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ASPECTS OF MULTIVARIATE COMPLEX QUADRATIC FORMS

by

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CHAPTER 1

INTRODUCTION, SUMMARY AND NOTATION

1.1 INTRODUCTION AND SUMMARY

In this study the distributional properties of certain multivariate complex quadratic forms and their characteristic roots are investigated.

Multivariate complex distribution theory was originally introduced by Wooding (1956), Turin (1960) and Goodman (1963 a) when they derived and studied the multivariate complex normal distribution. The multivariate complex normal distribution is the basis of complex distribution theory and plays an important role in various areas. In the area of multiple time-series the complex distribution theory is found very useful. For some discussions of this topic the reader is referred to Wahba (1968, 1971), Goodman and Dubman (1969), Hannan (1970), Smith (1972), Priestly, Subba Rao and Tong (1973), Brillinger (1974), Krishnaiah (1976), Steel (1979) and Shaman (1980). The complex normal distribution and related distributions are also found very useful in the field of nuclear physics especially in studying the energy levels of atomic nuclei. For further detail the reader is referred to Dyson (1962 a, 1962 b, 1962 c), Dyson and Mehta (1963 a, 1963 b) and Carmeli (1974).

As is the case in real multivariate statistical analysis, multivariate complex quadratic forms play an important role in complex multivariate statistical analysis. In many cases test statistics are functions of multivariate complex quadratic forms. Such functions appear for example in the analysis of variance of complex normal variates (cf. Khatri, 1970).

Various types of multivariate real quadratic forms are studied in the literature (cf. Khatri, 1966; Hayakawa, 1966, 1972 a, 1972 b; Crowther, 1972, 1975 and Underhill, 1973) and although some attention is given to certain equivalent multivariate complex quadratic forms in a number of these publications, it is felt that a systematic study of multivariate complex quadratic forms could still be very useful. All the multivariate complex quadratic forms studied in this thesis are random hermitian matrices and are such that the joint probability density functions (p.d.f.s) of their characteristic roots are symmetric functions of the roots. As a result of this property of the joint p.d.f.s of the roots, the derivation of certain marginal distributions of the roots is less complicated than in the case of a real symmetric matrix. Since this aspect of the characteristic roots of complex multivariate quadratic forms is not studied in the literature, it is also investigated in this thesis.

Hence, the purpose of this study is the following:

(i) To derive the p.d.f.s and moments of various multivariate complex quadratic forms,

(ii) to derive the joint p.d.f.s of the characteristic roots of these complex quadratic forms,

(iii) to derive certain marginal distributions of these characteristic roots and

(iv) to derive the p.d.f.s and cumulative distribution functions (c.d.f.s) of certain functions of these roots.

It is important to note that the p.d.f.s, moments and joint p.d.f.s of the characteristic roots of complex quadratic forms are derived in similar ways as in the real case. Despite this correspondence between the real and complex cases, the complex results are not merely a duplication of results in the real case and therefore the complete derivation of these complex results are given.

In order to derive the different p.d.f.s, moments and c.d.f.s, a thorough understanding of certain mathematical functions and techniques is essential. These mathematical functions and techniques are offered in chapter 2. Special attention will be given to hermitian matrices and zonal polynomials of hermitian matrices because of their important applications in multivariate complex statistical theory. Certain properties of these polynomials which lead to results that differ from the results for zonal polynomials of symmetric matrices will be studied in particular. Hypergeometric functions with hermitian matrix arguments, Hayakawa's polynomials with complex matrix arguments, Meijer's G-function, the symmetrised p.d.f. of a hermitian random matrix, the incomplete gamma- and beta functions and the complex multivariate normal- and Wishart distributions are the topics which will also be considered in this chapter.

By using certain results which involve zonal polynomials of hermitian matrices and hypergeometric functions with hermitian matrix arguments, the joint p.d.f.s of the characteristic roots of the complex quadratic forms can be written in a form which displays the random components in a general form. In chapter 3 integrals are considered with this general random component as integrand and which are needed to obtain:

(i) The c.d.f. of the extreme characteristic roots,

(ii) the c.d.f. of any intermediate characteristic root,

(iii) the c.d.f. of any two intermediate characteristic roots,

(iv) the joint p.d.f. of any few ordered characteristic roots and

(v) the joint p.d.f. of any few unordered characteristic roots.

In chapter 4 the multivariate complex quadratic form of complex normal variates is considered. The p.d.f., symmetrised p.d.f. and moments of this quadratic form are derived for different specifications of the parameter matrices. Two types of representations of the p.d.f. of this quadratic form are discussed, i.e. the power-series representations and the Γ-type representation. The relationship between these two representations will also be discussed. By using the theory given in chapter 2 and the integrals in chapter 3, the different marginal distributions of the characteristic roots of this quadratic form are derived. The multivariate compound quadratic form of complex normal variates is also considered in this chapter.

It is felt that it could be useful to give attention to multivariate complex beta distributions in this study since multivariate complex beta matrices are essentially complex quadratic forms. In chapter 6 and 7 extensions of these complex beta distributions are considered. Chapter 5 thus deals with complex multivariate beta distributions. The p.d.f.s , symmetrised p.d.f.s , the joint p.d.f.s of the characteristic roots and the moments of the multivariate complex beta type 1A-, 2A- and 2B random matrices are given. By using results of chapter 2 and the integrals of chapter 3, expressions for certain marginal distributions of the characteristic roots of these beta-matrices are derived. Some of these expressions are better from a computational point of view than existing expressions found in the literature. The generalised sample multiple coherence matrix has a complex multivariate beta distribution and therefore its distributional properties are also discussed in this chapter.

The complex beta type 1 matrix can be extended to the cases where:

(i) The complex Wishart matrix which appears in the numerator and the denominator of the complex beta matrix is replaced by a multivariate central complex quadratic form of normal variates,

(ii) both complex Wishart matrices which appear in the complex beta matrix are replaced by multivariate central complex quadratic forms of normal variates.

The distributional properties of these quadratic forms and their characteristic roots are investigated in chapter 6.

As above, the complex beta type 2 matrix can also be extended to the cases where:

(i) The complex Wishart matrix which appears in the numerator of the beta matrix is replaced by a multivariate central complex quadratic form of normal variates,

(ii) both complex Wishart matrices which appear in the beta matrix are replaced by multivariate central complex quadratic forms of normal variates.

In chapter 7 the distributional properties of these quadratic forms and their characteristic roots are investigated.

1.2 NOTATION

The following general notation will be used in this thesis:

$A:p\times q = (a_{ij})$	:	Real or complex matrix with p rows and q columns, the ij-th element being a_{ij}
A^{-1}	:	Inverse of the real or complex matrix $A:p\times p$
A'	:	Transpose of the real or complex matrix $A:p\times q$
$\lvert A\rvert = \lvert(a_{ij})\rvert$	:	Determinant of the real or complex matrix $A:p\times p$

tr(A)	:	Trace of the real or complex matrix A:p×p, i.e. $\sum_{i=1}^{p} a_{ii}$
exp(a)	:	e^a
etr(A)	:	Exp{tr(A)}
L.H.S.	:	Left-hand side
R.H.S.	:	Right-hand side
$\bar{A} = (\bar{a}_{ij})$	:	The conjugate of the complex matrix A:p×p, the ij-th element being $\bar{a}_{ij}$
$A:p\times p > 0$	:	The real or complex matrix A:p×p positive definite
$0 < A:p\times p < B:p\times p$	:	The real or complex matrix A:p×p positive definite and $B - A$ positive definite
A:p×p h.p.d.	:	The complex matrix A:p×p hermitian positive definite
$\tilde{a}_i$	:	The i-th characteristic root of the real or complex matrix A:p×p
$D_A = \begin{bmatrix} \tilde{a}_1 & 0 & \dots & 0 \\ 0 & \tilde{a}_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \tilde{a}_p \end{bmatrix} = \text{Diag}\,(\tilde{a}_1,\cdots,\tilde{a}_p)$	:	Diagonal matrix with diagonal elements the characteristic roots $\tilde{a}_1,\cdots,\tilde{a}_p$ of A:p×p

The following notation will be used in connection with real and complex random variables, random vectors, random matrices and characteristic roots of random matrices as well as the "values" of these variates:

X	: Real or complex random variable
x	: Value of X
$f_X(x)$	: P.d.f. of X
$\underline{X}:p\times 1 = \begin{bmatrix} X_1 \\ \vdots \\ X_p \end{bmatrix}$	: Real or complex random vector i.e. a p-dimensional vector with random variables as elements
$\underline{x}:p\times 1 = \begin{bmatrix} x_1 \\ \vdots \\ x_p \end{bmatrix}$	: Vector of values of $\underline{X}:p\times 1$
$f_{\underline{X}}(\underline{x})$	: P.d.f. of $\underline{X}:p\times 1$
$\underset{\sim}{X}:p\times q = \begin{bmatrix} X_{11} & \cdots & X_{1q} \\ \cdots & \cdots & \cdots \\ X_{p1} & \cdots & X_{pq} \end{bmatrix}$	: Real or complex random matrix i.e. a matrix of order $p\times q$ with real or complex random variables as elements
$X:p\times q = \begin{bmatrix} x_{11} & \cdots & x_{1q} \\ \cdots & \cdots & \cdots \\ x_{p1} & \cdots & x_{pq} \end{bmatrix}$	: Matrix of values of $\underset{\sim}{X}:p\times q$
$f_{\underset{\sim}{X}}(X)$	: P.d.f. of $\underset{\sim}{X}:p\times q$
$\tilde{X}_i$	: The i-th characteristic root of the real or complex random matrix $\underset{\sim}{X}:p\times p$

$\tilde{x}_i$	: Value of $\tilde{X}_i$
$f_{\tilde{X}_i}(\tilde{x}_i)$	: P.d.f. of $\tilde{X}_i$
$\underset{\sim}{D}_X = \begin{bmatrix} \tilde{X}_1 & 0 & \dots & 0 \\ 0 & \tilde{X}_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \tilde{X}_p \end{bmatrix}$	: Diagonal matrix with diagonal elements the characteristic roots $\tilde{X}_1,\cdots,\tilde{X}_p$ of $\underset{\sim}{X}:p\times p$
$D_X = \begin{bmatrix} \tilde{x}_1 & 0 & \dots & 0 \\ 0 & \tilde{x}_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \tilde{x}_p \end{bmatrix}$	: Matrix of values of $\underset{\sim}{D}_X$
$f_{\underset{\sim}{D}_X}(D_X)$	: Joint p.d.f. of the characteristic roots $\tilde{X}_1,\cdots,\tilde{X}_p$ of $\underset{\sim}{X}:p\times p$
$f_{tr\underset{\sim}{X}}(trX)$	: P.d.f. of $tr\underset{\sim}{X} = \sum_{i=1}^{p} X_{ii} = \sum_{i=1}^{p} \tilde{X}_i$
$f_{\lvert\underset{\sim}{X}\rvert}(\lvert X\rvert)$	: P.d.f. of $\lvert\underset{\sim}{X}\rvert = \prod_{i=1}^{p} X_{ii} = \prod_{i=1}^{p} \tilde{X}_i$

Remark 1.2.1

The letter "i" will as is usual be used as the imaginary constant, $i = \sqrt{-1}$, and will only be used as a subscript when there is no possibility of confusion.

CHAPTER 2

MATHEMATICAL CONCEPTS AND RELATED THEORY

2.1 INTRODUCTION

A thorough understanding of certain mathematical functions and techniques is essential when multivariate complex quadratic forms are considered. In this chapter results to be used in the subsequent chapters and which involve these functions and techniques are briefly discussed.

Hermitian matrices, zonal polynomials of hermitian matrices and certain important results involving these polynomials are considered in section 2.2. A few new results on these polynomials are proved in this section. In section 2.3 the hypergeometric function of a single variable and the hypergeometric function with hermitian matrix arguments are discussed. Hayakawa's polynomials with complex matrix arguments which are very useful in the study of complex quadratic forms are offered in section 2.4. Section 2.5 contains results which make it possible to derive certain p.d.f.s of certain random variables in terms of Meijer's G-function. Various marginal distributions of characteristic roots of complex random matrices can be expressed in terms of the incomplete gamma- and beta functions and therefore these functions are given in section 2.6. In section 2.7 the by now well-known symmetrised p.d.f. of a real random matrix variate is extended to the symmetrised p.d.f. of a hermitian random matrix variate. The complex multivariate normal and Wishart distributions are briefly discussed in section 2.8. Although these two distributions are widely discussed in the literature, it is felt that for completeness of this thesis attention has to be given to it.

2.2 ZONAL POLYNOMIALS OF HERMITIAN MATRICES

Zonal polynomials of real symmetric matrices play an important role in the real multivariate statistical theory. The theory in this regard can be found in James (1960, 1961, 1964) and Constantine (1963, 1966) while a large amount of results on zonal polynomials of real symmetric matrices and their applications to various branches of multivariate statistical analysis can be found in Subrahmaniam (1976).

As in the real case zonal polynomials of hermitian matrices have important applications in the multivariate complex statistical theory. James (1964, p. 487 - 491) gives a brief description of zonal polynomials of hermitian matrices. In this section zonal polynomials of hermitian matrices are discussed and certain properties of these polynomials which lead to results that differ from the results for zonal polynomials of real symmetric matrices are considered.

In section 2.2.1 the definition and some important properties of a hermitian matrix are given. A few jacobians of hermitian and complex matrix transformations are also considered. The definition and properties of zonal polynomials of a hermitian matrix are given in section 2.2.2. One of these properties is used to derive expressions for these polynomials in terms of the monomial symmetric functions of the hermitian matrix, the characteristic roots of the hermitian matrix and the sum of the g-th powers of the characteristic roots. Tables containing these expressions are given. In section 2.2.3 results involving zonal polynomials of hermitian matrices which are to be used in this thesis will be given.

2.2.1 Hermitian matrices

Consider the following definition:

<u>Definition 2.2.1</u> (Macduffee, 1946, p. 62)

The complex matrix $A:p\times p$, the $k\ell$-th element being

$a_{k\ell} = a_{k\ell R} + i\, a_{k\ell I}$, is defined as a hermitian matrix if $A = \bar{A}'$.

From definition 2.2.1 follows that

$$a_{k\ell} = \bar{a}_{\ell k} = a_{\ell k R} - i\, a_{\ell k I},$$

$$a_{k\ell R} = a_{\ell k R},$$

$$a_{k\ell I} = -a_{\ell k I}, \quad (k, \ell = 1, \cdots, p)$$

and hence the following properties of the hermitian matrix $A:p\times p$ can easily be verified:

Property 2.2.1

$$A = A_1 + i\, A_2$$

with

$A_1:p\times p$ a real symmetric matrix

and

$A_2:p\times p$ a real skewsymmetric matrix.

Property 2.2.2

$$\operatorname{tr}(A) = \operatorname{tr}(A_1)$$

Theorem 2.2.1

Let $A:p\times p$ be a hermitian matrix; then:

(i) There exists a unitary matrix U (i.e. $U\bar{U}' = \bar{U}'U = I_p$) such that $A = U D_A \bar{U}'$,

(ii) the characteristic roots $\tilde{a}_1, \cdots, \tilde{a}_p$ of $A:p\times p$ are all real,

(iii) if $A:p\times p$ is positive definite, it can be uniquely expressed as $A = T\bar{T}'$ where $T:p\times p$ is a lower triangular matrix with real positive diagonal elements,

(iv) if $A:p \times p$ is positive semi-definite, it can be uniquely expressed as $A = T\bar{T}'$ where $T:p \times p$ is a hermitian positive semi-definite matrix.

Proof

(i) Macduffee (1946, p. 75).

(ii) Macduffee (1946, p. 26).

(iii) Goodman (1957, p. 146).

(iv) Macduffee (1946, p. 77).

Remark 2.2.1

(i) As for a symmetric positive definite real matrix the roots of a hermitian positive definite matrix are all positive.

(ii) As for a symmetric positive semi-definite real matrix the roots of a hermitian positive semi-definite matrix are non-negative and at least one root is equal to zero.

In theorem 2.2.2 jacobians of complex and hermitian matrix transformations which will be used repeatedly in this thesis, are given.

Theorem 2.2.2

(i) Let $A:p \times r$ and $B:p \times r$ be complex matrices and $C:p \times p$ h.p.d.. Consider the transformation:

(2.2.1) $A = CB$.

The jacobian of (2.2.1) is given by

(2.2.2) $J(A \to B) = |C|^{2r}$.

(ii) Let $A:p \times r$ and $B:p \times r$ be complex matrices and $C:r \times r$ h.p.d.. Consider the transformation:

(2.2.3) $A = BC$.

The jacobian of (2.2.3) is given by

(2.2.4) $J(A \to B) = |C|^{2p}$.

(iii) Let $A:p\times p$, $B:p\times p$ and $C:p\times p$ be h.p.d.. Consider the transformation:

(2.2.5) $A = CBC$.

The jacobian of (2.2.5) is given by

(2.2.6) $J(A \to B) = |C|^{2p}$.

(iv) Let $A:p\times p$ and $B:p\times p$ be h.p.d.. Consider the transformation:

(2.2.7) $A = B^{-1}$.

The jacobian of (2.2.7) is given by

(2.2.8) $J(A \to B) = |B|^{-2p}$.

Proof

(i) Smith (1972, p. 90).

(ii) Smith (1972, p. 90).

(iii) Khatri (1965, p. 102).

(iv) Taking differentials of $A = B^{-1}$ leads to

(2.2.9) $dA = -B^{-1}\,dB\,B^{-1}$.

From Deemer and Olkin (1951, p. 366) and Smith (1972, p. 87) follows:

(2.2.10) $J(dA \to dB) = J(A \to B)$.

The application of (2.2.10) and (2.2.6) leads to (2.2.8).

2.2.2 Definition and properties of zonal polynomials of a hermitian matrix

Consider the following definition:

Definition 2.2.2

The zonal polynomial $\tilde{C}_\kappa(A)$ is defined as the component of $(\mathrm{tr}(A))^k$ in the irreducible invariant subspace V_κ which is the vector space of homogeneous polynomials $\phi(A)$ of degree k in the $n = \frac{1}{2}p(p+1)$ different complex elements of the hermitian matrix $A{:}p\times p$.

In definition 2.2.2 $\kappa = (k_1,\cdots,k_p)$ is a partition of the integer k i.e. a set of integers $k_1 \geqq \cdots \geqq k_p \geqq 0$ such that $\sum_{i=1}^{p} k_i = k$; $V_k = \oplus V_\kappa$ and $(\mathrm{tr}(A))^k$ has a unique decomposition in terms of zonal polynomials:

$$(2.2.11) \quad (\mathrm{tr}(A))^k = \sum_\kappa \tilde{C}_\kappa(A)\,.$$

The summation in (2.2.11) is taken over all partitions κ of k into not more than p parts. For $p = 1$ follows that

$$(2.2.12) \quad \tilde{C}_\kappa(A) = \tilde{C}_\kappa(a) = a^k\,.$$

A more complete definition of $\tilde{C}_\kappa(A)$ can be found in Steel (1979, p. 24 - 25).

Property 2.2.3

$$(2.2.13) \quad \tilde{C}_\kappa(bA) = b^k\,\tilde{C}_\kappa(A)$$

for b a scalar.

Property 2.2.4

The zonal polynomial $\tilde{C}_\kappa(A)$ is invariant under the unitary group, i.e.

$$(2.2.14) \quad \tilde{C}_\kappa(A) = \tilde{C}_\kappa(UA\bar{U}')$$

for every $U \varepsilon U(p)$ ($U(p)$ is the group of $p\times p$ unitary matrices) and is a symmetric homogeneous polynomial of degree k in the characteristic roots of $A:p\times p$.

<u>Definition 2.2.3</u>

For $A:p\times p$ a hermitian matrix, $R:p\times p$ h.p.d.:

$$(2.2.15) \quad \tilde{C}_\kappa(RA) = \tilde{C}_\kappa(R^{\frac{1}{2}} A R^{\frac{1}{2}})$$

where $R^{\frac{1}{2}}$ is the unique positive definite square root of R and where the expression $R^{\frac{1}{2}} A R^{\frac{1}{2}}$ really stands for $R^{\frac{1}{2}} A \bar{R}^{\frac{1}{2}\,'}$.

Definition 2.2.3 is justified by the fact that RA and $R^{\frac{1}{2}} A R^{\frac{1}{2}}$ have the same characteristic roots. It is clear that the definitions and properties of zonal polynomials of a hermitian matrix discussed above are analogous to those of zonal polynomials of a real symmetric matrix. A few definitions will be given below to understand property 2.2.5. This property in particular is of great importance in the case of zonal polynomials of a hermitian matrix.

<u>Definition 2.2.4</u> (James, 1964, p. 481)

The i-th monomial symmetric function of the characteristic roots of the matrix $A:p\times p$ is defined as

$$(2.2.16) \quad \alpha_i = \sum_{r_1+\cdots+r_p=i} \tilde{a}_1^{r_1}\cdots\tilde{a}_p^{r_p} \quad \text{if} \quad i=1,2,\cdots$$

$$= 1 \quad \text{if} \quad i=0$$

$$= 0 \quad \text{if} \quad i<0 ,$$

i.e. the sum of all monomials of $\tilde{a}_1,\cdots,\tilde{a}_p$ of degree i.

<u>Definition 2.2.5</u> (James, 1964, p. 481)

The i-th elementary symmetric function of the matrix $A:p\times p$ is defined as

$$(2.2.17)\quad \alpha_i^* = \sum_{1\leq r_1<\cdots<r_i\leq p} \tilde{a}_{r_1}\cdots\tilde{a}_{r_i} \quad \text{if} \quad i=1,2,\cdots,p$$

$$= 1 \quad \text{if} \quad i=0$$

$$= 0 \quad \text{if} \quad i<0\ .$$

<u>Definition 2.2.6</u> (Littlewood, 1940, p. 60)

The partition $\hat{\kappa} = (\hat{k}_1,\cdots,\hat{k}_p)$ which is conjugate to the partition $\kappa = (k_1,\cdots,k_p)$ is defined as

$$\hat{\kappa} = (p^{k_p}, (p-1)^{k_{p-1}-k_p}, (p-2)^{k_{p-2}-k_{p-1}}, \cdots, 1^{k_1-k_2})$$

$$\text{i.e.}\quad \hat{k}_i = (p-i+1)^{k_{p-i+1}-k_{p-i+2}},\ (i=1,\cdots,p)\ ,\ \hat{k}_{p+1} = 0\ .$$

<u>Definition 2.2.7</u> (James 1964, p. 491, Constantine 1963, p. 1272)

The two partitions $\kappa = (k_1,\cdots,k_p)$ and $\tau = (t_1,\cdots,t_p)$ of k are ordered lexicographically if $k_1 = t_1,\cdots,k_i = t_i$, $k_{i+1} > t_{i+1}$, $(i=0,\cdots,p-1)$ and is written as $\kappa > \tau$.

<u>Property 2.2.5</u> (James 1964, p. 487 - 488)

For $A:p\times p$ hermitian:

$$(2.2.18)\quad \tilde{C}_\kappa(A) = \chi_{[\kappa]}(1)\ \chi_{\{\kappa\}}(A)$$

where $\chi_{[\kappa]}(1)$ is the dimension of the representation $[\kappa]$ of the symmetric group and is given as

$$(2.2.19)\quad \chi_{[\kappa]}(1) = \frac{k!\ \prod_{i<j}^{p} (k_i-k_j-i+j)}{\prod_{i=1}^{p} (k_i+p-i)!}$$

and $\chi_{\{\kappa\}}(A)$ is the character of the representation $\{k\}$ of the linear group and is given as a symmetric function of the characteristic roots $\tilde{a}_1,\cdots,\tilde{a}_p$ of $A:p\times p$:

$$(2.2.20) \quad \chi_{\{\kappa\}}(A) = \frac{|(\tilde{a}_i^{k_j+p-j})|}{|(\tilde{a}_i^{p-j})|} = \frac{|(\tilde{a}_j^{k_i+p-i})|}{|(\tilde{a}_j^{p-i})|}$$

$$(2.2.21) \quad = |(\alpha_{k_j-j+i})|$$

$$(2.2.22) \quad = |(\alpha^*_{\hat{k}_j-j+i})|$$

$$(2.2.23) \quad = \alpha_1^{*\,k_1-k_2} \cdot \alpha_2^{*\,k_2-k_3} \cdots \alpha_p^{*\,k_p} + \text{"terms of lower weight".}$$

By "terms of lower weight" is meant monomials in α^*_j similar to the one displayed in (2.2.23) but corresponding to the partition $\tau < \kappa$.

<u>Theorem 2.2.3</u>

$$(2.2.24) \quad \tilde{C}_\kappa(I_p) = \{\chi_{[\kappa]}(1)\}^2 \frac{\tilde{\Gamma}_p(p,\kappa)}{k!\ \tilde{\Gamma}_p(p)}$$

where

$$(2.2.25) \quad \tilde{\Gamma}_p(t,\kappa) = \pi^{\frac{1}{2}p(p-1)} \prod_{i=1}^{p} \Gamma(t+k_i-i+1)$$

$$= \tilde{\Gamma}_p(t)\ [t]_\kappa$$

with

$$(2.2.26) \quad \tilde{\Gamma}_p(t) = \pi^{\frac{1}{2}p(p-1)} \prod_{i=1}^{p} \Gamma(t-i+1) ,$$

$$(2.2.27) \quad [t]_\kappa = \prod_{i=1}^{p} (t-i+1)_{k_i}$$

and

(2.2.28) $(a)_k = a(a+1) \cdots (a+k-1)$, $a_0 = 1$.

Proof

Khatri (1970, p. 66 - 67).

It will become clear in the rest of this thesis that the expression given in (2.2.20) along with (2.2.18) are very useful in the derivation of certain marginal distributions of the characteristic roots of certain random h.p.d. matrices. The expression given in (2.2.21) for $\chi_{\{\kappa\}}(A)$ is used to express $\tilde{C}_\kappa(A)$ in terms of α_i , $(i=1,\cdots,k)$. These formulae are given in table 2.2.1 for $k=1,2,3$ and 4.

TABLE 2.2.1

Zonal polynomials of a hermitian matrix in terms of α_i, $(i=1,\cdots,k)$

k	κ	$\tilde{C}_\kappa(A)$
1	1	α_1
2	1^2	$\alpha_1^2 - \alpha_2$
	2	α_2
3	1^3	$\alpha_1^3 - 2\alpha_1\alpha_2 + \alpha_3$
	21	$2(\alpha_1\alpha_2 - \alpha_3)$
	3	α_3
4	1^4	$\alpha_1^4 - 3\alpha_1^2\alpha_2 + \alpha_2^2 + 2\alpha_1\alpha_3 - \alpha_4$
	21^2	$3(\alpha_1^2\alpha_2 - \alpha_2^2 - \alpha_1\alpha_3 + \alpha_4)$
	2^2	$2(\alpha_2^2 - \alpha_1\alpha_3)$
	31	$3(\alpha_1\alpha_3 - \alpha_4)$
	4	α_4

The contents of table 2.2.1 and definition 2.2.4 are used to express $\tilde{C}_\kappa(A)$ in terms of the characteristic roots of $A{:}p\times p$. These formulae for $\tilde{C}_\kappa(A)$ are given in table 2.2.2 for $p=1,2,3$ and 4 and $k=1,2,3$ and 4.

TABLE 2.2.2

Zonal polynomials of a hermitian matrix in terms of the characteristic roots

p	k	κ	$\tilde{C}_\kappa(A)$	$\tilde{C}_\kappa(I_p)$
1	1	1	$\tilde{a}_1$	1
	2	2	$\tilde{a}_1^2$	1
	3	3	$\tilde{a}_1^3$	1
	4	4	$\tilde{a}_1^4$	1
2	1	1	$\tilde{a}_1 + \tilde{a}_2$	2
	2	1^2	$\tilde{a}_1\tilde{a}_2$	1
		2	$\tilde{a}_1^2 + \tilde{a}_2^2 + \tilde{a}_1\tilde{a}_2$	3
	3	21	$2(\tilde{a}_1^2\tilde{a}_2 + \tilde{a}_1\tilde{a}_2^2)$	4
		3	$\tilde{a}_1^3 + \tilde{a}_2^3 + \tilde{a}_1^2\tilde{a}_2 + \tilde{a}_1\tilde{a}_2^2$	4
	4	2^2	$2\tilde{a}_1^2\tilde{a}_2^2$	2
		31	$3(\tilde{a}_1^3\tilde{a}_2 + \tilde{a}_1\tilde{a}_2^3 + \tilde{a}_1^2\tilde{a}_2^2)$	9
		4	$\tilde{a}_1^4 + \tilde{a}_2^4 + \tilde{a}_1^3\tilde{a}_2 + \tilde{a}_1\tilde{a}_2^3 + \tilde{a}_1^2\tilde{a}_2^2$	5
3	1	1	$\tilde{a}_1 + \tilde{a}_2 + \tilde{a}_3$	3
	2	1^2	$\tilde{a}_1\tilde{a}_2 + \tilde{a}_1\tilde{a}_3 + \tilde{a}_2\tilde{a}_3$	3
		2	$\tilde{a}_1^2 + \tilde{a}_2^2 + \tilde{a}_3^2 + \tilde{a}_1\tilde{a}_2 + \tilde{a}_1\tilde{a}_3 + \tilde{a}_2\tilde{a}_3$	6
	3	1^3	$\tilde{a}_1\tilde{a}_2\tilde{a}_3$	1
		21	$2(\tilde{a}_1^2\tilde{a}_2 + \tilde{a}_1^2\tilde{a}_3 + \tilde{a}_2^2\tilde{a}_1 + \tilde{a}_2^2\tilde{a}_3 + \tilde{a}_3^2\tilde{a}_1$ $+\tilde{a}_3^2\tilde{a}_2 + 2\tilde{a}_1\tilde{a}_2\tilde{a}_3)$	16

continued/...

p	k	κ	$\tilde{C}_\kappa(A)$	$\tilde{C}_\kappa(I_p)$
		3	$\tilde{a}_1^3 + \tilde{a}_2^3 + \tilde{a}_3^3 + \tilde{a}_1^2\tilde{a}_2 + \tilde{a}_1^2\tilde{a}_3 + \tilde{a}_2^2\tilde{a}_1 + \tilde{a}_2^2\tilde{a}_3 + \tilde{a}_3^2\tilde{a}_1 + \tilde{a}_3^2\tilde{a}_2 + \tilde{a}_1\tilde{a}_2\tilde{a}_3$	10
	4	21^2	$3(\tilde{a}_1^2\tilde{a}_2\tilde{a}_3 + \tilde{a}_1\tilde{a}_2^2\tilde{a}_3 + \tilde{a}_1\tilde{a}_2\tilde{a}_3^2)$	9
		2^2	$2(\tilde{a}_1^2\tilde{a}_2^2 + \tilde{a}_1^2\tilde{a}_3^2 + \tilde{a}_2^2\tilde{a}_3^2 + \tilde{a}_1^2\tilde{a}_2\tilde{a}_3 + \tilde{a}_1\tilde{a}_2^2\tilde{a}_3 + \tilde{a}_1\tilde{a}_2\tilde{a}_3^2)$	12
		31	$3(\tilde{a}_1^3\tilde{a}_2 + \tilde{a}_1^3\tilde{a}_3 + \tilde{a}_2^3\tilde{a}_1 + \tilde{a}_2^3\tilde{a}_3 + \tilde{a}_3^3\tilde{a}_1 + \tilde{a}_3^3\tilde{a}_2 + \tilde{a}_1^2\tilde{a}_2^2 + \tilde{a}_1^2\tilde{a}_3^2 + \tilde{a}_2^2\tilde{a}_3^2 + 2\tilde{a}_1^2\tilde{a}_2\tilde{a}_3 + 2\tilde{a}_1\tilde{a}_2^2\tilde{a}_3 + 2\tilde{a}_1\tilde{a}_2\tilde{a}_3^2)$	45
		4	$\tilde{a}_1^4 + \tilde{a}_2^4 + \tilde{a}_3^4 + \tilde{a}_1^3\tilde{a}_2 + \tilde{a}_1^3\tilde{a}_3 + \tilde{a}_2^3\tilde{a}_1 + \tilde{a}_2^3\tilde{a}_3 + \tilde{a}_3^3\tilde{a}_1 + \tilde{a}_3^3\tilde{a}_2 + \tilde{a}_1^2\tilde{a}_2^2 + \tilde{a}_1^2\tilde{a}_3^2 + \tilde{a}_2^2\tilde{a}_3^2 + \tilde{a}_1^2\tilde{a}_2\tilde{a}_3 + \tilde{a}_1\tilde{a}_2^2\tilde{a}_3 + \tilde{a}_1\tilde{a}_2\tilde{a}_3^2$	15
4	1	1	$\tilde{a}_1 + \tilde{a}_2 + \tilde{a}_3 + \tilde{a}_4$	4
	2	1^2	$\tilde{a}_1\tilde{a}_2 + \tilde{a}_1\tilde{a}_3 + \tilde{a}_1\tilde{a}_4 + \tilde{a}_2\tilde{a}_3 + \tilde{a}_2\tilde{a}_4 + \tilde{a}_3\tilde{a}_4$	6
		2	$\tilde{a}_1^2 + \tilde{a}_2^2 + \tilde{a}_3^2 + \tilde{a}_4^2 + \tilde{a}_1\tilde{a}_2 + \tilde{a}_1\tilde{a}_3 + \tilde{a}_1\tilde{a}_4 + \tilde{a}_2\tilde{a}_3 + \tilde{a}_2\tilde{a}_4 + \tilde{a}_3\tilde{a}_4$	10
	3	1^3	$\tilde{a}_1\tilde{a}_2\tilde{a}_3 + \tilde{a}_1\tilde{a}_3\tilde{a}_4 + \tilde{a}_1\tilde{a}_2\tilde{a}_4 + \tilde{a}_2\tilde{a}_3\tilde{a}_4$	4
		21	$2(\tilde{a}_1^2\tilde{a}_2 + \tilde{a}_1^2\tilde{a}_3 + \tilde{a}_1^2\tilde{a}_4 + \tilde{a}_2^2\tilde{a}_1 + \tilde{a}_2^2\tilde{a}_3 + \tilde{a}_2^2\tilde{a}_4 + \tilde{a}_3^2\tilde{a}_1 + \tilde{a}_3^2\tilde{a}_2 + \tilde{a}_3^2\tilde{a}_4 + \tilde{a}_4^2\tilde{a}_1 + \tilde{a}_4^2\tilde{a}_2 + \tilde{a}_4^2\tilde{a}_3 + 2\tilde{a}_1\tilde{a}_2\tilde{a}_3 + 2\tilde{a}_1\tilde{a}_2\tilde{a}_4 + 2\tilde{a}_2\tilde{a}_3\tilde{a}_4 + 2\tilde{a}_1\tilde{a}_3\tilde{a}_4)$	40

continued/...

p	k	κ	$\tilde{C}_\kappa(A)$	$\tilde{C}_\kappa(I_p)$
		3	$\tilde{a}_1^3 + \tilde{a}_2^3 + \tilde{a}_3^3 + \tilde{a}_4^3 + \tilde{a}_1^2\tilde{a}_2 + \tilde{a}_1^2\tilde{a}_3 + \tilde{a}_1^2\tilde{a}_4 + \tilde{a}_2^2\tilde{a}_1 + \tilde{a}_2^2\tilde{a}_3 + \tilde{a}_2^2\tilde{a}_4 + \tilde{a}_3^2\tilde{a}_1 + \tilde{a}_3^2\tilde{a}_2 + \tilde{a}_3^2\tilde{a}_4 + \tilde{a}_4^2\tilde{a}_1 + \tilde{a}_4^2\tilde{a}_2 + \tilde{a}_4^2\tilde{a}_3 + \tilde{a}_1\tilde{a}_2\tilde{a}_3 + \tilde{a}_1\tilde{a}_2\tilde{a}_4 + \tilde{a}_2\tilde{a}_3\tilde{a}_4 + \tilde{a}_1\tilde{a}_3\tilde{a}_4$	20
	4	1^4	$\tilde{a}_1\tilde{a}_2\tilde{a}_3\tilde{a}_4$	1
		21^2	$3(\tilde{a}_1^2\tilde{a}_2\tilde{a}_3 + \tilde{a}_1^2\tilde{a}_2\tilde{a}_3 + \tilde{a}_1^2\tilde{a}_3\tilde{a}_4 + \tilde{a}_1\tilde{a}_2^2\tilde{a}_3 + \tilde{a}_2^2\tilde{a}_3\tilde{a}_4 + \tilde{a}_1\tilde{a}_2^2\tilde{a}_4 + \tilde{a}_1\tilde{a}_2\tilde{a}_3^2 + \tilde{a}_1\tilde{a}_3^2\tilde{a}_4 + \tilde{a}_2\tilde{a}_3^2\tilde{a}_4 + \tilde{a}_1\tilde{a}_2\tilde{a}_4^2 + \tilde{a}_1\tilde{a}_3\tilde{a}_4^2 + \tilde{a}_2\tilde{a}_3\tilde{a}_4^2 + 3\tilde{a}_1\tilde{a}_2\tilde{a}_3\tilde{a}_4)$	45
		2^2	$2(\tilde{a}_1^2\tilde{a}_2^2 + \tilde{a}_1^2\tilde{a}_3^2 + \tilde{a}_1^2\tilde{a}_4^2 + \tilde{a}_2^2\tilde{a}_3^2 + \tilde{a}_2^2\tilde{a}_4^2 + \tilde{a}_3^2\tilde{a}_4^2 + \tilde{a}_1^2\tilde{a}_2\tilde{a}_3 + \tilde{a}_1^2\tilde{a}_2\tilde{a}_4 + \tilde{a}_1^2\tilde{a}_3\tilde{a}_4 + \tilde{a}_1\tilde{a}_2^2\tilde{a}_3 + \tilde{a}_2^2\tilde{a}_3\tilde{a}_4 + \tilde{a}_1\tilde{a}_2^2\tilde{a}_4 + \tilde{a}_1\tilde{a}_2\tilde{a}_3^2 + \tilde{a}_1\tilde{a}_3^2\tilde{a}_4 + \tilde{a}_2\tilde{a}_3^2\tilde{a}_4 + \tilde{a}_1\tilde{a}_2\tilde{a}_4^2 + \tilde{a}_1\tilde{a}_3\tilde{a}_4^2 + \tilde{a}_2\tilde{a}_3\tilde{a}_4^2 + 2\tilde{a}_1\tilde{a}_2\tilde{a}_3\tilde{a}_4)$	40
		31	$3(\tilde{a}_1^3\tilde{a}_2 + \tilde{a}_1^3\tilde{a}_3 + \tilde{a}_1^3\tilde{a}_4 + \tilde{a}_2^3\tilde{a}_1 + \tilde{a}_2^3\tilde{a}_3 + \tilde{a}_2^3\tilde{a}_4 + \tilde{a}_3^3\tilde{a}_1 + \tilde{a}_3^3\tilde{a}_2 + \tilde{a}_3^3\tilde{a}_4 + \tilde{a}_4^3\tilde{a}_1 + \tilde{a}_4^3\tilde{a}_2 + \tilde{a}_4^3\tilde{a}_3 + \tilde{a}_1^2\tilde{a}_2^2 + \tilde{a}_1^2\tilde{a}_3^2 + \tilde{a}_1^2\tilde{a}_4^2 + \tilde{a}_2^2\tilde{a}_3^2 + \tilde{a}_2^2\tilde{a}_4^2 + \tilde{a}_3^2\tilde{a}_4^2 + 2(\tilde{a}_1^2\tilde{a}_2\tilde{a}_3 + \tilde{a}_1^2\tilde{a}_2\tilde{a}_4 + \tilde{a}_1^2\tilde{a}_3\tilde{a}_4 + \tilde{a}_1\tilde{a}_2^2\tilde{a}_3 + \tilde{a}_2^2\tilde{a}_3\tilde{a}_4 + \tilde{a}_1\tilde{a}_2^2\tilde{a}_4 + \tilde{a}_1\tilde{a}_2\tilde{a}_3^2 + \tilde{a}_1\tilde{a}_3^2\tilde{a}_4 + \tilde{a}_2\tilde{a}_3^2\tilde{a}_4 + \tilde{a}_1\tilde{a}_2\tilde{a}_4^2 + \tilde{a}_1\tilde{a}_3\tilde{a}_4^2 + \tilde{a}_2\tilde{a}_3\tilde{a}_4^2) + 3\tilde{a}_1\tilde{a}_2\tilde{a}_3\tilde{a}_4)$	135

continued/...

p	k	κ	$\tilde{C}_\kappa(A)$	$\tilde{C}_\kappa(I_p)$
		4	$\tilde{a}_1^4 + \tilde{a}_2^4 + \tilde{a}_3^4 + \tilde{a}_4^4 + \tilde{a}_1^3\tilde{a}_2 + \tilde{a}_1^3\tilde{a}_3$ $+ \tilde{a}_1^3\tilde{a}_4 + \tilde{a}_2^3\tilde{a}_1 + \tilde{a}_2^3\tilde{a}_3 + \tilde{a}_2^3\tilde{a}_4 + \tilde{a}_3^3\tilde{a}_1$ $+ \tilde{a}_3^3\tilde{a}_2 + \tilde{a}_3^3\tilde{a}_4 + \tilde{a}_4^3\tilde{a}_1 + \tilde{a}_4^3\tilde{a}_2 + \tilde{a}_4^3\tilde{a}_3$ $+ \tilde{a}_1^2\tilde{a}_2^2 + \tilde{a}_1^2\tilde{a}_3^2 + \tilde{a}_1^2\tilde{a}_4^2 + \tilde{a}_2^2\tilde{a}_3^2 + \tilde{a}_2^2\tilde{a}_4^2$ $+ \tilde{a}_3^2\tilde{a}_4^2 + \tilde{a}_1^2\tilde{a}_2\tilde{a}_3 + \tilde{a}_1^2\tilde{a}_3\tilde{a}_4 + \tilde{a}_1^2\tilde{a}_2\tilde{a}_4$ $+ \tilde{a}_1\tilde{a}_2^2\tilde{a}_3 + \tilde{a}_1\tilde{a}_2^2\tilde{a}_4 + \tilde{a}_2^2\tilde{a}_3\tilde{a}_4 + \tilde{a}_1\tilde{a}_2\tilde{a}_3^2$ $+ \tilde{a}_1\tilde{a}_3^2\tilde{a}_4 + \tilde{a}_2\tilde{a}_3^2\tilde{a}_4 + \tilde{a}_1\tilde{a}_2\tilde{a}_4^2 + \tilde{a}_1\tilde{a}_3\tilde{a}_4^2$ $+ \tilde{a}_2\tilde{a}_3\tilde{a}_4^2 + \tilde{a}_1\tilde{a}_2\tilde{a}_3\tilde{a}_4$	35

The contents of table 2.2.2 are used to express $\tilde{C}_\kappa(A)$ in terms of the quantities

$$a_g = \text{sum of g-th powers of the characteristic roots of } A{:}p\times p$$
$$= \operatorname{tr}(A^g)\ .$$

These formulae for $\tilde{C}_\kappa(A)$ are given in table 2.2.3 for $k = 1,2,3$ and 4.

TABLE 2.2.3

Zonal polynomials of $A: p\times p$ in terms of $a_g = \mathrm{tr}(A^g)$

k	κ	$\tilde{C}_\kappa(A)$	$\tilde{C}_\kappa(I_p)$
1	1	a_1	p
2	1^2	$\frac{1}{2}(a_1^2 - a_2)$	$\frac{1}{2}p(p-1)$
	2	$\frac{1}{2}(a_1^2 - a_2)$	$\frac{1}{2}p(p+1)$
3	1^3	$\frac{1}{6}(a_1^3 - 3a_1a_2 + 2a_3)$	$\frac{1}{6}p(p-2)(p-1)$
	21	$\frac{2}{3}(a_1^3 - a_3)$	$\frac{2}{3}p(p-1)(p+1)$
	3	$\frac{1}{6}(a_1^3 + 3a_1a_2 + 2a_3)$	$\frac{1}{6}p(p+2)(p+1)$
4	1^4	$\frac{1}{24}(a_1^4 - 6a_1^2a_2 + 3a_2^2 + 8a_1a_3 - 6a_4)$	$\frac{1}{24}p(p-1)(p-2)(p-3)$
	21^2	$\frac{3}{8}(a_1^4 - 2a_1^2a_2 - a_2^2 + 2a_4)$	$\frac{3}{8}p(p-2)(p-1)(p+1)$
	2^2	$\frac{1}{6}(a_1^4 + 3a_2^2 - 4a_1a_3)$	$\frac{1}{6}p^2(p-1)(p+1)$
	31	$\frac{3}{8}(a_1^4 + 2a_1^2a_2 - a_2^2 - 2a_4)$	$\frac{3}{8}p(p+2)(p-1)(p+1)$
	4	$\frac{1}{24}(a_1^4 + 6a_1^2a_2 + 3a_2^2 + 8a_1a_3 + 6a_4)$	$\frac{1}{24}p(p+1)(p+2)(p+3)$

It is important to note that the contents of table 2.2.3 are quite different from the contents of table 2 in Johnson and Kotz (1972, p. 171) in which the zonal polynomials of a real symmetric matrix are expressed in terms of the sums of the g-th powers of the characteristic roots of the real matrix. The contents of tables 2.2.1 and 2.2.2 will be used repeatedly in sections 2.2.3.

2.2.3 Important results involving zonal polynomials of a hermitian matrix

Consider the following theorem:

Theorem 2.2.4 (Fundamental property of zonal polynomials)

Let $S:p\times p$ and $T:p\times p$ be hermitian matrices; then

$$(2.2.29)\qquad \int_{U(p)} \tilde{C}_\kappa(S\,U\,T\,\bar{U}')\,dU = \frac{\tilde{C}_\kappa(S)\;\tilde{C}_\kappa(T)}{\tilde{C}_\kappa(I_p)}$$

where U is an unitary matrix and dU is the invariant Haar measure on the unitary group of $p\times p$ matrices, $U(p)$, normalised to make the volume of the group manifold unity, i.e.

$$(2.2.30)\qquad \int_{U(p)} dU = 1\ .$$

Proof

James (1960, p. 155), James (1964, p. 488), Hayakawa (1972 a, p. 2).

Remark 2.2.2

The normalising constant in (2.2.30) is given in Khatri (1965, p. 101) and Hayakawa (1972 a, p. 16) as

$$(2.2.31)\qquad \int_{U(p)} dU = \frac{\pi^{p(p-1)}}{\tilde{\Gamma}_p(p)}\ .$$

Theorem 2.2.5

Let $S:p\times p$ be h.p.d. and $T:p\times p$ be a hermitian matrix; then

$$(2.2.32)\qquad \int_{R=\bar{R}'>0} \operatorname{etr}(-RS)\,|R|^{t-p}\,\tilde{C}_\kappa(TR)\,dR$$

$$= \tilde{\Gamma}_p(t,\kappa)\;|S|^{-t}\,\tilde{C}_\kappa(TS^{-1})$$

where the integration is over the space of all h.p.d. matrices $R = \bar{R}' > 0$ and valid for all $\mathrm{Re}(t) > p-1$.

Proof

Hayakawa (1972 c, p. 232).

Theorem 2.2.6

Let $S:p\times p$ be h.p.d. and $T:p\times p$ be a hermitian matrix; then

$$(2.2.33) \qquad \int_{R=\bar{R}'>0} \mathrm{etr}(-RS)\,|R|^{t-p}\,\tilde{C}_\kappa(R^{-1}T)\,dR$$

$$= \tilde{\Gamma}_p(t,\,-\kappa)\,|S|^{-t}\,\tilde{C}_\kappa(TS)$$

where the integration is over the space of all h.p.d. matrices $R = \bar{R}' > 0$ and

$$(2.2.34) \qquad \tilde{\Gamma}_p(t,\,-\kappa) = \pi^{\frac{1}{2}p(p-1)} \prod_{i=1}^{p} \Gamma(t-p-k_i+i)$$

with $t > p-1+k_1$, k_1 is the largest element in the partition κ.

Proof

Khatri (1966, p. 477).

Theorem 2.2.7

Let $S:p\times p$ be h.p.d.; then:

$$(2.2.35) \qquad \int_{I_p>R=\bar{R}'>0} |R|^{t-p}|I_p-R|^{u-p}\,\tilde{C}_\kappa(RS)\,dR$$

$$= \frac{\tilde{\Gamma}_p(t,\kappa)\,\tilde{\Gamma}_p(u)}{\tilde{\Gamma}_p(t+u,\kappa)}\,\tilde{C}_\kappa(S) ,$$

$$(2.2.36) \qquad \int_{I_p > R = \bar{R}' > 0} |R|^{t-p} |I_p - R|^{u-p} \, \tilde{C}_\kappa((I_p - R)S) \, dR$$

$$= \frac{\tilde{\Gamma}_p(u,\kappa) \, \tilde{\Gamma}_p(t)}{\tilde{\Gamma}_p(t+u,\kappa)} \, \tilde{C}_\kappa(S) \, ,$$

$$(2.2.37) \qquad \int_{R = \bar{R}' > 0} |R|^{t-p} |I_p + R|^{-(t+u)} \, \tilde{C}_\kappa((I_p + R)^{-1} S) \, dR$$

$$= \frac{\tilde{\Gamma}_p(u,\kappa) \, \tilde{\Gamma}_p(t)}{\tilde{\Gamma}_p(t+u,\kappa)} \, \tilde{C}_\kappa(S) \, ,$$

$$(2.2.38) \qquad \int_{R = \bar{R}' > 0} |R|^{t-p} |I_p + R|^{-(t+u)} \, \tilde{C}_\kappa(SR(I_p + R)^{-1}) \, dR$$

$$= \frac{\tilde{\Gamma}_p(t,\kappa) \, \tilde{\Gamma}_p(u)}{\tilde{\Gamma}_p(t+u,\kappa)} \, \tilde{C}_\kappa(S) \, .$$

<u>Proof</u>

(<u>2.2.35</u>) De Waal (1968, p. 95), Hayakawa (1972 c, p. 232).

(<u>2.2.36</u>) De Waal (1968, p. 93 and p. 112).

(<u>2.2.37</u>)

In (2.2.36) make the transformation

$$(2.2.39) \qquad W = (I_p - R)^{-\frac{1}{2}} R (I_p - R)^{-\frac{1}{2}}$$

with inverse transformation

$$(2.2.40) \qquad R = I_p - (I_p + W)^{-1} \, .$$

The jacobian of (2.2.40) follows from (2.2.7) and (2.2.8) as

$$J(R \to W) = |I_p + W|^{-2p} \, .$$

From (2.2.39) follows that

$$|R| = |I_p - R||W|$$

$$= |I_p + W|^{-1}|W| .$$

Hence, the L.H.S. of (2.2.37) follows.

(2.2.38)

In (2.2.35) make the transformation (2.2.39) then the proof of (2.2.38) is similar to the proof of (2.2.37) given above.

Theorem 2.2.8

Let $A:p\times p$ be h.p.d. with characteristic roots $0 < \tilde{a}_1 < \cdots < \tilde{a}_p$; then:

(2.2.41) $$\prod_{i>j}^{p} (\tilde{a}_i - \tilde{a}_j)^2 = |(\tilde{a}_i^{\,p-j})|^2 = |(\tilde{a}_j^{\,p-i})|^2 ,$$

(2.2.42) $$\tilde{C}_\kappa(A) \prod_{i>j}^{p} (\tilde{a}_i - \tilde{a}_j)^2 = \chi_{[\kappa]}(1) \; |(\tilde{a}_j^{\,k_i+p-i})||(\tilde{a}_j^{\,p-i})| ,$$

(2.2.43) $$\tilde{C}_\kappa((I_p + A)^{-1}) \prod_{i>j}^{p} (\tilde{a}_i - \tilde{a}_j)^2$$

$$= \chi_{[\kappa]}(1)\,|((\tfrac{1}{1+\tilde{a}_j})^{k_i+p-i})| \; |((\tfrac{1}{1+\tilde{a}_j})^{p-i})| \prod_{i=1}^{p} (1+\tilde{a}_i)^{2(p-1)} ,$$

(2.2.44) $$\tilde{C}_\kappa(A(I_p + A)^{-1}) \prod_{i>j}^{p} (\tilde{a}_i - \tilde{a}_j)^2$$

$$= \chi_{[\kappa]}(1)\,|((\tfrac{\tilde{a}_j}{1+\tilde{a}_j})^{k_i+p-i})||((\tfrac{\tilde{a}_j}{1+\tilde{a}_j})^{p-i})| \prod_{i=1}^{p} (1+\tilde{a}_i)^{2(p-1)} .$$

Proof

(2.2.41)

The product $\prod_{i>j}^{p} (\tilde{a}_i - \tilde{a}_j)$ can be written as the Vandermonde determinant (cf. Pillai, 1956, p. 1106):

$$(2.2.45)\quad \prod_{i>j}^{p} (\tilde{a}_i - \tilde{a}_j) = |(\tilde{a}_{p-i+1}^{p-j})| = |(\tilde{a}_{p-j+1}^{p-i})| .$$

Hence,

$$(2.2.46)\quad \prod_{i>j}^{p} (\tilde{a}_i - \tilde{a}_j)^2 = |(\tilde{a}_{p-i+1}^{p-j})|^2 = |(\tilde{a}_{p-j+1}^{p-i})|^2 .$$

By interchanging the rows of $|(\tilde{a}_{p-i+1}^{p-j})|^2$ and the columns of $|(\tilde{a}_{p-j+1}^{p-i})|^2$ it follows that

$$(2.2.47)\quad |(\tilde{a}_{p-i+1}^{p-j})|^2 = (-1)^{2b}|(\tilde{a}_i^{p-j})|^2$$

and

$$(2.2.48)\quad |(\tilde{a}_{p-j+1}^{p-i})| = (-1)^{2b}|(\tilde{a}_j^{p-i})|^2$$

where

$b = \frac{1}{2}p$ when p is even,

$= \frac{1}{2}(p-1)$ when p is uneven.

Thus $2b$ is even for all p and hence (2.2.41) follows from (2.2.46), (2.2.47) and (2.2.48).

(<u>2.2.42</u>)

The result follows from (2.2.18), (2.2.20) and (2.2.41).

(<u>2.2.43</u>)

From (2.2.18) and (2.2.20) follows that

$$(2.2.49)\quad \tilde{C}_\kappa((I_p + A)^{-1}) = \chi_{[\kappa]}(1)\, \frac{|((\frac{1}{1+\tilde{a}_j})^{k_i+p-i})|}{|((\frac{1}{1+\tilde{a}_j})^{p-i})|} .$$

The symmetric function $\prod_{i>j}^{p} (\tilde{a}_i - \tilde{a}_j)^2$ can be written as

$$(2.2.50)\quad \prod_{i>j}^{p} (\tilde{a}_i - \tilde{a}_j)^2$$

$$= \prod_{i>j}^{p} ((1+\tilde{a}_i)(1+\tilde{a}_j))^2 ((\frac{1}{1+\tilde{a}_j})(\frac{\tilde{a}_i}{1+\tilde{a}_i}) - (\frac{\tilde{a}_j}{1+\tilde{a}_j})(\frac{1}{1+\tilde{a}_i}))^2$$

$$= \prod_{i>j}^{p} ((1+\tilde{a}_i)(1+\tilde{a}_j))^2 (\frac{1}{1+\tilde{a}_j}(1-\frac{1}{1+\tilde{a}_i}) - (\frac{1}{1+\tilde{a}_i})(1-\frac{1}{1+\tilde{a}_j}))^2$$

$$= \prod_{i=1}^{p} (1+\tilde{a}_i)^{2(p-1)} \prod_{i>j}^{p} (\frac{1}{1+\tilde{a}_j} - \frac{1}{1+\tilde{a}_i})^2$$

$$= \prod_{i=1}^{p} (1+\tilde{a}_i)^{2(p-1)} |((\frac{1}{1+\tilde{a}_j})^{p-i})|^2 .$$

Multiplication of (2.2.49) with (2.2.50) leads to (2.2.43).

(<u>2.2.44</u>)

From (2.2.18) and (2.2.20) follows that

$$(2.2.51)\quad \tilde{C}_\kappa(A(I_p + A)^{-1}) = \chi_{[\kappa]}(1) \frac{|((\frac{\tilde{a}_j}{1+\tilde{a}_j})^{k_i+p-i})|}{|((\frac{\tilde{a}_j}{1+\tilde{a}_j})^{p-i})|} .$$

The symmetric function $\prod_{i>j}^{p} (\tilde{a}_i - \tilde{a}_j)^2$ can be written as

$$(2.2.52)\quad \prod_{i>j}^{p} (\tilde{a}_i - \tilde{a}_j)^2$$

$$= \prod_{i>j}^{p} ((1+\tilde{a}_i)(1+\tilde{a}_j))^2 (\frac{\tilde{a}_i}{1+\tilde{a}_i} - \frac{\tilde{a}_j}{1+\tilde{a}_j})^2$$

$$= \prod_{i=1}^{p} (1+\tilde{a}_i)^{2(p-1)} \left| \left(\left(\frac{\tilde{a}_j}{1+\tilde{a}_j}\right)^{p-i} \right) \right|^2 .$$

Multiplication of (2.2.52) with (2.2.51) leads to (2.2.44).

The results given in theorems 2.2.4 - 2.2.7 are analogous to the corresponding results for real symmetric matrices and will be used in the derivation of the p.d.f.s and moments of certain complex quadratic forms. The results proved in theorem 2.2.8 follow direct from the important property of zonal polynomials of a hermitian matrix given in property 2.2.5. These results will be very useful in the derivation of, in particular certain marginal distributions of the characteristic roots of certain random hermitian matrices.

The following four theorems explain how the coefficients which appear in the expansions of $\tilde{C}_\kappa(A)\,\tilde{C}_\tau(A)$, $\tilde{C}_\kappa(I_p - A)$ and $\tilde{C}_\kappa(D_A^1)$ can be calculated where

$$(2.2.53) \quad D_A^1 = \begin{bmatrix} \tilde{a}_1 & 0 & \cdots\cdots & 0 \\ 0 & \tilde{a}_2 & \cdots\cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots\; \tilde{a}_{p-1} & 0 \\ 0 & 0 & \cdots\cdots & 1 \end{bmatrix} .$$

These coefficients are tabulated for small values of p and k.

<u>Theorem 2.2.9</u>

Let $A:p\times p$ be a hermitian matrix; then

$$(2.2.54) \quad \tilde{C}_\kappa(A)\,\tilde{C}_\tau(A) = \sum_\delta \tilde{g}^\delta_{\kappa,\tau}\ \tilde{C}_\delta(A)$$

where $\kappa \varepsilon P(k,p)$, $\tau \varepsilon P(t,p)$, $\delta \varepsilon P(k+t,p)$ and $P(k,p)$ is the set of partitions of the positive integer k into not more than p parts. The coefficients $\tilde{g}^\delta_{\kappa,\tau}$ are tabulated in table 2.2.4

for each partition of $d = k+t$ up to order 4.

Proof

The coefficients $\tilde{g}^{\delta}_{\kappa,\tau}$ can easily be obtained by using the contents of table 2.2.1. Only the obtaining of $\tilde{g}^{(31)}_{(1),(21)}$, $\tilde{g}^{(2^2)}_{(1),(21)}$ and $\tilde{g}^{(21^2)}_{(1),(21)}$ will be shown here.

From table 2.2.1 follows that

$$\begin{aligned}(2.2.55)\quad \tilde{C}_{(1)}(A)\,\tilde{C}_{(21)}(A) &= 2\alpha_1(\alpha_2\alpha_1 - \alpha_3)\\ &= 2\alpha_1^2\alpha_2 - 2\alpha_1\alpha_3\\ &= 2(\alpha_1\alpha_3 - \alpha_4) + 2(\alpha_2^2 - \alpha_1\alpha_3)\\ &\quad + 2(\alpha_1^2\alpha_2 - \alpha_2^2 - \alpha_1\alpha_3 + \alpha_4)\\ &= \tfrac{2}{3}\tilde{C}_{(31)}(A) + \tilde{C}_{(2^2)}(A) + \tfrac{2}{3}\tilde{C}_{(21^2)}(A)\ .\end{aligned}$$

Thus $\tilde{g}^{(1^4)}_{(1),(21)} = 0$, $\tilde{g}^{(21^2)}_{(1),(21)} = \tfrac{2}{3}$, $\tilde{g}^{(2^2)}_{(1),(21)} = 1$,

$\tilde{g}^{(31)}_{(1),(21)} = \tfrac{2}{3}$ and $\tilde{g}^{(4)}_{(1),(21)} = 0$.

TABLE 2.2.4

The values of $\tilde{g}^{\delta}_{\kappa,\tau}$

$\tilde{C}_\kappa(\cdot)\ \tilde{C}_\tau(\cdot)$	d = 2		d = 3		
	$\tilde{C}_{(2)}(\cdot)$	$\tilde{C}_{(1^2)}(\cdot)$	$\tilde{C}_{(3)}(\cdot)$	$\tilde{C}_{(21)}(\cdot)$	$\tilde{C}_{(1^3)}(\cdot)$
$\tilde{C}_{(1)}(\cdot)\ \tilde{C}_{(1)}(\cdot)$	1	1			
$\tilde{C}_{(1)}(\cdot)\ \tilde{C}_{(2)}(\cdot)$			1	$\frac{1}{2}$	
$\tilde{C}_{(1)}(\cdot)\ \tilde{C}_{(1^2)}(\cdot)$				$\frac{1}{2}$	1

continued/...

$\tilde{C}_{\kappa}(\cdot)\ \tilde{C}_{\tau}(\cdot)$	d = 4				
	$\tilde{C}_{(4)}(\cdot)$	$\tilde{C}_{(31)}(\cdot)$	$\tilde{C}_{(2^2)}(\cdot)$	$\tilde{C}_{(21^2)}(\cdot)$	$\tilde{C}_{(1^4)}(\cdot)$
$\tilde{C}_{(1)}(\cdot)\ \tilde{C}_{(3)}(\cdot)$	1	$\frac{1}{3}$			
$\tilde{C}_{(1)}(\cdot)\ \tilde{C}_{(21)}(\cdot)$		$\frac{2}{3}$	1	$\frac{2}{3}$	
$\tilde{C}_{(1)}(\cdot)\ \tilde{C}_{(1^3)}(\cdot)$				$\frac{1}{3}$	1
$\tilde{C}_{(2)}(\cdot)\ \tilde{C}_{(2)}(\cdot)$	1	$\frac{1}{3}$	$\frac{1}{2}$		
$\tilde{C}_{(2)}(\cdot)\ \tilde{C}_{(1^2)}(\cdot)$		$\frac{1}{3}$		$\frac{1}{3}$	
$\tilde{C}_{(1^2)}(\cdot)\ \tilde{C}_{(1^2)}(\cdot)$			$\frac{1}{2}$	$\frac{1}{3}$	1

Remark 2.2.3

It is important to note that the coefficients $\tilde{g}^{\delta}_{\kappa,\tau}$ are not identical to the coefficients $g^{\delta}_{\kappa,\tau}$ tabulated in Khatri and Pillai (1968, p. 218) and Gupta (1970) for $A{:}p\times p$, a real symmetric matrix.

Theorem 2.2.10

Let $A{:}p\times p$ be a hermitian matrix; then

$$(2.2.56)\quad \tilde{C}_{\kappa}(I_p - A) = \sum_{t=0}^{k} \sum_{\tau} \tilde{q}_{\tau}\, \tilde{C}_{\tau}(A)$$

where $\kappa \in P(k,p)$ and $\tau \in P(t,p)$. The coefficients $\tilde{q}_{\tau}$ are tabulated in table 2.2.4 for $p = 1,2,3$ and 4 and for each partition of $k = 1,2$ and 3.

Proof

The coefficients $\tilde{q}_{\tau}$ can easily be obtained by using the contents of table 2.2.2. Only the obtaining of $\tilde{q}_{\tau}$ in

$$\tilde{C}_{(21)}(I_p - A) = \sum_{t=0}^{3} \sum_{\tau} \tilde{q}_{\tau}\, \tilde{C}_{\tau}(A)$$

for $p = 3$ will be shown here.

From table 2.2.2 follows for $p = 3$ that

$$(2.2.57)\quad \tilde{C}_{(21)}(I_p - A)$$

$$= 2((1-\tilde{a}_1)^2(1-\tilde{a}_2)+(1-\tilde{a}_1)^2(1-\tilde{a}_3)+(1-\tilde{a}_2)^2(1-\tilde{a}_1)$$

$$+(1-\tilde{a}_2)^2(1-\tilde{a}_3)+(1-\tilde{a}_3)^2(1-\tilde{a}_1)+(1-\tilde{a}_3)^2(1-\tilde{a}_2)$$

$$+2(1-\tilde{a}_1)(1-\tilde{a}_2)(1-\tilde{a}_3))$$

$$= 16 - 16(\tilde{a}_1+\tilde{a}_1+\tilde{a}_2)+4(\tilde{a}_1^2 + \tilde{a}_2^2 + \tilde{a}_3^2 + 3\tilde{a}_1\tilde{a}_2 + 3\tilde{a}_1\tilde{a}_3 + 3\tilde{a}_2\tilde{a}_3)$$

$$-2(\tilde{a}_1^2\tilde{a}_2 + \tilde{a}_1^2\tilde{a}_3 + \tilde{a}_2^2\tilde{a}_1 + \tilde{a}_2^2\tilde{a}_3 + \tilde{a}_3^2\tilde{a}_1 + \tilde{a}_3^2\tilde{a}_2 + 2\tilde{a}_1\tilde{a}_2\tilde{a}_3)$$

$$= 16 - 16(\tilde{a}_1+\tilde{a}_2+\tilde{a}_3)+8(\tilde{a}_1\tilde{a}_2+\tilde{a}_1\tilde{a}_3+\tilde{a}_2\tilde{a}_3)+4(\tilde{a}_1^2 + \tilde{a}_2^2 + \tilde{a}_3^2$$
$$+\tilde{a}_1\tilde{a}_2+\tilde{a}_1\tilde{a}_3+\tilde{a}_2\tilde{a}_3)-2(\tilde{a}_1^2\tilde{a}_2 + \tilde{a}_1^2\tilde{a}_3 + \tilde{a}_2^2\tilde{a}_1 + \tilde{a}_2^2\tilde{a}_3 + \tilde{a}_3^2\tilde{a}_1$$
$$+\tilde{a}_3^2\tilde{a}_2 + 2\tilde{a}_1\tilde{a}_2\tilde{a}_3)$$

$$= \sum_{t=0}^{3} \sum_{\tau} \tilde{q}_\tau \, C_\tau(A)$$

where: $\tilde{q}_{(0)} = 16$, $\tilde{q}_{(1)} = -16$, $\tilde{q}_{(1^2)} = 8$, $\tilde{q}_{(2)} = 4$, $\tilde{q}_{(1^3)} = 0$,

$\tilde{q}_{(21)} = -1$, $\tilde{q}_{(3)} = 0$.

TABLE 2.2.5

The values of $\tilde{q}_\tau$

k	p	κ	$\tilde{q}_{(0)}$	$\tilde{q}_{(1)}$	$\tilde{q}_{(1^2)}$	$\tilde{q}_{(2)}$	$\tilde{q}_{(1^3)}$	$\tilde{q}_{(21)}$	$\tilde{q}_{(3)}$
1	1	1	1	-1					
	2	1	2	-1					
	3	1	3	-1					
	4	1	4	-1					
2	1	2	1	-2		1			
	2	1^2	1	-1	1				
		2	3	-3		1			
	3	1^2	1	-2	1				
		21	6	-4		1			
	4	1^2	6	-3	1				
		2^2	10	-5		1			
3	1	1	1	-3		3			-1
	2	21	4	-6	6	2		-1	
		3	4	-6		4			-1

continued/...

k	p	κ	$\tilde{q}_{(0)}$	$\tilde{q}_{(1)}$	$\tilde{q}_{(1^2)}$	$\tilde{q}_{(2)}$	$\tilde{q}_{(1^3)}$	$\tilde{q}_{(21)}$	$\tilde{q}_{(3)}$
	3	1^3	1	-1	1		-1		
		21	16	-16	8	4		-1	
		3	10	-10		5			-1
	4	1^3	4	-4	2		-1		
		21	40	-30	10	6		-1	
		3	20	-15		6			-1

Theorem 2.2.11

Let $A:p\times p$ be a hermitian matrix; then

$$(2.2.58)\quad \tilde{C}_\kappa(I_p - A) = \tilde{C}_\kappa(I_p) \sum_{t=0}^{k} \sum_{\tau} (-1)^t \tilde{q}_{\kappa,\tau} \frac{\tilde{C}_\tau(A)}{\tilde{C}_\tau(I_p)}$$

where $\kappa \in P(k,p)$ and $\tau \in P(t,p)$. The coefficients $\tilde{q}_{\kappa,\tau}$ are tabulated in table 2.2.6 for each partition of $k = 1,2,3$ and 4.

Proof

The coefficient $\tilde{q}_{\kappa,\tau}$ can easily be obtained by using the contents of table 2.2.2. Only the obtaining of $\tilde{q}_{\kappa,\tau}$ in

$$(2.2.59)\quad \tilde{C}_{(21)}(A) = \tilde{C}_{(21)}(I_p) \sum_{t=0}^{3} \sum_{\tau} (-1)^t \tilde{q}_{\kappa,\tau} \frac{\tilde{C}_\tau(A)}{\tilde{C}_\tau(I_p)}$$

for $p = 3$ will be shown here.

From table 2.2.2 follows for $p = 3$ that

$$\frac{\tilde{C}_{(21)}(I_p-A)}{\tilde{C}_{(21)}(I_p)}$$

$$= 1-(\tilde{a}_1+\tilde{a}_2+\tilde{a}_3)+\tfrac{1}{4}(\tilde{a}_1^2 + \tilde{a}_2^2 + \tilde{a}_3^2 + 3\tilde{a}_1\tilde{a}_2 + 3\tilde{a}_1\tilde{a}_3 + 3\tilde{a}_2\tilde{a}_3)$$

$$-\tfrac{1}{6}(\tilde a_1^2\tilde a_2 + \tilde a_1^2\tilde a_3 + \tilde a_2^2\tilde a_1 + \tilde a_2^2\tilde a_3 + \tilde a_3^2\tilde a_1 + \tilde a_3^2\tilde a_2 + 2\tilde a_1\tilde a_2\tilde a_3)$$

$$= 1 - 3\cdot\tfrac{1}{3}(\tilde a_1+\tilde a_2+\tilde a_3) + \tfrac{3}{2}\cdot\tfrac{1}{3}(\tilde a_1\tilde a_2 + \tilde a_1\tilde a_3 + \tilde a_2\tilde a_3)$$

$$+\tfrac{3}{2}\cdot\tfrac{1}{6}(\tilde a_1^2 + \tilde a_2^2 + \tilde a_3^2 + \tilde a_1\tilde a_2 + \tilde a_1\tilde a_3 + \tilde a_2\tilde a_3)$$

$$-1\cdot\tfrac{1}{8}(\tilde a_1^2\tilde a_2 + \tilde a_1^2\tilde a_3 + \tilde a_2^2\tilde a_1 + \tilde a_2^2\tilde a_3 + \tilde a_3^2\tilde a_1 + \tilde a_3^2\tilde a_2 + 2\tilde a_1\tilde a_2\tilde a_3)$$

$$= \sum_{t=0}^{3} \sum_{\tau} (-1)^t \, \tilde q_{\kappa,\tau} \frac{\tilde C_\tau(A)}{\tilde C_\tau(I_p)}$$

where: $\tilde q_{(21),(0)} = 1$, $\tilde q_{(21),(1)} = 3$, $\tilde q_{(21),(1^2)} = \tfrac{3}{2}$,

$\tilde q_{(21),(2)} = \tfrac{3}{2}$, $\tilde q_{(21),(1^3)} = 0$, $\tilde q_{(21),(21)} = 1$,

$\tilde q_{(21),(3)} = 0$.

TABLE 2.2.6

The values of $\tilde{q}_{\kappa,\tau}$

k	κ	τ											
		0	1	1^2	2	1^3	21	3	1^4	21^2	2^2	31	4
1	1	1	1										
2	1^2	1	2	1									
	2	1	2		1								
3	1^3	1	3	3		1							
	21	1	3	$\frac{3}{2}$	$\frac{3}{2}$		1						
	3	1	3		3			1					
4	1^4	1	4	6		4			1				
	21^2	1	4	4	2	$\frac{4}{3}$	$\frac{8}{3}$			1			
	2^2	1	4	3	3		4				1		
	31	1	4	2	4		$\frac{8}{3}$	$\frac{4}{3}$				1	
	4	1	4		6			4					1

Remark 2.2.4

(i) If $\tilde{C}_\kappa(I_p - A)$ is written in terms of the characteristic roots of $A{:}p\times p$ for different values of p by using the contents of table 2.2.2 it is easy to verify that the coefficients $\tilde{q}_{\kappa,\tau}$ remain the same, independently of the value of p, for given partitions κ of k and τ of t, $t \leq k$.

(ii) It is important to note that the coefficients $\tilde{q}_{\kappa,\tau}$ are not identical to the coefficients $a_{\kappa,\tau}$ tabulated in Constantine (1966, p. 224 - 225) for $A{:}p\times p$ a real symmetric matrix.

Theorem 2.2.12

Let $A{:}p\times p$ be a hermitian matrix; then

$$(2.2.60)\quad \tilde{C}_\kappa(D_A^1) = \sum_{t=0}^{k} \sum_{\tau} \tilde{b}_{\kappa,\tau}\, \tilde{C}_\tau(D_A^*)$$

where $\kappa \in P(k,p)$, $\tau \in P(t,p)$, D_A^1 is given by (2.2.53) and

$$(2.2.61)\quad D_A^* = \begin{bmatrix} \tilde{a}_1 & 0 \cdots & 0 \\ 0 & \tilde{a}_2 \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & 0 \cdots & \tilde{a}_{p-1} \end{bmatrix} .$$

The coefficients $\tilde{b}_{\kappa,\tau}$ are tabulated in table 2.2.7 for each partition of $k = 1,2,3$ and 4.

Proof

The coefficients $\tilde{b}_{\kappa,\tau}$ can easily be obtained by using the contents of table 2.2.2. Only the obtaining of $\tilde{b}_{\kappa,\tau}$ in

$$(2.2.62)\quad \tilde{C}_{(21)}(D_A^1) = \sum_{t=0}^{3} \sum_{\tau} \tilde{b}_{\kappa,\tau}\, \tilde{C}_\tau(D_A^*)$$

for $p = 4$ will be shown here.

From table 2.2.2 follows for $p = 4$ that

$$(2.2.63)\quad \tilde{C}_{(21)}(D_A^1) = 2(\tilde{a}_1^2\tilde{a}_2 + \tilde{a}_1^2\tilde{a}_3 + \tilde{a}_1^2 + \tilde{a}_2^2\tilde{a}_1 + \tilde{a}_2^2\tilde{a}_3 + \tilde{a}_2^2 + \tilde{a}_3^2\tilde{a}_1 + \tilde{a}_3^2\tilde{a}_2 + \tilde{a}_3^2 + \tilde{a}_1 + \tilde{a}_2 + \tilde{a}_3 + 2\tilde{a}_1\tilde{a}_2\tilde{a}_3 + 2\tilde{a}_1\tilde{a}_2 + 2\tilde{a}_2\tilde{a}_3 + 2\tilde{a}_1\tilde{a}_3)$$

$$= 2(\tilde{a}_1 + \tilde{a}_2 + \tilde{a}_3) + 2(\tilde{a}_1^2 + \tilde{a}_2^2 + \tilde{a}_3^2 + \tilde{a}_1\tilde{a}_2 + \tilde{a}_1\tilde{a}_3 + \tilde{a}_2\tilde{a}_3) + 2(\tilde{a}_1\tilde{a}_2 + \tilde{a}_1\tilde{a}_3 + \tilde{a}_2\tilde{a}_3) + 2(\tilde{a}_1^2\tilde{a}_2 + \tilde{a}_1^2\tilde{a}_3 + \tilde{a}_2^2\tilde{a}_1 + \tilde{a}_2^2\tilde{a}_3 + \tilde{a}_3^2\tilde{a}_1 + \tilde{a}_3^2\tilde{a}_2 + 2\tilde{a}_1\tilde{a}_2\tilde{a}_3)$$

$$= \sum_{t=0}^{3} \sum_{\tau} \tilde{b}_{\kappa,\tau}\, \tilde{C}_\tau(D_A^*)$$

where: $\tilde{b}_{(21),(0)} = 0$, $\tilde{b}_{(21),(1)} = 2$, $\tilde{b}_{(21),(2)} = 2$,

$\tilde{b}_{(21),(1^2)} = 2$, $\tilde{b}_{(21),(21)} = 1$, $\tilde{b}_{(21),(1^3)} = 0$.

TABLE 2.2.7

The values of $\tilde{b}_{\kappa,\tau}$

k	κ	τ											
		0	1	1^2	2	1^3	21	3	1^4	21^2	2^2	31	4
1	1	1	1										
2	1^2		1	1									
	2	1	1		1								
3	1^3			1		1							
	21		2	2	2		1						
	3	1	1		1			1					
4	1^4					1		1	1				
	21^2			3		1	$\frac{3}{2}$			1			
	2^2				2		1				1		
	31		3	3	3		$\frac{3}{2}$	3				1	
	4	1	1		1			1					1

Remark 2.2.5

(i) If $\tilde{C}_\kappa(D_A^1)$ is written in terms of the characteristic roots of $A:p\times p$ for different values of p by using the contents of table 2.2.2 it is clear that the coefficients $\tilde{b}_{\kappa,\tau}$ remain the same, independently of the value of p, for given partitions κ of k and τ of t, $t\leqq k$.

(ii) It is important to note that the coefficients $\tilde{b}_{\kappa,\tau}$ are not identical to the coefficients $b_{\kappa,\tau}$ tabulated in Khatri and Pillai (1968, p. 224 - 225) for $A:p\times p$, a real symmetric matrix.

Theorem 2.2.13 is analogous to a theorem proved by Khatri (1971) while the identities given in theorem 2.2.14 are analogous to identities proved by Troskie (1971), De Waal (1968) and Underhill (1973). These results given in these two theorems will be used repeatedly in this thesis.

Theorem 2.2.13

Let $A:p\times p$, $B:p\times p$ and $S:p\times p$ be hermitian matrices; then

$$(2.2.64)\qquad \int_{U(p)} \tilde{C}_\kappa(S\,U\,A\,\bar{U}')\,\tilde{C}_\tau(S\,U\,B\,\bar{U}')\,dU$$

$$= \sum_\delta \tilde{C}_\delta(S)\;\tilde{P}^\delta_{\kappa,\tau}\;(A\,,\,B)$$

where $\kappa\,\varepsilon\,P(k,p)$, $\tau\,\varepsilon\,P(t,p)$, $\delta\,\varepsilon\,P(k+t,p)$ and $\tilde{P}^\delta_{\kappa,\tau}(A,B)$ depends on A and B only.

Proof

From (2.2.23) follows that

$$(2.2.65)\qquad \tilde{C}_\kappa(A) = \chi_{[\kappa]}(1)\;(\alpha_1^{*\,k_1-k_2}.\alpha_2^{*\,k_2-k_3}\cdots\alpha_p^{*\,k_p}$$

$$+ \text{"terms of lower weight"}).$$

From Nel (1972, p. 8) follows that

(2.2.66) $\alpha_j^* = \mathrm{tr}_j(A)$, $(j = 1,\cdots,p)$

where

> $\mathrm{tr}_j(A)$ is the sum of the principal minors of order j of A, $(j = 1,\cdots,p)$.

Substitution of (2.2.66) into (2.2.65) leads to

(2.2.67) $\tilde{C}_\kappa(A) = \sum_{\kappa^*} \tilde{d}_{\kappa^*,\kappa} \prod_{i=1}^{p} (\mathrm{tr}_i(A))^{r_i}$

where the summation is over the partitions $\kappa^* \leqq \kappa$ and $r_1 = k_1^* - k_2^*$, $r_2 = k_2^* - k_3^*,\cdots,r_p = k_p^*$, i.e. $\sum_{i=1}^{p} ir_i = k$.
The constant $\tilde{d}_{\kappa^*,\kappa}$ depends on the partition κ^* and is not identical to the constant $d_{\kappa^*,\kappa}$ given in Khatri (1971, p. 211) for $A{:}p\times p$ a real symmetrix matrix.

As in Khatri (1971, p. 211) and without loss of generality $S{:}p\times p$ is taken as a diagonal matrix, and

(2.2.68) $\mathrm{tr}_i(SA) = \sum_{1\leqq j_1\leqq j_2\leqq,\cdots,\leqq j_i} (\tilde{s}_{j_1}\cdots\tilde{s}_{j_i})\,|(A)_{j_1,\cdots,j_i}|$

where $(A)_{j_1,\cdots,j_i}$ is the submatrix formed by $j_1,\cdots,j_i$ rows and $j_1,\cdots,j_i$ columns of $A{:}p\times p$. From the product $\tilde{C}_\kappa(S\,U\,A\,\bar{U}')\,\tilde{C}_\tau(S\,U\,B\,\bar{U}')$ the coefficients of $\tilde{s}_1^{t_1}\cdots\tilde{s}_p^{t_p}$ are collected. Integration over the unitary group gives the same value for any permutations of $t_1,\cdots,t_p$ which leads to (2.2.64).

<u>Theorem 2.2.14</u>

Let $A{:}p\times p$ be h.p.d.; then:

$$(2.2.69)\quad |I_p + A|^{-a}\ [a]_\kappa\ \tilde{C}_\kappa((I_p + A)^{-1})$$

$$= \sum_t \sum_\tau \sum_\delta (-1)^t\ \tilde{g}^{\delta}_{\kappa,\tau}\ [a]_\delta\ \frac{\tilde{C}_\tau(A)\ \tilde{C}_\delta(I_p)}{t!\ \tilde{C}_\tau(I_p)}\ ,$$

$$(2.2.70)\quad |I_p + A|^{-a}\ [a]_\kappa\ \tilde{C}_\kappa((I_p + A)^{-1}A)$$

$$= \sum_t \sum_\tau \sum_\delta \frac{(-1)^t}{t!}\ \tilde{g}^{\delta}_{\kappa,\tau}\ [a]_\delta\ \tilde{C}_\delta(A)$$

where a is an arbitrary constant such that $a > p - 1$.

Let $R{:}p\times p$, $S{:}p\times p$ and $T{:}p\times p$ be hermitian matrices; then:

$$(2.2.71)\quad \sum_k \sum_\kappa \sum_\mu \frac{(-1)^k}{k!}\ \frac{\tilde{C}_\kappa(I_p+T)\ \tilde{C}_\kappa(R)}{\tilde{C}_\kappa(I_p)\ \tilde{C}_\kappa(I_p)}\ \tilde{g}^{\mu}_{\kappa,\beta}\ \tilde{C}_\mu(I_p)\ [a]_\mu$$

$$= \sum_k \sum_\kappa \sum_t \sum_\tau \sum_\sigma \sum_\delta \frac{\tilde{g}^{\delta}_{\kappa,\tau}\ \tilde{g}^{\sigma}_{\delta,\beta}\ (-1)^{k+t}}{k!\ t!}\ \frac{\tilde{C}_\kappa(T)\ \tilde{C}_\delta(R)\ \tilde{C}_\sigma(I_p)\,[a]_\sigma}{\tilde{C}_\kappa(I_p)\ \tilde{C}_\delta(I_p)}\ ,$$

$$(2.2.72)\quad \sum_k \sum_\kappa \sum_\delta \frac{(-1)^k}{k!}\ \frac{\tilde{P}^{\delta}_{\kappa,\tau}(R,(I_p-R))}{\tilde{C}_\kappa(I_p)\ \tilde{C}_\tau(I_p)}\ [a]_\delta\ \tilde{C}_\kappa(d\,I_p + S)\ \tilde{C}_\tau(T)\ \tilde{C}_\delta(I_p)$$

$$= \sum_k \sum_\kappa \sum_b \sum_\beta \sum_\mu \sum_\sigma \frac{(-1)^{b+k}\ \tilde{g}^{\mu}_{\kappa,\beta}\ d^b}{b!k!\ \tilde{C}_\kappa(I_p)\ \tilde{C}_\tau(I_p)}\ \tilde{P}^{\sigma}_{\mu,\tau}(R,(I_p-R))\,[a]_\sigma$$

$$\tilde{C}_\kappa(S)\ \tilde{C}_\tau(T)\ \tilde{C}_\sigma(I_p)$$

where $a > p - 1$ and d are arbitrary constants.

<u>Proof</u>

These four identities are easily proved along the same lines as in the case where $R{:}p\times p$, $S{:}p\times p$ and $T{:}p\times p$ are real symmetric matrices and $A:p\times p$ is a real positive definite symmetric matrix by replacing:

Symmetric matrix with hermitian matrix;
orthogonal matrix with unitary matrix;
integrate over the orthogonal group with
integrate over the unitary group;
$a-\frac{1}{2}(p+1)$ with $a-p$,

in the proofs for the real matrices. Because of this similarity in the proofs, the proofs of the identities above are not given here. The proofs for $A:p\times p$, $R:p\times p$, $S:p\times p$ and $T:p\times p$ real matrices are given in:

(2.2.69) - Troskie (1971, p. 1754);
(2.2.70) - De Waal (1968, p. 67), Underhill (1973, p. 1.5);
(2.2.71) - Underhill (1973, p. 1.8);
(2.2.72) - Underhill (1973, p. 1.9).

The result given in the following theorem will be very useful in the derivation of the p.d.f.s of certain ratios of characteristic roots of certain random hermitian matrices:

Theorem 2.2.15

Let D_A^1 and D_A^* be defined as in (2.2.53) and (2.2.61) respectively; then

$$(2.2.73)\quad \underset{0<\tilde{a}_1<\cdots<\tilde{a}_{p-1}<1}{\int \cdots\cdots\cdots \int} |D_A^*|^{t-p}\, \tilde{C}_\kappa(D_A^1)\, |I_{p-1}-D_A^*|^2 \prod_{i>j}^{p-1} (\tilde{a}_i-\tilde{a}_j)^2\, d\tilde{a}_1\cdots d\tilde{a}_{p-1}$$

$$= \frac{\tilde{\Gamma}_p(t,\kappa)\,\{\tilde{\Gamma}_p(p)\}^2\,(pt+k)}{\pi^{p(p-1)}\,\tilde{\Gamma}_p(t+p,\,\kappa)}\,\tilde{C}_\kappa(I_p)\ .$$

Proof

Hayakawa (1972 a, p. 7).

2.3 HYPERGEOMETRIC FUNCTIONS

Consider the following definition of the generalised hypergeometric function of a single variable:

Definition 2.3.1 (Erdélyi, 1953, p. 182, Rainville, 1960, p. 73-74)

$$(2.3.1)\quad {}_mF_n(a_i\ ;\ b_j\ ;\ x) = {}_mF_n(a_1,\cdots,a_m\ ;\ b_1,\cdots,b_n\ ;\ x)$$

$$= \sum_{k=0}^{\infty} \frac{(a_1)_k\cdots(a_m)_k}{(b_1)_k\cdots(b_n)_k}\frac{x^k}{k!}$$

where x may be real or complex, $b_j > 0$, $(j = 1,\cdots,n)$ and if any $a_j \leq 0$, $(j = 1,\cdots,m)$, the series terminates.

Conditions for the convergence of the series in (2.3.1) are:

(i) If $m \leq n$, the series converges for all finite x,

(ii) if $m = n+1$, the series converges for $|x| < 1$ and diverges for $|x| > 1$,

(iii) if $m > n+1$, the series diverges for $x \neq 0$.

James (1964, p. 488) defined the hypergeometric function of a hermitian matrix variable analogous to the hypergeometric function of a single variable and the familiar hypergeometric function of a matrix variable, as follows:

Definition 2.3.2

$$(2.3.2)\quad {}_m\tilde{F}_n(a_i\ ;\ b_j\ ;\ S) = {}_m\tilde{F}_n(a_1,\cdots,a_m\ ;\ b_1,\cdots,b_n\ ;\ S)$$

$$= \sum_{k=0}^{\infty}\ \sum_{\kappa\varepsilon P(k,p)} \frac{[a_1]_\kappa\cdots[a_m]_\kappa}{[b_1]_\kappa\cdots[b_n]_\kappa}\frac{\tilde{C}_\kappa(S)}{k!}$$

where $S{:}p\times p$ is a hermitian matrix.

Conditions for the convergence of the series in (2.3.2) are:

(i) $m \leqq n+1$, otherwise the series may only converge for $S{:}p\times p = 0$,

(ii) for $m = n+1$, the series converges for $\|S\| < 1$ (where the norm $\|\cdot\|$ on $S{:}p\times p$ is defined as the maximum absolute value of the characteristic roots of $S{:}p\times p$),

(iii) for $m < n+1$ the series converges for all $S{:}p\times p$,

(iv) the parameters a_i and b_j, are arbitrary complex numbers, but none of the b_j is an integer or half integer $\leqq p-1$,

(v) if an a_i is a negative integer, say $-q$, then for $k \geqq pq+1$ all coefficients in (2.3.2) will vanish, so that the function reduces to a finite polynomial of degree pq.

Hypergeometric functions of two hermitian matrices $S{:}p\times p$ and $T{:}p\times p$ can also be defined:

<u>Definition 2.3.3</u>

(2.3.3) $${}_m\tilde{F}_n(a_i\ ;\ b_j\ ;\ S\ ,\ T)$$

$$= {}_m\tilde{F}_n(a_1,\cdots,a_m\ ;\ b_1,\cdots,b_n\ ;\ S\ ,\ T)$$

$$= \sum_{k=0}^{\infty} \sum_{\kappa\varepsilon P(k,p)} \frac{[a_1]_\kappa\cdots[a_m]_\kappa}{[b_1]_\kappa\cdots[b_n]_\kappa} \frac{\tilde{C}_\kappa(S)\ \tilde{C}_\kappa(T)}{\tilde{C}_\kappa(I_p)\quad k!} .$$

Conditions for the convergence of (2.3.3) are similar to the conditions for the convergence of (2.3.2) except that for $m = n+1$ the series converges for $\|S\| < 1$ or $\|T\| < 1$. When both $S{:}p\times p$ and $T{:}p\times p$ are such that $\|S\| < 1$ and $\|T\| < 1$, the series will converge more rapidly.

Some of the results of section 2.2.3 can be extended to the theory of hypergeometric functions:

<u>Theorem 2.3.1</u> (Corollary of theorem 2.2.4)

Let $S:p\times p$ and $T:p\times p$ be hermitian matrices; then

$$(2.3.4) \qquad \int_{U(p)} {}_m\tilde{F}_n(a_i\ ;\ b_j\ ;\ SUT\bar{U}')\,dU = {}_m\tilde{F}_n(a_i\ ;\ b_j\ ;\ S\,,T)\ .$$

<u>Theorem 2.3.2</u> (Corollary of theorem 2.2.5)

Let $S:p\times p$ be h.p.d. and $T:p\times p$ be a hermitian matrix; then

$$(2.3.5) \qquad \int_{R=\bar{R}'>0} \operatorname{etr}(-RS)\,|R|^{t-p}\,{}_m\tilde{F}_n(a_i\ ;\ b_j\ ;\ TR)\,dR$$

$$= \tilde{\Gamma}_p(t)\,|S|^{-t}\,{}_{m+1}\tilde{F}_n(a_i,\ t\ ;\ b_j\ ;\ TS^{-1})\ .$$

<u>Theorem 2.3.3</u> (Corollary of theorem 2.2.7)

Let $S:p\times p$ be h.p.d.; then:

$$(2.3.6) \qquad \int_{I_p>R=\bar{R}'>0} |R|^{t-p}|I_p-R|^{u-p}\,{}_m\tilde{F}_n(a_i\ ;\ b_j\ ;\ RS)\,dR$$

$$= \frac{\tilde{\Gamma}_p(t)\ \tilde{\Gamma}_p(u)}{\tilde{\Gamma}_p(t+u)}\ {}_{m+1}\tilde{F}_{n+1}(a_i,\ t\ ;\ b_j,\ t+u\ ;\ S)\ ,$$

$$(2.3.7) \qquad \int_{I_p>R=\bar{R}'>0} |R|^{t-p}|I_p-R|^{u-p}\,{}_m\tilde{F}_n(a_i\ ;\ b_j\ ;\ (I_p-R)S)\,dR$$

$$= \frac{\tilde{\Gamma}_p(u)\ \tilde{\Gamma}_p(t)}{\tilde{\Gamma}_p(u+t)}\ {}_{m+1}\tilde{F}_{n+1}(a_i,\ u\ ;\ b_j,\ t+u\ ;\ S)\ ,$$

$$(2.3.8)\qquad \int\limits_{R=\bar{R}'>0} |R|^{t-p}|I_p+R|^{-(t+u)}\,{}_m\tilde{F}_n(a_i\ ;\ b_j\ ;\ (I_p+R)^{-1}S)\ dR$$

$$= \frac{\tilde{\Gamma}_p(u)\ \tilde{\Gamma}_p(t)}{\tilde{\Gamma}_p(t+u)}\ {}_{m+1}\tilde{F}_{n+1}(a_i,\ u\ ;\ b_j,\ t+u\ ;\ S)\ ,$$

$$(2.3.9)\qquad \int\limits_{R=\bar{R}'>0} |R|^{t-p}|I_p+R|^{-(t+u)}\,{}_m\tilde{F}_n(a_i\ ;\ b_j\ ;\ SR(I_p+R)^{-1})\ dR$$

$$= \frac{\tilde{\Gamma}_p(u)\ \tilde{\Gamma}_p(t)}{\tilde{\Gamma}_p(t+u)}\ {}_{m+1}\tilde{F}_{n+1}(a_i,\ t\ ;\ b_j,\ t+u\ ;\ S)\ .$$

Theorem 2.3.4

Let $W:p\times n$, $p \leqq n$ be a complex matrix; then

$$(2.3.10)\qquad \int\limits_{U(n)} \operatorname{etr}(WU_1 + \overline{WU}_1)dU = {}_0\tilde{F}_1(n\ ;\ W\bar{W}')$$

where $U:n\times n = [U_1 \vdots U_2]$ with $U_1:n\times p$ a submatrix of U.

Proof

James (1964, p. 488).

Some important functions have simple expansions as hypergeometric functions with hermitian matrix arguments. In theorem 2.3.5 the exponential and binomial functions are considered:

Theorem 2.3.5

Let $A:p\times p$ be h.p.d.; then:

$$(2.3.11)\qquad \operatorname{etr}(A) = {}_0\tilde{F}_0(A)\ ,$$

$$(2.3.12)\qquad |I_p-A|^{-n} = {}_1\tilde{F}_0(n\ ;\ A)\ ,$$

(2.3.13) $$|I_p - A|^n = {}_1\tilde{F}_0(n\ ;\ -A(I_p-A)^{-1})\ .$$

Proof

(2.3.11), (2.3.12) James (1964, p. 488).

(2.3.14)

$$|I_p - A|^n = |(I_p-A)^{-\frac{1}{2}}(I_p-A+A)(I_p-A)^{-\frac{1}{2}}|^{-n}$$

$$= |I_p+(I_p-A)^{-\frac{1}{2}}A(I_p-A)^{-\frac{1}{2}}|^{-n}$$

$$= {}_1\tilde{F}_0(n\ ;\ -A(I_p-A)^{-1})\ .$$

The following very useful result for hypergeometric functions with hermitian matrix arguments is proved in Khatri (1970):

Theorem 2.3.6

Let $R:p\times p$ and $S:p\times p$ be hermitian matrices with characteristic roots $\tilde{r}_1 < \tilde{r}_2 < \cdots < \tilde{r}_p$ and $\tilde{s}_1 < \tilde{s}_2 < \cdots < \tilde{s}_p$; then

(2.3.14) $${}_m\tilde{F}_n(a_i\ ;\ b_j\ ;\ R,S)$$

$$= \frac{\tilde{\Gamma}_p(p) \prod_{i=1}^{p} \prod_{j=1}^{n} (b_j-i+1)^{i-1}\ |W|}{\pi^{\frac{1}{2}p(p-1)} \prod_{i>j}^{p} (\tilde{r}_i-\tilde{r}_j) \prod_{i>j}^{p} (\tilde{s}_i-\tilde{s}_j) \prod_{i=1}^{p} \prod_{j=1}^{m} (a_j-i+1)^{i-1}}$$

where

(2.3.15) $$w_{ij} = {}_mF_n(a_1-p+1,\cdots,a_m-p+1\ ;\ b_1-p+1,\cdots,b_n-p+1\ ;\ \tilde{r}_i\tilde{s}_j)\ ,$$

$$(i,j = 1,\cdots,p)\ .$$

Proof

Khatri (1970, p. 68).

Theorem 2.3.7 (Corollary of theorem 2.3.6)

(2.3.16) $${}_0\tilde{F}_0(R,S) = \frac{\tilde{\Gamma}_p(p)\ |(\exp(\tilde{r}_i\tilde{s}_j))|}{\pi^{\frac{1}{2}p(p-1)} \prod_{i>j}^{p} (\tilde{r}_i-\tilde{r}_j) \prod_{i>j}^{p} (\tilde{s}_i-\tilde{s}_j)} .$$

(2.3.17) $${}_1\tilde{F}_0(a\,;R,S) = \frac{\tilde{\Gamma}_p(p)\ |((1-\tilde{r}_i\tilde{s}_j)^{p-a-1})|}{\pi^{\frac{1}{2}p(p-1)} \prod_{i=1}^{p} (a-i+1)^{i-1} \prod_{i>j}^{p} (\tilde{r}_i-\tilde{r}_j) \prod_{i>j}^{p} (\tilde{s}_i-\tilde{s}_j)} .$$

2.4 HAYAKAWA'S POLYNOMIALS

The generalised Hermite polynomial (g.H.p.), $H_\kappa(T)$ and its extension $P_\kappa(T,A)$ are functions with real matrix arguments and are discussed in Hayakawa (1969). These functions play an important role in the derivation of the p.d.f.s of real multivariate quadratic forms. Hayakawa (1972 a, 1972 b) extends the polynomials $H_\kappa(T)$ and $P_\kappa(T,A)$ to the polynomials $\tilde{H}_\kappa(T)$ and $\tilde{P}_\kappa(T,A)$ which are functions with complex matrix arguments. The definitions and properties of $\tilde{H}_\kappa(T)$ and $\tilde{P}_\kappa(T,A)$ are considered in sections 2.4.1 and 2.4.2 respectively.

It is important to note that although the polynomials $\tilde{H}_\kappa(T)$ and $\tilde{P}_\kappa(T,A)$ have complex matrices as arguments it is still polynomials of real quantities. This essential property of these polynomials become clear if the calculations of the polynomials are considered.

2.4.1 The Polynomial $\tilde{H}_\kappa(T)$

Hayakawa (1972 a, p. 2 - 3) defined the g.H.p. $\tilde{H}_\kappa(T)$ with complex matrix argument in order to discuss the joint p.d.f. of the characteristic roots of the non-central complex Wishart matrix.

Definition 2.4.1

Let $T:p\times n$ and $W:p\times n$, $p \leqq n$, be arbitrary complex matrices; then the g.H.p. $\tilde{H}_\kappa(T)$ which corresponds to the partition κ of k is defined as

(2.4.1) $\tilde{H}_\kappa(T)$

$$= \operatorname{etr}(T\bar{T}')\ \pi^{-np} \int_W \operatorname{etr}(-i(T\bar{W}' + W\bar{T}'))\ \operatorname{etr}(-W\bar{W}')\ \tilde{C}_\kappa(-W\bar{W}')\ dW .$$

The following three properties of $\tilde{H}_\kappa(T)$ are proved in Hayakawa (1972 a , p. 3):

Property 2.4.1

(2.4.2) $\tilde{H}_\kappa(T) = \tilde{H}_\kappa(U_1 T) = \tilde{H}_\kappa(TU_2)$

where $U_1 \in U(p)$ and $U_2 \in U(n)$.

Property 2.4.2

(2.4.3) $|\tilde{H}_\kappa(T)| \leq \operatorname{etr}(T\bar{T}')\ [n]_\kappa\ \tilde{C}_\kappa(I_p)$.

Property 2.4.3

(2.4.4) $\tilde{H}_\kappa(0) = (-1)^k\ [n]_\kappa\ \tilde{C}_\kappa(I_p)$.

The generating function of $\tilde{H}_\kappa(T)$ is given in theorem 2.4.1:

Theorem 2.4.1

Let $S:p\times n$ and $T:p\times n$, $p \leqq n$, be arbitrary complex matrices; then the generating function of $\tilde{H}_\kappa(T)$ is given by

$$(2.4.5)\quad \int_{U(p)} \int_{U(n)} \operatorname{etr}(-S\bar{S}' + U_1 S U_2 \bar{T}' + T \bar{U}_2' \bar{S}' \bar{U}_1')\, dU_2\, dU_1$$

$$= \sum_{k=0}^{\infty} \sum_{\kappa \varepsilon P(k,p)} \frac{\tilde{H}_\kappa(T)\ \tilde{C}_\kappa(S\bar{S}')}{k!\ [n]_\kappa\ \tilde{C}_\kappa(I_p)}$$

where $U_1 \varepsilon U(p)$ and $U_2 \varepsilon U(n)$.

Proof

Hayakawa (1972 a , p. 3 - 4).

Hayakawa (1972 a) also gives the relation between the g.H.p.s and the generalised Laquerre polynomials with hermitian matrix arguments which he also defined.

2.4.2 The Polynomial $\tilde{P}_\kappa(T,A)$

Hayakawa (1972 b, p. 221) defined the polynomials $\tilde{P}_\kappa(T,A)$ with complex matrix arguments in order to discuss non-central multivariate complex quadratic forms.

Definition 2.4.2

Let $T{:}p\times n$ and $W{:}p\times n$, $p \leq n$, be arbitrary complex matrices and let $A{:}n\times n$ be h.p.d.; then $\tilde{P}_\kappa(T,A)$ which corresponds to the partition κ of k is defined as

$$(2.4.6)\quad \tilde{P}_\kappa(T,A)$$

$$= \operatorname{etr}(T\bar{T}')\ \pi^{-pn} \int_W \operatorname{etr}(-i(T\bar{W}' + W\bar{T}'))\ \operatorname{etr}(-W\bar{W}')\ \tilde{C}_\kappa(-WA\bar{W}')\, dW\ .$$

The following three properties of $\tilde{P}_\kappa(T,A)$ are given in Hayakawa (1972 b, p. 221):

Property 2.4.4

$$(2.4.7) \quad \tilde{P}_\kappa(0,A) = (-1)^k [n]_\kappa \frac{\tilde{C}_\kappa(A)\ \tilde{C}_\kappa(I_p)}{\tilde{C}_\kappa(I_n)} .$$

Property 2.4.5

$$(2.4.8) \quad \tilde{P}_\kappa(T, I_n) = \tilde{H}_\kappa(T) .$$

Property 2.5.6

$$(2.4.9) \quad |\tilde{P}_\kappa(T,A)| \leq \operatorname{etr}(T\bar{T}')\ [n]_\kappa \frac{\tilde{C}_\kappa(A)\ \tilde{C}_\kappa(I_p)}{\tilde{C}_\kappa(I_n)} .$$

The generating function of $\tilde{P}_\kappa(T,A)$ is given in the following theorem:

Theorem 2.4.2

Let $S{:}p\times n$ and $T{:}p\times n$, $p \leq n$, be arbitrary complex matrices and $A{:}n\times n$ be h.p.d.; then the generating function of $\tilde{P}_\kappa(T,A)$ is given by

$$(2.4.10) \quad \int_{U(p)} \int_{U(n)} \operatorname{etr}(-SU_2A\bar{U}_2'\bar{S}' + U_1SU_2A^{\frac{1}{2}}\bar{T}' + TA^{\frac{1}{2}}\bar{U}_2'\bar{S}'\bar{U}_1')\ dU_2\ dU_1$$

$$= \sum_{k=0}^{\infty} \sum_{\kappa\varepsilon P(k,p)} \frac{\tilde{P}_\kappa(T,A)\ \tilde{C}_\kappa(S\bar{S}')}{k!\ [n]_\kappa\ \tilde{C}_\kappa(I_p)}$$

where $U_1 \varepsilon U(p)$ and $U_2 \varepsilon U(n)$.

Proof

Hayakawa (1972 b, p. 222).

The calculations of $\tilde{P}_K(T,A)$ for each partition of $k = 1,2$ and 3 are given in Hayakawa (1972 b, p. 228 - 229).

2.5 G-FUNCTIONS

Greenacre (1972, p. 15 - 19) extended the results of Mathai (1971, p. 72) in such a way that it is possible, under certain conditions to write down the p.d.f. of a random variable in terms of Meijer's G-functions if the moment sequence of the random variable is given.

Definition 2.5.1

The G-function of a complex variable z is defined as

$$(2.5.1)\qquad G(z) = G^{m\,n}_{p\,q}\left[z\,\middle|\,\begin{matrix} a_1,\cdots,a_p \\ b_1,\cdots,b_q \end{matrix}\right]$$

$$= \frac{1}{2\pi i} \oint_C \frac{\prod_{j=1}^{m} \Gamma(b_j+s) \prod_{j=1}^{n} \Gamma(1-a_j-s)}{\prod_{j=m+1}^{q} \Gamma(1-b_j-s) \prod_{j=n+1}^{p} \Gamma(a_j+s)} z^{-s}\, ds$$

where

p, q, m and n are integers such that $0 \leqq n \leqq p$ and $1 \leqq m \leqq q$;

a_j and b_h are complex numbers such that
$b_h + v \neq a_j - 1 - r$ for $v,r = 0,1,\cdots$
$h = 1,\cdots,m$
$j = 1,\cdots,n$;

the contour C is such that the points:

$$(2.5.2)\qquad -s = b_h + v\ , \quad (h = 1,\cdots,m)\ , \quad (r = 0,1,\cdots)$$

are enclosed within it and separate from the points;

(2.5.3) $\quad -s = a_j - 1 - r \;, \quad (j = 1,\cdots,n) \;, \quad (r = 0,1,\cdots) \;;$

(2.5.2) and (2.5.3) are clearly the poles of the

functions $\prod_{j=1}^{m} \Gamma(b_j+s)$ and $\prod_{j=1}^{n} \Gamma(1-a_j-s)$ respectively.

For the applications in this thesis the a_j and b_h are always real and a simpler form of the G-function in which $m = q = p$ and $n = 0$ is used, i.e.

(2.5.4)
$$G_p\left[z\left|\begin{matrix} a_j \\ b_j \end{matrix}\right.\right] = \frac{1}{2\pi i} \oint_C \frac{\prod_{j=1}^{p} \Gamma(b_j+s)}{\prod_{j=1}^{p} \Gamma(a_j+s)} z^{-s}\, ds \;.$$

In (2.5.4) C is a limiting contour enclosing all the poles of $\prod_{j=1}^{p} \Gamma(b_j+s)$.

Given a moment sequence $E(X^h)$ of a random variable X it can be shown that a p.d.f. which has these moments exists in the form of the inverse Mellin transform

(2.5.5)
$$f_X(x) = \frac{1}{2\pi i} \oint E(x^h)\, x^{-h-1}\, dh \;.$$

The function $f_X(x)$ is the p.d.f. of X if it can be shown that the moments determine the p.d.f. uniquely. Greenacre (1972, p. 18) showed that if $0 \leq x \leq 1$ the series

(2.5.6)
$$\sum_{h=1}^{\infty} (E(X^{2h}))^{-\frac{1}{2h}}$$

is divergent and from Carleman's theorem (cf. Anderson, 1958, p. 172) follows that at most one distribution has the moment $E(X^h)$. Thus for $0 \leq x \leq 1$ the moment sequence $E(X^h)$ determines the p.d.f.

of X uniquely through the inverse Mellin transform.

For $0 \leq x \leq 1$ and K independent of h Greenacre (1972, p. 18) proved the following theorem:

Theorem 2.5.1

If

$$(2.5.7) \quad E(X^h) = KT^h \frac{\prod_{j=1}^{p} \Gamma(a_j+h)}{\prod_{j=1}^{p} \Gamma(b_j+h)}$$

then

$$(2.5.8) \quad f_X(x) = Kx^{-1} \; G_p\left[\frac{x}{T} \;\middle|\; \begin{matrix} b_j \\ a_j \end{matrix}\right] .$$

As a result of the difference between $\tilde{\Gamma}_p(a,\kappa)$ and

$\Gamma_p(a,\kappa) = \pi^{\frac{1}{4}p(p-1)} \prod_{i=1}^{p} \Gamma(a+k_j-\frac{1}{2}(j-1))$ the results, proved in the following two theorems, are slightly different from those proved in Greenacre (1972, p. 19).

Theorem 2.5.2

If

$$(2.5.9) \quad E(X^h) = KT^h \frac{\tilde{\Gamma}_p(a+h,\kappa)}{\tilde{\Gamma}_p(b+h,\tau)}$$

where K is independent of h ; then

$$(2.5.10) \quad f_X(x) = Kx^{-1} \; G_p\left[\frac{x}{T} \;\middle|\; \begin{matrix} b+t_j-j+1 \\ a+k_j-j+1 \end{matrix}\right] .$$

Proof

$$(2.5.11) \quad E(X^h) = KT^h \frac{\prod_{j=1}^{p} \Gamma(a+h+k_j-j+1)}{\prod_{j=1}^{p} \Gamma(b+h+t_j-j+1)} .$$

The result follows from theorem 2.5.1.

Theorem 2.5.3

If

$$(2.5.12) \quad E(X^h) = KT^h {}_m\tilde{F}_n(a_1,\cdots,a_r+h,\cdots,a_m \; ; \; b_1,\cdots,b_s+h,\cdots,b_n \; ; \; S) \frac{\tilde{\Gamma}_p(a_r+h)}{\tilde{\Gamma}_p(b_s+h)}$$

where K is independent of h ; then

$$(2.5.13) \quad f_X(x)$$

$$= Kx^{-1} \sum_k \sum_{\kappa \varepsilon P(k,p)} \frac{[a_1]_\kappa \cdots [a_{r-1}]_\kappa [a_{r+1}]_\kappa \cdots [a_m]_\kappa}{[b_1]_\kappa \cdots [b_{s-1}]_\kappa [b_{s+1}]_\kappa \cdots [b_n]_\kappa} \frac{\tilde{C}_\kappa(S)}{k!} G_p\left[\frac{x}{T} \middle| \begin{matrix} b_s + k_j - j + 1 \\ a_r + k_j - j + 1 \end{matrix}\right] .$$

Proof

Expand $E(X^h)$; then follow the factors depending on h as

$$(2.5.14) \quad \frac{\tilde{\Gamma}_p(a_r+h) \; [a_r+h]_\kappa}{\tilde{\Gamma}_p(b_s+h) \; [b_s+h]_\kappa} = \frac{\tilde{\Gamma}_p(a_r+h \; ; \; \kappa)}{\tilde{\Gamma}_p(b_s+h \; ; \; \kappa)} .$$

The result follows from theorem 2.5.2.

2.6 INCOMPLETE GAMMA- AND BETA FUNCTIONS

Various marginal distributions of the characteristic roots of hermitian random matrices can be expressed in terms of incomplete gamma- and beta functions. These functions will be used repeatedly in this thesis and therefore it is considered here.

Definition 2.6.1

The incomplete gamma function is defined as

$$(2.6.1) \quad \Gamma(t,n) = \int_0^t e^{-x} x^{n-1} \, dx \, , \quad n > 0, \quad t > 0 \, .$$

In definition 2.6.1 $\Gamma(\infty,n)$ is the complete gamma function $\Gamma(n)$. The function

$$(2.6.2) \quad I(u\,,\,n-1) = \frac{1}{\Gamma(n)} \int_0^{\sqrt{n}\,u} e^{-x} x^{n-1} \, dx$$

$$= \frac{\Gamma(\sqrt{n}\;u\,,\,n)}{\Gamma(n)}$$

is tabulated in Pearson (1934 a) for different values of u and n.

Definition 2.6.2

The incomplete beta function of the first type is defined as

$$(2.6.3) \quad B_t(a,b) = \int_0^t x^{a-1} (1-x)^{b-1} \, dx \, , \quad 0 < t \leq 1 \, , \quad a,b > 0 \, .$$

In definition 2.6.2 $B_1(a,b)$ is the complete beta function $B(a,b)$. The function

$$(2.6.4) \quad I_t(a,b) = \frac{1}{B(a,b)} \int_0^t x^{a-1} (1-x)^{b-1}\, dx$$

$$= \frac{B_t(a,b)}{B(a,b)}$$

is tabulated in Pearson (1934 b) for different values of t , a and b.

Definition 2.6.3

The incomplete beta function of the second type is defined as

$$(2.6.5) \quad B_t^*(a,b) = \int_0^t \frac{x^{a-1}}{(1+x)^{a+b}}\, dx , \quad t > 0 , \quad a,b > 0 .$$

Theorem 2.6.1

$$(2.6.6) \quad B_t^*(a,b) = B_{\frac{t}{1+t}}(a,b) , \quad t > 0 , \quad a,b > 0 .$$

Proof

In (2.6.5) make the transformation

$$(2.6.7) \quad y = \frac{x}{1+x}$$

with inverse transformation

$$(2.6.8) \quad x = \frac{y}{1-y} .$$

The jacobian of (2.6.8) follows as

$$(2.6.9) \quad J(x \to y) = (1-y)^{-2}$$

so that (2.6.5) can be written as

$$(2.6.10) \quad B_t^*(a,b) = \int_0^{\frac{t}{1+t}} y^{a-1} (1-y)^{b-1} \, dy = B_{\frac{t}{1+t}}(a,b) \, .$$

2.7 THE SYMMETRISED DISTRIBUTION OF A RANDOM POSITIVE DEFINITE HERMITIAN MATRIX

Greenacre (1973, p. 97) defined the symmetrised p.d.f. of a real positive definite symmetric matrix. This form of the p.d.f. is in many important respects equivalent to the actual p.d.f. and is also found very useful in the study of multivariate non-central distributions. In this section the concept of the symmetrised p.d.f. of a real positive definite symmetric matrix is extended to the symmetrised p.d.f. of a hermitian positive definite matrix.

<u>Definition 2.7.1</u>

The symmetrised p.d.f. of the positive definite hermitian matrix $\underset{\sim}{A}:p\times p$ is defined as

$$(2.7.1) \quad f_{csym}(A) = \int_{U(p)} f_{\underset{\sim}{A}}(U A \bar{U}') \, du$$

where $f_{\underset{\sim}{A}}(A)$ is the p.d.f. of $\underset{\sim}{A}:p\times p$.

The following important property of $f_{csym}(A)$ is proved analogous to the real case:

<u>Property 2.7.1</u> (Analogous to Greenacre, 1973, p. 97, theorem 2.1)

Let $g(\underset{\sim}{A})$ be a function of the random h.p.d. matrix $\underset{\sim}{A}:p\times p$ such that $g(A) = g(U A \bar{U}')$, $U \varepsilon U(p)$; then the distribution of $g(\underset{\sim}{A})$ is invariant with respect to symmetrisation of the p.d.f. $f_{\underset{\sim}{A}}(A)$.

Proof

Let M be a measurable set in the range of g and let $g^{-1}(M)$ be its preimage. Let $\mu(M)$ and $\nu(M)$ be respectively probabilities of M when $\underset{\sim}{A}:p\times p$ has the symmetrised and unsymmetrised p.d.f.s; then

$$(2.7.2)\quad \mu(M) = \int_{g^{-1}(M)} \left[\int_{U(p)} f_{\underset{\sim}{A}}(U A \bar{U}')\, dU \right] dA$$

$$= \int_{U(p)} \left[\int_{g^{-1}(M)} f_{\underset{\sim}{A}}(U A \bar{U}')\, dA \right] dU .$$

In (2.7.2) make the transformation

$$(2.7.3)\quad B = U A \bar{U}'$$

with inverse transformation

$$(2.7.4)\quad A = \bar{U}' B U .$$

The jacobian of (2.7.4) is

$$(2.7.5)\quad J(A \to B) = 1 .$$

Hence,

$$(2.7.6)\quad \mu(M) = \int_{U(p)} \left[\int_{U g^{-1}(M) \bar{U}'} f_{\underset{\sim}{B}}(B)\, dB \right] dU .$$

But $U g^{-1}(M) \bar{U}' = g^{-1}(M)$ because $g(M) = g(U M \bar{U}')$.

Thus

$$\mu(M) = \int_{U(p)} dU\left[\int_{g^{-1}(M)} f_{\underset{\sim}{B}}(B)\, dB\right]$$

$$= \int_{g^{-1}(M)} f_{\underset{\sim}{B}}(B)\, dB = \nu(M)$$

which proves the result.

Corollary 2.7.1

The p.d.f.s of $|\underset{\sim}{A}|$, $|I_p - \underset{\sim}{A}|$, $D_{\underset{\sim}{A}}$ and $tr(\underset{\sim}{A})$ are invariant with respect to symmetrisation of the p.d.f. of $\underset{\sim}{A}$:p×p.

Proof

These results follow clearly from property 2.7.1 and the fact that for all $U \varepsilon U(p)$:

$$|A| = |U A \bar{U}'| \; ;$$

$$|I_p - A| = |I_p - U A \bar{U}'| \; ;$$

$$|I_p - \tilde{a}_i A| = 0 \Rightarrow |I_p - \tilde{a}_i U A \bar{U}'| = 0 \; ;$$

$$tr(A) = tr(U A \bar{U}') \; .$$

Remark 2.7.1

From corollary 2.7.1 is it clear that $E(|A|^h)$, $E(|I_p - A|^h)$, and $E((tr(A))^h)$ are invariant with respect to symmetrisation of the p.d.f. $f_{\underset{\sim}{A}}(A)$.

2.8 THE COMPLEX NORMAL- AND COMPLEX WISHART DISTRIBUTIONS

The complex normal- and complex Wishart distributions will be used repeatedly in this thesis. In section 2.8.1 the p-variable complex normal distribution and its extension to the matrix variate complex normal distribution are discussed. The definition of a complex Wishart matrix and its p.d.f. in the central and non-central cases are given in section 2.8.2.

2.8.1 The complex normal distribution

The p-variable complex normal distribution was originally derived by Wooding (1956) and Goodman (1963 a). The theory and properties of this distribution are also discussed in Khatri (1965, p. 103), Smith (1972, p. 85 - 87), Gupta (1971, 1973, 1976) and in more detail in Steel (1979, p. 112 - 119).

Definition 2.8.1

Let $\underline{Z}_0:2p\times 1 = [X_1,\cdots,X_p,\ Y_1,\cdots,Y_p]' = [\underline{X},\ \underline{Y}]'$ have the 2p-variable real normal distribution with expected value

$$(2.8.1)\qquad E(\underline{Z}_0) = [E(\underline{X}'),\ E(\underline{Y}')]'$$

and covariance matrix

$$(2.8.2)\qquad \Sigma_0:2p\times 2p = \tfrac{1}{2}\begin{bmatrix}\Sigma_{01} & -\Sigma_{02}\\ \Sigma_{02} & \Sigma_{01}\end{bmatrix}$$

with $\Sigma_{01}:p\times p$ a real symmetric matrix and $\Sigma_{02}:p\times p$ a real skewsymmetric matrix. The p-variable complex normal vector is defined as

$$(2.8.3)\qquad \underline{Z}:p\times 1 = [Z_1,\cdots,Z_p]' = [X_1+iY_1,\cdots,X_p+iY_p]'$$

$$= \underline{X} + i\underline{Y}$$

with expected value

$$(2.8.4)\qquad E(\underline{Z}) = E(\underline{X}) + i\,E(\underline{Y}) = \underline{\mu}:p\times 1$$

and hermitian covariance matrix

$$(2.8.5)\qquad \Sigma:p\times p = E[\underline{Z}-\underline{\mu}]\,[\overline{\underline{Z}-\underline{\mu}}]'$$

$$= \Sigma_{01} + i\,\Sigma_{02}\ .$$

The p.d.f. of $\underline{Z}:p\times 1$ is given by

$$(2.8.6)\quad f_{\underline{Z}}(\underline{z}) = \pi^{-p}|\Sigma|^{-1}\exp(-(\overline{\underline{z}-\underline{\mu}})'\,\Sigma^{-1}(\underline{z}-\underline{\mu}))$$

with $\underline{z}\,\varepsilon\,\{\underline{z} = [z_1,\cdots,z_p]'\ ;\ z_j = x_j + iy_j\ ;\ -\infty < x_j\ ,\ y_j < \infty\ ,$

$j = 1,\cdots,p\}$.

The p.d.f. given in (2.8.6) will be denoted as $\underline{Z}:p\times 1 \sim CN(p,\ \underline{\mu},\ \Sigma)$.

Remark 2.8.1

(i) The structure of $\Sigma_0:2p\times 2p$ given in (2.8.2) is the most distinguishing characteristic of the p-variable complex normal distribution. This structure implies:

$$(2.8.7)\quad \mathrm{Cov}(X_j\ ,\ X_k) = \mathrm{Cov}(X_k\ ,\ X_j) = \mathrm{Cov}(Y_j\ ,\ Y_k) = \mathrm{Cov}(Y_k\ ,\ Y_j)\ ,\quad (j\ ,\ k = 1,\cdots,p)\ ,$$

$$(2.8.8)\quad \mathrm{Cov}(X_j\ ,\ Y_j) = 0\ ,\quad (j = 1,\cdots,p)$$

and

$$(2.8.9)\quad \mathrm{Cov}(X_j\ ,\ Y_k) = \mathrm{Cov}(Y_k\ ,\ X_j) = -\mathrm{Cov}(Y_j\ ,\ X_k) = -\mathrm{Cov}(X_k\ ,\ Y_j)\ ,\quad (j\ ,\ k = 1,\cdots,p\ ,\ j\neq k)\ .$$

These relations are of great importance because they are the basic assumptions originally made by Wooding (1956) and Goodman (1963 a) in their derivation of the p-variable complex normal distribution and are the basis of complex distribution theory.

(ii) From (2.8.5) follows that

$$(2.8.10)\quad \mathrm{Cov}(\underline{X}\ ,\ \underline{X}') = \mathrm{Cov}(\underline{Y}\ ,\ \underline{Y}') = \tfrac{1}{2}\mathrm{Re}(\Sigma)$$

and

$$(2.8.11)\quad \mathrm{Cov}(\underline{X}\ ,\ \underline{Y}') = -\mathrm{Cov}(\underline{Y}\ ,\ \underline{X}') = \tfrac{1}{2}\mathrm{Im}(\Sigma)$$

where $Re(\Sigma)$ and $Im(\Sigma)$ denote the real and imaginary matrix components of Σ respectively. It is also clear that

(2.8.12) $$Var(Z_j) = Var(X_j) + Var(Y_j) \ , \quad (j = 1,\cdots,p)$$

and

(2.8.13) $$Cov(Z_j , Z_k)$$

$$= Cov(X_j , X_k) + Cov(Y_j , Y_k) + i(Cov(Y_j , X_k) - Cov(X_j , Y_k)) ,$$

$$(j , k = 1,\cdots,p , j \neq k) \ .$$

The p-variable complex normal distribution can be extended to the complex matrix variate normal distribution and is discussed in Steel (1979, p. 130 - 140) and Tan (1968, p. 263 - 267). In definition 2.8.2 the complex normal matrix variate with dependent rows and independent columns is defined.

Definition 2.8.2

Let $\underline{Z}_j : p\times 1 = [Z_{1j},\cdots,Z_{pj}]' \sim CN(p, \underline{\mu}_j, \Sigma)$, $(j = 1,\cdots,n)$ where $\underline{Z}_j : p\times 1$ is independently distributed of $\underline{Z}_k : p\times 1$, $j \neq k$. The complex normal matrix variate with dependent rows and independent columns is defined as

(2.8.14) $$\underset{\sim}{Z} : p\times n = [\underline{Z}_1,\cdots,\underline{Z}_n] = \begin{bmatrix} Z_{11} & Z_{12} \cdots & Z_{1n} \\ Z_{21} & Z_{22} \cdots & Z_{2n} \\ \cdots & \cdots & \cdots \\ Z_{p1} & Z_{p2} \cdots & Z_{pn} \end{bmatrix}$$

$$= [\underline{X}_1 + i\,\underline{Y}_1 ,\cdots,\underline{X}_n + i\,\underline{Y}_n]$$

$$= \underset{\sim}{X} + i\,\underset{\sim}{Y}$$

with matrix of expected values

$$(2.8.15)\qquad E(\underset{\sim}{Z}) = [\underline{\mu}_1, \cdots, \underline{\mu}_n] = M{:}p\times n$$

and the covariance matrix

$$(2.8.16)\qquad \mathrm{Cov}(\underset{\sim}{Z})$$

$$= E\begin{bmatrix} (\underline{Z}_1-\underline{\mu}_1)(\overline{\underline{Z}_1-\underline{\mu}_1})' & (\underline{Z}_1-\underline{\mu}_1)(\overline{\underline{Z}_2-\underline{\mu}_2})' & \cdots & (\underline{Z}_1-\underline{\mu}_1)(\overline{\underline{Z}_n-\underline{\mu}_n})' \\ (\underline{Z}_2-\underline{\mu}_2)(\overline{\underline{Z}_1-\underline{\mu}_1})' & (\underline{Z}_2-\underline{\mu}_2)(\overline{\underline{Z}_2-\underline{\mu}_2})' & \cdots & (\underline{Z}_2-\underline{\mu}_2)(\overline{\underline{Z}_n-\underline{\mu}_n})' \\ \cdots & \cdots & \cdots & \cdots \\ (\underline{Z}_n-\underline{\mu}_n)(\overline{\underline{Z}_1-\underline{\mu}_1})' & (\underline{Z}_n-\underline{\mu}_n)(\overline{\underline{Z}_2-\underline{\mu}_2})' & \cdots & (\underline{Z}_n-\underline{\mu}_n)(\overline{\underline{Z}_n-\underline{\mu}_n})' \end{bmatrix}$$

$$= \begin{bmatrix} \Sigma & 0 & \cdots & 0 \\ 0 & \Sigma & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \Sigma \end{bmatrix}$$

$$= I_n \otimes \Sigma \ .$$

Since the n columns of $\underset{\sim}{Z}{:}p\times n$ are independently distributed, the p.d.f. of $\underset{\sim}{Z}{:}p\times n$ follows as

$$(2.8.17)\qquad f_{\underset{\sim}{Z}}(Z)$$

$$= \prod_{j=1}^{n} \pi^{-p}\,|\Sigma|^{-1} \exp(-(\overline{\underline{z}_j-\underline{\mu}_j})'\Sigma^{-1}(\underline{z}_j-\underline{\mu}_j))$$

$$= \pi^{-np}\,|\Sigma|^{-n}\,\mathrm{etr}[-(\overline{Z-M})'\Sigma^{-1}(Z-M)]$$

$$= \pi^{-np}\,|\Sigma|^{-n}\,\mathrm{etr}[-\Sigma^{-1}(Z-M)(\overline{Z-M})']$$

with

$$Z \,\varepsilon\, \{Z{:}p\times n = (z_{jk})\ ;\ z_{jk} = x_{jk} + iy_{jk}\ ;\ -\infty < x_{jk}, y_{jk} < \infty\ ,\quad j = 1,\cdots,p\ ,\ \ k = 1,\cdots,n\}\ .$$

This density will be denoted as $\underset{\sim}{Z}:p\times n \sim CMTN(p,\ n,\ M,\ I_n \otimes \Sigma)$.

Remark 2.8.2

The covariance matrix given in (2.8.16) can also be interpreted as follows:

Form the column vector

$$(2.8.18) \quad \underline{Z}_c:np\times 1 = \begin{bmatrix} Z_{11} \\ Z_{21} \\ \vdots \\ Z_{p1} \\ Z_{12} \\ \vdots \\ Z_{pn} \end{bmatrix}$$

by packing the columns of $\underset{\sim}{Z}:p\times n$ below one another into a single column vector; then follows that

$$(2.8.19) \quad \mathrm{Cov}(\underline{Z}_c\ ,\ \underline{Z}_c') = I_n \otimes \Sigma\ .$$

The distribution of $\underset{\sim}{Z}:p\times n$ can be generalised to the case where there is also dependence between the column vectors of $\underset{\sim}{Z}:p\times n$. In this case the covariance matrix is of the form $\Phi \otimes \Sigma$ where $\Phi:n\times n \neq I_n$ is a hermitian positive definite matrix. Let $\underline{m}_c:np\times 1 = [\underline{\mu}_1',\cdots,\underline{\mu}_n']'$ i.e. a column vector formed by packing the columns of $M:p\times n$ below one another, then the p.d.f. of $\underline{Z}_c:np\times 1$ follows as

$$(2.8.20) \quad f_{\underline{Z}_c}(\underline{z}_c)$$

$$= \pi^{-np}\ |\Phi \otimes \Sigma|^{-1}\ \exp(-(\overline{\underline{z}_c - \underline{m}_c})'(\Phi \otimes \Sigma)^{-1}(\underline{z}_c - \underline{m}_c))$$

$$= \pi^{-np} \, |\Phi|^{-p} |\Sigma|^{-n} \, \exp(-(\overline{\underline{z}_c - \underline{m}_c})' \Phi^{-1} \otimes \Sigma^{-1} \, (\underline{z}_c - \underline{m}_c))$$

$$= \pi^{-np} \, |\Phi|^{-p} |\Sigma|^{-n} \, \mathrm{etr}[-(Z - M)\Phi^{-1} (\overline{Z - M})' \Sigma^{-1}]$$

$$= \pi^{-np} \, |\Phi|^{-p} |\Sigma|^{-n} \, \mathrm{etr}[-\Sigma^{-1} (Z - M)\Phi^{-1} (\overline{Z - M})']$$

$$= f_{\underset{\sim}{Z}}(Z)$$

with

$$Z \, \varepsilon \, \{Z{:}p\times n = (z_{jk}) \; ; \; z_{jk} = x_{jk} + iy_{jk} \; ; \; -\infty < x_{jk}, y_{jk} < \infty \, , \; j = 1, \cdots, p \, , \, k = 1, \cdots, n\} \, .$$

This density will be denoted as $\underset{\sim}{Z}{:}p\times n \sim \mathrm{CMTN}(p, n, M, \Phi \otimes \Sigma)$.

Remark 2.8.3

The algebra regarding kronecker products used in the derivation of (2.8.20) can be found in Graybill (1969, p. 196 - 209), Macduffee (1946, p. 81 - 85) and Roy and Gnanandesikan (1959).

2.8.2 The complex Wishart distribution

The complex Wishart matrix was originally defined by Goodman (1963 a). Goodman (1963 a) also derived the central p.d.f. of the Wishart matrix. Discussions of the theory and properties of the central and non-central Wishart distributions can be found in Goodman (1963 b), James (1964, p. 489), Khatri (1965, p. 105 - 107), Tan (1968, p. 267 - 271), Smith (1972, p. 99 - 107) and Steel (1979, p. 47 - 51 and p. 147 - 148). In definition 2.8.3 the non-central complex Wishart matrix is defined and the p.d.f. of this complex random matrix is given.

Definition 2.8.3

Let $\underset{\sim}{Z}{:}p\times n = [\underline{Z}_1, \cdots, \underline{Z}_n] \sim \mathrm{CMTN}(p, n, M, I_n \otimes \Sigma)$; then for $n \geq p$ the non-central complex Wishart matrix is defined as

(2.8.21) $$\underset{\sim}{A}:p\times p = \underset{\sim}{Z}\underset{\sim}{\bar{Z}}' = \sum_{j=1}^{n} \underline{Z}_j \underline{\bar{Z}}'_j$$

$$= (\underset{\sim}{X} + i\,\underset{\sim}{Y})(\underset{\sim}{X} - i\,\underset{\sim}{Y})'$$

$$= \underset{\sim}{A}_1 + i\,\underset{\sim}{A}_2$$

where

(2.8.22) $$\underset{\sim}{A}_1:p\times p = \underset{\sim}{X}\underset{\sim}{X}' + \underset{\sim}{Y}\underset{\sim}{Y}'$$

is a real symmetric random matrix and

(2.8.23) $$\underset{\sim}{A}_2:p\times p = \underset{\sim}{Y}\underset{\sim}{X}' - \underset{\sim}{X}\underset{\sim}{Y}'$$

is a real skewsymmetric random matrix.

The hermitian matrix $A:p\times p$ has the non-central Wishart distribution with n degrees of freedom and non-centrality parameter

(2.8.24) $$\Omega:p\times p = \Sigma^{-1} M\bar{M}' \ .$$

The p.d.f. of $A:p\times p$ is given by James (1964, p. 489) as

(2.8.25) $$f_{\underset{\sim}{A}}(A)$$

$$= \frac{\operatorname{etr}(-\Omega)}{\tilde{\Gamma}_p(n)\ |\Sigma|^n}\ {}_0\tilde{F}_1(n\ ;\ \Omega\Sigma^{-1}A)\,|A|^{n-p}\operatorname{etr}[-\Sigma^{-1}A]\ ,\quad A = \bar{A}' > 0\ .$$

This density will be denoted as $\underset{\sim}{A}:p\times p \sim \mathrm{NCCW}(p,\ n,\ \Sigma,\Omega)$.

Let $M:p\times n$ be the null matrix, i.e. $\underset{\sim}{Z}:p\times n \sim \mathrm{CMTN}(p,\ n,\ 0,\ I_n \otimes \Sigma)$; then $\underset{\sim}{A}:p\times p = \underset{\sim}{Z}\underset{\sim}{\bar{Z}}'$ has the central complex Wishart distribution with n degrees of freedom. The p.d.f. of $\underset{\sim}{A}:p\times p$ is given as

(2.8.26) $$f_{\underset{\sim}{A}}(A) = \frac{|A|^{n-p}\operatorname{etr}(-\Sigma^{-1}A)}{\tilde{\Gamma}_p(n)\ |\Sigma|^n}\ ,\quad A = \bar{A}' > 0$$

$$= \frac{|A|^{n-p} \; {}_0\tilde{F}_0(-\Sigma^{-1}A)}{\tilde{\Gamma}_p(n) \; |\Sigma|^n} \quad , \quad A = \bar{A}' > 0$$

$$(2.8.27) \qquad = \frac{|A|^{n-p} \; \mathrm{etr}(-A) \; {}_0\tilde{F}_0((I_p - \Sigma^{-1})A)}{\tilde{\Gamma}_p(n) \; |\Sigma|^n} \quad , \quad A = \bar{A}' > 0 \; .$$

This density will be denoted as $\underset{\sim}{A}: p \times p \sim CW(p, n, \Sigma)$. The p.d.f. of the central and non-central Wishart matrices will also be considered in chapter 4.

The inverted complex Wishart distribution is derived by Shaman (1980). He also investigated the application of this distribution to spectral estimation.

CHAPTER 3

INTEGRALS FOR DERIVING CERTAIN MARGINAL DISTRIBUTIONS OF THE CHARACTERISTIC ROOTS OF A RANDOM HERMITIAN POSITIVE DEFINITE MATRIX

3.1 INTRODUCTION

If the p.d.f. of a random h.p.d. matrix is a symmetric function of the characteristic roots of this h.p.d. matrix, the joint p.d.f. of the characteristic roots is also a symmetric function of the roots. This essential property of the joint p.d.f.s of the characteristic roots of h.p.d. matrices will be discussed in section 3.2. As a result of this property, the obtaining of certain marginal distributions of the characteristic roots of a h.p.d. random matrix is less complicated than of the characteristic roots of a real positive definite symmetric random matrix. In sections 3.3 - 3.7 integrals which are needed to derive these marginal distributions of the characteristic roots of the h.p.d. random matrices, discussed in the subsequent chapters, will be considered.

The following definition of a determinant of a square matrix will be repeatedly used in the rest of this chapter:

Definition 3.1.1 (Anderson, 1958, p. 335)

Let $B{:}p\times p = (b_{ij})$, $(i, j = 1,\cdots,p)$, be any square matrix; then the determinant, $|B|$, is defined by

$$(3.1.1) \quad |B| = \Sigma(-1)^{f(j_1,\cdots,j_p)} \prod_{i=1}^{p} b_{ij_i} .$$

In definition 3.1.1 the summation is over all permutations $(j_1,\cdots,j_p)$ of the set of integers $(1,\cdots,p)$, and

$f(j_1,\cdots,j_p)$ is the number of transpositions required to change $(1,\cdots,p)$ into $(j_1,\cdots,j_p)$. A transposition consists of interchanging two numbers, and it can be shown that, although one can transform $(1,\cdots,p)$ into $(j_1,\cdots,j_p)$ by transpositions in many different ways, the number of transpositions required is always even or always odd so that $(-1)^{f(j_1,\cdots,j_p)}$ is consistently defined.

3.2 THE JOINT P.D.F. OF THE CHARACTERISTIC ROOTS OF A RANDOM H.P.D. MATRIX

Consider the following theorem:

Theorem 3.2.1

Let $\underset{\sim}{A}:p\times p$ be a h.p.d. random matrix with p.d.f.

$$(3.2.1) \quad f_{\underset{\sim}{A}}(A) = h(\tilde{a}_1,\cdots,\tilde{a}_p)$$

where $h(\tilde{a}_1,\cdots,\tilde{a}_p)$ is a function which depends only on the characteristic roots $0 < \tilde{a}_1 < \cdots < \tilde{a}_p$ of $A:p\times p$; then the joint p.d.f. of $\tilde{A}_1,\cdots,\tilde{A}_p$ is given by

$$(3.2.2) \quad f_{\underset{\sim}{D}_A}(D_A) = \frac{\pi^{p(p-1)}}{\tilde{\Gamma}_p(p)} \prod_{i>j}^{p} (\tilde{a}_i - \tilde{a}_j)^2 \, h(\tilde{a}_1,\cdots,\tilde{a}_p) \ ,$$

$$0 < \tilde{a}_1 < \cdots < \tilde{a}_p \ .$$

Proof

James (1964, p. 488), Khatri (1965 , p. 102 - 103).

It is clear from (3.2.2) that $f_{\underset{\sim}{D}_A}(D_A)$ is a symmetric function of $\tilde{a}_1,\cdots,\tilde{a}_p$ if $h(\tilde{a}_1,\cdots,\tilde{a}_p)$ is a symmetric function of these roots. The corresponding joint p.d.f. of the characteristic roots of a real symmetric random matrix is not a symmetric function of the roots because it contains the term $\prod_{i>j}^{p} (\tilde{a}_i - \tilde{a}_j)$, (cf. Anderson, 1958, p. 318 - 320). All the h.p.d. random matrices studied in this thesis are such that $h(\tilde{a}_1,\cdots,\tilde{a}_p)$ is a symmetric function of the roots so that for the purpose of this thesis $f_{\underset{\sim}{D}_A}(D_A)$ is a symmetric function.

It is important to note that all the non-central and some of the central p.d.f.s of the h.p.d. random matrices studied in the subsequent chapters involve zonal polynomials of these hermitian matrices or hypergeometric functions with these hermitian matrices as arguments. By using the results given in theorems 2.2.8, 2.3.6, 2.3.7 and 3.2.1 it will become clear in these chapters that the random component in the joint p.d.f. of the characteristic roots of the h.p.d. random matrix, $\underset{\sim}{A}:p\times p$, which has to be integrated to obtain the different marginal distributions of the roots, can be written as

$$(3.2.3) \qquad \prod_{i=1}^{p} g(\tilde{a}_i)\ \phi(\tilde{a}_1,\cdots,\tilde{a}_p)\ \psi(\tilde{a}_1,\cdots,\tilde{a}_p)$$

where

(i) $\prod_{i=1}^{p} g(\tilde{a}_i)$ is a symmetric function of $\tilde{a}_1,\cdots,\tilde{a}_p$,

(ii) $\phi(\tilde{a}_1,\cdots,\tilde{a}_p) = |(\phi_i(\tilde{a}_j))|$

and

(iii) $\psi(\tilde{a}_1,\cdots,\tilde{a}_p) = |(\psi_i(\tilde{a}_j))|$.

Thus in the rest of this chapter only integrals with (3.2.3) as integrand will be considered.

3.3 C.D.F. OF THE EXTREME CHARACTERISTIC ROOTS

In theorem 3.3.1 the integral which is needed to obtain $P(c < \tilde{A}_1 < \cdots < \tilde{A}_p < d)$, $0 \leqq c < d$, given (3.2.3), is solved:

Theorem 3.3.1

$$(3.3.1) \quad \int_{a < x_1 < \cdots < x_p < b} \cdots\cdots\cdots \int \prod_{i=1}^{p} g(x_i) \; \phi(x_1,\cdots,x_p) \; \psi(x_1,\cdots,x_p) \; dx_1 \cdots dx_p$$

$$= |(b_{ij})|$$

where

$$(3.3.2) \quad b_{ij} = \int_a^b g(x) \; \phi_i(x) \; \psi_j(x) \; dx \, , \quad (i \, , j = 1,\cdots,p) \; .$$

Proof

Khatri (1970, p. 67) proved:

$$(3.3.3) \quad \int_{a < x_1 < \cdots < x_p < b} \cdots\cdots\cdots \int \phi(x_1,\cdots,x_p) \; \psi(x_1,\cdots,x_p) \; dx_1 \cdots dx_p$$

$$= |(c_{ij})|$$

where

$$(3.3.4) \quad c_{ij} = \int_a^b \phi_i(x) \; \psi_j(x) \; dx \, , \quad (i \, , j = 1,\cdots,p) \; .$$

The product $\prod_{i=1}^{p} g(x_i) \; \phi(x_1,\cdots,x_p)$ in (3.3.1) can be written as

$$(3.3.5) \qquad \prod_{i=1}^{p} g(x_i)\ \phi(x_1,\cdots,x_p)$$

$$= \prod_{i=1}^{p} g(x_i)\ \Sigma(-1)^{f(j_1,\cdots,j_p)} \prod_{i=1}^{p} \phi_i(x_{j_i})$$

$$= \Sigma(-1)^{f(j_1,\cdots,j_p)} \prod_{i=1}^{p} \phi_i(x_{j_i})\ g(x_{j_i})$$

$$= |(\phi_i(x_j)\ g(x_j))|$$

$$= \phi^*(x_1,\cdots,x_p)$$

with

$$(3.3.6) \qquad \phi_i^*(x_j) = \phi_i(x_j)\ g(x_j)\ , \qquad (i\ ,\ j = 1,\cdots,p)\ .$$

By replacing $\phi(x_1,\cdots,x_p)$ with $\phi^*(x_1,\cdots,x_p)$ in (3.3.3) the result follows.

Remark 3.3.1

It is clear that $P(\tilde{A}_p < b)$ and $P(\tilde{A}_1 < a) = 1 - P(\tilde{A}_1 > a)$ can be obtained from (3.3.1) by taking $a = 0$ and $b = \infty$ respectively.

3.4 C.D.F. OF ANY INTERMEDIATE CHARACTERISTIC ROOT

The c.d.f. of the t-th characteristic root, $\tilde{A}_t$, of $\underset{\sim}{A}:p\times p$, $(t = 2,\cdots,p-1)$, follows as

$$(3.4.1) \qquad P(\tilde{A}_t < c)$$

$$= P(\tilde{A}_{t+1} < c) + P(0 < \tilde{A}_1 < \cdots < \tilde{A}_t < c < \tilde{A}_{t+1} < \cdots < \tilde{A}_p < \infty)\ .$$

The repeated application of (3.4.1) leads to

$$(3.4.2) \qquad P(\tilde{A}_t < c)$$

$$= P(A_{t+2} < c) + P(0 < \tilde{A}_1 < \cdots < \tilde{A}_{t+1} < c < \tilde{A}_{t+2} < \cdots < \tilde{A}_p < \infty)$$

$$+ P(0 < \tilde{A}_1 < \cdots < \tilde{A}_t < c < \tilde{A}_{t+1} < \cdots < \tilde{A}_p < \infty)$$

$$= \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$= \sum_{r=t}^{p-1} P(0 < \tilde{A}_1 < \cdots < \tilde{A}_r < c < \tilde{A}_{r+1} < \cdots < \tilde{A}_p < \infty) + P(\tilde{A}_p < c) \quad .$$

In theorem 3.4.1 the integral which is needed to obtain an expression for $P(0 < \tilde{A}_1 < \cdots < \tilde{A}_r < c < \tilde{A}_{r+1} < \cdots < \tilde{A}_p < \infty)$, given (3.2.3), is solved:

Theorem 3.4.1

$$(3.4.3) \qquad \int_{\Lambda_1} \cdots \int \prod_{i=1}^{p} g(x_i) \ \phi(x_1,\cdots,x_p) \ \psi(x_1,\cdots,x_p) \ dx_1 \cdots dx_p$$

$$= \Sigma_1 \ \Sigma_2 (-1)^{\Sigma\delta_i + \Sigma\alpha_i} \ |(b_{3ij})| \, |(b_{4ij})|$$

where

$\Lambda_1 = \{a < x_1 < \cdots < x_r < x < x_{r+1} < \cdots < x_p < b\}$,

$\delta_1 < \cdots < \delta_r$ is a subset of the integers $1,2,\cdots,p$,

$\nu_1 < \cdots < \nu_{p-r}$ is a subset complementary to $\delta_1,\cdots,\delta_r$,

Σ_1 denotes the summation over all $\binom{p}{r}$ possible choices of $\delta_1 < \cdots < \delta_r$,

$\alpha_1 < \cdots < \alpha_r$ is a subset of the integers $1,\cdots,p$,

$\beta_1 < \cdots < \beta_{p-r}$ is a subset complementary to $\alpha_1,\cdots,\alpha_r$,

Σ_2 denotes the summation over all $\binom{p}{r}$ possible choices of $\alpha_1 < \cdots < \alpha_r$,

$$(3.4.4)\quad b_{3ij} = \int_a^x g(y)\,\phi_{\delta_i}(y)\,\psi_{\alpha_j}(y)\,dy\,,\quad (i\,,j = 1,\cdots,r)$$

and

$$(3.4.5)\quad b_{4ij} = \int_x^b g(y)\,\phi_{\nu_i}(y)\,\psi_{\beta_j}(y)\,dy\,,\quad (i\,,j = 1,\cdots,p-r)\;.$$

<u>Proof</u>

For $h(x_1,\cdots,x_p)$ a symmetric function of $0 < x_1 < \cdots < x_p$, Krishnaiah (1976, p. 3) gives the following result:

$$(3.4.6)\quad \int_{\Lambda_1}\cdots\int h(x_1,\cdots,x_p)\,\phi(x_1,\cdots,x_p)\,\psi(x_1,\cdots,x_p)\,dx_1\cdots dx_p$$

$$= \Sigma_1\,\Sigma_2(-1)^{\Sigma\delta_i+\Sigma\alpha_i}\int_{\substack{a<x_i<x,(i=1,\cdots,r)\\ x<x_j<b,(j=r+1,\cdots,p)}}\cdots\cdots\int h(x_1,\cdots,x_p)\,|(b_{1ij})|\,|(b_{2ij})|\,dx_1\cdots dx_p$$

where

$$(3.4.7)\quad b_{1ij} = \phi_{\delta_i}(x_j)\,\psi_{\alpha_j}(x_j)\,,\quad (i\,,j = 1,\cdots,r)$$

and

$$(3.4.8)\quad b_{2ij} = \phi_{\nu_i}(x_{r+j})\,\psi_{\beta_j}(x_{r+j})\,,\quad (i\,,j = 1,\cdots,p-r)\;.$$

If $h(x_1,\cdots,x_p) = \prod_{i=1}^{p} g(x_i)$, the R.H.S. of (3.4.6) can be written as

$$(3.4.9) \qquad \Sigma_1 \; \Sigma_2 (-1)^{\Sigma\delta_i+\Sigma\alpha_i} \int_{\substack{a < x_i < x, \; (i=1,\cdots,r) \\ x < x_j < b, \; (j=r+1,\cdots,p)}} \cdots\cdots\cdots \int \prod_{i=1}^{p} g(x_i)$$

$$\Sigma(-1)^{f(j_1,\cdots,j_r)} \prod_{i=1}^{r} b_{1ij_i} \; \Sigma(-1)^{f(r+j_1,\cdots,r+j_{p-r})}$$

$$\prod_{i=1}^{p-r} b_{2ij_i} \; dx_1\cdots dx_p$$

$$= \Sigma_1 \; \Sigma_2 (-1)^{\Sigma\delta_i+\Sigma\alpha_i} \; \Sigma(-1)^{f(j_1,\cdots,j_r)} \prod_{i=1}^{r} \int_0^x b_{1ij_i} \; g(x_{j_i}) \; dx_{j_i}$$

$$\Sigma(-1)^{f(r+j_1,\cdots,r+j_{p-r})} \prod_{i=1}^{p-r} \int_x^b b_{2ij_i} \; g(x_{r+j_i}) \; dx_{r+j_i}$$

$$= \Sigma_1 \; \Sigma_2 (-1)^{\Sigma\delta_i+\Sigma\alpha_i} \; |(b_{3ij})| \, |(b_{4ij})|$$

which proves the theorem.

3.5 C.D.F. OF ANY TWO INTERMEDIATE CHARACTERISTIC ROOTS

The c.d.f. of any two intermediate characteristic roots, $\tilde{A}_t$ and $\tilde{A}_u$, $(1 < t < u < p)$, of $\underset{\sim}{A}:p\times p$ follows as

$$(3.5.1) \qquad P(c < \tilde{A}_t < \tilde{A}_u < d) = P(\tilde{A}_{t-1} < c < \tilde{A}_t < \tilde{A}_u < d < \tilde{A}_{u+1})$$
$$+P(\tilde{A}_{t-2} < c < \tilde{A}_{t-1} < \tilde{A}_t < \tilde{A}_u < d < \tilde{A}_{u+1})$$

$$+ P(\tilde{A}_{t-1} < c < \tilde{A}_t < \tilde{A}_u < \tilde{A}_{u+1} < d < \tilde{A}_{u+2})$$

$$+ P(c < \tilde{A}_{t-1} < \tilde{A}_t < \tilde{A}_u < \tilde{A}_{u+1} < d) \ .$$

The last term in (3.5.1) can be further evaluated by using (3.5.1) repeatedly. In theorem 3.5.1 the integral which is needed to obtain expressions for the terms in (3.5.1), given (3.2.3), is solved:

Theorem 3.5.1

$$(3.5.2) \qquad \int_{\Lambda_2} \cdots \int \prod_{i=1}^{p} g(x_i)\ \phi(x_1,\cdots,x_p)\ \psi(x_1,\cdots,x_p)\ dx_1\cdots dx_p$$

$$= \Sigma_3\ \Sigma_4\ \Sigma_5\ \Sigma_6 (-1)^{\Sigma\delta_i+\Sigma\delta_j^*+\Sigma\alpha_i+\Sigma\alpha_i^*}\ |(b_{8ij})|\,|(b_{9ij})|\,|(b_{10ij})|$$

where

$$\Lambda_2 = \{a < x_1 < \cdots < x_r < x < x_{r+1} < \cdots < x_{r+s} < y < x_{r+s+1} < \cdots < x_p < b\},$$

$\delta_1 < \cdots < \delta_r$ is a subset of the intergers $1,\cdots,p$,

$\nu_1 < \cdots < \nu_{p-r}$ is a subset complementary to $\delta_1,\cdots,\delta_r$,

Σ_3 denotes the summation over all $\binom{p}{r}$ possible choices of $\delta_1 < \cdots < \delta_r$,

$\alpha_1 < \cdots < \alpha_s$ is a subset of the integers $\nu_1,\cdots,\nu_{p-r}$,

$\beta_{r+s+1} < \cdots < \beta_p$ is a subset complementary to $\alpha_1,\cdots,\alpha_s$,

Σ_4 denotes the summation over all $\binom{p-r}{s}$ possible choices of $\alpha_1,\cdots,\alpha_s$,

$\delta_1^* < \cdots < \delta_r^*$ is a subset of the integers $1,\cdots,p$,

$\nu_1^* < \cdots < \nu_{p-r}^*$ is a subset complementary to $\delta_1^*, \cdots, \delta_r^*$,

Σ_5 denotes the summation over all $\binom{p}{r}$ possible choices of $\delta_1^*, \cdots, \delta_r^*$,

$\alpha_1^* < \cdots < \alpha_s^*$ is a subset of the integers $\nu_1^*, \cdots, \nu_{p-r}^*$,

$\beta_{r+s+1}^* < \cdots < \beta_p^*$ is a subset complementary to $\alpha_1^*, \cdots, \alpha_s^*$,

Σ_6 denotes the summation over all $\binom{p-r}{s}$ possible choices of $\alpha_1^*, \cdots, \alpha_s^*$,

$$(3.5.3) \quad b_{8ij} = \int_a^x g(z)\, \phi_{\delta_i}(z)\, \psi_{\delta_j^*}(z)\, dz\ , \quad (i\,,\,j = 1, \cdots, r)\ ,$$

$$(3.5.4) \quad b_{9ij} = \int_x^y g(z)\, \phi_{\alpha_i}(z)\, \psi_{\alpha_j^*}(z)\, dz\ , \quad (i\,,\,j = 1, \cdots, s)$$

and

$$(3.5.5) \quad b_{10ij} = \int_y^b g(z)\, \phi_{\beta_i}(z)\, \psi_{\beta_j^*}(z)\, dz\ , \quad (i\,,\,j = r+s+1, \cdots, p)\ .$$

Proof

Krishnaiah (1976, p. 5) proved:

$$(3.5.6) \quad \int \cdots \int_{\Lambda_2} \phi(x_1, \cdots, x_p)\, \psi(x_1, \cdots, x_p)\, dx_1 \cdots dx_p$$

$$= \Sigma_3\, \Sigma_4\, \Sigma_5\, \Sigma_6 (-1)^{\Sigma\delta_i + \Sigma\delta_i^* + \Sigma\alpha_i + \Sigma\alpha_i^*} |(b_{5ij})|\,|(b_{6ij})|\,|(b_{7ij})|$$

where

$$(3.5.7)\quad b_{5ij} = \int_a^x \phi_{\delta_i}(z)\,\psi_{\delta_j^*}(z)\,dz\ ,\quad (i\ ,\ j = 1,\cdots,r)\ ,$$

$$(3.5.8)\quad b_{6ij} = \int_x^y \phi_{\alpha_i}(z)\,\psi_{\alpha_j^*}(z)\,dz\ ,\quad (i\ ,\ j = 1,\cdots,s)$$

and

$$(3.5.9)\quad b_{7ij} = \int_y^b \phi_{\beta_i}(z)\,\psi_{\beta_j^*}(z)\,dz\ ,\quad (i\ ,\ j = r+s+1,\cdots,p)\ .$$

From (3.3.5) follows that

$$\prod_{i=1}^{p} g(x_i)\ \phi(x_1,\cdots,x_p) = \phi^*(x_1,\cdots,x_p)$$

with

$$\phi_i^*(x_j) = \phi_i(x_j)\ g(x_j)\ ,\quad (i\ ,\ j = 1,\cdots,p)\ .$$

By replacing $\phi(x_1,\cdots,x_p)$ with $\phi^*(x_1,\cdots,x_p)$ in (3.5.6) , the result follows.

3.6 THE JOINT P.D.F. OF ANY FEW ORDERED CHARACTERISTIC ROOTS

In theorem 3.6.1 the integral which is needed to obtain $f_{\tilde{A}_{r+1},\cdots,\tilde{A}_{r+s}}(\tilde{a}_{r+1},\cdots,\tilde{a}_{r+s})$, $0 < \tilde{a}_{r+1} < \cdots < \tilde{a}_{r+s}$, $(0 \leq r \leq r+s \leq p)$, given (3.2.3), is solved:

Theorem 3.6.1

$$(3.6.1)\quad \int\cdots\int_{\Lambda_3} \prod_{i=1}^{p} g(x_i)\ \phi(x_1,\cdots,x_p)\ \psi(x_1,\cdots,x_p)\ dx_1\cdots dx_r\ dx_{r+s+1}\cdots dx_p$$

$$= \Sigma_3\ \Sigma_4\ \Sigma_5\ \Sigma_6 (-1)^{\Sigma\delta_i + \Sigma\alpha_i + \Sigma\delta_i^* + \Sigma\alpha_i^*} |(b_{13ij})|\,|(b_{14ij})|$$

$$|(g(x_{r+j})\ \phi_{\alpha_i}(x_{r+j})\ ;\ i, j = 1,\cdots,s)|$$

$$|(\psi_{\alpha_i^*}(x_{r+j})\ ;\ i, j = 1,\cdots,s)|$$

where

$$\Lambda_3 = \{a < x_1 < \cdots < x_r < x_{r+1} < x_{r+s} < x_{r+s+1} < \cdots < x_p < b\} ,$$

$$(3.6.2) \quad b_{13ij} = \int_a^{x_{r+1}} g(z)\ \phi_{\delta_i}(z)\ \psi_{\delta_j^*}(z)\ dz , \quad (i, j = 1,\cdots,r) ,$$

$$(3.6.3) \quad b_{14ij} = \int_{x_{r+s}}^{b} g(z)\ \phi_{\beta_i}(z)\ \psi_{\beta_j^*}(z)\ dz , \quad (i, j = r+s+1,\cdots,p)$$

and the rest of the symbols are defined as in theorem 3.5.1.

Proof

Krishnaiah (1976, p. 5) proved:

$$(3.6.4) \quad \int_{\Lambda_3}\cdots\int \phi(x_1,\cdots,x_p)\ \psi(x_1,\cdots,x_p)\ dx_1\cdots dx_r\ dx_{r+s+1}\cdots dx_p$$

$$= \Sigma_3\ \Sigma_4\ \Sigma_5\ \Sigma_6 (-1)^{\Sigma\delta_i + \Sigma\alpha_i + \Sigma\delta_i^* + \Sigma\alpha_i^*} |(b_{11ij})|\,|(b_{12ij})|$$

$$|(\phi_{\alpha_i}(x_{r+j})\ ;\ i, j = 1,\cdots,s)|$$

$$|(\psi_{\alpha_i^*}(x_{r+j})\ ;\ i, j = 1,\cdots,s)|$$

where

$$(3.6.5)\quad b_{11ij} = \int_a^{x_{r+1}} \phi_{\delta_i}(z)\, \psi_{\delta_i^*}(z)\, dz \;, \quad (i\,,\, j = 1,\cdots,r)$$

and

$$(3.6.6)\quad b_{12ij} = \int_{x_{r+s}}^{b} \phi_{\beta_i}(z)\, \psi_{\beta_j^*}(z)\, dz \;, \quad (i\,,\, j = r+s+1,\cdots,p)\ .$$

By replacing $\phi(x_1,\cdots,x_p)$ with $\phi^*(x_1,\cdots,x_p)$ in (3.6.4) where $\phi_i^*(x_j)$ is given in (3.3.6), the result follows.

3.7 THE JOINT P.D.F. OF ANY FEW UNORDERED CHARACTERISTIC ROOTS

If the joint p.d.f. of the ordered characteristic roots $\tilde{A}_1,\cdots,\tilde{A}_p$ is given by $f_{\underset{\sim}{D}_A}(D_A)$, $0 < \tilde{a}_1 < \cdots < \tilde{a}_p$; then the joint p.d.f. of the unordered characteristic roots is given by

$$(3.7.1)\quad f_{\underset{\sim}{D}_A}(D_A) = \frac{1}{p!}\, f_{\underset{\sim}{D}_A}(D_A) \;, \quad a_i > 0 \;, \quad (i = 1,\cdots,p)\ .$$

In theorem 3.7.1 the integral which is needed to obtain $f_{\tilde{A}_1,\cdots,\tilde{A}_r}(\tilde{a}_1,\cdots,\tilde{a}_r)$, $\tilde{a}_i > 0$, $(i = 1,\cdots,r)$ given (3.7.1) and (3.2.3) is solved:

<u>Theorem 3.7.1</u>

$$(3.7.2)\quad \int_a^b \cdots \int_a^b \prod_{i=1}^{p} g(x_i)\, \phi(x_1,\cdots,x_p)\, \psi(x_1,\cdots,x_p)\, dx_{r+1}\cdots dx_p$$

$$= \Sigma_1\, \Sigma_2 (-1)^{\Sigma\delta_i + \Sigma\alpha_i}\, |(b_{17ij})|\,|(b_{18ij})|$$

where

$$(3.7.3)\quad b_{17ij} = \sum_{k=1}^{r} g(x_k)\ \phi_{\delta_i}(x_k)\ \psi_{\alpha_j}(x_k)\ , \quad (i\ ,\ j = 1,\cdots,r)\ ,$$

$$(3.7.4)\quad b_{18ij} = (p-r)! \int_a^b g(y)\ \phi_{\nu_i}(y)\ \psi_{\beta_j}(y)\ dy\ , \quad (i\ ,\ j = 1,\cdots,p-r)$$

and the rest of the symbols are defined as in theorem 3.4.1.

Proof

Krishnaiah (1976, p. 12) proved:

$$(3.7.5)\quad \int_a^b \cdots \int_a^b \phi(x_1,\cdots,x_p)\ \psi(x_1,\cdots,x_p)\ dx_{r+1}\cdots dx_p$$

$$= \Sigma_1\ \Sigma_2 (-1)^{\Sigma\delta_i + \Sigma\alpha_i}\ |(b_{15ij})|\,|(b_{16ij})|$$

where

$$(3.7.6)\quad b_{15ij} = \sum_{k=1}^{r} \phi_{\delta_i}(x_k)\ \psi_{\alpha_j}(x_k)\ , \quad (i\ ,\ j = 1,\cdots,r)$$

and

$$(3.7.7)\quad b_{16ij} = (p-r)! \int_a^b \phi_{\nu_i}(y)\ \psi_{\beta_j}(y)\ dy\ , \quad (i\ ,\ j = 1,\cdots,p-r)\ .$$

By replacing $\phi(x_1,\cdots,x_p)$ with $\phi^*(x_1,\cdots,x_p)$ in (3.7.5) where $\phi_i^*(x_j)$ is given in (3.3.6), the result follows.

CHAPTER 4

THE MULTIVARIATE COMPLEX QUADRATIC FORM OF COMPLEX NORMAL VARIATES

4.1 INTRODUCTION

In this chapter the multivariate complex quadratic form of complex normal variates and the multivariate complex compound quadratic form of complex normal variates are considered. The p.d.f.s, the joint p.d.f.s of the characteristic roots and the moments of these quadratic forms are derived, analogous to the real case, for different specifications of the parameter matrices. The results given in theorems 2.2.8, 2.3.6 and 2.3.7 are used to write the joint p.d.f.s of the characteristic roots of the quadratic forms in the form which makes it possible to derive certain marginal distributions of the roots by using the results given in chapter 3. These marginal distributions as well as the p.d.f.s of certain functions of the characteristic roots are thus also derived in this chapter.

4.2 THE QUADRATIC FORM $\underset{\sim}{S}:p\times p = \underset{\sim}{Z}\,L\,\underset{\sim}{\bar{Z}}'$ WHERE $\underset{\sim}{Z}:p\times n \sim \text{CMTN}(p,\ n,\ M,\ \Phi\otimes\Sigma)$

4.2.1 The p.d.f. and moments of $\underset{\sim}{S}:p\times p$ and $\underset{\sim}{D}_S$

In theorem 4.2.1 the p.d.f.s, symmetrised p.d.f.s and moments of $\underset{\sim}{S}:p\times p$ and $\underset{\sim}{D}_S$ are derived for different specifications of the parameter matrices. The central and non-central Wishart matrices are special cases of the quadratic form $\underset{\sim}{S}:p\times p = \underset{\sim}{Z}\,L\,\underset{\sim}{\bar{Z}}'$ and will therefore also be considered in theorem 4.2.1. Two types of representation of the p.d.f.s of $\underset{\sim}{S}:p\times p$ and $\underset{\sim}{D}_S$ are considered i.e. the power-series type representation and the Γ-series type representation. Some authors call these two types of representation "Hayakawa's form" and "Khatri's form" of the p.d.f. respectively. In remark 4.2.1 the relationship between these two representations will be discussed and it will also become clear in theorem 4.2.1 and the rest of this chapter which representation is the most convenient in studying a specific property of the quadratic form.

The following lemma will be used in theorem 4.2.1:

Lemma 4.2.1

Let $\underset{\sim}{X}:p\times n$ be a complex random matrix with p.d.f. $f_{\underset{\sim}{X}}(X) = f(X\bar{X}')$; then the p.d.f. of the hermitian random matrix $\underset{\sim}{B}:p\times p = \underset{\sim}{X}\underset{\sim}{\bar{X}}'$ is given by

$$(4.2.1) \quad f_{\underset{\sim}{B}}(B) = \frac{\pi^{pn} \, |B|^{n-p}}{\tilde{\Gamma}_p(n)} f(B) \ .$$

Proof

Srivastava (1965, p. 312 - 315).

Theorem 4.2.1

Let $\underset{\sim}{Z}:p\times n \sim \text{CMTN}(p, n, M, \Phi\otimes\Sigma)$ and let $L:n\times n$ be a hermitian matrix; then the p.d.f., symmetrised p.d.f. and moments of $\underset{\sim}{S}:p\times p = \underset{\sim}{Z}\, L\, \underset{\sim}{\bar{Z}}'$, $\underset{\sim}{D}_S$ and $\underset{\sim}{D}_B$, where $\underset{\sim}{B}:p\times p = \Sigma^{-\frac{1}{2}}\underset{\sim}{S}\,\Sigma^{-\frac{1}{2}}$, are given below for certain specifications of $M:p\times n$, $\Phi:n\times n$, $\Sigma:p\times p$ and $L:n\times n$.

(i) $M:p\times n \neq 0$

Power-series representation

$$(4.2.2) \qquad f_{\underset{\sim}{D}_B}(D_B)$$

$$= \frac{\pi^{p(p-1)} \operatorname{etr}[-\Sigma^{-1} M \Phi^{-1} \bar{M}']}{\tilde{\Gamma}_p(n)\, \tilde{\Gamma}_p(p)\, |L\,\Phi|^p} |D_B|^{n-p}$$

$$\prod_{i>j}^{p} (\tilde{b}_i - \tilde{b}_j)^2 \sum_{k=0}^{\infty} \sum_{\kappa} \frac{\tilde{P}_\kappa(\Sigma^{-\frac{1}{2}} M \Phi^{-1} L^{-\frac{1}{2}} C^{\frac{1}{2}}, C^{-1})}{k!\ [n]_\kappa\ \tilde{C}_\kappa(I_p)} \tilde{C}_\kappa(D_B) \ ,$$

$$0 < \tilde{b}_1 < \cdots < \tilde{b}_p \ ,$$

where

(4.2.3) $\quad C{:}n\times n = L^{\frac{1}{2}} \Phi L^{\frac{1}{2}}$.

<u>Γ-type representation</u>

(4.2.4) $\quad f_{\underset{\sim}{D}_B}(D_B)$

$$= \frac{\pi^{p(p-1)}\ \mathrm{etr}[-\Sigma^{-1} M \Phi^{-1} \bar{M}']}{\tilde{\Gamma}_p(n)\ \tilde{\Gamma}_p(p)\ |L\Phi|^p}\ \mathrm{etr}[-q^{-1} D_B]\ |D_B|^{n-p}$$

$$\prod_{i>j}^{p} (\tilde{b}_i - \tilde{b}_j)^2 \sum_{k=0}^{\infty} \sum_{\kappa} \frac{P_\kappa(\Sigma^{-\frac{1}{2}} M \Phi^{-1} L^{-\frac{1}{2}} T^{-\frac{1}{2}}, T)}{k!\ [n]_\kappa\ \tilde{C}_\kappa(I_p)} \tilde{C}_\kappa(D_B) \ ,$$

$$0 < \tilde{b}_1 < \cdots < \tilde{b}_p$$

where

(4.2.5) $\quad T{:}n\times n = C^{-1} - q^{-1} I_n$

and $\|C\| < q$, $(T{:}n\times n$ h.p.d.).

(ii) <u>$M{:}p\times n = 0$</u>

<u>Power-series representation</u>

(4.2.6) $\quad f_{\underset{\sim}{S}}(S) = \dfrac{|S|^{n-p}\ {}_0\tilde{F}_0(-L^{-1}\Phi^{-1}, \Sigma^{-1}S)}{|L\Phi|^p\ |\Sigma|^n\ \tilde{\Gamma}_p(n)}$, $\quad S = \bar{S}' > 0$.

(4.2.7) $f_{\text{csym}}(S)$

$$= \frac{|S|^{n-p}}{|L\Phi|^p \, |\Sigma|^n \, \tilde{\Gamma}_p(n)} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{\tilde{C}_\kappa(-L^{-1}\Phi^{-1}) \, \tilde{C}_\kappa(\Sigma^{-1}) \, \tilde{C}_\kappa(S)}{k! \, \tilde{C}_\kappa(I_n) \, \tilde{C}_\kappa(I_p)} ,$$

$$S = \bar{S}' > 0 .$$

(4.2.8) $f_{\underset{\sim}{D}_S}(D_S)$

$$= \frac{\pi^{p(p-1)} \, |D_S|^{n-p} \prod_{i>j}^{p} (\tilde{s}_i - \tilde{s}_j)^2}{\tilde{\Gamma}_p(p) \, |L\Phi|^p \, |\Sigma|^n \, \tilde{\Gamma}_p(n)}$$

$$\sum_{k=0}^{\infty} \sum_{\kappa} \frac{\tilde{C}_\kappa(-L^{-1}\Phi^{-1}) \, \tilde{C}_\kappa(\Sigma^{-1}) \, \tilde{C}_\kappa(D_S)}{k! \, \tilde{C}_\kappa(I_n) \, \tilde{C}_\kappa(I_p)} , \quad 0 < \tilde{s}_1 < \cdots < \tilde{s}_p .$$

<u>$\tilde{\Gamma}$-type representation</u>

(4.2.9) $f_{\underset{\sim}{S}}(S)$

$$= \frac{|S|^{n-p} \operatorname{etr}[-q^{-1}\Sigma^{-1}S]}{\tilde{\Gamma}_p(n) \, |L\Phi|^p \, |\Sigma|^n} \; {}_0\tilde{F}_0(-T, \Sigma^{-1}S) , \quad S = \bar{S}' > 0 .$$

(4.2.10) $E(|\underset{\sim}{S}|^h)$

$$= \frac{\tilde{\Gamma}_p(n+h) \, q^{(n+h)p} |\Sigma|^h}{\tilde{\Gamma}_p(n) \, |L\Phi|^p} \; {}_1\tilde{F}_0(n+h \,;\, -T, qI_p) .$$

(4.2.11) $f_{\underset{\sim}{D}_B}(D_B)$

$$= \frac{\pi^{p(p-1)} \operatorname{etr}[-q^{-1}D_B]}{\tilde{\Gamma}_p(n) \, \tilde{\Gamma}_p(p) \, |L\Phi|^p} \, |D_B|^{n-p} \prod_{i>j}^{p} (\tilde{b}_i - \tilde{b}_j)^2 \; {}_0\tilde{F}_0(-T, D_B) ,$$

$$0 < \tilde{b}_1 < \cdots < \tilde{b}_p ,$$

where $T:n\times n = C^{-1} - q^{-1} I_n$ and $q > 0$ in (4.2.9) and (4.2.11) and $\|T\| < 1$ or $0 < q < 1$ in (4.2.10).

(iii) $\underline{M:p\times n \neq 0\ ,\ \Phi:n\times n = I_n\ ,\ \Sigma:p\times p = I_p}$

Power-series representation

(4.2.12) $f_{csym}(S)$

$$= \frac{|S|^{n-p}\ \mathrm{etr}[-M\bar{M}']}{|L|^p\ \tilde{\Gamma}_p(n)} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{\tilde{P}_\kappa(M\ ,\ L^{-1})\ \tilde{C}_\kappa(S)}{k!\ [n]_\kappa\ \tilde{C}_\kappa(I_p)}\ ,\quad S = \bar{S}' > 0\ .$$

(4.2.13) $f_{\underset{\sim}{D}_S}(D_S)$

$$= \frac{\pi^{p(p-1)}\ \mathrm{etr}[-M\bar{M}']}{|L|^p\ \tilde{\Gamma}_p(n)\ \tilde{\Gamma}_p(p)}\ |D_S|^{n-p} \prod_{i>j}^{p} (\tilde{s}_i - \tilde{s}_j)^2$$

$$\sum_{k=0}^{\infty} \sum_{\kappa} \frac{\tilde{P}_\kappa(M\ ,\ L^{-1})\ \tilde{C}_\kappa(D_S)}{k!\ [n]_\kappa\ \tilde{C}_\kappa(I_p)}\ ,\quad 0 < \tilde{s}_1 < \cdots < \tilde{s}_p\ .$$

Γ-type representation

(4.2.14) $f_{csym}(S)$

$$= \frac{|S|^{n-p}\ \mathrm{etr}[-M\bar{M}']}{|L|^p\ \tilde{\Gamma}_p(n)}\ \mathrm{etr}[-q^{-1} S]$$

$$\sum_{k=0}^{\infty} \sum_{\kappa} \frac{\tilde{P}_\kappa(ML^{-\frac{1}{2}}\ T^{-\frac{1}{2}}\ ,\ T)}{k!\ [n]_\kappa\ \tilde{C}_\kappa(I_p)}\ \tilde{C}_\kappa(S)\ ,\quad S = \bar{S}' > 0$$

where

(4.2.15) $T:n\times n = L^{-1} - q^{-1} I_n$

and $||L|| < q$, $(T{:}n\times n$ h.p.d.) and $q > 0$.

(4.2.16) $E(|\underset{\sim}{S}|^h)$

$$= \frac{\text{etr}[-M\bar{M}'] \; q^{(n+h)p} \; \tilde{\Gamma}_p(n+h)}{|L|^p \; \tilde{\Gamma}_p(n)}$$

$$\sum_{k=0}^{\infty} \sum_{\kappa} \frac{\tilde{P}_\kappa(ML^{-\frac{1}{2}} \; T^{-\frac{1}{2}}, T) \; q^k \; [n+h]_\kappa}{k! \; [n]_\kappa} \; .$$

(4.2.17) $f_{\underset{\sim}{D}_S}(D_S)$

$$= \frac{\pi^{p(p-1)} \; \text{etr}[-M\bar{M}']}{|L|^p \; \tilde{\Gamma}_p(n) \; \tilde{\Gamma}_p(p)} \; |D_S|^{n-p} \; \text{etr}[-q^{-1} D_S] \prod_{i>j}^{p} (\tilde{s}_i-\tilde{s}_j)^2$$

$$\sum_{k=0}^{\infty} \sum_{\kappa} \frac{\tilde{P}_\kappa(ML^{-\frac{1}{2}} \; T^{-\frac{1}{2}}, T) \; \tilde{C}_\kappa(D_S)}{k! \; [n]_\kappa \; \tilde{C}_\kappa(I_p)} \; , \quad 0 < \tilde{s}_1 < \cdots < \tilde{s}_p \; .$$

(iv) <u>$M{:}p\times n \neq 0$, $\Phi{:}n\times n = I_n$, $L{:}n\times n = I_n$, i.e.</u>
<u>$\underset{\sim}{S}{:}p\times p \sim \text{NCCW}(p, n, \Sigma, \Omega)$</u>

<u>Power-series representation</u>

(4.2.18) $f_{\text{csym}}(S)$

$$= |S|^{n-p} \; \text{etr}[-\Omega] \sum_{k=0}^{\infty} \sum_{\kappa} \frac{\tilde{H}_\kappa(M) \; \tilde{C}_\kappa(S)}{k! \; [n]_\kappa \; \tilde{C}_\kappa(I_p)} \; , \quad S = \bar{S}' > 0 \; , \; \Sigma = I_p \; .$$

(4.2.19) $f_{\underset{\sim}{D}_B}(D_B)$

$$= \frac{\pi^{p(p-1)} \; \text{etr}[-\Omega]}{\tilde{\Gamma}_p(n) \; \tilde{\Gamma}_p(p)} \; |D_B|^{n-p} \prod_{i>j}^{p} (\tilde{b}_i-\tilde{b}_j)^2$$

$$\sum_{k=0}^{\infty} \sum_{\kappa} \frac{\tilde{H}_\kappa(\Sigma^{-\frac{1}{2}}M)\ \tilde{C}_\kappa(D_B)}{k!\ [n]_\kappa\ \tilde{C}_\kappa(I_p)} \quad , \quad 0 < \tilde{b}_1 < \cdots < \tilde{b}_p \ .$$

Γ-type representation

(4.2.20) $f_{\underset{\sim}{S}}(S)$

$$= (2.8.25) = \frac{\mathrm{etr}[-\Omega]\,{}_0\tilde{F}_1(n\ ;\ \Omega\Sigma^{-1}S)}{\tilde{\Gamma}_p(n)\ \ |\Sigma|^n} |S|^{n-p}\ \mathrm{etr}[-\Sigma^{-1}S] \ ,$$

$$S = \bar{S}' > 0 \ .$$

(4.2.21) $f_{csym}(S)$

$$= \frac{\mathrm{etr}[-\Omega]\,|S|^{n-p}}{\tilde{\Gamma}_p(n)}\ \mathrm{etr}[-S]\,{}_0\tilde{F}_1(n\ ;\ \Omega\ ,\ S) \ , \quad S = \bar{S}' > 0 \ , \quad \Sigma = I_p \ .$$

(4.2.22) $$E(|\underset{\sim}{S}|^h) = \frac{\tilde{\Gamma}_p(n+h)}{\tilde{\Gamma}_p(n)}\ \mathrm{etr}[-\Omega]\,{}_1\tilde{F}_1(n+h\ ;\ n\ ;\ \Omega)\,|\Sigma|^h \ .$$

(4.2.23) $f_{\underset{\sim}{D}_B}(D_B)$

$$= \frac{\pi^{p(p-1)}\ \ \mathrm{etr}[-\Omega]}{\tilde{\Gamma}_p(n)\ \ \tilde{\Gamma}_p(p)}\ |D_B|^{n-p}\ \mathrm{etr}[-D_B]$$

$$\prod_{i>j}^{p} (\tilde{b}_i-\tilde{b}_j)^2\ {}_0\tilde{F}_1(n\ ;\ \Omega\ ,\ D_B) \ , \quad 0 < \tilde{b}_1 < \cdots < \tilde{b}_p \ .$$

(v) <u>$M{:}p\times n = 0$, $\Phi{:}n\times n = I_n$, $L{:}n\times n = I_n$, i.e. $\underset{\sim}{S}{:}p\times p \sim CW(p,\ n,\ \Sigma)$</u>

Power-series representation

(4.2.24) $$f_{\underset{\sim}{S}}(S) = (2.8.26) = \frac{|S|^{n-p}\ \mathrm{etr}[-\Sigma^{-1}S]}{\tilde{\Gamma}_p(n)\ \ |\Sigma|^n} \ , \quad S = \bar{S}' > 0 \ .$$

$$(4.2.25) \qquad f_{csym}(S) = \frac{|S|^{n-p} \; {}_0\tilde{F}_0(-\Sigma^{-1}, S)}{\tilde{\Gamma}_p(n) \; |\Sigma|^n} \; , \quad S = \bar{S}' > 0 \; .$$

$$(4.2.26) \qquad f_{\underset{\sim}{D}_S}(D_S)$$

$$= \frac{\pi^{p(p-1)} \; |D_S|^{n-p}}{\tilde{\Gamma}_p(n) \; \tilde{\Gamma}_p(p) \; |\Sigma|^n} \prod_{i>j}^{p} (\tilde{s}_i - \tilde{s}_j)^2 \; {}_0\tilde{F}_0(-\Sigma^{-1}, D_S) \; ,$$

$$0 < \tilde{s}_1 < \cdots < \tilde{s}_p \; .$$

<u>Γ-type representation</u>

$$(4.2.27) \qquad f_{\underset{\sim}{S}}(S)$$

$$= (2.8.27) = \frac{|S|^{n-p} \; \mathrm{etr}[-S]}{\tilde{\Gamma}_p(n) \; |\Sigma|^n} \; {}_0\tilde{F}_0((I_p - \Sigma^{-1})S) \; , \quad S = \bar{S}' > 0 \; .$$

$$(4.2.28) \qquad f_{csym}(S) = \frac{|S|^{n-p} \; \mathrm{etr}[-S]}{\tilde{\Gamma}_p(n) \; |\Sigma|^n} \; {}_0\tilde{F}_0((I_p - \Sigma^{-1}), S) \; , \quad S = \bar{S}' > 0 \; .$$

$$(4.2.29) \qquad E(|\underset{\sim}{S}|^h) = \frac{\tilde{\Gamma}_p(n+h) \; {}_1\tilde{F}_0(n+h \, ; \, I_p - \Sigma^{-1})}{\tilde{\Gamma}_p(n) \; |\Sigma|^n} \; .$$

$$(4.2.30) \qquad f_{\underset{\sim}{D}_S}(D_S)$$

$$= \frac{\pi^{p(p-1)} \; |D_S|^{n-p} \; \mathrm{etr}[-D_S]}{\tilde{\Gamma}_p(n) \; \tilde{\Gamma}_p(p) \; |\Sigma|^n} \prod_{i>j}^{p} (\tilde{s}_i - \tilde{s}_j)^2 \; {}_0\tilde{F}_0(I_p - \Sigma^{-1}, D_S) \; ,$$

$$0 < \tilde{s}_1 < \cdots < \tilde{s}_p \; .$$

<u>Proof</u>

(<u>4.2.2</u>) Hayakawa (1972 b, p. 223).

(4.2.4) Hayakawa (1972b, p. 224)

(4.2.6)

The p.d.f. of $\underset{\sim}{Z}:p\times n$ is given by

$$(4.2.31) \quad f_{\underset{\sim}{Z}}(Z) = \pi^{-np} |\Phi|^{-p} |\Sigma|^{-n} \operatorname{etr}[-\Sigma^{-1} Z \Phi^{-1} \bar{Z}'] .$$

In (4.2.31) make the transformation

$$(4.2.32) \quad Y = Z L^{\frac{1}{2}}$$

with inverse transformation

$$(4.2.33) \quad Z = Y L^{-\frac{1}{2}} .$$

The jacobian of (4.2.33) follows from (2.2.6) as

$$(4.2.34) \quad J(Z \to Y) = |L|^{-p} .$$

Hence,

$$(4.2.35) \quad f_{\underset{\sim}{Y}}(Y) = \pi^{-np} |L\Phi|^{-p} |\Sigma|^{-n} \operatorname{etr}[-\Sigma^{-1} Y L^{-\frac{1}{2}} \Phi^{-1} L^{-\frac{1}{2}} \bar{Y}'] .$$

The matrix $L^{-\frac{1}{2}} \Phi^{-1} L^{-\frac{1}{2}}$ is a hermitian matrix, thus, after expanding the exponential function, transforming $L^{-\frac{1}{2}} \Phi^{-1} L^{-\frac{1}{2}} \to U L^{-\frac{1}{2}} \Phi^{-1} L^{-\frac{1}{2}} \bar{U}'$, where $U \in U(n)$, and integrating over the unitary group, using (2.2.29), the p.d.f. of $\underset{\sim}{Y}:p\times n$ follows as

$$(4.2.36) \quad f_{\underset{\sim}{Y}}(Y) = \pi^{-np} |L\Phi|^{-p} |\Sigma|^{-n} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{\tilde{C}_{\kappa}(-L^{-1}\Phi^{-1})\, \tilde{C}_{\kappa}(\Sigma^{-1} Y\bar{Y}')}{k!\, \tilde{C}_{\kappa}(I_n)} .$$

The application of lemma 4.2.1 leads to

$$(4.2.37) \quad f_{\underset{\sim}{S}}(S) = \frac{|S|^{n-p}}{\tilde{\Gamma}_p(n)\, |L\Phi|^p |\Sigma|^n} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{\tilde{C}_{\kappa}(-L^{-1}\Phi^{-1})\, \tilde{C}_{\kappa}(\Sigma^{-1} S)}{k!\, \tilde{C}_{\kappa}(I_n)}$$

where $\underset{\sim}{S}:p\times p = \underset{\sim}{Y}\bar{\underset{\sim}{Y}}'$, which proves (4.2.6).

(4.2.7)

The symmetrised p.d.f. of $\underset{\sim}{S}:p\times p$ follows from (2.7.1) and (2.2.29).

(4.2.8)

The application of theorem 3.2.1 and corollary 2.7.1 leads to (4.2.8).

(4.2.9) Khatri (1966, p. 477).

(4.2.10)

$$(4.2.38) \quad E(|\underset{\sim}{S}|^h) = (\tilde{\Gamma}_p(n)\ |L\,\Phi|^p\ |\Sigma|^n)^{-1}\ I^*$$

where

$$(4.2.39) \quad I^* = \int_{S=\bar{S}'>0} |S|^{n+h-p}\ \mathrm{etr}[-q^{-1}\ \Sigma^{-1}\ S]\ {}_0\tilde{F}_0(-T\ ,\ \Sigma^{-1}\ S)\ dS$$

$$= \tilde{\Gamma}_p(n+h)\ |q\,\Sigma|^{n+h}\ {}_1\tilde{F}_0(n+h\ ;\ -T,\ qI_p), \quad \text{(from (2.3.5))}$$

with ${}_1\tilde{F}_0(n+h\ ;\ -T,\ q\ I_p)$ convergent only for $||T|| < 1$ or $q < 1$ which proves (4.2.10).

(4.2.11)

In (4.2.9) make the transformation

$$(4.2.40) \quad B = \Sigma^{-\frac{1}{2}}\, S\, \Sigma^{-\frac{1}{2}}$$

with inverse transformation

$$(4.2.41) \quad S = \Sigma^{\frac{1}{2}}\, B\, \Sigma^{\frac{1}{2}}\ .$$

The jacobian of (4.2.41) follows from (2.2.6) as

$$(4.2.42) \quad J(S \to B) = |\Sigma|^p\ .$$

Hence,

$$(4.2.43)\quad f_{\underset{\sim}{B}}(B) = \frac{|B|^{n-p}\ \mathrm{etr}[-q^{-1}B]}{\tilde{\Gamma}_p(n)\ |L\Phi|^p}\ {}_0\tilde{F}_0(-T,\ B)\ ,\quad B=\bar{B}'>0\ .$$

The application of theorem 3.2.1 leads to (4.2.11).

Let $M=0$ in (4.2.4); then (4.2.11) also follows by using (2.4.7).

(<u>4.2.12</u>)

The p.d.f. of $\underset{\sim}{Z}:p\times n$ is given by

$$(4.2.44)\quad f_{\underset{\sim}{Z}}(Z) = \pi^{-np}\ \mathrm{etr}[-M\bar{M}']\ \mathrm{etr}[-Z\bar{Z}' + Z\bar{M}' + M\bar{Z}']\ .$$

In (4.2.44) make the transformation (4.2.32) with inverse transformation (4.2.33) and jacobian (4.2.34); then follows:

$$(4.2.45)\quad f_{\underset{\sim}{Y}}(Y) = \frac{\mathrm{etr}[-M\bar{M}']}{\pi^{np}\ |L|^p}\ \mathrm{etr}[-YL^{-1}\bar{Y}' + YL^{-\frac{1}{2}}\bar{M}' + ML^{-\frac{1}{2}}\bar{Y}']\ .$$

In (4.2.45) make the transformation

$$(4.2.46)\quad X = Y\bar{U}_2'$$

with inverse transformation

$$(4.2.47)\quad Y = XU_2$$

where $U_2\in U(n)$. The jacobian of (4.2.47) is 1.

Hence,

$$(4.2.48)\quad f_{\underset{\sim}{X}}(X)$$

$$= \frac{\mathrm{etr}[-M\bar{M}']}{\pi^{np}\ |L|^p}\int_{U(n)} \mathrm{etr}[-XU_2L^{-1}\bar{U}_2'\bar{X}' + XU_2L^{-\frac{1}{2}}\bar{M}' + ML^{-\frac{1}{2}}\bar{U}_2'\bar{X}']\ dU_2\ .$$

In (4.2.48) make the transformation

$$(4.2.49) \quad \underset{\sim}{Q} = \bar{U}_1' X$$

with inverse transformation

$$(4.2.50) \quad X = U_1 Q$$

where $U_1 \varepsilon U(p)$. The jacobian of (4.2.50) is 1.

Hence,

$$(4.2.51) \quad f_{\underset{\sim}{Q}}(Q) = \frac{\mathrm{etr}[-M\bar{M}']}{\pi^{np}\ |L|^p} \int_{U(n)} \int_{U(p)} \mathrm{etr}[-QU_2L^{-1}\bar{U}_2'\bar{Q}' + U_1QU_2L^{-\frac{1}{2}}\bar{M}' + ML^{-\frac{1}{2}}\bar{U}_2'\bar{Q}'\bar{U}_1']\, dU_2\, dU_1 \ .$$

From (2.4.10) follows:

$$(4.2.52) \quad f_{\underset{\sim}{Q}}(Q) = \frac{\mathrm{etr}[-M\bar{M}']}{\pi^{np}\ |L|^p} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{\tilde{P}_\kappa(M, L^{-1})\ \tilde{C}_\kappa(Q\bar{Q}')}{k!\ [n]_\kappa\ \tilde{C}_\kappa(I_p)} \ .$$

The application of lemma 4.2.1 leads to

$$(4.2.53) \quad f_{\underset{\sim}{A}}(A) = \frac{|A|^{n-p}\ \mathrm{etr}[-M\bar{M}']}{|L|^p\ \tilde{\Gamma}_p(n)} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{\tilde{P}_\kappa(M, L^{-1})\ \tilde{C}_\kappa(A)}{k!\ [n]_\kappa\ \tilde{C}_\kappa(I_p)}$$

where $\underset{\sim}{A}:p\times p = \underset{\sim}{Q}\underset{\sim}{\bar{Q}}'$, but $A = Q\bar{Q}' = \bar{U}_1'X\bar{X}'U_1 = \bar{U}_1'Y\bar{Y}'U_1 = \bar{U}_1'SU_1$ so that $f_{\underset{\sim}{A}}(A)$ is the p.d.f. of $\bar{U}_1'\underset{\sim}{S}U_1$ with U_1 integrated out which is the symmetrised p.d.f. of $\underset{\sim}{S}:p\times p$. Thus (4.2.12) is proved.

(<u>4.2.13</u>)

The application of theorem 3.2.1 and corollary 2.7.1 leads to (4.2.13). Let $\Sigma:p\times p = I_p$ and $\Phi:n\times n = I_n$ in (4.2.2); then (4.2.13) also follows.

(<u>4.2.14</u>)

The p.d.f. of $\underset{\sim}{Y}:p\times n = ZL^{\frac{1}{2}}$ given in (4.2.45) can be written as

$$(4.2.54)\quad f_{\underset{\sim}{Y}}(Y) = \frac{\operatorname{etr}[-M\bar{M}']}{\pi^{np}\ |L|^{p}} \operatorname{etr}[-q^{-1}\, Y\bar{Y}'] \operatorname{etr}[-YT\bar{Y}' + YT^{\frac{1}{2}}(T^{-\frac{1}{2}}L^{-\frac{1}{2}}\bar{M}') + (ML^{-\frac{1}{2}}T^{-\frac{1}{2}})T^{\frac{1}{2}}\bar{Y}']$$

where

$$(4.2.55)\quad T:n\times n = L^{-1} - q^{-1}\, I_n\ .$$

The rest of the proof is similar to the proof of (4.2.12).

(<u>4.2.16</u>)

$$(4.2.56)\quad E(|\underset{\sim}{S}|^{h}) = \frac{\operatorname{etr}[-M\bar{M}']}{|L|^{p}\ \tilde{\Gamma}_p(n)\ \tilde{\Gamma}_p(p)} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{\tilde{P}_{\kappa}(ML^{-\frac{1}{2}}\,T^{-\frac{1}{2}}\ ,\ T)}{k!\ [n]_{\kappa}\ \tilde{C}_{\kappa}(I_p)}\ I^{*}$$

where

$$(4.2.57)\quad I^{*} = \int_{S=\bar{S}'>0} |S|^{n+h-p} \operatorname{etr}[-q^{-1}\, S]\ \tilde{C}_{\kappa}(S)\ dS$$

$$= \tilde{\Gamma}_p(n+h,\ \kappa)\ q^{(n+h)p}\ \tilde{C}_{\kappa}(q\, I_p)\ ,\quad \text{(from (2.2.32))}$$

which proves (4.2.16).

(<u>4.2.17</u>)

The application of theorem 3.2.1 and corollary 2.7.1 leads to (4.2.17). Let $\Sigma:p\times p = I_p$ and $\Phi:n\times n = I_n$ in (4.2.4); then (4.2.17) also follows.

(<u>4.2.18</u>)

Let $L:n\times n = I_n$ in (4.2.12); then follows from (2.4.8) that

$$\tilde{P}_{\kappa}(M\ ,\ I_n) = \tilde{H}_{\kappa}(M)$$

so that the symmetrised non-central complex Wishart p.d.f. is given by (4.2.18).

(4.2.19) Hayakawa (1972 a, p. 8 - 9).

(4.2.21)

The symmetrised p.d.f. of $\underset{\sim}{S}:p\times p$ follows from (2.7.1) and (2.3.4).

(4.2.22)

The proof is similar to the proof of (4.2.10).

(4.2.23)

$|S - \tilde{s}\Sigma| = 0$ iff. $|B - \tilde{s}\, I_p| = 0$ where $B:p\times p \sim NCCW(p, n, I_p, \Omega)$. Thus (4.2.23) follows from (4.2.21), theorem 3.2.1 and corollary 2.7.1.

(4.2.25)

The symmetrised p.d.f. of $\underset{\sim}{S}:p\times p$ follows from (2.7.1) and (2.3.4).

(4.2.26)

The application of theorem 3.2.1 and corollary 2.7.1 leads to (4.2.26).

(4.2.28), (4.2.29), (4.2.30)

These results follow along the same lines as (4.2.25), (4.2.10) and (4.2.26) respectively.

Remark 4.2.1

(i) From the results given in theorem 4.2.1 it is clear that the Γ-type representation of the p.d.f. of $\underset{\sim}{S}:p\times p$ is more suitable for the derivation of the moments of $\underset{\sim}{S}:p\times p$. In the case where $M:p\times n = 0$, the power-series representation of the p.d.f. is more suitable for the derivation of $f_{csym}(S)$ and $f_{\underset{\sim}{D}_S}(D_S)$. By expanding $\mathrm{etr}[-q^{-1}\Sigma^{-1}S]$ in (4.2.9), the Γ-type representation, it is however possible to find expressions for $f_{csym}(S)$ and $f_{\underset{\sim}{D}_S}(D_S)$ which involve five summation signs. These expressions can not be considered Γ-type representations because they no more contain the term $\mathrm{etr}[-q^{-1}\Sigma^{-1}S]$.

(ii) In the non-central cases, i.e. $M:p\times n \neq 0$, it is only possible to derive:

(a) $f_{\underset{\sim}{D}_B}(D_B)$ when $\Sigma:p\times p$ is known and $\Phi:n\times n \neq I_n$,

(b) $f_{csym}(S)$, $f_{\underset{\sim}{D}_S}(D_S)$ and $E(|\underset{\sim}{S}|^h)$ when $\Sigma:p\times p = I_p$ and $\Phi:n\times n = I_n$,

(c) $f_{\underset{\sim}{S}}(S)$ and $E(|\underset{\sim}{S}|^h)$ when $L:n\times n = I_n$, $\Phi:n\times n = I_n$ and $\Sigma:p\times p \neq I_p$, i.e. $\underset{\sim}{S}:p\times p \sim NCCW(p, n, \Sigma, \Omega)$,

(d) $f_{\underset{\sim}{D}_B}(D_B)$ when $L:n\times n = I_n$, $\Phi:n\times n = I_n$ and $\Sigma:p\times p$ known, i.e. $\underset{\sim}{S}:p\times p \sim NCCW(p, n, \Sigma, \Omega)$,

(e) $f_{csym}(S)$ when $L:n\times n = I_n$, $\Phi:n\times n = I_n$ and $\Sigma:p\times p = I_p$, i.e. $\underset{\sim}{S}:p\times p \sim NCCW(p, n, I_p, \Omega)$.

It was not possible for the author to derive the p.d.f. of $\underset{\sim}{S}:p\times p$ or $\underset{\sim}{D}_S$ in the case where $M:p\times n \neq 0$, $L:n\times n \neq I_n$ and $\Sigma:p\times p \neq I_n$. It seems that the technique used by Crowther (1975, p. 29 - 30) for the derivation of the p.d.f. of $\underset{\sim}{S}:p\times p$, a real non-central quadratic form can not be used in this case for the derivation of the p.d.f. of $\underset{\sim}{S}:p\times p$ a non-central complex quadratic form. Crowther (1975, p. 29 - 30) used the result:

$$\int_X \pi^{-\frac{1}{2}np} \operatorname{etr}\left[-(X - \tfrac{i}{\sqrt{2}} M)(X - \tfrac{i}{\sqrt{2}} M)'\right] dX = 1$$

where $X: p \times n$ and $M: p \times n$ are real matrices. In the complex case the equivalent integral which has to be used is

$$\int_Z \pi^{-np} \operatorname{etr}\left[-(Z - \tfrac{i}{\sqrt{2}} M)(Z - \tfrac{i}{\sqrt{2}} M)'\right] dZ$$

where $Z: p \times n$ and $M: p \times n$ are complex matrices and this integral is not equal to 1.

(iii) Let $r = \frac{1}{q}$ in (4.2.4); then

$$\operatorname{etr}[-r\, D_B] \to 1$$

and

$$\tilde{P}_K(\Sigma^{-\frac{1}{2}} M \Phi^{-1} L^{-\frac{1}{2}} T^{-\frac{1}{2}}, T) \to \tilde{P}_K(\Sigma^{-\frac{1}{2}} M \Phi^{-1} L^{-\frac{1}{2}} C^{\frac{1}{2}}, C^{-1})$$

when $r \to 0$. This implies that (4.2.4) tends to (4.2.2) when $r \to 0$, i.e. $q \to \infty$.

(iv) Let $r = \frac{1}{q}$ in (4.2.9); then

$$\operatorname{etr}[-r\, \Sigma^{-1} S] \to 1$$

and

$$ {}_0\tilde{F}_0(-T, \Sigma^{-1} S) = {}_0\tilde{F}_0(-C^{-1} + r I_n ; \Sigma^{-1} S)$$

$$\to {}_0\tilde{F}_0(-L^{-1} \Phi^{-1}, \Sigma^{-1} S)$$

when $r \to 0$. This implies that (4.2.9) tends to (4.2.6) when $r \to 0$, i.e. $q \to \infty$.

(v) Along the same lines as in (iii) it follows that (4.2.14) tends to (4.2.12) when $r \to 0$, i.e. $q \to \infty$.

(vi) Crowther (1972, p. 21 - 27) derived the p.d.f. and moments of $\underset{\sim}{S} = W \underset{\sim}{Z} L \bar{\underset{\sim}{Z}}' \bar{W}'$ and $D_{\underset{\sim}{S}}$ where $\text{rank}(W{:}d\times p) = d$, $\text{rank}(L{:}n\times n) = r$, $L{:}n\times n$ is a hermitian positive semi-definite matrix, $r \geq p \geq d$ and $M = 0$. These results correspond with (4.2.6), (4.2.9), (4.2.8) and (4.2.10).

(vii) The results regarding the real multivariate quadratic form of normal variates which correspond with the results discussed in theorem 4.2.1 (i), (ii) and (iii) can be found in Khatri (1966), Hayakawa (1966, 1972 b) and Underhill (1973, p. 2.1 - p. 2.9). The results regarding the real central and non-central Wishart distributions which correspond with the results discussed in theorem 4.2.1 (iv) and (v) are widely discussed in the literature.

4.2.2 Certain marginal distributions of the characteristic roots of $\underset{\sim}{S}{:}p\times p$

In theorems 4.2.3 - 4.2.7 certain marginal distributions of the characteristic roots of $\underset{\sim}{S}{:}p\times p$ are derived by using the theorems in sections 3.3 - 3.7. For this purpose the joint p.d.f. of the characteristic roots is written in a form such that the random component is in the form given in (3.2.3). In theorem 4.2.2 it is shown how the joint p.d.f. of the roots of $\underset{\sim}{S}{:}p\times p$ for different specifications of $M{:}p\times n$, $\Sigma{:}p\times p$ and $\Phi{:}n\times n$ can be written in this convenient form by using the results given in theorems 2.2.8, 2.3.6 and 2.3.7.

Theorem 4.2.2

Let $\underset{\sim}{Z}:p\times n \sim CMTN(p, n, M, \Phi\otimes\Sigma)$ and let $L:n\times n$ be a hermitian matrix; then the p.d.f. of $\underset{\sim}{D}_S$ and $\underset{\sim}{D}_B$ where $\underset{\sim}{S}:p\times p = \underset{\sim}{Z}\,L\,\underset{\sim}{\bar{Z}}'$ and $\underset{\sim}{B}:p\times p = \Sigma^{-\frac{1}{2}}\,\underset{\sim}{S}\,\Sigma^{-\frac{1}{2}}$ respectively, are given below for certain specifications of $M:p\times n$, $\Phi:n\times n$, $\Sigma:p\times p$ and $L:n\times n$:

(i) $M:p\times n \neq 0$

Power-series representation

(4.2.58) $$f_{\underset{\sim}{D}_B}(D_B)$$

$$= \frac{\pi^{p(p-1)}\ \mathrm{etr}\left[-\Sigma^{-1} M\,\Phi^{-1}\,\bar{M}'\right]}{\tilde{\Gamma}_p(n)\ \tilde{\Gamma}_p(p)\ |L\,\Phi|^p}$$

$$\sum_{k=0}^{\infty}\sum_{\kappa}\frac{\tilde{P}_\kappa(\Sigma^{-\frac{1}{2}} M\,\Phi^{-1}\,L^{-\frac{1}{2}}\,C^{\frac{1}{2}},\ C^{-1})}{k!\ [n]_\kappa\ \tilde{C}_\kappa(I_p)}$$

$$\chi_{[\kappa]}(1)\prod_{i=1}^{p}\tilde{b}_i^{\,n-p}\ |(\tilde{b}_j^{\,k_i+p-i})|\,|(\tilde{b}_j^{\,p-i})|\,,$$

$$0<\tilde{b}_1<\cdots<\tilde{b}_p\,,\quad C:n\times n = L^{\frac{1}{2}}\,\Phi\,L^{\frac{1}{2}}\ .$$

Γ-type representation

(4.2.59) $$f_{\underset{\sim}{D}_B}(D_B)$$

$$= \frac{\pi^{p(p-1)}\ \mathrm{etr}\left[-\Sigma^{-1} M\,\Phi^{-1}\,\bar{M}'\right]}{\tilde{\Gamma}_p(n)\ \tilde{\Gamma}_p(p)\ |L\,\Phi|^p}$$

$$\sum_{k=0}^{\infty}\sum_{\kappa}\frac{\tilde{P}_\kappa(\Sigma^{-\frac{1}{2}} M\,\Phi^{-1}\,L^{-\frac{1}{2}}\,T^{-\frac{1}{2}},\ T)}{k!\ [n]_\kappa\ \tilde{C}_\kappa(I_p)}$$

$$\chi_{[\kappa]}(1)\prod_{i=1}^{p} e^{-q^{-1}\tilde{b}_i}\ \tilde{b}_i^{\,n-p}\ |(\tilde{b}_j^{\,k_i+p-i})|\,|(\tilde{b}_j^{\,p-i})|\,,$$

$0 < \tilde{b}_1 < \cdots < \tilde{b}_p$, $T{:}n\times n = C^{-1} - q^{-1} I_n$ and $\|C\| < q$.

(ii) <u>$M{:}p\times n = 0$</u>

<u>Power-series representation</u>

(4.2.60) $f_{\underset{\sim}{D}_S}(D_S)$

$$= \frac{\pi^{p(p-1)}}{\tilde{\Gamma}_p(p)\ \tilde{\Gamma}_p(n)\ |L\,\Phi|^p\ |\Sigma|^n} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{\tilde{C}_\kappa(-L^{-1}\,\Phi^{-1})\ \tilde{C}_\kappa(\Sigma^{-1})}{k!\ \tilde{C}_\kappa(I_n)\ \tilde{C}_\kappa(I_p)}$$

$$\chi_{[\kappa]}(1) \prod_{i=1}^{p} \tilde{s}_i^{n-p}\ |(\tilde{s}_j^{k_i+p-i})|\,|(\tilde{s}_j^{p-i})|,\quad 0 < \tilde{s}_1 < \cdots < \tilde{s}_p .$$

<u>Γ-type representation</u>

(4.2.61) $f_{\underset{\sim}{D}_B}(D_B)$

$$= \frac{\pi^{p(p-1)}}{\tilde{\Gamma}_p(n)\ |L\,\Phi|^p\ \tilde{\Gamma}_p(p)} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{\tilde{C}_\kappa(-T)}{k!\ \tilde{C}_\kappa(I_n)} \chi_{[\kappa]}(1)$$

$$\prod_{i=1}^{p} e^{-q^{-1}\tilde{b}_i}\ \tilde{b}_i^{n-p}\ |(\tilde{b}_j^{k_i+p-i})|\,|(\tilde{b}_j^{p-i})|,\quad 0 < \tilde{b}_1 < \cdots < \tilde{b}_p .$$

(iii) <u>$M{:}p\times n \neq 0$, $\Phi{:}n\times n = I_n$, $\Sigma{:}p\times p = I_p$</u>

<u>Power-series representation</u>

(4.2.62) $f_{\underset{\sim}{D}_S}(D_S)$

$$= \frac{\pi^{p(p-1)}\ \mathrm{etr}[-M\bar{M}']}{|L|^p\ \tilde{\Gamma}_p(n)\ \tilde{\Gamma}_p(p)} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{\tilde{P}_\kappa(M\,,L^{-1})}{k!\ [n]_\kappa\ \tilde{C}_\kappa(I_p)}$$

$$\chi_{[\kappa]}(1) \prod_{i=1}^{p} \tilde{s}_i^{n-p} \, |(\tilde{s}_j^{k_i+p-i})| \, |(\tilde{s}_j^{p-i})| \,, \quad 0 < \tilde{s}_1 < \cdots < \tilde{s}_p \,.$$

<u>Γ-type representation</u>

(4.2.63) $\quad f_{\underset{\sim}{D}_S}(D_S)$

$$= \frac{\pi^{p(p-1)} \operatorname{etr}[-M\bar{M}']}{|L|^p \, \tilde{\Gamma}_p(n) \, \tilde{\Gamma}_p(p)} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{\tilde{P}_\kappa(M L^{-\frac{1}{2}} T^{-\frac{1}{2}}, T)}{k! \, [n]_\kappa \, \tilde{C}_\kappa(I_p)}$$

$$\chi_{[\kappa]}(1) \prod_{i=1}^{p} e^{-q^{-1}\tilde{s}_i} \, \tilde{s}_i^{n-p} \, |(\tilde{s}_j^{k_i+p-i})| \, |(\tilde{s}_j^{p-i})| \,,$$

$$0 < \tilde{s}_1 < \cdots < \tilde{s}_p \,, \quad T{:}n\times n = L^{-1} - q\, I_n \,, \quad \|L\| < q \,.$$

(iv) $\quad$ <u>$\underset{\sim}{S}{:}p\times p \sim NCCW(p,\, n,\, \Sigma,\, \Omega)$</u>

<u>Power-series representation</u>

(4.2.64) $\quad f_{\underset{\sim}{D}_B}(D_B)$

$$= \frac{\pi^{p(p-1)} \operatorname{etr}[-\Omega]}{\tilde{\Gamma}_p(n) \, \tilde{\Gamma}_p(p)} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{\tilde{H}_\kappa(\Sigma^{-\frac{1}{2}} M) \, \chi_{[\kappa]}(1)}{k! \, [n]_\kappa \, \tilde{C}_\kappa(I_p)}$$

$$\prod_{i=1}^{p} \tilde{b}_i^{n-p} \, |(\tilde{b}_j^{k_i+p-i})| \, |(\tilde{b}_j^{p-i})| \,, \quad 0 < \tilde{b}_1 < \cdots < \tilde{b}_p \,.$$

<u>Γ-type representation</u>

(4.2.65) $\quad f_{\underset{\sim}{D}_B}(D_B)$

$$= \frac{\pi^{\frac{1}{2}p(p-1)} \operatorname{etr}[-\Omega]}{\tilde{\Gamma}_p(n) \prod_{i>j}^{p} (\tilde{\omega}_i - \tilde{\omega}_j)} \prod_{i=1}^{p} (n-i+1)^{i-1} \prod_{i=1}^{p} e^{-\tilde{b}_i} \, \tilde{b}_i^{n-p}$$

$$|({}_0F_1(n-p+1\,;\,\tilde{\omega}_i\,\tilde{b}_j))|\,|(\tilde{b}_j^{p-i})|\quad,\quad 0<\tilde{b}_1<\cdots<\tilde{b}_p$$

where $0<\tilde{\omega}_1<\cdots<\tilde{\omega}_p$ are the characteristic roots of $\Omega{:}p\times p$.

(v) $\underset{\sim}{S}{:}p\times p \sim CW(p,\ n,\ \Sigma)$

Power-series representation

(4.2.66) $f_{\underset{\sim}{D}_S}(D_S)$

$$= \frac{\pi^{\frac{1}{2}p(p-1)}\prod_{i=1}^{p}\tilde{s}_i^{n-p}}{\tilde{\Gamma}_p(n)\ |\Sigma|^n\prod_{i>j}^{p}\left(\frac{1}{\tilde{\sigma}_j}-\frac{1}{\tilde{\sigma}_i}\right)}\ |(\exp(-\frac{1}{\tilde{\sigma}_i}\ \tilde{s}_j))|\,|(\tilde{s}_j^{p-i})|\ ,$$

$$0<\tilde{s}_1<\cdots<\tilde{s}_p\ ,$$

where $0<\tilde{\sigma}_1<\cdots<\tilde{\sigma}_p$ are the characteristic roots of $\Sigma{:}p\times p$ so that $0<\frac{1}{\tilde{\sigma}_p}<\cdots<\frac{1}{\tilde{\sigma}_1}$ are the characteristic roots of $\Sigma^{-1}{:}p\times p$.

Γ-type representation

(4.2.67) $f_{\underset{\sim}{D}_S}(D_S)$

$$= \frac{\pi^{\frac{1}{2}p(p-1)}\prod_{i=1}^{p}\tilde{s}_i^{n-p}}{\tilde{\Gamma}_p(n)\ |\Sigma|^n\prod_{i>j}^{p}\left(\frac{1}{\tilde{\sigma}_j}-\frac{1}{\tilde{\sigma}_i}\right)}\ |(\exp(-\frac{1}{\tilde{\sigma}_i}\ \tilde{s}_j))|\,|(\tilde{s}_j^{p-i})|\ ,$$

$$0<\tilde{s}_1<\cdots<\tilde{s}_p\ .$$

<u>Proof</u>

(<u>4.2.58</u>), (<u>4.2.59</u>), (<u>4.2.60</u>), (<u>4.2.61</u>), (<u>4.2.62</u>), (<u>4.2.63</u>)
(<u>4.2.64</u>)

These seven p.d.f.s follow from (4.2.2), (4.2.4), (4.2.8), (4.2.11), (4.2.13), (4.2.17) and (4.2.19) respectively by using the result:

$$\tilde{C}_\kappa(S) \prod_{i>j}^{p} (\tilde{s}_i - \tilde{s}_j)^2 = \chi_{[\kappa]}(1) \;|(\tilde{s}_j^{k_i+p-i})|\,|(\tilde{s}_j^{p-i})| ,$$

for $S:p\times p$ h.p.d., which is proved in theorem 2.2.8.

(<u>4.2.65</u>)

Let $\underset{\sim}{S}:p\times p \sim \text{NCCW}(p,\ n,\ \Sigma,\ \Omega)$; then follows from (4.2.23) and (2.2.41) that

(4.2.68) $f_{\underset{\sim}{D}_B}(D_B)$

$$= \frac{\pi^{p(p-1)}\ \text{etr}[-\Omega]}{\tilde{\Gamma}_p(n)\ \tilde{\Gamma}_p(p)} \prod_{i=1}^{p} \tilde{b}_i^{n-p}\ e^{-\tilde{b}_i}\ |(\tilde{b}_j^{p-i})|^2\ {}_0\tilde{F}_1(n\ ;\ \Omega,\ D_B) .$$

From (2.3.14), (2.3.15) and (2.2.41) follows:

(4.2.69) ${}_0\tilde{F}_1(n\ ;\ \Omega,\ D_B)$

$$= \frac{\tilde{\Gamma}_p(p) \prod_{i=1}^{p} (n-i+1)^{i-1}\ |({}_0F_1(n-p+1\ ;\ \tilde{\omega}_i\, \tilde{b}_j))|}{\pi^{\frac{1}{2}p(p-1)} \prod_{i>j}^{p} (\tilde{\omega}_i - \tilde{\omega}_j)\ \ |(\tilde{b}_j^{p-i})|}$$

where $0 < \tilde{\omega}_1 < \cdots < \tilde{\omega}_p$ are the roots of $\Omega:p\times p$.

Substitution of (4.2.69) into (4.2.68) leads to (4.2.65).

(4.2.66)

Let $\underset{\sim}{S}:p\times p \sim CW(p, n, \Sigma)$; then follows from (4.2.26) and (2.2.41) that

$$(4.2.70) \qquad f_{\underset{\sim}{D}_S}(D_S) = \frac{\pi^{p(p-1)} \prod_{i=1}^{p} \tilde{s}_i^{n-p}}{\tilde{\Gamma}_p(n)\ \tilde{\Gamma}_p(p)\ |\Sigma|^n} \ |(\tilde{s}_j^{p-i})|^2 \ {}_0\tilde{F}_0(-\Sigma^{-1}, D_S) \ .$$

From (2.3.16) and (2.2.41) follows:

$$(4.2.71) \qquad {}_0\tilde{F}_0(-\Sigma^{-1}, D_S) = \frac{\tilde{\Gamma}_p(p)\ |\exp(-\frac{1}{\tilde{\sigma}_i}\tilde{s}_j)|}{\pi^{\frac{1}{2}p(p-1)} \prod_{i>j}^{p} (\frac{1}{\tilde{\sigma}_j} - \frac{1}{\tilde{\sigma}_i})\ |(\tilde{s}_j^{p-i})|}$$

where $-\frac{1}{\tilde{\sigma}_1} < -\frac{1}{\tilde{\sigma}_2} < \cdots < -\frac{1}{\tilde{\sigma}_p}$ are the characteristic roots of $-\Sigma^{-1}:p\times p$.

Substitution of (4.2.71) into (4.2.70) leads to (4.2.66).

(4.2.67)

The proof of (4.2.67) is similar to the proof of (4.2.66).

From theorem 4.2.2 it is clear that the random component in the p.d.f.s is:

(i) in (4.2.58), (4.2.60), (4.2.62) and (4.2.64) of the form:

$$(4.2.72) \qquad \prod_{i=1}^{p} \tilde{s}_i^{n-p} \ |(\tilde{s}_j^{k_i+p-i})|\,|(\tilde{s}_j^{p-i})| \ ,$$

(ii) in (4.2.59), (4.2.61) and (4.2.63) of the form:

$$(4.2.73) \qquad \prod_{i=1}^{p} \tilde{s}_i^{n-p} e^{-q^{-1}\tilde{s}_i} \left|(\tilde{s}_j^{k_i+p-i})\right| \left|(\tilde{s}_j^{p-i})\right| ,$$

(iii) in (4.2.66) and (4.2.67) of the form:

$$(4.2.74) \qquad \prod_{i=1}^{p} \tilde{s}_i^{n-p} \left|(\exp(-\frac{1}{\tilde{\sigma}_i}\tilde{s}_j))\right| \left|(\tilde{s}_j^{p-i})\right| ,$$

(iv) in (4.2.65) of the form:

$$(4.2.75) \qquad \prod_{i=1}^{p} e^{-\tilde{s}_i} \tilde{s}_i^{n-p} \left|({}_0F_1(n-p+1\,;\,\tilde{\omega}_i\,\tilde{s}_j))\right| \left|(\tilde{s}_j^{p-i})\right| .$$

Because of these similarities the c.d.f. of the extreme characteristic roots will be derived in theorem 4.2.3 only in the cases where the joint p.d.f. of the roots is given by (4.2.60), (4.2.63), (4.2.66) and (4.2.65).

<u>Theorem 4.2.3</u>

Let $\underset{\sim}{D}_S$ have the p.d.f. given in (4.2.60) ; then

$$(4.2.76) \qquad P(c < \tilde{S}_1 < \tilde{S}_p < d)$$

$$= \frac{\pi^{p(p-1)}}{\tilde{\Gamma}_p(p)\ \tilde{\Gamma}_p(n)\ |L\Phi|^n\ |\Sigma|^n} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{\tilde{C}_\kappa(-L^{-1}\Phi^{-1})\ \tilde{C}_\kappa(\Sigma^{-1})}{k!\ \tilde{C}_\kappa(I_n)\ \tilde{C}_\kappa(I_p)}$$

$$\chi_{[\kappa]}(1)\ \left|(b_{ij})\right| ,$$

where

$$(4.2.77) \qquad b_{ij} = \frac{d^{n+p+k_i-i-j+1} - c^{n+p+k_i-i-j+1}}{n+p+k_i-i-j+1} .$$

Let $\underset{\sim}{D}_S$ have the p.d.f. given in (4.2.63); then

(4.2.78) $\quad P(c < \tilde{S}_1 < \tilde{S}_p < d)$

$$= \frac{\pi^{p(p-1)}\ \mathrm{etr}[-M\bar{M}']}{|L|^p\ \tilde{\Gamma}_p(n)\ \tilde{\Gamma}_p(p)} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{\tilde{P}_\kappa(ML^{-\frac{1}{2}}T^{-\frac{1}{2}}, T)}{k!\ [n]_\kappa\ \tilde{C}_\kappa(I_p)} \chi_{[\kappa]}(1)\ |(b_{ij})|$$

where

(4.2.79) $$b_{ij} = q^{n+p+k_i-i-j+1}\{\Gamma(q^{-1}d\,,\, n+p+k_i-i-j+1) - \Gamma(q^{-1}c\,,\, n+p+k_i-i-j+1)\}$$

and $\Gamma(\cdot;\cdot)$ is defined in (2.6.1).

Let $\underset{\sim}{D}_S$ have the p.d.f. given in (4.2.66); then

(4.2.80) $$P(c < \tilde{S}_1 < \tilde{S}_p < d) = \frac{\pi^{\frac{1}{2}p(p-1)}\ |(b_{ij})|}{\tilde{\Gamma}_p(n)\ |\Sigma|^n \prod_{i>j}^{p} (\frac{1}{\tilde{\sigma}_j} - \frac{1}{\tilde{\sigma}_i})}$$

where

(4.2.81) $$b_{ij} = \tilde{\sigma}_i^{n-j+1}\{\Gamma(\frac{d}{\tilde{\sigma}_i}\,,\, n-j+1) - \Gamma(\frac{c}{\tilde{\sigma}_i}\,,\, n-j+1)\}\ .$$

Let $\underset{\sim}{D}_B$ have the p.d.f. given in (4.2.65); then

(4.2.82) $$P(c < \tilde{B}_1 < \tilde{B}_p < d) = \frac{\pi^{\frac{1}{2}p(p-1)}\ \mathrm{etr}[-\Omega]}{\tilde{\Gamma}_p(n) \prod_{i>j}^{p} (\tilde{\omega}_i - \tilde{\omega}_j)} \prod_{i=1}^{p} (n-i+1)^{i-1}\ |(b_{ij})|$$

where

(4.2.83) $$b_{ij} = \sum_{k=0}^{\infty} \frac{\tilde{\omega}_i^k}{(n-p+1)_k \, k!} \{\Gamma(d,\ n+k-j+1) - \Gamma(c,\ n+k-j+1)\} .$$

Proof

(4.2.76), (4.2.77)

To find (4.2.76) the following integral is solved:

(4.2.84) $$I^* = \int_{c<\tilde{s}_1<\cdots<\tilde{s}_p<d} \cdots\cdots\cdots \int \prod_{i=1}^{p} \tilde{s}_i^{n-p} \, |(\tilde{s}_j^{k_i+p-i})| \, |(\tilde{s}_j^{p-i})| \, d\tilde{s}_1 \cdots d\tilde{s}_p .$$

Let

(4.2.85) $$g(\tilde{s}_i) = \tilde{s}_i^{n-p} ,$$

(4.2.86) $$\phi_i(\tilde{s}_j) = \tilde{s}_j^{k_i+p-i}$$

and

(4.2.87) $$\psi_i(\tilde{s}_j) = \tilde{s}_j^{p-i} .$$

From theorem 3.3.1 follows that

(4.2.88) $$I^* = |(b_{ij})|$$

where

(4.2.89) $$b_{ij} = \int_c^d x^{n+p+k_i-i-j} \, dx$$

$$= \frac{d^{n+p+k_i-i-j+1} - c^{n+p+k_i-i-j+1}}{n+p+k_i-i-j+1}$$

which proves (4.2.76).

(4.2.78), (4.2.79)

To find (4.2.78) the following integral is solved:

$$(4.2.90) \qquad I^* = \int_{c < \tilde{s}_1 < \cdots < \tilde{s}_p < d} \cdots\cdots\cdots\cdots \int \prod_{i=1}^{p} e^{-q^{-1}\tilde{s}_i} \tilde{s}_i^{n-p} \, |(\tilde{s}_j^{k_i+p-i})| \, |(\tilde{s}_j^{p-i})| \, d\tilde{s}_1 \cdots d\tilde{s}_p \, .$$

Let

$$(4.2.91) \qquad g(\tilde{s}_i) = e^{-q^{-1}\tilde{s}_i} \tilde{s}_i^{n-p} \, ,$$

$$(4.2.92) \qquad \phi_i(\tilde{s}_j) = \tilde{s}_j^{p+k_i-i}$$

and

$$(4.2.93) \qquad \psi_i(\tilde{s}_j) = \tilde{s}_j^{p-i} \, .$$

From theorem 3.3.1 follows that

$$(4.2.94) \qquad I^* = |(b_{ij})|$$

where

$$(4.2.95) \qquad b_{ij} = \int_c^d e^{-q^{-1}x} x^{n+p+k_i-i-j} \, dx$$

$$= q^{n+p+k_i-i-j+1} \int_{q^{-1}c}^{q^{-1}d} e^{-y} y^{n+p+k_i-i-j} \, dy$$

$$= q^{n+p+k_i-i-j+1} \{\Gamma(q^{-1}d,\ n+p+k_i-i-j+1) - \Gamma(q^{-1}c,\ n+p+k_i-i-j+1)\}$$

which proves (4.2.78).

(4.2.80), (4.2.81)

To find (4.2.80) the following integral is solved:

$$(4.2.96) \qquad I^* = \int_{c < \tilde{s}_1 < \cdots < \tilde{s}_p < d} \cdots\cdots\cdots \int \prod_{i=1}^{p} \tilde{s}_i^{n-p} \, |(\exp(-\frac{1}{\tilde{\sigma}_i}\tilde{s}_j))| \, |(\tilde{s}_j^{p-i})| \, d\tilde{s}_1 \cdots d\tilde{s}_p \ .$$

Let

$$(4.2.97) \qquad g(\tilde{s}_i) = \tilde{s}_i^{n-p} \ ,$$

$$(4.2.98) \qquad \phi_i(\tilde{s}_j) = e^{-\frac{1}{\tilde{\sigma}_i}\tilde{s}_j}$$

and

$$(4.2.99) \qquad \psi_i(\tilde{s}_j) = \tilde{s}_j^{p-i} \ .$$

From theorem 3.3.1 follows that

$$(4.2.100) \qquad I^* = |(b_{ij})|$$

where

$$(4.2.101) \qquad b_{ij} = \int_c^d e^{-\frac{1}{\tilde{\sigma}_i}x} \, x^{n-j} \, dx$$

$$= \tilde{\sigma}_i^{n-j+1} \int_{\frac{c}{\tilde{\sigma}_i}}^{\frac{d}{\tilde{\sigma}_i}} e^{-y}\, y^{n-j}\, dy$$

$$= \tilde{\sigma}_i^{n-j+1} \{\Gamma(\tfrac{d}{\tilde{\sigma}_i},\; n-j+1) - \Gamma(\tfrac{c}{\tilde{\sigma}_i},\; n-j+1)\}$$

which proves (4.2.80).

<u>(4.2.82)</u>, <u>(4.2.83)</u>

To find (4.2.82) the following integral is solved:

$$(4.2.102) \quad I^* = \int_{c<\tilde{b}_1<\cdots<\tilde{b}_p<d} \cdots\cdots\cdots \int \prod_{i=1}^{p} e^{-\tilde{b}_i}\, \tilde{b}_i^{n-p}\, |({}_0F_1(n-p+1\,;\,\tilde{\omega}_i\,\tilde{b}_j))|\, |(\tilde{b}_j^{p-i})|\, d\tilde{b}_1\cdots d\tilde{b}_p \,.$$

Let

$$(4.2.103) \quad g(\tilde{b}_i) = e^{-\tilde{b}_i}\, \tilde{b}_i^{n-p} \,,$$

$$(4.2.104) \quad \phi_i(\tilde{b}_j) = {}_0F_1(n-p+1\,;\,\tilde{\omega}_i\,\tilde{b}_j)$$

and

$$(4.2.105) \quad \psi_i(\tilde{b}_j) = \tilde{b}_j^{p-i} \,.$$

From theorem 3.3.1 follows that

$$(4.2.106) \quad I^* = |(\tilde{b}_{ij})|$$

where

$$(4.2.107) \quad b_{ij} = \int_c^d e^{-x} x^{n-j} {}_0F_1(n-p+1\,;\,\tilde{\omega}_i x)\, dx$$

$$= \sum_{k=0}^{\infty} \frac{(\tilde{\omega}_i)^k}{(n-p+1)_k\, k!} \int_c^d e^{-x} x^{k+n-j}\, dx$$

$$= \sum_{k=0}^{\infty} \frac{(\tilde{\omega}_i)^k}{(n-p+1)_k\, k!} \{\Gamma(d,\ k+n-j+1) - \Gamma(c,\ k+n-j+1)\}$$

which proves (4.2.82).

The c.d.f. of $\tilde{S}_p$, the largest characteristic root of $\underset{\sim}{S}:p\times p$, for the different forms of the p.d.f. of $\underset{\sim}{D}_S$ follows now as a corollary of theorem 4.2.3. These different c.d.f.s are identical to the corresponding expressions for $P(c < \tilde{S}_1 < \tilde{S}_p < d)$, derived in theorem 4.2.3, except for the determinant $|(b_{ij})|$. Therefore only the corresponding expressions for b_{ij} will be given in corollary 4.2.1.

<u>Corollary 4.2.1</u>

Let $\underset{\sim}{D}_S$ have the p.d.f. given in (4.2.60); then

$$(4.2.108) \quad P(\tilde{S}_p < d) = (4.2.76)$$

where

$$(4.2.109) \quad b_{ij} = \frac{d^{n+p+k_i-i-j+1}}{n+p+k_i-i-j+1} .$$

Let $\underset{\sim}{D}_S$ have the p.d.f. given in (4.2.63); then

(4.2.110) $P(\tilde{S}_p < d) = (4.2.78)$

where

(4.2.111) $$b_{ij} = q^{n+p+k_i-i-j+1} \Gamma(q^{-1}d,\ n+p+k_i-i-j+1) .$$

Let $\underset{\sim}{D}_S$ have the p.d.f. given in (4.2.66); then

(4.2.112) $P(S_p < d) = (4.2.80)$

where

(4.2.113) $$b_{ij} = \tilde{\sigma}_i^{n-j+1} \Gamma(\frac{d}{\tilde{\sigma}_i},\ n-j+1) .$$

Let $\underset{\sim}{D}_B$ have the p.d.f. given in (4.2.65); then

(4.2.114) $P(B_p < d) = (4.2.82)$

where

(4.2.115) $$b_{ij} = \sum_{k=0}^{\infty} \frac{\tilde{\omega}_i^k \Gamma(d,\ n+k-j+1)}{(n-p+1)_k\ k!} .$$

<u>Proof</u>

It is clear that $P(\tilde{S}_p < d) = P(0 < \tilde{S}_1 < \tilde{S}_p < d)$. Thus by taking $c = 0$ (4.2.109), (4.2.111), (4.2.113) and (4.2.115) follow from (4.2.77), (4.2.79), (4.2.81) and (4.2.83) respectively.

<u>Remark 4.2.2</u>

(i) Let $\underset{\sim}{S}:p\times p \sim CW(p,\ n,\ I_p)$; then follows from (4.2.24), theorem 3.2.1 and (2.2.41) that

$$(4.2.116)\quad f_{\underset{\sim}{D}_S}(D_S) = \frac{\pi^{p(p-1)} \prod_{i=1}^{p} \tilde{s}_i^{n-p} e^{-\tilde{s}_i}}{\tilde{\Gamma}_p(n)\ \tilde{\Gamma}_p(p)} \left|(\tilde{s}_j^{p-i})\right|^2$$

$$0 < \tilde{s}_1 < \cdots < \tilde{s}_p .$$

By using theorem 3.3.1 it follows along the same lines as in theorem 4.2.3 and corollary 4.2.1 that

$$(4.2.117)\quad P(\tilde{S}_p < d) = \frac{\pi^{p(p-1)}}{\tilde{\Gamma}_p(n)\ \tilde{\Gamma}_p(p)} \left|(\Gamma(d,\ n+p-i-j+1))\right| .$$

Using mathematical induction it can be shown that

$$(4.2.118)\quad \left|(\Gamma(d,\ n+p-i-j+1))\right| = \left|(\Gamma(d,\ n-p+i+j-2))\right| ,$$

so that $P(\tilde{S}_p < d)$ can also be written as

$$(4.2.119)\quad P(\tilde{S}_p < d) = \frac{\pi^{p(p-1)}}{\tilde{\Gamma}_p(n)\ \tilde{\Gamma}_p(p)} \left|(\Gamma(d,\ n-p+i+j-2))\right| ,$$

which is a result proved by Khatri (1964) in a different way as above. By using (4.2.119) Pillai and Young (1971) calculated approximate percentage points of $\tilde{S}_p$. They tabulated d where d is such that

$$P(\tilde{S}_p < d) = 1 - \alpha$$

for

$$\alpha = 0,10\ ;\ 0,05\ ;\ 0,025\ ;\ 0,01\ ;\ 0,005\ ,$$

$$p = 2,3,\cdots,11$$

and

$$n-p = 0\ (1)\ 20\ (2)\ 30\ (5)\ 50\ (10)\ 100 \ .$$

(ii) For $\underset{\sim}{S}:p\times p \sim CW(p,\, n,\, \Sigma)$, Hirakawa (1975, p. 360) derived an expression for $P(\frac{\tilde{s}_p}{1+\tilde{s}_p} < d)$ which involves Laguerre polynomials. The expression given in (4.2.112) for $P(\underset{\sim}{S}_p < d)$ is however, from a computational point of view, better than the expression derived by Hirakawa (1975).

The c.d.f. of $\tilde{s}_1$, the smallest characteristic root of $\underset{\sim}{S}:p\times p$, follows for three of the four forms of the p.d.f. of $\underset{\sim}{D}_S$ as a corollary of theorem 4.2.3. It is clear that

$$P(\tilde{s}_1 < c) = 1 - P(c < \tilde{s}_1 < \tilde{s}_p < \infty) \ ,$$

thus by taking $d = \infty$ in the results proved in theorem 4.2.3, the c.d.f. of $\tilde{s}_1$ follows except when the random component in the joint p.d.f. of the roots is in the form:

$$\prod_{i=1}^{p} \tilde{s}_i^{n-p} \,|(\tilde{s}_j^{k_i+p-i})|\,|(\tilde{s}_j^{p-i})| \ .$$

From (4.2.76) and (4.2.77) it follows that this form leads to an improper integral which is divergent when $d \to \infty$ and which implies that the power-series representation of the p.d.f. of:

$\underset{\sim}{D}_B$ when $M:p\times n \neq 0$,

$\underset{\sim}{D}_S$ when $M:p\times n = 0$,

$\underset{\sim}{D}_S$ when $M:p\times n = 0$, $\Phi:n\times n = I_n$ and $\Sigma:p\times p = I_p$ and

$\underset{\sim}{D}_B$ when $\underset{\sim}{S}:p\times p \sim NCCW(p,\, n,\, \Sigma,\, \Omega)$

can not be used for the derivation of the c.d.f. of $\tilde{s}_1$. Consider now corollary 4.2.2 in which $P(\tilde{s}_1 < c)$ are given for the remaining three forms of the p.d.f. of $\underset{\sim}{D}_S$.

Corollary 4.2.2

Let $\underset{\sim}{D}_S$ have the p.d.f. given in (4.2.63); then

$$(4.2.120) \quad P(\tilde{S}_1 < c) = 1 - (4.2.78)$$

where

$$(4.2.121) \quad b_{ij} = q^{n+p+k_i-i-j+1} \{\Gamma(n+p+k_i-i-j+1) - \Gamma(q^{-1}c,\ n+p+k_i-i-j+1)\} .$$

Let $\underset{\sim}{D}_S$ have the p.d.f. given in (4.2.66); then

$$(4.2.122) \quad P(\tilde{S}_1 < c) = 1 - (4.2.80)$$

where

$$(4.2.123) \quad b_{ij} = \tilde{\sigma}_i^{n-j+1} \{\Gamma(n-j+1) - \Gamma(\frac{c}{\tilde{\sigma}_i},\ n-j+1)\} .$$

Let $\underset{\sim}{D}_B$ have the p.d.f. given in (4.2.65); then

$$(4.2.124) \quad P(\tilde{B}_1 < c) = 1 - (4.2.82)$$

where

$$(4.2.125) \quad b_{ij} = \sum_{k=0}^{\infty} \frac{\tilde{\omega}_i^k}{(n-p+1)_k\ k!} \{\Gamma(n+k-j+1) - \Gamma(c,\ n+k-j+1)\} .$$

Proof

Let $d \to \infty$ in (4.2.79), (4.2.81) and (4.2.83); then (4.2.121), (4.2.123) and (4.2.125) follow respectively.

Remark 4.2.3

For $\underset{\sim}{S}:p\times p \sim CW(p,\ n,\ \Sigma)$, Hirakawa (1975, p. 360) derived an expression for $P(\frac{\tilde{S}_1}{1+\tilde{S}_1} < d)$ which involves Laguerre polynomials. The expression given in (4.2.122) is however from a computational point of view better than the expression derived by Hirakawa (1975).

In the following four theorems expressions to obtain the c.d.f. of any intermediate characteristic root, c.d.f. of any two intermediate characteristic roots, the joint p.d.f. of any few ordered characteristic roots and the joint p.d.f. of any few unordered characteristic roots are derived when $\underset{\sim}{D}_S$ has the p.d.f. (4.2.63) and (4.2.66) and $\underset{\sim}{D}_B$ has the p.d.f. (4.2.65). As in the case of the derivation of the c.d.f. of $\tilde{S}_1$, only these three forms of the joint p.d.f. of the characteristic roots are considered because when the random component is

$$\prod_{i=1}^{p} \tilde{s}_i^{n-p} \, |(\tilde{s}_j^{k_i+p-i})| \, |(\tilde{s}_j^{p-i})| \ ,$$

the derivation of the above-mentioned distributions also leads to improper integrals which are divergent.

From (3.4.2) follows that

$$P(\tilde{S}_t < c) = \sum_{r=t}^{p-1} P(0<\tilde{S}_1<\cdots<\tilde{S}_r<c<\tilde{S}_{r+1}<\cdots<\tilde{S}_p<\infty) + P(\tilde{S}_p<c) \ ,$$

$$(t = 1,\cdots,p-1) \ .$$

In theorem (4.2.4) $P(0 < \tilde{S}_1 < \cdots < \tilde{S}_r < c < \tilde{S}_{r+1} < \cdots < \tilde{S}_p)$ will be considered.

Theorem 4.2.4

Let $\underset{\sim}{D}_S$ have the p.d.f. given in (4.2.63); then

(4.2.126) $$P(0 < \tilde{S}_1 < \cdots < \tilde{S}_r < c < \tilde{S}_{r+1} < \cdots < \tilde{S}_p)$$

$$= \frac{\pi^{p(p-1)} \operatorname{etr}[-M\bar{M}']}{|L|^p \tilde{\Gamma}_p(n) \tilde{\Gamma}_p(p)} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{\tilde{P}_\kappa(ML^{-\frac{1}{2}} T^{-\frac{1}{2}}, T)}{k! \; [n]_\kappa \tilde{C}_\kappa(I_p)}$$

$$\chi_{[\kappa]}(1) \; \Sigma_1 \Sigma_2 (-1)^{\Sigma\delta_i + \Sigma\alpha_i} |(b_{3ij})| |(b_{4ij})|$$

where

(4.2.127) $$b_{3ij} = q^{n+p+k_{\delta_i}-\delta_i-\alpha_j+1} \Gamma(cq^{-1}, n+p+k_{\delta_i}-\delta_i-\alpha_j+1) \; , \quad (i,j = 1,\cdots,r) \; ,$$

(4.2.128) $$b_{4ij} = q^{n+p+k_{\nu_i}-\nu_i-\beta_j+1} \{\Gamma(n+p+k_{\nu_i}-\nu_i-\beta_j+1) - \Gamma(cq^{-1}, n+p+k_{\nu_i}-\nu_i-\beta_j+1)\} \; , \quad (i,j = 1,\cdots,p-r)$$

and the rest of the symbols are defined as in theorem 3.4.1.

Let $\underset{\sim}{D}_S$ have the p.d.f. given in (4.2.66); then

(4.2.129) $$P(0 < \tilde{S}_1 < \cdots < \tilde{S}_r < c < \tilde{S}_{r+1} < \cdots < \tilde{S}_p)$$

$$= \frac{\pi^{\frac{1}{2}p(p-1)} \; \Sigma_1 \Sigma_2 (-1)^{\Sigma\delta_i + \Sigma\alpha_i}}{\tilde{\Gamma}_p(n) \; |\Sigma|^n \prod_{i>j}^{p} (\frac{1}{\tilde{\sigma}_i} - \frac{1}{\tilde{\sigma}_j})} |(b_{3ij})| |(b_{4ij})|$$

where

$$(4.2.130) \quad b_{3ij} = \tilde{\sigma}_{\delta_i}^{n-\alpha_j+1} \, \Gamma(\frac{c}{\tilde{\sigma}_{\delta_i}}, \, n-\alpha_j+1), \quad (i,j = 1,\cdots,r) \, ,$$

$$(4.2.131) \quad b_{4ij} = \tilde{\sigma}_{\nu_i}^{n-\beta_j+1} \, \{\Gamma(n-\beta_j+1) - \Gamma(\frac{c}{\tilde{\sigma}_{\nu_i}}, \, n-\beta_j+1)\} \, ,$$

$$(i,j = 1,\cdots,p-r)$$

and the rest of the symbols are defined as in theorem 3.4.1.

Let $\underset{\sim}{D}_B$ have the p.d.f. given in (4.2.65); then

$$(4.2.132) \qquad P(0 < \tilde{B}_1 < \cdots < \tilde{B}_r < c < \tilde{B}_{r+1} < \cdots < \tilde{B}_p)$$

$$= \frac{\pi^{\frac{1}{2}p(p-1)} \, \operatorname{etr}[-\Omega] \, \prod_{i=1}^{p} (n-i+1)^{i-1}}{\tilde{\Gamma}_p(n) \, \prod_{i>j}^{p} (\tilde{\omega}_i - \tilde{\omega}_j)} \, \Sigma_1 \, \Sigma_2 \, (-1)^{\Sigma\delta_i + \Sigma\alpha_i}$$

$$|(b_{3ij})| \, |(b_{4ij})|$$

where

$$(4.2.133) \quad b_{3ij} = \sum_{k=0}^{\infty} \frac{\tilde{\omega}_{\delta_i}^{k}}{k! \, (n-p+1)_k} \, \Gamma(c, \, n+k-\alpha_j+1) \, , \quad (i,j = 1,\cdots,r) \, ,$$

$$(4.2.134) \quad b_{4ij} = \sum_{k=0}^{\infty} \frac{\tilde{\omega}_{\nu_i}^{k}}{k! \, (n-p+1)_k} \, \{\Gamma(n+k-\beta_j+1) - \Gamma(c, \, n+k-\beta_j+1)\} \, ,$$

$$(i,j = 1,\cdots,p-r)$$

and the rest of the symbols are defined as in theorem 3.4.1.

Proof

(4.2.126), (4.2.127), (4.2.128)

The application of theorem 3.4.1 with $g(\tilde{s}_i)$, $\phi_i(\tilde{s}_j)$ and $\psi_i(\tilde{s}_j)$ given in (4.2.91), (4.2.92) and (4.2.93) respectively leads to (4.2.126) with

$$b_{3ij} = \int_0^c e^{-q^{-1}x}\, x^{n+p+k_{\delta_i}-\delta_i-\alpha_j}\, dx\,, \quad (i,j = 1,\cdots,r)$$

and

$$b_{4ij} = \int_c^\infty e^{-q^{-1}x}\, x^{n+p+k_{\nu_i}-\nu_i-\beta_j}\, dx\,, \quad (i,j = 1,\cdots,p-r)$$

and hence the theorem is proved.

(4.2.129), (4.2.130), (4.2.131)

The application of theorem 3.4.1 with $g(\tilde{s}_i)$, $\phi_i(\tilde{s}_j)$ and $\psi_i(\tilde{s}_j)$ given in (4.2.97), (4.2.98) and (4.2.99) respectively leads to (4.2.129) with

$$b_{3ij} = \int_0^c e^{-\frac{1}{\tilde{\sigma}_{\delta_i}}x}\, x^{n-\alpha_j}\, dx\,, \quad (i,j = 1,\cdots,r)$$

and

$$b_{4ij} = \int_c^\infty e^{-\frac{1}{\tilde{\sigma}_{\nu_i}}x}\, x^{n-\beta_j}\, dx\,, \quad (i,j = 1,\cdots,p-r)$$

and hence the theorem is proved.

(4.2.132), (4.2.133), (4.2.134)

The application of theorem 3.4.1 with $g(\tilde{s}_i)$, $\phi_i(\tilde{s}_j)$ and $\psi_i(\tilde{s}_j)$ given in (4.2.103), (4.2.104) and (4.2.105) respectively leads to (4.2.132) with

$$b_{3ij} = \int_0^c e^{-x} x^{n-\alpha_j} \, {}_0F_1(n-p+1 \,;\, \tilde{\omega}_{\delta_i} x)\, dx \,, \quad (i,j = 1,\cdots,r)$$

and

$$b_{4ij} = \int_c^\infty e^{-x} x^{n-\beta_j} \, {}_0F_1(n-p+1 \,;\, \tilde{\omega}_{\nu_i} x)\, dx \,, \quad (i,j = 1,\cdots,p-r)$$

and hence the theorem is proved.

Remark 4.2.4

(i) For $\underset{\sim}{S}:p\times p \sim NCCW(p,\, n,\, \Sigma,\, \Omega)$, Khatri (1969) proved:

(4.2.135) $$P(0 < \tilde{B}_1 < \cdots < \tilde{B}_r < c < \tilde{B}_{r+1} < \cdots < \tilde{B}_p)$$

$$= \frac{\pi^{p(p-1)} \; etr[-\Omega]}{\tilde{\Gamma}_p(n)\; \tilde{\Gamma}_p(p)} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{\tilde{C}_\kappa(\Omega)\; \tilde{C}_\kappa(I_p)}{[n]_\kappa \; k!} \chi_{[\kappa]}(1) \; \Sigma_1 \; |(b_{ij})| \,,$$

$$(r = 1,\cdots,p-1)$$

where

Σ_1 denotes the summation over the combination $(\delta_p < \cdots < \delta_{r+1})$ and $(\delta_r < \cdots < \delta_1)$

and

$$b_{ij} = \begin{cases} \Gamma(c,\ n+p+k_j-\delta_i-j+1), & (i=1,\cdots,r\ ;\ j=1,\cdots,p) \\ \Gamma(n+p+k_j-\delta_i-j+1) - \Gamma(c,\ n+p+k_j-\delta_i-j+1) & \\ & (i=r+1,\cdots,p\ ;\ j=1,\cdots,p)\ . \end{cases}$$

The expression for $P(0 < \tilde{B}_1 < \cdots < \tilde{B}_r < c < \tilde{B}_{r+1} < \cdots < \tilde{B}_p)$ when $\underset{\sim}{S}:p\times p \sim NCCW(p,\ n,\ \Omega)$, given in (4.2.132), is better from a computational point of view, than (4.2.135) because (4.2.135) involves zonal polynomials and (4.2.132) not.

(ii) For $\underset{\sim}{S}:p\times p \sim CW(p,\ n,\ I_p)$, an expression for $P(0 < \tilde{S}_1 < \cdots < \tilde{S}_r < c < \tilde{S}_{r+1} < \cdots < \tilde{S}_p)$ is given by Al-Ani (1972, p. 326) while Schuurman and Waikar (1974) tabulated c where c is such that

$$P(\tilde{S}_t < c) = 1-\alpha\ ,\ (t = 1,\cdots,p-1)$$

for

$\alpha = 0{,}10\ ;\ 0{,}05\ ;\ 0{,}025\ ;\ 0{,}01$,

$p = 3,\ 4$ and 5

and

different values of n between 3 and 104.

In theorem 4.2.5 expressions for $P(0 < \tilde{S}_1 < \cdots < \tilde{S}_r < c < \tilde{S}_{r+1} < \cdots < \tilde{S}_{r+t} < d < \tilde{S}_{r+t+1} < \cdots < \tilde{S}_p)$ are considered.

Theorem 4.2.5

Let $\underset{\sim}{D}_S$ have the p.d.f. given in (4.2.63); then

(4.2.136) $$P(0<\tilde{S}_1<\cdots<\tilde{S}_r<c<\tilde{S}_{r+1}<\cdots<\tilde{S}_{r+t}<d<\tilde{S}_{r+t+1}<\cdots<\tilde{S}_p)$$

$$= \frac{\pi^{p(p-1)}\ \mathrm{etr}[-M\bar{M}']}{|L|^p\ \tilde{\Gamma}_p(n)\ \tilde{\Gamma}_p(p)} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{\tilde{P}_\kappa(ML^{-\frac{1}{2}}\ T^{-\frac{1}{2}}, T)}{k!\ [n]_\kappa\ \tilde{C}_\kappa(I_p)} \chi_{[\kappa]}(1)$$

$$\Sigma_3\Sigma_4\Sigma_5\Sigma_6(-1)^{\Sigma\delta_i+\Sigma\delta_i^*+\Sigma\alpha_i+\Sigma\alpha_i^*} |(b_{8ij})||(b_{9ij})||(b_{10ij})|$$

where

(4.2.137) $$b_{8ij} = q^{n+p+k_{\delta_i}-\delta_i-\delta_j^*+1}\ \Gamma(cq^{-1},\ n+p+k_{\delta_i}-\delta_i-\delta_j^*+1)\ ,$$

$$(i,j = 1,\cdots,r)\ ,$$

(4.2.138) $$b_{9ij} = q^{n+p+k_{\alpha_i}-\alpha_i-\alpha_j^*+1}\ \{\Gamma(dq^{-1},\ n+p+k_{\alpha_i}-\alpha_i-\alpha_j^*+1)$$

$$-\Gamma(cq^{-1},\ n+p+k_{\alpha_i}-\alpha_i-\alpha_j^*+1)\}\ ,\quad (i,j = 1,\cdots,t)\ ,$$

(4.2.139) $$b_{10ij} = q^{n+p+k_{\beta_i}-\beta_i-\beta_j^*+1}\ \{\Gamma(n+p+k_{\beta_i}-\beta_i-\beta_j^*+1)$$

$$-\Gamma(dq^{-1},\ n+p+k_{\beta_i}-\beta_i-\beta_j^*+1)\ ,\quad (i,j = t+r+1,\cdots,p)$$

and the rest of the symbols are defined as in theorem 3.5.1.

Let $\underset{\sim}{D}_S$ have the p.d.f. given in (4.2.66); then

(4.2.140) $$P(0<\tilde{S}_1<\cdots<\tilde{S}_r<c<\tilde{S}_{r+1}<\cdots<\tilde{S}_{r+t}<d<\tilde{S}_{r+t+1}<\cdots<\tilde{S}_p)$$

$$= \frac{\pi^{\frac{1}{2}p(p-1)}}{\tilde{\Gamma}_p(n)\ |\Sigma|^n \prod_{i>j}^{p} (\frac{1}{\tilde{\sigma}_i} - \frac{1}{\tilde{\sigma}_j})} \Sigma_3\Sigma_4\Sigma_5\Sigma_6(-1)^{\Sigma\delta_i+\Sigma\delta_i^*+\Sigma\alpha_i+\Sigma\alpha_i^*}$$

$$|(b_{8ij})||(b_{9ij})||(b_{10ij})|$$

where

$$(4.2.141)\quad b_{8ij} = \tilde{\sigma}_{\delta_i}^{n-\delta_j^*+1}\ \Gamma\left(\frac{c}{\tilde{\sigma}_{\delta_i}},\ n-\delta_j^*+1\right),\quad (i,j = 1,\cdots,r),$$

$$(4.2.142)\quad b_{9ij} = \tilde{\sigma}_{\alpha_i}^{n-\alpha_j^*+1}\ \left\{\Gamma\left(\frac{d}{\tilde{\sigma}_{\alpha_i}},\ n-\alpha_j^*+1\right) - \Gamma\left(\frac{c}{\tilde{\sigma}_{\alpha_i}},\ n-\alpha_j^*+1\right)\right\},$$

$$(i,j = 1,\cdots,t),$$

$$(4.2.143)\quad b_{10ij} = \tilde{\sigma}_{\beta_i}^{n-\beta_j^*+1}\ \left\{\Gamma(n-\beta_j^*+1) - \Gamma\left(\frac{d}{\tilde{\sigma}_{\beta_i}},\ n-\beta_j^*+1\right)\right\},$$

$$(i,j = t+r+1,\cdots,p)$$

and the rest of the symbols are defined as in theorem 3.5.1.

Let $\underset{\sim}{D}_B$ have the p.d.f. given in (4.2.65); then

$$(4.2.144)\qquad P(0<\tilde{B}_1<\cdots<\tilde{B}_r<c<\tilde{B}_{r+1}<\cdots<\tilde{B}_{r+t}<d<\tilde{B}_{r+t+1}<\cdots<\tilde{B}_p)$$

$$= \frac{\pi^{\frac{1}{2}p(p-1)}\ \operatorname{etr}[-\Omega]\ \prod_{i=1}^{p}(n-i+1)^{i-1}}{\tilde{\Gamma}_p(n)\ \prod_{i>j}^{p}(\tilde{\omega}_i - \tilde{\omega}_j)}$$

$$\Sigma_3\Sigma_4\Sigma_5\Sigma_6(-1)^{\Sigma\delta_i+\Sigma\delta_i^*+\Sigma\alpha_i+\Sigma\alpha_i^*}\ |(b_{8ij})|\,|(b_{9ij})|\,|(b'_{10ij})|$$

where

$$(4.2.145)\quad b_{8ij} = \sum_{k=0}^{\infty} \frac{\tilde{\omega}_{\delta_i}^{k}}{k!\ (n-p+1)_k}\ \Gamma(c,\ n+k-\delta_j^*+1),\quad (i,j = 1,\cdots,r),$$

$$(4.2.146)\quad b_{9ij} = \sum_{k=0}^{\infty} \frac{\tilde{\omega}_{\alpha_i}^{k}}{k!\ (n-p+1)_k} \{\Gamma(d,\ n+k-\alpha_j^*+1) - \Gamma(c,\ n+k-\alpha_j^*+1)\},$$

$$(i,j = 1,\cdots,t),$$

$$(4.2.147)\quad b_{10ij} = \sum_{k=0}^{\infty} \frac{\tilde{\omega}_{\beta_i}^{k}}{k!\ (n-p+1)_k} \{\Gamma(n+k-\beta_j^*+1) - \Gamma(d,\ n+k-\beta_j^*+1)\},$$

$$(i,j = r+t+1,\cdots,p)$$

and the rest of the symbols are defined as in theorem 3.5.1.

Proof

(<u>4.2.136</u>), (<u>4.2.137</u>), (<u>4.2.138</u>), (<u>4.2.139</u>)

The application of theorem 3.5.1 with $g(\tilde{s}_i)$, $\phi_i(\tilde{s}_j)$ and $\psi_i(\tilde{s}_j)$ given in (4.2.91), (4.2.92) and (4.2.93) respectively leads to (4.2.136) with

$$b_{8ij} = \int_0^c e^{-q^{-1}x}\, x^{n+p+k_{\delta_i}-\delta_i-\delta_j^*}\, dx,\quad (i,j = 1,\cdots,r),$$

$$b_{9ij} = \int_c^d e^{-q^{-1}x}\, x^{n+p+k_{\alpha_i}-\alpha_i-\alpha_j^*}\, dx,\quad (i,j = 1,\cdots,t)$$

and

$$b_{10ij} = \int_d^{\infty} e^{-q^{-1}x}\, x^{n+p+k_{\beta_i}-\beta_i-\beta_j^*}\, dx,\quad (i,j = r+t+1,\cdots,p)$$

and hence the theorem is proved.

(4.2.140), (4.2.141), (4.2.142), (4.2.143)

The application of theorem 3.5.1 with $g(\tilde{s}_i)$, $\phi_i(\tilde{s}_j)$ and $\psi_i(\tilde{s}_j)$ given in (4.2.97), (4.2.98) and (4.2.99) respectively leads to (4.2.140) with

$$b_{8ij} = \int_0^c e^{-\frac{1}{\tilde{\sigma}_{\delta_i}} x} x^{n-\delta_j^*} dx , \quad (i,j = 1,\cdots,r) ,$$

$$b_{9ij} = \int_c^d e^{-\frac{1}{\tilde{\sigma}_{\alpha_i}} x} x^{n-\alpha_j^*} dx , \quad (i,j = 1,\cdots,t)$$

and

$$b_{10ij} = \int_d^\infty e^{-\frac{1}{\tilde{\sigma}_{\beta_i}} x} x^{n-\beta_j^*} dx , \quad (i,j = r+t+1,\cdots,p)$$

and hence the theorem is proved.

(4.2.144), (4.2.145), (4.2.146), (4.2.147)

The application of theorem 3.5.1 with $g(\tilde{s}_i)$, $\phi_i(\tilde{s}_j)$ and $\psi_i(\tilde{s}_j)$ given in (4.2.103), (4.2.104) and (4.2.105) respectively leads to (4.2.144) with

$$b_{8ij} = \int_0^c e^{-x} x^{n-\delta_j^*} {}_0F_1(n-p+1 ; \tilde{\omega}_{\delta_i} x) dx ; \quad (i,j = 1,\cdots,r) ,$$

$$b_{9ij} = \int_c^d e^{-x} x^{n-\alpha_j^*} {}_0F_1(n-p+1\,;\,\tilde{\omega}_{\alpha_i} x)\,dx\,, \qquad (i,j=1,\cdots,t)$$

and

$$b_{10ij} = \int_d^\infty e^{-x} x^{n-\beta_j^*} {}_0F_1(n-p+1\,;\,\tilde{\omega}_{\beta_i} x)\,dx\,, \qquad (i,j=r+t+1,\cdots,p)$$

and hence the theorem is proved.

In theorem 4.2.6 expressions for $f_{\tilde{S}_{r+1},\cdots,\tilde{S}_{r+t}}(\tilde{s}_{r+1},\cdots,\tilde{s}_{r+t})$, $0<\tilde{s}_{r+1}<\cdots<\tilde{s}_{r+t}$, $(0\leq r\leq r+t\leq p)$ are considered.

<u>Theorem 4.2.6</u>

Let $\underset{\sim}{D}_S$ have the p.d.f. given in (4.2.63); then

(4.2.148)
$$f_{\tilde{S}_{r+1},\cdots,\tilde{S}_{r+t}}(\tilde{s}_{r+1},\cdots,\tilde{s}_{r+t})$$

$$= \frac{\pi^{p(p-1)}\ \mathrm{etr}[-M\bar{M}']}{|L|^p\ \tilde{\Gamma}_p(n)\ \tilde{\Gamma}_p(p)} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{\tilde{P}_\kappa(ML^{-\frac{1}{2}}\ T^{-\frac{1}{2}}\,,\,T)}{k!\ [n]_\kappa\ \tilde{C}_\kappa(I_p)}\ \chi_{[\kappa]}(1)$$

$$\Sigma_3\Sigma_4\Sigma_5\Sigma_6(-1)^{\Sigma\delta_i+\Sigma\alpha_i+\Sigma\delta_i^*+\Sigma\alpha_i^*}\ |(b_{13ij})|\,|(b_{14ij})|$$

$$|(e^{-q^{-1}\tilde{s}_{r+j}}\ \tilde{s}_{r+j}^{\,n+k_{\alpha_i}-\alpha_i}\ ;\ i,j=1,\cdots,t)|$$

$$|(\tilde{s}_{r+j}^{\,p-\alpha_i^*}\ ;\ i,j=1,\cdots,t)|\ ,\quad 0<\tilde{s}_{r+1}<\cdots<\tilde{s}_{r+t}\ ,$$

where

$$(4.2.149)\quad b_{13ij} = q^{n+p+k_{\delta_i}-\delta_i-\delta_j^*+1}\,\Gamma(\tilde{s}_{r+1}q^{-1},\ n+p+k_{\delta_i}-\delta_i-\delta_j^*+1)\ ,$$

$$(i,j=1,\cdots,r)\ ,$$

$$(4.2.150)\quad b_{14ij} = q^{n+p+k_{\beta_i}-\beta_i-\beta_j^*+1}\{\Gamma(n+p+k_{\beta_i}-\beta_i-\beta_j^*+1)$$

$$-\Gamma(\tilde{s}_{r+t}q^{-1},\ n+p+k_{\beta_i}-\beta_i-\beta_j^*+1)\}\ ,$$

$$(i,j=r+t+1,\cdots,p)$$

and the rest of the symbols are defined as in theorem 3.5.1.

Let $\underset{\sim}{D}_S$ have the p.d.f. given in (4.2.66); then

$$(4.2.151)\quad f_{\tilde{s}_{r+1},\cdots,\tilde{s}_{r+t}}(\tilde{s}_{r+1},\cdots,\tilde{s}_{r+t})$$

$$= \frac{\pi^{\frac{1}{2}p(p-1)}}{\tilde{\Gamma}_p(n)\ |\Sigma|^n \prod\limits_{i>j}^{p}(\frac{1}{\tilde{\sigma}_i}-\frac{1}{\tilde{\sigma}_j})}\ \Sigma_3\Sigma_4\Sigma_5\Sigma_6(-1)^{\Sigma\delta_i+\Sigma\alpha_i+\Sigma\delta_i^*+\Sigma\alpha_i^*}$$

$$|(b_{13ij})|\,|(b_{14ij})|\,|(\tilde{s}_{r+j}^{n-p}\ e^{-\frac{1}{\tilde{\sigma}_{\alpha_i}}\tilde{s}_{r+j}}\ ;\ i,j=1,\cdots,t)|$$

$$|(\tilde{s}_{r+j}^{p-\alpha_j^*}\ ;\ i,j=1,\cdots,t)|\ ,\quad 0<\tilde{s}_{r+1}<\cdots<\tilde{s}_{r+t}\ ,$$

where

$$(4.2.152)\quad b_{13ij} = \tilde{\sigma}_{\delta_i}^{n-\delta_j^*+1}\ \Gamma(\frac{\tilde{s}_{r+1}}{\tilde{\sigma}_{\delta_i}}\ ,\ \ n-\delta_j^*+1)\ ,\ (i,j=1,\cdots,t)\ ,$$

$$(4.2.153) \quad b_{14ij} = \tilde{\sigma}_{\beta_i}^{n-\beta_j^*+1} \{\Gamma(n-\beta_j^*+1) - \Gamma(\frac{\tilde{s}_{r+t}}{\tilde{\sigma}_{\beta_i}}, n-\beta_j^*+1)\} ,$$

$$(i,j = r+t+1,\cdots,p)$$

and the rest of the symbols are defined as in theorem 3.5.1.

Let $\underset{\sim}{D}_B$ have the p.d.f. given in (4.2.65); then

$$(4.2.154) \quad f_{\tilde{B}_{r+1},\cdots,\tilde{B}_{r+t}}(\tilde{b}_{r+1},\cdots,\tilde{b}_{r+t})$$

$$= \frac{\pi^{\frac{1}{2}p(p-1)} \operatorname{etr}[-\Omega] \prod_{i=1}^{p} (n-i+1)^{i-1}}{\tilde{\Gamma}_p(n) \prod_{i>j}^{p} (\tilde{\omega}_i - \tilde{\omega}_j)}$$

$$\Sigma_3\Sigma_4\Sigma_5\Sigma_6(-1)^{\Sigma\delta_i+\Sigma\alpha_i+\Sigma\delta_i^*+\Sigma\alpha_i^*} |(b_{13ij})||(b_{14ij})|$$

$$|(e^{-\tilde{b}_{r+j}} \tilde{b}_{r+j}^{n-p} {}_0F_1(n-p+1 ; \tilde{\omega}_{\alpha_i} \tilde{b}_{r+j}) ; \quad i,j = 1,\cdots,t)|$$

$$|(\tilde{b}_{r+j}^{p-\alpha_i^*} ; \quad i,j = 1,\cdots,t)| , \quad 0 < \tilde{b}_{r+1} < \cdots < \tilde{b}_{r+t} ,$$

where

$$(4.2.155) \quad b_{13ij} = \sum_{k=0}^{\infty} \frac{\tilde{\omega}_{\delta_i}^k}{k! \ (n-p+1)_k} \Gamma(\tilde{b}_{r+1} , \ n+k-\delta_j^*+1) ,$$

$$(i,j = 1,\cdots,r) ,$$

$$(4.2.156) \quad b_{14ij} = \sum_{k=0}^{\infty} \frac{\tilde{\omega}_{\beta_i}^k}{k! \ (n-p+1)_k} \{\Gamma(n+k-\beta_j^*+1) - \Gamma(\tilde{b}_{r+t}, \ n+k-\beta_j^*+1)\} ,$$

$$(i,j = r+t+1,\cdots,p)$$

and the rest of the symbols are defined as in theorem 3.5.1.

Proof

(4.2.148), (4.2.149), (4.2.150)

The application of theorem 3.6.1 with $g(\tilde{s}_i)$, $\phi_i(\tilde{s}_j)$ and $\psi_i(\tilde{s}_j)$ given in (4.2.91), (4.2.92) and (4.2.93) respectively leads to (4.2.148) with

$$b_{13ij} = \int_0^{\tilde{s}_{r+1}} e^{-q^{-1}x} \, x^{n+p-k_{\delta_i}-\delta_i-\delta_j^*} \, dx \,, \quad (i,j = 1,\cdots,r)$$

and

$$b_{14ij} = \int_{\tilde{s}_{r+t}}^{\infty} e^{-q^{-1}x} \, x^{n+p-k_{\beta_i}-\beta_i-\beta_j^*} \, dx \,, \quad (i,j = r+t+1,\cdots,p)$$

and hence the theorem is proved.

(4.2.151), (4.2.152), (4.2.153)

The application of theorem 3.6.1 with $g(\tilde{s}_i)$, $\phi_i(\tilde{s}_j)$ and $\psi_i(\tilde{s}_j)$ given in (4.2.97), (4.2.98) and (4.2.99) respectively leads to (4.2.151) with

$$b_{13ij} = \int_0^{\tilde{s}_{r+1}} e^{-\frac{1}{\tilde{\sigma}_{\delta_i}}x} \, x^{n-\delta_j^*} \, dx \,, \quad (i,j = 1,\cdots,r)$$

and

$$b_{14ij} = \int_{\tilde{s}_{r+t}}^{\infty} e^{-\frac{1}{\tilde{\sigma}_{\beta_i}} x} x^{n-\beta_j^*} dx \ , \quad (i,j = r+t+1, \cdots, p)$$

and hence the theorem is proved.

(<u>4.2.154</u>), (<u>4.2.155</u>), (<u>4.2.156</u>)

The application of theorem 3.6.1 with $g(\tilde{s}_i)$, $\phi_i(\tilde{s}_j)$ and $\psi_i(\tilde{s}_j)$ given in (4.2.103), (4.2.104) and (4.2.105) respectively leads to (4.2.154) with

$$b_{13ij} = \int_{0}^{\tilde{b}_{r+1}} e^{-x} x^{n-\delta_j^*} {}_0F_1(n-p+1 \ ; \tilde{\omega}_{\delta_i} x) \, dx \ , \quad (i,j = 1, \cdots, t)$$

and

$$b_{14ij} = \int_{\tilde{b}_{r+t}}^{\infty} e^{-x} x^{n-\beta_j^*} {}_0F_1(n-p+1 \ ; \tilde{\omega}_{\beta_i} x) \, dx \ ,$$

$$(i,j = r+t+1, \cdots, p)$$

and hence the theorem is proved.

In theorem 4.2.7 expressions for the joint p.d.f. of r unordered roots i.e. $f_{\tilde{S}_1,\cdots,\tilde{S}_r}(\tilde{s}_1,\cdots,\tilde{s}_r)$, $(\tilde{s}_i > 0 \ ; i = 1,\cdots,r)$ are considered.

<u>Theorem 4.2.7</u>

Let $D_{\underset{\sim}{S}}$ have the p.d.f. given in (4.2.63); then the joint p.d.f. of any r unordered roots is given by

(4.2.157) $$f_{\tilde{S}_1,\cdots,\tilde{S}_r}(\tilde{s}_1,\cdots,\tilde{s}_r)$$

$$= \frac{\pi^{p(p-1)}\ \mathrm{etr}[-M\bar{M}']}{p!\ |L|^p\ \tilde{\Gamma}_p(n)\ \tilde{\Gamma}_p(p)} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{\tilde{P}_\kappa(ML^{-\frac{1}{2}}\ T^{-\frac{1}{2}}, T)}{k!\ [n]_\kappa\ \tilde{C}_\kappa(I_p)} \chi_{[\kappa]}(1)$$

$$\Sigma_1\Sigma_2(-1)^{\Sigma\delta_i+\Sigma\alpha_i} |(b_{17ij})||(b_{18ij})| ,$$

$$(\tilde{s}_i > 0\ ;\ i = 1,\cdots,r) ,$$

where

(4.2.158) $$b_{17ij} = \sum_{t=1}^{r} e^{-q^{-1}\tilde{s}_t}\ \tilde{s}_t^{\,n+p+k_{\delta_i}-\delta_i-\alpha_j} , \quad (i,j = 1,\cdots,r) ,$$

(4.2.159) $$b_{18ij} = (p-r)!\ q^{\,n+p+k_{\nu_i}-\nu_i-\beta_j+1}\ \Gamma(n+p+k_{\nu_i}-\nu_i-\beta_j+1) ,$$

$$(i,j = 1,\cdots,p-r)$$

and the rest of the symbols are defined as in theorem 3.4.1.

Let $\underset{\sim}{D}_S$ have the p.d.f. given in (4.2.66); then the joint p.d.f. of any r unordered roots is given by

(4.2.160) $$f_{\tilde{S}_1,\cdots,\tilde{S}_r}(\tilde{s}_1,\cdots,\tilde{s}_r)$$

$$= \frac{\pi^{\frac{1}{2}p(p-1)}}{p!\ \tilde{\Gamma}_p(n)\ |\Sigma|^n} \frac{\Sigma_1\Sigma_2(-1)^{\Sigma\delta_i+\Sigma\alpha_i}}{\prod_{i>j}^{p}\left(\frac{1}{\tilde{\sigma}_i}-\frac{1}{\tilde{\sigma}_j}\right)} |(b_{17ij})||(b_{18ij})| ,$$

$$(\tilde{s}_i > 0\ ;\ i = 1,\cdots,r) ,$$

where

$$(4.2.161) \quad b_{17ij} = \sum_{t=1}^{r} \tilde{s}_t^{\,n-\alpha_j} e^{-\frac{1}{\tilde{\sigma}_{\delta_i}} \tilde{s}_t} , \quad (i,j = 1,\cdots,r) ,$$

$$(4.2.162) \quad b_{18ij} = (p-r)!\, \tilde{\sigma}_{\nu_i}^{\,n-\beta_j-1} \Gamma(n-\beta_j+1) , \quad (i,j = 1,\cdots,p-r)$$

and the rest of the symbols are defined as in theorem 3.4.1.

Let $\underset{\sim}{D}_B$ have the p.d.f. given in (4.2.65); then the joint p.d.f. of any r unordered roots is given by

$$(4.2.163) \qquad f_{\tilde{B}_1,\cdots,\tilde{B}_r}(\tilde{b}_1,\cdots,\tilde{b}_r)$$

$$= \frac{\pi^{\frac{1}{2}p(p-1)} \operatorname{etr}[-\Omega]}{p!\, \tilde{\Gamma}_p(n) \prod_{i>j}^{p} (\tilde{\omega}_i - \tilde{\omega}_j)} \prod_{i=1}^{p} (n-i+1)^{i-1} \Sigma_1 \Sigma_2 (-1)^{\Sigma\delta_i+\Sigma\alpha_i}$$

$$|(b_{17ij})||(b_{18ij})| , \quad (\tilde{b}_i > 0 \; ; \; i = 1,\cdots,r) ,$$

where

$$(4.2.164) \quad b_{17ij} = \sum_{t=1}^{r} e^{-\tilde{b}_t} \tilde{b}_t^{\,n-\alpha_j} \, {}_0F_1(n-p+1 \; ; \; \tilde{\omega}_{\delta_i} \tilde{b}_t) ,$$

$$(i,j = 1,\cdots,r) ,$$

$$(4.2.165) \quad b_{18ij} = (p-r)! \sum_{k=0}^{\infty} \frac{\tilde{\omega}_{\nu_i}^k}{k!\, (n-p+1)_k} \Gamma(n+k-\beta_j+1) ,$$

$$(i,j = 1,\cdots,p-r)$$

and the rest of the symbols are defined as in theorem 3.4.1.

Proof

(4.2.157), (4.2.158), (4.2.159)

The application of theorem 3.7.1 with $g(\tilde{s}_i)$, $\phi_i(\tilde{s}_j)$ and $\psi_i(\tilde{s}_j)$ given in (4.2.91), (4.2.92) and (4.2.93) respectively leads to (4.2.157) with

$$b_{17ij} = \sum_{t=1}^{r} e^{-q^{-1}\tilde{s}_t} \tilde{s}_t^{\,n+p+k_{\delta_i}-\delta_i-\alpha_j} , \quad (i,j = 1,\cdots,r)$$

and

$$b_{18ij} = (p-r)! \int_0^\infty e^{-q^{-1}x} x^{n+p+k_{\nu_i}-\nu_i-\beta_j} dx , \quad (i,j = 1,\cdots,p-r)$$

and hence the theorem is proved.

(4.2.160), (4.2.161), (4.2.162)

The application of theorem 3.7.1 with $g(\tilde{s}_i)$, $\phi_i(\tilde{s}_j)$ and $\psi_i(\tilde{s}_j)$ given in (4.2.97), (4.2.98) and (4.2.99) respectively leads to (4.2.160) with

$$b_{17ij} = \sum_{t=1}^{r} \tilde{s}_t^{\,n-\alpha_j} e^{-\frac{1}{\tilde{\sigma}_{\delta_i}}\tilde{s}_t} , \quad (i,j = 1,\cdots,r)$$

and

$$b_{18ij} = (p-r)! \int_0^\infty e^{-\frac{1}{\tilde{\sigma}_{\nu_i}}x} x^{n-\beta_j+1} dx , \quad (i,j = 1,\cdots,p-r)$$

and hence the theorem is proved.

(4.2.163), (4.2.164), (4.2.165)

The application of theorem 3.7.1 with $g(\tilde{s}_i)$, $\phi_i(\tilde{s}_j)$ and $\psi_i(\tilde{s}_j)$ given in (4.2.103), (4.2.104) and (4.2.105) respectively leads to (4.2.163) with

$$b_{17ij} = \sum_{t=1}^{r} e^{-\tilde{b}_t} \tilde{b}_t^{n-\alpha_j} {}_0F_1(n-p+1 \; ; \tilde{\omega}_{\delta_i} b_t) , \quad (i,j = 1,\cdots,r)$$

and

$$b_{18ij} = (p-r)! \int_0^\infty e^{-x} x^{n-\beta_j} {}_0F_1(n-p+1 \; ; \tilde{\omega}_{\nu_i} x) \, dx , \quad (i,j = 1,\cdots,p-r)$$

and hence the theorem is proved.

Remark 4.2.5

For $\underset{\sim}{S}:p\times p \sim NCCW(p, n, \Sigma, \Omega)$, the joint p.d.f. of r unordered characteristic roots of $\underset{\sim}{B} = \Sigma^{-\frac{1}{2}} \underset{\sim}{S} \Sigma^{-\frac{1}{2}}$ is also given by Waikar, Chang and Krishnaiah (1972) as

$$f_{\tilde{B}_1,\cdots,\tilde{B}_r}(\tilde{b}_1,\cdots,\tilde{b}_r)$$

$$= \frac{\pi^{p(p-1)} \operatorname{etr}[-\Omega]}{\tilde{\Gamma}_p(n) \tilde{\Gamma}_p(p)} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{\tilde{C}_\kappa(\Omega)}{[n]_\kappa k! \tilde{C}_\kappa(I_p)}$$

$$\chi_{[\kappa]}(1) \Sigma_1 \Sigma_2 (-1)^{f(\alpha_1,\cdots,\alpha_p)} (-1)^{f(\delta_1,\cdots,\delta_p)}$$

$$\Sigma_3 \prod_{i=1}^{r} (\tilde{b}_{t_{\delta_i}})^{k_{\alpha_i}+2p-\alpha_i-\delta_i} e^{-\tilde{b}_i} \tilde{b}_i^{n-p}$$

$$(p-r)! \prod_{i=r+1}^{p} \Gamma(n+p+k_{\alpha_i}-\alpha_i-\delta_i+1)$$

where

Σ_1 denotes the summation over the combination $(\delta_1 < \cdots < \delta_r)$ and $(\delta_{r+1} < \cdots < \delta_p)$,

Σ_2 denotes the summation over all permutations $(\alpha_1,\cdots,\alpha_p)$ of $(1,\cdots,p)$ and

Σ_3 denotes the summation over all permutations $(t_{\delta_1},\cdots,t_{\delta_r})$ of $(1,\cdots,r)$.

It is clear that the expression for the joint p.d.f. of r unordered roots given in (4.2.163) is in a much simpler form than the expression given above.

4.2.3 P.d.f.s and c.d.f.s of functions of the characteristic roots of $\underset{\sim}{S}:p\times p$

In theorem 4.2.8 the p.d.f.s of $\operatorname{tr}\underset{\sim}{S}$ and $\operatorname{tr}\underset{\sim}{B}$ for different specifications of the parameter matrices are considered. Only the Γ-type representation of the p.d.f.s of $\underset{\sim}{D}_S$ and $\underset{\sim}{D}_B$ is used in the derivation of the p.d.f.s of $\operatorname{tr}\underset{\sim}{S}$ and $\operatorname{tr}\underset{\sim}{B}$ because it leads to p.d.f.s which are convergent for $\operatorname{tr}\underset{\sim}{S} > 0$ and $\operatorname{tr}\underset{\sim}{B} > 0$, while the power-series representation leads to p.d.f.s of $\operatorname{tr}\underset{\sim}{S}$ and $\operatorname{tr}\underset{\sim}{B}$ which are only convergent for $\operatorname{tr}\underset{\sim}{S} < 1$ and $\operatorname{tr}\underset{\sim}{B} < 1$.

The following lemma will be used in theorem 4.2.8:

Lemma 4.2.2

$$(4.2.166)\qquad \int_\Lambda \cdots \int \prod_{i=1}^{p} a_i^{n-p}\, \tilde{C}_\kappa(D_a) \prod_{i>j} (a_i - a_j)^2 \, da_1 \cdots da_{p-1}$$

$$= \frac{\tilde{\Gamma}_p(n)\ \tilde{\Gamma}_p(p)\ [n]_\kappa}{\pi^{p(p-1)}\ \Gamma(pn)\ (pn)_k}\ \tilde{C}_\kappa(I_p)$$

where

$$\Lambda = \{0 < a_1 < \cdots < a_{p-1} < a_p = 1-a_1 - \cdots - a_{p-1}\}$$

and

$$D_a = \begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_p \end{bmatrix} = \mathrm{Diag}(a_1,\cdots,a_p) \ .$$

Proof

The p.d.f. of $\mathrm{tr}\,\underset{\sim}{B} = \mathrm{tr}\,D_{\underset{\sim}{B}}$ when $D_{\underset{\sim}{B}}$ has the p.d.f. given in (4.2.19), is derived by Hayakawa (1972 a, p. 10) as

(4.2.167) $\quad f_{\mathrm{tr}\,\underset{\sim}{B}}(\mathrm{tr}\,B)$

$$= \frac{(\mathrm{tr}\,B)^{np-1}\,\mathrm{etr}[-\Omega]}{\Gamma(np)} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(\mathrm{tr}\,B)^k\ \tilde{H}_\kappa(\Sigma^{-\frac{1}{2}} M)}{k!\ (np)_k} \ .$$

Hayakawa (1972 a) derived (4.2.167) by using the inverse formula of the Laplace transform $g(t) = E(\mathrm{etr}[-tD_{\underset{\sim}{B}}])$; however, it is also possible to derive (4.2.167) by making the following transformation in (4.2.19):

(4.2.168) $\quad a_i = \dfrac{\tilde{b}_i}{\mathrm{tr}\,B}\ , \quad (i = 1,\cdots,p-1)$

with inverse transformation

(4.2.169) $\quad \tilde{b}_i = a_i\,\mathrm{tr}\,B\ , \quad (i = 1,\cdots,p-1)$

and

$$(4.2.170)\quad \tilde{b}_p = \operatorname{tr} B - \sum_{i=1}^{p-1} \tilde{b}_i$$

$$= \operatorname{tr} B\,(1 - \sum_{i=1}^{p-1} a_i)$$

$$= a_p \operatorname{tr} B \,.$$

The jacobian of (4.2.169) and (4.2.170) follows as

$$(4.2.171)\quad J(\tilde{b}_1,\cdots,\tilde{b}_p \to a_1,\cdots,a_{p-1}, \operatorname{tr} B) = (\operatorname{tr} B)^{p-1} .$$

After the transformation (4.2.168) is made in (4.2.19) it follows that

$$(4.2.172)\quad |D_B|^{n-p} = (\operatorname{tr} B)^{(n-p)p} \prod_{i=1}^{p} a_i^{n-p} ,$$

$$(4.2.173)\quad \prod_{i>j}^{p} (\tilde{b}_i - \tilde{b}_j)^2 = (\operatorname{tr} B)^{p(p-1)} \prod_{i>j}^{p} (a_i - a_j)^2$$

and

$$(4.2.174)\quad \tilde{C}_\kappa(D_B) = (\operatorname{tr} B)^k \; \tilde{C}_\kappa(D_a)$$

so that

$$(4.2.175)\quad f_{\operatorname{tr} \underset{\sim}{B}}(\operatorname{tr} B)$$

$$= \frac{\pi^{p(p-1)} \operatorname{etr}[-\Omega]}{\tilde{\Gamma}_p(n)\; \tilde{\Gamma}_p(p)} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{\tilde{H}_\kappa(\Sigma^{-\frac{1}{2}} M)}{k!\; [n]_\kappa \, \tilde{C}_\kappa(I_p)}$$

$$(\operatorname{tr} B)^{np+k-1} \int_{\Lambda} \prod_{i=1}^{p} a_i^{n-p} \, \tilde{C}_\kappa(D_a) \prod_{i>j}^{p} (a_i - a_j)^2 \, da_1 \cdots da_p \,.$$

The equating of the coefficients of $\tilde{H}_\kappa(\Sigma^{-\frac{1}{2}} M)$ in (4.2.167) and (4.2.175) leads to (4.2.166).

Theorem 4.2.8

Let $\underset{\sim}{Z}:p\times n \sim \mathrm{CMNT}(p,\ n,\ M,\ \Phi \otimes \Sigma)$ and let $L:n\times n$ be a hermitian matrix; then the p.d.f.s of $\mathrm{tr}\,\underset{\sim}{S}$ and $\mathrm{tr}\,\underset{\sim}{B}$ where $\underset{\sim}{S}:p\times p = \underset{\sim}{Z}\, L\, \bar{\underset{\sim}{Z}}'$ and $\underset{\sim}{B}:p\times p = \Sigma^{-\frac{1}{2}}\, \underset{\sim}{S}\, \Sigma^{-\frac{1}{2}}$ respectively, are given below for certain specifications of $M:p\times n$, $\Phi:n\times n$, $\Sigma:p\times p$ and $L:n\times n$.

(i) $M:p\times n \neq 0$

(4.2.176) $$f_{\mathrm{tr}\,\underset{\sim}{B}}(\mathrm{tr}\, B)$$

$$= \frac{\mathrm{etr}[-\Sigma^{-1} M\, \Phi^{-1} \bar{M}']\; e^{-q^{-1}\,\mathrm{tr}\, B}}{\Gamma(np)\ \ |L\,\Phi|^{p}}$$

$$\sum_{k=0}^{\infty} \sum_{\kappa} \frac{\tilde{P}_\kappa(\Sigma^{-\frac{1}{2}} M\, \Phi^{-1}\, L^{-\frac{1}{2}}\, T^{-\frac{1}{2}},\ T)}{k!\ \ (np)_k}\ (\mathrm{tr}\, B)^{np+k-1}\ ,$$

$\mathrm{tr}\, B > 0$, $T:n\times n = C^{-1} - q^{-1}\, I_n$ and $\|C\| < q$.

(ii) $M:p\times n = 0$

(4.2.177) $$f_{\mathrm{tr}\,\underset{\sim}{B}}(\mathrm{tr}\, B)$$

$$= \frac{e^{-\mathrm{tr}\, B}}{\Gamma(np)\ \ |L\,\Phi|^{p}} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{\tilde{C}_\kappa(-T)\ \ [n]_\kappa\ \ (\mathrm{tr}\, B)^{np+k-1}}{k!\ \ (np)_k}\ ,$$

$\mathrm{tr}\, B > 0$, $T:n\times n = C^{-1} - I_n$.

(iii) <u>$M:p\times n \neq 0, \quad \Phi:n\times n = I_n, \quad \Sigma:p\times p = I_p$</u> .

(4.2.178) $f_{\mathrm{tr}\,\underset{\sim}{S}}(\mathrm{tr}\,S)$

$$= \frac{\mathrm{etr}[-M\bar{M}']\; e^{-q^{-1}\mathrm{tr}\,S}}{\Gamma(np)\;|L|^p}$$

$$\sum_{k=0}^{\infty}\sum_{\kappa}\frac{\tilde{P}_{\kappa}(ML^{-\frac{1}{2}}T^{-\frac{1}{2}}, T)\;(\mathrm{tr}\,S)^{np+k-1}}{k!\;(np)_k} ,$$

$$\mathrm{tr}\,S > 0 , \quad T:n\times n = L^{-1} - q^{-1} I_n \quad \text{and} \quad \|L\| < q .$$

(iv) <u>$\underset{\sim}{S}:p\times p \sim NCCW(p, n, \Sigma, \Omega)$</u>

(4.2.179) $f_{\mathrm{tr}\,\underset{\sim}{B}}(\mathrm{tr}\,B)$

$$= \frac{\mathrm{etr}[-\Omega]\; e^{-\mathrm{tr}\,B}}{\Gamma(np)}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{\tilde{C}_{\kappa}(\Omega)\;(\mathrm{tr}\,B)^{np+k-1}}{k!\;(np)_k} , \quad \mathrm{tr}\,B > 0 .$$

(v) <u>$\underset{\sim}{S}:p\times p \sim CW(p, n, \Sigma)$</u>

(4.2.180) $f_{\mathrm{tr}\,\underset{\sim}{S}}(\mathrm{tr}\,S)$

$$= \frac{e^{-\mathrm{tr}\,S}}{|\Sigma|^n\;\Gamma(np)}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{\tilde{C}_{\kappa}(I_p - \Sigma^{-1})\;[n]_{\kappa}\;(\mathrm{tr}\,S)^{np+k-1}}{k!\;(np)_k} ,$$

$$\mathrm{tr}\,S > 0 .$$

<u>Proof</u>

These p.d.f.s follow from the Γ-type representation of the p.d.f.s of $\underset{\sim}{D}_S$ and $\underset{\sim}{D}_B$, derived in theorem 4.2.1, by using (4.2.166). Only (4.2.176) will be proved here, the proofs of (4.2.177), (4.2.178), (4.2.179) and (4.2.180) being similar.

In (4.2.4) make the transformation

$$a_i = \frac{\tilde{b}_i}{\operatorname{tr} B} \quad , \quad (i = 1,\cdots,p-1)$$

with inverse transformation

$$\tilde{b}_i = a_i \operatorname{tr} B \ , \quad (i = 1,\cdots,p-1)$$

and

$$\tilde{b}_p = \operatorname{tr} B \,(1 - \sum_{i=1}^{p-1} a_i)$$

$$= a_p \operatorname{tr} B \quad .$$

The jacobian follows as

$$J(\tilde{b}_1,\cdots,\tilde{b}_p \to a_1,\cdots,a_{p-1}, \operatorname{tr} B) = (\operatorname{tr} B)^{p-1} \quad .$$

As in lemma 4.2.2 it follows:

(4.2.181) $\quad f_{\operatorname{tr} \underset{\sim}{B}}(\operatorname{tr} B)$

$$= \frac{\pi^{p(p-1)} \operatorname{etr}[-\Sigma^{-1} M \Phi^{-1} \bar{M}'] \, e^{-q^{-1} \operatorname{tr} B}}{\tilde{\Gamma}_p(n) \ \tilde{\Gamma}_p(p) \ |L \Phi|^p}$$

$$\sum_{k=0}^{\infty} \sum_{\kappa} \frac{\tilde{P}_\kappa(\Sigma^{-\frac{1}{2}} M \Phi^{-1} L^{-\frac{1}{2}} T^{-\frac{1}{2}}, T) \ (\operatorname{tr} B)^{np+k-1}}{k! \ [n]_\kappa \ \tilde{C}_\kappa(I_p)}$$

$$\int_\Lambda \prod_{i=1}^{p} a_i^{n-p} \ \tilde{C}_\kappa(D_a) \prod_{i>j}^{p} (a_i - a_j)^2 \ da_1 \cdots da_{p-1} \ .$$

The p.d.f. of $\operatorname{tr} \underset{\sim}{B}$ follows from (4.2.181) by using (4.2.166).

Remark 4.2.6

Let $\Sigma : p\times p = I_p$ in (4.2.180); then the p.d.f. of $\mathrm{tr}\,\underset{\sim}{S}$ is given by

$$(4.2.182) \quad f_{\mathrm{tr}\,\underset{\sim}{S}}(\mathrm{tr}\,S) = \frac{e^{-\mathrm{tr}\,S}\,(\mathrm{tr}\,S)^{np-1}}{\Gamma(np)} \;, \quad \mathrm{tr}\,S > 0$$

which is the Gamma-p.d.f. with parameters np and 1.

Corollary 4.2.3

Let $\mathrm{tr}\,\underset{\sim}{B}$ have the p.d.f. given in (4.2.176); then

$$(4.2.183) \quad P(\mathrm{tr}\,\underset{\sim}{B} < c)$$

$$= \frac{\mathrm{etr}[-\Sigma^{-1}M\Phi^{-1}\bar{M}']}{\Gamma(np)\;|L\Phi|^{p}} \sum_{k=0}^{\infty}\sum_{\kappa} \frac{\tilde{P}_{\kappa}(\Sigma^{-\frac{1}{2}}M\Phi^{-1}L^{-\frac{1}{2}}T^{-\frac{1}{2}}, T)}{k!\;(np)_k}$$

$$q^{np+k}\,\Gamma(q^{-1}c,\; np+k) \;, \quad c > 0 .$$

Let $\mathrm{tr}\,\underset{\sim}{B}$ have the p.d.f. given in (4.2.177); then

$$(4.2.184) \quad P(\mathrm{tr}\,\underset{\sim}{B} < c)$$

$$= (\Gamma(np)\;|L\Phi|^{p})^{-1} \sum_{k=0}^{\infty}\sum_{\kappa} \frac{\tilde{C}_{\kappa}(-T)}{k!\;(np)_k}\,\Gamma(c,\; np+k) \;, \quad c > 0 .$$

Let $\mathrm{tr}\,\underset{\sim}{S}$ have the p.d.f. given in (4.2.178); then

$$(4.2.185) \quad P(\mathrm{tr}\,\underset{\sim}{S} < c)$$

$$= \frac{\mathrm{etr}[-M\bar{M}']}{\Gamma(np)\;|L|^{p}} \sum_{k=0}^{\infty}\sum_{\kappa} \frac{\tilde{P}_{\kappa}(ML^{-\frac{1}{2}}T^{-\frac{1}{2}}, T)}{k!\;(np)_k}\, q^{np+k}\,\Gamma(q^{-1}c,\; np+k) \;,$$

$$c > 0 .$$

Let $\operatorname{tr} \underset{\sim}{B}$ have the p.d.f. given in (4.2.179); then

(4.2.186) $$P(\operatorname{tr} \underset{\sim}{B} < c)$$

$$= \frac{\operatorname{etr}[-\Omega]}{\Gamma(np)} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{\tilde{C}_\kappa(\Omega)}{k!\ (np)_k} \Gamma(c,\ np+k)\ , \quad c > 0\ .$$

Let $\operatorname{tr} \underset{\sim}{S}$ have the p.d.f. given in (4.2.180); then

(4.2.187) $$P(\operatorname{tr} \underset{\sim}{S} < c)$$

$$= (|\Sigma|\ \Gamma(np))^{-1} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{\tilde{C}_\kappa(I_p - \Sigma^{-1})\ \ [n]_\kappa}{k!\ (np)_k} \Gamma(c,\ np+k)\ ,$$

$$c > 0\ .$$

<u>Proof</u>

The expressions (4.2.183), (4.2.184), (4.2.185), (4.2.186) and (4.2.187) follow from (4.2.176), (4.2.177), (4.2.178), (4.2.179) and (4.2.180) respectively by using:

(4.2.188) $$\int_0^c e^{-q^{-1}x}\ x^{np+k-1}\ dx = q^{np+k}\ \Gamma(q^{-1}c,\ np+k)\ .$$

For (4.2.184), (4.2.186) and (4.2.187) $q = 1$.

In theorem 4.2.9 the p.d.f. of $L_1 = 1 - \dfrac{\tilde{S}_1}{\tilde{S}_p}$ is considered. The p.d.f. of L_1 is derived only when $\underset{\sim}{S}{:}p{\times}p$ has the p.d.f. given in (4.2.30). The derivation of $L_1 = 1 - \dfrac{\tilde{S}_1}{\tilde{S}_p}$ (or $1 - \dfrac{\tilde{B}_1}{\tilde{B}_p}$) when $\underset{\sim}{D}_S$

has the p.d.f. given in (4.2.17) and $\underset{\sim}{D}_B$ has the p.d.f.s given in (4.2.4), (4.2.11) and (4.2.23) being very similar. It is also clear from (4.2.189) that the p.d.f. of $L_1 = 1 - \frac{\tilde{s}_1}{\tilde{s}_p}$ is in a very complicated form and therefore of limited practical value. To find the p.d.f. of L_1 the Γ-type representation of the p.d.f. of $\underset{\sim}{D}_S$ has to be used.

Theorem 4.2.9

Let $\underset{\sim}{D}_S$ have the p.d.f. given in (4.2.30), i.e. $\underset{\sim}{S}: p\times p \sim CW(p,\ n,\ \Sigma)$; then the p.d.f. of $L_1 = 1 - \frac{\tilde{s}_1}{\tilde{s}_p}$ is given by

(4.2.189)
$$f_{L_1}(\ell_1)$$

$$= \frac{\pi^{2(p-1)}\ \tilde{\Gamma}_{p-1}(p+1)\ \{\tilde{\Gamma}_{p-1}(p-1)\}^2}{\tilde{\Gamma}_p(n)\ \tilde{\Gamma}_p(p)\ \tilde{\Gamma}_{p-1}(2p)\ |\Sigma|^n} \sum_{k=0}^{\infty} \sum_{\kappa} \sum_{t=0}^{k} \sum_{\tau} \sum_{b=0}^{t} \sum_{\beta} \sum_{m=0}^{\infty} \sum_{\mu}$$

$$\sum_{r=0}^{\infty} \sum_{\rho} \sum_{\delta} \sum_{\nu} \frac{\tilde{C}_\kappa(I_p - \Sigma^{-1})\ (-1)^b\ \tilde{b}_{\kappa,\tau}\ \tilde{q}_{\tau,\beta}\ \tilde{g}^{\delta}_{\mu,\beta}\ \tilde{g}^{\nu}_{\rho,\delta}\ [p-n]_\rho}{k!\ m!\ r!\ \tilde{C}_\kappa(I_{p-1})\ [2p]_\nu\ \tilde{C}_\beta(I_{p-1})\ p^{np+k+m}}$$

$$[p+1]_\nu\ \Gamma((p-1)(p+1)+m+b+r)\ \Gamma(np+k+m)\ \tilde{C}_\tau(I_{p-1})\ \tilde{C}_\nu(I_{p-1})$$

$$\ell_1^{p^2+m+b+r-2}\ ,\quad 0 < \ell_1 < 1\ ,$$

where $\tilde{g}^{\delta}_{\kappa,\tau}$, $\tilde{q}_{\kappa,\tau}$ and $\tilde{b}_{\kappa,\tau}$ are tabulated in tables 2.2.4, 2.2.6 and 2.2.7 respectively and $\kappa \in P(k,p)$, $\tau \in P(t,\ p-1)$, $\beta \in P(b,\ p-1)$, $\mu \in P(m,\ p-1)$, $\rho \in P(r,\ p-1)$, $\delta \in P(m+b,\ p-1)$ and $\nu \in P(m+b+r,\ p-1)$.

Proof

In (4.2.30) make the transformation

$$(4.2.190)\quad \ell_i = \frac{\tilde{s}_p - \tilde{s}_i}{\tilde{s}_p} , \quad i = 1,\cdots,p-1$$

$$= 1 - \frac{\tilde{s}_i}{\tilde{s}_p} .$$

The inverse transformation follows as

$$(4.2.191)\quad \tilde{s}_i = \tilde{s}_p(1 - \ell_i) , \quad i = 1,\cdots,p-1$$

and

$$(4.2.192)\quad \tilde{s}_p = \tilde{s}_p .$$

The jacobian of the inverse transformation follows:

$$(4.2.193)\quad J(\tilde{s}_1,\cdots,\tilde{s}_p \to \ell_1,\cdots,\ell_{p-1}, \tilde{s}_p) = \tilde{s}_p^{p-1} .$$

After the transformation (4.2.190) is made in (4.2.30) it follows that

$$(4.2.194)\quad |D_S|^{n-p} = \tilde{s}_p^{(n-p)p} |I_{p-1} - D_\ell|^{n-p} ,$$

$$(4.2.195)\quad \mathrm{etr}[-D_S] = e^{-p\tilde{s}_p} \mathrm{etr}[\tilde{s}_p D_\ell] ,$$

$$(4.2.196)\quad \prod_{i>j}^{p} (\tilde{s}_i - \tilde{s}_j)^2 = \tilde{s}_p^{p(p-1)} |D_\ell|^2 \prod_{i>j}^{p-1} (\ell_i - \ell_j)^2$$

and

$$(4.2.197)\quad \tilde{C}_\kappa(D_S) = \tilde{C}_\kappa(\tilde{s}_p(I_{p-1} - D_\ell)^1)$$

$$= \tilde{s}_p^k \sum_{t=0}^{k} \sum_{\tau} \tilde{b}_{\kappa,\tau}\, \tilde{C}_\tau(I_{p-1} - D_\ell)$$

where

$$(4.2.198)\quad D_\ell = \mathrm{Diag}(\ell_1, \cdots, \ell_{p-1})$$

and $\tilde{b}_{\kappa,\tau}$ is tabulated in table 2.2.7.

Hence,

$$(4.2.199)\quad f_{L_1,\cdots,L_{p-1},\, \tilde{S}_p}(\ell_1,\cdots,\ell_{p-1},\, \tilde{s}_p)$$

$$= \frac{\pi^{p(p-1)}\, e^{-p\tilde{s}_p}}{\tilde{\Gamma}_p(n)\, \tilde{\Gamma}_p(p)\, |\Sigma|^n} \sum_{k=0}^{\infty} \sum_{\kappa} \sum_{t=0}^{k} \sum_{\tau} \frac{\tilde{C}_\kappa(I_p - \Sigma^{-1})\, \tilde{s}_p^{np+k-1}}{k!\, \tilde{C}_\kappa(I_p)}$$

$$\tilde{b}_{\kappa,\tau}\, \tilde{C}_\tau(I_{p-1} - D_\ell)\, |I_{p-1} - D_\ell|^{n-p}\, \mathrm{etr}[\tilde{s}_p D_\ell]\, |D_\ell|^2$$

$$\prod_{i>j}^{p-1} (\ell_i - \ell_j)^2\,, \quad 0 < \ell_{p-1} < \cdots < \ell_1 < 1\,, \quad \tilde{s}_p > 0\,.$$

From (2.2.58) follows:

$$(4.2.200)\quad \tilde{C}_\tau(I_{p-1} - D_\ell) = \tilde{C}_\tau(I_{p-1}) \sum_{b=0}^{t} \sum_{\beta} (-1)^b\, \tilde{q}_{\tau,\beta}\, \frac{\tilde{C}_\beta(D_\ell)}{\tilde{C}_\beta(I_{p-1})}$$

where $\beta \in P(b,\, p-1)$ and $\tilde{q}_{\tau,\beta}$ is tabulated in table 2.2.6.

From (2.3.11) follows:

$$(4.2.201) \quad \operatorname{etr}[\tilde{s}_p D_\ell] = \sum_{m=0}^{\infty} \sum_{\mu} \frac{\tilde{s}_p^m \tilde{C}_\mu(D_\ell)}{m!} .$$

From (2.3.12) follows:

$$(4.2.202) \quad |I_{p-1} - D_\ell|^{n-p} = \sum_{r=0}^{\infty} \sum_{\rho} \frac{[p-n]_\rho \tilde{C}_\rho(D_\ell)}{r!} .$$

From (2.2.54) follows:

$$(4.2.203) \quad \tilde{C}_\beta(D_\ell)\, \tilde{C}_\mu(D_\ell)\, \tilde{C}_\rho(D_\ell) = \sum_{\delta} \sum_{\nu} \tilde{g}^{\delta}_{\mu,\beta}\, \tilde{g}^{\nu}_{\rho,\delta}\, \tilde{C}_\nu(D_\ell)$$

where $\delta \in P(m+b,\ p-1)$, $\nu \in P(m+b+r,\ p-1)$ and $\tilde{g}^{\delta}_{\kappa,\tau}$ is tabulated in table 2.2.4.

Thus

$$(4.2.204) \quad f_{L_1,\cdots,L_{p-1},\ \tilde{S}_p}(\ell_1,\cdots,\ell_{p-1},\ \tilde{s}_p)$$

$$= \frac{\pi^{p(p-1)}\, e^{-p\tilde{s}_p}}{\tilde{\Gamma}_p(n)\, \tilde{\Gamma}_p(p)\, |\Sigma|^n} \sum_{k=0}^{\infty} \sum_{\kappa} \sum_{t=0}^{k} \sum_{\tau} \sum_{b=0}^{t} \sum_{\beta} \sum_{m=0}^{\infty} \sum_{\mu} \sum_{r=0}^{\infty} \sum_{\rho} \sum_{\delta} \sum_{\nu}$$

$$\frac{\tilde{C}_\kappa(I_p-\Sigma^{-1})\, \tilde{s}_p^{np+k+m-1}\, (-1)^b\, \tilde{b}_{\kappa,\tau}\, \tilde{q}_{\tau,\beta}\, \tilde{g}^{\delta}_{\mu,\beta}\, \tilde{g}^{\nu}_{\rho,\delta}}{k!\, \tilde{C}_\kappa(I_p)\, m!\, r!\, \tilde{C}_\beta(I_{p-1})}$$

$$[p-n]_\rho\, \tilde{C}_\tau(I_{p-1})\, \tilde{C}_\nu(D_\ell)\, |D_\ell|^2 \prod_{i>j}^{p-1} (\ell_i - \ell_j)^2 .$$

In (4.2.204) make the transformation

$$(4.2.205) \quad a_i = \frac{\ell_i}{\ell_1} , \quad i = 2,\cdots,p-1$$

with inverse transformation

(4.2.206) $\ell_i = \ell_1 a_i , \quad i = 2,\cdots,p-1$

and

(4.2.207) $\ell_1 = \ell_1 .$

The jacobian follows as

(4.2.208) $J(\ell_1, \ell_2,\cdots,\ell_{p-1} \to \ell_1, a_2,\cdots,a_{p-1}) = \ell_1^{p-2} .$

In terms of the new variables follows:

(4.2.209) $|D_\ell|^2 = \ell_1^{2(p-1)} \; |D_a|^2 ,$

(4.2.210) $\tilde{C}_\nu(D_\ell) = \ell_1^{m+b+r} \; \tilde{C}_\nu({}^1D_a)$

and

(4.2.211) $\prod_{i>j}^{p-1} (\ell_i - \ell_j)^2 = \ell_1^{(p-1)(p-2)} \; |I_{p-2} - D_a|^2 \prod_{i>j>1}^{p-1} (a_i - a_j)^2$

where

$$D_a = \mathrm{Diag}(a_2,\cdots,a_{p-1})$$

and

$${}^1D_a = \mathrm{Diag}(1, a_2,\cdots,a_{p-1}) .$$

Hence,

(4.2.212) $$f_{L_1,\, A_2,\cdots,A_{p-1},\, \tilde{S}_p}(\ell_1,\, a_2,\cdots,a_{p-1},\, \tilde{s}_p)$$

$$= \frac{\pi^{p(p-1)}\, e^{-p\tilde{s}_p}}{\tilde{\Gamma}_p(n)\;\tilde{\Gamma}_p(p)\;|\Sigma|^n} \sum_{k=0}^{\infty}\sum_{\kappa}\sum_{t=0}^{k}\sum_{\tau}\sum_{b=0}^{t}\sum_{\beta}\sum_{m=0}^{\infty}\sum_{\mu}\sum_{r=0}^{\infty}\sum_{\rho}\sum_{\delta}\sum_{\nu}$$

$$\frac{\tilde{C}_\kappa(I_p-\Sigma^{-1})\;\tilde{s}_p^{np+k+m-1}\;(-1)^b\;\tilde{b}_{\kappa,\tau}\;\tilde{q}_{\tau,\beta}\;\tilde{g}^{\delta}_{\mu,\beta}\;\tilde{g}^{\nu}_{\rho,\delta}}{k!\;m!\;r!\;\tilde{C}_\kappa(I_p)\;\tilde{C}_\beta(I_{p-1})}$$

$$[p-n]_\rho\;\tilde{C}_\tau(I_{p-1})\;\ell_1^{p^2+m+b+r-2}$$

$$|D_a|^2\;\tilde{C}_\nu({}^1D_a)\;|I_{p-2}-D_a|^2\prod_{i>j>1}^{p-1}(a_i-a_j)^2\;,$$

$$0<\ell_1<1\;,\quad 0<a_{p-1}<\cdots<a_2<1\;,\quad \tilde{s}_p<0\;.$$

Integration over $0<a_{p-1}<\cdots<a_2<1$ leads to the integral:

(4.2.213) $$\int_{0<a_{p-1}<\cdots<a_2<1}\cdots\cdots\cdots\int |D_a|^2\;\tilde{C}_\nu({}^1D_a)\;|I_{p-2}-D_a|^2$$

$$\prod_{i>j>1}^{p-1}(a_i-a_j)^2\;da_2\cdots da_{p-1}$$

$$= \frac{\tilde{\Gamma}_{p-1}((p+1),\;\nu)\;\{\tilde{\Gamma}_{p-1}(p-1)\}^2}{\pi^{(p-1)(p-2)}\;\tilde{\Gamma}_{p-1}(2p,\;\nu)}\;((p-1)(p+1)+m+b+r)$$

$$\tilde{C}_\nu(I_{p-1})\;,\quad \text{(from (2.2.73))}.$$

Thus the joint p.d.f. of L_1 and $\tilde{S}_p$ is given by

(4.2.214) $$f_{L_1, \tilde{s}_p}(\ell_1, \tilde{s}_p)$$

$$= \frac{\pi^{2(p-1)}\ \tilde{\Gamma}_{p-1}(p+1)\ \{\tilde{\Gamma}_{p-1}(p-1)\}^2\ e^{-p\tilde{s}_p}}{\tilde{\Gamma}_p(n)\ \tilde{\Gamma}_p(p)\ \tilde{\Gamma}_{p-1}(2p)\ |\Sigma|^n}$$

$$\sum_{k=0}^{\infty} \sum_{\kappa} \sum_{t=0}^{k} \sum_{\tau} \sum_{b=0}^{t} \sum_{\beta} \sum_{m=0}^{\infty} \sum_{\mu} \sum_{r=0}^{\infty} \sum_{\rho} \sum_{\delta} \sum_{\nu}$$

$$\frac{\tilde{C}_\kappa(I_p-\Sigma^{-1})\ \tilde{s}_p^{np+k+m-1}\ (-1)^b\ \tilde{b}_{\kappa,\tau}\ \tilde{q}_{\tau,\beta}\ \tilde{g}^{\delta}_{\mu,\beta}\ \tilde{g}^{\nu}_{\rho,\delta}}{k!\ m!\ r!\ \tilde{C}_\kappa(I_{p-1})\ [2p]_\nu\ \tilde{C}_\beta(I_{p-1})}$$

$$[p-n]_\rho\ [p+1]_\nu\ \tilde{C}_\tau(I_{p-r})\ \tilde{C}_\nu(I_{p-1})$$

$$((p-1)(p+1)+m+b+r)\ \ell_1^{p^2+m+b+r-2}\ , \quad 0 < \ell_1 < 1\ , \quad \tilde{s}_p > 0\ .$$

Integration w.r.t. $\tilde{s}_p$ leads to the integral

(4.2.215) $$\int_0^\infty e^{-p\tilde{s}_p}\ \tilde{s}_p^{np+k+m-1}\ d\tilde{s}_p = \frac{\tilde{\Gamma}(np+k+m)}{p^{np+k+m}}\ .$$

Substitution of (4.2.215) into (4.2.214) leads to (4.2.189).

<u>Corollary 4.2.4</u>

Let $L_1 = 1 - \dfrac{\tilde{s}_1}{\tilde{s}_p}$ have the p.d.f. given in (4.2.189); then the c.d.f. of L_1, i.e. $P(L_1 < c)$ follows clearly by using

(4.2.216) $$\int_0^c \ell_1^{p^2+m+b+r-2}\ d\ell_1 = \frac{c^{p^2+m+b+r-1}}{p^2+m+b+r-1}\ .$$

4.3 THE COMPOUND QUADRATIC FORM $\underset{\sim}{S}:p\times p = \underset{\sim}{Z}\,\underset{\sim}{L}\,\underset{\sim}{\bar{Z}}'$ WHERE $\underset{\sim}{Z}:p\times n \sim CMTN(p,\ n,\ 0,\ \Phi\otimes\Sigma)$ AND $\underset{\sim}{L}:n\times n \sim CW(n,\ m,\ \Psi)$

4.3.1 The p.d.f. and moments of $\underset{\sim}{S}$ and $D_{\underset{\sim}{S}}$

The compound p.d.f. of $\underset{\sim}{S}:p\times p = \underset{\sim}{Z}\,\underset{\sim}{L}\,\underset{\sim}{\bar{Z}}'$ is given by

$$(4.3.1)\qquad f^{c}_{\underset{\sim}{S}}(S) = \int_{L=\bar{L}'>0} f_{\underset{\sim}{S}|\underset{\sim}{L}}(S|L)\ g_{\underset{\sim}{L}|\Psi}(L|\Psi)\ dL\ .$$

In theorem 4.3.1 the compound p.d.f. and moments of $\underset{\sim}{S}:p\times p$ and $D_{\underset{\sim}{S}}$ are derived. As in theorem 4.2.1 the power-series representation and the Γ-type representation of the compound p.d.f. of $\underset{\sim}{S}:p\times p$ and $D_{\underset{\sim}{S}}$ will be considered. In remark 4.3.1 the relationship between these two representations will be discussed.

Theorem 4.3.1

Let $\underset{\sim}{Z}:p\times n \sim CMTN(p,\ n,\ 0,\ \Phi\otimes\Sigma)$ and let $\underset{\sim}{L}:n\times n \sim CW(n,\ m,\ \Psi)$; then the compound p.d.f. and moments of $\underset{\sim}{S}:p\times p = \underset{\sim}{Z}\,\underset{\sim}{L}\,\underset{\sim}{\bar{Z}}'$, $D_{\underset{\sim}{S}}$ and $D_{\underset{\sim}{B}}$ where $B:p\times p = \Sigma^{-\frac{1}{2}}\,S\,\Sigma^{-\frac{1}{2}}$ are given below:

Power-series representation

$$(4.3.2)\qquad f^{c}_{\underset{\sim}{S}}(S)$$

$$= \frac{|S|^{n-p}}{\tilde{\Gamma}_p(n)\ \tilde{\Gamma}_n(m)\ |\Phi\ \Psi|^{p}\ |\Sigma|^{n}} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{\tilde{C}_{\kappa}(-\Sigma^{-1}S)}{\tilde{C}_{\kappa}(I_n)\ k!}$$

$$\tilde{C}_{\kappa}(\Phi^{-1}\ \Psi^{-1})\ \tilde{\Gamma}_n(m-p\ ;\ -\kappa)\ ,\qquad S=\bar{S}'>0$$

where $\tilde{\Gamma}_p(t,\ -\kappa)$ is given in (2.2.34).

(4.3.3) $f^{c}_{csym}(S)$

$$= \frac{|S|^{n-p}}{\tilde{\Gamma}_p(n)\ \tilde{\Gamma}_n(m)\ |\Phi\ \Psi|^p\ |\Sigma|^n} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{\tilde{C}_\kappa(-\Sigma^{-1})\ \tilde{C}_\kappa(S)}{k!\ \tilde{C}_\kappa(I_n)\ \tilde{C}_\kappa(I_p)}$$

$$\tilde{C}_\kappa(\Phi^{-1}\ \Psi^{-1})\ \tilde{\Gamma}_n(m-p\,;-\kappa)\ , \quad S = \bar{S}' > 0\ .$$

(4.3.4) $f^{c}_{\underset{\sim}{D}_S}(D_S)$

$$= \frac{\pi^{p(p-1)} \prod_{i=1}^{p} \tilde{s}_i^{n-p} \prod_{i>j}^{p} (\tilde{s}_i - \tilde{s}_j)^2}{\tilde{\Gamma}_p(p)\ \tilde{\Gamma}_p(n)\ \tilde{\Gamma}_n(m)\ |\Phi\,\Psi|^p\ |\Sigma|^n} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{\tilde{C}_\kappa(-\Sigma^{-1})\ \tilde{C}_\kappa(D_S)}{\tilde{C}_\kappa(I_n)\ \tilde{C}_\kappa(I_p)\ k!}$$

$$\tilde{C}_\kappa(\Phi^{-1}\ \Psi^{-1})\ \tilde{\Gamma}_n(m-p\,;-\kappa)\ , \quad 0 < \tilde{s}_1 < \cdots < \tilde{s}_p\ .$$

<u>Γ-type representation</u>

(4.3.5) $f^{c}_{\underset{\sim}{S}}(S)$

$$= \frac{|S|^{n-p}\ \mathrm{etr}[-q^{-1}\ \Sigma^{-1}\ S]}{\tilde{\Gamma}_p(n)\ \tilde{\Gamma}_n(m)\ |\Phi\,\Psi|^p\ |\Sigma|^n} \sum_{k=0}^{\infty} \sum_{\kappa} \sum_{b=0}^{k} \sum_{\beta} \frac{(-1)^b\ \tilde{q}_{\kappa,\beta}}{\tilde{C}_\beta(I_n)\ k!}$$

$$\tilde{\Gamma}_n(m-p\,;-\beta)\ \tilde{C}_\kappa(q^{-1}\ \Sigma^{-1}\ S)\ \tilde{C}_\beta(q\ \Phi^{-1}\ \Psi^{-1})\ ,$$

$$S = \bar{S}' > 0 \quad \text{and} \quad q > 0\ .$$

(4.3.6) $f^{c}_{\underset{\sim}{D}_B}(D_B)$

$$= \frac{\pi^{p(p-1)}\ |D_B|^{n-p}\ \mathrm{etr}[-q^{-1}\ D_B] \prod_{i>j}^{p} (\tilde{b}_i - \tilde{b}_j)^2}{\tilde{\Gamma}_p(n)\ \tilde{\Gamma}_n(m)\ \tilde{\Gamma}_p(p)\ |\Phi\,\Psi|^p}$$

$$\sum_{k=0}^{\infty} \sum_{\kappa} \sum_{b=0}^{k} \sum_{\beta} \frac{(-1)^b \tilde{q}_{\kappa,\beta} \tilde{\Gamma}_n(m-p; -\beta) q^{-k}}{\tilde{C}_\beta(I_n) k!}$$

$$\tilde{C}_\kappa(D_B) \tilde{C}_\beta(q \Phi^{-1} \Psi^{-1}), \quad 0 < \tilde{b}_1 < \cdots < \tilde{b}_p .$$

(4.3.7) $E(|\underset{\sim}{S}|^h)$

$$= \frac{\tilde{\Gamma}_p(n+h) q^{(n+h)p} |\Sigma|^h}{\tilde{\Gamma}_p(n) \tilde{\Gamma}_n(m) |\Phi\Psi|^p} \sum_{k=0}^{\infty} \sum_{\kappa} \sum_{b=0}^{k} \sum_{\beta} \frac{(-1)^b \tilde{q}_{\kappa,\beta}}{\tilde{C}_\beta(I_n) k!}$$

$$\cdot \tilde{\Gamma}_n(m-p, -\beta) [n+h]_\kappa \tilde{C}_\beta(q \Phi \Psi) \tilde{C}_\kappa(I_p) .$$

<u>Proof</u>

<u>(4.3.2)</u>

The compound p.d.f. of $\underset{\sim}{S}:p\times p$ follows from (2.8.25), (4.2.6) and (4.3.1) as

(4.3.8) $$f^c_{\underset{\sim}{S}}(S) = (\tilde{\Gamma}_p(n) \tilde{\Gamma}_n(m) |\Phi|^p |\Sigma|^n |\Psi|^m)^{-1} |S|^{n-p}$$

$$\int_{L=\bar{L}'>0} {}_0\tilde{F}_0(-L^{-1} \Phi^{-1}, \Sigma^{-1} S) |L|^{m-n-p}$$

$$\operatorname{etr}[-\Psi^{-1} L] \, dL .$$

Expand the hypergeometric function; then

(4.3.9) $$f^c_{\underset{\sim}{S}}(S) = (\tilde{\Gamma}_p(n) \tilde{\Gamma}_n(m) |\Phi|^p |\Sigma|^n |\Psi|^m)^{-1} |S|^{n-p}$$

$$\sum_{k=0}^{\infty} \sum_{\kappa} \frac{\tilde{C}_\kappa(-\Sigma^{-1} S)}{k! \tilde{C}_\kappa(I_n)} I^*$$

where

$$(4.3.10) \qquad I^* = \int_{L=\bar{L}'>0} |L|^{m-n-p} \operatorname{etr}[-\Psi^{-1} L]\, \tilde{C}_\kappa(L^{-1}\,\Phi^{-1})\, dL$$

$$= \tilde{\Gamma}_n(m-p,\, -\kappa)\; |\Psi|^{m-p}\, \tilde{C}_\kappa(\Phi^{-1}\,\Psi^{-1}) \;, \quad \text{(from (2.2.33))} .$$

Substitution of (4.3.10) into (4.3.9) leads to (4.3.2).

(4.3.3)

The symmetrised p.d.f. of $\underset{\sim}{S}{:}p{\times}p$ follows from (2.7.1) and (2.2.29).

(4.3.4)

The application of theorem 3.2.1 and corollary 2.7.1 leads to (4.3.4).

(4.3.5)

The compound p.d.f. of $\underset{\sim}{S}{:}p{\times}p$ follows from (2.8.25), (4.2.9) and (4.3.1) as

$$(4.3.11) \qquad f^c_{\underset{\sim}{S}}(S)$$

$$= (\tilde{\Gamma}_p(n)\; \tilde{\Gamma}_n(m)\; |\Phi|^p\; |\Sigma|^n\; |\Psi|^m)^{-1}\; |S|^{n-p}$$

$$\operatorname{etr}[-q^{-1}\,\Sigma^{-1}\,S] \int_{L=\bar{L}'>0} {}_0\tilde{F}_0(q^{-1}\,I_n - L^{-\frac{1}{2}}\,\Phi^{-1}\,L^{-\frac{1}{2}}\,;\; \Sigma^{-1}\,S)$$

$$|L|^{m-n-p} \operatorname{etr}[-\Psi^{-1}\,L]\; dL \;.$$

Expand the hypergeometric function; then

(4.3.12) $\quad f^{c}_{\underset{\sim}{S}}(S)$

$$= (\tilde{\Gamma}_p(n)\ \tilde{\Gamma}_n(m)\ |\Phi|^p\ |\Sigma|^n\ |\Psi|^m)^{-1}\ |S|^{n-p}$$

$$\operatorname{etr}[-q^{-1}\Sigma^{-1}S] \int_{L=\bar{L}'>0} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{\tilde{C}_\kappa(q^{-1}\Sigma^{-1}S)}{k!\ \tilde{C}_\kappa(I_n)}$$

$$\tilde{C}_\kappa(I_n - q\,L^{-\frac{1}{2}}\,\Phi^{-1}\,L^{-\frac{1}{2}})\ |L|^{m+n-p}\ \operatorname{etr}[-\Psi^{-1}L]\ dL\ .$$

From (2.2.58) follows:

(4.3.13) $\quad \tilde{C}_\kappa(I_n - q\,L^{-\frac{1}{2}}\,\Phi^{-1}\,L^{-\frac{1}{2}})$

$$= \tilde{C}_\kappa(I_n) \sum_{b=0}^{k} \sum_{\beta} (-1)^b\ \tilde{q}_{\kappa,\beta}\ \frac{\tilde{C}_\beta(q\,L^{-1}\,\Phi^{-1})}{\tilde{C}_\beta(I_n)}\ .$$

Substitution of (4.3.13) into (4.3.12) leads to

(4.3.14) $\quad f^{c}_{\underset{\sim}{S}}(S)$

$$= (\tilde{\Gamma}_p(n)\ \tilde{\Gamma}_n(m)\ |\Phi|^p\ |\Sigma|^n\ |\Psi|^m)^{-1}\ |S|^{n-p}$$

$$\operatorname{etr}[-q^{-1}\Sigma^{-1}S] \sum_{k=0}^{\infty} \sum_{\kappa} \sum_{b=0}^{k} \sum_{\beta} \frac{(-1)^b\ \tilde{q}_{\kappa,\beta}\ \tilde{C}_\kappa(q^{-1}\Sigma^{-1}S)}{k!\ \tilde{C}_\beta(I_n)}\ I^*$$

where

(4.3.15) $$I^* = \int_{L=\bar{L}'>0} |L|^{m-n-p}\ \operatorname{etr}[-\Psi^{-1}L]\ \tilde{C}_\beta(q\,L^{-1}\,\Phi^{-1})\ dL$$

$$= \tilde{\Gamma}_n(m-p,\ -\beta)\ |\Psi|^{m-p}\ \tilde{C}_\beta(q\,\Psi^{-1}\,\Phi^{-1})\ , \quad \text{(from (2.2.33))}\ .$$

Substitution of (4.3.15) into (4.3.14) leads to (4.3.5).

(4.3.6)

In (4.3.5) make the transformation

$$(4.3.16) \qquad B = \Sigma^{-\frac{1}{2}} S \Sigma^{-\frac{1}{2}}$$

with inverse transformation

$$(4.3.17) \qquad S = \Sigma^{\frac{1}{2}} B \Sigma^{\frac{1}{2}} .$$

The jacobian of (4.3.17) follows from (2.2.6) as

$$(4.3.18) \qquad J(S \to B) = |\Sigma|^{p} .$$

Hence,

$$(4.3.19) \qquad f_{\underset{\sim}{B}}(B)$$

$$= \frac{|B|^{n-p} \operatorname{etr}[-q^{-1} B]}{\tilde{\Gamma}_p(n)\, \tilde{\Gamma}_n(m)\, |\Phi \Psi|^{p}} \sum_{k=0} \sum_{\kappa} \sum_{b=0}^{k} \sum_{\beta} \frac{(-1)^b\, \tilde{q}_{\kappa,\beta}}{\tilde{C}_\beta(I_n)\, k!}$$

$$\tilde{\Gamma}_n(m-p\,;\,-\beta)\, \tilde{C}_\kappa(q^{-1} B)\, \tilde{C}_\beta(q\, \Phi^{-1} \Psi^{-1}) .$$

The application of theorem 3.2.1 and corollary 2.7.1 leads to (4.3.6). It is clear that (4.3.19) also follows if $\Sigma{:}p\times p = I_p$ in (4.3.5).

(4.3.7)

$$(4.3.20) \qquad E(|\underset{\sim}{S}|^{h})$$

$$= (\tilde{\Gamma}_p(n)\, \tilde{\Gamma}_n(m)\, |\Phi \Psi|^{p}\, |\Sigma|^{n})^{-1} \sum_{k=0} \sum_{\kappa} \sum_{b=0} \sum_{\beta} \frac{(-1)^b\, \tilde{q}_{\kappa,\beta}}{k!\, \tilde{C}_\beta(I_n)}$$

$$\tilde{\Gamma}_n(m-p\,;-\beta)\ \tilde{C}_\beta(q\,\Phi^{-1}\,\Psi^{-1})\ I^*$$

where

$$(4.3.21)\qquad I^* = \int\limits_{S=\tilde{S}'>0} |S|^{n+h-p}\ \mathrm{etr}|-q^{-1}\,\Sigma^{-1}\,S|\ \tilde{C}_\kappa(q^{-1}\,\Sigma^{-1}\,S)\ dS$$

$$= \tilde{\Gamma}_p(n+h\,;\,\kappa)\ q^{(n+h)p}\ |\Sigma|^{n+h}\ \tilde{C}_\kappa(I_p)\ ,$$

(from (2.2.32)) .

Substitution of (4.3.21) into (4.3.20) leads to (4.3.7).

Remark 4.3.1

(i) It is clear that the Γ-type representation is more suitable for the derivation of the moments of $\underset{\sim}{S}:p\times p$ while the power-series representation is more suitable for the derivation of $f^c_{csym}(S)$ and $f_{\underset{\sim}{D}_S}(D_S)$. By expanding the exponential function $\mathrm{etr}[-q^{-1}\,\Sigma^{-1}\,S]$ in (4.3.5) it is however possible to find expressions for $f^c_{csym}(S)$ and $f_{\underset{\sim}{D}_S}(D_S)$ which involve seven summation signs. These expressions can not be considered Γ-type representations because they no more contain the term $\mathrm{etr}[-q^{-1}\,\Sigma^{-1}\,S]$.

(ii) Let $r = q^{-1}$; then the Γ-type representation of the p.d.f. of $\underset{\sim}{S}:p\times p$ given in (4.3.5) can be written as

$$(4.3.22)\qquad f^c_{\underset{\sim}{S}}(S)$$

$$= \frac{|S|^{n-p}\ \mathrm{etr}[-r\,\Sigma^{-1}\,S]}{\tilde{\Gamma}_p(n)\ \tilde{\Gamma}_n(m)\ |\Phi\,\Psi|^p\ |\Sigma|^n}\ \sum_{k=0}^{\infty}\ \sum_{\kappa}\ \sum_{b=0}^{k}\ \sum_{\beta}\ \frac{(-1)^b\ \tilde{q}_{\kappa,\beta}}{\tilde{C}_\beta(I_n)\ k!}$$

$$r^{k-b}\ \tilde{\Gamma}_n(m-p\,;-\beta)\ \tilde{C}_\kappa(\Sigma^{-1}\,S)\ \tilde{C}_\beta(\Phi^{-1}\,\Psi^{-1})\ .$$

When $r \to 0$ (4.3.22) exists only when $k = b$ because $k \geqq b$. When $k = b$ it follows from table 2.2.6 that

$$\tilde{q}_{\kappa,\beta} = \begin{cases} 1 & \text{when} \quad \kappa = \beta \\ 0 & \text{when} \quad \kappa \neq \beta \, . \end{cases}$$

Thus when $r \to 0$ (4.3.22) tends to (4.3.2), i.e. the Γ-type representation of the p.d.f. of $\underset{\sim}{S}:p\times p$ tends to the power-series representation of the p.d.f. of $\underset{\sim}{S}:p\times p$.

(iii) The results regarding the real multivariate quadratic form of normal variates which correspond with the results derived in theorem 4.3.1 can be found in Underhill (1973, p. 7.1 - 7.7).

4.3.2 Certain marginal distributions of the characteristic roots of $\underset{\sim}{S}:p\times p$

The random component in the joint p.d.f. of the characteristic roots of $\underset{\sim}{S}:p\times p$ and $\underset{\sim}{B}:p\times p$ has to be written in the form given in (3.2.3) to obtain certain marginal distributions of the roots. In theorem 4.3.2 this convenient form of the joint p.d.f. of the characteristic roots of $\underset{\sim}{S}:p\times p$ and $\underset{\sim}{B}:p\times p$ are given.

<u>Theorem 4.3.2</u>

Let $\underset{\sim}{Z}:p\times n \sim \text{CMTN}(p, n, 0, \Phi \otimes \Sigma)$ and let $\underset{\sim}{L}:n\times n \sim \text{CW}(n, m, \Psi)$; then the compound p.d.f. of $\underset{\sim}{D}_S$ and $\underset{\sim}{D}_B$ where $\underset{\sim}{S}:p\times p = \underset{\sim}{Z}\,\underset{\sim}{L}\,\bar{\underset{\sim}{Z}}'$ and $\underset{\sim}{B}:p\times p = \Sigma^{-\frac{1}{2}}\,\underset{\sim}{S}\,\Sigma^{-\frac{1}{2}}$ respectively, are given below:

<u>Power-series representation</u>

(4.3.23) $\quad f^{C}_{\underset{\sim}{D}_S}(D_S)$

$$= \frac{\pi^{p(p-1)}}{\tilde{\Gamma}_p(p)\;\tilde{\Gamma}_p(n)\;\tilde{\Gamma}_n(m)\;|\Psi\,\Phi|^p\;|\Sigma|^n} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{\tilde{C}_\kappa(-\Sigma^{-1})\;\tilde{C}_\kappa(\Phi^{-1}\;\Psi^{-1})}{\tilde{C}_\kappa(I_p)\;k!\;\tilde{C}_\kappa(I_n)}$$

$$\tilde{\Gamma}_n(m-p\,;\,-\kappa)\ \chi_{[\kappa]}(1)\ \prod_{i=1}^{p}\ \tilde{s}_i^{n-p}$$

$$|(\tilde{s}_j^{k_i+p-i})||(\tilde{s}_j^{p-i})|\ ,\quad 0<\tilde{s}_1<\cdots<\tilde{s}_p\ .$$

<u>Γ-type representation</u>

(4.3.24) $$f^C_{\underset{\sim}{D}_B}(D_B)$$

$$=\frac{\pi^{p(p-1)}}{\tilde{\Gamma}_p(n)\ \tilde{\Gamma}_p(p)\ \tilde{\Gamma}_n(m)\ \ |\Phi\,\Psi|^p}\ \sum_{k=0}^{\infty}\ \sum_{\kappa}\ \sum_{b=0}^{k}\ \sum_{\beta}\ \frac{(-1)^b\ \tilde{q}_{\kappa,\beta}}{\tilde{C}_\beta(I_n)\ k!}$$

$$\tilde{\Gamma}_n(m-p\,;-\beta)\ q^{-k}\ \tilde{C}_\beta(q\,\Phi^{-1}\,\Psi^{-1})\ \chi_{[\kappa]}(1)\ \prod_{i=1}^{p}\ e^{-\tilde{b}_i}\ \tilde{b}_i^{n-p}$$

$$|(\tilde{b}_j^{k_i+p-i})||(\tilde{b}_j^{p-i})|\ ,\quad 0<\tilde{b}_1<\cdots<\tilde{b}_p\ .$$

<u>Proof</u>

These two p.d.f.s follow from (4.3.2) and (4.3.6) respectively by using the result:

$$\tilde{C}_\kappa(S)\ \prod_{i>j}^{p}(\tilde{s}_i-\tilde{s}_j)^2=\chi_{[\kappa]}(1)\ \ |(\tilde{s}_j^{k_i+p-1})||(\tilde{s}_j^{p-i})|$$

for $S{:}p\times p$ h.p.d., which is proved in theorem 2.2.8.

It is clear that the marginal distributions of the characteristic roots of $\underset{\sim}{S}{:}p\times p$ and $\underset{\sim}{B}{:}p\times p$, when $\underset{\sim}{D}_S$ and $\underset{\sim}{D}_B$ have the compound p.d.f.s given in (4.3.23) and (4.3.24) respectively, can be obtained along the same lines as in the cases where $\underset{\sim}{D}_S$ has the p.d.f.s given in (4.2.60) and (4.2.63). Thus from the theorems in section 4.2.2 it follows that:

(i) $P(c<\tilde{S}_1<\tilde{S}_p<d)$ and $P(\tilde{S}_p<d)$ can be obtained when

$\underset{\sim}{D}_S$ has the compound p.d.f. given in (4.3.23),

(ii) $P(\tilde{S}_1 < c)$, $P(0 < \tilde{S}_1 < \cdots < \tilde{S}_r < c < \tilde{S}_{r+1} < \cdots < \tilde{S}_p)$, $P(0 < \tilde{S}_1 < \cdots < \tilde{S}_r < c < \tilde{S}_{r+1} < \cdots < \tilde{S}_{r+t} < d < \tilde{S}_{r+t+1} < \cdots \cdots < \tilde{S}_p)$, the joint p.d.f. of any few ordered characteristic roots and the joint p.d.f. of any few unordered characteristic roots can not be obtained when $\underset{\sim}{D}_S$ has the compound p.d.f. given in (4.3.23) because the derivation of these marginal distributions leads to improper integrals,

(iii) all the marginal distributions mentioned in (i) and (ii) can be obtained when $\underset{\sim}{D}_S$ has the compound p.d.f. given in (4.3.24).

4.3.3 The p.d.f. and the c.d.f. of $\mathrm{tr}(\Sigma^{-\frac{1}{2}} \underset{\sim}{S} \Sigma^{-\frac{1}{2}})$

In theorem 4.3.3 the p.d.f. of $\mathrm{tr}(\Sigma^{-\frac{1}{2}} \underset{\sim}{S} \Sigma^{-\frac{1}{2}})$ when $\underset{\sim}{S}{:}p\times p$ has the p.d.f. given in (4.3.5) is derived.

Theorem 4.3.3

Let $\underset{\sim}{Z}{:}p\times p \sim \mathrm{CMTN}(p,\ n,\ 0,\ \Phi \otimes \Sigma)$ and let $\underset{\sim}{L}{:}n\times n \sim \mathrm{CW}(n,\ m,\ \Psi)$; then the p.d.f. of $\mathrm{tr}\,\underset{\sim}{B}$ where $\underset{\sim}{S}{:}p\times p = \underset{\sim}{Z}\,\underset{\sim}{L}\,\bar{\underset{\sim}{Z}}'$ and $\underset{\sim}{B} = \Sigma^{-\frac{1}{2}} \underset{\sim}{S}\, \Sigma^{-\frac{1}{2}}$ is given by

(4.3.25) $$f_{\mathrm{tr}\,\underset{\sim}{B}}(\mathrm{tr}\,B)$$

$$= \frac{e^{-\mathrm{tr}\,B}}{|L\,\Phi|^{p}\ \tilde{\Gamma}_n(m)} \sum_{k=0}^{\infty} \sum_{\kappa} \sum_{b=0}^{k} \sum_{\beta} \frac{(-1)^b\ \tilde{q}_{\kappa,\beta}}{\tilde{C}_\beta(I_n)\ k!\ (np)_k}$$

$$\tilde{\Gamma}_n(m-p\,;-\beta)\ \tilde{C}_\beta(\Phi^{-1}\Psi^{-1})\ \tilde{C}_\kappa(I_p)\ [n]_\kappa\ (\mathrm{tr}\,B)^{np+k-1}\,,$$

$$\mathrm{tr}\,B > 0\,.$$

Proof

The proof of (4.3.25) is similar to the proof of (4.2.176).

Corollary 4.3.1

(4.3.26) $P(\operatorname{tr} \underset{\sim}{B} < c)$

$$= (|L\Phi|^p \, \tilde{\Gamma}_n(m))^{-1} \sum_{k=0}^{\infty} \sum_{\kappa} \sum_{b=0}^{k} \sum_{\beta} \frac{(-1)^b \, \tilde{q}_{\kappa,\beta}}{\tilde{C}_\beta(I_n) \; k! \; (np)_k}$$

$$\tilde{\Gamma}_n(m-p\,;\,-\beta) \; \tilde{C}_\beta(\Phi^{-1}\Psi^{-1}) \; \tilde{C}_\kappa(I_p) \; [n]_\kappa \; \Gamma(c,\, np+k) \quad , \qquad c > 0 \; .$$

Proof

The result follows from (4.3.25) by using

$$\int_0^c e^{-x} \, x^{np+k-1} \, dx = \Gamma(c,\, np+k) \; .$$

CHAPTER 5

MULTIVARIATE COMPLEX BETA DISTRIBUTIONS

5.1 INTRODUCTION

As this is a study of complex quadratic forms and since a beta matrix is a certain type of quadratic form it is felt that it will be useful to give attention to the complex multivariate beta distributions. In chapters 6 and 7 extensions of the complex multivariate beta distributions are considered.

The p.d.f.s, the joint p.d.f.s of the characteristic roots and the moments of the complex multivariate beta type 1B, 2A and 2B matrices are given. Most of these results can be found in the literature and will be given without proofs. The results given in theorems 2.2.8, 2.3.6 and 2.3.7 along with the integrals discussed in chapter 3 are used to derive certain marginal distributions of the characteristic roots of the complex beta matrices. The relationship between the different beta matrices and also between the roots of the different beta matrices are discussed. Little attention is given to the p.d.f.s of functions of the characteristic roots of the beta matrices.

The generalised sample multiple coherence matrix has a complex multivariate beta distribution and therefore the p.d.f. of this matrix as well as certain marginal distributions of its roots are considered.

5.2 THE MULTIVARIATE COMPLEX BETA TYPE 1 DISTRIBUTION

5.2.1 The p.d.f., moments and the joint p.d.f. of the characteristic roots of the complex beta type 1 matrix

Let $\underset{\sim}{A}:p\times p \sim CW(p, m, \Sigma)$, $\underset{\sim}{B}:p\times p \sim NCCW(p, n, \Phi, \Omega)$,

$\underset{\sim}{G}:p\times p = (\underset{\sim}{A}+\underset{\sim}{B})^{-\frac{1}{2}}\underset{\sim}{A}(\underset{\sim}{A}+\underset{\sim}{B})^{-\frac{1}{2}}$ and $\underset{\sim}{L}:p\times p = (\underset{\sim}{A}+\underset{\sim}{B})^{-\frac{1}{2}}\underset{\sim}{B}(\underset{\sim}{A}+\underset{\sim}{B})^{-\frac{1}{2}}$; then $\underset{\sim}{G}:p\times p$ and $\underset{\sim}{L}:p\times p$ have the non-central complex multivariate beta type 1A and 1B distributions respectively. These distributions will be denoted by:

$\underset{\sim}{G}:p\times p \sim NCCMB_{1A}(p, m, n, \Omega)$ and $\underset{\sim}{L}:p\times p \sim NCCMB_{1B}(p, m, n, \Omega)$ when $\Sigma:p\times p = \Phi:p\times p$;

$\underset{\sim}{L}:p\times p$ (or $\underset{\sim}{G}:p\times p$) $\sim CMB_1(p, m, n)$ when $\Omega:p\times p = 0$ and $\Sigma:p\times p = \Phi:p\times p$;

$\underset{\sim}{L}:p\times p \sim CMB_1(p, m, n, \Sigma, \Phi)$ when $\Omega:p\times p = 0$ and $\Sigma:p\times p \neq \Phi:p\times p$.

The p.d.f. and moments of $\underset{\sim}{L}:p\times p$ and $D_{\underset{\sim}{L}}$ are given by De Waal (1968, p. 92 - 97), Waikar, Chang and Krishnaiah (1972, p. 87) and Tan (1968, p. 271) for certain specifications of the parameter matrices. These results as well as the symmetrised p.d.f. of $\underset{\sim}{L}:p\times p$ and $D_{\underset{\sim}{L}}$ are given in theorem 5.2.1.

The relation between $L:p\times p$ and $G:p\times p$ is simply $L = I_p - G$ and since the jacobian of this transformation is 1, all the results for $\underset{\sim}{G}:p\times p$ follow simply by making this transformation in the results for $\underset{\sim}{L}:p\times p$. Therefore only the results for $\underset{\sim}{L}:p\times p$ will be considered.

Theorem 5.2.1

Let $\underset{\sim}{A}:p\times p \sim CW(p, m, \Sigma)$ and $\underset{\sim}{B}:p\times p \sim NCCW(p, n, \Phi, \Omega)$, rank $\Omega = p$ and $\underset{\sim}{A}:p\times p$ and $\underset{\sim}{B}:p\times p$ be independently distributed; then the p.d.f., symmetrised p.d.f. and moments of $\underset{\sim}{L}:p\times p = (\underset{\sim}{A}+\underset{\sim}{B})^{-\frac{1}{2}}\underset{\sim}{B}(\underset{\sim}{A}+\underset{\sim}{B})^{-\frac{1}{2}}$ and $D_{\underset{\sim}{L}}$ are given below for certain specifications of $\Sigma:p\times p$, $\Phi:p\times p$ and $\Omega:p\times p$.

(i) $\Sigma:p\times p \neq \Phi:p\times p$, $\Omega:p\times p = 0$, i.e. $\underset{\sim}{L} \sim CMB_1(p, m, n, \Sigma, \Phi)$

(5.2.1) $f_{\underset{\sim}{L}}(L)$

$$= \frac{|L|^{n-p} \; |I_p - L|^{m-p}}{\tilde{\Gamma}_p(m) \; \tilde{\Gamma}_p(n) \; |\Sigma|^m \; |\Phi|^n} \int_{T=\bar{T}'>0} \operatorname{etr}[-\Sigma^{-1} T] \; |T|^{m+n-p}$$

$${}_0\tilde{F}_0(-T^{-\frac{1}{2}}(\Phi^{-1} - \Sigma^{-1})T^{\frac{1}{2}} L) \, dT \; , \quad 0 < L = \bar{L}' < I_p \; .$$

(5.2.2) $f_{csym}(L)$

$$= \frac{\tilde{\Gamma}_p(m+n)}{\tilde{\Gamma}_p(m) \; \tilde{\Gamma}_p(n)} \; |\Sigma\,\Phi^{-1}|^n \; |L|^{n-p} \; |I_p - L|^{m-p}$$

$${}_1\tilde{F}_0(m+n \,; I_p - \Sigma\Phi^{-1}, \; L) \; , \quad 0 < L = \bar{L}' < I_p \; .$$

(5.2.3) $E(|\underset{\sim}{L}|^h)$

$$= \frac{\tilde{\Gamma}_p(m+n) \; \tilde{\Gamma}_p(n+h)}{\tilde{\Gamma}_p(n) \; \tilde{\Gamma}_p(m+n+h)} \; |\Sigma\Phi^{-1}|^n \; {}_2\tilde{F}_1(m+n, \; n+h \,; m+n+h \,; I_p - \Sigma\Phi^{-1}) \; ,$$

$$\|I_p - \Sigma\Phi^{-1}\| < 1 \; .$$

(5.2.4) $E(|I - \underset{\sim}{L}|^h)$

$$= \frac{\tilde{\Gamma}_p(m+n) \; \tilde{\Gamma}_p(m+h)}{\tilde{\Gamma}_p(m) \; \tilde{\Gamma}_p(m+n+h)} \; |\Sigma\Phi^{-1}|^n \; {}_2\tilde{F}_1(n, \; m+n \,; m+n+h \,; I_p - \Sigma\Phi^{-1}) \; ,$$

$$\|I_p - \Sigma\Phi^{-1}\| < 1 \; .$$

(5.2.5) $f_{\underset{\sim}{D}_L}(D_L)$

$$= \frac{\pi^{p(p-1)}\ \tilde{\Gamma}_p(m+n)}{\tilde{\Gamma}_p(m)\ \tilde{\Gamma}_p(n)\ \tilde{\Gamma}_p(p)}\ |\Sigma\Phi^{-1}|^n\ |D_L|^{n-p}\ |I_p - D_L|^{m-p}$$

$$\prod_{i>j}^{p} (\tilde{\ell}_i - \tilde{\ell}_j)^2\ {}_1\tilde{F}_0(m+n\ ;\ I_p - \Sigma\Phi^{-1},\ D_L)\ ,$$

$$0 < \tilde{\ell}_1 < \cdots < \tilde{\ell}_p < 1\ .$$

(ii) <u>$\Sigma{:}p\times p = \Phi{:}p\times p$, i.e. $\underset{\sim}{L} \sim NCCMB_{1B}(p,\ m,\ n,\ \Omega)$</u>

(5.2.6) $f_{\underset{\sim}{L}}(L)$

$$= \frac{\mathrm{etr}[-\Omega]}{\tilde{\Gamma}_p(m)\ \tilde{\Gamma}_p(n)\ |\Sigma|^{m+n}}\ |L|^{n-p}\ |I_p - L|^{m-p}$$

$$\int_{T=\bar{T}'>0} \mathrm{etr}[-\Sigma^{-1}T]\ |T|^{m+n-p}\ {}_0\tilde{F}_1(n\ ;\ \Omega\Sigma^{-1}\ T^{\frac{1}{2}}\ L\ T^{\frac{1}{2}})\ dT\ ,$$

$$0 < L = \bar{L}' < I_p\ .$$

(5.2.7) $f_{csym}(L)$

$$= \frac{\mathrm{etr}[-\Omega]\ \tilde{\Gamma}_p(m+n)}{\tilde{\Gamma}_p(m)\ \tilde{\Gamma}_p(n)}\ |L|^{n-p}\ |I_p - L|^{m-p}\ {}_1\tilde{F}_1(m+n\ ;\ n\ ;\ \Omega,\ L)\ ,$$

$$0 < L = \bar{L}' < I_p\ .$$

(5.2.8) $E(|\underset{\sim}{L}|^h)$

$$= \frac{\tilde{\Gamma}_p(n+h)\ \tilde{\Gamma}_p(m+n)}{\tilde{\Gamma}_p(n)\ \tilde{\Gamma}_p(m+n+h)}\ \mathrm{etr}[-\Omega]\ {}_2\tilde{F}_2(m+n,\ n+h\ ;\ n,\ m+n+h\ ;\ \Omega)\ .$$

(5.2.9) $$E(|I_p - \underset{\sim}{L}|^h)$$

$$= \frac{\tilde{\Gamma}_p(m+h)\ \tilde{\Gamma}_p(m+n)}{\tilde{\Gamma}_p(m)\ \tilde{\Gamma}_p(m+n+h)}\ \text{etr}[-\Omega]\ {}_1\tilde{F}_1(m+n\,;\,m+n+h\,;\,\Omega)\ .$$

(5.2.10) $$f_{\underset{\sim}{D}_L}(D_L)$$

$$= \frac{\pi^{p(p-1)}\ \tilde{\Gamma}_p(n+m)\ \text{etr}[-\Omega]}{\tilde{\Gamma}_p(p)\ \tilde{\Gamma}_p(m)\ \tilde{\Gamma}_p(n)}\ |D_L|^{n-p}\ |I_p - D_L|^{m-p}$$

$$\prod_{i>j}^{p} (\tilde{\ell}_i - \tilde{\ell}_j)^2\ {}_1\tilde{F}_1(m+n\,;\,n\,;\,\Omega,\ D_L)\ , \quad 0 < \tilde{\ell}_1 < \cdots < \tilde{\ell}_p < 1\ .$$

(iii) $\underline{\Sigma{:}p\times p = \Phi{:}p\times p\ , \quad \Omega{:}p\times p = 0 \quad \text{i.e.} \quad \underset{\sim}{L} \sim CMB_1(p,\ m,\ n)}$

(5.2.11) $$f_{\underset{\sim}{L}}(L)$$

$$= \frac{\tilde{\Gamma}_p(n+m)}{\tilde{\Gamma}_p(m)\ \tilde{\Gamma}_p(n)}\ |L|^{n-p}\ |I_p - L|^{m-p}\ , \quad 0 < L = \bar{L}' < I_p\ .$$

<u>Proof</u>

(<u>5.2.1</u>) De Waal (1968, p. 96).

(<u>5.2.2</u>)

(5.2.12) $$f_{csym}(L) = \int_{U(p)} f_{\underset{\sim}{L}}(U\,L\,\bar{U}')\ dU$$

$$= \frac{|L|^{n-p}\ |I_p - L|^{m-p}}{\tilde{\Gamma}_p(m)\ \tilde{\Gamma}_p(n)\ |\Sigma|^m\ |\Phi|^n}\quad I^*$$

where

$$(5.2.13) \quad I^* = \int_{U(p)} \int_{T=\bar{T}'>0} \operatorname{etr}[-\Sigma^{-1} T] \, |T|^{m+n-p}$$

$$_0\tilde{F}_0(-T^{\frac{1}{2}}(\Phi^{-1} - \Sigma^{-1}) \, T^{\frac{1}{2}} \, U \, L \, \bar{U}') \, dT \, dU \, .$$

Change the order of integration, then follows from (2.3.4) that

$$(5.2.14) \quad I^* = \int_{T=\bar{T}'>0} \operatorname{etr}[-\Sigma^{-1} T] \, |T|^{m+n-p} \, {}_0\tilde{F}_0(-T(\Phi^{-1} - \Sigma^{-1}), L) \, dT$$

$$= \tilde{\Gamma}_p(m+n) \, |\Sigma|^{m+n} \, {}_1\tilde{F}_0(m+n \, ; \, I_p - \Sigma\Phi^{-1}, L) \, ,$$

(from (2.3.5)) .

Substitution of (5.2.14) into (5.2.12) leads to (5.2.2).

(5.2.3) De Waal (1968, p. 97).

(5.2.4) De Waal (1968, p. 97).

(5.2.5)

The application of corollary 2.7.1 and theorem 3.2.1 leads to (5.2.5).

(5.2.6) De Waal (1968, p. 93).

(5.2.7)

$$(5.2.15) \quad f_{csym}(L) = \frac{\operatorname{etr}[-\Omega] \, |L|^{n-p} \, |I_p - L|^{m-p}}{\tilde{\Gamma}_p(m) \, \tilde{\Gamma}_p(n) \, |\Sigma|^{m+n}} \, I^*$$

where

$$(5.2.16) \quad I^* = \int_{U(p)} \int_{T=\bar{T}'>0} \operatorname{etr}[-\Sigma^{-1}T]\, |T|^{m+n-p}$$

$$ {}_0\tilde{F}_1(n\,;\,\Omega\Sigma^{-1}T^{\frac{1}{2}}\,U\,L\,\bar{U}'\,T^{\frac{1}{2}})\, dT\, dU\ .$$

Change the order of integration, then follows from (2.3.4) that

$$(5.2.17) \quad I^* = \int_{T=\bar{T}'>0} \operatorname{etr}[-\Sigma^{-1}T]\, |T|^{m+n-p}\ {}_0\tilde{F}_1(n\,;\,\Omega\Sigma^{-1}T\,,\,L)\, dT$$

$$= \tilde{\Gamma}_p(m+n)\ |\Sigma|^{m+n}\ {}_1\tilde{F}_1(m+n\,;\,n\,;\,\Omega,\ L)\ .$$

Substitution of (5.2.17) into (5.2.15) leads to (5.2.7).

(5.2.8) De Waal (1968, p. 94).

(5.2.9) De Waal (1968, p. 93).

(5.2.10)

The application of corollary 2.7.1 and theorem 3.2.1 leads to (5.2.10). The p.d.f. of $\underset{\sim}{D}_L$ is also given by De Waal (1968, p. 100 - 101) and Waikar, Chang and Krishnaiah (1972, p. 87).

(5.2.11)

Let $\Omega{:}p\times p = 0$ in (5.2.6); then

$$(5.2.18) \quad f_{\underset{\sim}{L}}(L) = \frac{|L|^{n-p}\ |I_p - L|^{m-p}}{\tilde{\Gamma}_p(m)\ \tilde{\Gamma}_p(n)\ |\Sigma|^{m+n}}\ I^*$$

where

$$(5.2.19) \quad I^* = \int_{T=\bar{T}'>0} \operatorname{etr}[-\Sigma^{-1} T] \, |T|^{m+n-p} \, dT$$

$$= \tilde{\Gamma}_p(m+n) \, |\Sigma|^{m+n} \, , \quad \text{(the complex Wishart integral)} .$$

Substitution of (5.2.19) into (5.2.18) leads to (5.2.11).

Remark 5.2.1

De Waal (1968, p. 106 - 110) also proved that if rank $\Omega < p$, then the p.d.f. of $\underset{\sim}{L}:p\times p$ can be written as a non-central multivariate beta type 1 p.d.f. multiplied with the product of independent single-variable beta type 1 distributions.

5.2.2 Certain marginal distributions of the characteristic roots of the complex beta type 1 matrix

In theorems 5.2.3 - 5.2.7 certain marginal distributions of the characteristic roots of $\underset{\sim}{L}:p\times p$ are derived by using the theorems in sections 3.3 - 3.7. Therefore the joint p.d.f. of the characteristic roots of $\underset{\sim}{L}:p\times p$ is written in a form such that the random component is in the form given in (3.2.3). In theorem 5.2.2 it is shown how the p.d.f. of $\underset{\sim}{D}_L$, when $\underset{\sim}{L}:p\times p \sim CMB_1(p, m, n, \Sigma, \Phi)$, $\underset{\sim}{L}:p\times p \sim NCCMB_{1B}(p, m, n, \Omega)$ and $\underset{\sim}{L}:p\times p \sim CMB_1(p, m, n)$, can be written in this convenient form by using theorems 2.2.8 and 2.3.6.

Theorem 5.2.2

Let $\underset{\sim}{L}:p\times p \sim CMB_1(p, m, n, \Sigma, \Phi)$; then

$$(5.2.20) \quad f_{\underset{\sim}{D}_L}(D_L)$$

$$= \frac{\pi^{\frac{1}{2}p(p-1)} \, \tilde{\Gamma}_p(m+n) \, |\Psi|^n}{\tilde{\Gamma}_p(m) \, \tilde{\Gamma}_p(n) \prod_{i>j}^{p} (\tilde{\psi}_{p-j+1} - \tilde{\psi}_{p-i+1}) \prod_{i=1}^{p} (m+n-i+1)^{i-1}}$$

$$\prod_{i=1}^{p} \tilde{\ell}_i^{n-p}(1-\tilde{\ell}_i)^{m-p} \, |({}_1F_0(m+n-p+1\,;\,(1-\tilde{\psi}_{p-i+1})\tilde{\ell}_j))|$$

$$|(\tilde{\ell}_j^{p-i})| \;, \quad 0 < \tilde{\ell}_1 < \cdots < \tilde{\ell}_p < 1 \;,$$

where $\Psi{:}p\times p = \Phi^{-1}\Sigma$ with characteristic roots $\tilde{\psi}_1 < \cdots < \tilde{\psi}_p$ so that $1-\tilde{\psi}_p < \cdots < 1-\tilde{\psi}_1$.

Let $\underset{\sim}{L}{:}p\times p \sim NCCMB_{1B}(p,\, m,\, n,\, \Omega)$; then

(5.2.21) $$f_{\underset{\sim}{D}_L}(D_L)$$

$$= \frac{\pi^{\frac{1}{2}p(p-1)}\,\tilde{\Gamma}_p(m+n)\,\mathrm{etr}[-\Omega]\,\prod_{i=1}^{p}(n-i+1)^{i-1}}{\tilde{\Gamma}_p(m)\,\tilde{\Gamma}_p(n)\,\prod_{i>j}^{p}(\tilde{\omega}_i-\tilde{\omega}_j)\,\prod_{i=1}^{p}(m+n-i+1)^{i-1}}$$

$$\prod_{i=1}^{p} \tilde{\ell}_i^{n-p}(1-\tilde{\ell}_i)^{m-p} \, |({}_1F_1(m+n-p+1\,;\,n-p+1\,;\,\tilde{\omega}_i\tilde{\ell}_j))|$$

$$|(\tilde{\ell}_j^{p-i})| \;, \quad 0 < \tilde{\ell}_1 < \cdots < \tilde{\ell}_p < 1 \;.$$

Let $\underset{\sim}{L}{:}p\times p \sim CMB_1(p,\, m,\, n)$; then

(5.2.22) $$f_{\underset{\sim}{D}_L}(D_L)$$

$$= \frac{\pi^{p(p-1)}\,\tilde{\Gamma}_p(n+m)\,\prod_{i=1}^{p}\tilde{\ell}_i^{n-p}(1-\tilde{\ell}_i)^{m-p}}{\tilde{\Gamma}_p(m)\,\tilde{\Gamma}_p(n)\,\tilde{\Gamma}_p(p)}\,|(\tilde{\ell}_j^{p-i})|^2 \;,$$

$$0 < \tilde{\ell}_1 < \cdots < \tilde{\ell}_p < 1 \;.$$

Proof

(5.2.20)

From (2.2.41) follows that (5.2.5) can be written as

(5.2.23) $f_{\underset{\sim}{D}_L}(D_L)$

$$= \frac{\pi^{p(p-1)}\ \tilde{\Gamma}_p(m+n)\ |\Psi|^n}{\tilde{\Gamma}_p(m)\ \tilde{\Gamma}_p(n)\ \tilde{\Gamma}_p(p)} \prod_{i=1}^{p} \tilde{\ell}_i^{n-p}(1-\tilde{\ell}_i)^{m-p}$$

$$|(\tilde{\ell}_j^{p-i})|^2\ {}_1\tilde{F}_0(m+n\ ;\ I_p - \Psi\ ,\ D_L)$$

where $\Psi{:}p\times p = \Phi^{-1}\Sigma$ with characteristic roots $\tilde{\psi}_1 < \cdots < \tilde{\psi}_p$ so that the roots of $I_p - \Psi$ are given as $1-\tilde{\psi}_p < \cdots < 1-\tilde{\psi}_1$.

From (2.2.41), (2.3.14) and (2.3.15) follows that

(5.2.24) ${}_1\tilde{F}_0(m+n\ ;\ I_p - \Psi\ ,\ D_L)$

$$= \frac{\tilde{\Gamma}_p(p)\ \ |({}_1F_0(m+n-p+1\ ;\ (1-\tilde{\psi}_{p-i+1})\tilde{\ell}_j)|}{\pi^{\frac{1}{2}p(p-1)} \prod_{i>j}^{p}(\tilde{\psi}_{p-j+1}-\tilde{\psi}_{p-i+1}) \prod_{i=1}^{p}(m+n-i+1)^{i-1}\,|(\tilde{\ell}_j^{p-i})|} \ .$$

Substitution of (5.2.24) into (5.2.23) leads to (5.2.20).

(5.2.21)

From (2.2.41) follows that (5.2.10) can be written as

(5.2.25) $f_{\underset{\sim}{D}_L}(D_L)$

$$= \frac{\pi^{p(p-1)}\ \tilde{\Gamma}_p(m+n)\ \mathrm{etr}[-\Omega]}{\tilde{\Gamma}_p(p)\ \tilde{\Gamma}_p(m)\ \tilde{\Gamma}_p(n)} \prod_{i=1}^{p} \tilde{\ell}_i^{n-p}(1-\tilde{\ell}_i)^{m-p}$$

$$|(\tilde{\ell}_j^{p-i})|^2\ {}_1\tilde{F}_1(m+n\ ;\ n\ ;\ \Omega\ ,\ D_L)\ .$$

From (2.2.41), (2.3.14) and (2.3.15) follow that

(5.2.26) $\quad {}_1\tilde{F}_1(m+n\,;\,n\,;\,\Omega\,,\,D_L)$

$$= \frac{\tilde{\Gamma}_p(p) \prod_{i=1}^{p} (n-i+1)^{i-1} \;|({}_1F_1(m+n-p+1\,;\,n-p+1\,;\,\tilde{\omega}_i\tilde{\ell}_j))|}{\pi^{\frac{1}{2}p(p-1)} \prod_{i>j}^{p} (\tilde{\omega}_i - \tilde{\omega}_j) \prod_{i=1}^{p} (m+n-i+1)^{i-1} \;|(\tilde{\ell}_j^{p-i})|}$$

Substitution of (5.2.26) into (5.2.25) leads to (5.2.21).

(5.2.22)

Let $\Omega:p\times p = 0$ in (5.2.10); then (5.2.22) follows from (2.2.41).

Remark 5.2.2

By using the result:

$$\tilde{C}_\kappa(D_L) \prod_{i>j}^{p} (\tilde{\ell}_i - \tilde{\ell}_j)^2 = \chi_{[\kappa]}(1) \;|(\tilde{\ell}_j^{k_i+p-1})|\,|(\tilde{\ell}_j^{p-i})| \;,$$

proved in theorem 2.2.8, it follows that the p.d.f. of $\underset{\sim}{D}_L$ when $\underset{\sim}{L}:p\times p \sim CMB_1(p,\, m,\, n,\, \Sigma,\, \Phi)$ can also be written as

(5.2.27) $\quad f_{\underset{\sim}{D}_L}(D_L)$

$$= \frac{\pi^{p(p-1)}\; \tilde{\Gamma}_p(m+n)\; |\Psi|^n}{\tilde{\Gamma}_p(m)\; \tilde{\Gamma}_p(n)\; \tilde{\Gamma}_p(p)} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{[m+n]_\kappa}{k!\; \tilde{C}_\kappa(I_p)}$$

$$\tilde{C}_\kappa(I_p - \Psi)\; \chi_{[\kappa]}(1) \prod_{i=1}^{p} \tilde{\ell}_i^{n-p}(1-\tilde{\ell}_i)^{m-p} \;|(\tilde{\ell}_j^{k_i+p-i})|$$

$$|(\tilde{\ell}^{p-i})| \;, \quad 0 < \tilde{\ell}_1 < \cdots < \tilde{\ell}_p < 1$$

and when $\underset{\sim}{L}:p\times p \sim NCCMB_{1B}(p,\, m,\, n,\, \Omega)$ as

(5.2.28) $f_{\underset{\sim}{D}_L}(D_L)$

$$= \frac{\pi^{p(p-1)}\ \tilde{\Gamma}_p(n+m)\ \text{etr}[-\Omega]}{\tilde{\Gamma}_p(p)\ \tilde{\Gamma}_p(m)\ \tilde{\Gamma}_p(n)} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{[m+n]_\kappa\ \tilde{C}_\kappa(\Omega)}{[n]_\kappa\ \tilde{C}_\kappa(I_p)}$$

$$\chi_{[\kappa]}(1) \prod_{i=1}^{p} \tilde{\ell}_i^{n-p}(1-\tilde{\ell}_i)^{m-p}\ |(\tilde{\ell}_j^{k_i+p-i})||(\tilde{\ell}_j^{p-i})|\ ,$$

$$0 < \tilde{\ell}_1 < \cdots < \tilde{\ell}_p < 1\ .$$

From a computational point of view (5.2.20) and (5.2.21) lead to better results than (5.2.27) and (5.2.28) because the former two expressions do not involve zonal polynomials. Expressions which contain the random component given in (5.2.27) and (5.2.28) will be discussed in chapter 6.

In the theorems that follow marginal distributions of the characteristic roots of $\underset{\sim}{L}:p\times p$ will be derived for $\underset{\sim}{L}:p\times p \sim NCCMB_{1B}(p,\ m,\ n,\ \Omega)$ and $\underset{\sim}{L}:p\times p \sim CMB_{1B}(p,\ m,\ n)$. It is clear that the random component in (5.2.20) is in a similar form as the random component in (5.2.21) and thus the marginal distributions of the roots of $\underset{\sim}{L}:p\times p \sim CMB_{1B}(p,\ m,\ n,\ \Sigma,\ \Phi)$ can be obtained along the same lines as in the case when $\underset{\sim}{L}:p\times p \sim NCCMB_{1B}(p,\ m,\ n,\ \Omega)$.

In theorem 5.2.3 the c.d.f. of the extreme characteristic roots of $\underset{\sim}{L}:p\times p$ will be derived.

<u>Theorem 5.2.3</u>

Let $\underset{\sim}{L}:p\times p \sim NCCMB_{1B}(p,\ m,\ n,\ \Omega)$; then

$$(5.2.29) \qquad P(c < \tilde{L}_1 < \tilde{L}_p < d)$$

$$= \frac{\pi^{\frac{1}{2}p(p-1)}\ \tilde{\Gamma}_p(m+n)\ \mathrm{etr}[-\Omega]\ \prod_{i=1}^{p}(n-i+1)^{i-1}}{\tilde{\Gamma}_p(m)\ \tilde{\Gamma}_p(n)\ \prod_{i>j}^{p}(\tilde{\omega}_i - \tilde{\omega}_j)\ \prod_{i=1}^{p}(m+n-i+1)^{i-1}}\ |(b_{ij})|$$

where

$$(5.2.30) \qquad b_{ij} = \sum_{k=0}^{\infty} \frac{(m+n-p+1)_k\ \tilde{\omega}_i^k}{(n-p+1)_k\ k!} \{B_d(n+k+1-j,\ m-p+1) - B_c(n+k-j+1,\ m-p+1)\}$$

and $B_t(\cdot\,,\cdot)$ is defined in (2.6.3).

Let $\underset{\sim}{L}:p\times p \sim CMB_1(p,\ m,\ n)$; then

$$(5.2.31) \qquad P(c < \tilde{L}_1 < \tilde{L}_p < d) = \frac{\pi^{p(p-1)}\ \tilde{\Gamma}_p(n+m)}{\tilde{\Gamma}_p(m)\ \tilde{\Gamma}_p(n)\ \tilde{\Gamma}_p(p)}\ |(b_{ij})|$$

where

$$(5.2.32) \qquad b_{ij} = B_d(n+p-i-j+1,\ m-p+1) - B_c(n+p-i-j+1,\ m-p+1)\ .$$

<u>Proof</u>

(<u>5.2.29</u>), (<u>5.2.30</u>)

To find (5.2.29) it follows from (5.2.21) that the following integral has to be solved:

$$(5.2.33) \qquad I^* = \int_{c<\tilde{\ell}_1<\cdots<\tilde{\ell}_p<d} \cdots\cdots\cdots\cdots \int \prod_{i=1}^{p} \tilde{\ell}_i^{n-p}(1-\tilde{\ell}_i)^{m-p}$$

$$|({}_1F_1(m+n-p+1\,;\,n-p+1\,;\,\tilde{\omega}_i\,\tilde{\ell}_j))|\,|(\tilde{\ell}_j^{p-i})|\,d\tilde{\ell}_1\cdots d\tilde{\ell}_p\ .$$

Let

(5.2.34) $\quad g(\tilde{\ell}_i) = \tilde{\ell}_i^{n-p}(1-\tilde{\ell}_i)^{m-p}$,

(5.2.35) $\quad \phi_i(\tilde{\ell}_j) = {}_1F_1(m+n-p+1\,;\,n-p+1\,;\,\tilde{\omega}_i\tilde{\ell}_j)$

and

(5.2.36) $\quad \psi_i(\tilde{\ell}_j) = \tilde{\ell}_j^{p-i}$.

From theorem 3.3.1 follows that

(5.2.37) $\quad I^* = |(b_{ij})|$

where

(5.2.38)
$$b_{ij} = \int_c^d x^{n-j}\,(1-x)^{m-p}\;{}_1F_1(m+n-p+1\,;\,n-p+1\,;\,\tilde{\omega}_i x)\,dx$$

$$= \sum_{k=0}^{\infty} \frac{(m+n-p+1)_k\;\tilde{\omega}_i^k}{(n-p+1)_k\;k!} \int_c^d x^{n+k-j}\,(1-x)^{m-p}\,dx$$

$$= \sum_{k=0}^{\infty} \frac{(m+n-p+1)_k\;\tilde{\omega}_i^k}{(n-p+1)_k\;k!} \{B_d(n+k-j+1,\ m-p+1)$$

$$- B_c(n+k-j+1,\ m-p+1)\}$$

which proves (5.2.29).

(5.2.31), (5.2.32)

To find (5.2.31) it follows from (5.2.22) that the following integral has to be solved:

$$(5.2.39) \quad I^* = \int_{c<\tilde{\ell}_1<\cdots<\tilde{\ell}_p<d} \cdots\cdots\cdots \int \prod_{i=1}^{p} \tilde{\ell}_i^{n-p}(1-\tilde{\ell}_i)^{m-p} \, |(\tilde{\ell}_j^{p-i})|^2 \, d\tilde{\ell}_1\cdots d\tilde{\ell}_p \, .$$

Let

$$(5.2.40) \quad g(\tilde{\ell}_i) = \tilde{\ell}_i^{n-p}(1-\tilde{\ell}_i)^{m-p} \, ,$$

$$(5.2.41) \quad \phi_i(\tilde{\ell}_j) = \tilde{\ell}_j^{p-i}$$

and

$$(5.2.42) \quad \psi_i(\tilde{\ell}_j) = \tilde{\ell}_j^{p-i} \, .$$

From theorem 3.3.1 follows that

$$(5.2.43) \quad I^* = |(b_{ij})|$$

where

$$(5.2.44) \quad b_{ij} = \int_c^d x^{n+p-i-j} \, (1-x)^{m-p} \, dx$$

$$= B_d(n+p-i-j+1,\ m-p+1) - B_c(n+p-i-j+1,\ m-p+1)$$

which proves (5.2.31).

The c.d.f.s of $\tilde{L}_p$ and $\tilde{L}_1$, the largest and smallest characteristic roots of $\underset{\sim}{L}:p\times p$, follow now as corollaries of theorem 5.2.3. The expressions for $P(\tilde{L}_p < d)$ and $P(\tilde{L}_1 < c)$ are identical to the corresponding expressions for $P(c < \tilde{L}_1 < \tilde{L}_p < d)$ except for the determinant $|(b_{ij})|$. Therefore only the corresponding expressions for b_{ij} will be given in corollaries 5.2.1 and 5.2.2.

Corollary 5.2.1

Let $\underset{\sim}{L}:p\times p \sim NCCMB_{1B}(p, m, n, \Omega)$; then

$$(5.2.45) \quad P(\tilde{L}_p < d) = (5.2.29)$$

where

$$(5.2.46) \quad b_{ij} = \sum_{k=0}^{\infty} \frac{(m+n-p+1)_k}{(n-p+1)_k \; k!} B_d(n+k+1-j,\ m-p+1) .$$

Let $\underset{\sim}{L}:p\times p \sim CMB_1(p, m, n)$; then

$$(5.2.47) \quad P(\tilde{L}_p < d) = (5.2.31)$$

where

$$(5.2.48) \quad b_{ij} = B_d(n+p-i-j+1) .$$

Proof

By taking $c = 0$ in (5.2.30) and (5.2.31) the expressions (5.2.46) and (5.2.48) follow respectively.

Remark 5.2.3

(i) Let $\underset{\sim}{L}:p\times p \sim NCCMB_{1B}(p, m, n, \Omega)$; then De Waal (1968, p. 101 - 103) derived an expression for the p.d.f. of $\tilde{L}_p$ which involves six summation signs and some zonal polynomials. This p.d.f. of $\tilde{L}_p$ leads to an expression for the c.d.f. of $\tilde{L}_p$ which also involves six summation signs and some zonal polynomials. The expression given in (5.2.45) for the c.d.f. of $\tilde{L}_p$ is from a computational point of view better than the expression given by De Waal (1968).

(ii) Approximate percentage points of the largest characteristic root $\tilde{L}_p$ of $\underset{\sim}{L}:p\times p \sim CMB_1(p, m, n)$ are given by Pillai and Jouris (1972).

(iii) Sugiyama (1972, p. 90) derived the following expression for the c.d.f. of $\tilde{L}_p$ when $\underset{\sim}{L}:p\times p \sim CMB_1(p, m, n)$:

(5.2.49) $$P(\tilde{L}_p < d)$$

$$= \frac{\tilde{\Gamma}_p(m+n)\ \tilde{\Gamma}_p(p)}{\tilde{\Gamma}_p(m)\ \tilde{\Gamma}_p(n+p)}\ d^{np}\ (1-d)^{mp}\ {}_2\tilde{F}_1(m+n,\ p\,;\ n+p,\ d\,I_p)\ .$$

The expression given in (5.2.47) is from a computational point of view better than (5.2.49).

<u>Corollary 5.2.2</u>

Let $\underset{\sim}{L}:p\times p \sim NCCMB_{1B}(p, m, n, \Omega)$; then

(5.2.50) $$P(\tilde{L}_1 < c) = 1 - (5.2.29)$$

where

(5.2.51) $$b_{ij} = \sum_{k=0}^{\infty} \frac{(m+n-p+1)_k\ \tilde{\omega}_i^k}{(n-p+1)_k\ k!}\ \{B(n+k+1-j,\ m-p+1) - B_c(n+k+1-j,\ m-p+1)\}\ .$$

Let $\underset{\sim}{L}:p\times p \sim CMB_1(p,\ m,\ n)$; then

(5.2.52) $\quad P(\tilde{L}_1 < c) \;=\; 1 - (5.2.31)$

where

(5.2.53) $\quad b_{ij} = B(n+p-i-j+1,\ m-p+1) - B_c(n+p-i-j+1,\ m-p+1)$.

Proof

By taking $d = 1$ in (5.2.29) and (5.2.31) the expressions (5.2.50) and (5.2.52) follow respectively.

Remark 5.2.4

Khatri (1964, p. 1810) derived the following expression for the c.d.f. of $\tilde{L}_1$ when $\underset{\sim}{L}:p\times p \sim CMB_1(p,\ m,\ n)$:

(5.2.54) $$P(\tilde{L}_1 < c) \;=\; 1 - \frac{\pi^{p(p-1)}\ \tilde{\Gamma}_p(m+n)}{\tilde{\Gamma}_p(p)\ \tilde{\Gamma}_p(m)\ \tilde{\Gamma}_p(n)}\ |(a_{i+j-2})|$$

where

(5.2.55) $$a_{i+j-2} = \int_0^{1-c} x^{m-p+i+j-2}\ (1-x)^{m-p}\ dx$$

$$= B_{1-c}(m-p+i+j-1,\ m-p+1)\ .$$

It can be shown that

$$|(a_{i+j-2})| \;=\; |(b_{ij})|$$

where b_{ij} is given by (5.2.53) so that (5.2.54) is identical to (5.2.52).

In theorem 5.2.4 expressions for $P(0<\tilde{L}_1<\cdots<\tilde{L}_r<c<\tilde{L}_{r+1}<\cdots<\tilde{L}_p<1)$ will be derived.

Theorem 5.2.4

Let $\underset{\sim}{L}:p\times p \sim NCCMB_{1B}(p,\ m,\ n,\ \Omega)$; then

(5.2.56) $$P(0<\tilde{L}_1<\cdots<\tilde{L}_r<c<\tilde{L}_{r+1}<\cdots<\tilde{L}_p<1)$$

$$= \frac{\pi^{\frac{1}{2}p(p-1)}\ \tilde{\Gamma}_p(m+n)\ \operatorname{etr}[-\Omega]\ \prod_{i=1}^{p}(n-i+1)^{i-1}}{\tilde{\Gamma}_p(m)\ \tilde{\Gamma}_p(n)\ \prod_{i>j}^{p}(\tilde{\omega}_i-\tilde{\omega}_j)\ \prod_{i=1}^{p}(m+n-i+1)^{i-1}}\ \Sigma_1\Sigma_2(-1)^{\Sigma\delta_i+\Sigma\alpha_i}\ |(b_{3ij})|\,|(b_{4ij})|$$

where

(5.2.57) $$b_{3ij} = \sum_{k=0}^{\infty}\frac{(m+n-p+1)_k\ \tilde{\omega}_{\delta_i}^{k}}{(n-p+1)_k\ k!}\ B_c(n+k-\alpha_j+1,\ m-p+1)\ ,\quad (i,j=1,\cdots,r)\ ,$$

(5.2.58) $$b_{4ij} = \sum_{k=0}^{\infty}\frac{(m+n-p+1)_k\ \tilde{\omega}_{\nu_i}^{k}}{(n-p+1)_k\ k!}\ \{B(n+k-\beta_j+1,\ m-p+1) - B_c(n+k-\beta_j+1,\ m-p+1)\}\ ,\quad (i,j=1,\cdots,p-r)$$

and the rest of the symbols are defined as in theorem 3.4.1.

Let $\underset{\sim}{L}:p\times p \sim CMB_1(p,\ m,\ n)$; then

$$(5.2.59)\qquad P(0<\tilde{L}_1<\cdots<\tilde{L}_r<c<\tilde{L}_{r+1}<\cdots<\tilde{L}_p<1)$$

$$= \frac{\pi^{p(p-1)}\ \tilde{\Gamma}_p(n+m)}{\tilde{\Gamma}_p(m)\ \tilde{\Gamma}_p(n)\ \tilde{\Gamma}_p(p)}\ \Sigma_1\Sigma_2(-1)^{\Sigma\delta_i+\Sigma\alpha_i}\ |(b_{3ij})||(b_{4ij})|$$

where

$$(5.2.60)\quad b_{3ij} = B_c(n+p-\delta_i-\alpha_j+1,\ m-p+1)\ ,\quad (i,j=1,\cdots,r)\ ,$$

$$(5.2.61)\quad b_{4ij} = B(n+p-\nu_i-\beta_j+1,\ m-p+1) - B_c(n+p-\nu_i-\beta_j+1,\ m-p+1)\ ,$$
$$(i,j=1,\cdots,p-r)$$

and the rest of the symbols are defined as in theorem 3.4.1.

Proof

(5.2.56), (5.2.57), (5.2.58)

The application of theorem 3.4.1 with $g(\tilde{\ell}_i)$, $\phi_i(\tilde{\ell}_j)$ and $\psi_i(\tilde{\ell}_j)$ given in (5.2.34), (5.2.35) and (5.2.36)′ respectively leads to (5.2.56) with

$$b_{3ij} = \int_0^c x^{n-\alpha_j}\ (1-x)^{m-p}\ {}_1F_1(m+n-p+1\ ;\ n-p+1\ ;\ \tilde{\omega}_{\delta_i}x)\ dx\ ,$$
$$(i,j=1,\cdots,r)$$

and

$$b_{4ij} = \int_c^1 x^{n-\beta_j}\ (1-x)^{m-p}\ {}_1F_1(m+n-p+1\ ;\ n-p+1\ ;\ \tilde{\omega}_{\nu_i}x)\ dx\ ,$$
$$(i,j=1,\cdots,p-r)$$

and hence the theorem is proved.

(5.2.59), (5.2.60), (5.2.61)

The application of theorem 3.4.1 with $g(\tilde{\ell}_i)$, $\phi_i(\tilde{\ell}_j)$ and $\psi_i(\tilde{\ell}_j)$ given in (5.2.40), (5.2.41) and (5.2.42) respectively leads to (5.2.59) with

$$b_{3ij} = \int_0^c x^{n+p-\delta_i-\alpha_j} (1-x)^{m-p} \, dx , \quad (i,j = 1,\cdots,r)$$

and

$$b_{4ij} = \int_c^1 x^{n+p-\nu_i-\beta_j} (1-x)^{m-p} \, dx , \quad (i,j = 1,\cdots,p-r)$$

and hence the theorem is proved.

Remark 5.2.5

An expression for $P(0<\tilde{L}_1<\cdots<\tilde{L}_r<c<\tilde{L}_{r+1}<\cdots<\tilde{L}_p<1)$ when $\underset{\sim}{L}:p\times p \sim CMB_1(p, m, n)$ is also derived by Al-Ani (1972). From a computational point of view is there no difference between (5.2.59) and the result given by Al-Ani (1972).

In theorem 5.2.5 expressions for $P(0<\tilde{L}_1<\cdots<\tilde{L}_r<c<\tilde{L}_{r+1}<\cdots<\tilde{L}_{r+t}<d<\tilde{L}_{r+t+1}<\cdots<\tilde{L}_p<1)$ will be derived.

Theorem 5.2.5

Let $\underset{\sim}{L}:p\times p \sim NCCMB_{1B}(p, m, n, \Omega)$; then

(5.2.62) $$P(0<\tilde{L}_1<\cdots<\tilde{L}_r<c<\tilde{L}_{r+1}<\cdots<\tilde{L}_{r+t}<d<\tilde{L}_{r+t+1}<\cdots<\tilde{L}_p<1)$$

$$= \frac{\pi^{\frac{1}{2}p(p-1)} \tilde{\Gamma}_p(m+n) \operatorname{etr}[-\Omega] \prod_{i=1}^{p} (n-i+1)^{i-1}}{\tilde{\Gamma}_p(m) \tilde{\Gamma}_p(n) \prod_{i>j}^{p} (\tilde{\omega}_i - \tilde{\omega}_j) \prod_{i=1}^{p} (m+n-i+1)^{i-1}}$$

$$\Sigma_3\Sigma_4\Sigma_5\Sigma_6(-1)^{\Sigma\delta_i+\Sigma\delta_i^*+\Sigma\alpha_i+\Sigma\alpha_i^*}\,|(b_{8ij})|\,|(b_{9ij})|\,|(b_{10ij})|$$

where

$$(5.2.63)\quad b_{8ij} = \sum_{k=0}^{\infty} \frac{(m+n-p+1)_k\ \tilde{\omega}_{\delta_i}^k}{(n-p+1)_k\ k!} B_c(n+k-\delta_j^*+1,\ m-p+1)\ ,$$

$$(i,j = 1,\cdots,r)\ ,$$

$$(5.2.64)\quad b_{9ij} = \sum_{k=0}^{\infty} \frac{(m+n-p+1)_k\ \tilde{\omega}_{\alpha_i}^k}{(n-p+1)_k\ k!} \{B_d(n+k-\alpha_j^*+1,\ m-p+1)$$

$$- B_c(n+k-\alpha_j^*+1,\ m-p+1)\}\ ,\quad (i,j = 1,\cdots,t)\ ,$$

$$(5.2.65)\quad b_{10ij} = \sum_{k=0}^{\infty} \frac{(m+n-p+1)_k\ \tilde{\omega}_{\beta_i}^k}{(n-p+1)_k\ k!} \{B(n+k-\beta_j^*+1,\ m-p+1)$$

$$- B_d(n+k-\beta_j^*+1,\ m-p+1)\ ,\quad (i,j = r+t+1,\cdots,p)$$

and the rest of the symbols are defined as in theorem 3.5.1.

Let $\underset{\sim}{L}:p\times p \sim CMB_1(p,\ m,\ n)$; then

$$(5.2.66)\qquad P(0<\tilde{L}_1<\cdots<\tilde{L}_r<c<\tilde{L}_{r+1}<\cdots<\tilde{L}_{r+t}<d<\tilde{L}_{r+t+1}<\cdots<\tilde{L}_p<1)$$

$$= \frac{\pi^{p(p-1)}\ \tilde{\Gamma}_p(m+n)}{\tilde{\Gamma}_p(m)\ \tilde{\Gamma}_p(n)\ \tilde{\Gamma}_p(p)} \Sigma_3\Sigma_4\Sigma_5\Sigma_6(-1)^{\Sigma\delta_i+\Sigma\delta_i^*+\Sigma\alpha_i+\Sigma\alpha_i^*}$$

$$|(b_{8ij})|\,|(b_{9ij})|\,|(b_{10ij})|$$

where

(5.2.67) $\quad b_{8ij} = B_c(n+p-\delta_i-\delta_j^*+1,\ m-p+1)\ ,\quad (i,j=1,\cdots,r)\ ,$

(5.2.68) $\quad b_{9ij} = B_d(n+p-\alpha_i-\alpha_j^*+1,\ m-p+1) - B_c(n+p-\alpha_i-\alpha_j^*+1,\ m-p+1)\ ,$

$(i,j=1,\cdots,t)\ ,$

(5.2.69) $\quad b_{10ij} = B(n+p-\beta_i-\beta_j^*+1,\ m-p+1) - B_d(n+p-\beta_i-\beta_j^*+1,\ m-p+1)\ ,$

$(i,j=r+t+1,\cdots,p)$

and the rest of the symbols are defined as in theorem 3.5.1.

Proof

(5.2.62), (5.2.63), (5.2.64), (5.2.65)

The application of theorem 3.5.1 with $g(\tilde{\ell}_i)$, $\phi_i(\tilde{\ell}_j)$ and $\psi_i(\tilde{\ell}_j)$ given in (5.2.34), (5.2.35) and (5.2.36) respectively leads to (5.2.62) with

$$b_{8ij} = \int_0^c x^{n-\delta_j^*}(1-x)^{m-p}\ {}_1F_1(m+n-p+1\ ;\ n-p+1\ ;\ \tilde{\omega}_{\delta_i}x)\ dx\ ,$$

$(i,j=1,\cdots,r)\ ,$

$$b_{9ij} = \int_c^d x^{n-\alpha_j^*}(1-x)^{m-p}\ {}_1F_1(m+n-p+1\ ;\ n-p+1\ ;\ \tilde{\omega}_{\alpha_i}x)\ dx\ ,$$

$(i,j=1,\cdots,t)$

and

$$b_{10ij} = \int_d^1 x^{n-\beta_j^*}(1-x)^{m-p}\ {}_1F_1(m+n-p\ ;\ n-p+1\ ;\ \tilde{\omega}_{\beta_i}x)\ dx\ ,$$

$(i,j=r+t+1,\cdots,p)$

and hence the theorem is proved.

(5.2.66), (5.2.67), (5.2.68), (5.2.69)

The application of theorem 3.5.1 with $g(\tilde{\ell}_i)$, $\phi_i(\tilde{\ell}_j)$ and $\psi_i(\tilde{\ell}_j)$ given in (5.2.40), (5.2.41) and (5.2.42) respectively leads to (5.2.66) with

$$b_{8ij} = \int_0^c x^{n+p-\delta_i-\delta_j^*} (1-x)^{m-p}\, dx \;, \quad (i,j = 1,\cdots,r) \;,$$

$$b_{9ij} = \int_c^d x^{n+p-\alpha_i-\alpha_j^*} (1-x)^{m-p}\, dx \;, \quad (i,j = 1,\cdots,t)$$

and

$$b_{10ij} = \int_d^1 x^{n+p-\beta_i-\beta_j^*} (1-x)^{m-p}\, dx \;, \quad (i,j = r+t+1,\cdots,p)$$

and hence the theorem is proved.

In theorem 5.2.6 expressions for $f_{\tilde{L}_{r+1},\cdots,\tilde{L}_{r+t}}(\tilde{\ell}_{r+1},\cdots,\tilde{\ell}_{r+t})$ $0 < \tilde{\ell}_{r+1} < \cdots < \tilde{\ell}_{r+t}$, $(0 \leq r < r+t \leq p)$ will be derived.

Theorem 5.2.6

Let $\underset{\sim}{L}: p\times p \sim NCCMB_{1B}(p,\, m,\, n,\, \Omega)$; then

(5.2.70) $$f_{\tilde{L}_{r+1},\cdots,\tilde{L}_{r+t}}(\tilde{\ell}_{r+1},\cdots,\tilde{\ell}_{r+t})$$

$$= \frac{\pi^{\frac{1}{2}p(p-1)}\, \tilde{\Gamma}_p(m+n)\, \mathrm{etr}[-\Omega] \prod_{i=1}^{p} (n-i+1)^{i-1}}{\tilde{\Gamma}_p(m)\, \tilde{\Gamma}_p(n) \prod_{i>j}^{p} (\tilde{\omega}_i - \tilde{\omega}_j) \prod_{i=1}^{p} (m+n-i+1)^{i-1}}$$

$$\Sigma_3\Sigma_4\Sigma_5\Sigma_6(-1)^{\Sigma\delta_i+\Sigma\alpha_i+\Sigma\delta_i^*+\Sigma\alpha_i^*}\,|(b_{13ij})|\,|(b_{14ij})|$$

$$|(\tilde{\ell}_{r+j}^{n-p}(1-\tilde{\ell}_{r+j})^{m-p}\;{}_1F_1(m+n-p+1\,;\,n-p+1\,;\,\tilde{\omega}_{\alpha_i}\,\tilde{\ell}_{r+j})\,;$$

$$i,j=1,\cdots,t)|\,|(\tilde{\ell}_{r+j}^{p-\alpha_i^*}\,;\,i,j=1,\cdots,t)|\;,$$

$$0<\tilde{\ell}_{r+1}<\cdots<\tilde{\ell}_{r+t}<1$$

where

$$(5.2.71)\quad b_{13ij}=\sum_{k=0}^{\infty}\frac{(m+n-p+1)_k\,\tilde{\omega}_{\delta_i}^k}{(n-p+1)_k\,k!}\,B_{\tilde{\ell}_{r+1}}(n+k-\delta_j^*+1,\;m-p+1)\;,$$

$$(i,j=1,\cdots,r)\;,$$

$$(5.2.72)\quad b_{14ij}=\sum_{k=0}^{\infty}\frac{(m+n-p+1)_k\,\tilde{\omega}_{\beta_i}^k}{(n-p+1)_k\,k!}\,\{B(n+k-\beta_j^*+1,\;m-p+1)$$

$$-B_{\tilde{\ell}_{r+t}}(n+k-\beta_j^*+1,\;m-p+1)\;,$$

$$(i,j=r+t+1,\cdots,p)$$

and the rest of the symbols are defined as in theorem 3.5.1.

Let $\underset{\sim}{L}:p\times p\sim CMB_1(p,\ m,\ n)$; then

$$(5.2.73)\quad f_{\tilde{L}_{r+1},\cdots,\tilde{L}_{r+t}}(\tilde{\ell}_{r+1},\cdots,\tilde{\ell}_{r+t})$$

$$=\frac{\pi^{p(p-1)}\,\tilde{\Gamma}_p(m+n)}{\tilde{\Gamma}_p(m)\,\tilde{\Gamma}_p(n)\,\tilde{\Gamma}_p(p)}\,\Sigma_3\Sigma_4\Sigma_5\Sigma_6(-1)^{\Sigma\delta_i+\Sigma\alpha_i+\Sigma\delta_i^*+\Sigma\alpha_i^*}$$

$$|(b_{13ij})|\,|(b_{14ij})|\,|(\tilde{\ell}_{r+j}^{n-\alpha_i}(1-\tilde{\ell}_{r+j})^{m-p}\,;\,i,j=1,\cdots,t)|$$

$$|(\tilde{\ell}_{r+j}^{p-\alpha_i^*}\ ;\ i,j=1,\cdots,t)|\ ,\quad 0<\tilde{\ell}_{r+1}<\cdots<\tilde{\ell}_{r+t}<1$$

where

$$(5.2.74)\quad b_{13ij}=B_{\tilde{\ell}_{r+1}}(n+p-\delta_i-\delta_j^*+1,\ m-p+1)\ ,\quad (i,j=1,\cdots,t)\ ,$$

$$(5.2.75)\quad b_{14ij}=B(n+p-\beta_i-\beta_j^*+1,\ m-p+1)-B_{\tilde{\ell}_{r+t}}(n+p-\beta_i-\beta_j^*+1,\ m-p+1)\ ,$$

$$(i,j=r+t+1,\cdots,p)$$

and the rest of the symbols are defined as in theorem 3.5.1.

Proof

(5.2.70), (5.2.71), (5.2.72)

The application of theorem 3.6.1 with $g(\tilde{\ell}_i)$, $\phi_i(\tilde{\ell}_j)$ and $\psi_i(\tilde{\ell}_j)$ given in (5.2.34), (5.2.35) and (5.2.36) respectively leads to (5.2.70) with

$$b_{13ij}=\int_0^{\tilde{\ell}_{r+1}} x^{n-\delta_j^*}(1-x)^{m-p}\ {}_1F_1(m+n-p+1\ ;\ n-p+1\ ;\ \tilde{\omega}_{\delta_i}x)\ dx\ ,$$

$$(i,j=1,\cdots,r)\ ,$$

and

$$b_{14ij}=\int_{\tilde{\ell}_{r+t}}^{1} x^{n-\beta_j^*}(1-x)^{m-p}\ {}_1F_1(m+n-p+1\ ;\ n-p+1\ ;\ \tilde{\omega}_{\beta_i}x)\ dx\ ,$$

$$(i,j=r+t+1,\cdots,p)$$

and hence the theorem is proved.

(5.2.73), (5.2.74), (5.2.75)

The application of theorem 3.6.1 with $g(\tilde{\ell}_i)$, $\phi_i(\tilde{\ell}_j)$ and $\psi_i(\tilde{\ell}_j)$ given in (5.2.40), (5.2.41) and (5.2.42) respectively leads to (5.2.73) with

$$b_{13ij} = \int_0^{\tilde{\ell}_{r+1}} x^{n+p-\delta_i-\delta_j^*} (1-x)^{m-p}\, dx \ , \quad (i,j = 1,\cdots,t)$$

and

$$b_{14ij} = \int_{\tilde{\ell}_{r+t}}^{1} x^{n+p-\beta_i-\beta_j^*} (1-x)^{m-p}\, dx \ , \quad (i,j = r+t+1,\cdots,p)$$

and hence the theorem is proved.

In theorem 5.2.7 expressions for the joint p.d.f. of r unordered characteristic roots i.e. $f_{\tilde{L}_1,\cdots,\tilde{L}_r}(\tilde{\ell}_1,\cdots,\tilde{\ell}_r)$, $(\tilde{\ell}_i > 0 \ ; \ i = 1,\cdots,r)$ will be derived.

Theorem 5.2.7

Let $\underset{\sim}{L}:p\times p \sim NCCMB_{1B}(p,\ m,\ n,\ \Omega)$; then the joint p.d.f. of any r unordered characteristic roots of $\underset{\sim}{L}:p\times p$ is given by

(5.2.76) $$f_{\tilde{L}_1,\cdots,\tilde{L}_r}(\tilde{\ell}_1,\cdots,\tilde{\ell}_r)$$

$$= \frac{\pi^{\frac{1}{2}p(p-1)}\ \tilde{\Gamma}_p(m+n)\ \mathrm{etr}[-\Omega]\ \prod_{i=1}^{p}(n-i+1)^{i-1}}{p!\ \tilde{\Gamma}_p(m)\ \tilde{\Gamma}_p(n)\ \prod_{i>j}^{p}(\tilde{\omega}_i - \tilde{\omega}_j)\ \prod_{i=1}^{p}(m+n-i+1)^{i-1}}$$

$$\Sigma_1 \Sigma_2 (-1)^{\Sigma\delta_i + \Sigma\alpha_i} \left|(b_{17ij})\right| \left|(b_{18ij})\right| ,$$

$$(0 < \tilde{\ell}_i < 1 \; ; \; i = 1, \cdots, r)$$

where

$$(5.2.77) \quad b_{17ij} = \sum_{t=1}^{r} \tilde{\ell}_t^{\,n-\alpha_j} (1-\tilde{\ell}_t)^{m-p} \; {}_1F_1(m+n-p+1 \, ; \, n-p+1 \, ; \, \tilde{\omega}_{\delta_i} \tilde{\ell}_t) ,$$

$$(i,j = 1, \cdots, r) ,$$

$$(5.2.78) \quad b_{18ij} = (p-r)! \sum_{k=0}^{\infty} \frac{(m+n-p+1)_k \, \omega_{\nu_i}^k}{(n-p+1)_k \, k!} B(n+k-\beta_j+1, \, m-p+1) ,$$

$$(i,j = 1, \cdots, p-r)$$

and the rest of the symbols are defined as in theorem 3.4.1.

Let $\underset{\sim}{L}:p \times p \sim CMB_1(p, m, n)$; then the joint p.d.f. of any r unordered characteristic roots of $\underset{\sim}{L}:p \times p$ is given by

$$(5.2.79) \quad f_{\tilde{L}_1, \cdots, \tilde{L}_r}(\tilde{\ell}_1, \cdots, \tilde{\ell}_r)$$

$$= \frac{\pi^{p(p-1)} \tilde{\Gamma}_p(m+n)}{p! \; \tilde{\Gamma}_p(m) \; \tilde{\Gamma}_p(n) \; \tilde{\Gamma}_p(p)} \Sigma_1 \Sigma_2 (-1)^{\Sigma\delta_i + \Sigma\alpha_i}$$

$$\left|(b_{17ij})\right| \left|(b_{18ij})\right| , \quad (0 < \tilde{\ell}_i < 1 , \quad i = 1, \cdots, r)$$

where

$$(5.2.80) \quad b_{17ij} = \sum_{t=1}^{r} \tilde{\ell}_t^{\,n+p-\delta_i-\alpha_j} (1-\tilde{\ell}_t)^{m-p} , \quad (i,j = 1, \cdots, r) ,$$

$$(5.2.81)\quad b_{18ij} = (p-r)!\ B(n+p-\nu_i-\beta_j+1,\ m-p+1)\ , \quad (i,j = 1,\cdots,p-r)$$

and the rest of the symbols are defined as in theorem 3.4.1.

Proof

<u>(5.2.76)</u>, <u>(5.2.77)</u>, <u>(5.2.78)</u>

The application of theorem 3.7.1 with $g(\tilde{\ell}_i)$, $\phi_i(\tilde{\ell}_j)$ and $\psi_i(\tilde{\ell}_j)$ given in (5.2.34), (5.2.35) and (5.2.36) respectively leads to (5.2.76) with

$$b_{17ij} = \sum_{t=1}^{r} \tilde{\ell}_t^{\,n-\alpha_j} (1-\tilde{\ell}_t)^{m-p}\ {}_1F_1(m+n-p+1\ ;\ n-p+1\ ;\ \tilde{\omega}_{\delta_i}\ \tilde{\ell}_t)\ ,$$

$$(i,j = 1,\cdots,r)$$

and

$$b_{18ij} = (p-r)! \int_0^1 x^{n-\beta_j} (1-x)^{m-p}\ {}_1F_1(m+n-p+1\ ;\ n-p+1\ ;\ \tilde{\omega}_{\nu_i} x)\ dx$$

and hence the theorem is proved.

<u>(5.2.79)</u>, <u>(5.2.80)</u>, <u>(5.2.81)</u>

The application of theorem 3.7.1 with $g(\tilde{\ell}_i)$, $\phi_i(\tilde{\ell}_j)$ and $\psi_i(\tilde{\ell}_j)$ given in (5.2.40), (5.2.41) and (5.2.42) respectively leads to (5.2.79) with

$$b_{17ij} = \sum_{t=1}^{r} \tilde{\ell}_t^{\,n+p-\delta_i-\alpha_j} (1-\tilde{\ell}_t)^{m-p}\ , \quad (i,j = 1,\cdots,r)$$

and

$$b_{18ij} = (p-r)! \int_0^1 x^{n+p-\nu_i-\beta_j} (1-x)^{m-p}\, dx \ , \quad (i,j = 1,\cdots,p-r)$$

and hence the theorem is proved.

<u>Remark 5.2.6</u>

Waikar, Chang and Krishnaiah (1972) derived the following expression for the joint p.d.f. of any r unordered characteristic roots when $\underset{\sim}{L}:p\times p \sim \mathrm{NCCMB}_{1B}(p,\ m,\ n,\ \Omega)$:

(5.2.82)
$$f_{\tilde{L}_1,\cdots,\tilde{L}_r}(\tilde{\ell}_1,\cdots,\tilde{\ell}_r)$$

$$= \frac{\pi^{p(p-1)}\ \tilde{\Gamma}_p(m+n)\ \mathrm{etr}[-\Omega]}{\tilde{\Gamma}_p(p)\ \tilde{\Gamma}_p(m)\ \tilde{\Gamma}_p(n)} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{[m+n]_\kappa\ \tilde{C}_\kappa(\Omega)}{k!\ [n]_\kappa\ \tilde{C}_\kappa(I_p)}$$

$$\chi_{[\kappa]}(1)\ \Sigma_1\Sigma_2(-1)^{f(\alpha_1,\cdots,\alpha_p)}(-1)^{f(\delta_1,\cdots,\delta_p)}$$

$$\Sigma_3 \prod_{i=1}^{r} \tilde{\ell}_{t_{\delta_i}}^{k_{\alpha_i}+2p-\alpha_i-\delta_i}\ \tilde{\ell}_i^{n-p}\ (1-\tilde{\ell}_i)^{m-p}$$

$$(p-r)!\ B(k_{\alpha_i}+p-\alpha_i-\delta_i+n+1,\ m-p+1)\ ,$$

$$(0 < \tilde{\ell}_i < 1\ ,\quad i = 1,\cdots,r)\ ,$$

where the symbols are defined as in remark 4.2.5. It is clear that the expression for the joint p.d.f. of any r unordered roots of $\underset{\sim}{L}:p\times p \sim \mathrm{NCCMB}_{1B}(p,\ m,\ n,\ \Omega)$ given in (5.2.76) is in a much simpler form than the expression given in (5.2.82).

5.2.3 P.d.f.s of functions of the characteristic roots of the complex beta type 1 matrix

In theorem 5.2.8 the p.d.f.s of $|\underset{\sim}{L}| = \prod_{i=1}^{p} \tilde{L}_i$ and $|I_p - \underset{\sim}{L}| = \prod_{i=1}^{p} (1 - \tilde{L}_i)$ for $\underset{\sim}{L}:p\times p \sim CMB_1(p, m, n, \Sigma, \Phi)$ and $\underset{\sim}{L}:p\times p \sim NCCMB_{1B}(p, m, n, \Omega)$ are derived in terms of Meijer's G-function.

Theorem 5.2.8

Let $\underset{\sim}{L}:p\times p \sim CMB_1(p, m, n, \Sigma, \Phi)$; then

(5.2.83)
$$f_{|\underset{\sim}{L}|}(|L|)$$
$$= \frac{\tilde{\Gamma}_p(m+n)}{\tilde{\Gamma}_p(n)} |\Sigma\Phi^{-1}|^n |L|^{-1} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{[m+n]_\kappa}{k!} \tilde{C}_\kappa(I_p - \Sigma\Phi^{-1})$$
$$G_p\left[|L| \left| \begin{matrix} m+n+k_j-j+1 \\ n+k_j-j+1 \end{matrix} \right.\right], \quad 0 < |L| < 1$$

and

(5.2.84)
$$f_{|I_p - \underset{\sim}{L}|}(|I_p - L|)$$
$$= \frac{\tilde{\Gamma}_p(m+n)}{\tilde{\Gamma}_p(m)} |\Sigma\Phi^{-1}|^n \; |I_p - L|^{-1}$$
$$\sum_{k=0}^{\infty} \sum_{\kappa} \frac{[n]_\kappa \; [m+n]_\kappa}{k!} \tilde{C}_\kappa(I_p - \Sigma\Phi^{-1})$$
$$G_p\left[|I_p - L| \left| \begin{matrix} m+n+k_j-j+1 \\ m-j+1 \end{matrix} \right.\right], \quad 0 < |I_p - L| < 1 .$$

Let $\underset{\sim}{L}:p\times p \sim NCCMB_{1B}(p, m, n, \Omega)$; then

(5.2.85) $$f_{|\underset{\sim}{L}|}(|L|)$$

$$= \frac{\tilde{\Gamma}_p(m+n)}{\tilde{\Gamma}_p(n)} \operatorname{etr}[-\Omega] \; |L|^{-1} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{[m+n]_\kappa \; \tilde{C}_\kappa(\Omega)}{[n]_\kappa \; k!}$$

$$G_p\left[|L| \left| \begin{matrix} m+n+k_j-j+1 \\ n+k_j-j+1 \end{matrix} \right.\right], \quad 0 < |L| < 1$$

and

(5.2.86) $$f_{|I_p - \underset{\sim}{L}|}(|I_p - L|)$$

$$= \frac{\tilde{\Gamma}_p(m+n)}{\tilde{\Gamma}_p(m)} \operatorname{etr}[-\Omega] \; |I_p - L|^{-1} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{[m+n]_\kappa}{k!} \tilde{C}_\kappa(\Omega)$$

$$G_p\left[|I_p - L| \left| \begin{matrix} m+n+k_j-j+1 \\ m-j+1 \end{matrix} \right.\right], \quad 0 < |L| < 1 \; .$$

Proof

These four results follow from (5.2.3), (5.2.4), (5.2.8) and (5.2.9) respectively by using theorem 2.5.3.

Remark 5.2.6

(i) The expressions for $f_{|I_p - \underset{\sim}{L}|}(|I_p - L|)$ given in (5.2.84) and (5.2.86) are also derived by Pillai and Jouris (1971, p. 518 - 520).

(ii) Asymptotic expansions for the p.d.f.s of $|\underset{\sim}{L}|$ and $|I_p - \underset{\sim}{L}|$ are derived by De Waal (1968, p. 97 - 100).

(iii) Money (1972, p. 91 - 93) showed how explicit series expressions for $f_{|\underset{\sim}{L}|}(|L|)$ and $f_{|I_p - \underset{\sim}{L}|}(|I_p - L|)$ (for $p = 2$ and $p = 3$) can be obtained in the non-central linear case i.e. when $\text{rank}(\Omega) = 1$.

(iv) As in the real case (cf. De Waal, 1968, p. 47 and Greenacre, 1972, p. 64 - 66) it is possible to derive the p.d.f.s of $\text{tr}\,\underset{\sim}{L}$, $\text{tr}(\underset{\sim}{L}(I_p - \underset{\sim}{L})^{-1})$ and $\text{tr}((I_p - \underset{\sim}{L})\underset{\sim}{L}^{-1})$. These p.d.f.s are however convergent only for $\text{tr}\,L < 1$, $\text{tr}(L(I_p - L)^{-1}) < 1$ and $\text{tr}((I_p - L)L^{-1}) < 1$ and will therefore not be considered here.

(v) The moment generating function of $\text{tr}\,\underset{\sim}{L}$ and $\text{tr}(I_p - \underset{\sim}{L})$ when $\underset{\sim}{L}:p\times p \sim NCCMB_{1B}(p, m, n, \Omega)$ and of $\text{tr}\,\underset{\sim}{L}$ when $\underset{\sim}{L}:p\times p \sim CMB_1(p, m, n, \Sigma, \Phi)$ are derived by De Waal (1968, p. 105).

5.3 THE GENERALISED SAMPLE MULTIPLE COHERENCE MATRIX $\underset{\sim}{R}:q\times q$

5.3.1 The definition, p.d.f. and moments of $\underset{\sim}{R}:q\times q$ and $\underset{\sim}{D}_R$

Let $\underline{Z}:p\times 1 \sim CN(p, \underline{\mu}, \Sigma)$ and let $\underline{Z}_{(1)},\cdots,\underline{Z}_{(N)}$ be a random sample of N observations on $\underline{Z}:p\times 1$, $(N > p)$; then

$$(5.3.1) \qquad \underset{\sim}{A}:p\times p = \sum_{\alpha=1}^{N} (\underline{Z}_\alpha - \underline{\overline{Z}})\overline{(\underline{Z}_\alpha - \underline{\overline{Z}})}' \;, \qquad \left(\underline{\overline{Z}} = \frac{1}{N}\sum_{\alpha=1}^{N} \underline{Z}_\alpha\right)$$

has the central complex Wishart distribution i.e. $\underset{\sim}{A}:p\times p \sim CW(p, n, \Sigma)$, $(n = N-1)$.

Let $\underline{Z}:p\times 1$ be partitioned into two sets of components:

$$(5.3.2) \qquad \underline{Z}:p\times 1 = \begin{bmatrix} \underline{Z}^{(1)}:q\times 1 \\ \underline{Z}^{(2)}:r\times 1 \end{bmatrix} .$$

Partition $\Sigma:p\times p$ and $\underset{\sim}{A}:p\times p$ accordingly:

$$(5.3.3) \qquad \Sigma:p\times p = \begin{bmatrix} \Sigma_{11}:q\times q & \Sigma_{12}:q\times r \\ \Sigma_{21}:r\times q & \Sigma_{22}:r\times r \end{bmatrix} , \quad \underset{\sim}{A}:p\times p = \begin{bmatrix} \underset{\sim}{A}_{11}:q\times q & \underset{\sim}{A}_{12}:q\times r \\ \underset{\sim}{A}_{21}:r\times q & \underset{\sim}{A}_{22}:r\times r \end{bmatrix} .$$

Definition 5.3.1 (Troskie, 1969, p. 119)

The generalised sample multiple coherence matrix between the two sets $\underline{Z}^{(1)}:q\times 1$ and $\underline{Z}^{(2)}:r\times 1$ is defined as

$$(5.3.4) \qquad \underset{\sim}{R}:q\times q = \underset{\sim}{A}_{11}^{-\frac{1}{2}} \underset{\sim}{A}_{12} \underset{\sim}{A}_{22}^{-1} \underset{\sim}{A}_{21} \underset{\sim}{A}_{11}^{-\frac{1}{2}} .$$

The generalised population multiple coherence matrix is defined accordingly as

$$(5.3.5) \qquad P:q\times q = \Sigma_{11}^{-\frac{1}{2}} \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21} \Sigma_{11}^{-\frac{1}{2}} .$$

The characteristic roots of $\underset{\sim}{R}:q\times q$, $(0 < \tilde{r}_1^2 < \cdots < \tilde{r}_q^2 < 1)$, and $P:q\times q$, $(0 < \tilde{\rho}_1^2 < \cdots < \tilde{\rho}_q^2 < 1)$, are the squares of the sample and population canonical correlation coefficients respectively.

The matrix $\underset{\sim}{R}:q\times q$ can be written as

$$(5.3.6) \qquad \underset{\sim}{R}:q\times q = (\underset{\sim}{E} + \underset{\sim}{F})^{-\frac{1}{2}} \underset{\sim}{F}(\underset{\sim}{E} + \underset{\sim}{F})^{-\frac{1}{2}}$$

where

$$(5.3.7) \qquad \underset{\sim}{E}:q\times q = \underset{\sim}{A}_{11} - \underset{\sim}{A}_{12}\, \underset{\sim}{A}_{22}^{-1}\, \underset{\sim}{A}_{21} \sim CW(q,\ n-r,\ \Sigma_{11.2}) ,$$

$$(5.3.8) \qquad \underset{\sim}{F}:q\times q = \underset{\sim}{A}_{12}\, \underset{\sim}{A}_{22}^{-1}\, \underset{\sim}{A}_{21} \sim NCCW(q,\ r,\ \Sigma_{11.2},\ \Omega) ,$$

$$(5.3.9) \qquad \Sigma_{11.2}:q\times q = \Sigma_{11} - \Sigma_{12}\, \Sigma_{22}^{-1}\, \Sigma_{21} ,$$

$$(5.3.10) \qquad B:q\times r = \Sigma_{12}\, \Sigma_{22}^{-1}$$

and

$$(5.3.11) \qquad \underset{\sim}{\Omega}:q\times q = \Sigma_{11.2}^{-1}\, B\, \underset{\sim}{A}_{22}\, \overline{B}' .$$

From (5.3.6), (5.3.7) and (5.3.8) follows that $\underset{\sim}{R}:q\times q$ given $\underline{Z}^{(2)} = \underline{z}^{(2)}$, (i.e. given $A_{22}:r\times r$) is conditionally distributed as a $NCCMB_{1B}(q,\ n-r,\ r,\ \Omega)$ - variate. In theorem 5.3.1 the unconditional p.d.f., the unconditional symmetrised p.d.f., the unconditional joint p.d.f. of the characteristic roots and the unconditional moments of $\underset{\sim}{R}:q\times q$ are considered. The following lemma will be used in the proof of theorem 5.3.1:

<u>Lemma 5.3.1</u>

Let $A:p\times p$, $B:p\times m$ and $C:m\times m$ be real or complex matrices with $|A| \neq 0$ and $|C| \neq 0$; then the following three identities hold:

$$(5.3.12) \qquad (A + BC\overline{B}')^{-1} = A^{-1} - A^{-1}B\,(C^{-1} + \overline{B}'A^{-1}B)\,\overline{B}'A^{-1} ,$$

$$(5.3.13) \qquad (A + BC\overline{B}')^{-1}B = A^{-1}B\,(C^{-1} + \overline{B}'A^{-1}B)^{-1}\,C^{-1} ,$$

$$(5.3.14) \qquad |A + BC\overline{B}'| = |A|\,|C|\,|C^{-1} + \overline{B}'A^{-1}B| .$$

Proof

(5.3.12), (5.3.13), (5.3.14) Morrison (1976, p. 68 - 69).

Theorem 5.3.1

Let $\underset{\sim}{R}:q\times q$ and $P:q\times q$ be defined as in definition 5.3.1; then:

The unconditional p.d.f. of $\underset{\sim}{R}:q\times q$ is given by

$$(5.3.15)\quad f_{\underset{\sim}{R}}(R) = \frac{\tilde{\Gamma}_q(n)\ |I_q-P|^n\ |R|^{r-q}\ |I_q-R|^{n-r-q}}{\tilde{\Gamma}_q(n-r)\ \tilde{\Gamma}_q(r)\ \tilde{\Gamma}_q(n)\ |\Sigma_{11.2}|^n} \int_{G=\bar{G}'>0} |G|^{n-q}\ \mathrm{etr}[-\Sigma_{11.2}^{-1}G]$$

$$_1\tilde{F}_1(n\ ;\ r\ ;\ \Sigma^{12}(\Sigma^{22})^{-1}\Sigma^{21}G^{\frac{1}{2}}RG^{\frac{1}{2}})\,dG\ ,$$

$$0 < R < I_q$$

where

$$(5.3.16)\quad \Sigma^{12}:q\times r = -\Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22.1}^{-1}\ ,$$

$$(5.3.17)\quad \Sigma^{22}:r\times r = \Sigma_{22.1}^{-1}$$

and

$$(5.3.18)\quad \Sigma^{21}:r\times q = -\Sigma_{22}^{-1}\Sigma_{21}\Sigma_{11.2}^{-1}\ ,$$

the unconditional symmetrised p.d.f. of $\underset{\sim}{R}:q\times q$ is given by

(5.3.19) $\quad f_{csym}(R)$

$$= \frac{\tilde{\Gamma}_q(n)\ |I_q-P|^n\ |R|^{r-q}\ |I_q-R|^{n-r-q}}{\tilde{\Gamma}_q(n-r)\ \tilde{\Gamma}_q(r)}$$

$${}_2\tilde{F}_1(n,\ n\ ;\ r\ ;\ P,\ R)\ ,\quad 0 < R = \overline{R}' < I_q\ ,$$

(5.3.20) $\quad E(|\underset{\sim}{R}|^h)$

$$= \frac{\tilde{\Gamma}_q(n)\ \tilde{\Gamma}_q(r+h)}{\tilde{\Gamma}_q(n+h)\ \tilde{\Gamma}_q(r)}\ |I_q - P|\ {}_3\tilde{F}_2(n,\ n,\ r+h\ ;\ n+h,\ r\ ;\ P)\ ,$$

(5.3.21) $\quad E(|I_q - \underset{\sim}{R}|^h)$

$$= \frac{\tilde{\Gamma}_q(n)\ \tilde{\Gamma}_q(n-r+h)}{\tilde{\Gamma}_q(n+h)\ \tilde{\Gamma}_q(n-r)}\ |I - P|^n\ {}_2\tilde{F}_1(n,\ n\ ;\ n+h\ ;\ P)\ ,$$

the unconditional p.d.f. of $\underset{\sim}{D}_R$ is given by

(5.3.22) $\quad f_{\underset{\sim}{D}_R}(D_R)$

$$= \frac{\pi^{q(q-1)}\ \tilde{\Gamma}_q(n)}{\tilde{\Gamma}_q(n-r)\ \tilde{\Gamma}_q(r)\ \tilde{\Gamma}_q(q)}\ |I_q - P|^n\ |D_R|^{r-q}\ |I_q - D_R|^{n-r-q}$$

$$\prod_{i>j}^{q} (\tilde{r}_i^2 - \tilde{r}_j^2)^2\ {}_2\tilde{F}_1(n,\ n\ ;\ r\ ;\ P,\ D_R)\ ,\quad 0 < \tilde{r}_1^2 < \cdots < \tilde{r}_q^2 < 1\ ,$$

where

(5.3.23) $\quad D_R = \mathrm{Diag}(\tilde{r}_1^2,\cdots,\tilde{r}_q^2)\ .$

Proof

(5.3.15) Troskie (1969, p. 120).

(5.3.19)

The symmetrised conditional p.d.f. of $\underset{\sim}{R}:q\times q$ follows from (5.3.6), (5.3.7), (5.3.8) and (5.2.7) as

(5.3.24) $$f_{csym}(R|A_{22})$$

$$= \frac{\text{etr}[-\Omega]\ \tilde{\Gamma}_q(n)}{\tilde{\Gamma}_q(n-r)\ \tilde{\Gamma}_q(r)} |R|^{r-q} |I_q - R|^{n-r-q} \ {}_1\tilde{F}_1(n\ ;\ r\ ;\ \Omega,\ R)\ .$$

Since $\underset{\sim}{A}_{22}:r\times r \sim CW(r,\ n,\ \Sigma_{22})$, multiply $f_{csym}(R|A_{22})$ with $f_{\underset{\sim}{A}_{22}}(A_{22})$ and integrate w.r.t. $A_{22}:r\times r$, i.e.

(5.3.25) $$f_{csym}(R)$$

$$= \frac{\tilde{\Gamma}_q(n)\ |R|^{r-q}\ |I_q-R|^{n-r-q}}{\tilde{\Gamma}_q(n-r)\ \tilde{\Gamma}_q(r)\ \tilde{\Gamma}_r(n)\ |\Sigma_{22}|^n}\ I^*$$

where

(5.3.26) $$I^* = \int_{A_{22}=\bar{A}'_{22}>0} \text{etr}[-(\Sigma_{22}^{-1} + \bar{B}'\ \Sigma_{11.2}^{-1}\ B)A_{22}]\ |A_{22}|^{n-r}$$

$$ {}_1\tilde{F}_1(n\ ;\ r\ ;\ \bar{B}'\ \Sigma_{11.2}^{-1}\ B\ A_{22},\ R)\ dA_{22}$$

$$= \tilde{\Gamma}_r(n)\ |\Sigma_{22}|^n\ |I_r + \bar{B}'\ \Sigma_{11.2}^{-1}\ B\ \Sigma_{22}|^{-n}$$

$$ {}_2\tilde{F}_1(n,\ n\ ;\ r\ ;\ \bar{B}'\ \Sigma_{11.2}^{-1}\ B(\Sigma_{22}^{-1} + \bar{B}'\ \Sigma_{11.2}^{-1}\ B)^{-1},\ R)\ ,$$

(from (2.3.5)).

Now it follows that

$$(5.3.27)\quad |I_r + \bar{B}'\,\Sigma_{11.2}^{-1}\,B\,\Sigma_{22}| = |I_r + \Sigma_{22}^{-1}\,\Sigma_{21}\,\Sigma_{11.2}^{-1}\,\Sigma_{12}|$$

$$= |\Sigma_{22}^{-1}|\,|\Sigma_{22} + \Sigma_{21}\,\Sigma_{11.2}^{-1}\,\Sigma_{12}|$$

$$= |\Sigma_{11.2}^{-1}|\,|\Sigma_{11.2} + \Sigma_{12}\,\Sigma_{22}^{-1}\,\Sigma_{21}|\ ,$$

(from (5.3.14))

$$= |\Sigma_{11.2}^{-1}|\,|\Sigma_{11}|$$

$$= |I_q - P|$$

and

$$(5.3.28)\quad \bar{B}'\,\Sigma_{11.2}^{-1}\,B\,(\Sigma_{22}^{-1} + \bar{B}'\,\Sigma_{11.2}^{-1}\,B)^{-1}$$

$$= \bar{B}'(\Sigma_{11} - \Sigma_{12}\,\Sigma_{22}^{-1}\,\Sigma_{21})^{-1}\,\Sigma_{12}\Sigma_{22}^{-1}(\Sigma_{22} - \Sigma_{22}\,B\,(\Sigma_{11.2} + B\,\Sigma_{22}\,\bar{B}')^{-1}$$

$$B\,\Sigma_{22})\ ,\quad \text{(from (5.3.12))}$$

$$= \bar{B}'(\Sigma_{11}^{-1}\,\Sigma_{12}(\Sigma_{21}\,\Sigma_{11}^{-1}\,\Sigma_{12} - \Sigma_{22})^{-1}(-\Sigma_{22})(\Sigma_{22})$$

$$(\Sigma_{22} - \Sigma_{21}\,\Sigma_{11}^{-1}\,\Sigma_{12})\ ,\quad \text{(from (5.3.13))}$$

$$= \Sigma_{22}^{-1}\,\Sigma_{21}\,\Sigma_{11}^{-1}\,\Sigma_{12}\ .$$

Substitution of (5.3.28) and (5.3.27) into (5.3.26) and (5.3.26) into (5.3.25) leads to (5.3.19).

(<u>5.3.20</u>), (<u>5.3.21</u>) Troskie (1969, p. 120)

(5.3.22)

The application of corollary 2.7.1 and theorem 3.2.1 leads to (5.3.22). James (1964, p. 491) originally derived (5.3.22).

Remark 5.3.1

The unconditional symmetrised p.d.f. of $\underset{\sim}{R}:q\times q$ follows also from (5.3.15) and (2.7.1) as

$$(5.3.29) \quad f_{csym}(R) = \frac{\tilde{\Gamma}_q(n)\ |I_q - P|^n\ |R|^{r-q}\ |I_q - R|^{n-r-q}}{\tilde{\Gamma}_q(n-r)\ \tilde{\Gamma}_q(r)\ \tilde{\Gamma}_q(n)\ |\Sigma_{11.2}|^n}\ I^*$$

where

$$(5.3.30) \quad I^* = \int_{U(p)} \int_{G=\bar{G}'>0} |G|^{n-q}\ \mathrm{etr}[-\Sigma_{11.2}^{-1} G]$$

$$ {}_1\tilde{F}_1(n\,;\,r\,;\,\Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\Sigma_{11.2}^{-1}\, G^{\frac{1}{2}}\, U R \bar{U}'\, G^{\frac{1}{2}})$$

$$dG\, dU\ .$$

Change the order of integration then follows from (2.3.4) that

$$(5.3.31) \quad I^* = \int_{G=\bar{G}'>0} |G|^{n-q}\ \mathrm{etr}[-\Sigma_{11.2}^{-1} G]$$

$$ {}_1\tilde{F}_1(n\,;\,r\,;\,\Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\Sigma_{11.2}\, G\,,\,R)\ dG$$

$$= \tilde{\Gamma}_q(n)\ |\Sigma_{11.2}|^n\ {}_2\tilde{F}_1(n,\ n\,;\,r\,;\,P,\ R)\ , \qquad (\text{from } (2.3.5)).$$

Substitution of (5.3.31) into (5.3.29) leads to (5.3.19).

5.3.2 Certain marginal distributions of the characteristic roots of $\underset{\sim}{R}:q\times q$

In theorem 5.3.2 it is shown how the central and non-central unconditional p.d.f.s of $\underset{\sim}{D}_R$ can be written in a form such that the random components are in the form given in (3.2.3), by using theorems 2.2.8 and 2.3.6.

Theorem 5.3.2

Let $\underset{\sim}{R}:q\times q$ and $P:q\times q$ be defined as in definition 5.3.1; then

$$(5.3.32) \quad f_{\underset{\sim}{D}_R}(D_R) = \frac{\pi^{\frac{1}{2}q(q-1)}\ \tilde{\Gamma}_q(n)\ |I_q-P|^n \prod_{i=1}^{q}(r-i+1)^{i-1}}{\tilde{\Gamma}_q(n-r)\ \tilde{\Gamma}_q(r) \prod_{i>j}^{q}(\tilde{\rho}_i^2-\tilde{\rho}_j^2)\{\prod_{i=1}^{q}(n-i+1)^{i-1}\}^2}$$

$$\prod_{i=1}^{q}(\tilde{r}_i^2)^{r-q}\ (1-\tilde{r}_i^2)^{n-r-q}$$

$$|({}_2F_1(n-q+1,\ n-q+1\ ;\ r-q+1\ ;\ \tilde{\rho}_i^2\,\tilde{r}_j^2))|\ |((\tilde{r}_j^2)^{q-i})|\ ,$$

$$0<\tilde{r}_1^2<\cdots<\tilde{r}_q^2<1\ .$$

Let $\underset{\sim}{R}:q\times q$ be defined as in definition 5.3.1 and $\Sigma_{12}:q\times r=0$ i.e. $P:q\times q=0$; then the central p.d.f. of $\underset{\sim}{D}_R$ follows as

$$(5.3.33) \quad f_{\underset{\sim}{D}_R}(D_R) = \frac{\pi^{q(q-1)}\ \tilde{\Gamma}_q(n) \prod_{i=1}^{q}(\tilde{r}_1^2)^{r-q}\ (1-\tilde{r}_i^2)^{n-r-q}}{\tilde{\Gamma}_q(n-r)\ \tilde{\Gamma}_q(r)\ \tilde{\Gamma}_q(q)}$$

$$|((\tilde{r}_j^2)^{q-i})|\ ,\quad 0<\tilde{r}_1^2<\cdots<\tilde{r}_q^2<1\ .$$

Proof

(5.3.32)

From (2.2.41) follows that (5.3.22) can be written as

$$(5.3.34)\quad f_{\underset{\sim}{D}_R}(D_R) = \frac{\pi^{q(q-1)}\,\tilde{\Gamma}_q(n)\ |I_q - P|^n \prod_{i=1}^{q} (\tilde{r}_i^2)^{r-q}\ (1-\tilde{r}_i^2)^{n-r-q}}{\tilde{\Gamma}_q(n-r)\ \tilde{\Gamma}_q(r)\ \tilde{\Gamma}_q(q)}$$

$$|((\tilde{r}_j^2)^{q-i})|^2\ {}_2\tilde{F}_1(n,\ n\ ;\ r\ ;\ P,\ D_R)\ .$$

From (2.2.41), (2.3.14) and (2.3.15) follows that

$$(5.3.35)\quad {}_2\tilde{F}_1(n,\ n\ ;\ r\ ;\ P,\ D_R)$$

$$= \frac{\tilde{\Gamma}_q(q) \prod_{i=1}^{q} (r-i+1)^{i-1}\ |({}_2F_1(n-q+1,\ n-q+1\ ;\ r-q+1\ ;\ \tilde{\rho}_i^2\ \tilde{r}_j^2))|}{\pi^{\frac{1}{2}q(q-1)} \prod_{i>j}^{q} (\tilde{\rho}_i^2 - \tilde{\rho}_j^2)\ |((\tilde{r}_j^2)^{q-i})|\ \{\prod_{i=1}^{q} (n-i+1)^{i-1}\}^2}\ .$$

Substitution of (5.3.35) into (5.3.34) leads to (5.3.22).

(<u>5.3.33</u>)

Let $\Sigma_{12}{:}q\times r = 0$ in (5.3.22); then (5.3.33) follows from (2.2.41).

<u>Remark 5.3.2</u>

By using (2.2.42), it follows that the non-central p.d.f. of $\underset{\sim}{D}_R$ can also be written as

$$(5.3.36)\quad f_{\underset{\sim}{D}_R}(D_R) = \frac{\pi^{q(q-1)}\ \tilde{\Gamma}_q(n)\ |I_q-P|^n}{\tilde{\Gamma}_q(n-r)\ \tilde{\Gamma}_q(r)\ \tilde{\Gamma}_q(q)} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{[n]_\kappa\ [n]_\kappa\ \tilde{C}_\kappa(P)}{[r]_\kappa\ k!\ \tilde{C}_\kappa(I_q)}$$

$$\chi_{[\kappa]}(1) \prod_{i=1}^{q} (\tilde{r}_i^2)^{r-q}\ (1-\tilde{r}_i^2)^{n-r-q}\ |((\tilde{r}_j^2)^{k_i+q-i})|$$

$$|((\tilde{r}_j^2)^{q-i})|\ ,\quad 0 < \tilde{r}_1^2 < \cdots < \tilde{r}_q^2 < 1\ .$$

From a computational point of view (5.3.32) leads to better results than (5.3.36) because (5.3.32) does not involve zonal polynomials.

The random component in (5.3.32), the non-central unconditional p.d.f. of $\underset{\sim}{D}_R$, corresponds with the random component in the p.d.f. of $\underset{\sim}{D}_L$, given in (5.2.21), i.e. when $\underset{\sim}{L}:p\times p \sim NCCMB_{1B}(p, m, n, \Omega)$. By replacing:

(i) p with q,

(ii) m with $n-r$,

(iii) n with r,

(iv) $\tilde{\ell}_j$ with $\tilde{r}_j^2$,

(v) $\tilde{\omega}_i$ with $\tilde{\rho}_i^2$,

(vi) ${}_1F_1(m+n-p+1\,;\, n-p+1\,;\, \tilde{\omega}_i\,\tilde{\ell}_j)$ with

$$ {}_2F_1(n-q+1,\ n-q+1\,;\, r-q+1\,;\, \tilde{\rho}_i^2\,\tilde{r}_j^2) $$

in the random component in (5.2.21) the random component in (5.3.32) follows. It is also clear that the random component in the central p.d.f. of $\underset{\sim}{D}_R$ corresponds with the random component in the p.d.f. of $\underset{\sim}{D}_L$ when $\underset{\sim}{L}:p\times p \sim CMB_1(p, m, n)$. Because of these similarities between the p.d.f.s of $\underset{\sim}{D}_L$ and $\underset{\sim}{D}_R$, the marginal distributions of the characteristic roots of $\underset{\sim}{R}:q\times q$ can be obtained from the marginal distributions of the roots of $\underset{\sim}{L}:p\times p$, derived in theorems 5.2.3 - 5.2.7 and corollaries 5.2.1 and 5.2.2. By replacing the constant and making the replacements, mentioned above, in the marginal distributions of the roots of $\underset{\sim}{L}:p\times p$, the marginal distributions of the roots of $\underset{\sim}{R}:q\times q$ follow. Thus the marginal distributions of the roots of $\underset{\sim}{R}:q\times q$ will not be derived here.

The p.d.f. of the largest characteristic root, the c.d.f. of any intermediate characteristic root and the p.d.f. of any s unordered characteristic roots of $\underset{\sim}{R}:q\times q$ are derived in the literature. In remark 5.3.3 these expressions are compared with the corresponding expressions which follow from the expressions derived in corollary 5.2.1 and theorems 5.2.4 and 5.2.7.

Remark 5.3.3

(i) De Waal (1968, p. 137 - 138) derived an expression for the p.d.f. of $\tilde{R}_q^2$, the largest characteristic root of $\underset{\sim}{R}:q\times q$, when $\underset{\sim}{D}_R$ has the p.d.f. given in (5.3.22). This expression involves five summation signs and some zonal polynomials and leads to an expression for the c.d.f. of $\tilde{R}_q^2$ which also involves five summation signs and some zonal polynomials. From (5.3.32) and along the same lines as in corollary 5.2.1 an expression for $P(\tilde{R}_q^2 < d)$ follows as

(5.3.37) $$P(\tilde{R}_q^2 < d)$$

$$= \frac{\pi^{\frac{1}{2}q(q-1)}\ \tilde{\Gamma}_q(n)\ |I_q - P|^n \prod_{i=1}^{q} (r-i+1)^{i-1}\ |(b_{ij})|}{\tilde{\Gamma}_q(n-r)\ \tilde{\Gamma}_q(r) \prod_{i>j}^{p} (\tilde{\rho}_i^2 - \tilde{\rho}_j^2)\ \{\prod_{i=1}^{q} (n-i+1)^{i-1}\}^2}$$

where

(5.3.38) $$b_{ij} = \sum_{k=0}^{\infty} \frac{(n-q+1)_k\ (n-q+1)_k\ (\tilde{\rho}_i^2)^k}{(r-q+1)_k\ k!} B_d(r+k+1-j,\ n-r-q+1)\ .$$

This expression for $P(\tilde{R}_q^2 < d)$ is from a computational point of view better than the result derived by De Waal (1968).

(ii) Khatri (1969) derived the following expression for $P(0 < \tilde{R}_1^2 < \cdots < \tilde{R}_t^2 < c < \tilde{R}_{t+1}^2 < \cdots < \tilde{R}_q^2 < 1)$ when $\underset{\sim}{D}_R$ has the p.d.f. given in 5.3.22:

(5.3.39) $$P(0 < \tilde{R}_1^2 < \cdots < \tilde{R}_t^2 < c < \tilde{R}_{t+1}^2 < \cdots < \tilde{R}_q^2 < 1)$$

$$= \frac{\tilde{\Gamma}_q(n)\ \pi^{q(q-1)}\ |I_q - P|^n}{\tilde{\Gamma}_q(n-r)\ \tilde{\Gamma}_q(r)\ \tilde{\Gamma}_q(q)} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{[n]_\kappa\ [n]_\kappa\ \tilde{C}_\kappa(P)}{[r]_\kappa\ \tilde{C}_\kappa(I_p)\ k!}$$

$$\chi_{[\kappa]}(1)\ \Sigma_1\ |(b_{ij})|$$

where

Σ_1 denotes the summation over the combination $(\delta_q < \cdots < \delta_{t+1})$ and $(\delta_t < \cdots < \delta_1)$

and

$$b_{ij} = \begin{cases} B_c(r+k_j+q-\delta_i-j+1,\ n-q-r+1)\ , & (i = 1,\cdots,t\ ;\ j = 1,\cdots,q) \\ B(r+k_j+q-\delta_i-j+1,\ n-q-r+1) - B_c(r+k_j+q-\delta_i-j+1,\ n-q-r+1)\ , & (i = t+1,\cdots,q\ ;\ j = 1,\cdots,q)\ . \end{cases}$$

From (5.3.32) and along the same lines as in theorem 5.2.4 an expression for $P(0 < \tilde{R}_1^2 < \cdots < \tilde{R}_t^2 < c < \tilde{R}_{t+1}^2 < \cdots < \tilde{R}_q^2 < 1)$ follows as

$$(5.3.40)\qquad P(0 < \tilde{R}_1^2 < \cdots < \tilde{R}_t^2 < c < \tilde{R}_{t+1}^2 < \cdots < \tilde{R}_q^2 < 1)$$

$$= \frac{\pi^{\frac{1}{2}q(q-1)}\ \tilde{\Gamma}_q(n)\ |I_q - P|^n \prod_{i=1}^{q} (r-i+1)^{i-1}}{\tilde{\Gamma}_q(n-r)\ \tilde{\Gamma}_q(r) \prod_{i>j}^{q} (\tilde{\rho}_i^2 - \tilde{\rho}_j^2)\ \{\prod_{i=1}^{q} (n-i+1)^{i-1}\}^2}$$

$$\Sigma_1 \Sigma_2 (-1)^{\Sigma\delta_i + \Sigma\alpha_i}\ |(b_{3ij})|\,|(b_{4ij})|$$

where

$$(5.3.41)\qquad b_{3ij} = \sum_{k=0}^{\infty} \frac{(n-q+1)_k\ (n-q+1)_k\ (\tilde{\rho}_{\delta_i}^2)^k}{(r-q+1)_k\ k!}\ B_c(r+k-\alpha_j+1,\ n-r-q+1)\ ,\qquad (i,j = 1,\cdots,t)\ ,$$

$$(5.3.42)\qquad b_{4ij} = \sum_{k=0}^{\infty} \frac{(n-q+1)_k\ (n-q+1)_k\ (\tilde{\rho}_{\nu_i}^2)^k}{(r-q+1)_k\ k!}\ \{B(r+k-\beta_j+1,\ n-r-q+1) - B_c(r+k-\beta_j+1,\ n-r-q+1)\ ,\qquad (i,j = 1,\cdots,q-t)$$

and the rest of the symbols are defined as in theorem 3.4.1. It is clear that the expression given in (5.3.40) is from a computational point of view better than the expression derived by Khatri (1969).

(iii) Waikar, Chang and Krishnaiah (1972) derived the following expression for the p.d.f. of any s unordered characteristic roots of $\underset{\sim}{R}:q\times q$ when $\underset{\sim}{D}_R$ has the p.d.f. given in (5.3.22):

(5.3.43) $$f_{\tilde{R}_1^2,\cdots,\tilde{R}_s^2}(\tilde{r}_1^2,\cdots,\tilde{r}_s^2)$$

$$= \frac{\pi^{q(q-1)}\ \tilde{\Gamma}_q(n)\ |I_q-P|^n}{\tilde{\Gamma}_q(n-r)\ \tilde{\Gamma}_q(r)\ \tilde{\Gamma}_q(q)} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{[n]_\kappa\ [n]_\kappa\ \tilde{C}_\kappa(P)}{[r]_\kappa\ k!\ \tilde{C}_\kappa(I_p)}$$

$$\chi_{[\kappa]}(1)\ \Sigma_1\Sigma_2(-1)^{f(\alpha_1,\cdots,\alpha_q)}\ (-1)^{f(\delta_1,\cdots,\delta_q)}$$

$$\Sigma_3 \prod_{i=1}^{q} (\tilde{r}^2_{t_{\delta_i}})^{k_{\alpha_i}+2q-\alpha_i-\delta_i}\ (\tilde{r}_i^2)^{r-q}\ (1-\tilde{r}_i^2)^{n-r-q}$$

$$(q-s)! \prod_{i=s+1}^{q} B(k_{\alpha_i}+q-\alpha_i-\delta_i+r+1,\ n-r-q+1)\ ,$$

$$(0 < \tilde{r}_i^2 < 1\ ,\quad i = 1,\cdots,s)$$

where the symbols are defined as in remark 4.2.5.

From (5.3.32) and along the same lines as in theorem 5.2.7 an expression for the p.d.f. of any s unordered roots of $\underset{\sim}{R}:q\times q$ follows as

(5.3.44) $$f_{\tilde{R}_1^2,\cdots,\tilde{R}_s^2}(\tilde{r}_1^2,\cdots,\tilde{r}_s^2)$$

$$= \frac{\pi^{\frac{1}{2}q(q-1)}\ \tilde{\Gamma}_q(n)\ |I_q-P|^n \prod_{i=1}^{q} (r-i+1)^{i-1}}{q!\ \tilde{\Gamma}_q(n-r)\ \tilde{\Gamma}_q(r) \prod_{i>j}^{q} (\tilde{\rho}_i^2 - \tilde{\rho}_j^2)\ \{\prod_{i=1}^{q} (n-i+1)^{i-1}\}^2}$$

$$\Sigma_1\Sigma_2(-1)^{\Sigma\delta_i+\Sigma\alpha_i}\ |(b_{17ij})|\,|(b_{18ij})|\ ,$$

$$(0 < \tilde{r}_i^2 < 1\ ,\quad i = 1,\cdots,s)$$

where

$$(5.3.45)\quad b_{17ij} = \sum_{t=1}^{s} \tilde{r}_t^{\,r-\alpha_j} (1-\tilde{r}_t)^{n-r-q}$$

$$ {}_2F_1(n-q+1,\ n-q+1\ ;\ r-q+1\ ;\ \tilde{\rho}^2_{\delta_i}\tilde{r}^2_t)\ ,\quad (i,j=1,\cdots,s)\ ,$$

$$(5.3.46)\quad b_{18ij} = (q-s)!\ \sum_{k=0}^{\infty} \frac{(n-q+1)_k\ (n-q+1)_k\ (\tilde{\rho}^2_{\nu_i})^k}{(r-q+1)_k\ k!}$$

$$B(r+k-\beta_j+1,\ n-r-q+1)\ ,\quad (i,j=1,\cdots,q-s)$$

and the rest of the symbols are defined as in theorem 3.4.1. It is clear that the expression in (5.3.44) is in a much simpler form than the expression given in (5.3.43).

5.3.3 P.d.f.s of functions of the characteristic roots of $\underset{\sim}{R}:q\times q$

Troskie (1969, p. 120) showed that the p.d.f.s of $|\underset{\sim}{R}| = \prod_{i=1}^{q} (\tilde{R}_i^2)$ and $|I_q-\underset{\sim}{R}| = \prod_{i=1}^{q} (1-\tilde{R}_i^2)$ can be written in the non-central linear case as the product of q independent Beta-variables. Pillai and Jouris (1971, p. 520 - 521) derived the non-central p.d.f. of $|I_q-\underset{\sim}{R}|$ in terms of Meijer's G-function. They also derived the central p.d.f. of $\operatorname{tr}\underset{\sim}{R}$ for $q=2$. Expressions for the exact non-central p.d.f.s of $|\underset{\sim}{R}|$ and $|I_q-\underset{\sim}{R}|$ are derived by Money (1972, p. 8.1 - 8.12) in the linear case for $q=2$ and $q=3$. Hart (1974, p. 9.1 - 9.3) gave an algorithm which can be used to calculate percentage points and powers when $|\underset{\sim}{R}|$ and $|I_q-\underset{\sim}{R}|$ are used as test criteria to test the hypothesis $H_0 : \Sigma_{12}:q\times r = 0$ in the linear case. Expressions for the asymptotic expansions of the p.d.f.s of $|\underset{\sim}{R}|$ and $|I_q-\underset{\sim}{R}|$ are derived by De Waal (1968, p. 135 - 138) in the non-central linear and non-linear cases.

5.4 THE COMPLEX MULTIVARIATE BETA TYPE 2 DISTRIBUTION

5.4.1 The p.d.f., moments and the joint p.d.f. of the characteristic roots of complex beta type 2 matrices

Let $\underset{\sim}{A}:p\times p \sim CW(p, m, \Sigma)$, $\underset{\sim}{B}:p\times p \sim NCCW(p, n, \Phi, \Omega)$, $\underset{\sim}{V}:p\times p = \underset{\sim}{B}^{-\frac{1}{2}} \underset{\sim}{A} \underset{\sim}{B}^{-\frac{1}{2}}$ and $\underset{\sim}{Z}:p\times p = \underset{\sim}{B}^{\frac{1}{2}} \underset{\sim}{A}^{-1} \underset{\sim}{B}^{\frac{1}{2}}$; then $\underset{\sim}{V}:p\times p$ and $\underset{\sim}{Z}:p\times p$ have the non-central complex multivariate beta type 2A and 2B distributions respectively. These distributions will be denoted by:

$$\underset{\sim}{V}:p\times p \sim NCCMB_{2A}(p, m, n, \Sigma, \Phi, \Omega)$$

and

$$\underset{\sim}{Z}:p\times p \sim NCCMB_{2B}(p, n, m, \Sigma, \Phi, \Omega) .$$

The p.d.f. and moments of $\underset{\sim}{V}:p\times p$ and $D_{\underset{\sim}{V}}$ are given by De Waal (1968, p. 111 - 113) for certain specifications of the parameter matrices. These results as well as the symmetrised p.d.f. of $\underset{\sim}{V}:p\times p$ and $D_{\underset{\sim}{V}}$ are given in theorem 5.4.1.

Theorem 5.4.1

Let $\underset{\sim}{A}:p\times p \sim CW(p, m, \Sigma)$ and $\underset{\sim}{B}:p\times p \sim NCCW(p, n, \Phi, \Omega)$, rank $\Omega = p$ and $\underset{\sim}{A}:p\times p$ and $\underset{\sim}{B}:p\times p$ be independently distributed; then the p.d.f., symmetrised p.d.f. and moments of $\underset{\sim}{V}:p\times p = \underset{\sim}{B}^{-\frac{1}{2}} \underset{\sim}{A} \underset{\sim}{B}^{-\frac{1}{2}}$ and $D_{\underset{\sim}{V}}$ are given below for certain specifications of $\Sigma:p\times p$, $\Phi:p\times p$ and $\Omega:p\times p$.

(i) <u>$\Sigma:p\times p \neq \Phi:p\times p$, i.e. $\underset{\sim}{V}:p\times p \sim NCCMB_{2A}(p, m, n, \Sigma, \Phi, \Omega)$</u>

(5.4.1) $f_{\underset{\sim}{V}}(V)$

$$= \frac{\operatorname{etr}[-\Omega]\ |V|^{m-p}}{\tilde{\Gamma}_p(m)\ \tilde{\Gamma}_p(n)\ |\Sigma|^m\ |\Phi|^n} \int_{B=\bar{B}'>0} \operatorname{etr}[-(\Phi^{-1} B + B^{\frac{1}{2}} \Sigma^{-1} B^{\frac{1}{2}} V)]$$

$$|B|^{m+n-p}\ {}_0\tilde{F}_1(n\ ;\ \Omega \Phi^{-1} B)\ dB\ , \quad V = \bar{V}' > 0\ .$$

(5.4.2) $E(|\underset{\sim}{V}|^h)$

$$= \frac{\tilde{\Gamma}_p(m+h)\ \tilde{\Gamma}_p(n-h)}{\tilde{\Gamma}_p(m)\ \tilde{\Gamma}_p(n)} \ |\Sigma\,\Phi^{-1}|^h \ \mathrm{etr}[-\Omega]\ {}_1\tilde{F}_1(n-h\,;\,n\,;\,\Omega) \ .$$

(ii) $\underline{\Sigma:p\times p \neq \Phi:p\times p, \quad \Omega:p\times p = 0, \quad \text{i.e.} \quad \underset{\sim}{V}:p\times p \sim CMB_{2A}(p,\ m,\ n,\ \Sigma,\ \Phi)}$

(5.4.3) $f_{\underset{\sim}{V}}(V)$

$$= \frac{|V|^{m-p}}{\tilde{\Gamma}_p(m)\ \tilde{\Gamma}_p(n)\ |\Sigma|^m\ |\Phi|^n} \int_{B=\overline{B}'>0} \mathrm{etr}[-(\Phi^{-1}B + B^{\frac{1}{2}}\,\Sigma^{-1}\,B^{\frac{1}{2}}\,V)]$$

$$|B|^{m+n-p}\,dB \ , \quad V = \overline{V}' > 0 \ .$$

(5.4.4) $f_{csym}(V)$

$$= \frac{\tilde{\Gamma}_p(m+n)}{\tilde{\Gamma}_p(m)\ \tilde{\Gamma}_p(n)} \ |\Sigma^{-1}\,\Phi|^m \ \ |V|^{m-p} \ {}_1\tilde{F}_0(m+n\,;\,-\Sigma^{-1}\,\Phi,\,V) \ ,$$

$$V = \overline{V}' > 0 \ , \quad ||\Sigma^{-1}\,\Phi|| < 1 \ .$$

(5.4.5) $f_{\underset{\sim}{D}_V}(D_V)$

$$= \frac{\pi^{p(p-1)}\ \tilde{\Gamma}_p(m+n)}{\tilde{\Gamma}_p(m)\ \tilde{\Gamma}_p(n)\ \tilde{\Gamma}_p(p)} \ |\Sigma^{-1}\,\Phi|^m \prod_{i=1}^{p} \tilde{v}_i^{m-p} \prod_{i>j}^{p} (\tilde{v}_i - \tilde{v}_j)^2$$

$${}_1\tilde{F}_0(m+n\,;\,-\Sigma^{-1}\,\Phi,\,D_V) \ , \quad 0 < \tilde{v}_1 < \cdots < \tilde{v}_p \quad \text{and} \quad ||\Sigma^{-1}\,\Phi|| < 1 \ .$$

(iii) $\Sigma : p\times p = \Phi : p\times p$, i.e. $\underset{\sim}{V} : p\times p \sim NCCMB_{2A}(p,\ m,\ n,\ \Omega)$

(5.4.6) $f_{\underset{\sim}{V}}(V)$

$$= \frac{\mathrm{etr}[-\Omega]\ |V|^{m-p}}{\tilde{\Gamma}_p(m)\ \tilde{\Gamma}_p(n)\ |\Sigma|^{m+n}} \int_{B=\bar{B}'>0} \mathrm{etr}[-(\Sigma^{-1} B + B^{\frac{1}{2}} \Sigma^{-1} B^{\frac{1}{2}} V)]$$

$$|B|^{m+n-p}\ {}_0\tilde{F}_1(n\ ;\ \Omega\, \Sigma^{-1} B)\ dB\ , \quad V = \bar{V}' > 0\ .$$

(5.4.7) $f_{csym}(V)$

$$= \frac{\tilde{\Gamma}_p(m+n)\ \mathrm{etr}[-\Omega]}{\tilde{\Gamma}_p(m)\ \tilde{\Gamma}_p(n)}\ |V|^{m-p}\ |I_p+V|^{-(m+n)}$$

$${}_1\tilde{F}_1(m+n\ ;\ n\ ;\ \Omega,\ (I_p+V)^{-1})\ , \quad V = \bar{V}' > 0\ .$$

(5.4.8) $E(|(I_p+\underset{\sim}{V})^{-1}|^h)$

$$= \frac{\tilde{\Gamma}_p(n+h)\ \tilde{\Gamma}_p(m+n)}{\tilde{\Gamma}_p(n)\ \tilde{\Gamma}_p(m+n+h)}\ \mathrm{etr}[-\Omega]\ {}_2\tilde{F}_2(n+h,\ m+n\ ;\ m+n+h,\ n\ ;\ \Omega)\ .$$

(5.4.9) $E(|\underset{\sim}{V}(I_p+\underset{\sim}{V})^{-1}|^h)$

$$= \frac{\tilde{\Gamma}_p(m+n)\ \tilde{\Gamma}_p(m+h)}{\tilde{\Gamma}_p(m)\ \tilde{\Gamma}_p(m+n+h)}\ \mathrm{etr}[-\Omega]\ {}_1\tilde{F}_1(m+n\ ;\ m+n+h\ ;\ \Omega)\ .$$

(5.4.10) $f_{\underset{\sim}{D}_V}(D_V)$

$$= \frac{\pi^{p(p-1)}\ \tilde{\Gamma}_p(m+n)\ \mathrm{etr}[-\Omega]}{\tilde{\Gamma}_p(p)\ \tilde{\Gamma}_p(m)\ \tilde{\Gamma}_p(n)}\ |D_V|^{m-p} |I_p+D_V|^{-(m+n)} \prod_{i>j}^{p} (\tilde{v}_i - \tilde{v}_j)^2$$

$${}_1\tilde{F}_1(m+n\ ;\ n\ ;\ \Omega,\ (I_p + D_V)^{-1})\ , \quad 0 < \tilde{v}_1 < \cdots < \tilde{v}_p\ .$$

(iv) $\underline{\Sigma:p\times p = \Phi:p\times p, \quad \Omega:p\times p = 0, \quad \text{i.e.} \quad \underset{\sim}{V}:p\times p \sim CMB_2(p,\ m,\ n)}$

$$(5.4.11) \quad f_{\underset{\sim}{V}}(V) = \frac{\tilde{\Gamma}_p(m+n)}{\tilde{\Gamma}_p(n)\ \tilde{\Gamma}_p(m)} |V|^{m-p}\ |I_p+V|^{-(m+n)}, \quad V = \bar{V}' > 0 .$$

Proof

(5.4.1) De Waal (1968, p. 111).

(5.4.2)

$$(5.4.12) \quad E(|\underset{\sim}{V}|^h) = \int_{V=\bar{V}'>0} |V|^h\ f_{\underset{\sim}{V}}(V)\ dV$$

$$= \frac{etr[-\Omega]}{\tilde{\Gamma}_p(m)\ \tilde{\Gamma}_p(n)\ |\Sigma|^m\ |\Phi|^n} I^*$$

where

$$(5.4.13) \quad I^* = \int_{V=\bar{V}'>0} |V|^{m+h-p} \int_{B=\bar{B}'>0} etr[-(\Phi^{-1} B + B^{\frac{1}{2}} \Sigma^{-1} B^{\frac{1}{2}} V)]\ |B|^{m+n-p}$$

$$ {}_0\tilde{F}_1(n\ ;\ \Omega\Phi^{-1} B)\ dB\ dV .$$

Changing of the order of integration and integration w.r.t. $V:p\times p$, using (2.3.5), leads to

$$(5.4.14) \quad I^* = \tilde{\Gamma}_p(m+h)\ |\Sigma|^{m+h} \int_{B=\bar{B}'>0} etr[-\Phi^{-1} B]\ |B|^{n-h-p}$$

$$ {}_0\tilde{F}_1(n\ ;\ \Omega\Phi^{-1} B)\ dB$$

$$= \tilde{\Gamma}_p(m+h)\ \tilde{\Gamma}_p(n-h)\ |\Sigma|^{m+h}\ |\Phi|^{n-h}\ {}_1\tilde{F}_1(n-h\ ;\ n\ ;\ \Omega) ,$$

(from (2.3.5)) .

Substitution of (5.4.14) into (5.4.12) leads to (5.4.2).

(5.4.3)

Let $\Omega{:}p\times p = 0$ in (5.4.1); then (5.4.3) follows.

(5.4.4)

$$(5.4.15)\quad f_{csym}(V) = \frac{|V|^{m-p}}{\tilde{\Gamma}_p(m)\ \tilde{\Gamma}_p(n)\ |\Sigma|^m\ |\Phi|^n}\ I^*$$

where

$$(5.4.16)\quad I^* = \int_{U(p)} \int_{B=\bar{B}'>0} \operatorname{etr}[-(\Phi^{-1} B + B^{\frac{1}{2}} \Sigma^{-1} B^{\frac{1}{2}} U V \bar{U}')]\ |B|^{m+n-p}\ dB\ dU\ .$$

Change the order of integration, then:

$$(5.4.17)\quad I^* = \int_{B=\bar{B}'>0} \operatorname{etr}[-\Phi^{-1} B]\ |B|^{m+n-p} \int_{U(p)} {}_0\tilde{F}_0(-B^{\frac{1}{2}} \Sigma^{-1} B^{\frac{1}{2}} U V \bar{U}')\ dU\ dB$$

$$= \int_{B=\bar{B}'>0} \operatorname{etr}[-\Phi^{-1} B]\ |B|^{m+n-p}\ {}_0\tilde{F}_0(-B\,\Sigma^{-1}, V)\ dB\ ,$$

(from (2.3.4))

$$= \tilde{\Gamma}_p(m+n)\ |\Phi|^{m+n}\ {}_1\tilde{F}_0(m+n\,;\, -\Sigma^{-1}\Phi, V)\ ,\quad \text{(from (2.3.5))}\ .$$

From definition 2.3.2 follows that ${}_1\tilde{F}_0(m+n\,;\,\Sigma^{-1}\Phi, V)$ is convergent for $\|\Sigma^{-1}\Phi\| < 1$ or $\|V\| < 1$. Substitution of (5.4.17) into (5.4.15) leads to (5.4.4).

(5.4.5)

The application of corollary 2.7.1 and theorem 3.2.1 leads to (5.4.5).

(5.4.6)

Let $\Sigma:p\times p = \Phi:p\times p$ in (5.4.1); then (5.4.6) follows.

(5.4.7)

$$(5.4.18) \quad f_{csym}(V) = \frac{\operatorname{etr}[-\Omega]\ |V|^{m-p}}{\tilde{\Gamma}_p(m)\ \tilde{\Gamma}_p(n)\ |\Sigma|^{m+n}}\ I^*$$

where

$$(5.4.19) \quad I^* = \int_{U(p)} \int_{B=\bar{B}'>0} \operatorname{etr}[-(\Sigma^{-1}B + B^{\frac{1}{2}}\Sigma^{-1}B^{\frac{1}{2}}UV\bar{U}')]\,|B|^{m+n-p}\ {}_0\tilde{F}_1(n\,;\,\Omega\Sigma^{-1}B)\ dB\ dU\ .$$

Changing of the order of integration and integration over the unitary group, using (2.3.11) and (2.3.4), leads to

$$(5.4.20) \quad I^* = \int_{B=\bar{B}'>0} |B|^{m+n-p}\ {}_0\tilde{F}_1(n\,;\,\Omega\Sigma^{-1}B)\ {}_0\tilde{F}_0(-B\Sigma^{-1},\,I_p+V)\ dB$$

$$= \int_{B=\bar{B}'>0} |B|^{m+n-p}\ {}_0\tilde{F}_1(n\,;\,\Omega\Sigma^{-1}B) \int_{U(p)} \operatorname{etr}[-B\Sigma^{-1}U(I_p+V)\bar{U}']\ dU\ dB\ .$$

Changing of the order of integration and integration w.r.t. $B:p\times p$, using (2.3.5), leads to

$$(5.4.21) \quad I^* = \tilde{\Gamma}_p(m+n)\ |\Sigma|^{m+n}\ |I_p+V|^{-(m+n)}$$

$$\int_{U(p)} {}_1\tilde{F}_1(m+n\,;\,n\,;\,\Omega\, U(I_p+V)^{-1}\bar{U}')\ dU$$

$$= \tilde{\Gamma}_p(m+n)\ |\Sigma|^{m+n}\ |I_p+V|^{-(m+n)}\ {}_1\tilde{F}_1(m+n\,;\,n\,;\,\Omega,\ (I_p+V)^{-1})\ .$$

Substitution of (5.4.21) into (5.4.18) leads to (5.4.7).

<u>(5.4.8)</u> De Waal (1968, p. 112).

<u>(5.4.9)</u>

Using (2.3.8), then (5.4.9) follows from (5.4.7).

<u>(5.4.10)</u>

The application of corollary 2.7.1 and theorem 3.2.1 leads to (5.4.10). This result is also derived by De Waal (1968, p. 113) and James (1964, p. 490).

<u>(5.4.11)</u>

Let $\Omega{:}p\times p = 0$ in (5.4.6), then (5.4.11) follows by using (2.3.5).

<u>Remark 5.4.1</u>

(i) Let $\Sigma{:}p\times p = I_p \neq \Phi{:}p\times p$; then by using (2.3.5), an expression for $f_{\underset{\sim}{V}}(V)$ follows from (5.4.1) as

$$(5.4.22) \quad f_{\underset{\sim}{V}}(V) = \frac{\tilde{\Gamma}_p(m+n)\ \operatorname{etr}[-\Omega]}{\tilde{\Gamma}_p(m)\ \tilde{\Gamma}_p(n)\ |\Phi|^n}\ |V|^{m-p}\ |\Phi^{-1}+V|^{-(m+n)}$$

$$ {}_1\tilde{F}_1(m+n\,;\,n\,;\,\Omega\,\Phi^{-1}\,(\Phi^{-1}+V)^{-1})\ ,\quad V=\bar{V}'>0\ .$$

(ii) De Waal (1968, p. 117 - 120) also proved that if rank $\Omega < p$, then the p.d.f. of $\underset{\sim}{V}{:}p\times p$ can be written as a non-central multivariate beta type 2 p.d.f. multiplied with the product of independent single-variable beta type 2 p.d.f.s .

In theorem 5.4.2 the p.d.f., symmetrised p.d.f. and moments of $\underset{\sim}{Z}:p\times p = \underset{\sim}{B}^{\frac{1}{2}} \underset{\sim}{A}^{-1} \underset{\sim}{B}^{\frac{1}{2}}$ and $\underset{\sim}{D}_Z$, for different specifications of the parameter matrices, are given. These different p.d.f.s of $\underset{\sim}{Z}:p\times p$ follow clearly from the corresponding p.d.f.s of $\underset{\sim}{V}:p\times p$ by making the transformation $Z = V^{-1}$.

Theorem 5.4.2

Let $\underset{\sim}{A}:p\times p \sim CW(p, m, \Sigma)$ and $\underset{\sim}{B}:p\times p \sim NCCW(p, n, \Phi, \Omega)$, rank $\Omega = p$ and $\underset{\sim}{A}:p\times p$ and $\underset{\sim}{B}:p\times p$ be independently distributed; then the p.d.f., symmetrised p.d.f. and moments of $\underset{\sim}{Z}:p\times p = \underset{\sim}{B}^{\frac{1}{2}} \underset{\sim}{A}^{-1} \underset{\sim}{B}^{\frac{1}{2}}$ and $\underset{\sim}{D}_Z$ are given below for certain specifications of $\Sigma:p\times p$, $\Phi:p\times p$ and $\Omega:p\times p$.

(i) $\underline{\Sigma:p\times p \neq \Phi:p\times p \text{, i.e. } \underset{\sim}{Z}:p\times p \sim NCCMB_{2B}(p, n, m, \Sigma, \Phi, \Omega)}$

(5.4.23) $f_{\underset{\sim}{Z}}(Z)$

$$= \frac{etr[-\Omega]\ |Z|^{-(m+p)}}{\tilde{\Gamma}_p(m)\ \tilde{\Gamma}_p(n)\ |\Sigma|^m\ |\Phi|^n} \int_{B=\bar{B}'>0} etr[-(\Phi^{-1} B + B^{\frac{1}{2}} \Sigma^{-1} B^{\frac{1}{2}} Z^{-1})]$$

$$|B|^{m+n-p}\ {}_0\tilde{F}_1(n\,;\,\Omega\,\Phi^{-1} B)\ dB\ , \quad Z = \bar{Z}' > 0\ .$$

(5.4.24) $$E(|\underset{\sim}{Z}|^h) = \frac{\tilde{\Gamma}_p(m-h)\ \tilde{\Gamma}_p(n+h)}{\tilde{\Gamma}_p(m)\ \tilde{\Gamma}_p(n)} |\Sigma^{-1}\,\Phi|^h\, etr[-\Omega]\,{}_1\tilde{F}_1(n+h\,;\,n\,;\,\Omega)\ .$$

(ii) $\underline{\Sigma:p\times p \neq \Phi:p\times p\text{, } \Omega:p\times p = 0\text{, i.e. } \underset{\sim}{Z}:p\times p \sim CMB_{2B}(p, n, m, \Sigma, \Phi)}$

(5.4.25) $f_{\underset{\sim}{Z}}(Z)$

$$= \frac{|Z|^{-(m+p)}}{\tilde{\Gamma}_p(m)\ \tilde{\Gamma}_p(n)\ |\Sigma|^m\ |\Phi|^n} \int_{B=\bar{B}'>0} etr[-(\Phi^{-1} B + B^{\frac{1}{2}} \Sigma^{-1} B^{\frac{1}{2}} Z^{-1})]$$

$$|B|^{m+n-p}\, dB \;, \; Z = \bar{Z}' > 0 \;.$$

(5.4.26) $$f_{csym}(Z) = \frac{\tilde{\Gamma}_p(m+n)\ |\Sigma^{-1}\Phi|^m}{\tilde{\Gamma}_p(m)\ \tilde{\Gamma}_p(n)}\ |Z|^{-(m+p)}$$

$$_1\tilde{F}_0(m+n\,;\,-\Sigma^{-1}\Phi,\ Z^{-1}) \;, \quad Z = \bar{Z}' > 0 \;, \quad \|\Sigma^{-1}\Phi\| < 1 \;.$$

(5.4.27) $$f_{\underset{\sim}{D}_Z}(D_Z) = \frac{\pi^{p(p-1)}\ \tilde{\Gamma}_p(m+n)}{\tilde{\Gamma}_p(m)\ \tilde{\Gamma}_p(n)\ \tilde{\Gamma}_p(p)}\ |\Sigma^{-1}\Phi|^m\ \ |D_Z|^{-(m+p)}$$

$$\prod_{i>j}^{p}(\tilde{z}_i - \tilde{z}_j)^2\ \ _1\tilde{F}_0(m+n\,;\,-\Sigma^{-1}\Phi,\ D_Z^{-1})$$

$$0 < \tilde{z}_1 < \cdots < \tilde{z}_p \;, \quad \|\Sigma^{-1}\Phi\| < 1 \;.$$

(iii) <u>$\Sigma{:}p\times p = \Phi{:}p\times p$, i.e. $\underset{\sim}{Z}{:}p\times p \sim NCCMB_{2B}(p,\ n,\ m,\ \Omega)$</u>

(5.4.28) $$f_{\underset{\sim}{Z}}(Z)$$

$$= \frac{\mathrm{etr}[-\Omega]\ \ |Z|^{-(m+p)}}{\tilde{\Gamma}_p(n)\ \tilde{\Gamma}_p(m)\ \ |\Sigma|^{m+n}} \int_{B=\bar{B}'>0} \mathrm{etr}[-(\Sigma^{-1}B + B^{\frac{1}{2}}\Sigma^{-1}B^{\frac{1}{2}}Z^{-1})]$$

$$|B|^{m+n-p}\ \ _0\tilde{F}_1(n\,;\,\Omega\,\Sigma^{-1}B)\ dB \;, \quad Z = \bar{Z}' > 0 \;.$$

(5.4.29) $$f_{csym}(Z) = \frac{\tilde{\Gamma}_p(m+n)\ \mathrm{etr}[-\Omega]}{\tilde{\Gamma}_p(m)\ \tilde{\Gamma}_p(n)}\ |Z|^{n-p}\ \ |I_p+Z|^{-(m+n)}$$

$$_1\tilde{F}_1(m+n\,;\,n\,;\,\Omega,\ Z(I_p+Z)^{-1}) \;, \quad Z = \bar{Z}' > 0 \;.$$

$$(5.4.30)\quad E(|(\underset{\sim}{I}_p+\underset{\sim}{Z})^{-1}|^h) = \frac{\tilde{\Gamma}_p(m+n)\ \mathrm{etr}[-\Omega]\ \tilde{\Gamma}_p(m+h)}{\tilde{\Gamma}_p(m)\ \tilde{\Gamma}_p(m+n+h)}$$

$$ {}_1\tilde{F}_1(m+n\ ;\ m+n+h\ ;\ \Omega)\ .$$

$$(5.4.31)\quad E(|\underset{\sim}{Z}(I_p+\underset{\sim}{Z})^{-1}|^h) = \frac{\tilde{\Gamma}_p(m+n)\ \tilde{\Gamma}_p(n+h)\ \mathrm{etr}[-\Omega]}{\tilde{\Gamma}_p(n)\ \tilde{\Gamma}_p(m+n+h)}$$

$$ {}_2\tilde{F}_2(m+n,\ n+h\ ;\ n,\ m+n+h\ ;\ \Omega)\ .$$

$$(5.4.32)\quad f_{\underset{\sim}{D}_Z}(D_Z) = \frac{\pi^{p(p-1)}\ \tilde{\Gamma}_p(m+n)\ \mathrm{etr}[-\Omega]}{\tilde{\Gamma}_p(m)\ \tilde{\Gamma}_p(n)\ \tilde{\Gamma}_p(p)}\ |D_Z|^{n-p}\ |I_p+D_Z|^{-(m+n)}$$

$$\prod_{i>j}^{p}(\tilde{z}_i-\tilde{z}_j)^2\ {}_1\tilde{F}_1(m+n\ ;\ n\ ;\ \Omega,\ D_Z(I_p+D_Z)^{-1})\ ,$$

$$0<\tilde{z}_1<\cdots<\tilde{z}_p\ .$$

(iv) <u>$\Sigma:p\times p = \Phi:p\times p\ ,\quad \Omega:p\times p = 0\ ,\quad$ i.e. $\quad \underset{\sim}{Z}:p\times p \sim CMB_2(p,\ n,\ m)$</u>

$$(5.4.33)\quad f_{\underset{\sim}{Z}}(Z) = \frac{\tilde{\Gamma}_p(m+n)}{\tilde{\Gamma}_p(m)\ \tilde{\Gamma}_p(n)}\ |Z|^{n-p}\ |I_p+Z|^{-(m+n)}\ .$$

<u>Proof</u>

(<u>5.4.23</u>), (<u>5.4.25</u>), (<u>5.4.26</u>), (<u>5.4.27</u>), (<u>5.4.28</u>), (<u>5.4.29</u>) (<u>5.4.32</u>), (<u>5.4.33</u>)

In (5.4.1), (5.4.3), (5.4.4), (5.4.5), (5.4.6), (5.4.7), (5.4.10) and (5.4.11) make the transformation

$$(5.4.34)\quad Z = V^{-1}$$

with inverse transformation

$$(5.4.35) \quad V = Z^{-1} .$$

The jacobian of (5.4.35) follows from (2.2.8) as

$$(5.4.36) \quad J(V \to Z) = |Z|^{-2p} .$$

Hence the results for $\underset{\sim}{Z}:p\times p$ follow.

(5.4.24)

$$(5.4.37) \quad E(|\underset{\sim}{Z}|^h) = \frac{\operatorname{etr}[-\Omega]}{\tilde{\Gamma}_p(m)\ \tilde{\Gamma}_p(n)\ |\Sigma|^m\ |\Phi|^n}\ I_1^*$$

where

$$(5.4.38) \quad I_1^* = \int_{Z=\bar{Z}'>0} |Z|^{h-(m+p)} \int_{B=\bar{B}'>0} \operatorname{etr}[-(\Phi^{-1}B + B^{\frac{1}{2}}\Sigma^{-1}B^{\frac{1}{2}}Z^{-1})]$$

$$|B|^{m+n-p}\ {}_0\tilde{F}_1(n\ ;\ \Omega\,\Phi^{-1}B)\ dB\ dZ .$$

Change the order of integration; then

$$(5.4.39) \quad I_1^* = \int_{B=\bar{B}'>0} \operatorname{etr}[-\Phi^{-1}B]\ {}_0\tilde{F}_1(n\ ;\ \Omega\,\Phi^{-1}B)\ |B|^{m+n-p}\ I_2^*\ dB$$

where

$$(5.4.40) \quad I_2^* = \int_{Z=\bar{Z}'>0} |Z|^{h-(m+p)}\ \operatorname{etr}[-B^{\frac{1}{2}}\Sigma^{-1}B^{\frac{1}{2}}Z^{-1}]\ dZ .$$

In (5.4.40) make the transformation $Y = Z^{-1}$ with inverse transformation $Z = Y^{-1}$ and jacobian $J(Z \to Y) = |Y|^{-2p}$; then

$$(5.4.41)\quad I_2^* = \int_{Y=\bar{Y}'>0} |Y|^{(m-h)-p}\ \mathrm{etr}[-B^{\frac{1}{2}}\Sigma^{-1}B^{\frac{1}{2}}Y]\ dY$$

$$= \tilde{\Gamma}_p(m-h)\ |\Sigma|^{m-h}\ |B|^{h-m}\ , \quad (\text{from } (2.3.5))\ .$$

Substitution of (5.4.41) into (5.4.39) leads to

$$(5.4.42)\quad I_1^* = \tilde{\Gamma}_p(m-h)\ |\Sigma|^{m-h} \int_{B=\bar{B}'>0} \mathrm{etr}[-\Phi^{-1}B]\ |B|^{n+h-p}$$

$${}_0\tilde{F}_1(n\ ;\ \Omega\,\Phi^{-1}B)\ dB$$

$$= \tilde{\Gamma}_p(m-h)\ |\Sigma|^{m-h}\ \tilde{\Gamma}_p(n+h)\ |\Phi|^{n+h}\ {}_1\tilde{F}_1(n+h\ ;\ n\ ;\ \Omega)\ ,$$

$$(\text{from } (2.3.5))\ .$$

Substitution of (5.4.42) into (5.4.37) leads to (5.4.24).

(<u>5.4.30</u>), (<u>5.4.31</u>)

These two expressions follow from (5.4.29) by using (2.3.9).

<u>Remark 5.4.2</u>

Let $\Sigma:p\times p = I_p \neq \Phi:p\times p$; then an expression for $f_{\underset{\sim}{Z}}(Z)$ follows from (5.4.23) as

$$(5.4.43)\quad f_{\underset{\sim}{Z}}(Z) = \frac{\tilde{\Gamma}_p(m+n)\ \mathrm{etr}[-\Omega]}{\tilde{\Gamma}_p(m)\ \tilde{\Gamma}_p(n)\ |\Phi|^n}\ |Z|^{-(m+p)}\,|\Phi^{-1}+Z^{-1}|^{-(m+n)}$$

$${}_1\tilde{F}_1(m+n\ ;\ n\ ;\ \Omega\Phi^{-1}(\Phi^{-1}+Z^{-1})^{-1})\ , \quad Z=\bar{Z}'>0\ .$$

5.4.2 <u>Certain marginal distributions of the characteristic roots of complex beta type 2 matrices</u>

In theorem 5.4.3 it is shown how the p.d.f.s of $D_{\underset{\sim}{V}}$ and $D_{\underset{\sim}{Z}}$, when $\underset{\sim}{V}:p\times p \sim \mathrm{NCCMB}_{2A}(p,\ m,\ n,\ \Omega)$, $\underset{\sim}{Z}:p\times p \sim \mathrm{NCCMB}_{2B}(p,\ n,\ m,\ \Omega)$

and $\underset{\sim}{V}:p\times p \sim CMB_2(p,\ m,\ n)$, can be written in a form such that the random components are in the form given in (3.2.3).

Theorem 5.4.3

Let $\underset{\sim}{V}:p\times p \sim NCCMB_{2A}(p,\ m,\ n,\ \Omega)$; then

$$(5.4.44)\quad f_{\underset{\sim}{D}_V}(D_V) = \frac{\pi^{\frac{1}{2}p(p-1)}\ \tilde{\Gamma}_p(m+n)\ \mathrm{etr}[-\Omega]\ \prod_{i=1}^{p}(n-i+1)^{i-1}}{\tilde{\Gamma}_p(m)\ \tilde{\Gamma}_p(n)\ \prod_{i>j}^{p}(\tilde{\omega}_i-\tilde{\omega}_j)\ \prod_{i=1}^{p}(m+n-p+1)^{i-1}}$$

$$\prod_{i=1}^{p} \tilde{v}_i^{m-p}\ (1+\tilde{v}_i)^{-(m+n)+2(p-1)}\ \left|\left(\left(\frac{1}{1+\tilde{v}_{p-j+1}}\right)^{p-i}\right)\right|$$

$$\left|\left({}_1F_1(m+n-p+1\ ;\ n-p+1\ ;\ \tilde{\omega}_i(1+\tilde{v}_{p-j+1})^{-1})\right)\right| ,$$

$$0 < \tilde{v}_1 < \cdots < \tilde{v}_p .$$

Let $\underset{\sim}{Z}:p\times p \sim NCCMB_{2B}(p,\ n,\ m,\ \Omega)$; then

$$(5.4.45)\quad f_{\underset{\sim}{D}_Z}(D_Z) = \frac{\pi^{\frac{1}{2}p(p-1)}\ \tilde{\Gamma}_p(m+n)\ \mathrm{etr}[-\Omega]\ \prod_{i=1}^{p}(n-i+1)^{i-1}}{\tilde{\Gamma}_p(m)\ \tilde{\Gamma}_p(n)\ \prod_{i>j}^{p}(\tilde{\omega}_i-\tilde{\omega}_j)\ \prod_{i=1}^{p}(m+n-i+1)^{i-1}}$$

$$\prod_{i=1}^{p} \tilde{z}_i^{n-p}\ (1+\tilde{z}_i)^{-(m+n)+2(p-1)}\ \left|\left(\left(\frac{\tilde{z}_j}{1+\tilde{z}_j}\right)^{p-i}\right)\right|$$

$$\left|\left({}_1F_1\left(m+n-p+1\ ;\ n-p+1\ ;\ \tilde{\omega}_i\ \frac{\tilde{z}_j}{1+\tilde{z}_j}\right)\right)\right| ,$$

$$0 < \tilde{z}_1 < \cdots < \tilde{z}_p .$$

Let $\underset{\sim}{V}:p\times p \sim CMB_2(p,\ m,\ n)$; then

$$(5.4.46)\quad f_{\underset{\sim}{D}_V}(D_V) = \frac{\pi^{p(p-1)}\,\tilde{\Gamma}_p(m+n)}{\tilde{\Gamma}_p(m)\,\tilde{\Gamma}_p(n)\,\tilde{\Gamma}_p(p)} \prod_{i=1}^{p} \tilde{v}_i^{m-p}\,(1+\tilde{v}_i)^{-(m+n)+2(p-1)}$$

$$\left|\left(\left(\frac{1}{1+\tilde{v}_j}\right)^{p-i}\right)\right|^2, \quad 0 < \tilde{v}_1 < \cdots < \tilde{v}_p .$$

Proof

(5.4.44)

From (2.2.50) and the fact that

$$\left|\left(\left(\frac{1}{1+\tilde{v}_j}\right)^{p-i}\right)\right|^2 = \left|\left(\left(\frac{1}{1+\tilde{v}_{p-j+1}}\right)^{p-i}\right)\right|^2,$$

it follows that (5.4.10) can be written as

$$(5.4.47)\quad f_{\underset{\sim}{D}_V}(D_V) = \frac{\pi^{p(p-1)}\,\tilde{\Gamma}_p(m+n)\,\mathrm{etr}[-\Omega]}{\tilde{\Gamma}_p(p)\,\tilde{\Gamma}_p(m)\,\tilde{\Gamma}_p(n)}$$

$$\prod_{i=1}^{p} \tilde{v}_i^{m-p}\,(1+\tilde{v}_i)^{-(m+n)+2(p-1)}\,\left|\left(\left(\frac{1}{1+\tilde{v}_{p-j+1}}\right)^{p-i}\right)\right|^2$$

$${}_1\tilde{F}_1(m+n\,;\,n\,;\,\Omega,(I_p+D_V)^{-1}) .$$

From (2.2.41), (2.3.14) and (2.3.15) follows that

$$(5.4.48)\quad {}_1\tilde{F}_1(m+n\,;\,n\,;\,\Omega,\,(I_p+D_V)^{-1})$$

$$= \frac{\tilde{\Gamma}_p(p)\,\prod_{i=1}^{p}(n-i+1)^{i-1}}{\pi^{\frac{1}{2}p(p-1)}\,\prod_{i>j}^{p}(\tilde{\omega}_i - \tilde{\omega}_j)}$$

$$\frac{|({}_1F_1(m+n-p+1\,;\,n-p+1\,;\,\tilde{\omega}_i(1+\tilde{v}_{p-j+1})^{-1}))|}{|((\frac{1}{1+\tilde{v}_{p-j+1}})^{p-i})| \prod_{i=1}^{p}(m+n-i+1)^{i-1}} .$$

Substitution of (5.4.48) into (5.4.47) leads to (5.4.44).

(<u>5.4.45</u>)

From (2.2.52) follows that (5.4.32) can be written as

(5.4.49) $f_{\underset{\sim}{D}_Z}(D_Z)$

$$= \frac{\pi^{p(p-1)}\,\tilde{\Gamma}_p(m+n)\;\mathrm{etr}[-\Omega]}{\tilde{\Gamma}_p(m)\;\tilde{\Gamma}_p(n)\;\tilde{\Gamma}_p(p)} \prod_{i=1}^{p} \tilde{z}_i^{n-p}\,(1+\tilde{z}_i)^{-(m+n)+2(p-1)}$$

$$|((\frac{\tilde{z}_j}{1+\tilde{z}_j})^{p-i})|^2 \;{}_1\tilde{F}_1(m+n\,;\,n\,;\,\Omega,\;D_Z(I_p+D_Z)^{-1}) .$$

From (2.2.41), (2.3.14) and (2.3.15) follows that

(5.4.50) ${}_1\tilde{F}_1(m+n\,;\,n\,;\,\Omega,\;D_Z(I_p+D_Z)^{-1})$

$$= \frac{\tilde{\Gamma}_p(p)\prod_{i=1}^{p}(n-i+1)^{i-1}\;|({}_1F_1(m+n-p+1\,;\,n-p+1\,;\,\tilde{\omega}_i(\frac{\tilde{z}_j}{1+\tilde{z}_j})))|}{\pi^{\frac{1}{2}p(p-1)}\prod_{i>j}^{p}(\tilde{\omega}_i-\tilde{\omega}_j)\;|((\frac{\tilde{z}_j}{1+\tilde{z}_j})^{p-i})|\;\prod_{i=1}^{p}(m+n-i+1)^{i-1}} .$$

Substitution of (5.4.50) into (5.4.49) leads to (5.4.45).

(<u>5.4.46</u>)

Let $\Omega{:}p\times p = 0$ in (5.4.10); then (5.4.46) follows from (2.2.50).

<u>Remark 5.4.3</u>

By using the result (2.2.43), it follows from (5.4.10) that the p.d.f. of $\underset{\sim}{D}_V$ when $\underset{\sim}{V}{:}p\times p \sim \mathrm{NCCMB}_{2A}(p,\,m,\,n,\,\Omega)$, can be written as

(5.4.51) $$f_{\underset{\sim}{D}_V}(D_V)$$

$$= \frac{\pi^{p(p-1)} \tilde{\Gamma}_p(m+n) \operatorname{etr}[-\Omega]}{\tilde{\Gamma}_p(p) \tilde{\Gamma}_p(m) \tilde{\Gamma}_p(n)} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{[m+n]_\kappa \tilde{C}_\kappa \Omega}{[n]_\kappa k! \tilde{C}_\kappa(I_p)}$$

$$\chi_{[\kappa]}(1) \prod_{i=1}^{p} \tilde{v}_i^{m-p} (1+\tilde{v}_i)^{2(p-1)-(m+n)} |((\frac{1}{1+\tilde{v}_j})^{k_i+p-i})|$$

$$|((\frac{1}{1+\tilde{v}_j})^{p-i})| , \quad 0 < \tilde{v}_1 < \cdots < \tilde{v}_p$$

and by using (2.2.44), it follows from (5.4.32) that the p.d.f. of $\underset{\sim}{D}_Z$ when $\underset{\sim}{Z}:p\times p \sim NCCMB_{2B}(p, n, m, \Omega)$, can be written as

(5.4.52) $$f_{\underset{\sim}{D}_Z}(D_Z)$$

$$= \frac{\pi^{p(p-1)} \tilde{\Gamma}_p(m+n) \operatorname{etr}[-\Omega]}{\tilde{\Gamma}_p(m) \tilde{\Gamma}_p(n) \tilde{\Gamma}_p(p)} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{[m+n]_\kappa \tilde{C}_\kappa(\Omega)}{[n]_\kappa k! \tilde{C}_\kappa(I_p)}$$

$$\chi_{[\kappa]}(1) \prod_{i=1}^{p} \tilde{z}_i^{n-p} (1+\tilde{z}_i)^{2(p-1)-(m+n)} |((\frac{\tilde{z}_j}{1+\tilde{z}_j})^{k_i+p-i})|$$

$$|((\frac{\tilde{z}_j}{1+\tilde{z}_j})^{p-i})| , \quad 0 < \tilde{z}_1 < \cdots < \tilde{z}_p .$$

From a computational point of view (5.4.44) and (5.4.45) lead to better results than (5.4.51) and (5.4.52). Expressions which contain the random component given in (5.4.52) will be discussed in chapter 7.

Let $\underset{\sim}{G}:p\times p = (\underset{\sim}{A}+\underset{\sim}{B})^{-\frac{1}{2}}\underset{\sim}{A}(\underset{\sim}{A}+\underset{\sim}{B})^{-\frac{1}{2}}$ be the complex multivariate beta type 1A matrix; then

$$(5.4.53)\quad |G-\tilde{g}I_p| = |(A+B)^{-\frac{1}{2}}A(A+B)^{-\frac{1}{2}} - \tilde{g}I_p| = 0$$

$$\text{iff. } |B^{-\frac{1}{2}}AB^{-\frac{1}{2}} - \frac{\tilde{g}}{1-\tilde{g}}I_p| = 0$$

$$\text{iff. } |V - \frac{\tilde{g}}{1-\tilde{g}}I_p| = 0 \ .$$

Because of this relation between the characteristic roots of $\underset{\sim}{G}:p\times p$ and $\underset{\sim}{V}:p\times p$, the p.d.f. of $D_{\underset{\sim}{V}}$ follows from the p.d.f. of $D_{\underset{\sim}{G}}$ by making the following transformation in the p.d.f. of $D_{\underset{\sim}{G}}$:

$$(5.4.54)\quad \tilde{v}_i = \frac{\tilde{g}_i}{1-\tilde{g}_i}\ , \quad (i=1,\cdots,p)$$

with inverse transformation

$$(5.4.55)\quad \tilde{g}_i = \frac{\tilde{v}_i}{1+\tilde{v}_i}\ , \quad (i=1,\cdots,p)\ .$$

The jacobian of (5.4.55) follows as

$$(5.4.56)\quad J(\tilde{g}_1,\cdots,\tilde{g}_p \rightarrow \tilde{v}_1,\cdots,\tilde{v}_p) = \prod_{i=1}^{p}(1+\tilde{v}_i)^{-2}\ .$$

Thus the marginal distributions of the characteristic roots of $\underset{\sim}{V}:p\times p$ can be obtained from the marginal distributions of the characteristic roots of $\underset{\sim}{G}:p\times p$. These marginal distributions of the roots of $\underset{\sim}{G}:p\times p$ were not considered in sections 5.2 but it follows clearly from the marginal distributions of the roots of $\underset{\sim}{L}:p\times p$ and thus the marginal distributions of the roots of $\underset{\sim}{V}:p\times p$

can also be obtained from the marginal distributions of $\tilde{L}_1, \cdots, \tilde{L}_p$. In theorems 5.4.4 - 5.4.8 the marginal distributions $\tilde{V}_1, \cdots, \tilde{V}_p$ will be given as corollaries of theorems 5.2.3 - 5.2.7.

<u>Theorem 5.4.4</u> (Corollary of theorem 5.2.3)

Let $\underset{\sim}{V}: p \times p \sim NCCMB_{2A}(p, m, n, \Omega)$; then

$$P(c < \tilde{V}_1 < \tilde{V}_p < d) = (5.2.29)$$

with

$$(5.4.57) \quad b_{ij} = \sum_{k=0}^{\infty} \frac{(m+n-p+1)_k \, \tilde{\omega}_i^k}{(n-p+1)_k \, k!} \{B_{\frac{d}{1+d}}(m-p+1, n+k+1-j) - B_{\frac{c}{1+c}}(m-p+1, n+k+1-j)\} .$$

Let $\underset{\sim}{V}: p \times p \sim CMB_2(p, m, n)$; then

$$P(c < \tilde{V}_1 < \tilde{V}_p < d) = (5.2.31)$$

with

$$(5.4.58) \quad b_{ij} = B_{\frac{d}{1+d}}(m-p+1, n+p-i-j+1) - B_{\frac{c}{1+c}}(m-p+1, n+p-i-j+1) .$$

<u>Corollary 5.4.1</u>

Let $\underset{\sim}{V}: p \times p \sim NCCMB_{2A}(p, m, n, \Omega)$; then

$$P(\tilde{V}_p < d) = (5.2.29)$$

with

$$(5.4.59) \quad b_{ij} = \sum_{k=0}^{\infty} \frac{(m+n-p+1)_k \, \tilde{\omega}_i^k}{(n-p+1)_k \, k!} B_{\frac{d}{1+d}}(m-p+1,\ n+k+1-j) \ .$$

Let $\underset{\sim}{V}:p\times p \sim CMB_2(p,\ m,\ n)$; then

$$P(\tilde{V}_p < d) = (5.2.31)$$

with

$$(5.4.60) \quad b_{ij} = B_{\frac{d}{1+d}}(m-p+1,\ n+p-i-j+1) \ .$$

<u>Proof</u>

Let $c = 0$ in (5.4.57) and (5.4.58); then (5.4.59) and (5.4.60) follow respectively.

<u>Corollary 5.4.2</u>

Let $\underset{\sim}{V}:p\times p \sim NCCMB_{2A}(p,\ m,\ n,\ \Omega)$; then

$$P(\tilde{V}_1 < c) = 1 - (5.2.29)$$

with

$$(5.4.61) \quad b_{ij} = \sum_{k=0}^{\infty} \frac{(m+n-p+1)_k \, \tilde{\omega}_i^k}{(n-p+1)_k \, k!} \{B(m-p+1,\ n+k+1-j) - B_{\frac{c}{1+c}}(m-p+1,\ n+k+1-j)\} \ .$$

Let $\underset{\sim}{V}:p\times p \sim CMB_2(p,\ m,\ n)$; then

$$P(\tilde{V}_1 < c) = 1 - (5.2.31)$$

with

$$(5.4.62)\quad b_{ij} = B(m-p+1,\ n+p-i-j+1) - B_{\frac{c}{1+c}}(m-p+1,\ n+p-i-j+1)\ .$$

Proof

Let $d \to \infty$ in (5.4.57) and (5.4.58); then (5.4.61) and (5.4.62) follow.

Theorem 5.4.5 (Corollary of theorem 5.2.4)

Let $\underset{\sim}{V}: p\times p \sim NCCMB_{2A}(p,\ m,\ n,\ \Omega)$; then

$$P(0 < \tilde{V}_1 < \cdots < \tilde{V}_r < c < \tilde{V}_{r+1} < \cdots < \tilde{V}_p) = (5.2.56)$$

with

$$(5.4.63)\quad b_{3ij} = \sum_{k=0}^{\infty} \frac{(m+n-p+1)_k\ \tilde{\omega}^k_{\delta_i}}{(n-p+1)_k\ k!} B_{\frac{c}{1+c}}(m-p+1,\ n+k+\alpha_j+1)\ ,$$

$$(i,j = 1,\cdots,r)$$

and

$$(5.4.64)\quad b_{4ij} = \sum_{k=0}^{\infty} \frac{(m+n-p+1)_k\ \tilde{\omega}^k_{\nu_i}}{(n-p+1)_k\ k!} \{B(m-p+1,\ n+k-\beta_j+1) - B_{\frac{c}{1+c}}(m-p+1,\ n+k-\beta_j+1)\}\ ,$$

$$(i,j = 1,\cdots,p-r)\ .$$

Let $\underset{\sim}{V}: p\times p \sim CMB_2(p,\ m,\ n)$; then

$$P(0 < \tilde{V}_1 < \cdots < \tilde{V}_r < c < \tilde{V}_{r+1} < \cdots < \tilde{V}_p) = (5.2.59)$$

with

$$(5.4.65)\quad b_{3ij} = B_{\frac{c}{1+c}}(m-p+1,\ n+p-\delta_i-\alpha_j+1)\ ,\quad (i,j=1,\cdots,r)$$

and

$$(5.4.66)\quad b_{4ij} = B(m-p+1,\ n+p-\nu_i-\beta_j+1) - B_{\frac{c}{1+c}}(m-p+1,\ n+p-\nu_i-\beta_j+1)\ ,$$

$$(i,j=1,\cdots,p-r)\ .$$

Theorem 5.4.6 (Corollary of theorem 5.2.5)

Let $\underset{\sim}{V}:p\times p \sim NCCMB_{2A}(p,\ m,\ n,\ \Omega)$; then

$$P(0 < \tilde{V}_1 < \cdots < \tilde{V}_r < c < \tilde{V}_{r+1} < \cdots < \tilde{V}_{r+t} < d < \tilde{V}_{r+t+1} < \cdots < \tilde{V}_p)$$

$$= (5.2.62)$$

with

$$(5.4.67)\quad b_{8ij} = \sum_{k=0}^{\infty} \frac{(m+n-p+1)_k\ \tilde{\omega}_{\delta_i}^k}{(n-p+1)_k\ k!} B_{\frac{c}{1+c}}(m-p+1,\ n+k-\delta_j^*+1)\ ,$$

$$(i,j=1,\cdots,r)\ ,$$

$$(5.4.68)\quad b_{9ij} = \sum_{k=0}^{\infty} \frac{(m+n-p+1)_k\ \tilde{\omega}_{\alpha_i}^k}{(n-p+1)_k\ k!} \{B_{\frac{d}{1+d}}(m-p+1,\ n+k-\alpha_j^*+1)$$

$$- B_{\frac{c}{1+c}}(m-p+1,\ n+k-\alpha_j^*+1)\}\ ,$$

$$(i,j=1,\cdots,t)\ ,$$

and

$$(5.4.69)\quad b_{10ij} = \sum_{k=0}^{\infty} \frac{(m+n-p+1)_k\, \tilde{\omega}_{\beta_i}^k}{(n-p+1)_k\, k!} \{B(m-p+1,\ n+k-\beta_j^*+1)$$

$$- B_{\frac{d}{1+d}}(m-p+1,\ n+k-\beta_j^*+1)\}\ ,$$

$$(i,j = r+t+1,\cdots,p)\ .$$

Let $\tilde{V}:p\times p \sim CMB_2(p,\ m,\ n)$; then

$$P(0 < \tilde{V}_1 < \cdots < \tilde{V}_r < c < \tilde{V}_{r+1} < \cdots < \tilde{V}_{r+t} < d < \tilde{V}_{r+t+1} < \cdots < \tilde{V}_p)$$

$$= (5.2.66)$$

with

$$(5.4.70)\quad b_{8ij} = B_{\frac{c}{1+c}}(m-p+1,\ n+p-\delta_i-\delta_j^*+1)\ ,\quad (i,j = 1,\cdots,r)\ ,$$

$$(5.4.71)\quad b_{9ij} = B_{\frac{d}{1+d}}(m-p+1,\ n+p-\alpha_i-\alpha_j^*+1) - B_{\frac{c}{1+c}}(m-p+1,\ n+p-\alpha_i-\alpha_j^*+1)\ ,$$

$$(i,j = 1,\cdots,t)$$

and

$$(5.4.72)\quad b_{10ij} = B(m-p+1,\ n+p-\beta_i-\beta_j^*+1) - B_{\frac{d}{1+d}}(m-p+1,\ n+p-\beta_i-\beta_j^*+1)\ ,$$

$$(i,j = r+t+1,\cdots,p)\ .$$

<u>Theorem 5.4.7</u> (Corollary of theorem 5.2.6)

Let $\underset{\sim}{V}:p\times p \sim NCCMB_{2A}(p,\ m,\ n,\ \Omega)$; then

(5.4.73)
$$f_{\tilde{v}_{r+1},\cdots,\tilde{v}_{r+t}}(\tilde{v}_{r+1},\cdots,\tilde{v}_{r+t})$$

$$= \frac{\pi^{\frac{1}{2}p(p-1)}\ \tilde{\Gamma}_p(m+n)\ \operatorname{etr}[-\Omega]\ \prod_{i=1}^{p}(n-i+1)^{i-1}}{\tilde{\Gamma}_p(m)\ \tilde{\Gamma}_p(n)\ \prod_{i>j}^{p}(\tilde{\omega}_i-\tilde{\omega}_j)\ \prod_{i=1}^{p}(m+n-i+1)^{i-1}}$$

$$\Sigma_3\Sigma_4\Sigma_5\Sigma_6(-1)^{\Sigma\delta_i+\Sigma\alpha_i+\Sigma\delta_i^*+\Sigma\alpha_i^*}\ |(b_{13ij})|\,|(b_{14ij})|$$

$$|(\tilde{v}_{p-j+1+r}^{m-p}(1+\tilde{v}_{p-j+1+r})^{-(m+n)+2(p-1)}$$

$${}_1F_1(m-p+1\ ;\ n-p+1\ ;\ \tilde{\omega}_i(1+\tilde{v}_{p-j+1+r})^{-1})\ ;$$

$$i=1,\cdots,t\ ,\quad j=p-t+1,\cdots,p)\,|\,|((\frac{1}{1+\tilde{v}_{p-j+1+r}})^{p-\alpha_i^*}\ ;$$

$$i=1,\cdots,t\ ,\quad j=p-t+1,\cdots,p)|\ ,\quad 0<\tilde{v}_{r+1}<\cdots<\tilde{v}_{r+t}\ ,$$

where

(5.4.74)
$$b_{13ij}=\sum_{k=0}^{\infty}\frac{(m+n-p+1)_k\ \tilde{\omega}_{\delta_i}^k}{(n-p+1)_k\ k!}\ B_{\frac{\tilde{v}_{r+1}}{1+\tilde{v}_{r+1}}}(m-p+1,\ n+k-\delta_j^*+1)\ ,$$

$$(i,j=1,\cdots,r)$$

and

(5.4.75)
$$b_{14ij}=\sum_{k=0}^{\infty}\frac{(m+n-p+1)_k\ \tilde{\omega}_{\beta_i}^k}{(n-p+1)_k\ k!}\ \{B(m-p+1,\ n+k-\beta_j^*+1)$$

$$-B_{\frac{\tilde{v}_{r+t}}{1+\tilde{v}_{r+t}}}(m-p+1,\ n+k-\beta_j^*+1)\}\ ,$$

$$(i,j=r+t+1,\cdots,p)\ .$$

Let $\underset{\sim}{V}:p\times p \sim CMB_2(p,\ m,\ n)$; then

(5.4.76) $$f_{\tilde{v}_{r+1},\cdots,\tilde{v}_{r+t}}(\tilde{v}_{r+1},\cdots,\tilde{v}_{r+t})$$

$$= \frac{\pi^{p(p-1)}\ \tilde{\Gamma}_p(m+n)}{\tilde{\Gamma}_p(m)\ \tilde{\Gamma}_p(n)\ \tilde{\Gamma}_p(p)}$$

$$\Sigma_3\Sigma_4\Sigma_5\Sigma_6(-1)^{\Sigma\delta_i+\Sigma\alpha_i+\Sigma\delta_i^*+\Sigma\alpha_i^*}\ |(b_{13ij})|\,|(b_{14ij})|$$

$$|(\tilde{v}_{r+j}^{m-\alpha_i}\ (1+\tilde{v}_{r+j})^{-(m+n)+2(p-1)}\ ;\ i,j = 1,\cdots,t)|$$

$$|((\frac{1}{1+\tilde{v}_{r+j}})^{p-\alpha_i^*}\ ;\ i,j = 1,\cdots,t)|\ ,\quad 0 < \tilde{v}_{r+1}<\cdots<\tilde{v}_{r+t}\ ,$$

where

(5.4.77) $$b_{13ij} = B_{\frac{\tilde{v}_{r+1}}{1+\tilde{v}_{r+1}}}(m-p+1,\ n+p-\delta_i-\delta_j^*+1)\ ,\quad (i,j = 1,\cdots,t)$$

and

(5.4.78) $$b_{14ij} = B(m-p+1,\ n+p-\beta_i-\beta_j+1) - B_{\frac{\tilde{v}_{r+t}}{1+\tilde{v}_{r+t}}}(m-p+1,\ n+p-\beta_i-\beta_j^*+1)\ ,$$

$$(i,j = r+t+1,\cdots,p)\ .$$

<u>Theorem 5.4.8</u> (Corollary of theorem 5.2.7)

Let $\underset{\sim}{V}:p\times p \sim NCCMB_{2A}(p,\ m,\ n,\ \Omega)$; then the joint p.d.f. of any r unordered characteristic roots of $\underset{\sim}{V}:p\times p$ is given by

(5.4.79) $$f_{\tilde{v}_1,\cdots,\tilde{v}_r}(\tilde{v}_1,\cdots,\tilde{v}_r)$$

$$= \frac{\pi^{\frac{1}{2}p(p-1)}\ \tilde{\Gamma}_p(m+n)\ \mathrm{etr}[-\Omega]\ \prod_{i=1}^{p}(n-i+1)^{i-1}}{p!\ \tilde{\Gamma}_p(m)\ \tilde{\Gamma}_p(n)\ \prod_{i>j}^{p}(\tilde{\omega}_i-\tilde{\omega}_j)\ \prod_{i=1}^{p}(m+n-i+1)^{i-1}}$$

$$\Sigma_1\Sigma_2(-1)^{\Sigma\delta_i+\Sigma\alpha_i}\ |(b_{17ij})|\,|(b_{18ij})|\ ,$$

$$(\tilde{v}_i>0\ ,\quad i=1,\cdots,r)\ ,$$

where

(5.4.80) $$b_{17ij} = \sum_{t=1}^{r} \tilde{v}_t^{m-p}\ (1+\tilde{v}_t)^{-(m+n)+p+\alpha_j-2}\ {}_1F_1(m+n-p+1\ ;\ n-p+1\ ;\ \tilde{\omega}_{\delta_i}(1+\tilde{v}_t)^{-1})\ ,\quad (i,j=1,\cdots,r)$$

and

(5.4.81) $$b_{18ij} = (p-r)!\ \sum_{k=0}^{\infty} \frac{(m+n-p+1)_k\ \tilde{\omega}_{\nu_i}^k}{(n-p+1)_k\ k!}\ B(m-p+1,\ n+k-\beta_j+1)\ ,$$

$$(i,j=1,\cdots,p-r)\ .$$

Let $\underset{\sim}{V}:p\times p \sim CMB_2(p,\ m,\ n)$; then the joint p.d.f. of any r unordered characteristic roots of $\underset{\sim}{V}:p\times p$ is given by

(5.4.82) $$f_{\tilde{v}_1,\cdots,\tilde{v}_r}(\tilde{v}_1,\cdots,\tilde{v}_r)$$

$$= \frac{\pi^{p(p-1)}\ \tilde{\Gamma}_p(m+n)}{p!\ \tilde{\Gamma}_p(m)\ \tilde{\Gamma}_p(n)\ \tilde{\Gamma}_p(p)}\ \Sigma_1\Sigma_2(-1)^{\Sigma\delta_i+\Sigma\alpha_i}\ |(b_{17ij})|\,|(b_{18ij})|\ ,$$

$$(\tilde{v}_i>0\ ,\quad i=1,\cdots,r)\ ,$$

where

$$(5.4.83)\quad b_{17ij} = \sum_{t=1}^{r} \tilde{v}_t^{m-p} (1+\tilde{v}_t)^{-(m+n+2)+\delta_i+\alpha_j} , \quad (i,j = 1,\cdots,r)$$

and

$$(5.4.84)\quad b_{18ij} = (p-r)!\ B(m-p+1,\ n+p-\nu_i-\beta_j+1) , \quad (i,j = 1,\cdots,p-r) .$$

Let $\underset{\sim}{L}:p\times p = (\underset{\sim}{A} + \underset{\sim}{B})^{-\frac{1}{2}} \underset{\sim}{B}(\underset{\sim}{A} + \underset{\sim}{B})^{-\frac{1}{2}}$ be the complex multivariate beta type 1B matrix; then

$$(5.4.85)\quad |L - \tilde{\ell}\, I_p| = |(A + B)^{-\frac{1}{2}} B(A + B)^{-\frac{1}{2}} - \tilde{\ell}\, I_p| = 0$$

$$\text{iff. } |B^{\frac{1}{2}} A^{-1} B^{\frac{1}{2}} - \frac{\tilde{\ell}}{1-\tilde{\ell}} I_p| = 0$$

$$\text{iff. } |Z - \frac{\tilde{\ell}}{1-\tilde{\ell}} I_p| = 0 .$$

Because of this relation between the characteristic roots of $\underset{\sim}{L}:p\times p$ and $\underset{\sim}{Z}:p\times p$, the p.d.f. of $\underset{\sim}{D}_Z$ follows from the p.d.f. of $\underset{\sim}{D}_L$ by making the following transformation in the p.d.f. of $\underset{\sim}{D}_L$:

$$(5.4.86)\quad \tilde{z}_i = \frac{\tilde{\ell}_i}{1-\tilde{\ell}_i} , \quad (i = 1,\cdots,p) .$$

The marginal distributions of $\tilde{Z}_1,\cdots,\tilde{Z}_p$ can thus be obtained from the marginal distributions of $\tilde{L}_1,\cdots,\tilde{L}_p$ by replacing $B_t(a, b)$ with $B_{\frac{t}{1+t}}(a, b)$ in the marginal distributions of $\tilde{L}_1,\cdots,\tilde{L}_p$ and also by making the transformation, $z_i = \dfrac{\tilde{\ell}_i}{1-\tilde{\ell}_i}$ in the marginal p.d.f.s of $\tilde{L}_1,\cdots,\tilde{L}_p$.

5.4.3 P.d.f.s of functions of the characteristic roots of complex beta type 2 matrices

In theorem 5.4.9 the p.d.f.s of $|(I_p + \underset{\sim}{V})^{-1}| = \prod_{i=1}^{p} (1 + \tilde{v}_i)^{-1}$ and $|\underset{\sim}{V}(I_p + \underset{\sim}{V})^{-1}| = \prod_{i=1}^{p} \frac{\tilde{v}_i}{1 + \tilde{v}_i}$ are derived in terms of Meijer's G-function.

Theorem 5.4.9

Let $\underset{\sim}{V}: p\times p \sim NCCMB_{2A}(p,\ m,\ n,\ \Omega)$; then:

(5.4.87)
$$f_{|(I_p + \underset{\sim}{V})^{-1}|}(|(I_p + V)^{-1}|)$$
$$= \frac{\tilde{\Gamma}_p(m+n)\ \operatorname{etr}[-\Omega]\ |I_p + V|}{\tilde{\Gamma}_p(n)} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{[m+n]_\kappa\ \tilde{C}_\kappa(\Omega)}{[n]_\kappa\ k!}$$
$$G_p\left[|(I_p + V)^{-1}| \,\middle|\, \begin{matrix} m+n+k_j-j+1 \\ n+k_j-j+1 \end{matrix}\right], \quad |(I_p + V)^{-1}| < 1 .$$

(5.4.88)
$$f_{|\underset{\sim}{V}(I_p + \underset{\sim}{V})^{-1}|}(|V(I_p + V)^{-1}|)$$
$$= \frac{\tilde{\Gamma}_p(m+n)\ \operatorname{etr}[-\Omega]\ |V^{-1}(I_p + V)|}{\tilde{\Gamma}_p(m)} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{[m+n]_\kappa\ \tilde{C}_\kappa(\Omega)}{k!}$$
$$G_p\left[|V(I_p + V)^{-1}| \,\middle|\, \begin{matrix} m+n+k_j-j+1 \\ m-j+1 \end{matrix}\right], \quad |V(I_p + V)^{-1}| < 1 .$$

Proof

These two results follow from (5.4.8) and (5.4.9) respectively by using theorem 2.5.3.

Remark 5.4.4

(i) By comparing $E(|(I_p + \underset{\sim}{Z})^{-1}|^h)$, given in (5.4.30), with $E(|\underset{\sim}{V}(I_p + \underset{\sim}{V})^{-1}|^h)$, given in (5.4.9), and $E(|\underset{\sim}{Z}(I_p + \underset{\sim}{Z})^{-1}|^h)$ given in (5.4.31) with $E(|(I_p + \underset{\sim}{V})^{-1}|^h)$ given in (5.4.8) it is clear that

$$f_{|(I_p + \underset{\sim}{Z})^{-1}|}(|(I_p + Z)^{-1}|) = f_{|\underset{\sim}{V}(I_p + \underset{\sim}{V})^{-1}|}(|V(I_p + V)^{-1}|)$$

and

$$f_{|\underset{\sim}{Z}(I_p + \underset{\sim}{Z})^{-1}|}(|Z(I_p + Z)^{-1}|) = f_{|(I_p + \underset{\sim}{V})^{-1}|}(|(I_p + V)^{-1}|) .$$

(ii) Expressions for the p.d.f.s of $\mathrm{tr}\,\underset{\sim}{V}$ and $\mathrm{tr}\,\underset{\sim}{V}^{-1}$ are derived by De Waal (1968, p. 114 - 117). These expressions, however are convergent only for $\mathrm{tr}\,V < 1$ and $\mathrm{tr}\,V^{-1} < 1$.

CHAPTER 6

EXTENSIONS OF THE MULTIVARIATE COMPLEX BETA TYPE 1 DISTRIBUTIONS

6.1 INTRODUCTION

In this chapter the multivariate complex beta type 1A distribution is extended to the cases where:

(i) The complex Wishart matrix which appears in the numerator and the denominator of the complex beta matrix is replaced by a central complex quadratic form of complex normal variates,

(ii) both Wishart matrices which appear in the complex beta matrix are replaced by central complex quadratic forms of complex normal variates.

In 6.2 the random hermitian matrix $\underset{\sim}{G}:p\times p = (\underset{\sim}{A} + \underset{\sim}{S})^{-\frac{1}{2}}\, \underset{\sim}{S}(\underset{\sim}{A} + \underset{\sim}{S})^{-\frac{1}{2}}$, where $\underset{\sim}{S}:p\times p$ is a central complex quadratic form and $\underset{\sim}{A}:p\times p$ is a non-central complex Wishart matrix, is considered. The symmetrised p.d.f. and moments of $\underset{\sim}{G}:p\times p$ and $D_{\underset{\sim}{G}}$ are derived in section 6.2.1 analogous to the real case (cf. Underhill, 1973, p. 4.1 - 4.21) for different specifications of the parameter matrices. In section 6.2.2 an indication will be given how certain marginal distributions of the characteristic roots of $\underset{\sim}{G}:p\times p$ can be derived. In section 6.2.3 the p.d.f.s of certain functions of the roots of $\underset{\sim}{G}:p\times p$ are considered.

In 6.3 the random hermitian matrix $\underset{\sim}{G}:p\times p = (\underset{\sim}{S} + \underset{\sim}{T})^{-\frac{1}{2}}\, \underset{\sim}{S}(\underset{\sim}{S} + \underset{\sim}{T})^{-\frac{1}{2}}$, where both $\underset{\sim}{S}:p\times p$ and $\underset{\sim}{T}:p\times p$ are central complex quadratic forms, is considered. Only the symmetrised p.d.f. of $\underset{\sim}{G}:p\times p$ are derived analogous to the real case (cf. Underhill, 1973, p. 6.1 - 6.5).

6.2 THE QUADRATIC FORM $\underset{\sim}{G}:p\times p = (\underset{\sim}{A}+\underset{\sim}{S})^{-\frac{1}{2}}\, \underset{\sim}{S}\,(\underset{\sim}{A}+\underset{\sim}{S})^{-\frac{1}{2}}$ WHEN $\underset{\sim}{S}:p\times p$ IS A CENTRAL COMPLEX QUADRATIC FORM AND $\underset{\sim}{A}:p\times p \sim NCCW(p,\ m,\ \Psi,\ \Omega)$

6.2.1 The symmetrised p.d.f. and moments of $\underset{\sim}{G}:p\times p$ and $D_{\underset{\sim}{G}}$

In theorem 6.2.1 the symmetrised p.d.f. and moments of $\underset{\sim}{G}:p\times p$ and $D_{\underset{\sim}{G}}$ are derived for different specifications of the parameter matrices. It does not seem possible to derive the actual p.d.f. of $\underset{\sim}{G}:p\times p$.

Theorem 6.2.1

Let the central complex quadratic form $\underset{\sim}{S}:p\times p = \underset{\sim}{Z}\, L\, \overline{\underset{\sim}{Z}}'$ have the p.d.f. given in (4.2.6), (power-series representation) or (4.2.9), (Γ-type representation) and let $\underset{\sim}{A}:p\times p \sim NCCW(p,\ m,\ \Psi,\ \Omega)$; then the symmetrised p.d.f. and moments of $\underset{\sim}{G}:p\times p = (\underset{\sim}{A}+\underset{\sim}{S})^{-\frac{1}{2}}\, \underset{\sim}{S}\,(\underset{\sim}{A}+\underset{\sim}{S})^{-\frac{1}{2}}$ and $D_{\underset{\sim}{G}}$ are given below for certain specifications of $\Omega:p\times p$, $\Psi:p\times p$ and $\Sigma:p\times p$.

(i) $\Sigma:p\times p = \Psi:p\times p$ i.e. $\underset{\sim}{A}:p\times p \sim NCCW(p,\ m,\ \Sigma,\ \Omega)$

The p.d.f. of $\underset{\sim}{S}:p\times p$ has the Γ-type representation

(6.2.1) $$f_{csym}(G)$$

$$= \frac{\tilde{\Gamma}_p(m+n)\ \mathrm{etr}[-\Omega]}{\tilde{\Gamma}_p(n)\ \tilde{\Gamma}_p(m)\ |L\,\Phi|^p}\ |G|^{n-p}\ |I_p - G|^{m-p}$$

$$\sum_{t=0}^{\infty}\sum_{\tau}\sum_{k=0}^{\infty}\sum_{\kappa}\sum_{\delta} \frac{(-1)^k\ \tilde{P}^{\delta}_{\kappa,\tau}\ (G,\ I_p - G)\ [m+n]_{\delta}}{k!\ t!\ \tilde{C}_{\kappa}(I_n)\ \tilde{C}_{\tau}(I_p)\ [m]_{\tau}}$$

$$\tilde{C}_{\kappa}(L^{-\frac{1}{2}}\ \Phi^{-1}\ L^{-\frac{1}{2}} - I_n)\ \tilde{C}_{\tau}(\Omega)\ \tilde{C}_{\delta}(I_p)\ ,\quad 0 < G = \overline{G}' < I_p\ .$$

<u>The p.d.f. of $\underset{\sim}{S}:p\times p$ has the power-series representation</u>

(6.2.2) $f_{csym}(G) = (6.2.1)$.

(ii) <u>$\Sigma:p\times p = \Psi:p\times p = I_p$ i.e. $\underset{\sim}{A}:p\times p \sim NCCW(p, m, I_p, \Omega)$</u>

(6.2.3) $f_{csym}(G) = (6.2.1)$.

(iii) <u>$\Omega:p\times p = 0$ i.e. $\underset{\sim}{A}:p\times p \sim CW(p, m, \Psi)$</u>

<u>The p.d.f. of $\underset{\sim}{S}:p\times p$ has the power-series representation</u>

(6.2.4) $f_{csym}(G)$

$$= \frac{\tilde{\Gamma}_p(m+n)\ |G|^{n-p}\ |I_p - G|^{m-p}}{\tilde{\Gamma}_p(n)\ \tilde{\Gamma}_p(m)\ |L\,\Phi|^p\ |\Sigma\,\Psi^{-1}|^n}$$

$$\sum_{k=0}^{\infty} \sum_{\kappa} \sum_{t=0}^{\infty} \sum_{\tau} \sum_{\delta} \frac{\tilde{g}^{\delta}_{\kappa,\tau}\ [m+n]_{\delta}\ \tilde{C}_{\kappa}(\Phi^{-1}\ L^{-1})\ \tilde{C}_{\kappa}(-\Sigma^{-1})}{\tilde{C}_{\kappa}(I_n)\ \tilde{C}_{\kappa}(I_p)\ \tilde{C}_{\tau}(I_p)\ \tilde{C}_{\delta}(I_p)\ k!\ t!}$$

$$\tilde{C}_{\tau}(\Psi^{-1})\ \tilde{C}_{\delta}(\Psi)\ \tilde{C}_{\delta}(G)\ ,\quad 0 < G = \overline{G}' < I_p\ .$$

(6.2.5) $f_{\underset{\sim}{D}_G}(D_G)$

$$= \frac{\pi^{p(p-1)}\ \tilde{\Gamma}_p(m+n)\ |D_G|^{n-p}\ |I_p - D_G|^{m-p}}{\tilde{\Gamma}_p(n)\ \tilde{\Gamma}_p(m)\ \tilde{\Gamma}_p(p)\ |L\,\Phi|\ |\Sigma\ \Psi^{-1}|^n} \prod_{i>j}^{p} (\tilde{g}_i - \tilde{g}_j)^2$$

$$\sum_{k=0}^{\infty} \sum_{\kappa} \sum_{t=0}^{\infty} \sum_{\tau} \sum_{\delta} \frac{\tilde{g}^{\delta}_{\kappa,\tau}\ [m+n]_{\delta}\ \tilde{C}_{\kappa}(\Phi^{-1}\ L^{-1})\ \tilde{C}_{\kappa}(-\Sigma^{-1})}{\tilde{C}_{\kappa}(I_n)\ \tilde{C}_{\kappa}(I_p)\ \tilde{C}_{\tau}(I_p)\ \tilde{C}_{\delta}(I_p)\ k!\ t!}$$

$$\tilde{C}_\tau(\Psi^{-1})\ \tilde{C}_\delta(\Psi)\ \tilde{C}_\delta(D_G)\ , \quad 0 < \tilde{g}_1 < \cdots < \tilde{g}_p < 1 \ .$$

(iv) $\underline{\Omega:p\times p = 0\ , \quad \Sigma:p\times p = \Psi:p\times p \quad \text{i.e.} \quad \underset{\sim}{A}:p\times p \sim CW(p,\ m,\ \Sigma)}$

$\underline{\text{The p.d.f. of } \underset{\sim}{S}:p\times p \text{ has the } \Gamma\text{-type representation}}$

(6.2.6) $\quad f_{csym}(G)$

$$= \frac{\tilde{\Gamma}_p(m+n)\ |G|^{n-p}\ |I_p - G|^{m-p}\ |I_p + (q^{-1}-1)\ G|^{-(m+n)}}{\tilde{\Gamma}_p(m)\ \tilde{\Gamma}_p(n)\ |L\,\Phi|^p}$$

$${}_1\tilde{F}_0(m+n\ ;\ -G(I_p + (q^{-1}-1)\ G)^{-1}\ ,\ L^{-\frac{1}{2}}\ \Phi^{-1}\ L^{-\frac{1}{2}} - q^{-1}\ I_n)\ ,$$

$$0 < G = \overline{G}' < I_p \quad \text{and} \quad q > 0 \ .$$

(6.2.7) $\quad f_{\underset{\sim}{D}_G}(D_G)$

$$= \frac{\pi^{p(p-1)}\ \tilde{\Gamma}_p(m+n)\ |D_G|^{n-p}\ |I_p - D_G|^{m-p}}{\tilde{\Gamma}_p(m)\ \tilde{\Gamma}_p(n)\ \tilde{\Gamma}_p(p)\ |L\,\Phi|^p}$$

$$|I_p + (q^{-1}-1)\ D_G|^{-(m+n)}\ \prod_{i>j}^{p} (\tilde{g}_i - \tilde{g}_j)^2$$

$${}_1\tilde{F}_0(m+n\ ;\ -D_G(I_p + (q^{-1}-1)\ D_G)^{-1}\ ,\ L^{-\frac{1}{2}}\ \Phi^{-1}\ L^{-\frac{1}{2}} - q^{-1}\ I_n)\ ,$$

$$0 < \tilde{g}_1 < \cdots < \tilde{g}_p < 1 \quad \text{and} \quad q > 0 \ .$$

(6.2.8) $\quad E(|\underset{\sim}{G}|^h)$

$$= \frac{\tilde{\Gamma}_p(m+n)\ \tilde{\Gamma}_p(n+h)}{\tilde{\Gamma}_p(n)\ \tilde{\Gamma}_p(m+n+h)} \sum_{k=0}^{\infty} \sum_{\kappa} \sum_{t=0}^{\infty} \sum_{\tau} \sum_{\delta} \frac{\tilde{g}^{\delta}_{\kappa,\tau}(-1)^{k+t}\ (q^{-1}-1)^t}{\tilde{C}_\kappa(I_n)\ k!\ t!}$$

$$\frac{[m+n]_\delta \; [n+h]_\delta}{[m+n+h]_\delta} \tilde{C}_\kappa(L^{-\frac{1}{2}} \, \Phi^{-1} \, L^{-\frac{1}{2}} - q^{-1} I_n) \, \tilde{C}_\delta(I_p) \; .$$

(6.2.9) $\quad E(|I_p - \underset{\sim}{G}|^h)$

$$= \frac{\tilde{\Gamma}_p(m+n) \; \tilde{\Gamma}_p(m+h)}{\tilde{\Gamma}_p(m) \; \tilde{\Gamma}_p(m+n+h)} \sum_{k=0}^{\infty} \sum_{\kappa} \sum_{t=0}^{\infty} \sum_{\tau} \sum_{\delta} \frac{\tilde{g}^{\delta}_{\kappa,\tau} (-1)^{k+t}}{\tilde{C}_\kappa(I_n) \; k! \; t!}$$

$$\frac{(q^{-1}-1)^t \; [m+n]_\delta \; [n]_\delta}{[m+n+h]_\delta} \tilde{C}_\kappa(L^{-\frac{1}{2}} \, \Phi^{-1} \, L^{-\frac{1}{2}} - q^{-1} I_n) \, \tilde{C}_\delta(I_p) \; .$$

The p.d.f. of $\underset{\sim}{S}{:}p{\times}p$ has the power-series representation

(6.2.10) $\quad f_{csym}(G)$

$$= \frac{\tilde{\Gamma}_p(m+n) \; |G|^{n-p} \; |I_p - G|^{-(p+n)}}{\tilde{\Gamma}_p(n) \; \tilde{\Gamma}_p(m) \; |L \, \Phi|^p}$$

$${}_1\tilde{F}_0(m+n \, ; \, \Phi^{-1} \, L^{-1} \, , \; - G(I_p - G)^{-1}) \; , \quad 0 < G = \bar{G}' < I_p \; .$$

(6.2.11) $\quad f_{\underset{\sim}{D}_G}(D_G)$

$$= \frac{\pi^{p(p-1)} \; \tilde{\Gamma}_p(m+n) \; |D_G|^{n-p} \; |I_p - D_G|^{-(p+n)}}{\tilde{\Gamma}_p(n) \; \tilde{\Gamma}_p(m) \; |L \, \Phi|^p}$$

$$\prod_{i>j}^{p} (\tilde{g}_i - \tilde{g}_j)^2 \; {}_1\tilde{F}_0(m+n \, ; \, \Phi^{-1} \, L^{-1} \, , \; - D_G(I_p - D_G)^{-1}) \; ,$$

$$0 < \tilde{g}_1 < \cdots < \tilde{g}_p < 1 \; .$$

Proof

(6.2.1)

The joint p.d.f. of $\underset{\sim}{S}:p\times p$ and $\underset{\sim}{A}:p\times p$ follows as

(6.2.12) $$f_{\underset{\sim}{S},\underset{\sim}{A}}(S,A)$$

$$= \frac{|S|^{n-p}\ \mathrm{etr}[-q^{-1}\ \Sigma^{-1}\ S]}{\tilde{\Gamma}_p(n)\ \ |L\,\Phi|^p\ |\Sigma|^n}\ {}_0\tilde{F}_0(L^{-\frac{1}{2}}\ \Phi^{-1}\ L^{-\frac{1}{2}} - q^{-1}\ I_n\ ,\ -\Sigma^{-1}\ S\,)$$

$$\frac{\mathrm{etr}[-\Omega]\ {}_0\tilde{F}_1(m\ ;\ \Omega\,\Sigma^{-1}\ A)\ \mathrm{etr}[-\Sigma^{-1}\ A]\ |A|^{m-p}}{\tilde{\Gamma}_p(m)\ \ |\Sigma|^m}\ .$$

In (6.2.12) make the transformations

(6.2.13) $W = A + S$

(6.2.14) $G = W^{-\frac{1}{2}}\,S\,W^{-\frac{1}{2}}$

with inverse set of transformations

(6.2.15) $A = W - W^{\frac{1}{2}}\,G\,W^{\frac{1}{2}}$

(6.2.16) $S = W^{\frac{1}{2}}\,G\,W^{\frac{1}{2}}$.

From Le Roux (1978, p. 187 - 188), Steel (1979, p. 30 - 31) and theorem 2.2.2 (iii) the jacobian of this inverse set of transformations follows as

(6.2.17) $J(S,A \rightarrow W,G) = |W|^p$.

The joint p.d.f. of $\underset{\sim}{G}:p\times p$ and $\underset{\sim}{W}:p\times p$ follows as

(6.2.18) $$f_{\underset{\sim}{G},\underset{\sim}{W}}(G,W)$$

$$= c|G|^{n-p}\ |I_p - G|^{m-p}\ |W|^{m+n-p}\ \mathrm{etr}[\,(1-q^{-1})\ \Sigma^{-1}\ W^{\frac{1}{2}}\,G\,W^{\frac{1}{2}}]$$

$$\text{etr}[-\Sigma^{-1} W] \quad {}_0\tilde{F}_1(m\,;\,\Omega\,\Sigma^{-1} W^{\frac{1}{2}} (I_p - G)\, W^{\frac{1}{2}})$$

$$ {}_0\tilde{F}_0(L^{-\frac{1}{2}}\, \Phi^{-1}\, L^{-\frac{1}{2}} - q^{-1} I_n\,,\, -\Sigma^{-1} W^{\frac{1}{2}}\, G\, W^{\frac{1}{2}})$$

with

$$(6.2.19) \qquad c = \frac{\text{etr}[-\Omega]}{\tilde{\Gamma}_p(n)\;\tilde{\Gamma}_p(m)\;|L\,\Phi|^p\;|\Sigma|^{m+n}}\;.$$

It is not possible to find the marginal p.d.f. of $\underset{\sim}{G}:p\times p$ because the integral

$$\int_{W=\overline{W}'>0} f_{\underset{\sim}{G},\underset{\sim}{W}}(G,W)\; dW$$

is not solvable, therefore the symmetrised p.d.f. of $\underset{\sim}{G}:p\times p$ is obtained, i.e.

$$(6.2.20) \qquad f_{csym}(G) = \int_{U(p)} \int_{W=\overline{W}'>0} f_{\underset{\sim}{G},\underset{\sim}{W}}(U\,G\,\overline{U}',\, W)\; dW\;.$$

Thc expansion of the first exponential function and the hypergeometric functions in (6.2.18) and the application of (2.2.54) lead to

$$(6.2.21) \qquad f_{\underset{\sim}{G},\underset{\sim}{W}}(G,W)$$

$$= c|G|^{n-p}\;|I_p - G|^{m-p}\;|W|^{m+n-p}\;\text{etr}[-\Sigma^{-1} W]$$

$$\sum_{k=0}^{\infty}\sum_{\kappa}\sum_{b=0}^{\infty}\sum_{\beta}\sum_{t=0}^{\infty}\sum_{\tau}\sum_{\mu} \frac{(q^{-1}-1)^b\;\tilde{g}^{\mu}_{\kappa,\beta}\;\tilde{C}_\mu(-\Sigma^{-1} W^{\frac{1}{2}}\, G\, W^{\frac{1}{2}})}{b!\;t!\;k!\;[m]_\tau\;\tilde{C}_\kappa(I_n)}$$

$$\tilde{C}_\kappa(L^{-\frac{1}{2}}\,\Phi^{-1}\,L^{-\frac{1}{2}} - q^{-1} I_n)\;\tilde{C}_\tau(\Omega\,\Sigma^{-1} W^{\frac{1}{2}}(I_p - G)\, W^{\frac{1}{2}})\;.$$

The zonal polynomial $\tilde{C}_\tau(\Omega\,\Sigma^{-1}\,W^{\frac{1}{2}}(I_p - G)\,W^{\frac{1}{2}})$ can be written as $\tilde{C}_\tau(\Sigma^{-\frac{1}{2}}\,M\bar{M}'\,\Sigma^{-\frac{1}{2}}\,\Sigma^{-\frac{1}{2}}\,W^{\frac{1}{2}}\,(I_p - G)\,W^{\frac{1}{2}}\,\Sigma^{-\frac{1}{2}})$. The matrix $\Sigma^{-\frac{1}{2}}\,M\bar{M}'\,\Sigma^{-\frac{1}{2}}$ is a hermitian matrix, thus after transforming $\Sigma^{-\frac{1}{2}}\,M\bar{M}'\,\Sigma^{-\frac{1}{2}} \to U\,\Sigma^{-\frac{1}{2}}\,M\bar{M}'\,\Sigma^{-\frac{1}{2}}\,\bar{U}'$ and integrating over the unitary group the symmetrised p.d.f. of $\underset{\sim}{G}:p\times p$ follows as

(6.2.22) $$f_{csym}(G)$$

$$= c|G|^{n-p}\,|I_p - G|^{m-p} \int_{U(p)} \int_{W=\bar{W}'>0} |W|^{m+n-p}\,\mathrm{etr}[-\Sigma^{-1}W]$$

$$\sum_{k=0}^{\infty}\sum_{\kappa}\sum_{b=0}^{\infty}\sum_{\beta}\sum_{t=0}^{\infty}\sum_{\tau}\sum_{\mu} \frac{(q^{-1}-1)^b\,\tilde{g}^{\mu}_{\kappa,\beta}\,\tilde{C}_\mu(-\Sigma^{-1}\,W^{\frac{1}{2}}\,U\,G\,\bar{U}'\,W^{\frac{1}{2}})}{b!\;t!\;k!\;[m]_\tau\,\tilde{C}_\tau(I_p)\,\tilde{C}_\kappa(I_n)}$$

$$\tilde{C}_\kappa(L^{-\frac{1}{2}}\,\Phi^{-1}\,L^{-\frac{1}{2}} - q^{-1}\,I_n)\,\tilde{C}_\tau(\Sigma^{-1}\,W^{\frac{1}{2}}\,U(I_p - G)\bar{U}'\,W^{\frac{1}{2}})\,\tilde{C}_\tau(\Omega)\,dW\,dU.$$

Change the order of integration; then integration over the unitary group, using (2.2.64), leads to

(6.2.23) $$f_{csym}(G)$$

$$= c|G|^{n-p}\,|I_p - G|^{m-p} \sum_{k=0}^{\infty}\sum_{\kappa}\sum_{b=0}^{\infty}\sum_{\beta}\sum_{t=0}^{\infty}\sum_{\tau}\sum_{\mu}\sum_{\sigma} \frac{(q^{-1}-1)^b}{b!\;t!\;k!}$$

$$\frac{(-1)^{k+b}\,\tilde{g}^{\mu}_{\kappa,\beta}\,\tilde{P}^{\sigma}_{\tau,\mu}\,(G,\,I_p - G)}{[m]_\tau\,\tilde{C}_\tau(I_p)\,\tilde{C}_\kappa(I_n)}$$

$$\tilde{C}_\tau(\Omega)\,\tilde{C}_\kappa(L^{-\frac{1}{2}}\,\Phi^{-1}\,L^{-\frac{1}{2}} - q^{-1}\,I_n)\quad I^*$$

where

$$(6.2.24) \quad I^* = \int_{W=\overline{W}'>0} |W|^{m+n-p} \operatorname{etr}[-\Sigma^{-1} W] \, \tilde{C}_\sigma(\Sigma^{-1} W) \, dW$$

$$= \tilde{\Gamma}_p(m+n) \, [m+n]_\sigma \, |\Sigma|^{m+n} \, \tilde{C}_\sigma(I_p) \, , \quad \text{(from (2.2.32))} .$$

Substitution of (6.2.24) into (6.2.23) leads to

$$(6.2.25) \quad f_{csym}(G)$$

$$= \frac{\tilde{\Gamma}_p(m+n) \operatorname{etr}[-\Omega] \, |G|^{n-p} \, |I_p - G|^{m-p}}{\tilde{\Gamma}_p(n) \, \tilde{\Gamma}_p(m) \, |L\Phi|^p}$$

$$\sum_{k=0}^{\infty} \sum_{\kappa} \sum_{b=0}^{\infty} \sum_{\beta} \sum_{t=0}^{\infty} \sum_{\tau} \sum_{\mu} \sum_{\sigma} \frac{(q^{-1}-1)^b \, \tilde{g}^{\mu}_{\kappa,\beta} \, (-1)^{k+b}}{b! \, t! \, k! \, [m]_\tau \, \tilde{C}_\tau(I_p) \, \tilde{C}_\kappa(I_n)}$$

$$\tilde{P}^{\sigma}_{\tau,\mu} \, (G, \, I_p - G) \, [m+n]_\sigma \, \tilde{C}_\kappa(L^{-\frac{1}{2}} \, \Phi^{-1} \, L^{-\frac{1}{2}} - q^{-1} I_n)$$

$$\tilde{C}_\tau(\Omega) \, \tilde{C}_\sigma(I_p) \, .$$

Application of (2.2.72) with $S = L^{-\frac{1}{2}} \Phi^{-1} L^{-\frac{1}{2}} - q^{-1} I_n$, $a = m + n$ and $d = q^{-1} - 1$ leads to (6.2.1).

If $q = 1$ it follows that

$$(q^{-1} - 1)^b = 1 \quad \text{if} \quad b = 0$$

$$= 0 \quad \text{if} \quad b > 0$$

and

$$\tilde{g}^{\mu}_{\kappa,\beta} = 1 \quad \text{if} \quad b = 0 \, ,$$

thus (6.2.1) also follows from (6.2.25) if $q = 1$.

(6.2.2)

The joint p.d.f. of $\underset{\sim}{S}:p\times p$ and $\underset{\sim}{A}:p\times p$ follows as

$$(6.2.26)\quad f_{\underset{\sim}{S},\underset{\sim}{A}}(S,A) = \frac{|S|^{n-p}\ {}_0\tilde{F}_0(-\Sigma^{-1}S,\ \Phi^{-1}L^{-1})\ \mathrm{etr}[-\Omega]}{\tilde{\Gamma}_p(n)\ |L\Phi|^p\ |\Sigma|^{m+n}\ \tilde{\Gamma}_p(m)}$$

$${}_0\tilde{F}_1(m;\Omega\Sigma^{-1}A)\ \mathrm{etr}[-\Sigma^{-1}A]\ |A|^{m-p}\ .$$

In (6.2.26) make the transformations (6.2.13) and (6.2.14), then the joint p.d.f. of $\underset{\sim}{G}:p\times p$ and $\underset{\sim}{W}:p\times p$ follows as

$$(6.2.27)\quad f_{\underset{\sim}{G},\underset{\sim}{W}}(G,W) = c|W|^{m+n-p}\ \mathrm{etr}[-\Sigma^{-1}W]\ |G|^{n-p}\ |I_p-G|^{m-p}$$

$$\mathrm{etr}[\Sigma^{-1}W^{\frac{1}{2}}GW^{\frac{1}{2}}]\ {}_0\tilde{F}_0(-\Sigma^{-1}W^{\frac{1}{2}}GW^{\frac{1}{2}},\ \Phi^{-1}L^{-1})$$

$${}_0\tilde{F}_1(m;\Omega\Sigma^{-1}W^{\frac{1}{2}}(I_p-G)W^{\frac{1}{2}})$$

where c is given in (6.2.19).

The symmetrised p.d.f. of $\underset{\sim}{G}:p\times p$ can be derived along the same lines as (6.2.25), thus

$$(6.2.28)\quad f_{csym}(G)$$

$$= \frac{\mathrm{etr}[-\Omega]\ \tilde{\Gamma}_p(m+n)\ |G|^{n-p}\ |I_p-G|^{m-p}}{\tilde{\Gamma}_p(n)\ \tilde{\Gamma}_p(m)\ |L\Phi|^p}$$

$$\sum_{k=0}^{\infty}\sum_{\kappa}\sum_{b=0}^{\infty}\sum_{\beta}\sum_{t=0}^{\infty}\sum_{\tau}\sum_{\mu}\sum_{\sigma}\frac{\tilde{g}^{\mu}_{\kappa,\beta}\ \tilde{P}^{\sigma}_{\tau,\mu}(G,\ I_p-G)\ (-1)^{k+b}\ (-1)^b}{b!\ k!\ t!\ [m]_\tau\ \tilde{C}_\kappa(I_n)\ \tilde{C}_\tau(I_p)}$$

$$[m+n]_\sigma\ \tilde{C}_\kappa(\Phi^{-1}L^{-1})\ \tilde{C}_\tau(\Omega)\ \tilde{C}_\sigma(I_p)\ .$$

Application of (2.2.72) with $S = \Phi^{-1} L^{-1}$, $a = m + n$ and $\lambda = -1$ leads to (6.2.2).

(6.2.3)

Let $\Sigma : p \times p = \Psi : p \times p$; then the proof of (6.2.3) is similar to the proof of (6.2.1).

(6.2.4)

The joint p.d.f. of $\underset{\sim}{S} : p \times p$ and $\underset{\sim}{A} : p \times p$ follows as

$$(6.2.29) \qquad f_{\underset{\sim}{S},\underset{\sim}{A}}(S,A) = \frac{|S|^{n-p} \; {}_0\tilde{F}_0(-\Sigma^{-1} S, \Phi^{-1} L^{-1}) \; \text{etr}[-\Psi^{-1} A] \; |A|^{m-p}}{\tilde{\Gamma}_p(n) \; |L \Phi|^p \; |\Sigma|^n \; \tilde{\Gamma}_p(m) \; |\Psi|^m} .$$

In (6.2.29) make the transformations (6.2.13) and (6.2.14), then the expansion of the hypergeometric function and the exponential function, which contains $G : p \times p$ leads to the following expression for the joint p.d.f. of $\underset{\sim}{G} : p \times p$ and $\underset{\sim}{W} : p \times p$:

$$(6.2.30) \qquad f_{\underset{\sim}{G},\underset{\sim}{W}}(G,W)$$

$$= c|G|^{n-p} \; |I_p - G|^{m-p} \; |W|^{m+n-p} \; \text{etr}[-\Psi^{-1} W]$$

$$\sum_{k=0}^{\infty} \sum_{\kappa} \sum_{t=0}^{\infty} \sum_{\tau} \frac{\tilde{C}_\kappa(\Phi^{-1} L^{-1}) \; \tilde{C}_\kappa(-\Sigma^{-1} W^{\frac{1}{2}} G W^{\frac{1}{2}}) \; \tilde{C}_\tau(\Psi^{-1} W^{\frac{1}{2}} G W^{\frac{1}{2}})}{\tilde{C}_\kappa(I_n) \; k! \; t!}$$

with

$$(6.2.31) \qquad c = (\tilde{\Gamma}_p(n) \; \tilde{\Gamma}_p(m) \; |L \Phi|^p \; |\Sigma|^n \; |\Psi|^m)^{-1} .$$

To find the symmetrised p.d.f. of $\underset{\sim}{G} : p \times p$ the matrices in $\tilde{C}_\kappa(-\Sigma^{-1} W^{\frac{1}{2}} G W^{\frac{1}{2}})$ and $\tilde{C}_\tau(\Psi^{-1} W^{\frac{1}{2}} G W^{\frac{1}{2}})$ have to be "split". The matrices $\Sigma^{-1} : p \times p$ and $\Psi^{-1} : p \times p$ are hermitian matrices, thus by transforming

$$\Sigma^{-1} \to U\,\Sigma^{-1}\,\overline{U}'$$

and

$$\Psi^{-1} \to U\,\Psi^{-1}\,\overline{U}'$$

and integrating over the unitary group after each transformation, the symmetrised p.d.f. of $\underset{\sim}{G}:p\times p$, after changing the order of integration, follows as

$$(6.2.32) \quad f_{csym}(G) = c|G|^{n-p}\,|I_p - G|^{m-p} \sum_{k=0}^{\infty} \sum_{\kappa} \sum_{t=0}^{\infty} \sum_{\tau} \sum_{\delta} \frac{\tilde{g}^{\delta}_{\kappa,\tau}}{k!\ t!} \frac{\tilde{C}_{\kappa}(\Phi^{-1}\ L^{-1})\ \tilde{C}_{\kappa}(-\Sigma^{-1})\ \tilde{C}_{\tau}(\Psi^{-1})\ \tilde{C}_{\delta}(G)}{\tilde{C}_{\delta}(I_p)\ \tilde{C}_{\kappa}(I_n)\ \tilde{C}_{\kappa}(I_p)\ \tilde{C}_{\tau}(I_p)}\ I^*$$

where

$$(6.2.33) \quad I^* = \int_{W=\overline{W}'>0} |W|^{m+n-p}\ \mathrm{etr}[-\Psi^{-1}W]\ \tilde{C}_{\delta}(W)\ dW$$

$$= \tilde{\Gamma}_p(m+n)\ [m+n]_{\delta}\ |\Psi|^{m+n}\ \tilde{C}_{\delta}(\Psi)\ .$$

Substitution of (6.2.33) into (6.2.32) leads to (6.2.4).

<u>(6.2.5)</u>

The application of theorem 3.2.1 and corollary 2.7.1 leads to (6.2.5).

<u>(6.2.6)</u>

Let $\Sigma:p\times p = \Psi:p\times p$ in (6.2.21); then the joint p.d.f. of $\underset{\sim}{G}:p\times p$ and $\underset{\sim}{W}:p\times p$ is given as

$$(6.2.34) \qquad f_{\underset{\sim}{G},\underset{\sim}{W}}(G,W)$$

$$= c|G|^{n-p} \; |I_p - G|^{m-p} \; |W|^{m+n-p} \operatorname{etr}[-\Sigma^{-1} W]$$

$$\sum_{k=0}^{\infty} \sum_{\kappa} \sum_{t=0}^{\infty} \sum_{\tau} \sum_{\delta} \frac{(q^{-1} - 1)^t \; \tilde{g}^{\delta}_{\kappa,\tau} \; \tilde{C}_{\kappa}(L^{-\frac{1}{2}} \; \Phi^{-1} \; L^{-\frac{1}{2}} - q^{-1} I_n)}{\tilde{C}_{\kappa}(I_n) \; k! \; t!}$$

$$\tilde{C}_{\delta}(\Sigma^{-1} W^{\frac{1}{2}} G W^{\frac{1}{2}}) \; (-1)^{k+t}$$

with

$$(6.2.35) \qquad c = (\tilde{\Gamma}_p(m) \; \tilde{\Gamma}_p(n) \; |L \Phi|^p \; |\Sigma|^{m+n})^{-1} \; .$$

The symmetrised p.d.f. of $\underset{\sim}{G}:p\times p$ follows as

$$(6.2.36) \qquad f_{csym}(G) = c|G|^{n-p} \; |I_p - G|^{m-p} \sum_{k=0}^{\infty} \sum_{\kappa} \sum_{t=0}^{\infty} \sum_{\tau} \sum_{\delta} \frac{(q^{-1}-1)^t (-1)^{k+t}}{\tilde{C}_{\kappa}(I_n)} ,$$

$$\frac{\tilde{g}^{\delta}_{\kappa,\tau} \; \tilde{C}_{\kappa}(L^{-\frac{1}{2}} \; \Phi^{-1} \; L^{-\frac{1}{2}} - q^{-1} I_n) \; \tilde{C}_{\delta}(G)}{k! \; t! \; \tilde{C}_{\delta}(I_p)} \; I^*$$

where

$$(6.2.37) \qquad I^* = \int_{W=\overline{W}'>0} \operatorname{etr}[-\Sigma^{-1} W] \; |W|^{m+n-p} \; \tilde{C}_{\delta}(\Sigma^{-1} W) \, dW$$

$$= [m+n]_{\delta} \; \tilde{\Gamma}_p(m+n) \; |\Sigma|^{m+n} \; \tilde{C}_{\delta}(I_p) \; , \quad (\text{from } (2.2.32)) \; .$$

Substitution of (6.2.37) into (6.2.36) leads to

(6.2.38) $f_{csym}(G)$

$$= \frac{\tilde{\Gamma}_p(m+n)\ |G|^{n-p}\ |I_p - G|^{m-p}}{\tilde{\Gamma}_p(m)\ \tilde{\Gamma}_p(n)\ |L\,\Phi|^p} \sum_{k=0}^{\infty} \sum_{\kappa} \sum_{t=0}^{\infty} \sum_{\tau} \sum_{\delta} \frac{(q^{-1}-1)^t\ (-1)^{k+t}}{\tilde{C}_\kappa(I_n)}$$

$$\frac{\tilde{g}^{\delta}_{\kappa,\tau}\ \tilde{C}_\kappa(L^{-\frac{1}{2}}\ \Phi^{-1}\ L^{-\frac{1}{2}} - q^{-1}\ I_n)\ \tilde{C}_\delta(G)\ [m+n]_\delta}{k!\ t!} .$$

It also follows that

(6.2.39) $$(q^{-1}-1)^t\ (-1)^k\ \tilde{C}_\delta(G) = (q^{-1}-1)^{-k}\ (-1)^{-k}\ \tilde{C}_\delta((q^{-1}-1)G)$$

$$= (1-q^{-1})^{-k}\ \tilde{C}_\delta((q^{-1}-1)G) .$$

Substitution of (6.2.39) into (6.2.38) and the application of (2.2.70) with $A = (q^{-1}-1)G$ and $a = m + n$ lead to (6.2.6).

<u>(6.2.7)</u>

The application of theorem 3.2.1 and corollary 2.7.1 leads to (6.2.7).

<u>(6.2.8)</u>, <u>(6.2.9)</u>

These results follow from (6.2.38) by using the integral given in (2.2.35).

<u>(6.2.10)</u>

Let $\Sigma:p\times p = \Psi:p\times p$ in (6.2.30); then the joint p.d.f. of $\underset{\sim}{G}:p\times p$ and $\underset{\sim}{W}:p\times p$ is given as

(6.2.40) $f_{\underset{\sim}{G},\underset{\sim}{W}}(G,W)$

$$= c|G|^{n-p}\ |I_p - G|^{m-p}\ |W|^{m+n-p}\ \mathrm{etr}[-\Sigma^{-1}\ W]$$

$$\sum_{k=0}^{\infty} \sum_{\kappa} \sum_{t=0}^{\infty} \sum_{\tau} \sum_{\delta} \frac{\tilde{g}_{\kappa,\tau}^{\delta}\ \tilde{C}_{\kappa}(\Phi^{-1} L^{-1})\ \tilde{C}_{\delta}(\Sigma^{-1} W^{\frac{1}{2}} G W^{\frac{1}{2}})\ (-1)^{k}}{\tilde{C}_{\kappa}(I_n)\ k!\ t!}$$

where

(6.2.41) $\quad c = (\tilde{\Gamma}_p(n)\ \tilde{\Gamma}_p(m)\ |L\Phi|^{p}\ \ |\Sigma|^{m+n})^{-1}$.

The symmetrised p.d.f. of $\underset{\sim}{G}:p\times p$ follows after integrating over the unitary group and with respect to $W:p\times p$, as

(6.2.42) $\quad f_{csym}(G)$

$$= \frac{\tilde{\Gamma}_p(m+n)\ |G|^{n-p}\ |I_p - G|^{m-p}}{\tilde{\Gamma}_p(n)\ \tilde{\Gamma}_p(m)\ |L\Phi|^{p}}$$

$$\sum_{k=0}^{\infty} \sum_{\kappa} \sum_{t=0}^{\infty} \sum_{\tau} \sum_{\delta} \frac{\tilde{g}_{\kappa,\tau}^{\delta}\ \tilde{C}_{\kappa}(\Phi^{-1} L^{-1})\ \tilde{C}_{\delta}(-G)\,(-1)^{t}\ [m+n]_{\delta}}{\tilde{C}_{\kappa}(I_n)\ k!\ t!} .$$

The application of (2.2.70) with $A = -G$ and $a = m+n$ leads to (6.2.10).

<u>(6.2.11)</u>

The application of theorem 3.2.1 and corollary 2.7.1 leads to (6.2.11).

<u>Remark 6.2.1</u>

(i) As in the real case (cf. Underhill, 1973, p. 4.17) it does not seem possible to find an expression for the symmetrised p.d.f. of $\underset{\sim}{G}:p\times p$ when $\Omega:p\times p \neq 0$ and $\Sigma:p\times p \neq \Psi:p\times p$. It also does not seem possible to find an expression for $f_{csym}(G)$ when the p.d.f. of $\underset{\sim}{S}:p\times p$ has the Γ-type representation, $\Omega:p\times p = 0$ and $\Sigma:p\times p \neq \Psi:p\times p$.

(ii) It is interesting to note that in the case when $\Omega:p\times p \neq 0$ and $\Sigma:p\times p = \Psi:p\times p$ the Γ-type representation and the power-series representation of the p.d.f. of $\underset{\sim}{S}:p\times p$ lead to similar expressions for $f_{csym}(G)$. It seems not possible to express $\tilde{P}_{\kappa,\tau}(G, I_p - G)$ in terms of the characteristic roots of $G:p\times p$ and therefore the joint p.d.f. of these roots can not be derived in this case.

(iii) It is clear that (6.2.6) tends to (6.2.10) when $\frac{1}{q} \to 0$, i.e. $q \to \infty$.

(iv) Let $\Phi:n\times n = L^{-1}:n\times n$ in (6.2.10); then

$$(6.2.43) \quad f_{csym}(G) = \frac{\tilde{\Gamma}_p(m+n)\ |G|^{n-p}\ |I_p - G|^{-(p+n)}}{\tilde{\Gamma}_p(n)\ \tilde{\Gamma}_p(m)}\ {}_1\tilde{F}_0(m+n\,;\ -G(I_p - G))\ .$$

From (2.3.13) follows:

$$(6.2.44) \quad {}_1\tilde{F}_0(m+n\,;\ -G(I_p - G)) = |I_p - G|^{(m+n)}\ .$$

Substitution of (6.2.44) into (6.2.43) leads to

$$(6.2.45) \quad f_{csym}(G) = \frac{\tilde{\Gamma}_p(m+n)\ |G|^{n-p}\ |I_p - G|^{m-p}}{\tilde{\Gamma}_p(m)\ \tilde{\Gamma}_p(n)}$$

which is the $CMB_1(p, m, n)$-p.d.f..

(v) Let $\Phi:n\times n = L^{-1}:n\times n$ in (6.2.1); then

$$(6.2.46) \quad f_{csym}(G) = \frac{\tilde{\Gamma}_p(m+n)\ \mathrm{etr}[-\Omega]\ |G|^{n-p}\ |I_p - G|^{m-p}}{\tilde{\Gamma}_p(n)\ \tilde{\Gamma}_p(m)}$$

$$\sum_{t=0}^{\infty} \sum_{\tau} \frac{\tilde{P}^{\tau}_{(0),\tau}(G,\ I_p - G)\ [m+n]_\tau\ \tilde{C}_\tau(\Omega)}{[m]_\tau\ t!} .$$

It is also clear that $\underset{\sim}{S}:p\times p \sim CW(p,\ n,\ \Sigma)$ if $\Phi:n\times n = L^{-1}:n\times n$ so that $\underset{\sim}{G}:p\times p = (\underset{\sim}{A} + \underset{\sim}{S})^{-\frac{1}{2}}\ \underset{\sim}{S}(\underset{\sim}{A} + \underset{\sim}{S})^{-\frac{1}{2}}$ $\sim NCCMB_{1A}(p,\ n,\ m,\ \Omega)$ i.e.

$$(6.2.47) \quad f_{csym}(G) = \frac{\tilde{\Gamma}_p(m+n)\ \mathrm{etr}[-\Omega]\ |G|^{n-p}\ |I_p - G|^{m-p}}{\tilde{\Gamma}_p(m)\ \tilde{\Gamma}_p(n)}$$

$$\sum_{t=0}^{\infty} \sum_{\tau} \frac{[m+n]_\tau\ \tilde{C}_\tau(\Omega)\ \tilde{C}_\tau(I_p - G)}{[m]_\tau\ t!\ \tilde{C}_\tau(I_p)} , \quad \text{(from (5.2.7))} .$$

The equating of (6.2.46) and (6.2.47) leads to

$$(6.2.48) \quad \tilde{P}^{\tau}_{(0),\tau}(G,\ I_p - G) = \frac{\tilde{C}_\tau(I_p - G)}{\tilde{C}_\tau(I_p)} .$$

6.2.2 Certain marginal distributions of the characteristic roots of $\underset{\sim}{G}:p\times p$

In theorem 6.2.2 it is shown how the p.d.f. of $\underset{\sim}{D}_G$ can be written in a form such that the random component is in the form given in (3.2.3).

Theorem 6.2.2

Let the p.d.f. of the central complex quadratic form $\underset{\sim}{S}:p\times p = \underset{\sim}{Z}\,L\,\overline{\underset{\sim}{Z}}'$ have the power-series representation, given in (4.2.6) and let $\underset{\sim}{A}:p\times p \sim CW(p,\ m,\ \Psi)$; then the p.d.f. of $\underset{\sim}{D}_G$ where $\underset{\sim}{G}:p\times p = (\underset{\sim}{A} + \underset{\sim}{S})^{-\frac{1}{2}}\ \underset{\sim}{S}(\underset{\sim}{A} + \underset{\sim}{S})^{-\frac{1}{2}}$ is given below for certain specifications of $\Psi:p\times p$ and $\Sigma:p\times p$.

(i) $\underline{\Psi:p\times p \neq \Sigma:p\times p \,, \quad \Omega:p\times p = 0}$

(6.2.49) $f_{\underset{\sim}{D}_G}(D_G)$

$$= \frac{\pi^{p(p-1)}\ \tilde{\Gamma}_p(m+n)}{\tilde{\Gamma}_p(n)\ \tilde{\Gamma}_p(m)\ \tilde{\Gamma}_p(p)\ |L\,\Phi|^p\ \ |\Sigma\,\Psi^{-1}|^n}$$

$$\sum_{k=0}^{\infty}\ \sum_{\kappa}\ \sum_{t=0}^{\infty}\ \sum_{\tau}\ \sum_{\delta}\ \frac{\tilde{g}^{\delta}_{\kappa,\tau}\ [m+n]_{\delta}\ \tilde{C}_{\kappa}(\Phi^{-1}\ L^{-1})\ \tilde{C}_{\kappa}(-\Sigma^{-1})}{\tilde{C}_{\kappa}(I_n)\ \tilde{C}_{\kappa}(I_p)\ \tilde{C}_{\tau}(I_p)\ \tilde{C}_{\delta}(I_p)}$$

$$\frac{\tilde{C}_{\tau}(\Psi^{-1})\ \tilde{C}_{\delta}(\Psi)}{k!\ t!}\ \ \chi_{[\kappa]}(1)\ \prod_{i=1}^{p}\ \tilde{g}_i^{n-p}\ (1-\tilde{g}_i)^{m-p}$$

$$|(\tilde{g}_j^{k_i+p-i})|\,|(\tilde{g}_j^{p-i})|\ ,\quad 0<\tilde{g}_1<\cdots<\tilde{g}_p<1\ .$$

(ii) $\underline{\Psi:p\times p = \Sigma:p\times p \,, \quad \Omega:p\times p = 0}$

(6.2.50) $f_{\underset{\sim}{D}_G}(D_G)$

$$= \frac{\pi^{p(p-1)}\ \tilde{\Gamma}_p(m+n)}{\tilde{\Gamma}_p(n)\ \tilde{\Gamma}_p(m)\ |L\,\Phi|^p}\ \sum_{k=0}^{\infty}\ \sum_{\kappa}\ \sum_{t=0}^{\infty}\ \sum_{\tau}\ \sum_{\delta}\ \frac{\tilde{g}^{\delta}_{\kappa,\tau}(-1)^k}{\tilde{C}_{\kappa}(I_n)\ k!\ t!}$$

$$\tilde{C}_{\kappa}(\Phi^{-1}\ L^{-1})\ [m+n]_{\delta}\ \ \chi_{[\kappa]}(1)\ \prod_{i=1}^{p}\ \tilde{g}_i^{n-p}\ (1-\tilde{g}_i)^{m-p}$$

$$|(\tilde{g}_j^{k_i+p-i})|\,|(\tilde{g}_j^{p-i})|\ ,\quad 0<\tilde{g}_1<\cdots<\tilde{g}_p<1\ .$$

Proof

(6.2.49)

By using the result in (2.2.42) the expression in (6.2.49) follows from (6.2.5).

(6.2.50)

By using theorem 3.2.1, corollary 2.7.1 and (2.2.42) the expression in (6.2.50) follows from (6.2.42).

Remark 6.2.2

By using (2.2.44) and (2.3.3) the random component of (6.2.11) can also be written in the form given in (3.2.3). This expression, however, leads to improper integrals if the theorems in chapter 3 are used to derive certain marginal distributions of the characteristic roots of $\underset{\sim}{G}:p\times p$ and therefore (6.2.50) will be considered here.

By comparing the random component in the $CMB_1(p, m, n)$ - p.d.f. given in (5.2.22), i.e.

$$\prod_{i=1}^{p} \tilde{\ell}_i^{n-p} (1-\tilde{\ell}_i)^{m-p} |(\tilde{\ell}_j^{p-i})| |(\tilde{\ell}_j^{p-i})|$$

with the random component in (6.2.49) and (6.2.50), i.e.

$$\prod_{i=1}^{p} \tilde{g}_i^{n-p} (1-\tilde{g}_i)^{m-p} |(\tilde{g}_j^{k_i+p-i})| |(\tilde{g}_j^{p-i})|$$

it is clear that the two components are similar except for the power of the elements of the first determinant. The different marginal distributions of the characteristic roots of $\underset{\sim}{G}:p\times p$ when $\underset{\sim}{D}_G$ has the p.d.f. given in (6.2.49) or (6.2.50) can thus be derived along the same lines as the different marginal distributions of the characteristic roots of the $CMB_1(p, m, n)$ - matrix. Consider the following example:

Let $\underset{\sim}{D}_G$ have the p.d.f. given in (6.2.50); then

(6.2.51) $\quad P(c < \tilde{G}_1 < \tilde{G}_p < d)$

$$= \frac{\pi^{p(p-1)} \tilde{\Gamma}_p(m+n)}{\tilde{\Gamma}_p(n)\ \tilde{\Gamma}_p(m)\ |L\,\Phi|^p} \sum_{k=0}^{\infty} \sum_{\kappa} \sum_{t=0}^{\infty} \sum_{\tau} \sum_{\delta} \frac{\tilde{g}^{\delta}_{\kappa,\tau}\ (-1)^k}{\tilde{C}_\kappa(I_n)\ k!\ t!}$$

$$\tilde{C}_\kappa(\Phi^{-1}\ L^{-1})\ [m+n]_\delta\ \ \chi_{[\kappa]}(1)\ \ |(b_{ij})|$$

where

(6.2.52) $\quad b_{ij} = B_d(n+p+k_i-i-j+1,\ m-p+1) - B_c(n+p+k_i-i-j+1,\ m-p+1)$.

By comparing (6.2.51) and (6.2.52) with (5.2.31) and (5.2.32) the similarity between the expressions for $P(c < \tilde{G}_1 < \tilde{G}_p < d)$ and $P(c < \tilde{L}_1 < \tilde{L}_p < d)$ becomes clear.

Because of these similarities between the two random components mentioned above, the different marginal distributions of $\tilde{G}_1,\cdots,\tilde{G}_p$ will not be considered here further.

6.2.3. P.d.f.s of functions of the characteristic roots of $\underset{\sim}{G}:p\times p$

In theorem 6.2.3 the p.d.f.s of $|\underset{\sim}{G}| = \prod_{i=1}^{p} \tilde{G}_i$ and

$|I_p - \underset{\sim}{G}| = \prod_{i=1}^{p} (1-\tilde{G}_i)$ when $\Omega:p\times p = 0$ and $\Sigma:p\times p = \Psi:p\times p$ are derived in terms of Meijer's G-function.

<u>Theorem 6.2.3</u>

Let $\underset{\sim}{G}:p\times p$ have the symmetrised p.d.f. given in (6.2.6); then the p.d.f. of $|\underset{\sim}{G}|$ is given by

(6.2.53) $$f_{|\underset{\sim}{G}|}(|G|)$$

$$= \frac{\tilde{\Gamma}_p(m+n)\ |G|^{-1}}{\tilde{\Gamma}_p(n)} \sum_{k=0}^{\infty} \sum_{\kappa} \sum_{t=0}^{\infty} \sum_{\tau} \sum_{\delta} \frac{\tilde{g}^{\delta}_{\kappa,\tau}\ (-1)^{k+t}\ (q^{-1}-1)^t}{\tilde{C}_\kappa(I_n)\ k!\ t!}$$

$$[m+n]_\delta\ \tilde{C}_\kappa(L^{-\frac{1}{2}}\ \Phi^{-1}\ L^{-\frac{1}{2}} - \underset{\sim}{q}^{-1}\ I_n)\ \tilde{C}_\delta(I_p)$$

$$G_p\left[|G| \,\middle|\, \begin{matrix} m+n+d_j-j+1 \\ n+d_j-j+1 \end{matrix}\right], \quad 0 < |G| < 1 \quad \text{and} \quad q > 0$$

and the p.d.f. of $|I_p - \underset{\sim}{G}|$ is given by

(6.2.54) $$f_{|I_p - \underset{\sim}{G}|}(|I_p - G|)$$

$$= \frac{\tilde{\Gamma}_p(m+n)\ |I_p - G|^{-1}}{\tilde{\Gamma}_p(m)} \sum_{k=0}^{\infty} \sum_{\kappa} \sum_{t=0}^{\infty} \sum_{\tau} \sum_{\delta} \frac{\tilde{g}^{\delta}_{\kappa,\tau}\ \tilde{C}_\delta(I_p)}{k!\ t!\ \tilde{C}_\kappa(I_n)}$$

$$(-1)^{k+t}\ (q^{-1}-1)^t\ [m+n]_\delta\ [n]_\delta\ \tilde{C}_\kappa(L^{-\frac{1}{2}}\ \Phi^{-1}\ L^{-\frac{1}{2}} - q^{-1}\ I_n)$$

$$G_p\left[|I_p - G| \,\middle|\, \begin{matrix} m+n+d_j-j+1 \\ m-j+1 \end{matrix}\right], \quad 0 < |I_p - G| < 1 \quad \text{and} \quad q > 0 .$$

<u>Proof</u>

<u>(6.2.53)</u>

The factor in $E(|\underset{\sim}{G}|^h)$, (6.2.8), which depends on h can be written as

(6.2.55) $$\frac{\tilde{\Gamma}_p(n+h)\ [n+h]_\delta}{\tilde{\Gamma}_p(m+n+h)\ [m+n+h]_\delta} = \frac{\tilde{\Gamma}_p(n+h,\ \delta)}{\tilde{\Gamma}_p(m+n+h,\ \delta)} .$$

The application of theorem 2.5.2 leads now to (6.2.53).

(6.2.54)

The factor in $E(|I_p - \underset{\sim}{G}|^h)$, (6.2.9), which depends on h can be written as

$$(6.2.56) \quad \frac{\tilde{\Gamma}_p(m+h)}{\tilde{\Gamma}_p(m+n+h)\ [m+n+h]_\delta} = \frac{\tilde{\Gamma}_p(m+h)}{\tilde{\Gamma}_p(m+n+h,\ \delta)} .$$

The application of theorem 2.5.2 leads now to (6.2.54).

In theorem 6.2.4 the p.d.f. of $\operatorname{tr} \underset{\sim}{G}$ for certain specifications of $\Omega{:}p\times p$, $\Sigma{:}p\times p$ and $\Psi{:}p\times p$ will be derived.

Theorem 6.2.4

(i) $\Omega{:}p\times p = 0$

$$(6.2.57) \quad f_{\operatorname{tr} \underset{\sim}{G}}(\operatorname{tr} G) = \frac{\tilde{\Gamma}_p(m+n)\ (\operatorname{tr} G)^{np-1}}{\tilde{\Gamma}_p(m)\ |L\,\Phi|^p\ |\Sigma\,\Psi^{-1}|^n\ \Gamma(np)}$$

$$\sum_{k=0}^{\infty} \sum_{\kappa} \sum_{t=0}^{\infty} \sum_{\tau} \sum_{\delta} \sum_{b=0}^{\infty} \sum_{\beta} \sum_{\sigma} \frac{\tilde{g}^{\delta}_{\kappa,\tau}\ \tilde{g}^{\sigma}_{\delta,\beta}}{b!\ k!\ t!}$$

$$\frac{[n]_\sigma\ [m+n]_\delta\ [p-m]_\beta\ \tilde{C}_\kappa(\Phi^{-1}\ L^{-1})\ \tilde{C}_\kappa(-\Sigma^{-1})}{\tilde{C}_\kappa(I_n)\ \tilde{C}_\kappa(I_p)\ \tilde{C}_\tau(I_p)\ \tilde{C}_\delta(I_p)\ (pn)_\sigma}$$

$$\tilde{C}_\tau(\Psi^{-1})\ \tilde{C}_\sigma(I_p)\ (\operatorname{tr} G)^{k+b+t} , \quad 0 < \operatorname{tr} G < 1 .$$

(ii) $\Omega{:}p\times p = 0$, $\Sigma{:}p\times p = \Psi{:}p\times p$

$$(6.2.58) \quad f_{\operatorname{tr} \underset{\sim}{G}}(\operatorname{tr} G) = \frac{\tilde{\Gamma}_p(m+n)\ (\operatorname{tr} G)^{np-1}}{\tilde{\Gamma}_p(m)\ \Gamma(np)\ |L\,\Phi|^p}$$

$$\sum_{k=0}^{\infty} \sum_{\kappa} \sum_{t=0}^{\infty} \sum_{\tau} \sum_{\delta} \sum_{b=0}^{\infty} \sum_{\beta} \sum_{\sigma} \frac{\tilde{g}^{\delta}_{\kappa,\tau}\ \tilde{g}^{\sigma}_{\delta,\beta}\ (-1)^k}{k!\ t!\ b!\ (np)_\sigma\ \tilde{C}_\kappa(I_n)}$$

$$\tilde{C}_\kappa(\Phi^{-1}\ L^{-1})\ [m+n]_\delta\ [n]_\sigma\ \tilde{C}_\sigma(I_p)$$

$$[p-m]_\beta\ (\mathrm{tr}\, G)^{k+b+t}\ ,\quad 0 < \mathrm{tr}\, G < 1\ .$$

Proof

Only (6.2.58) will be proved here, the proof of (6.2.57) being similar.

After the expanding of $|I_p - G|^{m-p}$ in a series of zonal polynomials, the p.d.f. of $\underset{\sim}{D}_G$ follows from (6.2.42) as

$$(6.2.59)\quad f_{\underset{\sim}{D}_G}(D_G) = \frac{\pi^{p(p-1)}\ \tilde{\Gamma}_p(m+n)\ |D_G|^{n-p}}{\tilde{\Gamma}_p(n)\ \tilde{\Gamma}_p(m)\ \tilde{\Gamma}_p(p)\ |L\,\Phi|^p} \prod_{i>j}^{p} (\tilde{g}_i - \tilde{g}_j)^2$$

$$\sum_{k=0}^{\infty} \sum_{\kappa} \sum_{t=0}^{\infty} \sum_{\tau} \sum_{\delta} \sum_{b=0}^{\infty} \sum_{\beta} \sum_{\sigma} \frac{\tilde{g}^{\delta}_{\kappa,\tau}\ \tilde{g}^{\sigma}_{\delta,\beta}\ (-1)^k}{k!\ t!\ b!\ \tilde{C}_\kappa(I_n)}$$

$$\tilde{C}_\kappa(\Phi^{-1}\ L^{-1})\ [m+n]_\delta\ [p-m]_\beta\ \tilde{C}_\sigma(D_G)\ .$$

In (6.2.59) make the transformation

$$a_i = \frac{\tilde{g}_i}{\mathrm{tr}\, G}\ ,\quad (i = 1,\cdots,p-1)$$

with inverse transformation

$$\tilde{g}_i = a_i\ \mathrm{tr}\, G\ ,\quad (i = 1,\cdots,p-1)$$

and

$$\tilde{g}_p = \operatorname{tr} G \,(1 - \sum_{i=1}^{p-1} a_i)$$

$$= a_p \operatorname{tr} G \,.$$

The jacobian follows as

$$J(\tilde{g}_1,\cdots,\tilde{g}_p \rightarrow a_1,\cdots,a_{p-1}, \operatorname{tr} G) = (\operatorname{tr} G)^{p-1} \,.$$

As in lemma 4.2.2 it follows:

$$(6.2.60) \quad f_{\operatorname{tr} \underset{\sim}{G}}(\operatorname{tr} G) = \frac{\pi^{p(p-1)}\, \tilde{\Gamma}_p(m+n)\, (\operatorname{tr} G)^{np-1}}{\tilde{\Gamma}_p(n)\, \tilde{\Gamma}_p(m)\, \tilde{\Gamma}_p(p)\, |L\,\Phi|^p}$$

$$\sum_{k=0}^{\infty} \sum_{\kappa} \sum_{t=0}^{\infty} \sum_{\tau} \sum_{\delta} \sum_{b=0}^{\infty} \sum_{\beta} \sum_{\sigma} \frac{\tilde{g}^{\delta}_{\kappa,\tau}\, \tilde{g}^{\sigma}_{\delta,\beta}\, (-1)^k}{k!\; t!\; \tilde{C}_\kappa(I_n)}$$

$$\tilde{C}_\kappa(\Phi^{-1}\, L^{-1})\, [m+n]_\delta\, (\operatorname{tr} G)^{k+b+t}\, [p-m]_\beta\; I^*$$

where

$$(6.2.61) \quad I^* = \int_{\Lambda} \prod_{i=1}^{p} a_i^{n-p}\, \tilde{C}_\sigma(D_a) \prod_{i>j} (a_i - a_j)^2\, da_1 \cdots da_{p-1}$$

$$= \frac{\tilde{\Gamma}_p(n)\, \tilde{\Gamma}_p(p)\, [n]_\sigma}{\pi^{p(p-1)}\, \Gamma(np)\, (pn)_\sigma}\, \tilde{C}_\sigma(I_p)$$

with $\Lambda = \{0 < a_1 < \cdots < a_{p-1} < a_p = 1-a_1- \cdots -a_{p-1}\}$.

Substitution of (6.2.61) into (6.2.60) leads to (6.2.58).

Remark 6.2.3

(i) It is important to note that the p.d.f.s of $\mathrm{tr}\, \underset{\sim}{G}$ given in (6.2.57) and (6.2.58) are convergent only for $0 < \mathrm{tr}\, G < 1$. This restriction and the fact that each p.d.f. involves eight summation signs, make it of limited practical value.

(ii) Khatri (1970, p. 75 – 77) derived expressions for $E((\mathrm{tr}\, G)^h)$ in the following cases:

$\underset{\sim}{S}:p\times p$ is a central complex quadratic form, $\Omega:p\times p = 0$ and $\Sigma:p\times p = \Psi:p\times p$;

$\underset{\sim}{S}:p\times p$ is a non-central complex quadratic form, $\Omega:p\times p = 0$ and $\Sigma:p\times p = \Psi:p\times p$;

$\underset{\sim}{S}:p\times p$ is a non-central complex compound quadratic form, $\Omega:p\times p = 0$ and $\Sigma:p\times p = \Psi:p\times p$.

These expressions for $E((\mathrm{tr}\, G)^h)$ are in terms of the characteristic roots of $S A^{-1}$ and are also in a very complicated form.

6.3 THE QUADRATIC FORM $\underset{\sim}{G}:p\times p = (\underset{\sim}{S} + \underset{\sim}{T})^{-\frac{1}{2}} \underset{\sim}{S}(\underset{\sim}{S} + \underset{\sim}{T})^{-\frac{1}{2}}$ WHEN BOTH $\underset{\sim}{S}:p\times p$ AND $\underset{\sim}{T}:p\times p$ ARE CENTRAL COMPLEX QUADRATIC FORMS

Consider the following theorem in which the symmetrised p.d.f. of $\underset{\sim}{G}:p\times p = (\underset{\sim}{S} + \underset{\sim}{T})^{-\frac{1}{2}} \underset{\sim}{S}(\underset{\sim}{S} + \underset{\sim}{T})^{-\frac{1}{2}}$ is derived. It is to be noted that it seems not possible to derive the actual p.d.f. of $\underset{\sim}{G}:p\times p$.

Theorem 6.3.1

Let $\underset{\sim}{Z}:p\times n \sim \mathrm{CMTN}(p,\ n,\ 0,\ \Phi\otimes\Sigma)$, $\underset{\sim}{Y}:p\times m \sim \mathrm{CMTN}(p,\ m,\ 0,\ \Psi\otimes\Sigma)$ and $L:n\times n$ and $Q:m\times m$ be h.p.d. matrices; then the symmetrised p.d.f. of $\underset{\sim}{G}:p\times p = (\underset{\sim}{S} + \underset{\sim}{T})^{-\frac{1}{2}} \underset{\sim}{S}(\underset{\sim}{S} + \underset{\sim}{T})^{-\frac{1}{2}}$, where $\underset{\sim}{S}:p\times p = \underset{\sim}{Z}\, L\, \overline{\underset{\sim}{Z}}'$ and $\underset{\sim}{T}:p\times p = \underset{\sim}{Y}\, Q\, \overline{\underset{\sim}{Y}}'$, is given below.

(i) The p.d.f.s of $\underset{\sim}{S}:p\times p$ and $\underset{\sim}{T}:p\times p$ have the Γ-type representation

(6.3.1) $$f_{csym}(G)$$

$$= \frac{\tilde{\Gamma}_p(m+n)\ |G|^{n-p}\ |I_p - G|^{m-p}\ r^{(m+n)p}}{\tilde{\Gamma}_p(n)\ \tilde{\Gamma}_p(m)\ |L\Phi|^p\ |Q\Psi|^p}$$

$$\sum_{k=0}^{\infty} \sum_{\kappa} \sum_{t=0}^{\infty} \sum_{\tau} \sum_{b=0}^{\infty} \sum_{\beta} \sum_{\delta} \sum_{\sigma} \frac{(-1)^{k+b}\ (r^{-1}-q^{-1})^t\ r^{k+t+b}\ \tilde{g}^{\delta}_{\kappa,\tau}}{t!\ k!\ b!\ \tilde{C}_{\kappa}(I_n)\ \tilde{C}_{\beta}(I_m)}$$

$$[m+n]_{\sigma}\ \tilde{C}_{\kappa}(L^{-\frac{1}{2}}\ \Phi^{-1}\ L^{-\frac{1}{2}} - q^{-1}\ I_n)\ \tilde{C}_{\beta}(Q^{-\frac{1}{2}}\ \Psi^{-1}\ Q^{-\frac{1}{2}} - r^{-1}\ I_m)$$

$$\tilde{C}_{\sigma}(I_p)\ \tilde{P}^{\sigma}_{\delta,\beta}(G,\ I_p - G)\ ,\quad 0 < G = \bar{G}' < I_p\ ,\quad q > 0 \text{ and } r > 0\ .$$

(ii) The p.d.f. of $\underset{\sim}{S}:p\times p$ has the power-series representation and the p.d.f. of $\underset{\sim}{T}:p\times p$ has the Γ-type representation

(6.3.2) $$f_{csym}(G)$$

$$= \frac{\tilde{\Gamma}_p(m+n)\ |G|^{n-p}\ |I_p - G|^{m-p}\ r^{(m+n)p}}{\tilde{\Gamma}_p(n)\ \tilde{\Gamma}_p(m)\ |L\Phi|^p\ |\Psi Q|^p}$$

$$\sum_{k=0}^{\infty} \sum_{\kappa} \sum_{t=0}^{\infty} \sum_{\tau} \sum_{b=0}^{\infty} \sum_{\beta} \sum_{\delta} \sum_{\sigma} \frac{(-1)^{k+b}\ r^{k+b}\ \tilde{g}^{\delta}_{\kappa,\tau}\ [m+n]_{\sigma}}{t!\ k!\ b!\ \tilde{C}_{\kappa}(I_n)\ \tilde{C}_{\beta}(I_m)}$$

$$\tilde{C}_{\kappa}(\Phi^{-1}\ L^{-1})\ \tilde{C}_{\beta}(Q^{-\frac{1}{2}}\ \Psi^{-1}\ Q^{-\frac{1}{2}} - r^{-1}\ I_m)\ \tilde{C}_{\sigma}(I_p)\ \tilde{P}^{\sigma}_{\delta,\beta}(G,\ I_p - G),$$

$$0 < G = \bar{G}' < I_p \text{ and } r > 0\ .$$

Proof

(6.3.1)

The joint p.d.f. of $\underset{\sim}{S}:p\times p$ and $\underset{\sim}{T}:p\times p$ follows as

$$(6.3.3) \qquad f_{\underset{\sim}{S},\underset{\sim}{T}}(S,T)$$

$$= c|S|^{n-p}\,\mathrm{etr}[-q^{-1}\,\Sigma^{-1}\,S]\;{}_0\tilde{F}_0(L^{-\frac{1}{2}}\,\Phi^{-1}\,L^{-\frac{1}{2}} - q^{-1}\,I_n\,,\; -\Sigma^{-1}\,S)$$

$$|T|^{m-p}\,\mathrm{etr}[-r^{-1}\,\Sigma^{-1}\,T]\;{}_0\tilde{F}_0(Q^{-\frac{1}{2}}\,\Psi^{-1}\,Q^{-\frac{1}{2}} - r^{-1}\,I_m\,,\; -\Sigma^{-1}\,T)$$

with

$$(6.3.4) \qquad c = (\tilde{\Gamma}_p(m)\;\tilde{\Gamma}_p(n)\;|\Sigma|^{m+n}\;|Q\,\Psi|^p\;|L\,\Phi|^p)^{-1}\,.$$

In (6.3.3) make the transformations (6.2.13) and (6.2.14) with $A:p\times p$ replaced by $T:p\times p$ in (6.2.13), then the joint p.d.f. of $\underset{\sim}{G}:p\times p$ and $\underset{\sim}{W}:p\times p$ follows as

$$(6.3.5) \qquad f_{\underset{\sim}{G},\underset{\sim}{W}}(G,W) = c|G|^{n-p}\;|I_p - G|^{m-p}\;|W|^{m+n-p}\;\mathrm{etr}[-r^{-1}\,\Sigma^{-1}\,W]$$

$$\mathrm{etr}[(r^{-1} - q^{-1})\,\Sigma^{-1}\,W^{\frac{1}{2}}\,G\,W^{\frac{1}{2}}]$$

$${}_0\tilde{F}_0[L^{-\frac{1}{2}}\,\Phi^{-1}\,L^{-\frac{1}{2}} - q^{-1}\,I_n\,,\; -\Sigma\;\;W^{\frac{1}{2}}\,G\,W^{\frac{1}{2}}]$$

$${}_0\tilde{F}_0[Q^{-\frac{1}{2}}\,\Psi^{-1}\,Q^{-\frac{1}{2}} - r^{-1}\,I_m\,,\; -\Sigma^{-1}\,W^{\frac{1}{2}}\,(I_p - G)\,W^{\frac{1}{2}}]\,.$$

The expansion of the second exponential function and the hypergeometric functions leads to

$$(6.3.6) \qquad f_{\underset{\sim}{G},\underset{\sim}{W}}(G,W)$$

$$= c|G|^{n-p}\;|I_p - G|^{m-p}\;|W|^{m+n-p}\;\mathrm{etr}[-r^{-1}\,\Sigma^{-1}\,W]$$

$$\sum_{k=0}^{\infty} \sum_{\kappa} \sum_{t=0}^{\infty} \sum_{\tau} \sum_{b=0}^{\infty} \sum_{\beta} \sum_{\delta} \frac{(-1)^{k+b} \, (r^{-1}-q^{-1})^{t} \, \tilde{g}^{\delta}_{\kappa,\tau}}{k! \; t! \; b! \; \tilde{C}_{\kappa}(I_n) \; \tilde{C}_{\beta}(I_m)}$$

$$\tilde{C}_{\kappa}(L^{-\frac{1}{2}} \, \Phi^{-1} \, L^{-\frac{1}{2}} - q^{-1} I_n) \, \tilde{C}_{\delta}(\Sigma^{-1} W^{\frac{1}{2}} G W^{\frac{1}{2}})$$

$$\tilde{C}_{\beta}(Q^{-\frac{1}{2}} \, \Psi^{-1} \, Q^{-\frac{1}{2}} - r^{-1} I_m) \, \tilde{C}_{\beta}(\Sigma^{-1} W^{\frac{1}{2}} (I_p - G) \, W^{\frac{1}{2}}) \; .$$

After changing the order of integration, the symmetrised p.d.f. of $\underset{\sim}{G}:p\times p$ follows as

(6.3.7) $\quad f_{csym}$

$$= c|G|^{n-p} \, |I_p - G|^{m-p} \int_{W=\overline{W}'>0} |W|^{m+n-p} \, \text{etr}[-r^{-1} \, \Sigma^{-1} W]$$

$$\sum_{k=0}^{\infty} \sum_{\kappa} \sum_{t=0}^{\infty} \sum_{\tau} \sum_{b=0}^{\infty} \sum_{\beta} \sum_{\delta} \frac{(-1)^{k+b} \, (r^{-1}-q^{-1})^{t} \, \tilde{g}^{\delta}_{\kappa,\tau}}{k! \; t! \; b! \; \tilde{C}_{\kappa}(I_n) \; \tilde{C}_{\beta}(I_m)}$$

$$\tilde{C}_{\kappa}(L^{-\frac{1}{2}} \, \Phi^{-1} \, L^{-\frac{1}{2}} - q^{-1} I_n) \, \tilde{C}_{\beta}(Q^{-\frac{1}{2}} \, \Psi^{-1} \, Q^{-\frac{1}{2}} - r^{-1} I_m) \quad I_1^*$$

where

$$(6.3.8) \quad I_1^* = \int_{U(p)} \tilde{C}_{\delta}(\Sigma^{-1} W^{\frac{1}{2}} U G \overline{U}' W^{\frac{1}{2}}) \, \tilde{C}_{\beta}(\Sigma^{-1} W^{\frac{1}{2}} U(I_p - G)\overline{U}' W^{\frac{1}{2}}) \, dU$$

$$= \sum_{\sigma} \tilde{C}_{\sigma}(\Sigma^{-1} W) \, \tilde{P}^{\sigma}_{\delta,\beta} \, (G, \, I_p - G) \; , \quad \text{(from (2.2.64))} .$$

Substitution of (6.3.8) into (6.3.7) and integration w.r.t. $W:p\times p$, using (2.2.32), lead to (6.3.1).

(<u>6.3.2</u>)

The joint p.d.f. of $\underset{\sim}{S}:p\times p$ and $\underset{\sim}{T}:p\times p$ follows as

$$(6.3.9) \quad f_{\underset{\sim}{S},\underset{\sim}{T}}(S,T) = c|S|^{n-p} \, {}_0\tilde{F}_0(-\Phi^{-1} L^{-1}, \Sigma^{-1} S) \, |T|^{m-p}$$

$$\operatorname{etr}[-r^{-1} \Sigma^{-1} T] \, {}_0\tilde{F}_0(Q^{-\frac{1}{2}} \Psi^{-1} Q^{-\frac{1}{2}} - r^{-1} I_m, -\Sigma^{-1} T)$$

with c given in (6.3.4).

In (6.3.9) make the transformations (6.2.13) and (6.2.14) with $\underset{\sim}{A}:p\times p$ replaced by $\underset{\sim}{T}:p\times p$ in (6.2.13), then the joint p.d.f. of $\underset{\sim}{G}:p\times p$ and $\underset{\sim}{W}:p\times p$ follows as

$$(6.3.10) \quad f_{\underset{\sim}{G},\underset{\sim}{W}}(G,W)$$

$$= c|G|^{n-p} \, |I_p - G|^{m-p} \, |W|^{m+n-p} \operatorname{etr}[-r^{-1} \Sigma^{-1} W]$$

$$\operatorname{etr}[r^{-1} \Sigma^{-1} W^{\frac{1}{2}} G W^{\frac{1}{2}}] \, {}_0\tilde{F}_0(-\Sigma^{-1} W^{\frac{1}{2}} G W^{\frac{1}{2}}, \Phi^{-1} L^{-1})$$

$${}_0\tilde{F}_0(Q^{-\frac{1}{2}} \Psi^{-1} Q^{-\frac{1}{2}} - r^{-1} I_m, -\Sigma^{-1} W^{\frac{1}{2}} (I_p - G) W^{\frac{1}{2}}) .$$

After the expansion of the second exponential function and the hypergeometric functions, the symmetrised p.d.f. of $\underset{\sim}{G}:p\times p$ can be derived along the same lines as (6.2.1).

<u>Remark 6.3.1</u>

(i) As in the real case (cf. Underhill, 1973, p. 6.5) it is not possible to derive the p.d.f. of $\underset{\sim}{G}:p\times p$ when the p.d.f. of $\underset{\sim}{T}:p\times p$ has the power-series representation because the exponential function $\operatorname{etr}[-r^{-1} \Sigma^{-1} W]$ is needed to perform the integration over $W:p\times p$.

(ii) It seems not possible to express $\tilde{P}^{\sigma}_{\delta,\beta}(G, I_p - G)$ in terms of the characteristic roots of $\underset{\sim}{G}:p\times p$ and therefore the joint p.d.f. of these roots can not be derived in this case.

(iii) It is clear that (6.3.1) tends to (6.3.2) if $q \to \infty$.

(iv) Let $\Psi: m \times m = Q^{-1}: m \times m$ and $r = 1$; then $\underset{\sim}{T}: p \times p \sim CW(p, m, \Sigma)$. If these substitutions are made in the results derived in theorem 6.3.1 it ought to reduce to certain results derived in theorem 6.2.1:

(a) Let $\Psi: m \times m = Q^{-1}: m \times m$ and $r = 1$ in (6.3.1); then by using (6.2.48) the expression in (6.3.1) leads to (6.2.6).

(b) Let $\Psi: m \times m = Q^{-1}: m \times m$ and $r = 1$ in (6.3.2); then by using (6.2.48) the expression in (6.3.2) leads to (6.2.42).

CHAPTER 7

EXTENSIONS OF THE MULTIVARIATE COMPLEX BETA TYPE 2 DISTRIBUTION

7.1 INTRODUCTION

In this chapter the multivariate complex beta type 2A distribution is extended to the cases where:

(i) The complex Wishart matrix which appears in the numerator of the beta matrix is replaced by a central complex quadratic form of normal variates,

(ii) both complex Wishart matrices which appear in the beta matrix are replaced by central complex quadratic forms of normal variates.

In 7.2 the random hermitian matrix $\underset{\sim}{V}:p\times p = \underset{\sim}{A}^{-\frac{1}{2}}\underset{\sim}{S}\,\underset{\sim}{A}^{-\frac{1}{2}}$, where $\underset{\sim}{S}:p\times p$ is a central complex quadratic form and $\underset{\sim}{A}:p\times p$ is a non-central complex Wishart matrix, is considered. The symmetrised p.d.f. and moments of $\underset{\sim}{V}:p\times p$ and $D_{\underset{\sim}{V}}$ are derived in section 7.2.1 analogous to the real case (cf. Underhill, 1973, p. 3.1 - 3.21) for different specifications of the parameter matrices. In section 7.2.2 an indication will be given how certain marginal distributions of the characteristic roots of $\underset{\sim}{V}:p\times p$ can be derived. In section 7.2.3 the p.d.f.s of $|\underset{\sim}{V}(I_p + q^{-1}\underset{\sim}{V})^{-1}|$ and $\mathrm{tr}\,\underset{\sim}{V}$ are derived.

In 7.3 the random hermitian matrix $\underset{\sim}{V}:p\times p = \underset{\sim}{T}^{-\frac{1}{2}}\underset{\sim}{S}\,\underset{\sim}{T}^{-\frac{1}{2}}$, where both $\underset{\sim}{S}:p\times p$ and $\underset{\sim}{T}:p\times p$ are central complex quadratic forms, is considered. In section 7.3.1 the symmetrised p.d.f. of $\underset{\sim}{V}:p\times p$ and $D_{\underset{\sim}{V}}$ are derived analogous to the real case (cf. Underhill, 1973, p. 5.1 - 5.9).

The obtaining of certain marginal distributions of the characteristic roots of $\underset{\sim}{V}:p\times p$ is discussed in section 7.3.2. The p.d.f. of $\mathrm{tr}\,\underset{\sim}{V}$ for certain specifications of the parameter matrices are derived in section 7.3.3.

7.2 THE QUADRATIC FORM $\underset{\sim}{V}:p\times p = \underset{\sim}{A}^{-\frac{1}{2}}\underset{\sim}{S}\,\underset{\sim}{A}^{-\frac{1}{2}}$ WHEN $\underset{\sim}{S}:p\times p$ IS A CENTRAL COMPLEX QUADRATIC FORM AND $\underset{\sim}{A}:p\times p \sim \mathrm{NCCW}(p,\ m,\ \Psi,\ \Omega)$

7.2.1 The symmetrised p.d.f. and moments of $\underset{\sim}{V}:p\times p$ and $D_{\underset{\sim}{V}}$

In theorem 7.2.1 the symmetrised p.d.f. and moments of $\underset{\sim}{V}:p\times p$ and $D_{\underset{\sim}{V}}$ are derived for different specifications of the parameter matrices. It does not seem possible to derive the actual p.d.f. of $\underset{\sim}{V}:p\times p$ except in the case when $\Omega:p\times p = 0$ and $\Sigma:p\times p = \Psi:p\times p = I_p$.

Theorem 7.2.1

Let the central complex quadratic form $\underset{\sim}{S}:p\times p = \underset{\sim}{Z}\,L\,\overline{\underset{\sim}{Z}}'$ have the p.d.f. given in (4.2.6), (power-series representation) or (4.2.9), (Γ-type representation) and let $\underset{\sim}{A}:p\times p \sim \mathrm{NCCW}(p,\ m,\ \Psi,\ \Omega)$; then the p.d.f., symmetrised p.d.f. and moments of $\underset{\sim}{V}:p\times p = \underset{\sim}{A}^{-\frac{1}{2}}\underset{\sim}{S}\,\underset{\sim}{A}^{-\frac{1}{2}}$ and D_V are given below for certain specifications of $\Omega:p\times p$, $\Psi:p\times p$ and $\Sigma:p\times p$.

(i) $\Sigma:p\times p \neq \Psi:p\times p$

The p.d.f. of $\underset{\sim}{S}:p\times p$ has the Γ-type representation

(7.2.1) $f_{csym}(V)$

$$= \frac{\tilde{\Gamma}_p(m+n)\ \mathrm{etr}[-\Omega]\ |V|^{n-p}}{\tilde{\Gamma}_p(n)\ \tilde{\Gamma}_p(m)\ |L\,\Phi|^p\ |\Sigma\,\Psi^{-1}|^n}$$

$$\sum_{k=0}^{\infty} \sum_{\kappa} \sum_{b=0}^{\infty} \sum_{\beta} \sum_{\delta} \sum_{t=0}^{\infty} \sum_{\tau} \sum_{\sigma} \frac{\tilde{g}_{\kappa,\beta}^{\delta}\ \tilde{g}_{\delta,\tau}^{\sigma}\ q^{-b}\ (-1)^{k+b}\ [m+n]_{\sigma}}{k!\ b!\ t!\ [m]_{\tau}\ \tilde{C}_{\kappa}(I_n)}$$

$$\frac{\tilde{C}_{\kappa}(L^{-\frac{1}{2}}\,\Phi^{-1}\,L^{-\frac{1}{2}} - q^{-1}\,I_n)\ \tilde{C}_{\tau}(\Omega\,\Psi^{-1})}{\tilde{C}_{\delta}(I_p)\ \tilde{C}_{\tau}(I_p)\ \tilde{C}_{\delta}(I_p)}\ \tilde{C}_{\delta}(\Sigma^{-1})\ \tilde{C}_{\sigma}(\Psi)\ \tilde{C}_{\delta}(V)\ ,$$

$$V = \bar{V}' > 0 \quad \text{and} \quad q > 0\ .$$

(7.2.2) $$f_{\underset{\sim}{D}_V}(D_V) = \frac{\pi^{p(p-1)}}{\tilde{\Gamma}_p(p)} \prod_{i>j}^{p} (\tilde{v}_i - \tilde{v}_j)^2\ f_{csym}(D_V)\ ,$$

$$0 < \tilde{v}_1 < \cdots < \tilde{v}_p \quad \text{and} \quad q > 0\ .$$

The p.d.f. of $\underset{\sim}{S}{:}p{\times}p$ has the power-series representation

(7.2.3) $$f_{csym}(V)$$

$$= \frac{\tilde{\Gamma}_p(m+n)\ \mathrm{etr}[-\Omega]\ |V|^{n-p}}{\tilde{\Gamma}_p(n)\ \tilde{\Gamma}_p(m)\ |L\,\Phi|^{p}\ |\Sigma\,\Psi^{-1}|^{n}} \sum_{k=0}^{\infty} \sum_{\kappa} \sum_{t=0}^{\infty} \sum_{\tau} \sum_{\delta} \frac{\tilde{g}_{\kappa,\tau}^{\delta}}{k!\ t!}$$

$$\frac{(-1)^{k}\ [m+n]_{\delta}\ \tilde{C}_{\kappa}(\Phi^{-1}\,L^{-1})\ \tilde{C}_{\kappa}(\Sigma^{-1})}{\tilde{C}_{\kappa}(I_p)\ \tilde{C}_{\kappa}(I_n)\ \tilde{C}_{\kappa}(I_p)\ \tilde{C}_{\tau}(I_n)}\ \tilde{C}_{\tau}(\Omega\,\Psi^{-1})\ \tilde{C}_{\delta}(\Psi)\ \tilde{C}_{\kappa}(V)\ ,$$

$$V = \bar{V}' > 0\ .$$

(7.2.4) $$f_{\underset{\sim}{D}_V}(D_V) = \frac{\pi^{p(p-1)}}{\tilde{\Gamma}_p(p)} \prod_{i>j}^{p} (\tilde{v}_i - \tilde{v}_j)^2\ f_{csym}(D_V)\ ,$$

$$0 < \tilde{v}_1 < \cdots < \tilde{v}_p\ .$$

(ii) $\underline{\Sigma{:}p\times p = \Psi{:}p\times p \quad \text{i.e.} \quad \underset{\sim}{A}{:}p\times p \sim NCCW(p,\ m,\ \Sigma,\ \Omega)}$

The p.d.f. of $\underset{\sim}{S}{:}p\times p$ has the Γ-type representation

$$(7.2.5) \quad f_{csym}(V) = \frac{\tilde{\Gamma}_p(m+n)\ \mathrm{etr}[-\Omega]\ |V|^{n-p}}{\tilde{\Gamma}_p(n)\ \tilde{\Gamma}_p(m)\ |L\,\Phi|^p} \sum_{k=0}^{\infty} \sum_{\kappa} \sum_{t=0}^{\infty} \sum_{\tau} \sum_{\delta} \frac{(-1)^k\ \tilde{g}^{\delta}_{\kappa,\tau}}{k!\ t!\ [m]_\tau}$$

$$\frac{[m+n]_\delta\ \tilde{C}_\kappa(L^{-1}\,\Phi^{-1})\ \tilde{C}_\tau(\Omega)}{\tilde{C}_\kappa(I_p)\ \tilde{C}_\kappa(I_n)\ \tilde{C}_\tau(I_p)}\ \tilde{C}_\delta(I_p)\ \tilde{C}_\kappa(V)\ ,$$

$$V = \bar{V}' > 0\ .$$

$$(7.2.6) \quad f_{\underset{\sim}{D}_V}(D_V) = \frac{\pi^{p(p-1)}}{\tilde{\Gamma}_p(p)} \prod_{i>j}^{p} (\tilde{v}_i - \tilde{v}_j)^2\ f_{csym}(D_V)\ ,$$

$$0 < \tilde{v}_1 < \cdots < \tilde{v}_p\ .$$

The p.d.f. of $\underset{\sim}{S}{:}p\times p$ has the power-series representation

$$(7.2.7) \quad f_{csym}(V) = (7.2.5)\ .$$

(iii) $\underline{\Omega{:}p\times p = 0\ , \quad \Sigma{:}p\times p \neq \Psi{:}p\times p \quad \text{i.e.} \quad \underset{\sim}{A}{:}p\times p \sim CW(p,\ m,\ \Psi)}$

The p.d.f. of $\underset{\sim}{S}{:}p\times p$ has the Γ-type representation

$$(7.2.8) \quad f_{csym}(V)$$

$$= \frac{\tilde{\Gamma}_p(m+n)\ |V|^{n-p}}{\tilde{\Gamma}_p(m)\ \tilde{\Gamma}_p(n)\ |L\,\Phi|^p\ |\Sigma\,\Psi^{-1}|^n} \sum_{k=0}^{\infty} \sum_{\kappa} \sum_{t=0}^{\infty} \sum_{\tau} \sum_{\delta} \frac{\tilde{g}^{\delta}_{\kappa,\tau}\ (-1)^{k+t}}{t!\ k!}$$

$$\frac{[m+n]_\delta\ (q^{-1})^t\ \tilde{C}_\kappa(L^{-\frac{1}{2}}\,\Phi^{-1}\,L^{-\frac{1}{2}} - q^{-1}\,I_n)}{\tilde{C}_\kappa(I_n)\ \tilde{C}_\delta(I_p)}\ \tilde{C}_\delta(\Sigma^{-1}\,\Psi)\ \tilde{C}_\delta(V)\ ,$$

$$V = \bar{V}' > 0 \quad \text{and} \quad q > 0\ .$$

$$\text{(7.2.9)} \quad f_{\underset{\sim}{D}_V}(D_V) = \frac{\pi^{p(p-1)}}{\tilde{\Gamma}_p(p)} \prod_{i>j}^{p} (\tilde{v}_i - \tilde{v}_j)^2 \quad f_{csym}(D_V) \ ,$$

$$0 < \tilde{v}_1 < \cdots < \tilde{v}_p \quad \text{and} \quad q > 0 \ .$$

The p.d.f. of $\underset{\sim}{S}:p\times p$ has the power-series representation

$$\text{(7.2.10)} \quad f_{csym}(V)$$

$$= \frac{\tilde{\Gamma}_p(m+n) \ |V|^{n-p}}{\tilde{\Gamma}_p(n) \ \tilde{\Gamma}_p(m) \ |\Sigma\Psi^{-1}|^n \ |L\Phi|^p} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(-1)^k \ [m+n]_\kappa}{k! \ \tilde{C}_\kappa(I_n)}$$

$$\frac{\tilde{C}_\kappa(\Phi^{-1} L^{-1})}{\tilde{C}_\kappa(I_p)} \ \tilde{C}_\kappa(\Sigma^{-1}\Psi) \ \tilde{C}_\kappa(V) \ , \quad V = \overline{V}' > 0 \ .$$

$$\text{(7.2.11)} \quad f_{\underset{\sim}{D}_V}(D_V) = \frac{\pi^{p(p-1)}}{\tilde{\Gamma}_p(p)} \prod_{i>j}^{p} (\tilde{v}_i - \tilde{v}_j)^2 \quad f_{csym}(D_V) \ ,$$

$$0 < \tilde{v}_1 < \cdots < \tilde{v}_p \ .$$

(iv) $\Omega:p\times p = 0$, $\Sigma:p\times p = \Psi:p\times p$ i.e. $\underset{\sim}{A}:p\times p \sim CW(p, m, \Sigma)$

The p.d.f. of $\underset{\sim}{S}:p\times p$ has the Γ-type representation

$$\text{(7.2.12)} \quad f_{csym}(V) = \frac{\tilde{\Gamma}_p(m+n) \ |V|^{n-p} \ |I_p + q^{-1}V|^{-(m+n)}}{\tilde{\Gamma}_p(m) \ \tilde{\Gamma}_p(n) \ |L\Phi|^p}$$

$${}_1\tilde{F}_0(m+n \ ; \ q^{-1} V(I_p + q^{-1}V)^{-1}, \ I_n - q \ L^{-\frac{1}{2}} \Phi^{-1} L^{-\frac{1}{2}}) \ ,$$

$$V = \overline{V}' > 0 \quad \text{and} \quad q > 0 \ .$$

(7.2.13) $$f_{\underset{\sim}{D}_V}(D_V) = \frac{\pi^{p(p-1)}}{\tilde{\Gamma}_p(p)} \prod_{i>j}^{p} (\tilde{v}_i - \tilde{v}_j)^2 \; f_{csym}(D_V) \,,$$

$$0 < \tilde{v}_1 < \cdots < \tilde{v}_p \quad \text{and} \quad q > 0 \,.$$

(7.2.14) $$E(|\underset{\sim}{V}|^h) = \frac{q^{p(n+h+p)} \; \tilde{\Gamma}_p(n+h) \; \tilde{\Gamma}_p(m-h)}{\tilde{\Gamma}_p(n) \; \tilde{\Gamma}_p(m) \; |L\,\Phi|^p}$$

$$ {}_1\tilde{F}_0(n+h \,;\, I_p,\; I_n - q\, L^{-\frac{1}{2}} \Phi^{-1} L^{-\frac{1}{2}}) \,, \quad q > 0 \quad \text{and}$$

$$\| I_n - q\, L^{-\frac{1}{2}} \Phi^{-1} L^{-\frac{1}{2}} \| < 1 \,.$$

(7.2.15) $$E(|I_p + q^{-1} \underset{\sim}{V}|^{-h})$$

$$= \frac{\tilde{\Gamma}_p(m+n) \; \tilde{\Gamma}_p(m+h) \; q^{p(p+n)}}{\tilde{\Gamma}_p(m) \; \tilde{\Gamma}_p(m+n+h) \; |L\,\Phi|^p}$$

$$ {}_2\tilde{F}_1(m+n,\; n \,;\, m+n+h \,;\, I_p,\; I_n - q\, L^{-\frac{1}{2}} \Phi^{-1} L^{-\frac{1}{2}}) \,, \quad q > 0 \quad \text{and}$$

$$\| I_n - q\, L^{-\frac{1}{2}} \Phi^{-1} L^{-\frac{1}{2}} \| < 1 \,.$$

(7.2.16) $$E(|\underset{\sim}{V}(I_p + q^{-1} \underset{\sim}{V})^{-1}|^h)$$

$$= \frac{\tilde{\Gamma}_p(m+n) \; \tilde{\Gamma}_p(n+h) \; q^{p(p+n+h)}}{\tilde{\Gamma}_p(n) \; \tilde{\Gamma}_p(m+n+h) \; |L\,\Phi|^p}$$

$$ {}_2\tilde{F}_1(m+n,\; n+h \,;\, m+n+h \,;\, I_p,\; I_n - q\, L^{-\frac{1}{2}} \Phi^{-1} L^{-\frac{1}{2}}) \,,$$

$$q > 0 \quad \text{and} \quad \| I_n - q\, L^{-\frac{1}{2}} \Phi^{-1} L^{-\frac{1}{2}} \| < 1 \,.$$

<u>The p.d.f. of $\underset{\sim}{S}:p\times p$ has the power-series representation</u>

(7.2.17) $$f_{csym}(V) = \frac{\tilde{\Gamma}_p(m+n)\ |V|^{n-p}}{\tilde{\Gamma}_p(n)\ \tilde{\Gamma}_p(m)\ |L\,\Phi|^p}\ {}_1\tilde{F}_0(m+n\,;\,V,\,-\Phi^{-1}L^{-1})\ ,$$

$$V=\bar{V}'>0 \quad \text{and} \quad \|\Phi^{-1}L^{-1}\|<1\ .$$

(7.2.18) $$f_{\underset{\sim}{D}_V}(D_V) = \frac{\pi^{p(p-1)}}{\tilde{\Gamma}_p(p)} \prod_{i>j}^{p} (\tilde{v}_i - \tilde{v}_j)^2\ f_{csym}(D_V)\ ,$$

$$0<\tilde{v}_1<\cdots<\tilde{v}_p \quad \text{and} \quad \|\Phi^{-1}L^{-1}\|<1\ .$$

(v) <u>$\Omega:p\times p = 0$, $\Sigma:p\times p = \Psi:p\times p = I_p$ i.e. $\underset{\sim}{A}:p\times p \sim CW(p,\,m,\,I_p)$</u>

<u>The p.d.f. of $\underset{\sim}{S}:p\times p$ has the Γ-type representation</u>

(7.2.19) $f_{\underset{\sim}{V}}(V) = (7.2.12)$.

<u>The p.d.f. of $\underset{\sim}{S}:p\times p$ has the power-series representation</u>

(7.2.20) $f_{\underset{\sim}{V}}(V) = (7.2.17)$.

<u>Proof</u>

(<u>7.2.1</u>)

The joint p.d.f. of $\underset{\sim}{S}:p\times p$ and $\underset{\sim}{A}:p\times p$ follows as

(7.2.21) $$f_{\underset{\sim}{S},\underset{\sim}{A}}(S,A)$$

$$= c\ |S|^{n-p}\ \mathrm{etr}[-q^{-1}\Sigma^{-1}S]\ {}_0\tilde{F}_0(L^{-\frac{1}{2}}\Phi^{-1}L^{-\frac{1}{2}} - q^{-1}I_n,\,-\Sigma^{-1}S)$$

$${}_0\tilde{F}_1(m\,;\,\Omega\,\Psi^{-1}A)\ \mathrm{etr}[-\Psi^{-1}A]\ |A|^{m-p}$$

with

$$(7.2.22) \quad c = \frac{\mathrm{etr}[-\Omega]}{\tilde{\Gamma}_p(n)\ \tilde{\Gamma}_p(m)\ |L\,\Phi|^p\ |\Sigma|^n\ |\Psi|^m} .$$

In (7.2.21) make the transformation

$$(7.2.23) \quad V = A^{-\frac{1}{2}}\, S\, A^{-\frac{1}{2}}$$

with inverse transformation

$$(7.2.24) \quad S = A^{\frac{1}{2}}\, V\, A^{\frac{1}{2}} .$$

From theorem 2.2.2 (iii), the jacobian of (7.2.24) follows as

$$(7.2.25) \quad J(S \to V) = |A|^p .$$

The joint p.d.f. of $\underset{\sim}{V}: p \times p$ and $\underset{\sim}{A}: p \times p$ follows as

$$(7.2.26) \quad f_{\underset{\sim}{V},\underset{\sim}{A}}(V,A)$$

$$= c\ |V|^{n-p}\ |A|^{m+n-p}\ \mathrm{etr}[-q^{-1}\,\Sigma^{-1}\,A^{\frac{1}{2}}\,V\,A^{\frac{1}{2}}]\ \mathrm{etr}[-\Psi^{-1}\,A]$$

$${}_0\tilde{F}_0(L^{-\frac{1}{2}}\,\Phi^{-1}\,L^{-\frac{1}{2}} - q^{-1}\,I_n,\ -\Sigma^{-1}\,A^{\frac{1}{2}}\,V\,A^{\frac{1}{2}})\ {}_0\tilde{F}_1(m\,;\,\Omega\,\Psi^{-1}\,A) .$$

The expansion of the hypergeometric functions and the first exponential function and the application of (2.2.54) lead to

$$(7.2.27) \quad f_{\underset{\sim}{V},\underset{\sim}{A}}(V,A)$$

$$= c\ |V|^{n-p}\ |A|^{m+n-p}\ \mathrm{etr}[-\Psi^{-1}\,A]$$

$$\sum_{k=0}^{\infty} \sum_{\kappa} \sum_{b=0}^{\infty} \sum_{\beta} \sum_{\delta} \sum_{t=0}^{\infty} \sum_{\tau} \frac{\tilde{g}^{\delta}_{\kappa,\beta}\ q^{-b}\ \tilde{C}_{\kappa}(L^{-\frac{1}{2}}\,\Phi^{-1}\,L^{-\frac{1}{2}} - q^{-1}\,I_n)}{k!\ b!\ t!\ [m]_{\tau}\ \tilde{C}_{\kappa}(I_n)}$$

$$\tilde{C}_{\delta}(-\Sigma^{-1}\,A^{\frac{1}{2}}\,V\,A^{\frac{1}{2}})\ \tilde{C}_{\tau}(\Omega\,\Psi^{-1}\,A) .$$

It is not possible to find the marginal p.d.f. of $\underset{\sim}{V}:p\times p$ because the integral

$$\int_{A=\bar{A}'>0} f_{\underset{\sim}{V},\underset{\sim}{A}}(V,A)\, dA$$

is not solvable, therefore the symmetrised p.d.f. of $\underset{\sim}{V}:p\times p$ is obtained, i.e.

$$(7.2.28) \qquad f_{csym}(V) = \int_{U(p)} \int_{A=\bar{A}'>0} f_{\underset{\sim}{V},\underset{\sim}{A}}(U V \bar{U}', A)\, dA\, dU .$$

Change the order of integration, then integration over the unitary group, using (2.2.29), leads to

$$(7.2.29) \qquad f_{csym}(V)$$

$$= c\, |V|^{n-p} \sum_{k=0}^{\infty} \sum_{\kappa} \sum_{b=0}^{\infty} \sum_{\beta} \sum_{\delta} \sum_{t=0}^{\infty} \sum_{\tau} \frac{\tilde{g}^{\delta}_{\kappa,\beta}\; (-1)^{k+b}\; q^{-b}}{k!\; b!\; t!\; [m]_{\tau}}$$

$$\frac{\tilde{C}_{\kappa}(L^{-\frac{1}{2}}\, \Phi^{-1}\, L^{-\frac{1}{2}} - q^{-1}\, I_n)}{\tilde{C}_{\delta}(I_p)\; \tilde{C}_{\kappa}(I_n)}\; \tilde{C}_{\delta}(V) \quad I^*$$

where

$$(7.2.30) \qquad I^* = \int_{A=\bar{A}'>0} |A|^{m+n-p}\, \mathrm{etr}[-\Psi^{-1} A]\; \tilde{C}_{\delta}(\Sigma^{-1} A)\; \tilde{C}_{\tau}(\Omega\, \Psi^{-1} A)\, dA .$$

The matrices Σ^{-1} and $\Omega\, \Psi^{-1} = \Psi^{-1}\, M \bar{M}'\, \Psi^{-1}$ are hermitian matrices. Thus, by transforming $\Sigma^{-1} \to U\, \Sigma^{-1}\, \bar{U}'$ and $\Omega\, \Psi^{-1} \to U\, \Omega\, \Psi^{-1}\, \bar{U}'$ and after integrating over the unitary group after each transformation, the integral in (7.2.30) can be written as

$$(7.2.31)\quad I^* = \frac{\tilde{C}_\delta(\Sigma^{-1})\ \tilde{C}_\tau(\Psi^{-1}\Omega)}{\tilde{C}_\delta(I_p)\ \tilde{C}_\tau(I_p)} \int_{A=\bar{A}'>0} \sum_\sigma g^\sigma_{\delta,\tau}\ |A|^{m+n-p}\ \mathrm{etr}[-\Psi^{-1}A]\ \tilde{C}_\sigma(A)\ dA$$

$$= \tilde{\Gamma}_p(m+n)\ |\Psi|^{m+n} \sum_\sigma \frac{g^\sigma_{\delta,\tau}\ \tilde{C}_\delta(\Sigma^{-1})\ \tilde{C}_\tau(\Psi^{-1}A)\ [m+n]_\sigma\ \tilde{C}_\sigma(\Psi)}{\tilde{C}_\delta(I_p)\ \tilde{C}_\tau(I_p)},$$

(from (2.2.32)).

Substitution of (7.2.31) into (7.2.29) leads to (7.2.1).

(7.2.2)

The application of theorem 3.2.1 and corollary 2.7.1 leads to (7.2.2).

(7.2.3)

The joint p.d.f. of $\underset{\sim}{S}:p\times p$ and $\underset{\sim}{A}:p\times p$ follows as

$$(7.2.32)\quad f_{\underset{\sim}{S},\underset{\sim}{A}}(S,A) = c\ |S|^{n-p}\ {}_0\tilde{F}_0(-\Sigma^{-1}S,\ \Phi^{-1}L^{-1})\ {}_0\tilde{F}_1(m;\ \Omega\Psi^{-1}A)\ \mathrm{etr}[-\Psi^{-1}A]\ |A|^{m-p}$$

where c is given in (7.2.22).

In (7.2.32) make the transformation (7.2.23), then the joint p.d.f. of $\underset{\sim}{V}:p\times p$ and $\underset{\sim}{A}:p\times p$ follows as

$$(7.2.33)\quad f_{\underset{\sim}{V},\underset{\sim}{A}}(V,A) = c\ |V|^{n-p}\ |A|^{m+n-p}\ {}_0\tilde{F}_0(-\Sigma^{-1}A^{\frac{1}{2}}VA^{\frac{1}{2}},\ \Phi^{-1}L^{-1})\ {}_0\tilde{F}_1(m;\ \Omega\Psi^{-1}A)\ \mathrm{etr}[-\Psi^{-1}A]\ .$$

It does not seem possible to integrate (7.2.33) with respect to $A:p\times p$, therefore the symmetrised p.d.f. of $\underset{\sim}{V}:p\times p$ is obtained. The expansion of the two hypergeometric functions and integration over the unitary group after changing the order of integration lead to

$$(7.2.34) \qquad f_{csym}(V)$$

$$= c|V|^{n-p} \sum_{k=0}^{\infty} \sum_{\kappa} \sum_{t=0}^{\infty} \sum_{\tau} \frac{\tilde{C}_\kappa(\Phi^{-1} L^{-1})\, \tilde{C}_\kappa(V)\, (-1)^k\, I^*}{k!\; t!\; [m]_\tau\, \tilde{C}_\kappa(I_n)\, \tilde{C}_\kappa(I_p)}$$

where

$$(7.2.35) \qquad I^* = \int_{A=\bar{A}'>0} |A|^{m+n-p} \operatorname{etr}[-\Psi^{-1} A]\, \tilde{C}_\tau(\Omega\,\Psi^{-1} A)\, \tilde{C}_\kappa(\Sigma^{-1} A)\, dA$$

$$= \tilde{\Gamma}_p(m+n)\; |\Psi|^{m+n} \sum_{\delta} \frac{\tilde{g}^{\delta}_{\kappa,\tau}\, \tilde{C}_\kappa(\Sigma^{-1})\, \tilde{C}_\tau(\Psi^{-1}\Omega)\, [m+n]_\delta\, \tilde{C}_\delta(\Psi)}{\tilde{C}_\kappa(I_p)\, \tilde{C}_\tau(I_p)},$$

(from (7.2.31)).

Substitution of (7.2.35) into (7.2.34) leads to (7.2.3).

(7.2.4)

The application of theorem 3.2.1 and corollary 2.7.1 leads to (7.2.4).

(7.2.5)

Let $\Sigma:p\times p = \Psi:p\times p$ in (7.2.29); then

$$(7.2.36) \qquad f_{csym}(V)$$

$$= c\; |V|^{n-p} \sum_{k=0}^{\infty} \sum_{\kappa} \sum_{b=0}^{\infty} \sum_{\beta} \sum_{\delta} \sum_{t=0}^{\infty} \sum_{\tau} \frac{\tilde{g}^{\delta}_{\kappa,\beta}\, (-1)^{k+b}\, q^{-b}}{k!\; b!\; t!\; [m]_\tau}$$

$$\frac{\tilde{C}_\kappa(L^{-\frac{1}{2}}\,\Phi^{-1}\,L^{-\frac{1}{2}} - q^{-1}\,I_n)\;\tilde{C}_\delta(V)\quad I^*}{\tilde{C}_\delta(I_p)\;\tilde{C}_\kappa(I_n)}\;,$$

where

$$(7.2.37)\qquad c = \frac{\operatorname{etr}[-\Omega]}{\tilde{\Gamma}_p(n)\;\tilde{\Gamma}_p(m)\;|L\,\Phi|^p\;|\Sigma|^{m+n}}$$

and

$$(7.2.38)\qquad I^* = \int_{A=\bar{A}'>0} |A|^{m+n-p}\;\operatorname{etr}[-\Sigma^{-1}A]\;\tilde{C}_\delta(\Sigma^{-1}A)\;\tilde{C}_\tau(\Omega\,\Sigma^{-1}A)\;dA$$

$$= \int_{A=\bar{A}'>0} |A|^{m+n-p}\;\operatorname{etr}[-\Sigma^{-1}A]\;\tilde{C}_\delta(\Sigma^{-1}A)$$

$$\tilde{C}_\tau(\Sigma^{-\frac{1}{2}}\,\Sigma^{-\frac{1}{2}}\,M\,\bar{M}'\,\Sigma^{-\frac{1}{2}}\,\Sigma^{-\frac{1}{2}}\,A)\;dA\;.$$

The matrix $\Sigma^{-\frac{1}{2}}\,M\,\bar{M}'\,\Sigma^{-\frac{1}{2}}$ is a hermitian matrix. Thus, by transforming $\Sigma^{-\frac{1}{2}}\,M\,\bar{M}'\,\Sigma^{-\frac{1}{2}} \to U\,\Sigma^{-\frac{1}{2}}\,M\,\bar{M}'\,\Sigma^{-\frac{1}{2}}\,\bar{U}'$, integrating over the unitary group and using (2.2.54), the integral in (7.2.38) can be written as

$$(7.2.39)\qquad I^* = \frac{\tilde{C}_\tau(\Omega)}{\tilde{C}_\tau(I_p)}\sum_\sigma \tilde{g}^{\sigma}_{\delta,\tau}\int_{A=\bar{A}'>0} |A|^{m+n-p}\;\operatorname{etr}[-\Sigma^{-1}A]\;\tilde{C}_\sigma(\Sigma^{-1}A)\;dA$$

$$= \frac{|\Sigma|^{m+n}\;\tilde{\Gamma}_p(m+n)\;\tilde{C}_\tau(\Omega)}{\tilde{C}_\tau(I_p)}\sum_\sigma \tilde{g}^{\sigma}_{\delta,\tau}\;[m+n]_\sigma\;\tilde{C}_\sigma(I_p)\;,$$

(from (2.2.32)).

Substitution of (7.2.39) into (7.2.36) leads to

(7.2.40) $\quad f_{csym}(V)$

$$= \frac{\tilde{\Gamma}_p(m+n)\ \mathrm{etr}[-\Omega]}{\tilde{\Gamma}_p(m)\ \tilde{\Gamma}_p(n)\ |L\,\Phi|^p} \sum_{k=0}^{\infty} \sum_{\kappa} \sum_{b=0}^{\infty} \sum_{\beta} \sum_{\delta} \sum_{t=0}^{\infty} \sum_{\tau} \sum_{\sigma} \frac{\tilde{g}^{\delta}_{\kappa,\beta}\ \tilde{g}^{\sigma}_{\delta,\tau}}{k!\ b!\ t!\ [m]_\tau}$$

$$\frac{(-1)^{k+b}\ q^{-b}\ [m+n]_\sigma\ \tilde{C}_\kappa(L^{-\frac{1}{2}}\,\Phi^{-1}\,L^{-\frac{1}{2}} - q^{-1}\,I_n)\ \tilde{C}_\delta(V)}{\tilde{C}_\tau(I_p)\ \tilde{C}_\kappa(I_n)\ \tilde{C}_\delta(I_p)}$$

$$\tilde{C}_\tau(\Omega)\ \tilde{C}_\delta(I_p)\ .$$

The application of (2.2.71) with $T = q\,L^{-\frac{1}{2}}\,\Phi^{-1}\,L^{-\frac{1}{2}} - I_n$ and $a = m+n$ leads to (7.2.5).

(7.2.6)

The application of theorem 3.2.1 and corollary 2.7.1 leads to (7.2.6).

(7.2.7)

Let $\Sigma{:}p\times p = \Psi{:}p\times p$ in (7.2.34); then

(7.2.41) $\quad f_{csym}(V)$

$$= c\ |V|^{n-p} \sum_{k=0}^{\infty} \sum_{\kappa} \sum_{t=0}^{\infty} \sum_{\tau} \frac{\tilde{C}_\kappa(\Phi^{-1}\,L^{-1})\ \tilde{C}_\kappa(V)\ \ (-1)^k\ \ I^*}{k!\ t!\ [m]_\tau\ \tilde{C}_\kappa(I_n)\ \tilde{C}_\kappa(I_p)}$$

where c is given in (7.2.37) and

(7.2.42) $$I^* = \int_{A=\bar{A}'>0} |A|^{m+n-p}\ \mathrm{etr}[-\Sigma^{-1}A]\ \tilde{C}_\tau(\dot{\Omega}\,\Sigma^{-1}A)\ \tilde{C}_\kappa(\Sigma^{-1}A)\,dA$$

$$= \frac{|\Sigma|^{m+n}\ \tilde{\Gamma}_p(m+n)\ \tilde{C}_\tau(\Omega)}{C_\tau(I_p)} \sum_{\delta} \tilde{g}^{\delta}_{\kappa,\tau}\ [m+n]_\delta\ \tilde{C}_\delta(I_p)\ ,$$

(from (7.2.38) and (7.2.39)).

Substitution of (7.2.42) into (7.2.41) leads to (7.2.7).

(7.2.8)

Let $\Omega{:}p\times p = 0$ in (7.2.29); then

$$(7.2.43)\quad f_{csym}(V) = c\ |V|^{n-p} \sum_{k=0}^{\infty} \sum_{\kappa} \sum_{b=0}^{\infty} \sum_{\beta} \sum_{\delta} \frac{\tilde{g}^{\delta}_{\kappa,\beta}\ (-1)^{k+b}\ q^{-b}}{k!\ b!}$$

$$\frac{\tilde{C}_{\kappa}(L^{-\frac{1}{2}}\ \Phi^{-1}\ L^{-\frac{1}{2}} - q^{-1}\ I_n)\ \tilde{C}_{\delta}(V)\quad I^*}{\tilde{C}_{\delta}(I_p)\ \tilde{C}_{\kappa}(I_n)}$$

where

$$(7.2.44)\quad c = (\tilde{\Gamma}_p(n)\ \tilde{\Gamma}_p(m)\ \ |L\,\Phi|^p\ \ |\Sigma|^n\ \ |\Psi|^m)^{-1}$$

and

$$(7.2.45)\quad I^* = \int_{A=\bar{A}'>0} |A|^{m+n-p}\ \mathrm{etr}[-\Psi^{-1}\ A]\ \tilde{C}_{\delta}(\Sigma^{-1}\ A)\ dA$$

$$= \tilde{\Gamma}_p(m+n)\ \ [m+n]_{\delta}\ \ |\Psi|^{m+n}\ \tilde{C}_{\delta}(\Sigma^{-1}\ \Psi)\ ,\qquad \text{(from (2.2.32))}.$$

Substitution of (7.2.45) into (7.2.44) leads to (7.2.8).

(7.2.9)

The application of theorem 3.2.1 and corollary 2.7.1 leads to (7.2.9).

(7.2.10)

Let $\Omega{:}p\times p = 0$ in (7.2.34); then

$$(7.2.46)\quad f_{csym}(V) = c\ |V|^{n-p} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{\tilde{C}_\kappa(\Phi^{-1} L^{-1})\ \tilde{C}_\kappa(V)\quad I^*}{k!\ \tilde{C}_\kappa(I_n)\ \tilde{C}_\kappa(I_p)}$$

where c is given in (7.2.44) and

$$(7.2.47)\quad I^* = \int_{A=\bar{A}'>0} |A|^{m+n-p}\ \mathrm{etr}[\Psi^{-1} A]\ \tilde{C}_\kappa(\Sigma^{-1} A)\ dA$$

$$= \tilde{\Gamma}_p(m+n)\ [m+n]_\kappa\ |\Psi|^{m+n}\ \tilde{C}_\delta(\Sigma^{-1} A)\ , \qquad \text{(from (2.2.32))}.$$

Substitution of (7.2.47) into (7.2.46) leads to (7.2.10).

<u>(7.2.11)</u>

The application of theorem 3.2.1 and corollary 2.7.1 leads to (7.2.11).

<u>(7.2.12)</u>

Let $\Sigma{:}p\times p = \Psi{:}p\times p$ in (7.2.43); then

$$(7.2.48)\quad f_{csym}(V) = c\ |V|^{n-p} \sum_{k=0}^{\infty} \sum_{\kappa} \sum_{b=0}^{\infty} \sum_{\beta} \sum_{\delta} \frac{\tilde{g}^{\delta}_{\kappa,\beta}\ (-1)^{k+b}\ q^{-b}}{k!\ b!\ \tilde{C}_\delta(I_p)\ \tilde{C}_\kappa(I_n)}$$

$$\tilde{C}_\kappa(L^{-\frac{1}{2}} \Phi^{-1} L^{-\frac{1}{2}} - q^{-1} I_n)\ \tilde{C}_\delta(V)\quad I^*$$

where

$$(7.2.49)\quad c = (\tilde{\Gamma}_p(n)\ \tilde{\Gamma}_p(m)\ |L\,\Phi|^p\ |\Sigma|^{m+n})^{-1}$$

and

$$(7.2.50)\quad I^* = \int_{A=\bar{A}'>0} |A|^{m+n-p}\ \mathrm{etr}[-\Sigma^{-1} A]\ \tilde{C}_\delta(\Sigma^{-1} A)\ dA$$

$$= \tilde{\Gamma}_p(m+n)\ [m+n]_\delta\ |\Sigma|^{m+n}\ \tilde{C}_\delta(I_p)\ , \qquad \text{(from (2.2.32))}.$$

Substitution of (7.2.50) into (7.2.48) leads to

(7.2.51) $f_{csym}(V)$

$$= \frac{\tilde{\Gamma}_p(m+n)\ |V|^{n-p}}{\tilde{\Gamma}_p(n)\ \tilde{\Gamma}_p(m)\ |L\,\Phi|^p} \sum_{k=0}^{\infty} \sum_{\kappa} \sum_{b=0}^{\infty} \sum_{\beta} \sum_{\delta} \frac{\tilde{g}^{\delta}_{\kappa,\beta}\ (-1)^{k+b}\ q^{-b}}{k!\ b!\ \tilde{C}_\kappa(I_n)}$$

$$[m+n]_\delta\ \tilde{C}_\kappa(L^{-\frac{1}{2}}\,\Phi^{-1}\,L^{-\frac{1}{2}} - q^{-1}\,I_n)\ \tilde{C}_\delta(V)\ .$$

The application of (2.2.70) with $A = q^{-1}V$ and $a = m+n$ leads to (7.2.12).

(7.2.13)

The application of theorem 3.2.1 and corollary 2.7.1 leads to (7.2.13).

(7.2.14), (7.2.15), (7.2.16)

Only (7.2.14) will be proved here, the proofs of (7.2.15) and (7.2.16) being similar.

(7.2.52) $E(|\underset{\sim}{V}|^h)$

$$= \frac{\tilde{\Gamma}_p(m+n)}{\tilde{\Gamma}_p(m)\ \tilde{\Gamma}_p(n)\ |L\,\Phi|^p} \int_{V=\bar{V}'>0} |V|^{n+h-p}\ |I_p + q^{-1}V|^{-(m+n)}$$

$$_0\tilde{F}_1(m+n\ ;\ q^{-1}V(I_p + q^{-1}V)^{-1},\ I_n - q^{-1}L^{-\frac{1}{2}}\,\Phi^{-1}\,L^{-\frac{1}{2}})\ dV\ .$$

In the integral above make the transformation

(7.2.53) $W = q^{-1}V$

with inverse transformation

(7.2.54) $V = qW\ .$

The jacobian of (7.2.54) follows from theorem 2.2.2 (i) as

$$(7.2.55)\quad J(V \to W) = q^{2p^2} .$$

Hence,

$$(7.2.56)\qquad E(|\underset{\sim}{V}|^h)$$

$$= \frac{\tilde{\Gamma}_p(m+n)\ q^{(p+n+h)p}}{\tilde{\Gamma}_p(m)\ \tilde{\Gamma}_p(n)\ |L\,\Phi|^p} \int\limits_{W=\bar{W}'>0} |W|^{n+h-p}\ |I_p + W|^{-(m+n)}$$

$${}_0\tilde{F}_1(m+n\ ;\ W(I_p + W)^{-1}\ ,\ I_n - q^{-1} L^{-\frac{1}{2}} \Phi^{-1} L^{-\frac{1}{2}})\ dW\ .$$

The application of (2.3.9) leads to (7.2.14).

(<u>7.2.17</u>)

Let $\Sigma{:}p\times p = \Psi{:}p\times p$ in (7.2.10); then (7.2.17) follows. It is important to note that from definition 2.3.2 it follows that ${}_1\tilde{F}_0(m+n\ ;\ V,\ -\Phi^{-1} L^{-1})$ is convergent only for $\|V\| < 1$ or $\|\Phi^{-1} L^{-1}\| < 1$.

(<u>7.2.18</u>)

The application of theorem 3.2.1 and corollary 2.7.1 leads to (7.2.18).

(<u>7.2.19</u>)

Let $\Omega{:}p\times p = 0$ and $\Sigma{:}p\times p = \Psi{:}p\times p = I_p$ in (7.2.26); then follows

$$(7.2.57)\qquad f_{\underset{\sim}{V}}(V)$$

$$= \frac{|V|^{n-p}}{\tilde{\Gamma}_p(n)\ \tilde{\Gamma}_p(m)\ |L\,\Phi|^p} \int\limits_{A=\bar{A}'>0} |A|^{m+n-p}\ \mathrm{etr}[-A(I_p + q^{-1} V)]$$

$${}_0\tilde{F}_0(A\,V,\ q^{-1} I_n - L^{-\frac{1}{2}} \Phi^{-1} L^{-\frac{1}{2}})\ dA\ .$$

The application of (2.3.5) leads to (7.2.19).

(7.2.20)

Let $\Omega:p\times p = 0$ and $\Sigma:p\times p = \Psi:p\times p = I_p$ in (7.2.33); then follows

$$(7.2.58)\quad f_{\underset{\sim}{V}}(V) = \frac{|V|^{n-p}}{\tilde{\Gamma}_p(n)\ \tilde{\Gamma}_p(m)\ |L\,\Phi|^p} \int_{A=\bar{A}'>0} |A|^{m+n-p}\ \mathrm{etr}[-A]$$

$$ {}_0\tilde{F}_0(-\Sigma^{-1}S,\ \Phi^{-1}L^{-1})\, dA\ .$$

The application of (2.3.5) leads to (7.2.20).

Remark 7.2.1

(i) In the following cases, the power-series representation of the p.d.f. of $\underset{\sim}{S}:p\times p$ leads to expressions for $f_{csym}(V)$ which involve lesser summation signs than when the Γ-type representation of the p.d.f. of $\underset{\sim}{S}:p\times p$ is used:

$$\Omega:p\times p \neq 0\ ,\quad \Sigma:p\times p \neq \Psi:p\times p\ ;$$
$$\Omega:p\times p = 0\ ,\quad \Sigma:p\times p \neq \Psi:p\times p\ .$$

The expressions for $f_{csym}(V)$ given in (7.2.3) and (7.2.10) are therefore from a computational point of view better that the expressions given in (7.2.1) and (7.2.8) respectively.

(ii) In the following cases, the power-series representation of the p.d.f. of $\underset{\sim}{S}:p\times p$ leads to expressions for $f_{csym}(V)$ which involve the same number of summation signs than when the Γ-type representation of the p.d.f. of $\underset{\sim}{S}:p\times p$ is used:

$$\Omega \neq 0\ ,\quad \Sigma:p\times p = \Psi:p\times p\ ;$$
$$\Omega = 0\ ,\quad \Sigma:p\times p = \Psi:p\times p\ .$$

The expression for $f_{csym}(V)$ given in (7.2.12) is therefore from a computational point of view similar to (7.2.17).

(iii) It is clear that (7.2.12) tends to (7.2.17) when $\frac{1}{q} \to 0$, i.e. $q \to \infty$.

(iv) If $\Omega: p\times p = 0$ and $\Sigma: p\times p \neq \Psi: p\times p$ the Γ-type representation of $\underset{\sim}{S}: p\times p$ leads to an expression for $f_{csym}(V)$, (7.2.8), which has the term $(-q)^{-t}$. As $\frac{1}{q} \to 0$, i.e. $q \to \infty$, this term tends to 0 for all t, except $t = 0$. The summation over t, τ and δ reduces respectively to a summation over 0, partitions of 0 (which is an empty set) and δ where δ is now a partition of k into not more than p parts i.e. $\kappa \varepsilon P(k,p)$. In this case $\tilde{g}^{\kappa}_{\kappa,(0)}$ is equal to one, thus (7.2.8) tends to (7.2.10) as $q \to \infty$.

(v) The actual p.d.f. of $\underset{\sim}{V}: p\times p$ can be derived only in the case when $\Omega: p\times p = 0$ and $\Sigma: p\times p = \Psi: p\times p = I_p$.

7.2.2 Certain marginal distributions of the characteristic roots of $\underset{\sim}{V}: p\times p$

In theorem 7.2.2 it is shown how the p.d.f. of $\underset{\sim}{D}_V$ can be written in a form such that the random component is in the form given in (3.2.3). Expand the hypergeometric function in (7.2.18); then the p.d.f.s of $\underset{\sim}{D}_V$ given in (7.2.2), (7.2.4), (7.2.6), (7.2.9), (7.2.11) and (7.2.18) have the same random component, therefore only the p.d.f.s of $\underset{\sim}{D}_V$ given in (7.2.13) and (7.2.18) will be considered in theorem 7.2.2.

Theorem 7.2.2

Let the central complex quadratic form $\underset{\sim}{S}: p\times p = \underset{\sim}{Z} L \overline{\underset{\sim}{Z}}'$ have the p.d.f. given in (4.2.6), (power-series representation) or (4.2.9),

(Γ-type representation) and let $\underset{\sim}{A}:p\times p \sim CW(p, m, \Sigma)$; then the p.d.f. of $\underset{\sim}{D}_V$ where $\underset{\sim}{V}:p\times p = \underset{\sim}{A}^{-\frac{1}{2}} \underset{\sim}{S} \underset{\sim}{A}^{-\frac{1}{2}}$ is given below.

<u>The p.d.f. of $\underset{\sim}{S}:p\times p$ has the Γ-type representation</u>

(7.2.59) $\quad f_{\underset{\sim}{D}_V}(D_V)$

$$= \frac{\tilde{\Gamma}_p(m+n)\ \pi^{p(p-1)}}{\tilde{\Gamma}_p(m)\ \tilde{\Gamma}_p(n)\ \tilde{\Gamma}_p(p)\ |L\,\Phi|^p} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{[m+n]_\kappa}{k!\ \tilde{C}_\kappa(I_n)}$$

$$\tilde{C}_\kappa(I_n - L^{-\frac{1}{2}}\,\Phi^{-1}\,L^{-\frac{1}{2}})\ \chi_{[\kappa]}(1)$$

$$\prod_{i=1}^{p} \tilde{v}_i^{n-p}\ (1+\tilde{v}_i)^{-(m+n)+2(p-1)}\ \left|\left(\left(\frac{\tilde{v}_j}{1+\tilde{v}_j}\right)^{k_i+p-i}\right)\right|$$

$$\left|\left(\left(\frac{\tilde{v}_j}{1+\tilde{v}_j}\right)^{p-i}\right)\right| , \quad 0 < \tilde{v}_1 < \cdots < \tilde{v}_p .$$

<u>The p.d.f. of $\underset{\sim}{S}:p\times p$ has the power-series representation</u>

(7.2.60) $\quad f_{\underset{\sim}{D}_V}(D_V)$

$$= \frac{\pi^{p(p-1)}\ \tilde{\Gamma}_p(m+n)}{\tilde{\Gamma}_p(n)\ \tilde{\Gamma}_p(m)\ \tilde{\Gamma}_p(p)\ |L\,\Phi|^p} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{[m+n]_\kappa\ \tilde{C}_\kappa(-\Phi^{-1}\,L^{-1})}{\tilde{C}_\kappa(I_n)\ k!}$$

$$\chi_{[\kappa]}(1) \prod_{i=1}^{p} \tilde{v}_i^{n-p}\ |(\tilde{v}_j^{k_i+p-i})|\,|(\tilde{v}_j^{p-i})| ,$$

$$0 < \tilde{v}_1 < \cdots < \tilde{v}_p \quad \text{and} \quad \|\Phi^{-1}\,L^{-1}\| < 1 .$$

Proof

(7.2.59)

Let $q=1$ in (7.2.13); then the expansion of the hypergeometric function leads to

(7.2.61) $f_{\underset{\sim}{D}_V}(D_V)$

$$= \frac{\pi^{p(p-1)}\,\tilde{\Gamma}_p(m+n)}{\tilde{\Gamma}_p(m)\,\tilde{\Gamma}_p(n)\,\tilde{\Gamma}_p(p)\,|L\,\Phi|^p} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{[m+n]_\kappa}{k!\,\tilde{C}_\kappa(I_n)}$$

$$\tilde{C}_\kappa(I_n - L^{-\frac{1}{2}}\,\Phi^{-1}\,L^{-\frac{1}{2}}) \prod_{i=1}^{p} \tilde{v}_i^{n-p}\,(1+\tilde{v}_i)^{-(m+n)}$$

$$\prod_{i>j}^{p} (\tilde{v}_i - \tilde{v}_j)^2\,\tilde{C}_\kappa(D_V(I_p + D_V)^{-1}) \;.$$

The application of (2.2.44) leads to (7.2.59).

(7.2.60)

The expansion of the hypergeometric function in (7.2.18) leads to

(7.2.62) $f_{\underset{\sim}{D}_V}(D_V)$

$$= \frac{\pi^{p(p-1)}\,\tilde{\Gamma}_p(m+n)}{\tilde{\Gamma}_p(n)\,\tilde{\Gamma}_p(m)\,\tilde{\Gamma}_p(p)\,|L\,\Phi|^p} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{[m+n]_\kappa\,\tilde{C}_\kappa(-\Phi^{-1}\,L^{-1})}{k!\,\tilde{C}_\kappa(I_n)}$$

$$\prod_{i=1}^{p} \tilde{v}_i^{n-p} \prod_{i>j}^{p} (\tilde{v}_i - \tilde{v}_j)^2\,\tilde{C}_\kappa(D_V) \;.$$

The application of (2.2.42) leads to (7.2.60).

As is the case in preceding chapters the marginal distributions of $\tilde{v}_1,\ldots,\tilde{v}_p$ when $\underset{\sim}{D}_V$ has the p.d.f. given in (7.2.59), can be obtained by using the theorems in chapter 3 with

$$\phi_i(\tilde{v}_j) = \left(\frac{\tilde{v}_j}{1+\tilde{v}_j}\right)^{k_i+p-i} ,$$

$$\psi_i(\tilde{v}_j) = \left(\frac{\tilde{v}_j}{1+\tilde{v}_j}\right)^{p-i}$$

and

$$g(\tilde{v}_j) = \tilde{v}_j^{n-p} \, (1+\tilde{v}_j)^{2(p-1)-(m+n)} .$$

Consider the following example:

Let $\underset{\sim}{D}_V$ have the p.d.f. given in (7.2.59); then

$$(7.2.63) \qquad P(d < \tilde{V}_1 < \tilde{V}_p < d)$$

$$= \frac{\tilde{\Gamma}_p(m+n) \; \pi^{p(p-1)}}{\tilde{\Gamma}_p(n) \; \tilde{\Gamma}_p(m) \; \tilde{\Gamma}_p(p) \; |L\,\Phi|^p} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{[m+n]_\kappa}{k! \; \tilde{C}_\kappa(I_n)}$$

$$\tilde{C}_\kappa(I_n - L^{-\frac{1}{2}}\,\Phi^{-1}\,L^{-\frac{1}{2}}) \; \chi_{[\kappa]}(1) \; |(b_{ij})|$$

where

$$(7.2.64) \quad b_{ij} = B_{\frac{d}{1+d}}(n+p+k_i-i-j+1,\ m-p+1) - B_{\frac{c}{1+c}}(n+p+k_i-i-j+1,\ m-p+1) .$$

The random component in (7.2.59) is similar to the random component of the $NCCMB_{2B}(p,\ n,\ m,\ \Omega)$ - p.d.f. given in (5.4.52). This similarity and the relationship, which exists between the characteristic roots of the complex beta type 1B and 2B matrices, lead to a certain correspondence between the marginal distributions of $\tilde{V}_1,\cdots,\tilde{V}_p$ when $\underset{\sim}{D}_V$ has the p.d.f. given in (7.2.59) and the marginal distributions of $\tilde{L}_1, \cdots, \tilde{L}_p$ when $\underset{\sim}{L}: p\times p \sim CMB_1(p, m, n)$, (cf. 5.2.2). Compare for example (7.2.63) and (7.2.64) with (5.2.31) and (5.2.32) respectively.

The random component in (7.2.60) is similar to the random component given in (4.2.72). The obtaining of $P(c < \tilde{V}_1 < \tilde{V}_p < d)$ and $P(\tilde{V}_p < d)$ when $\underset{\sim}{D}_V$ has the p.d.f. given in (7.2.60) is thus similar to the obtaining of $P(c < \tilde{S}_1 < \tilde{S}_p < d)$ and $P(\tilde{S}_p < d)$ when $\underset{\sim}{D}_S$ has the p.d.f. given in (4.2.60). Only these two marginal distributions of $\tilde{V}_1, \cdots, \tilde{V}_p$ can be derived because the use of theorems 3.4.1 - 3.7.1, given this random component, leads to improper integrals which are divergent.

7.2.3 P.d.f.s of functions of the characteristic roots of $\underset{\sim}{V}: p\times p$

In theorem 7.2.3 the p.d.f. of $|\underset{\sim}{V}(I_p + q^{-1}\underset{\sim}{V})^{-1}| = \prod_{i=1}^{p} \frac{\tilde{V}_i}{1+q^{-1}\tilde{V}_i}$ when $\Omega: p\times p = 0$ and $\Sigma: p\times p = \Psi: p\times p$, is derived in terms of Meijer's G-function.

<u>Theorem 7.2.3</u>

Let $\underset{\sim}{V}: p\times p$ have the symmetrised p.d.f. given in (7.2.12); then the p.d.f. of $|\underset{\sim}{V}(I_p + q^{-1}\underset{\sim}{V})^{-1}|$ is given by

$$(7.2.65) \qquad f_{|\underset{\sim}{V}(I_p + q^{-1}\underset{\sim}{V})^{-1}|}(|V(I_p + q^{-1}V)^{-1}|)$$

$$= \frac{\tilde{\Gamma}_p(m+n)\, q^{p(p+n+h)}\, |V(I_p + q^{-1}V)^{-1}|^{-1}}{\tilde{\Gamma}_p(n)\, |L\Phi|^p}$$

$$\sum_{k=0}^{\infty} \sum_{\kappa} \frac{[m+n]_\kappa\, \tilde{C}_\kappa(I_p)\, \tilde{C}_\kappa(I_n - q L^{-\frac{1}{2}} \Phi^{-1} L^{-\frac{1}{2}})}{\tilde{C}_\kappa(I_n)\, k!}$$

$$G_p\left[|V(I_p + q^{-1} V)^{-1}| \left\| \begin{matrix} m+n+k_j-j+1 \\ n+k_j-j+1 \end{matrix} \right\|\right] ,$$

$$|V(I_p + q^{-1} V)^{-1}| < 1 , \quad q > 0 \quad \text{and} \quad \|I_n - q L^{-\frac{1}{2}} \Phi^{-1} L^{-\frac{1}{2}}\| < 1 .$$

The application of theorem 2.5.3, given (7.2.16), leads to (7.2.65).

In theorem 7.2.4 the p.d.f. of $\operatorname{tr} \underset{\sim}{V}$ will be given for certain specifications of $\Omega{:}p\times p$, $\Psi{:}p\times p$ and $\Sigma{:}p\times p$.

Theorem 7.2.4

(i) $\underline{\Sigma{:}p\times p \neq \Psi{:}p\times p}$

(7.2.66) $f_{\operatorname{tr} \underset{\sim}{V}}(\operatorname{tr} V)$

$$= \frac{\tilde{\Gamma}_p(m+n) \ \operatorname{etr}[-\Omega] \ (\operatorname{tr} V)^{np-1}}{\tilde{\Gamma}_p(m) \ \ \Gamma(np) \ \ |L\,\Phi|^p \ \ |\Sigma\,\Psi^{-1}|^n}$$

$$\sum_{k=0}^{\infty} \sum_{\kappa} \sum_{t=0}^{\infty} \sum_{\tau} \sum_{\delta} \frac{\tilde{g}^{\delta}_{\kappa,\tau} \ (-1)^k \ [m+n]_\delta \ [n]_\kappa \ \tilde{C}_\kappa(\Phi^{-1} L^{-1})}{k! \ t! \ \tilde{C}_\kappa(I_p) \ \tilde{C}_\kappa(I_n) \ \tilde{C}_\tau(I_n) \ (np)_k}$$

$$\tilde{C}_\kappa(\Sigma^{-1}) \ \tilde{C}_\tau(\Omega\,\Psi^{-1}) \ \tilde{C}_\delta(\Psi) \ (\operatorname{tr} V)^k , \qquad 0 < \operatorname{tr} V < 1 .$$

(ii) $\underline{\Sigma{:}p\times p = \Psi{:}p\times p}$

(7.2.67) $f_{\operatorname{tr} \underset{\sim}{V}}(\operatorname{tr} V)$

$$= \frac{\tilde{\Gamma}_p(m+n) \ \operatorname{etr}[-\Omega] \ (\operatorname{tr} V)^{np-1}}{\tilde{\Gamma}_p(m) \ \Gamma(np) \ |L\,\Phi|^p}$$

$$\sum_{k=0}^{\infty} \sum_{\kappa} \sum_{t=0}^{\infty} \sum_{\tau} \sum_{\delta} \frac{\tilde{g}^{\delta}_{\kappa,\tau} \ (-1)^k \ [m+n]_{\delta} \ [n]_{\kappa} \ \tilde{C}_{\kappa}(L^{-1}\Phi^{-1}) \ \tilde{C}_{\tau}(\Omega)}{k! \ t! \ [m]_{\tau} \ \tilde{C}_{\kappa}(I_n) \ \tilde{C}_{\tau}(I_p)}$$

$$\tilde{C}_{\delta}(I_p) \ (\mathrm{tr}\, V)^k \ , \quad 0 < \mathrm{tr}\, V < 1 \ .$$

(iii) $\underline{\Omega{:}p\times p = 0 \ , \quad \Sigma{:}p\times p \neq \Psi{:}p\times p}$

(7.2.68) $\quad f_{\mathrm{tr}\, \underset{\sim}{V}}(\mathrm{tr}\, V)$

$$= \frac{\tilde{\Gamma}_p(m+n) \ (\mathrm{tr}\, V)^{np-1}}{\tilde{\Gamma}_p(m) \ \Gamma(np) \ |\Sigma\Psi^{-1}|^n \ |L\Phi|^p} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(-1)^k \ [m+n]_{\kappa} \ [n]_{\kappa}}{k! \ \tilde{C}_{\kappa}(I_n) \ (np)_k}$$

$$\tilde{C}_{\kappa}(\Phi^{-1}L^{-1}) \ \tilde{C}_{\kappa}(\Sigma^{-1}\Psi) \ (\mathrm{tr}\, V)^k \ , \quad 0 < \mathrm{tr}\, V < 1 \ .$$

(iv) $\underline{\Omega{:}p\times p = 0 \ , \quad \Sigma{:}p\times p = \Psi{:}p\times p}$.

(7.2.69) $\quad f_{\mathrm{tr}\, \underset{\sim}{V}}(\mathrm{tr}\, V)$

$$= \frac{\tilde{\Gamma}_p(m+n) \ (\mathrm{tr}\, V)^{np-1}}{\tilde{\Gamma}_p(m) \ \Gamma(np) \ |L\Phi|^p} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{[m+n]_{\kappa} \ [n]_{\kappa}}{k! \ \tilde{C}_{\kappa}(I_n) \ (np)_k}$$

$$\tilde{C}_{\kappa}(I_p) \ \tilde{C}_{\kappa}(-\Phi^{-1}L^{-1}) \ (\mathrm{tr}\, V)^k \ , \quad 0 < \mathrm{tr}\, V < 1 \ .$$

Proof

Only (7.2.69) will be proved here, the proofs of (7.2.66), (7.2.67) and (7.2.68) being similar.

The expansion of the hypergeometric function in (7.2.18) leads to

$$(7.2.70) \quad f_{\underset{\sim}{D_V}}(D_V) = \frac{\pi^{p(p-1)}\,\tilde{\Gamma}_p(m+n)\,|D_V|^{n-p}}{\tilde{\Gamma}_p(n)\,\tilde{\Gamma}_p(m)\,\tilde{\Gamma}_p(p)\,|L\,\Phi|^p}\prod_{i>j}^{p}(\tilde{v}_i-\tilde{v}_j)^2$$

$$\sum_{k=0}^{\infty}\sum_{\kappa}\frac{[m+n]_\kappa\,\tilde{C}_\kappa(-\Phi^{-1}L^{-1})\,\tilde{C}_\kappa(D_V)}{k!\,\tilde{C}_\kappa(I_n)} .$$

In (7.2.70) make the transformation

$$a_i = \frac{\tilde{v}_i}{\operatorname{tr} V} \quad , \quad (i = 1,\cdots,p-1)$$

with inverse transformation

$$\tilde{v}_i = a_i \operatorname{tr} V \quad , \quad (i = 1,\cdots,p-1)$$

and

$$\tilde{v}_p = \operatorname{tr} V\left(1 - \sum_{i=1}^{p-1} a_i\right)$$

$$= a_p \operatorname{tr} V \ .$$

The jacobian follows as

$$J(\tilde{v}_1,\cdots,\tilde{v}_p \to a_1,\cdots,a_{p-1}, \operatorname{tr} V) = (\operatorname{tr} V)^{p-1} .$$

Integration with respect to $0 < a_1 < \cdots < a_{p-1} < a_p = 1-a_1-\cdots-a_{p-1}$ using (4.2.166) leads to (7.2.69).

Remark 7.2.2

(i) It is important to note that the p.d.f.s of $\operatorname{tr} \underset{\sim}{V}$ derived in theorem 7.2.4 are convergent only for $0 < \operatorname{tr} V < 1$ which makes it of limited practical value.

(ii) Khatri (1970, p. 74) derived expressions for $E((\operatorname{tr} V)^h)$ in the following cases:

$\underset{\sim}{S}:p\times p$ is a non-central complex quadratic form, $\Omega:p\times p = 0$ and $\Sigma:p\times p = \Psi:p\times p$;

$\underset{\sim}{S}:p\times p$ is a non-central complex compound quadratic form, $\Omega:p\times p = 0$ and $\Sigma:p\times p = \Psi:p\times p$;

$\underset{\sim}{S}:p\times p$ is a central complex quadratic form, $\Omega:p\times p = 0$ and $\Sigma:p\times p = \Psi:p\times p$.

In the first two cases, the expressions for $E((\operatorname{tr} V)^h)$ are given in terms of generalised Laguerre polynomials in Hermitian matrices while in the third case the expression is given in terms of zonal polynomials of $\Sigma\,\Psi^{-1}$.

7.3 THE QUADRATIC FORM $\underset{\sim}{V}:p\times p = \underset{\sim}{T}^{-\frac{1}{2}}\,\underset{\sim}{S}\,\underset{\sim}{T}^{-\frac{1}{2}}$ WHEN BOTH $\underset{\sim}{S}:p\times p$ AND $\underset{\sim}{T}:p\times p$ ARE CENTRAL COMPLEX QUADRATIC FORMS

7.3.1 The symmetrised p.d.f. of $\underset{\sim}{V}:p\times p$ and $\underset{\sim}{D}_V$

Consider the following theorem in which the symmetrised p.d.f. of $\underset{\sim}{V}:p\times p = \underset{\sim}{T}^{-\frac{1}{2}}\,\underset{\sim}{S}\,\underset{\sim}{T}^{-\frac{1}{2}}$ and $\underset{\sim}{D}_V$ are derived for certain specifications of the parameter matrices. It is to be noted that it seems not possible to derive the actual p.d.f. of $\underset{\sim}{V}:p\times p$.

Theorem 7.3.1

Let $\underset{\sim}{Z}:p\times n \sim \mathrm{CMTN}(p,\, n,\, 0,\, \Phi\otimes\Sigma)$, $\underset{\sim}{Y}:p\times m \sim \mathrm{CMTN}(p,\, m,\, 0,\, \Psi\otimes\Xi)$ and $L:n\times n$ and $Q:m\times m$ be h.p.d. matrices; then the symmetrised p.d.f. of $\underset{\sim}{V}:p\times p = \underset{\sim}{T}^{-\frac{1}{2}}\,\underset{\sim}{S}\,\underset{\sim}{T}^{-\frac{1}{2}}$ and $\underset{\sim}{D}_V$, where $\underset{\sim}{S}:p\times p = \underset{\sim}{Z}\,L\,\overline{\underset{\sim}{Z}}'$ and $\underset{\sim}{T}:p\times p = \underset{\sim}{Y}\,Q\,\overline{\underset{\sim}{Y}}'$, are given below for certain specifications of $\Sigma:p\times p$ and $\Xi:p\times p$.

(i) $\underline{\Sigma:p\times p \neq \Xi:p\times p}$

<u>The p.d.f.s of $\underset{\sim}{S}:p\times p$ and $\underset{\sim}{T}:p\times p$ have the Γ-type representation</u>

(7.3.1) $\quad f_{csym}(V)$

$$= \frac{\tilde{\Gamma}_p(m+n)\ r^{(m+n)p}\ |V|^{n-p}}{\tilde{\Gamma}_p(m)\ \tilde{\Gamma}_p(n)\ |L\,\Phi|^p\ |Q\,\Psi|^p\ |\Sigma\,\Xi^{-1}|^n}$$

$$\sum_{k=0}^{\infty}\sum_{\kappa}\sum_{t=0}^{\infty}\sum_{\tau}\sum_{\delta}\sum_{b=0}^{\infty}\sum_{\beta}\sum_{\sigma}\frac{\tilde{g}^{\delta}_{\kappa,\tau}\ \tilde{g}^{\sigma}_{\delta,\beta}\ q^{-t}\ (-r)^{k+t+b}\ [m+n]_{\sigma}}{t!\ k!\ b!\ \tilde{C}_{\kappa}(I_n)\ \tilde{C}_{\delta}(I_p)\ \tilde{C}_{\delta}(I_p)}$$

$$\frac{\tilde{C}_{\kappa}(L^{-\frac{1}{2}}\,\Sigma^{-1}\,L^{-\frac{1}{2}} - q^{-1}\,I_n)\ \tilde{C}_{\delta}(\Sigma^{-1})}{\tilde{C}_{\beta}(I_m)\ \tilde{C}_{\beta}(I_p)}\ \tilde{C}_{\beta}(Q^{-\frac{1}{2}}\,\Psi^{-1}\,Q^{-\frac{1}{2}} - r^{-1}\,I_m)$$

$$\tilde{C}_{\beta}(\Xi^{-1})\ \tilde{C}_{\sigma}(\Xi^{-1})\ \tilde{C}_{\delta}(V)\ ,\quad V=\overline{V}'>0\ ,\quad q>0\ \text{ and }\ r>0\ .$$

(7.3.2) $$f_{\underset{\sim}{D}_V}(D_V) = \frac{\pi^{p(p-1)}}{\tilde{\Gamma}_p(p)}\prod_{i>j}^{p}(\tilde{v}_i - \tilde{v}_j)^2\ f_{csym}(D_V)\ ,$$

$$0<\tilde{v}_1<\cdots<\tilde{v}_p\ ,\quad q>0\ \text{ and }\ r>0\ .$$

<u>The p.d.f. of $\underset{\sim}{S}:p\times p$ has the power-series representation and the p.d.f. of $\underset{\sim}{T}:p\times p$ has the Γ-type representation</u>

(7.3.3) $\quad f_{csym}(V)$

$$= \frac{\tilde{\Gamma}_p(m+n)\ |V|^{n-p}\ r^{(m+n)p}}{\tilde{\Gamma}_p(n)\ \tilde{\Gamma}_p(m)\ |L\,\Phi|^p\ |Q\,\Psi|^p\ |\Sigma\,\Xi^{-1}|^n}$$

$$\sum_{k=0}^{\infty} \sum_{\kappa} \sum_{t=0}^{\infty} \sum_{\tau} \sum_{\delta} \frac{\tilde{g}^{\delta}_{\kappa,\tau} \; (-r)^{k+t} \; [m+n]_{\delta} \; \tilde{C}_{\kappa}(\Phi^{-1} L^{-1}) \; \tilde{C}_{\kappa}(\Sigma^{-1})}{k! \; t! \; \tilde{C}_{\kappa}(I_n) \; \tilde{C}_{\tau}(I_m) \; \tilde{C}_{\kappa}(I_p) \; \tilde{C}_{\tau}(I_p) \; \tilde{C}_{\kappa}(I_p)}$$

$$\tilde{C}_{\tau}(\Omega^{-\frac{1}{2}} \Psi^{-1} \Omega^{-\frac{1}{2}} - r^{-1} I_m) \; \tilde{C}_{\tau}(\Xi^{-1}) \; \tilde{C}_{\delta}(\Xi) \; \tilde{C}_{\kappa}(V) \; ,$$

$$V = \overline{V}' > 0 \quad \text{and} \quad r > 0 \; .$$

(7.3.4) $$f_{\underset{\sim}{D}_V}(D_V) = \frac{\pi^{p(p-1)}}{\tilde{\Gamma}_p(p)} \prod_{i>j}^{p} (\tilde{v}_i - \tilde{v}_j)^2 \; f_{csym}(D_V) \; ,$$

$$0 < \tilde{v}_1 < \cdots < \tilde{v}_p \quad \text{and} \quad r > 0 \; .$$

(ii) $\Sigma : p\times p = \Xi : p\times p$

The p.d.f.s of $\underset{\sim}{S} : p\times p$ and $T : p\times p$ have the Γ-type representation

(7.3.5) $$f_{csym}(V)$$

$$= \frac{\tilde{\Gamma}_p(m+n) \; r^{(m+n)p} \; |V|^{n-p}}{\tilde{\Gamma}_p(m) \; \tilde{\Gamma}_p(n) \; |L \Phi|^{p} \; |\Omega \Psi|^{p}}$$

$$\sum_{k=0}^{\infty} \sum_{\kappa} \sum_{t=0}^{\infty} \sum_{\tau} \sum_{b=0}^{\infty} \sum_{\beta} \sum_{\delta} \sum_{\sigma} \frac{\tilde{g}^{\delta}_{\kappa,\tau} \; \tilde{g}^{\sigma}_{\delta,\beta} \; q^{-t} \; (-r)^{k+t+b} \; [m+n]_{\sigma}}{t! \; k! \; b! \; \tilde{C}_{\kappa}(I_n) \; \tilde{C}_{\beta}(I_n) \; \tilde{C}_{\delta}(I_p)}$$

$$\tilde{C}_{\sigma}(I_p) \; \tilde{C}_{\kappa}(L^{-\frac{1}{2}} \Sigma^{-1} L^{-\frac{1}{2}} - q^{-1} I_n) \; \tilde{C}_{\beta}(\Omega^{-\frac{1}{2}} \Psi^{-1} \Omega^{-\frac{1}{2}} - r^{-1} I_m)$$

$$\tilde{C}_{\delta}(V) \; , \quad V = \overline{V}' > 0 \; , \quad q > 0 \quad \text{and} \quad r > 0 \; .$$

(7.3.6) $$f_{\underset{\sim}{D}_V}(D_V) = \frac{\pi^{p(p-1)}}{\tilde{\Gamma}_p(p)} \prod_{i>j}^{p} (\tilde{v}_i - \tilde{v}_j)^2 \; f_{csym}(D_V) \; ,$$

$$0 < \tilde{v}_1 < \cdots < \tilde{v}_p \; , \quad q > 0 \quad \text{and} \quad r > 0 \; .$$

<u>The p.d.f. of $\underset{\sim}{S}:p\times p$ has the power-series representation and the p.d.f. of $\underset{\sim}{T}:p\times p$ has the Γ-type representation</u>

(7.3.7) $\quad f_{csym}(V)$

$$= \frac{\tilde{\Gamma}_p(m+n)\ r^{(m+n)p}\ |V|^{n-p}}{\tilde{\Gamma}_p(m)\ \tilde{\Gamma}_p(n)\ |L\,\Phi|^p\ |Q\,\Psi|^p}$$

$$\sum_{k=0}^{\infty}\sum_{\kappa}\sum_{t=0}^{\infty}\sum_{\tau}\sum_{\delta}\frac{\tilde{g}^{\delta}_{\kappa,\tau}\ (-r)^{k+t}\ [m+n]_{\delta}\ \tilde{C}_{\kappa}(\Phi^{-1}L^{-1})}{k!\ t!\ \tilde{C}_{\kappa}(I_n)\ \tilde{C}_{\kappa}(I_p)\ \tilde{C}_{\tau}(I_m)}$$

$$\tilde{C}_{\tau}(Q^{-\frac{1}{2}}\,\Psi^{-1}\,Q^{-\frac{1}{2}} - r^{-1}\,I_m)\ \tilde{C}_{\delta}(I_p)\ \tilde{C}_{\kappa}(V)\ ,$$

$$V = \bar{V}' > 0 \quad \text{and} \quad r > 0\ .$$

(7.3.8) $$f_{\underset{\sim}{D}_V}(D_V) = \frac{\pi^{p(p-1)}}{\tilde{\Gamma}_p(p)}\prod_{i>j}^{p}(\tilde{v}_i - \tilde{v}_j)^2\ f_{csym}(D_V)\ ,$$

$$0 < \tilde{v}_1 < \cdots < \tilde{v}_p \quad \text{and} \quad r > 0\ .$$

<u>Proof</u>

(<u>7.3.1</u>)

The joint p.d.f. of $\underset{\sim}{S}:p\times p$ and $\underset{\sim}{T}:p\times p$ follows as

(7.3.9) $\quad f_{\underset{\sim}{S},\underset{\sim}{T}}(S,T)$

$$= c\ |S|^{n-p}\ \mathrm{etr}[-q^{-1}\,\Sigma^{-1}\,S]\ {}_0\tilde{F}_0(L^{-\frac{1}{2}}\,\Phi^{-1}\,L^{-\frac{1}{2}} - q^{-1}\,I_n, -\Sigma^{-1}\,S)$$

$$|T|^{m-p}\ \mathrm{etr}[-r^{-1}\,\Xi^{-1}\,T]\ {}_0\tilde{F}_0(Q^{-\frac{1}{2}}\,\Psi^{-1}\,Q^{-\frac{1}{2}} - r^{-1}\,I_m, -\Xi^{-1}\,T)$$

with

(7.3.10) $$c = (\tilde{\Gamma}_p(m)\ \tilde{\Gamma}_p(n)\ |L\,\Phi|^p\ |Q\,\Psi|^p\ |\Sigma|^n\ |\Xi|^m)^{-1}\ .$$

In (7.3.9) make the transformation (7.2.23) with $A:p\times p$ replaced by $T:p\times p$ in (7.2.23), then the joint p.d.f. of $\underset{\sim}{V}:p\times p$ and $\underset{\sim}{T}:p\times p$ follows as

(7.3.11) $$f_{\underset{\sim}{T},\underset{\sim}{V}}(T,V)$$

$$= c\ |V|^{n-p}\ |T|^{m+n-p}\ \mathrm{etr}[-q^{-1}\Sigma^{-1}T^{\frac{1}{2}}VT^{\frac{1}{2}}]\ \mathrm{etr}[-r^{-1}\Xi^{-1}T]$$

$$ {}_0\tilde{F}_0(L^{-\frac{1}{2}}\Phi^{-1}L^{-\frac{1}{2}} - q^{-1}I_n, -\Sigma^{-1}T^{\frac{1}{2}}VT^{\frac{1}{2}})$$

$$ {}_0\tilde{F}_0(Q^{-\frac{1}{2}}\Psi^{-1}Q^{-\frac{1}{2}} - r^{-1}I_m, -\Xi^{-1}T)\ .$$

The expansion of the first exponential and the hypergeometric functions and the application of (2.2.54) lead to

(7.3.12) $$f_{\underset{\sim}{T},\underset{\sim}{V}}(T,V)$$

$$= c\ |V|^{n-p}\ |T|^{m+n-p}\ \mathrm{etr}[-r^{-1}\Xi^{-1}T]$$

$$\sum_{k=0}^{\infty}\sum_{\kappa}\sum_{t=0}^{\infty}\sum_{\tau}\sum_{\delta}\sum_{b=0}^{\infty}\sum_{\beta}\frac{\tilde{g}^{\delta}_{\kappa,\tau}\ q^{-t}\ \tilde{C}_{\kappa}(L^{-\frac{1}{2}}\Phi^{-1}L^{-\frac{1}{2}} - q^{-1}I_n)}{t!\ k!\ \tilde{C}_{\kappa}(I_n)\ \tilde{C}_{\beta}(I_m)}$$

$$\tilde{C}_{\delta}(-\Sigma^{-1}T^{\frac{1}{2}}VT^{\frac{1}{2}})\ \tilde{C}_{\beta}(Q^{-\frac{1}{2}}\Psi^{-1}Q^{-\frac{1}{2}} - r^{-1}I_m)\ \tilde{C}_{\beta}(-\Xi^{-1}T)\ .$$

After changing the order of integration, the symmetrised p.d.f. of $\underset{\sim}{V}:p\times p$ follows as

(7.3.13) $$f_{csym}(V)$$

$$= c\ |V|^{n-p}\sum_{k=0}^{\infty}\sum_{\kappa}\sum_{t=0}^{\infty}\sum_{\tau}\sum_{\delta}\sum_{b=0}^{\infty}\sum_{\beta}\frac{\tilde{g}^{\delta}_{\kappa,\tau}\ q^{-t}\ (-1)^{k+t+b}}{t!\ k!\ b!\ \tilde{C}_{\kappa}(I_n)\ \tilde{C}_{\beta}(I_m)}$$

$$\frac{\tilde{C}_{\delta}(V)}{\tilde{C}_{\delta}(I_p)}\ \tilde{C}_{\kappa}(L^{-\frac{1}{2}}\Phi^{-1}L^{-1} - q^{-1}I_n)\ \tilde{C}_{\beta}(Q^{-\frac{1}{2}}\Psi^{-1}Q^{-\frac{1}{2}} - r^{-1}I_m)\ I^{*}\ ,$$

(from (2.2.29)).

where

$$(7.3.14) \quad I^* = \int_{T=\bar{T}'>0} |T|^{m+n-p}\, \tilde{C}_\delta(\Sigma^{-1}T)\, \tilde{C}_\beta(\Xi^{-1}T)\, \mathrm{etr}[-r^{-1}\Xi^{-1}T]\, dT .$$

The matrices Σ^{-1} and Ξ^{-1} are hermitian matrices. Thus by transforming $\Sigma^{-1} \to U\Sigma^{-1}\bar{U}'$ and $\Xi^{-1} \to U\Xi^{-1}\bar{U}'$ and after integrating over the unitary group after each transformation, the integral in (7.3.14) can be written as

$$(7.3.15) \quad I^* = \frac{\tilde{C}_\delta(\Sigma^{-1})\ \tilde{C}_\beta(\Xi^{-1})}{\tilde{C}_\delta(I_p)\ \tilde{C}_\beta(I_p)} \int_{T=\bar{T}'>0} \sum_\sigma \tilde{g}^{\sigma}_{\delta,\beta}\, |T|^{m+n-p}\, \tilde{C}_\sigma(T)$$

$$\mathrm{etr}[-r^{-1}\Xi^{-1}T]\, dT$$

$$= \frac{\tilde{C}_\delta(\Sigma^{-1})\ \tilde{C}_\beta(\Xi^{-1})}{\tilde{C}_\delta(I_p)\ \tilde{C}_\beta(I_p)}\, \tilde{\Gamma}_p(m+n)\ r^{(m+n)p}\ |\Xi|^{m+n}$$

$$\sum_\sigma \tilde{g}^{\sigma}_{\delta,\beta}\ [m+n]_\sigma\ \tilde{C}_\sigma(r\,\Xi)\ , \quad \text{(from (2.2.32))}.$$

Substitution of (7.3.15) into (7.3.13) leads to (7.3.1).

(7.3.2)

The application of theorem 3.2.1 and corollary 2.7.1 leads to (7.3.2).

(7.3.3)

The joint p.d.f. of $\underset{\sim}{S}:p\times p$ and $\underset{\sim}{T}:p\times p$ follows as

$$(7.3.16) \quad f_{\underset{\sim}{S},\underset{\sim}{T}}(S,T) = c\ |S|^{n-p}\ {}_0\tilde{F}_0(\Phi^{-1}L^{-1}, -\Sigma^{-1}S)\ |T|^{m-p}$$

$$\mathrm{etr}[-r^{-1}\Xi^{-1}T]\ {}_0\tilde{F}_0(Q^{-\frac{1}{2}}\Psi^{-1}Q^{-\frac{1}{2}} - r^{-1}I_m, -\Xi^{-1}T)$$

with c given in (7.3.10).

In (7.3.16) make the transformation (7.2.23) with $A:p\times p$ replaced by $T:p\times p$ in (7.2.23), then the joint p.d.f. of $\underset{\sim}{V}:p\times p$ and $\underset{\sim}{T}:p\times p$ follows as

$$(7.3.17) \quad f_{\underset{\sim}{T},\underset{\sim}{V}}(T,V) = c\ |V|^{n-p}\ |T|^{m+n-p}\ {}_0\tilde{F}_0(-\Sigma^{-1}\,T^{\frac{1}{2}}\,V\,T^{\frac{1}{2}},\ \Phi^{-1}\,L^{-1})$$

$$\operatorname{etr}[-r^{-1}\,\Xi^{-1}\,T]\ {}_0\tilde{F}_0(Q^{-\frac{1}{2}}\,\Psi^{-1}\,Q^{-\frac{1}{2}} - r^{-1}\,I_m,\ -\Xi^{-1}\,T)\ .$$

The expansion of the hypergeometric functions leads to

$$(7.3.18) \quad f_{\underset{\sim}{T},\underset{\sim}{V}}(T,V)$$

$$= c\ |V|^{n-p}\ |T|^{m+n-p}\ \operatorname{etr}[-r^{-1}\,\Xi^{-1}\,T]$$

$$\sum_{k=0}^{\infty}\sum_{\kappa}\sum_{t=0}^{\infty}\sum_{\tau}\frac{\tilde{C}_{\kappa}(-\Sigma^{-1}\,T^{\frac{1}{2}}\,V\,T^{\frac{1}{2}})\ \tilde{C}_{\kappa}(\Phi^{-1}\,L^{-1})}{\tilde{C}_{\kappa}(I_n)\ k!\ \tilde{C}_{\tau}(I_m)\ t!}$$

$$\tilde{C}_{\tau}(Q^{-\frac{1}{2}}\,\Psi^{-1}\,Q^{-\frac{1}{2}} - r^{-1}\,I_m)\ \tilde{C}_{\tau}(-\Xi^{-1}\,T)\ .$$

After changing the order of integration, the symmetrised p.d.f. of $\underset{\sim}{V}:p\times p$ follows as

$$(7.3.19) \quad f_{csym}(V)$$

$$= c\ |V|^{n-p}\sum_{k=0}^{\infty}\sum_{\kappa}\sum_{t=0}^{\infty}\sum_{\tau}\frac{(-1)^{k+t}\ \tilde{C}_{\kappa}(\Phi^{-1}\,L^{-1})}{k!\ t!\ \tilde{C}_{\kappa}(I_n)\ \tilde{C}_{\tau}(I_m)\ \tilde{C}_{\kappa}(I_p)}$$

$$\tilde{C}_{\tau}(Q^{-\frac{1}{2}}\,\Psi^{-1}\,Q^{-\frac{1}{2}} - r^{-1}\,I_m)\ \tilde{C}_{\kappa}(V)\quad I^*$$

where

$$(7.3.20) \quad I^* = \int_{T=\bar{T}'>0} |T|^{m+n-p} \tilde{C}_\kappa(\Sigma^{-1} T) \tilde{C}_\tau(\Xi^{-1} T) \operatorname{etr}[-r^{-1} \Xi^{-1} T] \, dT$$

$$= \frac{\tilde{C}_\kappa(\Sigma^{-1}) \ \tilde{C}_\tau(\Xi^{-1})}{\tilde{C}_\kappa(I_p) \ \tilde{C}_\tau(I_p)} \tilde{\Gamma}_p(m+n) \ r^{(m+n)p} \ |\Xi|^{m+n}$$

$$\sum_\delta \tilde{g}^\delta_{\kappa,\tau} \ [m+n]_\delta \ \tilde{C}_\delta(r\,\Xi) \ , \quad \text{(from (7.3.15))}.$$

Substitution of (7.3.20) into (7.3.19) leads to (7.3.3).

(<u>7.3.4</u>)

The application of theorem 3.2.1 and corollary 2.7.1 leads to (7.3.4).

(<u>7.3.5</u>)

Let $\Sigma{:}p\times p = \Xi{:}p\times p$ in (7.3.13); then

$$(7.3.21) \quad f_{csym}(V)$$

$$= c \ |V|^{n-p} \sum_{k=0}^{\infty} \sum_{\kappa} \sum_{t=0}^{\infty} \sum_{\tau} \sum_{\delta} \sum_{b=0}^{\infty} \sum_{\beta} \sum_{\sigma} \frac{\tilde{g}^\delta_{\kappa,\tau} \ \tilde{g}^\sigma_{\delta,\beta} \ (-1)^{k+t+b}}{t! \ k! \ b! \ \tilde{C}_\kappa(I_n)}$$

$$\frac{q^{-t} \ \tilde{C}_\delta(V) \ \tilde{C}_\kappa(L^{-\frac{1}{2}} \Phi^{-1} L^{-\frac{1}{2}} - q^{-1} I_n)}{\tilde{C}_\beta(I_m) \ \tilde{C}_\delta(I_p)}$$

$$\tilde{C}_\beta(Q^{-\frac{1}{2}} \Psi^{-1} Q^{-\frac{1}{2}} - r^{-1} I_m) \ I^*$$

where

$$(7.3.22) \quad c = (\tilde{\Gamma}_p(m) \ \tilde{\Gamma}_p(n) \ |L\,\Phi|^p \ |Q\,\Psi|^p \ |\Sigma|^{m+n})^{-1}$$

and

$$(7.3.23)\quad I^* = \int_{T=\bar{T}'>0} \tilde{C}_\sigma(\Sigma^{-1} T)\,|T|^{m+n-p}\,\mathrm{etr}[-r^{-1}\Sigma^{-1}T]\,dT$$

$$= \tilde{\Gamma}_p(m+n)\,[m+n]_\sigma\,|r\,\Sigma|^{m+n}\,\tilde{C}_\sigma(r\,I_p)\ ,\quad \text{(from (2.2.32))}.$$

Substitution of (7.3.23) into (7.3.21) leads to (7.3.5).

(7.3.6)

The application of theorem 3.2.1 and corollary 2.7.1 leads to (7.3.6).

(7.3.7)

Let $\Sigma:p\times p = \Xi:p\times p$ in (7.3.19); then

$$(7.3.24)\quad f_{csym}(V)$$

$$= c\,|V|^{n-p} \sum_{k=0}^{\infty} \sum_{\kappa} \sum_{t=0}^{\infty} \sum_{\tau} \sum_{\delta} \frac{\tilde{g}^{\delta}_{\kappa,\tau}\,(-1)^{k+t}\,\tilde{C}_\kappa(\Phi^{-1}L^{-1})}{k!\ t!\ \tilde{C}_\kappa(I_n)\,\tilde{C}_\tau(I_m)\,\tilde{C}_\kappa(I_p)}$$

$$\tilde{C}_\tau(Q^{-\frac{1}{2}}\Psi^{-1}Q^{-\frac{1}{2}} - r^{-1}I_m)\,\tilde{C}_\kappa(V)\quad I^*$$

where c is given in (7.3.22) and

$$(7.3.25)\quad I^* = \int_{T=\bar{T}'>0} |T|^{m+n}\,\tilde{C}_\delta(\Sigma^{-1} T)\,\mathrm{etr}[-r^{-1}\Sigma^{-1}T]\,dT$$

$$= \tilde{\Gamma}_p(m+n)\,[m+n]_\delta\,|r\,\Sigma|^{m+n}\,\tilde{C}_\delta(r\,I_p)\ ,\quad \text{(from (2.2.32))}.$$

Substitution of (7.3.25) into (7.3.24) leads to (7.3.7).

(7.3.8)

The application of theorem 3.2.1 and corollary 2.7.1 leads to (7.3.8).

Remark 7.3.1

(i) As in the real case (cf. Underhill, 1973, p. 5.6) it is not possible to derive the p.d.f. of $\underset{\sim}{V}:p\times p = \underset{\sim}{T}^{-\frac{1}{2}}\,\underset{\sim}{S}\,\underset{\sim}{T}^{-\frac{1}{2}}$ when the p.d.f. of $\underset{\sim}{T}:p\times p$ has the power-series representation because the exponential function $\operatorname{etr}[-r^{-1}\,\Xi^{-1}\,T]$ is needed to perform the integration over $T:p\times p$.

(ii) By using the same argument as in remark 7.2.1 (iv), it can be shown that (7.3.1) tends to (7.3.3) and (7.3.5) tends to (7.3.7) if $q \to \infty$ i.e. $\frac{1}{q} \to 0$.

(iii) Let $\Psi:m\times m = Q^{-1}:m\times m$, $\Xi:p\times p = \Sigma:p\times p$ and $r = 1$; then $\underset{\sim}{T}:p\times p \sim CW(p,\ m,\ \Sigma)$. If these substitutions are made in the results derived in theorem 7.3.1 it ought to reduce to certain results derived in theorem 7.2.1:

(a) Let $\Psi:m\times m = Q^{-1}:m\times m$, $\Xi:p\times p = \Sigma:p\times p$ and $r = 1$ in (7.3.5); then:

(7.3.26) $f_{csym}(V)$

$$= \frac{\tilde{\Gamma}_p(m+n)\ |V|^{n-p}}{\tilde{\Gamma}_p(m)\ \tilde{\Gamma}_p(n)\ |L\,\Phi|^p} \sum_{k=0}^{\infty} \sum_{\kappa} \sum_{t=0}^{\infty} \sum_{\tau} \sum_{\delta} \frac{\tilde{g}^{\delta}_{\kappa,\tau}\ q^{-t}}{t!\ k!\ \tilde{C}_\kappa(I_n)}$$

$$(-1)^{k+t}\ [m+n]_\delta\ \tilde{C}_\kappa(L^{-\frac{1}{2}}\,\Phi^{-1}\,L^{-\frac{1}{2}} - q^{-1}\,I_n)\ \tilde{C}_\delta(V)$$

$$= (7.2.51)\ ,$$

which leads to (7.2.12).

(b) Let $\Psi:m\times m = Q^{-1}:m\times m$, $\Xi:p\times p = \Sigma:p\times p$ and $r = 1$ in (7.3.7); then:

(7.3.27) $f_{csym}(V)$

$$= \frac{\tilde{\Gamma}_p(m+n)\ |V|^{n-p}}{\tilde{\Gamma}_p(m)\ \tilde{\Gamma}_p(n)\ |L\,\Phi|^p} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{[m+n]_\kappa\ \tilde{C}_\kappa(-\Phi^{-1}L^{-1})\ \tilde{C}_\kappa(V)}{k!\ \tilde{C}_\kappa(I_n)}$$

$$= (7.2.17)\ .$$

7.3.2 <u>Certain marginal distributions of the characteristic roots of $\underset{\sim}{V}$:p×p</u>

The p.d.f.s of $\underset{\sim}{D}_V$ given in (7.3.2), (7.3.4), (7.3.6) and (7.3.8) have the same random component. By using (2.2.44) these p.d.f.s can be written, similar to previous cases, in a form such that the random component is in the form (3.2.3). The p.d.f. given in (7.3.8) can be written for example as:

(7.3.28) $f_{\underset{\sim}{D}_V}(D_V)$

$$= \frac{\tilde{\Gamma}_p(m+n)\ r^{(m+n)p}\ \pi^{p(p-1)}}{\tilde{\Gamma}_p(p)\ \tilde{\Gamma}_p(m)\ \tilde{\Gamma}_p(n)\ |L\,\Phi|^p\ |Q\,\Psi|^p}$$

$$\sum_{k=0}^{\infty} \sum_{\kappa} \sum_{t=0}^{\infty} \sum_{\tau} \sum_{\delta} \frac{\tilde{g}^{\delta}_{\kappa,\tau}\ (-r)^{k+t}\ [m+n]_\delta\ \tilde{C}_\kappa(\Phi^{-1}L^{-1})}{k!\ t!\ \tilde{C}_\kappa(I_n)\ \tilde{C}_\kappa(I_p)\ \tilde{C}_\tau(I_m)}$$

$$\tilde{C}_\tau(Q^{-\frac{1}{2}}\Psi^{-1}Q^{-\frac{1}{2}} - r^{-1}I_m)\ \tilde{C}_\delta(I_p)\ \chi_{[\kappa]}(1)$$

$$\prod_{i=1}^{p} \tilde{v}_i^{n-p}\ |(\tilde{v}_j^{k_i+p-i})|\,|(\tilde{v}_j^{p-i})|\ ,\quad 0<\tilde{v}_1<\cdots<\tilde{v}_p \text{ and } r<0\ .$$

The random component in (7.3.28) is similar to the random component given in (4.2.72). The obtaining of $P(c<\tilde{V}_1<\tilde{V}_p<d)$ and $P(\tilde{V}_p<d)$ when $\underset{\sim}{D}_V$ has the p.d.f. given in (7.3.28) is thus similar to the obtaining of $P(c<\tilde{S}_1<\tilde{S}_p<d)$ and $P(\tilde{S}_p<d)$ when

$\underset{\sim}{D}_S$ has the p.d.f. given in (4.2.60). Only these two marginal distributions of $\tilde{V}_1, \cdots, \tilde{V}_p$ can be derived because the use of theorems 3.4.1 - 3.7.1, given this random component, leads to improper integrals which are divergent.

7.3.3 The p.d.f. of $\operatorname{tr} \underset{\sim}{V}$

In theorem 7.3.2 the p.d.f. of $\operatorname{tr} \underset{\sim}{V}$ will be given for certain specifications of $\Sigma{:}p\times p$ and $\Xi{:}p\times p$. The p.d.f.s of $\underset{\sim}{D}_V$ given in (7.3.4) and (7.3.8) are used to obtain the different p.d.f.s of $\operatorname{tr} \underset{\sim}{V}$ because these p.d.f.s involve lesser summation signs than the p.d.f.s of $\underset{\sim}{D}_V$ given in (7.3.2) and (7.3.6).

Theorem 7.3.2

(i) $\Sigma{:}p\times p \neq \Xi{:}p\times p$

(7.3.29)
$$f_{\operatorname{tr} \underset{\sim}{V}}(\operatorname{tr} V)$$

$$= \frac{\tilde{\Gamma}_p(m+n)\ (\operatorname{tr} V)^{np-1}}{\tilde{\Gamma}_p(m)\ \Gamma(np)\ |L\,\Phi|^p\ |Q\,\Psi|^p\ |\Sigma\,\Xi^{-1}|^n}$$

$$\sum_{k=0}^{\infty}\sum_{\kappa}\sum_{t=0}^{\infty}\sum_{\tau}\sum_{\delta} \frac{\tilde{g}^{\delta}_{\kappa,\tau}\ (-1)^{k+t}\ [m+n]_\delta\ [n]_\kappa\ \tilde{C}_\kappa(\Phi^{-1}L^{-1})}{k!\ t!\ \tilde{C}_\kappa(I_n)\ \tilde{C}_\tau(I_m)\ \tilde{C}_\tau(I_p)\ (np)_k}$$

$$\tilde{C}_\tau(Q^{-\frac{1}{2}}\Psi^{-1}Q^{-\frac{1}{2}} - I_m)\ \tilde{C}_\tau(\Xi^{-1})\ \tilde{C}_\delta(\Xi)\ (\operatorname{tr} V)^k ,$$

$$0 < \operatorname{tr} V < 1 .$$

(ii) $\Sigma{:}p\times p = \Xi{:}p\times p$

(7.3.30)
$$f_{\operatorname{tr} \underset{\sim}{V}}(\operatorname{tr} V)$$

$$= \frac{\tilde{\Gamma}_p(m+n)\ (\operatorname{tr} V)^{np-1}}{\tilde{\Gamma}_p(m)\ \Gamma(np)\ |L\,\Phi|^p\ |Q\,\Psi|^p} \sum_{k=0}^{\infty}\sum_{\kappa}\sum_{t=0}^{\infty}\sum_{\tau}\sum_{\delta} \frac{\tilde{g}^{\delta}_{\kappa,\tau}\ [n]_\kappa}{k!\ t!}$$

$$\frac{(-1)^{k+t}\,[m+n]_{\delta}\,\tilde{C}_{\kappa}(\Phi^{-1}L^{-1})\,\tilde{C}_{\tau}(Q^{-\frac{1}{2}}\Psi^{-1}Q^{-\frac{1}{2}}-I_m)}{(np)_k\,\tilde{C}_{\kappa}(I_n)\,\tilde{C}_{\tau}(I_m)}$$

$$\tilde{C}_{\delta}(I_p)\;(\mathrm{tr}\,V)^k\;, \qquad 0 < \mathrm{tr}\,V < 1\;.$$

Proof

The proofs of (7.3.29) and (7.3.30) being similar to the proof of theorem 7.2.4.

Remark 7.3.2

It is important to note that the p.d.f.s of $\mathrm{tr}\,\underset{\sim}{V}$ derived in theorem 7.3.2 are convergent only for $0 < \mathrm{tr}\,V < 1$ which makes it of limited practical value.

* * * * * * * * * * *

RÉSUMÉ

The distributional properties of a number of multivariate complex quadratic forms and their characteristic roots have been studied in this thesis. The probability density functions, moments and the joint probability density functions of the characteristic roots of these multivariate complex quadratic forms were derived in similar ways as in the real case. Despite this correspondence it is to be mentioned that these complex results are not merely a duplication of the results in the real case. All the multivariate complex quadratic forms studied in this thesis are random hermitian matrices and are such that the joint probability density functions of their characteristic roots are symmetric functions of the roots. As a result of this property of the joint probability density functions of the roots and by using certain properties of zonal polynomials of hermitian matrices and hypergeometric functions with hermitian matrix arguments, the derivation of certain marginal distributions of the roots is found to be less complicated than in the case of real symmetric matrices. In order to derive the different distributions, attention has also been given to certain important mathematical functions and techniques.

Although some of the results derived in this thesis are highly theoretical, it is to be hoped that it will stimulate future research to making it more tractable for practical purposes.

BIBLIOGRAPHY

Al-Ani, S. 1972. On the ith latent root of a complex matrix. *Canad. Math. Bull.*, vol. 15, 323 - 327.

Anderson, T.W. 1958. *An introduction to multivariate statistical analysis.* New York: Wiley.

Brillinger, D.R. 1974. *Time Series: data analysis and theory.* New York: Holt, Rinehart and Winston.

Carmeli, M. 1974. Statistical theory of energy levels and random matrices in physics. *J. Statist. Phys.*, vol. 10, 259 - 297.

Constantine, A.G. 1963. Some non-central distribution problems in multivariate analysis. *Ann. Math. Statist.*, vol. 34, 1270 - 1285.

Constantine, A.G. 1966. The distribution of Hotelling's generalized T_0^2. *Ann. Math. Statist.*, vol. 37, 215 - 225.

Crowther, N.A.S. 1972. Meervariante lineêre en kwadratiese vorme van normale vektore. Unpublished D.Sc. thesis, University of the Orange Free State.

Crowther, N.A.S. 1975. The exact non-central distribution of a quadratic form in normal vectors. *S.Afr. Statist. J.*, vol. 9, 27 - 36.

Deemer, W.L. and Olkin, I. 1951. The jacobians of certain matrix transformations useful in multivariate analysis. *Biometrika*, vol. 38, 345 - 367.

De Waal, D.J. 1968. Nie-sentrale meerveranderlike beta-verdelings. Unpublished Ph.D. thesis, University of Cape Town.

Dyson, F.J. 1962 a. Statistical theory of the energy levels of complex systems, I. *J. Math. Phys.*, vol. 3, 140 - 156.

Dyson, F.J. 1962 b. Statistical theory of the energy levels of complex systems, II. *J. Math. Phys.*, vol. 3, 157 - 165.

Dyson, F.J. 1962 c. Statistical theory of the energy levels of complex systems, III. *J. Math. Phys.*, vol. 3, 166 - 175.

Dyson, F.J. and Mehta, M.L. 1963 a. Statistical theory of the energy levels of complex systems, IV. *J. Math. Phys.*, vol. 4, 701 - 712.

Dyson, F.J. and Mehta, M.L. 1963 b. Statistical theory of the energy levels of complex systems, V. *J. Math. Phys.*, vol. 4, 713 - 719.

Erdélyi, A. 1953. *Higher transcendental functions vol. 1.* New York: McGraw-Hill.

Goodman, N.R. 1957. On the joint estimation of the spectra cospectrum and quadrature spectrum of a two-dimensional stationary Gaussian process. Paper No. 19, Engineering statistical laboratory, College of engineering, New York University.

Goodman, N.R. 1963 a. Statistical analysis based on a certain multivariate complex Gaussian distribution (an introduction). *Ann. Math. Statist.*, vol. 34, 152 - 177.

Goodman, N.R. 1963 b. The distribution of the determinant of a complex Wishart distributed matrix. *Ann. Math. Statist.*, vol. 34, 178 - 180.

Goodman, N.R. and Dubman, M.R. 1969. Theory of time-varying spectral analysis and complex Wishart processes, in *Multivariate analysis - II.* Edited by Krishnaiah, P.R. New York: Academic Press.

Graybill, F.A. 1969. *Introduction to matrices with applications in Statistics.* Belmont, California: Wadsworth.

Greenacre, M.J. 1972. Some noncentral distributions and applications in multivariate analysis. Unpublished M.Sc. dissertation, University of South Africa, Pretoria.

Greenacre, M.J. 1973. Symmetrised multivariate distributions. *S.Afr. Statist. J.*, vol. 7, 95 - 101.

Gupta, A.K. 1970. Evaluation of the coefficients $g^{\delta}_{X,\eta,i}$. *Mathematica*, vol. 12, 299 - 304.

Gupta, A.K. 1971. Distribution of Wilk's likelihood ratio criterion in the complex case. *Ann. Inst. Statist. Math.*, vol. 23, 77 - 87.

Gupta, A.K. 1973. On a test for reality of the covariance matrix of a complex Gaussian distribution. *J. Statist. Comput. Simul.*, vol. 2, 333 - 342.

Gupta, A.K. 1976. Nonnull distribution of Wilk's statistic for MANOVA in the complex case. *Comm. Statist.*, vol. B 5, 177 - 188.

Hannan, E.J. 1970. *Multiple time series.* New York: Wiley.

Hart, M.L. 1974. Exact powers of some multivariate test criteria. Unpublished Ph.D. thesis, University of Cape Town.

Hayakawa, T. 1966. On the distribution of a quadratic form in a multivariate normal sample. *Ann. Inst. Statist. Math.*, vol. 18, 191 - 201.

Hayakawa, T. 1969. On the distribution of the latent roots of a positive definite random symmetric matrix I. *Ann. Inst. Statist. Math.*, vol. 21, 1 - 21.

Hayakawa, T. 1972 a. On the distribution of the latent roots of a complex Wishart matrix (non-central case). *Ann. Inst. Statist. Math.*, vol. 24, 1 - 17.

Hayakawa, T. 1972 b. On the distribution of the multivariate quadratic form in multivariate normal samples. *Ann. Inst. Statist. Math.*, vol. 24, 205 - 230.

Hayakawa, T.	1972 c.	The asymptotic distributions of the statistics based on the complex Gaussian distribution. *Ann. Inst. Statist. Math.*, vol. 24, 231 - 244.
Hirakawa, F.	1975.	Some distributions of the latent roots of a complex Wishart matrix variate. *Ann. Inst. Statist. Math.*, vol. 27, 357 - 363.
James, A.T.	1960.	The distribution of the latent roots of the covariance matrix. *Ann. Math. Statist.*, vol. 31, 151 - 158.
James, A.T.	1961.	Zonal polynomials of the real positive definite symmetric matrices. *Ann. Math.*, vol. 74, 456 - 469.
James, A.T.	1964.	Distributions of matrix variates and latent roots derived from normal samples. *Ann. Math. Statist.*, vol. 35, 475 - 501.
Johnson, N.L. and Kotz, S.	1972.	*Distributions in statistics: Continuous multivariate distributions.* New York: Wiley.
Khatri, C.G.	1964.	Distribution of the largest or the smallest characteristic root under null hypothesis concerning complex multivariate normal populations. *Ann. Math. Statist.*, vol. 35, 1807 - 1810.
Khatri, C.G.	1965.	Classical statistical analysis based on a certain multivariate complex Gaussian distribution. *Ann. Math. Statist.*, vol. 36, 98 - 114.
Khatri, C.G.	1966.	On certain distribution problems based on positive definite quadratic functions in normal vectors. *Ann. Math. Statist.*, vol. 37, 468 - 479.
Khatri, C.G.	1969.	Non-central distributions of i th largest characteristic roots of three matrices concerning complex multivariate normal populations. *Ann. Inst. Statist. Math.*, vol. 21, 23 - 32.

Khatri, C.G. 1970. On the moments of traces of two matrices in three situations for complex multivariate normal populations. *Sankhyā A*, vol. 32, 65 - 80.

Khatri, C.G. 1971. Series representations of distributions of quadratic form in the normal vectors and generalised variance. *J. Multivariate Anal.*, vol. 1, 199 - 214.

Khatri, C.G. and Pillai, K.C.S. 1968. On the non-central distributions of two test criteria in multivariate analysis of variance. *Ann. Math. Statist.*, vol. 39, 215 - 226.

Krishnaiah, P.R. 1976. Some recent developments on complex multivariate distributions. *J. Multivariate Anal.*, vol. 6, 1 - 30.

Le Roux, N.J. 1978. The algebra of random matrices. Unpublished Ph.D. thesis, University of South Africa, Pretoria.

Littlewood, D.E. 1940. *The theory of group characters and matrix representations of groups.* Oxford: Clarendon Press.

Macduffee, C.C. 1946. *The theory of matrices.* New York: Chelsea.

Mathai, A.M. 1971. An expansion of Meijer's G-function and the distribution of products of independent beta variables. *S.Afr. Statist. J.*, vol. 5, 71 - 90.

Money, A.M. 1972. Noncentral multivariate beta distributions and their applications. Unpublished Ph.D. thesis, University of Cape Town.

Morrison, D.F. 1976. *Multivariate statistical methods.* 2nd edition. New York: McGraw-Hill.

Nel, D.G. 1972. Die elementêre simmetriese funksies van Wishart matrikse. Unpublished Ph.D. thesis, University of the Orange Free State.

Pearson, K.P. 1934 a. *Tables of the incomplete Γ-function.* Edited by Pearson, K.P. Proprietors of Biometrika.

Pearson, K.P. 1934 b. *Tables of the incomplete beta-functions.* Edited by Pearson, K.P. Proprietors of Biometrika.

Pillai, K.C.S. 1956. Some results useful in multivariate analysis. *Ann. Math. Statist.*, vol. 27, 1106 - 1114.

Pillai, K.C.S. and Jouris, G.M. 1971. Some distribution problems in the multivariate complex Gaussian case. *Ann. Math. Statist.*, vol. 42, 517 - 525.

Pillai, K.C.S. and Jouris, G.M. 1972. An approximation to the distribution of the largest root of a matrix in the complex Gaussian case. *Ann. Inst. Statist. Math.*, vol. 24, 61 - 70.

Pillai, K.C.S. and Young, D.L. 1971. An approximation to the distribution of the largest root of a complex Wishart matrix. *Ann. Inst. Statist. Math.*, vol. 23, 89 - 96.

Priestly, M.B. Subba Rao, T. and Tong, H. 1973. Identification of the structure of multivariable stochastic systems, in *Multivariate Analysis - III.* Edited by Krishnaiah, P.R. New York: Academic Press.

Rainville, E.D. 1960. *Special functions.* New York: The Macmillan Company.

Roy, S.N. and Gnanadesikan, R. 1959. Some contributions to anova in one or more dimensions: II. *Ann. Math. Statist.*, vol. 30, 318 - 340.

Schuurman, F.J. and Waikar, V.B. 1974. Upper percentage points of the individual roots of the complex Wishart matrix. *Sankhyā B*, vol. 36, 299 - 305.

Shaman, P. 1980. The inverted complex Wishart distribution and its application to spectral estimation. *J. Multivariate Anal.*, vol. 10, 51 - 59.

Smith, H.A. 1972. Complex distribution theory and its applications in multivariate spectral analysis. Unpublished M.Sc. thesis, University of Cape Town.

Srivastava, M.S. 1965. On the complex Wishart distribution. *Ann. Math. Statist.*, vol. 36, 312 - 315.

Steel, S.J. 1979. Aspekte van die komplekse verdelingsleer. Unpublished M.Sc. dissertation, University of Stellenbosch.

Subrahmaniam, K. 1976. Recent trends in multivariate normal distribution theory: On the zonal polynomials and other functions of matrix argument. *Sankhyā A*, vol. 38, 221 - 258.

Sugiyama, T. 1972. Distributions of the largest latent root of the multivariate complex Gaussian distribution. *Ann. Inst. Statist. Math.*, vol. 24, 87 - 94.

Tan, W.Y. 1968. Some distribution theory associated with complex Gaussian distribution. *Tamkang Journal*, vol. 7, 263 - 301.

Troskie, C.G. 1969. The generalised multiple correlation matrix. *S.Afr. Statist. J.*, vol. 3, 109 - 121.

Troskie, C.G. 1971. The distributions of some test criteria in multivariate analysis. *Ann. Math. Statist.*, vol. 42, 1752 - 1757.

Turin, G.L. 1960. The characteristic function of hermitian quadratic forms in complex normal variables. *Biometrika*, vol. 47, 199 - 201.

Underhill, L.G. 1973. Further distributions of multivariate quadratic forms. Unpublished Ph.D. thesis, University of Cape Town.

Wahba, G. 1968. On the distributions of some statistics useful in the analysis of jointly stationary time series. *Ann. Math. Statist.*, vol. 39, 1849 - 1862.

Wahba, G. 1971. Some tests of independence for stationary multivariate time series. *J. Roy Statist. Soc. B*, vol. 33, 153 - 166

Waikar, V.B. Chang, T.C. and Krishnaiah, P.R. 1972. Exact distributions of a few arbitrary roots of some complex random matrices. *Austr. J. Statist.*, vol. 14, 84 - 88.

Wooding, R.A. 1956. The multivariate distribution of complex normal variables. *Biometrika*, vol. 43, 212 - 215.

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