\mathbb{N}}(x_{ni} - x_i)^+ + x_i^+ \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

If $1 \leq n < n_\epsilon$, then there is $i_\epsilon^n \in \mathbb{N}$ such that $|x_{ni}| < \frac{\epsilon}{2}$ for $i \geq i_\epsilon^n$, since $(x_{ni}) \in c_0$. Let $i'_\epsilon = \max_{1\leq n<n_\epsilon} i_\epsilon^n$ and $j_\epsilon = \max\{i'_\epsilon, i_\epsilon\}$. If $i \geq j_\epsilon$, then $x_{ni}^+ \leq |x_{ni}| < \epsilon$ for all $n \in \mathbb{N}$.

Conversely, suppose $x_{ni} \xrightarrow{q} x_i$ for all $i \in \mathbb{N}$ and $\sup_{n\in\mathbb{N}} \sup_{i\geq m} x_{ni}^+ \to 0$ as $m \to \infty$. Let $\epsilon > 0$. Then there is $i_1^\epsilon \in \mathbb{N}$ such that $\sup_{n\in\mathbb{N}} \sup_{i\geq i_1^\epsilon} x_{ni}^+ < \frac{\epsilon}{3}$, and there is $i_2^\epsilon \in \mathbb{N}$ such that $|x_i| < \frac{\epsilon}{3}$ for $i \geq i_2^\epsilon$ (since $\mathbf{x} \in c_0$). Let $i_\epsilon = \max\{i_1^\epsilon, i_2^\epsilon\}$. Then for every n we have

$$\begin{aligned} p(\mathbf{x}_n - \mathbf{x}) &= \sup_{i\in\mathbb{N}}(x_{ni} - x_i)^+ \\ &\leq \sup_{1\leq i<i_\epsilon}(x_{ni} - x_i)^+ + \sup_{i\geq i_\epsilon}(x_{ni} - x_i)^+ \\ &\leq \sup_{1\leq i<i_\epsilon}(x_{ni} - x_i)^+ + \sup_{i\geq i_\epsilon} x_{ni}^+ + \sup_{i\geq i_\epsilon} x_i^-, \text{ since } (-x_i)^+ = x_i^- \\ &\leq \sup_{1\leq i<i_\epsilon}(x_{ni} - x_i)^+ + \sup_{i\geq i_\epsilon} x_{ni}^+ + \sup_{i\geq i_\epsilon} |x_i|, \text{ since } x_i^- \leq |x_i| \\ &< \sup_{1\leq i<i_\epsilon}(x_{ni} - x_i)^+ + \frac{\epsilon}{3} + \frac{\epsilon}{3}. \end{aligned}$$

Since for each $i \in \mathbb{N}$, $(x_{ni} - x_i)^+ \to 0$ as $n \to \infty$, we can choose an $n_\epsilon^i \in \mathbb{N}$ such that

$$(x_{ni} - x_i)^+ < \frac{\epsilon}{3} \text{ for } n \geq n_\epsilon^i.$$

Let $l_\epsilon = \max_{1\leq i<i_\epsilon} n_\epsilon^i$. Then

$$\sup_{1\leq i<i_\epsilon}(x_{ni} - x_i)^+ < \frac{\epsilon}{3} \text{ for } n \geq l_\epsilon.$$

Hence

$$p(\mathbf{x}_n - \mathbf{x}) < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \text{ for } n \geq l_\epsilon,$$

and hence $\mathbf{x}_n \xrightarrow{p} \mathbf{x}$ as $n \to \infty$. $\square$

Recall that we define ℓ_p, $1 \leq p < \infty$, to be the set of all sequences $(x_n)_{n=1}^{\infty}$ of real numbers such that $\sum_{i=1}^{\infty} |x_i|^p < \infty$ with the norm defined by

$$\|\mathbf{x}\|_p = \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{\frac{1}{p}}.$$

Then $(\ell_p, \|.\|_p)$ is a Banach lattice.

Proposition 2.3.3. *Let $(\mathbf{x}_n)_{n=1}^{\infty}$ be a sequence in ℓ_p and $\mathbf{x} \in \ell_p$,$1 \leq p < \infty$. Then*

$$\mathbf{x}_n \xrightarrow{p} \mathbf{x}$$

if and only if $x_{ni} \xrightarrow{q} x_i$ for all $i \in \mathbb{N}$ and

$$\sup_{n \in \mathbb{N}} \left(\sum_{i=m}^{\infty} (x_{ni}^{+})^p\right)^{\frac{1}{p}} \to 0 \text{ as } m \to \infty.$$

Proof. Let $(\mathbf{x}_n)_{n=1}^{\infty}$ and $\mathbf{x}$ be a sequence and a point respectively, in ℓ_p, $1 \leq p < \infty$. Suppose $\mathbf{x}_n \xrightarrow{p} \mathbf{x}$. Let $\epsilon > 0$, then there is $n_\epsilon \in \mathbb{N}$ such that

$$p(\mathbf{x}_n - \mathbf{x}) = \left(\sum_{i \in \mathbb{N}} ((x_{ni} - x_i)^{+})^p\right)^{\frac{1}{p}} < \frac{\epsilon}{2} \text{ for all } n \geq n_\epsilon.$$

Therefore, for each $i \in \mathbb{N}$, $(x_{ni} - x_i)^{+} < \frac{\epsilon}{2}$ for all $n \geq n_\epsilon$. Thus $x_{ni} \xrightarrow{q} x_i$ as $n \to \infty$ for all $i \in \mathbb{N}$. We show that for every $\epsilon > 0$, there is $i_\epsilon \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, for every $i \geq i_\epsilon$,

$$\left(\sum_{i \geq i_\epsilon} (x_{ni}^{+})^p\right)^{\frac{1}{p}} < \epsilon.$$

Since $(x_i) \in \ell_p$, there is $i_\epsilon \in \mathbb{N}$ such that $\left(\sum_{i \geq i_\epsilon} |x_i|^p\right)^{\frac{1}{p}} < \frac{\epsilon}{2}$, hence $\left(\sum_{i \geq i_\epsilon} (x_i^{+})^p\right)^{\frac{1}{p}} < \frac{\epsilon}{2}$. If $n \geq n_\epsilon$

$$\begin{aligned}
\left(\sum_{i \geq i_\epsilon} (x_{ni}^{+})^p\right)^{\frac{1}{p}} &\leq \left(\sum_{i \geq i_\epsilon} ((x_{ni} - x_i)^{+})^p\right)^{\frac{1}{p}} + \left(\sum_{i \geq i_\epsilon} (x_i^{+})^p\right)^{\frac{1}{p}} \text{ by Minskowski's inequality} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

For every n such that $1 \leq n < n_\epsilon$, there is an $i_\epsilon^n \in \mathbb{N}$ such that

$$\left(\sum_{i \geq i_\epsilon^n} |x_{ni}|^p\right) < \epsilon,$$

since $(x_{ni}) \in \ell_p$. Let $i'_\epsilon = \max_{1 \leq n < n_\epsilon} i_\epsilon^n$ and $j_\epsilon = \max\{i'_\epsilon, i_\epsilon\}$, then for all $n \in \mathbb{N}$,

$$\left(\sum_{i \geq j_\epsilon} (x_{ni}^+)^p\right)^{\frac{1}{p}} \leq \left(\sum_{i \geq j_\epsilon} |x_{ni}|^p\right)^{\frac{1}{p}} < \epsilon.$$

Conversely, suppose that $x_{ni} \xrightarrow{q} x_i$ as $n \to \infty$ for all $i \in \mathbb{N}$ and

$$\sup_{n \in \mathbb{N}} \sum_{i=m}^{\infty} (x_{ni}^+)^p \to 0 \text{ as } m \to \infty.$$

Let $\epsilon > 0$. Then there is $i_1^\epsilon \in \mathbb{N}$ such that $\sup_{n \in \mathbb{N}} \left(\sum_{i \geq i_1^\epsilon}^{\infty} (x_{ni}^+)^p\right)^{\frac{1}{p}} < \frac{\epsilon}{3}$, and there is $i_2^\epsilon \in \mathbb{N}$ such that $\left(\sum_{i \geq i_2^\epsilon} |x_i|^p\right)^{\frac{1}{p}} < \frac{\epsilon}{3}$. Let $i_\epsilon = \max\{i_1^\epsilon, i_2^\epsilon\}$. For every n

$$\begin{aligned}
p(\mathbf{x}_n - \mathbf{x}) &= \left(\sum_{i=1}^{\infty} ((x_{ni} - x_i)^+)^p\right)^{\frac{1}{p}} \\
&\leq \left(\sum_{1 \leq i < i_\epsilon} ((x_{ni} - x_i)^+)^p\right)^{\frac{1}{p}} + \left(\sum_{i \geq i_\epsilon} ((x_{ni} - x_i)^+)^p\right)^{\frac{1}{p}} \text{ by Minskowski's inequality} \\
&\leq \left(\sum_{1 \leq i < i_\epsilon} ((x_{ni} - x_i)^+)^p\right)^{\frac{1}{p}} + \left(\sum_{i \geq i_\epsilon} (x_{ni}^+)^p\right)^{\frac{1}{p}} + \left(\sum_{i \geq i_\epsilon} (x_i^-)^p\right)^{\frac{1}{p}} \text{ since } (-x_i)^+ = x_i^- \\
&\leq \left(\sum_{1 \leq i < i_\epsilon} ((x_{ni} - x_i)^+)^p\right)^{\frac{1}{p}} + \left(\sum_{i \geq i_\epsilon} (x_{ni}^+)^p\right)^{\frac{1}{p}} + \left(\sum_{i \geq i_\epsilon} |x_i|^p\right)^{\frac{1}{p}} \text{ since } x_i^- \leq |x_i| \\
&< (\sum_{1 \leq i < i_\epsilon} ((x_{ni} - x_i)^+)^p)^{\frac{1}{p}} + \frac{2\epsilon}{3}.
\end{aligned}$$

Since for each $i \in \mathbb{N}$, $x_{ni} \xrightarrow{q} x_i$, we can choose $n_\epsilon^i \in \mathbb{N}$ such that

$$(x_{ni} - x_i)^+ < \frac{\epsilon}{3i_\epsilon} \text{ for } n \geq n_\epsilon^i.$$

Let $l_\epsilon = \max_{1 \leq i < i_\epsilon} n_\epsilon^i$, then for $n \geq l_\epsilon$,

$$\begin{aligned} p(\mathbf{x}_n - \mathbf{x}) &\leq \left((i_\epsilon - 1) \left(\frac{\epsilon}{3 i_\epsilon} \right)^p \right)^{\frac{1}{p}} + \frac{2\epsilon}{3} \\ &= (i_\epsilon - 1)^{\frac{1}{p}} \frac{\epsilon}{3 i_\epsilon} + \frac{2\epsilon}{3} \\ &< \frac{\epsilon}{3} + \frac{2\epsilon}{3} = \epsilon. \end{aligned}$$

Hence $\mathbf{x}_n \xrightarrow{p} \mathbf{x}$. □

Chapter 3

Left-K-sequential completeness in asymmetrically normed lattices

In quasi-metric spaces there are a large number of different notions of completeness. In this chapter, we study the left-K-sequential completeness of asymmetrically normed lattices. We have chosen left-K-sequential completeness since our main goal of the thesis is to study p-compactness of the subsets of asymmetrically normed lattice and p-compactness is characterized by left-K-sequential completeness and p-precompactness.

We characterize the bounded above left-K-sequentially complete subsets of $\mathbb{R}^m$. We show that the asymmetrically normed lattices $\mathbb{R}^m$, $C(\Omega)$, ℓ_∞, c_0 and ℓ_p, $1 \leq p < \infty$, are left-K-sequentially complete. We provide an example to show that not every asymmetrically normed lattice is left-K-sequentially complete, and an example of a left-K-sequentially complete space (X, p) which is not p^s-complete. Some of the results in this chapter have been published in [13].

3.1 Left-K-Cauchy sequences in asymmetrically normed lattices

In this section, we consider left-K-Cauchy sequences in an asymmetrically normed lattice (X, p). We derive a number of useful results about such sequences in $(\mathbb{R}, q)$.

Definition 3.1.1. *([37, Definition 1]) A sequence $(x_n)_{n=1}^{\infty}$ in an asymmetrically normed space (X, p) is called a* **left** *(respectively* **right***)-K-Cauchy sequence if for each $\epsilon > 0$ there is $n_\epsilon \in \mathbb{N}$ such that $p(x_n - x_k) < \epsilon$ (respectively $p(x_k - x_n) < \epsilon$) for all $n \geq k \geq n_\epsilon$.*

It is mentioned in [10] that a p-convergent sequence $(x_n)_{n=1}^{\infty}$ is not necessarily left-K-Cauchy. We give an example below.

Example 3.1.2. *In $(\mathbb{R}, q)$, let $x_n = (-1)^n$. Then by Proposition 2.2.5, $x_n \xrightarrow{q} 1$. We next show that $(x_n)_{n=1}^{\infty}$ is not left-K-Cauchy. Let $\epsilon = 1$. Then for every $n_0 \in \mathbb{N}$, choose an even number n and an odd number k such that $n > k \geq n_0$. Then*

$$(x_n - x_k)^+ = 2 > 1.$$

Therefore $(x_n)_{n=1}^{\infty}$ is not a left-K-Cauchy sequence.

Proposition 3.1.3. *([13, Proposition 3.2]) Any left-K-Cauchy sequence $(x_n)_{n=1}^{\infty}$ in an asymmetrically normed space (X, p) is p-bounded.*

Proof. Suppose that $(x_n)_{n=1}^{\infty}$ is a left-K-Cauchy sequence in X. Then we choose a $k \in \mathbb{N}$ such that

$$p(x_n - x_k) < 1, \text{ for all } n \geq k.$$

Then for $n \geq k$,

$$\begin{aligned} p(x_n) &= p(x_n - x_k + x_k) \\ &\leq p(x_n - x_k) + p(x_k) \\ &< 1 + p(x_k). \end{aligned}$$

Therefore

$$p(x_n) \leq \max\{p(x_1), p(x_2), ..., p(x_{k-1}), 1 + p(x_k)\}, \text{ for all } n.$$

Thus $(x_n)_{n=1}^{\infty}$ is p-bounded. $\square$

Lemma 3.1.4. *Any decreasing sequence $(x_n)_{n=1}^{\infty}$ in an asymmetrically normed lattice (X, p) is a left-K-Cauchy sequence.*

Proof. To see this, let $\epsilon > 0$, then for all $n, k \in \mathbb{N}$ such that $n \geq k$, we obtain

$$x_n - x_k \leq 0,$$

since $(x_n)_{n=1}^{\infty}$ is decreasing. Therefore

$$p(x_n - x_k) = 0 < \epsilon.$$

Thus a decreasing sequence is a left-K-Cauchy sequence. $\square$

Corollary 3.1.5. *Any increasing sequence $(x_n)_{n=1}^{\infty}$ in an asymmetrically normed lattice (X, p) is a right-K-Cauchy sequence.*

Proposition 3.1.6. *([12, Proposition 1.2.1]) If a sequence $(\mathbf{x}_n)_{n=1}^{\infty}$ in an asymmetrically normed space (X, p) is p^s-Cauchy, then it is a left-K-Cauchy sequence.*

The converse is not true in general as the next example shows.

Example 3.1.7. *In $(\mathbb{R}, q)$, let $x_n = (-n)_{n=1}^{\infty}$. Then by Lemma 3.1.4, $(x_n)_{n=1}^{\infty}$ is a left-K-Cauchy sequence, since it is decreasing. But clearly, it is not q^s-Cauchy, since it is not bounded below.*

Proposition 3.1.8. *([12, Proposition 1.2.4]) Let $(x_n)_{n=1}^{\infty}$ be a left-K-Cauchy sequence in an asymmetrically normed space (X, p).*

(a) If $(x_n)_{n=1}^{\infty}$ has a subsequence which is p-convergent to x, then $(x_n)_{n=1}^{\infty}$ is p-convergent to x.

(b) If $(x_n)_{n=1}^{\infty}$ has a subsequence which p^{-1}-convergent to x, then $(x_n)_{n=1}^{\infty}$ is p^{-1}-convergent to x.

Corollary 3.1.9. *(cf. [12, Proposition 1.2.4]) Let $(x_n)_{n=1}^{\infty}$ be a left-K-Cauchy sequence in an asymmetrically normed space (X, p). If $(x_n)_{n=1}^{\infty}$ has a subsequence which is p^s-convergent to x, then $(x_n)_{n=1}^{\infty}$ is p^s-convergent to x.*

Proof. Take any left-K-Cauchy sequence $(x_n)_{n=1}^{\infty}$ in X. Suppose there is a subsequence $(x_{n_k})_{k=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ such that $x_{n_k} \xrightarrow{p^s} x \in X$. Then by Proposition 2.1.2, $x_{n_k} \xrightarrow{p} x$ and $x_{n_k} \xrightarrow{p^{-1}} x$. Hence by Proposition 3.1.8, $x_n \xrightarrow{p} x$ and $x_n \xrightarrow{p^{-1}} x$. Therefore, by Proposition 2.1.2, $x_n \xrightarrow{p^s} x$. □

Corollary 3.1.10. *Let $(x_n)_{n=1}^{\infty}$ be a left-K-Cauchy sequence in an asymmetrically normed space (X, p). If $(x_n)_{n=1}^{\infty}$ has a subsequence which is p^s-convergent to x, then $(x_n)_{n=1}^{\infty}$ is a p^s-Cauchy sequence.*

Corollary 3.1.11. *Let $(x_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$. If $\limsup_n x_n$ exists and $(x_n)_{n=1}^{\infty}$ is not bounded below, then $(x_n)_{n=1}^{\infty}$ is not left-K-Cauchy in $(\mathbb{R}, q)$.*

Proof. Let $(x_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$ which is not bounded below and such that $\limsup_n x_n$ exists. We show that $(x_n)_{n=1}^{\infty}$ is not a left-K-Cauchy sequence. Suppose it is left-K-Cauchy. Since $x = \limsup_n x_n$ exists, there is a subsequence of $(x_n)_{n=1}^{\infty}$ that q^s-converges to x. Since $(x_n)_{n=1}^{\infty}$ is left-K-Cauchy, it follows from Corollary 3.1.9 that it q^s-converges to x. Hence it is q^s-bounded. But $(x_n)_{n=1}^{\infty}$ is not bounded below, hence not q^s-bounded, a contradiction. Therefore $(x_n)_{n=1}^{\infty}$ is not a left-K-Cauchy sequence. □

We mention the contrapositive of the above result as a corollary below.

Corollary 3.1.12. *If a sequence $(x_n)_{n=1}^{\infty}$ in $\mathbb{R}$ is left-K-Cauchy, then either the* $\limsup_n x_n$ *does not exist or $(x_n)_{n=1}^{\infty}$ is bounded below.*

Lemma 3.1.13. *([13, Proposition 3.3]) Let $(x_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$ which is not bounded below, then*

(a) for every $x \in \mathbb{R}$, there is a subsequence $(x_{n_k})_{k=1}^{\infty}$ such that $x_{n_k} \xrightarrow{q} x$.

(b) if $(x_n)_{n=1}^{\infty}$ is a left-K-Cauchy sequence, then $x_n \xrightarrow{q} x$, for all $x \in \mathbb{R}$.

Proof. (a) Let $x \in \mathbb{R}$. Since $(x_n)_{n=1}^{\infty}$ is not bounded below, there is a subsequence $(x_{n_k})_{k=1}^{\infty}$ such that $x_{n_k} \leq -k + x$, for every $k \in \mathbb{N}$. Then

$$q(x_{n_k} - x) = (x_{n_k} - x)^+ \leq (-k)^+ = 0.$$

Therefore,

$$x_{n_k} \xrightarrow{q} x.$$

(b) The statement follows from (a) and Proposition 3.1.8. □

Proposition 3.1.14. *([13, Proposition 3.5]) Let $(x_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$. If $(x_n)_{n=1}^{\infty}$ is bounded below and left-K-Cauchy, then it is q^s-convergent.*

Proof. Let $(x_n)_{n=1}^{\infty}$ be bounded below and a left-K-Cauchy sequence in $\mathbb{R}$, then by Proposition 3.1.3, it is bounded above and hence bounded. Therefore

$$u = \limsup_n x_n$$

and

$$l = \liminf_n x_n$$

exist. By Proposition 2.2.5 and Corollary 2.2.7,

$$x_n \xrightarrow{q} u, \text{ and } x_n \xrightarrow{q^{-1}} l \text{ as } n \to \infty.$$

We show that $(x_n)_{n=1}^{\infty}$ is q^s-convergent. It suffices to show that $l \geq u$. Since l exists, there is a subsequence $(x_{n_k})_{k=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ such that $x_{n_k} \xrightarrow{q^s} l$, hence $x_{n_k} \xrightarrow{q} l$. Since $(x_n)_{n=1}^{\infty}$ is left-K-Cauchy, it follows from Proposition 3.1.8, that $x_n \xrightarrow{q} l$. Since l is a q-limit of $(x_n)_{n=1}^{\infty}$, it follows from Proposition 2.2.5 that $l \geq u$. Thus $l = u$ and so $(x_n)_{n=1}^{\infty}$ is q^s-convergent. $\square$

Corollary 3.1.15. *([13, Corollary 3.6]) Let $(\mathbf{x}_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{R}^m$. If $(\mathbf{x}_n)_{n=1}^{\infty}$ is a bounded below left-K-Cauchy sequence, then it is p^s-convergent.*

3.2 Left-K-sequential completeness in asymmetrically normed lattices

In this section, we study the left-K-sequentially complete subsets of an asymmetrically normed lattice.

Definition 3.2.1. *([37, Definition 3]) A subset A of an asymmetrically normed space (X, p) is* **left** *(respectively* **right***)-K-***sequentially complete** *if every left (respectively right)-K-Cauchy sequence $(x_n)_{n=1}^{\infty}$ in A p-converges to some point in A.*

Definition 3.2.2. *A topological space X is of the* **second category in itself** *if the intersection of every countable family of dense open sets is non-empty.*

Theorem 3.2.3. *([26, Theorem 2.11]) Let (X, p) be an asymmetrically normed space. Then (X, p) is of second category in itself if (X, p^{-1}) is right-K-sequentially complete.*

Lemma 3.2.4. *([4, p. 787]) The non-empty p-open (p^{-1}-closed) subsets of an asymmetrically normed lattice (X, p) are p-dense.*

Definition 3.2.5. *A topological space X is called a* **Baire space** *if the intersection of every countable family of dense open sets is also dense.*

It follows that every Baire space is of the second category in itself.

We show that no p-open subset of an asymmetrically normed lattice (X, p) is a Baire space. We first prove the following useful lemma,

Lemma 3.2.6. *Let (X, p) be an asymmetrically normed lattice. If $x_1, x_2 \in X$ are such that $x_1 \leq x_2$, and $\epsilon > 0$, then*

$$B^p_\epsilon(x_1) \subseteq B^p_\epsilon(x_2).$$

Proof. Clearly,

$$x_1 - z \leq x_2 - z, \text{ for every } z \in X,$$

since $x_1 \leq x_2$. Hence by Theorem 1.3.9

$$(x_2 - z)^- \leq (x_1 - z)^-.$$

This implies

$$||(x_2 - z)^-|| \leq ||(x_1 - z)^-||,$$

since X is a normed lattice. For $\epsilon > 0$, let $z \in B^p_\epsilon(x_1)$. Then using the fact that $x^+ = (-x)^-$ for every $x \in X$,

$$\begin{aligned} ||(z - x_2)^+|| &= ||(x_2 - z)^-||. \\ &\leq ||(x_1 - z)^-||. \\ &= ||(z - x_1)^+||. \\ &< \epsilon, \text{ since } z \in B^p_\epsilon(x_1). \end{aligned}$$

Thus $z \in B^p_\epsilon(x_2)$. Hence

$$B^p_\epsilon(x_1) \subseteq B^p_\epsilon(x_2).$$

□

The proof of the following result is modelled on that of ([4, Proposition 1]).

Proposition 3.2.7. *A non-empty p-open subset A of a non-trivial asymmetrically normed lattice (X, p) is never of second category in itself and therefore not a Baire space.*

Proof. Let A be a non-empty p-open subset of X, then by Lemma 3.2.4, it is p-dense. We show that A contains a negative element. Since $A \neq \emptyset$, there is an $x \in A$. If $x = 0$, there is a $y \in A$, $y < 0$, since A is decreasing and p-dense. If $x \neq 0$, then $-x^- \in A$ ($-x^-$ is negative since $x^- = (-x) \vee 0$ is positive), since

$$x = x^+ - x^- \geq -x^-,$$

and A is decreasing. So let $x \in A$, $x < 0$. Since A is p-open, there is $\epsilon > 0$, such that $B_\epsilon^p(x) \subseteq A$. The sequence $(B_\epsilon^p(nx))_{n=1}^\infty \subseteq A$, since for each n, $nx \leq x$. Therefore by Lemma 3.2.6,

$$B_\epsilon^p(nx) \subseteq B_\epsilon^p(x) \subseteq A.$$

We show that this family has empty intersection. Suppose

$$y \in \bigcap_{n=1}^\infty B_\epsilon^p(nx), \text{ then } \|(y - nx)^+\| < \epsilon, \text{ for all } n.$$

So for all n, we obtain

$$\begin{aligned} n\|x\| &= \|nx\| = \|-nx\| \\ &= \|(-nx)^+\| \\ &= \|(y - nx - y)^+\| \\ &\leq \|(y - nx)^+\| + \|(-y)^+\| \\ &< \epsilon + \|(-y)^+\|. \end{aligned}$$

Therefore it follows that

$$\|x\| < \frac{1}{n}[\epsilon + \|(-y)^+\|] \text{ for all } n$$

and therefore $\|x\| = 0$ which is a contradiction since $x \neq 0$. Consequently, $\{B_\epsilon^p(nx))_{n=1}^\infty\}$ is a countable family of p-open subsets of A whose intersection is empty, and hence A is not of the second category in itself. $\square$

Corollary 3.2.8. *([4, Proposition 1]) An asymmetrically normed lattice (X,p) is never of the second category in itself and therefore not a Baire space.*

Corollary 3.2.9. *Let (X,p) be an asymmetrically normed lattice.*

(a) (X,p^{-1}) is not right-K-sequentially complete.

(b) No p-open subset A of (X,p^{-1}) is right-K-sequentially complete.

Proof. (a) We show that (X,p^{-1}) is not right-K-sequentially complete. Suppose it is right-K-sequentially complete, then by Theorem 3.2.3, (X,p) is of the second category in itself. But by Corollary 3.2.8, (X,p) is not of the second category. Therefore (X,p^{-1}) is not right-K-sequentially complete.
(b) This follows as in (a). □

Proposition 3.2.10. *([13, Proposition 3.7]) If A is a subset of an asymmetrically normed lattice (X,p) such that the supremum of A exists and is an element of A, then A is left-K-sequentially complete.*

Proof. Let A be a subset of X such that $x = \sup A \in A$. Take any left-K-Cauchy sequence $(x_n)_{n=1}^{\infty}$ in A, then by Proposition 2.1.5, $x_n \xrightarrow{p} x$ and $x \in A$. Therefore A is left-K-sequentially complete. □

Corollary 3.2.11. *A set A which contains its infimum is left-K-sequentially complete in (X,p^{-1}).*

Proof. The proof is similar to that of Proposition 3.2.10. □

Proposition 3.2.12. *Let (X,p) be a left-K-sequentially complete asymmetrically normed space. If a subset A of X is p-closed, then it is left-K-sequentially complete.*

Proof. Take any left-K-Cauchy sequence $(x_n)_{n=1}^{\infty}$ in A, then there is $x \in X$ such that

$$x_n \xrightarrow{p} x \text{ as } n \to \infty,$$

since X is left-K-sequentially complete. But A is p-closed, so $x \in A$. Hence A is left-K-sequentially complete. $\square$

The converses of Propositions 3.2.10 and 3.2.12 do not hold in general (as we shall see in Example 3.3.6).

Theorem 3.2.13. *([13, Theorem 3.11]) Let (X, p) be an asymmetrically normed lattice such that every p-bounded sequence is bounded above in the lattice.*

(a) Any increasing subset A of X is left-K-sequentially complete in (X, p).

(b) Any decreasing subset A of X is left-K-sequentially complete in (X, p^{-1}).

In particular, (X, p) and (X, p^{-1}) are left-K-sequentially complete.

Proof. (a) Suppose $A \subseteq X$ is an increasing set. Take any left-K-Cauchy sequence $(x_n)_{n=1}^{\infty}$ in A, then by Proposition 3.1.3, $(x_n)_{n=1}^{\infty}$ is p-bounded. By assumption, $(x_n)_{n=1}^{\infty}$ is bounded above (say by $c \in X$). Therefore, by Proposition 2.1.5, $x_n \xrightarrow{p} c$. Since A is increasing and $x_n \leq c$ for all n, it follows that $c \in A$. Thus A is left-K-sequentially complete.

(b) A left-K-Cauchy sequence $(x_n)_{n=1}^{\infty}$ in (X, p^{-1}) is p^{-1}-bounded, and it follows from the assumption that it will be bounded below in X, and therefore, by Proposition 2.1.5, it is p^{-1}-convergent to any lower bound. If A is decreasing, these lower bounds will be in A. Hence A is left-K-sequentially complete. $\square$

Corollary 3.2.14. *Let (X, p) be an asymmetrically normed lattice such that every p-bounded sequence is bounded above in the lattice, then any p^{-1}-open subset A of X is left-K-sequentially complete in (X, p).*

Proof. If $A \subseteq X$ is p^{-1}-open, then by Lemma 1.4.4, it is increasing. Therefore by Theorem 3.2.13, it is left$K$-sequentially complete in (X, p). $\square$

We recall that in a normed lattice X, the set

$$X^+ = \{x \in X : x \geq 0\}$$

is called the positive cone of X.

Proposition 3.2.15. *The set X^+ is left-K-sequentially complete in (X, p^{-1}).*

Proof. This follows from Corollary 3.2.11. $\square$

3.3 Left-K-sequential completeness in finite dimensional spaces

In this section we show that the finite dimensional space $\mathbb{R}^m$ is left-K-sequentially complete and characterize the left-K-sequential completeness of subsets of this space that are bounded above.

We will use the asymmetric norm p derived from the maximum norm.

Proposition 3.3.1. *([13, Corollary 3.12] The asymmetrically normed lattices $(\mathbb{R}^m, p)$ and $(\mathbb{R}^m, p^{-1})$ are left-K-sequentially complete.*

Proof. The statement follows from Proposition 2.2.2 and Theorem 3.2.13. $\square$

We give a characterization of the left-K-sequentially complete subsets of $\mathbb{R}$.

Proposition 3.3.2. *([13, Proposition 3.9] Let A be a subset of* $(\mathbb{R}, q)$*.*

(a) If A is bounded above, then it is left-K-sequentially complete if and only if it contains its supremum.

(b) If A is not bounded above, then it is left-K-sequentially complete.

Proof. (a) Suppose that A is left-K-sequentially complete. Since A is bounded above, $c = \sup A$ exists and there is an increasing sequence $(x_n)_{n=1}^{\infty}$ in A such that $x_n \xrightarrow{q^s} c$ and

$$c = \sup x_n = \lim_n \sup x_n.$$

Since $(x_n)_{n=1}^{\infty}$ is q^s-convergent, it is q^s-Cauchy, and hence left-K-Cauchy (see Proposition 3.1.6). Since A is left-K-sequentially complete, $(x_n)_{n=1}^{\infty}$ q-converges to some $x \in A$. Therefore by Lemma 2.2.5, $x \geq c$. But since $x \in A$ and $c = \sup A$, $c = x \in A$. The converse follows from Proposition 3.2.10.

(b) Suppose $A \subseteq \mathbb{R}$ is not bounded above. If $(x_n)_{n=1}^{\infty}$ is a left-K-Cauchy sequence in A, then by Proposition 3.1.3 it is q-bounded and so bounded above, and hence by Proposition 2.1.5, it is q-convergent to any of its upper bounds. Let c be an upper bound of $(x_n)_{n=1}^{\infty}$, then c is not an upper bound of A (since A is not bounded above). So there is $a \in A$ such that $a > c$. Then by Proposition 2.1.4, $x_n \xrightarrow{q} a$. Thus A is left-K-sequentially complete. □

Corollary 3.3.3. *([13, Corollary 3.10] Any q^s-closed subset of* $(\mathbb{R}, q)$ *is left-K-sequentially complete.*

Proof. Let A be a q^s-closed subset of $\mathbb{R}$. If A is bounded above, then it contains its supremum. Hence by part (a) of Proposition 3.3.2, it is left-Ksequentially complete. If A is not bounded above, it follows from part (b) of Proposition 3.3.2 that A is left-K-sequentially complete. □

This statement is no longer true in $\mathbb{R}^2$ as the next example shows.

Example 3.3.4. *([13, p. 10] Let*

$$A = \left\{ (x, y) \in \mathbb{R}^2 : x < 0, y = \frac{1}{x} \right\}.$$

Then A is p^s-closed, since it is the graph of a p^s-continuous function, but it is not left-K-sequentially complete. To see this, consider the left-K-Cauchy sequence $(-\frac{1}{n}, -n)_{n=1}^{\infty}$ (this is left-K-Cauchy in $\mathbb{R}^2$, since both $(-\frac{1}{n})$ and $(-n)$ for $n \in \mathbb{N}$ are left-K-Cauchy in $\mathbb{R}$) in A. It follows from Proposition 2.2.6 that the set of limit points of A is

$$\{(x, y) \in \mathbb{R}^2 : x \geq 0\},$$

which is disjoint from A.

Proposition 3.3.5. *([13, Proposition 3.8]) Let A be a subset of $\mathbb{R}^m$. If A is bounded below and p^s-closed, then it is left-K-sequentially complete.*

Proof. Let A be a p^s-closed bounded below subset of $\mathbb{R}^m$. Take any left-K-Cauchy sequence $(\mathbf{x}_n)_{n=1}^{\infty}$ in A, then $(\mathbf{x}_n)_{n=1}^{\infty}$ is bounded below (since A is bounded below). Therefore by Corollary 3.1.15, it is p^s-convergent to some $\mathbf{x} \in A$, since A is p^s-closed. Thus A is left-K-sequentially complete. $\square$

The following example show that the converses of Propositions 3.3.5, 3.2.10 and 3.2.12 are not true in general.

Examples 3.3.6. *([13, p. 9 and 10])*

(a) *A left-K-sequentially complete set need not contain its supremum. Consider the set*

$$A = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0, x^2 + y^2 \leq 1\}.$$

It is bounded below and p^s-closed, hence by Proposition 3.3.5 it is left-K-sequentially complete, but $\sup A = (1, 1) \notin A$.

(b) *The set of rational numbers* $\mathbb{Q}$ *is a left-*K*-sequentially complete subset of* $(\mathbb{R}, q)$ *(Proposition 3.3.2). But it is well known that* $\mathbb{Q}$ *is not a* q^s*-closed subset of* $(\mathbb{R}, q^s)$*. Hence by Corollary 1.4.5, it is not* q*-closed. It is also not bounded below.*

Remark 3.3.7. *The set* $\mathbb{Q}$ *is a left-*K*-sequentially complete subset of* $(\mathbb{R}, q^{-1})$ *which is not* q^s*-closed. Hence by Corollary 1.4.6, it is not* q^{-1}*-closed.*

Definition 3.3.8. *Let* X *be a normed lattice and* A *be a subset of* X*. Then we say that* A *has the* B*-property if and only if for every* $x \in \overline{A}$*, there is an* $a \in A$ *such that* $a \geq x$*, that is, for all* $x \in \overline{A}$*,*

$$\uparrow x \cap A \neq \emptyset.$$

Examples 3.3.9. (a) *Any norm-closed set* A *has the* B*-property.*

(b) *Any subset of* $\mathbb{R}$ *which is not bounded above has the* B*-property in* $\mathbb{R}$*.*

Proposition 3.3.10. *If* A *is a left-*K*-sequentially complete subset of* $\mathbb{R}^m$*, then* A *has the* B*-property in* $\mathbb{R}^m$*.*

Proof. Let A be a left-Ksequentially complete subset of $\mathbb{R}^m$. Take a point $\mathbf{x} \in \overline{A}^{p^s}$, then there is a sequence $(\mathbf{x}_n)_{n=1}^{\infty}$ in A such that $\mathbf{x}_n \xrightarrow{p^s} \mathbf{x}$. Therefore, $(\mathbf{x}_n)_{n=1}^{\infty}$ is p^s-Cauchy and hence by Proposition 3.1.6, left-K-Cauchy. Since A is left-K-sequentially complete, there is $\mathbf{y} \in A$ such that $\mathbf{x}_n \xrightarrow{p} \mathbf{y}$. Since $\mathbf{x}_n \xrightarrow{p^s} \mathbf{x}$, it follows that $\mathbf{x}_n \xrightarrow{p} \mathbf{x}$ and hence, by Proposition 2.2.4,

$$x_{ni} \xrightarrow{q} x_i = \limsup_n x_{ni} \text{ for all } i.$$

It follows from Proposition 2.2.6 that $\mathbf{y} \geq \mathbf{x}$. Therefore, A has the B-property in $\mathbb{R}^m$. $\square$

Recall that we denote by $\mathbb{R}^{\mathbb{N}}$ the space of all real sequences. The proof above shows that the result also holds in any normed Riesz subspace X of the Riesz space $\mathbb{R}^{\mathbb{N}}$, equipped with the pointwise ordering, provided that convergence in the norm of X implies pointwise convergence.

The converse of this result is not true in general. See Example 3.3.4 for an example of a set which is p^s-closed and hence it has the B-property but is not left-K-sequentially complete.

Let $m \in \mathbb{N}$ and put $I_m = \{1, 2, ..., m\}$. For $\emptyset \neq I \subseteq I_m$, let k be the number of elements in I, write

$$I = \{i_1, i_2, ..., i_k\}$$

and define the function $P_I : \mathbb{R}^m \to \mathbb{R}^k$ by

$$P_I((x_1, x_2, ..., x_m)) = (x_{i_1}, x_{i_2}, ..., x_{i_k}).$$

For $A \subseteq \mathbb{R}^m$, let

$$I_A = \{i \in I_m : \{x_i : (x_1, x_2, ..., x_m) \in A\} \text{ is bounded above in } \mathbb{R}\}.$$

Examples 3.3.11. *(a) If $A \subseteq \mathbb{R}^2$ has the B-property in $\mathbb{R}^2$, $P_{\{1\}}(A)$ and $P_{\{2\}}(A)$ need not have the B-property in $\mathbb{R}$. To see this, let*

$$A = \left\{(x, y) \in \mathbb{R}^2 : x < 0, y \leq \frac{1}{x}\right\}.$$

Since A is a p^s-closed subset of $\mathbb{R}^2$, it follows that A has the B-property in $\mathbb{R}^2$. But $P_{\{1\}}(A)$ does not have the B-property in $\mathbb{R}$, since

$$\overline{P_{\{1\}}(A)}^{q^s} = (-\infty, 0] \text{ and } P_{\{1\}}(A) = (-\infty, 0),$$

and so $0 \in \overline{P_{\{1\}}(A)}^{q^s}$, but there is no $x \in P_{\{1\}}(A)$ such that $x \geq 0$. Therefore, $P_{\{1\}}(A)$ does not have the B-property in $\mathbb{R}$. Similarly, $P_{\{2\}}(A)$ does not have the B-property in $\mathbb{R}$.

(b) If $A \subseteq \mathbb{R}^2$ is such that $P_{\{1\}}(A)$ and $P_{\{2\}}(A)$ both have the B-property in $\mathbb{R}$, then A need not have the B-property in $\mathbb{R}^2$. To see this, let

$$A = \left\{(x, y) \in \mathbb{R}^2 : x < 0, y < \frac{1}{x}\right\} \cup \{(-2, 0)\} \cup \{(0, -1)\}.$$

Then $P_{\{1\}}(A) = (-\infty, 0]$ and $P_{\{2\}}(A) = (-\infty, 0]$ both have the B-property in $\mathbb{R}$. However A does not have the B-property in $\mathbb{R}^2$, since $(-1,-1) \in \overline{A}^{p^s}$, but there is no $(x, y) \in A$ such that $(x, y) \geq (-1, -1)$. Therefore, A does not have the B-property in $\mathbb{R}^2$.

In the proofs of the following theorems, p^s will be used for the maximum norm in $\mathbb{R}^k$, for any $k \geq 2$.

Proposition 3.3.12. *Let A be a non-empty subset of $\mathbb{R}^m$. If for every non-empty subset I of I_m, $P_I(A)$ has the B-property in $\mathbb{R}^k$, where k is the number of elements in I, then A is left-K-sequentially complete.*

Proof. Let A be a non-empty subset of $\mathbb{R}^m$. Take any left-K-Cauchy sequence $(\mathbf{x}_n)_{n=1}^{\infty}$ in A and let

$$J_1 = \{i : (x_{ni}) \text{ is bounded below } \}$$

and

$$J_2 = I_m \setminus J_1.$$

We consider the following cases:
Case 1: Suppose $J_2 = I_m$, then $\limsup_n x_{ni}$ does not exists for $i = 1, 2, ..., m$. We choose any $(x_1, x_2, ..., x_m) \in A$, then by Proposition 2.2.6,

$$(x_{n1}, x_{n2}, ..., x_{nm}) \xrightarrow{p} (x_1, x_2, ..., x_m) \in A.$$

Case 2: Suppose $J_1 = I_m$, then by Proposition 3.1.14 and Proposition 2.2.5, $x_{ni} \xrightarrow{q^s} x_i = \limsup_n x_{ni}$ for $i = 1, 2, ..., m$. Hence $\mathbf{x}_n \xrightarrow{p^s} \mathbf{x}$, where $\mathbf{x} = (x_1, x_2, ..., x_m)$, and so $\mathbf{x} \in \overline{A}^{p^s}$. Since A has the B-property, there is an $\mathbf{y} \in A$ such that $\mathbf{y} \geq \mathbf{x}$. Hence by Proposition 2.1.4, $\mathbf{x}_n \xrightarrow{p} \mathbf{y} \in A$, since $\mathbf{x}_n \xrightarrow{p} \mathbf{x}$.
Case 3: Suppose $J_1 \neq \emptyset$ and $J_2 \neq \emptyset$, then $x_i = \limsup_n x_{ni}$ exists for all $i \in J_1$. Consider $P_{J_1}(\mathbf{x}_n)$, then $P_{J_1}(\mathbf{x}_n) \xrightarrow{p^s} \mathbf{x} \in \mathbb{R}^k$, where k is the number of elements in J_1 and $\mathbf{x} = (x_i)$, with $x_i = \limsup_n x_{ni}$ for all $i \in J_1$, and hence $\mathbf{x}$ is in $\overline{P_{J_1}(A)}^{p^s}$.

Since $P_{J_1}(A)$ has the B-property, there is $\mathbf{y}$ in $P_{J_1}(A)$ such that $\mathbf{y} \geq \mathbf{x}$. Therefore by Proposition 2.1.4, $P_{J_1}(\mathbf{x}_n) \xrightarrow{p} \mathbf{y}$. Since $\mathbf{y}$ is in $P_{J_1}(A)$, there is $\mathbf{z}$ in $P_{J_2}(A)$ such that $\mathbf{w}$ (formed by combining the coordinates of $\mathbf{y}$ and $\mathbf{z}$ in the correct order) is in A, and by Proposition 2.2.6 $P_{J_2}(\mathbf{x}_n) \xrightarrow{p} \mathbf{z} \in \mathbb{R}^l$, where l is the number of elements in J_2. Hence $\mathbf{x}_n \xrightarrow{p} \mathbf{w} \in A$. Thus A is left-K-sequentially complete. $\square$

We first state and prove the following well known result in real analysis.

Lemma 3.3.13. *If a sequence $(x_n)_{n=1}^{\infty}$ in $\mathbb{R}$ is bounded above and $\limsup_n x_n$ does not exist, then there is a decreasing subsequence $(x_{n_k})_{k=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ such that $x_{n_k} \to -\infty$.*

Proof. Let $(x_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$. Suppose $(x_n)_{n=1}^{\infty}$ is bounded above and $\limsup_n x_n$ does not exist. Then the sequence $(\sup_{k\geq n} x_k)_{n=1}^{\infty}$ is not bounded below. Hence, for every $k \in \mathbb{N}$, there is $n_k \in \mathbb{N}$ such that $x_{n_k} \leq \sup_{i\geq n_k} x_i < -k$. Since $(x_n)_{n=1}^{\infty}$ is not bounded below, there is no loss of generality in assuming that $x_{n_k} < x_{x_{n_{k-1}}}$ and $x_{n_k} < -k$. Hence $x_{n_k} \to -\infty$. $\square$

Theorem 3.3.14. *Let A be a non-empty subset of $\mathbb{R}^m$ which is bounded above. Then A is left-K-sequentially complete if and only if for every non-empty subset I of I_m, $P_I(A)$ has the B-property in $\mathbb{R}^k$, where k is the number of elements in I.*

Proof. We give a proof for subsets of $\mathbb{R}^3$.
Let A be a left-K-sequentially complete subset of $\mathbb{R}^3$, then by Proposition 3.3.10, A has the B-property in $\mathbb{R}^3$.
We show that $P_{\{1,2\}}(A)$, $P_{\{1,3\}}(A)$ and $P_{\{2,3\}}(A)$ have the B-property in $\mathbb{R}^2$. Take $(x, y) \in \overline{P_{\{1,2\}}(A)}^{p^s}$, then there is a sequence $((x_n, y_n))_{n=1}^{\infty}$ in $P_{\{1,2\}}(A)$ such that

$$(x_n, y_n) \xrightarrow{p^s} (x, y).$$

Hence $x_n \xrightarrow{q^s} x$ and $y_n \xrightarrow{q^s} y$. Since $((x_n, y_n))_{n=1}^{\infty}$ is in $P_{\{1,2\}}(A)$, there is a sequence $(z_n)_{n=1}^{\infty}$ in $\mathbb{R}$ such that $((x_n, y_n, z_n))_{n=1}^{\infty} \in A$. Since A is bounded above and

$((x_n, y_n, z_n))_{n=1}^{\infty}$ is in A, $(z_n)_{n=1}^{\infty}$ is bounded above. If $z = \limsup_n z_n$ exists, then there is a subsequence $(z_{n_k})_{k=1}^{\infty}$ of $(z_n)_{n=1}^{\infty}$ such that $z_{n_k} \xrightarrow{q^s} z$. Since $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ are q^s-convergent, $x_{n_k} \xrightarrow{q^s} x$ and $y_{n_k} \xrightarrow{q^s} y$ (a subsequence of convergent sequence converges to the same limit). Therefore

$$(x_{n_k}, y_{n_k}, z_{n_k}) \xrightarrow{p^s} (x, y, z)$$

and hence $((x_{n_k}, y_{n_k}, z_{n_k}))_{k=1}^{\infty}$ is left-K-Cauchy. Since A is left-K-sequentially complete,

$$(x_{n_k}, y_{n_k}, z_{n_k}) \xrightarrow{p} (a, b, c) \in A$$

and by Proposition 2.2.5 $(a, b, c) \geq (x, y, z)$, since

$$a \geq \limsup_n x_{n_k} = \lim_k x_{n_k} = \lim_n x_n = x$$

and similarly $b \geq y$. Therefore $(a, b) \geq (x, y)$. If $\limsup_n z_n$ does not exist, then by Lemma 3.3.13 there is a decreasing subsequence $(z_{n_k})_{k=1}^{\infty}$ of $(z_n)_{n=1}^{\infty}$ such that $z_{n_k} \to -\infty$, since $(z_n)_{n=1}^{\infty}$ is bounded above in $\mathbb{R}$. Since $(z_{n_k})_{k=1}^{\infty}$ is decreasing. Hence by Lemma 3.1.4, it is left-K-Cauchy. Therefore $((x_{n_k}, y_{n_k}, z_{n_k}))_{k=1}^{\infty}$ is left-K-Cauchy, since $(x_{n_k}, y_{n_k})_{k=1}^{\infty}$ is p^s-convergent, hence p^s-Cauchy, therefore left-K-Cauchy. Therefore, there is an $(a, b, c) \in A$ such that

$$(x_{n_k}, y_{n_k}, z_{n_k}) \xrightarrow{p} (a, b, c),$$

since A is left-K-sequentially complete. Hence as above, $a \geq x$ and $b \geq y$, so $(a, b, c) \geq (x, y, c)$. Hence $(a, b) \in P_{\{1,2\}}(A)$ and $(a_1, b_1) \geq (x, y)$ and so $P_{\{1,2\}}(A)$ has the B-property in $\mathbb{R}^2$. The proofs for $P_{\{2,3\}}(A)$ and $P_{\{1,3\}}(A)$ follow similarly. We next show that $P_{\{1\}}(A)$, $P_{\{2\}}(A)$ and $P_{\{3\}}(A)$ have the B-property in $\mathbb{R}$. Take $x \in \overline{P_{\{1\}}(A)}^{q^s}$, then there is a sequence $(x_n)_{n=1}^{\infty}$ in $P_{\{1\}}(A)$ such that $x_n \xrightarrow{q^s} x$. Since $(x_n)_{n=1}^{\infty}$ is in $P_{\{1\}}(A)$, there are sequences $(y_n)_{n=1}^{\infty}$ and $(z_n)_{n=1}^{\infty}$ in $\mathbb{R}$ such that $((x_n, y_n, z_n))_{n=1}^{\infty}$ is in A. Since A is bounded above and $((x_n, y_n, z_n))_{n=1}^{\infty}$ is in A, $(y_n)_{n=1}^{\infty}$ and $(z_n)_{n=1}^{\infty}$ are bounded above. We consider the following cases: Case 1. If $\limsup_n y_n$ exists, then there is a subsequence $(y_{n_k})_{k=1}^{\infty}$ of $(y_n)_{n=1}^{\infty}$

such that $y_{n_k} \xrightarrow{q^s} y = \limsup_n y_n$. Now consider the subsequence $(z_{n_k})_{k=1}^{\infty}$ of $(z_n)_{n=1}^{\infty}$. Suppose $z = \limsup_n z_{n_k}$ exists, then there is a subsequence $(z_{n_{k_l}})_{l=1}^{\infty}$ of $(z_{n_k})_{k=1}^{\infty}$ such that $z_{n_{k_l}} \xrightarrow{q^s} z$. Therefore, $(x_{n_{k_l}}, y_{n_{k_l}}, z_{n_{k_l}}) \xrightarrow{p^s} (x, y, z)$ and hence $((x_{n_{k_l}}, y_{n_{k_l}}, z_{n_{k_l}}))_{l=1}^{\infty}$ is left-K-Cauchy, since it is p^s-Cauchy. Now it follows as before, that for some $(a, b, c) \in A$,

$$(x_{n_{k_l}}, y_{n_{k_l}}, z_{n_{k_l}}) \xrightarrow{p} (a, b, c)$$

and $a \geq x$. Suppose that $\limsup_n z_{n_k}$ does not exist, then by Lemma 3.3.13, there is a decreasing subsequence $(z_{n_{k_l}})_{l=1}^{\infty}$ of $(z_{n_k})_{k=1}^{\infty}$ such that $z_{n_{k_l}} \to -\infty$. Hence by Lemma 3.1.4, $(z_{n_{k_l}})_{l=1}^{\infty}$ is left-K-Cauchy, since it decreasing. Therefore, $((x_{n_{k_l}}, y_{n_{k_l}}, z_{n_{k_l}}))_{l=1}^{\infty}$ is left-K-Cauchy, since $(x_{n_{k_l}}, y_{n_{k_l}})_{l=1}^{\infty}$ is p^s-convergent. Hence as above, $a \geq x$.

Case 2: If $\limsup_n y_n$ does not exist, then by Lemma 3.3.13, there is a decreasing subsequence $(y_{n_k})_{k=1}^{\infty}$ such that $y_{n_k} \to -\infty$ and hence by Lemma 3.1.4, $(y_{n_k})_{k=1}^{\infty}$ is left-K-Cauchy, since is decreasing. Now consider the subsequence $(z_{n_k})_{k=1}^{\infty}$ of $(z_n)_{n=1}^{\infty}$. If $z = \limsup_n z_{n_k}$ exists, then there is a subsequence $(z_{n_{k_l}})_{l=1}^{\infty}$ of $(z_{n_k})_{k=1}^{\infty}$ such that $z_{n_{k_l}} \xrightarrow{q^s} z$. Therefore $((x_{n_{k_l}}, y_{n_{k_l}}, z_{n_{k_l}}))_{l=1}^{\infty}$ is left-K-Cauchy, since $(x_{n_{k_l}}, z_{n_{k_l}})_{l=1}^{\infty}$ is p^s-convergent and by Lemma 3.1.4, $(y_{n_{k_l}})_{l=1}^{\infty}$ is left-K-Cauchy, it is decreasing. Then $a \geq x$ as above. If $\limsup_n z_{n_k}$ does not exists, then by Lemma 3.3.13, there is a decreasing subsequence $(z_{n_{k_l}})_{l=1}^{\infty}$ of $(z_{n_k})_{k=1}^{\infty}$ such that $z_{n_{k_l}} \to -\infty$. Hence by Lemma 3.1.4, $(z_{n_k})_{k=1}^{\infty}$ is left-K-Cauchy, since is decreasing, and therefore $((x_{n_{k_l}}, y_{n_{k_l}}, z_{n_{k_l}}))_{l=1}^{\infty}$ is left-K-Cauchy, since $(x_{n_{k_l}})_{l=1}^{\infty}$ is q^s-convergent. Similar procedure as above shows that $a \geq x$. The proofs for $P_{\{2\}}(A)$ and $P_{\{3\}}(A)$ are similar.

The converse is Proposition 3.3.12. $\square$

We have not succeeded in characterizing the left-K-sequentially complete subsets of $\mathbb{R}^2$ which are not bounded above. This could be an interesting question of further research. For our purpose of investigating p-compact sets in $\mathbb{R}^m$, we are only interested in the sets that are bounded above, since p-compact sets are bounded above. Below we give some sufficient conditions for a subset of $\mathbb{R}^2$ which is not bounded above to be left-K-sequentially complete.

Remark 3.3.15. *If $A \subseteq \mathbb{R}^2$ has the B-property in $\mathbb{R}^2$ and $I_A = \emptyset$, then A is left-K-sequentially complete.*

To see this, suppose $A \subseteq \mathbb{R}^2$ is not bounded above, has the B-property in $\mathbb{R}^2$ and $I_A = \emptyset$. Since $I_A = \emptyset$, $P_{\{1\}}(A)$ and $P_{\{2\}}(A)$ are not bounded above in $\mathbb{R}$, therefore they both have the B-property in $\mathbb{R}$. Since A has the B-property in $\mathbb{R}^2$, it follows from Proposition 3.3.12, that A is left-K-sequentially complete.

Proposition 3.3.16. *If $A \subseteq \mathbb{R}^2$ is not bounded above, has the B-property in $\mathbb{R}^2$ and $P_{\{i\}}(A)$ is bounded below for all $i \notin I_A$, then it is left-K-sequentially complete.*

Proof. Let A be a subset of $\mathbb{R}^2$ which has the B-property in $\mathbb{R}^2$ and is not bounded above. We consider the case $I_A \neq \emptyset$. The case for $I_A = \emptyset$ follows from Remark 3.3.15. Suppose $P_{\{1\}}(A)$ is bounded above and $P_{\{2\}}(A)$ is bounded below and not bounded above. Since $P_{\{2\}}(A)$ is not bounded above in $\mathbb{R}$, it has the B-property in $\mathbb{R}$. We now show that A is left-K-sequentially complete. To see this, take any left-K-Cauchy sequence $(\mathbf{x}_n)_{n=1}^{\infty}$ in A, then by Proposition 3.1.3, it is p-bounded and hence by Proposition 2.2.2, $(x_{n1})_{n=1}^{\infty}$ and $(x_{n2})_{n=1}^{\infty}$ are bounded above in $\mathbb{R}$. We first consider the left-K-Cauchy sequence $(x_{n2})_{n=1}^{\infty}$. Since $P_{\{2\}}(A)$ is bounded below in $\mathbb{R}$, it follows from Proposition 3.1.14, that $x_{n2} \xrightarrow{q^s} x_2$ (hence $x_{n2} \xrightarrow{q} x_2$) for some $x_2 \in \mathbb{R}$ and therefore $x_2 \in \overline{P_{\{2\}}(A)}^{q^s}$. Since $P_{\{2\}}(A)$ has the B-property in $\mathbb{R}$, there is a $y_2 \in P_{\{2\}}(A)$ such that $y_2 \geq x_2$ and hence there is a $y_1 \in \mathbb{R}$ such that $(y_1, y_2) \in A$. Therefore by Proposition 2.1.4, $x_{n2} \xrightarrow{q} y_2$. Now consider $(x_{n1})_{n=1}^{\infty}$. If $x_1 = \limsup_n x_{n1}$ exists, then by Corollary 3.1.9, $x_{n1} \xrightarrow{q^s} x_1$. Hence $(x_{n1}, x_{n2}) \xrightarrow{p^s} (x_1, x_2)$ and $(x_1, x_2) \in \overline{A}^{p^s}$. Since A has the B-property in $\mathbb{R}^2$, there is $(a, b) \in A$ such that $(a, b) \geq (x_1, x_2)$. Therefore by Proposition 2.1.4, $(x_{n1}, x_{n2}) \xrightarrow{p} (a, b) \in A$. If $\limsup_n x_{n1}$ does not exist, then by Proposition 2.2.5, $x_{n1} \xrightarrow{q} y_1 \in \mathbb{R}$. Hence $(x_{n1}, x_{n2}) \xrightarrow{p} (y_1, y_2) \in A$. Thus A is left-K-sequentially complete. $\square$

We show that there is a left-K-sequentially complete subset A of $\mathbb{R}^2$ which has the B-property in $\mathbb{R}^2$ and $P_{\{2\}}(A)$ has the B-property in $\mathbb{R}$ but $P_{\{1\}}(A)$ does not have the B-property in $\mathbb{R}$.

Example 3.3.17. *Let*

$$A = \left\{(x, y) \in \mathbb{R}^2 : x < 0, y = -\frac{1}{x}\right\}.$$

Then A is p^s-closed, since it is the graph of a p^s-continuous function. Therefore it has the B-property in $\mathbb{R}^2$, and $P_{\{2\}}(A) = (0, \infty)$ has the B-property in $\mathbb{R}$, since $P_{\{2\}}(A)$ is not bounded above. But $P_{\{1\}}(A) = (-\infty, 0)$ does not have the B-property in $\mathbb{R}$, since $0 \in \overline{P_{\{1\}}(A)}^{q^s}$ and there is no $x \in P_{\{1\}}(A)$ such that $x \geq 0$. Therefore, by Proposition 3.3.16 A is left-K-sequentially complete.

3.4 Left-K-sequential completeness for some infinite dimensional spaces

In this section we show that some classical normed lattices are left$K$-sequentially complete and give an example to show that not all normed lattices are left-K-sequentially complete.

Let Γ be a non-empty set and $B(\Gamma)$ the set of all bounded functions $f : \Gamma \to \mathbb{R}$, with the norm $\|f\| = \sup_{x\in\Gamma} |f(x)|$. Then $B(\Gamma)$ is a normed lattice with the pointwise ordering. Let Ω be a compact Hausdorff topological space. We denote by $C(\Omega)$ the vector lattice of all real-valued continuous functions on Ω with pointwise ordering. We consider the supremum norm on this space.

Proposition 3.4.1. *Let Γ be an arbitrary set and Ω a compact Hausdorff space. If (X, p) is one of the asymmetrically normed lattices $(B(\Gamma), p)$ or $(C(\Omega), p)$, then every p-bounded sequence in X is bounded above.*

Proof. Take any p-bounded sequence $(f_n)_{n=1}^{\infty}$ in $B(\Gamma)$. Then there is $m \in \mathbb{R}^+$ such that

$$p(f_n) = \sup_{x \in \Gamma} f_n^+(x) \leq m \text{ for all } n \in \mathbb{N}.$$

This implies that $f_n^+(x) \leq m$ for all $n \in \mathbb{N}$. Hence $f_n(x) \leq m$ for all $n \in \mathbb{N}$. Thus $(f_n(x))_{n=1}^{\infty}$ is bounded above in $B(\Gamma)$, since the function g defined by $g(x) = m$ for all $x \in \Gamma$ is in $B(\Gamma)$. $\square$

Proposition 3.4.2. *([13, Corollary 3.12]) The asymmetrically normed lattices $(C(\Omega), p)$, $(C(\Omega), p^{-1})$, (ℓ_∞, p) and (ℓ_∞, p^{-1}) are left-K-sequentially complete.*

Proof. The statement follows from Proposition 3.4.1 and Theorem 3.2.13, since if $X = \mathbb{N}$, then $B(X) = \ell_\infty$. $\square$

Recall that we denote by c the set of all real convergent sequences, and that it is a closed subspace of ℓ_∞ when the latter is equipped with its usual supremum norm.

Corollary 3.4.3. *Let $X = \ell_\infty$ or $X = c$. Then any p-closed subset A of X is left-K-sequentially complete.*

Proof. We have $\ell_\infty = B(\mathbb{N})$. Furthermore, the space c may be identified with $C(\Omega)$, where Ω is the compact subset of $\mathbb{R}$ given by

$$\Omega = \left\{ \frac{1}{n} : n \in \mathbb{R} \right\} \cup \{0\}.$$

To see this, for $\mathbf{x} \in c$ define $f_{\mathbf{x}} : \Omega \to \mathbb{R}$ by $f_{\mathbf{x}}(\frac{1}{n}) = x_n$, $n \in \mathbb{N}$, and $f_{\mathbf{x}}(0) = \lim_n x_n$. Then $f_{\mathbf{x}} \in C(\Omega)$ and the correspondence $\mathbf{x} \mapsto f_{\mathbf{x}}$ is an isometric and order isomorphism between the Banach lattices c and $C(\Omega)$. The result now follows from Proposition 3.2.12 and Theorem 3.2.13. $\square$

The asymmetrically normed lattices (c_0, p) and (c, p) are not p-closed subspaces of (ℓ_∞, p). This follows at once from the fact that $\ell_\infty \setminus c$ contains positive elements, and that neither c nor c_0 are therefore increasing. Therefore we cannot use Proposition 3.2.12 and Theorem 3.2.13 to conclude that c_0 and c are left-K-sequentially complete.

Note that a p-bounded sequence in the asymmetrically normed lattice (c_0, p) is not necessarily bounded above in c_0 as the following example shows. Therefore we cannot use Theorem 3.2.13 to conclude that c_0 is left-K-sequentially complete.

Example 3.4.4. *Consider the sequence $(\mathbf{x}_n)_{n=1}^{\infty}$ in c_0^+, with $\mathbf{x}_n$ the sequence with the first n terms equal to 1, and all other terms equal to 0, then $p(\mathbf{x}_n) = 1$ for all $n \in \mathbb{N}$, but $(\mathbf{x}_n)_{n=1}^{\infty}$ is not bounded above in c_0. For suppose $\mathbf{x} = (x_k) \in c_0$, $\mathbf{x}_n \leq \mathbf{x}$, for all $n \in \mathbb{N}$, then $x_k \geq 1$ for all $k \in \mathbb{N}$. Hence $\lim_k x_k \neq 0$, a contradiction.*

Lemma 3.4.5. *([13, Lemma 3.13]) If $(\mathbf{x}_m)_{m=1}^{\infty}$ is a left-K-Cauchy sequence in c_0, then*

$$\sup_{m \in \mathbb{N}} \sup_{i \geq n} x_{mi}^+ \to 0 \text{ as } n \to \infty.$$

Proof. Take any left-K-Cauchy sequence $(\mathbf{x}_m)_{m=1}^{\infty}$ in c_0. We show that for every $\epsilon > 0$, there is $i_\epsilon \in \mathbb{N}$ such that for all $m \in \mathbb{N}$, for every $i \in \mathbb{N}$,

$$i \geq i_\epsilon \Rightarrow x_{mi}^+ < \epsilon.$$

Since $(\mathbf{x}_m)_{m=1}^{\infty}$ is a left-K-Cauchy sequence, there is an $n_\epsilon \in \mathbb{N}$ such that for all $m \geq n \geq n_\epsilon$

$$p(\mathbf{x}_m - \mathbf{x}_n) = \sup_{i \in \mathbb{N}} (x_{mi} - x_{ni})^+ < \frac{\epsilon}{2}.$$

In particular for $m \geq n_\epsilon$,

$$p(\mathbf{x}_m - \mathbf{x}_{n_\epsilon}) = \sup_{i \in \mathbb{N}} (x_{mi} - x_{n_\epsilon i})^+ < \frac{\epsilon}{2}.$$

Since $(x_{n_\epsilon i}) \in c_0$, there is $i_\epsilon \in \mathbb{N}$ such that $|x_{n_\epsilon i}| < \frac{\epsilon}{2}$ for all $i \geq i_\epsilon$, hence $x_{n_\epsilon i}^+ < \frac{\epsilon}{2}$

for all $i \geq i_\epsilon$. Therefore for $m \geq n_\epsilon$ and $i \geq i_\epsilon$,

$$\begin{aligned} x_{mi}^+ &\leq (x_{mi} - x_{n_\epsilon i})^+ + x_{n_\epsilon i}^+ \\ &\leq \sup_{i\in\mathbb{N}}(x_{mi} - x_{n_\epsilon i})^+ + x_{n_\epsilon i}^+ \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

If $1 \leq m < n_\epsilon$, then there is $i_\epsilon^m \in \mathbb{N}$ such that $|x_{mi}| < \frac{\epsilon}{2}$ for $i \geq i_\epsilon^m$, since $(x_{mi}) \in c_0$. Let $i'_\epsilon = \max_{1\leq m<n_\epsilon} i_\epsilon^m$ and $j_\epsilon = \max\{i'_\epsilon, i_\epsilon\}$. If $i \geq i_\epsilon$, then $x_{mi}^+ \leq |x_{mi}| < \epsilon$. □

Theorem 3.4.6. *([13, Theorem 3.14]) The asymmetrically normed lattice (c_0, p) is left-K-sequentially complete.*

Proof. If $(\mathbf{x}_n)_{n=1}^\infty$ is a left-K-Cauchy sequence in c_0 and $\epsilon > 0$, then there is an $n_\epsilon \in \mathbb{N}$ such that for all $n \geq m \geq n_\epsilon$

$$p(\mathbf{x}_n - \mathbf{x}_m) = \sup_{i\in\mathbb{N}}(x_{ni} - x_{mi})^+ < \frac{\epsilon}{2}. \tag{3.1}$$

For each $i \in \mathbb{N}$, $(x_{ni})_{n=1}^\infty$ is a left-K-Cauchy sequence in $\mathbb{R}$. By Corollary 3.1.11, $(x_{ni})_{n=1}^\infty$ is either bounded below or $\limsup_n x_n$ exists. If $(x_{ni})_{n=1}^\infty$ is bounded below, then by Proposition 3.1.14, $(x_{ni})_{n=1}^\infty$ is q^s-convergent. We form a sequence $\mathbf{x} = (x_i)$, where $x_i = y_i^+$ and

$$y_i = \begin{cases} q^s - \lim_{n\to\infty} x_{ni}, & \text{if } (x_{ni})_{n=1}^\infty \text{ is bounded below} \\ 0 \text{ otherwise.} \end{cases}$$

Let $A = \{i \in \mathbb{N} : (x_{ni})_{n=1}^\infty \text{ is bounded below}\}$. If $i \in A$, $x_{ni} \xrightarrow{q^s} y_i$, and so $x_{ni} \xrightarrow{q} y_i$. If $i \in \mathbb{N} \setminus A$, then by Lemma 3.1.13, $x_{ni} \xrightarrow{q} x$ for any $x \in \mathbb{R}$, in particular, $x_{ni} \xrightarrow{q} 0 = y_i$. Hence $x_{ni} \xrightarrow{q} y_i$ for every $i \in \mathbb{N}$. Since $x_i = y_i^+ \geq y_i$, $x_{ni} \xrightarrow{q} x_i$ for every $i \in \mathbb{N}$ (Proposition 2.1.4). We show that $(x_i) \in c_0$. If A is finite, x_i is eventually 0, and so $(x_i) \in c_0$. If A is infinite, we may write

$A = \{i_k : k \in \mathbb{N}\}$, with (i_k) a strictly increasing sequence in $\mathbb{N}$. Since $x_{ni_k} \xrightarrow{q^s} y_{i_k}$ as $n \to \infty$ for every k, it follows from (3.1) and the q^s-continuity of the map $y \mapsto y^+$ that

$$(y_{i_k} - x_{n_\epsilon i_k})^+ \leq \frac{\epsilon}{2} \text{ for all } k \in \mathbb{N}.$$

Since $(x_{n_\epsilon i_k})_{k=1}^{\infty} \in c_0$, there is an $m_\epsilon \in \mathbb{N}$ such that

$$x^+_{n_\epsilon i_k} \leq |x_{n_\epsilon i_k}| < \frac{\epsilon}{2} \text{ for } k \geq m_\epsilon.$$

If $i \geq i_{m_\epsilon}$ and $i \in A$, then $i = i_k$ for some k, and $k \geq m_\epsilon$. Therefore

$$|x_i| = x_{i_k} = y^+_{i_k} \leq (y_{i_k} - x_{n_\epsilon i_k})^+ + x^+_{n_\epsilon i_k} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

If $i \geq i_{m_\epsilon}$ and $i \notin A$, $x_i = 0$. Hence $|x_i| < \epsilon$ and so $|x_i| < \epsilon$ for all $i \geq i_{m_\epsilon}$, showing that $(x_i) \in c_0$. Since $(\mathbf{x}_n)_{n=1}^{\infty}$ is a left-K-Cauchy sequence, it follows from Lemma 3.4.5 that

$$\sup_{n \in \mathbb{N}} \sup_{i \geq m} x^+_{ni} \to 0 \text{ as } m \to \infty.$$

Therefore, by Proposition 2.3.2,

$$\mathbf{x}_n \xrightarrow{p} \mathbf{x} \text{ as } n \to \infty.$$

Consequently (c_0, p) is left-K-sequentially complete. $\square$

The proof of Theorem 3.4.6 also shows that:

Corollary 3.4.7. *If a left-K-Cauchy sequence $(\mathbf{x}_n)_{n=1}^{\infty}$ in (c_0, p) is positive, then $\mathbf{x}_n \xrightarrow{p} \mathbf{x}$, where $\mathbf{x} = (x_i)$ with $x_i = q^s - \lim_{n \to \infty} x_{ni}$ for every $i \in \mathbb{N}$.*

Lemma 3.4.8. *([13, Proposition 3.15]) If $(\mathbf{x}_m)_{m=1}^{\infty}$ is a left-K-Cauchy sequence in ℓ_p, $1 \leq p < \infty$, then*

$$\sup_{m \in \mathbb{N}} \sum_{k=n}^{\infty} (x^+_{mk})^p \to 0 \text{ as } n \to \infty.$$

Proof. Take any left-K-Cauchy sequence $(\mathbf{x}_m)_{m=1}^{\infty}$ in ℓ_p. We show that for every $\epsilon > 0$, there is $k_\epsilon \in \mathbb{N}$ such that for all $m \in \mathbb{N}$, for every $k \geq k_\epsilon$,

$$\sum_{i \geq k_\epsilon} (x_{mi}^{+})^p < 2\epsilon^p.$$

Since $(\mathbf{x}_m)_{m=1}^{\infty}$ is a left-K-Cauchy sequence, for every $\epsilon > 0$, there is $n_\epsilon \in \mathbb{N}$ such that for all $m \geq n \geq n_\epsilon$

$$p(\mathbf{x}_m - \mathbf{x}_n) = (\sum_{i \in \mathbb{N}} ((x_{mi} - x_{ni})^{+})^p)^{\frac{1}{p}} < \frac{\epsilon}{2}.$$

In particular for $m \geq n_\epsilon$

$$\sum_{i \in \mathbb{N}} ((x_{mi} - x_{n_\epsilon i})^{+})^p < (\frac{\epsilon}{2})^p.$$

Since $(x_{n_\epsilon i}) \in \ell_p$, there is a $i_\epsilon \in \mathbb{N}$ such that $\sum_{i \geq i_\epsilon} |x_{n_\epsilon i}|^p < (\frac{\epsilon}{2})^p$, hence $\sum_{i \geq i_\epsilon} (x_{n_\epsilon i}^{+})^p < (\frac{\epsilon}{2})^p$. Therefore for $m \geq n_\epsilon$,

$$\begin{aligned}
\sum_{i \geq k_\epsilon} (x_{mi}^{+})^p &\leq 2^p(\sum_{i \geq k_\epsilon} ((x_{mi} - x_{n_\epsilon i})^{+})^p + \sum_{k \geq k_\epsilon} (x_{n_\epsilon i}^{+})^p) \\
&< 2^p((\frac{\epsilon}{2})^p + (\frac{\epsilon}{2})^p) \\
&= 2\epsilon^p.
\end{aligned}$$

If $1 \leq m < n_\epsilon$, then there is $i_\epsilon^m \in \mathbb{N}$ such that

$$\sum_{i \geq i_\epsilon^m} (x_{mi}^{+})^p < (\frac{\epsilon}{2})^p \leq \sum_{i \geq i_\epsilon^m} |x_{mi}|^p < (\frac{\epsilon}{2})^p,$$

since $(x_{mi}) \in \ell_p$. Let $i'_\epsilon = \max_{1 \leq m < n_\epsilon} i_\epsilon^m$ and $k_\epsilon = \max\{i'_\epsilon, i_\epsilon\}$, then for all $m \in \mathbb{N}$,

$$\sum_{i \geq k_\epsilon} (x_{mi}^{+})^p \leq \sum_{i \geq k_\epsilon} |x_{mi}|^p \leq \max\{2\epsilon^p, (\frac{\epsilon}{2})^p\} \leq 2\epsilon^p.$$

□

Theorem 3.4.9. *([13, Proposition 3.16]) The sequence space (ℓ_p, p) is left-K-sequentially complete.*

Proof. Take any left-K-Cauchy sequence $(\mathbf{x}_n)_{n=1}^{\infty}$ in ℓ_p and let $\epsilon > 0$. There is an $n_\epsilon \in \mathbb{N}$ such that for all $n \geq m \geq n_\epsilon$

$$p(\mathbf{x}_n - \mathbf{x}_m) = \left(\sum_{i \in \mathbb{N}} ((x_{ni} - x_{mi})^+)^p \right)^{\frac{1}{p}} < \epsilon. \tag{3.2}$$

For each $i \in \mathbb{N}$, $(x_{ni})_{n=1}^{\infty}$ is a left-K-Cauchy sequence in $\mathbb{R}$. By Corollary 3.1.11, $(x_{ni})_{n=1}^{\infty}$ is either bounded below or $\lim_n \sup x_n$ exists. If $(x_{ni})_{n=1}^{\infty}$ is bounded below, then by Proposition 3.1.14, $(x_{ni})_{n=1}^{\infty}$ is q^s-convergent. We form a sequence $\mathbf{x} = (x_i)$, where $x_i = y_i^+$ and

$$y_i = \begin{cases} q^s - \lim_{n\to\infty} x_{ni}, & \text{if } (x_{ni})_{n=1}^{\infty} \text{ is bounded below} \\ 0 \text{ otherwise.} \end{cases}$$

Let

$$A = \{i \in \mathbb{N} : (x_{ni})_{n=1}^{\infty} \text{ is bounded below}\}.$$

If $i \in A$, $x_{ni} \xrightarrow{q^s} y_i$, and so $x_{ni} \xrightarrow{q} y_i$. If $i \in \mathbb{N} \setminus A$, then by Lemma 3.1.13, $x_{ni} \xrightarrow{q} x$ for any $x \in \mathbb{R}$, in particular, $x_{ni} \xrightarrow{q} 0 = y_i$. Hence $x_{ni} \xrightarrow{q} y_i$ for every $i \in \mathbb{N}$. Since $x_i = y_i^+ \geq y_i$, $x_{ni} \xrightarrow{q} x_i$ for every $i \in \mathbb{N}$ (Proposition 2.1.4). If A is finite, clearly $\sum_{i=1}^{\infty} |x_i|^p < \infty$, therefore $(x_i) \in \ell_p$. If A is infinite, let

$$A = \{i_k : k \in \mathbb{N}\},$$

with (i_k) strictly increasing. For every $m \in \mathbb{N}$,

$$\left(\sum_{k=1}^{m} ((x_{ni_k} - x_{n_\epsilon i_k})^+)^p \right)^{\frac{1}{p}} \xrightarrow{q^s} \left(\sum_{k=1}^{m} ((y_{i_k} - x_{n_\epsilon i_k})^+)^p \right)^{\frac{1}{p}} \text{ as } n \to \infty, \tag{3.3}$$

since the maps

$$x \mapsto x^+, \; x \mapsto x^p \text{ and } x \mapsto x^{\frac{1}{p}}, \text{ and addition are } q^s\text{-continuous}$$

and $x_{ni} \xrightarrow{q^s} x_i$, and

$$\begin{aligned}\left(\sum_{i=1}^{i_m}|x_i|^p\right)^{\frac{1}{p}} &= \left(\sum_{k=1}^{m}|x_{i_k}|^p\right)^{\frac{1}{p}} \\ &= \left(\sum_{k=1}^{m}(y_{i_k}^+)^p\right)^{\frac{1}{p}} \\ &\leq \left(\sum_{k=1}^{m}((y_{i_k} - x_{n_\epsilon i_k})^+)^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{m}(x_{n_\epsilon i_k}^+)^p\right)^{\frac{1}{p}} \\ &\leq \epsilon + \|\mathbf{x}_{n_\epsilon}\|_p,\end{aligned}$$

using (3.2) and (3.3). It follows that any partial sum of $\sum_{i=1}^{\infty}|x_i|^p$ is bounded above by $(\epsilon + \|\mathbf{x}_{n_\epsilon}\|_p)^p$, and hence $(x_i) \in \ell_p$. Since $\mathbf{x} \in \ell_p$, it follows from Lemma 3.4.8 and Proposition 2.3.3 that $\mathbf{x}_n \xrightarrow{p} \mathbf{x}$. Thus (ℓ_p, p) is left-K-sequentially complete. □

The proof of Theorem 3.4.9 also shows that:

Corollary 3.4.10. *If a left-K-Cauchy sequence $(\mathbf{x}_n)_{n=1}^{\infty}$ in (ℓ_p, p) is positive, then $\mathbf{x}_n \xrightarrow{p} \mathbf{x}$, where $\mathbf{x} = (x_i)$ with $x_i = q^s - \lim_{n\to\infty} x_{ni}$ for every $i \in \mathbb{N}$.*

Not every asymmetrically normed lattice (X, p) is left-K-sequentially complete, as the following example shows. Recall that the space ℓ_0 of all real sequences with at most a finite number of non-zero terms is not complete with respect to the supremum norm.

Example 3.4.11. *([13, Example 3.18]) The space (ℓ_0, p), where $p((x_n)_{n=1}^{\infty}) = \sup_{n\in\mathbb{N}} x_n^+$ is not left-K-sequentially complete. We consider the sequence $(\mathbf{x}_n)_{n=1}^{\infty}$ in ℓ_0 with the n-term given by*

$$\mathbf{x}_n = \left(1, \frac{1}{2}, \frac{1}{2^2}, ..., \frac{1}{2^{n-1}}, 0, 0, ...\right).$$

This sequence is convergent in (ℓ_∞, p^s), so it is p^s-Cauchy, hence left-K-Cauchy by Proposition 3.1.6. If $\mathbf{x}_n \xrightarrow{p} \mathbf{x}$, then by Proposition 2.3.1, $x_{ni} \xrightarrow{q} x_i$ for every

$i \in \mathbb{N}$. Therefore by Lemma 2.2.5,

$$x_i \geq \limsup_n x_{ni} = 2^{-(i-1)} > 0 \textit{ for every } i \in \mathbb{N}.$$

Therefore $\mathbf{x} \notin \ell_0$ for every limit $\mathbf{x}$ of $(\mathbf{x}_n)_{n=1}^{\infty}$. Thus ℓ_0 is not left-K-sequentially complete.

In [37, Example 3], it was shown by an example that if a quasi-metric space (X, d) is left-K-sequentially complete, then (X, d^{-1}) need not be right-K-sequentially complete. We give an example to show that the same can happen in an asymmetrically normed lattice.

Example 3.4.12. *([13, Example 3.17]) Let $X = \mathbb{R}$. Then $(\mathbb{R}, q)$ is left-K-sequentially complete (see Theorem 3.3.1). But the space $(\mathbb{R}, q^{-1})$ is not right-K-sequentially complete. Let $x_n = (-n)_{n=1}^{\infty}$ in $\mathbb{R}$, then $(x_n)_{n=1}^{\infty}$ is right-K-Cauchy in $(\mathbb{R}, q^{-1})$. But by Corollary 2.2.7, it is not q^{-1}-convergent, since it is not bounded below. Hence $(\mathbb{R}, q^{-1})$ is not right-K-sequentially complete. A similar argument shows that $(\mathbb{R}, q)$ is not right-K-sequentially complete.*

In [1, Corollary 6], it is claimed that for an asymmetrically normed lattice (X, p), neither (X, p) nor (X, p^{-1}) is left-K-sequentially complete. However, we have shown that there are asymmetrically normed lattices X such that (X, p) and (X, p^{-1}) are left-K-sequentially complete (see Corollary 3.4.2, Theorem 3.4.6 and Theorem 3.4.9). The proof in [1] uses the fact that if (X, p^{-1}) is right-K-sequentially complete, then (X, p) is of second category in itself (Theorem 3.2.3) and the fact that (X, p) is not of second category in itself (Proposition 3.2.8) to conclude that (X, p^{-1}) is not right-K-sequentially complete. However, it does not follow from this that (X, p) is not left-K-sequentially complete, as Example 3.4.12 shows.

3.5 Smyth completeness in asymmetrically normed lattices

In this section we study the Smyth completeness of positive cones of various asymmetrically normed lattices.

Definition 3.5.1. *([24, p. 8]) A subset A of an asymmetrically normed space (X,p) is* **Smyth complete** *if and only if every left-K-Cauchy sequence $(x_n)_{n=1}^{\infty}$ in A has a p^s-limit point in A.*

Remark 3.5.2. *If A is a p^s-closed bounded below subset of $\mathbb{R}^m$, then A is Smyth complete.*

This statement follows from the proof of Proposition 3.3.5.

Corollary 3.5.3. *The cone $((\mathbb{R}^m)^+, p)$ is Smyth complete.*

Proposition 3.5.4. *The cone (c_0^+, p) is Smyth complete.*

Proof. Take any left-K-Cauchy sequence $(\mathbf{x}_n)_{n=1}^{\infty}$ in c_0^+. For every $\epsilon > 0$, there is $n_\epsilon \in \mathbb{N}$ such that

$$p(\mathbf{x}_n - \mathbf{x}_m) = \sup_{i\in\mathbb{N}}(x_{ni} - x_{mi})^+ < \epsilon \text{ for all } n \geq m \geq n_\epsilon. \tag{3.4}$$

For each $i \in \mathbb{N}$, $(x_{ni})_{n=1}^{\infty}$ is a left-K-Cauchy sequence in $\mathbb{R}$, and $(x_{ni})_{n=1}^{\infty}$ is bounded below by 0. Hence by Proposition 3.1.14, it is q^s-convergent, say $x_{ni} \xrightarrow{q^s} x_i$ as $n \to \infty$. Let $\mathbf{x} = (x_i)$. Then by Corollary 3.4.7, $\mathbf{x}_n \xrightarrow{p} \mathbf{x} \in c_0^+$, since $(\mathbf{x}_n)_{n=1}^{\infty}$ is in c_0^+. We show that $\mathbf{x}_n \xrightarrow{p^{-1}} \mathbf{x}$. In (3.4), let $m \geq n_\epsilon$ and let $n \to \infty$, then

$$\sup_{i\in\mathbb{N}}(x_i - x_{mi})^+ \leq \epsilon,$$

since the map

$$x \mapsto x^+ \text{ is } q^s\text{-continuous and } x_{ni} \xrightarrow{q^s} x_i \text{ as } n \to \infty.$$

Thus

$$p(\mathbf{x} - \mathbf{x}_m) = \sup_{i \in \mathbb{N}} (x_i - x_{mi})^+ \leq \epsilon \text{ for } m \geq n_\epsilon.$$

Thus $\mathbf{x}_m \xrightarrow{p^{-1}} \mathbf{x}$. Therefore $\mathbf{x}_n \xrightarrow{p^s} \mathbf{x}$. Hence (c_0^+, p) is Smyth complete. $\square$

Proposition 3.5.5. *The cone (ℓ_p^+, p) is Smyth complete.*

Proof. Take any left-K-Cauchy sequence $(\mathbf{x}_n)_{n=1}^\infty$ in ℓ_p^+. For every $\epsilon > 0$, there is $n_\epsilon \in \mathbb{N}$ such that

$$p(\mathbf{x}_n - \mathbf{x}_m) = \left(\sum_{i=1}^{\infty} ((x_{ni} - x_{mi})^+)^p \right)^{\frac{1}{p}} < \epsilon \text{ for all } n \geq m \geq n_\epsilon. \tag{3.5}$$

For each $i \in \mathbb{N}$, $(x_{ni})_{n=1}^\infty$ is a left-K-Cauchy sequence in $\mathbb{R}$, and it is bounded below by 0. Hence by Proposition 3.1.14, it is q^s-convergent, say $x_{ni} \xrightarrow{q^s} x_i$ as $n \to \infty$. Let $\mathbf{x} = (x_i)$. Then by Corollary 3.4.10, $\mathbf{x}_n \xrightarrow{p} \mathbf{x} \in \ell_p^+$, since $(\mathbf{x}_n)_{n=1}^\infty$ is in ℓ_p^+. We show that $\mathbf{x}_n \xrightarrow{p^{-1}} \mathbf{x}$. To end this, for each $k \in \mathbb{N}$, we have

$$\left(\sum_{i=1}^{k} ((x_{ni} - x_{mi})^+)^p \right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{\infty} ((x_{ni} - x_{mi})^+)^p \right)^{\frac{1}{p}} < \epsilon,$$

that is,

$$\sum_{i=1}^{k} ((x_{ni} - x_{mi})^+)^p < \epsilon^p \text{ for all } k \in \mathbb{N}.$$

For $k \in \mathbb{N}$ and $m \geq n_\epsilon$, let $n \to \infty$, then

$$\sum_{i=1}^{k} ((x_i - x_{mi})^+)^p \leq \epsilon^p,$$

since the maps

$$x \mapsto x^+ \text{ and } x \mapsto x^p, \text{ and addition are } q^s\text{-continuous}$$

and $x_{ni} \xrightarrow{q^s} x_i$. Now let $k \to \infty$, then for all $m \geq n_\epsilon$,

$$\sum_{i=1}^{\infty} ((x_i - x_{mi})^+)^p \leq \epsilon^p.$$

Thus

$$p(\mathbf{x} - \mathbf{x}_m) = \left(\sum_{i=1}^{\infty} ((x_i - x_{mi})^+)^p \right)^{\frac{1}{p}} \leq \epsilon \text{ for } m \geq n_\epsilon.$$

Thus $\mathbf{x}_m \xrightarrow{p^{-1}} \mathbf{x}$. Therefore $\mathbf{x}_n \xrightarrow{p^s} \mathbf{x}$. Hence (ℓ_p^+, p) is Smyth complete. $\square$

Examples 3.5.6. *The positive cones of c, ℓ_∞ and $C([0,1])$ are not Smyth complete.*

(a) *Consider the sequence $(\mathbf{x}_n)_{n=1}^{\infty}$ in c^+, with $\mathbf{x}_n$ the sequence with the first $n-1$ terms equal to 0, and all other terms equal to 2. Then by Lemma 3.1.4, $(\mathbf{x}_n)_{n=1}^{\infty}$ is a left-K-Cauchy sequence, since it is decreasing. But $(\mathbf{x}_n)_{n=1}^{\infty}$ is not p^s-Cauchy. To see this, let $n > m$ and $\epsilon = 1$, then*

$$p^s(\mathbf{x}_n - \mathbf{x}_m) = 2 > 1.$$

Therefore, $(\mathbf{x}_n)_{n=1}^{\infty}$ is not p^s-Cauchy, therefore not p^s-convergent. Thus (c^+, p) is not Smyth complete. The same sequence shows that (ℓ_∞^+, p) is not Smyth complete.

(b) *Take the sequence $(f_n)_{n=1}^{\infty}$ in $C^+([0,1])$, where f_n is defined by*

$$f_n(x) = \begin{cases} 1 - nx, & 0 \leq x < \frac{1}{n} \\ 0, & \frac{1}{n} \leq x \end{cases}$$

Then by Lemma 3.1.4, (f_n) is a left-K-Cauchy sequence, since it is decreasing and (f_n) converges pointwise to the function f defined by

$$f(x) = \begin{cases} 1, & x = 0 \\ 0, & x > 0. \end{cases}$$

Since the limit function f is not continuous, (f_n) does not converge uniformly to f on $[0,1]$. Therefore $p^s(f_n - f) \not\to 0$ as $n \to \infty$. Thus f_n does not p^s-converge. Hence $C^+([0,1])$ is not Smyth complete.

Chapter 4

Compactness in asymmetrically normed lattices

In this chapter, we investigate the p-compact and p-precompact subsets of an asymmetrically normed lattice (X, p). We first recall some known results and definitions about p-compactness and p-precompactness in asymmetrically normed spaces and mention their counterparts in normed spaces. We show that any subset of an asymmetrically normed lattice which contains its supremum is compact and we characterize the precompact subsets of a finite dimensional space. In general outside precompactness and precompactness are not equivalent in an asymmetrically normed space. We show that in finite dimensional asymmetrically normed lattices the two notions are equivalent.

We investigate the relationship between almost order boundedness, outside precompactness and precompactness in asymmetrically normed lattices. We show that in any asymmetrically normed lattice a subset is outside-p-precompact if and only if it is almost order bounded above and we show that for a subset of finite dimensional asymmetrically normed lattice, the three notions are equivalent.

In the final section we investigate the relationship between p-compact and strongly p-compact sets in finite dimensional asymmetrically normed lattices.

4.1 p-compact and p-precompact sets in asymmetrically normed spaces

In this section, we recall some known results and definitions about p-compactness and p-precompactness in asymmetrically normed spaces and mention their counterparts in normed spaces.

Definition 4.1.1. *([2, Definition 3]) A subset A of an asymmetrically normed space (X, p) is p-***precompact** *if for every $\epsilon > 0$ we can find a finite set of points $\{a_1, ..., a_n\}$ in A such that $A \subseteq \bigcup_{i=1}^{n} B_\epsilon^p(a_i)$.*

It is well known that if p is a norm, we obtain an equivalent definition if we only require the finite set of points to be in X, rather than in A. This is no longer the case if p is not a norm (see Example 4.2.2). This necessitates the following definition:

Definition 4.1.2. *([2, Definition 3]) A subset A of an asymmetrically normed space (X, p) is* **outside-p-precompact** *if for every $\epsilon > 0$, there is a finite set $\{x_1, ...x_n\}$ in X such that $A \subseteq \bigcup_{i=1}^{n} B_\epsilon^p(x_i)$.*

It is clear that in an asymmetrically normed space p-precompactness implies outside-p-precompactness. The converse is not true in general (as we shall see in Example 4.2.2). The next result clarifies the relationship between the two concepts:

Proposition 4.1.3. *([2, Proposition 4]) A subset A of an asymmetrically normed space (X, p) is p-precompact if and only if for every $\epsilon > 0$ there is a finite set*

$\{x_1, x_2, ..., x_n\}$ *in* X *such that*

$$A \subseteq \bigcup_{n=1}^{\infty} B^p_\epsilon(x_i) \text{ and } B^{p^{-1}}_\epsilon(x_i) \cap A \neq \emptyset \text{ for all } 1 \leq i \leq n.$$

In a normed space a set is precompact if and only its closure is precompact. This is no longer the case in an asymmetrically normed space (see Example 4.3.7). But we do have the following result:

Proposition 4.1.4. *([2, Proposition 9]) A subset* A *of an asymmetrically normed space* (X, p) *is* p*-precompact if and only if the* p^{-1}*-closure of* A *is* p*-precompact.*

The following propositions list some properties of precompact sets in normed spaces which continue to hold in asymmetrically normed spaces.

Proposition 4.1.5. *([2, Proposition 8]) Let* (X, p) *be an asymmetrically normed space.*

(a) The finite sum and the finite union of p*-precompact sets is* p*-precompact.*

(b) The convex hull of a p*-precompact set is* p*-precompact.*

Proposition 4.1.6. *([12, Proposition 1.2.19(2)]) If a sequence* $(x_n)_{n=1}^{\infty}$ *in an asymmetrically normed space* (X, p) *is* p*-convergent, then* $\{x_n : n \in \mathbb{N}\}$ *is outside-*p*-precompact. If* $x_n \xrightarrow{p} x \in X$*, then* $\{x_n : n \in \mathbb{N}\} \cup \{x\}$ *is* p*-precompact.*

In a normed space, a subset of a precompact set is precompact, but this need not be the case in an asymmetrically normed space (see Example 4.2.4). However, a subset of an outside-p-precompact set is outside-p-precompact. We also have the following:

Proposition 4.1.7. *([2, Corollary 10]) Let* A *and* B *be two subsets of an asymmetrically normed space* (X, p) *such that* $A \subseteq B$ *and* B *is* p*-precompact. If* A *is* p^{-1}*-dense in* B*, then* A *is* p*-precompact.*

The above condition is a sufficient condition but not a necessary condition (as we shall see in Example 4.3.8).

Definition 4.1.8. *([39, p. 80]) A topological space X is called* **compact** *if every open cover of X has a finite subcover.*

We now mention some properties of compact sets in normed spaces which continue to hold in asymmetrically normed spaces.

Proposition 4.1.9. *([2, p. 530]) If a subset of an asymmetrically normed space (X, p) is p-compact, then it is p-bounded. A p-closed subset of a p-compact set is p-compact.*

Proposition 4.1.10. *([24, p. 849]) If a subset A of an asymmetrically normed space (X, p) is p^s-compact, then it is p-compact.*

The converse of the above result is not true (as we shall see in Example 4.2.5).

Theorem 4.1.11. *([32, Theorem 2]) A subset of an asymmetrically normed space (X, p) is p-compact if and only if it is p-precompact and left-K-sequentially complete.*

Note that since an asymmetrically normed space (X, p) need not be Hausdorff, a p-compact set need not be p-closed (see Example 4.2.4).

Recall that the set θ_0^p in asymmetrically normed space (X, p) is defined by

$$\theta_0^p = \{y \in X : p(y) = 0\}.$$

Proposition 4.1.12. *([24, Proposition 6]) Let A be a subset of an asymmetrically normed space (X, p). Then A is p-compact if and only if $A + \theta_0^p$ is p-compact.*

We introduce a class of p-compact sets which first appeared in [24] and was extensively studied in [2]. The terminology is ours.

Definition 4.1.13. *([2, Proposition 11]) We say that a subset A of an asymmetrically normed space (X,p) is* **strongly p-compact** *if there is a p^s-compact subset A_0 of A such that $A_0 \subseteq A \subseteq A_0 + \theta_0^p$.*

Proposition 4.1.14. *([24, Proposition 11]) Any strongly p-compact subset of an asymmetrically normed space (X,p) is p-compact.*

The converse is not true in general (as we shall see in Example 4.4.13).

4.2 p-compact and p-precompact sets in asymmetrically normed lattices

In this section, we study the p-compact and p-precompact subsets of an asymmetrically normed lattice (X,p).

Proposition 4.2.1. *If a subset A of an asymmetrically normed lattice (X,p) is bounded above, then it is outside-p-precompact.*

Proof. Suppose $A \subseteq X$ is bounded above, then there is $y \in X$ such that $x \leq y$, for all $x \in A$. Thus if $x \in A$, $x \in B^p_\epsilon(y)$ for every $\epsilon > 0$. Hence

$$A \subseteq B^p_\epsilon(y) = y + B^p_\epsilon(0).$$

Therefore A is outside-p-precompact. □

We show that an outside-p-precompact set is not necessarily p-precompact.

Example 4.2.2. *In the asymmetrically normed lattice (ℓ_∞, p), with*

$$p(\mathbf{x}) = \sup x_n^+, \; \mathbf{x} = (x_n)_{n=1}^\infty \in \ell_\infty,$$

let $A = \{\mathbf{e}_n : n \in \mathbb{N}\}$, where

$$\mathbf{e}_n = (0,0,0,0,...,1,0,0,..),$$

is the sequence with n-th term 1 and 0 for all other terms. Then the sequence $(\mathbf{e}_n)_{i=1}^{\infty}$ p-converges by Proposition 2.1.6, since it is bounded above in ℓ_∞. Therefore by Proposition 4.1.6 A is outside p-precompact. We show that A is not p-precompact. Let $\epsilon = \frac{1}{2}$. If A is p-precompact, there is a finite subset $\{n_1, n_2, ..., n_k\}$ of $\mathbb{N}$ such that

$$\{\mathbf{e}_n : n \in \mathbb{N}\} \subseteq \bigcup_{i=1}^{k} B^p_{\frac{1}{2}}(e_{n_i}).$$

If $n > \max\{n_1, n_2, ..., n_k\}$, then

$$p(\mathbf{e}_n - \mathbf{e}_{n_i}) = 1 \textit{ for } 1 \leq i \leq k.$$

This shows that $\{\mathbf{e}_n : n \in \mathbb{N}\}$ is not contained in $\bigcup_{i=1}^{n_k} B^p_{\frac{1}{2}}(\mathbf{e}_{n_i})$, which is a contradiction. Therefore A is not p-precompact.

Proposition 4.2.3. *Any subset A of an asymmetrically normed lattice (X, p) which contains its supremum is p-compact.*

Proof. Let A be a subset of X. Suppose $c = \sup A \in A$ and that $\{G_i : i \in I\}$ is a p-open cover of A, that is

$$A \subseteq \bigcup_{i \in I} G_i.$$

Since $c \in A$, there is $i_0 \in I$ such that $c \in G_{i_0}$. Since G_{i_0} is p-open, there is an $\epsilon > 0$ such that $B^p_\epsilon(c) \subseteq G_{i_0}$. Since $a \leq c$ for all $a \in A$, it follows that

$$p(a - c) = 0 < \epsilon.$$

Therefore every element of A is an element of $B^p_\epsilon(c)$. Consequently

$$A \subseteq B^p_\epsilon(c) \subseteq G_{i_0}.$$

Hence A is p-compact, since every p-open cover of A has a finite subcover. □

We now give some of the examples promised in section one.

Examples 4.2.4. *(a) A subset of a p-precompact set need not be p-precompact. In the asymmetrically normed lattice* (ℓ_∞, p)*, let*

$$A = \{\mathbf{e}_n : n \in \mathbb{N}\} \cup \{\mathbf{1}\},$$

where $\mathbf{e}_n$ *is defined as in Example 4.2.2 and* $\mathbf{1} = (1, 1, 1, ...)$*. Then by Proposition 4.2.3, A is p-compact, hence p-precompact, since it contains its supremum. But* $B = \{\mathbf{e}_n : n \in \mathbb{N}\} \subseteq A$ *is not p-precompact (see Example 4.2.2).*

(b) A p-compact set need not be p^s*-closed and therefore need not be p-closed. In the asymmetrically normed lattice* $(\mathbb{R}, q)$*, let*

$$A = (-1, 1].$$

Then A is q-compact by Proposition 4.2.3, since it contains its supremum. But A is not q^s*-closed, since* $A = (-1, 1] \neq [-1, 1] = \overline{A}^{q^s}$*.*

It is well known that the closed unit ball of an infinite dimensional normed space is never compact. However, the unit ball $B_1^p[0]$ of an infinite dimensional asymmetrically normed latices (X, p) can be p-compact.

Example 4.2.5. *Let* $X = \ell_\infty$ *or* $X = C(\Omega)$*, then the ball* $B_1^p[0]$ *of the infinite dimensional asymmetrically normed lattice* (X, p) *is p-compact, since it contains its supremum (see Proposition 4.2.3).*

4.3 p-precompact sets in finite dimensional spaces

In this section, we characterize the p-precompact subsets of the finite dimensional asymmetrically normed lattice $(\mathbb{R}^m, p)$.

Let $k \in \mathbb{N}$. We write p_k for the asymmetric norm on $\mathbb{R}^k$ defined by

$$p_k(\mathbf{x}) = \max\{x_1^+, x_2^+, \ldots, x_k^+\}, \quad \mathbf{x} \in \mathbb{R}^k.$$

We will need the following result:

Proposition 4.3.1. *A p_k^s-bounded subset A of $\mathbb{R}^k$ is p_k-precompact.*

Proof. This follows at once from the facts that A is p_k^s-precompact and $B_\epsilon^{p_k^s}(\mathbf{x}) \subseteq B_\epsilon^{p_k}(\mathbf{x})$ for every $\epsilon > 0$ and $\mathbf{x} \in \mathbb{R}^k$. $\square$

For $m, k \in \mathbb{N}$ with $1 \le k \le m$ the projection $P_k : \mathbb{R}^m \to \mathbb{R}$ is defined by

$$P_k(\mathbf{x}) = x_k, \quad \mathbf{x} \in \mathbb{R}^m.$$

Proposition 4.3.2. *A subset A of $\mathbb{R}$ is p_1-precompact if and only if it is bounded above.*

Proof. Let $a = \sup\{x : x \in A\}$. If $\epsilon > 0$, there is a $b \in A$ such that $a < b + \epsilon$. Hence if $x \in A$, $x \le a < b + \epsilon$, from which it follows that $A \subseteq (-\infty, b+\epsilon) = B_\epsilon^{p_1}(b)$. $\square$

Proposition 4.3.3. *A subset A of $\mathbb{R}^2$ is p_2-precompact if and only if it is p_2-bounded (equivalently, order bounded above).*

Proof. Let $a_k = \sup P_k(A), k = 1, 2$ and $\epsilon > 0$. Then for $k = 1, 2$ we can find $\mathbf{b}_k \in A$ such that $a_k < b_{kk} + \epsilon$. Put

$$R = \{\mathbf{x} \in \mathbb{R}^2 : b_{21} \le x_1 \le a_1, b_{12} \le x_2 \le a_2\}.$$

Then $A \cap R$ is p_2^s-bounded and hence by Proposition 4.3.1 we can find $\mathbf{a}_1, \ldots, \mathbf{a}_n \in A \cap R$ such that $A \cap R \subseteq \bigcup_{i=1}^n B_\epsilon^{p_2}(\mathbf{a}_i)$.

If $\mathbf{x} \in A \setminus R$, then either $x_1 < b_{21}$ or $x_2 < b_{12}$.

If $x_1 < b_{21}$, then $(x_1 - b_{21})^+ = 0, (x_2 - b_{22})^+ \le (a_2 - b_{22}) < \epsilon$. It follows that $\mathbf{x} \in B_\epsilon^{p_2}(\mathbf{b}_2)$.

If $x_2 < b_{12}$, then $(x_1 - b_{11})^+ \leq (a_1 - b_{11}) < \epsilon$ and $(x_2 - b_{12})^+ = 0$; hence $\mathbf{x} \in B_\epsilon^{p_2}(\mathbf{b}_1)$. Therefore

$$A \subseteq \cup_{i=1}^n B_\epsilon^{p_2}(\mathbf{a}_i) \cup B_\epsilon^{p_2}(\mathbf{b}_1) \cup B_\epsilon^{p_2}(\mathbf{b}_2).$$

□

Lemma 4.3.4. *Let $A \subseteq \mathbb{R}^3$ and suppose there are real numbers $k_2, k_3, \ell_1, \ell_2, \ell_3$ such that*

$$A \subseteq R = \{\mathbf{x} \in \mathbb{R}^3 : -\infty < x_1 \leq \ell_1, k_2 \leq x_2 \leq \ell_2, k_3 \leq x_3 \leq \ell_3\}.$$

Then A is p_3-precompact.

Proof. Let $\epsilon > 0$ and choose partitions of the intervals $[k_2, \ell_2]$ and $[k_3, \ell_3]$ into n and m subintervals respectively, all of the same length d, with $d < \epsilon$. This gives a subdivision of R into $n \times m$ squares $S_{ij}, 1 \leq i \leq n, 1 \leq j \leq m$, each with side lengths less that ϵ. Put

$$R_{ij} = \{\mathbf{x} \in \mathbb{R}^3 : -\infty < x_1 \leq \ell_1, (x_2, x_3) \in S_{ij}\}.$$

Then

$$A = \cup\{A \cap R_{ij} : 1 \leq i \leq n, 1 \leq j \leq m\}.$$

For every i, j such that $A \cap R_{ij} \neq \emptyset$, we put $r_{ij} = \sup P_1(A \cap R_{ij})$ and then choose $\mathbf{a}(i,j) \in A \cap R_{ij}$ such that $r_{ij} < a_1(i,j) + \epsilon$, where $a_1(i,j) = P_1(\mathbf{a}(i,j))$. Then $A \cap R_{ij} \subseteq B_\epsilon^{p_3}(\mathbf{a}(i,j))$. To see this, note that if $\mathbf{x} \in A \cap R_{ij}$, then $x_1 \leq r_{ij} < a_1(i,j) + \epsilon$, and $x_2 - a_2(i,j) < \epsilon, x_3 - a_3(i,j) < \epsilon$ since $(x_2, x_3) \in S_{ij}$. (We write $a_2(i,j)$ and $a_3(i,j)$ for $P_2(\mathbf{a}(i,j))$ and $P_3(\mathbf{a}(i,j))$ respectively.) It now follows that

$$A \subseteq \cup\{B_\epsilon^{p_3}(\mathbf{a}(i,j)) : 1 \leq i \leq n, 1 \leq j \leq m, A \cap R_{ij} \neq \emptyset\}.$$

□

Theorem 4.3.5. *Let $m \in \mathbb{N}$. A subset A of $\mathbb{R}^m$ is p_m-precompact if and only if it is p_m-bounded.*

Proof. The cases $m = 1, 2$ have been proved above. We give a proof for the case $m = 3$; the proofs in higher dimensions are similar, but the notation more involved. Let $\epsilon > 0$. For $k = 1, 2, 3$ put $a_k = \sup P_k(A)$ and choose $\mathbf{b}_k \in A$ such that $a_k < b_{kk} + \epsilon$. Now put $c_k = \min\{b_{1k}, b_{2k}, b_{3k}\}$ and let

$$\begin{aligned} R &= \{\mathbf{x} \in \mathbb{R}^3 : c_1 \leq x_1 \leq a_1, c_2 \leq x_2 \leq a_2, c_3 \leq x_3 \leq a_3\} \\ R_{12} &= \{\mathbf{x} \in \mathbb{R}^3 : c_1 \leq x_1 \leq a_1, c_2 \leq x_2 \leq a_2, -\infty \leq x_3 \leq a_3\} \\ R_{13} &= \{\mathbf{x} \in \mathbb{R}^3 : c_1 \leq x_1 \leq a_1, -\infty \leq x_2 \leq a_2, c_3 \leq x_3 \leq a_3\} \\ R_{23} &= \{\mathbf{x} \in \mathbb{R}^3 : -\infty \leq x_1 \leq a_1, c_2 \leq x_2 \leq a_2, c_3 \leq x_3 \leq a_3\} \end{aligned}$$

We claim that

$$A \subseteq R \cup R_{12} \cup R_{13} \cup R_{23} \cup B_\epsilon^{p_3}(\mathbf{b}_1) \cup B_\epsilon^{p_3}(\mathbf{b}_2) \cup B_\epsilon^{p_3}(\mathbf{b}_3).$$

To see this, let $\mathbf{x} \in A$.

If $x_1 \geq c_1, x_2 \geq c_2, x_3 \geq c_3$, then $\mathbf{x} \in R$.

If $x_1 < c_1, x_2 \geq c_2, x_3 \geq c_3$, then $-\infty < x_1 < c_1, c_2 \leq x_2 \leq a_2, c_3 \leq x_3 \leq a_3$ and so $\mathbf{x} \in R_{23}$.

If $x_1 \geq c_1, x_2 < c_2, x_3 \geq c_3$, then $c_1 \leq x_1 \leq a_1, -\infty < x_2 < c_2, c_3 \leq x_3 \leq a_3$ and so $\mathbf{x} \in R_{13}$.

If $x_1 \geq c_1, x_2 \geq c_2, x_3 < c_3$, then $c_1 \leq x_1 \leq a_1, c - 2 \leq x_2 \leq a_2, -\infty < x_2 < c_3$ and so $\mathbf{x} \in R_{12}$.

If $x_1 < c_1, x_2 < c_2, x_3 \geq c_3$, then $(x_1 - b_{31})^+ \leq (c_1 - b_{31})^+ = 0$, $(x_2 - b_{32})^+ \leq (c_2 - b_{32})^+ = 0$ and $(x_3 - b_{33})^+ \leq (a_3 - b_{33})^+ < \epsilon$. It follows that $\mathbf{x} \in B_\epsilon^{p_3}(\mathbf{b}_3)$. If $x_1 < c_1, x_2 < c_2, x_3 < c_3$, a similar argument shows that $\mathbf{x} \in B_\epsilon^{p_3}(\mathbf{b}_1) \cap B_\epsilon^{p_3}(\mathbf{b}_2) \cap B_\epsilon^{p_3}(\mathbf{b}_3)$. The other cases are treated similarly.

Since it follows from Proposition 4.3.1 and Lemma 4.3.4 that each of the sets $R \cap A, R_{12} \cap A, R_{13} \cap A$ and $R_{23} \cap A$ are p_3-compact, we can now deduce that A itself is p_3-compact. $\square$

Corollary 4.3.6. *A subset A of asymmetrically normed lattice $(\mathbb{R}^m, p)$ is p-precompact if and only if it is outside-p-precompact.*

Proof. Every outside-p-precompact set is p-bounded in $\mathbb{R}^m$ and p-bounded sets are bounded above in $\mathbb{R}^m$ (see Proposition 2.2.2). $\square$

We now show that the p-closure of a p-precompact subset in an asymmetrically normed lattice need not be p-precompact.

Example 4.3.7. *In the asymmetrically normed lattice $(\mathbb{R}, q)$, let $A = (-\infty, 1)$, then by Proposition 4.3.5, A is q-precompact, since it is bounded above. But $\overline{A}^q = \mathbb{R}$ is not q-precompact.*

We end this section by showing that the condition in Proposition 4.1.7 is not a necessary condition.

Example 4.3.8. *In the asymmetrically normed lattice $(\mathbb{R}, q)$, let*

$$A = \{1\}, \;\; B = (-\infty, 2).$$

Then $A \subseteq B$ and B is p-precompact by Theorem 4.3.5. But

$$\overline{A}^{q^{-1}} = (-\infty, 1] \neq (-\infty, 2) = B,$$

so A is not q^{-1}-dense in B. But A is p-precompact, since it is p^s-compact.

4.4 Almost order bounded above, p-precompact and outside-p-precompact subsets of an asymmetrically normed lattice

In this section, we study the relationship between almost order bounded above, outside-p-precompact and p-precompact subsets of an asymmetrically normed lattice. We show that almost order boundedness above is equivalent to outside-p-precompactness, and hence equivalent to p-precompactness in a finite dimensional asymmetrically normed lattice.

In a large class of Banach lattices, almost order boundedness can be used to characterize compactness (see, for example, Corollary 7.3 in [14], where the term order precompactness is used for almost order boundedness). We consider almost order bounded above subsets of an asymmetrically normed lattice and show that in a finite dimensional space order boundedness above is equivalent to precompactness.

Definition 4.4.1. *([41, p. 501]) Let E be a normed lattice. A subset S of E is called* **almost order bounded** *if for every $\epsilon > 0$ there exists an order interval $A_\epsilon = [p, q]$ such that*

$$S \subseteq A_\epsilon + B_\epsilon(0),$$

where

$$B_\epsilon(0) = \{x \in E : \|x\| < \epsilon\}.$$

Any precompact or order bounded subset of a normed lattice is an almost order bounded set (see [41, Theorem 122.2]).

Lemma 4.4.2. *A subset A of an asymmetrically normed lattice (X, p) is bounded above if and only if there is $x \in X$ such that*

$$A \subseteq \theta^p_x = \{x\} + \theta^p_0 = \downarrow x.$$

Proof. The proof follows from the definition. □

Definition 4.4.3. *A subset A of an asymmetrically normed lattice (X, p) is* **almost order-bounded above** *if for every $\epsilon > 0$, there is $x_\epsilon \in X$ such that $A \subseteq \theta^p_{x_\epsilon} + B^p_\epsilon(x_\epsilon)$.*

Lemma 4.4.4. *([24, Lemma 3]) Let (X,p) be a asymmetrically normed space and $x \in X$. Then*

$$B^p_\epsilon(x) = B^p_\epsilon(x) + \theta^p_0.$$

Note that for every $\epsilon > 0$, for every $x_\epsilon \in X$,

$$\begin{aligned}\theta^p_{x_\epsilon} + B^p_\epsilon(0) &= \{x_\epsilon\} + \theta^p_0 + B^p_\epsilon(0)\\ &= \{x_\epsilon\} + B^p_\epsilon(0) \text{ by Lemma 4.4.4}\\ &= B^p_\epsilon(x_\epsilon).\end{aligned}$$

Proposition 4.4.5. *A subset A of an asymmetrically normed lattice (X,p) is almost order bounded above if and only if for $\epsilon > 0$, there is $x_\epsilon \in X$ such that $A \subseteq B^p_\epsilon(x_\epsilon)$.*

Proof. The statement follows from the above equality. □

Proposition 4.4.6. *Let A be a subset of an asymmetrically normed lattice (X,p). Then A is outside p-precompact if and only if it is almost order bounded above.*

Proof. Suppose A is outside p-precompact, then for every $\epsilon > 0$, there is a finite set $\{x^\epsilon_1, ..., x^\epsilon_{n_\epsilon}\} \subseteq X$ such that $A \subseteq \bigcup_{i=1}^{n_\epsilon} B^p_\epsilon(x^\epsilon_i)$. Thus $x^\epsilon = \bigvee_{i=1}^{n_\epsilon}(x^\epsilon_i) \in X$, since X is a lattice, and $x^\epsilon \geq x^\epsilon_i$, for $i = 1, 2, ..., n_\epsilon$. By Lemma 3.2.6,

$$B^p_\epsilon(x^\epsilon_i) \subseteq B^p_\epsilon(x^\epsilon), \text{ for } 1 \leq i \leq n_\epsilon.$$

Hence,

$$A \subseteq \bigcup_{i=1}^{n_\epsilon} B^p_\epsilon(x^\epsilon_i) \subseteq B^p_\epsilon(x^\epsilon).$$

Conversely, suppose A is almost order bounded above, then A is outside p-precompact, since for every $\epsilon > 0$, there is $x_\epsilon \in X$ such that $A \subseteq B^p_\epsilon(x_\epsilon)$. □

Let A be a subset of an asymmetrically normed lattice (X,p). We denote the decreasing hull of A by $\downarrow A$, where

$$\downarrow A = \{y \in X : \text{ there exists an } x \in A \text{ with } y \leq x\}.$$

The decreasing hull is the smallest decreasing set containing A.

Proposition 4.4.7. *If $A \subseteq X$ is outside p-precompact, then $\downarrow A$ is outside p-precompact.*

Proof. Suppose A is outside p-precompact, then by Proposition 4.4.6, it is almost order bounded above. Hence for every $\epsilon > 0$, there is an $x_\epsilon \in X$ such that

$$A \subseteq B^p_\epsilon(x_\epsilon). \tag{4.1}$$

But since $B^p_\epsilon(x_\epsilon)$ is decreasing, it follows that

$$\downarrow A \subseteq B^p_\epsilon(x_\epsilon).$$

Thus $\downarrow A$ is almost order bounded above, hence outside-p-precompact. □

Proposition 4.4.8. *If a subset A of an asymmetrically normed lattice (X, p) is p-precompact, then it is almost order bounded above.*

Proof. If A is p-precompact, then it is outside p-precompact. Therefore by Proposition 4.4.6, it is almost order bounded above. □

The converse of the above result is not true in general (see Example 4.2.2).

Proposition 4.4.9. *Let A be a subset of an asymmetrically normed lattice (X, p). If A bounded above, then it is almost order bounded above.*

Proof. Suppose A is bounded above, then by Lemma 4.4.2, there is $x \in X$ such that

$$A \subseteq \theta^p_x.$$

But for every $\epsilon > 0$,

$$\theta^p_x \subseteq B^p_\epsilon(x).$$

Therefore by Proposition 4.4.5, A is almost order bounded above. □

The converse of this proposition is not true in general as the next example shows.

Example 4.4.10. *([41, Exercise 122.5]) In the asymmetrically normed lattice (ℓ_2, p), let $(\mathbf{e}_n)_{n=1}^{\infty}$ be the sequence with n-th term 1 and 0 for all other terms and $(\eta_n)_{n=1}^{\infty}$ be a sequence of positive numbers such that*

$$\eta_n \to 0 \text{ and } \sum_{n=1}^{\infty} \eta_n^2 = \infty.$$

Let $\mathbf{u}_n = \eta_n \mathbf{e}_n$ and

$$A = \{\mathbf{u}_n : n \in \mathbb{N}\} \cup \{\mathbf{0}\},$$

where $\mathbf{0} = (0, 0, 0, ...)$. Then $\mathbf{u}_n \xrightarrow{p} \mathbf{0}$ and so A is p-precompact by Proposition 4.1.6, since it contains one of its p-limits. Hence by Proposition 4.4.8, A is almost order bounded above. We now show that A is not bounded above. Suppose A is bounded above, then there is an element $\mathbf{y} \in \ell_2$ such that for all $\mathbf{x} \in A$, $\mathbf{y} \geq \mathbf{x}$. But then $(y_i) = \mathbf{y} \geq \mathbf{u}_n$ for all $n \in \mathbb{N}$ and $\mathbf{y} \geq \mathbf{0}$. This implies that $y_i \geq u_{ii} = \eta_i$ which implies that

$$\sum_{n=1}^{\infty} y_n^2 \geq \sum_{n=1}^{\infty} \eta_n^2 = \infty,$$

contradicting the fact that $\mathbf{y} \in \ell_2$. Hence A is not bounded above.

In some asymmetrically normed lattices a set is bounded above if and only if it is almost order bounded above. Recall that for a non-empty set Γ, $B(\Gamma)$ denotes the set of all bounded functions $f : \Gamma \to \mathbb{R}$, and for a compact Hausdorff space Ω, $C(\Omega)$ denotes the set of all continuous real valued functions. Equipped with the supremum norm and the usual pointwise ordering, these spaces are Banach lattices.

Proposition 4.4.11. *Let $F \subseteq X$, where $X = B(\Gamma)$ or $X = C(\Omega)$. Then F is bounded above if and only if it is almost order bounded above in (X, p), where $p(f) = \|f^+\|$.*

Proof. The forward implication follows from Proposition 4.4.9.
Conversely, suppose A is almost order bounded above, then taking $\epsilon = 1$, there

is an $f_1 \in B(\Gamma)$ such that

$$F \subseteq B_1^p(f_1).$$

That is, for every $f \in F$,

$$f(x) - f_1(x) \leq (f(x) - f_1(x))^+ < 1, \text{ for every } x \in \Gamma$$

which implies that for every $f \in F$ and $x \in \Gamma$,

$$f(x) < f_1(x) + 1.$$

Therefore $F \subseteq \theta_h^p$, where $h(x) = f_1(x) + 1$ for every $x \in X$, and $h \in B(\Gamma)$, since f_1 and the constant function 1 are bounded. Thus F is bounded above. □

Corollary 4.4.12. *A set in (ℓ_∞, p) is order bounded above if and only if it is almost order bounded above.*

Proof. The statement follows, since $B(\Gamma) = \ell_\infty$ when $\Gamma = \mathbb{N}$. □

Recall from Proposition 4.1.14 that a strongly p-compact (Definition 4.1.13) set is p-compact. In [2, Example 12] an example of a subset of ℓ_1 that is p-compact but not strongly p-compact was given. We show that this example remains valid in a larger class of sequence spaces. Later we will also give an example in a finite dimensional space (Example 4.5.18). Let $\mathbb{R}^{\mathbb{N}}$ be the space of all real sequences (a vector lattice under the coordinatewise ordering), and let ℓ_0 be the space of all real sequences with only finitely many non-zero terms (a Riesz subspace of $\mathbb{R}^{\mathbb{N}}$).

Example 4.4.13. *Let X be a Riesz subspace of $\mathbb{R}^{\mathbb{N}}$ containing ℓ_0 and*

$$A = \{\mathbf{x}_n : n \in \mathbb{N}\} \cup \{\mathbf{0}\}, \tag{4.2}$$

where

$$\mathbf{x}_n = (0, 0, 0, ..., -1, 0, 0, ..),$$

the sequence with n-th term -1 and all other terms 0 and

$$\mathbf{0} = (0, 0, 0, ...).$$

Then obviously $A \subseteq \ell_0 \subseteq \mathbb{R}^{\mathbb{N}}$*. Suppose* $\|\cdot\|$ *is a norm on* X *such that* $(X, \|\cdot\|)$ *is a normed Riesz space and such that* $\inf_n \|e_n\| > 0$*, where*

$$\mathbf{e}_n = (0, 0, 0, ..., 1, 0, 0, ..),$$

with 1 *in the* n*-th position. Then by Proposition 4.2.3,* A *is* p*-compact, since it contains its supremum. We show that* A *is not strongly* p*-compact. Suppose on the contrary that there is a* p^s*-compact set* A_0 *such that,*

$$A_0 \subseteq A \subseteq A_0 + \theta_0^p.$$

Let $\mathbf{x} \in A$*. Then* $\mathbf{x} = \mathbf{0}$ *or* $\mathbf{x} = \mathbf{x}_m$ *for some* $m \in \mathbb{N}$*. There is an* $n_m \in \mathbb{N}$ *such that* $\mathbf{x}_{n_m} \in A_0$ *and* $\mathbf{x}_m = \mathbf{x}_{n_m} + \mathbf{z}$*, where* $\mathbf{z} \in \theta_0^p$*. Therefore*

$$p(\mathbf{x}_m - \mathbf{x}_{n_m}) = p(\mathbf{z}) = 0. \tag{4.3}$$

If $m \neq n_m$*, then*

$$p(\mathbf{x}_m - \mathbf{x}_{n_m}) = \|(\mathbf{x}_m - \mathbf{x}_{n_m})^+\| = \|(0, 0, 0, ..., 1, 0, ..)\| = \|\mathbf{e}_{n_m}\| \geq \inf_n \|\mathbf{e}_n\| > 0.$$

This contradicts (4.3). Therefore $m = n_m$ *and so* $\mathbf{x} = \mathbf{x}_m = \mathbf{x}_{n_m} \in A_0$*. Hence* $A \subseteq A_0 \cup \{\mathbf{0}\}$*. But* $A_0 \cup \{\mathbf{0}\} \subseteq A$*. Then* $A = A_0 \cup \{\mathbf{0}\}$*, therefore* A *must be* p^s*-compact. For* $n \neq m$*,*

$$\begin{aligned} p^s(\mathbf{x}_n - \mathbf{x}_m) &= p(\mathbf{x}_n - \mathbf{x}_m) \vee p(\mathbf{x}_m - \mathbf{x}_n) \\ &= 1. \end{aligned}$$

Therefore no subsequence of $(\mathbf{x}_n)_{n=1}^{\infty}$ *can be* p^s*-Cauchy. This contradicts the fact that* A_0 *is* p^s*-precompact.*

The following definition was introduced in ([2, Definition 17]) in order to give a sufficient condition for a p-compact set to be strongly p-compact.

Definition 4.4.14. *([2, Definition 17]) Let* (X, p) *be an asymmetrically normed space. If* $A \subseteq X$*, then* A *has the* $B(A)$*-property if there is a function* $\kappa : A \times \mathbb{R}^+ \to \mathbb{R}^+$ *such that;*

(a) *For every pair* $x, y \in A$ *and for all* $t \in \mathbb{R}^+$, $\kappa(x,t) \leq \kappa(y,t)$ *whenever* $x \in y + \theta_0^p$, *and*

(b) $B^p_{\kappa(x,t)}(x) \cap A \subseteq (B^{p^s}_t(x) \cap A) + \theta_0^p$, *for all* $t \in \mathbb{R}^+$ *and for every* $x \in A$.

We recall (Definition 1.2.13) that if (X,p) is an asymmetrically normed space such that for every $\epsilon > 0$

$$B^p_\epsilon(0) \subseteq B^p_\epsilon(0) + \theta_0^p$$

or equivalently

$$B^p_\epsilon[0] \subseteq B^p_\epsilon[0] + \theta_0^p,$$

then (X,p) is called right bounded with constant $r = 1$; it is easy to check that in this case the inclusions above are actually equalities ([2, p. 536]).

Proposition 4.4.15. *([2, Corollary 22]) Let* A *be a* p^s*-closed subset of an asymmetrically normed space* (X,p) *which is right-bounded with constant* $r = 1$ *such that* A *is* p*-compact and has the* $B(A)$*-property. Then* A *is strongly* p*-compact.*

Note that in [2] it is assumed that the asymmetrically normed space (X,p) is right-bounded with constant $r = 1$. In the case where A is a subset of an asymmetrically normed lattice (X,p), the $B(A)$-property means the following: A has the $B(A)$-property if there is a function $\kappa : A \times \mathbb{R}^+ \to \mathbb{R}^+$ such that

(a) κ is increasing for its first variable.

(b) For every $x \in X$ and $t \in \mathbb{R}^+$, if $y \in A$ satisfies that $p(y - x) < \kappa(x,t)$, then there is $z \in A$ such that $y \leq z$ and $p^s(z - x) < t$.

Note 4.4.16. *([2, p. 538]) If a subset* A *of an asymmetrically normed lattice* (X,p) *is such that, for every pair of elements* x, y *of* A, *the supremum* $x \vee y$ *is in* A, *then* A *has the* $B(A)$*-property. This can be seen by letting* $\kappa(x,t) = t$ *for all* $x \in A$ *and* $t \in \mathbb{R}^+$.

Theorem 4.4.17. *If A is a subset of an asymmetrically Banach lattice (X, p) which is p-precompact and closed under finite suprema, then there is an $x \in X$ such that*

$$A \subseteq \{x\} + \theta_0^p = \downarrow x,$$

that is, A is bounded above.

Proof. We first show that there is an increasing sequence $(x_n)_{n=1}^{\infty}$ in A such that

$$A \subseteq B^p_{\frac{1}{n}}(x_n), \text{ for all } n \in \mathbb{N}.$$

The proof is by induction on n. By Lemma 4.4.6 and Proposition 4.4.5, there is an $x_1 \in A$ such that $A \subseteq B^p_1(x_1)$. We can also find an $x_2' \in A$ such that $A \subseteq B^p_{\frac{1}{2}}(x_2')$. Let $x_2 = x_2' \vee x_1$. Then

$$x_2 = x_2' \vee x_1 \Rightarrow x_2 \geq x_1, \ x_2 \geq x_2',$$

and since A is closed under finite suprema, $x_2 \in A$. By Lemma 3.2.6 $A \subseteq B^p_{\frac{1}{2}}(x_2') \subseteq B^p_{\frac{1}{2}}(x_2)$. We now assume that there is an $x_n \in A$ such that $A \subseteq B^p_{\frac{1}{n}}(x_n)$. Again, by Lemma 4.4.6, there is an $x_{n+1}' \in A$ such that $A \subseteq B^p_{\frac{1}{n+1}}(x_{n+1}')$. We let $x_{n+1} = x_{n+1}' \vee x_n$. This implies, as before that $x_{n+1} \geq x_n$ and $x_{n+1} \in A$. Also

$$A \subseteq B^p_{\frac{1}{n+1}}(x_{n+1}') \subseteq B^p_{\frac{1}{n+1}}(x_{n+1}).$$

Hence, by induction, there is an increasing sequence $(x_n)_{n=1}^{\infty}$ in A such that,

$$A \subseteq B^p_{\frac{1}{n}}(x_n).$$

We now show that the sequence $(x_n)_{n=1}^{\infty}$ is p^s-Cauchy in A. Let $\epsilon > 0$ and choose $n \in \mathbb{N}$ such $\frac{1}{n} < \epsilon$. If $i > j \geq n$

$$\begin{aligned} p(x_i - x_j) &\leq p(x_i - x_n) + p(x_n - x_j). \\ &< \|(x_i - x_n)^+\| + ||(x_n - x_j)^+||. \\ &= \|(x_i - x_n)^+\| + 0, \text{ since } x_n \leq x_j. \\ &< \frac{1}{n} \text{ (since } x_i \in A \subseteq B^p_{\frac{1}{n}}(x_n) \\ &< \epsilon. \end{aligned}$$

Also,

$$\begin{aligned} p^{-1}(x_i - x_j) &= p(x_j - x_i) \\ &= ||(x_j - x_i)^+|| \\ &= 0, \text{ since } x_j \leq x_i. \end{aligned}$$

Therefore,

$$\begin{aligned} p^s(x_i - x_j) &= p(x_i - x_j) \vee p^{-1}(x_j - x_i) \\ &= p(x_i - x_j) \\ &< \epsilon. \end{aligned}$$

Thus, the sequence $(x_n)_{n=1}^{\infty}$ is p^s-Cauchy. Since $(x_n)_{n=1}^{\infty}$ is p^s-Cauchy and X is p^s-complete, there is $x \in X$ such that

$$x_n \xrightarrow{p^s} x.$$

The point x is the supremum of $(x_n)_{n=1}^{\infty}$ in X, since X is a Banach lattice and $(x_n)_{n=1}^{\infty}$ is an increasing convergent sequence ([34, Proposition 1.1.6(iii)]). We show that $A \subseteq \theta_x^p$. Let $z \in A$. Then we show that $z \leq x$ (equivalently $p(z-x) = 0$). For every $n \in \mathbb{N}^+$,

$$\begin{aligned} p(z - x) &= ||(z - x)^+||. \\ &\leq ||(z - x_n)^+|| + ||(x_n - x)^+||. \\ &< ||(z - x_n)^+|| + 0, \text{ since } x_n \leq x. \\ &< \frac{1}{n} \text{ (since } z \in A \subseteq B^p_{\frac{1}{n}}(x_n)). \end{aligned}$$

But since

$$0 \leq p(z - x) < \frac{1}{n}, \text{ for every } n,$$

then $p(z - x) = 0$, and therefore $z \leq x$, showing that $A \subseteq \theta_x^p$. □

Corollary 4.4.18. *If a subset A of an asymmetrically Banach lattice (X, p) is p^s-closed, p-precompact and is closed under finite suprema, then there is an $x \in A$ such that*

$$\{x\} \subseteq A \subseteq \{x\} + \theta_0^p,$$

and hence A is strongly p-compact.

Proof. Suppose $A \subseteq X$ is p^s-closed, p-precompact and is closed under finite suprema, then by Theorem 4.4.17, there is an increasing p^s-Cauchy sequence $(x_n)_{n=1}^{\infty}$ in A and $x \in X$ such that $x_n \xrightarrow{p^s} x$ and $A \subseteq \{x\} + \theta_x^p$. Since A is p^s-closed, it follows that $x \in A$. Therefore

$$\{x\} \subseteq A \subseteq \{x\} + \theta_0^p.$$

It follows that A is strongly p-compact. □

Proposition 4.4.19. *Let A be a subset of an asymmetrically Banach lattice (X, p). If A is p-precompact, closed under finite suprema and p^s-closed, then $\downarrow A$ is strongly p-compact.*

Proof. The statement follows from Corollary 4.4.18 and the fact that

$$\downarrow A \subseteq \downarrow x.$$

□

4.5 p-compact subsets of a finite dimensional asymmetrically normed lattice

In this section we characterize the q-compact subset of the asymmetrically normed lattice $\mathbb{R}$, and show that a subset of $\mathbb{R}$ is q-compact if and only if it is strongly q-compact. We show that the same is true for p^s-closed subsets of $\mathbb{R}^m$.

In the finite dimensional normed lattices $\mathbb{R}^m$, the closure of a bounded subset A of $(\mathbb{R}^m, p^s)$ is p^s-compact. Recall that we use the norm $\|\mathbf{x}\| = \max |x_i|$ on $\mathbb{R}^m$ and $p((x_1, x_2, \ldots, x_m)) = ||(x_1^+, x_2^+, \ldots, x_m^+)||$.

Remark 4.5.1. *In an asymmetrically normed lattice (X, p) the p-closure of a bounded above subset A of X is not bounded above.*

This follows since by Lemma 1.4.5, it is increasing. Hence it is not p-compact, since p-compact sets are bounded above and therefore not increasing. Note that we do not have a Heine-Borel theorem in the asymmetrically normed lattice $(\mathbb{R}^m, p)$, since there are sets which are p-compact but not p-closed (see Example 4.2.4) and there are no p-closed bounded above subsets of $\mathbb{R}^m$.

It is known that in a normed lattice X, the closed unit ball of X is compact if and only if X is finite dimensional.

Theorem 4.5.2. *([24, Theorem 13]) The unit p-ball $B_1^p[0]$ of a T_1 asymmetrically normed space (X, p) is p-compact if and only if X is finite dimensional.*

An asymmetrically normed lattice is never T_1 and the above characterization is no longer true (see Example 4.2.5).

In [24, Proposition 17], it is shown that if (X, p) is a finite dimensional asymmetrically normed space, $B_1^p[\mathbf{0}]$ is p-compact. We give a proof for the special case where (X, p) is an asymmetrically normed lattice. We recall from [2, p. 536] and [24, Corollary 20] that if an asymmetrically normed space (X, p) is right bounded with constant $r = 1$, we have that,

$$B_\epsilon^p[x] = B_\epsilon^{p^s}[x] + \theta_0^p, \tag{4.4}$$

for every $\epsilon > 0$ and every $x \in X$.

Proposition 4.5.3. *([24, Proposition 17]) For every $\mathbf{x} \in \mathbb{R}^m$, $\epsilon > 0$, the ball $B_\epsilon^p[\mathbf{x}]$ in the asymmetrically normed lattice $(\mathbb{R}^m, p)$ is p-compact.*

Proof. Let $\epsilon > 0$ and $\mathbf{x} \in \mathbb{R}^m$. The p^s-closed ball $B_\epsilon^{p^s}[\mathbf{x}]$ is p^s-compact by the Heine-Borel theorem, since it is p^s-bounded and p^s-closed. Therefore by Proposition 4.1.10, it is p-compact. Hence by Proposition 4.1.12, $B_\epsilon^{p^s}[\mathbf{x}]+\theta_{\mathbf{0}}^p$, is p-compact.

Since $(\mathbb{R}^m, p)$ is right bounded with $r = 1$ (see Proposition 1.4.3), then by (4.4)

$$B^p_\epsilon[\mathbf{x}] = B^{p^s}_\epsilon[\mathbf{x}] + \theta^p_{\mathbf{0}}.$$

Thus $B^p_\epsilon[\mathbf{x}]$ is p-compact. $\square$

Remark 4.5.4. *In an asymmetrically normed lattice $(\mathbb{R}^m, p)$, the p-closed ball $B^{p^{-1}}_\epsilon[\mathbf{x}]$, $\mathbf{x} \in \mathbb{R}^m$, $\epsilon > 0$ of $\mathbb{R}^m$ is never p-compact.*

This follows since by Proposition 1.4.5, it is increasing and hence not bounded above.

Theorem 4.5.5. *A subset A of $\mathbb{R}$ is q-compact if and only if it is bounded above and the supremum c of A is in A.*

Proof. Suppose A is q-compact, then it is q-bounded and hence bounded above. Therefore the supremum c of A exists in $\mathbb{R}$. We prove by contradiction that $c \in A$. Suppose $c \notin A$. Then we show that A is not q-compact, that is, there is a q-open cover $\{G_i : i \in I\}$ of A without a finite subcover. For each $i \in \mathbb{N}$, let

$$G_i = \left(-\infty, c - \frac{1}{i}\right).$$

Then G_i is p-open. We show that $\{G_i : i \in \mathbb{N}\}$ is a p-open cover of A. If $x \in A$, then $c - x > 0$, since $c \notin A$. Choose $i \in \mathbb{N}$ such that

$$\frac{1}{i} < c - x.$$

Then $x \in G_i$. For all k, $c - \frac{1}{k} < c$, so there is $a_k \in A$ such that $a_k > c - \frac{1}{k}$, that is, $a_k \notin (-\infty, c - \frac{1}{k})$. Therefore, for each $n \in \mathbb{N}$

$$\begin{aligned} A &\not\subseteq \bigcup_{k=1}^{n} \left(-\infty, c - \frac{1}{k}\right) \\ &= \bigcup_{k=1}^{n} G_k. \end{aligned}$$

Consequently $\{G_i : i \in \mathbb{N}\}$ has no finite subcover, showing that A is not q-compact, a contradiction. Therefore the supremum c of A must be in A.
The converse follows from Proposition 4.2.3. $\square$

Corollary 4.5.6. *Let $A \subseteq \mathbb{R}$. If A is bounded above and A q^s-closed, then A is q-compact.*

Proof. This follows, since any set in $\mathbb{R}$ which is bounded above and q^s-closed, contains its supremum. $\square$

Corollary 4.5.7. *A subset A of an asymmetrically normed lattice $(\mathbb{R}, q)$ is q-compact if and only if it is strongly q-compact.*

Proof. Let A be a subset of $\mathbb{R}$ and suppose A is q-compact, then by Theorem 4.5.5, $x = \sup A \in A$. Let $A_0 = \{x\}$, then A_0 is q^s-compact, and

$$\{x\} \subseteq A \subseteq \{x\} + \theta_0^q.$$

The converse follows from Proposition 4.1.14. $\square$

In a normed space, any closed subset of a compact subset is compact. In an asymmetrically normed lattice (X, p), a p-compact set cannot have any non-empty p-closed subsets, since such sets must be increasing and therefore cannot be bounded above. But p-compact sets are bounded above. We therefore consider p^{-1}-closed and p^s-closed subsets of a p-compact sets in X.

Corollary 4.5.8. *Let A, $B \subseteq \mathbb{R}$. If $A \subseteq B$, B q-compact and A is q^s-closed, then A is q-compact.*

Proof. The statement follows from Corollary 4.5.6. $\square$

Proposition 4.5.9. *Let A be a bounded above subset of the asymmetrically normed lattice $(\mathbb{R}, q)$, then the q^{-1}-closure $\overline{A}^{q^{-1}}$ is q-compact.*

Proof. Suppose $A \subseteq \mathbb{R}$ is bounded above, then by Proposition 4.3.5, it is q-precompact. Hence by Proposition 4.1.4, $\overline{A}^{q^{-1}}$ is q-precompact. Since $\overline{A}^{q^{-1}}$ is q^{-1}-closed, it is q^s-closed (see Corollary 1.4.6). Therefore by Corollary 3.3.3 $\overline{A}^{q^{-1}}$ is left-K-sequentially complete. Hence by Theorem 4.1.11, it is q-compact. $\square$

The above result is not true in general for subsets of $\mathbb{R}^2$ as the next example shows.

Example 4.5.10. *In the asymmetrically normed lattice* $(\mathbb{R}^2, p)$*, let*

$$A = \left\{(x, y) \in \mathbb{R}^2 : x < 0, y \leq \frac{1}{x}\right\}$$

and $B = \theta_{\mathbf{0}}^p$. *Then* $A \subseteq B$ *and by Proposition 4.2.3, B is p-compact, since it contains its supremum. The set A is p^s-closed and decreasing. Hence it is p^{-1}-closed but is not left-K-sequentially complete (see Example 3.3.4). Hence by Theorem 4.1.11, A is not p-compact.*

Remark 4.5.11. *Note that in* $\mathbb{R}^m$ *not all p-compact sets contains their supremum (see Example 3.3.6).*

Theorem 4.5.12. *Let A be a p^s-closed subset of* $\mathbb{R}^m$. *Then A is p-compact if and only if it is strongly p-compact.*

Proof. We give a proof in $\mathbb{R}^2$. Suppose $A \subseteq \mathbb{R}^2$ is p^s-closed and p-compact. Then A is left-K-sequentially complete and hence it has the B-property. Let $x_0 = \sup P_{\{1\}}(A)$, then $x_0 \in \overline{P_{\{1\}}(A)}^{q^s}$. Since $P_{\{1\}}(A)$ has the B-property, there is $u \in P_{\{1\}}(A)$ such that $u \geq x_0$. Since x_0 is the supremum of $P_{\{1\}}(A)$, it follows that $x_0 = u$. Let

$$y_0 = \sup\{y \in \mathbb{R} : (x_0, y) \in A\}.$$

There is a sequence $(y_n)_{n=1}^{\infty}$ that q^s-converges to y_0 such that $(x_0, y_n) \in A$ for every n. Since A is p^s-closed, $(x_0, y_0) \in A$. We now let $y_0' = \sup P_{\{2\}}(A)$. Then $y_0' \in \overline{P_{\{2\}}(A)}^{q^s}$. Since $P_{\{2\}}(A)$ has the B-property, there is $u' \in P_{\{2\}}(A)$ such that

$u' \geq y_0'$. Since y_0' is the supremum of $P_{\{2\}}(A)$, it follows that $u' = y_0'$. Let

$$x_0' = \sup\{x \in \mathbb{R} : (x, y_0') \in A\}.$$

There is a sequence $(x_n)_{n=1}^{\infty}$ that q^s-converges to x_0' such that $(x_n, y_0') \in A$ for every n. Since A is p^s-closed, $(x_0', y_o') \in A$. Let

$$R = \{(x, y) \in \mathbb{R}^2 : x_0' \leq x \leq x_0, y_0 \leq y \leq y_0'\}, \tag{4.5}$$

then R is p^s-closed and p^s-bounded. Now let

$$A_0 = R \cap A.$$

Then A_0 is p^s-closed, since A and R are p^s-closed (the intersection of p^s-closed sets is p^s-closed) and obviously A_0 is bounded. Hence by the Heine Borel theorem, A_0 is p^s-compact. Clearly, $A_0 \subseteq A$. We show that $A \subseteq A_0 + \theta_{\mathbf{0}}^p$. Take $\mathbf{x} = (x, y) \in A$. Then $x \leq x_0$ and $y \leq y_0'$. If $\mathbf{x} \in A_0$, then $\mathbf{x} \in A_0 + \theta_{\mathbf{0}}^p$, since $\mathbf{0} \in \theta_{\mathbf{0}}^p$ and $\mathbf{x} = \mathbf{x} + \mathbf{0}$. If $\mathbf{x} \notin A_0$, then either $x \leq x_0'$ or $y \leq y_0$. Suppose $x \leq x_0'$, then $(x, y) \leq (x_0', y_0')$. Therefore,

$$(x, y) \in \theta_{(x_0', y_0')}^p = \{(x_0', y_0')\} + \theta_{\mathbf{0}}^p \subseteq A_0 + \theta_{\mathbf{0}}^p,$$

since $(x_0', y_0') \in A_0$. Now suppose $y \leq y_0$, then $(x, y) \leq (x_0, y_0)$. Therefore,

$$(x, y) \in \theta_{(x_0, y_0)}^p = \{(x_0, y_0)\} + \theta_{\mathbf{0}}^p \subseteq A_0 + \theta_{\mathbf{0}}^p.$$

Therefore, $A \subseteq A_0 + \theta_0^p$. $\square$

The following example shows that there is a subset of $\mathbb{R}^2$ that is not p^s-closed but is strongly p-compact.

Example 4.5.13. *In the asymmetrically normed lattice* $(\mathbb{R}^2, p)$*, let*

$$A = \left\{ \left(\frac{1}{n}, \frac{1}{n} \right) : n \in \mathbb{N} \right\}.$$

By Proposition 4.2.3, A is p-compact, since $\sup A = (1, 1) \in A$. *Let* $A_0 = \{(1, 1)\}$, *then A_0 is p^s-compact and*

$$A_0 \subseteq A \subseteq A_0 + \theta_{\mathbf{0}}^p.$$

But A is not p^s-closed, since

$$\left(\frac{1}{n}, \frac{1}{n}\right) \xrightarrow{p^s} (0,0) \notin A.$$

The following example shows that there is a p-compact subset A of $\mathbb{R}^2$ that does not contain its supremum and is not p^s-closed but is strongly p-compact.

Example 4.5.14. *In the asymmetrically normed lattice $(\mathbb{R}^2, p)$, let*

$$A = \left\{(x,y) \in \mathbb{R}^2 : x < 0, y < \frac{1}{x}\right\} \cup \{(-1,0)\} \cup \{(0,-1)\}.$$

Then A is bounded above and hence by Proposition 4.3.5, it is p-precompact. To see that A is left-K-sequentially complete, note that

$$P_{\{1\}}(A) = (-\infty, 0] \text{ and } P_{\{2\}}(A) = (-\infty, 0]$$

both have the B-property in R, since they are q^s-closed. A also has the B-property in $\mathbb{R}^2$, since every point $(a,b) \in \overline{A}^{p^s}$ is bounded above by either $(-1,0)$ or $(0,-1)$. Therefore by Theorem 3.3.14 A is left-K-sequentially complete. Hence by Theorem 4.1.11, it is p-compact. Let

$$A_0 = \{(-1,0)\} \cup \{(0,-1)\}.$$

Then A_0 is p^s-compact and $A_0 \subseteq A$. We show that $A \subseteq A_0 + \theta^p_{\mathbf{0}}$. Take $\mathbf{x} = (x_2, y_2) \in A$. If $\mathbf{x} \in A_0$, then $\mathbf{x} = \mathbf{x} + \mathbf{0} \in A_0 + \theta^p_{\mathbf{0}}$. If $\mathbf{x} \notin A_0$, then either $x_2 \leq -1$ or $y_2 \leq -1$. Suppose $x_2 \leq -1$, then $(x_2, y_2) \leq (-1, 0)$ and therefore

$$(x_2, y_2) \in \theta^p_{(-1,0)} \subseteq A_0 + \theta^p_{\mathbf{0}}.$$

Similarly for $y_2 \leq -1$

$$(x_2, y_2) \in \theta^p_{(0,-1)} \subseteq A_0 + \theta^p_{\mathbf{0}}.$$

Thus

$$A_0 \subseteq A \subseteq A_0 + \theta^p_{\mathbf{0}}.$$

Remark 4.5.15. *Let A be a p-compact subset of $\mathbb{R}^m$ and R be defined as in* (4.5)*. If $R \cap A$ is p^s-closed, then A is strongly p-compact.*

We give an example of set in $\mathbb{R}^2$ which is not p^s-closed but satisfies the above condition.

Example 4.5.16. *In the asymmetrically normed lattice $(\mathbb{R}^2, p)$, let*

$$A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\},$$

$$B = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0, x^2 + y^2 = 1\}$$

and

$$C = A \cup B.$$

Then C is not p^s-closed, since

$$C \neq \overline{C}^{p^s} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}.$$

Also, $\sup P_{\{1\}}(C) = 1 = \sup P_{\{2\}}(C)$ *and*

$$\sup\{x \in \mathbb{R} : (x, 1) \in C\} = 0 = \sup\{y \in \mathbb{R} : (1, y) \in C\}.$$

The points $(1, 0)$ and $(0, 1)$ are in C and correspond to the points (x_0, y_0) and (x_0', y_0') in the proof of Theorem 4.5.12. We define the set R by

$$R = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1, 0 \leq y \leq 1\}.$$

Then

$$R \cap C = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0, x^2 + y^2 \leq 1\}$$

is p^s-closed and p^s-bounded. Hence by the Heine Borel theorem it is p^s-compact. Let $C_0 = R \cap C$, then

$$C_0 \subseteq C \subseteq C_0 + \theta^p_{\mathbf{0}}.$$

We show that there are p-compact sets of $\mathbb{R}^m$ that are not strongly p-compact.

Example 4.5.17. *In the asymmetrically normed lattice* $(\mathbb{R}^2, p)$, *let*

$$A = \left(\{(x,y) \in \mathbb{R}^2 : x \geq 0, y \geq 0, x^2 + y^2 = 1\} \cup \left\{\left(1, \frac{\sqrt{3}}{2}\right)\right\}\right) \setminus \left\{\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)\right\}.$$

Then A *has the* B*-property in* $\mathbb{R}^2$, *since* $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) < \left(1, \frac{\sqrt{3}}{2}\right)$ *and* $\left(1, \frac{\sqrt{3}}{2}\right) \in A$, *and* $P_{\{1\}}(A)$ *and* $P_{\{2\}}(A)$ *have the* B*-property in* $\mathbb{R}$, *since*

$$P_{\{1\}}(A) = [0,1] = P_{\{2\}}(A)$$

is q^s*-closed. Therefore, by Proposition 3.3.12* A *is left-*K*-sequentially complete. Since* A *is bounded above, it is* p*-precompact (see Proposition 4.3.5). Hence by Theorem 4.1.11,* A *is* p*-compact. Now let*

$$B = \left\{(x,y) \in \mathbb{R}^2 : 0 \leq x < \frac{1}{2}, x^2 + y^2 = 1\right\}$$

and $C = B \cup \left\{\left(1, \frac{\sqrt{3}}{2}\right)\right\}$, *then* $C \subseteq A \subseteq C + \theta^p_{\mathbf{0}}$. *We show that there is no* p^s*-closed subset* A_0 *such that*

$$A_0 \subseteq A \subseteq A_0 + \theta^p_{\mathbf{0}}.$$

Suppose there is $A_0 \subseteq A$, A_0 p^s*-closed and*

$$A_0 \subseteq A \subseteq A_0 + \theta^p_{\mathbf{0}}.$$

Then we must have $C \subseteq A_0$. *To see this, suppose* $C \nsubseteq A_0$, *then there is* $x \in C$ *such that* $x \notin A_0$. *Therefore* $x \notin A_0 + \theta^p_{\mathbf{0}}$ *which is a contradiction, since*

$$C \subseteq A \subseteq A_0 + \theta^p_{\mathbf{0}}.$$

Since $C \subseteq A_0$, *we have* $\overline{C}^{p^s} \subseteq \overline{A}_0^{p^s}$. *Therefore* $\overline{A}_0^{p^s} = A_0$, *since* A_0 *is* p^s*-closed. But* $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \in A_0$. *Therefore* $A_0 \nsubseteq A$, *since* $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \notin A$. *Therefore,* A *is not strongly* p*-compact.*

We now show that there is a p-compact set in $\mathbb{R}^2$ which does not have the $B(C)$-property (see Definition 4.4.14) but is strongly p-compact.

Example 4.5.18. *([2, Example 19]) In the asymmetrically normed lattice* $(\mathbb{R}^2, p)$*, let*

$$A = \{(x, y) \in \mathbb{R}^2 : (x^+)^2 + (y^+)^2 \leq 1\}.$$

Now let

$$C = A \cup \left\{\left(1, \frac{1}{2}\right)\right\},$$

then C *does not have the* $B(C)$*-property (see [2, Example 19]). We show that* C *is* p*-compact by showing that it is* p*-precompact and left-*K*-sequentially complete (see Theorem 4.1.11). Since* C *is bounded above, then by Proposition 4.3.5 it is* p*-precompact. Since* C *is* p^s*-closed (the union of two* p^s*-closed sets is* p^s*-closed), it has the* B*-property in* $\mathbb{R}^2$ *and*

$$P_{\{1\}}(C) = (-\infty, 1],$$

$$P_{\{2\}}(C) = (-\infty, 1]$$

have B*-property in* $\mathbb{R}$*, since they are* q^s*-closed. Therefore by Proposition 3.3.14,* C *is left-*K*-sequentially complete, hence* p*-compact. Since* C *is* p^s*-closed and* p*-compact, by Theorem 4.5.12 it is strongly* p*-compact.*

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