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An Approach to Coincidence Theory through Universal Covering Spaces

by

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CONTENTS.

Introduction.

The close relationship between the theory of fixed points and the theory of coincidences of maps is well known. This presentation is aimed at recording one of the less well documented approaches to fixed point theory as extended to the more general situation of coincidences. The approach referred to is that by way of the Universal Covering Spaces.

The existing theory of coincidences is geometrically well realised in this setting and after some consideration, the necessary extensions and generalizations of the techniques as utilized in fixed point theory lead to an appealing conceptual notion of " essentiality of coincidence classes ".

Many hints have been made in the literature (see [1] and "On the sharpness of the Δ_2 and Δ_1 Nielsen Numbers" by Robin Brooks, J.Reine Angew. Math. 259,(1973), 101-108.) that lifts of mappings and the theory of fibres and related topics lend themselves to coincidence theory . It is the intention of this presentation to follow some of the basic properties through this approach and to show, wherever it is thought desirable, the ties between this and two of the existing approaches - for example, in the definition of the Nielsen Number, which is fundamental to both fixed point theory and coincidence theory.

This work is largely inspired by a paper by Jiang Bo-ju and one by W.Franz (see [7] and [4]) .

In the first section, some existing properties of fixed point classes in absolute neighbourhood retracts are extended to coincidence classes for use in later sections. The definition of a covering space and the notion of Universal Covering Spaces and their properties are dealt with in section 2.

Having established the necessary background, a classification of lift-pairs is introduced in section 3, and is used throughout the following sections . Sections 4 and 5 are concerned with the number of lift-pair classes and how the classes may be affected by deformations of their maps.

Sections 6 and 7 compare the number of lift-pair classes with the number of coincidence classes obtained in two of the best known methods of arriving at the Nielsen Number.

In the last section, an estimate is made of the Nielsen Number using the first integral homology group.

Most of the results obtained here are extensions of those contained in [7] where the fixed point case is discussed.

Throughout this presentation, theorems (T), corollaries (Cor), lemmas (L), and propositions (P) are denoted by the appropriate symbols shown in brackets, preceeded by the section number and their number within the section, e.g. 3.5 P: represents the fifth listed part of section 3 which is a proposition.

New terms and terms considered essential to this presentation are listed as definitions (D), and numbered as above.

I wish to take this opportunity to thank my supervisor Dr.H.Schlagbauer for his constant encouragement and invaluable help in the preparation of this project .

1.2 D: We define an equivalence relation on $K(f,g)$ by setting $x \equiv x'$ for $x, x' \in K(f,g)$ if and only if there is a path ω from x to x' in X such that $f\omega$ is fixed end point homotopic to $g\omega$. $f\omega$ and $g\omega$ are paths in Y from $f(x) = g(x)$ to $f(x') = g(x')$ given by

$$f\omega(t) = f(\omega(t)) \quad \text{for} \quad t \in I$$
$$g\omega(t) = g(\omega(t)) \quad \text{for} \quad t \in I.$$

This equivalence relation classifies the coincidence points of (f,g) and we will denote by $[x]$ the class of all $x' \in K(f,g)$ which have $x \equiv x'$.

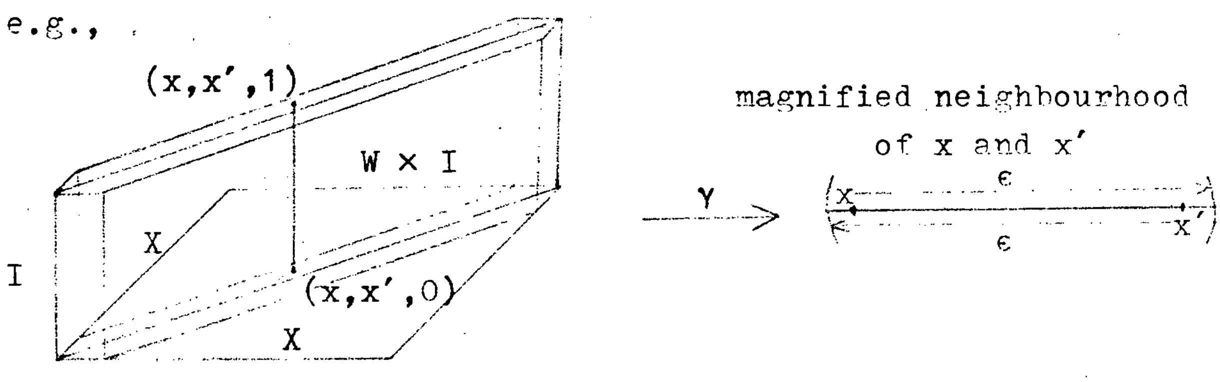
1.3 D: A compact metric space X with metric d is uniformly locally contractible if given $\epsilon > 0$ there exists $\delta > 0$ such that if

$W = \{ (x,x') \in X \times X : d(x,x') < \delta \}$ then there is a map $\gamma : W \times I \longrightarrow X$ such that

$$\gamma(x,x',0) = x$$
$$\gamma(x,x',1) = x'$$
$$\gamma(x,x,t) = x \quad \text{for all} \quad t \in I$$

and $\operatorname{diam}(\gamma(x,x') \times I) < \epsilon$ for all $(x,x') \in W$, (where $\operatorname{diam}(A)$, the diameter of the set A, is the supremum of the set $\{ d(x,x') : x,x' \in A \}$).

e.g.,



On $W \times 0$, γ acts as projection onto the first co-ordinate axis.

On $W \times 1$, γ acts as projection onto the second co-ordinate axis.

γ maps any line $(x,x') \times I$ so that any two image points are within ϵ of each other. So, in the diagram above, $\gamma((x,x') \times I)$ is a proper subset of the neighbourhood shown.

A subset A of X is called a <u>neighbourhood retract</u> of X if there is an open set U of X containing A and a map $r : U \longrightarrow A$ such that the restriction of r to A, $r/A = 1_A$ the identity map on A.

r is a <u>retraction</u> of U onto A.

1.4 D: We call a compact metric space X a <u>compact absolute neighbourhood retract</u> (compact ANR) if and only if there is an embedding $i : X \longrightarrow I^\infty$ such that $i(X)$ is a neighbourhood retract of I^∞ ; where I^∞ represents the Hilbert Cube.

The following well known results are stated without proof:

1.5 : A compact ANR is uniformly locally contractible. (see [3], page 39.)

1.6 : X is locally contractible, implies X is locally path-connected.
(see [6], page 89.)
(X is locally contractible if for every point $x \in X$ and every neighbourhood U of x, there is a neighbourhood $V \subset U$ of x such that there is a homotopy $H : V \times I \longrightarrow X$ with $H(v,0) = v$ and $H(v,1) = c$ constant in X, $\forall\, v \in V$.
X is locally path-connected if for every point $x \in X$ and every neighbourhood U of x there is a neighbourhood $V \subset U$ of x such that for every pair of points v, v' in V there is a path ω in U with $\omega(0) = v$, $\omega(1) = v'$.)

Hence,

1.7 : A compact ANR is locally path-connected.

1.8 T: If X is a compact ANR and ω, ω' are paths in X with the same initial point and terminal point, then there exists $\alpha > 0$ such that

$$d(\omega(t),\omega'(t)) < \alpha \quad \text{for all} \quad t \in I \quad \text{implies}$$

ω and ω' are fixed end point homotopic.
We call α a homotopy barrier for X.

Proof : X is uniformly locally contractible, so there exists $\alpha > 0$ so that if

$$W = \{ (x,x') \in X \times X : d_X(x,x') < \alpha \}$$

then there is a map $\gamma : W \times I \longrightarrow X$ such that

$$\gamma(x,x',0) = x ,$$
$$\gamma(x,x',1) = x',$$
$$\gamma(x,x,t) = x \text{ for all } t \in I.$$

Given that $(\omega(t),\omega'(t)) \in W$ for all $t \in I$ by hypothesis, we define $H : I \times I \longrightarrow X$ by

$$H(s,t) = \gamma(\omega(s),\omega'(s),t).$$

Then $H(s,0) = \gamma(\omega(s),\omega'(s),0) = \omega(s)$;

$$H(s,1) = \gamma(\omega(s),\omega'(s),1) = \omega'(s) \text{ for all } s \in I ;$$
$$H(0,t) = \gamma(\omega(0),\omega'(0),t) = \omega(0) = \omega'(0) \text{ and}$$
$$H(1,t) = \gamma(\omega(1),\omega'(1),t) = \omega(1) = \omega'(1), \text{ for all } t \in I.$$

H is the composition

$$(s,t) \xrightarrow{\varphi} (\omega(s),\omega'(s),t) \xrightarrow{\gamma} \gamma(\omega(s),\omega'(s),t).$$

φ is continuous :

Given $\epsilon > 0$,

$$\begin{aligned} d_{W\times I}(\varphi(s,t),\varphi(s',t)) &= d_{W\times I}((\omega(s),\omega'(s),t),(\omega(s'),\omega'(s'),t)) \\ &\leqq d_{W\times I}((\omega(s),\omega'(s),t),(\omega(s),\omega'(s'),t)) + \\ &\quad d_{W\times I}((\omega(s),\omega'(s'),t),(\omega(s'),\omega'(s'),t)) \\ &= d_{X\times X}(\omega'(s),\omega'(s')) + d_{X\times X}(\omega(s),\omega(s')) \\ &\leqq \epsilon/2 \text{ (continuity of } \omega') + \epsilon/2 \text{ (continuity of } \omega) \\ &= \epsilon. \end{aligned}$$

Therefore, H is continuous and is a fixed end point homotopy between ω and ω'.

1.9 Cor: For X any space and Y a compact ANR, if $x,x' \in K(f,g)$ there exists $\alpha > 0$ such that if ω is a path in X from x to x' satisfying

$$d(f\omega(t),g\omega(t)) < \alpha \text{ for all } t \in I, \text{ then } [x] = [x'].$$

Note: If X is also an ANR, then the existence of a path satisfying the above is guaranteed whenever we know $d(x,x')$ is sufficiently small. In this case we have that if two coincidence points are close enough then they are equivalent.

1.10 D: Let X be a topological space and let $x \in X$.

<u>The fundamental group of X based at x</u>, denoted by $\pi(X,x)$ is the group of path classes with x as initial and terminal point. Two closed paths, or loops, at x in X (i.e. paths in X which have x as initial and terminal point) are in the same path class if and only if they are fixed end point homotopic.

This group, whose elements are path classes has a group operation defined as follows.

Let α, β be elements of $\pi(X,x)$ with $\omega \in \alpha$ and $\omega' \in \beta$ (i.e. $[\omega] = \alpha$, $[\omega'] = \beta$, where the brackets denote the path classes.)

Then $\alpha.\beta = [\omega.\omega']$ where
$$\omega.\omega'(t) = \omega(2t) \quad 0 \leqq t \leqq 1/2$$
$$= \omega'(2t) \quad 1/2 \leqq t \leqq 1.$$

The fundamental group plays a large part in the development of the next sections since it is closely allied to the Universal Covering Spaces of X that are discussed.

If X is path-connected, $\pi(X,x)$ is isomorphic to $\pi(X,x')$ for any $x,x' \in X$, so if we select an arbitrary base point, say $x \in X$, we loose no generality by considering only $\pi(X,x)$.

If f is a map from X to Y, then f induces a homomorphism $f_* : \pi(X,x) \longrightarrow \pi(Y,f(x))$ if we define $f_*[\omega] = [f\omega]$ for any closed path ω at x in X.

Covering Spaces.

2. This section reviews some properties of covering spaces and lifts of maps which lead in a natural way to another classification of coincidence points comparable to that given in section 1. The classification by " lift-pairs " gives a clear geometric notion of the classification procedure and establishes, as is shown in section 4, the finiteness of the number of coincidence classes.

We consider topological spaces and maps between these spaces and develop the theory of coincidences through the Universal Covering Spaces placing restrictions on the structure of the base spaces when it is found to be necessary.

2.0 : All spaces considered in subsequent arguments are path-connected.

2.1 D: Let X be a topological space.
A covering space of X is a pair $(\tilde{X},p)$ where $\tilde{X}$ is a space and $p : \tilde{X} \longrightarrow X$ a map such that :
for each $x \in X$, there is a path-connected open neighbourhood U of x such that every path component of $p^{-1}(U)$ is mapped homeomorphically onto U by p.

Any open neighbourhood, with the properties of U as described, is called an elementary neighbourhood. p is referred to as a projection. X is called the base space of $(\tilde{X},p)$.

If X is locally path-connected and if $(\tilde{X},p)$ is a covering space of X, then $\tilde{X}$ is locally path-connected since p is a local homeomorphism. (i.e. each point $\tilde{x} \in \tilde{X}$ has an open neighbourhood V so that pV is open and p maps V homeomorphically onto pV.)

p is clearly an open map.

We require then that the spaces under consideration be locally path-connected since this property has the further effect that the path components of each space are open sets which coincide with the components of the space.

A topological space X is <u>simply connected</u> if and only if it is path-connected and $\pi(X,x)$ is trivial for every base point $x \in X$.

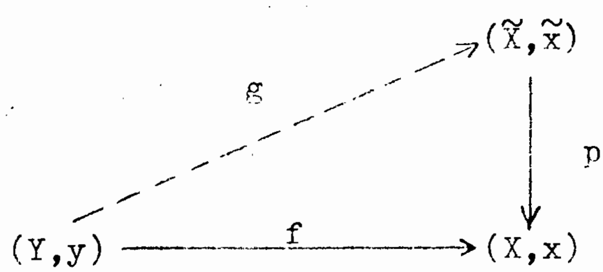
2.2 D: A covering space $(\tilde{X},p)$ of X is called a <u>Universal Covering Space of X</u> if and only if $\tilde{X}$ is simply connected.

We shall make use of the following notation when discussing spaces with predetermined base points.

If X and Y are topological spaces, $x \in X$ and $y \in Y$ then $f : (X,x) \longrightarrow (Y,y)$ means that f is a map of X into Y such that $f(x) = y$.

To approach coincidence theory through Universal Covering Spaces we need to be able to lift the paths in the base spaces and maps between the base spaces up to the Universal Covering Spaces. The criterion involved is as follows:

Given spaces X, Y and $f : (Y,y) \longrightarrow (X,x)$ and $(\tilde{X},p)$ a covering space of X, under what conditions will there exist a map $g : (Y,y) \longrightarrow (\tilde{X},\tilde{x})$ so that the diagram



commutes?

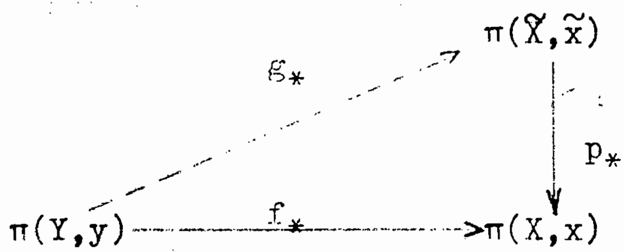
Clearly, $\tilde{x} \in p^{-1}(x)$ is assumed.

When such a map g exists, g is called a lift of f to $\tilde{X}$.

If we consider the algebraic implications of the commutativity of the above diagram it is obvious that a necessary condition for commutativity is that

$$f_*\pi(Y,y) \subset p_*\pi(\tilde{X},\tilde{x});$$

where f_* and p_* are the induced group homomorphisms in the corresponding commutative diagram



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It is possible to show that the condition above is also sufficient (see [8] page 156.)

so we get:

2.3 T: If $(\tilde{X},p)$ is a covering space of X, $y \in Y$, $\tilde{x} \in \tilde{X}$ and $x = p(\tilde{x})$, then a map $f : (Y,y) \longrightarrow (X,x)$ has a lift $g : (Y,y) \longrightarrow (\tilde{X},\tilde{x})$ if and only if

$$f_*\pi(Y,y) \subset p_*\pi(\tilde{X},\tilde{x}).$$

The following result is a form of uniqueness lemma regarding lifted maps.

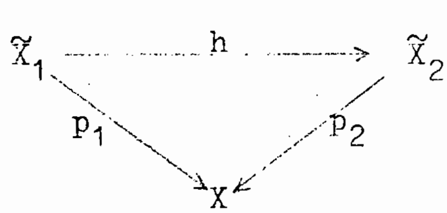
2.4 L: Let $(\tilde{X},p)$ be a covering space of X.
If $f, f' : Y \longrightarrow \tilde{X}$ are maps such that $pf = pf'$, then if there exists $y \in Y$ with $f(y) = f'(y)$,

f is identical to f'.

Proof: (see [8] page 152.)

2.5 D: If $(\tilde{X}_1,p_1)$ and $(\tilde{X}_2,p_2)$ are covering spaces of X, a <u>homomorphism</u> of $(\tilde{X}_1,p_1)$ into $(\tilde{X}_2,p_2)$ is a map

$h : \tilde{X}_1 \longrightarrow \tilde{X}_2$ such that the diagram



commutes.

A homomorphism h of $(\tilde{X}_1, p_1)$ into $(\tilde{X}_2, p_2)$ is an <u>isomorphism</u> if there exists a homomorphism k of $(\tilde{X}_2, p_2)$ to $(\tilde{X}_1, p_1)$ such that the compositions hk and kh yield the identity maps.

Since Universal Covering Spaces of X are simply connected we get:

2.6 Cor: Any two Universal Covering Spaces of X are isomorphic.

We, thus, talk about <u>the</u> Universal Covering Space of X implying uniqueness up to isomorphism.

For Universal Covering Spaces to be ensured we require the base spaces to be connected, locally path-connected and semi-locally simply connected. (see [8] and [9].)

X is semi-locally simply connected if every point $x \in X$ has a neighbourhood U such that the homomorphism $\pi(U,x) \longrightarrow \pi(X,x)$, induced by inclusion, is trivial.

We thus require all base spaces to be connected, locally path-connected, semi-locally simply connected topological spaces. <u>These properties will be assumed forthwith without further restatement</u>.

We will denote by $\tilde{X}$ the Universal Covering Space of X and endow $\tilde{X}$ with the topology arising from the construction of $\tilde{X}$ as given below in 2.9.

2.7 D: Given any map $f : X \longrightarrow Y$, because $\tilde{X}$ is simply connected, pf can be lifted to $\tilde{f}$ giving commutativity of the diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{f}} & \tilde{Y} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{f} & Y \end{array} .$$

We will call the map $\tilde{f}$ <u>a lift of f</u> into the Universal Covering Spaces of X and Y.

2.8 D: Given any pair (f,g) of maps from X to Y, a <u>lift-pair</u> $(\tilde{f},\tilde{g})$ of (f,g) is a pair of maps from $\tilde{X}$ to $\tilde{Y}$ where $\tilde{f}$ is a lift of f and $\tilde{g}$ is a lift of g.

The following note on the construction of $\tilde{X}$ from any connected, locally path-connected and semi-locally simply connected space X, is to clarify the structure of $\tilde{X}$ and to assure us that the operation of any element $\alpha \in \pi(X,x)$ on $\tilde{X}$, described in the next section, is a continuous map, under this topology on $\tilde{X}$, of $\tilde{X}$ into itself. In section 3 we also observe how the elements of the group of cover transformations of $\tilde{X}$ can be related to $\pi(X,x_0)$ for some base point $x_0 \in X$.

2.9 Construction : Choose a base point $x_0 \in X$, X connected, locally path-connected and semi-locally simply connected.

Define $\tilde{X}$ to be the set of all equivalence classes of paths in X which have x_0 as initial point. The equivalence relation used is that of fixed end point homotopy.

Define $p : \tilde{X} \longrightarrow X$ by

$p(\alpha)$ = common terminal point of the elements of the class α.

Our hypotheses on X, imply that the topology on X has as basis all open sets U such that U is path-connected and the homomorphism $\pi(U,x) \longrightarrow \pi(X,x)$ induced by inclusion, is trivial.

Equivalently we can characterize the basic open sets of X as those open sets U in which every closed path (in U) is equivalent to a constant path in X.

Given any class $\alpha \in \tilde{X}$ and any basic open set U that contains $p(\alpha)$, let the pair (α,U) represent the set of all those $\beta \in \tilde{X}$ such that

$\beta = \alpha.\gamma$ for some γ a path class within U.

(α,U) is a subset of $\tilde{X}$ and to topologize $\tilde{X}$ we

Lift-pair Classes.

3. This section recounts the action of the group $\pi(X,x)$ on the set $p^{-1}(x)$ and establishes those results required to make the definition of lift-pair classes precise.

3.1 P: Let $(\tilde{X},p)$ be the Universal Covering Space of X and $\tilde{x} \in \tilde{X}$. Let $\omega, \omega' : I \longrightarrow \tilde{X}$ be paths in $\tilde{X}$ with initial point $\tilde{x}$, then

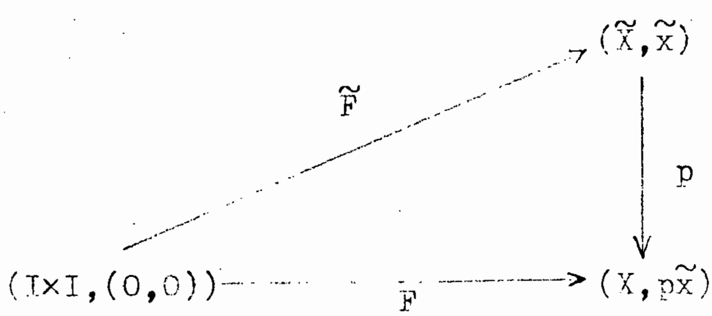
$p\omega$ and $p\omega'$ are fixed end point homotopic if and only if ω and ω' are fixed end point homotopic.

(The second statement is equivalent to saying that $\omega(1) = \omega'(1)$ since $\tilde{X}$ is simply connected.)

Proof : The implication one way is trivial so we only prove that $p\omega$ and $p\omega'$ fixed end point homotopic implies that ω and ω' are also.

Let $F : I \times I \longrightarrow X$ be a fixed end point homotopy between $p\omega$ and $p\omega'$.

Since $I \times I$ is simply connected, consider the diagram



We can find a map $\tilde{F} : I \times I \longrightarrow \tilde{X}$ such that $p\tilde{F} = F$ and $\tilde{F}(t,0) = \omega(t)$ for all $t \in I$.

Now $\tilde{F}(0,t)$ and $\tilde{F}(1,t)$ are contained in $p^{-1}(p\omega(0))$ and $p^{-1}(p\omega(1))$ respectively.

Since there can be no non-constant path in $p^{-1}(x)$ for any $x \in X$, - any such path would be a lift of the constant path at x and as such is unique - we must have

$\tilde{F}(0,t)$ is a single point and

$\tilde{F}(1,t)$ is a single point.

$\tilde{F}(t,1)$ is a path ω'' with $\omega''(0) = \omega(0)$ and $\omega''(1) = \omega(1)$, with $\tilde{F}$ a fixed end point homotopy between ω and ω''.

We also have $p\omega'' = p\omega'$.

Since ω'' and ω' have common origin at $\tilde{x}$ and are both lifts of $p\omega'$, by 2.4 we must have $\omega'' = \omega'$.

Hence ω and ω' are fixed end point homotopic.

In particular, $\omega(1) = \omega'(1)$.

This last statement relies upon the fact that :

3.2 L: If ω and ω' are paths in X from x to x' then ω is fixed end point homotopic to ω' if and only if the closed path $\omega.\omega'$ at x is fixed end point homotopic to the constant path at x . i.e. $[\omega.\omega'^{-1}] = [\epsilon_x]$.

Proof : Let F be a fixed end point homotopy from ω to ω'.

Denote $F(t,s)$ by $f_s(t)$

then $f_o(t) = \omega(t)$

$f_1(t) = \omega'(t)$ for all $t \in I$.

Define $G : I \times I \longrightarrow X$ by

$$G(t,s) = f_{2s}.f_1^{-1}(t) \qquad 0 \leqq s \leqq 1/2$$

$$= g_{\omega'(2-2s)}.g_{\omega'(2-2s)}^{-1}(t) \qquad 1/2 \leqq s \leqq 1$$

where $f_i(I)$ is a path from x to x', $i \in I$;

$g_{\omega'(s)}(t)$ represents the path from x to $\omega'(s)$ in ω' given by

$$g_{\omega'(s)}(t) = \omega'(t) \quad \text{for } t \leqq s$$

$$\omega'(s) \quad \text{for } t > s .$$

Then G is a fixed end point homotopy between the closed paths $\omega.\omega'^{-1}$ and the constant path at x .

Conversely: Suppose $G : I \times I \longrightarrow X$ is a fixed end point homotopy between $\omega.\omega'^{-1}$ and the constant path at x .

Denote $G(t,s)$ by $g_s(t)$ as before.

Define $F : I \times I \longrightarrow X$ by

$$F(t,s) = g_s.\omega'(t) \quad \text{for all } s,t \in I ,$$

then F is a fixed end point homotopy between $\omega.\omega'^{-1}.\omega'$ and ω'.

But $\omega.\omega'^{-1}.\omega'$ is fixed end point homotopic to ω, thus by the transitivity of the relation " fixed end point homotopic " ω is fixed end point homotopic to ω'.

3.3 Cor: If $\tilde{x} \in p^{-1}(x)$ and $\alpha \in \pi(X,x)$ there exists a unique path class $\tilde{\alpha}$ of paths in $\tilde{X}$ such that $p_*(\tilde{\alpha}) = \alpha$ and the initial point of elements of $\tilde{\alpha}$ is $\tilde{x}$.

This is a direct consequence of 2.4 and 3.1 .

We are led to the definition of the action of the group $\pi(X,x)$ on $p^{-1}(x)$ as follows :

3.4 D: There is an anti-homomorphism φ from $\pi(X,x)$ to the group of all permutations H of $p^{-1}(x)$ given by

$\varphi(\alpha) = h_\alpha$ for any $\alpha \in \pi(X,x)$ where h_α is as follows.

For any $\tilde{x} \in p^{-1}(x)$, there is a unique path class $\tilde{\alpha}$ in $\tilde{X}$ such that $p_*(\tilde{\alpha}) = \alpha$ where the initial point of $\tilde{\alpha}$ is $\tilde{x}$, (3.3). The terminal point of $\tilde{\alpha}$, $\tilde{x}'$ is defined as the image of $\tilde{x}$ under h_α.

i.e. $(\tilde{x})h_\alpha = \tilde{x}' \in p^{-1}(x)$.

That φ is an anti-homomorphism is easily established (see [5] page 261.)

whence, $\varphi(\alpha\beta) = h_{\alpha\beta} = h_\beta h_\alpha$ for all $\alpha,\beta \in \pi(X,x)$.

Now $\pi(X,x)$ itself acts on $p^{-1}(x)$ on the left as follows.

For $\alpha \in \pi(X,x)$ and $\tilde{x} \in p^{-1}(x)$

$\alpha\tilde{x} = (\tilde{x})h_\alpha$.

So we get $\alpha\beta(\tilde{x}) = (\tilde{x})h_{\alpha\beta} = (\tilde{x})h_\beta h_\alpha = \alpha(\beta(\tilde{x}))$.

We also have $1(\tilde{x}) = \tilde{x}$ for 1 the constant loop class at x, so it is established that $\pi(X,x)$ forms a group of left-operators on $p^{-1}(x)$.

We can go further than this and define the operation of $\pi(X,x)$ on the whole of $\tilde{X}$, in case X is path-connected.

Since we are insisting that X is path-connected, we can fix a path from the base point x to every point $x' \in X$.

In this way we induce an isomorphism between $\pi(X,x)$ and $\pi(X,x')$ for all $x' \in X$.

Suppose ω is the fixed path from x to x', then to $\alpha \in \pi(X,x)$ we relate the class

$$[\omega a \omega^{-1}] \quad \text{where} \quad a \in \alpha .$$

Then we write for any $x' \in X$, $\tilde{x}' \in p^{-1}(x)$, $\alpha \in \pi(X,x)$

$$\alpha\tilde{x}' = [\omega a \omega^{-1}]\tilde{x}' = (\tilde{x}')h_{[\omega a \omega^{-1}]} .$$

In this way we have defined the operation of $\pi(X,x)$ on $\tilde{X}$. It necessarily transforms $p^{-1}(x')$ into itself for any x' in X, as a permutation of that set. [5],[8].

We now agree to fix some base point $x_0 \in X$ and $\tilde{x}_0 \in p^{-1}(x_0)$; similarly, where relevant, $f(x_0) \in Y$ and $\tilde{y}_0 \in p^{-1}(f(x_0))$; and in future we may write $\pi(X)$ in place of $\pi(X,x_0)$ without loss of generality.

In case we write $\pi(X,x)$ we are referring to the image of the isomorphism of the group $\pi(X,x_0)$, which is induced by the path from the base point to x.

3.5 P: Since $\tilde{X}$ is a path-connected space it is clear that for any two points $\tilde{x}$ and $\tilde{x}'$ in $p^{-1}(x)$ there is a path class $\alpha \in \pi(X,x)$ with $\alpha\tilde{x} = \tilde{x}'$.

One need only take the projection of the path class in $\tilde{X}$ that has initial point $\tilde{x}$ as α and observe that by 2.4 and 3.2 this α is as required.

Given any $\alpha \in \pi(X,x)$ and $\beta \in \pi(Y,f(x))$, if $\tilde{f}$ is a lift of f, then so is $\tilde{f}\alpha$, $\beta\tilde{f}$, and $\beta\tilde{f}\alpha$; where we define $\tilde{f}\alpha(\tilde{x}) = \tilde{f}(\alpha\tilde{x}) = \tilde{f}((\tilde{x})h_\alpha)$

and $\beta\tilde{f}(\tilde{x}) = \beta(\tilde{f}(\tilde{x})) = (\tilde{f}(\tilde{x}))h_\beta$.

3.6 P: For each $x \in K(f,g)$

$$\tilde{x} \in p^{-1}(x) \quad \text{and} \quad \tilde{y} \in p^{-1}(f(x))$$

there is a <u>unique</u> lift-pair $(\tilde{f},\tilde{g})$ of (f,g) having

$$\tilde{f}(\tilde{x}) = \tilde{g}(\tilde{x}) = \tilde{y} :$$

i.e. having $\tilde{x} \in K(\tilde{f},\tilde{g})$ with common image at $\tilde{y}$.

Notice the rôle of $\tilde{y}$. There are many lift-pairs of (f,g) having coincidence at $\tilde{x}$ and images of $\tilde{x}$ in $p^{-1}(f(x))$.

The result follows immediately from 2.4 and the fact that such lifts do exist by 2.3 .

3.7 P: If $\tilde{x} \in K(\tilde{f},\tilde{g})$ with $\tilde{f}(\tilde{x}) = \tilde{g}(\tilde{x}) = \tilde{y}$ then the lift-pair having $\alpha\tilde{x}$ as coincidence with common image $\tilde{y}$ will be $(\tilde{f}\alpha^{-1},\tilde{g}\alpha^{-1})$

where $\alpha \in \pi(X)$ and

$$\tilde{f}\alpha^{-1}(\tilde{x}') = \tilde{f}(\alpha^{-1}\tilde{x}') \quad \text{for} \quad \tilde{x}' \in p^{-1}(x) .$$

This follows from the uniqueness of such a pair having $\tilde{f}\alpha^{-1}(\alpha\tilde{x}) = \tilde{g}\alpha^{-1}(\alpha\tilde{x}) = \tilde{y}$.

α^{-1} represents the class $[\omega^{-1}]$ where $\omega \in \alpha$ and $\omega^{-1}(t) = \omega(1-t)$.

We define an equivalence relation on the set of all lift-pairs of (f,g) as follows :

3.8 D: If $(\tilde{f},\tilde{g})$ and $(\tilde{f}',\tilde{g}')$ are lift-pairs of (f,g) then $(\tilde{f},\tilde{g})$ is equivalent to $(\tilde{f}',\tilde{g}')$ if and only if there exists $\alpha \in \pi(X)$,

and $\beta \in \pi(Y)$, so that

$$\tilde{f}' = \beta\tilde{f}\alpha$$

and $\tilde{g}' = \beta\tilde{g}\alpha$.

We write $(\tilde{f},\tilde{g}) \equiv (\tilde{f}',\tilde{g}')$ to denote the equivalence which divides the collection of all lift-pairs of (f,g) into equivalence classes. $[\tilde{f},\tilde{g}]$ will denote the class of all lift-pairs equivalent to $(\tilde{f},\tilde{g})$; so

$(\tilde{f},\tilde{g}) \equiv (\tilde{f}',\tilde{g}')$ is indicated by $[\tilde{f},\tilde{g}] = [\tilde{f}',\tilde{g}']$

From the above definition we can establish the following fact regarding lift-pairs.

3.9 P: If two lift-pairs $(\tilde{f},\tilde{g})$ and $(\tilde{f}',\tilde{g}')$ of (f,g) have a coincidence in $p^{-1}(x)$ then $(\tilde{f},\tilde{g}) \equiv (\tilde{f}',\tilde{g}')$.

That is, given $\tilde{f}(\tilde{x}) = \tilde{g}(\tilde{x}) = \tilde{y}$ for $\tilde{x} \in p^{-1}(x)$

and $\tilde{f}'(\tilde{x}') = \tilde{g}'(\tilde{x}') = \tilde{y}'$ for $\tilde{x}' \in p^{-1}(x)$

there exists $\alpha \in \pi(X,x)$

and $\beta \in \pi(Y,f(x))$ such that

$\alpha\tilde{x} = \tilde{x}'$ and $\beta\tilde{y} = \tilde{y}'$

and $\tilde{f}' = \beta\tilde{f}\alpha^{-1}$

$\tilde{g}' = \beta\tilde{g}\,\alpha^{-1}$.

Further, we get $K(\tilde{f}',\tilde{g}') = \alpha K(\tilde{f},\tilde{g})$

where for any subset A of $\tilde{X}$,

$\alpha A = \{ \alpha a : a \in A \}$.

Proof : The existence of α and β is a direct consequence of $p^{-1}(x)$ and $p^{-1}(f(x))$ being homogenous left $\pi(X,x)$ - and $\pi(Y,f(x))$ - spaces respectively.

We also have $\beta\tilde{f}$ is a lift of f with

$\beta\tilde{f}(\tilde{x}) = \beta\tilde{y} = \tilde{y}'$ and $\tilde{f}'\alpha$ is a lift of f with

$\tilde{f}'\alpha(\tilde{x}) = \tilde{y}'$.

Therefore, by uniqueness,

$\beta\tilde{f} = \tilde{f}'\alpha$ and so

$\tilde{f}' = \beta\tilde{f}\alpha^{-1}$.

Similarly,

$\tilde{g}' = \beta\tilde{g}\alpha^{-1}$,

Thus, $[\tilde{f},\tilde{g}] = [\tilde{f}',\tilde{g}']$.

Finally,

$$\begin{aligned}
&\tilde{x} \in K(\tilde{f}',\tilde{g}') \\
\longleftrightarrow\ &\tilde{f}'(\tilde{x}) = \tilde{g}'(\tilde{x}) \\
\longleftrightarrow\ &\beta\tilde{f}\alpha^{-1}(\tilde{x}) = \beta\tilde{g}\alpha^{-1}(\tilde{x}) \\
\longleftrightarrow\ &\beta\tilde{f}(\alpha^{-1}\tilde{x}) = \beta\tilde{g}(\alpha^{-1}\tilde{x}) \\
\overset{(a)}{\longleftrightarrow}\ &\tilde{f}(\alpha^{-1}\tilde{x}) = \tilde{g}(\alpha^{-1}\tilde{x}) \\
\longleftrightarrow\ &\alpha^{-1}\tilde{x} \in K(\tilde{f},\tilde{g}) .
\end{aligned}$$

(a) follows since $p^{-1}(f(x))$ is a left- $\pi(Y,f(x))$ space and we know that for any $\beta \in \pi(Y,f(x))$ the map of $p^{-1}(f(x)) \longrightarrow p^{-1}(f(x))$ given by $\tilde{y} \longrightarrow \beta\tilde{y}$ is a permutation of $p^{-1}(f(x))$.

The next result is a powerful corollary to 3.9 , and the results so far established in this section.

3.10 Cor : $pK(\tilde{f},\tilde{g}) \cap pK(\tilde{f}',\tilde{g}') \neq \emptyset$
$\longrightarrow (\tilde{f},\tilde{g}) \equiv (\tilde{f}',\tilde{g}')$
$\longrightarrow pK(\tilde{f},\tilde{g}) = pK(\tilde{f}',\tilde{g}')$.

Therefore,

$$K(f,g) = \bigcup_{(\tilde{f},\tilde{g})} pK(\tilde{f},\tilde{g}) = \bigcup_{[\tilde{f},\tilde{g}]} pK(\tilde{f},\tilde{g}) .$$

We are now in a position to define coincidence classes of a pair (f,g) in terms of lift-pair classes.

3.11 D: We shall call $pK(\tilde{f},\tilde{g})$ <u>the coincidence class of (f,g) determined by the lift-pair class $[\tilde{f},\tilde{g}]$</u> .

According to 3.8 and the last definition, we have as many coincidence classes of (f,g) as there are lift-pair classes of (f,g) . By 3.10 , they are mutually disjoint and their union is precisely the set of all coincidence points of f and g .

3.12 Note: Notice that it is necessary to distinguish between the empty coincidence classes , $pK(\tilde{f},\tilde{g})$ and $pK(\tilde{f}',\tilde{g}')$ in case

$K(\tilde{f},\tilde{g}) = \emptyset$ and $K(\tilde{f}',\tilde{g}') = \emptyset$ and $[\tilde{f},\tilde{g}] \neq [\tilde{f}',\tilde{g}']$.

Finiteness of the number of non-trivial lift-pair classes of (f,g).

4. In this section we justify the nomenclature used in 3.10 and investigate the classification of coincidence points given in 3.10 to establish that there are only finitely many such classes in the case where X and Y are compact ANRs .

4.1 P: The following statements are equivalent :

Given $x, x' \in X$,

(i) x and x' are in the same coincidence class of f and g as defined in 1.2 .

(ii) x and x' are in the same coincidence class of (f,g) as defined in 3.11 .

Proof : (i) ⟶ (ii) .

Let $\omega : I \longrightarrow X$ be a path in X from x to x' so that $f\omega$ and $g\omega$ are fixed end point homotopic.

There is a unique lift-pair $(\tilde{f},\tilde{g})$ of (f,g) having $\tilde{f}(\tilde{x}) = \tilde{g}(\tilde{x}) = \tilde{y}$, where $\tilde{x} \in p^{-1}(x)$ and $\tilde{y} \in p^{-1}(f(x))$ are predetermined elements.

Lift $f\omega$ to $(\widetilde{f\omega})$ and $g\omega$ to $(\widetilde{g\omega})$ in $\tilde{Y}$ with initial point $\tilde{y}$.

By uniqueness, $(\widetilde{f\omega}) = \tilde{f}\tilde{\omega}$ and

$(\widetilde{g\omega}) = \tilde{g}\tilde{\omega}$ where

$\tilde{\omega}$ is the unique lift of ω which has initial point $\tilde{x}$.

Now $p(\tilde{f}\tilde{\omega})$ and $p(\tilde{g}\tilde{\omega})$ are fixed end point homotopic and so by 3.1

$$\tilde{f}\tilde{\omega}(1) = \tilde{g}\tilde{\omega}(1)$$

i.e. $\tilde{\omega}(1) \in K(\tilde{f},\tilde{g})$.

But $\tilde{\omega}(1) \in p^{-1}(x')$ and so

$x \in pK(\tilde{f},\tilde{g})$ and

$x' \in pK(\tilde{f},\tilde{g})$

Thus x and x' are in the coincidence class of (f,g) determined by $[\tilde{f},\tilde{g}]$.

(ii) ⟶ (i) .

Let x and x' be in $pK(\tilde{f},\tilde{g})$ with $\tilde{x} \in \tilde{X}$ and

$\tilde{x}' \in \tilde{X}$ satisfying $p(\tilde{x}) = x$ and $p(\tilde{x}') = x'$, and also

$$\tilde{f}(\tilde{x}) = \tilde{g}(\tilde{x}) \text{ and}$$
$$\tilde{f}(\tilde{x}') = \tilde{g}(\tilde{x}') .$$

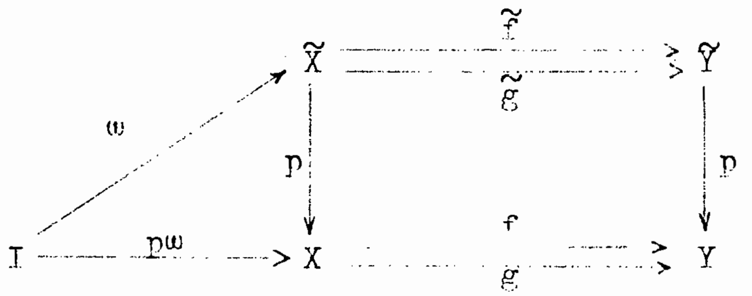
Let ω be a path in $\tilde{X}$ from $\tilde{x}$ to $\tilde{x}'$.

Consider $\tilde{f}\omega$ and $\tilde{g}\omega$.

These two paths begin at $\tilde{f}(\tilde{x}) = \tilde{g}(\tilde{x})$ and end at $\tilde{f}(\tilde{x}') = \tilde{g}(\tilde{x}')$ and have $[\tilde{f}\omega(\tilde{g}\omega)^{-1}] = [\epsilon_{\tilde{f}(\tilde{x})}] \in \pi(\tilde{Y},\tilde{f}(\tilde{x}))$ since $\tilde{Y}$ is simply connected.

$p\omega$ is a path from x to x' in X and has the property $fp\omega = p\tilde{f}\omega$ and

$gp\omega = p\tilde{g}\omega$ which is seen from the commutative diagram



p_* is a monomorphism by 3.1 and so

$$p_*[\tilde{f}\omega(\tilde{g}\omega)^{-1}] = [\epsilon_{f(x)}] \in \pi(Y,f(x)) .$$

We then have

$$[p\tilde{f}\omega(p\tilde{g}\omega)^{-1}] = [f(p\omega)(g(p\omega))^{-1}] = [\epsilon_{f(x)}] .$$

Therefore, $fp\omega$ and $gp\omega$ are fixed end point homotopic by 3.2 and so x and x' are equivalent in the sense of 1.2 .

This justifies the nomenclature of 3.11 and gives us another way to view coincidence points, by investigating their corresponding lift-pair classes.

The next Lemma is a reinterpretation of 1.8 to fit into the lift-pair situation. It does require that the image space under consideration should be uniformly locally contractible (1.3) so we take Y to be a compact ANR . Little is to be lost at this stage by insisting that Y be a compact ANR instead of just uniformly locally contractible as the incidence of ANRs amongst uniformly locally contractible spaces , in examples that we meet , is fairly high.

4.2 L: If $f,g : X \longrightarrow Y$ where Y is a compact ANR , there exists $\alpha > 0$ such that

if $[\tilde{f},\tilde{g}]$ and $[\tilde{f}',\tilde{g}']$ are lift-pair classes of (f,g) and $\tilde{f}(\tilde{x}) = \tilde{g}(\tilde{x})$, $\tilde{f}'(\tilde{x}') = \tilde{g}'(\tilde{x}')$ and if

$\omega : I \longrightarrow \tilde{X}$ is a path from $\tilde{x}$ to $\tilde{x}'$ in $\tilde{X}$, then

$d_Y(p\tilde{f}\omega(t),p\tilde{g}\omega(t)) = d_Y(p\tilde{f}'\omega(t),p\tilde{g}'\omega(t)) < \alpha$ for all $t \in I$, implies that

$$[\tilde{f},\tilde{g}] = [\tilde{f}',\tilde{g}'] .$$

Proof : Follows immediately from 1.8 ; α is a homotopy barrier for Y .

As in the case of 1.8 , if X is also a compact ANR, the existence of a path satisfying the above "closeness"-property is guaranteed for any pair of "sufficiently close" coincidence points of f and g .

Lemma 4.2 plays an important part in establishing the next result. However, the next result requires compactness of X in addition to the hypotheses on Y due to 4.2 . Once more it is felt that little of value is lost here if we assume both X and Y to be compact ANRs .

4.3 T: For X and Y compact ANRs and $f,g : X \longrightarrow Y$ there are at most finitely many non-trivial lift-pair classes of (f,g) .

Proof : Let $\mathcal{U} = \{ \tilde{x} \in \tilde{X} : d_Y(p\tilde{f}(\tilde{x}),p\tilde{g}(\tilde{x})) < \alpha : (\tilde{f},\tilde{g})$ any lift-pair of $(f,g) \}$,

(α is a homotopy barrier for Y.)

(Perhaps, in this new setting we could look upon α as a " lift-class barrier " for Y.)

$\mathcal{U}$ is a union of open sets and is thus an open set in $\tilde{X}$ and contains $\bigcup_{[\tilde{f},\tilde{g}]} K(\tilde{f},\tilde{g})$ where the union is over all lift-pair classes of (f,g) .

$\tilde{X}$ is locally path-connected and so, therefore, is $\mathcal{U}$ and so the path components in $\mathcal{U}$ coincide with the components of $\mathcal{U}$ and are thus open in $\mathcal{U}$ and $\tilde{X}$.

Let $\mathcal{C}_{\mathcal{U}}$ be the collection of all components of $\mathcal{U}$.

p is an open map and thus, for all $C \in \mathcal{C}_{\mathcal{U}}$, pC is open in X.

$\bigcup_{C \in \mathcal{C}_{\mathcal{U}}} pC$ is an open cover for $K(f,g)$ so there is a finite set $\{ C_i : C_i \in \mathcal{C}_{\mathcal{U}} ; i = 1,2,\ldots\ldots,n \}$ which has

$$K(f,g) \subset \bigcup_{i=1}^{n} pC_i \ .$$

We now require a Lemma :

4.4 L: Each pC_i contains elements of $pK[\tilde{f},\tilde{g}]$ for at most one lift-pair class $[\tilde{f},\tilde{g}]$.

Proof of Lemma : We prove that if $\tilde{x} \in K(\tilde{f},\tilde{g})$ and $\tilde{x}' \in K(\tilde{f}',\tilde{g}')$ and if $p\tilde{x}$ and $p\tilde{x}'$ are in pC_i, then $[\tilde{f},\tilde{g}] = [\tilde{f}',\tilde{g}']$.

Find $\tilde{x}_1$ and $\tilde{x}_1'$ in C_i such that $\tilde{x}_1 \in K[\tilde{f},\tilde{g}]$ and $\tilde{x}_1' \in K[f',g']$.

Say $\tilde{x}_1 \in K(\tilde{f}_1,\tilde{g}_1)$ and $[\tilde{f}_1,\tilde{g}_1] = [\tilde{f},\tilde{g}]$

and $\tilde{x}_1' \in K(\tilde{f}_1',\tilde{g}_1')$ and $[\tilde{f}_1',\tilde{g}_1'] = [\tilde{f}',\tilde{g}']$.

Let ω be a path in C_i from $\tilde{x}_1$ to $\tilde{x}_1'$.

We have $d_Y(p\tilde{f}_1\omega(t),p\tilde{g}_1\omega(t)) = d_Y(p\tilde{f}_1'\omega(t),p\tilde{g}_1'\omega(t)) < \alpha$ and hence by 4.2 we get

$$[\tilde{f}_1,\tilde{g}_1] = [\tilde{f}_1',\tilde{g}_1']$$

$$\text{so} \quad [\tilde{f},\tilde{g}] = [\tilde{f}',\tilde{g}'] \ .$$

Hence pC_i contains points of $pK[\tilde{f},\tilde{g}]$ for at most one lift-pair class $[\tilde{f},\tilde{g}]$ of (f,g).

We can now conclude the proof of the theorem 4.3 .

We have $\bigcup_{[\tilde{f},\tilde{g}]} pK(f,g) = K(f,g) \subset \bigcup_{i=1}^{n} pC_i$

and each pC_i contains elements of $pK[\tilde{f},\tilde{g}]$ for at most one class $[\tilde{f},\tilde{g}]$, therefore there can be at most finitely many non-trivial lift-pair classes of (f,g).

4.5 Example : In order to appreciate the geometric interpretation of the lift-pair classification a simple example is described.

For X take S^1 and for Y take S^1 also.
Let $f : S^1 \longrightarrow S^1$ be the identity map,
$g : S^1 \longrightarrow S^1$ be the " triple rotation map ",
that is $g(e^{i\theta}) = e^{i3\theta}$ $0 \leqq \theta \leqq 2\pi$.

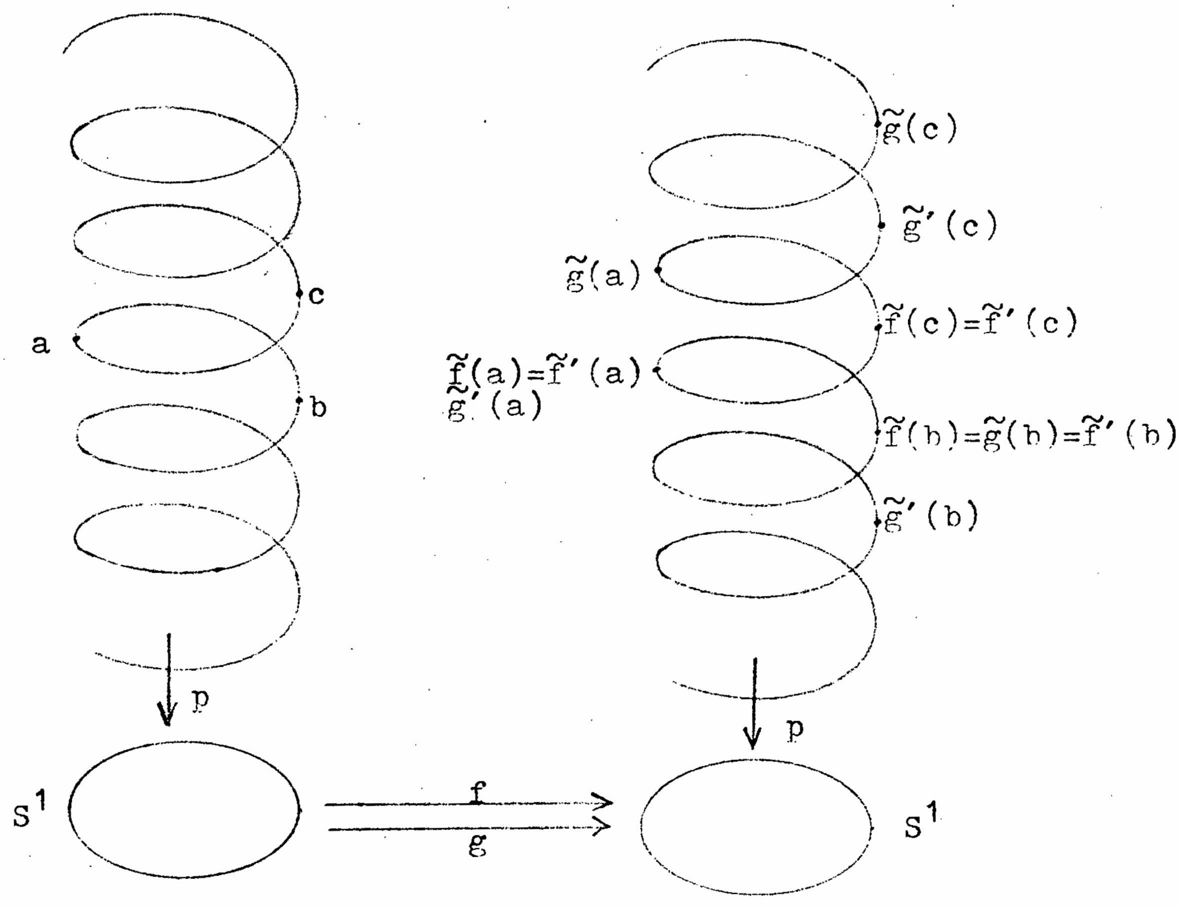
Consider the Universal covering space of S^1 given by $\mathbb{R}^1$ realised in 3-space as a spiral (see sketch).

This can be represented by :
let $p : \mathbb{R}^1 \longrightarrow S^1$ be defined by
$p(t) = (\cos(t), \sin(t))$ for any $t \in \mathbb{R}^1$.

Then the pair $(\mathbb{R}^1, p)$ is a covering space of S^1.
(Being simply connected, $\mathbb{R}^1$ is the Universal covering space of S^1.)

Take as lift-pairs, firstly $(\tilde{f}, \tilde{g})$ where
$\tilde{f}$ is the identity on $\tilde{S}^1$ and
$\tilde{g}$ coincides with $\tilde{f}$ at b , and secondly $(\tilde{f}', \tilde{g}')$
where $\tilde{f}' = \tilde{f}$ is the identity on $\tilde{S}^1$ and
$\tilde{g}'$ coincides with $\tilde{f}'$ at a .

To avoid confusion we draw the image of one sheet only under the lifts.



The lift-pairs $(\tilde{f},\tilde{g})$ and $(\tilde{f}',\tilde{g}')$ are clearly in different classes since for $\beta\tilde{f}\alpha$ to equal $\tilde{f}$, $\alpha,\beta \in \pi(S^1)$, we would require $\beta = \alpha =$ unit path class of S^1.

In this case, we would not get $\tilde{g}' = \beta\tilde{g}\alpha$.

However, if we consider the lift-pair $(\tilde{f}'',\tilde{g}'')$ where $\tilde{f}''$ is the map raising each sheet up one level, then clearly $(\tilde{f}'',\tilde{g}'') \equiv (\tilde{f},\tilde{g})$ as they differ everywhere in image by a level of the covering space.

Choosing α as the unit path class of S^1 and β as the class of ω, where $\omega(t) = e^{i2\pi t}$ is a path from b to b in S^1 we have the conditions for equivalence as required in 3.8 D .

It is easy to verify that $p(a)$ and $p(b)$ are in different coincidence classes of f and g in the sense of 1.2 .

Proof of 5.2 : We only mention here that we consider (H,K) as a pair of maps from $X \times I \longrightarrow Y$; since I is simply connected, we can choose $\tilde{X} \times I$ as $(X \tilde{\times} I)$ - that is as Universal covering space of $(X \times I)$ - because $\tilde{X} \times I$ is clearly a covering space of $X \times I$.

Then H can be lifted to $\tilde{H} : \tilde{X} \times I \longrightarrow \tilde{Y}$ sending $(\tilde{x}',0)$ to $\tilde{f}(\tilde{x}')$ for some $\tilde{x}' \in \tilde{X}$

and K can be lifted to $\tilde{K} : \tilde{X} \times I \longrightarrow \tilde{Y}$ sending $(\tilde{x}',0)$ to $\tilde{g}(\tilde{x}')$.

Now $\tilde{H}(\tilde{x},0)$ must coincide with $\tilde{f}(\tilde{x})$ for all $\tilde{x} \in \tilde{X}$ by uniqueness, and similarly

$\tilde{K}(\tilde{x},0) = \tilde{g}(\tilde{x})$ for all $\tilde{x} \in \tilde{X}$.

$(\tilde{H},\tilde{K})$ is a unique lift-pair of (H,K) , so $\tilde{H}(\tilde{x},1)$ and $\tilde{K}(\tilde{x},1)$ are uniquely determined for all $\tilde{x} \in \tilde{X}$ and are $\tilde{f}'$ and $\tilde{g}'$ respectively.

After some reflection, we can see that the following result must hold :

5.3 P: If $(H,K) : (\tilde{f},\tilde{g}) \simeq (\tilde{f}',\tilde{g}')$ where $(\tilde{f},\tilde{g})$ is a lift-pair of (f,g) and $(\tilde{f}',\tilde{g}')$ is a lift-pair of (f',g') with $(H,K) : (f,g) \simeq (f',g')$ then for any $\alpha \in \pi(X)$ and $\gamma_1, \gamma_2 \in \pi(Y)$ we have

$(H,K) : (\gamma_1\tilde{f}\alpha, \gamma_2\tilde{g}\alpha) \simeq (\gamma_1\tilde{f}'\alpha, \gamma_2\tilde{g}'\alpha)$.

Proof : Immediate from 5.2 . In this direction we mention that the pair $(\tilde{H}_1,\tilde{K}_1) : (\gamma_1\tilde{f},\gamma_2\tilde{g}) \simeq (\gamma_1\tilde{f}',\gamma_2\tilde{g}')$ is just

$(\gamma_1\tilde{H},\gamma_2\tilde{K})$ where $(\tilde{H},\tilde{K}) : (\tilde{f},\tilde{g}) \simeq (\tilde{f}',\tilde{g}')$

and $\gamma_1\tilde{H}$ means the following ;

$\gamma_1\tilde{H}(\tilde{x},t) = \gamma_1\tilde{H}_t(\tilde{x})$ where for each $t \in I$, $\tilde{H}_t : \tilde{X} \longrightarrow \tilde{Y}$.

In a similar way, we interpret $\tilde{H}\alpha(\tilde{x},t)$ to be $\tilde{H}(\alpha\tilde{x},t)$ due to our choice of Universal covering space for $X \times I$.

These last two propositions lead immediately to the next result.

5.4 Cor : If $(H,K) : (f,g) \simeq (f',g')$, then the (H,K)-induced 1-1 transformation from the lift-pairs of (f,g) onto those of (f',g') preserves the equivalence relation between lift-pairs, and thus induces a 1-1 transformation from the lift-pair classes of (f,g) onto those of (f',g').

Proof : Let $(\tilde{f},\tilde{g})$ and $(\tilde{f}_0,\tilde{g}_0)$ be equivalent lift-pairs of (f,g) and $(H,K) : (\tilde{f},\tilde{g}) \simeq (\tilde{f}',\tilde{g}')$;

$$(H,K) : (\tilde{f}_0,\tilde{g}_0) \simeq (\tilde{f}'_0,\tilde{g}'_0) .$$

Then there exists $\alpha \in \pi(X)$ and $\beta \in \pi(Y)$ such that

$$\tilde{f}_0 = \beta\tilde{f}\alpha$$
$$\tilde{g}_0 = \beta\tilde{g}\alpha .$$

We have $(H,K) : (\beta^{-1}\tilde{f}_0\alpha^{-1},\beta^{-1}\tilde{g}_0\alpha^{-1}) \simeq (\beta^{-1}\tilde{f}'_0\alpha^{-1},\beta^{-1}\tilde{g}'_0\alpha^{-1})$

or $(H,K) : (\tilde{f},\tilde{g}) \simeq (\beta^{-1}\tilde{f}'_0\alpha^{-1},\beta^{-1}\tilde{g}'_0\alpha^{-1})$

But from (H,K) and $(\tilde{f},\tilde{g})$ we get a unique lift-pair of (f',g') namely $(\tilde{f}',\tilde{g}')$.

Therefore we must have

$$(\tilde{f}',\tilde{g}') = (\beta^{-1}\tilde{f}'_0\alpha^{-1},\beta^{-1}\tilde{g}'_0\alpha^{-1})$$

or
$$\tilde{f}'_0 = \beta\tilde{f}'\alpha$$
$$\tilde{g}'_0 = \beta\tilde{g}'\alpha .$$

That is $(\tilde{f}',\tilde{g}')$ and $(\tilde{f}'_0,\tilde{g}'_0)$ are equivalent lift-pairs of (f',g') .

We have thus established a 1-1 correspondence between the coincidence classes of (f,g) and those of (f',g') indicating that the number of coincidence classes of a pair (f,g) is a homotopy invariant .

However, many of the coincidence classes correspond to trivial lift-pair classes (those lift-pair classes that have no coincidences) and are thus void .

It may therefore be possible to begin with a non-

-trivial lift-pair class and deform it by homotopies on the maps to a trivial lift-pair class . Such a lift-pair class will be called inessential .

5.5 D: If $[\tilde{f},\tilde{g}]$ is a lift-pair class of (f,g) and if for any homotopy couple $(H,K) : (f,g) \simeq (f',g')$ the lift-pair class $[\tilde{f}',\tilde{g}']$ associated with $[\tilde{f},\tilde{g}]$ by the 1-1 transformation induced by (H,K) has $pK[\tilde{f}',\tilde{g}'] \neq \emptyset$ then $[\tilde{f},\tilde{g}]$ is said to be an <u>essential</u> lift-pair class of (f,g) .

Since there are only finitely many non-trivial lift-pair classes of (f,g) , there are only finitely many essential lift-pair classes of (f,g) .

5.6 D: The number of essential lift-pair classes of (f,g) is called the Nielsen number of (f,g) and is denoted $N(f,g)$.

Obviously,

5.7 P: $N(f,g)$ is a homotopy invariant .

The Δ_1-Nielsen Number.

6. In this and the following section, a survey is done of the two best known methods of arriving at the Nielsen number and a justification is given in each case of the definition 5.6 .

The first method involves the essentiality of coincidence classes as presented by R.Brooks and R.F.Brown in a paper on " A lower bound for the Δ-Nielsen number." [2].

The second method is taken from a definition of essentiality of coincidence classes described by W.Franz in a paper on " Mindestzahlen von Koinzidenzpunkten." [4].

The equivalence of these two forms of classification can be established independently of the lift-pair theory; however, to make use of the geometry of the lift-pair theory, the two definitions of essentiality are shown to be equivalent to that given in 5.6 directly . This gives some indication of the notions involved in arguments when using the approach to coincidence theory through the Universal Covering Spaces.

Given a map $f : X \longrightarrow Y$ and a homotopy $F : X \times I \longrightarrow Y$ we again use the notation f_t (for any $t \in I$) to represent the map of X into Y given by

$$f_t(x) = F(x,t) \quad \text{for all } x \in X .$$

6.1 D: If $f, g : X \longrightarrow Y$ are maps and F, G are homotopies $F, G : X \times I \longrightarrow Y$ with $f_0 = f$, $g_0 = g$, then a coincidence point x_0 of f and g , in the sense of 1.2 , is <u>F,G-related</u> to a coincidence point x_1 of f_1 and g_1 if and only if there is a path ω in X from x_0 to x_1 with the property that

$[\{f_t.\omega(t)\}] = [\{g_t.\omega(t)\}]$ as classes of paths in Y based at $f(x_0) = g(x_0)$ with terminal point $f_1(x_1) = g_1(x_1)$.

That is, $\{f_t.\omega(t)\}$ is fixed end point homotopic to $\{g_t.\omega(t)\}$. One can think of these paths as the images of the path $(\omega(t),t)$ from $(x_0,0)$ to $(x_1,1)$ in $X \times I$, under the maps F and $G : X \times I \longrightarrow Y$.

It is not difficult to show that if one coincidence point in a class $[x]$ of coincidences of f and g is F,G-related to a coincidence point in a class $[x']$ of coincidences of f_1 and g_1, then every coincidence point in $[x]$ is F,G-related to every coincidence point in $[x']$.

In this way we identify coincidence classes of f and g with coincidence classes of f_1 and g_1.

In [2], homotopies between $f, g : X \longrightarrow Y$ are considered as paths in $Map(X,Y)$, (with the compact- open topology). Every path in $Map(X,Y)$ can likewise be considered as a homotopy between the maps which are its initial and end points.

Δ represents a class of ordered pairs of paths in $Map(X,Y)$ that is closed under pairwise partitioning and multiplication.

A coincidence class $[x]$ of f and g is said to be <u>Δ-essential</u> if and only if whenever $(F,G) \in \Delta$ and $(F(0),G(0)) = (f,g)$, there is an $x' \in K(F(1),G(1))$ such that $[x]$ is F,G-related to $[x']$.

An F,G-relation for $(F,G) \in \Delta$ defines a 1-1 transformation of the Δ-essential elements of the coincidence classes of $F(0)$ and $G(0)$ onto those of $F(1)$ and $G(1)$.

The number of Δ-essential coincidence classes of f and g is called the Δ-Nielsen number of f and g, and is denoted by $N(f,g, \ )$.

In case we allow Δ to be the class of all pairs (F,G) of paths F and G in $Map(X,Y)$, we denote this class by Δ_1 and then we have

6.2 D: A coincidence point of f and g, and the class containing it, is called Δ_1-essential if and only if for any homotopies F and G, from f and g respectively, there is a coincidence of f_1 and g_1 to which it is F,G-related , where $F(1) = f_1$ and $G(1) = g_1$.

The number of Δ_1-essential coincidence classes of f and g is invariant under homotopies of both f and g .

6.3 D: The number of Δ_1-essential coincidence classes of f and g is called the Δ_1-Nielsen number of f and g and is denoted by $N(f,g,\Delta_1)$.

We now show that this number and the number defined in 5.6 are the same .

6.4 T: $[\tilde{f},\tilde{g}]$ is essential $\Longleftrightarrow$ $[x]$ corresponding to $[\tilde{f},\tilde{g}]$ is Δ_1-essential .

Proof : Suppose $[x]$ is Δ_1-essential .

Then there exists a homotopy couple $(F,G) : (f,g)\simeq(f',g')$ such that for every path ω in X from x to $x' \in K(f',g')$ we have

$$[\{f_t.\omega(t)\}] \neq [\{g_t.\omega(t)\}] . \qquad (*)$$

Let us assume that $(\tilde{f},\tilde{g})$ coincide at $\tilde{x} \in p^{-1}(x)$ and (F,G) defines $(\tilde{F},\tilde{G}) : (\tilde{f},\tilde{g}) \simeq (\tilde{f}',\tilde{g}')$ a lift-pair of (f',g') ;

suppose $K(\tilde{f}',\tilde{g}') \neq \emptyset$. $\qquad (**)$

Let $\tilde{x}' \in K(\tilde{f}',\tilde{g}')$ and $p\tilde{x}' = x' \in X$.

Now $(x,0)$ and $(x',1)$ in $(X \times I)$ are coincidence points of $(F,G) : X \times I \longrightarrow Y$.

Let $\tilde{\omega}$ be a path in $(\tilde{X} \times I) \equiv (X \tilde{\times} I)$ from $(\tilde{x},0)$ to $(\tilde{x}',1)$, and consider $\tilde{F}(\tilde{\omega})$ and $\tilde{G}(\tilde{\omega})$ in $\tilde{Y}$.

$p\tilde{\omega} = \omega$ is a path in $X \times I$ from $(x,0)$ to $(x',1)$ and ω in turn defines a path ω_0 , given by projection onto X , in X from x to x', where

$$F\omega(t) = \{f_t.\omega_0(t)\} \text{ and}$$
$$G\omega(t) = \{g_t.\omega_0(t)\} .$$

Now $F\omega$ and $G\omega$ can be lifted uniquely to $\tilde{Y}$ with initial point $\tilde{F}(\tilde{\omega}(0)) = \tilde{G}(\tilde{\omega}(0))$, and so must coincide with $\tilde{F}(\tilde{\omega})$ and $\tilde{G}(\tilde{\omega})$ respectively , since these last two are obviously lifts of $F\omega$ and $G\omega$.

Therefore, $\widetilde{F\omega}$ and $\widetilde{G\omega}$ have terminal points

$\widetilde{F\omega}(1)$ and $\widetilde{G\omega}(1)$. Further

$$\widetilde{F\omega}(1) = \widetilde{F}(\widetilde{x}',1) = \widetilde{f}'(\widetilde{x}') = \widetilde{g}'(\widetilde{x}') = \widetilde{G}(\widetilde{x}',1) = \widetilde{G\omega}(1) .$$

So the lifts of $F\omega$ and $G\omega$ have terminal points coincident thus in the same sheet of $\widetilde{Y}$.

Thus the paths $F\omega$ and $G\omega$ are fixed end point homotopic .

That is $[\{f_t.\omega_o(t)\}] = [\{g_t.\omega_o(t)\}]$ which contradicts the statement (*) .

Therefore, (**) must be false and $K(\widetilde{f}',\widetilde{g}') = \emptyset$.

Hence by definition, since $pK(\widetilde{f}',\widetilde{g}') = \emptyset$,

$[\widetilde{f},\widetilde{g}]$ is inessential .

Conversely : (where we write $pK[f,g]$, we obviously mean $\bigcup_{(\widetilde{f}',\widetilde{g}') \in [\widetilde{f},\widetilde{g}]} pK(\widetilde{f}',\widetilde{g}')$. There need actually be no distinction since $pK(\widetilde{f},\widetilde{g}) = pK[\widetilde{f},\widetilde{g}]$.

If $x \in pK[\widetilde{f},\widetilde{g}]$, there exist $\widetilde{x} \in p^{-1}(x)$ such that

$$\widetilde{f}'(\widetilde{x}) = \widetilde{g}'(\widetilde{x}) ; \qquad \widetilde{f}' = \beta\widetilde{f}\alpha$$
$$\widetilde{g}' = \beta\widetilde{g}\alpha .$$

But then clearly $\alpha\widetilde{x} \in K(\widetilde{f},\widetilde{g})$ so $p(\alpha\widetilde{x}) = x \in pK(\widetilde{f},\widetilde{g})$.)

For the converse of the theorem, let us assume that $[\widetilde{f},\widetilde{g}]$ is inessential, and that $\widetilde{x} \in K(\widetilde{f},\widetilde{g})$ with $p\widetilde{x} = x$.

By inessentiality there exists a homotopy couple

$(F,G) : (f,g) \simeq (f',g')$ say, with

$(\widetilde{F},\widetilde{G}) : (\widetilde{f},\widetilde{g}) \simeq (\widetilde{f}',\widetilde{g}')$ the unique lift-pair of (F,G) which with $(\widetilde{f},\widetilde{g})$ determines uniquely $(\widetilde{f}',\widetilde{g}')$, a lift-pair of (f',g') ; and where $pK(\widetilde{f}',\widetilde{g}') = \emptyset$. (*)

Suppose $[x]$ is Δ_1-essential .

Then we can find $x' \in K(f',g')$ and a path ω_o in X from x to x' with

$$[\{f_t.\omega_o(t)\}] = [\{g_t.\omega_o(t)\}] \text{ or,}$$
$$[F(\omega_o(t),t)] = [G(\omega_o(t),t)] . \qquad (**)$$

i.e. $(\omega_o(0),0)$ and $(\omega_o(1),1) \in K(F,G)$, $F,G : X\times I \longrightarrow Y$.

$\omega = (\omega_o(t),t)$ is a path in $X \times I$ so can be lifted uniquely to a path $\tilde{\omega}$ in $\tilde{X} \times I \equiv (X \tilde{\times} I)$ with initial point $(\tilde{x},0)$.

Let $(\tilde{x}',1)$ be the terminal point and $p\tilde{x}' = x'$.
If we map $\tilde{\omega}$ by $\tilde{F}$ and $\tilde{G}$, we get the unique lifts of $F\omega$ and $G\omega$ with initial point

$$\tilde{F}(\tilde{x},0) = \tilde{f}(\tilde{x}) = \tilde{g}(\tilde{x}) = \tilde{G}(\tilde{x},0) ,$$

and terminal points

$$\tilde{F}(\tilde{x}',1) = \tilde{f}'(\tilde{x}')$$
$$\tilde{G}(\tilde{x}',1) = \tilde{g}'(\tilde{x}')$$

where $p\tilde{f}'(\tilde{x}') = f'(x') = g'(x') = p\tilde{g}'(\tilde{x}')$.

But from (**) the lifted paths must end in the same sheet of $\tilde{Y}$, being fixed end point homotopic (3.1)

Thus $\widetilde{F\omega}(1) = \widetilde{G\omega}(1)$

or $\tilde{F}(\tilde{x}',1) = \tilde{G}(\tilde{x}',1)$

i.e. $\tilde{f}'(\tilde{x}') = \tilde{g}'(\tilde{x}')$

which contradicts (*) .

Therefore, we must have, $[x]$ is Δ_1-inessential .

So, the Δ_1-Nielsen number $N(f,g,\Delta_1)$ is just the Nielsen number of (f,g) as defined in 5.6 .

6.5 Cor : $N(f,g,\Delta_1)$ of 6.3 $=$ $N(f,g)$ of 5.6 .

Essentiality of coincidence classes using $\pi(Y,y_o)$.

7. Franz [4] has defined essentiality of coincidence classes by means of the fundamental group of the image space.

With each coincidence point of f and g we associate a particular element in the fundamental group of the image space of f and g. We then define an equivalence relation on the fundamental group so that all points in a coincidence class of f and g corresponds, under the above association, to the same equivalence class of elements in the fundamental group.

This same relation enables us to associate coincidence classes of f and g with those of f' and g' in case (f,g) and (f',g') are homotopic. The method of deformation of (f,g) used to arrive at (f',g') is significant in that the association of a coincidence class of f and g with a coincidence class of f' and g' due to homotopies F and G can differ from the association under homotopies F' and G'.

We now make this precise.

Let $f,g : X \longrightarrow Y$ be maps.

Let $x_o \in X$ and $y_o \in Y$ be base points of the spaces.

Fix paths u and v in Y from y_o to $f(x_o)$ and from y_o to $g(x_o)$ respectively.

7.1 D: For any $\gamma \in \pi(X,x_o)$ we define $f(\gamma)$ and $g(\gamma)$ as follows :

let $\omega \in \gamma$ then

$f(\gamma) = [uf\omega u^{-1}]$ similarly,

$g(\gamma) = [vg\omega v^{-1}]$.

Given any $x \in K(f,g)$,

let ω_1 be a path from x_o to x in X .

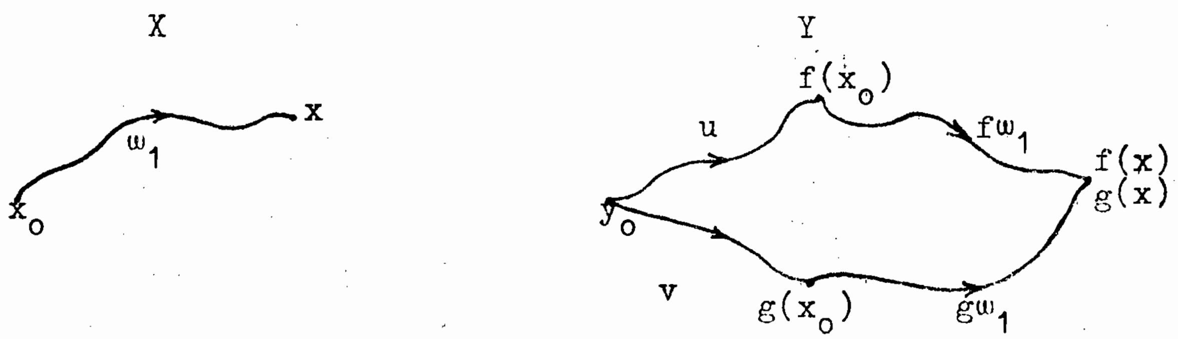
7.2 D: We relate an element of $\pi(Y,y_o)$ to x as follows :

$$x \longrightarrow \delta(x) = \delta = [uf\omega_1 g\omega_1^{-1} v^{-1}]$$

where obviously $g\omega_1^{-1}$ is not ambiguous since

$$g(\omega_1{}^{-1}) = (g\omega_1)^{-1} \quad \text{with} \quad g(\omega_1{}^{-1})(t) = g(\omega_1(1-t))$$
$$= g\omega_1(1-t)$$
$$= (g\omega_1)^{-1}(t) \quad \forall t \in I.$$

Consider the sketches below :



The choice of ω_1 influences $\delta = \delta(x)$ in the following way .

If ω_2 is any other path from x_0 to x, then ω_2 has the form $\omega_2 \equiv \omega\omega_1$ where ω is a loop at x_0, (e.g. $\omega = \omega_2\omega_1{}^{-1}$).

Suppose $[\omega] = \gamma \in \pi(X,x_0)$ then

$$\begin{aligned}
\delta_2 = \delta_2(x) &= [uf\omega_2 g\omega_2{}^{-1}v^{-1}] \\
&= [uf(\omega\omega_1)g(\omega\omega_1)^{-1}v^{-1}] \\
&= [uf\omega f\omega_1 g\omega_1{}^{-1} g\omega^{-1} v^{-1}] \\
&= [uf\omega u^{-1} uf\omega_1 g\omega_1{}^{-1} v^{-1} vg\omega^{-1} v^{-1}] \\
&= [uf\omega u^{-1}][uf\omega_1 g\omega_1{}^{-1}v^{-1}][vg\omega^{-1}v^{-1}] \\
&= f(\gamma)\ \delta\ g(\gamma^{-1}) .
\end{aligned}$$

Such a relation between elements of $\pi(Y,y_0)$ will be formally recognised .

7.3 D: For $\delta_1, \delta_2 \in \pi(Y,y_0)$ we say that δ_1 is <u>(f,g)-conjugate</u> to δ_2 if and only if there exists $\gamma \in \pi(X,x_0)$ such that

$$\delta_1 = f(\gamma)\ \delta_2\ g(\gamma^{-1}) .$$

The relation described is clearly a symmetric, reflexive and transitive relation between elements of $\pi(Y,y_0)$ and we can, therefore, classify the elements of

$\pi(Y,y_o)$ by (f,g)-conjugacy into (f,g)-conjugacy classes.

We write $[\delta_1] = [\delta_2]$ when δ_1 and δ_2 are (f,g)-conjugate elements of $\pi(Y,y_o)$. $[\delta_1]$ denotes the (f,g)-conjugacy class determined by δ_1 .

We now consider the effect of deformation on f and g. Suppose F and G are homotopies with $f_o = f$ and $g_o = g$.

We notice that the paths u and v will be transformed into paths u' and v' from y_o to $f_1(x_o)$ and from y_o to $g_1(x_o)$ respectively . The way we arrive at u' and v' is as follows .

7.4 D: If u and v are paths in Y from y_o to $f(x_o)$ and from y_o to $g(x_o)$, and if $F : f \simeq f'$ and $G : g \simeq g'$ are homotopies , then the paths u and v are "deformed" into paths u', v' from y_o to $f'(x_o)$ and from y_o to $g'(x_o)$ by

$$u' = u\{f_t(x_o)\}$$
$$v' = v\{g_t(x_o)\} \quad \text{where}$$

$\{f_t(x_o)\}$ and $\{g_t(x_o)\}$ represent the paths traced out by the image of (x_o,t), under the homotopies, from $f(x_o)$ to $f'(x_o)$ and from $g(x_o)$ to $g'(x_o)$.

Due to 7.4 , we notice that the homotopies can influence which (f',g')-conjugacy class is to be associated with a certain coincidence class of f' and g'.

If for two homotopies F and F' of $f \simeq f'$, $F(x_o,t)$ and $F'(x_o,t)$ are not fixed end point homotopic, then for any coincidence point x of f' and g' we will get a different $\delta(x)$ under F and G from that which we get under F' and G .

7.5 Note : If F and G , F' and G' are pairs of homotopies between f, g and f', g', a sufficient condition for F and G to give the same relation between coincidences of f' and g' and the (f',g')-conjugacy classes of $\pi(Y,y_o)$

as that given by F' and G' is that $F \simeq F'$ and $G \simeq G'$ relative to $(x_0,0)$ and $(x_0,1)$.

7.6 D: If $x \in K(f,g)$ and $x' \in K(f',g')$ where $F : f \simeq f'$ and $G : g \simeq g'$ then we say x is (F,G)-related to x' if and only if $\delta(x)$ and $\delta(x')$ are (f,g)-conjugate .

It is understood that "(F,G)-related" is relative to the homotopies employed .

For completeness we prove the following fact.

7.7 P: For $x \in K(f,g)$ and $x' \in K(f',g')$, if x is (F,G)-related to x' and

$F : f \simeq f'$

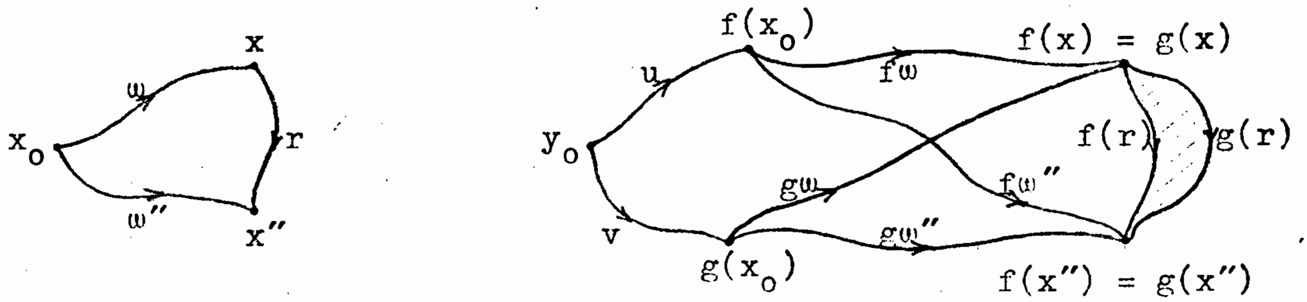
$G : g \simeq g'$

then $x'' \in [x] \longrightarrow x''$ is (F,G)-related to x'.

This indicates that the association of coincidence classes across homotopies is a good one.

Proof : We need only show that

$x'' \in [x] \longrightarrow x''$ is (F,G)-related to x since (F,G)-related is an equivalence relation and is therefore transitive.



By hypothesis, there exists a path r from x to x'' with $[fr] = [gr]$.

We have,

$$\delta(x) = [uf\omega g\omega^{-1}v^{-1}] \ .$$

Now $\delta(x'') = [uf\omega'' g\omega'^{-1} v^{-1}]$
$= [uf\omega'' fr^{-1} f\omega^{-1} u^{-1}][uf\omega g\omega^{-1} v^{-1}][vg\omega grg\omega''^{-1} v^{-1}]$
since $[fr^{-1} f\omega^{-1} u^{-1} uf\omega g\omega^{-1} v^{-1} vg\omega gr] = [\epsilon_{f(x'')=g(x'')}]$
(notice $[fr^{-1} gr] = [\epsilon_{f(x'')=g(x'')}]$.)

Thus $\delta(x'') = f[\omega'' r^{-1} \omega^{-1}]\ \delta(x)\ g[(\omega'' r^{-1} \omega^{-1})^{-1}]$
where $\omega'' r^{-1} \omega^{-1} \in \pi(X, x_0)$.

Hence $\delta(x'')$ is (f,g)-conjugate to $\delta(x)$ and so to $\delta(x')$.

7.8 D: A coincidence point x of f and g is <u>(essential)</u> if and only if for any homotopies F and G with
$f_0 = f$
$g_0 = g$ there exists some coincidence point x' of f_1 and g_1 such that x and x' are (F,G)-related, i.e. $[\delta(x)] = [\delta(x')]$.

7.9 T: $[\tilde{f},\tilde{g}]$ is essential $\Longleftrightarrow$ $[x]$ corresponding to $[\tilde{f},\tilde{g}]$ is (essential) (7.8) .

Proof : Given any homotopies F and G starting at f and g ,
suppose there exists $x' \in K(f_1, g_1)$ and that
$[\delta(x)] = [\delta(x')]$ where $F : f \simeq f_1$
$G : g \simeq g_1$,
then we can find $\gamma \in \pi(X, x_0)$ so that
$\delta(x') = f(\gamma)\ \delta(x)\ g(\gamma^{-1})$.
Let $\omega \in \gamma$.
Then $\delta(x') = [uf\omega u^{-1}][ufrgr^{-1} v^{-1}][vg\omega^{-1} v^{-1}]$
where r is a path from x_0 to x .
Let s be a path from x_0 to x'.
Then $\delta(x') = [u' f_1 s g_1 s^{-1} v'^{-1}]$ or
$= [u\{f_t(0)\} f_1 s g_1 s^{-1} \{g_{1-t}(0)\} v^{-1}]$
so we have
$[u\{f_t(0)\} f_1 s g_1 s^{-1} \{g_{1-t}(0)\} v^{-1}] = [uf\omega frgr^{-1} g\omega^{-1} v^{-1}]$
which implies that
$[\{f_t(0)\} f_1 s g_1 s^{-1} \{g_{1-t}(0)\}] = [f\omega frgr^{-1} g\omega^{-1}]$.

Rearranging ;

$$[fr^{-1}f\omega^{-1}\{f_t(0)\}f_1 s g_1 s^{-1}\{g_{1-t}(0)\}g\omega g r] = [\epsilon_{f(x)}] .$$

Let $\sigma = (r^{-1}(t),0)\omega^{-1}(r^{-1}(1),q)(s(1),1)$ be the path from $(x,0)$ to $(x',1)$ in $X \times I$ where

$(r^{-1}(t),0)$ is the path from $(x,0)$ to $(x_0,0)$

$(r^{-1}(1),q)$ is the path from $(x_0,0)$ to $(x_0,1)$

ω is the loop at x_0, already defined , and

$(s(1),1)$ is the path from $(x_0,1)$ to $(x',1)$.

We have $[F(\sigma)] = [G(\sigma)]$.

Lift F and G to $\tilde{F}$ and $\tilde{G}$ coinciding at

$(\tilde{x},0) \in K(\tilde{f},\tilde{g})$; $\tilde{x} \in p^{-1}(x)$ with

$\tilde{F}(\tilde{x},0) = \tilde{f}(\tilde{x}) = \tilde{g}(\tilde{x}) = \tilde{G}(\tilde{x},0)$.

Then lift $F(\sigma)$ and $G(\sigma)$ to begin at $\tilde{F}(\tilde{x},0) = \tilde{G}(\tilde{x},0)$ and to have terminal points common .

Lift σ to $\tilde{\sigma}$ with initial point $(\tilde{x},0)$ and terminal point in $p^{-1}(x')$, say $\tilde{x}'$.

By uniqueness, $\tilde{F}(\tilde{\sigma}) = (\widetilde{F\sigma})$, since $(\widetilde{F\sigma})$ starts at $F(x,0)$, and

$$\tilde{G}(\tilde{\sigma}) = (\widetilde{G\sigma}) .$$

So, $\tilde{F}(\tilde{x},1)$ which is the terminal point of the lift of $F\sigma$ is the same as the terminal point of the lift of $G\sigma$.

Since there is a unique homotopy couple starting at $(\tilde{f},\tilde{g})$, we have

$\tilde{F}(\tilde{x},1) = \tilde{f}'(\tilde{x}')$ and

$\tilde{G}(\tilde{x},1) = \tilde{g}'(\tilde{x}')$.

Thus we have proved that $\tilde{x}' \in K(\tilde{f}',\tilde{g}')$ and

$[\tilde{f},\tilde{g}]$ is essential .

Conversely : Suppose that $[\tilde{f},\tilde{g}]$ is essential.

Let (F,G) be any homotopy couple, say

$(F,G) : (f,g) \simeq (f',g')$.

Choose $\tilde{x} \in p^{-1}(x)$ such that $\tilde{f}(\tilde{x}) = \tilde{g}(\tilde{x})$.

Lift (F,G) to $(\tilde{F},\tilde{G})$ where

$\tilde{F}(\tilde{x},0) = \tilde{G}(\tilde{x},0) = \tilde{f}(\tilde{x}) = \tilde{g}(\tilde{x})$.

Call $\tilde{F}(x,1) = \tilde{f}'(x)$ and

$\tilde{G}(x,1) = \tilde{g}'(x)$ for all $x \in \tilde{X}$.

We know that there exists $\tilde{x}' \in \tilde{X}$ where $\tilde{f}'(\tilde{x}') = \tilde{g}'(\tilde{x}')$. (If $\tilde{\omega}$ is any path in $\tilde{X} \times I \equiv (X \tilde{\times} I)$ from $(\tilde{x},0)$ to $(\tilde{x}',1)$, $\tilde{F}\tilde{\omega}$ and $\tilde{G}\tilde{\omega}$ start at $\tilde{f}(\tilde{x}) = \tilde{g}(\tilde{x})$ and end at $\tilde{f}'(\tilde{x}') = \tilde{g}'(\tilde{x}')$. (*)

Applying the projections we get a path ω in $X \times I$ from $(x,0)$ to $(x',1)$ where $x' = p\tilde{x}' \in K(f',g')$.)

In particular choose $\tilde{\omega}$ as follows :

Join $(\tilde{x},0)$ to $(\tilde{x}_o,0)$ in $\tilde{X} \times \{0\}$ by $\tilde{\sigma}_1$, join $(\tilde{x}_o,0)$ to $(\tilde{x}'_o,0)$ in the sheet of $(\tilde{x}',0)$ in $\tilde{X} \times \{0\}$ by $\tilde{\sigma}_2$ where

$$\tilde{x}'_o \in p^{-1}(p\tilde{x}_o) = p^{-1}(x_o)\ ,\ x_o \text{ the base point of } X.$$

Join $(\tilde{x}'_o,0)$ to $(\tilde{x}'_o,1)$ by $\{(\tilde{x}'_o,t):t\in I\} = \tilde{\sigma}_3$, then join $(\tilde{x}'_o,1)$ to $(\tilde{x}',1)$ by $\tilde{\sigma}_4$ in $\tilde{X} \times \{1\}$;

let $\tilde{\omega} = (\tilde{\sigma}_1.\tilde{\sigma}_2.\tilde{\sigma}_3.\tilde{\sigma}_4)$.

On projecting into $X \times I$ we get

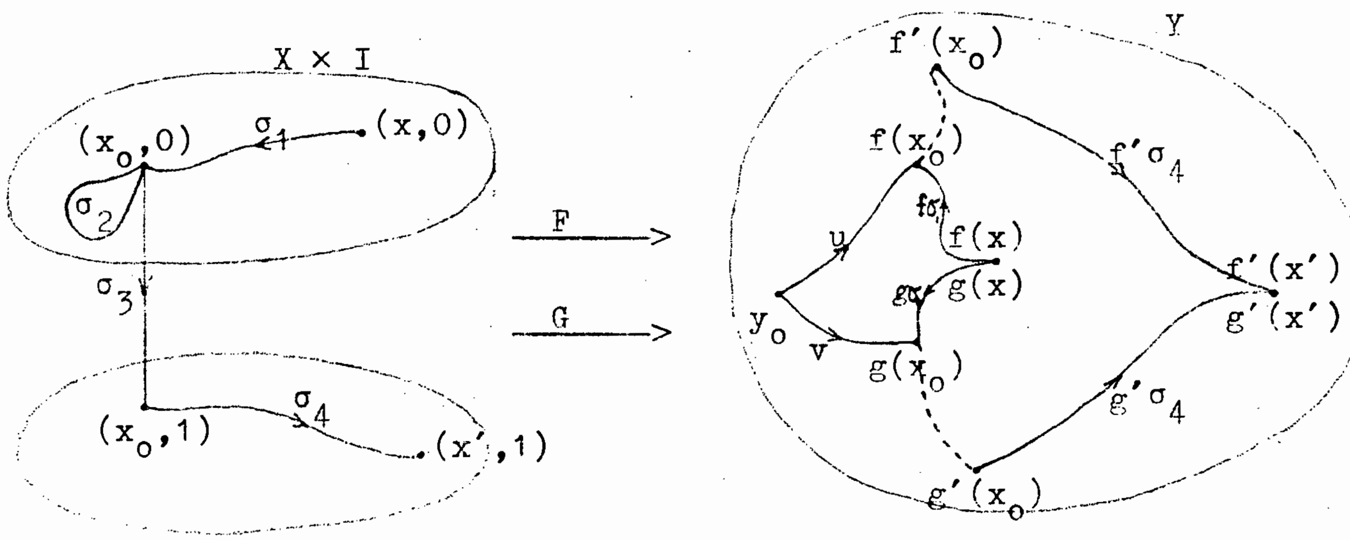
σ_1 : $(x,0)$ to $(x_o,0)$ in $X \times 0$

σ_2 : loop at $(x_o,0)$ in $X \times 0$

σ_3 : $(x_o,0)$ to $(x_o,1)$ in $x_o \times I$

σ_4 : $(x_o,1)$ to $(x',1)$ in $X \times 1$.

$$\omega = p\tilde{\omega} = (\sigma_1.\sigma_2.\sigma_3.\sigma_4)$$



Mapping across by F and G into Y we have , since $F\omega$ and $G\omega$ are fixed end point homotopic by (*), that

$$[f\sigma_1 f\sigma_2 \{f_t(x_o)\} f\sigma_4] = [g\sigma_1 g\sigma_2 \{g_t(x_o)\} g\sigma_4]$$

or, rearranging,

$$[\{f_t(x_o)\} f\sigma_4 g\sigma_4^{-1} \{g_{1-t}(x_o)\}] = [f\sigma_2^{-1} f\sigma_1^{-1} g\sigma_1 g\sigma_2]$$

so that

$$[u\{f_t(x_o)\} f\sigma_4 g\sigma_4^{-1} \{g_{1-t}(x_o)\} v^{-1}] = [uf\sigma_2^{-1} f\sigma_1^{-1} g\sigma_1 g\sigma_2 v^{-1}] .$$

That is,

$$[u' f\sigma_4 g\sigma_4^{-1} v'^{-1}] = [uf\sigma_2^{-1} u^{-1}][uf\sigma_1^{-1} g\sigma_1 v^{-1}][vg\sigma_2 v^{-1}]$$

or,

$$\delta'(x') = f[\sigma_2^{-1}]\ \delta'(x)\ g[\sigma_2]$$

where $\delta'(x)$ is in the same (f,g)-conjugacy class as $\delta(x)$ - the choice of σ_1 means we may have $\delta'(x)$ and $\delta(x)$ (defined by some fixed path to x from x_o) differing, but only by conjugacy ; and similarly for $\delta'(x')$.

Nevertheless, $[\delta(x')] = [\delta(x)]$.

So we have proved that $[x]$ is (essential) (7.8).

Hence :

7.10 Cor : The number of (essential) coincidence classes of f and g is equal to the Nielsen number of (f,g) as defined in 5.6 and is thus just the Δ_1-Nielsen number $N(f,g,\Delta_1)$ as defined in 6.3 .

Estimation of the number of lift-pair classes.

8. We define a homomorphism of $\pi(X)$ into $\pi(Y) \oplus \pi(Y)$ to establish a classification of the group elements of $\pi(Y) \oplus \pi(Y)$ into $\tilde{f}_{\#} \times \tilde{g}_{\#}$-conjugacy classes. It turns out that the number of lift-pair classes - hence coincidence classes - of (f,g) is just the number of $\tilde{f}_{\#} \times \tilde{g}_{\#}$-conjugacy classes, for any fixed lift-pair $(\tilde{f},\tilde{g})$. This assists in the estimation of the number of coincidence classes of f and g.

8.1 D: Let $(\tilde{f},\tilde{g})$ be a lift-pair of (f,g) then for any element $\alpha \in \pi(X)$, the pair $(\tilde{f}\alpha,\tilde{g}\alpha)$ is a lift-pair of (f,g) and so there exist

$$\gamma_1, \gamma_2 \in \pi(Y) \text{ such that}$$
$$(\gamma_1\tilde{f},\gamma_2\tilde{g}) = (\tilde{f}\alpha,\tilde{g}\alpha).$$

We denote such elements $\gamma_1 = \tilde{f}_{\#}(\alpha)$ and $\gamma_2 = \tilde{g}_{\#}(\alpha)$.

It is clear that the pair $(\gamma_1,\gamma_2) = \tilde{f}_{\#} \times \tilde{g}_{\#}(\alpha)$ is unique.

In this way we obtain a homomorphism

$$\tilde{f}_{\#} \times \tilde{g}_{\#} : \pi(X) \longrightarrow \pi(Y) \oplus \pi(Y).$$

8.2 L: If $\tilde{\omega}_1$ is any path in $\tilde{Y}$ from $\tilde{y}_o$ to $\tilde{f}(\tilde{x}_o)$ and $\tilde{\omega}_2$ is any path in $\tilde{Y}$ from $\tilde{y}_o$ to $\tilde{g}(\tilde{x}_o)$ with ω_1 and ω_2 their projections in Y from y to $f(x_o)$ and $g(x_o)$ respectively, then the following diagram commutes.

$$\begin{array}{ccccc}
\pi(X,x_o) & \xrightarrow{f_* \times g_*} & \pi(Y,f(x_o)) \oplus \pi(Y,g(x_o)) & \xrightarrow{\omega_{1*} \times \omega_{2*}} & \pi(Y,y_o) \oplus \pi(Y,y_o) \\
\| & & & & \| \\
\pi(X) & & \xrightarrow{\quad \tilde{f}_{\#} \times \tilde{g}_{\#} \quad} & & \pi(Y) \oplus \pi(Y)
\end{array}$$

where $f_* \times g_*(\alpha) = (f_*(\alpha),g_*(\alpha)) \qquad \alpha \in \pi(X,x_o)$

and $\omega_{1*} \times \omega_{2*}(\gamma_1,\gamma_2) = (\omega_{1*}(\gamma_1),\omega_{2*}(\gamma_2))$

for $(\gamma_1,\gamma_2) \in \pi(Y,f(x_o)) \oplus \pi(Y,g(x_o))$ or,

if $a_1 \in \gamma_1$ and $a_2 \in \gamma_2$,

$\omega_{1*} \times \omega_{2*}(\gamma_1,\gamma_2) = ([\omega_1 a_1 \omega_1^{-1}],[\omega_2 a_2 \omega_2^{-1}])$, where [ ] indicates path-classes in $\pi(Y)$.

Proof : From the definition of the homomorphism $\tilde{f}_{\#} \cdot \tilde{g}_{\#}$ and by observing that the uniqueness of path-lifting assures the fact that

$$\tilde{f}(\alpha\tilde{x}_0) = (f_*\alpha)(\tilde{f}\tilde{x}_0) ;$$
$$\tilde{g}(\alpha\tilde{x}_0) = (g_*\alpha)(\tilde{g}\tilde{x}_0) .$$

We have already proved that if we have a homotopy couple $(\tilde{F},\tilde{G}) : (\tilde{f},\tilde{g}) \simeq (\tilde{f}',\tilde{g}')$ where $(\tilde{f},\tilde{g})$ and $(\tilde{f}',\tilde{g}')$ are lift-pairs of (f,g) and (f',g') with

$$(F,G) : (f,g) \simeq (f',g')$$

then (F,G) induces a 1-1 transformation of the lift-pairs of (f,g) into those of (f',g') ; and then for any $\alpha \in \pi(X)$, $(\gamma_1,\gamma_2) \in \pi(Y) \oplus \pi(Y)$ we have $(\tilde{F},\tilde{G}) : (\gamma_1\tilde{f}\alpha,\gamma_2\tilde{g}\alpha) \simeq (\gamma_1\tilde{f}'\alpha,\gamma_2\tilde{g}'\alpha)$.

We can now directly establish

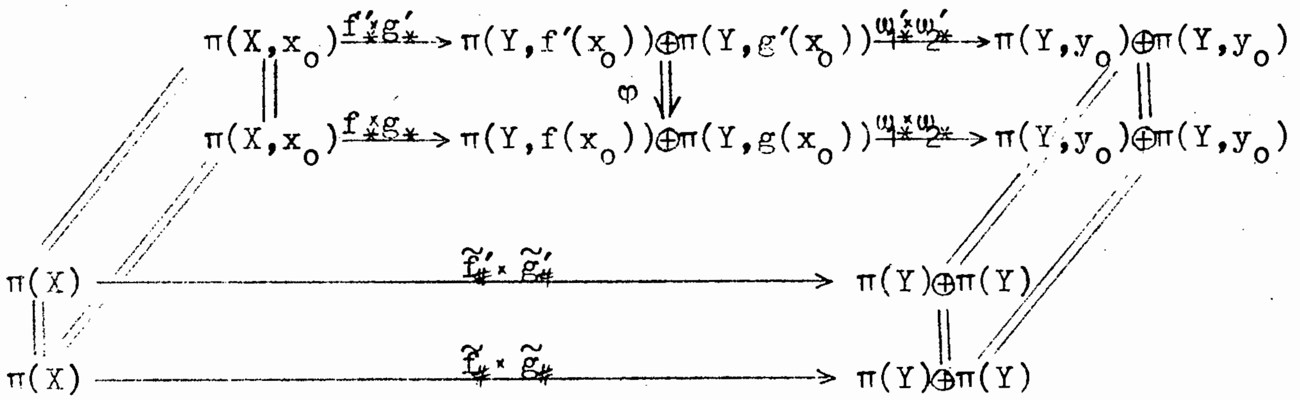
8.3 Cor : If $(F,G) : (f,g) \simeq (f',g')$ is a homotopy couple , then

$$(\tilde{F},\tilde{G}) : (\tilde{f},\tilde{g}) \simeq (\tilde{f}',\tilde{g}') \text{ and}$$
$$\tilde{f}_{\#} \times \tilde{g}_{\#} = \tilde{f}'_{\#} \times \tilde{g}'_{\#} : \pi(X) \longrightarrow \pi(Y) \oplus \pi(Y) .$$

Proof : $f \simeq f' \longrightarrow f_* = f'_*$

$g \simeq g' \longrightarrow g_* = g'_*$.

Result is immediate from diagram .



where φ is the isomorphism $\{f_t(x_0)\}_* \ \{g_t(x_0)\}_*$.

8.4 D: We will say that the pair $(\gamma_1,\gamma_2) \in \pi(Y) \oplus \pi(Y)$ is $\tilde{f}_\# \times \tilde{g}_\#$-conjugate to $(\gamma_1',\gamma_2') \in \pi(Y) \oplus \pi(Y)$ if and only if there exists $\alpha \in \pi(X)$ and $\beta \in \pi(Y)$ such that

$$(\gamma_1',\gamma_2') = \tilde{f}_\# \times \tilde{g}_\#(\alpha)\ (\gamma_1,\gamma_2)\ (\beta,\beta)$$

where $(\delta_1,\delta_2)\ (\gamma_1,\gamma_2)\ (\beta_1,\beta_2)$ means $(\delta_1\gamma_1\beta_1,\delta_2\gamma_2\beta_2)$ for (δ_1,δ_2), (γ_1,γ_2), $(\beta_1,\beta_2) \in \pi(Y) \oplus \pi(Y)$ as expected.

Thus, $\pi(Y) \oplus \pi(Y)$ is divided into $\tilde{f}_\# \times \tilde{g}_\#$-conjugacy classes. (It is easy to see that the relation described is symmetric, transitive and reflexive.)

We immediately obtain the result :

8.5 T: If $(\tilde{f},\tilde{g})$ is a lift-pair of (f,g) and (γ_1,γ_2), (δ_1,δ_2) are elements from $\pi(Y) \oplus \pi(Y)$ then (γ_1,γ_2) is $\tilde{f}_\# \times \tilde{g}_\#$-conjugate to (δ_1,δ_2) if and only if $(\gamma_1^{-1}\tilde{f},\gamma_2^{-1}\tilde{g}) \equiv (\delta_1^{-1}\tilde{f},\delta_2^{-1}\tilde{g})$.

Proof : If (γ_1,γ_2) is $\tilde{f}_\# \times \tilde{g}_\#$-conjugate to (δ_1,δ_2), then there exists $\alpha \in \pi(X)$ and $\beta \in \pi(Y)$ such that

$$(\delta_1,\delta_2) = \tilde{f}_\# \times \tilde{g}_\#(\alpha)\ (\gamma_1,\gamma_2)\ (\beta,\beta).$$

Now suppose that $\tilde{f}_\# \times \tilde{g}_\#(\alpha) = (\alpha_1,\alpha_2)$

then $$(\delta_1,\delta_2) = (\alpha_1\gamma_1\beta,\alpha_2\gamma_2\beta).$$

Thus we have,

$$\delta_1 = \alpha_1\gamma_1\beta \text{ and } \delta_2 = \alpha_2\gamma_2\beta \text{ or}$$
$$\delta_1^{-1} = \beta^{-1}\gamma_1^{-1}\alpha_1^{-1} \text{ and } \delta_2^{-1} = \beta^{-1}\gamma_2^{-1}\alpha_2^{-1}$$

so that

$$\delta_1^{-1}\tilde{f} = \beta^{-1}\gamma_1^{-1}\underline{\alpha_1^{-1}\tilde{f}} \quad , \quad \delta_2^{-1}\tilde{g} = \beta^{-1}\gamma_2^{-1}\underline{\alpha_2^{-1}\tilde{g}}$$
$$= \beta^{-1}\gamma_1^{-1}\underline{\tilde{f}\alpha^{-1}} \quad , \quad = \beta^{-1}\gamma_2^{-1}\underline{\tilde{g}\alpha^{-1}}$$

Therefore, $(\gamma_1^{-1}\tilde{f},\gamma_2^{-1}\tilde{g}) \equiv (\delta_1^{-1}\tilde{f},\delta_2^{-1}\tilde{g})$.

Conversely, suppose $\gamma_1^{-1}f = \beta\delta_1^{-1}f\alpha = \beta\delta_1^{-1}\alpha_1 f$ and $\gamma_2^{-1} = \beta\delta_2^{-1}g\alpha = \beta\delta_2^{-1}\alpha_2 g$

where $f_\# \times g_\#(\alpha) = (\alpha_1,\alpha_2)$.

So, by uniqueness, we have

$$\begin{aligned}(\gamma_1,\gamma_2) &= (\alpha_1^{-1}\delta_1\beta^{-1},\alpha_2^{-1}\delta_2\beta^{-1}) \\ &= \tilde{f}_\# \times \tilde{g}_\# (\alpha^{-1})(\delta_1,\delta_2)(\beta^{-1},\beta^{-1})\end{aligned}$$

and so (γ_1,γ_2) is $\tilde{f}_\# \times \tilde{g}_\#$-conjugate to (δ_1,δ_2) .

Hence, after a lift-pair $(\tilde{f},\tilde{g})$ of (f,g) has been chosen, we get

8.6 Cor : The $\tilde{f}_\# \times \tilde{g}_\#$-conjugacy classes are in 1-1 correspondence with the lift-pair classes of (f,g) .

Let $H_1(X)$ be the first integral homology group of X.

8.7 T: The number of lift-pair classes of (f,g) is at least Ord Coker$(f_{*1}-g_{*1})$, where Ord denotes the order of a group and f_{*1} and g_{*1} are the homomorphisms between $H_1(X)$ and $H_1(Y)$ induced by f and g .

Proof : Let $(\tilde{f},\tilde{g})$ be a lift-pair of (f,g) .

The number of lift-pair classes of (f,g) is equal to the number of $\tilde{f}_\# \times \tilde{g}_\#$-conjugacy classes of (f,g) by 8.6.

Let $h_X : \pi(X) \longrightarrow H_1(X)$ be the Hurewicz homomorphism which is a natural homomorphism . (see [9] page 387.)

(Recall $\ker h_X$ is the commutator subgroup of $\pi(X)$ and since we have assumed X to be path-connected, it is 0-connected and hence, h_X is surjective.)

Given $f \times g : X \longrightarrow Y \times Y$ defined by

$f \times g(x) = (f(x),g(x))$ and by the naturality of h_X we have commutativity of the following diagram.

$$\begin{array}{ccccccc}
\pi(X) & \xrightarrow{\tilde{f}_\# \times \tilde{g}_\# = (f_* \times g_*)(\omega_{1*} \times \omega_{2*})} & \pi(Y)\oplus\pi(Y) & & & & \\
\Big\downarrow h_X & & \Big\downarrow h_{Y\times Y} & & & & \\
H_1(X) & \xrightarrow[(f\times g)_{*1}]{} & H_1(Y)\oplus H_1(Y) & \xrightarrow[k]{} & H_1(Y) & \xrightarrow[\eta]{} & \dfrac{H_1(Y)}{\mathrm{Im}(f_{*1}-g_{*1})}
\end{array}$$

Define $k(y_1,y_2) = y_1-y_2$, then k is surjective, and denote by $(f_{*1}-g_{*1})$ the homomorphism from $H_1(X)$ to $H_1(Y)$ given by $k(f\times g)_{*1}$.

Let η be the quotient map

$$\eta : H_1(Y) \longrightarrow \frac{H_1(Y)}{\operatorname{Im}(f_{*1}-g_{*1})} = \operatorname{Coker}(f_{*1}-g_{*1})$$

which takes

$$y \longrightarrow y[\operatorname{Coker}(f_{*1}-g_{*1})] ,$$ the coset of y in the quotient group.

Since $kh_{Y\times Y}(\tilde{f}_{\#}\times \tilde{g}_{\#})(\alpha)(\gamma_1,\gamma_2)(\beta,\beta)$

$$= k(f\times g)_{*1}h_X(\alpha) + kh_{Y\times Y}(\gamma_1,\gamma_2) + kh_{Y\times Y}(\beta,\beta)$$

$$= (f_{*1}-g_{*1})h_X(\alpha) + kh_{Y\times Y}(\gamma_1,\gamma_2) ,$$

and because $(f_{*1}-g_{*1})h_X(\alpha) \in \operatorname{Im}(f_{*1}-g_{*1})$, $\tilde{f}_{\#}\times\tilde{g}_{\#}$-conjugate elements of $\pi(Y)\oplus\pi(Y)$ go to the same coset under $\eta kh_{Y\times Y}$; that is, $\tilde{f}_{\#}\times\tilde{g}_{\#}$-conjugate elements coincide under $\eta kh_{Y\times Y}$ in $\operatorname{Coker}(f_{*1}-g_{*1})$.

Therefore, the number of $\tilde{f}_{\#}\times\tilde{g}_{\#}$-conjugacy classes, and hence the number of lift-pair classes, of (f,g) must be at least Ord $\operatorname{Coker}(f_{*1}-g_{*1})$.

8.8 P: If $\pi(Y)$ is abelian, then the number of $\tilde{f}_{\#}\times\tilde{g}_{\#}$-conjugacy classes, hence the number of lift-pair classes, of (f,g) is exactly Ord $\operatorname{Coker}(f_{*1}-g_{*1})$.

Proof : Consider the diagram

$$\begin{array}{ccccccc}
\pi(X) & \xrightarrow{\tilde{f}_{\#}\times\tilde{g}_{\#}} & \pi(Y)\oplus\pi(Y) & \xrightarrow{k'} & \pi(Y) & \xrightarrow{\eta'} & \dfrac{\pi(Y)}{\operatorname{Im}\varphi} \\
h_X \downarrow & (*) & \downarrow h_{Y\times Y} & (**) & \downarrow h_Y & & \downarrow h' \\
H_1(X) & \xrightarrow{(f\times g)_{*1}} & H_1(Y)\oplus H_1(Y) & \xrightarrow{k} & H_1(Y) & \xrightarrow{\eta} & \dfrac{H_1(Y)}{\operatorname{Im}(f_{*1}-g_{*1})}
\end{array}$$

Since $\pi(Y)$ is abelian, $h_{Y\times Y}$ and h_Y are isomorphisms. Let k' be the homomorphism such that $k'(\gamma_1,\gamma_2) = \gamma_1\gamma_2^{-1}$, then $(*)$ and $(**)$ commute .

Let φ be $k'(\tilde{f}_{\#}\times\tilde{g}_{\#})$, and η' be the quotient homomorphism of $\pi(Y)$ onto $\pi(Y) \bmod \mathrm{Im}\varphi$; we show that then $\eta'k'$ maps distinct $\tilde{f}_{\#} \times \tilde{g}_{\#}$-conjugacy classes to distinct cosets and that $\eta'k'$ maps $\tilde{f}_{\#} \times \tilde{g}_{\#}$-conjugate elements of $\pi(Y) \oplus \pi(Y)$ to the same coset of $\pi(Y) \bmod \mathrm{Im}\varphi$:

Firstly, assume $\eta'k'$ does not distinguish between (γ_1,γ_2) and (δ_1,δ_2) and that (γ_1,γ_2) is not $\tilde{f}_{\#} \times \tilde{g}_{\#}$-conjugate to (δ_1,δ_2) .

Then there exists $\alpha \in \pi(X)$ such that

$$k'(\delta_1,\delta_2) - k'(\gamma_1,\gamma_2) = \alpha_1 - \alpha_2 \qquad \text{(I)}$$

where $(\tilde{f}_{\#} \times \tilde{g}_{\#})(\alpha) = (\alpha_1,\alpha_2)$.

Since (δ_1,δ_2) is not $\tilde{f}_{\#} \times \tilde{g}_{\#}$-conjugate to (γ_1,γ_2) , for all $\beta \in \pi(Y)$,

$$(\delta_1,\delta_2) \neq (\tilde{f}_{\#} \times \tilde{g}_{\#})(\alpha) + (\gamma_1,\gamma_2) + (\beta,\beta) .$$

(Since $\pi(Y)$ is abelian we use additive notation.)

i.e. $(\delta_1,\delta_2) = (\alpha_1+\gamma_1+\beta, \alpha_2+\gamma_2+\beta) + (\epsilon_1,\epsilon_2) \quad \epsilon_1 \neq \epsilon_2$.

But then (I) gives

$$k'((\alpha_1+\gamma_1+\beta,\alpha_2+\gamma_2+\beta) + (\epsilon_1,\epsilon_2)) - k'(\gamma_1,\gamma_2) = \alpha_1-\alpha_2$$

or $\alpha_1+\gamma_1+\beta+\epsilon_1-\alpha_2-\gamma_2-\beta-\epsilon_2-\gamma_1+\gamma_2 = \alpha_1 - \alpha_2$

which is impossible since $\epsilon_1 \neq \epsilon_2$.

So, $\eta'k'$ sends distinct $\tilde{f}_{\#} \times \tilde{g}_{\#}$-conjugate elements to distinct cosets in $\pi(Y) \bmod \mathrm{Im}\varphi$.

Secondly, if

$$(\delta_1,\delta_2) = (\tilde{f}_{\#} \times \tilde{g}_{\#})(\alpha)\ (\gamma_1,\gamma_2)\ (\beta,\beta)$$

then $k'(\delta_1,\delta_2) - k'(\gamma_1,\gamma_2) = (\alpha_1+\gamma_1+\beta) - (\alpha_2+\gamma_2+\beta) - (\gamma_1,\gamma_2)$

$$= \alpha_1 - \alpha_2 \in \mathrm{Im}\varphi .$$

So $\tilde{f}_{\#} \times \tilde{g}_{\#}$-conjugate elements go to the same coset of $\pi(Y) \bmod \mathrm{Im}\varphi$.

Since $\eta'k'$ is onto $\pi(Y) \bmod \mathrm{Im}\varphi$,the number of $\tilde{f}_{\#} \times \tilde{g}_{\#}$-conjugacy classes must be Ord Coker φ .

We have $h_Y(\mathrm{Im}\, k'(\tilde{f}_{\#} \times \tilde{g}_{\#})) = \mathrm{Im}\, k(f \times g)_{*1}$ since h_X is a monomorphism .

Trivially, $\ker h_Y \subseteq \mathrm{Im}\, k'(\tilde{f}_{\#} \times \tilde{g}_{\#})$, so by the first isomorphism theorem of groups , h' is an isomorphism.

Thus, the number of $\tilde{f}_{\#} \times \tilde{g}_{\#}$-conjugacy classes, and hence the number of lift-pair classes, of (f,g) is equal to Ord Coker$(f_{*1}-g_{*1})$.

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