Thesis Presented for the degree of Doctor of Philosophy

A Study of Solutions and Perturbations of Spherically Symmetric Spacetimes in Fourth Order Gravity

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Declaration

The work presented in this thesis is partly based on collaborations with my supervisor Prof. Peter Dunsby (University of Cape Town) together with Prof. George F. R. Ellis (University of Cape Town), Dr. Rituparno Goswami (University of KwaZulu-Natal), Dr. Timothy Clifton (Queen Mary University of London) and Dr. Sante Carloni (European Space Agency - Advanced Concepts Team).

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I hereby declare that the presented thesis has not previously been submitted to this or any other university for a degree and that it represents my own work.

Anne Marie Nzioki

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Now the day has dawned
and the lamp that lit my dark corner is out.
A summons has come
and I am ready for my journey.

- Rabindranath Tagore

Abstract

In this thesis we use the 1+1+2 covariant approach to General Relativity to study exact solutions and perturbations of rotationally symmetric spacetimes in $f(R)$ gravity, one of the most widely studied classes of fourth order gravity.

We begin by introducing $f(R)$ theories of gravity and present the general equations for these theories. We investigate the problem of matching different regions of spacetime, shedding light on the problem of constructing realistic inhomogeneous cosmologies in the context of $f(R)$ gravity. We also studying strong lensing in these fourth order theories of gravity derive the lens mass and magnification for the gravitational lens system.

We provide an extensive review of both the 1+3 and 1+1+2 covariant approaches to $f(R)$ theories of gravity and give the full system of evolution, propagation and constraint equations of LRS spacetimes. We then determine the conditions for the existence of spherically symmetric vacuum solutions of these fourth order field equations and prove a Jebsen-Birkhoff like theorem for $f(R)$ theories of gravity and the necessary conditions required for the existence of Schwarzschild solution in these theories.

In order to study the perturbations of Schwarzschild black holes in this context, we apply the 1+1+2 perturbative procedure to determine a complete set of gauge-invariant perturbation variables. A reduced set of frame independent master variables, which obey two closed wave equations are then found - one for the transverse, trace-free (tensor) perturbations and the other for the additional scalar degree of freedom, which is a feature of forth-order theories of gravity. We show that for the tensor modes, the underlying dynamics in $f(R)$ gravity is governed by a modified Regge-Wheeler tensor which obeys the same Regge-Wheeler equation as in General Relativity. For the quasinormal modes (QNMs) that follow from the scalar perturbations, we find that the possible sources of scalar QNMs for the lower multipoles are from primordial Black Holes, while higher mass, stellar black holes are associated with extremely high multipoles, which can only be produced in the first stage of black hole formation. Since the scalar QNMs are short ranged, this scenario makes their detection beyond the range of current experiments.

Keywords: $f(R)$ gravity, Spherically symmetric solutions, Birkhoff's theorem, Regge-Wheeler equation, Matching, Lensing

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Conventions and Abbreviations

Sign conventions

Signature:	$[-,+,+,+]$.
Geometrised units:	$8\pi G = c = 1$.
Latin indices:	0, 1, 2, 3.

Sign conventions follow Ellis [1] and Ellis and van Elst [2].
For a tensor $T^{a..b}{}_{c..d..e..f}$ we have:

symmetrization:	$T_{(ab)} = \frac{1}{2}\left(T_{ab} + T_{ba}\right)$,
antisymmetrization:	$T_{[ab]} = \frac{1}{2}\left(T_{ab} - T_{ba}\right)$,

over the indexes of the tensor. The symbol ∇ represents the usual covariant derivative and ∂ corresponds to partial differentiation.
The Riemann tensor is defined by

$$R^a{}_{bcd} = \Gamma^a{}_{bd,c} - \Gamma^a{}_{bc,d} + \Gamma^e{}_{bd}\Gamma^a{}_{ce} - \Gamma^e{}_{bc}\Gamma^a{}_{de} \ ,$$

and $\Gamma^a{}_{bd}$ are the Christoffel symbols (i.e. symmetric in the lower indices), defined by

$$\Gamma^a_{bd} = \frac{1}{2} g^{ae}\left(g_{be,d} + g_{ed,b} - g_{bd,e}\right) \ .$$

The Ricci tensor is obtained by contracting the *first* and the *third* indices

$$R_{ab} = g^{cd} R_{acbd} \ .$$

Abbreviations

BH	Black Hole
CDM	Cold dark matter
CMBR	Cosmic Microwave Background Radiation
DE	Dark energy
DEC	Dominant energy condition
EFEs	Einstein field equations
EGS	Ehlers-Geren-Sachs
FLRW	Friedmann-Lemaître-Robertson-Walker
FOG	Fourth order gravity
GR	General relativity
ISW	Integrated Sachs-Wolfe
LRS	Local rotational symmetry
PSTF	Projected symmetric and trace-free

SEC	Strong energy condition
TT	Transverse-traceless
WEC	Weak energy condition

Chapter 1

Introduction

1.1 $f(R)$ gravity

Einstein's theory of General Relativity (GR) [3] is widely accepted to be a fundamental theory for modern physics, describing well the standard model of gravitation and cosmology. Just three years after Einstein developed his theory, in 1918, Herman Weyl [4] began to consider modifications of GR by including higher order invariants in its action. Motivated by the desire to obtain a unified field theory, he extended the geometrical representation of GR to account not only for gravitational but also electromagnetic fields. In 1921, Arthur Eddington also began to consider fourth order theories of gravity [5, 6] and he followed this up by publishing *The Mathematical Theory of Relativity* that contained his work on generalised versions of Weyl's theory. Since then there have been a great number of proposed higher order theories of gravity that propose modification of GR.

The surge of interest in modified theories of gravity in the 60s, 70s and 80s was primarily due to limitations in GR when considering strong gravity regimes. Utiyama and DeWitt [7] showed that renormalisation of GR at the one-loop quantum level required that the field equations should be higher than second order. Modifications of the GR by supplementing the Einstein-Hilbert action with higher order curvature invariants were at the time limited to the early universe and provided, for example, a nice geometrical explanation for inflation [8] in cosmology. More recently, however, the corrections to GR have been introduced to accommodate recent observations and more so to account for the "dark sector" of the universe. The number counts of clusters of galaxies [9], measurements of type Ia supernovae [10–13] and the cosmic microwave background (CMB) anisotropies [14–16], indicate that of the energy density budget of the universe, 5% comprises ordinary matter (baryons, radiation and neutrinos), while the rest, which does not interact electromagnetically, consists of 27% dark matter and 68% dark energy (DE) [17]. Dark matter is responsible for the gravitational clumping of galaxies, galaxy clusters and large scale structures and the requirement of its existence had been known for some years [18]. Dark energy is a label for the relativistic energy density with negative pressure required to explain the inferred late-time accelerated expansion of the universe. If GR is the correct theory of the gravitational action then its application to cosmology should incorporate these observations. The implication of this description is that we live in a flat Friedmann-Lemaître-Robertson-Walker (FLRW) universe that is dominated by cold dark matter (CDM) and DE in the form of a positive cosmological constant. This model of the universe is the best fit so far and is based on the hypothesis that the universe is homogenous on large scale. It is commonly referred to as the *ΛCDM (or concordance) model*.

The question naturally arises: what is the importance of considering alternative theories of gravity to GR, as possible explanations to the observations if the ΛCDM model agrees well with the observations. One of the main motivations for the search for alternative theories of gravity arises from the obscure nature of DE candidates. The alternative possibility is to conjecture that the apparent need for DE could simply be because the application of Einstein's equations at cosmological scales is ill-suited. Some of the modified theories of gravity that provide a late time acceleration for the universe without the need for the presence of any exotic fluids are Scalar-tensor theories, Dvali-Gabadadze-Porrati (DGP) braneworld model [19], TeVeS (Tensor-Vector-Scalar) [20] and and Hořava-Lifschitz gravity [21–23]. One such theoretical proposal that has recently attracted a considerable amount of attention is *fourth order gravity* (FOG) that can accelerate at late times without the presence of DE [24–28]. In particular, dynamical systems analysis shows that for FLRW models, there exist classes of fourth order theories which admit a transient decelerated expansion phase that is important for structure growth, followed by one with an accelerated expansion rate [29]. These cosmic evolutions therefore mimic the standard ΛCDM cosmic history. Another feature of these FOG theories is that they are also able to account for the rotation curves of spiral galaxies without the need for dark matter [30]. See [31–33] for detailed reviews.

A complete understanding of the consequences of such a radical shift away from the standard approach to cosmology is still far from complete. We attempt to contribute to this understanding by considering the construction of inhomogeneous cosmological models within the framework of $f(R)$ theories of gravity. This will be done by attempting to match together existing solutions. In particular, we will attempt to construct Swiss cheese models by matching spherically symmetric vacuum solutions with FLRW solutions. The motivation for this study is to understand both the effect of cosmological expansion on the gravitational fields of astrophysical bodies, as well as the large-scale expansion that emerges in a universe with large density contrasts. These questions have been carefully studied in Einstein's theory, where the aforementioned constructions have proven to be useful devices for understanding them. Fourth-order theories are considerably more complicated than Einstein's theory, but by applying the same constructions we should expect to gain some insights into these questions. These extra complications include the absence of Jebsen-Birkhoff's theorem, so that spherically symmetric vacuum spacetimes are not unique [34], as well as more complicated junction conditions [35].

Further motivation for this study comes specifically from the work of Mignemi and Wiltshire [36], where these authors used a dynamical systems approach to perform a non-perturbative study of the static, spherically symmetric solutions of analytic $f(R)$ theories. They found that these solutions are generically not asymptotically flat, and that boundary conditions could therefore be important in determining the gravitational fields of isolated massive bodies. Similar results have been found for non-analytic $f(R)$ theories [37]. These effects are entirely absent if one assumes asymptotic flatness from the beginning, as is standard in most approaches to studying weak gravitational fields [38–43]. The construction of inhomogeneous cosmological models, as outlined above, provides a way to implement appropriate boundary conditions, and therefore allows the validity of standard weak-field approaches to be investigated.

A major point of interest with any theory of gravity is the degree to which the physics is consistent with *both* cosmological and solar system scales. Measurements coming from post-Newtonian tests like the precession of planetary orbits, the dragging of inertial frames and the deflection of light represent critical tests for any theory of gravity. One of the triumphs of GR is its prediction of the gravitational deflection of light, a feature that was confirmed by results from Arthur Eddington's solar eclipse expedition in 1919. Since then gravitational lensing has been a key tool for mapping the mass distribution of galaxies and galaxy clusters and putting constraints on scales as small as stars (microlensing) to large-scale structures and cosmological parameters [44]. Given that the lensing effect is dependent on the underlying theory of gravity, the consequences of deviating from Einstein's theory would result in deviations from the standard expression of the deflection angle and is worth investigating. In this thesis, we study strong gravitational lensing effects in $f(R)$ gravity where we consider in particular R^n gravity and find the deviations of the mass and magnification quantities from GR.

1.2 Covariant approach

Spacetime can be described using tetrad formalisms or metric (or coordinate) based approaches. The tetrad formalisms range from the Newman-Penrose null tetrad method [45], to the *1+3 covariant approach* developed by Ehlers and Ellis [1, 2, 46] which includes both a full tetrad approach and a 'partial' covariant approach where only one timelike tetrad vector is chosen. The formalism is based on a 1+3 threading of the spacetime manifold with respect to a timelike congruence, such that spacetime is decomposed into space and time.

The 1+3 formalism has been a useful tool for understanding of many aspects of relativistic fluid flows, whether it is applied in terms of fully nonlinear GR effects or the gauge invariant, covariant perturbation formalism. The 1+3 approach to perturbation theory was developed by Ellis, Bruni and Dunsby [47–49], building on early work by Hawking [50], Lyth and Mukherjee [51] and Ellis and Bruni [52]. The covariant perturbation formalism employs kinematic and dynamical variables to describe nature, that have both physical and geometric significance and remain valid in all coordinate systems. This is unlike the metric based approach which is plagued by gauge modes arising from the choice of reference coordinate system. Further work in the formalism has been in its implementation to the physics of the CMB [53–55].

More recently, linear perturbation theory has been developed for fourth order theories of gravity (FOG) using the 1+3 covariant approach [29, 56–59], providing important features that differentiate the structure growth in FOG from the GR results. It was found that the evolution of density perturbations is determined by a fourth order differential equation rather than a second order one, which in turn implies that the number of modes of the density perturbations increases from two to four. Other findings were that the perturbations in FOG are not scale-invariant as in GR but instead depend on the scale for any equation of state for standard matter and that in contrast with what one finds in GR, the growth of large scale density fluctuations can also occur in backgrounds in which the expansion rate is increasing in time. The latter finding effectively leads to a time-varying gravitational potential and puts tight constraints on the Integrated Sachs-Wolfe (ISW) effect for these models.

As an application of the 1+3 approach we consider the role that shear plays in the relationship between Newtonian and relativistic cosmologies. The presence/or lack of shear relates to the way information is conveyed along geodesic congruences. It is expected that since Newtonian gravity is a limiting form of GR, then the properties of Newtonian gravity should follow from those of GR as demonstrated by Ellis in 1967 [60]. He showed that if the four velocity vector field of a barotropic perfect fluid with vanishing pressure is shear-free, then either the expansion or the rotation of the fluid vanishes. The shear-free result has been extended to general barotropic fluids for a number of special cases by Senovilla [61] and there has been an attempt to prove the result for shear-free perfect fluid solutions with linear equations of state [62]. We consider whether the result holds in situations where the hydrodynamic and gravitational equations have been linearised about a Friedmann-Lemaître-Robertson-Walker (FLRW) background [63] and also whether it extends to the more general setting of FOG [64].

In this thesis we employ the *1+1+2 formalism* developed recently by Clarkson and Barrett [65] which is a natural extension to the 1+3 approach, optimised for problems which have spherical symmetry, including the Schwarzschild solution, Lemaître-Tolman-Bondi (LTB) models and many classes of Bianchi models. The approach involves a 'semi-tetrad' where, in addition to the timelike vector field of the 1+3 approach, a spatial vector is introduced. In GR, the 1+1+2 formalism has been applied to the study of perturbations of locally rotationally symmetric (LRS) spacetimes [65–71] and strong lensing studies [72]. It has also been introduced to the study of LRS spacetimes in the context of $f(R)$ gravity [73, 74].

The advantage of using the 1+1+2 formalism for spacetimes with preferred direction is that the 1+3 equations in these cases usually become intractable. As an example, in the astrophysical black hole setting a 1+3 decomposition results in the presence of non-zero vectors and tensors in the background spacetime and as a result all the equations have vector-tensor and tensor-tensor coupling in them, rendering them intractable. However, applying the 1+1+2 approach to these systems results in all projected vectors and tensors being of first-order, such that the aforementioned coupling in the background doesn't occur. After harmonic decomposition, the system of equations constitutes scalar quantities in the perturbed spacetime for which the solution can be found [65].

1.3 Spherically symmetric spacetimes

In GR, spherically symmetric vacuum spacetimes have an extra symmetry: they are either locally static or spatially homogeneous. This rigidity of spherically symmetric vacuum solutions is the essence of *Jebsen-Birkhoff theorem* [75–78]. This theorem makes the Schwarzschild solution crucially important in astrophysics and underlies the way local astronomical systems decouple from the global expansion of the universe. In essence, the Schwarzschild solution is the unique spherically symmetric solution of the vacuum Einstein field equations (EFEs) and represents the spacetime geometry of the Solar System, and the spacetime geometry outside spherically symmetric matter distributions to very good approximation. Moreover, it was recently shown [79, 80], that in GR, the rigidity of spherical vacuum solutions of Einstein's field equations continues even in the perturbed scenario: almost spherical symmetry and/or almost vacuum implies almost static or almost spatially homogeneous. The rigidity embodied in this property of the EFEs is specific to vacuum GR solutions, or those with a trace-free matter tensor and is known not to hold for theories with extra degrees of freedom (for example, $f(R)$ theories of gravity or other scalar-tensor theories [81, 82]).

Using the 1+1+2 covariant approach we outline the general conditions for the existence of certain types of static spherically symmetric solutions in $f(R)$ theories. In this framework we investigate the extra conditions required for a Jebsen-Birkhoff-like theorem in spherically symmetric spacetimes to hold in $f(R)$ gravity. The important result that emerges covariantly from our investigation is that, there exist a non-zero measure in the parameter space of these FOG theories, for which the Jebsen-Birkhoff like theorem remains stable under generic perturbations. Furthermore, our result is a local result and hence does not depend on specific boundary conditions used for solving the perturbation equations.

1.4 Covariant perturbations of Schwarzschild black holes

The interest in studying of black hole (BH) perturbations comes from the important role they play in gravitational wave physics. There are various ways by which a black hole can be perturbed: by incident gravitational waves, by objects falling into it or by aspherical gravitational collapse. The understanding of perturbations of black holes therefore provides insight into different number of areas of interest in gravitational radiation studies. Contributions to the investigation of BH properties in FOG theories include an extensive study of the Schwarzschild de Sitter BH in [83, 84], Schwarzschild BH perturbations in $f(R,G)$ gravity in [85] and a stability analysis of the Schwarzschild BH in [86] where they make a transformation from $f(R)$ gravity to the scalar-tensor theory for their analysis.

Perturbations of Schwarzschild BH at linear order in GR have been studied through metric perturbations, the Newman-Penrose (NP) formalism [87] as well as the 1+1+2 covariant formalism [65]. these studies found that the perturbations are governed by two second order wave equations with an effective potential namely, the *Regge-Wheeler equation* (derived initially by Regge and Wheeler [88]) for the odd (axial) perturbations and the *Zerilli equation* (first derived by Zerilli [89]) for the even (polar) perturbations. Using the 1+1+2 approach, Clarkson and Barrett [65] demonstrated that both the odd and even parity perturbations may be unified in a covariant wave equation equivalent to the Regge-Wheeler equation. This wave equation is characterised by a single a covariant, frame- and gauge-invariant, transverse-traceless tensor.

In this thesis we apply the 1+1+2 approach to the analysis of the perturbation of Schwarzschild BH in $f(R)$ gravity, following steps as given in [65, 69]. Due to the extra degree of freedom inherent in these FOG theories, one has to additionally consider the linearised Ricci scalar wave equation in the investigation. Gauge invariance is assured in the analysis via the Stewart-Walker lemma [90] which states that a perturbation variable is gauge invariant if it vanishes in the background. The linearisation procedure applies this criterion by considering these variables as first order and consequently neglecting their products. Harmonic functions can then be introduced in the background which results in two decoupled parities reflecting the invariance of the background spacetime under parity transformation. The introduction of harmonics means that finding a solution simply involves solving a linear system of algebraic equations. After introducing the harmonic functions, the main objective will be to find a reduced set of master variables which obey a closed set of wave equations.

The initial perturbations of the BH eventually get decay exponentially (ringing) at frequencies that are characteristic of the BH and independent of the source of the perturbation as was first discovered by Vishveshwara in 1970 [91]. These complex valued frequencies satisfy boundary conditions for purely outgoing waves at infinity and purely ingoing waves at the BH horizon. The solutions to the perturbation wave equations that are constructed from these frequencies are known as *quasinormal modes* which we discuss in the context of $f(R)$ gravity.

1.5 Thesis outline

In Chapter 2 we introduce $f(R)$ theories of gravity and present the general equations for these theories. Following this, in Chapter 3 we investigate the problem of matching different regions of spacetime in order to construct inhomogeneous cosmological models in the context of these theories. We also analyse the behaviour of the general expression for the deflection angle for spherically symmetric spacetimes in the case of $f(R) = R^n$ gravity and derive the lens mass and magnification for the gravitational lens system.

In Chapter 4 we outline the 1+3 covariant method in $f(R)$ gravity and hence provide a covariant (gauge invariant) description of spacetime. The approach is then applied to shear-free perturbations of FLRW universes for both GR and FOG cases in Chapter 5.

In Chapter 6 we present the full system of 1+1+2 decomposed field equations equations in $f(R)$ gravity.

Chapter 7 is devoted to proving a Jebsen-Birkhoff-like theorem for $f(R)$ theories of gravity, to find the necessary conditions required for the existence of a Schwarzschild solution in these theories. We discuss under what circumstances we can covariantly set up a scale in the problem. We then perturb the vacuum spacetime with respect to this covariant scale to find the stability of the theorem.

In Chapter 8 we present the vacuum field equations linearised around a Schwarzschild black hole background using the 1+1+2 formalism. We discuss the spherical and time harmonics which are introduced to the system of equations which allow us to write the equations in scalar form. We then derive a covariant and gauge-invariant wave equation which describes the perturbations of the Schwarzschild BH spacetime. This equation is the covariant form of the Regge-Wheeler equation, corresponding to a master variable that constitutes a gauge and frame invariant transverse-traceless (TT) tensor. We also investigate the stability of the BH to external perturbations and as part of the perturbative analysis we discuss quasinormal modes.

Chapter 9 focuses on the method of solution to the perturbation equations using matrix methods where we demonstrate the significance of the freedom of choice of frame basis.

Chapter 10 contains our conclusions and an outlook for extensions of the work we have presented.

Useful relations utilised in our work are contained in the appendix.

Chapter 2

$f(R)$ Gravity

In this chapter we introduce $f(R)$ theories of gravity and present the general equations for these theories (see [31–33] for detailed reviews).

2.1 Action and field equations

In GR the Einstein-Hilbert action is given as

$$\mathcal{S} = \frac{1}{2}\int dV \left[\sqrt{-g}\,(R-2\Lambda) + 2\,\mathcal{L}_M(g_{ab},\psi)\right], \tag{2.1}$$

where $\mathcal{L}_M$ is the Lagrangian density of the matter fields ψ, R is the Ricci scalar and Λ is the cosmological constant. The invariant volume element is given by the expression $\sqrt{-g}\,dV$ and the gravitational Lagrangian density as $\mathcal{L}_g = \sqrt{-g}\,(R-2\Lambda)$, where g is the determinant of the metric tensor g_{ab}. A generalisation of this action is done by replacing R in (2.1) with a C^2 function of the quadratic contractions of the Riemann curvature tensor R^2, $R_{ab}R^{ab}$, $R_{abcd}R^{abcd}$ and $\varepsilon^{klmn}R_{klst}R^{st}{}_{mn}$ where ε^{klmn} is the antisymmetric 4-volume element. In fact, in the quantum field picture, the effects of renormalisation are expected to add such terms to the Lagrangian in order to give a first approximation to some quantised theory of gravity [92, 93]. The Lagrangian density that can be constructed from the generalisation is of the form

$$\mathcal{L}_g = \sqrt{-g}\; f(R, R_{ab}\,R^{ab}, R_{abcd}\,R^{abcd})\;. \tag{2.2}$$

It is a well known result that [94–96],

$$(\delta/\delta g_{ab})\int dV \left(R_{abcd}\,R^{abcd} - 4R_{ab}\,R^{ab} + R^2\right) = 0\;, \tag{2.3}$$

$$(\delta/\delta g_{ab})\int dV\,\varepsilon^{klmn}R_{klst}\,R^{st}{}_{mn} = 0\;, \tag{2.4}$$

that is, the functional derivative of the Gauss-Bonnet invariant $R_{abcd}\,R^{abcd} - 4R_{ab}\,R^{ab} + R^2$ and $\epsilon^{iklm}R_{ikst}\,R^{st}{}_{lm}$ vanishes with respect to g_{ab}. If we consider the function f to be linear in $R_{abcd}R^{abcd}$, we can use this symmetry to rewrite $R_{abcd}R^{abcd}$ in terms of the other two invariants and as a result the action for FOG can be written as:

$$\mathcal{S} = \frac{1}{2}\int dV \left[\sqrt{-g}\left(c_0\,R + c_1\,R^2 + c_2\,R_{ab}\,R^{ab}\right) + 2\,\mathcal{L}_M(g_{ab},\psi)\right] \;. \tag{2.5}$$

where the coefficients c_0, c_1 and c_2 have the appropriate dimensions. Similarly, if the spacetime is homogeneous and isotropic, then because of the following identity,

$$(\delta/\delta g_{ab})\int dV\;\left(3R_{ab}\,R^{ab} - R^2\right) = 0\;, \tag{2.6}$$

the term $R_{ab}\,R^{ab}$ can always be rewritten in terms of the variation of R^2. It then follows that the "effective" fourth-order Lagrangian for these highly symmetric spacetimes contain only powers of R and we can, without loss of generality, write the action as

$$\mathcal{S} = \frac{1}{2}\int dV\left[\sqrt{-g}\,f(R) + 2\,\mathcal{L}_M(g_{ab},\psi)\right]\;. \tag{2.7}$$

This action is the simplest generalisation of the Einstein-Hilbert gravity. Though in our later analysis we do not always consider isotropic spacetimes, the action (2.7) still remains quite general as it represents the only ghost-free higher order theory. Demanding that the action be invariant under some symmetry ensures that the resulting field equations also respect that symmetry. That being the case, since the Lagrangian is a function R only, and R is a generally covariant and locally Lorentz invariant scalar quantity, then the field equations derived from the action (2.7) are generally covariant and Lorentz invariant.

There are different variational principles that can be applied to the action $\mathcal{S}$ in order to obtain the field equations. One approach is the *standard metric formalism* where variation of the action is with respect to the metric g_{ab} and the connection Γ^a_{bc} in this case is the Levi-Civita one, that is, the metric connection

$$\Gamma^a{}_{bc} = \frac{1}{2}\,g^{ad}\left(g_{bd,c} + g_{dc,b} - g_{bc,d}\right)\;. \tag{2.8}$$

In the *Palatini formalism*, the metric and the connection are treated as independent fields and the action is varied with respect to each of them. A third procedure is the *metric-affine* approach which uses the Palatini variation but without the assumption that the matter action is a function of the connection as well as the metric. Unlike in the Einstein-Hilbert case where both the metric and Palatini approach lead to the same field equations for the action, the field equations that one obtains from (2.7) depend on the variational principle used. The versions of $f(R)$ gravity as a result of this are the *standard metric f(R) gravity*, the *Palatini f(R) gravity* and additionally, the *metric-affine f(R) gravity.*

2.1.1 Metric formalism

Varying the action (2.7) with respect to the metric g_{ab} over a 4-volume yields:

$$\delta\mathcal{S} \;\; = \;\; -\frac{1}{2}\int dV\,\sqrt{-g}\left[\frac{1}{2}f\,g_{ab}\,\delta g^{ab} - f'\,\delta R + T^{M}_{ab}\,\delta g^{ab}\right] \;, \tag{2.9}$$

where $'$ denotes differentiation with respect to R, and T^{M}_{ab} is the matter *energy momentum tensor* (EMT) defined as

$$T^{M}_{ab} = -\frac{2}{\sqrt{-g}}\,\frac{\delta\mathcal{L}_M}{\delta g^{ab}} \;. \tag{2.10}$$

Writing the Ricci scalar as $R = g^{ab}\,R_{ab}$ and assuming the connection is the Levi-Civita one, we can write

$$f'\,\delta R \simeq \delta g^{ab}\left(f'\,R_{ab} + g_{ab}\,\Box f' - \nabla_a\nabla_b f'\right) \;, \tag{2.11}$$

where the $\simeq$ sign denotes equality up to surface terms and $\Box \equiv \nabla_c\nabla^c$. By demanding that the action be stationary, so that $\delta\mathcal{S} = 0$ with respect to variations in the metric, one has finally

$$f'\left(R_{ab} - \frac{1}{2}g_{ab}\,R\right) = \frac{1}{2}g_{ab}\,(f - R\,f') + \nabla_a\nabla_b f' - g_{ab}\,\Box f' + T^{M}_{ab} \;. \tag{2.12}$$

It can be seen that for the special case $f = R$, the equations reduce to the standard Einstein field equations.

It is convenient to write (2.12) in the form of effective Einstein equations as

$$G_{ab} = \left(R_{ab} - \frac{1}{2}g_{ab}\,R\right) = \tilde{T}^{M}_{ab} + T^{R}_{ab} = T_{ab} \;, \tag{2.13}$$

where we define T_{ab} as the total EMT comprising

$$\tilde{T}^{M}_{ab} = \frac{T^{M}_{ab}}{f'} \tag{2.14}$$

and

$$T^{R}_{ab} = \frac{1}{f'}\left[\frac{1}{2}g_{ab}\,(f - R\,f') + \nabla_a\nabla_b f' - g_{ab}\,\Box f'\right] \;. \tag{2.15}$$

The components of the T_{ab} can be considered to represent two effective “fluids” [24,28,30,97]: the *curvature “fluid”* (associated with T^{R}_{ab}) and the *effective matter “fluid”* (associated with $\tilde{T}^{M}_{ab}$). This allows us to adapt more easily techniques from the “covariant approach” (see, [2, 49, 52, 65, 98]), to study a wide range of problems in $f(R)$ gravity that were originally devised for GR.

The field equations (2.13) are fourth order in derivatives of the metric, which can be seen from the existence of the $\nabla_a\nabla_b f'$ term in (2.15). This result also follows directly from a ramification of Lovelock's theorem [99, 100] which requires, in a four-dimensional Riemannian manifold, that the construction of a metric theory of modified gravity admits higher than second order derivatives to the field equations. This is generally thought of as an undesirable feature in a Lagrangian based theory as it can lead to Ostrogradski instabilities [101] in the solutions of the field equations. The $f(R)$ theories, however, are a special case in which this instability can be avoided [102], due to the existence of an equivalence with scalar-tensor theories.

In order to help avoid confusion later, we point out that we use the superscripts ^{M} and ^{R} to denote quantities relating to the standard matter fluid and curvature fluid respectively and that the unbarred dynamic quantities with no superscripts are derived from the total effective EMT.

2.1.2 Palatini formalism

In the Palatini formalism, the metric g_{ab} and connection Γ^a_{bc} are treated as independent fields and the variation of the action is performed with respect to each of them separately. For the GR case, varying the Einstein-Hilbert action with respect to the connection, assuming the manifold is torsionless, results in the connection being the Levi-Civita connection and the variation with respect to the metric gives the usual Einstein field equations. For the $f(R)$ case, however, the resulting field equations from the Palatini approach differ from those obtained using the metric approach in these theories.

We denote the Ricci tensor as $\mathcal{R}_{ab}$ and, in this case, it is constructed with an independent connection and $\mathcal{R}$ is given as $g^{ab}\mathcal{R}_{ab}$.

Varying (2.7) with respect to the metric and the connection over a 4-volume yields, respectively,

$$f'\,\mathcal{R}_{ab} - \frac{1}{2}g_{ab}\,f \;=\; T^M_{ab}\;, \tag{2.16}$$

$$\nabla_c\left(\sqrt{-g}\,g^{ab}\,f'\right) \;=\; 0\;, \tag{2.17}$$

where the matter energy momentum tensor T^M_{ab} is defined the usual way and the covariant derivative is taken with respect to the independent connection. We see here that taking the condition $f(\mathcal{R}) = \mathcal{R}$, which implies $f'(\mathcal{R}) = 1$ yields (2.17) as the metricity condition of the Levi-Civita connection and hence the connection becomes the Levi-Civita one. It then follows that $\mathcal{R}_{ab} = R_{ab}$, $\mathcal{R} = R$ and from (2.16) we recover Einstein's field equations.

Serious shortcomings of the Palatini formalism include the introduction of non-perturbative corrections to the matter fields and strong couplings between gravity and matter at low energies [103, 104]. Furthermore, the nature of the Cauchy problem for $f(R)$ gravity in the Palatini formalism is not well formulated in the presence of matter. Without a well-posed initial value problem, Palatini $f(R)$ lacks the predictive power that is required of any physical theory [105].

2.1.3 Metric-affine formalism

In the Palatini formalism, the matter action $\mathcal{S}_{\mathcal{M}} = \int \mathcal{L}_M(g_{ab}, \psi)$ is assumed to be dependent only on the the metric and matter fields. In the metric-affine formalism, one considers the metric and connection to be independent field as in the Palatini approach, but in addition, the matter action is a function of the metric, the matter fields and the connection. The action of this theory then becomes [106],

$$\mathcal{S} = \frac{1}{2}\int dV \left[\sqrt{-g}\, f(\mathcal{R}) + 2\mathcal{L}_M(g_{ab}, \Gamma^a{}_{bc}, \psi)\right] \ . \tag{2.18}$$

where $\mathcal{R} = g^{ab}\,\mathcal{R}_{ab}$ and the Ricci tensor $\mathcal{R}_{ab}$ is constructed with an independent connection as in the Palatini approach.

If we consider that the Ricci scalar is invariant under projective transformation, $\Gamma^c{}_{de} \to \Gamma^c{}_{de} + \lambda_d\,\delta^c{}_e$, then any action built from a function of $\mathcal{R}$, and this includes the Einstein-Hilbert action, is projective invariant in metric-affine gravity. However, since the matter fields do not exhibit this type of invariance, this can lead to inconsistency of the field equations. One way to get around this problem is by adding an action term containing a Lagrange multiplier term B^a which has the form

$$\mathcal{S}_L = \int dV\, \sqrt{-g}\, B^a\, \Gamma^b{}_{[ba]} \ . \tag{2.19}$$

Varying the action with respect to the metric, the connection and the Lagrange multiplier results in, respectively,

$$f'\,\mathcal{R}_{ab} - \frac{1}{2}g_{ab}\, f = T^M_{ab} \ , \tag{2.20}$$

$$\Gamma^a{}_{[ab]} \ = \ 0 \ , \tag{2.21}$$

$$\frac{1}{\sqrt{-g}}\left[\nabla_c\left(\sqrt{-g}\, f'\, g^{ac}\right)\,\delta^b{}_d - \nabla_d\left(\sqrt{-g}\, f'\, g^{ab}\right)\right] + 2\, f'\, g^{ac}\,\Gamma^b{}_{[cd]} = \frac{\chi}{2}\left[\Delta_d{}^{ab} - \frac{2}{3}\,\Delta_c{}^{c[b}\delta^{a]}{}_d\right] \ , \tag{2.22}$$

where

$$\Delta_a{}^{bc} \equiv -\frac{2}{\sqrt{-g}} \frac{\delta \mathcal{S}_M}{\delta \Gamma^a{}_{bc}} \tag{2.23}$$

is the tensor defined by the variation of the matter action with respect to the connection. By splitting (2.22) into a symmetric and antisymmetric part and performing contractions and manipulations, it can be show that $\Delta_a{}^{(bc)} \neq 0$ will introduce non-metricity and $\Delta_a{}^{[bc]} = 0$ corresponds to the vanishing of torsion, respectively, with the Palatini $f(R)$ gravity belonging to the latter.

The metric approach to the $f(R)$ theories will be the focus of the thesis. For studies of the Palatini and metric-affine approaches and the results that follow, the reader is referred to [31, 106].

Chapter 3

Some results in $f(R)$ Gravity

In order to investigate the impact of modifications to gravity, in this chapter we consider the problem of matching different regions of spacetime so as to construct inhomogeneous cosmological models in the context of Lagrangian theories of gravity constructed from general analytic functions $f(R)$, and from non-analytic theories with $f(R) = R^n$. The junction conditions that need to be satisfied when matching together different solutions in $f(R)$ theories are discussed. We also discuss what we mean by 'the weak-field limit' which includes taking Minkowski space to be the solution around which weak-field expansions are performed. We then attempt to make a Swiss-cheese-like construction in which we match the usual weak-field solutions to FLRW solutions in theories with analytic $f(R)$ and proceed to try and match some known exact solutions, including here some theories with non-analytic $f(R)$.

Additionally, we study gravitational lensing which has proven to be a powerful tool in astrophysics and cosmology where it has been used to used to determine the mass distribution of galaxies and galaxy clusters and to put constraints on cosmological parameters. Given that the lensing effect is dependent on the underlying theory of gravity, investigating modifications of GR would result in deviations from the standard expression of the deflection angle which is worth investigating. On that account we study strong lensing of spherically symmetric spacetimes in the case of $f(R) = R^n$ gravity by analysing the behaviour of the general expression for the deflection angle within this context. We subsequently derive the lens mass and magnification for the gravitational lens system.

3.1 Junction conditions for $f(R)$ gravity

Matching together different regions of spacetime in $f(R)$ theories of gravity is a problem that has been considered in [35, 107], and it requires a set of junction conditions, analogous to the Israel-Darmois junction conditions from GR [108, 109]. We will briefly recap the relevant results here.

The central requirement in [35] is that if one allows delta functions in the matter part of the field equations (that is, if one allows matter fields to be localised on the boundary hyper-surface), then delta functions should occur at most linearly in the parts of the field equations that involve geometry only. Here we are interested in the case in which there is no brane located at the boundary. We therefore require that there should be *no* delta functions in the part of the field equations containing geometry only.

Now, in a Gaussian normal coordinate system, $ds^2 = dy^2 + \gamma_{\mu\nu}\, dx^\mu\, dx^\nu$, where the boundary is located at $y = 0$, the Ricci scalar can be written as

$$R = 2\,\partial_y K - K^*_{\mu\nu}\, K^{*\mu\nu} - \frac{4}{3}K^2 + \bar{R}\ , \tag{3.1}$$

where ∂_y is the normal covariant derivative with respect to the boundary, $\bar{R}$ is the Ricci curvature constructed from $\gamma_{\mu\nu}$, $K_{\mu\nu} = -\frac{1}{2}\,\mathcal{L}_y \gamma_{\mu\nu}$ (that is, the Lie derivative of $\gamma_{\mu\nu}$ with respect to the normal) is the extrinsic curvature of the boundary, K and $K^*_{\mu\nu}$ are the trace and trace-free parts of this quantity, respectively.

It can be seen from the field equations (2.13) that R must be continuous at the boundary. This is because the curvature fluid contains terms like $\partial_y f'(R)$, which can be expanded as

$$\partial_y f'(R) = f''(R)\,\partial_y R\ . \tag{3.2}$$

If R is not continuous then the second term above would contain a factor of $\delta(y)$. This is not allowed unless $f''(R) = 0$, which is just Einstein's equations. We can then see from (3.1) that $\gamma_{\mu\nu}$ must also be continuous, otherwise $K_{\mu\nu}$ would contain a factor of $\delta(y)$, and R would contain factors of $(\delta(y))^2$. This is not allowed, as $K_{\mu\nu}$ and R occur directly in the field equations. We therefore have that γ_{ab} and R must both be continuous across the boundary.

The yy and ya components of (2.13) are then given by

$$\partial_y\left[\left(K_{ab} - K\gamma_{\mu\nu}\right) f'(R) + \gamma_{\mu\nu}\, f''(R)\,\partial_y R\right] = 0. \tag{3.3}$$

Integrating this across the boundary one then finds

$$\left[\left(K_{\mu\nu} - K\gamma_{\mu\nu}\right)\ f'(R) + \gamma_{\mu\nu}\, f''(R)\,\partial_y R\right]^+_- = 0, \tag{3.4}$$

where the $[\dots]^+_-$ notation means the difference of the quantity in the brackets on either side of the boundary. Similarly, one can integrate R across the boundary to find, from (3.1) that $[R]^+_- = 0$, and hence that $[2\,\partial_y K - K^*_{\mu\nu}\, K^{*\mu\nu}]^+_- = 0$. The trace and trace-free parts of (3.4) are then given by

$$f''(R)\,[\partial_y R]^+_- = 0\,, \tag{3.5}$$
$$f'(R)\,[K^*_{ab}]^+_- = 0\,, \tag{3.6}$$
$$[K]^+_- = 0\,, \tag{3.7}$$

which, together with

$$[\gamma_{\mu\nu}]^+_- = 0\,, \tag{3.8}$$
$$[R]^+_- = 0\,, \tag{3.9}$$

form the junction conditions in $f(R)$ theories in which $f''(R) \neq 0$. For further details the reader is referred to [35].

3.2 Bottom-up construction of a cosmological model

One of the oldest ways of trying to construct inhomogeneous cosmological models is to join Friedmann-Lemaître-Robertson-Walker (FLRW) solutions, at some appropriate boundary, to the spherically symmetric spacetimes that are expected to exist around individual isolated objects. This was famously achieved by Einstein and Straus for the case of the Schwarzschild solution and the Einstein-de Sitter universe [110]. It is also possible to join the Lemaître-Tolman-Bondi solutions of Einstein's equations to FLRW at a spherical boundary [111]. These models are often referred to as 'Swiss cheese', as this is what the global structure starts to look like if one can keep removing regions of the FLRW 'cheese', and replacing it with either Schwarzschild or Lemaître-Tolman-Bondi holes.

The gravitational fields around isolated objects, and the FLRW solutions of $f(R)$ theories, have both been extensively studied in the literature (see [31–33, 112, 113] for reviews). In this section we do not intend to contribute further to the study of either of these fields, but instead to the ways in which one can construct cosmological models that contain massive astrophysical bodies. This will be done by attempting to match together existing solutions. In particular, we will attempt to construct 'Swiss cheese' models by matching spherically symmetric vacuum solutions with FLRW solutions.

At present, much of the current literature assumes that in $f(R)$ theories the evolution of the FLRW 'background' cosmology proceeds independently of the growth of structure within it. The motivation for this within Einstein's theory comes, in large part, from the studies of inhomogeneous cosmologies. It also comes, however, from the correspondence between Newtonian cosmology and the FLRW solutions of Einstein's equations during dust domination: The rate at which nearby astrophysical bodies fall away from each other can be considered as being due to a Newtonian force (up to the usual accuracy this implies), or due to the expansion of the universe. Both are reasonable descriptions on small enough scales. If one attempts to use $f(R)$ as an explanation of dark energy, however, then one wants the cosmological expansion to be *different* to that of a dust dominated universe. The usual interpretation of the motion of nearby astrophysical bodies as being describable (up to some accuracy) within Newtonian theory is therefore lost, and the intuition we have gained on this subject from studying the solutions of Einstein's equations must be re-evaluated.

In order to evaluate the existence or not of a weak-field limit, and the emergence of FLRW-like behaviour on large scales, one cannot begin by assuming the existence of either of these things. Any realistic investigation, however, needs to make some assumptions, and here we will begin by assuming that the gravitational fields around astrophysical bodies can be described by known solutions (either weak-field or exact). We will then proceed to see which FLRW solutions these can be matched with, or which FLRW behaviours emerge, given this assumption. Throughout this section we will use Latin letters a, b, c, etc. to denote spacetime indices, and Greek letters to denote coordinates on a boundary. When it is required, the letters i, j, k, etc. will be reserved for spatial indices.

3.2.1 Viability of FLRW geometry in $f(R)$

The spatially homogenous and isotropic FLRW model has been important in our understanding of the nature of the universe within the context of GR. In this model the metric, in spherical coordinates, is given by

$$ds^2 = -d\hat{t}^2 + a^2(\hat{t}) \left[\frac{d\hat{r}^2}{1 - k\hat{r}^2} + \hat{r}^2\, d\hat{\theta}^2 + \hat{r}^2\, \sin^2\hat{\theta}\, d\hat{\phi}^2 \right] , \tag{3.10}$$

where $a(\hat{t})$ is the scale factor and $k = -1,\ 0,\ 1$ for negative, zero, and positive curvature respectively.

The establishment of the spacetime as FLRW may be taken from the observable isotropy of the CMBR (assuming isotropy holds everywhere) together with the *Ehlers-Geren-Sachs therorem* (EGS) [114] which states

> *If a family of freely-falling observers measure self-gravitating background radiation to be everywhere exactly isotropic in the case of non-interacting matter and radiation, then the universe is exactly homogenous.*

However, the CMBR is not exactly isotropic, implying that it has been almost spatially homogenous and isotropic since decoupling of matter and radiation. Stoeger, Maartens and Ellis [115] proved the stability of the EGS results by showing that spacetimes that are close to satisfying the EGS conditions are almost-FLRW. These results are summarised in the *Almost-EGS theorem* which states

> *If the Einstein-Liouville equations are satisfied* [1] *in an expanding universe, where there is present pressure-free matter with a timelike 4-velocity vector field u^a such that (freely-propagating) background radiation is everywhere almost-isotropic relative to u^a in some domain U, then space-time is almost-FLRW in U.*

In terms of viability of FLRW geometry in $f(R)$, the validity of the EGS theorem has been extended to $f(R)$ theories by Rippl, van Elst, Tavakol, and Taylor in [118]. They generalised the results of Maartens and Taylor [119] to show that for metric $f(R)$ theories, a perfect fluid spacetime with vanishing vorticity, shear and acceleration is FLRW only if the fluid has in addition a barotropic equation of state. Accordingly, the EGS theorem and its almost extension are valid for general $f(R)$ theories as well. An independent proof of this result was demonstrated recently by Faraoni [120] where he went on to prove the validity of the EGS theorem for Palatini $f(R)$ gravity.

3.2.2 Matching weak-field geometries to FLRW

The simplicity of the Swiss cheese approach and the degree to which it has influenced the development of inhomogeneous cosmology in GR, makes it a natural place to begin studying the relationship between weak-field systems and cosmology in $f(R)$ theories of gravity.

By "weak-field" we mean that in extended regions of the Universe that are small compared to the Hubble scale, but large compared to the Schwarzschild radius of any compact objects that may exist within it, that the geometry of spacetime within the region (but outside of the compact objects) can be well described by small fluctuations around Minkowski space, such that

$$g_{ab} \simeq \eta_{ab} + h_{ab} \, , \tag{3.11}$$

where η_{ab} is the metric of Minkowski space, and there exists a coordinate system in which each of the components of h_{ab} is $\ll 1$ and slowly varying. The description given by (3.11), and the corresponding physics, is what is meant by 'the weak-field limit'.

There are a number of points in this explanation that require further clarification. Firstly, what we mean by 'Hubble scale' here is the quantity $c\,H^{-1}$ when considering space-like separations, and H^{-1} when considering time-like separations (here H is the Hubble constant, as measured by observers using the recessional velocity of nearby objects).

[1] It is assumed that Einstein equations are satisfied and radiation obeys the Liouville's equation $L(f) = 0$ where L is the Liouville operator from kinetic theory [46, 116, 117]

For a region to be 'small' compared to the Hubble scale then means that any two points on the boundary of that region that are space-like separated should be $\ll c\,H^{-1}$ apart, and that any two points that are time-like separated should be $\ll H^{-1}$ apart. This definition requires H to be reasonably uniform throughout each small region, which we will assume to be true. The criterion that these regions should be much larger than the Schwarzschild radius of any compact objects, and that (3.11) should not be taken to describe the regions inside (or near) compact objects, are simply intended to remove from our consideration the regions near black holes and neutron stars.

Let us now further consider equation (3.11). The crucial point here is that the geometry of spacetime in the region under consideration can be taken to be close to that of Minkowski space. In this case one can decompose the tensor h_{ab} according to how its various parts transform under spatial rotations in the background Minkowski space. In general, one can then write h_{ab} as (see [121])

$$h_{ab}\,dx^a\,dx^b = 2\Phi\,c^2\,dt^2 - 2B_i\,c\,dt\,dx^i + 2\,(\Psi\,\delta_{ij} + H_{ij})\,dx^i\,dx^j \ .$$

The divergence of B_i and the trace of H_{ij} can be set to zero by an appropriate choice of coordinates, and the divergence-less part of B_i and the trace-free part of H_{ij} can be consistently ignored. This leaves

$$ds^2 \simeq -(1-2\Phi)\,c^2\,dt^2 + (1+2\Psi)\delta_{ij}\,dx^i\,dx^j \ , \tag{3.12}$$

where ϕ and Ψ are both $\ll 1$ and slowly varying. We refer to this as 'the Newtonian limit' if Φ behaves like a Newtonian potential, and satisfies $\nabla^2\Phi \simeq -\,4\pi\,G\,\mu^M$.

Finally, we can make the concepts of 'small' and 'slow' precise by introducing a dimensionless order-of-smallness parameter, ϵ. Velocities, $v^i = dx^i/dt$, are then said to be 'small' if $v/c \sim O(\epsilon)$, and quantities are said to be 'slowly varying' if acting on them with a time-derivative adds an extra $O(\epsilon)$ of smallness when compared to a spatial derivative (the order of smallness of time derivatives and velocities are expected to be similar because the evolution of gravitating systems are typically governed by the motion of their constituents). From the field equations and equations of motion it can be seen that the lowest order parts of Φ and Ψ, and the matter energy density μ^M, are given by

$$\Phi \sim \Psi \sim G\,\mu^M \sim \frac{v^2}{c^2} \sim \epsilon^2 \ .$$

The field equations and equations of motion within the region under consideration can then be expanded order-by-order in ϵ, with the 'weak field' limit of equation (3.12) corresponding to the expansion up to $O(\epsilon^2)$. The $\simeq$ sign will be used in what follows to mean 'equal up to terms of $O(\epsilon^3)$ and smaller'. This is the same expansion in ϵ that is routinely used in the standard parameterised post-Newtonian (PPN) approach to gravitational physics in weak-field systems [122].

The weak-field limit of $f(R)$ theories of gravity has been studied extensively in the literature (see e.g. [31–33, 112, 113], and references therein), with the full post-Newtonian limit of theories with analytic $f(R)$ that admit Minkowski space as a solution being found in [43]. There the Lagrangian function is expanded in a Taylor series as

$$f(R) = f(0) + f'(0)\,R + \frac{1}{2}f''(0)\,R^2 + O(R^3)\ , \tag{3.13}$$

where primes denote differentiation with respect to R. One may note here that the expansion is being performed as a series around $R = 0$, in keeping with our assumption that Minkowski space is a suitable background about which we can perform an analysis of the weak field. This limits our consideration to theories in which $f(0)$, $f'(0)$ *etc.* are finite, which is certainly not true for all theories [25]. One is, of course, at liberty to consider expanding around other backgrounds, with non-zero Ricci curvature, R_0 (see, e.g., [123]). In this case, however, one must deal with the complexity of solving the full non-linear Einstein equations in order to find the background, which is both difficult and likely to result in many different possibilities. We will consider this further for some simple theories in Subsection 3.2.3.

To the order required here, and taking Minkowski space as the background geometry, the metric is given by (3.12) with [43]

$$\Phi = \frac{1}{f_0'}\left(V + \frac{1}{2}f_0''\,R\right)\ , \tag{3.14}$$

$$\Psi = \frac{1}{f_0'}\left(U - \frac{1}{2}f_0''\,R\right)\ , \tag{3.15}$$

where we have used the abbreviations $f_0' = f'(0)$ and $f_0'' = f''(0)$, and where U, V and R satisfy

$$\nabla^2 U = -\,4\,\pi\,\mu^M + \frac{f_0}{4}, \qquad \nabla^2 V = -4\pi\,\mu^M - \frac{f_0}{2} \tag{3.16}$$

and

$$\nabla^2 R - \frac{f_0'}{3f_0''}\,R = -\frac{8\pi}{3f_0''}\,\mu^M + \frac{f_0}{6f_0''}\ , \tag{3.17}$$

where $f_0 = f(0)$.

Assuming the existence of a weak-field limit, these theories can be seen to have a Newtonian limit if $f''R \ll U$. Unlike in the PPN treatment, we will not insist that the solutions of (3.16) and (3.17) approach zero at asymptotically large distances, but will instead enforce boundary conditions using cosmology.

We will now try to match the weak-field geometry to an FLRW geometry (3.10) that is filled with a perfect fluid. Within the FLRW spacetime we will excise a region interior to the sphere $\hat{r} = \hat{\Sigma}$, and replace it with a region of spacetime that is spherically symmetric, and that is well described by the weak-field geometry given in (3.12). In this case it is convenient to write the spatial metric in spherical polar coordinates, so that $\delta_{ij} dx^i dx^j = dr^2 + r^2\, d\theta^2 + r^2\, \sin^2\theta^2\, d\phi^2$. We can then identify the angular coordinates in both regions, which we will refer to as Region I and Region II, respectively.

Without loss of generality, we consider the boundary to be comoving with the fluid. As there are no spatial gradients in Region I, the boundary must be static with respect to the hypersurfaces of homogeneity that exist in the FLRW geometry. In Region II, however, the boundary is free to move in the radial direction. The first fundamental form of the boundary, on either side, is then given by

$$\begin{aligned} \gamma^{I}_{\mu\nu}\, dx^\mu\, dx^\nu &= -d\hat{t}^2 + a^2(\hat{t})\, \hat{\Sigma}^2 d\Omega^2\,, \\ \gamma^{II}_{\mu\nu}\, dx^\mu\, dx^\nu &\simeq -\left(1 - 2\Phi - \dot{\Sigma}^2\right) dt^2 + (1+2\Psi)\Sigma^2\, d\Omega^2\,, \end{aligned}$$

where the boundary is at $r = \Sigma$ in Region II, and where we have used the notation $\simeq$ to mean equal up to terms of post-Newtonian order (that is, up to $O(\epsilon^3)$). The junction condition (3.8) then gives the conditions

$$(1+\Psi)\Sigma \simeq a(\hat{t})\,\hat{\Sigma}\,, \tag{3.18}$$

$$\frac{d\hat{t}}{dt} \simeq 1 - \Phi - \frac{1}{2}\dot{\Sigma}^2\,. \tag{3.19}$$

Now let us consider the extrinsic curvature. To calculate this we need to know the normal to the boundary, which is given in each region by

$$n^{I}_{a} = \frac{a(\hat{t})\, \delta^{\hat{r}}{}_{a}}{\sqrt{1 - k\,\hat{r}^2}}\,, \tag{3.20}$$

$$n^{II}_{a} \simeq \left(1 + \Psi + \frac{1}{2}\dot{\Sigma}^2\right) \delta^{r}{}_{a} - \dot{\Sigma}\, \delta^{t}{}_{a}\,. \tag{3.21}$$

The second fundamental form on the boundary is then given by

$$K_{\mu\nu} = \frac{\partial x^a}{\partial x^\mu} \frac{\partial x^b}{\partial x^\nu}\, n_{a;b}\,, \tag{3.22}$$

which for the two spacetimes we are considering is

$$K^I_{\mu\nu}\, dx^\mu\, dx^\nu \;\;\simeq\;\; r\left(1-\frac{k\,r^2}{2\,a^2(\hat{t})}+\frac{1}{f_0'}\,U-\frac{f_0''}{2f_0'}\,R\right)d\Omega^2, \tag{3.23}$$

and

$$\begin{aligned} K^{II}_{\mu\nu}\, dx^\mu\, dx^\nu \;\;\simeq\;\; & \left(\frac{1}{f_0'}\,V_{,r}+\frac{f_0''}{2f_0'}\,R_{,r}-\ddot{\Sigma}\right)d\hat{t}^2 \\ & +r\left(1-\frac{f_0''}{2f_0'}\,R+\frac{1}{2}\dot{\Sigma}^2-\frac{f_0''}{2f_0'}\,r\,R_{,r}+\frac{1}{f_0'}\,U+\frac{r}{f_0'}\,U_{,r}\right)d\Omega^2\ , \end{aligned} \tag{3.24}$$

where we have already used the junction conditions (3.18) and (3.19).

The junction conditions (3.6) and (3.7) then give

$$\frac{\dot{\Sigma}^2}{\Sigma^2} \;\;\simeq\;\; -\frac{2\,U_{,r}|_\Sigma}{f_0'\Sigma}-\frac{k\,\hat{\Sigma}^2}{\Sigma^2}+\frac{f_0''}{f_0'}\,\frac{R_{,r}|_\Sigma}{\Sigma}\,, \tag{3.25}$$

$$\frac{\ddot{\Sigma}}{\Sigma} \;\;\simeq\;\; \frac{V_{,r}|_\Sigma}{f_0'\,\Sigma}+\frac{f_0''}{2f_0'}\,\frac{R_{,r}|_\Sigma}{\Sigma}\,. \tag{3.26}$$

These look very much like the Friedmann equations derived from Einstein's equations, with the terms containing the Newtonian potential U acting like the matter terms, and with the term involving the spatial curvature k playing its usual role. Here, however, we also have two additional terms containing derivatives of the Ricci scalar, R. These extra terms can be seen to contain all of the new behaviour that one obtains by generalising the gravitational Lagrangian from R to $f(R)$.

So far we have only applied the junction conditions that exist in Einstein's equations: That the first and second fundamental forms on the boundary must be continuous if we are to avoid a surface layer of matter. Let us now apply the additional junction condition (3.5). The spacetime in Region I is homogeneous, so in this case we must have

$$\partial_y R = \frac{\sqrt{1-k\hat{r}^2}}{a(\hat{t})}\,R_{,\hat{r}} = 0\ . \tag{3.27}$$

Applying the junction condition (3.5) then gives

$$R_{,r}|_\Sigma \simeq 0\ , \tag{3.28}$$

where we have used $k\,\hat{\Sigma}^2 \sim O(\epsilon^2)$, as can be seen from (3.25). This means that the last terms on the right-hand side of both (3.25) and (3.26) must vanish at $O(\epsilon^2)$, so that we are left with exactly the same equations as in Einstein's theory (up to the presence of f_0'

in the denominator of the terms involving U, which can be absorbed into constants, and the terms involving f_0 in (3.16) and (3.17), which act like Λ).

This treatment appears to show that the only Swiss cheese solutions that exist in $f(R)$ theories of gravity must either have FLRW regions that behave in the same way they do in Einstein's theory (possibly with Λ, and up to possible small corrections), or it must be the case that the spacetime within the excised spheres cannot be described using the weak-field geometry given in (3.12).

3.2.3 Matching exact solutions

We have so far considered joining weak-field geometries to FLRW, in theories in which $f(R)$ is an analytic function. This has shown that acceleration in the resulting cosmological model cannot occur in any new ways if the junction conditions given in Section 3.1 are to be obeyed. One must therefore either allow for gravitational fields to be rapidly varying, or give up on a description of the regions around astrophysical objects as small fluctuations about Minkowski space. The latter of these two possibilities suggests that it may be useful for us to consider exact solutions.

Unfortunately, the complexity of the field equations (2.13) make exact solutions difficult to find. However we know that *for all functions $f(R)$ which are of class C^3 at $R = 0$ and $f(0) = 0$ while $f'(0) \neq 0$, the Schwarzschild solution is the only vacuum solution with vanishing Ricci scalar* [73]. It therefore seems natural to try and match a spherical region with Schwarzschild geometry to an exterior FLRW spacetime. In the context of Einstein's theory this corresponds to the well-known Einstein-Straus approach described earlier [110]. Furthermore, if we restrict our considerations to $f(R) = R^{1+\delta}$ then there are two known exact solutions (other than the vacuum solutions of Einstein's equations, that is, which are also solutions of these theories). A static spherically symmetric vacuum solution with non-trivial asymptotics was found in [37], and more recently via an independent method in [73]. A time-dependent spherically symmetric vacuum solution was found in [34]. In what follows, we will also try and join these two solutions to FLRW geometries.

3.2.3.1 An Einstein-Straus-like construction

The constructions we consider here consist of point-like masses at the centre of otherwise empty spherical regions, whose geometry is described by the Schwarzschild metric, and that are embedded in FLRW geometry at appropriate boundaries. Such constructions were originally considered by Einstein and Straus [110], and were introduced to address the question of whether or not the expansion of the universe can affect local mechanical phenomena, such as planetary orbits. Since the spacetime near the central mass is Schwarzschild, the planetary orbits are given by the usual time-like geodesics of this geometry, and the cosmic expansion does not affect them. Let us now investigate whether such a construction can be performed in $f(R)$ gravity.

We begin by writing the Schwarzschild solution as

$$ds^2 = -A(r)\,dt^2 + \frac{dr^2}{A(r)} + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\ , \tag{3.29}$$

where

$$A(r) = \left(1 - \frac{2M}{r}\right)\ . \tag{3.30}$$

Let us now try and embed this solution in an FLRW geometry, as specified in (3.10). To do this, consider a boundary at $\hat{r} = \hat{\Sigma}$ in the FLRW spacetime and $r = \Sigma$ in the Schwarzschild solution. The first fundamental form on the boundary is then given in the vacuum region by

$$\gamma_{\mu\nu}\,dx^\mu\,dx^\nu = -\left(A - \frac{\dot{\Sigma}^2}{A}\right)dt^2 + \Sigma^2 d\Omega^2 \tag{3.31}$$

and in the FLRW region by

$$\gamma_{\mu\nu}\,dx^\mu\,dx^\nu = -d\hat{t}^2 + a^2(\hat{t})\,\hat{\Sigma}^2\,d\Omega^2\ , \tag{3.32}$$

where we have identified angular coordinates in the two different regions at the boundary and where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. The junction condition (3.8) then gives

$$\Sigma \;=\; a(\hat{t})\hat{\Sigma}\ , \tag{3.33}$$

$$\frac{d\hat{t}}{dt} \;=\; \sqrt{A - \frac{\dot{\Sigma}^2}{A}}. \tag{3.34}$$

To calculate the second fundamental form we need the space-like unit vector normal to the boundary. In the vacuum region this is given by

$$n_a = \frac{\sqrt{A}}{\sqrt{A^2 - \dot{\Sigma}^2}}\left(-\dot{\Sigma}, 1, 0, 0\right), \tag{3.35}$$

while in the FLRW region it is

$$n_a = \frac{a(\hat{t})\,\delta^r{}_a}{\sqrt{1 - k\,\hat{\Sigma}^2}}\ . \tag{3.36}$$

The second fundamental form on the FLRW side of the boundary is then

$$K_{\mu\nu}\,dx^\mu\,dx^\nu = a(\hat{t})\,\hat{\Sigma}\sqrt{1 - k\,\hat{\Sigma}^2}\,d\Omega^2\ , \tag{3.37}$$

while on the vacuum side of the boundary it is given by

$$K_{tt} = \frac{3A\,A_{,r}\,\dot{\Sigma}^2 - A^3\,A_{,r} - 2A^2\,\ddot{\Sigma}}{2\sqrt{A}\,(A^2 - \dot{\Sigma}^2)^{3/2}}\,, \tag{3.38}$$

$$K_{\theta\theta} = \sqrt{\frac{\Sigma^2\,A^3}{(A^2 - \dot{\Sigma}^2)}}\,, \tag{3.39}$$

where all quantities should be evaluated at the boundary. Matching $K^{\pm}_{\theta\theta}$ at the boundary we obtain

$$\dot{\Sigma}^2 = A^2\left[1 - \frac{A}{(1 - k\,\Sigma^2/a^2)}\right]. \tag{3.40}$$

Writing the above equation in the coordinates $(\hat{t}, \hat{r}, \theta, \phi)$, and using (3.33), (3.34) together with the form of the function $A(r)$, we find

$$\hat{\Sigma}^3\,a(\hat{t})\left[k + \left(\frac{da(\hat{t})}{d\hat{t}}\right)^2\right] = 2M\,. \tag{3.41}$$

The left-hand side of the above equation is the usual definition of the Cahill-Macvitte mass function [124] in FLRW spacetimes.

Differentiating (3.41) with respect to $\hat{t}$ gives $G^1{}_1 = 0$. This implies that the total pressure (standard matter and curvature fluid) must vanish on the boundary, but as the pressure in the FLRW region is homogeneous, this means that the total pressure should vanish throughout the FLRW region. In this case, equating the time component of the extrinsic curvature will not give any new information.

If we now impose the requirement that R should be the same on either side of the boundary, from (3.9), then we must have

$$6\left(\frac{1}{a(\hat{t})}\frac{d^2a(\hat{t})}{d\hat{t}^2} + \frac{1}{a(\hat{t})^2}\left(\frac{da(\hat{t})}{d\hat{t}}\right)^2 + \frac{k}{a(\hat{t})^2}\right) = 0\,. \tag{3.42}$$

The above equation combined with the condition of vanishing total pressure, then implies vanishing total density (curvature fluid and standard matter) in the FLRW region. What is more, putting $R = 0$ in (2.13) shows that the effective energy-momentum tensor of the curvature fluid must be proportional to g_{ab}. It then follows that the energy-momentum tensor of standard matter, T^m_{ab}, must also be proportional to g_{ab}, and so can only be a vacuum energy. It also follows that the FLRW region can only be Minkowski spacetime (in Milne coordinates, if $k = -1$). Finally, from (3.5) we see that the normal gradients automatically match identically, as $R = 0$ on both sides.

We note that the situation remains the same if, instead of a Schwarzschild interior we have a Schwarzschild-de Sitter, or anti-de Sitter, interior. In these cases the interior region has a constant, non-zero Ricci scalar. As R must be matched across the boundary, the FLRW region must also have a constant Ricci scalar, and from (2.13) it can be easily seen that the effective energy-momentum tensor of the curvature fluid must be proportional to g_{ab}. Furthermore, matching the second fundamental form now gives ${G^1}_1$ = constant in the FLRW region, which implies that the total pressure must be constant. Taken together, these two conditions imply that the total energy density should also be constant, and that the energy-momentum tensor of matter in the FLRW region must have $T^M_{ab} \propto g_{ab}$, which is nothing other than vacuum energy. The only solution in this case is therefore a spacetime that is de Sitter everywhere.

It is a curious result that the Schwarzschild solution cannot be embedded in any FLRW spacetime (other than the trivial case of Minkowski space) in $f(R)$ theories of gravity, unless the theory is linear in R. However, this conclusion is natural from the junction conditions because the conditions that the Ricci scalar and its first derivative should match across the boundary make the non-trivial $f(R)$ theories qualitatively different from GR, where R can be discontinuous. If a spherically symmetric object is joined to a FLRW geometry in $f(R)$ theories, then one must expect an evolution of the boundary values of R and $\dot{R}$, which is something that pure Schwarzschild or Schwarzschild-de Sitter solutions cannot satisfy. Hence, in the following sections, we will explore some other exact non-GR solutions in $f(R)$ gravity, in order to check whether Einstein-Straus-like constructions are possible with them.

3.2.3.2 A static solution in R^n gravity

An exact static, spherically symmetric vacuum solution of $f(R) = R^{1+\delta}$, that was found in [37] and later found using the 1+1+2 covariant approach in [73], is given by [37, 73]

$$ds^2 = -A(r)\,dt^2 + \frac{dr^2}{B(r)} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\ , \tag{3.43}$$

where

$$\begin{aligned} A(r) &= r^{\frac{2\delta(1+2\delta)}{(1-\delta)}} + \frac{c_1}{r^{\frac{(1-4\delta)}{(1-\delta)}}}\ , \\ B(r) &= \frac{(1-\delta)^2}{(1-2\delta+4\delta^2)(1-2\delta-2\delta^2)}\left(1 + \frac{c_1}{r^{\frac{(1-2\delta+4\delta^2)}{(1-\delta)}}}\right)\ . \end{aligned}$$

The Ricci scalar for this solution is

$$R = -\frac{6\,\delta(1+\delta)}{(1-2\delta-2\delta^2)\,a^2\,r^2}\ . \tag{3.44}$$

We will now try and embed this solution in an FLRW geometry. To do this, consider a boundary at $r = \Sigma$ in the vacuum region. The first fundamental form on the boundary is then given in the vacuum region by

$$\gamma_{\mu\nu}\,dx^\mu\,dx^\nu = -\left(A - \frac{\dot{\Sigma}^2}{B}\right)dt^2 + \Sigma^2\,d\Omega^2\ . \tag{3.45}$$

Matching the first fundamental forms then gives

$$\Sigma = a(\hat{t})\,\hat{\Sigma} \tag{3.46}$$

$$\frac{d\hat{t}}{dt} = \sqrt{A - \frac{\dot{\Sigma}^2}{B}}\ . \tag{3.47}$$

In the vacuum region the spacelike unit vector normal to the boundary is given by

$$n_a = \frac{\sqrt{A}}{\sqrt{A\,B - \dot{\Sigma}^2}}\left(-\dot{\Sigma}, 1, 0, 0\right)\ , \tag{3.48}$$

The second fundamental form of the vacuum side is

$$K_{tt} = \frac{2B\,A_{,r}\,\dot{\Sigma}^2 + A\,B_{,r}\,\dot{\Sigma}^2 - A\,B^2\,A_{,r} - 2A\,B\,\ddot{\Sigma}}{2\sqrt{A}(A\,B - \dot{\Sigma}^2)^{3/2}}\ , \tag{3.49}$$

$$K_{\theta\theta} = \sqrt{\frac{\Sigma^2\,B^2\,A}{(A\,B - \dot{\Sigma}^2)}}\ , \tag{3.50}$$

where all quantities should be evaluated at the boundary.

The junction conditions (3.6) and (3.7) are then satisfied if

$$\dot{\Sigma}^2 = A\,B\left[1 - \frac{B}{(1 - k\,\Sigma^2/a^2)}\right] \tag{3.51}$$

$$\ddot{\Sigma} = \frac{(A_{,r}\,B + B_{,r}\,A)}{2} - \frac{B(2A_{,r}\,B + B_{,r}\,A)}{2(1 - k\,\Sigma^2/a^2)}\ . \tag{3.52}$$

Consistency of these equations requires

$$\frac{(A\,B_{,r} - A_{,r}\,B)}{(1 - kr^2)} = 0\ . \tag{3.53}$$

Substitution from (3.43) shows that this can be achieved only if $\delta = 0$ or $-1/2$.

If we now impose the requirement that R should be the same on either side of the boundary, from (3.9), then we get

$$\frac{1}{a(\hat{t})}\frac{d^2a(\hat{t})}{d\hat{t}^2}+\frac{1}{a(\hat{t})^2}\left(\frac{da(\hat{t})}{d\hat{t}}\right)+\frac{k}{a(\hat{t})^2}=-\frac{\delta(1+\delta)}{(1-2\delta-2\delta^2)\,a(\hat{t})^2\,\hat{\Sigma}^2}\ . \tag{3.54}$$

This strongly constrains the allowed form of $a(t)$. Finally, from (3.5), we find that we must have

$$\frac{\sqrt{A}\,B\,R_{,r}}{\sqrt{A\,B-\dot{\Sigma}^2}}=0\ , \tag{3.55}$$

as there are no spatial gradients in the FLRW region. This means that we must also require $R_{,r}=0$ at the boundary in the vacuum region. This is only satisfied if $\delta=0$ or -1, as can be seen from the right-hand side of (3.54).

It is therefore the case that the junction conditions from Section 3.1 can only be satisfied if $\delta=0$, in which case the field equations (2.13) simply reduce to Einstein's equations. In this case the vacuum solution given in (3.43) reduces to the Schwarzschild solution, and (3.54) no longer needs to be satisfied as $f''=0$, and the right-hand side of (3.2) vanishes automatically. The vacuum solution (3.43) cannot, therefore, be used to model the gravitational field of an astrophysical object embedded in an FLRW universe in these theories, unless $f(R)$ is linear in R. This is despite the fact that this solution is the asymptotic attractor of all spherically symmetric, static, vacuum solutions of theories with $f(R)=R^{1+\delta}$ [37], suggesting that the spacetime around astrophysical objects that are embedded in FLRW should be time dependent.

3.2.3.3 A non-static solution in R^n gravity

An exact solution for time-dependent, spherically symmetric vacuum situations in $f(R)=R^{1+\delta}$ theories is given by [34]:

$$ds^2=-A(r)\,dt^2+q^2(t)\,B(r)\left(dr^2+r^2\,d\Omega^2\right)\ , \tag{3.56}$$

where $q(t)=t^{\frac{\delta(1+2\delta)}{(1-\delta)}}$, and

$$\begin{aligned} A(r) &= \left[\frac{1-\frac{c_2}{r}}{1+\frac{c_2}{r}}\right]^{2/\sigma}\ , \\ B(r) &= \left(1+\frac{c_2}{r}\right)^4 A^{\sigma+2\delta-1}\ , \end{aligned}$$

where $\sigma^2=1-2\delta+4\delta^2$. The Ricci scalar in this case is given by

$$R=-\frac{6\,\delta(1+\delta)(1+2\delta)(1-4\delta)}{(1-\delta)^2\,t^2\,A}\ . \tag{3.57}$$

Again, to match this solution with an FLRW exterior, consider a boundary at $r = \Sigma$ in this solution. The first fundamental form on the boundary is then given in the vacuum region by

$$\gamma_{\mu\nu}\,dx^\mu\,dx^\nu = -\left(A - q^2 B\,\dot{\Sigma}^2\right)dt^2 + q^2\,B\,\Sigma^2\,d\Omega^2\,. \tag{3.58}$$

Matching the first fundamental forms then gives

$$q\sqrt{B}\,\Sigma = a(\hat{t})\,\hat{\Sigma}\,, \tag{3.59}$$

$$\frac{d\hat{t}}{dt} = \sqrt{A - q^2\,B\,\dot{\Sigma}^2}\,. \tag{3.60}$$

and the unit vectors tangent and normal to the boundary are given by

$$u^a = \frac{1}{\sqrt{A - B\,q^2\,\dot{\Sigma}^2}}(1, \dot{\Sigma}, 0, 0)\,, \tag{3.61}$$

$$n_a = \frac{\sqrt{A\,B}\,q}{\sqrt{A - B\,q^2\,\dot{\Sigma}^2}}(-\dot{\Sigma}, 1, 0, 0). \tag{3.62}$$

Calculating the second fundamental form for the matching surface we get

$$K_{tt} = \frac{2\dot{q}\,q(B^2\,q^2\,\dot{\Sigma}^3 - 2A\,B\,\dot{\Sigma}) + q^2\,\dot{\Sigma}^2(2A_r\,B - B_r\,A) - 2q^2\,A\,B\ddot{\Sigma} - A\,A_r}{2\sqrt{A\,B}\,q(A - B\,q^2\,\dot{\Sigma}^2)^{3/2}}\,, \tag{3.63}$$

$$K_{\theta\theta} = \frac{q\,\Sigma(A\,B_r\,\Sigma + 2A\,B + 2\dot{q}\,\dot{\Sigma}\,B^2\,q\,\Sigma)}{2\sqrt{A\,B}\sqrt{A - B\,q^2\,\dot{\Sigma}^2}}. \tag{3.64}$$

Matching the Ricci scalar then gives

$$\frac{1}{a(\hat{t})}\frac{d^2a(\hat{t})}{d\hat{t}^2} + \frac{1}{a(\hat{t})^2}\left(\frac{da(\hat{t})}{d\hat{t}}\right)^2 + \frac{k}{a(\hat{t})^2} = -\frac{\delta(1+\delta)(1+2\delta)(1-4\delta)}{(1-\delta)^2\,t^2\,A(\Sigma)}. \tag{3.65}$$

Finally, the boundary condition $n \cdot \nabla R = 0$, gives

$$\dot{\Sigma} = -\frac{2\,c_2}{\sigma\,\Sigma^2\left(1 - \frac{c_2}{\Sigma}\right)^3\left(1 + \frac{c_2}{\Sigma}\right)^3}\left(\frac{1 - \frac{c_2}{\Sigma}}{1 + \frac{c_2}{\Sigma}}\right)^{\frac{4(1-\delta)}{\sigma}} t^{\frac{(1-3\delta-4\delta^2)}{(1-\delta)}}\,, \tag{3.66}$$

unless $\delta = 1/4,\ 0,\ -1$ or $-1/2$, in which case $n \cdot \nabla R = 0$ automatically. We can now construct an algebraic constraint for Σ by equating K_{tt} on either side of the boundary and using Eqs. (3.59) and (3.66) to remove $a(\hat{t})$ and $\dot{\Sigma}$. This gives

$$\frac{q}{A^{\frac{1-2\delta}{2}}}\left[\frac{\left(\Sigma^2+c_2^2\right)(1-\delta)k-2\,c_2\,\sigma^2\,\Sigma}{(1-\delta)\sqrt{\sigma^2-4\,c_2^2\,t^{\frac{2(1-2\delta-2\delta^2)}{(1-\delta)}}\,\Sigma^{8(1-\delta)/\sigma}(\Sigma^2-c_2^2)^2\,A^{2(1-\delta)}}}\right.$$
$$\left.-\sqrt{1-k\,\hat{\Sigma}^2}\left(\Sigma^2-c_2^2\right)\right]=0\ . \qquad (3.67)$$

This equation must be satisfied at all times, but is clearly very difficult to solve for Σ directly. We can, however, perform a series expansion in c_2. To zero order we then have the constraint

$$\sqrt{1-k\,\hat{\Sigma}^2}=1+O(c_2), \qquad (3.68)$$

so that $k\simeq 0$. This says that the FLRW geometry in which we are embedding must be close to spatially flat. Using this in the first order equation then gives

$$\frac{\sigma}{(1-\delta)}\,c_2=0+O(c_2^2), \qquad (3.69)$$

so that the only possible solutions would appear to require either $\sigma=0+O(c_2)$, or $c_2=0$. The first of these possibilities requires δ to be complex, in which we are not interested here, and the second is the requirement that the central mass vanishes. The matching of this latter situation to FLRW is trivial, as the geometry in (3.56) can itself be seen to reduce to FLRW when $c_2\to 0$. Once again we therefore appear to be unable to match solutions to FLRW, except when $\delta=0$, or when the entire spacetime is FLRW anyway.

The anomalous cases that remain are those in which $\delta = 1/4$, -1 or $-1/2$, as in these cases (3.66) can no longer be used. Of these $\delta = -1$ seems problematic as it corresponds to a Lagrangian density $\mathcal{L}$ =constant, which can hardly be said to be a Lagrangian for gravity at all. The cases $\delta = -1/2$ and $\delta = 1/4$ also seem problematic, as in these cases the field equations (2.13) contain terms that are ill-defined, with both numerator and denominator reducing to zero.

In all of these cases the Ricci scalar must vanish, so the only exterior FLRW geometry that one could match to would have to be Milne anyway. We do not, therefore, consider them to be of any interest for our current purposes.

We therefore find that even for this non-trivial non-static solution, a matching with a FLRW exterior is not possible. This is true even though the solution itself approaches FLRW asymptotically.

3.2.4 Discussion

After a discussion of the junction conditions that need to be satisfied when matching together different solutions in $f(R)$ theories, a number of attempts were made to construct inhomogeneous cosmological models by matching different regions of spacetime. This was done both for theories with general analytic functions $f(R)$ and for non-analytic theories with $f(R) = R^n$. In all cases studied, it was found that it is impossible to satisfy the required junction conditions without the large-scale behaviour reducing to what is found from Einstein's equations with a cosmological constant. For theories with analytic $f(R)$ this suggests that the usual treatment of weak-field systems as perturbations around Minkowski space may not be compatible with late-time acceleration that is driven by anything other an effective cosmological constant given by $f(0)$. For theories with $f(R) = R^n$, we found that a number of well-known spherically symmetric vacuum solutions could not be matched to an expanding FLRW background, including the well-known Einstein-Straus-like embeddings of the Schwarzschild exterior solution in FLRW spacetimes.

The absence of these constructions represents a crucial difference between $f(R)$ theories and scalar-tensor theories of gravity. In the latter it is already known that Einstein-Straus-like embeddings are indeed possible, both in cosmological and astrophysical gravitational collapse scenarios (see for example [125]). This is true despite the extra junction conditions that are required in scalar-tensor theories, where the scalar field and its normal derivative must be matched at the boundary. These two conditions may initially seem quite similar to the extra conditions required in $f(R)$ gravity, that is, matching the Ricci scalar and its normal derivative). However, it turns out that the conditions in $f(R)$ theories are much more restrictive, and give much stronger constraints on the spacetimes allowed on either side of the boundary. This is due to R taking a very specific form once an ansatz has been made for the metric (by specifying it should be give by Eqs. (3.12) or (3.29), for example), which is in general not true for scalar-tensor theories.

These results are quite different to what is suggested by using linear perturbation theory around an FLRW background in $f(R)$ theories. In that case there seems to be little impediment to including large density contrasts by allowing $\delta\mu^M$ to become large, while ϕ and ψ are required to stay small. This difference could indicate that while the weak-field solutions we have considered here are problematic, there may be ways of obtaining useful (approximate) spacetime geometries from the perturbed FLRW approach. This would, in fact, appear to be quite similar to the approach that is taken in [123], where the expansion of $f(R)$ is performed around a time-dependent, but spatially homogeneous and isotropic background geometry with $R = R_0(t)$. In this case small regions of spacetime can still be approximated as being close to Minkowski space, but the emergence of cosmological evolution on large scales cannot be studied in the same way, as it is, at least to some degree, being assumed from the outset. This does not in any way diminish the potential validity of such an approach, but it does appear to require knowledge about the geometry of the entire observable universe in order to model the spacetime around a single astrophysical object (it would also appear to require a re-think of the current framework for interpreting precision tests of gravity). Alternatively, it may be the case that the difference between the bottom-up constructions attempted here, and the top-down construction of perturbed FLRW, could be indicating that cosmological back-reaction is large in $f(R)$ theories. This is certainly plausible, and should probably be expected when "screening mechanisms" such as the chameleon effect come into play.

3.3 Lensing

Gravitational lensing has been a powerful tool used to determine the mass distribution of galaxies and galaxy clusters and to put constraints on scales as small as stars to large-scale structures and cosmological parameters. Given that the lensing effect is dependent on the underlying theory of gravity, investigating modifications of GR would result in deviations from the standard expression of the deflection angle and is consequently worth investigating.

The derivation of the form of the lensing angle for $f(R)$ theory has been presented in [126] and [127]. In this section we illustrate the effects of strong lensing in the case of an exact static spherically symmetric spacetime of $f(R) = R^n$ gravity as carried out in [74].

3.3.1 The bending angle

In the presence of a strong gravitational field such as a black hole, a photon experiences a deflection about the centre of symmetry. For $f(R) = R^n$ gravity, the form of the *deflection angle* in the static, spherically symmetric spacetime defined by the solution (3.43) is,

$$\alpha = \left| \int_{r_0}^{r_1} \mathcal{O}(r)\, dr \right| + \left| \int_{r_2}^{r_0} \mathcal{O}(r)\, dr \right| - \pi \ ; \tag{3.70}$$

where

$$\mathcal{O}(r) = L^{-1} \frac{J}{r^2} \left[r^{\frac{(4n^2-6n+2)}{n-2}} - J^2 \left(r^{-2} - 2\, M\, r^{\frac{(4n^2-12n+11)}{n-2}} \right) \right]^{-\frac{1}{2}} .$$

and $n = \delta + 1$. Here, M is the *effective mass* of the lensing object, J is the *impact parameter* and L is a constant. For an asymptotically flat solution, the total change in $\alpha(r)$ is twice the change from infinity to r_0.

It is important to remember that the metric (3.43) can be used only for $n < (1+\sqrt{3})/2 \approx 1.23$. Beyond this value of n the signature changes and the solution should be considered unphysical in this context. In the limit $n \to 1$ the form of the deflection angle (3.70) would be the standard form in GR [128] for a Schwarzschild spacetime.

We now analyse the behaviour of the deflection angle α by computing it against n for different distances from the source r_1 and for different values of the distance of closest approach r_0 as shown in Fig 3.1 and Fig 3.2 respectively. As a fiducial system, the distances r_0 are in units of the Schwarzschild radius.

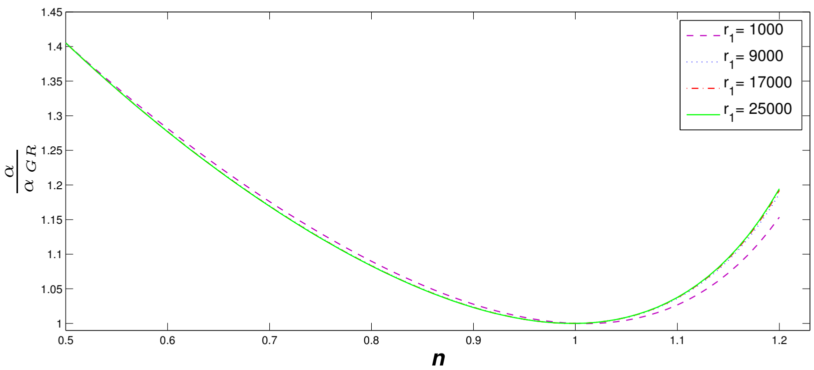

Figure 3.1: Plot of the bending angle α compared to the bending angle in general relativity against n corresponding to different values of r_1, with $r_0 = 100$ and $r_2 = 30000$.

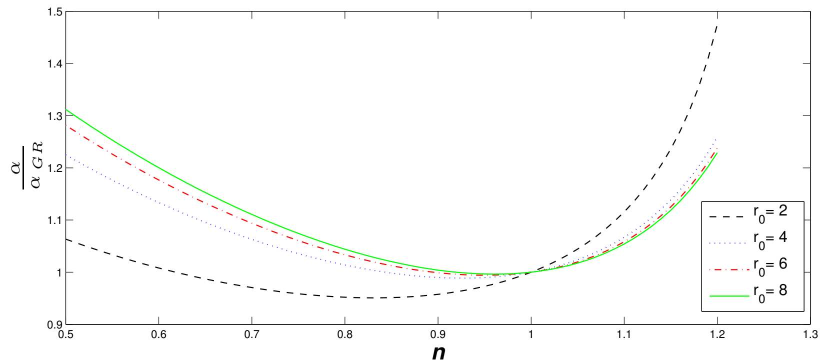

Figure 3.2: Plot of the bending angle α compared to the bending angle in general relativity against n corresponding to different values or r_0 with $r_1 = 25000$ and $r_2 = 10000$).

The divergence of the curves in both plots is indicative of the deviation from the standard GR bending angles values. In Fig 3.1, the deflection angle is dependent of the distance from the source r_1. There is increased bending when r_1 decreases for $n < 1$ and when r_1 increases for $n > 1$. In Fig 3.2, one can see that for a fixed n, the deflection angle varies for different values of distance of closest approach r_0. In particular for $n < 1$ there is more bending as the values of r_0 increase, whereas for $n > 1$ more bending occurs as r_0

decreases. Fig 3.2 also tells us that, at fixed r_0 the bending angle first decreases with n for $n < 1$ and then, for $n > 1$, starts increasing. However, our conclusions are only valid for $n < 1.23$ because of the limits on the validity of (3.43).

3.3.2 Einstein ring positions and lens mass

We consider the trajectory of a photon whose orbit is described by (3.70). Also, in this subsection we will only consider the values of $n \geq 1$. Hence, due to the limits of validity of the solution we have $1 \leq n < 1.23$. Following the approach given in Weinberg's book [128], to first order of approximation in $n-1$ (which is a small quantity as $1 \leq n < 1.23$) and M/r_0, the deflection angle can be obtained as

$$\alpha = 4\,M\left[\frac{1}{r_0} - \frac{1}{2r_1} - \frac{1}{2r_2} + (n-1)\left(\frac{(1+ln(r_0))}{r_0} - \frac{1+ln(r_1)}{2r_1} - \frac{1+ln(r_2)}{2r_2}\right)\right] \; . \tag{3.71}$$

We have kept the terms involving r_1 and r_2 as the solution is not asymptotically flat. It can be noted that even up to first order in $(n-1)$, this makes a considerable difference from the GR value. For $n = 1$, we recover the usual expression for the bending angle in GR, taking into consideration asymptotic flatness and neglecting the terms involving M/r_1 and M/r_2 (as we know in the weak deflection limit of GR, if M/r_0 is of the order of ϵ, then M/r_1 or M/r_2 is of order of ϵ^2):

$$\alpha_{GR} = \frac{4M}{r_0} \; . \tag{3.72}$$

The additional terms in (3.71) can be interpreted as the correction term to the classical lens equation and this correction depends on $r_1,\ r_0,\ r_2$ and the parameter n.

The basic geometric setup for a gravitational lens system is illustrated in Fig (3.3). The light ray emitted by the source S is deflected by the lens L and the image is seen by the observer O at S_1. β is the angular position of the source; θ the angular position of the image; and $D_L,\ D_S$ and D_{LS} are the observer-lens, observer-source and lens-source angular diameter distances, respectively.

From the Fig 3.3, the following relations hold:

$$\beta = \theta - \alpha \tag{3.73}$$

$$\alpha = \frac{D_{LS}}{D_S}\,\tilde{\alpha} \tag{3.74}$$

where α is the reduced deflection angle and is related to the actual deflection angle $\tilde{\alpha}$ through the relation (3.74), the assumption here being that the angles are small, that is,

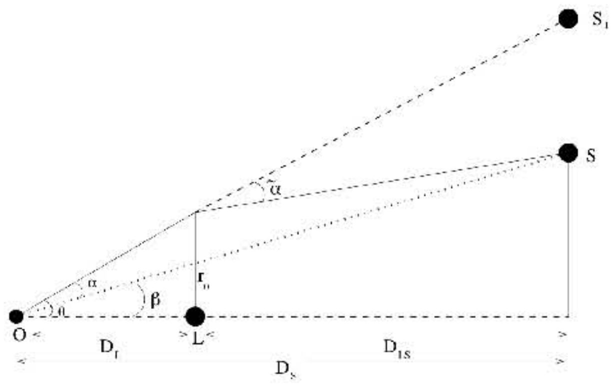

Figure 3.3: The basic geometric setup for a gravitational lens system with corresponding angles and angular diameters.

$\theta,\ \beta,\ \tilde{\alpha} << 1$. Explicitly the bending angle is

$$\beta = \theta - \frac{4\,M\,D_{LS}}{D_S}\left[\frac{1}{r_0} - \frac{1}{2r_1} - \frac{1}{2r_2} + (n-1)\left(\frac{1+ln(r_0)}{r_0} - \frac{1+ln(r_1)}{2r_1} - \frac{1+ln(r_2)}{2r_2}\right)\right]\ , \tag{3.75}$$

and in GR, (3.75) is simply

$$\beta = \theta - \frac{D_{LS}}{D_S}\frac{4M}{r_0}\ , \tag{3.76}$$

where M is the *lens mass*.

We require the Hubble expansion rate $H(z)$ in order to calculate the distances $D_L,\ D_S$ and D_{LS}, which are associated with observed redshifts[2]. We identify the overall background spacetime as homogeneous and isotropic described by the FLRW metric with curvature parameter κ, and assume the universe to be filled with a perfect fluid of pressure p^M and density μ^M (equation of state $p = \omega\mu^M$). For R^n theory the generalised Friedmann equation is given by [37]

$$H^2 + (n-1)H\,\frac{\dot{R}}{R} - \frac{(n-1)R}{6\,n} = \frac{8\,\pi\,G(2-n)}{3(3-2n)}\frac{R_0^{n-1}}{R^{n-1}}\,\mu^M\ , \tag{3.77}$$

where R_0 is the value of the Ricci tensor at the present epoch. For a flat universe, $\kappa = 0$, (3.77) has the power-law solution:

$$a(t) = t^{\frac{2n}{3(1+\omega)}}\ . \tag{3.78}$$

[2]We assume in this section that the embedding of the spacetime (3.43) in a surrounding FLRW region can exist without satisfying all the matching conditions

During the matter-domination era, the evolution of the scale factor (3.78) gives the results[3]

$$
\begin{aligned}
a(t) &= a_0 \left(\frac{t}{t_0}\right)^{\frac{2n}{3}} , \\
H_0 &= \frac{2n}{3t_0} , \\
\mu^M &= \frac{3\,H_0^2}{16\,\pi\,G} \frac{(3-2n)(3-13n+8n^2)}{n^3-2n^2} , \\
R(t) &= \frac{4(4n^2-3n)}{3t^2} .
\end{aligned}
\tag{3.79}
$$

Using (3.79) in the field equation (3.77) to solve for $H(z)$ to first order in $n-1$ yields

$$
H(z) \simeq (1+z)\, H_0\, \sqrt{2}\, \sqrt{1+z}^{\frac{3-2n}{n}} \left(1 - 2.686(n-1)\right) , \tag{3.80}
$$

Given (3.80) the angular luminosity distance is evaluated as

$$
d_A(z) = \frac{1}{(1+z)} \int_0^z \frac{d\hat{z}}{H(\hat{z})} . \tag{3.81}
$$

Fig 3.4 shows plots of dimensionless angular luminosity distance $(1/H_0)^{-1} d_A(z)$ for various values of n, where the small residual radiation effects have been neglected. From the plot we see that the value of $d_A(z)$ for R^n models increases with increasing n.

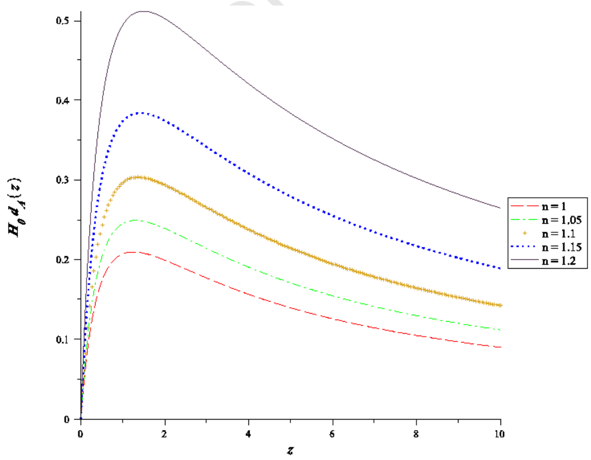

Figure 3.4: The variation of dimensionless angular luminosity distance as functions of redshift, for different values n

We now consider the special case of gravitational lensing where the source, lens and the observer are perfectly aligned so that $\beta = 0$ in (3.75). In this case, θ is the *Einstein*

[3] These solutions hold exactly from the time of matter-radiation equality up until the present

radius, which is the angular radius of an Einstein ring. Using (3.71), the lens equation is now given by

$$\theta = \frac{4\,M\,D_{LS}}{D_S}\left[\frac{1}{r_0} - \frac{1}{2r_1} - \frac{1}{2r_2} + (n-1)\left(\frac{1+ln(r_0)}{r_0} - \frac{1+ln(r_1)}{2r_1} - \frac{1+ln(r_2)}{2r_2}\right)\right] . \tag{3.82}$$

Taking $r_0 = \theta\,D_L,\ r_1 = D_{LS}$ and $r_2 = D_L$, we can now re-write the lens equation (3.82) as:

$$\begin{aligned}\theta = \frac{4\,M D_{LS}}{D_S}\Bigg[&\frac{1}{\theta\,D_L} - \frac{1}{2D_{LS}} - \frac{1}{2D_L}\\ &+ (n-1)\left(\frac{1+ln(\theta\,D_L)}{\theta\,D_L} - \frac{1+ln(D_{LS})}{2D_{LS}} - \frac{1+ln(D_L)}{2D_L}\right)\Bigg] .\end{aligned} \tag{3.83}$$

The position,

$$\theta_E = \sqrt{4M\frac{D_{LS}}{D_S\,D_L}}\ , \tag{3.84}$$

corresponding to the classic GR case (3.76) is the *Einstein angle*.

The *lens mass* M for the R^n case is given by

$$M = -\,\frac{D_L\,D_S\,\theta^2}{2\,\left[n\,(D_{LS}\,(\theta-2) + \theta\,D_L) + (n-1)\,\chi\right]}\ , \tag{3.85}$$

where

$$\chi = D_{LS}\,(\theta-2)\,ln(D_L) + \theta\,D_L\,ln(D_{LS}) - 2\,D_{LS}\,ln(\theta)\ .$$

and for GR ($n = 1$),

$$M_{GR} = -\,\frac{D_L\,D_S\,\theta^2}{2\,(D_{LS}(\theta-2) + D_L\theta)}\ . \tag{3.86}$$

As an example we consider the observed Einstein ring case [129] which is found to be an almost perfect ring. The system consists of a quasar as the background source at a redshift $z = 0.68$ which is lensed by a dwarf spheroid galaxy at a redshift of $z = 0.0375$ forming an almost perfect 360^o of radius 6".

We plot in Fig 3.41 the ratio of the mass of the lensing galaxy in (3.85) to the GR mass against n and find that the lens mass is higher than the classical GR mass. The value of the mass increases exponentially with increasing n. Thus even a small deviation from $n = 1$ would make the lens mass different from GR value.

Now the *radius of the ring* is obtained by solving for θ in (3.83) for different values of n. Solving numerically using *MAPLE*, we obtain two images of the background source

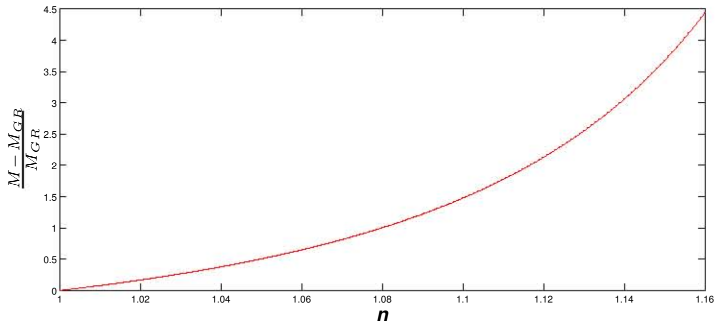

Figure 3.5: Plot of the ratio of the mass of the lensing galaxy to it's GR mass against n

as in the classical case and we recover the classical GR value for $n = 1$. In Fig 3.6 we plot the image positions of the Einstein radius against n for different source positions D_S. The image positions are sensitive to both n and D_S. We see that the value of the angular separation between the images decreases with increasing n and for a fixed value of n the image position increases in value with increasing D_S in agreement with Fig 3.1. The decrease in the value of angular separation converges at $n \simeq 1.16$, therefore no rings are expected to form for models that span $1.16 \leq n \leq 1.23$. A second ring also forms at $1.07 \leq n \leq 1.16$, and this becomes larger with increasing n. Possibilities of this occurring in the GR case are the existence of a second companion source, a star forming region or lensing by a singular isothermal sphere in two planes.

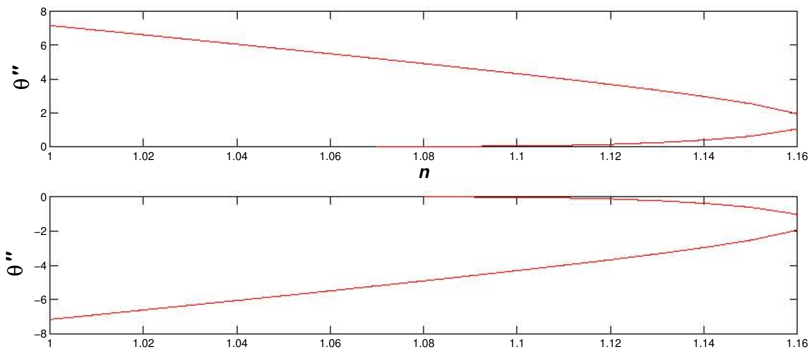

Figure 3.6: The image positions of the Einstein radius against n. Two images of the background source are obtained as in the classical GR case which is recovered at the value of $n = 1$

We can now take into account the lens induced *magnification* which is defined as the ratio of the lensed flux to the unlensed flux or as the ratio of the lensed and unlensed solid angles [44]:

$$\mathcal{M} = \left| \frac{\beta \, d\beta}{\theta \, d\theta} \right|^{-1} . \tag{3.87}$$

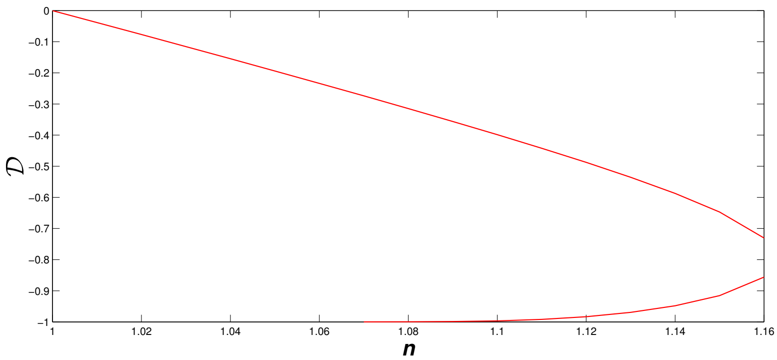

Figure 3.7: Plot of deviation of the Einstein angle θ from the classical GR value against n. The Einstein angle decreases with respect to the classical angle.

In units of the Einstein angle θ_E, that is, by setting

$$\bar{\beta} = \frac{\beta}{\theta_E} \ , \ \Theta = \frac{\theta}{\theta_E} \ ,$$

the lens equation (3.75) may be written as

$$\bar{\beta}\,\Theta = \Theta^2 - 1 - (n-1)\left(C_1 - C_2\,\Theta + ln(\Theta)\right) \tag{3.88}$$

where

$$\begin{aligned} C_1 &= 1 + ln(D_L) + ln\theta_E \ , \\ C_2 &= \frac{\theta_E}{2\,D_{LS}}\left[D_{LS} + D_L\left(1 + \frac{1}{(n-1)}\right) + D_{LS}\,ln(D_L) + D_L\,ln(D_{LS})\right] \end{aligned}$$

From (3.88) we obtain

$$\mathcal{M} = \left|1 - \frac{1}{\Theta^4} + \frac{\Upsilon}{2\,\Theta^4}\,(n-1)\right|^{-1} \tag{3.89}$$

where

$$\begin{aligned} \Upsilon = \ & \theta_E\,\phi\,\left[\Theta^2 + (n-1)\,ln(\theta_E)\,(D_{LS} + D_L)\,ln(D_L)\,(1 + \Theta^2) + 1\right] \\ & - 4\left[ln(\theta_E) + ln(D_L) + ln(\Theta)\right] - 2\,\Theta^2 - 2 \\ & + \frac{\theta_E\,(D_{LS} + D_L)\,\Theta\,\left[\Theta^2\,(n-1) + 1\right] + (n-1)\left[\Theta\,ln(\Theta) + ln(D_L)\right]}{(n-1)\,D_{LS}} \\ & + \frac{\theta_E\,D_L\,\Theta\left[\Theta^2 + (n-1)\,ln(D_{LS})\right]}{D_{LS}} \end{aligned}$$

Given the positions obtained numerically in (3.83), the magnification is found to remain constant with varying n.

3.3.3 Comparison with other models

In [126] it was found that the deflection angle is defined for $1 \leq n$ and that the value of lensing strength decreases with increasing n . The relative deviation of the image position angles with respect to the standard GR angle, in their case, takes on both negative and positive values indicating that R^n gravity may increase or decrease the Einstein angle with respect to the classical result. However, they found the value of the relative deviation to be negative over almost the full parameter space.

On computing the relative deviation of the images position angles from the standard GR case, we found that the deflection angle is defined for only $1 \leq n \leq 1.16$. This deviation as a function of n is negative over the parameter space and is always reduced for the outer ring while it increases for the second inner ring. The solution (3.43) used in the calculations is an exact spherically symmetric vacuum solution of the $f(R)$ field equations while the solution used in [126] is an approximate one. This may account for the difference in the results found here from the results in [126] in the deviation. Solar system tests provide stronger constraints on n to the extent that in [37], perihelion precession observations provide a stronger bound of $n = 1 + (2.7 \pm 4.5) \times 10^{-19}$. That being so, we see that the constraints on n weaken on cosmological scales.

Our results also show that the correction of the bending angle leads to the lensing mass being higher than that of the GR case, confirming what was found in [126]. In the case of [127], upon increasing the distance of closest approach, more bending is expected. Their results are limited to values of $n < 1$ and this agrees with the results we obtain in Fig 3.2, where for $n < 1$, as the distance of closest approach r_o increases, so does the bending.

3.3.4 Conclusion

In this section we have studied strong lensing in $f(R) = R^n$ gravity. The key features that emerged from this analysis are as follows:

1. It was shown that the bending angle is dependent of the details of the theory of gravity, (in this case the value of the parameter n), and also the geometry of the lens system (the values r_0 , r_1 , r_2), that is, the bending angle depends on the position of the observer, source and the distance of closest approach.

2. The lens mass as calculated for a small deviation from GR increases exponentially with increasing n.

3. The radius of the Einstein ring decreases with increasing n, and there exists multiple rings for certain intervals of n, which is a novel feature of fourth order gravity and cannot be accounted in GR without assuming the existence of a second companion

source, a star forming region or lensing by a singular isothermal sphere in two planes. The magnification of the ring, however remains unchanged up to small deviations from GR.

Chapter 4

The 1+3 covariant approach in $f(R)$ gravity

The 1+3 covariant approach provides a covariant description of spacetime in terms of scalars, 3-vectors and projected symmetric trace-free (PSTF) 3-tensors and the equations governing their dynamics, based on the Ricci and Bianchi identities. These quantities have a physical or direct geometrical meaning, which have a natural interpretation for comoving observers.

In this chapter, we adapt the 1+3 covariant approach based on [1, 46, 130–132] to FOG. A comprehensive review of the formalism in GR can be found in [2]. We present for the first time a complete 1+3 decomposition of the field equations in $f(R)$ gravity.

4.1 Frame choice

The covariant approach presented here is based on the introduction of a partial frame in the tangent space of each point. Once the frame has been chosen, a complete set of covariantly defined (that is, gauge invariant) exact variables, all of which vanish in the background are obtained, that set up equations describing the true spacetime. However, since the true spacetime lacks the symmetry of the background there is, in general, no unique covariant definition of the frame vectors and one is free to specify a choice of frame. In what follows, 'frame invariant' describes invariance under the choice of frame vectors.

We begin the analysis with a suitable choice of frame, that is, one corresponding to the 4-velocity u^a of an observer in spacetime. There are a number of natural choices for u^a. The *energy frame* (or Landau frame [133]) u^a_E, which is defined to be a timelike eigenvector of the energy momentum tensor. For observers following the energy frame, the energy flux vanishes. There is the *particle frame* (or Eckart frame [134]) u^a_N which is derived from the particle flux vector N_a; observers in this frame see no particle flux. There is also the *entropy frame* u^a_S, that is defined by the entropy flux vector S_a. The aim of these frame choices is to simplify the calculations by way of restructuring the equations and to better their interpretation, for example, by choosing the energy frame, the total energy is alway zero [49].

Equation (2.13) allows us to define a two effective fluid where the physical components are only the standard matter components, while the curvature fluid is a mathematical construction, present due to additional gravitational degrees of freedom. Choosing a frame corresponding to the total matter/curvature fluid would be physically unmatchable to observations as the energy conditions of the curvature fluid and effective standard matter are not necessarily satisfied [77]. This makes the choice of frames u^a_E, u^a_N and u^a_S, in general, not suitable. The most natural choice of frame is therefore the one associated with standard matter $u^a = u^a_M$ which remains thermodynamically well defined, whatever the behaviour of the effective fluid is. This choice is also physically motivated by the fact that the real observers are attached to galaxies and these galaxies follow the standard matter geodesics [97].

4.2 Kinematics

The non-intersecting timelike family of worldlines (associated with *fundamental observers* comoving with the cosmological fluid) form a congruence in spacetime $(\mathcal{M}, \mathbf{g})$ representing the average motion of matter at each point. In each case their *four-velocity* is

$$u^a = \frac{dx^a}{d\tau}, \qquad \text{with} \qquad u_a\, u^a = -1\ , \tag{4.1}$$

where τ is the proper time along the worldline of any fundamental observer. This vector field u^a provides a timelike threading for the spacetime. Given the four-velocity u^a, there are defined unique *projection tensors*

$$U^a{}_b = -u^a\, u_b\,, \tag{4.2}$$

$$h_{ab} = g_{ab} + u_a\, u_b\,, \tag{4.3}$$

where (4.2) projects parallel to u^a and (4.3) projects onto the rest space orthogonal to u^a and it follows that

$$\begin{aligned} &U^a{}_c\, U^c{}_b = -U^a{}_b\ , \quad U^a{}_b\, u^b = u^a\ , \quad U^a{}_a = 1\ , \\ &h_{ab}\, u^b = 0\ , \qquad h^a{}_c\, h^c{}_b = h^a{}_b\ , \quad h^a{}_a = 3\ . \end{aligned} \tag{4.4}$$

The effective *volume element* for the rest space of the comoving observer is given by

$$\varepsilon_{abc} = \eta_{abcd}\, u^d \;, \qquad \text{where} \qquad \varepsilon_{abc} = \varepsilon_{[abc]} \;\text{ and }\; \varepsilon_{abc}\, u^c = 0 \;. \tag{4.5}$$

Here, η_{abcd} is the four-dimensional volume element ($\eta_{abcd} = \sqrt{|\det g\,|}\,)\delta^0{}_{[a}\,\delta^1{}_b\,\delta^2{}_c\,\delta^3{}_{d]})$ thus,

$$\eta_{abcd} = 2u^{[a}\varepsilon_{b]cd} - 2\varepsilon_{ab[c}\, u_{d]} \;. \tag{4.6}$$

Since η_{abcd} is totally skew-symmetric $\eta_{abcd} = \eta_{[abcd]}$, it follows that the following contractions hold

$$\begin{aligned}
\varepsilon_{abc}\,\varepsilon^{def} &= 3!h^d{}_{[a}\,h^e{}_b\,h^f{}_{c]} \;, \\
\varepsilon_{abc}\,\varepsilon^{dec} &= 2h^d{}_{[a}\,h^e{}_{b]} \;, \\
\varepsilon_{abc}\,\varepsilon^{dbc} &= 2h^d{}_a \;, \\
\varepsilon_{abc}\,\varepsilon^{abc} &= 3! \;.
\end{aligned} \tag{4.7}$$

Moreover, two derivatives can be defined: the four-velocity u^a is used to define the *covariant time derivative* (denoted with a dot - ' $\dot{}$ ') along the observers' worldlines, where for any tensor $Z^{a..b}{}_{c..d}$

$$\dot{Z}^{a..b}{}_{c..d} = u^e\,\nabla_e Z^{a..b}{}_{c..d} \;, \tag{4.8}$$

and the spatial projection tensor h_{ab} is used to define the fully orthogonally projected *covariant spatial derivative* - 'D', such that,

$$\mathrm{D}_e Z^{a..b}{}_{c..d} = h^r{}_e\, h^p{}_c \ldots h^q{}_d\, h^a{}_f \ldots h^b{}_g\, \nabla_r Z^{f..g}{}_{p..q} \;, \tag{4.9}$$

with projection on all the free indices.

Any spacetime 4-vector v^a may be covariantly split into a scalar, V, which is the part of the vector parallel to u^a, and a 3-vector, V^a, lying orthogonal to u^a;

$$v_a = -u_a\, V + V_a \;, \qquad \text{where} \qquad V = v_b\, u^b \;\text{ and }\; V^a = h^a{}_b\, v^b \;. \tag{4.10}$$

Any projected rank-2 tensor S_{cd} can be split as

$$S_{ab} = S_{\langle ab\rangle} + \frac{1}{3} S\, h_{ab} + \varepsilon_{abc}\, S^c \;, \tag{4.11}$$

where $S = h_{cd}S^{cd}$ is the spatial trace, $S_{\langle ab\rangle}$ is the orthogonally *projected symmetric trace-free* PSTF part of the tensor defined as

$$S_{\langle ab\rangle} = \left(h^c{}_{(a}\, h^d{}_{b)} - \frac{1}{3} h_{ab}\, h^{cd}\right) S_{cd} \;, \tag{4.12}$$

and finally $S_{[ab]}$ is the skew part of the rank-2 tensor which is spatially dual to the spatial vector S^c

$$S_{[ab]} = \varepsilon_{abc}\, S^c \quad \Leftrightarrow \quad S_a = \frac{1}{2}\varepsilon_{abc}\, S^{[bc]} \ . \tag{4.13}$$

We use angle brackets to represent the PSTF tensors and also to denote orthogonal projections of covariant time derivatives along u^a:

$$\dot{V}^{\langle a \rangle} = h^a{}_b\, \dot{V}^b \ , \qquad \dot{S}_{\langle ab \rangle} = \left(h^c{}_{(a}\, h^d{}_{b)} - \frac{1}{3} h_{ab}\, h^{cd} \right) \dot{S}_{cd} \ . \tag{4.14}$$

By these definitions, for the derivatives of the projection tensors and the 3-volume element one obtains

$$\mathrm{D}_a\, U_{bc} = \mathrm{D}_a\, h_{bc} = \mathrm{D}_a\, \varepsilon_{bc} = 0 \ , \tag{4.15}$$

$$\dot{U}_{\langle ab \rangle} = \dot{h}_{\langle ab \rangle} = \dot{\varepsilon}_{\langle abc \rangle} = 0 \ , \tag{4.16}$$

$$\dot{h}_{ab} = 2\, u_{(a}\, \dot{u}_{b)} \ , \tag{4.17}$$

$$\dot{\varepsilon}_{abc} = 3\, \dot{u}^d\, \varepsilon_{d[ab}\, u_{c]} \ . \tag{4.18}$$

In analogy to vector analysis in three dimensions, we introduce the covariant spatial divergence and curl that generalises these Newtonian operators to curved spacetimes [132, 135]. The covariant spatial divergence and curl for projected vectors and fully projected rank-2 tensors are,

$$\begin{aligned} \mathsf{div}\, V &= \mathrm{D}^a V_a \ , \qquad & (\mathsf{div}\, S)_a &= \mathrm{D}^b S_{ab} \ ; \\ \mathsf{curl}\, V_a &= \varepsilon_{abc}\, \mathrm{D}^b V^c \ , \qquad & \mathsf{curl}\, S_{ab} &= \varepsilon_{cd(a}\, \mathrm{D}^c S^d{}_{b)} \ . \end{aligned} \tag{4.19}$$

For a symmetric rank-2 tensor,

$$S_{ab} = S_{(ab)} \qquad \rightarrow \qquad \mathsf{curl}\, S_{ab} = \mathsf{curl}\, S_{\langle ab \rangle} \ , \tag{4.20}$$

since curl $(kh_{ab}) = 0$ for any k. Note that unlike in the Euclidian case, div curl is *not* in general zero, for vectors or rank-2 tensors.

The covariant decomposition of the derivative of a scalar Ψ is:

$$\nabla_a \Psi = -\, u_a \dot{\Psi} + \mathrm{D}_a \Psi \ , \tag{4.21}$$

while the exact form of the covariant decomposition of the derivative of the 4-vector (4.10) is

$$\begin{aligned}\nabla_a v_b &= -V\left(-u_a\,\dot{u}_b + \frac{1}{3}h_{ab}\,\theta + \sigma_{ab} + \omega_{ab}\right) + u_b\left(\frac{1}{3}\theta\,V_a + \sigma^c{}_a\,V_c + \omega^c{}_a\,V_c\right)\\ &\quad - u_a\left(\dot{V}_{\langle b\rangle} + u_b\,\dot{u}_c\,V^c\right) + \frac{1}{3}\left(\mathsf{div}\,V\right)\,h_{ab} + \mathrm{D}_{\langle a}V_{b\rangle} + \frac{1}{2}\varepsilon_{abc}\,\mathsf{curl}\,V^c\\ &\quad - u_b\,\nabla_a V\ ,\end{aligned} \tag{4.22}$$

and that of the orthogonally projected rank-2 tensor (4.11) is

$$\begin{aligned}\nabla_c S_{ab} &= -u_c\left(\dot{S}_{\langle ab\rangle} + 2u_{(a}\,S_{b)d}\,\dot{u}^d\right) + 2u_{(a}\left(\frac{1}{3}\Theta S_{b)c} + S^d{}_{b)}\left(\sigma_{cd} - \varepsilon_{cde}\,\omega^e\right)\right)\\ &\quad + \frac{3}{5}(\mathsf{div}\,S)_{\langle a}h_{b\rangle c} - \frac{2}{3}\varepsilon_{dc(a}\,\mathsf{curl}\,S^d{}_{b)} + \mathrm{D}_{\langle a}S_{bc\rangle}\ .\end{aligned} \tag{4.23}$$

The algebraic terms $\Theta,\ \omega_{ab},\ \sigma_{ab},\ \dot{u}_a$, in (4.22) and (4.23) are kinematic quantities arising from the relative motion of comoving observers. The trace term defined as

$$\Theta = \mathrm{D}^a u_a\ , \tag{4.24}$$

is the *expansion scalar* (volume expansion) and represents the rate of expansion of the fluid. The *shear tensor*

$$\sigma_{ab} = \mathrm{D}_{\langle a}u_{b\rangle} \tag{4.25}$$

is the symmetric trace-free part of the spatial change of the four-velocity with the properties

$$\sigma_{ab} = \sigma_{(ab)}\ , \quad \sigma_{ab}\,u^b = 0\ , \quad \sigma^a{}_a = 0\ . \tag{4.26}$$

This tensor determines the distortion arising in the matter flow, leaving the volume invariant. The shear magnitude is expressed as

$$\sigma^2 = \frac{1}{2}\sigma^{ab}\,\sigma_{ab} \geq 0 \qquad \text{and} \qquad \sigma^2 = 0 \quad \Leftrightarrow \quad \sigma_{ab} = 0\ . \tag{4.27}$$

The anti-symmetric *vorticity tensor*

$$\omega_{ab} = \mathrm{D}_{[a}u_{b]}\ , \tag{4.28}$$

describes the rigid rotation of matter relative to a non-rotating frame with

$$\omega_{ab} = \omega_{[ab]}\ , \qquad \omega_{ab}\,u^b = 0\ . \tag{4.29}$$

The vorticity tensor may also be represented by the *vorticity vector* ω^a, where

$$\begin{aligned} \omega^a &= \frac{1}{2}\eta^{abcd}\, u_d\, \omega_{bc} = \frac{1}{2}\varepsilon^{abc}\, \omega_{bc} = \frac{1}{2}\,\mathsf{curl}\, u^a \qquad \Leftrightarrow \qquad \omega_{ab} = \varepsilon_{abc}\, \omega^c \; ; \\ \omega^a\, u_a &= \omega^a\, \omega_{ab} = 0 \; . \end{aligned} \tag{4.30}$$

The vorticity magnitude is given by

$$\omega^2 = \frac{1}{2}\omega^a\, \omega_a = \omega^{ab}\, \omega_{ab} \geq 0 \quad \text{and} \quad \omega = 0 \qquad \Leftrightarrow \; \omega_a = 0 \qquad \Leftrightarrow \qquad \omega_{ab} = 0. \tag{4.31}$$

Finally, the *four-acceleration* $\dot{u}_b = u^c\, \nabla_c u_b$, represents the degree to which the matter moves under forces other than gravity (a free-falling observer has vanishing acceleration in her rest-frame, that is, moves under gravity and inertia alone).

The variation of the velocity with position and time is of interest here and therefore we define the covariant derivative of the four velocity using (4.22) as

$$\nabla_a u_b = \sigma_{ab} + \omega_{ab} + \frac{1}{3}\Theta\, h_{ab} - u_a\, \dot{u}_b \; . \tag{4.32}$$

4.3 Riemann curvature

Any given vector field v^a defined on a manifold must obey the *Ricci identity*

$$2\nabla_{[a}\nabla_{b]} v_c = R^d{}_{abc}\, v_d \; , \tag{4.33}$$

where

$$R^d{}_{abc} = \Gamma^d{}_{ac,b} - \Gamma^d{}_{ab,c} + \Gamma^e{}_{ac}\, \Gamma^d{}_{eb} - \Gamma^e{}_{ab}\, \Gamma^d{}_{ec} \; , \tag{4.34}$$

with $\Gamma^a{}_{bc}$ being the *Levi-Civita connection.* The tensor $R^d{}_{abc}$ is the *Riemann curvature tensor* and represents the curvature of the spacetime manifold. This tensor possesses the following symmetry properties

$$R_{a[bcd]} = 0 \; , \tag{4.35}$$

known as the *first Bianchi identities*, and

$$R_{abcd} = R_{[ab][cd]} = R_{cdab} \; . \tag{4.36}$$

By contraction one obtains from (4.34) the *Ricci tensor* $R_{ab} = R^c{}_{acb} = R_{ba}$ and a further contraction yields the *Ricci scalar* (or *curvature scalar*) $R = R^a{}_a$.

The *second Bianchi identities* are defined as

$$\nabla_{[e}\, R_{ab]cd} = 0 \; . \tag{4.37}$$

Applying a twofold contraction to (4.37) gives the *twice-contracted Bianchi identity*

$$\nabla_a R^a{}_c + \nabla_b R^b{}_c - \nabla_c R = 0 \qquad \Longleftrightarrow \qquad \nabla^a G_{ab} = 0 \ . \tag{4.38}$$

4.4 The energy momentum tensor

The total *energy momentum tensor* (EMT) T_{ab} as defined in (2.13) can be decomposed relative to u^a by splitting it up into parts parallel and orthogonal to u^a as follows:

$$T_{ab} = \mu\, u_a\, u_b + q_a\, u_b + u_a\, q_b + p\, h_{ab} + \pi_{ab}; \tag{4.39}$$

where μ is the total effective *energy density* relative to u^a, p the total *isotropic pressure*, q_a the total *energy flux* (momentum density) relative to u^a and π_{ab} the and PSTF total *anisotropic stress*, such that[1]

$$\mu = T_{ab}\, u^a\, u^b = \tilde{\mu}^M + \mu^R \ , \tag{4.40}$$

$$p = \frac{1}{3} T_{ab}\, h^{ab} = \tilde{p}^M + p^R \ , \tag{4.41}$$

$$q_a = -\, T_{bc}\, u^c\, h^b{}_a = \tilde{q}^M_a + q^R_a \ , \tag{4.42}$$

$$\pi_{ab} = T_{cd}\, h^c{}_{\langle a}\, h^d{}_{b\rangle} = \tilde{\pi}^M_{ab} + \pi^R_{ab} \ , \tag{4.43}$$

with

$$\tilde{\mu}^M = \frac{\mu^M}{f'}, \quad \tilde{p}^M = \frac{p^M}{f'}, \quad \tilde{q}^M_a = \frac{q^M_a}{f'}, \quad \tilde{\pi}^M_{ab} = \frac{\pi^M_{ab}}{f'} \ . \tag{4.44}$$

The following properties hold for these dynamic quantities :

$$\begin{gathered} q_a\, u^a = 0 \ , \quad \pi_{ab}\, u^b = 0 \ , \quad \pi_{ab} = \pi_{(ab)} \ , \\ \pi^a{}_a = 0 \ , \quad q_a = q_{\langle a\rangle} \ , \quad \pi_{ab} = \pi_{\langle ab\rangle} \ . \end{gathered} \tag{4.45}$$

The physical behaviour of the matter present, that is, the relativistic energy, momentum and stresses associated with a matter field are represented in general by T^M_{ab}. The pressure p^M is induced by the random thermal motions, q^M_a is such that energy might be transmitted by heat conduction and it will carry a momentum (or is a heat conduction term in the instantaneous rest frame) and π^M_{ab} is due to processes such as viscosity. These quantities are related by an *equation of state* in order to capture the physics; for example, in the perfect fluid case where the total EMT is characterised by the equation

$$T_{ab} = \mu\, u_a\, u_b + p\, h_{ab} \ , \tag{4.46}$$

[1] As a reminder, we use the superscripts ^{M} and ^{R} to denote quantities relating to the standard matter fluid and curvature fluid respectively and that the unbarred dynamic quantities with none of these superscripts are derived from the total effective EMT.

the standard matter quantities p^M and μ^M are related by the equation of state $p^M = p^M(\mu^M, s)$, where s is the entropy density.

4.4.1 Energy conditions

The description of standard matter and radiation in the universe is such that at least one of the following conditions [77] is obeyed:

1. The weak energy condition (WEC): The energy momentum tensor T^M_{ab} at each $p \in \mathcal{M}$ obeys
$$T^M_{ab}\, u^a\, u^b \geq 0, \tag{4.47}$$
for any timelike vectors u^a. This means that the energy density as measured by any observer is non-negative. For a perfect fluid, WEC will hold if
$$\mu^M = T^M_{ab}\, u^a\, u^b \geq 0 \quad \text{and} \quad \mu^M + p^M \geq 0\ . \tag{4.48}$$
The expression $\mu^M + p^M \geq 0$ implies that matter will tend to move in the direction of a pressure gradient applied to it.

2. The dominant energy condition (DEC): This says that for every timelike vector u^a, then $T^M_{ab}\, u^a\, u^b \geq 0$ and that $T^M_{ab}\, u^a$ is a non-spacelike vector, that is, it is a future-directed timelike or null vector. For a perfect fluid case, this holds if $\mu^M \geq 0$ and $-\mu^M \leq p^M \leq \mu^M$. It then follows that the *isentropic speed of sound*
$$c_s^2 = (\partial p/\partial \mu)_{s=\mathrm{const}} \tag{4.49}$$
obeys,
$$0 \leq c_s^2 \leq 1 \qquad \Leftrightarrow \qquad 0 \leq (\partial p/\partial \mu)_{s=\mathrm{const}} \leq 1\ , \tag{4.50}$$
that is, the flow of T^M_{ab} as measured by the observer does not exceed the speed of light. This guarantees local stability of matter (lower bound) and causality (upper bound), respectively.

3. The strong energy condition (SEC): The EMT obeys the inequality
$$T^M_{ab}\, u^a\, u^b \geq \frac{1}{2} T^M\, u^d\, u_d, \tag{4.51}$$
for all timelike vectors u_c. This will hold for a perfect fluid if
$$\mu^M + p^M \geq 0 \quad \text{and} \quad \mu^M + 3p^M \geq 0\ , \tag{4.52}$$
and would be violated by a negative energy density or a large negative pressure. The SEC is related to the attractiveness of gravity.

From the twice contracted Bianchi identities (4.38), we know that the divergence of the left hand side of (2.13) is identically zero, making the divergence of the right hand side zero and as a result T_{ab} is conserved. This reveals that if standard matter is conserved, then the total fluid is also conserved even though the curvature fluid may in general possess off-diagonal terms [119, 136, 137], that is,

$$\nabla^b T_{ab} = 0 \tag{4.53}$$

It is worth noting here that even though the standard matter still follows the usual conservation equations $\nabla^b T^M_{ab} = 0$, the individual effective tensors are not conserved [97],

$$\nabla^b \tilde{T}^M_{ab} = -\nabla^b T^R_{ab} = -\frac{f''}{f'^2}\, T^M_{ab} \nabla^b R\ . \tag{4.54}$$

Furthermore, the fluids with T^R_{ab} and $\tilde{T}^M_{ab}$ defined above are *effective* and consequently can admit features that one would normally consider unphysical for a standard matter field. Thus, all the thermodynamical quantities associated with the curvature defined previously should be considered *effective* and not bound by matter constraints.

The curvature fluid and the effective matter do not necessarily satisfy the WEC. This relation is the key hypothesis which allows the timelike vectors u^a_E, u^a_N, u^a_S to exist and is, in general, a very reasonable assumption [29]. The violation of this condition means that the energy frame of the matter u^a_M is the natural choice of frame as standard matter maintains its thermodynamical properties, regardless.

4.4.2 Curvature energy momentum tensor

The *curvature EMT* as given in equation (2.15) is defined as

$$T^R_{ab} = \frac{1}{f'}\left[\frac{1}{2} g_{ab}\left(f - R\, f'\right) + g_{ca}\, g_{db}\, \nabla^c \nabla^d f' - g_{ab}\, \Box f'\right]\ , \tag{4.55}$$

and the derivative terms can be decomposed into time and spatial parts resulting in the curvature EMT taking the form

$$\begin{aligned} T^R_{ab} \;=\; & \frac{1}{f'}\Bigg[\frac{1}{2} g_{ab}\left(f - R\, f'\right) - \dot{f}'\left(\frac{1}{3} h_{ab}\,\theta + \sigma_{ab} + \omega_{ab}\right) + \frac{1}{3} h_{ab}\,\mathrm{D}^2 f' \\ & + \mathrm{D}_{\langle a}\mathrm{D}_{b\rangle} f' + \frac{1}{2}\varepsilon_{abc}\,\mathsf{curl}\,\mathrm{D}^c f' - u_a\left(h_{cb}\left(\mathrm{D}^c f'\right)^{\dot{}} + \dot{u}_c\, u_b\, \mathrm{D}^c f' - \dot{f}'\, \dot{u}_b\right) \\ & + u_b\left(\frac{1}{3}\theta\, \mathrm{D}_a f' + \sigma_a{}^c\, \mathrm{D}_c f' + \omega_a{}^c\, \mathrm{D}_c f' + u_a\, \ddot{f}' - \mathrm{D}_a \dot{f}'\right) \\ & - g_{ab}\left(\dot{u}_c\, \mathrm{D}^c f' - \theta\, \dot{f}'' - \ddot{f}' + \mathrm{D}^2 f'\right)\Bigg]\ . \end{aligned} \tag{4.56}$$

In this way, the *curvature thermodynamical quantities* $\mu^R = T^R_{ab}u^a u^b$, $p^R = \frac{1}{3}T^R_{ab}h^{ab}$, $\pi^R_{ab} = T^R_{cd}h^c{}_{\langle a}h^d{}_{b\rangle}$ and $q^R_a = -T^R_{bc}h^b{}_a u^c$ can be written in terms of 1+3 variables as

$$\mu^R = \frac{1}{f'}\left[\frac{1}{2}(R\,f' - f) + f'''\,\mathrm{D}^a R\,\mathrm{D}_a R + f''\,\mathrm{D}^2 R - \Theta\, f''\,\dot{R}\right]; \tag{4.57}$$

$$\begin{aligned} p^R &= \frac{1}{f'}\left[\frac{1}{2}(f - R\,f') - \frac{2}{3}f''\,\mathrm{D}^2 R - \frac{2}{3}f'''\,D^a R\,\mathrm{D}_a R + \frac{2}{3}\Theta\, f''\,\dot{R}\right. \\ &\quad \left. + f'''\,\dot{R}^2 + f''\,\ddot{R} - \dot{u}_c\, f''\,\mathrm{D}^c R\right]; \end{aligned} \tag{4.58}$$

$$\pi^R_{ab} = \frac{1}{f'}\left[f'''\,\mathrm{D}_{\langle a}R\,\mathrm{D}_{b\rangle}R + f''\,\mathrm{D}_{\langle a}\mathrm{D}_{b\rangle}R - \sigma_{ab}\, f''\,\dot{R}\right]; \tag{4.59}$$

$$q^R_a = -\frac{1}{f'}\left[f'''\,\dot{R}\,\mathrm{D}_a R + f''\,\mathrm{D}_a\dot{R} - \frac{1}{3}\Theta\, f''\,\mathrm{D}_a R - \sigma_{ac}\, f''\,\mathrm{D}^c R - \omega_{ac}\, f''\,\mathrm{D}^c R\right]. \tag{4.60}$$

Given that the field equation (2.13) can be written in its trace reverse form as

$$R_{ab} = T_{ab} - \frac{1}{2}g_{ab}\,T\ , \tag{4.61}$$

taking the the trace of equation (4.61) we find an expression for the Ricci scalar in terms of the total thermodynamical quantities

$$R \ = \ -T = -(\widetilde{T}^M + T^R) \ = \ \mu - 3p\ . \tag{4.62}$$

Using (4.62) in (2.13) and from the decomposition of the EMT (4.39), we obtain an expression for the 1+3 split of the Ricci tensor R_{ab} as

$$R_{ab} = \frac{1}{2}\,(\mu + 3p)\,u_a\,u_b + \frac{1}{2}\,(\mu - p)\,h_{ab} + 2u_{(a}\,q_{b)} + \pi_{ab}\ , \tag{4.63}$$

in terms of the total thermodynamical quantities.

Now for the effective matter fluid, the trace term $\widetilde{T}^M$ is given as

$$\widetilde{T}^M = \frac{1}{f'}\,g^{ab}\,T^M_{ab} = \frac{1}{f'}\left(3\,p^M - \mu^M\right)\ , \tag{4.64}$$

and from taking the trace of (4.56) we have

$$\begin{aligned} T^R = g^{ab}\,T^R_{ab} &= \frac{1}{f'}\left[2(f - R\,f') - 3\left(f''\,\mathrm{D}^2 R + f'''\,\mathrm{D}^a R\,\mathrm{D}_a R\right.\right. \\ &\quad \left.\left. - f'''\,\dot{R}^2 - f''\,\ddot{R} + \dot{u}_c\, f''\,\mathrm{D}^c R - f''\,\theta\,\dot{R}\right)\right]\ . \end{aligned} \tag{4.65}$$

Substituting (4.64) and (4.65) into equation (4.62) results in

$$\begin{aligned} R\,f' - 2f &+ \mu^M - 3\,p^M \\ &= -3\left(f''\,\mathrm{D}^2 R + f'''\,\mathrm{D}^a R\,\mathrm{D}_a R - f'''\,\dot{R}^2 - f''\,\ddot{R} + \dot{u}_c\,f''\,\mathrm{D}^c R - f''\,\theta\,\dot{R}\right)\ . \end{aligned} \tag{4.66}$$

This allows us to write the *curvature trace equation* as

$$R\,f' - 2f = -3\left(f''\,\mathrm{D}^2 R + f'''\,\mathrm{D}^a R\,\mathrm{D}_a R - f'''\,\dot{R}^2 - f''\,\ddot{R} + \dot{u}_c\,f''\mathrm{D}^c R - f''\,\theta\,\dot{R}\right)\ . \tag{4.67}$$

4.5 Weyl curvature

The locally free gravitational field is given by the *Weyl curvature tensor* C_{abcd} defined by the equation

$$C^{ab}{}_{cd} = R^{ab}{}_{cd} - 2g^{[a}{}_{[c}\,R^{b]}{}_{d]} + \frac{1}{3}R\,g^{[a}{}_{[c}\,g^{b]}{}_{d]}\ . \tag{4.68}$$

This is the part of the spacetime curvature which is not directly determined locally by matter. Following from the definition, the Weyl tensor has the symmetry properties (4.35) and (4.36) of the Riemann tensor with the additional property that it is trace-free on all its indices

$$C^c{}_{acb} = 0\ . \tag{4.69}$$

Thus we may think of the Ricci tensor R_{ab} as the trace of R_{abcd}, and of C_{abcd} as its trace-free part. The Weyl tensor can be split relative to u^a as

$$E_{ab} = C_{abcd}\,u^b\,u^d\ , \qquad \to E^a{}_a = 0,\quad E_{ab} = E_{(ab)},\quad E_{ab}\,u^b = 0\ , \tag{4.70}$$

$$H_{ab} = \frac{1}{2}\varepsilon_{ade}C^{de}{}_{bc}\,u^c\ , \qquad \to H^a{}_a = 0,\quad H_{ab} = H_{(ab)},\quad H_{ab}\,u^b = 0\ , \tag{4.71}$$

where we label E_{ab} and H_{ab} as the *'electric'* and *'magnetic' Weyl curvature* parts respectively, in analogy to the 1+3 split of the Maxwell field strength tensor [138]. On that account we may write C_{abcd} as

$$C_{abcd} = C^E_{abcd} + C^H_{abcd}, \tag{4.72}$$

where

$$\begin{aligned} C^E_{abcd} &= \left(4\,g_{a[p}\,g_{q]b}\,g_{c[r}\,g_{s]d} - \eta_{abpq}\,\eta_{cdrs}\right)u^p\,u^r\,E^{qs}\ , \\ C^H_{abcd} &= 2\left(\eta_{abpq}\,g_{c[r}\,g_{s]d} + g_{a[p}\,g_{q]b}\,\eta_{cdrs}\right)u^p\,u^r\,H^{qs}\ . \end{aligned}$$

The Bianchi identities (4.37) are integrability conditions relating the Ricci tensor to the Weyl tensor, enabling the action at a distance of the gravitational field (tidal forces, gravitational radiation) and influencing the motion of matter and radiation through the geodesic deviation equation for timelike and null vectors, respectively [139–141].

A fully 1+3 decomposed form of the Riemann curvature tensor R_{abcd} can now be obtained by inserting the equations (4.72), (4.63) and (4.62) into equation (4.68) giving

$$\begin{aligned}
R^{ab}{}_{cd} &= R^{ab}_{P\,cd} + R^{ab}_{I\,cd} + R^{ab}_{E\,cd} + R^{ab}_{B\,cd} \; ; \\
R^{ab}_{P\,cd} &= \frac{2}{3}\,(\mu+3p)\,u^{[a}\,u_{[c}\,h^{b]}{}_{d]} + \frac{2}{3}\,\mu\,h^{[a}{}_{[c}\,h^{b]}{}_{d]} \; , \\
R^{ab}_{I\,cd} &= -\,2\,u^{[a}\,h^{b]}{}_{[c}\,q_{d]} - 2\,u_{[c}\,h^{[a}{}_{d]}\,q^{b]} - 2\,u^{[a}\,u_{[c}\,\pi^{b]}{}_{d]} + 2\,h^{[a}{}_{[c}\,\pi^{b]}{}_{d]} \; , \\
R^{ab}_{E\,cd} &= 4\,u^{[a}\,u_{[c}\,E^{b]}{}_{d]} + 4\,h^{[a}{}_{[c}\,E^{b]}{}_{d]} \; , \\
R^{ab}_{H\,cd} &= 2\,\varepsilon^{abe}\,u_{[c}\,H_{d]e} + 2\,\varepsilon_{cde}\,u^{[a}\,H^{b]e} \; .
\end{aligned} \tag{4.73}$$

where P represents the perfect fluid part, I the imperfect fluid part and E and H are the parts due to the electric and magnetic Weyl tensor, respectively.

4.6 The field equations

We now look at the dynamical relations for an arbitrary spacetime in the 1+3 formulation of FOG. This spacetime may be completely characterised by the following set of geometrical quantities

$$\{R,\,\Theta,\,\dot{u}_a,\,\sigma_{ab},\,\omega_{ab},\,E_{ab},\,H_{ab}\} \; , \tag{4.74}$$

together with the set of thermodynamic matter variables as described in Section 4.4,

$$\{\mu^M,\,p^M,\,q^M_a,\,\pi^M_{ab}\} \; , \tag{4.75}$$

provided an equation of state which relates the thermodynamic variables is prescribed. The propagation, evolution and constraint equations for the above covariant variables can be obtained from the field equations (2.13) and its associated integrability conditions.

4.6.1 The Ricci identities

The first set of propagation equations arises from the Ricci identities (4.33) for the fundamental timelike vector field u^a, that is,

$$2\,\nabla_{[a}\nabla_{b]}u^c = R_{ab}{}^c{}_d\,u^d, \tag{4.76}$$

on substituting in from (4.32) and (4.73). The propagation equations are obtained by contracting (4.76) with u^a, separating out the orthogonally projected part into trace, skew symmetric and symmetric trace-free parts, respectively:

1. The expansion propagation equation (*generalized Raychaudhuri equation* [142])

$$\dot{\Theta} - \mathrm{D}_a \dot{u}^a = -\frac{1}{3}\,\Theta^2 + \dot{u}_a\,\dot{u}^a - \sigma_{ab}\,\sigma^{ab} + 2\,\omega_a\,\omega^a - \frac{1}{2}\,(\mu + 3p)\ , \tag{4.77}$$

demonstrates the attractive nature of the matter present [1,46,130].

2. The *vorticity propagation equation*

$$\dot{\omega}^{\langle a\rangle} - \frac{1}{2}\,\varepsilon^{abc}\mathrm{D}_b\dot{u}_c = -\frac{2}{3}\Theta\,\omega^a + \sigma^a{}_b\,\omega^b\ . \tag{4.78}$$

3. The *shear propagation equation*

$$\dot{\sigma}^{\langle ab\rangle} - \mathrm{D}^{\langle a}\dot{u}^{b\rangle} = -\frac{2}{3}\Theta\,\sigma^{ab} + \dot{u}^{\langle a}\,\dot{u}^{b\rangle} - \sigma^{\langle a}{}_c\,\sigma^{b\rangle c} - \omega^{\langle a}\,\omega^{b\rangle} - (E^{ab} - \frac{1}{2}\pi^{ab})\ , \tag{4.79}$$

shows how the gravitational field E_{ab} (the tidal force) directly induces shear, which then determines the vorticity propagation and also by (4.77), induces deceleration.

Three sets of constraint equations can be obtained by first projecting (4.76) orthogonally and then:

4. the *divergence equation for rate of shear* is obtained by contracting over indices b and c:

$$0 = (C_1)^a = \mathrm{D}_b\,\sigma^{ab} - \frac{2}{3}\mathrm{D}^a\Theta + \varepsilon^{abc}\,[\ \mathrm{D}_b\,\omega_c + 2\dot{u}_b\,\omega_c\ ] + q^a\ ; \tag{4.80}$$

5. the *divergence equation for vorticity* is obtained by multiplying with ε^{abc}:

$$0 = (C_2) = \mathrm{D}_a\omega^a - \dot{u}_a\,\omega^a\ ; \tag{4.81}$$

6. The H *constraint* is obtained by multiplying with ε^{abc} and taking the PSTF part:

$$0 = (C_3)^{ab} = H^{ab} + 2\,\dot{u}^{\langle a}\,\omega^{b\rangle} + \mathrm{D}^{\langle a}\omega^{b\rangle} - \varepsilon^{cd\langle a}\mathrm{D}_c\sigma^{b\rangle}{}_d\ ; \tag{4.82}$$

characterising the magnetic Weyl tensor as being constructed from the vorticity 'distortion' and the 'curl' of the shear.

4.6.2 The second Bianchi identities

I. From the equations (4.38), (4.39) and (4.32), we can rewrite (4.53) as

$$\dot{\mu} + \mathrm{D}_a q^a = -\,\Theta\,(\mu + p) - 2\,\dot{u}_a\,q^a - \sigma_{ab}\,\pi^{ab}\ , \tag{4.83}$$

for the component parallel to u^a and

$$\dot{q}^{\langle a\rangle} + \mathrm{D}^a p + \mathrm{D}_b\pi^{ab} = -\frac{4}{3}\Theta\,q^a - \sigma^a{}_b\,q^b - (\mu + p)\,\dot{u}^a - \dot{u}_b\,\pi^{ab} - \varepsilon^{abc}\,\omega_b\,q_c\ , \tag{4.84}$$

for the component orthogonal to u^a.

In the standard matter case, the evolution equation for $\dot{\mu}^M$ represents the *matter energy conservation equation* and determines the rate of change of relativistic energy along the fundamental world lines. The $\dot{q}^M$ equation gives the *momentum conservation equation* determining the acceleration caused by various pressure contributions. When we consider a perfect fluid case, the conservation equations for standard matter reduce to

$$\begin{aligned} \dot{\mu}^M &= -\Theta\left(\mu^M + p^M\right), && (4.85)\\ \mathrm{D}_a p^M &= -\left(\mu^M + p^M\right)\dot{u}_a, && (4.86) \end{aligned}$$

showing for (4.85), that $\left(\mu^M + p^M\right)$ is the initial mass density and also governs the conservation of energy. The relation (4.86) connects the acceleration $\dot{u}_a$ to μ^M and p^M.

II. Another set of equations arises from contracting the Bianchi identities (4.37) once, giving an additional pair of propagation equations and a further pair of constraint equations when covariantly decomposed. The propagation equations are

1. the *Gravito-electric* ($\dot{E}$) *propagation equation*:

$$\begin{aligned} \dot{E}^{\langle ab\rangle} &+ \frac{1}{2}\,\dot{\pi}^{\langle ab\rangle} - \varepsilon^{cd\langle a}\mathrm{D}_c H^{b\rangle}{}_d + \frac{1}{2}\,\mathrm{D}^{\langle a}q^{b\rangle} \\ &= -\frac{1}{2}\,(\mu+p)\,\sigma^{ab} - \Theta\left(E^{ab} + \frac{1}{6}\,\pi^{ab}\right) + 3\,\sigma^{\langle a}{}_c\left(E^{b\rangle c} - \frac{1}{6}\,\pi^{b\rangle c}\right) \\ &\quad - \dot{u}^{\langle a}\,q^{b\rangle} + \varepsilon^{cd\langle a}\left[2\,\dot{u}_c\,H^{b\rangle}{}_d + \omega_c\left(E^{b\rangle}{}_d + \frac{1}{2}\,\pi^{b\rangle}{}_d\right)\right], \qquad (4.87) \end{aligned}$$

2. and the *Gravito-magnetic* ($\dot{H}$) *propagation equation*:

$$\begin{aligned} \dot{H}^{\langle ab\rangle} + \varepsilon^{cd\langle a}\mathrm{D}_c\left(E^{b\rangle}{}_d - \frac{1}{2}\,\pi^{b\rangle}{}_d\right) &= -\Theta\,H^{ab} + 3\,\sigma^{\langle a}{}_c\,H^{b\rangle c} + \frac{3}{2}\,\omega^{\langle a}\,q^{b\rangle} \\ &\quad - \varepsilon^{cd\langle a}\left[2\,\dot{u}_c\,E^{b\rangle}{}_d - \frac{1}{2}\,\sigma^{b\rangle}{}_c\,q_d - \omega_c\,H^{b\rangle}{}_d\right], \end{aligned} \qquad (4.88)$$

respectively. These equations show how gravitational radiation arises by taking the time derivative of the equations, which gives a wave equation for E_{ab} as well as H_{ab}. The constraint equations derived from the once-contracted Bianchi identities (4.36) are the

3. *Gravito-electric* ($\mathsf{div}\, E$) *divergence equation*:

$$0 \;=\; (C_4)^a \;=\; \mathrm{D}_b\left(E^{ab}+\frac{1}{2}\,\pi^{ab}\right)-\frac{1}{3}\,\mathrm{D}^a\mu+\frac{1}{3}\,\Theta\,q^a-\frac{1}{2}\,\sigma^a{}_b\,q^b$$
$$-3\,\omega_b\,H^{ab}-\;\varepsilon^{abc}\left[\sigma_{bd}\,H^d{}_c-\frac{3}{2}\,\omega_b\,q_c\right]\;, \tag{4.89}$$

4. and the *Gravito-magnetic* ($\mathsf{div}\, H$) *divergence*:

$$0 \;=\; (C_5)^a \;=\; \mathrm{D}_b\,H^{ab}+(\mu+p)\;\omega^a+3\,\omega_b\left(E^{ab}-\frac{1}{6}\,\pi^{ab}\right)$$
$$+\;\varepsilon^{abc}\left[\frac{1}{2}\,\mathrm{D}_b q_c+\sigma_{bd}\,(E^d{}_c+\frac{1}{2}\,\pi^d{}_c)\right]\;. \tag{4.90}$$

We note here that the equations (4.80), (4.89) and (4.90) are not constraints in the real sense of the word as we have spatial and time derivatives of the curvature in the thermodynamic terms. These equations become constraints when $f(R)=R$, which is just the GR case.

4.6.3 Evolving the constraints

Propagating the constraints (4.80)–(4.82), (4.89) and (4.90) along u^a [132,143] leads to the following system of equations [2]:

$$(\dot{C}_1)^{\langle a\rangle} \;=\; -\,\Theta\,(C_1)^a-\frac{3}{2}\sigma^a{}_b\,(C_1)^b+\frac{1}{2}\varepsilon^{abc}\,\omega_b\,(C_1)_c-\frac{8}{3}\,\omega^a\,(C_2)$$
$$-\,\varepsilon^{abc}\sigma_{bd}\,(C_3)_c{}^d-3\omega_b\,(C_3)^{ab}-(C_4)^a\;; \tag{4.91}$$
$$(\dot{C}_2) \;=\; -\,\Theta\,(C_2)\;; \tag{4.92}$$
$$(\dot{C}_3)^{\langle ab\rangle} \;=\; -\,\Theta\,(C_3)^{ab}+3\sigma^{\langle a}{}_c\,(C_3)^{b\rangle c}+\varepsilon^{cd\langle a}\,\omega_c\,(C_3)^{b\rangle}{}_d$$
$$+\,\frac{1}{2}\varepsilon^{cd\langle a}\,\sigma^{b\rangle}{}_c\,(C_1)_d+\frac{3}{2}\omega^{\langle a}\,(C_1)^{b\rangle}\;; \tag{4.93}$$
$$(\dot{C}_4)^{\langle a\rangle}-\frac{1}{2}\varepsilon^{abc}\,\mathrm{D}_b(C_5)_c \;=\; -\,\frac{4}{3}\Theta\,(C_4)^a+\frac{1}{2}\sigma^a{}_b\,(C_4)^b-\frac{1}{2}\varepsilon^{abc}\,\omega_b\,(C_4)_c$$
$$-\,\frac{1}{2}\,(\mu+p)\;(C_1)^a-\frac{1}{2}\pi^a{}_b\,(C_1)^b$$
$$+\,2\varepsilon^{abc}\,E_{bd}\,(C_3)_c{}^d+\frac{3}{2}\varepsilon^{abc}\,\dot{u}_b\,(C_5)_c\;; \tag{4.94}$$
$$(\dot{C}_5)^{\langle a\rangle}+\frac{1}{2}\varepsilon^{abc}\,\mathrm{D}_b(C_4)_c \;=\; -\,\frac{4}{3}\Theta\,(C_5)^a+\frac{1}{2}\sigma^a{}_b\,(C_5)^b-\frac{1}{2}\varepsilon^{abc}\,\omega_b\,(C_5)_c$$
$$-\,\frac{1}{2}q_b\,(C_1)_c-\frac{2}{3}q^a\,(C_2)+2\varepsilon^{abc}\,H_{bd}\,(C_3)_c{}^d$$
$$-\,\frac{3}{2}\varepsilon^{abc}\,\dot{u}_b\,(C_4)_c\;. \tag{4.95}$$

[2] Derivation of these equations requires application of the commutation relations given hereafter.

If the constraints are satisfied at an initial instant (on the local 3-space) surface, it follows from (4.91) - (4.95) that the constraints vanish identically when propagated along u^a and therefore are satisfied for all time. This verifies that the constraint equations are preserved under evolution.

4.6.4 Irrotational flow

According to the Frobenius theorem, a vector field ξ^a is hypersurface orthogonal if and only if it satisfies

$$\xi_{[a}\nabla_b\,\xi_{c]} = 0 \ . \tag{4.96}$$

If u^a is hypersurface orthogonal, we have

$$\omega_{ab} = 0 \quad \Longleftrightarrow \quad 0 = u_{[a}\nabla_b u_{c]} = u_{[a}\,\mathrm{D}_b u_{c]} = u_{[a}\,\omega_{bc]} \tag{4.97}$$

therefore the timelike congruence u^a irrotational. By Frobenius theorem, it follows that the distribution of the rest spaces (3-vector spaces) is integrable. These instantaneous rest spaces, defined at each point by h_{ab}, 'fit together' to constitute 3-surfaces in spacetime orthogonal to u^a. The curvature tensor of the 3-spaces ${}^{(3)}R_{abcd}$, is defined by the three-dimensional version of the Ricci identity

$$2\mathrm{D}_{[a}\mathrm{D}_{b]}V_c = {}^{(3)}R_{abc}{}^{d}\,V_d \ , \tag{4.98}$$

for any 3-vector field V_a on the three dimensional manifold Σ. The intrinsic 3-curvature tensor is related to the Riemann curvature tensor R_{abcd} by the *Gauss equation* [77]:

$${}^{(3)}R_{abcd} = (R_{abcd})_{\perp} - K_{ac}\,K_{bd} + K_{bc}\,K_{ad} \ , \tag{4.99}$$

where $\perp$ means projection with h_{ab} on all indices and K_{ab} is the *extrinsic curvature* (second fundamental form),

$$K_{ab} = \mathrm{D}_a u_b = \frac{1}{3}\Theta\,h_{ab} + \sigma_{ab} \ . \tag{4.100}$$

The 1+3 decomposition (4.73) of the Riemann tensor yields

$$\left(R^{ab}{}_{cd}\right)_{\perp} = \frac{2}{3}\mu\,h^{[a}{}_{[c}\,h^{b]}{}_{d]} + 2\,h^{[a}{}_{[c}\,\pi^{b]}{}_{d]} + 4\,h^{[a}{}_{[c}\,E^{b]}{}_{d]} \ . \tag{4.101}$$

Using this in (4.99) and contracting reveals an expression for the 3-Ricci tensor,

$${}^{(3)}R_{ab} = \left(\frac{2}{3}\mu - \frac{2}{9}\,\Theta^2\right)h_{ab} - \frac{1}{3}\,\Theta\,\sigma_{ab} + E_{ab} + \frac{1}{2}\,\pi_{ab} + \sigma_{ac}\,\sigma^{c}{}_{b} \ . \tag{4.102}$$

The 3-Ricci tensor (4.102) can be divided into a trace-free and trace part as

$$^{(3)}R_{ab} = {}^{(3)}S_{ab} + \frac{1}{3}\,{}^{(3)}R\, h_{ab} \;, \tag{4.103}$$

where $^{(3)}S_{ab}$ denotes the trace-free part (which is essentially equivalent to E_{ab}) and R is the 3-Ricci scalar obtained by contracting (4.102),

$$^{(3)}R = 2\mu - \frac{2}{3}\Theta^2 + 2\sigma^2 \;, \tag{4.104}$$

which is the generalised Friedmann equation. The trace-free and the trace parts of $^{(3)}R_{ab}$ in defined in (4.103) are related to each other by the Bianchi identities for the 3-surfaces

$$\mathrm{D}_b\, {}^{(3)}S^b{}_a = \frac{1}{2}\mathrm{D}_a\, {}^{(3)}R \;, \tag{4.105}$$

which is equivalent to the (4.89) because of (4.102).

Moreover, we mention that the relation between the extrinsic curvature and the 3-Ricci tensor is given by the *Codacci-Mainardi equation*,

$$\mathrm{D}_a K^a{}_b - \mathrm{D}_b K^a{}_a = R_{cd}\, u^d\, h^c{}_b \;. \tag{4.106}$$

which is equivalent to the constraint (4.80) when the vorticity vanishes.

4.7 Commutation relations

In general the two derivatives - ‘ $\dot{}$ ’ and -‘D’ do not commute and therefore give rise to various commutator relations which play an integral part in all partial frame formalisms. This is a manifestation of spacetime curvature which is derived from the Ricci identities for spacetime scalars Z, 3-vectors V^a and rank-2 tensors S^{ab}, respectively [144]:

$$\nabla_{[a}\nabla_{b]}f = 0, \tag{4.107}$$

$$2\nabla_{[a}\nabla_{b]}V_c = R_{abcd}\, V^d \tag{4.108}$$

$$2\nabla_{[a}\nabla_{b]}Z_{cd} = -\,R_{abec}\, S^e{}_d - R_{abed}\, S^e{}_c. \tag{4.109}$$

The 3-space commutator relations orthogonal to the congruence u^a, follow by successively writing out the 3-commutators explicitly and then using the Ricci identities (4.107)–(4.109), the splitting (4.32) of $\nabla_a u_b$ and the generalised Gauss equation (4.99).

4.7.1 3-scalar derivatives

For scalar functions Z one obtains:

$$\mathrm{D}_{[a}\mathrm{D}_{b]}Z = \varepsilon_{abc}\omega^c\,\dot{Z} \quad \Longleftrightarrow \quad \varepsilon^{abc}\mathrm{D}_b\mathrm{D}_c Z = 2\,\omega^a\,\dot{Z}\ , \tag{4.110}$$

$$\mathrm{D}_a\dot{Z} - (\mathrm{D}_a Z)^{\cdot}_{\perp} = -\,\dot{u}_a\,\dot{Z} + \left(\frac{1}{3}\Theta\,h_{ab} + \sigma_{ab} + \varepsilon_{abc}\omega^c\right)\,\mathrm{D}^b Z\ . \tag{4.111}$$

4.7.2 3-vector derivatives

For the 3-vectors V^a:

$$\begin{aligned}
\mathrm{D}_{[a}\mathrm{D}_{b]}V_c =& \Bigg[\left(E_{c[a} + \frac{1}{2}\pi_{c[a}\right) - \frac{1}{3}\Theta\,\sigma_{c[a} + \frac{1}{3}\Theta\,\omega^d\,\varepsilon_{dc[a} + \omega_c\,\omega_{[a} \\
&+ \frac{1}{3}\left(\mu - \frac{1}{3}\theta^2 - 3\omega_d\,\omega^d\right)\,h_{c[a}\Bigg]\,V_{b]} + \Bigg[h_{c[a}\left(E_{b]d} + \frac{1}{2}\pi_{b]d}\right) \\
&- \frac{1}{3}\Theta\,h_{c[a}\,\sigma_{b]d} - \sigma_{c[a}\,\sigma_{b]d} - \frac{1}{3}\Theta\,h_{c[a}\,\varepsilon_{b]de}\omega^e - \sigma_{c[a}\,\varepsilon_{b]de}\omega^e \\
&+ \sigma_{d[a}\,\varepsilon_{b]ce}\omega^e + h_{c[a}\,\omega_{b]}\,\omega_d\Bigg]\,V^d + \varepsilon_{abd}\omega^d\,\dot{V}_{\langle c\rangle}\ ,
\end{aligned} \tag{4.112}$$

$$\begin{aligned}
\mathrm{D}_a\dot{V}_b - (\mathrm{D}_a V_b)^{\cdot}_{\perp} =& -\dot{u}_a\,\dot{V}_{\langle b\rangle} + \left(\frac{1}{3}\Theta\,h_{ac} + \sigma_{ac} + \varepsilon_{acd}\omega^d\right)(\mathrm{D}^c V_b + V^c\,\dot{u}_b) \\
&- H_a{}^d\,\varepsilon_{dbc}V^c - \frac{1}{2}h_{ab}\,q_c\,V^c + \frac{1}{2}V_a\,q_b\ .
\end{aligned} \tag{4.113}$$

4.7.3 3-tensor derivatives

For the second-rank 3-tensors S_{ab}, the following holds:

$$\begin{aligned}
\mathrm{D}_{[a}\mathrm{D}_{b]}S^{cd} =& 2\Bigg[\left(E^c{}_{[a} + \frac{1}{2}\pi^c{}_{[a}\right) - \frac{1}{3}\Theta\,\sigma^{(c}{}_{[a} + \frac{1}{3}\Theta\omega^e\,\varepsilon_{e[a}{}^c + \omega^{(c}\,\omega_{[a} \\
&+ \frac{1}{3}\left(\mu - \frac{1}{3}\theta^2 - 3\omega_e\,\omega^e\right)\,h^c{}_{[a}\Bigg]\,S^{d)}{}_{b]} + 2\Bigg[h^{(c}{}_{[a}\left(E_{b]e} + \frac{1}{2}\pi_{b]e}\right) \\
&- \frac{1}{3}\Theta\,h^{(c}{}_{[a}\,\sigma_{b]e} - \sigma^{(c}{}_{[a}\,\sigma_{b]e} - \frac{1}{3}\Theta\,h^{(c}{}_{[a}\,\varepsilon_{b]ef}\omega^f - \sigma^{(c}{}_{[a}\,\varepsilon_{b]ef}\omega^f \\
&- \omega^f\,\varepsilon_{f[a}{}^{(c}\,\sigma_{b]e} + h^c{}_{[a}\,\omega_{b]}\,\omega_e\Bigg]\,S^{d)e} + \varepsilon_{abe}\omega^e\,\dot{S}^{\langle cd\rangle}\ ,
\end{aligned} \tag{4.114}$$

$$\begin{aligned}
\mathrm{D}_a\dot{S}_{bc} - (\mathrm{D}_a S_{bc})^{\cdot}_{\perp} =& \left(\frac{1}{3}\Theta\,h_{ad} + \sigma_{ad} + \omega_{ad}\right)\left(\dot{u}_b\,S^d{}_c + \dot{u}_c\,S^d{}_b + \mathrm{D}^d S_{bc}\right) - \dot{u}_a\left(\dot{S}_{bc}\right)_{\perp} \\
&+ \left[h_{a[e}\,q_{b]} - \varepsilon_{ebd}H^d{}_a\right]\,S^e{}_c + \left[h_{a[e}\,q_{c]} - \varepsilon_{ecd}H^d{}_a\right]\,S^e{}_b\ .
\end{aligned} \tag{4.115}$$

Chapter 5

Shear-free perturbations of FLRW Universes

We showed in Chapter 4 how the differential properties of timelike geodesics are described by the kinematic quantities, expansion Θ, shear (or distortion) σ_{ab}, rotation ω_c, and acceleration $\dot{u}_a$. Of particular interest is the role that shear plays in the relationship between Newtonian and relativistic cosmologies. For example, it has been known for some time that quasi-Newtonian descriptions of cosmology, the so-called *Silent models*, may be constructed for observers moving along geodesics which are both shear-free and irrotational [145]. The intricate relationship between the kinematic quantities in Newtonian and relativistic fluid flows in GR is most strikingly seen in a remarkable result obtained by Ellis in 1967 [60]. In this paper it was found that,

> *If the four velocity vector field of a barotropic perfect fluid with vanishing pressure is shear-free, then either the expansion or the rotation of the fluid vanishes.*

This is a purely local result to which no corresponding Newtonian equivalent appears to hold, as counter-examples can be explicitly constructed. Given that this shear-free theorem and its extensions appear to hold for arbitrarily weak fields and for fluids of arbitrarily low density, one needs to understand why the Newtonian approximation fails. It is expected that since Newtonian gravity is a limiting form of GR, the properties of Newtonian gravity should follow from those of GR. Considerable work has been put into the nature of shear-free congruences [61, 62]. It is of considerable interest to ask whether the result holds in situations where the hydrodynamic and gravitational equations have been linearised about a Friedmann-Lemaître-Robertson-Walker (FLRW) background [63] and also whether it extends to the more general setting of FOG [64].

In this chapter, we illustrate the features of the 1+3 covariant approach by applying it to shear-free perturbations of FLRW universes for both GR [63] and FOG cases [64].

5.1 Gauge invariance

Lifshitz, in 1946 [146] pioneered the work on classical relativistic theory of cosmological perturbations, a study that has since been plagued by gauge issues. In the standard approach to investigating perturbations, any tensorial quantity $\mathcal{Q}$ can be split into a background part $\mathcal{Q}_0$ and a small perturbation $\delta\mathcal{Q}$.

$$\mathcal{Q} = \mathcal{Q}_0 + \delta\mathcal{Q} \tag{5.1}$$

To define the perturbations a gauge choice has to be made. This is the choice of specification of the mapping Φ between the observable (true) universe defined by the manifold $\mathcal{M}$ and a fiducial (background) manifold $\bar{\mathcal{M}}$. The existence of arbitrary numbers of mappings corresponds to the gauge freedom of theory and herein lies the problem of choosing the best way to carry out the mapping or correspondence, also known as the "fitting problem" in cosmology [52]. In terms of coordinate choice, for a given coordinate system in $\mathcal{M}$, there is a large choice of possible coordinate systems in $\bar{\mathcal{M}}$. The perturbed quantities are not invariant under a gauge transformation and are required to obey the transformation, whereas the background quantities remain unchanged. If a quantity is invariant under the choice of mapping, then it is gauge invariant .

An alternative definition of gauge invariance is described by the Stewart-Walker lemma [90]: the perturbation $\delta\mathcal{Q}$ to the geometrical background quantity $\mathcal{Q}_0$ on $\bar{\mathcal{M}}$ is gauge invariant if and only if $\mathcal{Q}_0$ either

i. vanishes,

ii. is a constant scalar,

iii. is a constant linear combination of products of Kronecker deltas with constant coefficients.

The definition of gauge invariance we use here is from the first two options. In this case the mapped quantity will be constant regardless of choice of mapping Φ, which defines the same perturbation $\delta\mathcal{Q}$.

Following previous work by Gelarch and Sengupta in 1978 [147], Bardeen in 1980 [121], developed a fully gauge invariant theory of cosmological linear perturbations. However, since most of the gauge invariant variables in this seminal paper are defined with respect to a particular coordinate system, they tend to have an obscure physical and geometrical meaning unless a particular hypersurface condition is specified [148, 149].

In what follows, by 'gauge invariant' we mean the invariance of the equations under the mapping between the true and background spacetimes.

5.2 Linearised field equations about FLRW background

To perturb the FLRW spacetime, we use the standard 1+3 covariant perturbation theory in [47–49, 52, 55, 98, 150], where the Hubble scale sets the characteristic scale for the perturbations. Furthermore, we consider the case of shear-free perturbations and hence the shear tensor (σ_{ab}) vanishes identically. The remaining quantities that vanish in the background spacetime

$$\{\omega_{ab},\, \dot{u}_a,\, E_{ab},\, H_{ab},\, q^M_a,\, \pi^M_{ab}\}\ ,$$

along with their derivatives and the spatial derivatives of $\{\Theta,\, p^M,\, \mu^M,\, R\}$ are considered to be first order and are automatically gauge-invariant by virtue of the Stewart and Walker lemma. In the linearisation procedure, we neglect all products of first order quantities in (4.76) – (4.90) and since we consider shear-free perturbations, the shear tensor vanishes identically. The standard matter is considered to be a perfect fluid in the perturbed spacetime and as a result q^M_a and π^M_{ab} are zero.

The effective thermodynamical quantities for the curvature fluid are now

$$\mu^R = \frac{1}{f'}\left[\frac{1}{2}(R\,f' - f) + f''\,\mathrm{D}^2 R - \Theta\, f''\,\dot{R}\right]; \tag{5.2}$$

$$p^R = \frac{1}{f'}\left[\frac{1}{2}(f - R\,f') + f''\,\ddot{R} + f'''\,\dot{R}^2 + \frac{2}{3}\,f''\left(\Theta\,\dot{R} - \mathrm{D}^2 R\right)\right]\ , \tag{5.3}$$

$$\pi^R_{ab} = \frac{f''}{f'}\,\mathrm{D}_{\langle a}\mathrm{D}_{b\rangle} R\ , \tag{5.4}$$

$$q^R_a = -\frac{1}{f'}\left(f'''\,\dot{R}\,\mathrm{D}_a R + f''\,\mathrm{D}_a\dot{R} - \frac{1}{3}\Theta\, f''\,\mathrm{D}_a R\right)\ , \tag{5.5}$$

With these conditions, the linearised field equations are then as follows:

Propagation equations

$$\dot{\Theta} - \mathsf{div}\,\dot{u} = -\frac{1}{3}\Theta^2 - \frac{1}{2}(\mu + 3p)\ , \tag{5.6}$$

$$\dot{\omega}_{\langle a\rangle} - \mathsf{curl}\,\dot{u}_a = -\,\frac{2}{3}\Theta\,\omega_a\ , \tag{5.7}$$

$$\dot{H}_{\langle ab\rangle} + \mathsf{curl}\,E_{ab} - \frac{1}{2}\mathsf{curl}\,\pi^R_{ab} = -\,\Theta\,H^{ab}\ , \tag{5.8}$$

$$\dot{E}_{\langle ab\rangle} + \frac{1}{2}\dot{\pi}^R_{ab} - \mathsf{curl}\,H_{ab} + \frac{1}{2}\mathrm{D}_{\langle a}\,q^R_{b\rangle} = -\Theta\,E_{ab} - \frac{1}{6}\Theta\,\pi^R_{ab}\ , \tag{5.9}$$

$$\dot{\mu}^{M} = -\,\Theta\,(\mu^{M} + p^{M})\ , \tag{5.10}$$

$$\dot{\mu} + \mathsf{div}\, q^{R} = -\,\Theta\,(\mu + p)\ , \tag{5.11}$$

$$\dot{q}^{R}_{\langle a\rangle} + \mathrm{D}^{a} p + \mathrm{D}^{b}\pi^{R}_{ab} = -\,\frac{4}{3}\Theta\, q^{R}_{a} - (\mu + p)\,\dot{u}_{a}\ , \tag{5.12}$$

Constraint equations

$$0 = (C_0)_{ab} = E_{ab} - \mathrm{D}_{\langle a}\dot{u}_{b\rangle} - \frac{1}{2}\pi^{R}_{ab}\ , \tag{5.13}$$

$$0 = (C_1)_{a} = \mathrm{D}_{a}\Theta - \frac{3}{2}\varepsilon_{abc}\,\mathrm{D}^{b}\,\omega^{c} + q^{R}_{a}\ , \tag{5.14}$$

$$0 = (C_2) = \mathsf{div}\,\omega\ , \tag{5.15}$$

$$0 = (C_3)_{ab} = H_{ab} + \mathrm{D}_{\langle a}\,\omega_{b\rangle}\ , \tag{5.16}$$

$$0 = (C_4)_{a} = \mathrm{D}_{a}\,p^{M} + (\mu^{M} + p^{M})\,\dot{u}_{a}\ , \tag{5.17}$$

$$0 = (C_5)_{a} = \mathrm{D}^{b}\left(E_{ab} + \frac{1}{2}\pi^{R}_{ab}\right) - \frac{1}{3}\mathrm{D}_{a}\,\mu + \frac{1}{3}\Theta\, q^{R}_{a}\ , \tag{5.18}$$

$$0 = (C_6)_{a} = \mathrm{D}^{b}H_{ab} + (\mu + p)\,\omega_{a} + \frac{1}{2}\mathsf{curl}\, q^{R}_{a}\ . \tag{5.19}$$

The linearised commutation relations for shear-free congruences are now:
For any scalar 'V'

$$\begin{aligned}
[\mathrm{D}_a\mathrm{D}_b - \mathrm{D}_b\mathrm{D}_a]\,V &= 2\varepsilon_{abc}\omega^{c}\,\dot{V}\ ,\\
\varepsilon^{abc}\mathrm{D}_b\mathrm{D}_c V &= 2\omega^{a}\,\dot{V}\ .
\end{aligned} \tag{5.20}$$

If the gradient of the scalar is of the first order, we then have

$$\left[\mathrm{D}^{a}\mathrm{D}_{b}\mathrm{D}_{a} - \mathrm{D}_{b}\mathrm{D}^{2}\right]V = \frac{2}{3}\left(\mu - \frac{1}{3}\Theta^{2}\right)\mathrm{D}_{b}V\ , \tag{5.21}$$

$$\left[\mathrm{D}^{2}\mathrm{D}_{b} - \mathrm{D}_{b}\mathrm{D}^{2}\right]V = \frac{2}{3}\left(\mu - \frac{1}{3}\Theta^{2}\right)\mathrm{D}_{b}V + 2\varepsilon_{dbc}D^{d}(\omega^{c}\dot{V})\ , \tag{5.22}$$

Also for any first order 3-vector V^a, we have

$$(\mathrm{D}^a \mathrm{D}_b - \mathrm{D}_b \mathrm{D}^a)\, V_a = \frac{2}{3}\left(\mu - \frac{1}{3}\Theta^2\right) h^a{}_{[a} V_{b]} \;, \tag{5.23}$$

$$h^a{}_c\, h^d{}_b\, (\mathrm{D}_d V^c)^{\cdot} = \mathrm{D}_b \dot{V}^{\langle a\rangle} - \frac{1}{3}\Theta\, \mathrm{D}_b V^a \;, \tag{5.24}$$

$$h^a{}_c\, (\mathrm{D}^2 V^c)^{\cdot} = \mathrm{D}_b (\mathrm{D}^{\langle b} V^{a\rangle})^{\cdot} - \frac{1}{3}\Theta\, \mathrm{D}^2 V^a \;. \tag{5.25}$$

The spatial curvature (4.104) is now

$$^{(3)}R = 2\left[\mu - (1/3)\,\Theta^2\right] \;. \tag{5.26}$$

up to linear order. Using the field equations and identities of this section we will now investigate the compatibility of the new constraints with the existing ones in terms of the consistency up to the linear order of their spatial and temporal propagation for both GR [63] and FOG cases [64].

5.3 Consistency of the new constraints: The GR case

We now take $f' = 1$ and $f'' = 0$ in (5.2)–(5.19) in order to recover the field equations in GR. We note that the constraints $(C_1)^a$, (C_2), $(C_3)^{ab}$, $(C_5)^a$ and $(C_6)^a$ are the constraints of the Einstein field equations for general matter motion specialised to the shear-free case and are known to be consistently *time propagated* along u^a locally. However the conditions $\sigma_{ab} = 0$ and $q^M_a = 0$ give the two new constraints $(C_0)^{ab}$ and $(C_4)^a$ respectively. Furthermore, we assume the matter to have a barotropic equation of state $p^M = p^M(\mu^M)$ satisfying the weak and dominant energy conditions. We exclude the vacuum case and therefore the energy conditions (4.48) and (4.50) will be

$$\mu^M > 0 \;\; ; \;\; \mu^M + p^M > 0 \;\; ; \;\; \mu^M \geq |p^M| \tag{5.27}$$

for both the background spacetime and the perturbed solution (the Minkowski and De Sitter backgrounds will not occur) and the speed of sound is (4.49).

The conditions of shear-free perturbations and the matter being a perfect fluid in the perturbed spacetime give rise to two new constraints $(C_0)^{ab}$ and $(C_4)^a$ respectively. To check their compatibility with the linearised existing constraints of Einstein field equations (henceforth all the equations are up to the linear order), we plug $(C_0)_{bd}$ in $(C_5)_b$ to get

$$\mathrm{D}^d\, \mathrm{D}_{\langle b} \dot{u}_{d\rangle} - \frac{1}{3}\mathrm{D}_b\, \mu^M = 0 \;. \tag{5.28}$$

Now from the constraint $(C_4)_b$ we have

$$\dot{u}_b = -\frac{c_s^2}{\mu^M + p^M}\,\mathrm{D}_b\,\mu^M \tag{5.29}$$

Using equation (5.29) in (5.28) we get the constraint

$$(C_7)_b := \frac{c_s^2}{\mu^M + p^M}\mathrm{D}^d\,\mathrm{D}_{\langle b}\mathrm{D}_{d\rangle}\mu^M + \frac{1}{3}D_b\,\mu^M = 0\ . \tag{5.30}$$

For the new constraints $(C_0)^{ab}$ and $(C_4)^a$ to be compatible with the existing ones, the constraint $(C_7)_b$ must be satisfied.

To check the spatial consistency of $(C_7)_b$ on any initial hypersurface we take the curl of (5.30) to get

$$\frac{c_s^2}{\mu^M + p^M}\,\varepsilon^{acb}\mathrm{D}_c\,\mathrm{D}^d\mathrm{D}_{\langle b}\mathrm{D}_{d\rangle}\mu^M + \frac{1}{3}\,\varepsilon^{acb}\mathrm{D}_c\mathrm{D}_b\,\mu^M = 0\ , \tag{5.31}$$

which using (5.20) gives

$$\frac{c_s^2}{\mu^M + p^M}\,\varepsilon^{acb}\mathrm{D}_c\,\mathrm{D}^d\mathrm{D}_{\langle b}\mathrm{D}_{d\rangle}\mu^M + \frac{2}{3}\omega^a\,\dot{\mu}^M = 0\ . \tag{5.32}$$

Breaking the PSTF part according to equation (4.12) and using the commutators (5.21), (5.22) we have

$$\begin{aligned}\frac{c_s^2}{\mu^M + p^M}\,\varepsilon^{acb}\Bigg[\frac{2}{3}\mathrm{D}_c\,\mathrm{D}_b\mathrm{D}^2\mu^M + \frac{2}{3}\left(\mu^M - \frac{1}{3}\Theta^2\right)\mathrm{D}_c\mathrm{D}_b\,\mu^M \\ + \mu^{\dot{M}}\,\varepsilon_{dbk}\mathrm{D}_c\mathrm{D}^d\omega^k\Bigg] + \frac{2}{3}\omega^a\,\dot{\mu}^M = 0\ .\end{aligned} \tag{5.33}$$

Again using (5.20) and (4.7) in the above equation we get

$$\frac{c_s^2}{\mu^M + p^M}\left[\frac{4}{3}\left(\mu^M - \frac{1}{3}\Theta^2\right)\omega^a\,\mu^{\dot{M}} - \mu^{\dot{M}}\mathrm{D}_k\mathrm{D}^a\omega^k + \dot{\mu}^M\,\mathrm{D}^2\omega^a\right] + \frac{2}{3}\omega^a\,\dot{\mu}^M = 0\ . \tag{5.34}$$

Now from the relation (5.22) and using (5.15) we know

$$\mathrm{D}_k\mathrm{D}^a\omega^k = \frac{2}{3}\left(\mu^M - \frac{1}{3}\Theta^2\right)\omega^a\ , \tag{5.35}$$

Plugging (5.35) and (5.10) in (5.34) and simplifying we finally get

$$(C_8)^a := \Theta\left[\frac{2}{3}\omega^a\,Y + c_s^2\,\mathrm{D}^2\omega^a\right] = 0\ , \tag{5.36}$$

where
$$Y = \mu^M + p^M + c_s^2\left(\mu^M - \frac{1}{3}\Theta^2\right) . \tag{5.37}$$

From $(C_8)^a$ we can immediately see that for matter with constant pressure (p^M = constant $\to$ $c_s^2 = 0$), shear-free perturbations are consistent if and only if $\Theta\,\omega^a = 0$ (as according to the second condition of (5.27), $\mu^M + p^M > 0$). That is, if the geodesics of the matter congruence in the perturbed spacetime are shear-free then they should be either expansion-free or vorticity-free (or both). This shows that *the results of [60] and [61] for pressure-free matter are true for the linearised theory.* However for a general equation of state, all we can say from the equation (5.36) is, either the matter congruence is expansion free ($\Theta = 0$), or the vorticity vector must satisfy

$$(C_9)^a := \frac{2}{3}\omega^a\, Y + c_s^2\, \mathrm{D}^2\omega^a = 0\ , \tag{5.38}$$

for the new constraints to be spatially consistent on any initial hypersurface.

Now let us check the temporal consistency of the constraint (5.38). Propagating it along u^a we get
$$\left(c_s^2\,\mathrm{D}^2\omega^a\right)^{\cdot} + \frac{2}{3}\left(\omega^a\, Y\right)^{\cdot} = 0\ . \tag{5.39}$$

We can easily see that
$$\dot{c}_s^2 = -\Theta\left(\mu^M + p^M\right)\frac{d^2p^M}{d(\mu^M)^2}\quad . \tag{5.40}$$

Now from (5.25) we have

$$c_s^2\left(\mathrm{D}^2\omega^a\right)^{\cdot} = c_s^2\left[\mathrm{D}_b(\mathrm{D}^{\langle b}\omega^{a\rangle})^{\cdot} - \frac{1}{3}\Theta\,\mathrm{D}^2\omega^a\right]\ . \tag{5.41}$$

We know from the constraint (5.15) that

$$\mathrm{D}_b\,(\mathrm{D}^{\langle b}\omega^{a\rangle})^{\cdot} = \frac{1}{2}\,\mathrm{D}_b\left[(\mathrm{D}^b\omega^a)^{\cdot} + (\mathrm{D}^a\omega^b)^{\cdot}\right]\ . \tag{5.42}$$

Using (5.24) the equation (5.42) becomes

$$\mathrm{D}_b\,(\mathrm{D}^{\langle b}\omega^{a\rangle})^{\cdot} = \frac{1}{2}\mathrm{D}_b\left[\mathrm{D}^b\dot{\omega}^{\langle a\rangle} - \frac{1}{3}\Theta\,\mathrm{D}^b\omega^a + \mathrm{D}^a\dot{\omega}^{\langle b\rangle} - \frac{1}{3}\Theta\,\mathrm{D}^a\omega^b\right]\ . \tag{5.43}$$

Simplifying the above equation using (5.7), (5.17) and (5.20), we get

$$\mathrm{D}_b\,(\mathrm{D}^{\langle b}\omega^{a\rangle})^{\cdot} = -\frac{1}{2}\Theta\left(1 - c_s^2\right)\left(\mathrm{D}^2\omega^a + \mathrm{D}_b\mathrm{D}^a\omega^b\right)\ . \tag{5.44}$$

Putting equation (5.44) in (5.41), we have

$$c_s^2\,(\mathrm{D}^2\omega^a)^{\cdot} = -\Theta\,\alpha\,c_s^2\,\mathrm{D}^2\omega^a - \Theta\,\beta\,\mathrm{D}_b\mathrm{D}^a\omega^b\ , \tag{5.45}$$

where

$$\alpha = -\frac{c_s^2}{2} + \frac{5}{6}\ \ ;\ \ \beta = \frac{c_s^2}{2}\left(1 - c_s^2\right)\ . \tag{5.46}$$

Using (5.38) and (5.35), (5.45) becomes

$$c_s^2\left(\mathrm{D}^2\omega^a\right)^{\cdot} = \frac{2}{3}\omega^a\,\Theta\left[\alpha\,Y - \beta\left(\mu^M - \frac{1}{3}\Theta^2\right)\right]\ . \tag{5.47}$$

Combining (5.40) and (5.47) and using (5.38) we get

$$\left(c_s^2\,\mathrm{D}^2\omega^a\right)^{\cdot} = \frac{2}{3}\omega^a\,\Theta\left[\frac{Y}{c_s^2}\left(\mu^M + p^M\right)\frac{d^2p^M}{d(\mu^M)^2} + \alpha\,Y - \beta\left(\mu^M - \frac{1}{3}\Theta^2\right)\right]\ . \tag{5.48}$$

Also from (5.6), (5.10) and (5.40) we have

$$\dot{Y} = -\Theta\left[\left(\mu^M + p^M\right)\left(\mu^M - \frac{1}{3}\Theta^2\right)\frac{d^2p^M}{d(\mu^M)^2} + Z\right]\ , \tag{5.49}$$

where

$$Z = \left(\mu^M + p^M\right)\left(1 + c_s^2\right) + \frac{2}{3}\,c_s^2\left(\mu^M - \frac{1}{3}\Theta^2\right)\ . \tag{5.50}$$

Now using (5.7), (5.17), (5.20) and (5.49) we get

$$\frac{2}{3}(\omega^a\,Y)^{\cdot} = -\frac{2}{3}\omega^a\,\Theta\left[\left(-c_s^2 + \frac{2}{3}\right)Y + (\mu^M + p^M)\left(\mu^M - \frac{1}{3}\Theta^2\right)\frac{d^2p^M}{d(\mu^M)^2} + Z\right] \tag{5.51}$$

Finally using (5.48) and (5.51) in (5.39) and simplifying, we get

$$\begin{aligned}\frac{2}{3}\,\omega^a\,\Theta\left(\mu^M + p^M\right)\Bigg[&\left(\mu^M + p^M\right)\frac{d^2p^M}{d(\mu^M)^2}\\ &- c_s^2\left(\frac{5}{6} + \frac{c_s^2}{2}\right) - \frac{{}^{(3)}R}{2\left(\mu^M + p^M\right)}\,c_s^4\left(1 - c_s^2\right)\Bigg] = 0\ .\end{aligned} \tag{5.52}$$

where ${}^{(3)}R = 2\left[\mu^M - (1/3)\,\Theta^2\right]$. In FLRW spacetimes it can be written in terms of the scale factor '$a(t)$' as,

$${}^{(3)}R = \frac{k}{a(t)^2} = k\,\exp\left\{\frac{2}{3}\int\frac{d\mu^M}{\mu^M + p^M}\right\}\ , \tag{5.53}$$

where $k = -1, 0, +1$ denotes open, flat and closed universes respectively. Thus we can easily see that for the new constraints to be spatially and temporally consistent we must have either $\omega^a\,\Theta = 0$ or the barotropic equation of state must satisfy the following non-linear higher order differential equation:

$$\left(\mu^M + p^M\right) \frac{d^2 p^M}{d(\mu^M)^2} - \frac{dp^M}{d\mu^M}\left(\frac{5}{6} + \frac{1}{2}\frac{dp^M}{d\mu^M}\right) - k\frac{\exp\left\{\frac{2}{3}\int \frac{d\mu^M}{\mu^M + p^M}\right\}}{2\left(\mu^M + p^M\right)}\left(\frac{dp^M}{d\mu^M}\right)^2\left(1 - \frac{dp^M}{d\mu^M}\right) = 0\,. \tag{5.54}$$

We see that *the shear-free results of [60] and [61] are avoided, at least at the linearised level, if the equation of state of the matter solves (5.54).* However, *a priori* it seems highly unlikely that any realistic barotropic equation of state will obey this extremely non-linear equation. We now try to find solutions of this equation, under various simplified assumptions or realistic initial conditions, to confirm it is nonphysical.

1. Flat universe ($k = 0$) with $c_s^2 = \text{constant} \neq 0$: This is the simplest case in which the equation (5.54) reduces to a simple algebraic equation

$$\left(\frac{5}{6} + \frac{1}{2}c_s^2\right) = 0\,, \tag{5.55}$$

which gives $c_s^2 = -5/3$. This is physically not possible as the lower bound on the local sound speed (4.49) is violated, implying that the matter will be locally unstable. This will then make the perturbations grow and the linearised equations will no longer be valid.

2. Closed/open universe with $c_s^2 = \text{constant} \neq 0$: In this case also, the equation (5.54) reduces to an algebraic equation, and we get the relation

$${}^{(3)}R = -2\frac{\left(\frac{5}{6} + \frac{1}{2}c_s^2\right)}{c_s^2\left(1 - c_s^2\right)}\left(\mu^M + p^M\right) \tag{5.56}$$

Differentiating (5.56) with respect to μ^M and using (5.53) we get

$$\frac{2}{3}\frac{{}^{(3)}R}{\left(\mu^M + p^M\right)} = -2\frac{\left(\frac{5}{6} + \frac{1}{2}c_s^2\right)}{c_s^2\left(1 - c_s^2\right)}\left(1 + c_s^2\right)\,. \tag{5.57}$$

Eliminating ${}^{(3)}R/\left(\mu^M + p^M\right)$ from (5.56) and (5.57) we get the solution $c_s^2 = -1/3$, which again violates the lower bound of the local sound speed, making the matter locally unstable and the perturbations will grow beyond the scope of linearised regime.

3. Flat universe with varying sound speed: In this case the equation (5.54) becomes

$$\left(\mu^M + p^M\right) \frac{d^2 p^M}{d(\mu^M)^2} - \frac{dp^M}{d\mu^M}\left(\frac{5}{6} + \frac{1}{2}\frac{dp^M}{d\mu^M}\right) = 0\,. \tag{5.58}$$

To solve (5.58), if we choose the initial epoch ($\mu^M = \mu_0^M$) to be a radiation dominated one (which is quite realistic in view of our current understanding of the universe) with $c_s^2 \approx 1/3$, then from (5.58) we can easily see that c_s^2 monotonically increases with μ^M. And in the interval $(\mu_0^M \leq \mu^M < \infty)$ the function $p^M(\mu^M)$ is concave upwards. Therefore there must exist an earlier epoch at which $p(\mu^M) > \mu^M$, which violates (5.27). Furthermore if we consider $p(\mu)$ to be a C^∞ function, we can easily see from (5.58) that at the matter dominated epoch (where $p(\mu) = 0$ and $c_s^2 = 0$), all the higher derivatives of $p(\mu)$ with respect to μ vanish, implying that the sound speed would be constant ($c_s^2 = 0$) for all $\mu \in [0, \infty)$. Hence any solution of (5.58) with varying sound speed can never pass through the matter dominated $c_s^2 = 0$ phase.

4. Closed/open universe with varying sound speed: This being the most general case, we try to find a solution with similar initial conditions as the previous case. Since we know that very early universe was radiation dominated, let us suppose that there exists an epoch ($a_0 << 1$) with density μ_0^M and pressure p_0^M where $(\mu_0^M, p_0^M) \approx 1/a_0^4$. As we have already seen, ${}^{(3)}R \approx 1/a_0^2$, hence the last term on the LHS of (5.54) becomes suppressed and in this case one can also easily show that c_s^2 monotonically increases with μ^M. Therefore there must exist an earlier epoch $a_1 < a_0$ with $\mu_1^M > \mu_0^M$, where $p^M(\mu^M) > \mu^M$ and (5.27) is violated. In other words, no solution satisfying (5.27) exists for (5.54) that gives a radiation dominated era in the early universe. In this case, we can easily show (as in the previous case) that there exists no solution of (5.54) with varying sound speed that can pass through matter dominated $c_s^2 = 0$ phase. This makes the equation of state (with varying sound speed) which solves (5.54) unphysical, as we know from our present understanding of the universe that it must pass through a matter dominated epoch.

Hence for any physically realistic barotropic equation of state, if the new constraints are to be consistently propagated, we must have $\omega^a\,\Theta = 0$. We thus proved an important theorem for shear-free perturbations of FLRW spacetimes:

> *For an "almost" homogeneous and isotropic Universe filled with a barotropic perfect fluid subject to a physically realistic equation of state, if the fluid congruence is shear-free in a domain U, then it must be either vorticity-free or expansion-free in U.*

5.4 Consistency of the new constraints: The $f(R)$ case

For $f(R)$ gravity [64], in addition to considering the linearised field equations (5.2)–(5.19), the standard matter in this case will be assumed to have a barotropic linear equation of state $p^M = w\,\mu^M$. As with the GR case, to check the compatibility of the new constraints $(C_0)^{ab}$ and $(C_4)^a$ with the existing constraints of the field equations, we begin by substituting

$(C_0)_{bd}$ into $(C_5)_b$ and use

$$\dot{u}_b = -\frac{w}{w+1}\,\mathrm{D}_b\psi\ , \tag{5.59}$$

where $\psi = \ln(\mu^M)$ in the resulting equation to obtain the constraint

$$\frac{w}{w+1}\mathrm{D}^d\,\mathrm{D}_{\langle b}\mathrm{D}_{d\rangle}\psi + \frac{1}{3}\mathrm{D}_b\,\mu - \mathrm{D}^d\pi^R_{bd} - \frac{1}{3}\Theta\,q^R_b = 0\ . \tag{5.60}$$

To check for the spatial consistency of the new constraints (5.60), we follow the steps (5.28)-(5.36) which in $f(R)$ results in

$$\frac{2}{3}\Theta\left\{\omega^a\left[\left(\frac{w}{2}+\frac{f''}{3f'}\,Q\right){}^{(3)}R + \frac{(1+w)\,\mu^M}{f'}\right] + \left(\frac{f''}{f'}Q + \frac{3w}{2}\right)\mathrm{D}^2\omega^a\right\} = 0\ , \tag{5.61}$$

where

$$Q = \frac{1}{3}\Theta^2\ (j-q-2) + {}^{(3)}R\ , \tag{5.62}$$

and the expansion Θ, acceleration q, jerk j and snap s parameters are defined by the following relations

$$\Theta = 3\,\frac{\dot{a}}{a}\ , \qquad q = -\frac{\ddot{a}a}{\dot{a}^2},$$
$$j = \frac{\dddot{a}\,a^2}{\dot{a}^3}\ , \qquad s = \frac{a^3}{\dot{a}^4}\,\frac{d^4a}{dt^4}\ , \tag{5.63}$$

in terms of the scale factor $a(t)$ of an FLRW spacetime, such that the Ricci scalar R can be written as

$$R = \frac{2}{3}\Theta^2\ (1-q) + {}^{(3)}R\ , \quad \to \quad \dot{R} = \frac{2}{3}\Theta\,Q\ , \tag{5.64}$$

which is useful to obtain the form of (5.61).

To check for temporal consistency of the new constraint (5.60), it is propagated along u^a which after a little manipulation ([64] gives a detailed derivation) results in

$$\Theta\,\omega^a\left\{\left[{}^{(3)}R\frac{(1-w)P}{3} + \frac{(1+w)}{f'}\,\frac{(3w+5)f' + 4f''Q}{6f'}\mu_m\right] + \frac{Z}{P}\left[\left(\frac{1+w}{f'}\right)\mu_m\right]\right\} = 0\ . \tag{5.65}$$

where

$$\begin{aligned} P &\equiv \frac{f''}{f'}Q + \frac{3w}{2}\ ,\\ Z &= \frac{2}{3}\left[\frac{f'''}{f'} - \left(\frac{f''}{f'}\right)^2\right]Q^2 + \frac{f''}{9f'}\left[(4+5q+j+j\,q+s)\,\Theta^2 + 6\,{}^{(3)}R\right]\ . \end{aligned} \tag{5.66}$$

We can see from (5.61) and (5.65) that for the new constraints to be spatially and temporally consistent, either $\omega^a\,\Theta = 0$ or the expression in the curly brackets must vanish. Interestingly enough, in (5.65) if the 3-curvature vanishes, the result of Section 5.3 can always be avoided for vacuum universes ($\mu^M = 0$). This implies, that

A shear-free, spatially flat vacuum universe in any $f(R)$ theory can rotate and expand simultaneously in the linearised regime.

The non-vacuum case, if a flat Milne universe is considered, for example, where the matter energy density is given by $\mu^M = \frac{\mu_0}{a^{3(1+w)}}$, we have

$$\dot{\Theta} = -\frac{1}{3}\,\Theta^2\,, \qquad R = \tfrac{2}{3}\,\Theta^2,$$
$$a(R) = \frac{1}{\sqrt{R}}\,, \qquad \dot{R} = -\sqrt{\tfrac{2}{3}}\,R^{\frac{3}{2}}\,. \tag{5.67}$$

Substituting these quantities into the Friedmann equation (5.26) yields

$$-R^2\,\frac{d^2 f(R)}{dR^2} + \frac{f(R)}{2} - \frac{\mu_0}{a(R)^{3(1+w)}} = 0\,, \tag{5.68}$$

which has the following general solution:

$$f(R) = C_1\,R^{\frac{1+\sqrt{3}}{2}} + C_2\,R^{\frac{1-\sqrt{3}}{2}} - \frac{4\,\mu_0}{1+12\,w+9\,w^2}R^{\frac{3(1+w)}{2}}. \tag{5.69}$$

Considering the particular solution (the last term of (5.69)), which is an R^n theory of gravity, the corresponding flat Milne universe in R^n gravity in (5.65) reduces the term in the curly brackets to

$$\frac{(1+w)\mu^M}{6f'}\,[3\,w + 9 - 4n] = 0. \tag{5.70}$$

Comparing solutions (5.70) with the particular solution of (5.69) and taking $n = 3(1+w)/2$, we get the result that $w = 1$ if $\mu^M \neq 0$. In other words:

For a stiff fluid in R^3 gravity, there exists a flat Milne-universe solution which can rotate and expand simultaneously at the level of linearised perturbation theory.

5.5 Discussion

These results give an interesting scenario. In GR the linearised shear-free solutions do not have the same behaviour as shear-free Newtonian solutions. This may affect simple structure formation scenarios for rotating matter. We would like to emphasise again that this local result of linearised Einstein field equations about an FLRW universe is only valid for isentropic perfect fluids in GR. For non-isentropic fluids, fluids with anisotropic stress (for example, collision free neutrinos in an anisotropic space-time) or for gravity theories where we have an extra degree of freedom, this result can be avoided as demonstrated in the $f(R)$ case and a shear-free fluid congruence may rotate and expand simultaneously.

Chapter 6

The 1+1+2 covariant approach in $f(R)$ gravity

The 1+3 covariant approach has been successful in its application to cosmology. Most cosmological models are based on the cosmological principle which is the hypothesis that the universe, at least on large scales, is isotropic and homogenous. This means that the only essential coordinate is time. However, if the spacetime considered admits less symmetry, for example if it is an inhomogenous spherically symmetric system, the resulting 1+3 equations are messy tensorial partial differential equations that become intractable. The 1+1+2 approach developed recently by Clarkson and Barrett [65] is ideally suited to investigate such systems in the sense that it includes an additional frame vector, assuming the background spacetime has some preferred direction, while keeping the benefits the 1+3 approach. This formalism has been applied in various areas in the context of GR [66, 71] and in $f(R)$ gravity in [73, 74] . A similar approach was introduced in [151] and further developed in [143,152,153] with previous studies mostly based on the context of symmetries of solutions of EFEs [153–155].

Following [69], this chapter presents for the first time the full system of 1+1+2 equations in $f(R)$ gravity.

6.1 Kinematics

In the 1+3 decomposition a timelike unit vector u^a is split in the form $R \otimes V$, where R denotes the timeline along a timelike unit vector u^a ($u^a\, u_a = -1$) and V is the 3-space perpendicular to u^a. In the 1+1+2 approach, we further split the 3-space V, by introducing the unit vector e^a orthogonal to u^a such that

$$e_a\, u^a = 0\,, \quad e_a\, e^a = 1\,. \tag{6.1}$$

Then the *projection tensor*

$$N_a{}^b \equiv h_a{}^b - e_a\, e^b = g_a{}^b + u_a\, u^b - e_a\, e^b \ , \tag{6.2}$$

projects vectors onto 2-spaces orthogonal to e^a and u^a which, following [65, 66, 69], we refer to as *sheets*. It thus follows from this that

$$e^a\, N_{ab} = 0 = u^a\, N_{ab} \ , \quad N^a{}_a = 2 \ . \tag{6.3}$$

Any spacetime 3-vector ψ^a can now be irreducibly split into a scalar, Ψ, which is the component along e^a and a 2-vector, Ψ^a, which is a sheet component orthogonal to e^a, i.e.,

$$\psi^a = \Psi\, e^a + \Psi^a \ , \qquad \text{where} \quad \Psi \equiv \psi_a\, e^a \ \text{ and } \ \Psi^a \equiv N^{ab}\, \psi_b \equiv \psi^{\bar{a}} \ , \tag{6.4}$$

where the bar on a particular index denotes projection with N_{ab} on that index. A similar decomposition can be done for a PSTF 3-tensor, ψ_{ab}, which can be split into scalar, 2-vector and 2-tensor parts as follows:

$$\psi_{ab} = \psi_{\langle ab \rangle} = \Psi \left(e_a\, e_b - \frac{1}{2} N_{ab} \right) + 2\Psi_{(a}\, e_{b)} + \Psi_{ab} \ , \tag{6.5}$$

where

$$\begin{aligned} \Psi &\equiv e^a\, e^b\, \psi_{ab} = -N^{ab} \psi_{ab} \ , \\ \Psi_a &\equiv N_a{}^b\, e^c\, \psi_{bc} \ , \\ \Psi_{ab} &\equiv \psi_{\{ab\}} \equiv \left(N_{(a}{}^c\, N_{b)}{}^d - \frac{1}{2} N_{ab}\, N^{cd} \right) \psi_{cd} \ . \end{aligned} \tag{6.6}$$

The curly brackets denote the part of a tensor which is PSTF with respect to e^a. We also have that,

$$h_{\{ab\}} = 0 = N_{\{ab\}} \ , \qquad N_{\langle ab \rangle} = -e_{\langle a}\, e_{b \rangle} = N_{ab} - \frac{2}{3} h_{ab} \ . \tag{6.7}$$

The sheet carries a natural 2-volume element, the alternating Levi-Civita 2-tensor

$$\varepsilon_{ab} \equiv \varepsilon_{abc}\, e^c = \eta_{dabc}\, e^c\, u^d \ , \tag{6.8}$$

induced by the volume element ε_{abc} of the 3-spaces. From the definition of ε_{ab} and N_{ab}, the following relations hold

$$
\begin{aligned}
\varepsilon_{ab}\, e^b &= 0 = \varepsilon_{(ab)} \ , && (6.9)\\
\varepsilon_{abc} &= e_a\, \varepsilon_{bc} - e_b\, \varepsilon_{ac} + e_c\, \varepsilon_{ab} \ , && (6.10)\\
\varepsilon_{ab}\, \varepsilon^{cd} &= N_a{}^c\, N_b{}^d - N_a{}^d\, N_b{}^c \ , && (6.11)\\
\varepsilon_a{}^c\, \varepsilon_{bc} &= N_{ab} \ , && (6.12)\\
\varepsilon^{ab}\, \varepsilon_{ab} &= 2 \ . && (6.13)
\end{aligned}
$$

From these definitions it follows that any object can be split in the 1+1+2 setting into scalars, 2-vectors in the sheet and PSTF 2-tensors (also defined in the sheet) .

Apart from the 'time' (dot) derivative of an object (scalar, vector or tensor), which is the derivative along the timelike congruence u^a, we introduce two new derivatives which the congruence e^a defines for any tensor $\psi_{a..b}{}^{c..d}$:

$$
\begin{aligned}
\hat{\psi}_{a..b}{}^{c..d} &\equiv e^f\, \nabla_f M_{a..b}{}^{c..d} \ ,\\
\delta_f \psi_{a..b}{}^{c..d} &\equiv N_f{}^j\, N_a{}^l \ldots N_b{}^g\, N_h{}^c \ldots N_i{}^d\, \mathrm{D}_j \psi_{l..g}{}^{h..i} \ . && (6.14)
\end{aligned}
$$

The hat-derivative is the spatial derivative along the vector-field e^a in the surfaces orthogonal to u^a. (We note that the congruence u^a retains the primary importance it has in the 1+3 covariant approach). The δ-derivative is the projected spatial derivative onto the orthogonal 2-sheet, with the projection on every free index. By these definitions, one obtains the following relations for the derivatives of the sheet-projection N_{ab} and the sheet volume element ε_{ab} :

$$
\begin{aligned}
\dot{N}_{ab} &= 2u_{(a}\, \dot{u}_{b)} - 2e_{(a}\, \dot{e}_{b)} = 2u_{(a}\, \mathcal{A}_{b)} - 2e_{(a}\, \alpha_{b)} \ ,\\
\hat{N}_{ab} &= -2e_{(a}\, a_{b)} \ ,\\
\delta_c N_{ab} &= 0 \ ,\\
\dot{\varepsilon}_{ab} &= -2u_{[a}\, \varepsilon_{b]c}\, \mathcal{A}^c + 2e_{[a}\, \varepsilon_{b]c}\, \alpha^c \ ,\\
\hat{\varepsilon}_{ab} &= 2e_{[a}\, \varepsilon_{b]c}\, a^c \ ,\\
\delta_c \varepsilon_{ab} &= 0 \ , && (6.15)
\end{aligned}
$$

where $\mathcal{A}_a \equiv \dot{u}_{\bar{a}}$, $\alpha_a \equiv \dot{e}_{\bar{a}}$ and $a_a \equiv e^c\, \mathrm{D}_c\, e_a = \hat{e}_a$.

Taking e^a to be arbitrary, the 1+3 kinematical and Weyl quantities can be split in accordance with the decompositions (6.4) and (6.5), respectively. The 4-acceleration, vorticity and shear, split irreducibly as

$$
\begin{aligned}
\dot{u}^a &= \mathcal{A}\, e^a + \mathcal{A}^a \,, &(6.16)\\
\omega^a &= \Omega\, e^a + \Omega^a \,, &(6.17)\\
\sigma_{ab} &= \Sigma \left(e_a\, e_b - \frac{1}{2} N_{ab} \right) + 2\, \Sigma_{(a}\, e_{b)} + \Sigma_{ab} \,. &(6.18)
\end{aligned}
$$

For shear scalar, σ one arrives at

$$
\sigma^2 \equiv \frac{1}{2}\, \sigma_{ab}\, \sigma^{ab} = \frac{3}{4}\, \Sigma^2 + \Sigma_a\, \Sigma^a + \frac{1}{2}\, \Sigma_{ab}\, \Sigma^{ab} \,, \qquad (6.19)
$$

and for the electric and magnetic Weyl tensors one gets

$$
\begin{aligned}
E_{ab} &= \mathcal{E} \left(e_a\, e_b - \frac{1}{2} N_{ab} \right) + 2\, \mathcal{E}_{(a}\, e_{b)} + \mathcal{E}_{ab} \,, &(6.20)\\
H_{ab} &= \mathcal{H} \left(e_a\, e_b - \frac{1}{2} N_{ab} \right) + 2\, \mathcal{H}_{(a}\, e_{b)} + \mathcal{H}_{ab} \,. &(6.21)
\end{aligned}
$$

From equation (4.22) and using the relations (see Appendix A) we can obtain the exact form of the covariant decomposition of the derivative of the 3-vector (6.4) as

$$
\begin{aligned}
\nabla_a \psi_b \;=\; & -u_a \left[\left(\dot{\Psi} - \Psi_c\, \alpha^c \right) e_b + \Psi\, \alpha_b + \dot{\Psi}_{\bar{b}} \right] - u_a u_b \left(\mathcal{A}\, \Psi + \mathcal{A}_c \Psi^c \right) \\
& + u_b \Big[\left(\frac{1}{3}\theta + \Sigma \right) \Psi\, e_a + \left(\frac{1}{3}\theta - \frac{1}{2}\Sigma \right) \Psi_a + \Sigma_a \Psi + \Sigma^c \Psi_c\, e_a \\
& + \Sigma_a{}^c \Psi_c + \Omega\, \varepsilon_a{}^c\, \Psi_c - \varepsilon_a{}^c\, \Omega_c\, \Psi + e_a\, \varepsilon^{cd} \Psi_c\, \Omega_d \Big] \\
& + \frac{1}{3} \left(\hat{\Psi} + \Psi\, \phi - \Psi_c\, a^c + \delta_c \Psi^c \right) \left(N_{ab} + e_a\, e_b \right) \\
& + \frac{1}{3} \left(2\hat{\Psi} - \phi \Psi - 2\Psi_c a^c - \delta_c \Psi^c \right) \left(e_a\, e_b - \frac{1}{2} N_{ab} \right) \\
& + \left[\Psi a_{(a} + \delta_{(a} \Psi + \hat{\Psi}_{(\bar{a}} - \frac{1}{2}\phi\, \Psi_{(a} + \Psi^c \left(\xi\, \varepsilon_{c(a} - \zeta_{c(a)} \right) \right] e_{b)} \\
& + \Psi\, \zeta_{ab} + \delta_{\{a} \Psi_{b\}} + \frac{1}{2} \varepsilon_{ab} \left(2\Psi\, \xi + \varepsilon^{cd}\, \delta_c \Psi_d \right) + e_{[a}\, \varepsilon_{b]c}\, \Psi^c\, \xi \\
& - e_{[a} \left(-\Psi\, a_{b]} + \delta_{b]} \Psi - \hat{\Psi}_{b]} - \frac{1}{2}\phi\, \Psi_{b]} - \zeta_{b]c}\, \Psi^c \right) \,. \qquad (6.22)
\end{aligned}
$$

where $\phi \equiv \delta_a e^a$, $\mathcal{A} \equiv e^a\, \dot{u}_a$, $\xi \equiv \frac{1}{2}\, \epsilon^{ab} \delta_a e_b$ and $\zeta_{ab} \equiv \delta_{\{a} e_{b\}}$. An analogous relation for the rank-2 tensors holds by applying (4.23) and using the Appendix A.

Thus by (6.22) the expression for the full covariant derivative of e^a in its irreducible form is

$$\nabla_a e_b = -\mathcal{A}\, u_a\, u_b - u_a\, \alpha_b + \left(\frac{1}{3}\Theta + \Sigma\right) e_a\, u_b + \left(\Sigma_a - \varepsilon_{ac}\, \Omega^c\right)\, u_b$$
$$+ e_a\, a_b + \frac{1}{2}\phi\, N_{ab} + \xi\, \varepsilon_{ab} + \zeta_{ab}\ , \tag{6.23}$$

from which we can obtain the spatial derivative of e^a as

$$\mathrm{D}_a e_b = e_a\, a_b + \frac{1}{2}\phi\, N_{ab} + \xi\, \varepsilon_{ab} + \zeta_{ab}\ . \tag{6.24}$$

The other derivative of e^a is its change along u^a,

$$\dot{e}_a = \mathcal{A}\, u_a + \alpha_a\ . \tag{6.25}$$

Similar to the kinematical variables of u^a in the 1+3 approach (which also appear here), the new kinematic variables a_a, ϕ, ξ, ζ_{ab}, $\mathcal{A}$ and α_a are fundamental objects in spacetime, and their dynamics give us information about the spacetime geometry. From equation (6.24) we see that along the spatial direction e^a, ϕ represents the *expansion of the sheet*, ζ_{ab} is the *shear of* e^a (i.e., the distortion of the sheet) and a_a its *acceleration*, while ξ represents the *vorticity* associated with e^a ('twisting' of the sheet).

We include here the expression for the 1+1+2 split of the full covariant derivative of u^a

$$\nabla_a u_b = -u_a \left(\mathcal{A}\, e_b + \mathcal{A}_b\right) + e_a\, e_b \left(\frac{1}{3}\theta + \Sigma\right) + e_a \left(\Sigma_b + \varepsilon_{bc}\Omega^c\right)$$
$$+ \left(\Sigma_a - \varepsilon_{ac}\Omega^c\right) e_b + N_{ab}\left(\frac{1}{3}\theta - \frac{1}{2}\Sigma\right) + \Omega\, \varepsilon_{ab} + \Sigma_{ab}\ , \tag{6.26}$$

from which we can derive the useful relation

$$\hat{u}_a = \left(\frac{1}{3}\theta + \Sigma\right) e_a + \Sigma_a + \varepsilon_{ab}\Omega^b\ , \tag{6.27}$$

for the calculation of the Ricci identities.

Furthermore, we may decompose the different parts of spatial derivative of a scalar Ψ and a 2-vector $\Psi_a = \Psi_{\bar{a}}$, respectively, as follows

$$\mathrm{D}_a \Psi = \hat{\Psi}\, e_a + \delta_a \Psi\ , \tag{6.28}$$
$$\mathrm{D}_a \Psi_b = -e_a\, e_b\, \Psi_c\, a^c + e_a \hat{\Psi}_{\bar{b}} - e_b \left(\frac{1}{2}\phi\, \Psi_a + \left(\xi\, \varepsilon_{ac} + \zeta_{ac}\right) \Psi^c\right) + \delta_a \Psi_b \tag{6.29}$$

Similarly for a PSTF 2-tensor $\Psi_{ab} = \Psi_{\{ab\}}$, we have

$$\mathrm{D}_a\Psi_{bc} = -2\, e_a\, e_{(b}\, \Psi_{c)d}\, a^d + e_a\, \hat{\Psi}_{bc} - 2e_{(c}\left[\frac{1}{2}\phi\, \Psi_{c)a} + \Psi_{c)}{}^{d}\, (\xi\, \varepsilon_{ad} + \zeta_{ad})\right] + \delta_a \Psi_{bc}\ . \quad (6.30)$$

6.2 The energy momentum tensor

Given that the anisotropic fluid variables q_a and π_{ab} split as

$$q_a = Q\, e^a + Q_a\ , \quad (6.31)$$
$$\pi_{ab} = \Pi\left(e_a\, e_b - \frac{1}{2}N_{ab}\right) + 2\Pi_{(a}\, e_{b)} + \Pi_{ab}\ , \quad (6.32)$$

in terms of the 1+1+2 variables, the total energy momentum tensor (4.39) is

$$T_{ab} = \mu\, u_a\, u_b + p\, h_{ab} + 2u_{(a}\left[Q\, e_{b)} + Q_a\right] + \Pi\left(e_a\, e_b - \frac{1}{2}N_{ab}\right) + 2\,\Pi_{(a}\, e_{b)} + \Pi_{ab}\ ; \quad (6.33)$$

recalling that the thermodynamic quantities as presented in (6.33) are representative of the total combination of the standard matter and curvature quantities. Moreover, in terms of 1+1+2 variables the curvature thermodynamic quantities are obtained from decomposition of the 1+3 quantities (4.57)-(4.60) as

$$\begin{aligned}\mu^R \equiv T^R_{ab}\, u^a\, u^b &= \frac{1}{f'}\left[\frac{1}{2}(R\, f' - f) - \theta f''\, \dot{R} + f'''\, X^2 + f'''\, \delta^a R\, \delta_a R\right.\\ &\left.\quad + f''\, \hat{X} + \phi f''\, X - a^a\, f''\, \delta_a R + f''\, \delta^a \delta_a R\right]\ ,\end{aligned} \quad (6.34)$$

$$\begin{aligned}p^R \equiv \frac{1}{3}T^R_{ab}\left(N^{ab} - e^a\, e^b\right) &= \frac{1}{f'}\left[\frac{1}{2}(f - R\, f') + \frac{2}{3}\theta\, f''\dot{R} + f'''\, \dot{R}^2 + f''\, \ddot{R} - \mathcal{A}\, f''\, X\right.\\ &\quad - \mathcal{A}^a f''\, \delta_a R - \frac{2}{3}(\phi\, f''\, X + f'''\, \delta^a R\, \delta_a R + f''\, \delta^a \delta_a R\\ &\left.\quad + f'''\, X^2 + f''\, \hat{X} - a_a\, f''\, \delta^a R)\right]\ ,\end{aligned} \quad (6.35)$$

$$Q^R \equiv e^a\, q^R_a = -\frac{1}{f'}\left[f'''\, \dot{R}\, X + f''\left(\dot{X} - \mathcal{A}\, \dot{R}\right) - \alpha^a\, f''\, \delta_a R\right]\ , \quad (6.36)$$

$$\begin{aligned}Q^R_a \equiv N_a{}^b\, q^R_b &= \frac{1}{f'}\left[\left(\frac{1}{3}\theta - \frac{1}{2}\Sigma\right) f''\, \delta_a R + \left(\Sigma_a - \varepsilon_a{}^b \Omega_b\right) f''\, X\right.\\ &\left.\quad + \left(\Sigma_a{}^b + \varepsilon_a{}^b\Omega\right) f''\, \delta_b R - \dot{R}\, f'''\, \delta_a R - f''\, \delta_a \dot{R}\right]\ ,\end{aligned} \quad (6.37)$$

$$\begin{aligned}\Pi^R \equiv e^a\, e^b \pi^R_{ab} \;=\; & \frac{1}{f'}\left[\frac{1}{3}\Big(2f'''\, X^2 + 2f''\,\hat{X} - 2\mathcal{A}_a\, f''\,\delta^a R - \phi\, f''\, X\right. \\ & \left. - f'''\,\delta^a R\,\delta_a R - f''\,\delta^a\delta_a R\Big) - \Sigma\, f''\,\dot{R}\right] ,\end{aligned} \tag{6.38}$$

$$\begin{aligned}\Pi^R_a \equiv N_a{}^b\, e^c\, \pi^R_{bc} \;=\; & \frac{1}{f'}\left[-\Sigma_a\, f''\,\dot{R} + X\, f'''\,\delta_a R + f''\,\delta_a X - \frac{1}{2}\phi\, f''\,\delta_a R\right. \\ & \left. + \left(\xi\,\varepsilon_a{}^b - \zeta_a{}^b\right) f''\,\delta_b R - \frac{1}{2}\left(\Sigma_a + \varepsilon_a{}^b\Omega_b\right)\, f''\,\dot{R}\right] ,\end{aligned} \tag{6.39}$$

$$\begin{aligned}\Pi^R_{ab} \equiv \left(N_{(a}{}^c\, N^d_{b)} - \frac{1}{2}N_{ab}\, N^{cd}\right)\pi^R_{cd} \;=\; & \frac{1}{f'}\Big(-\Sigma_{ab}\, f''\,\dot{R} + \zeta_{ab}\, f''\, X \\ & + f'''\,\delta_{\{a}R\,\delta_{b\}}R + f''\,\delta_{\{a}\delta_{b\}}R\Big) ,\end{aligned} \tag{6.40}$$

where we have defined $\hat{R} = X$. Additionally, the 1+1+2 split of the *curvature trace equation* (4.67) results in

$$\begin{aligned}R\, f' - 2f = 3\Big(& f''\,\theta\,\dot{R} - f'''\, X^2 - f'''\,\delta^a R\,\delta_a R - (\mathcal{A}+\phi)f''\, X \\ & - f''\,\hat{X} - f''\,\delta^a\delta_a R + f'''\,\dot{R}^2 + f''\,\ddot{R}\Big) .\end{aligned} \tag{6.41}$$

6.3 Derivatives and Commutators

In general the three derivatives defined so far, dot - '$\dot{}$', hat - '$\hat{}$' and delta - 'δ_a', do not commute. The commutations relations for these derivatives of any scalar ψ are

$$\hat{\dot{\psi}} - \dot{\hat{\psi}} = -\mathcal{A}\,\dot{\psi} + \left(\frac{1}{3}\Theta + \Sigma\right)\hat{\psi} + \left(\Sigma_a + \varepsilon_{ab}\Omega^b - \alpha_a\right)\delta^a\psi , \tag{6.42}$$

$$\begin{aligned}\delta_a\dot{\psi} - (\delta_a\psi)^{\cdot}_{\perp} = & -\mathcal{A}_a\,\dot{\psi} + \left(\alpha_a + \Sigma_a - \varepsilon_{ab}\Omega^b\right)\hat{\psi} + \left(\frac{1}{3}\Theta - \frac{1}{2}\Sigma\right)\delta_a\psi \\ & + \left(\Sigma_{ab} + \Omega\,\varepsilon_{ab}\right)\delta^b\psi ,\end{aligned} \tag{6.43}$$

$$\delta_a\hat{\psi} - (\delta_a\psi)^{\hat{}}_{\perp} = -2\,\varepsilon_{ab}\Omega^b\,\dot{\psi} + a_a\,\hat{\psi} + \frac{1}{2}\phi\,\delta_a\psi + \left(\zeta_{ab} + \xi\,\varepsilon_{ab}\right)\delta^b\psi , \tag{6.44}$$

$$\delta_{[a}\delta_{b]}\psi = \varepsilon_{ab}\left(\Omega\,\dot{\psi} - \xi\,\hat{\psi}\right) . \tag{6.45}$$

Here, and in the work that follows, the symbol $\perp$ denotes projection onto the sheet (it had previously been used in 1+3 to mean projection onto the observer's rest-space). From the above relations it is clear that the 2-sheet is a genuine 2-surface (instead of just a collection of tangent planes) if and only if:

– The commutator of the time and hat derivative do not depend on any sheet component, that is, when Greenberg's vector

$$\Sigma_a + \varepsilon_{ab}\Omega^b - \alpha_a \ , \tag{6.46}$$

vanishes [151,155]. Accordingly, the two vector fields u^a and e^a are 2-surface forming.

– The sheet derivatives commute (specifically, the derivative δ^a will be a true covariant derivative on this surface), that is, when $\xi = \Omega = 0$.

The commutation relations for 2-vectors ψ_a are

$$\begin{aligned} \hat{\dot{\psi}}_{\bar{a}} - \dot{\hat{\psi}}_{\bar{a}} \ &= \ -\mathcal{A}\,\dot{\psi}_{\bar{a}} + \left(\frac{1}{3}\Theta + \Sigma\right)\hat{\psi}_{\bar{a}} + \left(\Sigma_b + \varepsilon_{bc}\,\Omega^c - \alpha_b\right)\delta^b\psi_a \\ &\quad +\mathcal{A}_a\left(\Sigma_b + \varepsilon_{bc}\,\Omega^c\right)\psi^b + \mathcal{H}\,\varepsilon_{ab}\,\psi^b \ , \end{aligned} \tag{6.47}$$

$$\begin{aligned} \delta_a\dot{\psi}_b - \left(\delta_a\psi_b\right)^{\cdot}_{\perp} \ &= \ -\mathcal{A}_a\,\dot{\psi}_b + \left(\alpha_a + \Sigma_a - \varepsilon_{ac}\,\Omega^c\right)\hat{\psi}_{\bar{b}} + \left(\frac{1}{3}\Theta - \frac{1}{2}\Sigma\right)\left(\delta_a\psi_b + \psi_a\,\mathcal{A}_b\right) \\ &\quad + \left(\Sigma_{ac} + \Omega\,\varepsilon_{ac}\right)\left(\delta^c\psi_b + \psi^c\,\mathcal{A}_b\right) + \frac{1}{2}\left(\psi_a\,Q_b - N_{ab}\,\psi^c\,Q_c\right) \\ &\quad - \left(\frac{1}{2}\phi\,N_{ac} + \xi\,\varepsilon_{ac} + \zeta_{ac}\right)\psi^c\,\alpha_b + \mathcal{H}_a\,\varepsilon_{bc}\,\psi^c \ , \end{aligned} \tag{6.48}$$

$$\begin{aligned} \delta_a\hat{\psi}_b - \left(\delta_a\psi_b\right)^{\hat{}}_{\perp} \ &= \ -2\,\varepsilon_{ac}\,\Omega^c\,\dot{\psi}_{\bar{b}} + a_a\,\hat{\psi}_{\bar{b}} + \frac{1}{2}\phi\left(\delta_a\psi_b - \psi_a\,a_b\right) + \left(\zeta_{ac} + \xi\,\varepsilon_{ac}\right)\left(\delta^c\psi_b - \psi^c\,a_b\right) \\ &\quad -2\left(\Omega\,\varepsilon_{a[b} + \Sigma_{a[b}\right)\left(\Sigma_{c]} + \varepsilon_{c]d}\Omega^d\right)\psi^c \\ &\quad -\psi_a\left[\left(\frac{1}{2}\Sigma - \frac{1}{3}\Theta\right)\left(\Sigma_b + \varepsilon_{bc}\,\Omega^c\right) + \frac{1}{2}\Pi_b + \mathcal{E}_b\right] \\ &\quad +N_{ab}\left[\left(\frac{1}{2}\Sigma - \frac{1}{3}\Theta\right)\left(\Sigma_c + \varepsilon_{cd}\,\Omega^d\right) + \frac{1}{2}\Pi_c + \mathcal{E}_c\right]\psi^c \ , \end{aligned} \tag{6.49}$$

$$\begin{aligned} \delta_{[a}\delta_{b]}\psi^c \ &= \ \left[\left(\frac{1}{3}\Theta - \frac{1}{2}\Sigma\right)^2 - \frac{1}{4}\phi^2 + \frac{1}{2}\Pi + \mathcal{E} - \frac{1}{3}\mu\right]\psi_{[a}N_{b]}{}^c \\ &\quad -\psi_{[a}\left[-\left(\frac{1}{3}\Theta - \frac{1}{2}\Sigma\right)\left(\Sigma_{b]}{}^c + \Omega\,\varepsilon_{b]}{}^c\right) + \frac{1}{2}\phi\left(\zeta_{b]}{}^c + \xi\,\varepsilon_{b]}{}^c\right) + \frac{1}{2}\Pi_{b]}{}^c + \mathcal{E}_{b]}{}^c\right] \\ &\quad +N_{[a}{}^c\left[-\left(\frac{1}{3}\Theta - \frac{1}{2}\Sigma\right)\left(\Sigma_{b]d} + \Omega\,\varepsilon_{b]d}\right) + \frac{1}{2}\phi\left(\zeta_{b]d} + \xi\,\varepsilon_{b]d}\right) + \frac{1}{2}\Pi_{b]d} + \mathcal{E}_{b]d}\right]\psi^d \\ &\quad -\left[\left(\Sigma_{[a}{}^c + \Omega\,\varepsilon_{[a}{}^c\right)\left(\Sigma_{b]d} + \Omega\,\varepsilon_{b]d}\right) - \left(\zeta_{[a}{}^c + \xi\,\varepsilon_{[a}{}^c\right)\left(\zeta_{b]d} + \xi\,\varepsilon_{b]d}\right)\right]\psi^d \\ &\quad +\varepsilon_{ab}\left(\Omega\,\dot{\psi}^{\bar{c}} - \xi\,\hat{\psi}^{\bar{c}}\right) \ . \end{aligned} \tag{6.50}$$

Analogous relations for second-rank tensors hold but are more complicated.

6.4 The field equations

The key variables of the 1+1+2 formalism of FOG are the irreducible set of geometric variables,

$$\{R,\, \Theta,\, \mathcal{A},\, \Omega,\, \Sigma,\, \mathcal{E},\, \mathcal{H},\, \phi, \xi,\, \mathcal{A}_a,\, \Omega_a,\, \Sigma_a,\, \alpha_a,\, a_a,\, \mathcal{E}_a,\, \mathcal{H}_a,\, \Sigma_{ab},\, \zeta_{ab},\, \mathcal{E}_{ab},\, \mathcal{H}_{ab}\}\ , \tag{6.51}$$

together with the set of irreducible thermodynamic matter variables,

$$\{\mu^M,\, p^M,\, Q^M,\, \Pi^M,\, Q^M_a,\, \Pi^M_a,\, \Pi^M_{ab}\}\ , \tag{6.52}$$

for a given equation of state. The full 1+1+2 equations for the above covariant variables can be obtained by applying the 1+1+2 decomposition procedure to the 1+3 equations (Appendix A), and in addition, by covariantly splitting the Ricci identities for e^a:

$$R_{abc} \equiv 2\nabla_{[a}\nabla_{b]}e_c - R_{abcd}e^d = 0\ , \tag{6.53}$$

where R_{abcd} is the Riemann curvature tensor. By splitting this third-rank tensor using the two vector fields u^a and e^a, we obtain the *evolution* equations (along u^a) and *propagation* equations (along e^a) for α_a, a_a, ϕ, ξ and ζ_{ab}.

The full set of 1+1+2 equations for arbitrary spacetimes as given in [69] are:

6.4.1 The evolution equations

The evolution equations for the ϕ, ξ and ζ_{ab} are obtained from the projection $u^a\, R_{abc}$.

$u^a\, N^{bc} R_{abc}$:

$$\begin{aligned}\dot{\phi} &= \left(\frac{2}{3}\theta - \Sigma\right)\left(\mathcal{A} - \frac{1}{2}\phi\right) + 2\,\xi\,\Omega + \delta_a\alpha^a + \mathcal{A}^a\,(\alpha_a - a_a) \\ &\quad + (a^a - \mathcal{A}^a)\left(\Sigma_a - \varepsilon_{ab}\Omega^b\right) - \zeta^{ab}\,\Sigma_{ab} + Q\ ;\end{aligned} \tag{6.54}$$

$u^a\, \varepsilon^{bc} R_{abc}$:

$$\begin{aligned}\dot{\xi} &= \left(\frac{1}{2}\Sigma - \frac{1}{3}\theta\right)\xi + \left(\mathcal{A} - \frac{1}{2}\phi\right)\Omega + \frac{1}{2}\,(a^a + \mathcal{A}^a)\left[\Omega_a + \varepsilon_{ab}\left(\alpha^b + \Sigma^b\right)\right] \\ &\quad + \frac{1}{2}\varepsilon_{ab}\delta^a\alpha^b - \frac{1}{2}\varepsilon_{ca}\zeta_b{}^c\,\Sigma^{ab} + \frac{1}{2}\mathcal{H}\ ;\end{aligned} \tag{6.55}$$

$u^c\, R_{c\{ab\}}$:

$$\begin{aligned}\dot{\zeta}_{\{ab\}} &= \left(\frac{1}{2}\Sigma - \frac{1}{3}\theta\right)\zeta_{ab} + \Omega\,\varepsilon_{c\{a}\zeta_{b\}}{}^{c} + \left(\mathcal{A} - \frac{1}{2}\phi\right)\Sigma_{ab} - \xi\,\varepsilon_{c\{a}\Sigma_{b\}}{}^{c} - \zeta_{c\{a}\,\Sigma_{b\}}{}^{c} \\ &\quad + \delta_{\{a}\alpha_{b\}} + \left(\mathcal{A}_{\{a} - a_{\{a}\right)\,\alpha_{b\}} - \left(\mathcal{A}_{\{a} + a_{\{a}\right)\left(\Sigma_{b\}} - \varepsilon_{b\}d}\Omega^{d}\right) - \varepsilon_{c\{a}\mathcal{H}_{b\}}{}^{c}\ .\end{aligned} \tag{6.56}$$

A 1+1+2 decomposition of the standard 1+3 evolution equations gives us the remaining evolution equations, as not all information needed to determine the full 1+1+2 equations is contained in R_{abc}.

Vorticity evolution equation:

$$\dot{\Omega} = \frac{1}{2}\varepsilon_{ab}\delta^{a}\mathcal{A}^{b} + \mathcal{A}\,\xi + \Omega\left(\Sigma - \frac{2}{3}\theta\right) + \Omega_{a}\left(\Sigma^{a} + \alpha^{a}\right)\ ; \tag{6.57}$$

Shear evolution:

$$\begin{aligned}\dot{\Sigma}_{\{ab\}} &= \delta_{\{a}\mathcal{A}_{b\}} + \mathcal{A}_{\{a}\,\mathcal{A}_{b\}} - \Sigma_{\{a}\left[\Sigma_{b\}} + 2\,\alpha_{b\}}\right] - \Omega_{\{a}\,\Omega_{b\}} + \mathcal{A}\,\zeta_{ab} \\ &\quad - \left(\frac{2}{3}\theta + \frac{1}{2}\Sigma\right)\Sigma_{ab} - \Sigma_{c\{a}\,\Sigma_{b\}}{}^{c} - \mathcal{E}_{ab} + \frac{1}{2}\Pi_{ab}\ .\end{aligned} \tag{6.58}$$

6.4.2 Mixture of propagation and evolution

$u^a\, e^b\, R_{ab\bar{c}} = e^a\, u^b\, R_{ab\bar{c}}$:

$$\begin{aligned}\hat{\alpha}_{\bar{a}} - \dot{a}_{\bar{a}} &= -\left(\frac{1}{2}\phi + \mathcal{A}\right)\alpha_{a} - \xi\,\varepsilon_{ab}\alpha^{b} + \left(\frac{1}{3}\theta + \Sigma\right)\left(\mathcal{A}_{a} - a_{a}\right) + \left(\frac{1}{2}\phi - \mathcal{A}\right)\left(\Sigma_{a} + \varepsilon_{ab}\Omega^{b}\right) \\ &\quad - \xi\left(\varepsilon_{ab}\Sigma^{b} - \Omega_{a}\right) + \zeta_{ab}\left(-\alpha^{b} + \Sigma^{b} + \varepsilon^{bc}\Omega_{c}\right) + \frac{1}{2}Q_{a} - \varepsilon_{ab}\mathcal{H}^{b}\ ;\end{aligned} \tag{6.59}$$

$u^a\, e^b\, u^c R_{abc} = -e^a\, u^b\, u^c\, R_{abc}$:

$$\begin{aligned}\hat{\mathcal{A}} - \frac{1}{3}\dot{\theta} - \dot{\Sigma} &= -\mathcal{A}^{2} + \left(\frac{1}{3}\theta + \Sigma\right)^{2} - 2\,\alpha_{a}\,\Sigma^{a} + \Sigma_{a}\Sigma^{a} - \Omega_{a}\,\Omega^{a} - a_{a}\,\mathcal{A}^{a} \\ &\quad + \varepsilon_{ab}\alpha^{a}\,\Omega^{b} + \frac{1}{6}\left(\mu + 3p\right) + \mathcal{E} - \frac{1}{2}\Pi\ ;\end{aligned} \tag{6.60}$$

Raychaudhuri equation:

$$\begin{aligned} \hat{\mathcal{A}} - \dot{\theta} &= -\,\delta_a \mathcal{A}^a - (\mathcal{A} + \phi)\,\mathcal{A} + (a_a - \mathcal{A}_a)\,\mathcal{A}^a + \frac{1}{3}\theta^2 + \frac{3}{2}\Sigma^2 - 2\,\Omega^2 \\ &\quad + 2\,\Sigma_a\,\Sigma^a - 2\,\Omega_a\,\Omega^a + \Sigma_{ab}\,\Sigma^{ab} + \frac{1}{2}\,(\mu + 3p)\ \ ; \end{aligned} \tag{6.61}$$

Vorticity evolution:

$$\begin{aligned} \dot{\Omega}_{\bar{a}} + \frac{1}{2}\varepsilon_{ab}\hat{\mathcal{A}}^b &= -\left(\frac{2}{3}\theta + \frac{1}{2}\Sigma\right)\Omega_a + \frac{1}{2}\varepsilon_{ab}\left(\delta^b \mathcal{A} - \mathcal{A}\,a^b - \frac{1}{2}\phi\mathcal{A}^b\right) \\ &\quad + \Omega\,(\Sigma_a - \alpha_a) + \frac{1}{2}\xi\,\mathcal{A}_a - \frac{1}{2}\varepsilon_{ab}\zeta^{bc}\,\mathcal{A}_c + \Sigma_{ab}\,\Omega^b\ . \end{aligned} \tag{6.62}$$

Shear evolution:

$$\begin{aligned} \dot{\Sigma} - \frac{2}{3}\hat{\mathcal{A}} &= \frac{1}{3}\,(2\mathcal{A} - \phi)\,\mathcal{A} - \left(\frac{2}{3}\theta + \frac{1}{2}\Sigma\right)\Sigma - \frac{2}{3}\,\Omega^2 + \Sigma_a\left(2\,\alpha^a - \frac{1}{3}\Sigma^a\right) \\ &\quad - \frac{1}{3}\delta_a\mathcal{A}^a - \frac{1}{3}\mathcal{A}_a\,(2\,a^a - \mathcal{A}^a) + \frac{1}{3}\Omega_a\,\Omega^a + \frac{1}{3}\Sigma_{ab}\,\Sigma^{ab} - \mathcal{E} + \frac{1}{2}\Pi\ , \end{aligned} \tag{6.63}$$

$$\begin{aligned} \dot{\Sigma}_{\bar{a}} - \frac{1}{2}\hat{\mathcal{A}}_{\bar{a}} &= \frac{1}{2}\delta_a\mathcal{A} + \left(\mathcal{A} - \frac{1}{4}\phi\right)\mathcal{A}_a - \left(\frac{2}{3}\theta + \frac{1}{2}\Sigma\right)\Sigma_a + \frac{1}{2}\mathcal{A}\,a_a - \frac{3}{2}\Sigma\,\alpha_a \\ &\quad - \Omega\,\Omega_a - \frac{1}{2}\,(\xi\,\varepsilon_{ab} + \zeta_{ab})\,\mathcal{A}^b + \Sigma_{ab}\left(\alpha^b - \Sigma^b\right) - \mathcal{E}_a + \frac{1}{2}\Pi_a\ . \end{aligned} \tag{6.64}$$

Energy conservation:

$$\begin{aligned} \dot{\mu} + \hat{Q} &= -\,\theta\,(\mu + p) - (\phi + 2\mathcal{A})\,Q - \frac{3}{2}\Sigma\,\Pi + (a_a - 2\mathcal{A}_a)\,Q^a \\ &\quad - \delta_a Q^a - 2\,\Sigma_a\,\Pi^a - \Sigma_{ab}\,\Pi^{ab}\ ; \end{aligned} \tag{6.65}$$

Momentum conservation:

$$\begin{aligned} \dot{Q} + \hat{p} + \hat{\Pi} &= -\delta_a\Pi^a - \left(\frac{3}{2}\phi + \mathcal{A}\right)\Pi - \left(\frac{4}{3}\theta + \Sigma\right)Q - (\mu + p)\,\mathcal{A} \\ &\quad + \left(\alpha_a - \Sigma_a + \varepsilon_{ab}\Omega^b\right)Q^a + (2\,a_a - \mathcal{A}_a)\,\Pi^a + \zeta_{ab}\,\Pi^{ab}\ , \end{aligned} \tag{6.66}$$

$$
\begin{aligned}
\dot{Q}_{\bar{a}} + \hat{\Pi}_{\bar{a}} &= -\delta_a p + \frac{1}{2}\delta_a \Pi - \delta^b \Pi_{ab} - Q\left(\alpha_a + \Sigma_a + \varepsilon_{ab}\Omega^b\right) - \frac{3}{2}\Pi\, a_a \\
&- \left(\frac{4}{3}\theta - \frac{1}{2}\Sigma\right) Q_a + \Omega\, \varepsilon_{ab} Q^b - \left(\frac{3}{2}\phi + \mathcal{A}\right)\Pi_a + \xi\, \varepsilon_{ab}\Pi^b \\
&- \left(\mu + p - \frac{1}{2}\Pi\right)\mathcal{A}_a - \Sigma_{ab}\, Q^b - \zeta_{ab}\Pi^b + \Pi_{ab}\left(a^b - \mathcal{A}^b\right)\ ; \qquad (6.67)
\end{aligned}
$$

Electric Weyl evolution:

$$
\begin{aligned}
\dot{\mathcal{E}} + \frac{1}{2}\dot{\Pi} + \frac{1}{3}\hat{Q} &= \varepsilon_{ab}\delta^a \mathcal{H}^c + \frac{1}{6}\delta_a Q^a + \left(\frac{3}{2}\Sigma - \theta\right)\mathcal{E} - \frac{1}{2}\left(\frac{1}{3}\theta + \frac{1}{2}\Sigma\right)\Pi \\
&+ \frac{1}{3}\left(\frac{1}{2}\phi - 2\mathcal{A}\right) Q + 3\,\xi\,\mathcal{H} - \frac{1}{2}(\mu + p)\,\Sigma + \frac{1}{3}\left(a_a + \mathcal{A}_a\right) Q^a \\
&+ \left(2\,\alpha_a + \Sigma_a - \varepsilon_{ab}\Omega^b\right)\mathcal{E}^a + \left(\alpha_a - \frac{1}{6}\Sigma_a - \frac{1}{2}\varepsilon_{ab}\Omega^b\right)\Pi^a \\
&+ 2\varepsilon_{ab}\mathcal{A}^a\,\mathcal{H}^c - \Sigma_{ab}\left(\mathcal{E}^{ab} + \frac{1}{2}\Pi^{ab}\right) + \varepsilon_{ab}\mathcal{H}^{bc}\,\zeta^{a}{}_{c}\ , \qquad (6.68)
\end{aligned}
$$

$$
\begin{aligned}
\dot{\mathcal{E}}_{\bar{a}} + \frac{1}{2}\varepsilon_{ab}\hat{\mathcal{H}}^b + \frac{1}{2}\dot{\Pi}_{\bar{a}} + \frac{1}{4}\hat{Q}_{\bar{a}} &= \frac{3}{4}\varepsilon_{ab}\delta^b\mathcal{H} + \frac{1}{2}\varepsilon_{bc}\delta^b\mathcal{H}^c{}_a - \frac{1}{4}\delta_a Q + \frac{3}{4}\left(\mathcal{E} + \frac{1}{2}\Pi\right)\varepsilon_{ab}\Omega^b \\
&- \frac{1}{2}\left(\mu + p - \frac{3}{2}\mathcal{E} + \frac{1}{4}\Pi\right)\Sigma_a - \frac{1}{2}Q\,\mathcal{A}_a + \frac{3}{2}\mathcal{H}\,\varepsilon_{ab}\mathcal{A}^b \\
&- \frac{3}{2}\left(\mathcal{E} + \frac{1}{2}\Pi\right)\alpha_a - \frac{1}{4}Q\, a_a - \frac{3}{4}\mathcal{H}\,\varepsilon_{ab}a^b - \frac{1}{2}\Omega\,\varepsilon_{ab}\mathcal{E}^b \\
&+ \left(\frac{3}{4}\Sigma - \theta\right)\mathcal{E}_a + \frac{5}{2}\xi\,\mathcal{H}_a - \left(\frac{1}{4}\phi + \mathcal{A}\right)\varepsilon_{ab}\mathcal{H}^b + \frac{1}{4}\xi\,\varepsilon_{ab}Q^b \\
&+ \frac{1}{2}\left(\frac{1}{4}\phi - \mathcal{A}\right) Q_a - \frac{1}{2}\left(\frac{1}{3}\theta + \frac{1}{4}\Sigma\right)\Pi_a - \frac{1}{4}\Omega\,\varepsilon_{ab}\Pi^b \\
&+ \frac{1}{2}\Sigma_{ab}\left(3\mathcal{E}^b - \frac{1}{2}\Pi^b\right) + \frac{1}{2}\left(3\mathcal{E}_{ab} - \frac{1}{2}\Pi_{ab}\right)\Sigma^b - \mathcal{H}_{ab}\,\varepsilon^{bc}\mathcal{A}_c \\
&- \left(\mathcal{E}_{ab} + \frac{1}{2}\Pi_{ab}\right)\left(\alpha^b + \frac{1}{2}\varepsilon^{bc}\Omega_c\right) + \frac{1}{2}\zeta_{ab}\left(\varepsilon^{bc}\mathcal{H}_c + Q^b\right)\ ,
\end{aligned}
$$
(6.69)

$$
\begin{aligned}
\dot{\mathcal{E}}_{\{ab\}} - \varepsilon_{c\{a}\hat{\mathcal{H}}_{b\}}{}^{c} + \frac{1}{2}\dot{\Pi}_{\{ab\}} \;=\; & -\varepsilon_{c\{a}\delta^{c}\mathcal{H}_{b\}} - \frac{1}{2}\delta_{\{a}Q_{b\}} - \frac{1}{2}\left(\mu + p + 3\mathcal{E} - \frac{1}{2}\Pi\right)\Sigma_{ab} \\
& - \frac{1}{2}Q\,\zeta_{ab} - \frac{3}{2}\mathcal{H}\,\varepsilon_{c\{a}\zeta_{b\}}{}^{c} - \left(\theta + \frac{3}{2}\Sigma\right)\mathcal{E}_{ab} + \Omega\,\varepsilon_{c\{a}\mathcal{E}_{b\}}{}^{c} \\
& - \left(\frac{1}{6}\theta - \frac{1}{4}\Sigma\right)\Pi_{ab} + \frac{1}{2}\Omega\,\varepsilon_{c\{a}\Pi_{b\}}{}^{c} + \xi\,\mathcal{H}_{ab} \\
& + \left(\frac{1}{2}\phi + 2\mathcal{A}\right)\varepsilon_{c\{a}\mathcal{H}_{b\}}{}^{c} - \mathcal{A}_{\{a}Q_{b\}} + 2\varepsilon_{c\{a}\mathcal{H}_{b\}}\left(a^{c} - \mathcal{A}^{c}\right) \\
& - \left(\alpha_{\{a} + \frac{1}{2}\varepsilon_{c\{a}\Omega^{c}\right)\left(2\mathcal{E}_{b\}} + \Pi_{b\}}\right) + \Sigma_{\{a}\left(3\mathcal{E}_{b\}} - \frac{1}{2}\Pi_{b\}}\right) \\
& + \Sigma_{c\{a}\left(3\mathcal{E}_{b\}}{}^{c} - \frac{1}{2}\Pi_{b\}}{}^{c}\right) + \varepsilon_{c\{a}\mathcal{H}_{b\}d}\zeta^{cd}\;; \qquad (6.70)
\end{aligned}
$$

Magnetic Weyl evolution:

$$
\begin{aligned}
\dot{\mathcal{H}} \;=\; & -\varepsilon_{ab}\delta^{a}\mathcal{E}^{b} + \frac{1}{2}\varepsilon_{ab}\delta^{a}\Pi^{b} - 3\xi\,\mathcal{E} + \left(\theta + \frac{3}{2}\Sigma\right)\mathcal{H} + \Omega\,Q + \frac{3}{2}\xi\,\Pi - 2\varepsilon_{ab}\mathcal{A}^{a}\,\mathcal{E}^{b} \\
& + \left(2\,\alpha_{a} + \Sigma_{a} - \varepsilon_{ab}\Omega^{b}\right)\mathcal{H}^{a} - \frac{1}{2}\left(\Omega_{a} + \varepsilon_{ab}\Sigma^{b}\right)Q^{a} - \Sigma_{ab}\,\mathcal{H}^{ab} - \frac{1}{2}\varepsilon_{ab}\mathcal{E}^{bc}\,\zeta^{a}{}_{c}\;, \qquad (6.71)
\end{aligned}
$$

$$
\begin{aligned}
\dot{\mathcal{H}}_{\bar{a}} - \frac{1}{2}\varepsilon_{ab}\hat{\mathcal{E}}^{b} + \frac{1}{4}\varepsilon_{ab}\hat{\Pi}^{b} \;=\; & -\frac{3}{4}\varepsilon_{ab}\delta^{b}\mathcal{E} + \frac{3}{8}\varepsilon_{ab}\delta^{b}\Pi - \frac{1}{2}\varepsilon_{bc}\delta^{b}\mathcal{E}^{c}{}_{a} + \frac{1}{4}\varepsilon_{bc}\delta^{b}\Pi^{c}{}_{a} + \frac{3}{4}\mathcal{H}\,\Sigma_{a} \\
& + \frac{1}{4}Q\,\varepsilon_{ab}\Sigma^{b} + \frac{3}{4}Q\,\Omega_{a} + \frac{3}{4}\mathcal{H}\,\varepsilon_{ab}\Omega^{b} - \frac{3}{2}\mathcal{E}\,\varepsilon_{ab}\mathcal{A}^{b} - \frac{3}{2}\mathcal{H}\,\alpha_{a} \\
& + \frac{3}{4}\left(\mathcal{E} - \frac{1}{2}\Pi\right)\varepsilon_{ab}a^{b} - \frac{5}{2}\xi\mathcal{E}_{a} + \left(\frac{1}{4}\phi + \mathcal{A}\right)\varepsilon_{ab}\mathcal{E}^{b} - \frac{1}{8}\phi\,\varepsilon_{ab}\Pi^{b} \\
& + \left(\frac{3}{4}\Sigma - \theta\right)\mathcal{H}_{a} - \frac{1}{2}\Omega\varepsilon_{ab}\mathcal{H}^{b} + \frac{3}{4}\Omega\,Q_{a} - \frac{3}{8}\Sigma\,\varepsilon_{ab}Q^{b} + \frac{5}{4}\xi\,\Pi_{a} \\
& + \Sigma_{ab}\left(\frac{3}{2}\mathcal{H}^{b} + \frac{1}{4}\varepsilon^{bc}Q_{c}\right) + \frac{3}{2}\,\varepsilon_{ab}\zeta^{bc}\left(\mathcal{E}_{c} - \frac{1}{2}\Pi_{c} + \frac{2}{3}\mathcal{A}_{c}\right) \\
& + \mathcal{H}_{ab}\left(\alpha^{b} + \frac{3}{2}\Sigma^{b} - \frac{1}{2}\varepsilon^{bc}\Omega_{c}\right)\;, \qquad (6.72)
\end{aligned}
$$

$$
\begin{aligned}
\dot{\mathcal{H}}_{\{ab\}} + \varepsilon_{c\{a}\hat{\mathcal{E}}_{b\}}{}^{c} - \frac{1}{2}\varepsilon_{c\{a}\hat{\Pi}_{b\}}{}^{c} \quad = \quad & \varepsilon_{c\{a}\delta^{c}\mathcal{E}_{b\}} - \ 12\varepsilon_{c\{a}\delta^{c}\Pi_{b\}} - \frac{3}{2}\mathcal{H}\,\Sigma_{ab} + \frac{1}{2}Q\,\varepsilon_{c\{a}\Sigma_{b\}}{}^{c} \\
& + \frac{3}{2}\left(\mathcal{E} - \frac{1}{2}\Pi\right)\varepsilon_{c\{a}\zeta_{b\}}{}^{c} - \xi\mathcal{E}_{ab} - \left(\theta + \frac{3}{2}\Sigma\right)\mathcal{H}_{ab} \\
& - \left(\frac{1}{2}\phi + 2\mathcal{A}\right)\varepsilon_{c\{a}\mathcal{E}_{b\}}{}^{c} - \Omega\,\varepsilon_{c\{a}\mathcal{H}_{b\}}{}^{c} + \frac{1}{2}\xi\,\Pi_{ab} \\
& + \frac{1}{4}\phi\,\varepsilon_{c\{a}\Pi_{b\}}{}^{c} + \Sigma_{\{a}\left(3\mathcal{H}_{b\}} - \varepsilon_{b\}c}Q^{c}\right) - 2\,\alpha_{\{a}\mathcal{H}_{b\}} \\
& + \Omega_{\{a}\left(\frac{3}{2}Q_{b\}} - \varepsilon_{b\}c}H^{c}\right) + \mathcal{E}_{\{a}2\varepsilon_{b\}c}\left(a^{c} + \mathcal{A}^{c}\right) \\
& - \Pi_{\{a}\varepsilon_{b\}c}a^{c} + 3\Sigma_{c\{a}\mathcal{H}_{b\}}{}^{c} - \varepsilon_{c\{a}\zeta^{cd}\left(\mathcal{E}_{b\}d} - \frac{1}{2}\Pi_{b\}d}\right) \ .
\end{aligned} \tag{6.73}
$$

6.4.3 Propagation equations

Propagation and constraint equations are formed from either projecting R_{abc} as shown in this subsection, or from projections of the 1+3 constraint equations in Section 4.6.

$e^{a}\,N^{bc}\,R_{abc}$:

$$
\begin{aligned}
\hat{\phi} \quad = \quad & -\frac{1}{2}\phi^{2} + 2\xi^{2} + \left(\frac{1}{3}\theta + \Sigma\right)\left(\frac{2}{3}\theta - \Sigma\right) + \delta_{a}a^{a} - a_{a}\,a^{a} \\
& - \zeta_{ab}\,\zeta^{ab} + 2\varepsilon_{ab}\alpha^{a}\,\Omega^{b} - \Sigma_{a}\,\Sigma^{a} + \Omega_{a}\,\Omega^{a} - \frac{2}{3}\mu - \frac{1}{2}\Pi - \mathcal{E} \ ;
\end{aligned} \tag{6.74}
$$

$e^{a}\,\varepsilon^{bc}R_{abc}$:

$$
\hat{\xi} \quad = \quad -\phi\xi + \left(\frac{1}{3}\theta + \Sigma\right)\Omega + \frac{1}{2}\varepsilon_{ab}\delta^{a}a^{b} + \frac{1}{2}\varepsilon_{ab}\Sigma^{a}\,a^{b} + \left(\frac{1}{2}a_{a} + \alpha_{a}\right)\Omega^{a} \ ; \tag{6.75}
$$

$e^{a}\,R_{a\{bc\}}$:

$$
\begin{aligned}
\hat{\zeta}_{\{ab\}} \quad = \quad & -\phi\,\zeta_{ab} - \zeta^{c}{}_{\{a}\zeta_{b\}c} + \delta_{\{a}a_{b\}} - a_{\{a}\,a_{b\}} + 2\alpha_{\{a}\,\varepsilon_{b\}c}\Omega^{c} - \Omega_{\{a}\,\Omega_{b\}} \\
& - \Sigma_{\{a}\,\Sigma_{b\}} + \left(\frac{1}{3}\theta + \Sigma\right)\Sigma_{ab} - \frac{1}{2}\Pi_{ab} - \mathcal{E}_{ab} \ ;
\end{aligned} \tag{6.76}
$$

Shear divergence $(C_{1})^{a}\,e_{a}$:

$$
\hat{\Sigma} - \frac{2}{3}\hat{\theta} \quad = \quad -\frac{3}{2}\phi\,\Sigma - 2\xi\,\Omega - \delta_{a}\Sigma^{a} - \varepsilon_{ab}\delta^{a}\Omega^{b} + 2\Sigma_{a}\,a^{a} - 2\varepsilon_{ab}\mathcal{A}^{a}\,\Omega^{b} + \Sigma_{ab}\,\zeta^{ab} - Q \ , \tag{6.77}
$$

$(C_1)_{\bar{a}}$:

$$\begin{aligned}
\hat{\Sigma}_{\bar{a}} - \varepsilon_{ab}\hat{\Omega}^b \;=\;& \frac{1}{2}\delta_a\Sigma + \frac{2}{3}\delta_a\theta - \varepsilon_{ab}\delta^b\Omega - \frac{3}{2}\phi\Sigma_a + \xi\,\varepsilon_{ab}\Sigma^b - \xi\,\Omega_a - \frac{3}{2}\Sigma\,a_a \\
&+ \left(\frac{1}{2}\phi + 2\mathcal{A}\right)\varepsilon_{ab}\Omega^b + \Omega\,\varepsilon_{ab}\left(a^b - 2\mathcal{A}^b\right) - \delta^b\Sigma_{ab} - \zeta_{ab}\,\Sigma^b \\
&+ \Sigma_{ab}\,a^b + \varepsilon_{ab}\zeta^{bc}\,\Omega_c - Q_a\;; \qquad (6.78)
\end{aligned}$$

Vorticity divergence equation (C_2):

$$\hat{\Omega} \;=\; -\delta_a\Omega^a + (\mathcal{A} - \phi)\,\Omega + (a_a + \mathcal{A}_a)\,\Omega^a\;, \qquad (6.79)$$

$(C_3)_{\{ab\}}$:

$$\begin{aligned}
\hat{\Sigma}_{\{ab\}} \;=\;& \delta_{\{a}\Sigma_{b\}} - \varepsilon_{c\{a}\delta^c\Omega_{b\}} - \frac{1}{2}\phi\,\Sigma_{ab} + \xi\,\varepsilon_{c\{a}\Sigma_{b\}}{}^c + \frac{3}{2}\Sigma\,\zeta_{ab} - \Omega\,\varepsilon_{c\{a}\zeta_{b\}}{}^c \\
&- 2\Sigma_{\{a}\,a_{b\}} - 2\varepsilon_{c\{a}\mathcal{A}^c\,\Omega_{b\}} - \Sigma_{c\{a}\,\zeta_{b\}}{}^c - \varepsilon_{c\{a}\mathcal{H}_{b\}}{}^c\;; \qquad (6.80)
\end{aligned}$$

Electric Weyl Divergence $(C_4)^a\,e_a$:

$$\begin{aligned}
\hat{\mathcal{E}} - \frac{1}{3}\hat{\mu} + \frac{1}{2}\hat{\Pi} \;=\;& -\delta_a\mathcal{E}^a - \frac{1}{2}\delta_a\Pi^a - \frac{3}{2}\phi\left(\mathcal{E} + \frac{1}{2}\Pi\right) + \left(\frac{1}{2}\Sigma - \frac{1}{3}\theta\right)Q + 3\,\Omega\,\mathcal{H} \\
&+ (2\mathcal{E}_a + \Pi_a)\,a^a + \frac{1}{2}\Sigma_a\,Q^a + 3\Omega_a\,\mathcal{H}^a - \frac{3}{2}\varepsilon_{ab}\Omega^a\,Q^b + \varepsilon_{ab}\Sigma^{ac}\,\mathcal{H}_c{}^b \\
&+ \left(\mathcal{E}_{ab} + \frac{1}{2}\Pi_{ab}\right)\zeta^{ab}\;, \qquad (6.81)
\end{aligned}$$

$(C_4)_{\bar{a}}$:

$$\begin{aligned}
\hat{\mathcal{E}}_{\bar{a}} + \frac{1}{2}\hat{\Pi}_{\bar{a}} \;=\;& \frac{1}{2}\delta_a\mathcal{E} + \frac{1}{3}\delta_a\mu + \frac{1}{4}\delta_a\Pi - \delta^b\mathcal{E}_{ab} - \frac{1}{2}\delta^b\Pi_{ab} + \frac{1}{2}Q\,\Sigma_a + \mathcal{H}\,\varepsilon_{ab}\Sigma^b \\
&- \frac{3}{2}\mathcal{H}\,\Omega_a - \frac{3}{2}Q\,\varepsilon_{ab}\Omega^b - \frac{3}{2}\left(\mathcal{E} + \frac{1}{2}\Pi\right)a_a - \frac{3}{2}\phi\,\left(\mathcal{E}_a + \frac{1}{2}\Pi_a\right) + \frac{3}{2}\Omega\,\varepsilon_{ab}Q^b \\
&+ \xi\,\varepsilon_{ab}\left(\mathcal{E}^b + \frac{1}{2}\Pi^b\right) + 3\Omega\,\mathcal{H}_a - \Sigma\,\varepsilon_{ab}\mathcal{H}^b - \left(\frac{1}{3}\theta + \frac{1}{4}\Sigma\right)Q_a + \frac{1}{2}\Sigma_{ab}\,Q^b \\
&- \zeta_{ab}\left(\mathcal{E}^b + \frac{1}{2}\Pi^b\right) + \left(\mathcal{E}_{ab} + \frac{1}{2}\Pi_{ab}\right)a^b + 3\mathcal{H}_{ab}\,\Omega^b\;; \qquad (6.82)
\end{aligned}$$

Magnetic Weyl divergence $(C_5)^a\, e_a$:

$$\begin{aligned}\hat{\mathcal{H}} &= -\delta_a\mathcal{H}^a - \frac{1}{2}\varepsilon_{ab}\delta^a Q^b - \frac{3}{2}\phi\,\mathcal{H} - \left(3\mathcal{E} + \mu + p - \frac{1}{2}\Pi\right)\Omega - Q\,\xi \\ &\quad + 2\,\mathcal{H}_a\,a^a - 3\Omega_a\left(\mathcal{E}^a - \frac{1}{6}\Pi^a\right) + \zeta_{ab}\,\mathcal{H}^{ab} - \varepsilon_{ab}\Sigma^a{}_c\left(\mathcal{E}^{bc} + \frac{1}{2}\Pi^{bc}\right)\ ,\end{aligned} \tag{6.83}$$

$(C_5)_{\bar{a}}$:

$$\begin{aligned}\hat{\mathcal{H}}_{\bar{a}} - \frac{1}{2}\varepsilon_{ab}\hat{Q}^b &= \frac{1}{2}\delta_a\mathcal{H} - \delta^b\mathcal{H}_{ab} - \frac{1}{2}\varepsilon_{ab}\delta^b Q - \frac{3}{2}\left(\mathcal{E} + \frac{1}{2}\Pi\right)\varepsilon_{ab}\Sigma^b - \frac{3}{2}\phi\,\mathcal{H}_a \\ &\quad - \left(-\frac{3}{2}\mathcal{E} + \mu + p + \frac{1}{4}\Pi\right)\Omega_a - \frac{3}{2}\mathcal{H}\,a_a + \frac{1}{2}Q\,\varepsilon_{ab}a^b - 3\Omega\,\mathcal{E}_a \\ &\quad + \frac{3}{2}\Sigma\,\varepsilon_{ab}\mathcal{E}^b + \xi\,\varepsilon_{ab}\mathcal{H}^b - \frac{1}{2}\xi Q_a + \frac{1}{4}\phi\varepsilon_{ab}Q^b + \frac{1}{2}\Omega\,\Pi_a + \frac{3}{4}\Sigma\,\varepsilon_{ab}\Pi^b \\ &\quad + \mathcal{H}_{ab}\,a^b - \zeta_{ab}\,\mathcal{H}^b - 3\left(\mathcal{E}_{ab} - \frac{1}{6}\Pi_{ab}\right)\Omega^b + \frac{1}{2}\varepsilon_{ab}\zeta^{bc}\,Q_c\ .\end{aligned} \tag{6.84}$$

6.4.4 Constraints

$\varepsilon^{ab}u^c\,R_{abc}$:

$$\delta_a\Omega^a + \varepsilon_{ab}\delta^a\Sigma^b = (2\mathcal{A} - \phi)\,\Omega - 3\xi\,\Sigma + \varepsilon_{ab}\zeta^{ac}\Sigma^b{}_c + \mathcal{H}\ ; \tag{6.85}$$

$N^{bc}\,R_{\bar{a}bc}$:

$$\begin{aligned}\frac{1}{2}\delta_a\phi - \varepsilon_{ab}\delta^b\xi - \delta^b\zeta_{ab} &= -\Omega\left(\Omega_a + \varepsilon_{ab}\Sigma^b - 2\varepsilon_{ab}\alpha^b\right) - \left(\frac{1}{3}\theta - \frac{1}{2}\Sigma\right)\left(\Sigma_a - \varepsilon_{ab}\Omega^b\right) \\ &\quad -2\xi\,\varepsilon_{ab}a^b - \left(\Sigma^b - \varepsilon^{bc}\Omega_c\right)\Sigma_{ab} - \frac{1}{2}\Pi_a - \mathcal{E}_a\ ;\end{aligned} \tag{6.86}$$

From $(C_3)_{ab}\,e^b$ and $(C_1)_{\bar{a}}$, or $e^a\,u^c\,R_{a\bar{b}c}$

$$\begin{aligned}\delta_a\Sigma - \frac{2}{3}\delta_a\theta + 2\varepsilon_{ab}\delta^b\Omega + 2\delta^b\Sigma_{ab} &= -\phi\left(\Sigma_a - \varepsilon_{ab}\Omega^b\right) - 2\xi\left(\Omega_a - 3\varepsilon_{ab}\Sigma^b\right) - 4\Omega\,\varepsilon_{ab}\mathcal{A}^b \\ &\quad +2\zeta_{ab}\Sigma^b + 2\varepsilon_{ab}\zeta^{bc}\,\Omega_c + \Sigma_{ab}\,a^b - 2\varepsilon_{ab}\mathcal{H}^b - Q_a\ .\end{aligned} \tag{6.87}$$

Finally, we note that equations (6.86) and (6.87) are not real constraints due to the curvature thermodynamic terms that have spatial and temporal derivatives of the curvature. Furthermore, the equation formed from $(C_3)_{ab}\,e^a\,e^b$ is equivalent to (6.79) and (6.85).

We also draw attention to equation (6.60), which can be written in terms of (6.61) and (6.63) [(6.60)= $\frac{1}{3}$(7.11)−(6.63)]. The redundancy in the field equations is due to the fact that some of the information contained in R_{abc} is already contained in the 1+3 equations. We also note that there are no evolution equations for $\mathcal{A}$, $\mathcal{A}_a$, α_a, and there is no propagation equation for a_a; these must all be determined by specifying a choice of frame.

Chapter 7

Spherically symmetric spacetimes and the Jebsen-Birkhoff theorem in $f(R)$ gravity

It was recently shown in [79, 80], that in GR, the rigidity of spherical vacuum solutions of Einstein's field equations continues even in the perturbed scenario: almost spherical symmetry and/or almost vacuum implies almost static or almost spatially homogeneous. This is an important reason for the stability of the solar system and of black hole spacetimes and is interesting from the point of view that the universe expands globally though it is made up of locally spherically symmetric objects imbedded in vacuum regions whose local spacetime domains is required to be static by Jebsen-Birkhoff's theorem. A similar study of local stability is required for the spherically symmetric solutions in modified gravity theories, to see if these theories are physically viable.

In this chapter, we prove a Jebsen-Birkhoff-like theorem for $f(R)$ theories of gravity, to find the necessary conditions required for the existence of Schwarzschild solution in these theories. We discuss under what circumstances we can covariantly set up a scale in the problem. We then perturb the vacuum spacetime with respect to this covariant scale to find the stability of the theorem. We do this in two steps: (a) First we keep the spherical symmetry and perturb the Ricci scalar around $R = 0$ to find the necessary conditions on the spatial and temporal derivatives of the Ricci scalar for the spacetime to be almost Schwarzschild. (b) We then define the notion of *almost spherical symmetry* with respect to the covariant scale and perturb the spherical symmetry to prove the stability of the theorem.

7.1 1+1+2 equations for LRS-II spacetimes

Locally Rotationally Symmetric (LRS) spacetimes posses a continuous isotropy group at each point and hence a multi-transitive isometry group acting on the spacetime manifold [153]. These spacetimes exhibit locally (at each point) a unique preferred spatial direction, covariantly defined, for example, by either vorticity vector field or a non-vanishing non-gravitational acceleration of the matter fluids. The 1+1+2 formalism is therefore ideally suited for covariant description of these spacetimes, yielding a complete deviation in terms of invariant scalar quantities that have physical or direct geometrical meaning [66]. The preferred spatial direction in the LRS spacetimes constitutes a local axis of symmetry and in this case e^a is just a vector pointing along the axis of symmetry and is thus called a 'radial' vector. Since LRS spacetimes are defined to be isotropic, this allows for the vanishing of *all* 1+1+2 vectors and tensors, such that there are no preferred directions in the sheet. Thus, all the non-zero 1+1+2 variables are covariantly defined scalars. The variables that fully describe LRS spacetimes are

$$\textbf{LRS}: \qquad \left\{R,\, \mathcal{A},\, \Theta,\, \phi,\, \xi,\, \Sigma,\, \Omega,\, \mathcal{E},\, \mathcal{H},\, \mu^M,\, p^M,\, Q^M,\, \Pi^M\right\}\ , \tag{7.1}$$

and are what is solved for in the 1+1+2 approach. A detailed discussion of the covariant approach to LRS perfect fluid space-times can be found in [153].

A subclass of the LRS spacetimes, called LRS-II, contains all the LRS spacetimes that are rotation free. As a consequence, in LRS-II spacetimes the variables Ω, ξ and $\mathcal{H}$ are identically zero and the variables

$$\textbf{LRS class II}: \qquad \left\{R, \mathcal{A},\, \Theta,\, \phi,\, \Sigma,\, \mathcal{E},\, \mu^M,\, p^M,\, Q^M,\, \Pi^M\right\}\ , \tag{7.2}$$

fully characterise the kinematics. The propagation, evolution and constraint equations (as described in the previous chapter) become simplified for these variables and are given by:

Propagation equations:

$$\hat{\phi} = -\frac{1}{2}\phi^2 + \left(\frac{1}{3}\Theta + \Sigma\right)\left(\frac{2}{3}\Theta - \Sigma\right) - \frac{2}{3}\mu - \frac{1}{2}\Pi - \mathcal{E}\ , \tag{7.3}$$

$$\hat{\Sigma} - \frac{2}{3}\hat{\Theta} = -\frac{3}{2}\phi\,\Sigma - Q\ , \tag{7.4}$$

$$\hat{\mathcal{E}} - \frac{1}{3}\hat{\mu} + \frac{1}{2}\hat{\Pi} = -\frac{3}{2}\phi\left(\mathcal{E} + \frac{1}{2}\Pi\right) + \left(\frac{1}{2}\Sigma - \frac{1}{3}\Theta\right)Q\ . \tag{7.5}$$

Evolution equations:

$$\begin{aligned}
\dot{\phi} &= -\left(\Sigma-\frac{2}{3}\Theta\right)\left(\mathcal{A}-\frac{1}{2}\phi\right)+Q\ , && (7.6)\\
\dot{\Sigma}-\frac{2}{3}\dot{\Theta} &= -\mathcal{A}\,\phi+2\left(\frac{1}{3}\Theta-\frac{1}{2}\Sigma\right)^2+\frac{1}{3}\,(\mu+3p)-\mathcal{E}+\frac{1}{2}\Pi\ , && (7.7)\\
\dot{\mathcal{E}}-\frac{1}{3}\dot{\mu}+\frac{1}{2}\dot{\Pi} &= \left(\frac{3}{2}\Sigma-\Theta\right)\mathcal{E}+\frac{1}{4}\left(\Sigma-\frac{2}{3}\Theta\right)\Pi+\frac{1}{2}\phi\,Q-\frac{1}{2}\,(\mu+p)\left(\Sigma-\frac{2}{3}\Theta\right)\ . && (7.8)
\end{aligned}$$

Propagation/Evolution equations:

$$\begin{aligned}
\dot{\mu}+\hat{Q} &= \Theta\,(\mu+p)-(\phi+2\mathcal{A})\,Q-\frac{3}{2}\Sigma\,\Pi\ , && (7.9)\\
\dot{Q}+\hat{p}+\hat{\Pi} &= -\left(\frac{3}{2}\phi+\mathcal{A}\right)\Pi-\left(\frac{4}{3}\Theta+\Sigma\right)Q-(\mu+p)\,\mathcal{A}\ , && (7.10)\\
\hat{\mathcal{A}}-\dot{\Theta} &= -\,(\mathcal{A}+\phi)\,\mathcal{A}+\frac{1}{3}\Theta^2+\frac{3}{2}\Sigma^2+\frac{1}{2}\,(\mu+3p)\ . && (7.11)
\end{aligned}$$

where

$$\mu=\frac{1}{f'}\left[\mu^M+\frac{1}{2}(Rf'-f)-\theta\,f''\,\dot{R}+f'''\,X^2+f''\,\hat{X}+\phi\,f''\,X\right]\ , \tag{7.12}$$

$$\begin{aligned}
p &= \frac{1}{f'}\left[p^M+\frac{1}{2}(f-R\,f')+f'''\,\dot{R}^2+f''\,\ddot{R}-\mathcal{A}\,f''\,X\right.\\
&\quad\left.+\frac{2}{3}\left(\theta\,f''\,\dot{R}-\phi\,f''\,X-f'''\,X^2-f''\,\hat{X}\right)\right]\ ,
\end{aligned} \tag{7.13}$$

$$Q = -\frac{1}{f'}\left[Q^M+f'''\,\dot{R}\,X+f''\left(\dot{X}-\mathcal{A}\,\dot{R}\right)\right]\ , \tag{7.14}$$

$$\Pi = \frac{1}{f'}\left[\Pi^M+\frac{1}{3}\left(2f'''\,X^2+2f''\,\hat{X}-\phi\,f''\,X\right)-\Sigma\,f''\,\dot{R}\right]\ . \tag{7.15}$$

Commutation relation:

$$\hat{\dot{\psi}}-\dot{\hat{\psi}}=-\mathcal{A}\,\dot{\psi}+\left(\frac{1}{3}\Theta+\Sigma\right)\hat{\psi}\ . \tag{7.16}$$

Due to the additional degrees of freedom, equations (7.3)-(7.16) are not closed and we have to add the *curvature trace equation* (which corresponds to the trace of the modified field equations):

$$R\, f' - 2f = 3\left(f''\,\theta\,\dot{R} - f''\,\hat{X} + f''\,\ddot{R} - (\phi+\mathcal{A})\,f''X - f'''\,X^2 + f'''\,\dot{R}^2\right)\ . \tag{7.17}$$

Since the vorticity vanishes, the unit vector field u^a is hypersurface-orthogonal to the spacelike 3-surfaces whose intrinsic curvature can be calculated from the *Gauss equation* for u^a (4.99). With the additional constraint of the vanishing of the sheet distortion ξ, that is, the sheet is a genuine 2-surface, the Gauss equation for e^a together with the 3-Ricci identities determine the 3-Ricci curvature tensor of the spacelike 3-surfaces orthogonal to u^a to be [66]:

$$^3R_{ab} = -\left[\hat{\phi} + \frac{1}{2}\phi^2\right] e_a\, e_b - \left[\frac{1}{2}\hat{\phi} + \frac{1}{2}\phi^2 - K\right] N_{ab}\ . \tag{7.18}$$

This gives the 3-Ricci-scalar as

$$^3R = -\,2\left[\frac{1}{2}\hat{\phi} + \frac{3}{4}\phi^2 - K\right]\ , \tag{7.19}$$

where K is the *Gaussian curvature* of the 2-sheet and is related to the two dimensional Riemann curvature tensor and two dimensional Ricci tensor as

$$^{(2)}R^a{}_{bcd} = K\,(N^a{}_c\,N_{bd} - N^a{}_d\,N_{bc})\ , \quad \Longrightarrow \quad {}^2R_{ab} = K\,N_{ab}\ . \tag{7.20}$$

From (7.19) and (7.3) an expression for K is obtained in the form

$$K = \frac{1}{3}\mu - \mathcal{E} - \frac{1}{2}\Pi + \frac{1}{4}\phi^2 - \left(\frac{1}{3}\Theta - \frac{1}{2}\Sigma\right)^2\ . \tag{7.21}$$

From (7.3)-(7.8), the evolution and propagation equations of K can be determined as

$$\dot{K} \;=\; -\,\frac{2}{3}\left(\frac{2}{3}\Theta - \Sigma\right)K, \tag{7.22}$$

$$\hat{K} \;=\; -\,\phi\,K. \tag{7.23}$$

From equation (7.22), it follows that whenever the Gaussian curvature of the sheet is non-zero and constant in time, then the shear is always proportional to the expansion:

$$K \neq 0 \quad \text{and} \quad \dot{K} = 0 \quad \Longrightarrow \quad \Sigma = \frac{2}{3}\Theta\ . \tag{7.24}$$

7.2 Vacuum LRS II spacetimes

Following [79], we covariantly investigate the geometry of the vacuum ($\mu^M = p^M = Q^M = \Pi^M = 0$) LRS-II spacetime by trying to solve the *Killing equation* for a Killing vector of the form $\xi_a = \Psi\, u_a + \Phi\, e_a$, where Ψ and Φ are scalars. The Killing equation gives

$$\nabla_a(\Psi\, u_b + \Phi\, e_b) + \nabla_b(\Psi\, u_a + \Phi\, e_a) = 0\ . \tag{7.25}$$

Using equations (6.23) and (6.26), and multiplying the Killing equation by $u^a\, u^b$, $u^a\, e^b$, $e^a\, e^b$ and N^{ab} results in the following differential equations and constraints:

$$\dot{\Psi} + \mathcal{A}\,\Phi = 0\ , \tag{7.26}$$

$$\hat{\Psi} - \dot{\Phi} - \Psi\,\mathcal{A} + \Phi\left(\Sigma + \frac{1}{3}\Theta\right) = 0\ , \tag{7.27}$$

$$\hat{\Phi} + \Psi\left(\frac{1}{3}\Theta + \Sigma\right) = 0\ , \tag{7.28}$$

$$\Psi\left(\frac{2}{3}\Theta - \Sigma\right) + \Phi\,\phi = 0\ . \tag{7.29}$$

Considering that $\xi_a\, \xi^a = -\Psi^2 + \Phi^2$, if ξ^a is timelike (that is, $\xi_a\, \xi^a < 0$), then because of the arbitrariness in choosing the vector u^a, we can always make $\Phi = 0$, while if ξ^a is spacelike (that is $\xi_a\, \xi^a > 0$), then we can make $\Psi = 0$.

Let us first assume that ξ^a is timelike and $\Phi = 0$, then (7.26) - (7.29) reduce to

$$\dot{\Psi} = 0\ , \tag{7.30}$$

$$\hat{\Psi} - \Psi\,\mathcal{A} = 0\ , \tag{7.31}$$

$$\Psi\left(\frac{1}{3}\Theta + \Sigma\right) = 0\ , \tag{7.32}$$

$$\Psi\left(\frac{2}{3}\Theta - \Sigma\right) = 0\ . \tag{7.33}$$

Looking at (7.30) and (7.31), we know that their solutions always exists. For a non trivial Ψ, the constraints (7.32) and (7.33) together imply, that in general $\Theta = \Sigma = 0$, that is, the expansion and shear of a unit vector field along the timelike Killing vector vanishes. We also see that the time derivatives of all the quantities in the field equations (7.3)-(7.17) vanish and hence the spacetime is *static*.

Now if ξ^a is spacelike and $\Psi = 0$, then (7.26) - (7.29) reduce to

$$\begin{aligned} \mathcal{A}\,\Phi &= 0\,, & (7.34)\\ -\dot{\Phi} + \Phi(\Sigma + \frac{1}{3}\Theta) &= 0\,, & (7.35)\\ \hat{\Phi} &= 0\,, & (7.36)\\ \Phi\,\phi &= 0\,. & (7.37) \end{aligned}$$

The solution of equations (7.35) and (7.36) always exists and the constraints (7.34) and (7.37) in this case together imply that in general, (for a non trivial Φ), $\phi = \mathcal{A} = 0$.

If we impose further the condition,

$$R = R_0 = const. \quad \text{and} \quad f_0' \neq 0\,,$$

which in turn implies

$$\begin{aligned} \Pi &= 0\,, & (7.38)\\ \mu &= \frac{1}{f_0'}\left[\frac{1}{2}(R_0\,f_0' - f_0)\right]\,, & (7.39)\\ p &= \frac{1}{f_0'}\left[\frac{1}{2}(f_0 - R_0\,f_0')\right]\,, & (7.40)\\ R_0\,f_0' - 2f_0 &= 0\,, & (7.41) \end{aligned}$$

where $f'(R_0) = f_0'$, then all the spatial derivatives of all the quantities in (7.3)-(7.17) vanish. From this we see that *homogeneity* is only achieved if $R = constant$, otherwise inhomogeneity is admitted for non-constant R. This result is unlike that of GR where the spacetime is spatially homogenous upon setting $\phi = \mathcal{A} = 0$ in the list of LRS II equations.

We can now say that :

> *There always exists a Killing vector in the local $[u, e]$ plane for a vacuum LRS-II spacetime in $f(R)$ gravity. If the Killing vector is timelike then the spacetime is locally static. If the Killing vector is spacelike, the spacetime is locally spatially homogeneous if and only if $R = R_0 = const.$ and $f_0' \neq 0$.*

If we apply the conditions $R = R_0 = const.$ and $f_0' \neq 0$, to the system of equations (7.3)-(7.17), then from (7.5), (7.8), (7.22) and (7.23) we get:

$$\mathcal{E} = C\,K^{3/2}\,. \qquad (7.42)$$

That is, the 1+1+2 scalar of the electric part of the Weyl tensor is always proportional to the (3/2)th power of the Gaussian curvature of the 2-sheet. The proportionality constant

C sets up a scale in the problem in this particular case.

7.3 Spherically symmetric spacetimes in higher order gravity

Let us now turn to the case of spherically symmetric spacetimes which belong naturally to LRS class II.

7.3.1 Locally static vacuum spacetimes

As already discussed in the previous section, the condition of staticity implies that the dot derivatives of all the quantities vanish. Furthermore we have $\Theta = \Sigma = 0$, then $\dot{K} = 0$ in (7.22) and from equation (7.6) we have that the heat flux Q vanishes identically in these spacetimes. With this choice, and after a bit of manipulation, the set of 1+1+2 equations which describe the spacetime reduces to the following four coupled first-order equations [73],

$$f' \left[\hat{\phi} + \phi\left(\frac{1}{2}\phi - \mathcal{A}\right)\right] = \frac{1}{3}R\, f' - \frac{2}{3}f + (\phi + 2\mathcal{A})\, f''\, X\ , \tag{7.43}$$

$$f' \left[\hat{\mathcal{A}} + \mathcal{A}(\mathcal{A} + \phi)\right] = \frac{1}{6}f - \frac{1}{3}R\, f' - f''\, X\, \mathcal{A}\ , \tag{7.44}$$

$$\hat{R} = X\ , \tag{7.45}$$

$$f''\, \hat{X} = -\frac{1}{3}R\, f' + \frac{2}{3}f - f'''\, X^2 - (\phi + \mathcal{A})\, f''\, X\ . \tag{7.46}$$

If we then choose coordinates to make the Gaussian curvature 'K' of the spherical sheets proportional to the inverse square of the radius co-ordinate 'r', (such that this coordinate becomes the *area radius* of the sheets), then this geometrically relates the '*hat*' derivative with the radial coordinate 'r'. As we have already seen, $\hat{K} = -\phi K$, where the hat derivative, defined in terms of the derivative with respect to the co-ordinate 'r', depends on the specific choice of e^a (orthogonal to u^a and the sheet). If we choose the 'radial' co-ordinate as the area radius of the spherical sheets, then the most natural way to define the hat derivative of any scalar M would be

$$\hat{M} = \frac{1}{2}\, r\, \phi \frac{dM}{dr}\ , \tag{7.47}$$

for a static spacetime.

From the structure of (7.43)-(7.46) we can already deduce some important results for spherically symmetric static solutions in a general $f(R)$ gravity in an absolutely co-ordinate independent manner. These results are important because they can be used as guidelines to understand the behaviour of any proposed $f(R)$ model in this setting and to design new ones.

7.3.1.1 Necessary condition for existence of solutions with vanishing Ricci scalar.

It is evident from the equations (7.43)-(7.46) above, the function f must be of class C^3 at $R=0$, which implies,

$$|f'(0)|<+\infty\,,\quad |f''(0)|<+\infty\,,\quad |f'''(0)|<+\infty \quad . \tag{7.48}$$

Also, we impose the conditions

$$f(0)=0, \quad R=0 \tag{7.49}$$

Note that the condition of vanishing of the Ricci scalar throughout the manifold automatically implies $X=0$.

Now there are two possibilities:

1. $f'(0)\neq 0$: In this case we see the system reduces to the following:

$$\hat{\phi} = -\frac{1}{2}\phi^2-\mathcal{E}\,, \tag{7.50}$$

$$\hat{\mathcal{E}} = -\frac{3}{2}\phi\mathcal{E}\,, \tag{7.51}$$

$$\hat{\mathcal{A}} = -\mathcal{A}\,(\phi+\mathcal{A})\ , \tag{7.52}$$

together with the constraint:

$$\mathcal{E}+\mathcal{A}\phi=0\ . \tag{7.53}$$

Also, the local Gaussian curvature of the 2-sheets is given as

$$K=-\,\mathcal{E}+\frac{1}{4}\phi^2\ . \tag{7.54}$$

The parametric solutions for these variables (when $K>0$) are

$$\phi=\frac{2}{r}\sqrt{1-\frac{2m}{r}}\quad ,\qquad \mathcal{A}=\frac{m}{r^2}\left[1-\frac{2m}{r}\right]^{-\frac{1}{2}}\quad ,$$
$$\mathcal{E}=\frac{2m}{r^3}\quad ,\qquad K=\frac{1}{r^2}\ , \tag{7.55}$$

where m is the constant of integration. Solving for the metric using the definition of these geometrical quantities we get [66]

$$ds^2=-\left(1-\frac{2m}{r}\right)dt^2+\frac{dr^2}{(1-\frac{2m}{r})}+r^2\,d\Omega^2\ , \tag{7.56}$$

which is the metric of a static Schwarzschild exterior.

It is also interesting to note that the above result is consistent with the conditions $f' > 0$ and $f'' > 0$, which guarantee the attractive nature of the gravitational interaction and the absence of tachyons [27]. This shows that there may be a connection between this solution and the very nature of the gravitational interaction.

The presence of this solution can have interesting consequences on the validity of these models on the Solar System level. In particular if one concludes that the Sun behaves very close to a Schwarzschild solution, the experimental data of the solar system would help constraining these models.

2. $f'(0) = 0$, $f(0) = 0$: In this case (7.43)-(7.46) are identically satisfied for all values of ϕ and $\mathcal{A}$ that guarantees $R = 0$ and hence $X = 0$ [1]. Hence for all models with $f'(0) = 0$, any solution with vanishing Ricci Scalar in GR would be a solution to the above system. This is interesting as it shows that fourth-order gravity in this context can present the same solutions of GR plus additional solutions. For example, the Reissner-Nordström solution which represent the space time outside a spherically symmetric charged body, is a solution to the system (7.43)-(7.46) even if no electric charge is present. Similarly a static spherically symmetric solution for a perfect fluid interior with equation of state $p = (1/3)\mu^M$ (for example Hajj-Boutros solution [161] or the special case of Whittaker solution [162]) can be a solution of this system in the absence of any standard fluid.

The presence of solutions of the type in paragraph (2) shows that when the conditions given in paragraph (1) are not satisfied the Schwarzschild solution is not a unique static spherically symmetric solution. Such results hint towards a violation of the general Jebsen-Birkhoff theorem in its classical form for fourth-order gravity.

7.3.1.2 Necessary condition for existence of solutions with constant scalar curvature

Solutions with constant Ricci scalar are characterised by the fact that $R = R_0 = const.$ and, as consequence, X, $\hat{X} = 0$. Imposing these conditions on (7.43)-(7.46) and supposing

[1]It has been noted by several authors that the situation $f(0) = f'(0) = 0$ is somewhat pathological, since the scalar degree of freedom of this theory, $f'(R)$ corresponds to a Brans-Dicke scalar field in the equivalent Brans-Dicke representation, with Brans-Dicke parameter $\omega = 0$, it also corresponds (apart from a constant) to the inverse effective gravitational coupling of the theory. Therefore, $f' = 0$ corresponds to infinite gravitational coupling $G_{effective} = G/f'$ and to a singularity of the field equations. However, one can formally set $f' \equiv 0$ and look for solutions of the field equations with this constant value of f'. A similar situation has been pointed out to occur in scalar-tensor gravity [156–160]

it to be at least of class C^3 in $R = R_0$ one obtains

$$f_0' \left[\hat{\phi} + \phi\left(\frac{1}{2}\phi - \mathcal{A}\right)\right] = \frac{1}{3}R_0\, f_0' - \frac{2}{3}f_0\ , \tag{7.57}$$

$$f_0' \left[\hat{\mathcal{A}} + \mathcal{A}(\mathcal{A}+\phi)\right] = \frac{1}{6}f_0 - \frac{1}{3}R_0\, f'\ , \tag{7.58}$$

$$-R_0\, f_0' + 2f_0 = 0\ , \tag{7.59}$$

where $f'(R_0) = f_0'$ *etc.* A first solution exists if

$$f_0' \neq 0\ , \quad f_0 \neq 0\ , \quad 2f_0 - R_0\, f_0' = 0\,. \tag{7.60}$$

If we take instead $f_0' \neq 0$, $f_0 = 0$ one obtains again the Schwarzschild solution ($R_0 = 0$). Finally another solution can be achieved if

$$f_0' = 0\ , \quad f_0 = 0\ , \quad R = R_0\ , \quad X,\ \hat{X} = 0\ , \tag{7.61}$$

is satisfied. As in the previous Subsection 7.3.1.1, in this case also, any constant Ricci scalar solution in GR would identically be a solution of the system.

The relation (7.60) was already found by Barrow and Ottewill [163] in the cosmological context and later rediscovered in [164]. It relates the value of the constant Ricci scalar with the universal constants in the action. For example if we have the Lagrangian as $R - 2\Lambda$, which is the Lagrangian for GR with the cosmological constant, we must have, as is well known, the relation $R_0 = 4\Lambda$.

7.3.1.3 The curious case of R^2 gravity.

As we have already explained, the condition for existence of solutions with covariantly constant scalar curvature connects the constant curvature with the universal constants of the Lagrangian. However, this is not the case for $f(R) = K\,R^2$. In fact for this type of Lagrangian the third condition of (7.60) is identically satisfied. This means that we can have a constant curvature solution for any value of the curvature. Thus for R^2 gravity, the '*cosmological*' constant term in a Schwarzschild-dS/AdS spacetime becomes a local constant of integration just like the mass. Hence in this theory we can have two distant stars behaving like two different Schwarzschild-dS/AdS object with different values of the constant. Unfortunately this case is rather pathological since it corresponds to the case in which the trace of the field equations in vacuum, $3\Box f' + f'\,R - 2f = 0$ is satisfied *identically* for constant Ricci scalar, whereas usually it may be satisfied for special values of R. In any case this model is ruled out by solar system experiments (see [165, 166]).

7.3.2 Locally spatially homogenous vacuum spacetimes

Now if we consider the case when $\phi = \mathcal{A} = 0$ with $R = 0$, $f(0) = 0$ and $f'(0) \neq 0$, and choosing $u^a = \sqrt{\frac{2m}{t} - 1}\,\delta^a_t$, where m is a constant, solving (7.3)-(7.17) results in the unique solution

$$\Theta = \frac{3m - 2t}{t\sqrt{t(2m-t)}} \quad , \qquad \Sigma = -\frac{2}{3}\,\frac{3m - t}{t\sqrt{t(2m-t)}}, \tag{7.62}$$

$$\mathcal{E} = -\frac{2m}{t^3} \quad , \qquad K = \frac{1}{t^2}\ . \tag{7.63}$$

Again solving for the metric components we get

$$ds^2 = -\frac{dt^2}{(\frac{2m}{t} - 1)} + \left(\frac{2m}{t} - 1\right) dr^2 + t^2\, d\Omega^2, \tag{7.64}$$

which is a part of the Schwarzschild solution inside the Schwarzschild radius.

7.3.3 Jebsen-Birkoff like theorem in $f(R)$ gravity

We can now give a generalisation of the *Jebsen-Birkhoff-like theorem in* $f(R)$ gravity:

> *For $f(R)$ gravity, where the function f is of class C^3 at $R = 0$,with $f(0) = 0$ and $f'(0) \neq 0$, the only spherically symmetric solution with vanishing Ricci scalar in empty space in an open set $\mathcal{S}$, is one that is locally equivalent to part of maximally extended Schwarzschild solution in $\mathcal{S}$.*

It is also interesting to note that the covariant scale defined by equation (7.42) is equal to the Schwarzschild mass m.

7.4 Spherically symmetric spacetime with an almost vanishing Ricci scalar

From the previous section we know that for $f(R)$ gravity with $R = 0$, $f(0) = 0$ and $f'_0 \neq 0$, all spherically symmetric vacuum spacetimes are locally isomorphic to a part of Schwarzschild solution. In [80], the vacuum LRS II spacetime was perturbed by putting in a small amount of general matter that obeys WEC and DEC, to find out the amount of matter that can be introduced to the spacetime for the Jebsen-Birkhoff theorem to remain approximately true. Analogously, we investigate here the necessary conditions on the magnitude and spatial and temporal derivatives of the Ricci scalar, for the above theorem to remain approximately true. In this section we only deal with the static exterior background as it is astrophysically more interesting.

We have seen that the vacuum spherically symmetric spacetime with vanishing Ricci scalar has a covariant scale given by the Schwarzschild radius which sets up the scale for perturbation. Going by our description of the energy momentum tensor for vacuum LRS II spacetime in $f(R)$ gravity as consisting of curvature terms μ^R, p^R, Π^R and Q^R and taking a static Schwarzschild background, then the set $\{R,\,\Theta,\,\Sigma\}$, describes the first-order quantities (according to the Stewart and Walker lemma [90]). Performing a series expansion of $f(R)$ in the neighbourhood of $R=0$ gives $f(R)=f_0'\,R$ as a first-order term. Neglecting the higher order quantities in (7.12)-(7.15), we get the following equations

$$\mu = \frac{f_0''}{f_0'}\left(\hat{X}+\phi\,X\right)\,, \tag{7.65}$$

$$p = \frac{f_0''}{f_0'}\left(\ddot{R}-\mathcal{A}\,X-\frac{2}{3}\phi\,X-\frac{2}{3}\hat{X}\right)\,, \tag{7.66}$$

$$Q = -\frac{f_0''}{f_0'}\left(\dot{X}-\mathcal{A}\,\dot{R}\right)\,, \tag{7.67}$$

$$\Pi = \frac{f_0''}{3f_0'}\left(2\hat{X}-\phi\,X\right)\,. \tag{7.68}$$

and

$$R\,f_0' = 3f_0''\left(\hat{X}+(\mathcal{A}+\phi)X-\ddot{R}\right) \tag{7.69}$$

for the trace. Thus we see that by perturbing the Ricci scalar in the neighbourhood of $R=0$ background, we are actually generating a '*curvature fluid*' having the above thermodynamic quantities. Therefore the situation here is similar to introducing small amount of matter on a Schwarzschild background in GR. In [80] the sufficient conditions for the *smallness* of these matter perturbations in order for the spacetime to remain almost Schwarzschild are given. These conditions in our case become

$$\left[\frac{|R|}{K^{(3/2)}},\frac{f_0''^{(1/2)}\,|\dot{R}|}{K^{(3/2)}},\frac{f_0''\,|\hat{R}|}{K^{(3/2)}},\frac{f_0''^{(1/2)}\,|X|}{K^{(3/2)}},\frac{f_0''\,|\hat{X}|}{K^{(3/2)}},\frac{f_0''\,|\dot{X}|}{K^{(3/2)}}\right] << C, \tag{7.70}$$

and

$$\left[\frac{f_0''^{3/2}\,|\ddot{R}|}{K^{(3/2)}},\frac{f_0''^{3/2}\,|\ddot{X}|}{K^{(3/2)}}\right] << C\,. \tag{7.71}$$

Similar to [80], we also need to specify in what domain these equations will hold. This is important because eventually we will reach a radius r where these inequalities may no longer hold; but this will be unphysical. On the basis that in the real universe asymptotically flat regions are always of finite size, being replaced at larger scales by galactic and cosmological conditions, we will describe the local domain where our results will apply by [79],

– Defining *finite infinity* $\mathcal{F}$ as a 2-sphere of radius $R_{\mathcal{F}}\gg C$ surrounding the star: this is infinity for all practical purposes [167, 168].

– Assuming the relations (7.70), (7.71) hold in the domain $D_{\mathcal{F}}$ defined by $r_S < r < R_{\mathcal{F}}$ where $r_S > C$ is the radius of the surface of the star.

We now linearise the field equations (7.3)-(7.17) by neglecting the higher order quantities and we obtain the following equations for the first-order quantities

$$\hat{\Sigma} - \frac{2}{3}\hat{\Theta} \approx -\frac{3}{2}\phi\,\Sigma + \frac{f_0''}{f_0'}\left(\dot{X} - \mathcal{A}\,\dot{R}\right) , \tag{7.72}$$

$$\dot{\Theta} \approx -\frac{f_0''}{2f_0'}\left(3\ddot{R} - \hat{X} - (3\,\mathcal{A} + \phi)X\right) , \tag{7.73}$$

$$\dot{\Sigma} - \frac{2}{3}\dot{\Theta} \approx \frac{f_0''}{f_0'}\left[\ddot{R} - X\left(\mathcal{A} + \frac{1}{2}\phi\right)\right] , \tag{7.74}$$

$$\dot{\phi} \approx \left(\Sigma - \frac{2}{3}\Theta\right)\left(\mathcal{A} - \frac{1}{2}\phi\right) - \frac{f_0''}{f_0'}(\dot{X} - \mathcal{A}\,\dot{R}) , \tag{7.75}$$

$$\dot{\mathcal{E}} \approx \left(\frac{3}{2}\Sigma - \Theta\right)\mathcal{E} + \phi\,\mathcal{A}\,\frac{f_0''}{2f_0'}\,\dot{R} , \tag{7.76}$$

$$\frac{1}{3}R\,f_0' \approx f_0''\,\hat{X} - f_0''\,\ddot{R} + (\phi + \mathcal{A})\,f_0''\,X . \tag{7.77}$$

From these equations we can see that if (7.70) and (7.71) are locally satisfied at any epoch, within the domain $D_{\mathcal{F}}$, then the spatial and temporal variation of the expansion Θ and the shear Σ are of same order of smallness as the perturbations and derivatives of the Ricci scalar. In that case a timelike vector will not exactly solve the Killing equations (7.26)-(7.29) in general, although it may do so approximately. To see this explicitly, let us set $\Phi = 0$ in the *Killing equation* (7.25)

$$\nabla_a(\Psi\,u_b) + \nabla_b(\Psi\,u_a) = 0 . \tag{7.78}$$

and we once again try to solve the equation for a Killing vector of the form $\xi_a = \Psi\,u_a$ with an aim to see how close the vector $\xi_a = \Psi\,u_a$ is to Killing vector in the perturbed scenario.

We consider the scalars constructed by multiplying the killing equation by the vectors u^a, e^a, the projection tensor N^{ab} and utilise equation (6.23) and (6.26) to facilitate the calculation. We know that multiplying the Killing equation by $u^a\,u^b$ and $u^a\,e^b$ results in equations for which the solution of the scalar Ψ always exists. The constraints obtained from multiplying the Killing equation by $e^a\, e^b$ and N^{ab} only vanish if $\Theta = \Sigma = 0$, however, we are considering here the perturbed case which is characterised by non-zero Θ and Σ. As a result not all the equations are completely solved in general. If we set up (7.78) as a symmetric tensor

$$K_{ab} := \nabla_a(\Psi\, u_b) + \nabla_b(\Psi\, u_a)\ . \tag{7.79}$$

we can instead say that there always exists a non-trivial solution of the scalar Ψ for which $|K_{ab}\, u^a\, u^b|$ and $|K_{ab}\, u^a e^b|$ vanishes and that $|K_{ab}\, e^a\, e^b|^2$ and $|K_{ab}\, N^{ab}|^2$ are non-zero since Θ and Σ are non-zero. However, if the conditions

$$\left[\frac{|K_{ab}\, u^a\, u^b|^2}{K^{3/2}}, \frac{|K_{ab}\, u^a\, e^b|^2}{K^{3/2}}, \frac{|K_{ab}\, e^a\, e^b|^2}{K^{3/2}}, \frac{|K_{ab}\, N^{ab}|^2}{K^{3/2}}\right] << C \tag{7.80}$$

are satisfied, then we can say that $\xi_a = \Psi\, u_a$ is close to a Killing vector and that the spacetime is approximately static.

Subtracting the background equation (7.54) from (7.21), we get

$$\left(\frac{1}{3}\Theta - \frac{1}{2}\Sigma\right)^2 \approx \frac{f_0''}{2f_0'}\phi X\ . \tag{7.81}$$

Similarly subtracting (7.50) from (7.3) we get

$$\left(\frac{1}{3}\Theta + \Sigma\right)\left(\frac{2}{3}\Theta - \Sigma\right) \approx \frac{f_0''}{2f_0'}\left(2\hat{X} + \phi\, X\right)\ . \tag{7.82}$$

Using the above equations (7.81) and (7.82), we immediately see that if (7.70) is locally satisfied, then the following conditions are satisfied

$$|K_{ab}\, e^a\, e^b|^2 = \Psi^2\left(\frac{1}{3}\Theta + \Sigma\right)^2 \ll C\, K^{3/2}\ , \tag{7.83}$$

$$|K_{ab}\, N^{ab}|^2 = \Psi^2\left(\frac{2}{3}\Theta - \Sigma\right)^2 \ll C\, K^{3/2}\ . \tag{7.84}$$

It follows that there always exists a timelike vector that satisfies (7.80). This vector then almost solves the Killing equations in $\mathcal{S}$ and hence the spacetime is *almost* static in $\mathcal{S}$. Moreover, the resultant field equations are the zeroth-order equations (7.50)-(7.53) with an addition of $\mathcal{O}(\epsilon)$ terms.

7.5 Almost spherically symmetric spacetimes

In order to define the notion of an *almost spherically symmetric* spacetime, we begin by writing the *geodesic deviation equation* for a family of closely spaced geodesics on the 2-

sheets with tangent vectors $\psi^a(v)$ and separation vectors $\eta^a(v)$ (where 'v' is the parameter which labels the different geodesics) as [169],

$$\psi^e\,\delta_e(\psi^f\,\delta_f\eta^a) = K(\psi^a\,\psi_d\,\eta^d - \eta^a\,\psi_c\,\psi^c)\ . \tag{7.85}$$

We have used here the definition of the two dimensional Riemann curvature tensor equation (7.20).

We now define a vector V^a by

$$V^a = \psi^e\,\delta_e(\psi^f\,\delta_f\eta^a) - K_0(\psi^a\,\psi_d\,\eta^d - \eta^a\,\psi_c\,\psi^c)\ , \tag{7.86}$$

where K_0 is the Gaussian curvature for a spherical sheet at any point P, which can be fixed by making the vector $V^a = 0$ at that point. This vector vanishes for exact spherical 2-sheets in any open neighbourhood of P but doesn't for non-spherical sheets. As a result, from the magnitude of $V^a (= \sqrt{V_a\,V_a})$ we obtain a covariant measure of the deviation from the spherical symmetry.

We can now define an *almost spherically symmetric* spacetime in following the way [169]:

> *Any C^3 spacetime with positive Gaussian curvature everywhere, which admits a local 1+1+2 splitting at every point is called an* almost spherically symmetric *spacetime, if and only if the following quantities are either zero or much smaller than the scale defined by the modulus of the proportionality constant in equation (7.42):*
>
> – *The magnitude of all the 2-vectors (defined by $\sqrt{\psi_a\psi^a}$) and PSTF 2-tensors (defined by $\sqrt{\psi_{ab}\psi^{ab}}$) described in equation (6.51).*
>
> – *The magnitude of vector V^a defined above in (7.86).*

7.6 Almost spherically symmetric spacetime with vanishing Ricci scalar

We have seen in Section 7.4, that subject to the conditions (7.70) and (7.71), on any spherically symmetric local domain $D_{\mathcal{F}}$, the spacetime remains "almost" Schwarzschild for all the $f(R)$-theories that admit a Schwarzschild background, (that is, a background characterised by a vanishing Ricci scalar with $f(0) = 0$ and $f'_0 \neq 0$). We now wish to see to what extent the conditions hold when we perturb this geometry.

As previously stated, the sheet will be a genuine two surface if and only if the commutator of the time and hat derivative do not depend on any sheet component and the sheet derivatives commute in (6.42) and (6.45). Following from the definition of almost spherical symmetry, in the perturbed scenario we will require the sheet to be an almost genuine 2-surface such that the commutator of the time and hat derivative *almost* do not depend on any sheet component and the sheet derivatives *almost* commute. In that case we see from (6.42) and (6.45) that the scalars Ω and ξ will be of the same order of smallness as the other vectors and PSTF 2-tensors on the sheet. Furthermore, from the constraint equation (6.85), we see that the scalar $\mathcal{H}$ is of the same order of smallness. Dealing once again with the static exterior background, we now have it that the set of 1+1+2 variables

$$[R,\, \Theta,\, \Sigma,\, \Omega,\, \mathcal{H},\, \xi,\, \mathcal{A}^a,\, \Omega^a,\, \Sigma^a,\, \alpha^a,\, a^a,\, \mathcal{E}^a,\, \mathcal{H}^a,\, \Sigma_{ab},\, \mathcal{E}_{ab},\, \mathcal{H}_{ab},\, \zeta_{ab}]\ , \tag{7.87}$$

are all of $\mathcal{O}(\epsilon)$ with respect to the invariant scale. We shall treat these variables along with their derivatives and the dot - '$\dot{}$' and delta - 'δ' derivatives of $\{\mathcal{A},\, \mathcal{E},\, \phi\}$ as first-order relative to the background terms.

Performing a series expansion of $f(R)$ in the neighbourhood of $R = 0$ and neglecting the higher order quantities in (6.34)-(6.40), we obtain

$$\mu \approx \frac{f_0''}{f_0'}\left(\hat{X} + \phi\, X + \delta^2 R\right)\ , \tag{7.88}$$

$$p \approx \frac{f_0''}{f_0'}\left[\ddot{R} - \mathcal{A}\, X - \frac{2}{3}\left(\phi\, X + \hat{X} + \delta^2 R\right)\right]\ , \tag{7.89}$$

$$Q \approx -\frac{f_0''}{f_0'}\left(\dot{X} - \mathcal{A}\, \dot{R}\right)\ , \tag{7.90}$$

$$Q_a \approx -\frac{f_0''}{f_0'}\, \delta_a \dot{R}\ , \tag{7.91}$$

$$\Pi \approx \frac{f_0''}{3f_0'}\left(2\hat{X} - \phi\, X - \delta^2 R\right)\ , \tag{7.92}$$

$$\Pi_a \approx \frac{f_0''}{f_0'}\left(\delta_a X - \frac{1}{2}\phi\, \delta_a R\right)\ , \tag{7.93}$$

$$\Pi_{ab} \approx \frac{f_0''}{f_0'}\, \delta_{\{a}\delta_{b\}} R\ . \tag{7.94}$$

Linearising the field equations (6.54)-(6.84) and substituting in equations (7.88) - (7.94) we obtain:

Evolution equations

The evolution equations for ξ and $\zeta_{\{ab\}}$ are:

$$\dot{\xi} = \left(\mathcal{A} - \frac{1}{2}\phi\right)\Omega + \frac{1}{2}\varepsilon_{ab}\delta^a\alpha^b + \frac{1}{2}\mathcal{H}\ ; \tag{7.95}$$

$$\dot{\zeta}_{\{ab\}} = \left(\mathcal{A} - \frac{1}{2}\phi\right)\Sigma_{ab} + \delta_{\{a}\alpha_{b\}} - \varepsilon_{c\{a}\mathcal{H}_{b\}}{}^{c} \; ; \tag{7.96}$$

Vorticity evolution equation:

$$\dot{\Omega} = \frac{1}{2}\varepsilon_{ab}\delta^{a}\mathcal{A}^{b} + \mathcal{A}\,\xi \; , \tag{7.97}$$

$$\dot{\Omega}_{\bar{a}} + \frac{1}{2}\varepsilon_{ab}\hat{\mathcal{A}}^{b} = \frac{1}{2}\varepsilon_{ab}\left(\delta^{b}\mathcal{A} - \mathcal{A}\,a^{b} - \frac{1}{2}\phi\,\mathcal{A}^{b}\right) \; ; \tag{7.98}$$

Shear evolution:

$$\dot{\Sigma}_{\bar{a}} - \frac{1}{2}\hat{\mathcal{A}}_{a} = \frac{1}{2}\delta_{a}\mathcal{A} + \left(\mathcal{A} - \frac{1}{4}\phi\right)\mathcal{A}_{a} + \frac{1}{2}\mathcal{A}\,a_{a} - \mathcal{E}_{a} + \frac{f_0''}{2f_0'}\left(\delta_{a}X - \frac{1}{2}\phi\,\delta_{a}R\right) \; , \tag{7.99}$$

$$\dot{\Sigma}_{\{ab\}} = \delta_{\{a}\mathcal{A}_{b\}} + \mathcal{A}\,\zeta_{ab} - \mathcal{E}_{ab} + \frac{f_0''}{2f_0'}\,\delta_{\{a}\delta_{b\}}R \; ; \tag{7.100}$$

Magnetic Weyl evolution:

$$\dot{\mathcal{H}} = -\,\varepsilon_{ab}\delta^{a}\mathcal{E}^{b} - 3\xi\,\mathcal{E} \; , \tag{7.101}$$

$$\dot{\mathcal{H}}_{\bar{a}} = -\frac{3}{2}\mathcal{E}\,\varepsilon_{ab}\mathcal{A}^{b} - \frac{1}{2}\varepsilon_{ab}\delta^{b}\mathcal{E} - \frac{1}{2}\left(\phi - 2\mathcal{A}\right)\varepsilon_{ab}\mathcal{E}^{b} + \varepsilon_{c\{d}\delta^{d}\mathcal{E}_{a\}}{}^{c} - \mathcal{E}\,\frac{f_0''}{4f_0'}\,\varepsilon_{ab}\delta^{b}R \; , \tag{7.102}$$

$$\dot{\mathcal{H}}_{\{ab\}} + \varepsilon_{c\{a}\hat{\mathcal{E}}_{b\}}{}^{c} = \varepsilon_{c\{a}\delta^{c}\mathcal{E}_{b\}} + \frac{3}{2}\mathcal{E}\,\varepsilon_{c\{a}\zeta_{b\}}{}^{c} - \left(\frac{1}{2}\phi + 2\mathcal{A}\right)\varepsilon_{c\{a}\mathcal{E}_{b\}}{}^{c} \; ; \tag{7.103}$$

Electric Weyl evolution:

$$\begin{aligned}\dot{\mathcal{E}}_{\bar{a}} + \frac{1}{2}\varepsilon_{ab}\hat{\mathcal{H}}^{b} \;&=\; \frac{3}{4}\mathcal{E}\left(\varepsilon_{ab}\Omega^{b} + \Sigma_{a} - 2\alpha_{a}\right) - \left(\frac{1}{4}\phi + \mathcal{A}\right)\varepsilon_{ab}\mathcal{H}^{b} \\ &\quad + \frac{3}{4}\varepsilon_{ab}\delta^{b}\mathcal{H} + \frac{1}{2}\varepsilon_{bc}\delta^{b}\mathcal{H}^{c}{}_{a} \; ,\end{aligned} \tag{7.104}$$

$$\dot{\mathcal{E}}_{\{ab\}} - \varepsilon_{c\{a}\hat{\mathcal{H}}_{b\}}{}^{c} \;=\; -\,\varepsilon_{c\{a}\delta^{c}\mathcal{H}_{b\}} + \left(\frac{1}{2}\phi + 2\mathcal{A}\right)\varepsilon_{c\{a}\mathcal{H}_{b\}}{}^{c} - \frac{3}{2}\mathcal{E}\,\Sigma_{ab} \; ; \tag{7.105}$$

Evolution equation for $\hat{e}_a$:

$$\dot{a}_{\bar{a}} - \hat{\alpha}_{\bar{a}} = \left(\frac{1}{2}\phi + \mathcal{A}\right)\alpha_{a} - \left(\frac{1}{2}\phi - \mathcal{A}\right)\left(\Sigma_{a} + \varepsilon_{ab}\Omega^{b}\right) + \varepsilon_{ab}\mathcal{H}^{b} + \frac{f_0''}{2f_0'}\,\delta_{a}\dot{R} \; . \tag{7.106}$$

Propagation equations

$$\hat{\xi} = -\,\phi\,\xi + \frac{1}{2}\varepsilon_{ab}\delta^{a}a^{b} \; ; \tag{7.107}$$

$$\hat{\zeta}_{\{ab\}} = -\,\phi\,\zeta_{ab} + \delta_{\{a}a_{b\}} - \mathcal{E}_{ab} - \frac{f_0''}{2f_0'}\,\delta_{\{a}\delta_{b\}}R \; ; \tag{7.108}$$

Shear divergence:

$$\begin{aligned}\hat{\Sigma}_{\bar{a}} - \varepsilon_{ab}\hat{\Omega}^b &= \frac{1}{2}\delta_a\Sigma + \frac{2}{3}\delta_a\theta - \varepsilon_{ab}\delta^b\Omega - \frac{3}{2}\phi\,\Sigma_a \\ &+ \left(\frac{1}{2}\phi + 2\mathcal{A}\right)\varepsilon_{ab}\Omega^b - \delta^b\Sigma_{ab} + \frac{f_0''}{f_0'}\,\delta_a\dot{R}\;, \end{aligned} \tag{7.109}$$

$$\hat{\Sigma}_{\{ab\}} = \delta_{\{a}\Sigma_{b\}} - \varepsilon_{c\{a}\delta^c\Omega_{b\}} - \frac{1}{2}\phi\,\Sigma_{ab} - \varepsilon_{c\{a}\mathcal{H}_{b\}}{}^c\;; \tag{7.110}$$

Vorticity divergence equation:

$$\hat{\Omega} = -\,\delta_a\Omega^a + (\mathcal{A} - \phi)\,\Omega\;; \tag{7.111}$$

Electric Weyl Divergence:

$$\hat{\mathcal{E}}_{\bar{a}} = \frac{1}{2}\delta_a\mathcal{E} - \delta^b\mathcal{E}_{ab} - \frac{3}{2}\mathcal{E}\,a_a - \frac{3}{2}\phi\,\mathcal{E}_a + \mathcal{E}\,\frac{f_0''}{4f_0'}\,\delta_a R\;; \tag{7.112}$$

Magnetic Weyl divergence:

$$\hat{\mathcal{H}} = -\,\delta_a\mathcal{H}^a - \frac{3}{2}\phi\,\mathcal{H} - 3\mathcal{E}\,\Omega\;, \tag{7.113}$$

$$\hat{\mathcal{H}}_{\bar{a}} = \frac{1}{2}\delta_a\mathcal{H} - \delta^b\mathcal{H}_{ab} + \frac{3}{2}\mathcal{E}\left(\Omega_a - \varepsilon_{ab}\Sigma^b\right) - \frac{3}{2}\phi\,\mathcal{H}_a\;. \tag{7.114}$$

Together with the linearised curvature trace equation

$$\frac{1}{3}R = \frac{f_0''}{f_0'}\left[\hat{X} - \ddot{R} + (\phi + \mathcal{A})\,X + \delta^2 R\right]\;. \tag{7.115}$$

From the evolution equations (7.95) - (7.106), it is evident that if the background is static with $\Sigma = \Theta = 0$ or "almost static" with $\Sigma = \Theta = \mathcal{O}(\epsilon)$, the time derivatives of the first-order quantities at a given point are all of the same order of smallness as themselves. Hence if at a given epoch these quantities are of $\mathcal{O}(\epsilon)$, then there exists an open set $\mathcal{S}$ in the domain $D_{\mathcal{F}}$ where these quantities continue to be of the same order.

This time if we project the *Killing equation* (7.25) for a Killing vector of the form $\xi_a = \Psi\,u_a$, with $N_c^a\,u^b$, $N_c^a\,e^b$ and $N_c^a\,N_d^b$, we obtain the following additional constraints on the 2-sheet:

$$-\delta_c\Psi + \Psi\,\mathcal{A}_c = 0\;, \tag{7.116}$$

$$\Psi\,\Sigma_c = 0\;, \tag{7.117}$$

$$\Psi\,\Sigma_{cd} = 0\;. \tag{7.118}$$

The solution of (7.116) always exists and as we have just seen, the LHS of equations (7.117) and (7.118) remains $\mathcal{O}(\epsilon)$ in $\mathcal{S}$. Hence a timelike vector almost solves the Killing equations, making the spacetime almost static.

We have therefore demonstrated an important result: For any $f(R)$ theory of gravity which admits a Schwarzschild background, if (7.70) and (7.71) are locally satisfied at any epoch, (within the domain $D_{\mathcal{F}}$) and the sheet derivatives of these scalars are of the same order of smallness as themselves, then there exists an open set $\mathcal{S}$ in $D_{\mathcal{F}}$ where the conditions continue to hold. The size of the open set $\mathcal{S}$ depends on the parameters of theory (namely the quantity $f''(0)$) and the covariant scale (which is the Schwarzschild mass of the star) and we can always tune the parameters of the theory such that the perturbations continue to remain small for a time period which is greater than the age of the universe. In that case the local spacetime around almost spherical stars will be stable in the regime of linear perturbations. The results of a more rigorous analysis of the perturbation equations (done in the next chapter) is consistent with the above result.

Chapter 8

Perturbations around a Schwarzschild black hole in $f(R)$ gravity

The interest in studying black hole (BH) perturbations comes from the important role they play in gravitational wave physics. There are various ways by which a black hole can be perturbed: by incident gravitational waves, by objects falling into it or by aspherical gravitational collapse. The understanding of perturbations of black holes therefore provides insight into a different number of areas of interest in gravitational radiation studies.

Perturbations of Schwarzschild BH at linear order in GR have been studied through metric perturbations, the Newman-Penrose (NP) formalism [87] as well as the 1+1+2 covariant formalism [65]. In the metric approach, fluctuations of the spacetime geometry are determined by closed wave equations: the Regge-Wheeler equation for odd parity and the Zerilli equation in the even parity. These wave equations act on linear combinations of the functions (and their derivatives) appearing in the perturbed metric, but these functions do not determine directly the gravitational waves which they represent, nor are they frame independent, as a general co-ordinate transformation would not preserve the wave equation. Using the 1+1+2 approach, Clarkson and Barrett [65] demonstrated that both the odd and even parity perturbations may be unified in a covariant wave equation equivalent to the Regge-Wheeler equation. This wave equation is characterised by a single covariant, frame- and gauge-invariant, transverse-traceless tensor.

In this chapter we present the complete set of 1+1+2 covariant and gauge invariant evolution, propagation and constraint equations linearised around the Schwarzschild background in $f(R)$ gravity. As in the previous chapter, we keep in mind that gauge invariance holds for set of variables (7.87) that vanish in the background and we neglect the products of these $\mathcal{O}(\epsilon)$ variables in (6.54)-(6.87).

Furthermore, we also derive a covariant and gauge-invariant wave equation which describes the perturbations of a Schwarzschild black hole spacetime in FOG. This equation is the covariant form of the Regge-Wheeler equation, corresponding to a master variable that constitutes a gauge and frame invariant transverse-traceless (TT) tensor.

8.1 Linearised field equations

The linearised field equations (evolution, propagation and constraint) around a Schwarzschild background (with vanishing Ricci scalar) for $f(R)$- gravity are as follows:

Evolution equations:

$$\dot{\phi} = \left(\frac{2}{3}\Theta - \Sigma\right)\left(\mathcal{A} - \frac{1}{2}\phi\right) + \delta_a \alpha^a + \frac{f_0''}{f_0'}\left(\mathcal{A}\,\dot{R} - \dot{X}\right)\,, \tag{8.1}$$

$$\dot{\xi} = \left(\mathcal{A} - \frac{1}{2}\phi\right)\Omega + \frac{1}{2}\varepsilon_{ab}\delta^a\alpha^b + \frac{1}{2}\mathcal{H}\,, \tag{8.2}$$

$$\dot{\Omega} = \frac{1}{2}\varepsilon_{ab}\delta^a\mathcal{A}^b + \mathcal{A}\,\xi\,, \tag{8.3}$$

$$\dot{\Sigma} - \frac{2}{3}\dot{\Theta} = -\phi\,\mathcal{A} - \delta_a\mathcal{A}^a - \mathcal{E} - \frac{f_0''}{2f_0'}\left(\delta^2 R + (\phi + 2\mathcal{A})\,X - 2\ddot{R}\right)\,, \tag{8.4}$$

$$\dot{\mathcal{E}} = \left(\frac{3}{2}\Sigma - \Theta\right)\mathcal{E} + \varepsilon_{ab}\delta^a\mathcal{H}^b + \phi\,\mathcal{A}\,\frac{f_0''}{2f_0'}\,\dot{R}\,, \tag{8.5}$$

$$\dot{\mathcal{H}} = -\,\varepsilon_{ab}\delta^a\mathcal{E}^b - 3\xi\,\mathcal{E}\,, \tag{8.6}$$

$$\dot{\Sigma}_{\bar{a}} - \varepsilon_{ab}\dot{\Omega}^b = \delta_a\mathcal{A} + \left(\mathcal{A} - \frac{1}{2}\phi\right)\mathcal{A}_a - \mathcal{E}_a + \frac{f_0''}{2f_0'}\left(\delta_a X - \frac{1}{2}\phi\,\delta_a R\right)\,, \tag{8.7}$$

$$\dot{\mathcal{E}}_{\bar{a}} + \frac{1}{2}\varepsilon_{ab}\hat{\mathcal{H}}^b = \frac{3}{4}\,\mathcal{E}\left(\varepsilon_{ab}\Omega^b + \Sigma_a - 2\alpha_a\right) - \left(\frac{1}{4}\phi + \mathcal{A}\right)\varepsilon_{ab}\mathcal{H}^b + \frac{3}{4}\,\varepsilon_{ab}\delta^b\mathcal{H} + \frac{1}{2}\,\varepsilon_{bc}\delta^b\mathcal{H}^c{}_a\ , \tag{8.8}$$

$$\dot{\mathcal{H}}_{\bar{a}} = -\frac{3}{2}\mathcal{E}\,\varepsilon_{ab}\mathcal{A}^b - \frac{1}{2}\varepsilon_{ab}\delta^b\mathcal{E} - \frac{1}{2}\left(\phi - 2\mathcal{A}\right)\varepsilon_{ab}\mathcal{E}^b + \varepsilon_{c\{d}\delta^d\mathcal{E}_{a\}}{}^c - \mathcal{E}\,\frac{f_0''}{4f_0'}\,\varepsilon_{ab}\delta^b R\ , \tag{8.9}$$

$$\dot{\zeta}_{\{ab\}} = \left(\mathcal{A} - \frac{1}{2}\phi\right)\Sigma_{ab} + \delta_{\{a}\alpha_{b\}} - \varepsilon_{c\{a}\mathcal{H}_{b\}}{}^c\ , \tag{8.10}$$

$$\dot{\Sigma}_{\{ab\}} = \delta_{\{a}\mathcal{A}_{b\}} + \mathcal{A}\,\zeta_{ab} - \mathcal{E}_{ab} + \frac{f_0''}{2f_0'}\,\delta_{\{a}\delta_{b\}}R\ , \tag{8.11}$$

$$\frac{f_0''}{f_0'}\,\delta_a\dot{R} = \delta_a\Sigma - \frac{2}{3}\delta_a\theta + 2\,\varepsilon_{ab}\delta^b\Omega + 2\,\delta^b\Sigma_{ab} + \phi\left(\Sigma_a + \varepsilon_{ab}\Omega^b\right) + 2\varepsilon_{ab}\mathcal{H}^b\ . \tag{8.12}$$

Propagation equations:

$$\hat{\phi} = -\frac{1}{2}\phi^2 - \mathcal{E} + \delta_a a^a - \frac{f_0''}{2f_0'}\left(2\hat{X} + \phi\,X + \delta^2 R\right)\ , \tag{8.13}$$

$$\hat{\xi} = -\,\phi\,\xi + \frac{1}{2}\varepsilon_{ab}\delta^a a^b\ , \tag{8.14}$$

$$\hat{\Omega} = -\,\delta_a\Omega^a + \left(\mathcal{A} - \phi\right)\Omega\ , \tag{8.15}$$

$$\hat{\mathcal{A}} - \dot{\Theta} = -\delta_a\mathcal{A}^a - \left(\mathcal{A} + \phi\right)\mathcal{A} + \frac{f_0''}{2f_0'}\left[3\ddot{R} - \delta^2 R - \hat{X} - (3\mathcal{A} + \phi)X\right]\ , \tag{8.16}$$

$$\hat{\Sigma} - \frac{2}{3}\hat{\Theta} = -\frac{3}{2}\phi\,\Sigma - \delta_a\Sigma^a - \varepsilon_{ab}\delta^a\Omega^b + \frac{f_0''}{f_0'}\left(\dot{X} - \mathcal{A}\,\dot{R}\right)\ , \tag{8.17}$$

$$\hat{\mathcal{E}} = -\frac{3}{2}\phi\,\mathcal{E} - \delta_a\mathcal{E}^a - \mathcal{E}\,\frac{f_0''}{2f_0'}\,X\ , \tag{8.18}$$

$$\hat{\mathcal{H}} = -\,\delta_a\mathcal{H}^a - \frac{3}{2}\phi\,\mathcal{H} - 3\mathcal{E}\,\Omega\ , \tag{8.19}$$

$$\dot{a}_{\bar{a}} - \hat{\alpha}_{\bar{a}} = \left(\frac{1}{2}\phi + \mathcal{A}\right)\alpha_a - \left(\frac{1}{2}\phi - \mathcal{A}\right)\left(\Sigma_a + \varepsilon_{ab}\Omega^b\right) + \varepsilon_{ab}\mathcal{H}^b + \frac{f_0''}{2f_0'}\,\delta_a\dot{R}\ , \tag{8.20}$$

$$\begin{aligned}\hat{\Sigma}_{\bar{a}} - \varepsilon_{ab}\hat{\Omega}^b =& \frac{1}{2}\delta_a\Sigma + \frac{2}{3}\delta_a\theta - \varepsilon_{ab}\delta^b\Omega - \frac{3}{2}\phi\,\Sigma_a \\ &+ \left(\frac{1}{2}\phi + 2\mathcal{A}\right)\varepsilon_{ab}\Omega^b - \delta^b\Sigma_{ab} + \frac{f_0''}{f_0'}\,\delta_a\dot{R}\ ,\end{aligned} \tag{8.21}$$

$$\hat{\mathcal{A}}_a - 2\dot{\Sigma}_a = -\,\delta_a\mathcal{A} - 2\left(\mathcal{A} - \frac{1}{4}\phi\right)\mathcal{A}_a - \mathcal{A}\,a_a + 2\mathcal{E}_a - \frac{f_0''}{f_0'}\left(\delta_a X - \frac{1}{2}\phi\,\delta_a R\right)\ . \tag{8.22}$$

$$\hat{\mathcal{E}}_{\bar{a}} = \frac{1}{2}\delta_a\mathcal{E} - \delta^b\mathcal{E}_{ab} - \frac{3}{2}\mathcal{E}\,a_a - \frac{3}{2}\phi\,\mathcal{E}_a + \mathcal{E}\,\frac{f_0''}{4f_0'}\,\delta_a R\ , \tag{8.23}$$

$$\hat{\mathcal{H}}_{\bar{a}} = \frac{1}{2}\delta_a\mathcal{H} - \delta^b\mathcal{H}_{ab} + \frac{3}{2}\mathcal{E}\left(\Omega_a - \varepsilon_{ab}\Sigma^b\right) - \frac{3}{2}\phi\,\mathcal{H}_a\ , \tag{8.24}$$

$$\hat{\zeta}_{\{ab\}} = -\,\phi\,\zeta_{ab} + \delta_{\{a}a_{b\}} - \mathcal{E}_{ab} - \frac{f_0''}{2f_0'}\,\delta_{\{a}\delta_{b\}}R\ , \tag{8.25}$$

$$\hat{\Sigma}_{\{ab\}} = \delta_{\{a}\Sigma_{b\}} - \varepsilon_{c\{a}\delta^c\Omega_{b\}} - \frac{1}{2}\phi\,\Sigma_{ab} - \varepsilon_{c\{a}\mathcal{H}_{b\}}{}^{c}\ , \tag{8.26}$$

$$\dot{\mathcal{E}}_{\{ab\}} - \varepsilon_{c\{a}\hat{\mathcal{H}}_{b\}}{}^{c} = -\,\varepsilon_{c\{a}\delta^{c}\mathcal{H}_{b\}} + \left(\frac{1}{2}\phi + 2\mathcal{A}\right)\varepsilon_{c\{a}\mathcal{H}_{b\}}{}^{c} - \frac{3}{2}\mathcal{E}\,\Sigma_{ab}\ , \tag{8.27}$$

$$\dot{\mathcal{H}}_{\{ab\}} + \varepsilon_{c\{a}\hat{\mathcal{E}}_{b\}}{}^{c} = \varepsilon_{c\{a}\delta^{c}\mathcal{E}_{b\}} + \frac{3}{2}\mathcal{E}\,\varepsilon_{c\{a}\zeta_{b\}}{}^{c} - \left(\frac{1}{2}\phi + 2\mathcal{A}\right)\varepsilon_{c\{a}\mathcal{E}_{b\}}{}^{c}\ . \tag{8.28}$$

$$\frac{f_0''}{2f_0'}\left(\delta_a X - \frac{1}{2}\phi\delta_a R\right) = -\frac{1}{2}\delta_a\phi + \varepsilon_{ab}\delta^{b}\xi + \delta^{b}\zeta_{ab} - \mathcal{E}_a\ , \tag{8.29}$$

Trace equation:

$$f_0''(\hat{X} - \ddot{R}) = \frac{1}{3}R\,f_0' - f_0''\left[\delta^2 R + (\phi + \mathcal{A})\,X\right]\ . \tag{8.30}$$

Constraints:

$$\delta_a\Omega^a + \varepsilon_{ab}\delta^a\Sigma^b = (2\mathcal{A} - \phi)\,\Omega + \mathcal{H}\ , \tag{8.31}$$

It is important to notice here the freedom of choice of frame vectors demonstrated in absence of evolution equations for $\mathcal{A}$, $\mathcal{A}_a$, and α_a, along with the absence of a propagation equation for a_a in the preceding equations. This holds true in any spacetime, as one can *choose* the frame vectors at any point, whose motion cannot be uniquely determined and must be put into the equations by hand [65].

8.2 Gauge invariant variables

Not all the set of covariant equations in the previous section comply with the Stewart and Walker criterion [90] due to the isolated zeroth-order background terms that appear in them. By taking the angular derivatives of the background variables $\{\mathcal{E},\ \phi,\ \mathcal{A}\}$, we introduce the following set of gauge invariant characters

$$
\begin{aligned}
W_a &= \delta_a \mathcal{E} \;, & (8.32)\\
Y_a &= \delta_a \phi \;, & (8.33)\\
Z_a &= \delta_a \mathcal{A} \;, & (8.34)
\end{aligned}
$$

that vanish in background and are therefore gauge invariant. Applying the commutation relations (6.43) and (6.44) and substituting for the subsequent equations, we obtain the linearised propagation and evolution equations for the variables defined as

$$
\begin{aligned}
\dot{W}_a &= \frac{3}{2}\phi\,\mathcal{E}\left(\alpha_a + \Sigma_a - \varepsilon_{ab}\Omega^b\right) + \frac{3}{2}\mathcal{E}\left(\delta_a\Sigma - \frac{2}{3}\delta_a\Theta\right)\\
&\quad + \varepsilon_{bc}\delta_a\delta^b\mathcal{H}^c + \mathcal{A}\,\phi\,\frac{f_0''}{2f_0'}\,\delta_a\dot{R} \;,\\
\dot{Y}_a &= \left(\frac{1}{2}\phi^2 + \mathcal{E}\right)\left(\alpha_a + \Sigma_a - \varepsilon_{ab}\Omega^b\right) + \delta_a\delta_c\alpha^c\\
&\quad + \left(\frac{1}{2}\phi - \mathcal{A}\right)\left(\delta_a\Sigma - \frac{2}{3}\delta_a\Theta\right) + \frac{f_0''}{f_0'}\left(\mathcal{A}\,\delta_a\dot{R} - \delta_a\dot{X}\right) \;, & (8.35)\\
\hat{W}_a &= -2\phi\,W_a - \frac{3}{2}\mathcal{E}\,Y_a + \frac{3}{2}\phi\,\mathcal{E}\,a_a - \delta_a\delta_b\mathcal{E}^b - \mathcal{E}\,\frac{f_0''}{2f_0'}\,\delta_a X \;, & (8.36)\\
\hat{Y}_a &= -W_a - \frac{3}{2}\phi\,Y_a + \left(\frac{1}{2}\phi^2 + \mathcal{E}\right)a_a + \delta_a\delta_b a^b - \frac{1}{3}\delta_a R\\
&\quad + \frac{f_0''}{f_0'}\left[\left(\mathcal{A} + \frac{1}{2}\phi\right)\delta_a X + \frac{1}{2}\left(\mathcal{E} - \frac{1}{4}\phi^2\right)\delta_a R + \frac{1}{2}\delta^2\delta_a R - \delta_a\ddot{R}\right] \;, & (8.37)\\
\hat{Z}_a &= -\left(\frac{3}{2}\phi + 2\mathcal{A}\right)Z_a - \mathcal{A}\,Y_a + \mathcal{A}\left(\phi + \mathcal{A}\right)a_a + \delta_a\dot{\Theta}\\
&\quad - \delta_a\delta_b\mathcal{A}^b + \frac{f_0''}{f_0'}\left(\delta_a\ddot{R} - \mathcal{A}\,\delta_a\dot{X}\right) \;. & (8.38)
\end{aligned}
$$

These equations add no new information to what has already been given in the previous section however, they are now gauge invariant. We can then replace the equations (8.5), (8.1), (8.18), (8.13) and (8.16) with (8.35), (8.35), (8.36), (8.37) and (8.38) respectively.

The following additional constraints are obtained by application of the commutation relation (6.44) to the variables $\mathcal{E}$, ϕ and $\mathcal{A}$, respectfully,

$$
\begin{aligned}
\varepsilon_{ab}\delta^a W^b &= 3\phi\,\mathcal{E}\,\xi \;, & (8.39)\\
\varepsilon_{ab}\delta^a Y^b &= \left(\phi^2 + 2\mathcal{E}\right)\xi \;, & (8.40)\\
\varepsilon_{ab}\delta^a Z^b &= 2\mathcal{A}\left(\phi + \mathcal{A}\right)\xi \;. & (8.41)
\end{aligned}
$$

It is also useful to replace (8.4) with

$$\delta_a\dot{\Sigma}-\frac{2}{3}\delta_a\dot{\theta} \;=\; -W_a-\mathcal{A}\,Y_a-\phi\,Z_a-\delta_a\delta_b\mathcal{A}^b-\frac{f_0''}{2f_0'}\left[\delta^2\delta_a R-2\delta_a\ddot{R}\right.$$
$$\left.+\left(\mathcal{E}-\frac{1}{4}\phi^2\right)\delta_a R+(\phi+2\mathcal{A})\,\delta_a X\right]\;. \tag{8.42}$$

Introducing the new variables eliminates possible spherically symmetric perturbations (for which they are automatically zero) but since all the vacuum spherically symmetric static spacetimes are Schwarzschild, we do not lose any true degrees of freedom by adding them [65].

8.3 Commutation relations

The following are the relevant commutation relations for the derivatives of first-order scalars, vectors and tensors.

Scalars:

$$\dot{\hat{\Psi}}-\hat{\dot{\Psi}} \;=\; \mathcal{A}\,\dot{\Psi}\;, \tag{8.43}$$
$$\delta_a\dot{\Psi}-(\delta_a\Psi)^{\cdot} \;=\; 0\;, \tag{8.44}$$
$$\delta_a\hat{\Psi}-\widehat{(\delta_a\Psi)} \;=\; \frac{1}{2}\phi\,\delta_a\Psi\;, \tag{8.45}$$
$$\delta_{[a}\delta_{b]}\Psi \;=\; 0\;; \tag{8.46}$$

Vectors:

$$\dot{\hat{\Psi}}_{\bar{a}}-\hat{\dot{\Psi}}_{\bar{a}} \;=\; \mathcal{A}\,\dot{\Psi}_{\bar{a}}\;, \tag{8.47}$$
$$\delta_{[a}\delta_{b]}\Psi_c \;=\; \left(\frac{1}{4}\phi^2-\mathcal{E}\right)N_{c[a}\Psi_{b]}\;; \tag{8.48}$$

Tensors:

$$\dot{\hat{\Psi}}_{\{ab\}}-\hat{\dot{\Psi}}_{\{ab\}} \;=\; \mathcal{A}\,\dot{\Psi}_{\{ab\}}\;, \tag{8.49}$$
$$\delta_{[a}\delta_{b]}\Psi_{cd} \;=\; \left(\frac{1}{4}\phi^2-\mathcal{E}\right)\left(N_{c[a}\Psi_{b]d}+N_{d[a}\Psi_{b]c}\right)\;. \tag{8.50}$$

8.3.1 Harmonics

In order to solve the equations, it is standard procedure to decompose the first order variables harmonically (see, [50, 121]). The perturbations can be described by a linear system of ODEs by introducing spherical and time harmonics.

8.3.2 Spherical harmonics

In analogy with the decomposition of perturbations into scalar, vector and tensor modes in FLRW models [2, 48], the perturbations of the Schwarzschild geometry fall into two distinct classes based on how they transform on the surfaces of spherically symmetry: *even* (electric) and *odd* (magnetic) modes [1]. Given the spherical symmetry of the background, we can naturally choose spherical harmonics as our basis functions. This allows us to write the first-order variables as an infinite sum of the basis functions such that the scalars can be expanded as a sum of even modes, and the vectors and tensors can be expanded in sums over both the even and odd modes. Moreover, the angular derivatives appearing in the equations are effectively replaced by a harmonic coefficient. The presentation in this section follows [65] where the harmonics were introduced in a covariant manner.

We introduce the set of dimensionless spherical harmonic functions $Q = Q^{(\ell,m)}$, with $m = -\ell, \cdots, \ell$, defined in the background, as being eigenfunctions of the spherical laplacian operator such that

$$\delta^2 Q = -\frac{\ell(\ell+1)}{r^2} Q \ , \tag{8.51}$$

and Q is covariantly constant, $\hat{Q} = 0 = \dot{Q}$. The function r is covariantly defined by

$$\frac{\hat{r}}{r} = \frac{1}{2}\phi \ , \qquad \dot{r} = 0 = \delta_a r \ , \tag{8.52}$$

and gives a natural length scale to the spacetime. It is included in the definition (8.51) so that the equation propagates (and evolves as well). The factor r is defined up to an arbitrary constant, which reflects our freedom in choosing a particular normalisation of the spherical harmonic functions. We will find it most useful for our purposes to fix this freedom by identifying r with the usual Schwarzschild parameter through covariantly defining

$$r \equiv \left(\frac{1}{4}\phi^2 - \mathcal{E}\right)^{-1/2} \ . \tag{8.53}$$

We stress that these relations and harmonics are defined in the background only; we only expand first-order variables, so zeroth-order equations are sufficient.

We now look successfully at the expansion of first order scalars, vectors and tensors in spherical harmonics and the replacements which must be made in the equations.

[1] Alternatively, as first depicted in Chandrasekhars book [87], odd perturbations are called *axial* and even perturbations are called *polar*.

Scalar harmonics

We can now expand any first order scalar Ψ in terms of these functions as

$$\Psi = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{m=\ell} \Psi_{\mathsf{S}}^{(\ell,m)} Q^{(\ell,m)} = \Psi_{\mathsf{S}}\, Q, \tag{8.54}$$

where the sum over ℓ and m is implicit in the last equality. We use the subscript S to remind us that Ψ is a scalar, and that a spherical harmonic expansion has been made. Due to the spherical symmetry of the background, we can drop m in the equations.

The replacements which must be made for scalars when expanding the equations in spherical harmonics are

$$\Psi = \Psi_{\mathsf{S}}\, Q \;, \tag{8.55}$$
$$\delta_a \Psi = r^{-1} \Psi_{\mathsf{S}}\, Q_a \;, \tag{8.56}$$
$$\varepsilon_{ab}\delta^b \Psi = r^{-1} \Psi_{\mathsf{S}}\, \bar{Q}_a \;, \tag{8.57}$$

where the sums over ℓ and m is implicit.

Vector harmonics

We define the *even* (electric) parity vector spherical harmonics for $\ell \geq 1$ as

$$Q_a^{(\ell)} = r\, \delta_a Q^{(\ell)} \tag{8.58}$$

in order to have

$$\hat{Q}_a = 0 = \dot{Q}_a \;. \tag{8.59}$$

The vector harmonic (8.58) obeys

$$\delta^2 Q_a = \left(1 - \ell\,(\ell+1)\right) r^{-2} Q_a \;, \tag{8.60}$$

where the (ℓ) superscript is implicit. Similarly we define *odd* (magnetic) parity vector spherical harmonics as

$$\bar{Q}_a^{(\ell)} = r\, \varepsilon_{ab}\delta^b Q^{(\ell)} \quad \Rightarrow \quad \hat{\bar{Q}}_a = 0 = \dot{\bar{Q}}_a \;, \quad \delta^2 \bar{Q}_a = \left(1 - \ell\,(\ell+1)\right) r^{-2} \bar{Q}_a \;. \tag{8.61}$$

Note that $\bar{Q}_a = \varepsilon_{ab} Q^b \Leftrightarrow Q_a = -\,\varepsilon_{ab}\bar{Q}^b$, so that ε_{ab} is a parity operator. The crucial difference between these two types of vector spherical harmonics is that $\bar{Q}_a$ is solenoidal [2], so

$$\delta^a \bar{Q}_a = 0 \;, \quad \text{while} \quad \delta^a Q_a = -\ell\,(\ell+1)\, r^{-1} Q \;. \tag{8.62}$$

[2] Unlike Q_a, we cannot construct a scalar from $\bar{Q}_a$.

Note also that

$$\varepsilon_{ab}\delta^a Q^b = 0 \ , \quad \text{and} \quad \varepsilon_{ab}\delta^a \bar{Q}^b = \ell\,(\ell+1)\,r^{-1}Q \ . \tag{8.63}$$

The harmonics are orthogonal: $Q^a\,\bar{Q}_a = 0$ (for each ℓ), which implies that any first-order vector Ψ_a may be expanded as

$$\Psi_a = \sum_{\ell=1}^{\infty} \Psi_{\mathsf{V}}^{(\ell)}\,Q_a^{(\ell)} + \bar{\Psi}_{\mathsf{V}}^{(\ell)}\,\bar{Q}_a^{(\ell)} = \Psi_{\mathsf{V}}\,Q_a + \bar{\Psi}_{\mathsf{V}}\,\bar{Q}_a \ . \tag{8.64}$$

Again, we implicitly assume a sum over ℓ in the last equality, and the V reminds us that Ψ^a is a vector expanded in spherical harmonics.

As in the scalar case, the replacements to be made for vectors when expanding the equations in spherical harmonics are

$$\begin{aligned}
\Psi_a &= \Psi_{\mathsf{V}}\,Q_a + \bar{\Psi}_{\mathsf{V}}\,\bar{Q}_a \ , &(8.65)\\
\varepsilon_{ab}\Psi^b &= -\,\bar{\Psi}_{\mathsf{V}}\,Q_a + \Psi_{\mathsf{V}}\,\bar{Q}_a \ , &(8.66)\\
\delta^a\Psi_a &= -\,\ell\,(\ell+1)\,r^{-1}\Psi_{\mathsf{V}}\,Q \ , &(8.67)\\
\varepsilon_{ab}\delta^a\Psi^b &= \ell\,(\ell+1)\,r^{-1}\bar{\Psi}_{\mathsf{V}}\,Q \ , &(8.68)\\
\delta_{\{a}\Psi_{b\}} &= r^{-1}\left(\Psi_{\mathsf{V}}\,Q_{ab} - \bar{\Psi}_{\mathsf{V}}\,\bar{Q}_{ab}\right) \ , &(8.69)\\
\varepsilon_{c\{a}\delta^c\Psi_{b\}} &= r^{-1}\left(\bar{\Psi}_{\mathsf{V}}\,Q_{ab} + \Psi_{\mathsf{V}}\,\bar{Q}_{ab}\right) \ . &(8.70)
\end{aligned}$$

Tensor harmonics

We define even and odd tensor spherical harmonics for $\ell \geq 2$ as

$$Q_{ab} = r^2\,\delta_{\{a}\delta_{b\}}Q, \quad \Rightarrow \quad \hat{Q}_{ab} = 0 = \dot{Q}_{ab}, \quad \delta^2 Q_{ab} = \left[\phi^2 - 4\mathcal{E} - \ell\,(\ell+1)\,r^{-2}\right]Q_{ab} \ , \tag{8.71}$$

$$\bar{Q}_{ab} = r^2\,\varepsilon_{c\{a}\delta^c\delta_{b\}}Q \ , \quad \Rightarrow \quad \hat{\bar{Q}}_{ab} = 0 = \dot{\bar{Q}}_{ab}, \quad \delta^2\bar{Q}_{ab} = \left[\phi^2 - 4\mathcal{E} - \ell\,(\ell+1)\,r^{-2}\right]\bar{Q}_{ab}, \tag{8.72}$$

and as in the vector case they are orthogonal: $Q_{ab}\,\bar{Q}^{ab} = 0$, and parity inversions of one another: $Q_{ab} = -\varepsilon_{c\{a}\bar{Q}_{b\}}{}^c \Leftrightarrow \bar{Q}_{ab} = \varepsilon_{c\{a}Q_{b\}}{}^c$. Any first-order tensor may be expanded as

$$\Psi_{ab} = \sum_{\ell=2}^{\infty} \Psi_{\mathsf{T}}^{(\ell)}\,Q_{ab}^{(\ell)} + \bar{\Psi}_{\mathsf{T}}^{(\ell)}\,\bar{Q}_{ab}^{(\ell)} = \Psi_{\mathsf{T}}\,Q_{ab} + \bar{\Psi}_{\mathsf{T}}\,\bar{Q}_{ab} \ . \tag{8.73}$$

For the tensors, the following replacements must be made when expanding the equations in spherical harmonics:

$$\Psi_{ab} = \Psi_{\mathsf{T}}\, Q_{ab} + \bar{\Psi}_{\mathsf{T}}\, \bar{Q}_{ab} \ , \tag{8.74}$$

$$\varepsilon_{c\{a}\Psi_{b\}}{}^{c} = -\,\bar{\Psi}_{\mathsf{T}}\, Q_{ab} + \Psi_{\mathsf{T}}\, \bar{Q}_{ab} \ , \tag{8.75}$$

$$\delta^{b}\Psi_{ab} = \left[1 - \frac{1}{2}\ell(\ell+1)\right] r^{-1}\left(\Psi_{\mathsf{T}}\, Q_{a} - \bar{\Psi}_{\mathsf{T}}\, \bar{Q}_{a}\right) \ , \tag{8.76}$$

$$\varepsilon_{c\{d}\delta^{d}\Psi_{a\}}{}^{c} = -\left[1 - \frac{1}{2}\ell(\ell+1)\right] r^{-1}\left(\bar{\Psi}_{\mathsf{T}}\, Q_{a} + \Psi_{\mathsf{T}}\, \bar{Q}_{a}\right) \ . \tag{8.77}$$

Odd and even parity perturbations

Expanding the perturbations into spherical harmonics, leads to two independent subsets, namely:
Odd perturbations :

$$\begin{aligned} \mathbf{V}_{\mathsf{O}} \equiv & \{\bar{\mathcal{E}}_{\mathsf{T}},\ \mathcal{H}_{\mathsf{T}},\ \bar{\Sigma}_{\mathsf{T}},\ \bar{\zeta}_{\mathsf{T}}\} \ , \\ & \{\bar{\mathcal{E}}_{\mathsf{V}},\ \mathcal{H}_{\mathsf{V}},\ \bar{\Sigma}_{\mathsf{V}},\ \Omega_{\mathsf{V}},\ \bar{\mathcal{A}}_{\mathsf{V}},\ \bar{\alpha}_{\mathsf{V}},\ \bar{a}_{\mathsf{V}},\ \bar{X}_{\mathsf{V}},\ \bar{Y}_{\mathsf{V}},\ \bar{Z}_{\mathsf{V}}\} \ , \\ & \{\mathcal{H}_{\mathsf{S}},\ \Omega_{\mathsf{S}},\ \xi_{\mathsf{S}}\} \ ; \end{aligned} \tag{8.78}$$

Even perturbations :

$$\begin{aligned} \mathbf{V}_{\mathsf{E}} \equiv & \{\mathcal{E}_{\mathsf{T}},\ \bar{\mathcal{H}}_{\mathsf{T}},\ \Sigma_{\mathsf{T}},\ \zeta_{\mathsf{T}}\} \ , \\ & \{\mathcal{E}_{\mathsf{V}},\ \bar{\mathcal{H}}_{\mathsf{V}},\ \Sigma_{\mathsf{V}},\ \bar{\Omega}_{\mathsf{V}},\ \mathcal{A}_{\mathsf{V}},\ \alpha_{\mathsf{V}},\ a_{\mathsf{V}},\ X_{\mathsf{V}},\ Y_{\mathsf{V}},\ Z_{\mathsf{V}}\} \ , \\ & \{\Sigma_{\mathsf{S}},\ \theta_{\mathsf{S}}\ R_{\mathsf{S}}\} \ ; \end{aligned} \tag{8.79}$$

whose resulting equations are decoupled from each other as presented in Appendix B. We remark on the 'parity switching' which occurs between the sets of variables. We see in the equations that these terms always appear alongside a 'ε_{ab}' factor relative to other variables (e.g., $\mathcal{H}_{ab}$ and Ω^{a} appear alongside 'ε_{ab}' relative to variables such as $\mathcal{E}_{ab}$ and Σ^{a}).

8.3.3 Time harmonics

Since the background is static, we can resolve the perturbations into temporal harmonics. We do this by performing a Fourier analysis of the time derivatives of the first order quantities by decomposing them into their Fourier components. This corresponds to assuming a harmonic time dependence $e^{i\omega\tau}$ for the first order variables.

We define the time harmonic functions $T^{(\omega)}$ in the background by

$$\dot{T}^{(\omega)} = i\,\omega\, T^{(\omega)}, \quad \hat{T}^{(\omega)} = 0 = \delta_{a}T^{(\omega)}; \quad \dot{\omega} = 0 = \delta_{a}\omega \ , \tag{8.80}$$

and from the commutation relation between the *dot*- and *hat*- derivatives this must satisfy

$$\hat{\dot{T}} + \mathcal{A}\dot{T} = 0 \ , \tag{8.81}$$

which in turn implies

$$\hat{\omega} = -\mathcal{A}\,\omega \ , \tag{8.82}$$

in the background. Integrating (8.82) in terms of r, gives

$$\omega = \sigma\left(1 - \frac{2m}{r}\right)^{-1/2} = \frac{2\sigma}{\phi\, r} \ , \tag{8.83}$$

where σ is a constant. Then any first order variable Ψ may be expanded as

$$\Psi = \sum_{\omega} \Psi^{(\omega)} T^{(\omega)} = \Psi^{(\omega)} T^{(\omega)} \ , \tag{8.84}$$

and the dot - '$\cdot$' derivatives of these first order quantities can be replaced by factors of $i\,\omega$.

8.4 The Regge-Wheeler equation

In GR, the gravitational perturbations of Schwarzschild black holes are governed by a single second-order wave equation, namely the Regge-Wheeler equation [88] describing the odd perturbations and the Zerilli equation [89] describing the even perturbations. Both the equations satisfy a Schrödinger-like equation and it was demonstrated in [170] that the effective potentials of these equations have the same spectra. The aim of this section is to perform an analysis of the perturbation of Schwarzschild BH in $f(R)$ gravity and find a reduced set of master variables which obey a closed set of wave equations for these theories.

8.4.1 Gravitational perturbations

If we consider very large distances from the source ($\mathcal{A} = \phi = 0$), the gravitational perturbations should be well approximated by a plane wave, with e^a lying in the direction of propagation. On imposing the condition that R vanishes at infinity, the plane gravitational waves are described by the 1+1+2 transverse-traceless tensors $\mathcal{E}_{ab}$, $\mathcal{H}_{ab}$, Σ_{ab} and ζ_{ab} only, as in GR. Otherwise there is coupling with the scalar waves which can produce other scalar and vector modes. The tensors $\mathcal{E}_{ab}$ and $\mathcal{H}_{ab}$ represent the tidal and gravitational waves effects in analogy with the propagation of electromagnetic waves. However, the wave equations for these two tensors do not close in the general frame.

If we now consider the general case, apart from the four TT tensors, a number of other TT tensors can be constructed from the δ- derivatives of vectors and scalars in general, for example, $\delta_{\{a}W_{b\}}$, $\delta_{\{a}a_{b\}}$, $\delta_{\{a}\delta_{b\}}\Omega$, etc. The wave equations for these tensors can be calculated by applying the wave operator $\ddot{\Psi}_{\{ab\}} - \hat{\hat{\Psi}}_{\{ab\}}$ for a tensor Ψ_{ab} [65]. The aim here is to calculate all such possible wave equations involving these tensors and systematically eliminating unwanted terms until a closed equation is obtained. In particular, calculating the wave operator for ζ_{ab} and $\delta_{\{a}W_{b\}}$, we notice that they contain similar terms.

We consider the case of the wave operator for ζ_{ab}, that is, $\ddot{\zeta}_{\{ab\}} - \hat{\hat{\zeta}}_{\{ab\}}$, where we apply the following steps:

– Take the dot- derivative across (8.10), for which the resulting evolution equations are substituted.

– Substitute for a_a from (8.36) and α_a from (8.35) (while utilising the constraints (8.39), (8.12),(8.41), (8.29) and (8.40) to substitute for ξ, Σ, Z_a even Y_a and odd Y_a respectively).

What follows is an expression consisting of only $\delta_{\{a}W_{b\}}$ and ζ_{ab}, for the odd harmonics and $\delta_{\{a}X_{b\}}$, ζ_{ab} and $\delta_{\{a}\delta_{b\}}R$ for the even harmonics. We can recast this result as the wave equation,

$$\ddot{M}_{\{ab\}} - \hat{\hat{M}}_{\{ab\}} - \mathcal{A}\,\hat{M}_{\{ab\}} + \left(\phi^2 + \mathcal{E}\right) M_{ab} - \delta^2 M_{ab} = 0 \ , \tag{8.85}$$

where we have introduced the dimensionless, gauge-invariant, frame-invariant, transverse-traceless tensor M_{ab} defined as

$$M_{ab} = \frac{1}{2}\phi\, r^2\, \zeta_{ab} - \frac{1}{3} r^2\, \mathcal{E}^{-1}\, \delta_{\{a}W_{b\}} + \frac{f_0''}{3\, f_0'}\, r^2\, \delta_{\{a}\delta_{b\}}R \ . \tag{8.86}$$

The even part of (8.86) is coupled to the curvature term and as a result we have to include the trace equation (8.30) to achieve closure. On the other hand, the curvature term vanishes for the odd part of M_{ab} and this leaves the tensor in exactly the same form as in the GR case [65].

We can expand (8.85) into scalar harmonics as

$$\ddot{M} - \hat{\hat{M}} - \mathcal{A}\,\hat{M} + \left[\frac{\ell\,(\ell+1)}{r^2} + 3\mathcal{E}\right] M = 0 \ , \tag{8.87}$$

where we let $M = \{M_{\mathsf{T}}, \overline{M}_{\mathsf{T}}\}$. In appropriate coordinates the wave equation (8.87) is the *Regge-Wheeler equation.*

We note that both the odd and even parity parts of M_{ab} satisfy the same wave equation (8.87). We convert to the parameter r, using (7.47) and (7.55) and use the time harmonics in (8.87) to get

$$\kappa^2 M - \frac{2m}{r^2}\left[\frac{2m-r}{r}\right]\frac{dM}{dr} + \left(\frac{2m-r}{r}\right)^2\frac{d^2M}{dr^2} + \left(\frac{2m-r}{r}\right)\left[\frac{\ell\,(\ell+1)}{r^2} - \frac{6m}{r^3}\right] M = 0\ . \tag{8.88}$$

We then make a change to the 'tortoise' coordinate r_*, which is related to r by

$$r_* = r + 2m\,\ln\left(\frac{r}{2m} - 1\right)\ , \tag{8.89}$$

thus, (8.88) can be written in the form

$$\left(\frac{d^2}{dr_*^2} + \kappa^2 - V_{\mathsf{T}}\right) M = 0\ , \tag{8.90}$$

with the effective potential V_{T}

$$V_{\mathsf{T}} = \left(1 - \frac{2m}{r}\right)\left[\frac{\ell\,(\ell+1)}{r^2} - \frac{6m}{r^3}\right]\ , \tag{8.91}$$

which is the *Regge-Wheeler potential* for *gravitational perturbations*.

8.4.2 Scalar perturbations

The trace equation (8.30), which is a wave equation in R, corresponds to scalar modes that are not present in standard GR but occur in $f(R)$ theories of gravity due to the extra scalar degree of freedom. The equation constitutes the same generalised Regge-Wheeler equation for massive scalar perturbations on LRS background spacetimes in GR with

$$U^2 = \frac{f_0'}{3\,f_0''}\ , \tag{8.92}$$

as the effective mass of the scalar.

To obtain the familiar Regge-Wheeler equation we first rescale R as $R = r^{-1}\,\mathcal{R}$ and use (8.52) and (7.50) to rewrite equation (8.30) in the form

$$\ddot{\mathcal{R}} - \hat{\hat{\mathcal{R}}} - \mathcal{A}\,\hat{\mathcal{R}} - \left(\mathcal{E} - U^2 + \delta^2\right)\mathcal{R} = 0\ . \tag{8.93}$$

Proceeding as in the previous case, we introduce scalar spherical harmonics to (8.93) resulting in

$$\ddot{\mathcal{R}}_{\mathsf{S}} - \hat{\hat{\mathcal{R}}}_{\mathsf{S}} - \mathcal{A}\,\hat{\mathcal{R}}_{\mathsf{S}} - \left[\mathcal{E} - \tilde{U}^2 - \frac{\ell(\ell+1)}{r^2}\right]\mathcal{R}_{\mathsf{S}} = 0\ . \tag{8.94}$$

where $\tilde{U}^2 = \mathcal{C}_1/(3\,\mathcal{C}_2)$. Converting to the parameter r and then the tortoise coordinate, we get

$$\left(\frac{d^2}{dr_*^2} + \kappa^2 - V_{\mathsf{S}}\right)\mathcal{R} = 0\ , \tag{8.95}$$

where

$$V_{\mathsf{S}} = \left(1 - \frac{2m}{r}\right)\left[\frac{\ell\,(\ell+1)}{r^2} + \frac{2m}{r^3} + \tilde{U}^2\right]\ . \tag{8.96}$$

The expression (8.96) is the *Regge-Wheeler potential* for the *scalar perturbations.*

8.4.3 Potential profile

The form of the wave equations (8.90) and (8.95) describing black hole perturbation is similar to a one dimensional Schrödinger equation and hence their potentials correspond to a single potential barrier. We consider the potential profile of the effective potentials V_{T} and V_{S} in a Schwarzschild BH case for the gravitational and the scalar fields respectively. The Regge-Wheeler equations (8.90) and (8.95) can be made dimensionless by dividing through by the BH mass m. In this way the potential (8.91) and (8.96) become

$$V_{\mathsf{T}} \;=\; \left(1 - \frac{2}{r}\right)\left[\frac{\ell\,(\ell+1)}{r^2} - \frac{6}{r^3}\right]\ , \tag{8.97}$$

$$V_{\mathsf{S}} \;=\; \left(1 - \frac{2}{r}\right)\left[\frac{\ell\,(\ell+1)}{r^2} + \frac{2}{r^3} + u^2\right]\ , \tag{8.98}$$

where we have defined (and dropped the primes),

$$\kappa' = m\,\kappa\ , \qquad r' = \frac{r}{m}\ , \qquad u = m\,\tilde{U}\ . \tag{8.99}$$

For the gravitational perturbations and the scalar perturbations with $u = 0$, the derivative of the potential has two roots with one in the unphysical region $r < 0$ and the other one in the region $r > 0$ corresponding to a maximum of the potential. For the scalar perturbations with $u \neq 0$, the potential has three extrema: one in the unphysical region $r < 0$, a local maximum at r_{max} and local minimum at r_{min} in the region $r > 0$ such that $2 < r_{max} < r_{min}$.

Fig 8.1 shows a plot of the potential for the gravitational field for different ℓ as a function of the Schwarzschild radial coordinate r (a) and the tortoise coordinates r_* (b). In this case the potential decays exponentially near the horizon and as $1/r^2$ at spatial infinity.

Fig 8.2 shows the potential profile for the scalar field for several values of $u =$ at $\ell = 2$ (a) and at $\ell = 3$ (b). We see that the effect of the massive term $\tilde{U}$ is to move the asymptotic value of the potential of scalar perturbations up by u^2 and to cause the potential to approach the asymptotic value slowly. Moreover, increasing the value of u causes the peak of the potential to broaden as the peak value decreases relative to the asymptotic value.

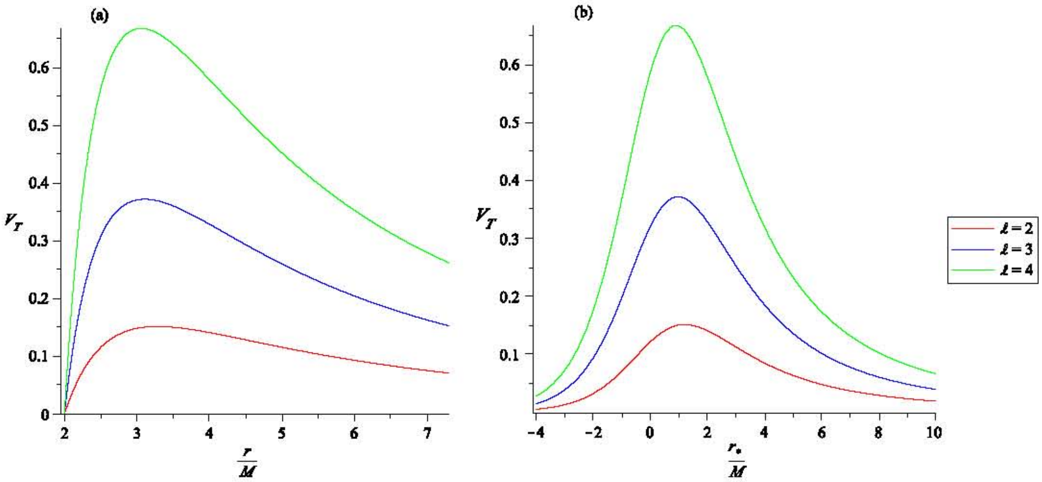

Figure 8.1: The potential for the gravitational field for $\ell = 2, 3, 4$ as a function of r (a) and r_* (b).

The peak eventually disappears altogether when u exceeds a certain value.

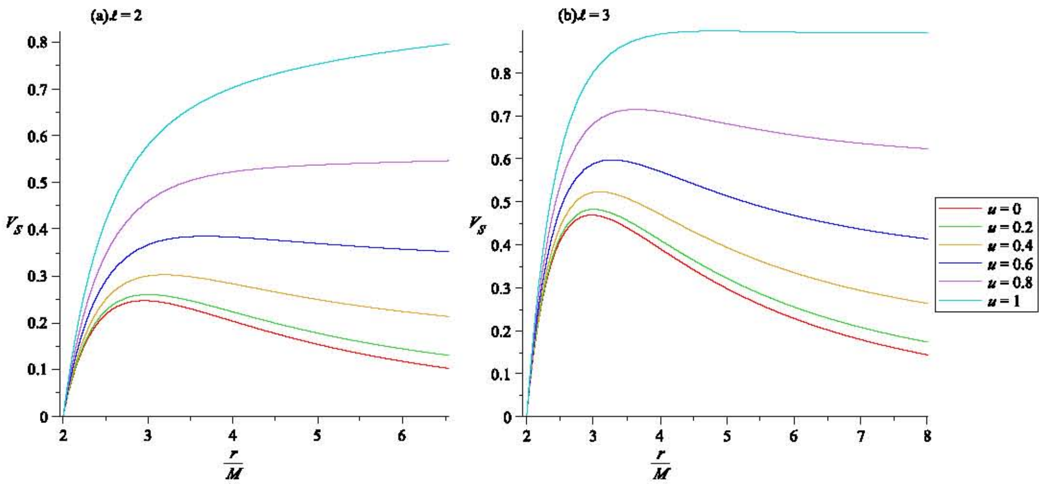

Figure 8.2: The potential for the scalar field for different u as a function of r for $\ell = 2$ (a) and $\ell = 3$ (b).

8.4.4 Black hole stability

We now investigate the stability of the BH to external perturbations which is pegged on the BH remaining bounded in time as it evolves.

The asymptotic behaviour of the solutions to (8.90) is given as

$$\psi \sim e^{\pm i\kappa r_*} \ , \tag{8.100}$$

both at the horizon and at spatial infinity. If we consider purely imaginary solutions such that we set $\kappa = -i\alpha$, then the time dependence of the perturbations becomes $e^{\alpha t}$ which is unstable owing to the fact that they grow exponentially with time. For regularity, we require the perturbation to fall off to zero at spatial infinity and therefore choose

$$\psi \sim e^{-\alpha r_*} \; . \tag{8.101}$$

If (8.101) is to be matched to the solution that goes to zero at the horizon, then $d\psi/dr_* < 0$, $d^2\psi/dr_*^2 < 0$ within the range $-\infty$ to ∞. However, this is not the case since the potential is positive definite and as a result (8.90) never becomes negative in this range. Since the solutions cannot be matched, this rules out perturbations that grow exponentially with time. This proof of stability of a BH was first provided by [91]. Later on [171,172] provided a more rigorous proof using the energy integral. This can be derived by first considering the time dependent version of (8.90)

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r_*^2} + V_{\mathsf{T}}\right) M = 0 \; . \tag{8.102}$$

(recalling that the time dependence was replaced by the factor $e^{i\omega t}$ when we considered time harmonics). Multiplying (8.102) by the partial derivative of the complex conjugate M^* with respect to time and then adding the resulting equation to its complex conjugate we get

$$\frac{\partial}{\partial r_*}\left(\frac{\partial M^*}{\partial t}\frac{\partial M}{\partial r_*} + \frac{\partial M^*}{\partial t}\frac{\partial M}{\partial r_*}\right) = \frac{\partial}{\partial t}\left(\left|\frac{\partial M}{\partial t}\right|^2 + \left|\frac{\partial M}{\partial r_*}\right|^2 + V_{\mathsf{T}}\,|M|^2\right) \; . \tag{8.103}$$

After integration by parts over r_* from $-\infty$ to ∞, the left-hand side of (8.103) vanishes and we obtain the *energy integral*,

$$\int_{-\infty}^{\infty}\left(\left|\frac{\partial M}{\partial t}\right|^2 + \left|\frac{\partial M}{\partial r_*}\right|^2 + V_{\mathsf{T}}\,|M|^2\right) dr_* = constant \; . \tag{8.104}$$

Since V_{T} is positive definite, the integral (8.104) bounds the integral of $|\partial M/\partial t|^2$ and it therefore excludes exponential growing solutions to (8.90). The above energy integral argument for stability falls short of a complete proof as it does not rule out perturbations that grow linearly with t. Also, since we have only provided the bounds for integrals of M, the perturbation may still blow up as $r \to \infty$.

The proof of stability for the scalar perturbations depends on $\tilde{U}$. The potential V_{S} in (8.96) remains positive definite subject to the condition

$$\tilde{U}^2 = \frac{\mathcal{C}_1}{3\,\mathcal{C}_2} \geq 0 \; . \tag{8.105}$$

A different type of instability will be the tachyonic instabilities associated with these modes if $\mathcal{C}_1 \leq 0$. Both these instabilities do not arise, however, as we have shown in Chapter 7 that the necessary conditions for the existence of a Schwarzschild BH solution in $f(R)$ theories are consistent with the requirement that $\mathcal{C}_1 > 0$ and $\mathcal{C}_2 > 0$.

8.4.5 Quasinormal modes

The gravitational wave radiation from a perturbed BH can in general be divided into three components:

(i) an initial pulse emitted directly by the perturbation source depending on the initial conditions;

(ii) an exponentially damped oscillation (ringing) at intermediate times characterised by a single complex frequency, which doesn't depend on the source but is characteristic of the BH parameters;

(iii) a power-law tail that develops after the ringing at very late times.

The ringing phase is due to a superposition of *quasinormal modes* (QNMs) of the BH. We see from (8.90) and (8.95) that for the $f(R)$ Schwarzschild black hole, the linearised equations lead to the same equations as for GR for gravitational and scalar perturbations respectively. Comprehensive reviews on BH and QNMs can be found in [173–176].

The gravitational QNMs are solutions to the Regge-Wheeler equation (8.90) subject to the boundary conditions

$$M \sim \begin{cases} e^{i\kappa r_*} & \text{for} \quad r_* \to -\infty \\ e^{-i\kappa r_*} & \text{for} \quad r_* \to +\infty \ . \end{cases} \tag{8.106}$$

These boundary conditions (8.106) represent purely outgoing waves at infinity ($r \sim r_* \to \infty$) and purely ingoing waves at the horizon ($r \to 2m$, $r_* \to -\infty$). In other words we want to discard unwanted contributions at the event horizon and at spatial infinity, as we do not want gravitational radiation entering the spacetime from infinity to continue to perturb the BH, nor do we want waves coming from the vicinity of the horizon due to external sources like accretion of matter.

To obtain solutions to (8.90) and (8.95) requires discrete values of the frequency parameter κ called *quasinormal frequencies* and the solutions constructed from them are the quasinormal modes. The quasinormal frequencies have both a real and imaginary part which we write as

$$\kappa = \Re(\kappa) + \Im(\kappa) \ . \tag{8.107}$$

Since QNMs are characterised by the parameters of the BH [91], we expect the imaginary part to be damped with time for each value of r_* due to energy being radiated to infinity or the horizon. If we then consider that in (8.90) and (8.95) that the time dependence has been replaced by the factor $e^{i\omega t}$, we expect to have $\psi \sim e^{i\kappa(t-r_*)}$ at spatial infinity. We see from this that $\Im(\kappa) < 0$ corresponds to a bound state since the solution (8.106) vanishes exponentially for $r_* \to +\infty$. This option for a negative imaginary part is excluded since the potential V_{T} decays towards spatial infinity and therefore disallows these bound states. We can therefore only have $\Im(\kappa) > 0$ which corresponds to the solution being damped with time but diverges exponentially as $r_* \to +\infty$ on a hypersurface of constant time; the same holds for the horizon. This consequence of divergence is balanced out by the fact that it takes the signal an infinite time to reach, for example, spatial infinity.

The scalar QNMs correspond to solutions of (8.95) with

$$\mathcal{R} \sim \begin{cases} e^{i\chi r_*} & \text{for} \quad r_* \to -\infty \\ e^{-i\chi r_*} & \text{for} \quad r_* \to +\infty \,, \end{cases} \tag{8.108}$$

where $\chi = \sqrt{\kappa^2 - \tilde{U}^2}$ for the scalar field. For the choices $\Im(\kappa) \approx 0$ and $\kappa \leq \tilde{U}$, there will be no energy radiating into infinity. The sign of χ is chosen so as to be in the same complex surface quadrant as κ.

8.4.5.1 Methods for computing quasinormal frequencies

There have been numerous attempts to calculate QNMs to high accuracy using numerical and semi-analytical methods. Difficulties arise from, for example, the admixture of the solutions such that the exponentially growing required solution gets contaminated by traces of the unwanted solution which decreases exponentially as we approach the boundaries. In 1975, Chandrasekhar and Detweiler [170] computed numerically the first few modes and in 1985, Leaver [177] proposed the most accurate method to date. We list here some of the methods that have been employed:

- Continued fraction method by Leaver [177], which was later improved by Nollert [178] to cater for quasinormal frequencies with very large imaginary parts. This is based on the observation that the Teukolsky equation is a special case of a class of spheroidal wave equations that appear in the determination of the eigenvalues of the H_2^+ ion. The quasinormal frequencies are calculated from the recurrence relations constructed for the coefficients of the series representation of the solutions of the equations governing the perturbations.

- Laplace transforms approach by Nollert and Schmidt [179] where the QNMs are regarded as the poles of the Green's function for the Laplace transformed solution of the time-dependent equations governing the perturbations.

– The inverted BH effective potentials approach by Mashhoon and Ferrari [180–182]. They provided an analytical approach to the problem by approximating the Regge-Wheeler potential in the wave equation governing the perturbations with other potentials. The parameters of these potentials are adjusted to obtain a good fit to the Regge-Wheeler potential near its maximum. This method doesn't allow for the determination of frequencies with large imaginary parts as these highly damped modes are more sensitive to changes in the potential far away from its maximum.

– WKB approach by Schutz, Will and Iyer [183–185]. This semi-analytical procedure is based on reducing QNM problem into the standard JWKB treatment of scattering of waves on the peak of the potential barrier in quantum mechanics. It involves relating matching of the asymptotic WKB solutions at spatial infinity and the event horizon with the Taylor expansion near the top of the potential barrier across the turning points. A QNM is expected to have a frequency such that the square of the frequency is approximately equal to the peak of the potential. The method works best for modes with relatively small imaginary parts.

Other methods include the phase integral approach [186] and the monodromy technique [187].

8.4.5.2 Results on gravitational field quasinormal modes

The low lying frequencies for gravitational QNMs start with comparatively large real parts and small imaginary parts. The imaginary part grows, while the real part decreases until it becomes almost zero at an overtone index $n = 9$ when $\ell = 2$, to $n = 41$ when $\ell = 3$. This point corresponds to a mode whose frequency is (almost) purely imaginary with n increasing with ℓ and is very close to the so-called algebraically special mode [188] located at

$$m\,\kappa \approx \pm\, i(\ell-1)\ell(\ell+1)(\ell+2)/12\ . \tag{8.109}$$

This algebraically special mode approximately marks the onset of the asymptotic high damping regime, such that the real part of the modes higher than (8.109) starts growing and approaches its asymptotical value. Fig 8.3 shows the low gravitational QNMs [3] of Schwarzschild black holes, calculated using the continued fraction method [177, 178].

Weakly damped modes: Mashhoon [180], Schutz and Will [183] have shown that the complex frequency for the fundamental quasinormal frequency ($n = 0$) and frequencies with small imaginary parts (small n) can be estimated from the relation

$$(2m\,\kappa)^2 \approx 4V(r_m) - 4\,i\,\left(n+\frac{1}{2}\right)\left(-2\,\frac{d^2V(r_m)}{dr_*^2}\right)^{1/2} \tag{8.110}$$

[3] Numerical data of 1000 QNMs is available from http://qnms.way.to

where the peak of the potential barrier is at r_m.

Highly damped modes: Using a variation of Leaver's method, Nollert [178] showed that the real part of the gravitational quasinormal frequencies approaches a constant value. Various other numerical and analytical techniques [187, 189] confirm his results which show

$$|\kappa_{\Im}| \to \infty \ , \quad \text{while} \quad \kappa_{\Re} \to T \ \ln 3 \ , \tag{8.111}$$

where $T = (8\pi\, m)^{-1}$ is the Hawking temperature.

Modes with large ℓ: The large multipole limit of QNMs has been determined analytically as

$$2m\,\kappa \approx \frac{1}{3\sqrt{3}}\left[2\ell + 1 + i\,(2n+1)\right] \tag{8.112}$$

in [181, 182, 185, 190].

8.4.5.3 Results on scalar field quasinormal modes

For the scalar field perturbations, studies have shown that the mass of the field has crucial influence on the damping rate of the QNMs. Using the WKB approximation [191–193], it was found that when the massive term u of the scalar field increases, the damping rate decreases. The WKB method that was used in this analysis is valid for $n < \ell$ and within this restriction, the approximation breaks down for large u. This is due to the potential losing its maximum as it drops relative to the asymptotic value (see Fig 8.2). The procedure requires modification [194] to avoid this problem.

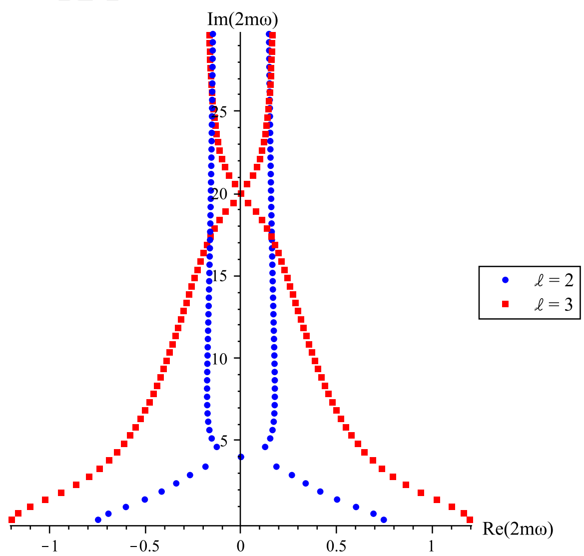

Figure 8.3: Plot of the first 60 QNMs for $\ell = 2$ and $\ell = 3$.

Later calculations using Leaver's method showed that as a result of the decreasing damping rates, for certain values of u, there are QNM oscillations with arbitrary long life [195, 196]. These 'almost' purely real modes are called *quasiresonant modes*, a term originally coined by Ohashi and Sakagami [195]. It has also been found that there is a threshold value of u above which the QNMs may disappear, at least for the lower overtones only. The higher overtones will continue to decay with time [196].

It is important to note that the massive u term affects the lower QNMs only as was observed in [196]. They showed that for asymptotically high overtones ($n \to \infty$), the real part of the frequencies approaches the same asymptotical value $\ln 3(8\pi\, m)^{-1}$ as in the gravitational field case (8.111).

In GR the possible sources of massive scalar QNMs are from the collapse of objects made up of self-gravitating scalar fields ('boson' stars) [197–199], in situations where the massless field gains an effective mass [200] or as scalar field dark matter [201]. In order to illustrate what these results mean for $f(R)$ theories of gravity we restrict our attention to the $\ell = 0$ multipole of the field. From [195], the cut-off mass at which the QNMs disappear for these modes is approximately at $m\tilde{U} = 0.4 - 0.5$ and from PPN constraints [43] for these theories we obtain the bound for $\tilde{U}$ as

$$\tilde{U}^2 = \frac{\mathcal{C}_1}{3\mathcal{C}_2} \gg \frac{2}{L^2} \tag{8.113}$$

where L is the smallest length scale on which Newtonian gravity has been observed. Recent results [202] place at $L \sim 10\,\mu m$ and using this we can set (8.113) as

$$\tilde{U} \gg 1.4 \times 10^5\, m^{-1} \tag{8.114}$$

Given these details, we can estimate that the mass of the BH associated with the disappearance of the QNMs

$$\text{BH mass} \ll 4\,\mu m\ . \tag{8.115}$$

Such a BH could only have been formed from density fluctuations in the early universe [203, 204]. What is more, if these primordial BH are to be detected now, they would have to have an initial mass of subatomic scales ($\sim 10^{-16} m$) [205]. These results apply to QNMs at lower overtones and even then, QNMs are short-ranged, making their detection currently unfeasible [176].

Chapter 9

Solutions to the perturbation equations

9.1 Structure of the equations

The structure of the system of governing equations for the perturbations is divided into three distinct types of equations: evolution, propagation, and constraint equations. The true degrees of freedom of this system is governed by the reduced set of master variables M and R, which obey the covariant, gauge-invariant tensorial equations (8.85) and (8.30), respectively. All other variables are then related to these master variables by quadrature, plus frame degrees of freedom. Harmonic expansion of the perturbation equations allows us, at any radial position from the black hole, to structure the equations in matrix form. The harmonic variables in (8.78) and (8.79) can then be treated as 'coordinates', that is, as dictating a 34-dimensional vector space $\mathcal{V}_{34}$. We then analyse the system of equations to obtain solutions. In this section, we itemise the procedure for this analysis, as set out in [65].

– After adopting spherical harmonic decomposition (see Appendix B), the number of variables in the system of equations is 34 in total. Let $\mathbf{V}$ denote the 34-dimensional vector consisting of these odd $\mathbf{V}_{\mathsf{O}}$ and even $\mathbf{V}_{\mathsf{E}}$ variables as presented in (8.78) and (8.79) respectively, such that

$$\mathbf{V} = (\text{Odd variables} \mid \text{Even variables}) = (\mathbf{V}_{\mathsf{O}}, \mathbf{V}_{\mathsf{E}}) \; . \tag{9.1}$$

– Further, inserting time harmonics into these equations (as discussed in Subsection 8.3.3) so that dot derivatives are everywhere replaced by $i\omega$, results in:

  • 29 propagation equations which constitute a linear system of ODE's

$$\hat{\mathbf{V}}_{29} = \mathbf{P}\,\mathbf{V} \; , \tag{9.2}$$

where $\mathbf{V}_{29}$ denotes the vector consisting of the 29 elements of $\mathbf{V}$ which have a propagation equation and $\mathbf{P}$ is the 29×34 propagation matrix, which contains $i\omega$ terms comprising evolution equations which have hat derivatives in them.

- 26 algebraic relations between the variables, comprising 18 equations from the evolution equations without a propagation derivative in them and 7 constraints. These take the form, in matrix notation

$$\mathbf{F}\,\mathbf{V} = \mathbf{0} \;, \tag{9.3}$$

where $\mathbf{F}$ is a 25×34 matrix. In view of the fact that the constraints propagate and evolve consistently, except for the constraint (8.41), the rows that make up the constraints are just linear combinations of the 18 rows that make up the algebraic relations derived from the evolution equations. The constraint (8.41) is excluded because it is *not* represented in the evolution equations owing to the fact that there is no equation for Z^a. This means that 6 of the rows in $\mathbf{F}$ give no additional information, resulting in $\mathbf{F}$ being of rank 19.

– So far, the formulation has resulted in 34 unknowns and 19 algebraic relations in the system which corresponds to $34 - 19 = 15$ degrees of freedom. This means that there are 15 variables to be solved, which we denote by $\mathbf{v}$, and write

$$\mathbf{V} = \mathbf{C}\,\mathbf{v} \;, \tag{9.4}$$

where $\mathbf{C}$ is a 34×15 matrix of the form

$$\left(\begin{array}{c|c} \longleftarrow \quad 6 \quad \longrightarrow & \longleftarrow \quad 9 \quad \longrightarrow \\ & \\ \text{ODD} & \text{EVEN} \\ & \end{array} \right) . \tag{9.5}$$

– We now split the vector $\mathbf{v}$ into two parts: $\mathbf{v} = (\mathbf{v}_D, \mathbf{v}_F)$, a 'determined' part containing 10 variables which have an individual propagation equation and a part containing $15 - 10 = 5$ variables which do not, corresponding to 5 frame degrees of freedom. If we insert (9.4) into the propagation equation, (9.2) we get the underlying propagation equation for the solution vector as

$$\hat{\mathbf{v}}_D = \mathbf{B}\,\mathbf{v}_D + \mathbf{A}\,\mathbf{v}_F, \tag{9.6}$$

where $\mathbf{B}$ is a 10×10 matrix and $\mathbf{A}$ is 10×5.

– Finally, since we can arbitrarily choose the 5 frame degrees of freedom ($\mathbf{v}_F$), we find that there are only $10 - 5 = 5$ true dynamical propagation equations. This means that only 5 components of $\mathbf{v}_D$ are unknowns for which we have propagation equations.

9.2 Determining the full solution

9.2.1 Odd

9.2.1.1 General frame

The problem of finding a solution is in deciding which variables to choose as the basis. If we don't specify a frame choice, and choose our solution vector as, say, $\mathbf{v}_{D_\mathsf{O}} = (\bar{\Sigma}_\mathsf{T},\ \bar{\zeta}_\mathsf{T},\ \bar{W}_\mathsf{V},\ \bar{\mathcal{A}}_\mathsf{V})$, then there are two undetermined variables, which we can choose to be $\mathbf{v}_{F_\mathsf{O}} = (\Omega_\mathsf{V},\ \bar{a}_\mathsf{V})$; the 4-dimensional dynamical system in general is therefore

$$\hat{\mathbf{v}}_D = \mathbf{B}^{\mathsf{g}}{}_\mathsf{O}\,\mathbf{v}_D + \mathbf{A}_\mathsf{O}\,\mathbf{v}_F\ . \tag{9.7}$$

where the 'g' stands for 'general frame'. The remaining variables are linear combinations of elements of $\mathbf{v}_{D_\mathsf{O}}$, except $\bar{\Sigma}_\mathsf{V}$, which depends on Ω_V and nothing depends on $\bar{a}_\mathsf{V}$.

9.2.1.2 Specific frame

To concur with [65] for the GR case, we will choose the frame such that $\bar{Y}_\mathsf{V} = \bar{\mathcal{A}}_\mathsf{V} = 0$ which implies that $\xi_\mathsf{S} = \Omega_\mathsf{S} = \bar{a}_\mathsf{V} = \overline{W}_\mathsf{V} = \bar{Z}_\mathsf{V} = \Omega_\mathsf{V} = 0$. The basis vector for the solution is chosen to be

$$\mathbf{v} = \begin{pmatrix} \overline{M}_\mathsf{T} \\ \hat{\overline{M}}_\mathsf{T} \end{pmatrix}; \tag{9.8}$$

that is, the governing DE will be the Regge-Wheeler equation. With regard to (9.4), the remaining variables in terms of this solution basis vector are given by

$$\begin{pmatrix} \bar{\mathcal{E}}_\mathsf{T} \\ \mathcal{H}_\mathsf{T} \\ \bar{\Sigma}_\mathsf{T} \\ \bar{\zeta}_\mathsf{T} \\ \bar{\mathcal{E}}_\mathsf{V} \\ \mathcal{H}_\mathsf{V} \\ \bar{\Sigma}_\mathsf{V} \\ \Omega_\mathsf{V} \\ \bar{\mathcal{A}}_\mathsf{V} \\ \bar{\alpha}_\mathsf{V} \\ \bar{a}_\mathsf{V} \\ \overline{W}_\mathsf{V} \\ \bar{Y}_\mathsf{V} \\ \bar{Z}_\mathsf{V} \\ \mathcal{H}_\mathsf{S} \\ \Omega_\mathsf{S} \\ \xi_\mathsf{S} \end{pmatrix} = \begin{pmatrix} -J/2\phi^2 r^4 & -2/\phi r^2 \\ \left(-4L + J + 8r^2\omega^2 + 16\right)/4i\omega\phi r^4 & -\,J/2i\omega\phi^2 r^4 \\ 1/i\omega r^2 & 2/i\omega\phi r^2 \\ 2/\phi r^2 & 0 \\ l/\phi r^3 & 0 \\ 0 & -l/i\omega\phi r^3 \\ -l/i\omega\phi r^3 & 0 \\ 0 & 0 \\ 0 & 0 \\ l/i\omega\phi r^3 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -Ll/i\omega\phi r^4 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \overline{M}_\mathsf{T} \\ \hat{\overline{M}}_\mathsf{T} \end{pmatrix} \qquad (9.9)$$

where for the sake of brevity we have used the aliases

$$L = \ell(\ell+1) \ , \qquad (9.10)$$

$$l = (\ell-1)(\ell+2) = L - 2 \ . \qquad (9.11)$$

9.2.2 Even

9.2.2.1 General frame

Without specifying a frame choice, we can choose the set $\mathbf{v}_{D_\mathsf{E}} = (\Sigma_\mathsf{T},\ \zeta_\mathsf{T},\ \Sigma_\mathsf{V} + \bar{\Omega}_\mathsf{V},\ a_\mathsf{V}\ W_\mathsf{V},\ \mathcal{R}_\mathsf{S})$, in terms of which we can solve the system of equations. In this case there are three undetermined variables which we can choose to be $\mathbf{v}_{F_\mathsf{E}} = (\bar{\Omega}_\mathsf{V},\ a_\mathsf{V},\ \Theta_\mathsf{S})$; following which our 6-dimensional differential equation in general is

$$\hat{\mathbf{v}}_E = \mathbf{B^g}_\mathsf{E}\,\mathbf{v}_D + \mathbf{A}_\mathsf{E}\,\mathbf{v}_F \ . \qquad (9.12)$$

The remaining variables are linear combinations of elements of $\mathbf{v}_{D_\mathsf{E}}$, except Σ_V, which depends on $\bar{\Omega}_\mathsf{V}$ and Σ_S, which depends on Θ_S.

9.2.2.2 Specific frame

As in the odd case, we choose our solution vector, $\mathcal{A}_{\mathsf{V}} = Y_{\mathsf{V}} = 0$ (and hence $Z_{\mathsf{V}} = 0$). In this case we will choose M_{T}, $\hat{M}_{\mathsf{T}}$, R_{S} and $\hat{R}_{\mathsf{S}}$ as the basis vector for the full solution. The expressions for the obtained solutions are rather huge and so in the interest of brevity we introduce the variable $\mathcal{M}$ as a function of the basis variables such that

$$\begin{aligned}\mathcal{M} \;=\; & \frac{1}{24c_3\mathcal{C}_1\,(L^2\,l^2 - \mathcal{A}^2(4L+4-c_3)^2 r^4\,\omega^2)}\Big\{-i\,\omega\,\phi\,r[96\,L\,l(L+1) - 3(8l\,(L+4) \\ & +3(8L-16-c_3)c_3)\phi^2 r^2]\mathcal{C}_1\,M_{\mathsf{T}} - 72\,i\omega\,\mathcal{A}\,\phi^3\,r^5\,c_3\,\mathcal{C}_1\,\hat{M}_{\mathsf{T}} \\ & -i\,\omega\,\phi\,r[(8\,l(L+4) + (8L-16-c_3)c_3)\phi^2\,r^2 - 32L\,l(L+1)]\mathcal{C}_2\,R_{\mathsf{S}} \\ & +(24\,i\,\omega\,\mathcal{A}\,\phi^3 r^5)c_3\,\mathcal{C}_2\,\hat{R}_{\mathsf{S}}\Big\}\;. \end{aligned} \tag{9.13}$$

Utilising (9.13), we now have for our basis vector

$$\mathbf{v} = \begin{pmatrix} \mathcal{M} \\ \hat{M}_{\mathsf{T}} \\ R_{\mathsf{S}} \\ \hat{R}_{\mathsf{S}} \end{pmatrix}, \tag{9.14}$$

with the solution given by,

EVEN	$\mathcal{M}$	M_{T}	R_{S}	X_{S}
$\mathcal{E}_{\mathsf{T}}$	$-\frac{3LlJ+(J-8)(J+4)\omega^2r^2}{6i\omega\phi^3r^5}$	$\frac{(4lc_3-32l(L+1)+c_3^2)}{2(8L+8+c_3)\phi^2r^4}$	$-\frac{\Delta}{18Llr^2c_3^2\mathcal{C}_1}$	$\frac{(4L-8-c_3)(3r-2)\mathcal{C}_2}{9\phi r^3c_3\mathcal{C}_1}$
$\bar{\mathcal{H}}_{\mathsf{T}}$	$-\frac{(J-4L)(J+4l)}{8\phi^2r^3}$	$-\frac{2i\omega}{\phi^2r^2}$	0	0
Σ_{T}	$-\frac{2Ll+(J-8)\omega^2r^2}{2\omega^2\phi r^3}$	$\frac{4l}{i\omega c_3r^2}$	$-\frac{2[3(l-c_3)\mathcal{C}_2-r(3r-2)(\mathcal{C}_1+6\mathcal{C}_2\omega^2)]}{9i\omega r^2c_3\mathcal{C}_1}$	$\frac{2(J-8)\mathcal{C}_2}{9i\omega\phi r^3c_3\mathcal{C}_1}$
ζ_{T}	$\frac{2Ll}{i\omega r^3\phi^2}$	$\frac{2}{\phi r^2}$	$\frac{2(4L+4-c_3)\mathcal{C}_2}{3\phi r^2c_3\mathcal{C}_1}$	$-\frac{4\mathcal{C}_2}{c_3\mathcal{C}_1}$
$\mathcal{E}_{\mathsf{V}}$	$-\frac{Ll^2}{i\omega\phi^2r^4}$	$\frac{l}{\phi r^3}$	$\frac{(4L+4-c_3)(c_3-4L+8)\mathcal{C}_2}{12\phi r^3c_3\mathcal{C}_1}$	$\frac{(4L-8-c_3)\mathcal{C}_2}{2rc_3\mathcal{C}_1}$
$\bar{\mathcal{H}}_{\mathsf{V}}$	$\frac{l(J-8)(L+2\omega^2r^2)}{8\omega^2\phi r^4}$	$-\frac{l(J-8)}{2i\omega r^3(4+c_3)}$	$\frac{(J-8)[3(l-c_3)\mathcal{C}_2-r(3r-2)(\mathcal{C}_1+6\omega^2\mathcal{C}_2)]}{36i\omega r^3c_3^2\mathcal{C}_1}$	$\frac{(J-8)\mathcal{A}\mathcal{C}_2}{3i\omega r^2c_3\mathcal{C}_1}$
Σ_{V}	$-\frac{Ll[(2L-4)c_3+(J-8)\mathcal{E}r^2]}{2\omega^2\phi^2r^4c_3}$	$\frac{l}{i\omega\phi r^3}$	$\frac{\Pi}{6i\omega\phi r^3c_3^2\mathcal{C}_1}$	$\frac{[c_3(8\mathcal{A}^2+(2L-4-c_3)(4\mathcal{E}-\phi^2))-16l\mathcal{E}]\mathcal{C}_2}{i\omega r(4\mathcal{E}-\phi^2)c_3^2\mathcal{C}_1}$
$\bar{\Omega}_{\mathsf{V}}$	$-\frac{Ll(J-8)\mathcal{A}}{2c_3\omega^2r^2\phi}$	0	$\frac{\mathcal{A}(2c_3r^2\mathcal{C}_1+[8l(L+1)+((L+10)c_3-6c_3\omega^2\phi^2)]\mathcal{C}_2)}{i\omega 3rc_3^2\mathcal{C}_1}$	$\frac{\mathcal{A}[(J-8)c_3-8l(J+4)]\mathcal{C}_2}{6i\omega\phi rc_3^2\mathcal{C}_1}$
$\mathcal{A}_{\mathsf{V}}$	0	0	0	0
α_{V}	$-\frac{Ll((4L-c_3))}{4\omega^2\phi^2r^4}$	$\frac{l(4L-c_3)}{i\omega\phi r^3c_3}$	$\frac{\Psi}{36i\omega\phi r^3c_3^2\mathcal{C}_1}$	$-\frac{[\mathcal{A}(4L-c_3)(4\mathcal{A}+\phi)-24\omega^2r]\mathcal{C}_2}{6i\omega c_3\mathcal{C}_1}$
a_{V}	$-\frac{Ll(J-8)}{i\omega c_3r^2\phi}$	0	$-\frac{4c_3r^2\mathcal{C}_1+2(8l(1+L)+(L+10)c_3-6c_3\phi^2\omega^2)\mathcal{C}_2}{3rc_3^2\mathcal{C}_1}$	$\frac{[8l(J+4)-c_3(J-8)]\mathcal{C}_2}{3r\phi c_3^2\mathcal{C}_1}$
W_{V}	$\frac{Ll(J-8)}{4i\omega r^4\phi}$	0	$\frac{(J-8)(L+1)\mathcal{C}_2}{3r^3c_3\mathcal{C}_1}$	$-\frac{(J-8)\phi\mathcal{C}_2}{2rc_3\mathcal{C}_1}$
Y_{V}	0	0	0	0
Z_{V}	0	0	0	0
Σ_{S}	$\frac{Ll(J-8)[3c_3-8(\mathcal{A}^2-\mathcal{E})r^2]}{12\omega^2\phi r^3c_3}$	0	$\frac{\chi}{i\omega r^2c_3^2\mathcal{C}_1}$	$-\frac{[(32l(J+4)-4(J-8)c_3)(\mathcal{A}^2-\mathcal{E})+9\phi^2(8-J+c_3)c_3]\mathcal{C}_2}{i\omega 18c_3^2\mathcal{C}_1\phi}$
θ_{S}	$-\frac{Ll\ell(J-8)(\mathcal{A}^2-\mathcal{E})}{\omega^2r\phi c_3}$	0	$\frac{\Gamma}{i\omega rc_3^2\mathcal{C}_1}$	$\frac{4\mathcal{E}(12\mathcal{A}^2r^3c_3+[6l(J+8)-(J-8)c_3]r-3c_3^2)\mathcal{C}_2}{i\omega(4+J)(4\mathcal{A}+\phi)rc_3^2\mathcal{C}_1}$

where

$$
\begin{aligned}
J &= 3\phi^2\, r^2 - 4 \;, && (9.15)\\
\Gamma &= c_3^2\mathcal{C}_1 + 4(\mathcal{A}^2 - \mathcal{E})[(2l(J+4) + 3(L+2)c_3)\mathcal{C}_2 + \frac{8c_3\mathcal{C}_1}{\phi^2 - 4\mathcal{E}}]r \\
&\quad + \frac{2[(4+J)\mathcal{E} + 3c_3 r]\omega^2 c_3\mathcal{C}_2}{r} \;, && (9.16)\\
\chi &= 1/72\{36(L(c_3+8) + 8 - c_3)c_3\mathcal{C}_2 + 64(\mathcal{A}^2 - \mathcal{E})r^2(l(J+4)\mathcal{C}_2 + c_3\mathcal{C}_1 r^2) \\
&\quad - 3[(J+4)(8L+8-c_3)\mathcal{C}_2 - 4(c_3\mathcal{C}_1 - \mathcal{C}_2\mathcal{E}(16 + 8L + 3c_3 - 16\omega^2\phi^2) \\
&\quad + 8\mathcal{A}^2\mathcal{C}_2(2 + l\omega^2\phi^2))r^2]c_3\} \;, && (9.17)\\
\Pi &= (2l - c_3)(4L + 4 - c_3)c_3\mathcal{C}_2 + 2\mathcal{E}[2c_3\mathcal{C}_1 r^4 + (2l(4+J) + 3(2+L)c_3)r^2\mathcal{C}_2 \\
&\quad + 2c_3(-4 - 4L + c_3)\mathcal{C}_2\omega^2] \;, && (9.18)\\
\Psi &= 3(4L - c_3)[4(J - 4L)(L+1) + (J + 2L)c_3 - c_3^2]\mathcal{C}_2 - 2c_3 r(3r) - 2[(4L - c_3)\mathcal{C}_1 \\
&\quad + 2(8 - 4L + c_3)\mathcal{C}_2\omega^2] \;, && (9.19)\\
\Delta &= 3r^2\{4Llc_3\mathcal{C}_1 + 6Ll(-6 - 6L + c_3)\phi^2\mathcal{C}_2 + [16(L+1)^3 + 8(2(L-4)L - 1)c_3 \\
&\quad + (L+1)c_3^2]\omega^2\mathcal{C}_2\} + 6Ll(4 + 4L - c_3)(6 + 6L - c_3)\mathcal{C}_2 - 8lLc_3 r(\mathcal{C}_1 + 6\mathcal{C}_2\omega^2) \;. \\
& && (9.20)
\end{aligned}
$$

Chapter 10

Conclusions and outlook

In this thesis we have used the 1+1+2 covariant approach to General Relativity (GR) to study exact solutions and perturbations of rotationally symmetric spacetimes in $f(R)$ gravity, one of the most widely studied classes of fourth order gravity.

We began in Chapter 2 by introducing $f(R)$ theories of gravity and presenting the general equations for these theories. We then considered the problem of matching different regions of spacetime in Chapter 3. The aim was to construct inhomogeneous cosmological models, shedding light on the problem of constructing realistic inhomogeneous cosmologies in the context of $f(R)$ gravity. In all of the cases studied, we found that it is impossible to satisfy the required junction conditions without the large-scale behaviour reducing to that expected from Einstein's equations with a cosmological constant. For theories with analytic $f(R)$, this suggests that the usual treatment of weak-field systems as perturbations about Minkowski space may not be compatible with late-time acceleration driven by anything other than a constant term of the form $f(0)$, which acts as a cosmological constant. In the absence of Minkowski space as a suitable background for weak-field systems, one must then choose and justify some other solution around which to perform perturbative analyses. For theories with $f(R) = R^n$ we find that no known spherically symmetric vacuum solutions can be matched to an expanding FLRW background. This includes the absence of any Einstein-Straus-like embeddings of the Schwarzschild exterior solution in FLRW spacetimes. On this note it would be interesting to study the physical consequences of '*jumps*' in the Ricci scalar and/or in the normal derivative of the Ricci scalar across the boundary. As is well known from the Israel-Darmois junction conditions, a jump in the second fundamental form gives rise to surface stress-energy and surface tension on the matching surface that can, for example, be used to stabilise gravitational vacuum condensate stars [206]. In a similar way, it is plausible that relaxing the extra matching conditions in $f(R)$ theories could give rise to surface terms that might be of physical interest. This has been studied in the context of brane-world cosmology in [35].

Chapter 3, was also devoted to studying strong lensing in $f(R) = R^n$ gravity. We showed that the bending angle is dependent of the details of the theory of gravity, (in this case the value of the parameter n). We also showed that the lens mass as calculated for a small deviation from GR increases exponentially with increasing n. The radius of the Einstein ring was found to decrease with increasing n, and it was also found that multiple rings exist for certain intervals of n. The multiple rings are a novel feature of fourth order gravity and cannot be accounted for in GR without assuming the existence of a second companion source, a star forming region or lensing by a singular isothermal sphere in two planes. The magnification of the ring, however, remains unchanged up to small deviations from GR. The aforementioned conclusions are valid for $n < 1.23$ but from [37] we see that the solar system constraints limit $(n-1) < 10^{-19}$ and hence put stronger constraints for the theory of gravity than strong lensing. However, pedagogically it is interesting to find some novel observational signatures of strong lensing in higher order gravity theories, that would help to obtain constraints on the function f and consequently test the nature of gravitational interaction in the strong field regime.

In Chapter 4 and 6 we provided an extensive review of both the 1+3 and 1+1+2 covariant approaches to $f(R)$ theories of gravity. In the 1+3 formalism, a time-like flow u^a is introduced which splits spacetime into 'time' and 'space'. The 1+1+2 further decomposes the '3-space' relative to a preferred spatial vector e^a. The full system of field equations (evolution, propagation and constraint) of spacetime is derived from the Bianchi and Ricci identities in the formalisms in a gauge invariant (co-ordinate independent) manner. From the structure of these equations we can already obtain some important information about the spacetime in general since the covariant decomposition of the spacetime introduces quantities that have a clear physical or geometrical meaning, which gives a better understanding of the underlying physics which sometimes remains obscure in the metric approach.

Furthermore, in Chapter 5 we used the 1+3 formalism to study the role that shear plays in the relationship between Newtonian and relativistic cosmologies in GR. Linearised shear-free solutions are almost universally used to study the formation of structure by gravitational instability in the expanding universe, and are believed to result in standard local Newtonian theory. We found an that an exact result for the Einstein field equations is that if pressure-free matter is moving in a shear-free way, then it must be either expansion-free or rotation-free (valid for isentropic perfect fluids). This result had been previously suggested to be true for any barotropic perfect fluid, but a proof has remained elusive. We considered the case of barotropic perfect-fluid solutions linearised about an FLRW geometry, and proved that the result remains true except for the case of a specific highly nonlinear equation of state. We argue that this equation of state is nonphysical, and hence the result is true in the linearised case for all physically realistic barotropic perfect fluids. This result, which is not true in Newtonian cosmology, demonstrates that the linearised solutions, believed to result in standard local Newtonian theory, do not always give the usual behaviour of Newtonian solutions. We also presented work from [64] which shows that these results do not always hold true a general $f(R)$ theory of gravity. They demonstrated that in these theories there is at least one physically realistic non-vacuum case in which both rotation and expansion is simultaneously possible. This result suggests that there are situations where linearised FOG posses properties with Newtonian theory that are not valid in GR. Another interesting point that emerged from our analysis is that there exists a class of barotropic equation of state (however unphysical that may be) for which the usual shear-free result can be avoided in the linearised case. It would be an interesting problem to see whether this same class of equations of state (or some similar class) allows shear-free rotating and expanding solutions for the full nonlinear Einstein equations for a barotropic perfect fluid.

In Chapter 7 we proceeded to apply the 1+1+2 covariant approach to determine the conditions for the existence of spherically symmetric vacuum solutions of these fourth order field equations. We proved a Jebsen-Birkhoff like theorem for $f(R)$ theories of gravity and set the necessary conditions required for the existence of Schwarzschild solution in these theories. In order to study the perturbations of Schwarzschild black holes in this context, we discussed under what circumstances we can covariantly set up a scale in the problem. We showed that subject to certain conditions holding true the spacetime remains: a) "almost" Schwarzschild when we perturb the spacetime by keeping spherical symmetry and perturbing the Ricci scalar around $R = 0$; b) "almost spherically symmetric" with respect to the covariant scale when we perturbed the spherical symmetry. The size of the open set $\mathcal{S}$ where this holds depends on the parameters of theory (namely the quantity $f''(0)$) and the covariant scale (which is the Schwarzschild mass of the star). As a result of this analysis we can make the deduction that one can always tune the parameters of the theory such that the perturbations continue to remain small for a time period which is greater than the age of the universe. In that case, the local spacetime around almost spherical stars will be stable in the regime of linear perturbations.

Having set up the scale for the perturbations, we applied the 1+1+2 perturbative procedure to study the perturbations of Schwarzschild black holes in $f(R)$ gravity. From the Stewart-Walker lemma which states that a variable is gauge invariant in the perturbed spacetime if it vanishes in the background, it follows that since the exact Schwarzschild black hole involves only scalars then all the vector and tensor quantities are gauge invariant under linear perturbations. We were able to obtain a frame invariant TT tensor M_{T} which satisfies the Regge-Wheeler equation irrespective of parity. We showed that for the tensor modes, the underlying dynamics in $f(R)$ gravity is governed by a modified Regge-Wheeler tensor which obeys the same Regge-Wheeler equation as in GR. In order to close the system the Ricci scalar wave equation is included which corresponds to scalar perturbations that are not present in GR. Since the Regge-Wheeler equation governs the odd (axial) perturbations the premise then would be to then work out a Zerilli tensor for the even (polar) perturbations. The analysis would be involve rigorous mathematical manipulation (and sufficient ingenuity) in FOG if one goes by the fact that its derivation in GR (which is second order) is considerably more complicated than the derivation of the Regge-Wheeler tensor due to the larger number of functions involved. Using the 1+1+2 formalism in GR, [65] obtained a Zerilli tensor which satisfies the Zerilli equation. They also showed that the Zerilli variable can expressed as a linear combination of the Regge-Wheeler tensor and it's derivative. This agrees with the results of [170] where they showed that Zerilli and Regge-Wheeler equations are representations of the same physical situation. Using results from Myung et al in [86] where using the metric method they derived the Zerilli equation for the even gravitational perturbations in $f(R)$ gravity, we can safely say at this point (albeit cautiously) that the Regge-Wheeler tensor is the more fundamental one of the two in $f(R)$ gravity. The main difference between GR and $f(R)$ gravities is the appearance of the scalar perturbations. For the quasinormal modes (QNMs) that follow from the scalar perturbations, we find that the possible sources of scalar QNMs for the lower multipoles are from primordial Black Holes. Higher mass, stellar black holes are associated with extremely high multipoles, which can only be produced in the first stage of black hole formation. Since the scalar QNMs are short ranged, this scenario makes their detection beyond the range of current experiments.

Finally, Chapter 9 is devoted to finding the solution to the perturbation equations. The introduction of harmonics reduced the system into a linear system of algebraic equations which simplified things and we were able to find the solution of the system using matrix methods, while employing the freedom to choice of frame vectors.

As a final comment we would like to point out that there are a number of other areas of application of the 1+1+2 perturbation approach that are worth pursuing in the context of $f(R)$ gravity. The violation of Birkhoff's theorem in its general form in FOG means that Schwarzschild is not the only exact types of static spherically symmetric solution in these theories. We can therefore consider extending the work done in this thesis to exploring the application of gravitational wave propagation of these other spacetimes. Following successful results in GR [66, 67], it would be interesting to use the 1+1+2 formalism to investigate whether a covariant Regge-Wheeler master equation can be found for electromagnetic perturbations of the spacetime in FOG. We leave these and other developments for future work.

Appendix A

Useful relations for decomposing

In this appendix we present useful expressions from [65] for decomposing 1+3 quantities to 1+1+2 variables which were employed in Chapter 6. Given any relation in 1+3 notation, these relations may be utilised to aid decomposition.

Any 1+3 spacetime 3-vectors x^a, y^a and PSTF 3-tensors ψ_{ab}, ϕ_{ab}, may be decomposed as

$$
\begin{aligned}
x^a &= X e^a + X^a \ , && \text{(A.1)}\\
y^a &= Y e^a + Y^a \ , && \text{(A.2)}\\
\psi_{ab} &= \psi_{\langle ab\rangle} = \Psi\left(e_a e_b - \frac{1}{2}N_{ab}\right) + 2\Psi_{(a}e_{b)} + \Psi_{ab} \ , && \text{(A.3)}\\
\phi_{ab} &= \phi_{\langle ab\rangle} = \Phi\left(e_a e_b - \frac{1}{2}N_{ab}\right) + 2\Phi_{(a}e_{b)} + \Phi_{ab} \ . && \text{(A.4)}
\end{aligned}
$$

The following expansions from 1+3 quantities $\longrightarrow$ 1+1+2 variables may be performed:

$$
\begin{aligned}
x_a\, x^a &= X^2 + X_a\, X^a \ , && \text{(A.5)}\\
\eta_{abc} x^b\, y^c &= \left(\varepsilon_{bc} X^b\, Y^c\right) e_a + \varepsilon_{ab}\left(Y\, X^b - X\, Y^b\right) \ , && \text{(A.6)}\\
x_{\langle a}\, y_{b\rangle} &= \frac{1}{3}\left(2X\, Y - X_c\, Y^c\right)\left(e_a e_b - \frac{1}{2}N_{ab}\right) + \left[X\, Y_{(a} + Y\, X_{(a}\right] e_{b)} + X_{\{a}\, Y_{b\}} \ , && \\
& && \text{(A.7)}\\
\psi_{ab}\, x^b &= \left(X\,\Psi + X_b\,\Psi^b\right) e_a - \frac{1}{2}\Psi\, X_a + X\,\Psi_a + \Psi_{ab}\, X^b, && \text{(A.8)}\\
\eta_{cd\langle a} x^c\, \psi_{b\rangle}{}^d &= \varepsilon_{cd} X^c\,\Psi^d\left(e_a e_b - \frac{1}{2}N_{ab}\right) + X\varepsilon_{c\{a}\,\Psi_{b\}}{}^c - X^c\varepsilon_{c\{a}\,\Psi_{b\}} \ , && \\
&+ \left[\left(X\,\Psi^c - \frac{3}{2}\Psi\, X^c\right)\varepsilon_{c(a} + \varepsilon_{cd} X^c\,\Psi^d{}_{(a}\right] e_{b)} \ , && \text{(A.9)}
\end{aligned}
$$

$$\psi_{ab}\,\psi^{ab} = \frac{3}{2}\Psi^2 + 2\Psi_a\,\Psi^a + \Psi_{ab}\,\Psi^{ab}\ , \tag{A.10}$$

$$\begin{aligned}\psi_{c\langle a}\,\phi_{b\rangle}{}^{c} &= \left(\frac{1}{2}\Psi\,\Phi + \frac{1}{3}\Psi_c\,\Phi^c - \frac{1}{3}\Psi_{cd}\,\Phi^{cd}\right)\left(e_a e_b - \frac{1}{2}N_{ab}\right)\\ &\quad + \left[\frac{1}{2}\Psi\,\Phi_{(a} + \frac{1}{2}\Phi\,\Psi_{(a} + \Psi^c\,\Phi_{c(a} + \Phi^c\,\Psi_{c(a}\right] e_{b)}\\ &\quad - \frac{1}{2}\Psi\,\Phi_{ab} - \frac{1}{2}\Phi\,\Psi_{ab} + \Psi_{\{a}\,\Phi_{b\}} + \Psi_{c\{a}\Phi_{b\}}{}^{c}\ ,\end{aligned} \tag{A.11}$$

$$\eta_{abc}\psi^{b}{}_{d}\,\phi^{dc} = e_a\varepsilon_{bc}\Psi^b{}_d\,\Phi^{dc} + \frac{3}{2}\varepsilon_{ab}\left(\Phi\,\Psi^b - \Psi\,\Phi^b\right)\ . \tag{A.12}$$

For 1+3 covariant time derivative ' $\dot{}$ ' and the fully orthogonally projected covariant spatial derivative 'D' we find:

$$\dot{x}_{\langle a\rangle} = \left(\dot{X} - X_b\,\alpha^b\right)e_a + X\,\alpha_a + \dot{X}_{\bar{a}}\ , \tag{A.13}$$

$$\begin{aligned}\dot{\psi}_{\langle ab\rangle} &= \left(\dot{\Psi} - 2\Psi_c\,\alpha^c\right)e_a e_b - \frac{1}{2}\dot{\Psi}N_{ab} + \left[3\Psi\,\alpha_{(a} + 2\dot{\Psi}_{(\bar{a}} - 2\alpha^c\,\Psi_{c(a}\right] e_{b)}\\ &\quad + 2\Psi_{(a}\,\alpha_{b)} + \dot{\Psi}_{\{ab\}}\ ,\end{aligned} \tag{A.14}$$

$$\mathrm{D}_a x^a = \hat{X} + X\,\phi - X_a\,a^a + \delta_a X^a\ , \tag{A.15}$$

$$\eta_{abc}\mathrm{D}^b x^c = \left(2X\,\xi + \varepsilon_{bc}\delta^b X^c\right)e_a + \xi\,X_a + \varepsilon_{ab}\left[-X\,a^b + \delta^b X - \hat{X}^b - \frac{1}{2}\phi\,X^b - \zeta^{bc}\,X_c\right]\ , \tag{A.16}$$

$$\begin{aligned}\mathrm{D}_{\langle a}x_{b\rangle} &= \frac{1}{3}\left[2\hat{X} - \phi X - 2X_c\,a^c - \delta_c X^c\right]\left(e_a e_b - \frac{1}{2}N_{ab}\right) + X\,\zeta_{ab} + \delta_{\{a}X_{b\}}\\ &\quad + \left[X\,a_{(a} + \delta_{(a}X + \hat{X}_{(\bar{a}} - \frac{1}{2}\phi\,X_{(a} + X^c\left(\xi\varepsilon_{c(a} - \zeta_{c(a}\right)\right] e_{b)}\ ,\end{aligned} \tag{A.17}$$

$$\begin{aligned}\mathrm{D}^b\psi_{ab} &= \left(\hat{\Psi} + \frac{3}{2}\phi\,\Psi - 2\Psi_b\,a^b + \delta_b\Psi^b - \Psi_{bc}\,\zeta^{bc}\right)e_a + \hat{\Psi}_{\bar{a}} + \frac{3}{2}\phi\,\Psi_a + \frac{3}{2}\Psi\,a_a - \frac{1}{2}\delta_a\Psi\\ &\quad - \Psi_{ab}\,a^b + \left[-\xi\,\varepsilon_{ab} + \zeta_{ab}\right]\Psi^b + \delta^b\Psi_{ab}\ ,\end{aligned} \tag{A.18}$$

$$\begin{aligned}\eta_{cd\langle a}\mathrm{D}^c\psi_{b\rangle}{}^{d} &= \left[3\xi\,\Psi + \varepsilon_{cd}\delta^c\Psi^d - \varepsilon_{cd}\Psi^{de}\,\zeta^c{}_e\right]\left(e_a e_b - \frac{1}{2}N_{ab}\right)\\ &\quad + \left(-\frac{3}{2}\delta^c\Psi + \frac{3}{2}\Psi a^c + \hat{\Psi}^c + \frac{1}{2}\phi\,\Psi^c + 2\Psi_d\,\zeta^{cd}\right)\varepsilon_{c(a}e_{b)}\\ &\quad + \left[5\xi\,\Psi_{(a} + \varepsilon^{cd}\left(\Psi_d\,\zeta_{c(a} + \delta_c\Psi_{d(a}\right)\right] e_{b)} - \varepsilon_{c\{a}\delta^c\Psi_{a\}} + 2\varepsilon_{c\{a}a^c\,\Psi_{b\}}\\ &\quad + \varepsilon_{c\{a}\hat{\Psi}^c{}_{b\}} + \frac{1}{2}\phi\,\varepsilon_{c\{a}\Psi^c{}_{b\}} - \frac{3}{2}\Psi\,\varepsilon_{c\{a}\zeta^c{}_{b\}} + \xi\,\Psi_{ab} + \varepsilon_{c\{a}\Psi_{b\}d}\zeta^{cd}\ .\end{aligned} \tag{A.19}$$

Appendix B

Harmonically decomposed equations

We present the system of harmonically decomposed ordinary differential equations. Each vector and tensor equation produces two harmonics equations for each ℓ, one of odd parity and one of even parity, due to the orthogonality of the vector and tensor harmonics. We implicitly assume a sum over ℓ in the equations, and the S, V, T subscripts indicate respectively, scalar, vector and tensor terms. These remind us that a spherical harmonic expansion has been made.

B.1 Propagation and evolution equations

$$\dot{\xi}_{\mathsf{S}} = \left(\mathcal{A} - \frac{1}{2}\phi\right)\Omega_{\mathsf{S}} + \frac{\ell\,(\ell+1)}{2r}\,\bar{\alpha}_{\mathsf{V}} + \frac{1}{2}\mathcal{H}_{\mathsf{S}}\ , \tag{B.1}$$

$$\dot{\Omega}_{\mathsf{S}} = \frac{\ell\,(\ell+1)}{2r}\,\bar{\mathcal{A}}_{\mathsf{V}} + \mathcal{A}\,\xi_{\mathsf{S}}\ , \tag{B.2}$$

$$\dot{\mathcal{H}}_{\mathsf{S}} = -\frac{\ell\,(\ell+1)}{r}\,\mathcal{E}_{\mathsf{V}} - 3\,\mathcal{E}\,\xi_{\mathsf{S}}\ , \tag{B.3}$$

$$\begin{aligned}\dot{\Sigma}_{\mathsf{S}} - \frac{2}{3}r\,\hat{Z}_{\mathsf{V}} = {}& \frac{2}{3}r\left[2\mathcal{A}\,Z_{\mathsf{V}} - \mathcal{A}(\phi+\mathcal{A})a_{\mathsf{V}} + \frac{\ell\,(\ell+1)}{2r^2}\,\mathcal{A}_{\mathsf{V}} - \frac{3}{2}W_{\mathsf{V}} - \frac{1}{2}\mathcal{A}\,Y_{\mathsf{V}}\right]\\ &+ \frac{1}{6}R_{\mathsf{S}} - \frac{\mathcal{C}_2}{2\,\mathcal{C}_1}\left[\phi\,X_{\mathsf{S}} + \mathcal{E}\,R_{\mathsf{S}} - \frac{1}{4}\phi^2\,R_{\mathsf{S}} + \frac{1-\ell\,(\ell+1)}{r^2}\,R_{\mathsf{S}}\right]\ ,\end{aligned} \tag{B.4}$$

$$\dot{\Theta}_{\mathsf{S}} - r\,\hat{Z}_{\mathsf{V}} = r\left[\left(\frac{3}{2}\phi + 2\mathcal{A}\right)Z_{\mathsf{V}} + \mathcal{A}\,Y_{\mathsf{V}} - \mathcal{A}(\phi+\mathcal{A})a_{\mathsf{V}} - \frac{\ell\,(\ell+1)}{r^2}\,\mathcal{A}_{\mathsf{V}}\right] + \frac{1}{6}R_{\mathsf{S}} - \frac{\mathcal{C}_2}{\mathcal{C}_1}\left(\ddot{R}_{\mathsf{S}} - \mathcal{A}\,X_{\mathsf{S}}\right)\;, \tag{B.5}$$

$$\dot{\Sigma}_{\mathsf{V}} + \dot{\bar{\Omega}}_{\mathsf{V}} = -\mathcal{E}_{\mathsf{V}} + Z_{\mathsf{V}} + \left(\mathcal{A} - \frac{1}{2}\phi\right)\mathcal{A}_{\mathsf{V}} + \frac{\mathcal{C}_2}{2\,r\,\mathcal{C}_1}\left(X_{\mathsf{S}} - \frac{1}{2}\phi\,R_{\mathsf{S}}\right)\;, \tag{B.6}$$

$$\dot{\bar{\Sigma}}_{\mathsf{V}} - \dot{\Omega}_{\mathsf{V}} = -\,\bar{\mathcal{E}}_{\mathsf{V}} + \bar{Z}_{\mathsf{V}} + \left(\mathcal{A} - \frac{1}{2}\phi\right)\bar{\mathcal{A}}_{\mathsf{V}}\;, \tag{B.7}$$

$$\dot{\mathcal{E}}_{\mathsf{V}} - \frac{1}{2}\hat{\bar{\mathcal{H}}}_{\mathsf{V}} = \frac{3}{4}\,\mathcal{E}\left(\Sigma_{\mathsf{V}} - 2\alpha_{\mathsf{V}} - \bar{\Omega}_{\mathsf{V}}\right) + \left(\frac{1}{4}\phi + \mathcal{A}\right)\bar{\mathcal{H}}_{\mathsf{V}} + \frac{2-\ell\,(\ell+1)}{4r}\bar{\mathcal{H}}_{\mathsf{T}}\;, \tag{B.8}$$

$$\dot{\bar{\mathcal{E}}}_{\mathsf{V}} + \frac{1}{2}\hat{\mathcal{H}}_{\mathsf{V}} = \frac{3}{4}\,\mathcal{E}\left(\bar{\Sigma}_{\mathsf{V}} - 2\,\bar{\alpha}_{\mathsf{V}} + \Omega_{\mathsf{V}}\right) - \left(\frac{1}{4}\phi + \mathcal{A}\right)\mathcal{H}_{\mathsf{V}} + \frac{3}{4\,r}\,\mathcal{H}_{\mathsf{S}} + \frac{2-\ell\,(\ell+1)}{4r}\,\mathcal{H}_{\mathsf{T}}\;, \tag{B.9}$$

$$\dot{\mathcal{H}}_{\mathsf{V}} = \frac{3}{2}\mathcal{E}\,\bar{\mathcal{A}}_{\mathsf{V}} + \frac{1}{2}\bar{W}_{\mathsf{V}} + \frac{1}{2}\left(\phi - 2\mathcal{A}\right)\bar{\mathcal{E}}_{\mathsf{V}} + \frac{\ell\,(\ell+1)-2}{2r}\,\bar{\mathcal{E}}_{\mathsf{T}}\;, \tag{B.10}$$

$$\dot{\bar{\mathcal{H}}}_{\mathsf{V}} = -\frac{3}{2}\mathcal{E}\,\mathcal{A}_{\mathsf{V}} - \frac{1}{2}\,W_{\mathsf{V}} - \frac{1}{2}\left(\phi - 2\mathcal{A}\right)\mathcal{E}_{\mathsf{V}} - \frac{\ell\,(\ell+1)-2}{2r}\,\mathcal{E}_{\mathsf{T}} - \mathcal{E}\,\frac{\mathcal{C}_2}{4\,r\,\mathcal{C}_1}\,R_{\mathsf{S}}\;, \tag{B.11}$$

$$\dot{W}_{\mathsf{V}} = \frac{3}{2}\phi\,\mathcal{E}\left(\alpha_{\mathsf{V}} + \Sigma_{\mathsf{V}} + \bar{\Omega}_{\mathsf{V}}\right) - \frac{\mathcal{E}}{r}\left(\Theta_{\mathsf{S}} - \frac{3}{2}\Sigma_{\mathsf{S}}\right) + \frac{\ell\,(\ell+1)}{r^2}\bar{\mathcal{H}}_{\mathsf{V}} + \mathcal{A}\,\phi\,\frac{\mathcal{C}_2}{2\,r\,\mathcal{C}_1}\,\dot{R}_{\mathsf{S}}\;, \tag{B.12}$$

$$\dot{\bar{W}}_{\mathsf{V}} = \frac{3}{2}\phi\,\mathcal{E}\left(\bar{\alpha}_{\mathsf{V}} + \bar{\Sigma}_{\mathsf{V}} - \Omega_{\mathsf{V}}\right)\;, \tag{B.13}$$

$$\begin{aligned}\dot{Y}_{\mathsf{V}} &= \left(\frac{1}{2}\phi^2+\mathcal{E}\right)\left(\alpha_{\mathsf{V}}+\Sigma_{\mathsf{V}}+\bar{\Omega}_{\mathsf{V}}\right)-\frac{\ell\,(\ell+1)}{r^2}\,\alpha_{\mathsf{V}} \\ &\quad +\frac{1}{r}\left(\frac{1}{2}\phi-\mathcal{A}\right)\left(\Sigma_{\mathsf{S}}-\frac{2}{3}\Theta_{\mathsf{S}}\right)-\frac{\mathcal{C}_2}{r\,\mathcal{C}_1}\,\hat{\bar{R}}_{\mathsf{S}}\ ,\end{aligned} \tag{B.14}$$

$$\dot{\bar{Y}}_{\mathsf{V}} = \left(\frac{1}{2}\phi^2+\mathcal{E}\right)\left(\bar{\alpha}_{\mathsf{V}}+\bar{\Sigma}_{\mathsf{V}}-\Omega_{\mathsf{V}}\right)\ , \tag{B.15}$$

$$\dot{\zeta}_{\mathsf{T}} = \left(\mathcal{A}-\frac{1}{2}\phi\right)\Sigma_{\mathsf{T}}+\frac{\alpha_{\mathsf{V}}}{r}+\bar{\mathcal{H}}_{\mathsf{T}}\ , \tag{B.16}$$

$$\dot{\bar{\zeta}}_{\mathsf{T}} = \left(\mathcal{A}-\frac{1}{2}\phi\right)\bar{\Sigma}_{\mathsf{T}}-\frac{\bar{\alpha}_{\mathsf{V}}}{r}-\mathcal{H}_{\mathsf{T}}\ , \tag{B.17}$$

$$\dot{\Sigma}_{\mathsf{T}} = \frac{\mathcal{A}_{\mathsf{V}}}{r}+\mathcal{A}\,\zeta_{\mathsf{T}}-\mathcal{E}_{\mathsf{T}}+\frac{\mathcal{C}_2}{2\,r^2\,\mathcal{C}_1}\,R_{\mathsf{S}}\ , \tag{B.18}$$

$$\dot{\bar{\Sigma}}_{\mathsf{T}} = -\frac{\bar{\mathcal{A}}_{\mathsf{V}}}{r}+\mathcal{A}\,\bar{\zeta}_{\mathsf{T}}-\bar{\mathcal{E}}_{\mathsf{T}}\ , \tag{B.19}$$

$$\dot{\mathcal{E}}_{\mathsf{T}}+\hat{\bar{\mathcal{H}}}_{\mathsf{T}} = -\frac{\bar{\mathcal{H}}_{\mathsf{V}}}{r}-\frac{3}{2}\mathcal{E}\,\Sigma_{\mathsf{T}}-\left(\frac{1}{2}\phi+2\mathcal{A}\right)\bar{\mathcal{H}}_{\mathsf{T}}\ , \tag{B.20}$$

$$\dot{\bar{\mathcal{E}}}_{\mathsf{T}}-\hat{\mathcal{H}}_{\mathsf{T}} = -\frac{\mathcal{H}_{\mathsf{V}}}{r}-\frac{3}{2}\mathcal{E}\,\bar{\Sigma}_{\mathsf{T}}+\left(\frac{1}{2}\phi+2\mathcal{A}\right)\mathcal{H}_{\mathsf{T}}\ , \tag{B.21}$$

$$\dot{\mathcal{H}}_{\mathsf{T}}-\hat{\bar{\mathcal{E}}}_{\mathsf{T}} = \frac{\bar{\mathcal{E}}_{\mathsf{V}}}{r}-\frac{3}{2}\mathcal{E}\,\bar{\zeta}_{\mathsf{T}}+\left(\frac{1}{2}\phi+2\mathcal{A}\right)\bar{\mathcal{E}}_{\mathsf{T}}\ . \tag{B.22}$$

$$\dot{\bar{\mathcal{H}}}_{\mathsf{T}}+\hat{\mathcal{E}}_{\mathsf{T}} = \frac{\mathcal{E}_{\mathsf{V}}}{r}+\frac{3}{2}\mathcal{E}\,\zeta_{\mathsf{T}}-\left(\frac{1}{2}\phi+2\mathcal{A}\right)\mathcal{E}_{\mathsf{T}}\ . \tag{B.23}$$

$$\hat{\xi}_{\mathsf{S}} = -\,\phi\,\xi_{\mathsf{S}} + \frac{\ell\,(\ell+1)}{2r}\,\bar{a}_{\mathsf{V}}\ , \tag{B.24}$$

$$\hat{\Omega}_{\mathsf{S}} = \frac{\ell\,(\ell+1)}{r}\Omega_{\mathsf{V}} + (\mathcal{A}-\phi)\,\Omega_{\mathsf{S}}\ , \tag{B.25}$$

$$\hat{\mathcal{H}}_{\mathsf{S}} = \frac{\ell\,(\ell+1)}{r}\mathcal{H}_{\mathsf{V}} - \frac{3}{2}\phi\,\mathcal{H}_{\mathsf{S}} - 3\mathcal{E}\,\Omega_{\mathsf{S}}\ , \tag{B.26}$$

$$\hat{\Sigma}_{\mathsf{S}} - \frac{2}{3}\hat{\Theta}_{\mathsf{S}} = -\,\frac{3}{2}\phi\,\Sigma_{\mathsf{S}} + \frac{\ell\,(\ell+1)}{r}\left(\Sigma_{\mathsf{V}} - \bar{\Omega}_{\mathsf{V}}\right) + \frac{\mathcal{C}_2}{\mathcal{C}_1}\,\hat{\dot{R}}_{\mathsf{S}}\ , \tag{B.27}$$

$$\begin{aligned}\hat{\Sigma}_{\mathsf{V}} + \hat{\bar{\Omega}}_{\mathsf{V}} =& \frac{1}{2r}\left(\Sigma_{\mathsf{S}} + \frac{4}{3}\theta_{\mathsf{S}}\right) - \frac{3}{2}\phi\,\Sigma_{\mathsf{V}} - \left(\frac{1}{2}\phi + 2\mathcal{A}\right)\bar{\Omega}_{\mathsf{V}}\\ &+ \frac{\ell\,(\ell+1)-2}{2r}\,\Sigma_{\mathsf{T}} + \frac{\mathcal{C}_2}{r\,\mathcal{C}_1}\,\dot{R}_{\mathsf{S}}\ ,\end{aligned} \tag{B.28}$$

$$\hat{\bar{\Sigma}}_{\mathsf{V}} - \hat{\Omega}_{\mathsf{V}} = -\,\frac{\Omega_{\mathsf{S}}}{r} - \frac{3}{2}\phi\bar{\Sigma}_{\mathsf{V}} + \left(\frac{1}{2}\phi + 2\mathcal{A}\right)\Omega_{\mathsf{V}} - \frac{\ell\,(\ell+1)-2}{2r}\bar{\Sigma}_{\mathsf{T}}\ , \tag{B.29}$$

$$\hat{\mathcal{A}}_{\mathsf{V}} - 2\dot{\Sigma}_{\mathsf{V}} = -\,Z_{\mathsf{V}} - 2\left(\mathcal{A} - \frac{1}{4}\phi\right)\mathcal{A}_{\mathsf{V}} - \mathcal{A}\,a_{\mathsf{V}} + 2\mathcal{E}_{\mathsf{V}} - \frac{\mathcal{C}_2}{r\,\mathcal{C}_1}\left(X_{\mathsf{S}} - \frac{1}{2}\phi\,R_{\mathsf{S}}\right)\ , \tag{B.30}$$

$$\hat{\bar{\mathcal{A}}}_{\mathsf{V}} - 2\dot{\bar{\Sigma}}_{\mathsf{V}} = -\bar{Z}_{\mathsf{V}} - 2\left(\mathcal{A} - \frac{1}{4}\phi\right)\bar{\mathcal{A}}_{\mathsf{V}} - \mathcal{A}\,\bar{a}_{\mathsf{V}} + 2\bar{\mathcal{E}}_{\mathsf{V}}\ , \tag{B.31}$$

$$\hat{\alpha}_{\mathsf{V}} - \dot{a}_{\mathsf{V}} = \bar{\mathcal{H}}_{\mathsf{V}} - \left(\frac{1}{2}\phi + \mathcal{A}\right)\alpha_{\mathsf{V}} + \left(\frac{1}{2}\phi - \mathcal{A}\right)\left(\Sigma_{\mathsf{V}} - \bar{\Omega}_{\mathsf{V}}\right) - \frac{\mathcal{C}_2}{2r\,\mathcal{C}_1}\,\dot{R}_{\mathsf{S}}\ , \tag{B.32}$$

$$\hat{\bar{\alpha}}_{\mathsf{V}} - \dot{\bar{a}}_{\mathsf{V}} = -\,\mathcal{H}_{\mathsf{V}} - \left(\frac{1}{2}\phi + \mathcal{A}\right)\bar{\alpha}_{\mathsf{V}} + \left(\frac{1}{2}\phi - \mathcal{A}\right)\left(\bar{\Sigma}_{\mathsf{V}} + \Omega_{\mathsf{V}}\right)\ , \tag{B.33}$$

$$\hat{\mathcal{E}}_{\mathsf{V}} = \frac{1}{2}W_{\mathsf{V}} + \frac{\ell\,(\ell+1)-2}{2r}\,\mathcal{E}_{\mathsf{T}} - \frac{3}{2}\mathcal{E}\,a_{\mathsf{V}} - \frac{3}{2}\phi\,\mathcal{E}_{\mathsf{V}} + \mathcal{E}\,\frac{\mathcal{C}_2}{4\,r\,\mathcal{C}_1}R_{\mathsf{S}}\;, \tag{B.34}$$

$$\hat{\bar{\mathcal{E}}}_{\mathsf{V}} = \frac{1}{2}\bar{W}_{\mathsf{V}} - \frac{\ell\,(\ell+1)-2}{2r}\,\bar{\mathcal{E}}_{\mathsf{T}} - \frac{3}{2}\mathcal{E}\,\bar{a}_{\mathsf{V}} - \frac{3}{2}\phi\,\bar{\mathcal{E}}_{\mathsf{V}}\;, \tag{B.35}$$

$$\hat{\mathcal{H}}_{\mathsf{V}} = \frac{1}{2r}\mathcal{H}_{\mathsf{S}} + \frac{\ell\,(\ell+1)-2}{2r}\mathcal{H}_{\mathsf{T}} + \frac{3}{2}\mathcal{E}\left(\Omega_{\mathsf{V}} + \bar{\Sigma}_{\mathsf{V}}\right) - \frac{3}{2}\phi\,\mathcal{H}_{\mathsf{V}}\;, \tag{B.36}$$

$$\hat{\bar{\mathcal{H}}}_{\mathsf{V}} = -\frac{3}{2}\phi\,\bar{\mathcal{H}}_{\mathsf{V}} - \frac{\ell\,(\ell+1)-2}{2r}\bar{\mathcal{H}}_{\mathsf{T}} + \frac{3}{2}\mathcal{E}\left(\bar{\Omega}_{\mathsf{V}} - \Sigma_{\mathsf{V}}\right)\;, \tag{B.37}$$

$$\hat{W}_{\mathsf{V}} = -\,2\phi\,W_{\mathsf{V}} - \frac{3}{2}\mathcal{E}\,Y_{\mathsf{V}} + \frac{3}{2}\phi\,\mathcal{E}\,a_{\mathsf{V}} + \frac{\ell\,(\ell+1)}{r^2}\mathcal{E}_{\mathsf{V}} - \mathcal{E}\frac{\mathcal{C}_2}{2\,r\,\mathcal{C}_1}\,X_{\mathsf{S}} \tag{B.38}$$

$$\hat{\bar{W}}_{\mathsf{V}} = -\,2\phi\,\bar{W}_{\mathsf{V}} - \frac{3}{2}\mathcal{E}\,\bar{Y}_{\mathsf{V}} + \frac{3}{2}\phi\,\mathcal{E}\,\bar{a}_{\mathsf{V}} \tag{B.39}$$

$$\begin{aligned}\hat{Y}_{\mathsf{V}} = &-W_{\mathsf{V}} - \frac{3}{2}\phi\,Y_{\mathsf{V}} + \left(\frac{1}{2}\phi^2 + \mathcal{E}\right)a_{\mathsf{V}} - \frac{\ell\,(\ell+1)}{r^2}a_{\mathsf{V}} - \frac{1}{3r}R_{\mathsf{S}}\\ &+\frac{\mathcal{C}_2}{r\,\mathcal{C}_1}\left[\left(\mathcal{A} + \frac{1}{2}\phi\right)X_{\mathsf{S}} + \frac{1}{2}\left(\mathcal{E} - \frac{1}{4}\phi^2\right)R_{\mathsf{S}} + \frac{1-\ell\,(\ell+1)}{r^2}R_{\mathsf{S}} - \ddot{R}_{\mathsf{S}}\right]\end{aligned} \tag{B.40}$$

$$\hat{\bar{Y}}_{\mathsf{V}} = -\bar{W}_{\mathsf{V}} - \frac{3}{2}\phi\,\bar{Y}_{\mathsf{V}} + \left(\frac{1}{2}\phi^2 + \mathcal{E}\right)\bar{a}_{\mathsf{V}} \tag{B.41}$$

$$\hat{\zeta}_{\mathsf{T}} = -\,\phi\,\zeta_{\mathsf{T}} + \frac{a_{\mathsf{V}}}{r} - \mathcal{E}_{\mathsf{T}} - \frac{\mathcal{C}_2}{2\,r^2\,\mathcal{C}_1}R_{\mathsf{S}}\;, \tag{B.42}$$

$$\hat{\bar{\zeta}}_{\mathsf{T}} = -\,\phi\,\bar{\zeta}_{\mathsf{T}} - \frac{\bar{a}_{\mathsf{V}}}{r} - \bar{\mathcal{E}}_{\mathsf{T}}\;, \tag{B.43}$$

$$\hat{\Sigma}_{\mathsf{T}} = \frac{1}{r}\left(\Sigma_{\mathsf{V}} - \bar{\Omega}_{\mathsf{V}}\right) - \frac{1}{2}\phi\,\Sigma_{\mathsf{T}} + \bar{\mathcal{H}}_{\mathsf{T}}\ , \tag{B.44}$$

$$\hat{\bar{\Sigma}}_{\mathsf{T}} = -\frac{1}{r}\left(\bar{\Sigma}_{\mathsf{V}} + \Omega_{\mathsf{V}}\right) - \frac{1}{2}\phi\,\bar{\Sigma}_{\mathsf{T}} - \mathcal{H}_{\mathsf{T}}\ , \tag{B.45}$$

$$\frac{1}{2}Y_{\mathsf{V}} + \frac{\ell\,(\ell+1)-2}{2r}\zeta_{\mathsf{T}} + \mathcal{E}_{\mathsf{V}} = -\frac{\mathcal{C}_2}{4r\,\mathcal{C}_1}\left(\phi\,R_{\mathsf{S}} - 2X_{\mathsf{S}}\right)\ , \tag{B.46}$$

$$\hat{R}_{\mathsf{S}} = \frac{1}{2}\phi\,R_{\mathsf{S}} + \frac{2r\,\mathcal{C}_1}{\mathcal{C}_2}\left[\frac{1}{2}Y_{\mathsf{V}} + \frac{\ell\,(\ell+1)-2}{2r}\zeta_{\mathsf{T}} + \mathcal{E}_{\mathsf{V}}\right]\ , \tag{B.47}$$

$$\dot{R}_{\mathsf{S}} = \frac{\mathcal{C}_1}{\mathcal{C}_2}\left[\Sigma_{\mathsf{S}} - \frac{2}{3}\theta_{\mathsf{S}} - \left(\ell\,(\ell+1)-2\right)\Sigma_{\mathsf{T}} + \phi\,r\left(\Sigma_{\mathsf{V}} + \bar{\Omega}_{\mathsf{V}}\right) - 2\,r\,\bar{\mathcal{H}}_{\mathsf{V}}\right]\ , \tag{B.48}$$

B.2 Trace equation

$$\mathcal{C}_2(\hat{X}_{\mathsf{S}} - \ddot{R}_{\mathsf{S}}) = \frac{1}{3}R_{\mathsf{S}}\,\mathcal{C}_1 - \mathcal{C}_2\left[\left(\phi + \mathcal{A}\right)X_{\mathsf{S}} - \frac{\ell\,(\ell+1)}{r^2}R_{\mathsf{S}}\right]\ . \tag{B.49}$$

B.3 Constraints

$$\mathcal{H}_{\mathsf{S}} = \frac{\ell\,(\ell+1)}{r}\left(\bar{\Sigma}_{\mathsf{V}} - \Omega_{\mathsf{V}}\right) - \left(2\mathcal{A} - \phi\right)\Omega_{\mathsf{S}}\ , \tag{B.50}$$

$$\frac{1}{2}\bar{Y}_{\mathsf{V}} - \frac{\xi_{\mathsf{S}}}{r} - \frac{\ell\,(\ell+1)-2}{2r}\,\bar{\zeta}_{\mathsf{T}} = -\,\bar{\mathcal{E}}_{\mathsf{V}}\ , \tag{B.51}$$

$$\mathcal{H}_{\mathsf{V}} = -\,\frac{\Omega_{\mathsf{S}}}{r} - \frac{\ell\,(\ell+1)-2}{2r}\,\bar{\Sigma}_{\mathsf{T}} - \frac{1}{2}\phi\left(\bar{\Sigma}_{\mathsf{V}} - \Omega\right)\ . \tag{B.52}$$

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