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Baer Sum Systems and Generalized Extensions

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CONTENTS

INTRODUCTION

A short exact sequence of R-modules $A \rightarrowtail B \twoheadrightarrow C$ is called an extension of A by C. The set of extensions of A by C, classified by a suitable equivalence relation, forms an abelian group Ext (C,A). Ext becomes a bifunctor from the category of R-modules to the category of abelian groups. Ext, and the functors Ext^n, are central in the theory of Homological Algebra ((15) or (14)).

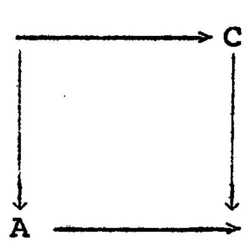
If E and E' are extensions of A by C, their sum in the abelian group Ext (C,A) is defined by the formula:

$$E + E' = \nabla(E \oplus E')\Delta \quad .$$

This composition is called the Baer Sum. Baer defined it in 1934 (1). Addition in the abelian group Hom (A,C) is also by the Baer Sum.

This thesis began with the observation that the Baer Sum may be used to define addition among diagrams other than diagrams of short exact sequences. For example, working in an abelian category $\underline{\underline{A}}$, we may define addition in the class of commutative squares fixed at A and C:

(see Chapter 3)



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In Chapter 1 we make this notion precise. We construct notation which makes possible the definition of a "Baer Sum System" in an abelian category $\underline{\underline{A}}$. Loosely, a Baer Sum System consists of:

1. The form of a diagram, which contains distinguished objects (l) and (r). In the case of Ext, this is the form of a short exact sequence

$$A \longrightarrow B \longrightarrow\!\!\!\!\rightarrow C$$

C is the distinguished object (l) and A is the distinguished object (r).

2. An equivalence relation among all diagrams fixed at (l) and (r). In the case of Ext this is the congruence relation among all extensions of A by C.

3. An induced bifunctor. The value of this bifunctor at objects A and C is the class of equivalence classes of diagrams of the selected form.

   In the case of Ext the induced bifunctor is Ext.

   Such induced bifunctors are characterized as "rich bifunctors". The values of these rich bifunctors are abelian groups. Many of the familiar bifunctors of Homology and Category Theory are shown to be rich bifunctors. The reader will recognise the proof of the converse of Proposition 1.13 as imitating proofs of the classical theory.

In Chapter 2 we show that Ext is, in particular, a "rich bifunctor" induced by a Baer Sum System. So too, are the functors Hom and Ext^n $(n > 2)$. We prove two Lemmas which simplify manipulations of the equivalence relation of a Baer Sum System.

In Chapter 3 we turn to the investigation of new Baer Sum Systems. Z is a very simple Baer Sum System, which induces the zero bifunctor. ES is another Baer Sum System, the diagrams of which are "exact squares". Once again, the induced bifunctor is the zero bifunctor. However the construction of ES is interesting in itself. Here we mention a curious fact which seems to demonstrate that the Baer Sum is central, not only to Homological Algebra, but also to other parts of mathematics. The section on exact squares is based on a paper by Hilton (7). In this paper Hilton writes that he was led to the definition of addition of "relations" (exact squares) partly "by the work of a group teaching fractions to children under the guise of 'stretchers and shrinkers'". In fact there is a strong analogy between the addition of "relations" and the addition of fractions. And we show that Hilton's addition is actually the Baer Sum.

Finally, we discover a remarkable family of Baer Sum Systems, which we call $\underline{\underline{P}}$-Hom. $\underline{\underline{P}}$ is a subcategory of an abelian category $\underline{\underline{A}}$. Substituting different "values" for $\underline{\underline{P}}$ we induce the Hom bifunctor, the zero bifunctor, and a tensor bifunctor. We obtain an interesting description of $\underline{\underline{P}}$-Hom in the case that $\underline{\underline{P}}$ is a reflective subcategory of $\underline{\underline{A}}$.

In Chapter 4 we begin the investigation of generalized extensions which we call 2Extensions. The system of 2Extensions forms a trivial Baer Sum System and so it is not as a Baer Sum System that we examine 2Extensions. Much of the interest of this chapter lies in the comparison of our theory of 2Extensions with the classical theory of extensions.

In Chapter 5 we examine the structure of 2Extensions at a greater depth, ending with the statement of a result which imitates the familiar long exact sequence:

$$\mathrm{Hom}\,(D,A) \longrightarrow \mathrm{Hom}\,(D,B) \longrightarrow \mathrm{Hom}\,(D,C) \longrightarrow$$
$$\longrightarrow \mathrm{Ext}\,(D,A) \longrightarrow \mathrm{Ext}\,(D,B) \longrightarrow \mathrm{Ext}\,(D,C) \longrightarrow$$

It seems to the author that a number of good questions about Baer Sum Systems remain to be asked. Firstly, we note that an infinite number of new Baer Sum Systems are possible, although in this thesis we have constructed only three. Baer Sum Systems seem to occupy a natural position in Homology and Category Theory. The nature of the bifunctors "P-Hom" is evidence for this. Secondly, we speculate that a formal description and construction of the satellites of the rich bifunctor induced by a Baer Sum System would be of interest. The construction would be a construction in terms of the Baer Sum System. A special case of such a description, if it is possible, would be the sequence of functors:

$$\text{Hom}\ (C,\ ),\ \text{Ext}^1\ (C,\ ),\ \text{Ext}^2\ (C,\ ),$$

Thirdly, we suggest that the analogy between the theory of ES and the theory of 2Extensions might be made formal. That is: it might be possible to describe a system, like a Baer Sum System, which, under a finer equivalence relation than that of a Baer Sum System, induced a bifunctor to the category of abelian monoids. Finally, a systematic study of the bifunctors P-Hom might be rewarding. This would entail varying the contents of the category P in a systematic way.

Throughout, except where otherwise stated, I have used the notation of (5).

All the results contained in this thesis are my original work, except where I have quoted theorems and sources. As far as I know the results are new.

I should like to thank my supervisor, Dr. K.A. Hardie, for all his assistance during the course of this project and during the time of preparation which preceded it. I am particularly grateful for the loan of a large number of mathematical papers.

I should like to thank Miss L. Jennings, of the Physics Department, who (skillfully) did the (difficult) typing.

CHAPTER ONE

BAER SUM SYSTEMS

Let $\underline{\underline{R}}$ be a unitary ring. Let A and B be R-modules. Hom, Ext, and Ext^n are group-valued bifunctors defined on the category of R-modules ((15) or (14)). In each abelian group Hom(A,B), Ext(A,B), $Ext^n(A,B)$, whether f and g are Homomorphisms, extensions or n-fold extensions, addition is defined by

$$f + g = \nabla(f \oplus g)\Delta,$$ which is the Baer Sum.

This observation provides a hint which leads to an abstract description of a "Baer Sum System" in the language of category theory. Each "Baer Sum System" will induce a group valued bifunctor (or a big group valued bifunctor). Hom, Ext and Ext^n will turn out to be bifunctors induced by "Baer Sum Systems".

In the construction that follows the reader should keep in mind "Hom" as an example of a Baer Sum System.

Throughout, $\underline{\underline{G}}$ will denote the category of abelian groups.

Let S be a small category containing two distinguished objects, (l) and (r). Let $\underline{\underline{A}}$ be an abelian category. $(S,\underline{\underline{A}})$ is the category of functors $F : S \longrightarrow \underline{\underline{A}}$.

Definition 1.1

A Pre-Baer Sum System over $\underline{\underline{A}}$ is a subcategory $\underline{\underline{B}}$ of $(S,\underline{\underline{A}})$ closed with respect to the formation of direct sums. (See Note 1.3). It will be denoted $\underline{\underline{B}} \subset (S,\underline{\underline{A}})$, and called a PBS System. In all our examples of PBS Systems $\underline{\underline{B}}$ will be a full subcategory of $(S,\underline{\underline{A}})$.

Example 1.2

Let Hom be the small category:

$$(1) \longrightarrow (r) .$$

Let $\underline{\underline{B}} = (\text{Hom},\underline{\underline{A}})$. Clearly $\underline{\underline{B}}$ is a PBS System.

Example 1.2.1

We give an example to illustrate the use of the subcategory $\underline{\underline{B}}$. In this example $\underline{\underline{B}}$ is a proper subcategory of $(S,\underline{\underline{A}})$.

Let Ext be the small category:

$$(r) \longrightarrow (i) \longrightarrow (1)$$

Let $\underline{\underline{B}} \subset (\text{Ext},\underline{\underline{A}})$ be the full subcategory of functors with values short exact sequences. Clearly (see (15) or (14)) $\underline{\underline{B}}$ is a PBS System.

Note 1.3

Direct sums exist in the category $(S,\underline{\underline{A}})$, for they may be defined "pointwise". Suppose that F_1, F_2 are functors, $F_1 : S \longrightarrow \underline{\underline{A}}$, $F_2 : S \longrightarrow \underline{\underline{A}}$. Define $F_1 \oplus F_2$ by

$$F_1 \oplus F_2(i) = F_1(i) \oplus F_2(i) \qquad (i \in S)$$

$$F_1 \oplus F_2(i \xrightarrow{\phi} j) = F_1(i) \oplus F_2(i) \xrightarrow{\begin{bmatrix} F_1(\phi) & 0 \\ 0 & F_2(\phi) \end{bmatrix}} F_1(j) \oplus F_2(j)$$

The matrix notation is explained in (5).

Definition 1.4

Given A and B, objects of $\underline{\underline{A}}$, let $S'(A,B) = \{F \mid F \in \underline{\underline{B}}, F(1) = A, F(r) = B\}$. $S'(A,B)$ is a class of functors, and not necessarily a set.

Example 1.5

$\text{Hom}'(A,B)$ is the set of functors F with values:

$$A = F(1) \xrightarrow{F(\phi)} F(r) = B .$$

We intend to place an equivalence relation on the class $S'(A,B)$. This equivalence relation imitates the congruence relation defined on $\mathrm{Ext}^n(A,B)$ (see (14)). From now on we will denote by $\underline{F}$ a particular functor $\underline{F} : S \longrightarrow \underline{\underline{A}}$. The equivalence class of which $\underline{F}$ is a representative element will be denoted F.

Definition 1.6

If $\underline{F_1} \in S'(A,B)$, and $\underline{F_2} \in S'(A',B')$, then $\underline{F_1} \xrightarrow{\alpha\eta\beta} \underline{F_2}$ will indicate that η is a natural transformation, and that $\eta(1) = \alpha$, and $\eta(r) = \beta$, where $A \xrightarrow{\alpha} A'$ and $B \xrightarrow{\beta} B'$.

Definition 1.7

If $\underline{F} \in S'(A,B)$ and $\underline{F'} \in S'(A',B')$, $\underline{F} \xrightarrow{\{\alpha\eta\beta\}} \underline{F'}$ will indicate that there is some integer k, and functors $\underline{F} = \underline{F_o}, \underline{F_1}, \ldots, \underline{F_{2k-1}}, \underline{F_{2k}} = \underline{F'}$, and natural transformations, $\eta^1, \eta^2, \ldots, \eta^{2k}$, making up the diagram:

$$\underline{F} = \underline{F_o} \xrightarrow{\eta^1} \underline{F_1} \xleftarrow{\eta^2} \underline{F_2} \cdots\cdots \underline{F_{2k-2}} \xrightarrow{\eta^{2k-1}} \underline{F_{2k-1}} \xleftarrow{\eta^{2k}} \underline{F_{2k}} = \underline{F'} .$$

(The natural transformations run alternately to the right and the left.)

Where $\eta^j = 1\eta 1$ except if

$$\eta^j = \underline{F_i} \xrightarrow{k\eta k'} \underline{F_{i+1}}$$

(Note the direction of $k\eta k'$) and the composition of all such ks is α, and the composition of all such k's is β. $\{\alpha\eta\beta\}$ will sometimes be called a natural transformation.

Definition 1.8

We define an equivalence relation on $S'(A,B)$ by:

$F_1 \equiv F_2$ if and only if

$$\exists\, \underline{F_1} \xrightarrow{\{1\eta 1\}} \underline{F_2} \quad \text{and} \quad \exists\, \underline{F_2} \xrightarrow{\{1\eta 1\}} \underline{F_1}\,.$$

The class of equivalence classes of $S'(A,B)$ will be denoted $S(A,B)$.

<u>Note</u>: $\underline{F_1} \xrightarrow{\alpha\eta\beta} \underline{F_2}$ implies that

$\underline{F_1} \xrightarrow{\{\alpha\eta\beta\}} \underline{F_2}$ but the converse does not hold.

<u>Note 1.9</u>

If $\underline{F} \xrightarrow{\{\alpha\eta\beta\}} \underline{F'}$, and $\underline{F} \equiv \underline{F_1}$, and $\underline{F'} \equiv \underline{F'_1}$, then $\underline{F_1} \xrightarrow{\{\alpha\eta\beta\}} \underline{F'_1}$. From now on we denote by F, the congruence class of $\underline{F}$. We may then define the notion:

$$F \xrightarrow{\{\alpha\eta\beta\}} F' \text{ by}$$

$F \xrightarrow{\{\alpha\eta\beta\}} F'$ if and only if $\underline{F} \xrightarrow{\{\alpha\eta\beta\}} \underline{F'}$. This is clearly well-defined.

<u>Example 1.10</u>

We show later that congruence in $\mathrm{Hom}'(A,B)$ is just identity, so that $\mathrm{Hom}'(A,B) = \mathrm{Hom}(A,B)$.

S is a (generalized) function defined upon $\underline{\underline{A}} \times \underline{\underline{A}}$, with values the classes $S(A,B)$. We shall be interested in special Pre Baer Sum Systems. In these, certain conditions will make possible the definition of $S(\alpha,B)$ and $S(A,\beta)$, given morphisms $A \xrightarrow{\alpha} A'$, and $B \xrightarrow{\beta} B'$ of $\underline{\underline{A}}$, in order to turn S into a bifunctor.

A counter example, illustrating a method which fails to make S a bifunctor, may be helpful. If $\underline{\underline{B}}<(\mathrm{Ext},\underline{\underline{A}})$ is the PBS System defined in 1.2.1, where $S = \mathrm{Ext}$, then $S'(A,B)$ is the class of functors with values short exact sequences of the form:

$$B \longrightarrow C \longrightarrow A\,.$$

S(A,B) is a class of equivalence classes of such short exact sequences (see 1.8). If $A \xrightarrow{\alpha} A'$ is a morphism of $\underline{\underline{A}}$, it seems we may not make S a bifunctor covariant in the first and contravariant in the second variable. For, there is no construction, in general, which allows us to fill in the object C' in the diagram which follows, in which both rows are short exact sequences:

$$\begin{array}{ccccc} B & \longrightarrow & C & \longrightarrow & A \\ {\scriptstyle 1}\downarrow & & \downarrow & & \downarrow{\scriptstyle \alpha} \\ B & \longrightarrow & C' & \longrightarrow & A' \end{array}$$

This is because, in the usual notation, ((15) or (14)), $\text{Ext}(\alpha,B)$ is not in general an epimorphism. Hence we are unable to define $S(\alpha,B)$.

We may, however, make S a bifunctor contravariant in the first and covariant in the second variable. This construction is made in Chapter 2, and S becomes the bifunctor Ext.

The reader will recall that classically, the construction of the bifunctor Ext is achieved by defining, where E is an extension of A by B, and $A \xrightarrow{\alpha} A'$, $B' \xrightarrow{\beta} B$ are morphisms of $\underline{\underline{A}}$, the induced extensions αE and $E\beta$. Further we recall that if $E_1 \xrightarrow{(\alpha,\cdot,\beta)} E_1'$ is a morphism of extensions, we obtain a congruence $\alpha E \equiv E'_{\beta}$. For the moment let us call these constructions, and the congruence, the "three conditions".

In Proposition 1.13 we show, in the abstract setting of a Pre Baer Sum System, that a generalization of these "three conditions" is sufficient to turn S into a bifunctor with (object) values the classes S(A,B). Moreover, in such cases, we discover that S has an additional property, which we shall isolate and define. This is the property of "richness". And in Proposition 1.13 we show, conversely, that the functorial property and the property of "richness" imply the "three conditions".

An additional consequence of the "three conditions" is that S(A,B) is a (big) abelian group.

We make a note on foundational problems. In general, S(A,B) may not be a set. If S(A,B) is not a set, then, in the cases where S(A,B) is equipped with the structure of an abelian group, we do not call S(A,B) an abelian group. Instead, S(A,B) becomes a big abelian group. A big abelian group is defined in the same way as an abelian group, except that the underlying class need not be a set. The "category of big abelian groups" cannot exist, however, because the class of morphisms between a given pair of big groups may not be a set. However we use the language of functors, and say that S is a big abelian group valued bifunctor. In such a case, and in similar cases, we shall sometimes abuse the language by saying that S defines a bifunctor

$S : \underline{\underline{A}} \times \underline{\underline{A}} \longrightarrow \underline{\underline{G}}$. In general, our discussion of S is not inhibited at all by these problems. Our conventions are standard practice, and we refer the reader to (14).

Our exposition will be simplified by the following definition. It seems valuable, too, to isolate the property of "richness".

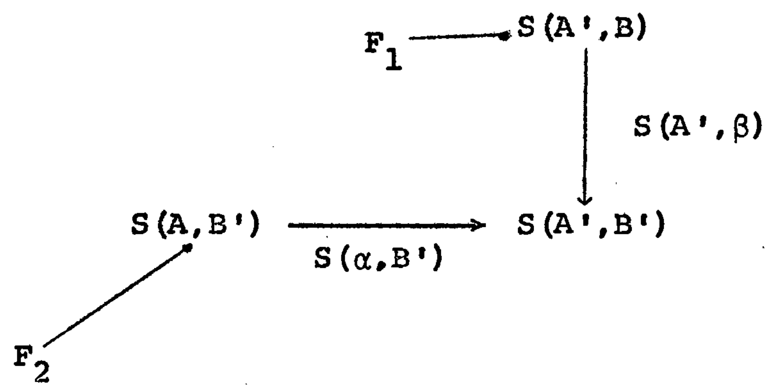
<u>Definition 1.11</u>

Let $\underline{\underline{B}} \langle (S, \underline{\underline{A}})$ be a PBS System over an abelian category $\underline{\underline{A}}$. Then S is a (generalized) function, defined upon $\underline{\underline{A}} \times \underline{\underline{A}}$, with values the classes S(A,B).

If S, in addition, can be made a bifunctor, S will be called a <u>rich bifunctor</u>, relative to $\underline{\underline{B}}$, if

1. S is contravariant in the first variable and covariant in the second variable.

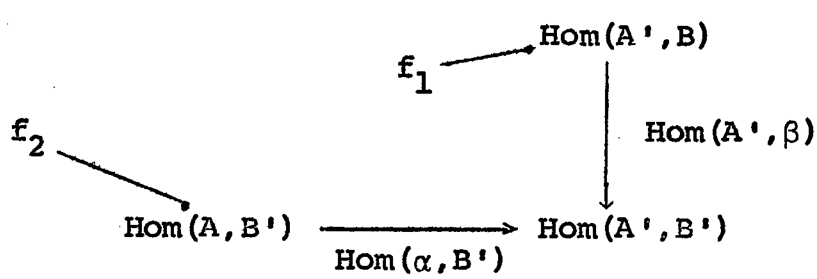
2. If $A' \xrightarrow{\alpha} A$ and $B \xrightarrow{\beta} B'$ are morphisms of $\underline{\underline{A}}$. then in the diagram:



where $F_1 \in S(A',B)$, $F_2 \in S(A,B')$,
$S(A',\beta)(F_1) = S(\alpha,B')(F_2)$ if and only if $\exists F_1 \xrightarrow{\{\alpha\eta\beta\}} F_2$.

<u>Example 1.12</u>

Hom is a rich bifunctor relative to the PBS System $\underline{\underline{B}} = (\mathrm{Hom}, \underline{\underline{A}})$ that we have already defined. We identify a functor $F : \mathrm{Hom} \longrightarrow \underline{\underline{A}}$ with the morphism $F(\phi)$. Clearly Hom satisfies property 1. (Hom is, in addition, additive in both variables.) Further, Hom satisfies property 2. If $A' \xrightarrow{\alpha} A$ and $B \xrightarrow{\beta} B'$ are morphisms of $\underline{\underline{A}}$, we have:



If $A' \xrightarrow{f_1} B$, and $A \xrightarrow{f_2} B'$, then
$\mathrm{Hom}(A',\beta)(f_1) = \mathrm{Hom}(\alpha,B')(f_2)$ if and only if

$A' \xrightarrow{f_1} B \xrightarrow{\beta} B' = A' \xrightarrow{\alpha} A \xrightarrow{f_2} B'$ if and only if

$$\begin{array}{ccc} A' & \xrightarrow{f_1} & B \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ A & \xrightarrow[f_2]{} & B' \end{array}$$

commutes,

if and only if $\exists\, f_1 \xrightarrow{\alpha\eta\beta} f_2$.

The proof that $\exists f_1 \xrightarrow{\alpha\eta\beta} f_2$ if and only if $\exists\, f_1 \xrightarrow{\{\alpha\eta\beta\}} f_2$ is not difficult, and, in any case, is given in Chapter 2.

Proposition 1.13

Let $\underline{\underline{B}}<(S,\underline{\underline{A}})$ be a PBS System over an abelian category $\underline{\underline{A}}$. Then S is a (generalized) function defined upon $\underline{\underline{A}} \times \underline{\underline{A}}$, with values the classes $S(A,B)$.

S gives rise to a rich (big) abelian group valued bifunctor, which is additive in both variables, if and only if:

1. For every $A' \xrightarrow{\alpha} A$ in $\underline{\underline{A}}$, and every $F \in S(A,B)$, $\exists F\alpha \in S(A',B)$ and a natural transformation $F\alpha \xrightarrow{\{\alpha\eta 1\}} F$.

2. For every $B \xrightarrow{\beta} B'$ in $\underline{\underline{A}}$, and every $F \in S(A,B)$, $\exists \beta F \in S(A,B')$ and a natural transformation $F \xrightarrow{\{1\eta\beta\}} \beta F$.

3. Given $F_1 \xrightarrow{\{\alpha\eta\beta\}} F_2$, where $F_1 \in S(A',B)$ and $F_2 \in S(A,B')$, then $\beta F_1 \equiv F_2\alpha$.

<u>Proof</u>

<u>Three Preliminary Notes</u>

(a) Given 1, 2 and 3, we may show that $1F \equiv F$, and $F1 \equiv F$. For we have natural transformations $F1 \xrightarrow{\{1\eta 1\}} F$ and $F \xrightarrow{\{1\eta 1\}} 1F$ by 1 and 2. By 3 the natural transformation $F \xrightarrow{\{1\eta 1\}} F$ yields $1F \equiv F1$. This congruence implies the existence of natural transformations $1F \xrightarrow{\{1\eta 1\}} F1$ and $F1 \xrightarrow{\{1\eta 1\}} 1F$. Hence there exists $F \xrightarrow{\{1\eta 1\}} 1F \xrightarrow{\{1\eta 1\}} F1$, which is $F \xrightarrow{\{1\eta 1\}} F1$, so that $F \equiv F1$. Similarly $1F \equiv F$.

(b) Given 1, 2 and 3, $F\alpha$ in 1 and βF in 2 are unique with respect to the existence of the natural transformations $\{\alpha\eta 1\}$ and $\{1\eta\beta\}$. For if F_* is such that there is $F_* \xrightarrow{\{\alpha\eta 1\}} F$, then $F_* \equiv 1F_* \equiv F\alpha$ by 3 and the note above. Similarly βF is unique.

(c) In "either direction" of the proof, we may simplify the equivalence relation defined in 1.8. In fact $F_1 \equiv F_2$ if and only if $\exists F_1 \xrightarrow{\{1\eta 1\}} F_2$ <u>or</u> $\exists F_2 \xrightarrow{\{1\eta 1\}} F_1$. For, suppose S is a rich bifunctor, then consider the diagram:

$$\begin{array}{ccc} & & S(A,B) \\ & & \downarrow \; S(1,B) \\ S(A,B) & \xrightarrow{S(A,1)} & S(A,B) \end{array}$$

Clearly, if $F_1, F_2 \in S(A,B)$

$F_1 \equiv F_2$ if and only if $S(A,1)(F_1) = S(1,B)(F_2)$

if and only if $\exists F_1 \xrightarrow{\{1\eta 1\}} F_2$ <u>or</u>

$\exists F_2 \xrightarrow{\{1\eta 1\}} F_1$.

Conversely, suppose 1, 2 and 3. Then $F_1 \xrightarrow{\{1\eta 1\}} F_2$ implies that $1F_1 \equiv F_2 1$ by 3, which implies that $F_1 \equiv F_2$.

Throughout the proof, and in subsequent work, we use this simplified version of the equivalence relation.

<u>Now</u> Suppose $S : \underline{\underline{A}} \times \underline{\underline{A}} \longrightarrow \underline{\underline{G}}$ is a rich bifunctor.

1. Given $A' \xrightarrow{\alpha} A$ in $\underline{\underline{A}}$, we obtain $S(A,B) \xrightarrow{S(\alpha,B)} S(A',B)$. Define, for any $F \in S(A,B)$, $F\alpha = S(\alpha,B)(F)$. We have:

$$\begin{array}{ccc} & & S(A',B) \leftarrow F\alpha \\ & & \Big\downarrow S(1,1) \\ F \rightarrow S(A,B) & \xrightarrow[S(\alpha,B)]{} & S(A',B) \end{array}$$

so that $\exists\, F\alpha \xrightarrow{\{\alpha\eta 1\}} F$.

2. Given $B \xrightarrow{\beta} B'$ in $\underline{\underline{A}}$, we obtain $S(A,B) \xrightarrow{S(A,\beta)} S(A,B')$. Define, for any $F \in S(A,B)$, $S(A,\beta)(F) = \beta F$. We have:

$$\begin{array}{ccc} & & F \rightarrow S(A,B) \\ & & \Big\downarrow S(A,\beta) \\ \beta F \rightarrow S(A,B') & \xrightarrow[S(1,1)]{} & S(A,B') \end{array}$$

so that $\exists\, F \xrightarrow{\{1\eta\beta\}} \beta F$.

Finally,

3. Given $A' \xrightarrow{\alpha} A$ and $B \xrightarrow{\beta} B'$ we obtain

$$
\begin{array}{ccc}
 & & S(A,B') \leftarrow F_2 \\
 & & \big\downarrow S(\alpha,1) \\
F_1 \nearrow S(A',B) & \xrightarrow[S(1,\beta)]{} & S(A',B')
\end{array}
$$

By definition, if $F_1 \in S(A',B)$, and $F_2 \in S(A,B')$, and there is a natural transformation

$$F_1 \xrightarrow{\{\alpha\eta\beta\}} F_2\,,\ \text{then } \beta F_1 \equiv F_2\alpha.$$

<u>Conversely</u>

Suppose 1, 2 and 3. Our proof has five sections.

<u>Section I</u>

For any $F \in S(A,B)$

(i) $1F \equiv F$ (see the preliminary note (a))

(i)* $F \equiv F1.$

(ii) $(\beta'\beta)F \equiv \beta'(\beta F)$ where $B \xrightarrow{\beta} B' \xrightarrow{\beta'} B''.$

By definition

$$\exists\ F \xrightarrow{\{1\eta\beta'\beta\}} (\beta'\beta)F\ , \text{ and}$$

$$\exists\ F \xrightarrow{\{1\eta\beta\}} \beta F \xrightarrow{\{1\eta\beta'\}} \beta'(\beta F),$$

providing $F \xrightarrow{\{1\eta\beta'\beta\}} \beta'(\beta F)$, so that, by uniqueness, $(\beta'\beta)F \equiv \beta'(\beta F)$.

(ii)* $F(\alpha\alpha') \equiv (F\alpha)\alpha'$, dually.

(iii) Given $A' \xrightarrow{\alpha} A$, and $B \xrightarrow{\beta} B'$, and $F \in S(A,B)$, there are natural transformations $F\alpha \xrightarrow{\{\alpha\eta 1\}} F \xrightarrow{\{1\eta\beta\}} \beta F$, providing $F\alpha \xrightarrow{\{\alpha\eta\beta\}} \beta F$, whence $(\beta F)\alpha \equiv \beta(F\alpha)$.

Section II

Δ_A, or simply Δ, will denote the morphism $(1,1) = \Delta : A \longrightarrow A\oplus A$. ∇_A, or ∇, will denote the morphism $\begin{bmatrix}1\\1\end{bmatrix} = \nabla : A\oplus A \longrightarrow A$, in the notation of (5).

If F_1, F_2 are elements of $S(A,B)$, containing representative elements $\underline{F}_1$, $\underline{F}_2$, we may validly define $F_1\oplus F_2$ to be the congruence class in the equation:

$$F_1\oplus F_2 = \{\underline{E}_1\oplus\underline{E}_2 \mid \underline{E}_1\equiv\underline{F}_1 \text{ and } \underline{E}_2\equiv\underline{F}_2\}.$$

For, given $\underline{E}_1 \equiv \underline{F}_1$ and $\underline{E}_2 \equiv \underline{F}_2$, there are finite sequences of natural transformations:

$$\underline{E}_1 \xrightarrow{1\eta 1} \cdots\cdots\cdots \xleftarrow{1\eta 1} \underline{F}_1$$

$$\underline{E}_2 \xrightarrow{1\eta' 1} \cdots\cdots\cdots \xleftarrow{1\eta' 1} \underline{F}_2 .$$

A direct sum of natural transformations is a natural transformation. We have a finite sequence of natural transformations:

$$\underline{E}_1\oplus\underline{E}_2 \xrightarrow{1\eta 1\oplus 1\eta' 1} \cdots\cdots \xleftarrow{1\eta 1\oplus 1\eta' 1} \underline{F}_1\oplus\underline{F}_2 \text{ , providing}$$

$$\underline{E}_1\oplus\underline{E}_2 \xrightarrow{\{1\eta 1\}} \underline{F}_1\oplus\underline{F}_2 .$$ Our result follows.

We define addition on the congruence classes of $S(A,B)$ as follows: If F_1, $F_2 \in S(A,B)$

$$F_1 + F_2 = \nabla_B(F_1\oplus F_2)\Delta_A$$

We may prove:

(i) If $D \xrightarrow{\alpha} A$, $D' \xrightarrow{\alpha'} A'$, are morphisms of $\underline{\underline{A}}$:

$$(\alpha\oplus\alpha')(F\oplus F') \equiv \alpha F\oplus\alpha' F'.$$

Let $\underline{\alpha F}$, $\underline{\alpha' F'}$, $\underline{F}$, $\underline{F'}$ be representative elements of αF, $\alpha' F'$, F, F', respectively.

Then $\exists\ \alpha F \xrightarrow{\{\alpha\eta 1\}} F$ and $\alpha' F' \xrightarrow{\{\alpha'\eta 1\}} F'$ if and only if $\exists\ \underline{\alpha F} \xrightarrow{\{\alpha\eta 1\}} \underline{F}$ and $\underline{\alpha' F'} \xrightarrow{\{\alpha'\eta 1\}} \underline{F'}$ if and only if $\exists\ \underline{\alpha F} \oplus \underline{\alpha' F'} \xrightarrow{\{\alpha\eta 1\oplus\ \alpha'\eta 1\}} \underline{F} \oplus \underline{F'}$ and, by the observation above, this follows if and only if $\exists\ \alpha F \oplus \alpha' F' \xrightarrow{\{\alpha\eta 1\ \oplus\ \alpha'\eta 1\}} F\oplus F'$ which implies that $\exists\ \alpha F \oplus \alpha' F' \xrightarrow{\{\alpha\oplus\alpha'\eta 1\}} F\oplus F'$ whence, by the uniqueness assertion in 1,

$$\alpha F\oplus\alpha' F' \equiv (\alpha\oplus\alpha')(F\oplus F').$$

(i)* Given $B \xrightarrow{\beta} D$ and $B' \xrightarrow{\beta'} D'$, $F \in S(A,B)$, $F' \in S(A',B')$, $(F\oplus F')(\beta\oplus\beta') \equiv (F\beta\oplus F'\beta')$, dually.

(ii) $(\alpha+\alpha')F \equiv \alpha F + \alpha' F$.

<u>Proof</u>

Consider any $F \in S(A,B)$ and any $\underline{F}$.

There is a natural transformation $\underline{F} \xrightarrow{\Delta\eta\Delta} \underline{F\oplus F}$, defined by $\Delta\eta\Delta(i) = \Delta_{F(i)}$, since, given $i \xrightarrow{\phi} j$, in S:

$$\begin{array}{ccc}
F(i) & \xrightarrow{F\phi} & F(j) \\
\Big\downarrow \Delta_{F(i)} & & \Big\downarrow \Delta_{F(j)} \\
F(i)\oplus F(i) & \xrightarrow[\begin{bmatrix} F(\phi) & 0 \\ 0 & F(\phi)\end{bmatrix}]{} & F(j)\oplus F(j)
\end{array}$$

commutes.

Hence, for any F, we obtain, keeping in mind the observation at the beginning of this section:

$$F \xrightarrow{\{\Delta\eta\Delta\}} F\oplus F \text{ , and, by 3,}$$

$$\Delta F \equiv (F\oplus F)\Delta$$

Then $(\alpha+\alpha')F \equiv (\nabla(\alpha\oplus\alpha')\Delta F)$

$\equiv \nabla(\alpha\oplus\alpha')(F\oplus F)\Delta$

$\equiv \nabla(\alpha F\oplus\alpha' F)\Delta$ (Section II, part (i))

$\equiv \alpha F + \alpha' F$

(ii)* Dually, $F(\gamma + \gamma') \equiv F\gamma + F\gamma'$.

<u>Section III</u>

S(A,B) is an abelian group, or, possibly, a big abelian group.

We note that, given F, F', F" ∈ S(A,B), and $\underline{F}$, $\underline{F'}$, $\underline{F''}$, there is a natural transformation

$$\underline{F} \oplus (\underline{F'}\oplus\underline{F''}) \xrightarrow{1\eta 1} (\underline{F}\oplus\underline{F'}) \oplus \underline{F''}$$

giving: $F \oplus (F'\oplus F'') \xrightarrow{\{1\eta 1\}} (F\oplus F') \oplus F''$, which implies that

$F \oplus (F'\oplus F'') \equiv (F\oplus F') \oplus F''$.

Then: $F + (F'+F'') = F + \nabla(F'\oplus F'')\Delta$

$= \nabla(F\oplus\nabla(F'\oplus F'')\Delta)\Delta$

$= \nabla(1\oplus\nabla(F\oplus(F'\oplus F''))1\oplus\Delta)\Delta$

$= \nabla(1\oplus\nabla((F\oplus F')\oplus F''))1\oplus\Delta)\Delta$

$= (F+F') + F''$.

+ is associative.

Let $\tau_{A\oplus A'}$, or simply τ, be the "twisting isomorphism" $\tau_A : A\oplus A' \longrightarrow A'\oplus A$, where $\tau_A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, in the notation of (5). Given F, F' ∈ S(A,B), and hence $\underline{F}$ and $\underline{F'}$, there is a natural transformation $\underline{F} \oplus \underline{F'} \xrightarrow{\tau\eta\tau} \underline{F'} \oplus \underline{F}$, defined by

$\tau\eta\tau(i) = \tau_{\underline{F}(i)\oplus\underline{F}'(i)}$, since, given $i \xrightarrow{\phi} j \in S$:

$$\begin{array}{ccc}
\underline{F}(i) \oplus \underline{F}'(i) & \xrightarrow{\tau_{F(i)\oplus F'(i)}} & \underline{F}'(i) \oplus \underline{F}(i) \\
\Big\downarrow {\scriptstyle \underline{F}(\phi)\oplus\underline{F}'(\phi)} & & \Big\downarrow {\scriptstyle \underline{F}'(\phi)\oplus\underline{F}(\phi)} \\
\underline{F}(j) \oplus \underline{F}'(j) & \xrightarrow[\tau_{F(j)\oplus F'(j)}]{} & \underline{F}'(j) \oplus \underline{F}(j)
\end{array}$$

commutes.

We have, then: $F\oplus F' \xrightarrow{\{\tau\eta\tau\}} F'\oplus F$

so that: $\tau(F\oplus F') \equiv (F'\oplus F)\tau$

Then $F+F' \equiv \nabla(F\oplus F')\Delta \equiv \nabla\tau(F\oplus F')\Delta \equiv \nabla(F'\oplus F)\tau\Delta = \nabla(F'\oplus F)\Delta$

$+$ is commutative.

The Zero

Given any $F, F' \in S(A,B)$ and hence $\underline{F}$, $\underline{F}'$, there is a natural transformation $\underline{F} \xrightarrow{o\eta o} \underline{F}'$ defined by $o\eta o(i) = o$. And so we have $F \xrightarrow{\{o\eta o\}} F'$, and hence $oF \equiv F'o$.

Then $oF \equiv F'o \equiv oF' \equiv Fo$, and this congruence class becomes the zero, for $F + Fo = F1 + Fo = F(1+o) = F1 = F$.

Inverses $(-1)F$ is the inverse of F.

Section IV

$S(A,B) : \underline{\underline{A}} \times \underline{\underline{A}} \longrightarrow \underline{\underline{G}}$ is a bifunctor, contravariant in A, covariant in B. For, given $A' \xrightarrow{\alpha} A$, define

$$S(A,B) \xrightarrow[S(\alpha,B)]{} S(A',B)$$

by $S(\alpha,B)(F) = F\alpha$. This is well-defined and †$(F+F')\alpha = F\alpha+F'\alpha$, so that $S(\alpha,B)$ is a group homomorphism. Further $(S\beta\alpha,1) = S(\beta,1)S(\alpha,1)$ by (ii)* Section I, and $S(1,B) = 1_{S(A,B)}$.

†(This follows from $\Delta\alpha = (\alpha \oplus \alpha)\Delta$)

Dually, Define, given $B \xrightarrow{\beta} B'$, $S(A,B) \xrightarrow[S(A,\beta)]{} S(A,B')$,

by $S(A,\beta)(F) = \beta F$.

Then

$$\begin{array}{ccc} S(A,B) & \xrightarrow{S(\alpha,B)} & S(A',B) \\ {\scriptstyle S(A,\beta)}\downarrow & & \downarrow{\scriptstyle S(A',\beta)} \\ S(A,B') & \xrightarrow{S(\alpha,B')} & S(A',B') \end{array}$$ commutes,

Since:

Clockwise: $F \longrightarrow F\alpha \longrightarrow \beta(F\alpha)$

Anticlockwise: $F \longrightarrow \beta F \longrightarrow (\beta F)\alpha$.

But $\beta(F\alpha) \equiv (\beta F)\alpha$, by (iii), Section I.

Additivity

$S(\alpha_1+\alpha_2,B)(F) = F(\alpha_1+\alpha_2) = F\alpha_1+F\alpha_2$ (by (ii)* of Section II) $= S(\alpha_1,B)(F) + S(\alpha_2,B)(F)$. Hence S is additive in the first variable. Dually S is additive in the second variable.

Section V

$S : \underline{\underline{A}} \times \underline{\underline{A}} \longrightarrow \underline{\underline{G}}$ is rich.

Suppose $A' \xrightarrow{\alpha} A$ and $B \xrightarrow{\beta} B'$ and then

$$\begin{array}{cccc} & & & S(A,B') \longleftarrow F_2 \\ & & & \downarrow{\scriptstyle S(\alpha,1)} \\ F_1 \longrightarrow & S(A',B) & \xrightarrow[S(1,\beta)]{} & S(A',B') \end{array}$$

Suppose $F_1 \in S(A',B)$, $F_2 \in S(A,B')$

Then $S(1,\beta)F_1 = S(\alpha,1)F_2$

if and only if $\beta F_1 \equiv F_2\alpha$.

then $\exists$ natural transformations

$$F_1 \xrightarrow{\{1\eta\beta\}} \beta F_1 \xrightarrow{\{1\eta 1\}} F_2\alpha \xrightarrow{\{\alpha\eta 1\}} F_2,$$ so that

$$\exists\ F_1 \xrightarrow{\{\alpha\eta\beta\}} F_2 \ .$$

Conversely

Given $F_1 \xrightarrow{\{\alpha\eta\beta\}} F_2$, by property 3 $\beta F_1 \equiv F_2\alpha$.

Definition 1.14

A PBS System $\underline{\underline{B}} \subset (S,\underline{\underline{A}})$ over an abelian category $\underline{\underline{A}}$ is a Baer Sum System if, and only if, it satisfies the conditions of Proposition 1.13.

We refer to a Baer Sum System by the name of the induced bifunctor.

Duality 1.15

In the notation of (5), if $\underline{\underline{B}} \subset (S,\underline{\underline{A}})$ is a Baer Sum System over an abelian category $\underline{\underline{A}}$, then $\underline{\underline{B}}^* \subset (S^*,\underline{\underline{A}}^*)$ is a Baer Sum System over $\underline{\underline{A}}^*$.

Proof

Clearly $\underline{\underline{B}}^*$ is a PBS System. $\underline{\underline{B}}^*$ is a Baer Sum System, because properties 1, 2, 3 of Proposition 1.13, are together self-dual. That is $1^* = 2$, $2^* = 1$, $3^* = 3$.

Note 1.16

In subsequent work in Baer Sum Systems we use the simplified definition of the equivalence relation, which is:

$$F_1 \equiv F_2 \text{ if and only if } \exists\ F_1 \xrightarrow{\{1\eta 1\}} F_2 \ \underline{\text{or}}$$
$$\exists\ F_2 \xrightarrow{\{1\eta 1\}} F_1 \ .$$

(See the preliminary note to Proposition 1.13.)

CHAPTER TWO

HOM AND EXT^n

We show that Hom and Ext^n are rich bifunctors induced by Baer Sum Systems. Two Lemmas will simplify the construction of Baer Sum Systems.

Lemma 2.1

In constructing Baer Sum Systems, in order to verify condition 3 of Proposition 1.13, i.e. that $F \xrightarrow{\{\alpha\eta\beta\}} F'$ implies that $\beta F \equiv F'\alpha$, it is sufficient to show that, if $\underline{F}$, $\underline{F'}$ are particular functors, and representative elements of F, F', then

$$\underline{F} \xrightarrow{\alpha\eta\beta} \underline{F'} \text{ implies that } \beta F \equiv F'\alpha$$

Proof

Suppose $\exists\, F \xrightarrow{\{\alpha\eta\beta\}} F'$, then there are finite number of natural transformations: (See Definition 1.7)

$$\underline{F} = \underline{F}_o \longrightarrow \underline{F}_1 \cdots \quad \cdots \underline{F}_{2k-1} \longleftarrow \underline{F}_{2k} = \underline{F'}$$

Suppose that $\underline{F}_{\alpha 1} \xrightarrow{K_1\eta K_1'} \underline{F}_{\alpha 2}, \ldots\ldots, \underline{F}_{\alpha n-1} \xrightarrow{K_{n-1}\eta K_{n-1}'} \underline{F}_{\alpha n}$, are the natural transformations, which are such that $K_i \neq 1$ or $K_i' \neq 1$. By our assumption, we have, reading from left to right:

$$F \equiv F_o \equiv F_1 \cdots \equiv F_{\alpha 1}, \text{ (and } F_{\alpha n} \equiv \cdots \equiv F_{2k} \equiv F'.)$$

$$K_1'F_{\alpha 1} \equiv F_{\alpha 2}K_1 \text{ and } F_{\alpha 2} \equiv F_{\alpha 3}$$

$$K_3'F_{\alpha 3} \equiv F_{\alpha 4}K_3 \text{ and } F_{\alpha 4} \equiv F_{\alpha 5},$$

which implies that $K_3'K_1'F_{\alpha 1} \equiv F_{\alpha 4}K_3K_1$.

Continuing in this way, we obtain

$$K_{n-1}'K_{n-2}' \cdots\cdot K_3'K_1'F_{\alpha 1} \equiv F_{\alpha n}K_{n-1}K_{n-2} \cdots K_3K_1$$

whence $\beta F_{\alpha 1} \equiv F_{\alpha n}\alpha$

and since $F_{\alpha 1} \equiv F$ and $F_{\alpha n} \equiv F'$

$\beta F \equiv F'\alpha.$

Lemma 2.2

In constructing Baer Sum Systems, in order to verify condition 1 of Proposition 1.13, it is sufficient to show that for every $A' \xrightarrow{\alpha} A$ of $\underline{\underline{A}}$, and every particular functor $\underline{F} \in S'(A,B)$, $\exists$ $\underline{F}\alpha \in S'(A',B)$ and $\underline{F}\alpha \xrightarrow{\alpha\eta 1} \underline{F}$.

Proof

For any equivalence class $F \in S(A,B)$, we define $F\alpha$ to be the equivalence class of $\underline{F}\alpha$, where $\underline{F}$ is any representative of F.

$F\alpha$ is well-defined: for, suppose $\underline{F}$ and $\underline{F*}$ are both representatives of F. By our assumption we have

$\underline{F}\alpha \xrightarrow{\alpha\eta 1} \underline{F}$ and

$\underline{F*}\alpha \xrightarrow{\alpha\eta 1} \underline{F}$.

If $\{\underline{F}\alpha\}$ and $\{\underline{F*}\alpha\}$ denote the equivalence classes of $\underline{F}\alpha$ and $\underline{F*}\alpha$, respectively, we have

$\{\underline{F}\alpha\} \xrightarrow{\{\alpha\eta 1\}} F$ and

$\{\underline{F*}\alpha\} \xrightarrow{\{\alpha\eta 1\}} F.$

By the uniqueness assertion contained in 3 of Proposition 1.13, $F\alpha = \{\underline{F}\alpha\} = \{\underline{F*}\alpha\}$.

Note

Clearly the dual of this Lemma applies to condition 2 of Proposition 1.13.

Proposition 2.3

Let $\underline{\underline{A}}$ be an abelian category. Hom is a rich bifunctor induced by a Baer Sum System.

Proof

We have already defined the small category Hom:

$$(1) \xrightarrow{\phi} (r)$$

We have seen that $\underline{\underline{B}} = (\text{Hom}, \underline{\underline{A}})$ is a PBS System.

Equivalence in this PBS System is identity; for, suppose, that $\underline{F}$, $\underline{F}'$ are particular functors $\underline{F}, \underline{F}' : \text{Hom} \longrightarrow \underline{\underline{A}}$. Then, if $\exists\ \underline{F} \xrightarrow{1\eta 1} \underline{F}'$, we have a commutative diagram

$$\begin{array}{ccc} \underline{F}(1) & \xrightarrow{\underline{F}(\phi)} & \underline{F}(r) \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle 1} \\ \underline{F}'(1) & \xrightarrow[\underline{F}'(\phi)]{} & \underline{F}'(r) \end{array}$$, whence

clearly $\underline{F} = \underline{F}'$.

Suppose, then, that $\exists\ \underline{F} \xrightarrow{\{1\eta 1\}} \underline{F}'$, then there are a finite number of natural transformations:

$$\underline{F} = \underline{F_o} \longrightarrow \underline{F_1} \ \dots\ \underline{F_{2k-1}} \longleftarrow \underline{F_{2k}} = \underline{F}'.$$

Suppose $\underline{F_{\alpha 1}} \xrightarrow{K_1 \eta K_1'} \underline{F_{\alpha 2}}, \ \dots,\ \underline{F_{\alpha n-1}} \xrightarrow{K_{n-1} \eta K_n'} \underline{F_{\alpha n}}$, are the exceptional natural transformations, in order, which are such that $K_i \neq 1$ or $K_i' \neq 1$. In the following paragraph we identify a functor $\underline{F}$ with the morphism $\underline{F}(\phi)$. The compositions are compositions of morphisms. (The reader should draw a diagram!)

$$\underline{F} = \underline{F_{\alpha 1}} \text{ and } \underline{F_{\alpha n}} = \underline{F}'$$

$$K_1' \underline{F_{\alpha 1}} = \underline{F_{\alpha 2}} K_1 \text{ and } \underline{F_{\alpha 2}} = \underline{F_{\alpha 3}}$$

$$K_3' \underline{F_{\alpha 3}} = \underline{F_{\alpha 4}} K_4 \text{ and } \underline{F_{\alpha 4}} = \underline{F_{\alpha 5}}$$

which implies that $K_3'K_1'\underline{F_{\alpha 1}} = \underline{F_{\alpha 4}}K_4K_1$.

Continuing in this way we obtain

$$K_{n-1}'K_{n-2}' \cdots K_3'K_1'\underline{F_{\alpha 1}} = \underline{F_{\alpha n}}K_{n-1}K_{n-2} \cdots K_3K_1$$

whence $\underline{F_{\alpha 1}} = \underline{F_{\alpha n}}$

whence $\underline{F} = \underline{F'}$

<u>Conditions</u> 1, 2, 3 of Proposition 1.13 are fulfilled.

1. Given $A' \xrightarrow{\alpha} A$ and $F \in \mathrm{Hom}(A,B)$:

$$\begin{array}{l} A' \\ \downarrow \alpha \\ A \xrightarrow{F(\phi)} B. \end{array}$$

Define $F\alpha : \mathrm{Hom} \longrightarrow \underline{\underline{A}}$

by $F\alpha(1) = A'$, $F\alpha(r) = B$, $F\alpha(\phi) = F(\phi).\alpha$.

Clearly $\exists\, F\alpha \xrightarrow{\{\alpha\eta 1\}} F$.

2. Given $B \xrightarrow{\beta} B'$ and $F \in \mathrm{Hom}(A,B)$,

$$\begin{array}{lcl} A & \xrightarrow{F(\phi)} & B \\ & & \downarrow \beta \\ & & B' \, . \end{array}$$

Define $\beta F : \mathrm{Hom} \longrightarrow \underline{\underline{A}}$, by $\beta F(1) = A$, $\beta F(r) = B'$,

$(\beta F)(\phi) = \beta(F(\phi))$.

Clearly $\exists\, F \xrightarrow{\{1\eta\beta\}} \beta F$.

3. Given $F \xrightarrow{\alpha\eta\beta} F'$, clearly $\beta F = F'\alpha$.

The result follows by Lemma 2.1.

Proposition 2.4

Let $\underline{\underline{A}}$ be an abelian category. Ext^n is a rich bifunctor induced by a Baer Sum System. $(n \geq 1)$

Proof

Let Ext^n be the small category:

$$(r) \longrightarrow (i) \longrightarrow \ldots\ldots \longrightarrow (j) \longrightarrow (l)$$

in which are n + 2 objects. Let $\underline{\underline{B}} \subset (Ext^n, \underline{\underline{A}})$ be the class of functors $F : Ext^n \longrightarrow \underline{\underline{A}}$ which have values n-fold exact sequences. Clearly $\underline{\underline{B}}$ is a PBS System.

Equivalence

The reader will recall (Proposition 5.2, page 84, (14)) that, if S and S' are two n-fold exact sequences starting at B and ending at A, then $S \underset{m}{\equiv} S'$ if and only if there is an integer k and 2k morphisms of n-fold exact sequences

$$S = S_o \longrightarrow S_1 \longrightarrow \ldots \longleftarrow S_{2k-1} \longleftarrow S_{2k} = S',$$

running alternately to the left and to the right, all starting with 1_B and ending with 1_A.

If we denote by $\equiv$, the congruence relation defined in Definition 1.8, we may show that $\equiv$ is $\underset{m}{\equiv}$.

Suppose $\underline{F}, \underline{F'} \in Ext^n(A,B)$.

If $\underline{F} \underset{m}{\equiv} \underline{F'}$, clearly $\exists\ \underline{F} \xrightarrow{\{1\eta 1\}} \underline{F'}$,

and $\exists\ \underline{F'} \xrightarrow{\{1\eta 1\}} \underline{F}$, so $\underline{F} \equiv \underline{F'}$.

Suppose, conversely, $\exists\ \underline{F} \xrightarrow{\{\alpha\eta\beta\}} \underline{F'}$. Then there is a finite sequence of natural transformations:

$$\underline{F} = \underline{F_o} \longrightarrow \underline{F_1} \ldots \underline{F}_{2k-1} \longleftarrow \underline{F_{2k}} = \underline{F'} .$$

Reading from left to right, denote the exceptional natural transformations, which are such that $K_i \neq 1$ or $K_i' \neq 1$, by

$$\underline{F_{\alpha 1}} \xrightarrow{K_1 \eta K_1'} \underline{F_{\alpha 2}} , \ldots\ldots\ldots \underline{F_{\alpha n-1}} \xrightarrow{K_{n-1} \eta K_{n-1}'} \underline{F_{\alpha n}} .$$

The reader will recall (Proposition 5.1, page 84, (14)) that a morphism $\Gamma : S \longrightarrow S'$ of n-fold exact sequences S and S', starting with α and ending with γ, yields a congruence $\alpha S \underset{m}{\equiv} S'\gamma$. We then have, just as we did in the proof of Lemma 2.1,

$$\underline{F} \underset{m}{\equiv} \underline{F_{\alpha 1}} \text{ and } \underline{F_{\alpha n}} \underset{m}{\equiv} \underline{F'}$$

$$K_1'\underline{F_{\alpha 1}} \underset{m}{\equiv} \underline{F_{\alpha 2}}K_1 \text{ and } \underline{F_{\alpha 2}} \underset{m}{\equiv} \underline{F_{\alpha 3}}$$

$$K_3'\underline{F_{\alpha 3}} \underset{m}{\equiv} \underline{F_{\alpha 4}}K_3 \text{ , and so on, from which we obtain}$$

that

$$K_{n-1}' \cdots\cdots\cdot K_3'K_1'\underline{F_{\alpha 1}} \underset{m}{\equiv} \underline{F_{\alpha n}}K_{n-1}K_{n-2} \cdots\cdot K_1$$

so that $\beta\underline{F_{\alpha 1}} \underset{m}{\equiv} \underline{F_{\alpha n}}\alpha$

whence $\beta\underline{F} \underset{m}{\equiv} \underline{F}'\alpha$.

In particular, if $\beta = \alpha = 1$, $\underline{F} \underset{m}{\equiv} \underline{F}'$.

<u>Conditions</u> 1, 2, 3 of Proposition 1.13 are fulfilled.

1. Given $A' \xrightarrow{\alpha} A$, and $F \in \text{Ext}^n(A,B)$, define $F\alpha$ to be the equivalence class of $\underline{F}\alpha$ defined in (14) (page 83).

   We have $F\alpha \xrightarrow{\{\alpha\eta 1\}} F$. (See Lemma 2.2)

2. Given $B \xrightarrow{\beta} B'$, and $F \in \text{Ext}^n(A,B)$, define βF to be the equivalence class of $\beta\underline{F}$ defined in (14) (page 83, too).

   We have $F \xrightarrow{\{1\eta\beta\}} \beta F$. (See Lemma 2.2)

3. Given $\underline{F_1} \xrightarrow{\{\alpha\eta\beta\}} \underline{F_2}$, we have shown, just above, that $\beta\underline{F_1} \equiv \underline{F_2}\alpha$, whence $\beta F_1 \equiv F_2\alpha$. Alternatively, we might use Lemma 2.1.

<u>Note</u>

A slight modification of Lemma 2.1 shows that

if $\underline{F} \xrightarrow{\alpha\eta\beta} \underline{F}'$ implies $\beta\underline{F} \equiv \underline{F}'\alpha$ (**)

then $\underline{F} \xrightarrow{\{\alpha\eta\beta\}} \underline{F}'$ implies $\beta\underline{F} \equiv \underline{F}'\alpha$.

This assures us that we may use Lemma 2.2 and condition (**) (which is then equivalent to Lemma 2.3) simultaneously, to construct a Baer Sum System.

CHAPTER THREE

OTHER BAER SUM SYSTEMS

We construct three new Baer Sum Systems. Many common diagrams in abelian categories form Baer Sum Systems. For example, the system of commutative squares forms a Baer Sum System. (The construction of this system is the same as the construction of ES, described in this chapter.) But often, as in this case, the induced bifunctor is the zero bifunctor, and we obtain a "Trivial Baer Sum System". One of the simplest of these is Z.

Z 3.1

Let $\underline{\underline{A}}$ be an abelian category.

Let Z be the small category:

(1) (r) , the only maps of which are identity maps. Let $\underline{\underline{B}} = (Z, \underline{\underline{A}})$. Clearly $\underline{\underline{B}}$ is a PBS System.

Equivalence

Equivalence is identity. For suppose $\underline{F}$, $\underline{F'}$ are particular functors $\underline{F}, \underline{F'} : Z \longrightarrow \underline{\underline{A}}$, and $\exists$ $\underline{F} \xrightarrow{\{1\eta 1\}} \underline{F'}$. We have a diagram for some integer 2k:

$$\underline{F}(r) \xrightarrow{1} \underline{F_1}(r) \;\ldots\ldots\; \underline{F_{2k-1}}(r) \xleftarrow{1} \underline{F'}(r)$$

$$\underline{F}(1) \xrightarrow{1} \underline{F_1}(1) \;\ldots\ldots\; \underline{F_{2k-1}}(1) \xleftarrow{1} \underline{F'}(1)$$

Discard any transformation of the form:

$$\underline{F_i} \xleftarrow{1\eta 1} \underline{F_{i+1}}$$

The remaining natural transformations compose to give a diagram:

$$\underline{F}(r) \xrightarrow{1} \underline{F'}(r)$$

$$\underline{F}(1) \xrightarrow{1} \underline{F'}(1) \text{ , whence } \underline{F} = \underline{F'}$$

<u>Conditions</u> 1, 2, 3 of Proposition 1.13.

1. Given $A' \xrightarrow{\alpha} A$, and $F \in Z(A,B)$, define αF by $\alpha F(1) = A'$, $\alpha F(r) = B$.

$$\begin{array}{ccccc} A' & & ; & \alpha F(1) & \alpha F(r) \\ \alpha\downarrow & & & \alpha\downarrow & \downarrow 1 \\ A & B & & F(1) & F(r) \end{array}$$

$$\exists \; \alpha F \xrightarrow{\{\alpha\eta 1\}} F.$$

2. Given $B \xrightarrow{\beta} B'$, and $F \in Z(A,B)$, define $F\beta$ by $F\beta(1) = A \quad F\beta(r) = B'$.

$$\exists \; F \xrightarrow{\{1\eta\beta\}} F\beta.$$

3. Given $F \xrightarrow{\{\alpha\eta\beta\}} F'$, we have a diagram:

$$\begin{array}{cc} F(1) & F(r) \\ \alpha\downarrow & \downarrow\beta \\ F'(1) & F'(r) \; . \end{array}$$

Obviously, $\alpha F' = F\beta$.

$Z(A,B)$ contains a single element, for all $A,B \in \underline{\underline{A}}$.

$Z : \underline{\underline{A}} \times \underline{\underline{A}} \longrightarrow \underline{\underline{G}}$ is the zero bifunctor.

<u>Exact Squares 3.2</u>

The system of exact squares forms a Baer Sum System. This we shall call ES. But the bifunctor we derive, alas, is the zero bifunctor. We shall nevertheless devote some space to the proof of this. And for these reasons:

We shall have illustrated the remark at the beginning of this chapter by showing how the equivalence relation of a PBS System often reduces S(A,B) to zero.

We shall obtain an interesting comparison between the theory of relations in an abelian category, which Hilton develops in (7), and the Baer Sum System ES. Now Hilton shows that the class $\tilde{\underline{\underline{A}}}$ (A,B) of relations between objects A and B of an abelian category $\underline{\underline{A}}$ forms a commutative semigroup with zero. We show that this class, under the equivalence relation of a Baer Sum System, forms a zero abelian group. That is, we might say, if we insist on making $\tilde{\underline{\underline{A}}}$ (A,B) an abelian group, we obtain only a trivial group.

And finally, in making this comparison, we shall make apparent a result concealed in Hilton's theory - that Hilton's definition of addition is in fact the Baer Sum.

We use the terms "pullback", "pushout" and "bicartesian", which are applicable to commutative squares. These terms are equivalent to "cartesian", "cocartesian" and "bicartesian", respectively, and are defined in (5), (2), or (8).

In Part I of this account we develop as much of Hilton's theory (7) as we need. Our account is derived from (7), (2), and (11). It is self-contained, though we do not give all the proofs.

In Part II we will construct the Baer Sum System ES, and make the comparisons we have promised.

We work in an abelian category $\underline{\underline{A}}$.

The reader interested in generalizations and additions to Hilton's theory is referred to (8), (9), (10), and (3).

Part I

Definition 3.3 ((2), page 41)

A square:

$$\begin{array}{ccc} R & \xrightarrow{\beta} & B \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\psi} \\ A & \xrightarrow[\phi]{} & X \end{array}$$

is exact, if it is commutative, and if, in the following diagram:

$$\begin{array}{ccccccc} \odot & \xrightarrow{\text{Ker }\beta} & \odot & \xrightarrow{\beta} & \odot & \xrightarrow{\text{Cok }\beta} & \odot \\ {\scriptstyle\bar{\alpha}}\downarrow & & {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\bar{\psi}} \\ \odot & \xrightarrow[\text{Ker }\phi]{} & \odot & \xrightarrow[\phi]{} & \odot & \xrightarrow[\text{Cok }\phi]{} & \odot \end{array}$$

$\bar{\alpha}$ is an epimorphism, and $\bar{\psi}$ is a monomorphism. The square is exact if, and only if, the sequence:

$$R \xrightarrow{(\alpha,\beta)} A\oplus B \xrightarrow{\begin{bmatrix}\psi\\ -\phi\end{bmatrix}} X \text{ is exact.}$$

Examples of exact squares are pullbacks, pushouts, and bicartesian squares. Exact squares are called "smooth" in (11).

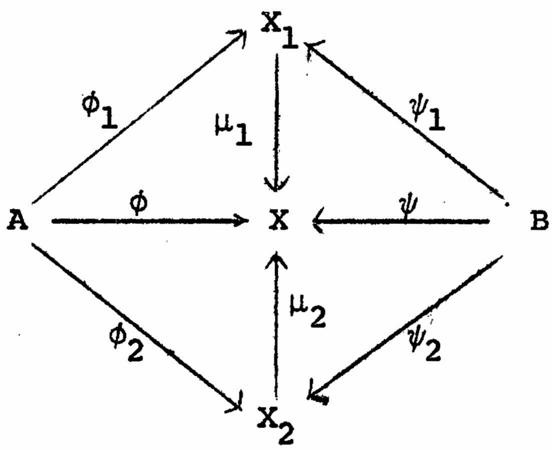
Definition 3.4 ((7), page 261)

Let A,B be objects of $\underline{\underline{A}}$. A relation from A to B is an equivalence class of pairs of $\underline{\underline{A}}$-morphisms:

$$A \xrightarrow{\phi} X \xleftarrow{\psi} B$$

under the equivalence relation:

$(\phi_1,\psi_1) \sim (\phi_2,\psi_2)$ if and only if there are monomorphisms μ_1 and μ_2, with $(\mu_1\phi_1,\mu_1\psi_1) = (\mu_2\phi_2,\mu_2\psi_2)$. We then have the following diagram:



<u>Note 3.5</u>

In (7) the notion of a relation is made self-dual. This is done by considering pairs of $\underline{\underline{A}}$-morphisms:

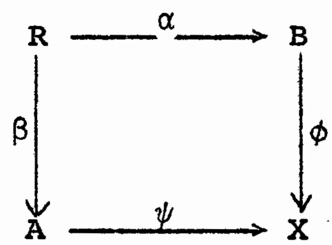
$$A \xleftarrow{\alpha} R \xrightarrow{\beta} B$$

under the equivalence relation

$(\alpha_1, \beta_1) \sim (\alpha_2, \beta_2)$ if and only if there are epimorphisms ϵ_1 and ϵ_2, and $(\alpha_1 \epsilon_1, \beta_1 \epsilon_1) = (\alpha_2 \epsilon_2, \beta_2 \epsilon_2)$.

Then $A \xrightarrow{\phi} X \xleftarrow{\psi} B$ and

$A \xleftarrow{\alpha} R \xrightarrow{\beta} B$ are identified if and only if the square



is exact.

In order that this identification should be well-defined, we need the theorems:

In the following diagram:

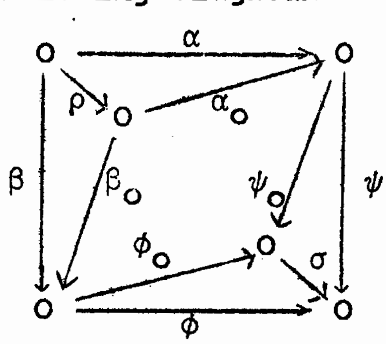


Diagram 3.6

<u>Theorem 3.7</u> ((11), page 63)

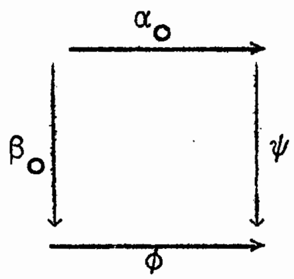
If the diagram commutes, and if the outer square is exact, then the inner square is exact.

<u>Theorem 3.8</u> ((11), page 63)

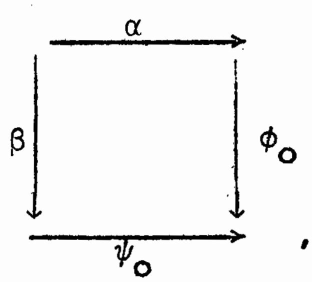
If the diagram commutes, and if the inner square is exact, and if ρ is an epimorphism, and σ is a monomorphism, then the outer square is exact.

<u>Theorem 3.9</u> ((14), page 257, and (11), page 63)

If the inner square is constructed from an exact outer square by the construction of a pullback



, and a pushout



, then ρ is an epimorphism, σ is a monomorphism, the inner square is bicartesian, and the diagram commutes.

Returning to our examination of the identification, we have:

If (ϕ_1,ψ_1) is identified with (α_1,β_1), and

$(\phi_1,\psi_1) \sim (\phi_2,\psi_2)$, and

$(\alpha_1,\beta_1) \sim (\alpha_2,\beta_2)$,

we have the diagrams:

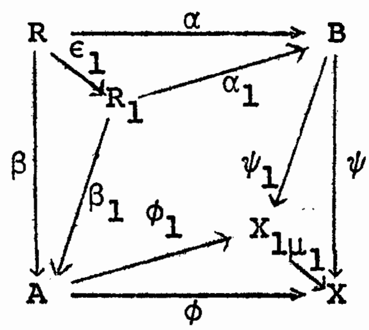


Diagram 3.10

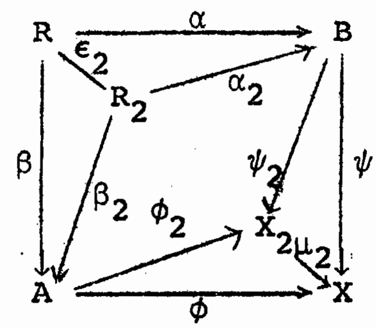


Diagram 3.11

The diagrams commute. ϵ_1 and ϵ_2 are epimorphisms, μ_1 and μ_2 are monomorphisms.

(α_1, β_1) is identified with (ϕ_1, ψ_1), so the inner square of diagram 3.10 is exact.

The outer square is exact by Theorem 3.8.

Hence: (α, β) is identified with (ϕ, ψ).

Hence, by Theorem 3.7, in diagram 3.11, (α_2, β_2) is identified with (ϕ_2, ψ_2). Hence the identification is well-defined.

We use Theorem 3.9 to obtain a representative of each equivalence class of exact squares. The representative of the equivalence class of the outer exact square, in diagram 3.6, is the inner bicartesian square, constructed with the aid of Theorem 3.9. Up to isomorphism, it is a unique minimal representative.

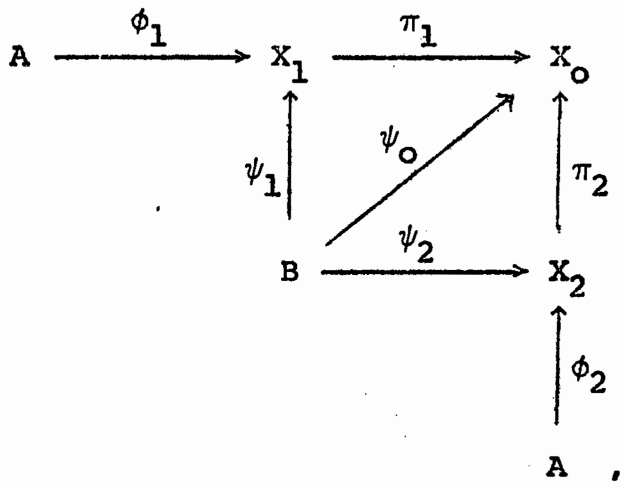
Note 3.12

Addition

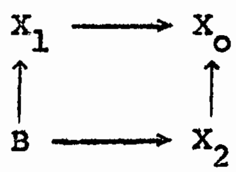
It will not be necessary to develop all the theory of (7), in order to define addition on the class of relations between A and B.

Given relations $A \xrightarrow{\phi_1} X_1 \xleftarrow{\psi_1} B$ and

$A \xrightarrow{\phi_2} X_2 \xleftarrow{\psi_2} B$, we construct the diagram:



where the square



is a pushout

(see (7), 267)

Then $(\phi_1, \psi_1) + (\phi_2, \psi_2) = (\phi_o, \psi_o)$

where $\phi_o = \pi_1\phi_1 + \pi_2\phi_2$ and

$$\psi_o = \pi_1\psi_1 = \pi_2\psi_2 .$$

Clearly, this definition has a dual. Hilton (7) shows that this addition, and the dual definition, coincide under the identification above. We quote this result without proof. This means that we may add exact squares together simply by adding the relations which they represent.

Under this addition, the class of relations between objects A and B, denoted $\underline{\underline{\tilde{A}}}$ (A,B), becomes a commutative semigroup with zero.

Part II

The Baer Sum System ES 3.13

We quote two well-known lemmas, which will be needed for the construction of this Baer Sum System. The first will be very useful in subsequent chapters. We omit proofs.

Lemma 3.14 ((5), page 54)

In an abelian category $\underline{\underline{A}}$:
Suppose that the commutative diagram:

$$\begin{array}{ccccccc} A' & \longrightarrow & A & \longrightarrow & B & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ & & C & \longrightarrow & D & & \end{array}$$

has an exact top row. Then the square is a pushout if and only if $A' \longrightarrow C \longrightarrow D \longrightarrow 0$ is exact.

Lemma 3.15. The Product Lemma

((2), page 45, also (4). See also (12))

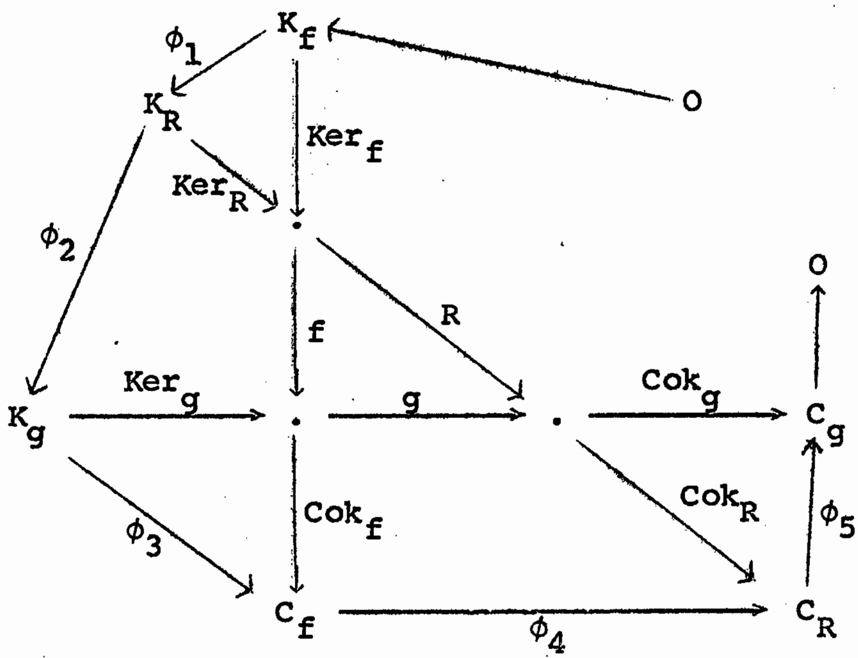
In an abelian category $\underline{\underline{A}}$:
Suppose that $R = gf$, in the diagram below. Then there are induced morphisms $\phi_1, \phi_2, \phi_3, \phi_4, \phi_5$, which are unique with respect to commutativity in the diagram, and which provide an exact sequence:

$$0 \longrightarrow K_f \xrightarrow{\phi_1} K_R \xrightarrow{\phi_2} K_g \xrightarrow{\phi_3} C_f \xrightarrow{\phi_4} C_R \xrightarrow{\phi_5} C_g \xrightarrow{\phi_6} 0,$$

where:

$K\alpha$ denotes the kernel of α, $\alpha = f,R,g$, and

$C\alpha$ denotes the cokernel of α, $\alpha = f,R,g$.

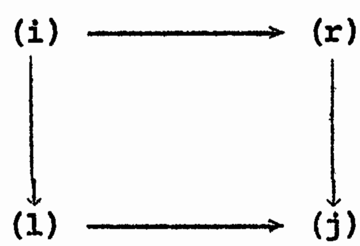


<u>Note</u>

ϕ_1, ϕ_2 are induced by the properties of kernels. ϕ_4, ϕ_5 are induced by the properties of cokernels. ϕ_3 is the composition $Cok_f . Ker_g$.

<u>The Construction of ES</u>

Let ES be the small category:



. There is a single morphism (i) $\longrightarrow$ (j) .

Let $\underline{\underline{B}}_{\mathcal{C}}(ES,\underline{\underline{A}})$ be the class of functors F : ES $\longrightarrow$ $\underline{\underline{A}}$, with values exact squares. It is easy to prove that a direct sum of exact squares is an exact square, from the definition 3.3. We have a PBS System.

<u>Conditions</u> 1, 2, 3 of Proposition 1.13.

1. If
$$\begin{array}{ccc} R = \underline{F}(i) & \longrightarrow & \underline{F}(r) = B \\ \downarrow & & \downarrow \\ A = \underline{F}(1) & \longrightarrow & \underline{F}(j) = X \end{array}$$

is an exact square, and $A' \xrightarrow[\gamma]{} A$ an $\underline{\underline{A}}$-morphism, make the following construction, in which 1 is a pullback:

$$\begin{array}{ccccc} R' & \xrightarrow{\gamma'} & R & \xrightarrow{\alpha} & B \\ \beta' \downarrow & 1 & \downarrow \beta & 2 & \downarrow \psi \\ A' & \xrightarrow[\gamma]{} & A & \xrightarrow[\phi]{} & X \end{array}$$

Then the composite square is exact : for, if we denote by K_f the kernel of a map f, and C_f the cokernel of a map f, the Product Lemma 3.15 provides the following commutative diagram:

$$\begin{array}{ccccccccccccccc} 0 & \longrightarrow & K\gamma' & \longrightarrow & K\alpha\gamma' & \longrightarrow & K\alpha & \longrightarrow & C\gamma' & \longrightarrow & C\alpha\gamma' & \longrightarrow & C\alpha & \longrightarrow & 0 \\ \delta_1 \downarrow & & \delta_2 \downarrow & & \delta_3 \downarrow & & \delta_4 \downarrow & & \delta_5 \downarrow & & \delta_6 \downarrow & & \delta_7 \downarrow & & \delta_8 \downarrow \\ 0 & \longrightarrow & K\gamma & \longrightarrow & K\phi\gamma & \longrightarrow & K\phi & \longrightarrow & C\gamma & \longrightarrow & C\phi\gamma & \longrightarrow & C\phi & \longrightarrow & 0 \end{array}$$

The rows are exact, by the Product Lemma. The vertical maps are those induced by Kernels and Cokernels. The reader will recall that a pullback (in particular 1, above) is an exact square.

Then: δ_5 is a monomorphism, because 1 is exact.
δ_2 is an epimorphism, because 1 is exact.
δ_4 is an epimorphism, because 2 is exact.

By the "Five Lemma" (see, for example (14), page 14)

δ_3 is an epimorphism.

Similarly, δ_6 is a monomorphism.

Hence the composite square is exact.

Define $\underline{F}\gamma \in ES'(A,B)$ to be the functor with value the composite exact square. We have a natural transformation

$$\underline{F}\gamma \xrightarrow{\gamma\eta 1} \underline{F} .$$

By Lemma 2.2 the equivalence class of $\underline{F}\gamma$ will serve as $F\gamma$, and we have

$$F\gamma \xrightarrow{\{\gamma\eta 1\}} F .$$

2. Given $\underline{F} \in ES'(A,B)$ and $B \xrightarrow{\delta} B'$, we define $\delta\underline{F}$, dually. That is, it is the composite square in the construction:

$$\begin{array}{ccccc}
\underline{F}(i) & \longrightarrow & \underline{F}(r) = B & \xrightarrow{\delta} & B' \\
\downarrow & 2 & \downarrow & 1 & \downarrow \\
A = \underline{F}(1) & \longrightarrow & \underline{F}(j) & \longrightarrow & X'
\end{array}$$

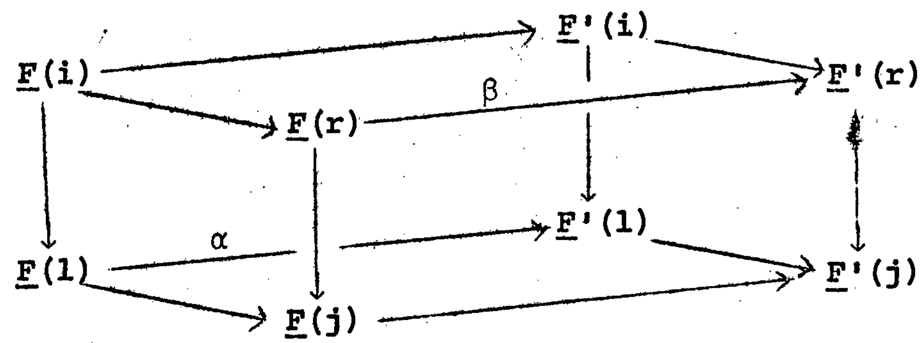
where 1 is a pushout. The equivalence class of $\delta\underline{F}$ serves as δF, and we have

$$F \xrightarrow{\{1\eta\delta\}} \delta F .$$

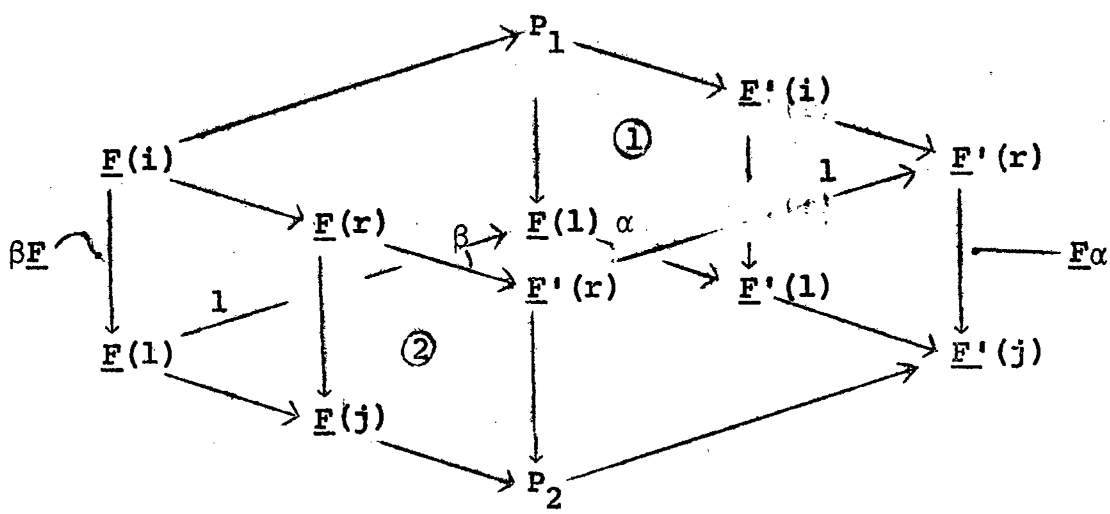
3. We use Lemma 2.1 to verify condition 3. Given a morphism of exact squares:

$$\underline{F} \xrightarrow{\alpha\eta\beta} \underline{F'} ,$$ we have

a commutative diagram:



Constructing $\beta\underline{F}$ and $\underline{F}'\alpha$, we obtain, by the properties of pullbacks and pushouts, induced maps $\underline{F}(i) \longrightarrow P_1$ and $P_2 \longrightarrow \underline{F}'(j)$, in the diagram below:



where square 1 is a pullback, and square 2 is a pushout. It is easy to check that the diagram commutes.

We have $\beta\underline{F} \xrightarrow{1\eta 1} \underline{F}\alpha$; it is then easy to prove that $\beta F \equiv F\alpha$.

We have a Baer Sum System.

Addition

To make our comparison of ES with Hilton's theory (7), we show, as we promised to, that addition, as defined in Note 3.12, coincides with addition as defined in the Baer Sum System.

Given elements

$$\begin{array}{ccc} R_1 & \longrightarrow & B \\ \downarrow & & \downarrow \psi_1 \\ A & \xrightarrow[\phi_1]{} & X_1 \end{array} \quad , \quad \begin{array}{ccc} R_2 & \longrightarrow & B \\ \downarrow & & \downarrow \psi_2 \\ A & \xrightarrow[\phi_2]{} & X_2 \end{array}$$

their sum in ES(A,B) is the composite square in the diagram:

$$\begin{array}{ccccccc} P_1 & \longrightarrow & R_1 \oplus R_2 & \longrightarrow & B \oplus B & \xrightarrow{\nabla} & B \\ \downarrow & 1 & \downarrow & & \downarrow \psi_1 \oplus \psi_2 & 2 & \downarrow \\ A & \xrightarrow{\Delta} & A \oplus A & \xrightarrow{\phi_1 \oplus \phi_2} & X_1 \oplus X_2 & \longrightarrow & P_2 \end{array}$$

where 1 is a pullback, and 2 is a pushout. But, referring to Note 3.12, in the diagram:

$$\begin{array}{ccccccc} B & \xrightarrow{(-1,1)} & B \oplus B & \xrightarrow{\nabla} & B & \longrightarrow & 0 \\ & & \psi_1 \oplus \psi_2 \downarrow & 3 & \downarrow \psi_0 & & \\ & & X_1 \oplus X_2 & \xrightarrow{\begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix}} & X_0 & \longrightarrow & 0 \end{array}$$

the square 3 is a pushout, by Lemma 3.14. Thus square 2 is square 3 , and

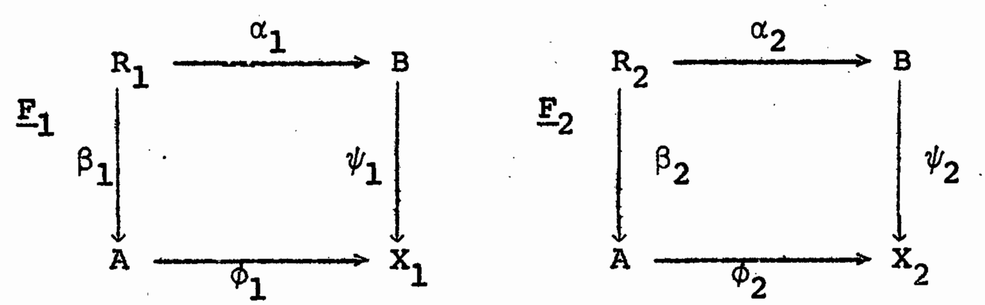
$$\begin{array}{l} A \longrightarrow P_2 \longleftarrow B \\ = A \xrightarrow{\pi_1 \phi_1 + \pi_2 \phi_2} X_0 \xleftarrow{\psi_0} B \; . \end{array}$$

Dually, the relation: $A \longleftarrow P_1 \longrightarrow B$ is the sum defined in the dual of Note 3.12. The result is proved.

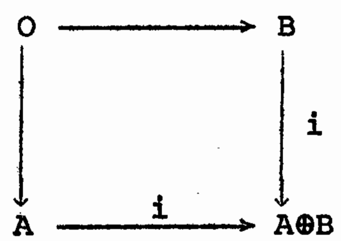
<u>Equivalence</u>

We show that the induced bifunctor is the zero bifunctor, as we foretold. The equivalence relation of the Baer Sum System is much coarser than the equivalence relation of definition 3.4, and ES(A,B) has a single element for all A and B.

For, given any two exact squares of ES'(A,B):

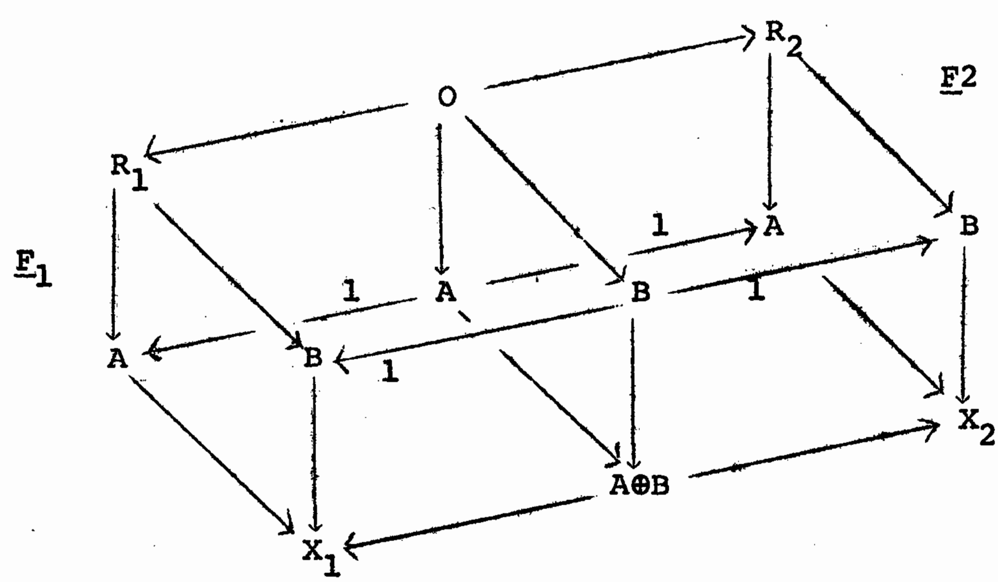


We note that:



is an exact square,

and we have morphisms:



providing $\underline{F}_1 \xrightarrow{\{1\eta 1\}} \underline{F}_2$, so that $\underline{F}_1 \equiv \underline{F}_2$.

<u>P-Hom 3.16</u>

We discover a new family of non-trivial Baer Sum Systems. Let $\underline{\underline{P}}$ be a full subcategory of an abelian category $\underline{\underline{A}}$ closed with respect to the formation of direct sums.

Let $\underline{\underline{P}}$-Hom be the small category:

$$(1) \xrightarrow{\theta_1} (i) \xrightarrow{\theta_2} (r)$$

Let $\underline{\underline{B}} \subset (\underline{\underline{P}}\text{-Hom}, \underline{\underline{A}}) = \{F \mid F(i) \in \underline{\underline{P}}\}$.

That is: $\underline{\underline{B}}$ is the class of functors with values of the form:

$A \longrightarrow P \longrightarrow B$, where P is an object of $\underline{\underline{P}}$. Clearly $\underline{\underline{B}}$ is a PBS System.

<u>$\underline{\underline{B}}$ is also a Baer Sum System</u>

We verify:

<u>Conditions</u> 1, 2, 3 of Proposition 1.13.

1. Given $A' \xrightarrow{\alpha} A$, a morphism of $\underline{\underline{A}}$, and $\underline{F} = A \xrightarrow{\phi_1} P \xrightarrow{\phi_2} B \in \underline{\underline{P}}\text{-Hom}'(A,B)$, define $\underline{F}\alpha$ by $\underline{F}\alpha(1) = A'$, $\underline{F}\alpha(\theta_1) = \phi_1\alpha$, $\underline{F}\alpha(i) = P$, $\underline{F}\alpha(\theta_2) = \phi_2$, $\underline{F}\alpha(r) = B$. We then have a natural transformation $\underline{F}\alpha \xrightarrow{\alpha\eta 1} \underline{F}$, described by the commutative diagram:

$$\begin{array}{ccccccc}
A' & \xrightarrow{\alpha} & A & \xrightarrow{\phi_1} & P & \xrightarrow{\phi_2} & B \\
\downarrow{\scriptstyle\alpha} & & & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} \\
A & & \xrightarrow{\quad\phi_1\quad} & & P & \xrightarrow{\phi_2} & B
\end{array}$$

By Lemma 2.2, the equivalence class of $\underline{F}\alpha$ serves as $F\alpha$, and we have

$$F\alpha \xrightarrow{\{\alpha\eta 1\}} F.$$

2. Given $B \xrightarrow{\beta} B'$, a morphism of $\underline{\underline{A}}$, and $\underline{F} = A \xrightarrow{\phi_1} P \xrightarrow{\phi_2} B \in \underline{\underline{P}}\text{-Hom}'(A,B)$, we define $\beta\underline{F}$ by $\beta\underline{F}(1) = A$, $\beta\underline{F}(\theta_1) = \phi_1$, $\beta\underline{F}(i) = P$, $\beta\underline{F}(\theta_2) = \beta\phi_2$,

$\beta\underline{F}(r) = B'$. $\beta\underline{F}$ is the functor with value:

$$A \xrightarrow{\phi 1} P \xrightarrow{\beta\phi 2} B'$$

βF is defined as the equivalence class of $\beta\underline{F}$ and we obtain:

$$\beta F \xrightarrow{\{1\eta\beta\}} F.$$

3. We use Lemma 2.1. Given particular functors $\underline{F}$, $\underline{F}' \in \underline{\underline{B}}$, and a natural transformation

$$\underline{F} \xrightarrow{\alpha\eta\beta} F',$$ we have a commutative diagram:

$$\begin{array}{cccccccc}
\underline{F} & : & A & \xrightarrow{\phi_1} & P & \xrightarrow{\phi_2} & B \\
 & & \alpha\downarrow & & \downarrow\gamma & & \downarrow\beta \\
\underline{F}' & : & A' & \xrightarrow{\phi_1'} & P' & \xrightarrow{\phi_2'} & B'
\end{array}$$

which provides a commutative diagram:

$$\begin{array}{cccccccc}
\beta\underline{F} & : & A & \xrightarrow{\phi_1} & P & \xrightarrow{\beta\phi 2} & B' \\
 & & 1\downarrow & & \downarrow\gamma & & \downarrow 1 \\
\underline{F}'\alpha & : & A & \xrightarrow{\phi_1'\alpha} & P' & \xrightarrow{\phi_2'} & B'
\end{array}$$

so that $\exists\ \beta\underline{F} \xrightarrow{1\eta 1} \underline{F}'\alpha$, and it is easy to prove that $\beta F \equiv F'\alpha$.

We have a Baer Sum System.

<u>Note</u> $\underline{\underline{P}}$-Hom(A,B) is a set if (but not only if) $\underline{\underline{P}}$ is small.

The significance of the equivalence relation placed on $\underline{\underline{P}}$-Hom(A,B) is not immediately clear. But, by considering special subcategories $\underline{\underline{P}}$ of $\underline{\underline{A}}$, we obtain results that illuminate the structure of $\underline{\underline{P}}$-Hom(A,B).

Proposition 3.17

If $\underline{\underline{P}} = \underline{\underline{A}}$, the functors $\underline{\underline{P}}$-Hom and Hom are naturally equivalent; that is:

$$\underline{\underline{P}}\text{-Hom} \approx \text{Hom}.$$

Proof

Let $\underline{F}, \underline{F}' \in \underline{\underline{P}}\text{-Hom}'(A,B)$, then $\underline{F} : A \xrightarrow{\phi} P \xrightarrow{\psi} B \equiv \underline{F}' : A \xrightarrow{\phi'} P' \xrightarrow{\psi'} B$ if, and only if, $\psi\phi = \psi'\phi'$.

For, $\underline{F} \equiv \underline{F}' \Rightarrow \exists\, \underline{F} \xrightarrow{\{1\eta 1\}} \underline{F}'$.

Then there exists a finite sequence of functors and natural transformations:

$$\underline{F} = \underline{F}_o \longrightarrow \underline{F}_1 \longleftarrow \underline{F}_2 \longrightarrow \cdots \longrightarrow \underline{F}_{2k-1} \longleftarrow \underline{F}_{2k} = \underline{F}'.$$

Let $\underline{F}\alpha_1 \xrightarrow{K_1\eta K_1'} \underline{F}\alpha_2$, ..., $\underline{F}\alpha_{n-1} \xrightarrow{K_{n-1}\eta K_{n-1}'} \underline{F}\alpha_n$ be the exceptional natural transformations, which are such that $K_i \neq 1$ or $K_i' \neq 1$. If ϕ_i, ψ_i denote the two morphisms of $\underline{F}_i$ then clearly:

$$\psi\phi \;(= \psi_o\phi_o) = \psi\alpha_1\phi\alpha_1$$

and

$$\psi'\phi' \;(= \psi_{2k}\phi_{2k}) = \psi\alpha_n\phi\alpha_n\,.$$

By considering the commutative diagram:

$$\begin{array}{ccccc} A & \xrightarrow{\phi\alpha_1} & P\alpha_1 & \xrightarrow{\psi\alpha_1} & B \\ {\scriptstyle K_1}\downarrow & & \downarrow & & \downarrow{\scriptstyle K_1'} \\ A & \xrightarrow{\phi\alpha_2} & P\alpha_2 & \xrightarrow{\psi\alpha_2} & B \end{array}$$

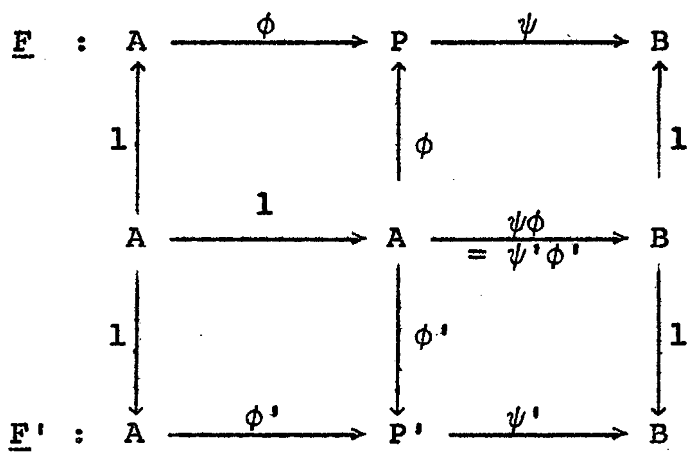
we obtain: $\psi\alpha_2\phi\alpha_2K_1 = K_1'\psi\alpha_1\phi\alpha_1$. By a technique entirely analogous to that of Lemma 2.1, we prove that

$$\psi\alpha_1\phi\alpha_1 = \psi\alpha_n\phi\alpha_n \text{ , so that}$$
$$\psi\phi = \psi'\phi' .$$

Conversely

Suppose $\psi\phi = \psi'\phi'$,

then the following diagram commutes:



providing $\underline{F} \xrightarrow{\{1\eta 1\}} \underline{F}'$, so $\underline{F} \equiv \underline{F}'$.

It is easy to check that the equivalence between $\underline{\underline{P}}$-Hom (A,B) and Hom (A,B) is natural.

Proposition 3.18

If $\underline{\underline{P}} = \{o\}$, $\underline{\underline{P}}$-Hom is the zero bifunctor. The result is obvious.

These two results are combined in the proposition that follows. First, we remind the reader of the definition of a reflective subcategory.

Definition 3.19 (see (5), page 79, or (6), page 76)

Let $\underline{\underline{P}}$ be a subcategory of $\underline{\underline{A}}$. $\underline{\underline{P}}$ is a reflective subcategory if for every object A of $\underline{\underline{A}}$ there is an object $\bar{A}$ of $\underline{\underline{P}}$, and a map $A \xrightarrow{r_A} \bar{A}$ which satisfies the following condition: For any P of $\underline{\underline{P}}$ and map $A \longrightarrow P$ there is a unique map $\bar{A} \longrightarrow P$ such that

$$\begin{array}{ccc} A & \xrightarrow{r_A} & \bar{A} \\ & \searrow & \downarrow \\ & & P \end{array}$$

commutes.

We quote the following results without proof:

If $\underline{\underline{P}}$ is a reflective subcategory of $\underline{\underline{A}}$:

1. (See (6), page 76) There is a functor $R : \underline{\underline{A}} \longrightarrow \underline{\underline{P}}$ defined by $R(A) = \bar{A}$

$R(f) = \bar{f}$, where $\bar{f}$ is the unique map making the following diagram commute:

$$\begin{array}{ccc} A & \longrightarrow & \bar{A} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \bar{f}} \\ A_1 & \longrightarrow & \bar{A}_1 \end{array}$$

R is termed the reflector.

2. (See (6), page 88) $\underline{\underline{P}}$ is closed with respect to the formation of limits in $\underline{\underline{A}}$.

Examples of reflective subcategories $\underline{\underline{P}}$ of $\underline{\underline{A}}$ are: (see (6), page 83, and elsewhere) $\underline{\underline{A}}$ itself; $\{0\}$, if $\underline{\underline{A}}$ has a zero object; the category of torsion free groups in the category of abelian groups. There are many other examples.

In the next proposition we assume that $\underline{\underline{P}}$ is a reflective subcategory of an abelian category $\underline{\underline{A}}$. The category $\underline{\underline{B}} < (\underline{\underline{P}}\text{-Hom}, \underline{\underline{A}})$ will then be closed with respect to the formation of direct sums, automatically, by our remark above.

<u>Proposition 3.20</u>

If $\mathbf{P}$ is a full reflective subcategory and $R : \mathbf{A} \longrightarrow \mathbf{P}$ is the reflector, and $E : \mathbf{P} \longrightarrow \mathbf{A}$ the embedding functor, then $\mathbf{P}$-Hom is naturally equivalent to the composition of functors:

$$\mathbf{A}\times\mathbf{A} \xrightarrow{R\times 1} \mathbf{P}\times\mathbf{A} \xrightarrow{E\times 1} \mathbf{A}\times\mathbf{A} \xrightarrow{Hom} \mathbf{G}\,.$$

That is: there is a natural isomorphism:

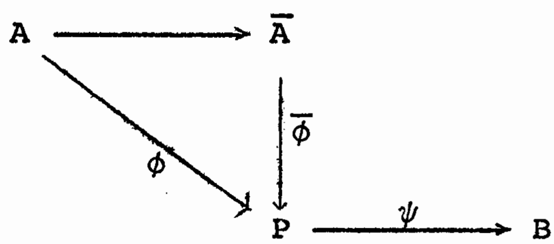
$$\mathbf{P}\text{-Hom}(A,B) \cong Hom(R(A),B).$$

<u>Proof</u>

Let $\bar{A}$ denote $R(A)$.

Then for any objects A,B of $\mathbf{A}$, we may define

$\bar{\Phi} : \mathbf{P}\text{-Hom}(A,B) \longrightarrow Hom(\bar{A},B)$ by considering the following diagram:



$\bar{\phi}$ is the unique map induced by ϕ.

Define $\bar{\Phi}'$ $(A \xrightarrow{\phi} P \xrightarrow{\psi} B) = \psi\bar{\phi}$.

Then define, where $\{A \xrightarrow{\phi} P \xrightarrow{\psi} B\}$ denotes the equivalence class of which $A \xrightarrow{\phi} P \xrightarrow{\psi} B$ is a representative:

$$\bar{\Phi}\,(\{A \xrightarrow{\phi} P \xrightarrow{\psi} B\}) =$$
$$\bar{\Phi}'\,(A \xrightarrow{\phi} P \xrightarrow{\psi} B) = \psi\bar{\phi}.$$

<u>$\bar{\Phi}$ is well defined.</u> We make three preliminary notes:

Note 1

Suppose $\underline{F}$, $\underline{F}' \in \underline{\underline{P}}\text{-Hom}'(A,B)$, and we have a commutative diagram:

$$\begin{array}{lcccccc}
\underline{F} : & A & \xrightarrow{\phi} & P & \xrightarrow{\psi} & B \\
 & \downarrow 1 & & \downarrow \pi & & \downarrow 1 \\
\underline{F}' : & A & \xrightarrow{\phi'} & P' & \xrightarrow{\psi'} & B & .
\end{array}$$

This is a situation which we will describe as $\underline{F} \equiv_* \underline{F}'$.

We obtain:

$$\begin{array}{lcccccc}
 & & & \bar{A} & & \\
 & & \nearrow r_A & \downarrow \bar{\phi} & & \\
\underline{F} : & A & \xrightarrow{\phi} & P & \xrightarrow{\psi} & B \\
 & \downarrow 1 & & \downarrow \pi & & \downarrow 1 \\
\underline{F}' : & A & \xrightarrow{\phi'} & P' & \xrightarrow{\psi'} & B & .
\end{array}$$

There is a unique map $\bar{\phi}' : \bar{A} \longrightarrow P'$ so that $\bar{\phi}' r_A = \phi'$.

This is $\pi\bar{\phi}$.

Then $\quad \bar{\phi}'(\underline{F}) = \psi\bar{\phi} \equiv \psi'\pi\bar{\phi}$.

$\quad\quad\quad \bar{\phi}'(\underline{F}') = \psi'\pi\bar{\phi}$.

Note 2

Given a commutative diagram:

$$\begin{array}{lcccccc}
\underline{F}\alpha_1 : & A & \xrightarrow{\phi\alpha_1} & P\alpha_1 & \xrightarrow{\psi\alpha_1} & B \\
 & \downarrow K & & \downarrow & & \downarrow K' \\
\underline{F}\alpha_2 : & A & \xrightarrow{\phi\alpha_2} & P\alpha_2 & \xrightarrow{\psi\alpha_2} & B
\end{array}$$

$\bar{\phi}'(\underline{F}\alpha_2 K) = \bar{\phi}'(\underline{F}\alpha_2)R(K)$.

sequence of natural transformations and functors

$$\underline{F} = \underline{F}_o \longrightarrow \underline{F}_1 \longleftarrow \cdots \quad \cdots \longrightarrow \underline{F}_{2k} \longleftarrow \underline{F}' ,$$ where

$\underline{F}\alpha_1 \xrightarrow{K_1\eta K_1'} \underline{F}\alpha_2$, ..., $\underline{F}\alpha_{n-1} \xrightarrow{K_{n-1}\eta K_{n-1}'} F\alpha_n$ shall denote the exceptional natural transformations, which are such that $K_i \neq 1$ or $K_i' \neq 1$.

Then, by Note 1,

$$\overline{\Phi}'(\underline{F}) = \overline{\Phi}'(\underline{F}_o) = \overline{\Phi}'(\underline{F}\alpha_1)$$

and $$\overline{\Phi}'(\underline{F}\alpha_n) = \overline{\Phi}'(\underline{F}').$$

By Notes 1, 2 and 3, we have:

$$\underline{F}\alpha_2 K_1 \equiv_* K_1'\underline{F}\alpha_1$$

$$\underline{F}\alpha_4 K_3 \equiv_* K_3'\underline{F}\alpha_3$$

$$\vdots$$

$$\underline{F}\alpha_n\, K_{n-1} \equiv_* K_{n-1}'\underline{F}\alpha_{n-1}$$

whence $$\overline{\Phi}'(\underline{F}\alpha_2)R(K_1) = K_1'\ \overline{\Phi}'(\underline{F}\alpha_1)$$

$$\overline{\Phi}'(\underline{F}\alpha_4)R(K_3) = K_3'\ \overline{\Phi}'(\underline{F}\alpha_3)$$

$$\vdots$$

$$\overline{\Phi}'(\underline{F}\alpha_n)R(K_{n-1}) = K_{n-1}'\ \overline{\Phi}'(\underline{F}\alpha_{n-1})$$

By the technique used in Lemma 2.1, we obtain:

$$K_{n-1}' \ \cdots\ K_3'K_1'\ \overline{\Phi}'(\underline{F}\alpha_1) = \overline{\Phi}'(\underline{F}\alpha_n)R(K_{n-1})\ \cdots\ R(K_1)$$

whence $$\overline{\Phi}'(\underline{F}\alpha_1) = \overline{\Phi}'(\underline{F}\alpha_n)$$

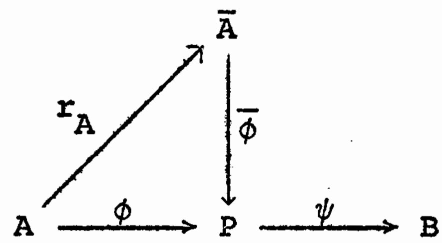
whence $$\overline{\Phi}'(\underline{F}) = \overline{\Phi}'(\underline{F}').$$

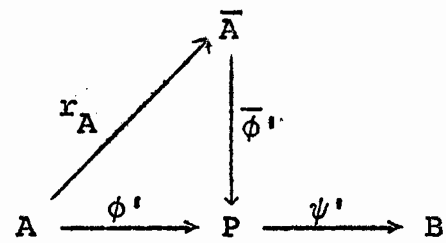
Our result follows.

<u>$\overline{\phi}$ is 1-1</u>

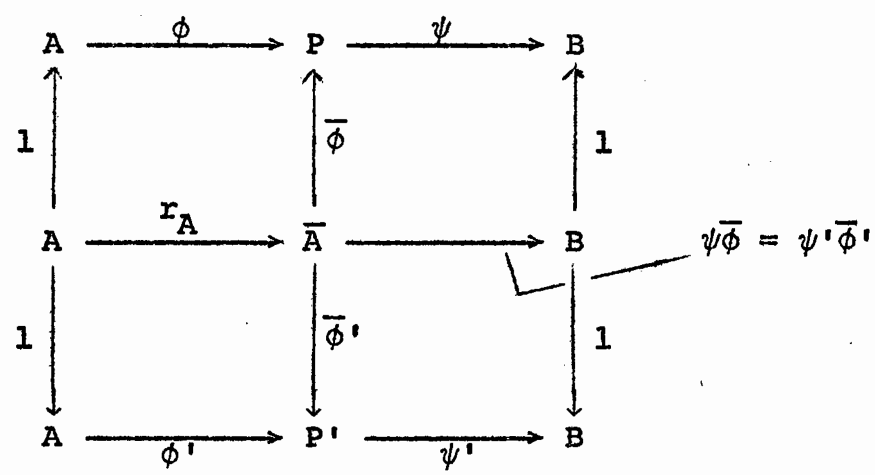
Suppose $\overline{\phi}\,(\{A \xrightarrow{\phi} P \xrightarrow{\psi} B\}) = \overline{\phi}\,(\{A \xrightarrow{\phi'} P \xrightarrow{\psi'} B\})$

Then, in the following diagrams:





$\psi\overline{\phi} = \psi'\overline{\phi}'$. We obtain a commutative diagram:



and $\{A \xrightarrow{\phi} P \xrightarrow{\psi} B\} \equiv \{A \xrightarrow{\phi'} P \xrightarrow{\psi'} B\}$.

<u>$\overline{\phi}$ is onto</u> Given $\overline{A} \xrightarrow{\gamma} B$ in Hom $(\overline{A}, B)$

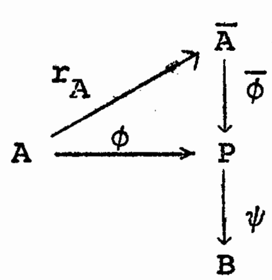
$\overline{\phi}\,(\{A \longrightarrow \overline{A} \xrightarrow{\gamma} B\}) = \gamma.$

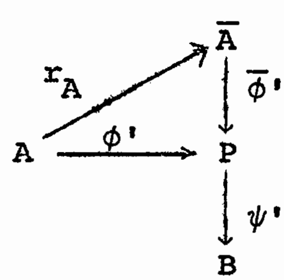
<u>$\overline{\phi}$ preserves the addition</u>

Suppose $\underline{F} = A \xrightarrow{\phi} P \xrightarrow{\psi} B$

$\underline{F}' = A \xrightarrow{\phi'} P' \xrightarrow{\psi'} B$, induce the

diagrams





Then, noting that a representative of the equivalence class of $\underline{F}$ is $A \xrightarrow{r_A} \bar{A} \xrightarrow{\psi\bar{\phi}} B$, we have:

$$
\begin{aligned}
& \bar{\underline{\phi}}\,(\{A \xrightarrow{\phi} P \xrightarrow{\psi} B \;+\; A \xrightarrow{\phi'} P \xrightarrow{\psi'} B\}) \\
= \;& \bar{\underline{\phi}}\,(\{A \xrightarrow{r_A} \bar{A} \xrightarrow{\psi\bar{\phi}} B \;+\; A \xrightarrow{r_A} \bar{A} \xrightarrow{\psi'\bar{\phi}'} B\}) \\
= \;& \bar{\underline{\phi}}\,(\{A \xrightarrow{\Delta} A\oplus A \xrightarrow{r_A\oplus r_A} \bar{A}\oplus\bar{A} \xrightarrow{\psi\bar{\phi}\oplus\psi'\bar{\phi}'} B\oplus B \xrightarrow{\nabla} B\})
\end{aligned}
$$

And, since the following diagram commutes:

$$
\begin{array}{ccccc}
A & \xrightarrow{r_A} & \bar{A} & & \\
\downarrow{\scriptstyle\Delta} & & \downarrow{\scriptstyle\Delta} & & \\
A\oplus A & \xrightarrow{r_A\oplus r_A} & \bar{A}\oplus\bar{A} & & \\
& \searrow{\scriptstyle\psi\phi+\psi'\phi'} & \downarrow{\scriptstyle\psi\bar{\phi}\oplus\psi'\bar{\phi}'} & & \\
& & B\oplus B & \xrightarrow{\nabla} & B
\end{array}
$$

$$= \quad \psi\bar{\phi} + \psi'\bar{\phi}' \, .$$

And, $\quad \bar{\underline{\phi}}\,(\{A \xrightarrow{\phi} P \xrightarrow{\psi} B\}) + \bar{\underline{\phi}}\,(\{A \xrightarrow{\phi'} P' \xrightarrow{\psi'} B\})$

$$= \quad \psi\bar{\phi} + \psi'\bar{\phi}' \, .$$

The result is proved.

<u>$\bar{\underline{\phi}}$ is natural</u>

Suppose $A' \xrightarrow{\alpha} A$ is a morphism of $\underline{\underline{A}}$; we have the diagram:

$$
\begin{array}{ccc}
\underline{\underline{P}}\text{-Hom}(A,B) & \xrightarrow{\bar{\phi}} & \text{Hom}(\bar{A},B) \\
\downarrow{\scriptstyle \underline{\underline{P}}\text{-Hom}(\alpha,B)} & & \downarrow{\scriptstyle \text{Hom}(R(\alpha),B)} \\
\underline{\underline{P}}\text{-Hom}(A',B) & \xrightarrow{\bar{\phi}} & \text{Hom}(\bar{A}',B)
\end{array}
$$

The diagram commutes.

Consider

$$\{A \xrightarrow{\phi} P \xrightarrow{\psi} B\} \equiv \{A \xrightarrow{r_A} \bar{A} \xrightarrow{\psi\bar{\phi}} B\}.$$

Travelling in a clockwise direction:

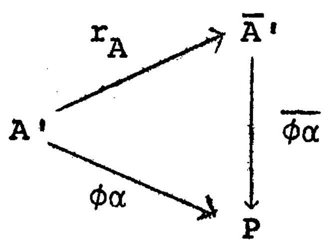
$$\{A \xrightarrow{\phi} P \xrightarrow{\psi} B\} \longrightarrow \bar{A} \xrightarrow{\psi\bar{\phi}} B \longrightarrow \psi\bar{\phi}.R(\alpha).$$

Anticlockwise:

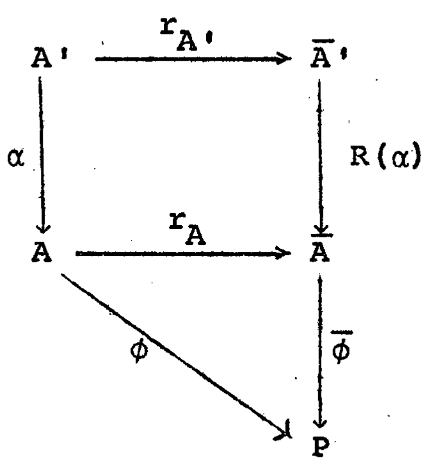
$$\{A \xrightarrow{\phi} P \xrightarrow{\psi} B\} \longrightarrow \{A' \xrightarrow{\phi\alpha} P \xrightarrow{\psi} B\}$$

$$\equiv \quad \{A \longrightarrow \bar{A}' \xrightarrow{\psi\overline{\phi\alpha}} B\} \longrightarrow \psi\overline{\phi\alpha}\ ,$$

where $\overline{\phi\alpha}$ is the unique map making the following diagram commute:



Now



commutes,

so that

$\psi\bar{\phi}R(\alpha) = \psi\overline{\phi\alpha}$. We have proved naturality in the first variable. Naturality in the second variable may be proved in a similar way.

The equivalence relation placed upon $\underline{\underline{P}}$-Hom(A,B) is analogous to the relations which define the tensor product of two modules. Surprisingly, we discover that $\underline{\underline{P}}$-Hom, under certain restrictions, becomes the tensor functor.

Proof

Define scalar multiplication by composition. That is, for $f \in (A,P)$, $\lambda \in End(P)$, the product $\lambda.f$ is the composition λf.

Note 3.24

For every object A of $\underline{\underline{A}}$ the set (P,A) is a right End(P)-module, with scalar multiplication defined by composition. That is: $f.\lambda = f\lambda$.

Note 3.25

$(P,-)$ is a covariant functor $(P,-) : \underline{\underline{A}} \longrightarrow {}_{\underline{\underline{G}}}End(P)$.

Similarly $(-,P)$ is a contravariant functor

$$(-,P) : \underline{\underline{A}} \longrightarrow End(P)_{\underline{\underline{G}}}.$$

In order to prove the first statement we need only show that, given a morphism

$A \xrightarrow{\alpha} A'$ of $\underline{\underline{A}}$, $(P,A) \xrightarrow{(P,\alpha)} (P,A')$ is a morphism of modules. Now, for $P \xrightarrow{f} A$, and $\lambda \in End(P)$

$$\{(P,\alpha)\}(f\lambda) = \alpha f \lambda$$
$$= \{(P,\alpha)(f)\}\lambda.$$

The second statement is proved in a similar way.

Definition 3.26

Given a category $\underline{\underline{A}}$, and a collection of objects $O \subset \underline{\underline{A}}$, the full subcategory generated by O is the subcategory consisting of all the maps between the objects of O.

Proposition 3.27

If $\underline{\underline{P}}$ is the subcategory of $\underline{\underline{A}}$ which is the full subcategory generated by a single object P, and all its (finite) direct sums, then there is a natural isomorphism.

$$\omega : (P,B) \underset{End(P)}{\otimes} (A,P) \longrightarrow \underline{\underline{P}}\text{-Hom}(A,B),$$

for objects A,B of $\underline{\underline{A}}$. That is: $\underline{\underline{P}}$-Hom is naturally equivalent to the composition:

$$\underline{\underline{A}} \times \underline{\underline{A}} \xrightarrow{(P,-)\times(-,P)} \underline{\underline{G}}^{End(P)} \times {}^{End(P)}\underline{\underline{G}} \xrightarrow{\otimes} \underline{\underline{G}}$$

Proof

We outline the form of the proof:

Let F be the free group generated by symbols $\psi \otimes \phi$, where $\psi \in (P,B)$, $\phi \in (A,P)$.

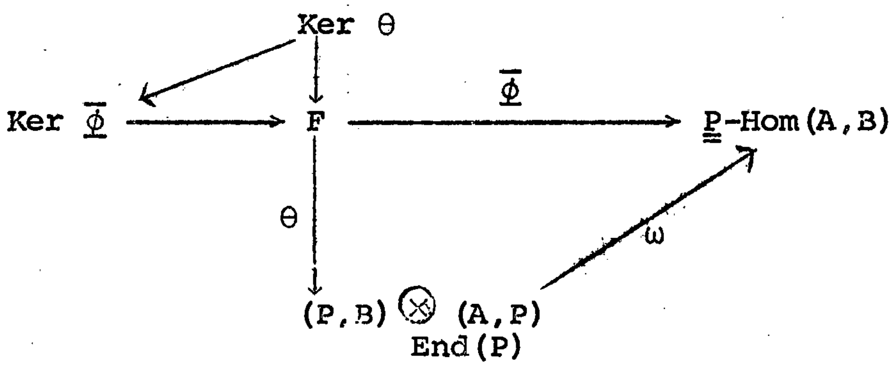
Let θ be the epimorphism

$$F \xrightarrow{\theta} (P,B) \underset{End(P)}{\otimes} (A,P).$$

We shall define $\overline{\underline{\Phi}}' : (P,B) \times (A,P) \longrightarrow \underline{\underline{P}}\text{-Hom}(A,B)$ which will induce an epimorphism

$$\overline{\underline{\Phi}} : F \longrightarrow \underline{\underline{P}}\text{-Hom}(A,B).$$

In the diagram that follows, we prove that Ker $\theta \subset$ Ker $\overline{\underline{\Phi}}$, thus inducing a map ω. It remains only to prove that ω is 1-1, in order to prove that ω is an isomorphism



Now, define $\overline{\underline{\Phi}}' : (P,B) \times (A,P) \longrightarrow \underline{\underline{P}}\text{-Hom}(A,B)$ by $\overline{\underline{\Phi}}'(\psi,\phi) = \{A \xrightarrow{\phi} P \xrightarrow{\psi} B\}$, where the latter symbol represents the equivalence class of $A \xrightarrow{\phi} P \xrightarrow{\psi} B$ in $\underline{\underline{P}}$-Hom(A,B). We obtain an induced mapping $\overline{\underline{\Phi}} : F \longrightarrow$ P-Hom(A,B).

<u>$\overline{\Phi}$ is an epimorphism</u>

We shall have proved this statement when we have shown that $\underline{\underline{P}}$-Hom(A,B) is generated by elements of the form
$\{A \xrightarrow{\phi} P \xrightarrow{\psi} B\}$.

Any element of $\underline{\underline{P}}$-Hom(A,B) has the form

$$\{A \xrightarrow{\begin{bmatrix}\phi_1\\ \vdots\\ \phi_n\end{bmatrix}} \underset{i=1..n}{\oplus} P_i \xrightarrow{(\psi_1,\ldots\psi_n)} B\}$$

where $P_i = P$, $i = 1 \ldots n$.

Now,

$$\{A \xrightarrow{\begin{bmatrix}\phi_1\\ \vdots\\ \phi_{n-1}\end{bmatrix}} \underset{i=1..n-1}{\oplus} P_i \xrightarrow{(\psi_1\ldots\psi_{n-1})} B\} + \{A \xrightarrow{\phi_n} P \xrightarrow{\psi_n} B\}$$

$$= \quad \{A \xrightarrow{\Delta} A\oplus A \xrightarrow{\begin{bmatrix}\phi_1\\ \vdots\\ \phi_{n-1}\end{bmatrix}\oplus\phi_n} \underset{i=1..n}{\oplus} P_i \xrightarrow{(\psi_1\ldots\psi_{n-1})\oplus\psi_n} B\oplus B \xrightarrow{\nabla} B\}$$

$$= \quad \{A \xrightarrow{\begin{bmatrix}\phi_1\\ \vdots\\ \phi_n\end{bmatrix}} \underset{i=1..n}{\oplus} P_i \xrightarrow{(\psi_1\ldots\psi_n)} B\} .$$

By induction,

$$\{A \xrightarrow{\begin{bmatrix}\phi_1\\ \vdots\\ \phi_n\end{bmatrix}} \oplus P_i \xrightarrow{(\psi_1\ldots\psi_n)} B\}$$

$$= \quad \{A \xrightarrow{\phi_1} P_1 \xrightarrow{\psi_1} B\} + \ldots. + \{A \xrightarrow{\phi_n} P_n \xrightarrow{\psi_n} B\}$$

<u>Ker $\theta \subset$ Ker $\overline{\Phi}$</u>

It is sufficient to show that every generator of Ker θ is an element of Ker $\overline{\Phi}$.

$$\overline{\Phi}\,((\psi_1 \otimes \psi_2) \otimes \phi - \psi_1 \otimes \phi - \psi_2 \otimes \phi) = 0,$$

since

$$\overline{\Phi}\,(\psi_1 \otimes \phi) + \overline{\Phi}\,(\psi_2 \otimes \phi)$$

$$= \{A \xrightarrow{\phi} P \xrightarrow{\psi_1} B\} + \{A \xrightarrow{\phi} P \xrightarrow{\psi_2} B\}$$

$$= \{A \xrightarrow{\Delta} A\oplus A \xrightarrow{\phi\oplus\phi} P\oplus P \xrightarrow{\psi_1\oplus\psi_2} B\oplus B \xrightarrow{\nabla} B\}$$

and, since the following diagram commutes:

$$\begin{array}{ccccccccc} A & & \xrightarrow{\quad\phi\quad} & & P & & \xrightarrow{\quad\psi_1+\psi_2\quad} & & B \\ {\scriptstyle 1}\downarrow & & & & \downarrow{\scriptstyle\Delta} & & & & \downarrow{\scriptstyle 1} \\ A & \xrightarrow{\Delta} & A\oplus A & \xrightarrow{\phi\oplus\phi} & P\oplus P & \xrightarrow{\psi_1\oplus\psi_2} & B\oplus B & \xrightarrow{\nabla} & B \end{array},$$

$$= \quad \bar{\Phi}\,((\psi_1 + \psi_2) \otimes \phi) \ .$$

A similar argument establishes that:

$$\bar{\Phi}\,(\psi \otimes (\phi_1 + \phi_2) - \psi \otimes \phi_1 - \psi \otimes \phi_2) = 0 ,$$

since

$$\begin{array}{ccccccccc} A & \xrightarrow{\Delta} & A\oplus A & \xrightarrow{\phi_1\oplus\phi_2} & P\oplus P & \xrightarrow{\psi\oplus\psi} & B\oplus B & \xrightarrow{\nabla} & B \\ {\scriptstyle 1}\downarrow & & & & \downarrow{\scriptstyle\nabla} & & & & \downarrow{\scriptstyle 1} \\ A & & \xrightarrow{\quad\phi_1+\phi_2\quad} & & P & & \xrightarrow{\quad\psi\oplus\psi\quad} & & B \end{array}$$

commutes.

Finally, $\bar{\Phi}\,(\psi\lambda \otimes \phi) \qquad (\lambda \in \mathrm{End}(P))$

$$= \quad A \xrightarrow{\phi} P \xrightarrow{\psi\lambda} B \ .$$

And, since the following diagram commutes:

$$\begin{array}{ccccc} A & \xrightarrow{\phi} & P & \xrightarrow{\psi\lambda} & B \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle\lambda} & & \downarrow{\scriptstyle 1} \\ A & \xrightarrow{\lambda\phi} & P & \xrightarrow{\psi} & B \end{array} .$$

$= \quad \bar{\Phi}\,(\psi \otimes \lambda\phi)$, which provides that $\bar{\Phi}\,(\psi\lambda \otimes \phi - \psi \otimes \lambda\phi) = 0$.

<u>ω is 1-1</u>

The map ω induced by the inclusion $\mathrm{Ker}\ \theta \subset \mathrm{Ker}\ \underline{\overline{\Phi}}$ is defined by

$$\omega(\psi \otimes \phi) = \underline{\overline{\Phi}}\,(\psi \otimes \phi) = A \xrightarrow{\phi} P \xrightarrow{\psi} B\ .$$

Suppose that $\omega(\psi \otimes \phi) \equiv \omega(\psi' \otimes \phi')$.

i.e. $\quad \underline{F} : A \xrightarrow{\phi} P \xrightarrow{\psi} B \equiv \underline{F}' : A \xrightarrow{\phi'} P \xrightarrow{\psi'} B.$

Then there are a finite sequence of natural transformations:

$$\underline{F} = \underline{F}_o \longrightarrow \underline{F}_1 \longrightarrow \cdots \longrightarrow \underline{F}_{2K-1} \longleftarrow \underline{F}_{2K} = \underline{F}'.$$

Let $\underline{F}\alpha_1 \xrightarrow{K_1 \eta K_1'} \underline{F}\alpha_2\,,\ \ldots\ldots,\ \underline{F}\alpha_{n-1} \xrightarrow{K_{n-1}\eta K_{n-1}'} \underline{F}\alpha_n$, denote the exceptional natural transformations, which are such that $K_i \neq 1$ or $K_i' \neq 1$.

We have a commutative diagram:

$$\begin{array}{cccccc} \underline{F}_o : & A & \xrightarrow{\phi_o} & P & \xrightarrow{\psi_o} & B \\ & \downarrow 1 & & \downarrow \lambda_1 & & \downarrow 1 \\ \underline{F}_1 : & A & \xrightarrow{\phi_1} & P & \xrightarrow{\psi_1} & B \end{array}$$

so that $\quad \lambda_1\phi_o = \phi_1\ ; \quad \psi_1\lambda_1 = \psi_o\ .$

Hence $\quad \psi_o \otimes \phi_o = \psi_1\lambda_1 \otimes \phi_o = \psi_1 \otimes \lambda_1\phi_o$

$$= \psi_1 \otimes \phi_1\ .$$

Hence $\quad \psi_o \otimes \phi_o = \psi_1 \otimes \phi_1 = \ldots\ldots = \psi\alpha_1 \otimes \phi\alpha_1\ .$

We have a commutative diagram:

$$\begin{array}{cccccc} \underline{F}\alpha_1 & A & \xrightarrow{\phi\alpha_1} & P & \xrightarrow{\psi\alpha_1} & B \\ & K_1 \downarrow & & \downarrow \lambda\alpha_1 & & \downarrow K_1' \\ \underline{F}\alpha_2 & A & \xrightarrow{\phi\alpha_2} & P & \xrightarrow{\psi\alpha_2} & B \end{array}$$

so that $\lambda\alpha_1 \ \phi\alpha_1 = \phi\alpha_2 K_1 \ ; \ K_1' \psi\alpha_1 = \psi\alpha_2 \lambda\alpha_1$.

Hence $((P,K_1') \otimes 1)(\psi\alpha_1 \otimes \phi\alpha_1) = K_1' \psi\alpha_1 \otimes \phi\alpha_1$

$= \psi\alpha_2 \lambda\alpha_1 \otimes \phi\alpha_1 = \psi\alpha_2 \otimes \lambda\alpha_1 \phi\alpha_1$

$= \psi\alpha_2 \otimes \phi\alpha_2 K_1$

$= (1 \otimes (K_1,P))(\psi\alpha_2 \otimes \phi_2)$

and, similarly,

$$((P,K_1') \otimes 1)(\psi\alpha_1 \otimes \phi\alpha_1) = (1 \otimes (K_1,P))(\psi\alpha_2 \otimes \phi\alpha_2)$$

$$\psi\alpha_2 \otimes \phi\alpha_2 = \psi\alpha_3 \otimes \phi\alpha_3$$

$$((P,K_3') \otimes 1)(\psi\alpha_3 \otimes \phi\alpha_3) = (1 \otimes (K_3,P))(\psi\alpha_3 \otimes \phi\alpha_3)$$

Continuing in this way, and using the technique of Lemma 2.1, again, we obtain

$$((P,K_{n-1}' \dots K_3' K_1') \otimes 1)(\psi\alpha_1 \otimes \phi\alpha_1)$$

$$= (1 \otimes (K_{n-1} \dots K_3 K_1, P))(\psi\alpha_n \otimes \phi\alpha_n)$$

Hence $\psi\alpha_1 \otimes \phi\alpha_1 = \psi\alpha_n \otimes \phi\alpha_n = \psi' \otimes \phi'$.

Our result follows.

<u>ω is natural</u>

Given $A' \xrightarrow{\alpha} A$, a morphism of $\underline{\underline{A}}$, we obtain the diagram

$$\begin{array}{ccc} \underline{\underline{P}}\text{-Hom}(A,B) & \xrightarrow{\underline{\underline{P}}\text{-Hom}(\alpha,B)} & \underline{\underline{P}}\text{-Hom}(A',B) \\ \uparrow \omega & & \uparrow \omega \\ (P,B) \otimes (A,P) & \xrightarrow{1 \otimes (\alpha,P)} & (P,B) \otimes (A',P) \end{array}$$

It is routine to verify that the diagram commutes, and that ω is natural in the second variable.

Corollary 3.28

If $\underline{\underline{A}}$ is the category of right R-modules $\underline{\underline{G}}^R$, and P is chosen as the object R, then there is a natural isomorphism.

$$\underline{\underline{P}}\text{-Hom}(A,B) \cong B \otimes A^*$$

for objects, $A,B \in \underline{\underline{G}}^R$. A* denotes the 'dual of A', which is the left R-module (A,R).

Proof

It is easy to prove that the ring of endomorphisms of the right R-module R, End(R), is ring isomorphic to R . Further, the right End(R)-module (R,B) is isomorphic to the right R-module B under the identification of End(R) and R.

CHAPTER FOUR

2EXTENSIONS - THE PRELIMINARY THEORY

We begin the investigation of generalized extensions, which we call 2Extensions. This is a departure from the theme of the earlier chapters. For, although the system of 2Extensions forms a Baer Sum System, it forms a trivial Baer Sum System, and is not of much interest as such. Instead we investigate the structure of 2Extensions under a finer equivalence relation than the equivalence relation of a Baer Sum System. And, in this way, we obtain a bifunctor, 2Ext, from any abelian category $\underline{\underline{A}}$ to the category of abelian monoids. The reader will notice an analogy between this situation and the situation described in the section on relations and exact squares.

The research for this thesis began with the theory of 2Extensions, and the induced bifunctor 2Ext. Given objects C and A of $\underline{\underline{A}}$, it was discovered that 2Ext (C,A) failed to be a (non-trivial) abelian group under every possible equivalence relation. An investigation of the reason for this failure led to the definition of a Baer Sum System and the characterization of rich bifunctors, such as Ext. Then the failure of 2Ext to be a rich bifunctor became simply evident.

We work in an abelian category $\underline{\underline{A}}$, and make frequent references to the theory of extensions, developed in (14) page 63 ff., or (15) (page 161 ff.).

Kerf, or Kf, will denote the kernel of a map f; Cokf, or Cf, will denote the cokernel of a map f. The symbol $\rightarrowtail$ will denote a monomorphism, and $\twoheadrightarrow$ will denote an epimorphism. As in 3.2, we use the terms pullback, pushout, and bicartesian. $\mathbb{Z}$ denotes the group of integers, and $\mathbb{Z}k$ denotes the group of integers modulo k.

$A \xrightarrow{i_A} A\oplus C$, $A\oplus C \xrightarrow{p_A} A$, denote the injection, and projection, associated with a direct sum. Imf will denote the image of a map f. Foundational problems are treated in the way discussed in Chapter 1.

<u>Definition 4.1</u>

A 2Extension of A by C is a sequence

$$A \rightarrowtail^{x} B_1 \xrightarrow{\rho} B_2 \twoheadrightarrow^{\sigma} C,$$

where $x = \text{Ker }(\sigma\rho)$,

$\sigma = \text{Cok }(\rho x)$.

The next lemma expresses the nature of a 2Extension in a different form:

<u>Lemma 4.2</u>

The commutative diagram below, in which both rows are short exact sequences, represents a 2Extension if, and only if, square 1 is a pullback and square 2 is a pushout.

$$\begin{array}{ccccc} A & \overset{x}{\rightarrowtail} & B_1 & \twoheadrightarrow & C_x \\ \downarrow & 1 & \downarrow & 2 & \downarrow \\ K_\sigma & \longrightarrow & B_2 & \overset{\sigma}{\longrightarrow} & C \end{array}$$

<u>Proof</u>

The proof follows immediately by Lemma 3.14, and its dual.

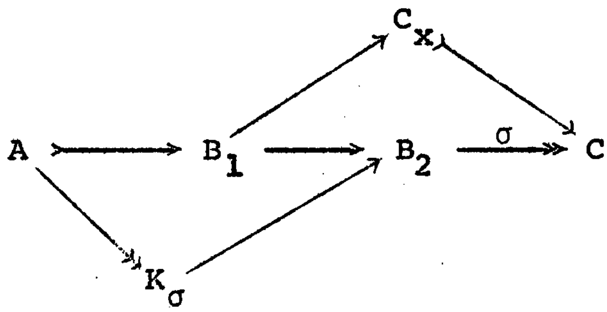
<u>Note 4.3</u>

A converse to Lemma 4.2 is the observation that a 2Extension is naturally represented by two short exact sequences,

$$A \longrightarrow B_1 \longrightarrow C_x \quad \text{and}$$

$$K_\sigma \longrightarrow B_2 \longrightarrow C \quad \text{, in the commutative diagram}$$

below:



where the induced map $A \longrightarrow K_\sigma$ is an epimorphism, and the induced map $C_x \longrightarrow C$ is a monomorphism.

Lemma 4.4 ((6), page 34)

"Cancellation of Limits".

If

$$\begin{array}{ccc} \rho & \xrightarrow{\alpha} & B \\ \beta\downarrow & & \downarrow \\ A & \longrightarrow & C \end{array}$$

is a pullback, and $\rho^1 \xrightarrow{f} \rho$, $\rho^1 \xrightarrow{g} \rho$, are two morphisms of $\underline{\underline{A}}$, then:

$$\alpha f = \alpha g \quad \text{and} \quad \beta f = \beta g$$

$$\Longrightarrow \quad f = g$$

Proof

The proof is immediate, by the properties of pullbacks. We shall frequently refer to the effect of this Lemma, or its dual, as "Cancellation of Limits".

Definition 4.5

2Ext (C,A) will denote the class of 2Extensions of A by C, under the equivalence relation defined by: $E \equiv E'$ if there is a commutative diagram:

$$\begin{array}{ccccccccc} E & = & A & \xrightarrow{x} & B_1 & \xrightarrow{\rho} & B_2 & \xrightarrow{\sigma} & C \\ & & \downarrow 1 & & \downarrow \beta_1 & & \downarrow \beta_2 & & \downarrow 1 \\ E' & = & A & \xrightarrow{x'} & B_1' & \xrightarrow{\rho'} & B_2' & \xrightarrow{\sigma'} & C \end{array}$$

where β_1 and β_2 are isomorphisms. The vertical maps in such a diagram will together be called a morphism of 2Extensions, and we shall write:

$$E \xrightarrow{(1,\beta_1,\beta_2,1)} E'$$

Note 4.6

The reader will recall ((14), page 64) that if E and E' are extensions of A by C, then the existence of a morphism $E \xrightarrow{(1,\beta,1)} E'$, implies that β is an isomorphism. The proof of this statement uses the "Five Lemma" ((14), page 14). The theory of 2Extensions does not have the benefit of the "Five Lemma", and in fact we may have a morphism $E \xrightarrow{(1,\beta_1,\beta_2,1)} E'$, as in the diagram below, in which neither β_1 nor β_2 is an isomorphism.

$$\begin{array}{ccccccccc} E & = & \mathbb{Z}_2 & \xrightarrow{1} & \mathbb{Z}_2 & \xrightarrow{i_1} & \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \xrightarrow{p_2} & \mathbb{Z}_2 \\ & & \downarrow 1 & & \downarrow i_1 & & \downarrow p_2 & & \downarrow 1 \\ E' & = & \mathbb{Z}_2 & \xrightarrow{i_1} & \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \xrightarrow{p_2} & \mathbb{Z}_2 & \xrightarrow{1} & \mathbb{Z}_2 \end{array}$$

The theory of 2Extensions differs, therefore, in a fundamental way, from the theory of extensions, and the theory of a Baer Sum System.

Example 4.7

We illustrate our definitions with some examples of distinct 2Extensions of the class 2Ext $(\mathbb{Z}_2, \mathbb{Z}_2)$:

$$\mathbb{Z}_2 \rightarrowtail \mathbb{Z}_4 \twoheadrightarrow \mathbb{Z}_2 \xrightarrow{1} \mathbb{Z}_2$$

$$\mathbb{Z}_2 \xrightarrow{1} \mathbb{Z}_2 \rightarrowtail \mathbb{Z}_4 \twoheadrightarrow \mathbb{Z}_2$$

$$\mathbb{Z}_2 \rightarrowtail \mathbb{Z}_4 \xrightarrow{1} \mathbb{Z}_4 \twoheadrightarrow \mathbb{Z}_2$$

$$\mathbb{Z}_2 \rightarrowtail \mathbb{Z}_2 \oplus \mathbb{Z}_2 \twoheadrightarrow \mathbb{Z}_2 \xrightarrow{1} \mathbb{Z}_2$$

$$\mathbb{Z}_2 \xrightarrow{1} \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \xrightarrow{1} \mathbb{Z}_2$$

There are other elements of the class. An example of a 2Extension in which the middle morphism is neither a monomorphism, an epimorphism, or a zero map, is

$$\mathbb{Z}_4 \xrightarrow{i} \mathbb{Z}_4 \oplus \mathbb{Z}_3 \xrightarrow{\begin{pmatrix} p & o \\ o & i \end{pmatrix}} \mathbb{Z}_2 \oplus \mathbb{Z}_6 \xrightarrow{p} \mathbb{Z}_6$$

By analogy with the theory of Baer Sum Systems we define, for every $E \in$ 2Ext (C,A), and morphism $C' \xrightarrow{\gamma} C$, a 2Extension $E\gamma$.

Proposition 4.8

If $E \in$ 2Ext (C,A), and $C' \xrightarrow{\gamma} C$ is a morphism of $\underline{\underline{A}}$, $\exists\, E\gamma \in$ 2Ext (C',A), and a morphism of 2Extensions

$$E\gamma \xrightarrow{(1,\beta_1,\beta_2,\gamma)} E\ .$$

Proof

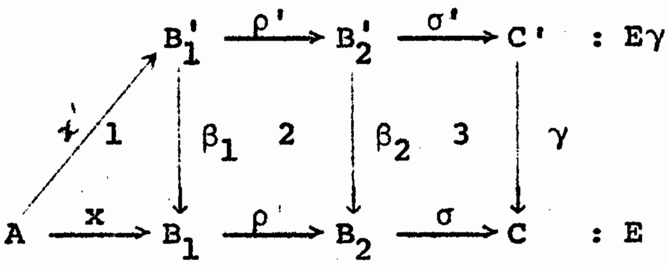
We make an explanation concerning our method of proof. Our results will be valid in any abelian category $\underline{\underline{A}}$. But in the following proof, and in subsequent proofs, we assume that $\underline{\underline{A}}$ is the abelian category ${}^R\underline{\underline{G}}$ of R-modules, for some ring R.

We make this assumption without loss of generality, for the following reason. The "Embedding Theorem" due to Freyd (5), Lubkin (13), and Mitchell (15) or (16), asserts that any statement concerning the exactness and existence of sequences of maps of an abelian category, which is true for all categories of R-modules, is true for all abelian categories.

We are thus permitted proofs by "diagram chasing"!

If E is the 2Extension

$A \xrightarrow{x} B_1 \xrightarrow{\rho} B_2 \xrightarrow{\sigma} C$, make the construction:



in which square 3 is a pullback, and the composite square 23 is a pullback. Then square 2 is a pullback, and the morphism $A \longrightarrow B_1'$ is induced by the zero map $A \xrightarrow{0} C'$.

It is well known, and routine to verify, that B_1', B_2' are described in ${}^R\underline{\underline{G}}$ by:

$$B_2' \subset B_2 \oplus C' = \{(b_2, c') \mid \sigma(b_2) = \gamma(c')\}$$
$$B_1' \subset B_1 \oplus C' = \{(b_1, c') \mid \sigma\rho(b_1) = \gamma(c')\}$$

and the maps σ', ρ', β_1, β_2, x' are described by

$$\sigma'(b_2, c') = c'$$
$$\rho'(b_1, c') = (\rho(b_1), c')$$
$$\beta_1(b_1, c') = b_1$$
$$\beta_2(b_2, c') = b_2$$
$$x'(a) = (x(a), 0)$$

Then it is easy to verify that the diagram commutes. Further, the sequence

$$E \;:\; A \xrightarrow{x'} B_1' \xrightarrow{\rho'} B_2' \xrightarrow{\sigma'} C' \text{ is a 2Extension,}$$

for

$$\begin{aligned} \operatorname{Ker} \sigma'\rho' &= \{(b_1,c') \mid \sigma'\rho'(b_1,c') = o\} \\ &= \{(b_1,o) \mid \sigma\rho(b_1) = \gamma(o) = o\} \\ &= \{(x(a),o) \mid a \in A\} \\ &= \operatorname{Im} x' \end{aligned}$$

and

$$\begin{aligned} \operatorname{Im} x'\rho' &= \{(\rho x(a),o)\} \\ &= \{(k,o) \mid k \in \ker\sigma\} \\ &= \operatorname{Ker} \sigma' \end{aligned}$$

<u>Proposition 4.9</u>

If $E \in 2\text{Ext}\ (C,A)$ and $A \xrightarrow{\alpha} A'$ is a morphism of $\underline{\underline{A}}$, $\exists\, \alpha E \in 2\text{Ext}\ (C,A')$ and a morphism

$$E \xrightarrow{(\alpha,\beta_1,\beta_2,1)} \alpha E$$

<u>Proof</u>

The proof is dual to Proposition 4.8. Make the construction:

$$\begin{array}{ccccccccc} E & : & A & \xrightarrow{x} & B_1 & \xrightarrow{\rho} & B_2 & \xrightarrow{\sigma} & C \\ & & \downarrow{\scriptstyle\alpha} & 1 & \downarrow{\scriptstyle\beta_1} & 2 & \downarrow{\scriptstyle\beta_2} & 3 \nearrow {\scriptstyle\sigma'} & \\ \alpha E & : & A' & \xrightarrow{x'} & B_1' & \xrightarrow{\rho'} & B_2' & & \end{array}$$

in which square 1, and the composite square 12 are pushout squares. We note that B_1', and B_2' are described in ${}^{R}\underline{\underline{G}}$ by

$$B_1' = A'\oplus B_1/N_1, \text{ where } N_1 = \{(-\alpha(a),x(a) \mid a \in A\}$$

$$B_2' = A'\oplus B_2/N_2, \text{ where } N_2 = \{(-\alpha(a),\rho x(a) \mid a \in A\}.$$

The result follows.

The next proposition describes the construction of αE in terms of the description of E set out in Note 4.3. This description is essential for the important Proposition 4.13, which follows.

Proposition 4.10

If $A \xrightarrow{\alpha} A'$ is a morphism of $\underline{\underline{A}}$, and the 2Extension

$$E : A \rightarrowtail B_1 \longrightarrow B_2 \twoheadrightarrow C$$ is represented,

following Note 4.3, by the short exact sequences

$$E_1 : A \rightarrowtail B_1 \twoheadrightarrow C_x,$$

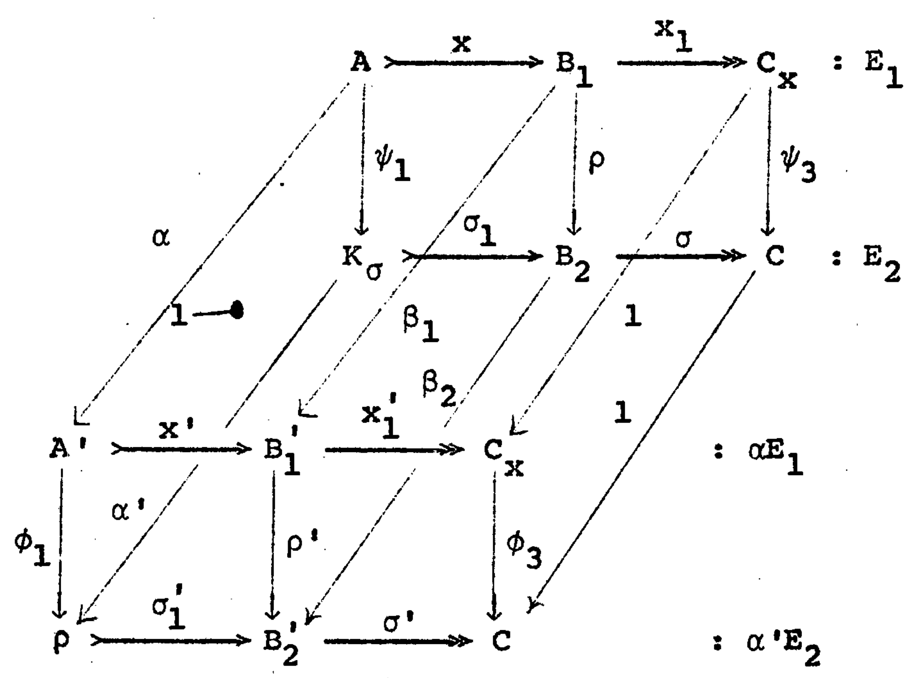
$$E_2 : K\sigma \rightarrowtail B_2 \twoheadrightarrow C ,$$

then αE may be constructed, as in the diagram that follows, by:

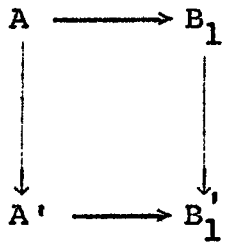
1. Constructing αE_1.
2. Constructing the pushout 1, and inducing the map α'.
3. Constructing $\alpha' E_2$.

αE_1, $\alpha' E_2$ are the compositions defined in (14), page 66, and previously used in Chapter 2.

Proof



The reader will recall that



is a pushout. Then, since $\beta_2 \rho x = \sigma_1' \phi_1 \alpha$, the morphism ρ' is induced, and $\rho' \beta_1 = \beta_2 \rho$; $\rho' x' = \sigma_1' \phi_1$.

Then $\phi_3 x_1' \beta_1 = \sigma' \rho' \beta_1$

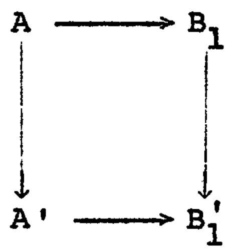
and $\phi_3 x_1' x' = \sigma' \rho' x'$ (= o)

By cancellation of limits (Lemma 4.4)

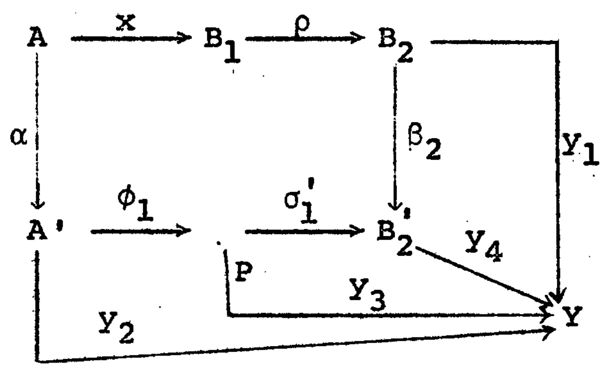
$$\phi_3 x_1' = \sigma' \rho' .$$

$\phi_3 = \psi_3$, and the diagram is commutative.

Furthermore, the nearest rectangle represents the 2Extension αE. Since



is a pushout, the object B_1' and the map x' are the corresponding object and map of αE. We show that the rectangle in the diagram:



is a pushout. To this end, suppose that the object Y, and the maps y_1, y_2 are given so that $y_1 \rho x = y_2 \alpha$. Since 1 is a pushout, by construction, $\exists\, y_3$, and $y_3\phi_1 = y_2$, and $y_3\alpha' = y_1\sigma_1$. Since

$$\begin{array}{ccc} K_\sigma & \longrightarrow & B_2 \\ \downarrow & & \downarrow \\ P & \longrightarrow & B_2' \end{array}$$

is a pushout, $\exists\, y_4$, and $y_4\sigma_1' = y_3$, and $y_4\beta_2 = y_1$. Hence $y_4\sigma_1'\phi_1 = y_2$, and $y_4\beta_2 = y_1$, proving the result.

We have shown now that B_2' and ρ' are the corresponding object and map of αE. σ' is the unique map induced by $A' \xrightarrow{0} C$, and is the corresponding map of αE. This completes the proof.

Note 4.11

The dual of this construction provides a description of $E\gamma$, where $C' \xrightarrow{\gamma} C$ is a morphism of $\underline{\underline{A}}$.

In Note 4.6 we pointed to a fundamental difference between our theory and the theory of extensions and Baer Sum Systems. This difference appears in the next remark. The reader should compare this result with condition 3 of Proposition 1.13.

Remark 4.12

A morphism $(\alpha,\beta_1,\beta_2,\gamma) : E \longrightarrow E'$ of 2Extensions does not always imply that $\alpha E \equiv E'\gamma$. For, let E and E' be the top and bottom rows in the commutative diagram:

$$\begin{array}{lccccccc} E: & \mathbb{Z}_2 & \xrightarrow{i} & \mathbb{Z}_2\oplus\mathbb{Z}_2 & \xrightarrow{p} & \mathbb{Z}_2 & \xrightarrow{1} & \mathbb{Z}_2 \\ & \downarrow 0 & & \downarrow 0\oplus i & & \downarrow i & & \downarrow i \\ E': & \mathbb{Z}_3 & \xrightarrow{i} & \mathbb{Z}_3\oplus\mathbb{Z}_4 & \xrightarrow{p} & \mathbb{Z}_4 & \xrightarrow{1} & \mathbb{Z}_4 \end{array}$$

Then OE is $\mathbb{Z}_3 \xrightarrow{i} \mathbb{Z}_3 \oplus \mathbb{Z}_2 \xrightarrow{1} \mathbb{Z}_3 \oplus \mathbb{Z}_2 \xrightarrow{p} \mathbb{Z}_2$
and $E'i$ is $\mathbb{Z}_3 \xrightarrow{i} \mathbb{Z}_3 \oplus \mathbb{Z}_2 \xrightarrow{p} \mathbb{Z}_2 \xrightarrow{1} \mathbb{Z}_2$, as the reader can verify. Clearly

$$OE \not\equiv E'i \, .$$

We recover some of this loss in the next proposition, which will be indispensable in subsequent work:

<u>Proposition 4.13</u>

A morphism $(\alpha, \beta_1, \beta_2, \gamma) : E \longrightarrow E'$ of the 2Extensions:

$$E \ : \ A \xrightarrow{x} B_1 \xrightarrow{\rho} B_2 \xrightarrow{\sigma} C$$
$$E' \ : \ A \xrightarrow{x'} B_1' \xrightarrow{\rho'} B_2' \xrightarrow{\sigma'} C'$$

implies that $\alpha E \equiv E'\gamma$ if, and only if, the induced map $K\rho x \longrightarrow K\rho' x'$ is an epimorphism and the induced map $C\sigma\rho \longrightarrow C\sigma'\rho'$ is a monomorphism.

<u>Proof</u>

Represent E by the pair of short exact sequences (E_1, E_2), and E' by the pair (E_1', E_2').

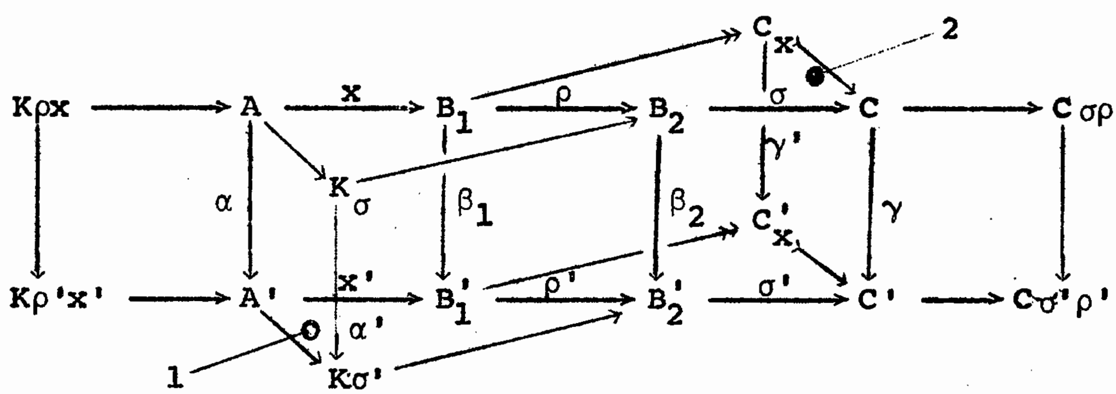
Then E_1 is $A \longrightarrow B_1 \longrightarrow C_x$

E_2 is $K_\sigma \longrightarrow B_2 \longrightarrow C$

E_1' is $A' \longrightarrow B_1' \longrightarrow C_x'$

E_2' is $K_\sigma' \longrightarrow B_2' \longrightarrow C'$

The morphism $(\alpha, \beta_1, \beta_2, \gamma)$ induces the commutative diagram:



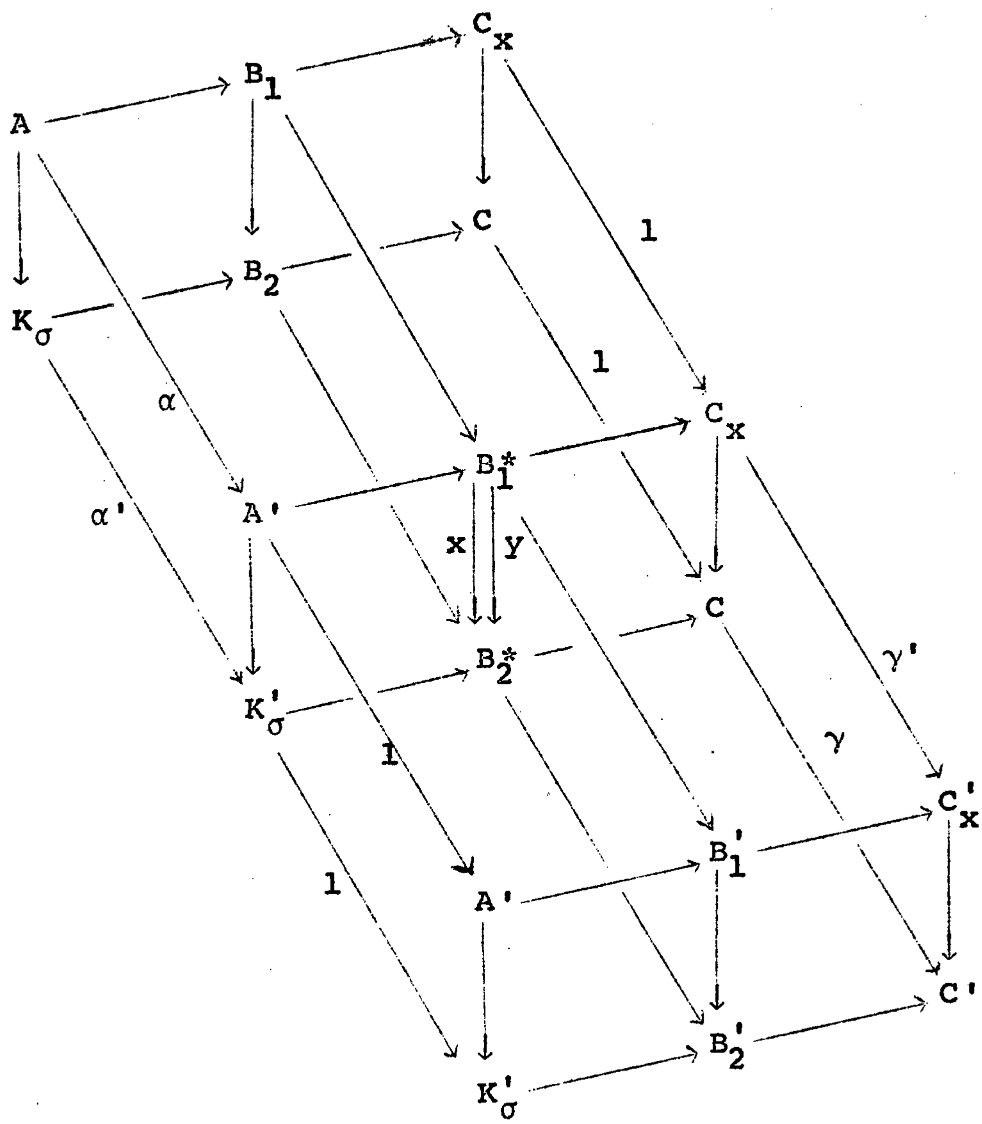
1. $\alpha E \equiv E\gamma'$ if and only if 1 is a pushout and 2 is a pullback.

For, if $\alpha E \equiv E\gamma'$, 1 is a pushout and 2 is a pullback, by the construction of Proposition 4.4 and its dual.

Conversely, if 1 is a pushout and 2 is a pullback, we have, by the commutativity of the diagram:

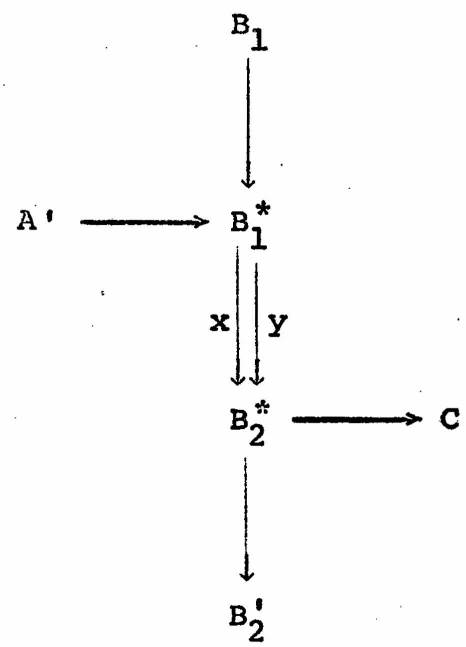
$$\alpha E_1 \equiv E_1'\gamma'$$
$$\alpha E_2 \equiv E_2'\gamma \ .$$

We may now draw the diagram:



The rectangle in the middle, together with the left map, x, represents αE. The same rectangle, together with the right map, y, represents $E\gamma'$. When we have proved that x = y, we shall have proved our result.

From the diagram above we may extract the diagram:



There is a single morphism

$A' \longrightarrow C$, $A' \longrightarrow B_2'$, $B_1 \longrightarrow C$, $B_1 \longrightarrow B_2'$.

Applying Lemma 4.4 twice we obtain, by cancellation of limits, that

$$x = y \ .$$

2. 1 is a pushout and 2 a pullback if and only if $\alpha(K\rho x) = K\rho' x'$ and Im $(Cx \longrightarrow C)$ = Ker $(C \xrightarrow{\gamma} C' \longrightarrow C\sigma'\rho')$.

For, recalling that $A \longrightarrow K\sigma$ is an epimorphism, and $Cx \longrightarrow C$ is a monomorphism, we have exact sequences.

$$K\rho x \longrightarrow A \longrightarrow K\sigma \longrightarrow 0 \quad \text{and}$$

$$0 \longrightarrow Cx' \longrightarrow C' \longrightarrow C\sigma'\rho' \quad .$$

The result follows immediately, by Lemma 3.14, and its dual.

3. $\alpha(K\rho x) = K\rho' x'$ and Im $(Cx \longrightarrow C)$ = Ker $(C \xrightarrow{\gamma} C' \longrightarrow C\sigma'\rho')$ $\Longleftrightarrow$

$K\rho x \longrightarrow K\rho' x'$ is an epimorphism and

$C\sigma\rho \longrightarrow C\sigma'\rho'$ is a monomorphism.

This follows by an easy diagram chase. This completes the proof.

<u>Proposition 4.14</u>

If E is the 2Extension, $A \xrightarrow{x} B_1 \xrightarrow{\rho} B_2 \xrightarrow{\sigma} C$, and $A \xrightarrow{\alpha} A'$ amd $C' \xrightarrow{\gamma} C$ are morphisms of $\underline{\underline{A}}$, then

$(\alpha E)\gamma \equiv \alpha(E\gamma)$.

<u>Proof</u>

Represent E by the pair of short exact sequences (E_1, E_2). If α' is the map defined by the pushout

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & A' \\ \downarrow & & \downarrow \\ K_\sigma & \xrightarrow{\alpha'} & P_1 \end{array}$$

, and γ' is the map defined by the pullback

$$\begin{array}{ccc} P_2 & \xrightarrow{\gamma'} & C_x \\ \downarrow & & \downarrow \\ C' & \xrightarrow{\gamma} & C \end{array}$$

, we have

$$\begin{aligned} \alpha E &= (\alpha E_1, \alpha' E_2) \\ (\alpha E)\gamma &= ((\alpha E_1)\gamma', (\alpha' E_2)\gamma) \\ E\gamma &= (E_1\gamma', E_2\gamma) \\ \alpha(E\gamma) &= (\alpha(E_1\gamma'), \alpha'(E_2\gamma)) \end{aligned}$$

and

$$(\alpha E_1)\gamma' \equiv \alpha(E_1\gamma') \;;\quad (\alpha' E_2)\gamma \equiv \alpha'(E_2\gamma) .$$

By drawing a diagram, and applying Lemma 4.4 twice, the reader can prove that this pair of congruences represents a congruence $(\alpha E)\gamma \equiv \alpha(E\gamma)$.

<u>Proposition 4.15</u>

Let S be the category of abelian monoids.

2Ext is an additive bifunctor

$$2\text{Ext} : \underline{\underline{A}} \times \underline{\underline{A}} \longrightarrow S ,$$

contravariant in the first and covariant in the second variable.

<u>Proof</u>

Let $E, E' \in 2Ext(C,A)$.

Part 1. (i) $1E \equiv E$ (Clear)

(i)* $E1 \equiv E$ (Clear)

(ii) $(\alpha'\alpha)E \equiv \alpha'(\alpha E)$.

We have a commutative diagram:

$$\begin{array}{lcccccccc}
E & : & A & \xrightarrow{x} & B_1 & \xrightarrow{\rho} & B_2 & \xrightarrow{\sigma} & C \\
 & & \downarrow{\scriptstyle\alpha} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle 1} \\
\alpha E & : & A' & \xrightarrow{x'} & B_1' & \xrightarrow{\rho'} & B_2' & \xrightarrow{\sigma'} & C \\
 & & \downarrow{\scriptstyle\alpha'} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle 1} \\
\alpha'(\alpha E) & : & A'' & \xrightarrow{x''} & B_1'' & \xrightarrow{\rho''} & B_2'' & \xrightarrow{\sigma''} & C
\end{array}$$

$K\rho x \longrightarrow K\rho'x'$, and $K\rho'x' \longrightarrow K\rho''x''$ are epimorphisms, by Proposition 4.13. Hence $K\rho x \longrightarrow K\rho''x''$ is an epimorphism. By 4.13, $\alpha'(\alpha E) \equiv (\alpha'\alpha)E$.

(ii)* $E(\gamma\gamma') \equiv (E\gamma)\gamma'$, dually.

Part 2. (i) $(\alpha\oplus\alpha')(E\oplus E') \equiv \alpha E\oplus\alpha'E'$

$$\begin{array}{lcccccccc}
\text{Let } E & : & A & \xrightarrow{x} & B_1 & \xrightarrow{\rho} & B_2 & \xrightarrow{\sigma} & C \\
\alpha E & : & D & \xrightarrow{w} & F_1 & \xrightarrow{y} & F_2 & \xrightarrow{z} & G \\
E' & : & A' & \xrightarrow{x'} & B_1' & \xrightarrow{\rho'} & B_2' & \xrightarrow{\sigma'} & C' \\
\alpha'E' & : & D' & \xrightarrow{w'} & F_1' & \xrightarrow{y'} & F_2' & \xrightarrow{z'} & G'
\end{array}$$

We have a commutative diagram:

$$\begin{array}{lccccccc} E\oplus E' : & A\oplus A' & \xrightarrow{x\oplus x'} & B_1\oplus B_1' & \xrightarrow{\rho\oplus\rho'} & B_2\oplus B_2' & \xrightarrow{\sigma\oplus\sigma'} & C\oplus C' \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \alpha E\oplus\alpha' E' : & D\oplus D' & \xrightarrow[w\oplus w']{} & F_1\oplus F_1' & \xrightarrow[y\oplus y']{} & F_2\oplus F_2' & \xrightarrow[z\oplus z']{} & G\oplus G' \end{array}$$

Noting that Ker $(\rho\oplus\rho'.x\oplus x')$ = Ker $(\rho x)\oplus$ Ker $(\rho' x')$

and Ker $(y\oplus y'.w\oplus w')$ = Ker $(yw)\oplus$ Ker $(y'w')$,

and that Ker $(\rho x) \longrightarrow$ Ker (yw)

and Ker $(\rho' x') \longrightarrow$ Ker $(y'w')$ are epimorphisms,

we conclude that

Ker $(\rho\oplus\rho'.x\oplus x') \longrightarrow$ Ker $(y\oplus y.w\oplus w')$ is an epimorphism. Dually

Cok $(\sigma\oplus\sigma'.\rho\oplus\rho') \longrightarrow$ Cok $(z\oplus z'.y\oplus y')$ is a monomorphism. The result follows by Proposition 4.13.

(i)* $(E\oplus E')(\gamma\oplus\gamma') \equiv E\gamma\oplus E'\gamma'$, dually.

Define an addition in 2Ext (C,A), by

$E + E' = \nabla(E\oplus E')\Delta$. It is clear that addition is well defined.

(ii) $\alpha(E + E') \equiv \alpha E + \alpha E'$.

$\alpha(E + E') = \alpha\nabla(E\oplus E')\Delta = \nabla(\alpha\oplus\alpha)(E\oplus E')\Delta$

$= \nabla(\alpha E\oplus\alpha E')\Delta = \alpha E + \alpha E'$.

(ii)* $(E + E')\gamma \equiv E\gamma + E\gamma'$, dually.

Part 3.

2Ext (C,A) is an abelian monoid, i.e. a commutative semigroup with a zero.

The proof that '+' is associative and commutative is the same as the proof of Section (III) of 1.13. We need remark only that the morphism

$(\tau A, \tau B_1, \tau B_2, \tau C) : E \oplus E' \longrightarrow E' \oplus E$ satisfies the conditions of 4.13, and provides that

$$\tau A\ (E \oplus E') \equiv (E' \oplus E)\tau C\ .$$

The zero of 2Ext (C,A) is the element:

$$E_o \ : \ A \xrightarrow{i} A \oplus C \xrightarrow{1} A \oplus C \xrightarrow{p} C.$$

This may be proved by computation, with the assistance of 4.13. We omit the details, which are tedious.

Part 4.

2Ext is a bifunctor.

Given $A \xrightarrow{\alpha} A'$, define 2Ext $(1,\alpha)(E) = \alpha E$.

Dually given, $C' \xrightarrow{\gamma} C$, define 2Ext $(\gamma,1)E = E\gamma$.

Part (1) provides that 2Ext is a functor in each variable, and 4.14 that 2Ext is a bifunctor. Part 2 proves 2Ext additive.

<u>Note 4.15</u>

2Ext (C,A) is never a (non-trivial) abelian group. For the element

$$E^* \ : \ A \xrightarrow{1} A \xrightarrow{o} C \xrightarrow{1} C$$

has the property that, for any $E \in$ 2Ext (C,A) $E + E^* = E^*$, as computation will verify. Clearly such an element cannot exist in a (non-trivial) abelian group. And the existence of this element proves that under any possible equivalence relation 2Ext (C,A) fails to be a (non-trivial) abelian group.

2Ext is a Baer Sum System. 2Ext can be made into a Baer Sum System in the obvious way. αE and $E\gamma$ are defined as we have just defined them and, under the coarser equivalence relation, a morphism $(\alpha, \beta_1, \beta_2, \gamma) : E \longrightarrow E'$ implies that $\alpha E \equiv E\gamma$. The last statement may be verified trivially, since for any $E \in 2\mathrm{Ext}\,(C,A)\ \exists\ E_o \xrightarrow{\{1\eta 1\}} E$, for

$$\begin{array}{cccccccc} & A & \xrightarrow{\ i\ } & A\oplus C & \xrightarrow{(1,o)} & A\oplus B_2 & \xrightarrow{(o,\sigma)} & C \\ & \Big\downarrow{\scriptstyle 1} & & \Big\downarrow{\scriptstyle (x,o)} & & \Big\downarrow{\scriptstyle (\rho x,1)} & & \Big\downarrow{\scriptstyle 1} \\ E: & A & \xrightarrow{\ x\ } & B_1 & \xrightarrow{\ \rho\ } & B_2 & \xrightarrow{\ \sigma\ } & C \end{array}$$

and

$$\begin{array}{cccccccc} & A & \xrightarrow{\ i\ } & A\oplus C & \xrightarrow{(1,o)} & A\oplus B_2 & \xrightarrow{(o,\sigma)} & C \\ & \Big\downarrow{\scriptstyle 1} & & \Big\downarrow{\scriptstyle (1,o)} & & \Big\downarrow{\scriptstyle (1,\sigma)} & & \Big\downarrow{\scriptstyle 1} \\ E_o: & A & \xrightarrow{\ i\ } & A\oplus C & \xrightarrow{\ 1\ } & A\oplus C & \xrightarrow{\ p\ } & C \end{array}$$

commute.

The reader should compare this result with the theory for exact squares (ES).

CHAPTER FIVE

THE STRUCTURE OF 2EXTENSIONS

Given objects D and A of $\underline{\underline{A}}$, we obtain a family of natural monomorphisms:

$$\{D'_*\} : \text{Ext}\ (D,A) \longrightarrow \text{2Ext}\ (D,A).$$

The family $\{D'_*\}$ is in 1-1 correspondence with the family $\{D'\}$ of subobjects of D.

If $A \rightarrowtail B \twoheadrightarrow C$ is a short exact sequence in $\underline{\underline{A}}$, the reader will recall (14) that:

$\text{Hom}\ (D,A) \rightarrowtail \text{Hom}\ (D,B) \longrightarrow \text{Hom}\ (D,C) \longrightarrow \text{Ext}\ (D,A) \longrightarrow \text{Ext}\ (D,B)$ $\longrightarrow \text{Ext}\ (D,C) \longrightarrow$ is a long exact sequence of abelian groups. In analogy, we define a functor LHom, and obtain a sequence:

$\text{LHom}\ (D,A) \rightarrowtail \text{LHom}\ (D,B) \longrightarrow \text{LHom}\ (D,C) \longrightarrow \text{2Ext}\ (D,A) \longrightarrow$ $\text{2Ext}\ (D,B) \longrightarrow \text{2Ext}\ (D,C) \longrightarrow$ which we discuss briefly. The result we obtain is weaker than the classical result.

Definition 5.1

If A and C are objects of $\underline{\underline{A}}$, LHom (A,C) is the class of diagrams:

$$E \ : \ A' \overset{j}{\rightarrowtail} A \longrightarrow C,$$ where j is a monomorphism,

under the equivalence relation

$$A' \overset{j}{\rightarrowtail} A \longrightarrow C \ \equiv \ A'' \overset{j''}{\rightarrowtail} A \longrightarrow C$$

if, and only if, there is an isomorphism ϕ and a commutative diagram:

$$\begin{array}{ccccc} A' & \overset{j}{\rightarrowtail} & A & \longrightarrow & C \\ \phi\downarrow & & \downarrow 1 & & \downarrow 1 \\ A'' & \overset{j''}{\rightarrowtail} & A & \longrightarrow & C \end{array}$$

Proposition 5.2

Let S be the category of abelian monoids. LHom is an additive bifunctor

$$\mathrm{LHom} : \underline{\underline{A}} \times \underline{\underline{A}} \longrightarrow S ,$$

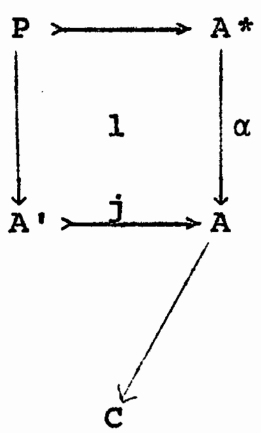
contravariant in the first, and covariant in the second variable.

Proof

Part 0.

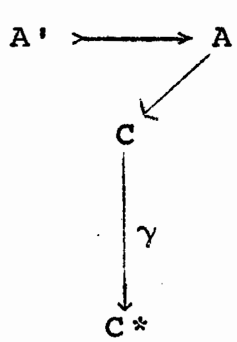
Let $E : A' \overset{j}{\rightarrowtail} A \longrightarrow C \quad \epsilon \quad \mathrm{LHom}(A,C)$.

If $A^* \xrightarrow{\alpha} A$ is a morphism of $\underline{\underline{A}}$, define $E\alpha$ to be $P \rightarrowtail A^* \longrightarrow C$, in the construction that follows, where square 1 is a pull-back.



We note that, by the properties of pullbacks, $P \rightarrowtail A^*$ is a monomorphism (see (5), page 52).

If $C \xrightarrow{\gamma} C^*$ is a morphism of $\underline{\underline{A}}$, define γE to be the diagram $A' \rightarrowtail A \longrightarrow C^*$ in the construction:



Part 1.

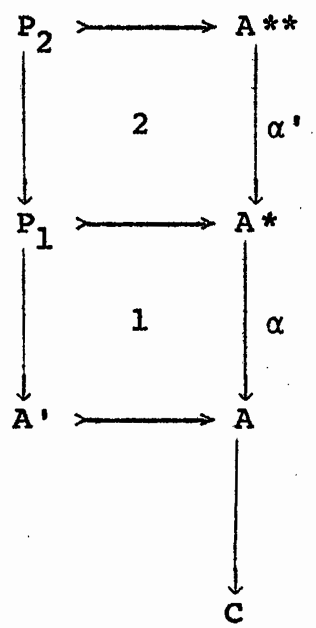
(i) $1E \equiv E$ (Clear)

(i)* $E1 \equiv E$ (Clear)

(ii) $(\gamma'\gamma)E \equiv \gamma'(\gamma E)$ (Clear)

(ii)* $E(\alpha\alpha') \equiv (E\alpha)\alpha'$.

In the diagram below, since 1 and 2 are pullbacks, the composite square 21 is a pullback, and the result follows:



(iii) $(\alpha E)\gamma \equiv \alpha(E\gamma)$

Part 2.

If $E_1 \in \text{LHom}(A_1,C_1)$ and $E_2 \in \text{LHom}(A_2,C_2)$ then $E_1 \oplus E_2 \in \text{LHom}(A_1 \oplus A_2,\ C_1 \oplus C_2)$, and

(i) $(\gamma_1 \oplus \gamma_2)(E_1 \oplus E_2) \equiv (\gamma_1 E_1 \oplus \gamma_2 E_2)$ (Clear)

(i)* $(E_1 \oplus E_2)(\alpha_1 \oplus \alpha_2) \equiv E_1\alpha_1 \oplus E_2\alpha_2$

For, suppose $E_1 : A_1' \rightarrowtail A_1 \twoheadrightarrow C_1$

$E_2 : A_2' \rightarrowtail A_2 \twoheadrightarrow C_2$

$E_1\alpha_1 : P_1 \rightarrowtail A_1^* \twoheadrightarrow C_1$

$E_2\alpha_2 : P_2 \rightarrowtail A_2^* \twoheadrightarrow C_2$, Then we shall have proved our result when we have shown that 1, in the diagram below, is a pullback:

$$
\begin{array}{ccc}
P_1 \oplus P_2 & \rightarrowtail & A_1^* \oplus A_2^* \\
\downarrow & 1 & \downarrow \alpha_1 \oplus \alpha_2 \\
A_1' \oplus A_2' & \rightarrowtail & A_1 \oplus A_2 \\
 & & \downarrow \\
 & & C_1 \oplus C_2
\end{array}
$$

1 is the direct sum of two pullbacks, and it is routine to verify that the direct sum of two pullbacks is a pullback.

Define an addition in LHom (A,C) by $E + E' = \nabla(E \oplus E')\Delta$. Addition is well-defined.

(ii) $\gamma(E + E') \equiv \gamma E + \gamma E'$. The proof is the same as the corresponding proof in Proposition 4.15.

(ii)* $(E + E')\gamma \equiv E\gamma + E'\gamma$.

Part 3.

LHom (A,C) is an abelian monoid. The proof is the same as the corresponding proof in 4.15. We note that $A \xrightarrow{1} A \xrightarrow{0} C$ is a zero for LHom (A,C), which is quickly proved by computation.

That $\tau_C\ (E_1 \oplus E_2) \equiv (E_1 \oplus E_2)\ \tau_A$ is easily established.

Part 4.

LHom is a bifunctor.

Given $A^* \xrightarrow{\alpha} A$ define LHom $(\alpha, 1)(E) = E\alpha$.

Given $C \xrightarrow{\gamma} C^*$ define LHom $(1, \gamma)(E) = \gamma E$.

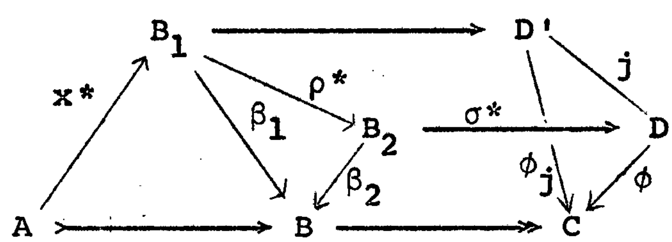
Note 5.3

If D is an object of $\underline{\underline{A}}$, and $E : A \rightarrowtail B \twoheadrightarrow C$ is a short exact sequence in $\underline{\underline{A}}$, there is a connecting morphism

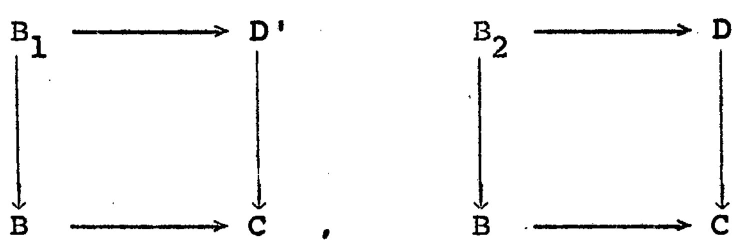
$$E_* : \text{LHom}(D,C) \longrightarrow \text{2Ext}(D,A),$$ which is a morphism of abelian monoids.

Proof

If $D' \overset{j}{\rightarrowtail} D \overset{\phi}{\longrightarrow} C \in \text{LHom}(D,C)$, define $E_*(D' \rightarrowtail D \longrightarrow C)$ to be the 2Extension $A \rightarrowtail B_1 \longrightarrow B_2 \twoheadrightarrow D$ in the construction:



where



are pullbacks, x^* is the (unique) map induced by $A \overset{0}{\longrightarrow} D'$. ρ^* is the map induced by the properties of the two pullbacks.

That $A \longrightarrow B_1 \longrightarrow B_2 \longrightarrow D$ is a 2Extension is proved by an easy diagram chase, assuming the embedding theorem mentioned in 4.8.

To prove that E_* preserves addition we need to make the following observations:

If $A \overset{\alpha}{\longrightarrow} A^*$, $D^* \overset{\gamma}{\longrightarrow} D$ are morphisms of $\underline{\underline{A}}$, and if $L_1, L_2 \in \text{LHom}(D,C)$, then

(i) $(\alpha E_*)(L_1) = \alpha(E_*(L_1))$

(ii) $(E \oplus E)_*(L_1 \oplus L_2) = E_*(L_1) \oplus E_*(L_2)$

(iii) $E_*(L_1\gamma) = (E_*(L_1))\gamma$

(iv) $E_*(\alpha L_1) = (E\alpha)_*(L_1)$

We omit the proofs of these facts and content ourselves with the remark that (i) follows by the properties of short exact sequences; that (ii) follows because a direct sum of pullbacks is a pullback; that (iii) follows by the properties of short exact sequences; and that (iv) is obvious.

$$\begin{aligned} \text{Then } E_*(L_1 + L_2) &= E_*(\nabla(L_1 \oplus L_2)\Delta) \\ &= (E\nabla)_*((L_1 \oplus L_2)\Delta) \\ &= (\nabla(E \oplus E))_*((L_1 \oplus L_2)\Delta) \\ &= \nabla(E_*(L_1) \oplus E_*(L_2))\Delta \\ &= E_*(L_1) + E_*(L_2) \end{aligned}$$

Proposition 5.4

Corresponding to every subobject D' of D there is a natural injection:

$$\text{Hom}\,(D,A) \xrightarrow{D^!_*} \text{LHom}\,(D,A)$$

of abelian monoids.

Proof

Define $D^!_*(D \xrightarrow{\phi} A) = D' \rightarrowtail D \xrightarrow{\phi} A$.

Then $D^!_*$ is clearly a monomorphism. Further, $D^!_*$ is a morphism of abelian monoids, since:

$$\begin{aligned} & D^!_*(D \xrightarrow{\phi_1} A + D \xrightarrow{\phi_2} A) \\ &= D^!_*(D \xrightarrow{\phi_1 + \phi_2} A) \\ &= D' \rightarrowtail D \xrightarrow{\phi_1 + \phi_2} A \end{aligned}$$

And: $$\begin{aligned} & D^!_*(D \xrightarrow{\phi_1} A) + D^!_*(D \xrightarrow{\phi_2} A) \\ &= D' \rightarrowtail D \xrightarrow{\phi_1} A + D' \rightarrowtail D \xrightarrow{\phi_2} A \end{aligned}$$

$$
= \quad \begin{array}{ccc}
D' & \rightarrowtail & D \\
\Delta \downarrow & 1 & \downarrow \Delta \\
D' \oplus D' & \rightarrowtail & D \oplus D \\
 & & \downarrow \phi_1 \oplus \phi_2 \\
 & & A \oplus A \\
 & & \downarrow \nabla \\
 & & A
\end{array}
\qquad \text{(1 is a pullback)}
$$

$$= D' \rightarrowtail D \xrightarrow{\phi_1 + \phi_2} A$$

Finally, D'_* is natural, for if $A \xrightarrow{\alpha} A^*$ is a morphism of $\underline{\underline{A}}$:

Then:

$$
\begin{array}{ccc}
\text{Hom }(D,A) & \xrightarrow{(D,\alpha)} & \text{Hom }(D,A^*) \\
D'_* \downarrow & & \downarrow D'_* \\
\text{LHom }(D,A) & \xrightarrow{(D,\alpha)} & \text{LHom }(D,A^*)
\end{array}
$$

commutes.

<u>Proposition 5.5</u>

Corresponding to every subobject D' of D there is a natural injection

$$\text{Ext }(D,A) \xrightarrow{D'_*} 2\text{Ext }(D,A)$$

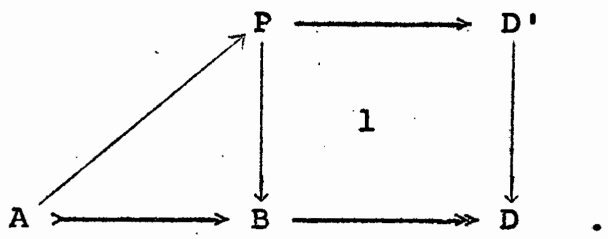
of abelian monoids.

<u>Proof</u>

Define:

$$D'_*(A \rightarrowtail B \twoheadrightarrow D) = A \rightarrowtail P \rightarrow B \twoheadrightarrow D,$$

in the following construction, where 1 is a pullback:



A diagram chase establishes that A $\rightarrowtail$ P $\longrightarrow$ B $\longrightarrow$ D is a 2Extension. (A $\xrightarrow{0}$ D' induces A $\longrightarrow$ P)

<u>$D^!_*$ preserves addition</u>

We prove several preliminary results:

If E_1 : A $\rightarrowtail$ B_1 $\twoheadrightarrow$ D, and

E_2 : A $\rightarrowtail$ B_2 $\twoheadrightarrow$ D are short exact sequences,

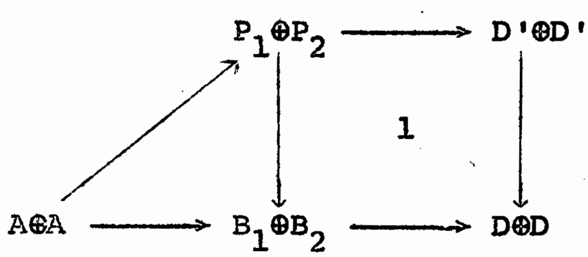
there is a map

$$D^!_{**} : \text{Ext}\ (D \oplus D,\ A \oplus A) \longrightarrow 2\text{Ext}\ (D \oplus D,\ A \oplus A)$$

which is the map, defined above, corresponding to the subgroup D'⊕D' of D⊕D.

(i) $$D^!_{**}(E_1 \oplus E_2) = D^!_*(E_1) \oplus D^!_*(E_2).$$

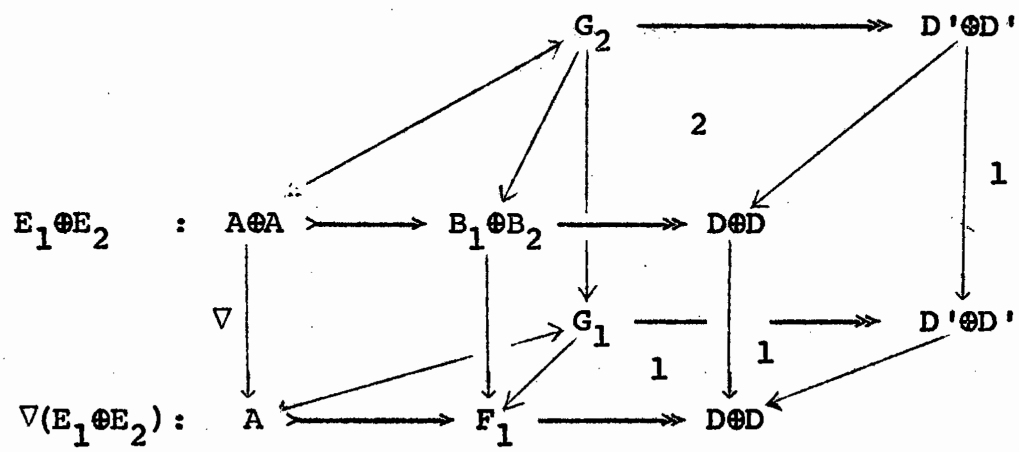
For, if P_1 and P_2 are the appropriate objects of $D^!_*(E_1)$ and $D^!_*(E_2)$, then in the following diagram, 1 is a pullback.



The result follows.

(ii) $$D^!_{**}(\nabla(E_1 \oplus E_2)) \equiv \nabla(D^!_{**}(E_1 \oplus E_2))$$

We construct the diagram which follows, in which 1 and 2 are pullbacks and $G_2 \longrightarrow G_1$ is induced by the properties of pullbacks.



$$D^!_{**}(E_1 \oplus E_2) = A \oplus A \longrightarrow G_2 \longrightarrow B_1 \oplus B_2 \longrightarrow D \oplus D$$

$$D^!_{**}(\nabla(E_1 \oplus E_2)) = A \longrightarrow G_1 \longrightarrow F_1 \longrightarrow D \oplus D$$

This induces the morphism of 2Extensions:

$$\begin{array}{ccccccccccc}
Ker(\beta_3 x_3) & \longrightarrow & A \oplus A & \xrightarrow{x_3} & G_2 & \xrightarrow{\beta_3} & B_1 \oplus B_2 & \xrightarrow{\sigma_3} & D \oplus D & \longrightarrow & Cok(\sigma_3 \beta_3) \\
\downarrow & & \downarrow \nabla & & \downarrow & & \downarrow & & \downarrow 1 & & \downarrow \\
Ker(\beta_2 x_2) & \longrightarrow & A & \xrightarrow{x_2} & G_1 & \xrightarrow{\beta_2} & F_1 & \xrightarrow{\sigma_2} & D \oplus D & \longrightarrow & Cok(\sigma_2 \beta_2)
\end{array}$$

$$Ker(\beta_3 x_3) \longrightarrow Ker(\beta_2 x_2)$$

is an epimorphism $(o \longrightarrow o)$

$$Cok(\sigma_3 \beta_3) \longrightarrow Cok(\sigma_2 \beta_2)$$

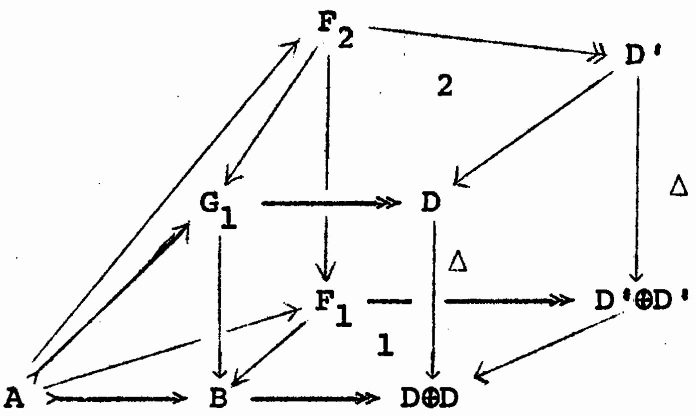
is a monomorphism (the identity map on $Cok(D' \oplus D' \longrightarrow D \oplus D)$)

The result follows by Proposition 4.13.

(iii) If $E \in Ext(D \oplus D, A)$, then

$$(D^!_{**}(E))\Delta \equiv D^!_*(E\Delta) \quad .$$

We construct the diagram which follows in which 1 and 2 are pullbacks and $F_2 \longrightarrow F_1$ is induced by the properties of pullbacks.



$$D^!_{**}(E) = A \longrightarrow F_1 \longrightarrow B \longrightarrow D\oplus D$$

$$D^!_*(E\Delta) = A \longrightarrow F_2 \longrightarrow G_1 \longrightarrow D$$

This induces the morphism of 2Extensions:

$$\begin{array}{ccccccccccc}
\mathrm{Ker}(\beta_2 x_2) & \longrightarrow & A & \xrightarrow{x_2} & F_2 & \xrightarrow{\beta_2} & G_1 & \xrightarrow{\sigma_2} & D & \longrightarrow & \mathrm{Cok}(\sigma_2\beta_2) \\
\downarrow & & \downarrow{\scriptstyle 1} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \Delta} & & \downarrow \\
\mathrm{Ker}(\beta_1 x_1) & \longrightarrow & A & \xrightarrow{x_1} & F_1 & \xrightarrow{\beta_1} & B & \xrightarrow{\sigma_1} & D\oplus D & \longrightarrow & \mathrm{Cok}(\sigma_1\beta_1)
\end{array}$$

$$\mathrm{Ker}(\beta_2 x_2) \longrightarrow \mathrm{Ker}(\beta_1 x_1)$$

is an epimorphism ($o \longrightarrow o$) . It is easy to check that $\mathrm{Cok}(\sigma_2\beta_2) \longrightarrow \mathrm{Cok}(\sigma_1\beta_1)$ is a monomorphism. The result follows by Proposition 4.13.

(iv) We have

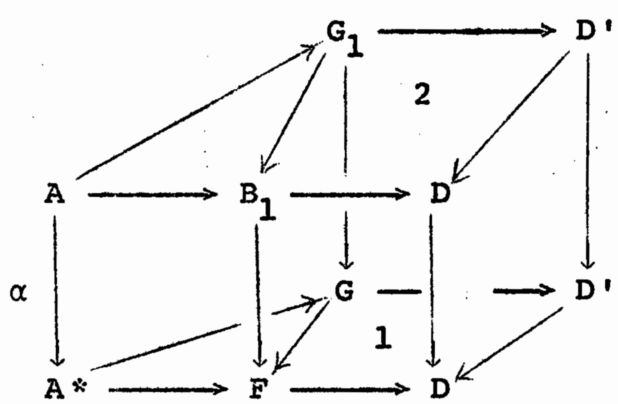
$$D^!_*(E_1 + E_2) = D^!_*(\nabla(E_1\oplus E_2)\Delta)$$
$$= \{D^!_{**}(\nabla(E_1\oplus E_2))\}\Delta = \{\nabla(D^!_{**}(E_1\oplus E_2))\}\Delta$$
$$= \{\nabla(D^!_*(E_1) \oplus D^!_*(E_2))\}\Delta$$
$$= D^!_*(E_1) + D^!_*(E_2)$$

<u>$D^!_*$ is natural</u>

For suppose $A \xrightarrow{\alpha} A^*$ is a morphism of $\underline{\underline{A}}$. We have the diagram:

$$\begin{array}{ccc} Ext(D,A) & \xrightarrow{Ext(D,\alpha)} & Ext(D,A^*) \\ {\scriptstyle D^!_*}\downarrow & & \downarrow{\scriptstyle D^!_*} \\ 2Ext(D,A) & \xrightarrow{2Ext(D,\alpha)} & 2Ext(D,A^*) \end{array}$$

Suppose $E : A \longrightarrow B_1 \longrightarrow D \in Ext(D,A)$. We can represent $D^!_*(Ext(D,\alpha)(E))$ and $D^!_*(E)$ in the diagram which follows. 1 and 2 are pullbacks, and $G_1 \longrightarrow G$ is induced by the properties of pullbacks.



$D^!_*(E) = A \longrightarrow G_1 \longrightarrow B_1 \longrightarrow D$

$D^!_*(Ext(D,\alpha)(E)) = A^* \longrightarrow G \longrightarrow F \longrightarrow D$

This induces a morphism of 2Extensions:

$$\begin{array}{ccccccccccc} Ker(\beta_4 x_4) & \longrightarrow & A & \xrightarrow{x_4} & G_1 & \xrightarrow{\beta_4} & B & \xrightarrow{\sigma_4} & D & \longrightarrow & Cok(\sigma_4\beta_4) \\ \downarrow & & {\scriptstyle \alpha}\downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle 1} & & \downarrow \\ Ker(\beta_3 x_3) & \longrightarrow & A^* & \xrightarrow{x_3} & G & \xrightarrow{\beta_3} & F & \xrightarrow{\sigma_3} & D & \longrightarrow & Cok(\sigma_3\beta_3) \end{array}$$

$Ker(\beta_4 x_4) \longrightarrow Ker(\beta_3 x_3)$ is an epimorphism $(o \longrightarrow o)$.

$Cok(\sigma_4\beta_4) \longrightarrow Cok(\sigma_3\beta_3)$ is a monomorphism (the identity on $Cok(D' \longrightarrow D)$).

The commutativity of the first diagram of this section, and hence the result, follows by Proposition 4.13.

<u>D'_* is a monomorphism</u>

Because Ext(D,A) is an abelian group the subset D'_*(Ext(D,A) of 2Ext(D,A) is an abelian group. The zero of this abelian group is the 2Extension:

$A \longrightarrow A\oplus D' \longrightarrow A\oplus D \longrightarrow D$ constructed in the diagram:

$$\begin{array}{ccccc} & & A\oplus D' & \xrightarrow{p} & D' \\ & \nearrow^{i} & \big\downarrow{\scriptstyle (1,j)} & 1 & \big\downarrow{\scriptstyle j} \\ A & \xrightarrow{i} & A\oplus D & \xrightarrow{p} & D \end{array}$$

in which 1 is a pullback.

Clearly if $E \neq A \longrightarrow A\oplus D \longrightarrow D$

$D'_*(E) \neq A \longrightarrow A\oplus D' \longrightarrow A\oplus D \longrightarrow D$

Our result follows.

<u>Proposition 5.6</u>

Let D' be a subobject of D. Given an exact sequence:

$E : A \rightarrowtail{x} B \xrightarrow{\sigma} C$

we obtain a commutative diagram:

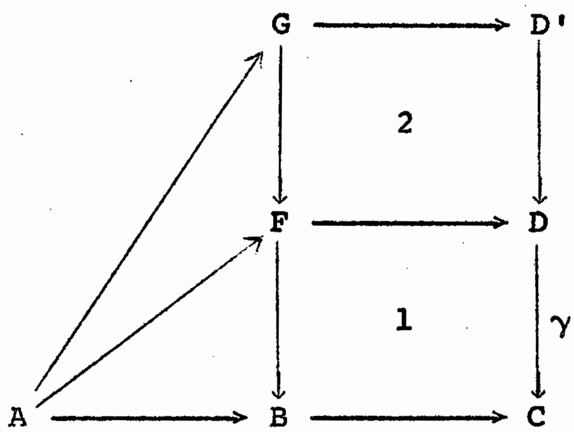
$$\begin{array}{ccccccccccc} \text{Hom}(D,A) & \xrightarrow{x_*} & \text{Hom}(D,B) & \xrightarrow{\sigma_*} & \text{Hom}(D,C) & \xrightarrow{E_*} & \text{Ext}(D,A) & \xrightarrow{x_*} & \text{Ext}(D,B) & \xrightarrow{\sigma_*} & \text{Ext}(D,C) \\ \big\downarrow{\scriptstyle D'_*} & & \big\downarrow{\scriptstyle D'_*} & & \big\downarrow{\scriptstyle D'_*} & 1 & \big\downarrow{\scriptstyle D'_*} & & \big\downarrow{\scriptstyle D'_*} & & \big\downarrow{\scriptstyle D'_*} \\ \text{LHom}(D,A) & \xrightarrow[x_*]{} & \text{LHom}(D,B) & \xrightarrow[\sigma_*]{} & \text{LHom}(D,C) & \xrightarrow[E_*]{} & \text{2Ext}(D,A) & \xrightarrow[x_*]{} & \text{2Ext}(D,B) & \xrightarrow[\sigma_*]{} & \text{2Ext}(D,C) \end{array}$$

where x_* represents (D,x), Ext(D,x), LHom(D,x), or 2Ext(D,x), according to context, and σ_* is defined similarly.

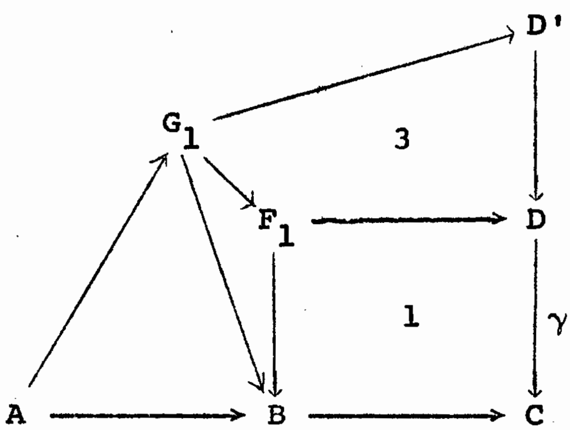
<u>Proof</u>

By 5.4 and 5.5, we need only show that 1 commutes. Let $\gamma \in$ Hom(D,C)

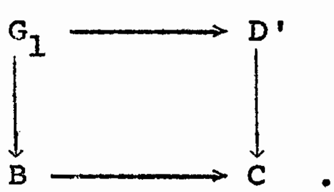
Clockwise: $D^!_*(E_*(\gamma))$ is the 2Extension $A \rightarrow G \rightarrow F \rightarrow D$ in the diagram which follows, in which 1 and 2 are pullbacks:



Anticlockwise: $E_*(D^!_*(\gamma))$ is the 2Extension $A \rightarrow G_1 \rightarrow F_1 \rightarrow D$ in the construction which follows, in which 1 and 2 are pullbacks:



2 is the square



But then 3 is a pullback, in the diagram above, and the uniqueness of pullbacks proves our result.

<u>Note 5.7</u>

We discuss a result concerning the "exactness" of the long sequence:

$$LHom(D,A) \longrightarrow LHom(D,B) \longrightarrow LHom(D,C) \longrightarrow$$
$$\longrightarrow 2Ext(D,A) \longrightarrow 2Ext(D,B) \longrightarrow 2Ext(D,C) \longrightarrow$$

The result is weaker than the classical result, and to obtain it we make manipulations of a somewhat arbitrary character. For this reason we have thought it appropriate to give only a brief account of the steps leading to the result, and to omit proofs. The proofs we omit are lengthy, but not difficult.

Considering the long sequence above we are immediately confronted with the fact that LHom(D,•) and 2Ext(D,•) are abelian monoids, and not abelian groups. We are therefore unable to prove, à priori, that the sequence is exact in the classical sense. But there are generalizations of the concept of exactness to categories other than abelian categories, and, in particular, to the category of pointed sets.

One of these generalizations is due to O. Wyler (17). In this theory "exactness" is defined in "Weakly Exact Categories"; one such category is the category of pointed sets. We refer the reader who is interested in the general theory to (17). Here we shall state the theory of (17) only in so far as it refers to the category of pointed sets.

A pointed set is a set A containing a distinguished point O_A. A morphism of pointed sets is a mapping, $f : A \longrightarrow B$ such that $f(O_A) = O_B$. S_O will denote the category of pointed sets.

The kernel of a map $f : A \longrightarrow B$ in S_O is $\{f^{-1}(O_B)\} \longrightarrow A$, denoted ker(f). The "normal image" of the map f is $f(A) \longrightarrow B$, denoted nim(f). The cokernel of a map $f : A \longrightarrow B$ is $B \longrightarrow B|f(A)$, denoted Cok(f).

The "normal coimage" of a map $f : A \longrightarrow B$ is, if it exists, $A \longrightarrow f(A)$, denoted conim(f). A map f has a normal coimage if, and only if, f, when restricted to the complement of $f^{-1}(O_B)$ in A, is 1-1.

Then, we make these definitions: A pair of maps $A \xrightarrow{f} B \xrightarrow{g} C$ is exact in S_O if, and only if,

$$\text{nim}(f) = \text{Ker}(g).$$

A pair of maps $A \xrightarrow{f} B \xrightarrow{g} C$ is coexact in S_O if g has a normal coimage and

$$\text{conim}(g) = \text{coker}(f).$$

And, more generally, a sequence:

$\xrightarrow{u_o} \ldots\ldots \xrightarrow{u_n}$ of maps in S_O is exact if the pairs $\xrightarrow{u_i} \xrightarrow{u_{i+1}}$ are exact $(o < i < n-1)$ and coexact if the pairs $\xrightarrow{u_i} \xrightarrow{u_{i+1}}$ are coexact $(o < i < n-1)$.

A sequence is biexact if it is exact and coexact. Exactness and coexactness are dual properties, but they are not equivalent.

Returning to the long sequence $\text{LHom}(D,A) \longrightarrow \ldots\ldots \longrightarrow 2\text{Ext}(D,C)$, we make the observation that neither LHom(D,O) nor 2Ext(D,O) are trivial abelian monoids. LHom(D,O) consists of diagrams:

$D' \rightarrowtail D \longrightarrow O$, where D' is a subobject of D, and 2Ext(D,O) consists of sequences

$O \longrightarrow D' \rightarrowtail D \longrightarrow D$, where D' is a subobject of D.

The morphism $O \longrightarrow A$, induces morphisms

$O_* : \text{LHom}(D.O) \longrightarrow \text{LHom}(D,A)$

$O_* : 2\text{Ext}(D,O) \longrightarrow 2\text{Ext}(D,A)$.

In order to prove our result, concerning the exactness of the long-sequence, we must insist that

$$LHom(D,0) = 0 \text{ and}$$

$$2Ext(D,0) = 0 .$$

More precisely, regarding LHom(D,A) simply as a pointed set, we "factor out" $0_*(LHom(D,0))$ to obtain the pointed set:

$$\frac{LHom(D,A)}{0_*(LHom(D,0))}$$

Similarly, regarding 2Ext(D,0) simply as a pointed set, we "factor out" $0_*(2Ext(D,0))$ to obtain the pointed set:

$$\frac{2Ext(D,A)}{0_*(2Ext(D,0))}$$

in neither of these factor sets, however, is the structure of an abelian monoid preserved.

We may then prove:

1. $\frac{LHom(D,A)}{0_*(LHom(D,0))}$ and $\frac{2Ext(D,A)}{0_*(2Ext(D,0))}$

define functors from $\underline{\underline{A}}$ to S_0.

2. The connecting morphism $LHom(D,C) \xrightarrow{E_*} 2Ext(D,A)$ induces a connecting morphism of pointed sets:

$$\frac{LHom(D,C)}{0_*(LHom(D,0))} \xrightarrow{E_*} \frac{2Ext(D,A)}{0_*(2Ext(D,0))}$$

in the obvious way.

3. The long sequence

$$\xrightarrow{0} \frac{LHom(D,A)}{0_*(LHom(D,0))} \longrightarrow \frac{LHom(D,B)}{0_*(LHom(D,0))} \xrightarrow{\sigma_*} \frac{LHom(D,C)}{0_*(LHom(D,0))}$$

$$\xrightarrow{E_*} \frac{2Ext(D,A)}{0_*(2Ext(D,0))} \longrightarrow \frac{2Ext(D,B)}{0_*(2Ext(D,0))} \longrightarrow \frac{2Ext(D,C)}{0_*(2Ext(D,0))}$$

is exact in S_0, in the sense we have just defined. The sequence is not co-exact, however, and hence not biexact. σ_*, in particular, fails in general to have a normal coimage.

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