

Fluid and gas models in FLRW and Almost FLRW universes

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Non sunt multiplicanda entia praeter necessitatem

William of Ockham

"No, I don't like work. I had rather laze about and think of all the fine things that can be done. I don't like work - no man does - but I like what is in the work, - the chance to find yourself. Your own reality - for yourself, not for others - what no other man can ever know. They can only see the mere show, and never can tell what it really means" - MARLOW.

Heart of Darkness, Joseph Conrad

The requisites of the definition of an uncreated thing are these :

- I It must exclude every cause; that is, the object must require for its explanation nothing but its own being.*
- II When the things definition has been given there must remain no room for the question, DOES IT EXIST.*
- III It must, as far as the mind is concerned, have no substantives which can be turned into adjectives; that is, it must not be explained through any abstractions.*
- IV finally; it is required that all its properties are inferred from its definition.*

[97]PART II Treatise on the correction of intellect. B. Spinoza¹ 1677

Pull out his eyes,
Apologize
Apologize
Pull out his eyes

Apologize
Pull out his eyes
Pull out his eyes
Apologize

James Joyce, A Portrait of an artist as a young man [pg 2]

"We trained hard— but it seemed that every time we were beginning to form up into teams we would be reorganized. I was to learn later in life that we tend to meet any situation by reorganizing; and a wonderful method it can be for creating the illusion of progress while producing confusion, inefficiency and demoralization."

Petronius circa 256 B.C.

¹ *Deus sive natura*

To my friends,

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Conventions and abbreviations

- **General or Tetrad basis** : early roman letters a, b, c, d, e, f, \dots , the spatial part is denoted by early greek, eg $p^a = (p^0, p^\alpha)$.
- **Coordinate components** : late roman i, j, k, l, m, n, \dots , the spatial part is denoted by late greek, eg $x^i = (x^0, x^\rho)$.
- **Completely antisymmetric tensor** :

$$\eta^{abcd} = \eta^{[abcd]}.$$

Defining $\delta_{efgh}^{[abcd]} \equiv \delta_e^a \delta_f^b \delta_g^c \delta_h^d$, this obeys the identities:

$$\eta^{abcd} \eta_{efgh} = -4! \delta_{efgh}^{[abcd]},$$

$$\eta^{abcd} \eta_{efgd} = -3! \delta_{efg}^{[abc]},$$

$$\eta^{abcd} \eta_{efcd} = -4! \delta_{ef}^{[ab]},$$

$$\eta^{abcd} \eta_{ebcd} = -3! \delta_e^a.$$

- $\epsilon_{abc} = \eta_{abcd} u^d$
- $(^3)\nabla_a = h^b_a \nabla_b \equiv \hat{\nabla}_a$
- $A \simeq B$ implies that $A = B$ to linear order, that is $O[1]$.
- $A \approx B$ implies that $A = B$ for small temperature anisotropies.
- $[]$ denotes an anti-symmetric tensor eg $T_{[ab]} = \frac{1}{2}(T_{ab} - T_{ba})$
- $()$ denotes a symmetric tensor eg $T_{(ab)} = \frac{1}{2}(T_{ab} + T_{ba})$
- $\langle \rangle$ denotes a Projected Symmetric Trace-free Tensor eg $T_{\langle ab \rangle} = T_{(ab)} - \frac{1}{3} h_{ab} T^c_c$.
- FLRW : Friedmann-Lemâitre-Robertson-Walker
- GIC : Gauge Invariant and Covariant.
- GI : Gauge Invariant
- RKT : Relativistic Kinetic Theory

Chapter 1

Introduction

1.1 Cosmology

On large scales, matter in the universe as we know it from observations currently takes the form of dust, with the galaxies themselves the fluid particles. These particles do not apparently interact very strongly with one another except through gravity, hence we may consider the fluid approximation to be good. In particular the perfect fluid approximation can be used where the effects of shear viscosity, bulk viscosity, and heat flow are ignored. In this respect perfect fluids can be taken to be a fairly good first order approximation when investigating the large scale structure of the universe. The apparent spatial homogeneity on large scales implies, via the Copernican principle, that it is not a bad approximation to consider FLRW geometry as the appropriate background. Such an approach should be treated as a useful entry point into cosmological models. It represents the type of direct cosmology that imposes rather than uncovers structure and then checks if they are within observational results. But it is worthy of study as it leads to insights with regard to the basic structure of space and time.

Recently the universe has been modeled in the covariant sense, in terms of fluid models [4] [3] [1] and perturbations thereof, leading to Gauge Invariant Covariant (GIC) perturbations of these fluid models [30] [31] [32] [28]. It is well known that kinetic theory provides a physically sound and consistent description of the matter and radiation in the universe [40] [29] [28] [35] [36], so a perturbative theory of gas models using kinetic theory would be most helpful. This has been done to a large degree in the Gauge Invariant (GI) Bardeen [28] approach to perturbation theory by studies of gases based on the relativistic Boltzmann equation [35] [34]. These treatments, however, were not fully covariant. The GI Bardeen approach is dependent on a co-ordinate choice [33], while in the full GIC perturbation theory full covariance is maintained along with gauge invariance by describing the the-

ory in a particular set of perturbation variables that differ from the Bardeen choice but can be related to the Bardeen variables [33]. The covariant formulation of the relativistic Boltzmann equation in terms of variables that are of use in the GIC theory for gases has been well described [24] [25]. In this thesis, I provide both a good introduction to the full GIC perturbation theory of a photon gas and matter fluid system in the linear theory as well as a solid grounding with respect to the exact FLRW fluid model upon which most of the original ideas and concepts of modern cosmology are based. The introduction to the exact FLRW model is done in the sense of the dynamical systems approach to cosmology which provides the easiest access to understanding the evolution of single and multi-fluid FLRW models.

A dynamical system on \mathbb{R}^n is a one-parameter group of onto maps $\mathbb{R}^n \xrightarrow{\psi_t} \mathbb{R}^n$. Where orbits through some coordinate x_0 are given by the curves $\alpha(x_0) = \{x_t = \psi_t(x_0) : t \in \mathbb{R}^n\}$, and these curves are solutions of the associated differential equations on \mathbb{R}^n , $dx/dt = f(x)$, $x \in \mathbb{R}^n$. The complete state of the physical system at an instant of time is given by the point $x \in \mathbb{R}^n$, the set of such points, (in the cosmological context) determines the state space (of cosmological models). For some autonomous set of $dx_i/dt = f_i(x_k)$, one can find phase-planes, which are given by $dx_i/dx_j = f_i(x_k)/f_j(x_k)$. Such phase planes fully characterize the state of the system (or cosmological model in our context).

The dynamical systems approach I adopt is based on that of Madsen et al [10], Stabell et al [8] and Gott et al [13]. The approach of Madsen et al [10] is generalized to include the work of Stabell et al [8]. The dynamical systems approach is helpful as it provides a general conceptual link between theory and observations without having to get bogged down in finding the exact solutions, in particular for the two and three fluid models. This approach is also used to demonstrate the use of a scalar field in FLRW models and to give a basic introduction and description of inflation, in particular the flat potential model discussed by Madsen et al [10] [11]. This is used to place the exact FLRW model in the context of general cosmology and a discussion of the measure problem using the Liouville measure [21] [18]. After the solid understanding of the FLRW universes is provided (Part A), the Relativistic Kinetic Theory (RKT) of Ellis et al [24] [25] is then developed from first principles. But first a return to Newtonian Cosmology.

It is useful to remember the essence of Newtonian dynamics and its application to models of the universe; this is helpful in drawing the connection between general relativistic cosmologies and the older more simplistic Newtonian paradigm. Consider an infinite gas cloud where the fluid elements are galaxies. Newton found that his dynamics and

gravitational theory caused the gravitational potential at a point in the cloud to be infinite. This does not happen in General Relativity. In the context of the Newtonian paradigm, this problem may be avoided by using a finite but large cloud.

Notice that a large cloud has a center, but an infinite one does not. It is more acceptable to avoid having to choose a unique center; rather one can adopt the notion of making the cloud uniform to the edge, isotropic about its center and much larger than any distance yet measured. This avoids the necessity of picking a unique center. Then any fluid element we observe, any galaxy, would see around itself a uniform isotropic universe. We would be unable to distinguish between a finite or infinite universe. The assumption of isotropy and uniformity not only allows any point in an observable region of the cloud to consider itself to be the center, but also simplifies the motion of the cloud.

Consider [49] now a point mass m , on the surface of a sphere with radius $S(t)$. Assume that the mass in the sphere is distributed uniformly with constant density μ . The mass of the sphere acts on m as if it were concentrated at the center, and is given by

$$M = \frac{4}{3}\pi S^3 \mu. \quad (1.1)$$

Newton's law $m\ddot{S} = -G_N \frac{mM}{S^2}$ gives that

$$S^2 \ddot{S} = -G_N M. \quad (1.2)$$

For constant M , multiply by (\dot{S}) and integrate to find

$$\frac{\dot{S}^2}{S^2} - \frac{8\pi}{3}G_N \mu = \frac{2E}{S^2}, \quad (1.3)$$

where E is the integration constant, understood as the total energy of the gravitationally bound particle-cloud system. This is the same as the Friedmann equation which we will obtain later from a relativistic derivation.

If the motion is in such a uniform and isotropic cloud, the velocity v of a particle relative to a radius vector r from a co-moving observer is then

$$v = f(t)r. \quad (1.4)$$

The position of the particle at time t , is $r = S(t)r_0$. Differentiating to find the velocity we find that $\dot{S} = S f(t)$, that is

$$\frac{\dot{S}}{S} = f(t) \quad (1.5)$$

Clearly $f(t)$ is the fractional expansion rate of the sphere defined by $S(t)r_0$, the radius vector to the observed particle from the comoving observer.

In the relativistic theory, one eliminates the Newtonian concept of external forces and deals primarily with point particles undergoing free-fall in some curved space-time in the usual manner [3], [49]. The curvature is characterized by the commutation of covariant derivatives.

Assume that the field equations are

$$G_{ab} = \kappa T_{ab}. \quad (1.6)$$

In an equivalent form, setting $\kappa = +1$ in natural units;

$$R_{ab} - \frac{1}{2}Rg_{ab} = T_{ab}, \quad (1.7)$$

$$R_{ab} = (T_{ab} - \frac{1}{2}Tg_{ab}). \quad (1.8)$$

Here R_{ab} is the usual Ricci tensor, T_{ab} is the usual energy momentum tensor and the Ricci scalar is $R = R_{ab}g^{ab}$. Assuming FLRW geometry ¹ and a comoving velocity u_a such that $u_a u^a = -1$, we can project orthogonal to u^a into the instantaneous rest space of the co-moving observer using the tensor $h_{ab} = g_{ab} + u_a u_b$. Using the field equations, we are able to relate the geometry to the dynamics.

I Geometry

Using the derived Ricci tensors for the FLRW space-time we find;

$$R_{ab}u^a u^b = -3\frac{\ddot{S}}{S}, \quad (1.9)$$

$$R_{ab}u^a h_c^b = 0, \quad (1.10)$$

$$R_{ab}h_c^a h_d^b = -\left(\frac{\ddot{S}}{S} + 2\frac{\dot{S}^2}{S^2} + \frac{2k}{S^2}\right). \quad (1.11)$$

II Dynamics

Any symmetric tensor T_{ab} can be decomposed as

$$T_{ab} = \mu u_a u_b + p h_{ab} + q_a u_b + u_a q_b + \pi_{ab}, \quad (1.12)$$

where $\pi_{[ab]} = 0 = \pi_{ab}u^b = \pi_a^a = q_a u^a$.

Using the field equations (1.8) one has that

$$R_{ab} = \left(\frac{1}{2}(\mu + 3p)u_a u_b + q_a u_b + u_a q_b + \pi_{ab} + \frac{1}{2}(\mu - p)h_{ab}\right). \quad (1.13)$$

¹see the Appendix A.2

Projecting in a similar way as in (1.9) (1.10) and (1.11), the above equations, with $u^a = \delta_0^a$, can be reduced to

$$R_{ab}u^a u^b = \frac{1}{2}(\mu + 3p), \quad (1.14)$$

$$R_{ab}u^a h_c^b = -q_c, \quad (1.15)$$

$$R_{ab}h_c^a h_d^b = \pi_{cd} + \frac{1}{2}(\mu - p)h_{cd}. \quad (1.16)$$

Using the equations (1.9) - (1.10) and (1.14) - (1.16) one gets the evolution equations for FLRW space-time;

$$-3\frac{\ddot{S}}{S} = \frac{1}{2}(\mu + 3p), \quad (1.17)$$

$$0 = q_c, \quad (1.18)$$

$$\left(\frac{\ddot{S}}{S} + 2\frac{\dot{S}^2}{S^2} + \frac{2k}{S^2}\right)h_{cd} = \pi_{cd} + \frac{1}{2}(\mu - p)h_{cd}. \quad (1.19)$$

If the d index is contracted out with g^{cd} and using $\pi_c^c = 0$, while noticing that h_{ab} is diagonal for FLRW universes, (1.19) then becomes

$$\left(\frac{\ddot{S}}{S} + 2\frac{\dot{S}^2}{S^2} + \frac{2k}{S^2}\right) = \frac{1}{2}(\mu - p), \quad (1.20)$$

$$\pi_{ab} = 0, \quad (1.21)$$

as expected. Hence we need only consider perfect fluids,

$$T_{ab} = \mu u_a u_b + p h_{ab}. \quad (1.22)$$

The most accessible technique through which to investigate such models is via the theory of dynamical systems, as this allows one to look at global trends, generic behaviour and behaviour of the solutions. I attempt to place some of the work that has been done, particularly in the area of FLRW models, in context and discuss where I think it should be going. This should be helpful in the sense that once one is able to understand classes of possible scenarios available within the boundaries of specific models it may be easier to apply the models to real world problems associated with understanding the available observational data.

1.2 PART A : Phase plane approach to FLRW models

In Part A, I discuss in broad terms the current status of FLRW phase planes² with particular interest in the observationally obtainable (H, Ω) planes as discussed in [13]. The ideal would be to look at the (Ω, H, q) hyperplanes, and investigate how the observational limits propagate backwards and forwards in time. These observational limits, I will call the current solution volume, in that the current values of these parameters, according to observations, lie somewhere in this volume in the (Ω, H, q) space. There are four possible ways of investigating models of the universe using phase planes, as presented in this thesis, I list these as the first 4 points below.

(1) One deals only with single fluid models, and may then look at (Ω, S) , (Ω, H) , (Ω, q) and (Ω, γ) and combinations of these. This would allow the data to be uniquely associated with the total density parameter as it is the only density contribution to the model. In this scenario one could easily look at the back and forward propagation of current data, and see what effects they have on various choices of single fluids. This has been done to some extent by several people; a good example is the (Ω, S) plane of Madsen and Ellis [10].

(2) One could look at effectively single fluid models, as these can easily be projected onto a 2 dimensional surface. By fixing all but one current fluid density, in a multi-fluid context, one could achieve a similar mapping. The latter method would be unsuitable unless one is very certain of the current density parameter values, which is only the case for radiation. Due to the nature of the resulting form of the total γ , one can still uniquely discuss the resulting (Ω, S) , (Ω, H) , (Ω, q) and (Ω, γ) planes, and combinations thereof, as there is still only one evolving energy density (that of the other fluid, be it radiation, matter, scalar field or whatever). This could not be done if one wishes to use these planes to discuss the measure problem, as such a mapping would change the canonical measure [21].

(3) (1) and (2) would allow an approximation to the true 2 fluid or 3 fluid noninteracting phase planes by using these descriptions in different epochs to represent the different stages in the evolution of the universe. For 2 fluid models with radiation and matter, for example, one knows that there would be a matter dominated phase in the latter evolution of the model and a radiation dominated phase in the early evolution of the universe. This could then be approximated by two single fluid type evolutions (that is with $\gamma = \text{constant}$ in each epoch). This would allow a uniquely defined phase plane of the type discussed

²The phase planes produced here were plotted on a Pentium running a Linux platform using a Linux port of the Cornell *dstool* software, this software was obtained from macomb.tn.cornell.edu.

above. Obviously one would have to match the junction between the two epochs in a satisfactory fashion [10].

(4) One cannot uniquely discuss the (Ω, S) plane for an arbitrary number of non-interacting fluids as this would be attempting to project down unique trajectories onto a sub-manifold of the full phase space. One can only discuss the $(\Omega_1, \Omega_2, \dots, \Omega_N; S)$ plane uniquely and the same goes for the $(\Omega_1, \Omega_2, \dots, \Omega_N; H)$ and $(\Omega_1, \Omega_2, \dots, \Omega_N; q)$ planes (q can be written in terms of the individual density parameters). It can, however, be argued that such projections are generic if the nature of the critical points are unchanged³, *i.e.* the projection of the full evolution space $(\Omega_1, \dots, \Omega_N)$ into some (Ω, S) sub-plane may be useful if such a projection is structurally stable, such as in the case when $\gamma = \gamma(S, M_i, \gamma_i)$ is used (*eg* for matter and radiation), as this reflects the higher dimensional evolution in a faithful fashion.

(5) It may also be useful to attempt to find generic trends in planes (where the time parameter is included as a coordinate), such as the proper time τ , H , Ω planes; that is the (τ, H, Ω) plane for single fluids or effectively single fluids, or $(\tau, H, \Omega_1, \Omega_2, \dots, \Omega_N)$ plane, for many fluid models. These are briefly considered, along with the plane equations for the standard candle, standard measuring rod phase planes, in terms of redshifts. These are included primarily as an indication of further use of a phase plane approach with regard to a better understanding of observational cosmology and its relation to the real universe (See Section A.3 in the Appendix).

Clearly (1) - (3) are simpler but one may be missing some interesting structure and behaviour. The traditional phase planes of Stabell and Refsdal [8] and others generally fall into the categories of (2) and occasionally (3). The treatment given by Ehlers and Rindler [9] is the 3 fluid version of this approach; they look basically at the $(\Omega_1, \Omega_2, \Omega_3)$ planes. They demonstrate the inevitability of the big bang within the framework of perfect fluids in FLRW space-times using the current solution volume and the evolution of the possible universes defined by these limits. It should also be noted that a subtle variation on this theme is to look at (Ω, q) for two fluids, which can be done as the invariant phase space (Ω_1, Ω_2) for the two fluids can be mapped uniquely into the (Ω, q) plane, this is what Stabell et al [8] have done. I call these types of models, transformed 2 fluid models.

The traditional method of looking at fixed γ in various epochs may be flawed in the sense that a running $\gamma(t)$ may be more fruitful in finding a natural way of getting an accelerating universe. This is rather tricky, as in a multicomponent fluid γ becomes a

³Private communication Prof G.F.R. Ellis

function not only of the total density but of the weighted individual density parameters too, in much the same way as when you move from single fluid field equations (where there are two first order equations in terms of the Hubble parameter and the energy density *i.e.* the $\dot{\mu}$ and \dot{H} equations with one constant defined by the equation of state) allowing a unique 2D representation of the evolution, to a two fluid model which requires at least a 2D representation in (Ω_1, Ω_2) plane (but higher dimensional if the total density is to be investigated in full generality). The two fluid model has three equations for three unique variables and two constants - hence one needs 3D phase-plane to investigate unique trajectories in the phase space, or some assumption that allows one to represent it as a single fluid model with a functional relation between the pressure and density rather than a constant as given in the usual equation of state.

The planes for the effective single fluid models are presented in Part A, that is the (Ω, S) , (Ω, q) and (Ω, H) planes and the problem of useful representation of the higher dimensional phase planes is then discussed, with the ultimate idea of trying to reverse propagate the current data back in a unique form. The problem of how good an approximation the effective single fluid models are, is left open to discussion, although the dynamical systems approach towards cosmology in the context of this problem is clearly the natural way of going even if it appears to be rather tricky in practice.

1.3 PART B : Kinetic Theory and Almost FLRW models

A brief review of basic relativistic kinetic theory is presented; that is the gas model approach to cosmology, in particular the use of Vlasov and Einstein-Boltzmann equations. This is presented and discussed in the context of the Cosmic Background Radiation (CBR). Part B deals with the following 3 issues;

(1) The Gauge Invariant Covariant (GIC) form of the Relativistic Kinetic Theory (RKT) is presented and the covariant gauge invariant Boltzmann equation is derived, motivated by Ellis et al [24] [25] and Maartens et al [40]. This is used to find the covariant gauge invariant equations that describe the evolution of the the temperature anisotropies in a linearized *Almost FLRW* universe [55] [56] with scalar perturbations. This can be easily generalized to any perturbed model universe with small temperature anisotropies, because the linearization procedure is independent of the metric, but has been shown [55] to be equivalent to the gauge invariant evolution equations that can be derived by linearizing about a perturbed FLRW universe using the covariant gauge invariant formalism [30] [31] [32]. This treatment also allows a natural generalization to second order effects something

that the Bardeen GI perturbation theory does not provide with such simplicity [33].

(2) The CBR, particular the CMBR (Cosmic Microwave Background Radiation) is of particular interest as it is the best directly observed data that has a direct link to the early universe (other than element abundances to nucleosynthesis), in particular the eras of pre-recombination, recombination and decoupling, which all occurred prior to the free-streaming regime. The electron-photon gas at decoupling has a black-body spectrum; this can be motivated by discussing the pre-recombination era in the sense of an exact FLRW universe [27], and indicates that to a large degree the coupled electron-photon system was in thermal equilibrium. The deformations inherent in the temperature maps we measure of the photon gas temperature on our sky are a direct link to the physics occurring at decoupling, in particular the so-called Doppler peaks or rather the Sakharov oscillations as described by the gauge invariant treatment given by [50] [52] [53] [54]. Hu [54] is able to reduce their version of the gauge invariant temperature Boltzmann like equations to a single second order in time equation using corrections to the tight coupling limit in what appears to be a fluid limit. I have derived a covariant equivalent of this equation. The equation that I have derived (as well as that in [50]) does not describe high l behaviour, where one expects true Doppler peak effects to exist in the spherical harmonic decomposition of the measures on-sky temperature anisotropies, but does demonstrate oscillations in the monopole temperature contribution at a given space-time position. The high l behaviour can be seen in the angular power spectrum, however at this time a clear formulation of the covariant auto-correlation function and its link to the angular power spectrum is unclear; this issue will have to be clarified if a transparent covariant treatment of Sakharov oscillations is to be achieved.

In deriving this equation several important issues have been raised, most of these issues are original to the best of my knowledge. The three most important of these (I feel) are;

- (A) The issue of decoupling the $l = 0$ and $l = 1$ GIC Boltzmann moment equations from the higher order moment equations, with particular emphasis on a possible consistency check on the validity of the multi-fluid approach [31] with regard to the evolution of GIC perturbations in the post decouple era. If the $l = 0$ and $l = 1$ Boltzmann equations can be consistently decoupled on a physical basis then the basic fluid perturbation equations can be used in a consistent and complete fashion. If this is not the case then it would appear that for a multi-fluid approach to be phys-

ically consistent with the GIC RKT, that some decoupling at least at $l = 2$ should occur. It should be pointed out that I am looking for a decoupling of the Boltzmann equations at some l value, not the truncation of the Harmonic expansion in F_{A_l} 's [24] which is extremely problematic. One method that is successful in decoupling the $l = 1$ Boltzmann equation from the $l = 2$ is that of a metric restriction, which involves setting $\pi_{ab} = 0$ (Section 3.5.4), but this is not satisfactory for two reasons : (i) the lack of a physical motivation for the metric choice (ii) the choice of metric compromises covariance and is hence not in line with the GIC approach.

- (B) The issue of using pressure at decoupling to ensure that in the ideal case of thermodynamic equilibrium between the photons and electrons, thermal exchange between the two gases is facilitated. This is followed for two reasons : (i) it provides a restricted form for $\Pi^a = \hat{\nabla}_b \pi^b_a$, which allows the $l = 0$ and $l = 1$ Boltzmann equations to be decoupled from the $l = 2$ Boltzmann equations and hence the rest of the infinite hierarchy of equations; (ii) In the tight-coupling limit (rather than near to tight-coupling) the mean scattering time of photons with electrons is zero, or rather that the scattering cross-section (in this case the Thompson cross-section) becomes arbitrarily large. It seems that the fluid equations may be sufficient to describe the physics of decoupling in this limit. If that is the case then the problems with velocity divergence between the photon fluid (not gas in this treatment) and the matter fluid may be removed with the inclusion of a pressure contribution to the matter⁴.
- (C) The Relativistic kinetic theory provides an excellent vehicle for looking at gases, that is a large number of particles not in equilibrium undergoing scattering processes. A novel note is that the usual Fokker-Planck method of constructing the scattering term is not only important in deriving the usual Compton-like scattering forms but can be easily modified to include Brownian- or Gaussian-like noise terms in the gas. What is interesting about this is that if the gas is Gaussian the scattering term is only a function of the two point correlations between the gas particles. This may be important as it seems that in the usual treatments of perturbations in the CMBR only Gaussian modes can be discussed, the choice of Gaussian perturbations ensures that the auto-correlation function of on-sky temperatures is in fact the two-point correlation function. One can recall that the Fourier transform of the two-point

⁴Private communication with P.K.S Dunsby.

correlation functions gives one the power spectrum (this is a useful trick). But to include other forms of noise or perturbations in the CMBR and to link these to the Boltzmann equations and hence the evolution equations may require Langevin and Fokker-Plank equations in a covariant formulation. The specific choice of Brownian noise can be used to generate Quantum Mechanics in the sense of the Schrödinger equation, the Nelson [68] [69] [70] form of Quantum mechanics. This can be used to derive not only the Schrödinger equation but also the Bohm Quantum potential, and has been generalized to curved manifolds [67]. This approach has not been very well investigated; it appears that using this approach and the GIC relativistic kinetic theory, a discussion of quantum effects in some GIC sense should be straight-forward. This should be treated as extremely speculative, but is included to demonstrate that the GIC RKT approach to cosmology may be very fruitful, beyond providing a covariant treatment of the CMBR. The suggestion of the inclusion of Quantum effects using the GIC RKT theory to find a Quantum Relativistic Kinetic Theory, in its current state, is very tentative as clearly three things need to be done to place it on a consistent footing : (i) a consistent derivation of a GIC Fokker-Plank equation in the sense of [24], (ii) a demonstration that the quantum theory of [68] is valid hence address the issues raised by [69] [70] and (iii) a demonstration that inclusion of Brownian motion in RKT can be reduced to the Klein Gordon and Schrödinger equations in the appropriate limits of GR. This may provide much useful physical insight and is included here as an indication of where I feel future work in RKT could be undertaken, much work has already been done in stochastic perturbations [71] and the issue of the quantum noise [72], but not in the GIC theory. This is not discussed further in this thesis.

Chapter 2

Phase plane approach to FLRW cosmology

2.1 Single perfect fluids with variable γ

We consider Friedmann-Lemaître-Robertson-Walker (FLRW) universes dominated by a perfect fluid with the usual energy-momentum tensor,

$$T_{ab} = \mu u_a u_b + p h_{ab}. \quad (2.1)$$

Here μ and p are the energy density and pressure of the fluid respectively. The field equations may be reduced to the Raychaudhuri, energy conservation, and Friedmann equations in the usual form, following the notation of [2] and [10]:

$$3\frac{\ddot{S}}{S} + \frac{1}{2}(\mu + 3p) - \Lambda = 0, \quad (2.2)$$

$$\dot{\mu} + (\mu + p)3\frac{\dot{S}}{S} = 0, \quad (2.3)$$

$$2\left(\frac{\dot{S}}{S}\right)^2 - \mu - \Lambda = -3K. \quad (2.4)$$

Here $S = S(t)$ is the scale factor and determines the physical conditions prevailing at each time. The cosmological constant Λ may be dropped in what follows, as it can be naturally introduced later as an extra fluid component ¹. The curvature is $K = +k/S^2$ for k a constant (normalized to -1, 0 or +1). Using the definition of *Hubble's parameter*,

$$H = \left(\frac{\dot{S}}{S}\right), \quad (2.5)$$

¹ $\kappa\mu = \Lambda = -\kappa p$ (we have used $\kappa = 1$) that is $\gamma = 0$

the field equations reduce to

$$3\dot{H} + 3H^2 + \frac{1}{2}(\mu + 3p) = 0, \quad (2.6)$$

$$\dot{\mu} + (\mu + p)3H = 0, \quad (2.7)$$

$$3H^2 - \mu = -3K. \quad (2.8)$$

The physics associated with a real fluid is completely specified by giving $\gamma = \gamma(\mu)$; as $\mu = \mu(S)$ this means that $\gamma = \gamma(S)$, where γ relates to the pressure and energy density of fluid in the usual way by

$$p = (\gamma - 1)\mu. \quad (2.9)$$

The most natural way of proceeding from here is to follow Jones [12] *i.e.* to investigate the $(H(t), \mu(t))$ plane. One can eliminate the explicit time dependence, resulting in autonomous equations, as required to produce phase planes, by dividing (2.6) by (2.7) and using (2.9) to get

$$\frac{\dot{H}}{\dot{\mu}} = \frac{dH}{d\mu} = \left(\frac{H^2 + \frac{\mu}{6}(3\gamma - 2)}{3H\gamma\mu} \right). \quad (2.10)$$

Once $\gamma = \gamma(\mu)$ has been stated the planes may be plotted. It is, however, useful to consider the evolution curves in terms of observationally more accessible variables, such as the density, deceleration and Hubble parameters.

The *density parameter* may be defined:

$$\Omega = \left(\frac{\mu}{3H^2} \right). \quad (2.11)$$

The relationship between curvature and density is obtained from the Friedmann equation (2.8):

$$K = H^2(\Omega - 1), \quad (2.12)$$

or equivalently

$$k = H^2 S^2 (\Omega - 1). \quad (2.13)$$

The *deceleration parameter* is

$$q = - \left(\frac{\ddot{S}}{S} \right) \frac{1}{H^2}. \quad (2.14)$$

From Raychaudhuri's equation (2.6) using (2.14) it can be shown that

$$q = \left(\frac{1}{2} \right) \Omega (3\gamma - 2). \quad (2.15)$$

Notice that $\gamma = \frac{2}{3}$ is a critical value separating the decelerating periods in the universe ($q > 0$) given by $\gamma > \frac{2}{3}$ from the accelerating periods ($q < 0$) given by $\gamma < \frac{2}{3}$. The universe is considered to be inflationary when $q < 0$. This enables the “horizons”² to grow large relative to the visible region of the universe.

We can follow Madsen et al [10] and find the time derivative of the density parameter (2.11) using (2.6) and (2.7) to obtain

$$\dot{\Omega} = H\Omega(1 - \Omega)(2 - 3\gamma). \quad (2.21)$$

Using (2.5),

$$\dot{S} = HS, \quad (2.22)$$

one obtains the (Ω, S) plane equation

$$\left(\frac{d\Omega}{dS}\right) = \left(\frac{(2 - 3\gamma)(1 - \Omega)\Omega}{S}\right). \quad (2.23)$$

The physics is determined by $\gamma = \gamma(S)$ and S can be seen to behave like a conformal time variable. Notice that $\Omega = 0, 1$ are solutions and that for $\gamma(S) = \frac{2}{3}$, $\Omega = \Omega_0 = \text{constant}$ is allowed for all Ω_0 . For $\Omega > 1$ the curves $\Omega(S)$ increase monotonically, while for $\Omega < 1$ they decrease, and $\Omega = 1$ is the separatrix between the increasing and decreasing curves.

Using (2.14) and (2.15), (2.5) becomes

$$\dot{H} = -H^2(q + 1) = -H^2\left(\frac{1}{2}\Omega(3\gamma - 2) + 1\right), \quad (2.24)$$

²Horizons Perhaps one would also like to include the proper time, the particle and event horizons on the phase planes. I would suggest doing this by using the definition of the proper time

$$\tau = \int_0^{S(\tau)} \frac{dS}{S}, \quad (2.16)$$

and the particle and event horizons, respectively :

$$\lambda_p = \int_0^{\tau} \frac{dt}{S(t)}, \quad (2.17)$$

$$\lambda_e = \int_{\tau}^{\infty} \frac{dt}{S(t)}. \quad (2.18)$$

The extra equations thus provided are:

$$\frac{d\lambda_p}{dS} = \frac{1}{HS^2}, \quad (2.19)$$

$$\frac{d\tau}{dS} = \frac{1}{HS}, \quad (2.20)$$

which could be solved numerically along with the usual phase-plane equations.

while using (2.21) the (Ω, H) plane equation is obtained:

$$\left(\frac{d\Omega}{dH}\right) = \left(\frac{(2-3\gamma)(1-\Omega)\Omega}{H(1/2\Omega(2-3\gamma)-1)}\right). \quad (2.25)$$

This then gives the (H, Ω) plane, which is the plane of interest with regard to the observational plane of Gott et al [13],

$$\left(\frac{dH}{d\Omega}\right) = \left(\frac{H}{\Omega}\right) \left(\frac{1}{2} \left(\frac{\Omega}{1-\Omega}\right) - \frac{1}{(1-\Omega)(2-3\gamma)}\right). \quad (2.26)$$

We also have from (2.24) and (2.22) that

$$\left(\frac{dH}{dS}\right) = -\left(\frac{H}{S}\right) \left(\frac{1}{2}\Omega(3\gamma-2)+1\right). \quad (2.27)$$

It is useful to transform the infinities of Ω , S , q and H into finite values so as to provide a more transparent representation of the evolution near the singularities at infinite parameter values. In this way we end up treating S as a sort of conformal time parameter. In the transformed³ coordinates, $\Omega = \infty$ becomes the boundary between the expanding and the contracting regions of the universe.

One can also include the \dot{K} equation by noticing that $\dot{K} = 2H\dot{H}(\Omega-1) + 2H^2\dot{\Omega}$.

2.2 Single fluids with constant γ

For a single fluid with $\gamma = \text{constant}$, one can obtain 2D phase planes, which allow easy visualization of the evolution curves of various initial conditions. Primarily they provide a useful access to back propagating experimental data from our current time to obtain the universes initial/final conditions. Unfortunately the current observational values are unclear, hence one should look at the back propagation of a solution volume rather than single points. The basic equation set used is :

$$\dot{\Omega} = H\Omega(1-\Omega)(2-3\gamma), \quad (2.31)$$

³The transformations used are $X(x) = \exp(\tan(x))$ so that the transformed variable is given in terms of the untransformed variable by $x = \arctan(\ln(X))$. To transform the derivatives, consider, for example, $Y(y) = \exp(\tan(y))$ and $X(x) = \exp(\tan(x))$, one finds respectively that

$$dY = \exp(\tan y)(1 + \tan^2 y)dy \quad (2.28)$$

$$dX = \exp(\tan x)(1 + \tan^2 x)dx \quad (2.29)$$

such that

$$\frac{dY}{dX} = F(X, Y) \longrightarrow \frac{dy}{dx} = \exp(\tan x - \tan y) \left(\frac{1 + \tan^2 x}{1 + \tan^2 y}\right) F(X(x), Y(y)). \quad (2.30)$$

$$\dot{\gamma} = 0 \quad (\Rightarrow \mu = \frac{M}{S^{3\gamma}}), \quad (2.32)$$

$$\dot{S} = HS, \quad (2.33)$$

$$\dot{H} = -H^2 \left(\frac{1}{2} \Omega (3\gamma - 2) + 1 \right), \quad (2.34)$$

$$\dot{q} = \frac{1}{2} \dot{\Omega} (3\gamma - 2). \quad (2.35)$$

There are two independent variables for single fluids.

2.2.1 (Ω, S) plane

The defining phase plane equation is (2.23):

$$\left(\frac{d\Omega}{dS} \right) = \left(\frac{(2 - 3\gamma)(1 - \Omega)\Omega}{S} \right), \quad (2.36)$$

Here there are clearly critical curves at $\Omega = 1, 0$ and $\gamma = 2/3$. This plane uniquely defines the evolution of the field equations for single fluids and effectively single fluids. In the transformed coordinates $\Omega = \exp(\tan(\omega))$ and $S = \exp(\tan(s))$ the (Ω, S) plane becomes

$$\left(\frac{d\omega}{ds} \right) = \frac{(1 + \tan^2(s))}{(1 + \tan^2(\omega))} (2 - 3\gamma)(1 - e^{\tan(\omega)}). \quad (2.37)$$

2.2.2 (Ω, H) plane

The defining equation is from (2.25),

$$\left(\frac{d\Omega}{dH} \right) = \left(\frac{(2 - 3\gamma)(1 - \Omega)\Omega}{H \left(\frac{1}{2} \Omega (2 - 3\gamma) - 1 \right)} \right). \quad (2.38)$$

This has the usual critical sets at $\gamma = 2/3$ and $\Omega = 1, 0$ with an additional critical point at $\frac{1}{2} \Omega (2 - 3\gamma) = 1$, that is at $q = -1$. This equation has solutions from (2.26),

$$H = C_0 (1 - \Omega)^{\left(\frac{3}{2} \frac{\gamma}{(3\gamma - 2)} \right)} e^{\frac{1}{2}(\Omega - 1)}. \quad (2.39)$$

Here C_0 is a constant determined by the initial conditions. In the transformed coordinates $\Omega = \exp(\tan(\omega))$ and $H = \exp(\tan(h))$ and the (H, Ω) plane becomes

$$\left(\frac{dh}{d\omega} \right) = \frac{(1 + \tan^2(\omega))}{(1 + \tan^2(h))} \left(\frac{1}{2} \left(\frac{e^{\tan(\omega)}}{(1 - e^{\tan(\omega)})} \right) - \left(\frac{1}{(1 - e^{\tan(\omega)})(2 - 3\gamma)} \right) \right). \quad (2.40)$$

The plane corresponding to the Gott, Gunn, Schramm and Tinsley [13] (H, ω) plane for perfect single fluids is given by

$$\left(\frac{dH}{d\omega} \right) = H \left(\frac{1}{2} \left(\frac{e^\omega}{1 - e^\omega} \right) - \frac{1}{(1 - e^\omega)(2 - 3\gamma)} \right). \quad (2.41)$$

Another interesting quirk about these planes is that for the choice of a false vacuum, *i.e.* $\gamma = 0$, the singularity at $\Omega = 1$ vanishes from the phase plane, this can be seen in the plane equation above (2.38). For small or almost zero choices of γ , this singularity would still exist. The (Ω, H, q) is redundant in the case of a single fluid with $\gamma = \text{constant}$, in so much as it can be fully represented by the (Ω, H) plane; this can be easily seen as $q(S) = \frac{1}{2}\Omega(S)(3\gamma - 2)$.

2.2.3 (H, μ) plane

These are the least complex planes to get at, in terms of analytical or numerical studies, but are difficult to use in relation to experimental data, and the evolution of such data to earlier and later times. The plane is defined from (2.10) by:

$$\frac{d\mu}{dH} = \frac{3H\gamma\mu}{H^2 + \frac{\kappa}{6}\mu(3\gamma - 2)}. \quad (2.42)$$

2.3 Multi component noninteracting adiabatic fluids

Using the single fluid treatment as the basis of the approach we can now consider extending the treatment to that of many noninteracting adiabatic fluids. That is, assume that instead of a single γ constant for a given epoch, corresponding to a simple adiabatic matter-field, we have the situation where the universe evolved when the energy-momentum content is the sum of simple fluids, each characterized by a specific γ_i , constant in a given epoch. The field equations (2.6) - (2.8) then retain their form but now we have the situation where the $\gamma = \gamma(S)$ or $\gamma = \gamma(\mu)$. Assume that the individual fluid energy densities μ_i and pressures p_i are related by

$$p_i = (\gamma_i - 1)\mu_i \quad (2.43)$$

for $\gamma_i = \text{constant}$. As the fluids are non-interacting, each component will obey a separate conservation equation,

$$\dot{\mu}_i + 3H(\mu_i + p_i) = 0. \quad (2.44)$$

This gives $\mu_i = M_i/S^{3\gamma_i}$ for M_i is representative of the “amount of matter” (Section 2.4). (2.9) still defines γ giving the total energy density μ and pressure p respectively as :

$$\mu = \sum_i \mu_i \quad (= \sum_i \frac{M_i}{S^{3\gamma_i}}), \quad (2.45)$$

$$p = \sum_i p_i \quad (= \sum_i (\gamma_i - 1) \frac{M_i}{S^{3\gamma_i}}). \quad (2.46)$$

Then it can be shown by summing over all the components in equation (2.43) and using (2.9), (2.45) and (2.46), that

$$\gamma\mu = \sum_i \gamma_i \mu_i \quad (= \sum_i \gamma_i \frac{M_i}{S^3 \gamma_i}). \quad (2.47)$$

This is important as it consistently allows the total energy momentum conservation equation to hold, *i.e.* from (2.44) by summing (2.7) is shown to be valid. This also gives us the functional form of γ for various physical conditions. Using (2.47) and $\dot{\mu} = -3H \sum_i \gamma_i \mu_i$ (that is equation (2.44)) we obtain the time derivative of γ :

$$\dot{\gamma} = 3H \left(\gamma^2 - \sum_i \frac{\gamma_i^2 \mu_i}{\mu} \right). \quad (2.48)$$

This is not necessary for the planes as we can substitute the functional form of γ into the evolution equations using (2.47) but it is a useful equation to see explicitly. We could then write down the (H, μ_i) phase plane equations as before, to get

$$\frac{\dot{H}}{\dot{\mu}_i} = \frac{dH}{d\mu_i} = \frac{3H}{3\gamma_i \mu_i} + \left(\frac{\kappa}{18} \right) \left(\frac{\mu(3\gamma - 2)}{H\gamma_i \mu_i} \right). \quad (2.49)$$

These can be used to plot the total energy density plane (H, μ) , using the definition of the total energy density and total $\gamma(\mu_i)$.

From the single component density parameters in natural units,

$$\Omega_i = \left(\frac{\mu_i}{3H^2} \right), \quad (2.50)$$

we can consistently define the total density parameter using (2.45)

$$\Omega = \sum_i \Omega_i. \quad (2.51)$$

Then by recalling how (2.21) was derived, using (2.50) and substituting in (2.6) for \dot{H} and (2.44) for $\dot{\mu}_i$ we obtain

$$\dot{\Omega}_i = (2 - 3\gamma)H\Omega_i \left(\left(\frac{2 - 3\gamma_i}{2 - 3\gamma} \right) - \Omega \right). \quad (2.52)$$

The individual $\dot{\Omega}_i$'s are not independent of other fluid density parameters as they are connected by their dependence on the Hubble parameter. This dependence is carried by the coupling with γ . By dividing (2.50) by (2.11) we get

$$\frac{\mu_i}{\mu} = \frac{\Omega_i}{\Omega} \quad (2.53)$$

and using this with (2.47) we have that

$$\gamma\Omega = \sum_i \gamma_i \Omega_i. \quad (2.54)$$

By summing over all the components (2.52) using (2.54) we get (2.21) ensuring consistency, as before, and giving us γ as a function of the individual Ω_i 's from the Raychaudhuri equation (2.6) and the definition of the deceleration parameter,

$$q = -\frac{1}{2}(2\Omega - 3 \sum_i \gamma_i \Omega_i). \quad (2.55)$$

This can be seen to be consistent with (2.15) by using (2.54).

We are now able to include the (Ω_i, S) and (Ω, γ) planes in the analysis. That is, using (2.52) and (2.5) we obtain

$$\left(\frac{d\Omega_i}{dS}\right) = (2 - 3\gamma) \frac{\Omega_i}{S} \left(\frac{(2 - 3\gamma_i)}{(2 - 3\gamma)} - \Omega\right). \quad (2.56)$$

We now have $N+4$ variables $(\Omega(S), H(S), \gamma(S), q(S), \Omega_i(S))$ and N constants (γ_i) corresponding to the N fluid components. Of these we need only solve equations (2.56) to get unique solutions in terms of the scale factor. We have the functional relations to obtain Ω , q and γ in terms of the Ω_i 's. A single curve in the resulting (Ω, S) plane would represent the implications of a set of possible individual present density parameters in a non-unique manner but the plane would be structurally stable in that $\gamma = \gamma(S, M_i, \gamma_i)$ would generate a (Ω, S) plane that has the singularity structure of the full $(\Omega_1, \dots, \Omega_N)$ plane. Ehlers and Rindler investigate this set of equations and the hyper-plane defined by them as $(\Omega, \omega, \lambda)$ (that is the $(\Omega_m, \Omega_r, \Omega_\Lambda)$ plane) for 3 fluid models (actually effectively 2 fluids as one of their fluids is a cosmological constant/false-vacuum, which has an energy density that is constant for all time within the evolving universe). We can transform all the many component variables as with the single component variables, to bring the infinities into finite values on the plots. In the transformed coordinates, we have :

$$\left(\frac{d\omega}{ds}\right) = \frac{(1 + \tan^2(s))}{(1 + \tan^2(\omega))} (2 - 3\gamma)(1 - e^{\tan(\omega)}), \quad (2.57)$$

$$\left(\frac{dh}{ds}\right) = \frac{(1 + \tan^2(\omega))}{(1 + \tan^2(h))} \left(\frac{1}{2} e^{\tan(\omega)} (3\gamma - 2) + 1\right), \quad (2.58)$$

$$\left(\frac{d\omega_i}{ds}\right) = (2 - 3\gamma) \frac{(1 + \tan^2(s))}{(1 + \tan^2(\omega_i))} \left(\frac{(2 - 3\gamma_i)}{(2 - 3\gamma)} - e^{\tan(\omega)}\right). \quad (2.59)$$

The last equation defines the unique phase space for the evolution of this class of models once we include (2.54).

2.4 $\gamma = \gamma(S, \mu_{0i}, \gamma_i)$ or $\gamma = \gamma(S, M_i, \gamma_i)$

At some time $t = t_0$, say the parameters Ω, S, q and H take the values Ω_0, S_0, q_0 and H_0 respectively. One can integrate (2.7) or rather (2.44) by using (2.5) and (2.43) to find that

$$\mu_i \propto S^{-3\gamma_i}. \quad (2.60)$$

This is equivalent to using the first law of thermodynamics, that is $d(\mu S^3) = -pd(S^3)$ or equivalently $d[S^3\gamma] = S^2 dp$. Using the current value of the density of the individual components and the scale parameter we have that :

$$\mu_i = \mu_{0i} \left(\frac{S}{S_0} \right)^{-3\gamma_i} \quad (2.61)$$

Using (2.47)

$$\gamma(S) = \sum_i \left(\frac{S}{S_0} \right)^{-3\gamma_i} \left(\frac{\mu_{0i}\gamma_i}{\mu} \right), \quad (2.62)$$

and using (2.45) while defining the “amount of matter” now to be

$$M_i = \left(\frac{\mu_i}{S^{-3\gamma_i}} \right)_{t=t_0}, \quad (2.63)$$

we find

$$\gamma(S) = \frac{\sum_i (M_i S^{-3\gamma_i} \gamma_i)}{\sum_i (M_i S^{-3\gamma_i})} = \langle \gamma \rangle. \quad (2.64)$$

We may also write M_i in terms of the present day values of the density parameter Ω using $y = S/S_0$; we have from (2.63) and (2.50) that

$$\gamma(y) = \frac{\sum_i (\Omega_{0i} y^{-3\gamma_i} \gamma_i)}{\sum_i (\Omega_{0i} y^{-3\gamma_i})}. \quad (2.65)$$

The density parameter can also be found in these traditional variables. This is easily done by evaluating the Friedmann equation in the form (2.13) at time $t = t_0$. This fixes k , dividing through by S^2 to get

$$K(S) = \frac{H_0^2(\Omega_0 - 1)}{(S/S_0)^2} = \frac{H_0^2(\Omega_0 - 1)}{y^2}. \quad (2.66)$$

Using this with (2.8 x $\frac{1}{\kappa\mu}$), (2.11) and (2.45), we find that :

$$\Omega(S) = \left(1 - \frac{(\Omega_0 - 1)}{\sum_i y^{2-3\gamma_i} \Omega_{0i}} \right)^{-1}, \quad (2.67)$$

where Ω_0 is the present day total density parameter. Similarly we can find the form for the acceleration parameter from (2.55) or equivalently (2.15) :

$$q(S) = \left(\frac{1}{2} \right) \Omega(S) (3\gamma(S) - 2), \quad (2.68)$$

using $\gamma(S)$ and $\Omega(S)$ from above.

2.5 Discussion on Phase-plane equations

It would appear that in the general case of multi component noninteracting fluids one cannot look at the single phase planes in isolation, specifically the planes including total Ω , γ or μ , as the individual components of the respective variables interact via γ . This is the gravitational interaction between the respective energy densities in the specific space-time. This makes a 2D phase plane representation rather tricky, and suggests that higher dimensional treatments would be more helpful, but this brings in the problem of representation. The higher dimensional phase hyper-planes can easily be found numerically but the question of how to obtain a useful representation of the resulting curves is very much a problem of “display-visualization” rather than physics or mathematics. Looking at the (Ω, S) evolution planes by writing down the explicit form of $\gamma(S)$ in terms of the current constant values of Ω for the individual fluid elements and then inserting this into the total density parameter equation and numerically grinding out the resulting plane, has been suggested. It would seem that this approach [10] may be misleading as one can only uniquely look at the (Ω, S) , (Ω, q) or (H, μ) planes for single fluids. One can only use the functional form of $\gamma(S)$ as given in section (2.4) for a *single* evolution curve in the (Ω, S) plane. That is, this γ should not be used to discuss the entire plane as it cannot be done in a generic sense with regard to the plane behavior, as one is projecting arbitrarily chosen trajectories in the full phase space down into some sub-manifold. This could lead to insights with regard to the generic evolution of a universe as represented by the entire set of defining equations without the exact knowledge of current parameters, if one is only concerned with the phase-plane behaviour near critical points, *i.e.* if one wished to use such projected multi-fluid planes under the assumption that the critical point behaviour is intrinsically the same (one should be very careful when using the resulting planes to comment on measures, for instance in the sense of the measure problem). Furthermore, if forms of the energy momentum tensor other than fluid models are used one cannot ensure the good behaviour of the critical points or the number of critical points (as an example). We can only work from the individual density parameters to the total, not the other way around in a fully general treatment, although for adiabatic fluid models it can be argued that uniqueness is not an important issue as the basic structure of the phase planes is given by the behaviour of the critical points. Hence we may map down from the higher dimensional planes without loss of generality in the sense suggested by [10].

The point here is that the invariant sub-plane of Ehlers et al would be the ideal plane to use with respect to the observational data as it is general, but in practice this is a

complicated plane to present. Hence it seems that it is helpful to then deal with the projected $(\Omega_1, \dots, \Omega_N)$ planes using $\gamma = \gamma(S, M_i, \gamma_i)$, this would generate a structurally stable representation of the full phase space that would faithfully reproduce in a (Ω, S) plane the full $(\Omega_1, \dots, \Omega_N)$ planes generic behaviour. The (Ω, S) plane found in this way would be structurally stable and generic in the sense that the critical point structure of the full evolution space would be retained and unchanged under the resulting mapping generated by using $\gamma = \gamma(S, M_i, \gamma_i)$. This is in fact what Madsen et al [10] have done.

Another possible way in which to construct a faithful projection of the full invariant phase space $(\Omega_1, \dots, \Omega_N)$ could be to look at N noninteracting fluids in some large N limit. This may be a useful approximation in the sense of ensuring that the projection down is itself unique as well as generic in the sense of preserving the critical point structure. In such an instance it may still be possible to discuss the reduced measure with some credibility if the N -fluid distribution function was well motivated physically; one may be able to write the weighted sum over individual density parameters in (2.54) as some integral. The essential point is that to discuss a current solution volume one cannot deal uniquely with sub-manifolds of the entire phase space without careful justification (such as the additional knowledge of current parameter values), particularly with regard to the motivation for the investigation of the planes in the first place. It is noted that to avoid such subtleties the $(\Omega_0, \Omega_1, \dots, \Omega_N)$ phase hyper-planes are the most appropriate planes to seriously investigate, in much the same way as [9] discuss the $(\Omega_m, \Omega_r, \Omega_\Lambda)$ hyper planes.

The defining equations for multi-fluid models are:

$$\dot{\Omega}_i = H\Omega_i(1 - \Omega)((2 - 3\gamma_i) - \Omega(2 - 3\gamma)), \quad (2.69)$$

$$\dot{\gamma} = 3H \left(\gamma^2 - \frac{\sum_i \gamma_i^2 \Omega_i}{\Omega} \right), \quad (2.70)$$

$$\dot{S} = HS, \quad (2.71)$$

$$\dot{H} = -H^2 \left(\frac{1}{2} \Omega (3\gamma - 2) + 1 \right), \quad (2.72)$$

$$\dot{q} = \frac{1}{2} (\dot{\Omega} (3\gamma - 2) + 3\Omega \dot{\gamma}), \quad (2.73)$$

$$\dot{\Omega} = H\Omega(1 - \Omega)(2 - 3\gamma). \quad (2.74)$$

2.5.1 $(\Omega_1, \Omega_2, \dots, S)$ hyper-plane

This plane is defined by :

$$\left(\frac{d\Omega_i}{dS}\right) = (2 - 3\gamma) \frac{\Omega_i}{S} \left(\frac{(2 - 3\gamma_i)}{(2 - 3\gamma)} - \Omega\right), \quad (2.75)$$

Here we use:

$$\gamma = \sum_i \frac{\gamma_i \Omega_i}{\Omega}, \quad (2.76)$$

$$\Omega = \sum_i \Omega_i. \quad (2.77)$$

It should be clear that this cannot be uniquely mapped into the (Ω, S) plane unless one either chooses the trivial case of only a single fluid, for example (2.76), or one is able in some large N limit to turn the sum in equation (2.76) into an integral, or some parameter values are well known. An additional constraint is then given that can be used to find an approximation or refinement of our choice of γ . One could equivalently look at this plane with any two density parameters replaced by the total density Ω and the total acceleration parameter q .

2.5.2 $(\Omega_1, \Omega_2, \dots, q)$ hyper-plane

The plane is defined by the N differential equations :

$$\frac{dq}{d\Omega_i} = \frac{2(q^2(\Omega - 1) + (q + \Omega)^2) - \frac{9}{2}\Omega \sum_i \gamma_i^2 \Omega_i}{\Omega \Omega_i (2 - 3\gamma_i + 2q)}, \quad (2.78)$$

where the usual definition for γ and Ω is used, as in the proceeding section. The above equation is obtained by first finding (dq/dS) and $(dS/d\Omega_i)$, *i.e.* by eliminating the dependence on H , dividing these (to eliminate the explicit S dependence) and substituting it for the form of q using the usual definition of $q = \frac{1}{2}\Omega(3\gamma - 2)$. It should be trivially noted that (Ω, q) cannot be found in terms of the total density parameter alone, as $\gamma = \gamma(\Omega_0(S), \Omega_1(S), \dots, \Omega_N(S))$ and not $\gamma(\Omega(S))$ *i.e.* $\gamma = \gamma(S, M_i, \gamma_i) \Rightarrow q = q(\Omega, M_i)$. This can be seen in the $\sum_i \gamma_i^2 \Omega_i$ term, which cannot be written in terms of Ω and γ alone unless some sort of approximation is considered. If this were the case one could easily write γ in terms of q and Ω and find the (Ω, q) plane. The plane equation giving the (Ω, q) sub-manifold slice of the complete phase space is :

$$\frac{dq}{d\Omega} = \frac{q}{\Omega} + \frac{1}{q(\Omega - 1)} \left(\frac{(q + \Omega)^2}{\Omega} - \frac{9}{4} \sum_i \gamma_i^2 \Omega_i \right), \quad (2.79)$$

where the sum $\sum \gamma_i^2 \Omega_i$ cannot be written trivially in terms of q and Ω . This is used later when the effectively single fluid case is discussed. These planes are redundant in the sense that $(\Omega_1, \dots, \Omega_{N-2}, \Omega, q)$ can be mapped into $(\Omega_1, \dots, \Omega_N)$.

2.5.3 $(\Omega_1, \Omega_2, \dots, H)$ hyper-plane

There is no generic (Ω, H) plane in the sense of the observational plane given by Gott, Gunn, Schramm and Tinsley [13], unless some sort of projection is constructed onto a 2-D plane (such as either dealing with the structurally stable (Ω, S) planes or some other projection) with which to make the comparison with the model dependent observational plane. The $(\Omega_1, \Omega_2, \dots, H)$ plane can in principle be investigated, but comparison to data would require additional constraints rather than merely the total density limits. If such limits were known the issue with this plane would again be reduced to the problem of representation. The plane equations are:

$$\frac{d\Omega_i}{dH} = \left(\frac{1 - \Omega}{H} \right) \left(\frac{2\Omega_i(2 - 3\gamma_i) - 2\Omega_i\Omega(2 - 3\gamma)}{\Omega(2 - 3\gamma) - 2} \right), \quad (2.80)$$

where we once again use

$$\gamma = \sum_i \frac{\gamma_i \Omega_i}{\Omega}, \quad (2.81)$$

$$\Omega = \sum_i \Omega_i. \quad (2.82)$$

2.5.4 $(\Omega_1, \Omega_2, \dots, \Omega_N)$ hyper-plane

We can consider $(d\Omega_i/dS)$ and $(d\Omega_j/dS)$ equations (2.56), and divide. The explicit S dependence may be eliminated, *i.e.* to find an autonomous form in terms of the conformal time parameter. We are then lead to the equations that describe the $(\Omega_1, \dots, \Omega_N)$ hyper-plane,

$$\frac{d\Omega_i}{d\Omega_j} = \frac{\Omega_i(2 - 3\gamma_i) - \Omega_i\Omega(2 - 3\gamma)}{\Omega_j(2 - 3\gamma_j) - \Omega_j\Omega(2 - 3\gamma)}, \quad (2.83)$$

where we use

$$\gamma = \sum_i \frac{\gamma_i \Omega_i}{\Omega}, \quad (2.84)$$

$$\Omega = \sum_i \Omega_i. \quad (2.85)$$

We can uniquely describe the behavior of the evolution curves of the universe for N non-interacting fluids by considering this plane. This has been done by Ehlers and Rindler

[9] for the 2 fluid + false vacuum case. Here it is generalized, we need only know the individual density parameter values at the current time to pin down the general evolution of a multi fluid universe.

H_0 and q_0 appear to be unnecessary with regard to fitting data to theory for multi fluid noninteracting universe evolution as the invariant sub-plane is given by $(\Omega_1, \Omega_2, \dots, \Omega_N)$. They are, however, useful in that H_0 would give the evolution of the energy densities if known well, and q_0 may provide a useful check through equation (2.15), of the way we choose the γ_i and number of fluid components in the sense of an extra constraint. q_0 is also an indication of whether the universe is open or closed, but can be used to find the bounds on a missing density parameter from the total density parameter.

One can eliminate the explicit dependence of the planes on any two single density parameters by replacing any (Ω_k, Ω_l) by (Ω, q) in the $(\Omega_1, \Omega_2, \Omega_3, \dots, \Omega_N)$ plane, *i.e.* we may transform

$$(\Omega_1, \Omega_2, \Omega_3, \dots, \Omega_N) \longrightarrow (\Omega_1, \Omega_2, \dots, \Omega_{k-1}, \Omega, \Omega_{k+1}, \dots, \Omega_{l-1}, q, \Omega_{l+1}, \dots, \Omega_N) \quad (2.86)$$

in a well defined manner.

General mapping from (Ω_k, Ω_l) sub-plane to (Ω, q) sub-plane

Using the definition of the total density parameter, that is $\Omega = \sum_{i=1}^N \Omega_i$ we have that,

$$\Omega_k = \Omega - \sum_{i=1, i \neq k}^N \Omega_i. \quad (2.87)$$

From the definition of the acceleration parameter,

$$q = \frac{1}{2}\Omega(3\gamma - 2) = \frac{1}{2}\Omega\left(3\sum_{i=1}^N \frac{\gamma_i \Omega_i}{\Omega} - 2\right) = \frac{1}{2}\sum_i \Omega_i(3\gamma_i - 2). \quad (2.88)$$

Clearly from this,

$$\frac{1}{2}\Omega_l(3\gamma_l - 2) = q - \frac{1}{2}\sum_{i \neq l, k} (3\gamma_i - 2)\Omega_i - \frac{1}{2}\Omega_k(3\gamma_k - 2). \quad (2.89)$$

From (2.87) and (2.89) it can be shown that

$$\Omega_l = \frac{q - \frac{1}{2}\Omega(3\gamma_k - 2) - \frac{3}{2}\sum_{i \neq l, k} \Omega_i(\gamma_i - \gamma_k)}{\frac{3}{2}(\gamma_l - \gamma_k)}. \quad (2.90)$$

We have found $\Omega_l = \Omega_l(\Omega, q, \Omega_i)$ for all $i \neq l$ or k . Similarly from this and (2.87), $\Omega_k = \Omega_k(\Omega, q, \Omega_i)$ can be found for all $i \neq l$ or k

$$\Omega_k = \Omega - \sum_{i \neq l, k} \Omega_i - \Omega_l. \quad (2.91)$$

Now we may rewrite the phase plane equation for $(\Omega_1, \Omega_2, \dots, \Omega_N)$ in terms of the pair (Ω, q) , instead of (Ω_k, Ω_l) . *i.e.* (i) from (2.83) replacing γ with some function of q and Ω , one obtains the $(d\Omega_i/d\Omega_j)$ equation for all $i \neq l$ or k , (ii) using the equation for $\dot{\Omega}$ and $\dot{\Omega}_i$ one can obtain $(d\Omega/d\Omega_i)$ for all $i \neq l$ or k and (iii) equation (2.78), for all $i \neq l$ or k , gives the (q, Ω_i) sub-plane. Finally (iv) the (q, Ω) sub-plane is obtain from equation (2.79) using (2.90) and (2.91). From equations (i) - (iv) derived as described above we get

$$(\Omega_1, \Omega_2, \dots, \Omega_{k-1}, \Omega, \Omega_{k+1}, \dots, \Omega_{l-1}, q, \Omega_{l+1}, \dots, \Omega_N). \quad (2.92)$$

plane.

2.5.5 $(\mu_1, \mu_2, \dots, \mu_N, H)$ hyper-plane

These are given by

$$\frac{dH}{d\mu_i} = \frac{3H}{3\gamma_i\mu_i} + \left(\frac{\kappa}{18}\right) \left(\frac{\mu(3\gamma - 2)}{H\gamma_i\mu_i}\right) \quad (2.93)$$

but are rather difficult to relate to experimental data *i.e.* the limits on Ω_0, H_0 and q_0 .

2.5.6 $(\gamma, S), (\gamma, \Omega)$ and (γ, Ω_i) planes

The (γ, S) plane is described by the plane equation

$$\left(\frac{d\gamma}{dS}\right) = \left(\frac{3}{S}\right) \left(\gamma^2 - \sum_i \gamma_i^2 \frac{\Omega_i}{\Omega}\right). \quad (2.94)$$

We use the usual definition for Ω and γ . The transformed plane is given by

$$\left(\frac{d\gamma}{ds}\right) = 3(1 + \tan^2(s)) \left(\gamma^2 - \sum_i \gamma_i \frac{e^{\tan(\omega_i)}}{e^{\tan(\omega)}}\right). \quad (2.95)$$

The (γ, Ω) and (γ, Ω_i) planes are given respectively by :

$$\left(\frac{d\Omega_i}{d\gamma}\right) = \frac{(2 - 3\gamma)\Omega_i(3 - 3\gamma_i)/(2 - 3\gamma) - \Omega}{3(\gamma^2 - \sum_i \gamma_i^2 \Omega_i/\Omega)}, \quad (2.96)$$

$$\left(\frac{d\Omega}{d\gamma}\right) = \frac{\Omega^2(3 - 3\gamma_i)(1 - \Omega)(2 - 3\gamma)}{3(\gamma^2\Omega - \sum_i \gamma_i^2 \Omega_i)}. \quad (2.97)$$

2.5.7 $(\Omega_1, \Omega_2, \dots, \Omega_N, H, q)$ hyper-planes

This would be a difficult plane to investigate, particularly when all the behavior is explicitly in the $(\Omega_1, \Omega_2, \dots, \Omega_N)$ plane, but may be useful as additional limits from H_0 and q_0 may be included along with those of all the fluid component density parameters. Going to this length without careful motivation seems unnecessarily complex as there is some implicit redundancy as discussed previously.

2.5.8 Multi-valued nature of $\gamma(S) \rightarrow \dot{\Omega}$

Consider defining the total density parameter in terms of the individual density parameter values. That is, consider the case of two possible $\gamma(S)$ functions say $\gamma^{(1)}(S)$ and $\gamma^{(2)}(S)$:

$$\gamma^{(1)}(S) = \frac{\sum_i (M_i^{(1)} S^{-3\gamma_i})}{\sum_i (M_i^{(1)} S^{-3\gamma_i})} = \frac{\sum_i (\Omega_{0i}^{(1)} y(S)^{-3\gamma_i})}{\sum_i (\Omega_{0i}^{(1)} y(S)^{-3\gamma_i})}, \quad (2.98)$$

$$\gamma^{(2)}(S) = \frac{\sum_i (M_i^{(2)} S^{-3\gamma_i})}{\sum_i (M_i^{(2)} S^{-3\gamma_i})} = \frac{\sum_i (\Omega_{0i}^{(2)} y(S)^{-3\gamma_i})}{\sum_i (\Omega_{0i}^{(2)} y(S)^{-3\gamma_i})}. \quad (2.99)$$

Here M_i and Ω_{0i} are defined as the usual current values of the ‘‘amount of matter’’ and the current density parameter values. We consider the case where

$$\sum_i M_i^{(1)} = \sum_i M_i^{(2)} \quad (2.100)$$

which is the case in, where the total ‘‘amount of matter’’ values coincide at the current time, some $t = t_0$. But we do not assume any restrictions in the manner in which these parameters are individually weighted. However the actual γ_i constants are predetermined for both cases. This is equivalent to the case of

$$\sum_i \Omega_{0i}^{(1)} = \sum_i \Omega_{0i}^{(2)} = \Omega_0. \quad (2.101)$$

If we wish to look at the (Ω, S) plane and claim that the plane can be generic and unique for curves in the phase plane, for fluid mixtures other than single fluid cases, then we would require that there be no multi-valued curves. That is, no two integral curves defined on the plane should intersect. Hence we consider the case in which a single total density parameter value is defined at the current time. Some (Ω_0, S_0) on the postulated (Ω, S) plane are chosen as the initial conditions for two integral curves defined by $\gamma^{(1)}$ and $\gamma^{(2)}$, as given above in equation (2.21).

That is, we have two sets of equations :

- for the $\Omega^{(1)}$ trajectory starting at some (Ω_0, S_0) :

$$\dot{\Omega}^{(1)} = H\Omega^{(1)}(1 - \Omega^{(1)})(2 - 3\gamma^{(1)}) \quad (2.102)$$

$$\dot{S} = HS \quad (2.103)$$

- for the $\Omega^{(2)}$ trajectory starting at some (Ω_0, S_0) :

$$\dot{\Omega}^{(2)} = H\Omega^{(2)}(1 - \Omega^{(2)})(2 - 3\gamma^{(2)}) \quad (2.104)$$

$$\dot{S} = HS. \quad (2.105)$$

This can be used to define

$$\frac{d\Omega^{(1)}}{d\Omega^{(2)}} = \frac{\Omega^{(1)}(1 - \Omega^{(1)})(2 - 3\gamma^{(1)})}{\Omega^{(2)}(1 - \Omega^{(2)})(2 - 3\gamma^{(2)})}. \quad (2.106)$$

Now if at some point for $\Omega_0 \neq 0, 1$ we have that $\Omega^{(1)} = \Omega^{(2)}$. We then have from equation (2.106) that at this point :

$$\frac{d\Omega^{(1)}}{d\Omega^{(2)}} = \frac{(2 - 3\gamma^{(1)})}{(2 - 3\gamma^{(2)})}. \quad (2.107)$$

If there is no intersection at some later or earlier time, for the initial condition (Ω_0, S_0) for the two orbits considered in the (Ω, S) plane, these trajectories are not unique, they intersect at the current values. If the curves do intersect at some later or earlier time, for the curves to be unique the trajectories must have the same derivatives, let's say at least up to second order. Then this gives us from (2.107) that

$$2 - 3\gamma^{(1)} = 2 - 3\gamma^{(2)}, \quad (2.108)$$

which implies that

$$\gamma^{(1)}(S) = \frac{\sum_i (M_i^{(1)} S^{-3\gamma_i} \gamma_i)}{\sum_i (M_i^{(1)} S^{-3\gamma_i})} = \frac{\sum_i (M_i^{(2)} S^{-3\gamma_i} \gamma_i)}{\sum_i (M_i^{(2)} S^{-3\gamma_i})} = \gamma^{(2)}(S). \quad (2.109)$$

This is an extra constraint that would allow the (Ω, S) plane for many fluid scenarios to make sense and is equivalent to defining some sort of effectively single fluid model. At any rate this can only be satisfied for arbitrary γ_i 's if all $M_i^{(1)} = M_i^{(2)}$. This is fairly obvious, but implies that we can only use such an approach, that is use $\gamma(S)$ as found in section (2.4), for a *single* unique trajectory. It cannot be used to define a (Ω, S) phase plane if one expects any sense or rigor with regard to the plane reflecting in a strong manner the evolution of particular universe, which is characterized by multi fluid scenarios. In a weak sense, one can claim that such phase planes are generic if one can ensure that new singularities do not arise in the reduced phase space under the mapping down from the full phase-space. That is, the number of critical points and their behaviour do not radically change under the mapping down into the sub-manifold of the full phase space, (the critical value is $\gamma = \frac{2}{3}$). What concerns me is that, for instance in the case of two fluids, the (Ω_1, Ω_2, S) plane is well defined, which implies that one could naturally map this plane into the (Ω, q, S) in an isomorphic sense. Can we then look at the (Ω, S) plane and claim coherence? The resulting (Ω, q) plane clearly has a very complex structure and behavior, which must in some sense be filtered out by the choice of $\gamma(S)$. That is, some subclass of the full set of trajectories is projected, and from these it is claimed that we understand

the full behavior of the phase space. This appears to be speculative, but is plausible in the weak sense as described above if one is only interested in the behaviour of the universe in the immediate neighborhood of critical points of perfect fluid models.

2.6 1-component fluid, $\gamma = \text{constant}$

For single fluid and effectively single fluid models there are basically two possible planes to be considered. These planes are the most widely investigated and discussed planes [10].

Clearly the greatest danger with a phase plane approach is that of producing graphic's for graphics sake, with this in mind I have tried not to go over board with regard to the production of phase planes. The planes I have included either highlight points I have made in the text or are novel in themselves. These planes can be easily reproduced on any numerical integration package from the equation sets described above.

Accelerating universe

The first single component case that can be numerically presented is for universes with $q < 0$, that is with the usual $\gamma < \frac{2}{3}$ which is the inflationary universe case, *i.e.* including the cosmological constant only case. This is the plane that is essentially used in a Hamiltonian FLRW formulation to discuss the measure problem (see figure 2.1).

Decelerating universe

The case for either radiation or matter is included in the case of decelerating universes, that is for $q > 0$ or equivalently $\gamma > \frac{2}{3}$. These can in principle be easily numerically solved, if they are of interest (see figure 2.2).

2.6.1 Effectively single fluid models with $\gamma_i = \text{constant}$

I will call a fluid model an effectively single fluid model if it satisfies condition (2.110), see equation (2.79).

$$\sum_i \gamma_i^2 \Omega_i = h(\Omega) \quad (2.110)$$

For example if we postulate a two fluid case with a density parameter that is constant for all time, that is $\Omega_2 = \alpha$ say, where $\alpha = \text{constant}$. Then equation (2.54) becomes

$$\gamma(\Omega_1) = \frac{\sum_{i=1}^2 \gamma_i \Omega_i(S)}{\sum_{i=1}^2 \Omega_i(S)} = \frac{\gamma_1 \Omega_1(S) + \gamma_2 \alpha}{\Omega_1(S) + \alpha}. \quad (2.111)$$

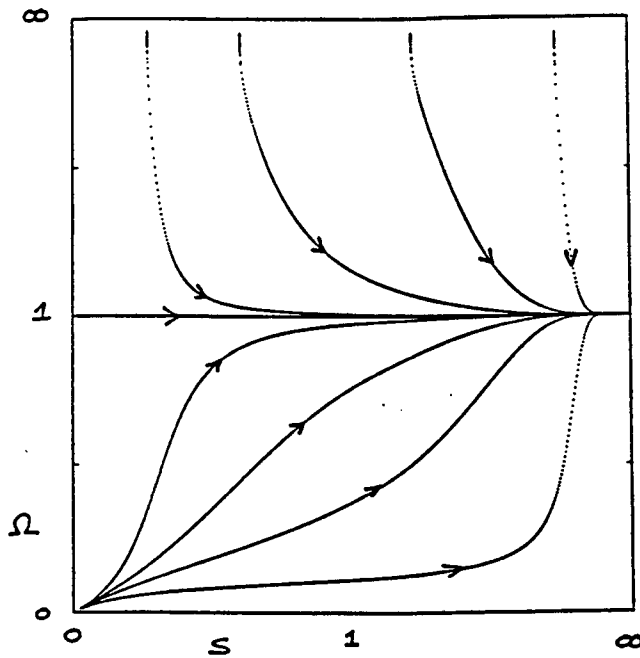


Figure 2.1: Transformed single fluid (Ω, S) Plane, $\gamma = 0$: This is the accelerating universe, that is an inflationary universe. The important point about these planes is that all the curves fall into $\Omega = 1$ after a sufficiently long time, but at almost all times one can find curves that are arbitrarily far from $\Omega = 1$. Ω is driven to one.

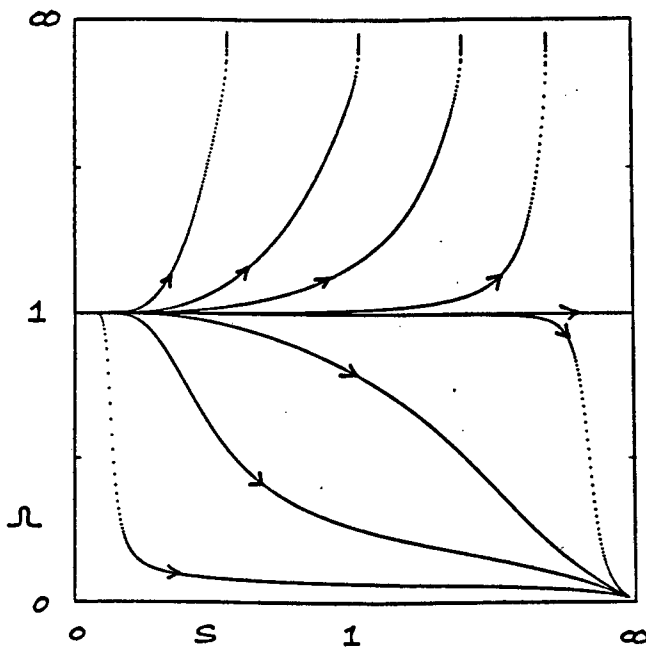


Figure 2.2: Transformed single fluid (Ω, S) Plane, $\gamma = 1$: The important point about these universes is that one can be arbitrarily close to $\Omega = 1$ for an arbitrarily long time, but that after a sufficiently long time Ω can be found arbitrarily far from $\Omega = 1$. Ω is driven from 1.

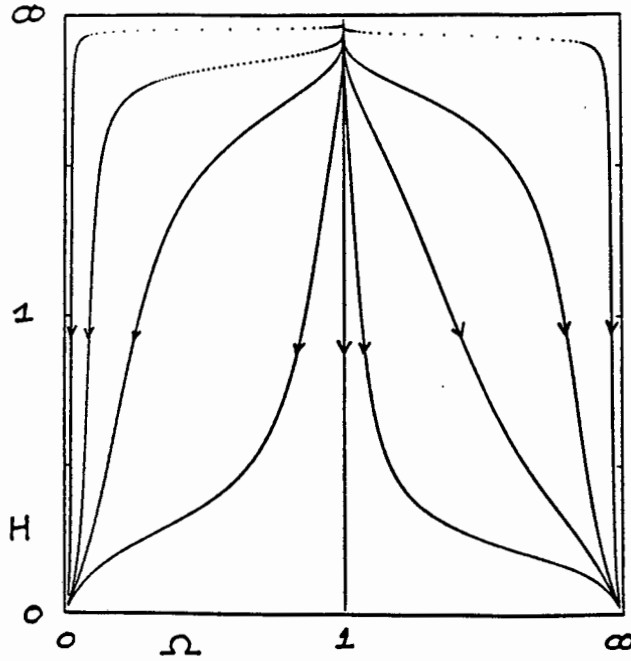


Figure 2.3: **Transformed** (Ω, H) , $\gamma = \frac{4}{3}$: Once again the $\Omega = 1$ separatrix is evident, dividing the open ($\Omega < 1$) from the closed ($\Omega > 1$) universes. The curves either fall into the turnover point, $(\Omega = \infty, H=0)$, *i.e.*, for the closed models, while the open models fall towards the $(0,0)$ point which is indicative of forever expanding models. This plane is generic for multifluid non-inflationary universes.

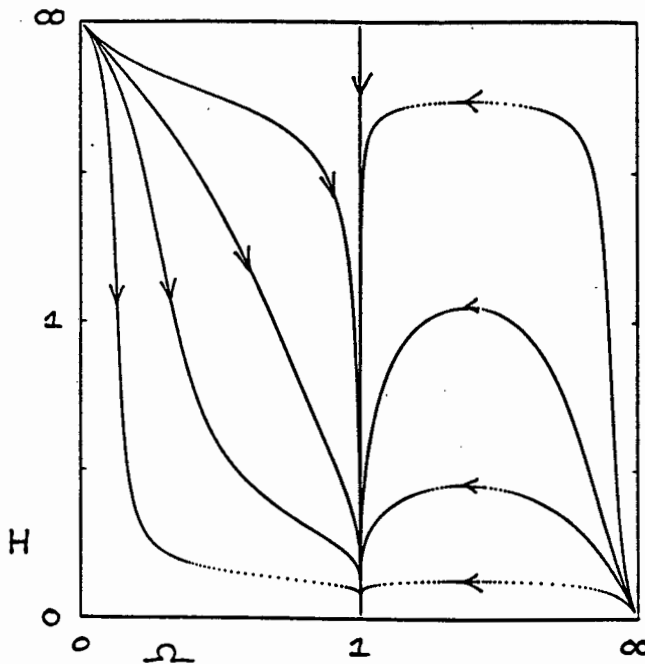


Figure 2.4: **Transformed** (Ω, H) , $\gamma = \frac{1}{3}$: This plane is for the case of an inflationary universe ($\gamma < \frac{2}{3}$). The curves all fall into the $(1,0)$ point after a sufficiently long time. One can see both the universes moving in from a turnover, *i.e.*, the case of non-zero energy density, but zero expansion rate; and the universes falling in from infinite expansion rate, H , but zero density parameter. All curves move towards the $\Omega = 1$ separatrix.

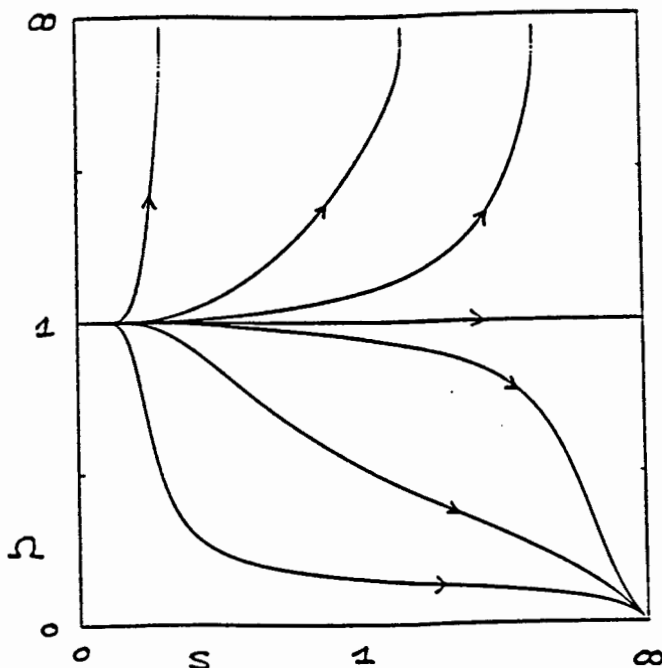


Figure 2.5: Generic Transformed (Ω, S) , $\gamma_1 = \frac{4}{3}$, $\gamma_2 = 1$: This is generic for $\gamma > \frac{2}{3}$, constructed using the form of $\gamma(S) = (\gamma_1 M_1 S^{-3\gamma_1} + \gamma_2 M_2 S^{-3\gamma_2}) / (M_1 S^{-3\gamma_1} + M_2 S^{-3\gamma_2})$, with $M_1 = M_2 = 0.1$. This demonstrates the evolution of radiation and dust FLRW models, notice that the curves fall away from $\Omega = 1$, but one may pick evolution curves arbitrarily close to the $\Omega = 1$ separatrix at almost all times in the models evolution.

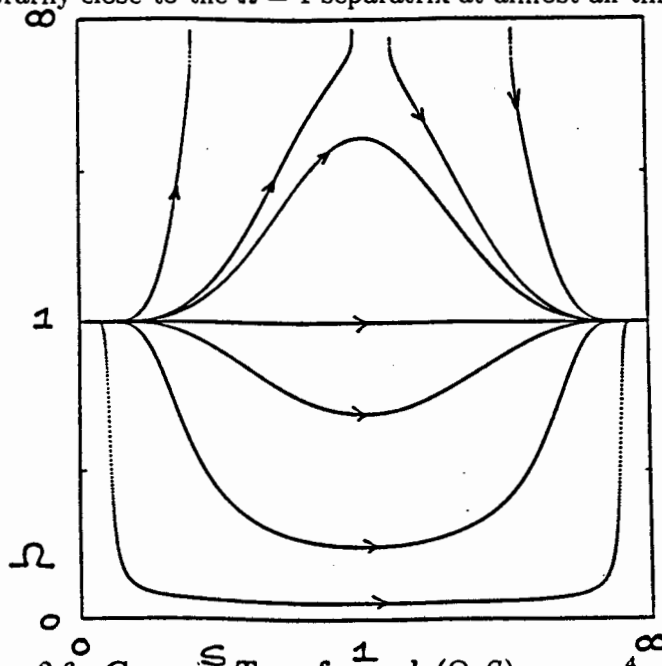


Figure 2.6: Generic Transformed (Ω, S) , $\gamma_1 = \frac{4}{3}$, $\gamma_2 = 0$: This is generic for a mixture of $\gamma < \frac{2}{3}$ and $\gamma > \frac{2}{3}$ FLRW models, notice that at early time the non-inflationary contribution (in this case radiation) dominates, the trajectories move away from the $\Omega = 1$ line, while at late times, the trajectories move back towards $\Omega = 1$, on such a plane it can see that one can move arbitrarily far from $\Omega = 1$ at almost all times while also being able to preserve the inflationary behaviour at late times, curves fall into $\Omega = 1$.

Here α is some arbitrary number and γ_2 is some number between 0 and 2. So there is only a dependence on Ω_1 , the first fluid's density parameter :

$$\sum_i \gamma_i^2 \Omega_i(S) = \gamma_1^2 \Omega(S) + \alpha(\gamma_2^2 - \gamma_1^2). \quad (2.112)$$

The total density sum can then be used to write the plane entirely in terms of an arbitrary constant α , where $\Omega_2 = \alpha$. That the total density becomes $\Omega(S) = \Omega_1(S) + \alpha$, hence the individual density parameter for the first fluid can be written as $\Omega_1(S) = (\Omega(S) - \alpha)$. This and (2.112) in (2.79) gives us the unique (Ω, q) plane for effectively single fluid models,

$$\frac{dq}{d\Omega} = \frac{q}{\Omega} + \frac{1}{q(\Omega - 1)} \left(\frac{(q + \Omega)^2}{\Omega} - \frac{9}{4}(\gamma_1^2 \Omega(S) + \alpha(\gamma_2^2 - \gamma_1^2)) \right) \quad (2.113)$$

This is not a particularly physical fluid but is included as a possible example of an effectively single fluid model.

2.7 2-component fluids, $\gamma_i = \text{constant}$.

There are three useful physical combinations for 2-component fluids models.

- (1) Consider :

pressure-free matter / dust : $p_1 = 0$ that is $\gamma_1 = 1$ and $\mu_1 \propto S^{-3}$

radiation : $p_2 = \mu_2/3$ that is $\gamma_2 = \frac{4}{3}$ and $\mu_2 \propto S^{-4}$

We consider a two component perfect fluid that consists only of noninteracting radiation and matter. This was considered by Ehlers and Rindler [9] who treated this as a special case of the three fluid scenario, of radiation, matter and false vacuum (see figures 2.5).

- (2) Consider :

radiation : $p_1 = \mu_1/3$ that is $\gamma_1 = \frac{4}{3}$ and $\mu_1 \propto S^{-4}$;

false vacuum : $p_2 = -\mu_2$ that is $\gamma_2 = 0$ and $\mu_2 = \text{constant}$.

We consider a two component perfect fluid that consists only of noninteracting radiation and cosmological constant. This is dealt with by Stabell and Refsdal [8] in the context of the (Ω, q) planes (see figures 2.6).

- (3) Consider :

pressure free matter / dust : $p_1 = 0$ that is $\gamma_1 = 1$ and $\mu_1 \propto S^{-3}$;

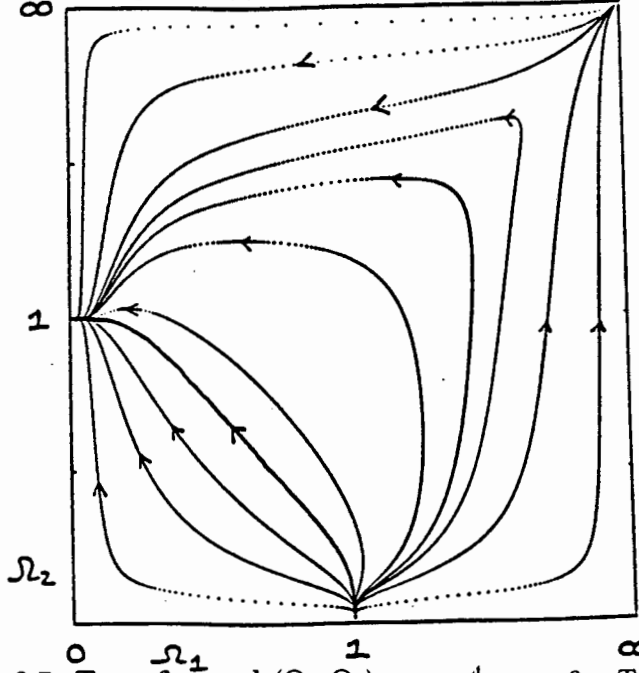


Figure 2.7: Transformed (Ω_1, Ω_2) , $\gamma_1 = \frac{4}{3}$ $\gamma_2 = 0$: This is the 2-fluid invariant phase plane for radiation and vacuum, *i.e.*, for inflationary universes, the line separating the open FLRW models from the closed FLRW models can be seen as the dark curve from $(1,0)$ to $(0,1)$, from radiation (Ω_1) dominated to vacuum (Ω_2) dominated, this is the trajectory on which the condition $\Omega_1 + \Omega_2 = \Omega = 1$ is satisfied. Notice that all 2-fluid inflationary universes end up arbitrarily close to $\Omega = 1$ after a sufficiently long time. The point (∞, ∞) is the turn-over point, $H=0$. Radiation dominated trajectories fall into this point and exit it in the vacuum dominated part of the plane and then fall into the $\Omega = 1$ attractor.

false vacuum : $p_2 = -\mu_2$ that is $\gamma_2 = 0$ $\mu_2 = \text{constant}$.

We consider a two component perfect fluid that consists only of noninteracting dust and cosmological constant. This can also be seen to be dealt with by Stabell and Refsdal [8] in the context of the (Ω, q) plane. Once again, they are able to construct their planes primarily because of the existence of the mapping from (Ω_1, Ω_2) into (Ω, q) this cannot be done following [8] for other cases (see figures 2.6).

What Stabell and Refsdal managed to do was to look at 2-fluid models (such as the transformed (Ω_1, Ω_2) planes) *i.e.* a single fluid plus a cosmological constant, where they were able to show that from (2.12) and (2.15) the cosmological constant can be treated as :

$$\Lambda = 3H_0^2((3\gamma_1 - 2)\sigma_0 - q_0). \quad (2.114)$$

Stabell and Refsdal are able to discuss the (Ω, q) ⁴ plane because they have effectively used a single fluid model. It should be realized that our universe is clearly not a single fluid

⁴ $\sigma_0 = \Omega_0/2$, can replace all H_0 , s_0 and q_0 by H , s and q to get my plane equations.

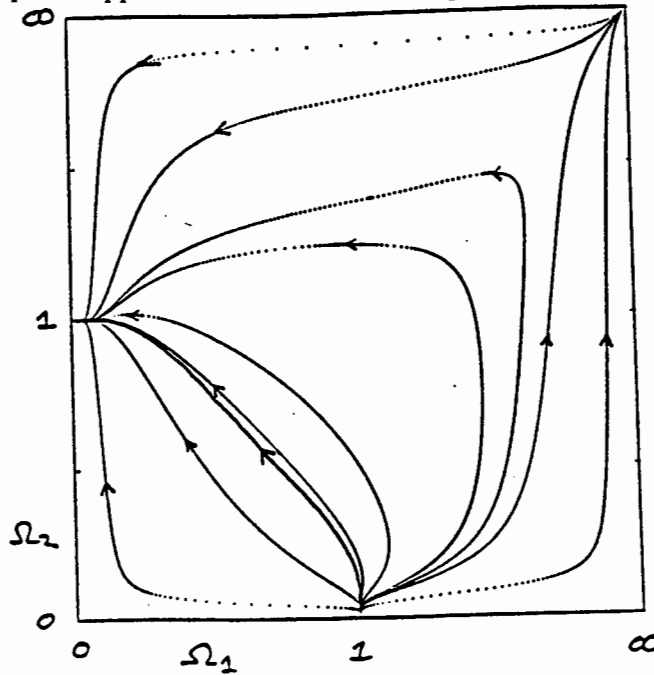


Figure 2.8: Transformed (Ω_1, Ω_2) Plane, $\gamma_1 = 1$ $\gamma_2 = 0$: This is the dust (Ω_1) and vacuum (Ω_2) plane for FLRW models, *i.e.*, inflationary universes, the structure is generically the same as for the radiation vacuum models. Once again one is able to find trajectories that are arbitrarily far from the $\Omega = 1$ separatrix for any time in the evolution of the universe.

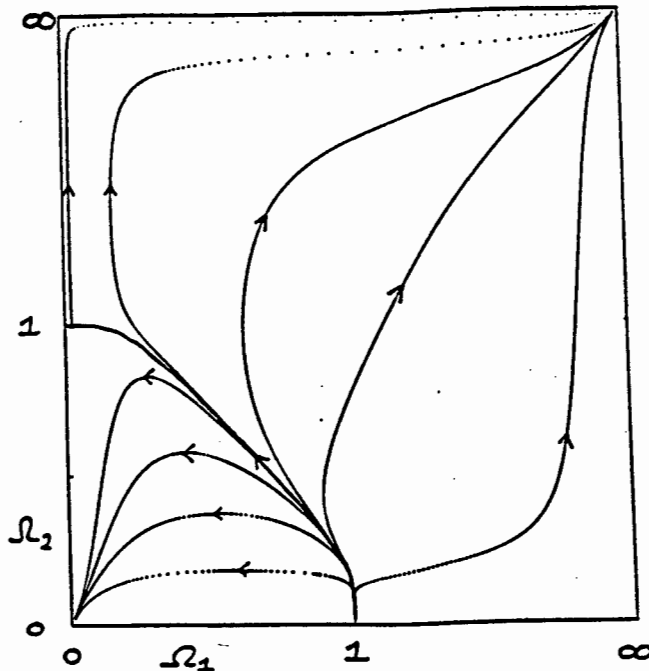


Figure 2.9: Transformed (Ω_1, Ω_2) Plane, $\gamma_1 = \frac{4}{3}$ $\gamma_2 = 1$: This is the invariant plane for radiation (Ω_1) and dust (Ω_2) . The $\Omega = 1, k=0$ separatrix is once again evident as the curve from $(1,0)$ to $(0,1)$. This divides the open $\Omega < 1$ FLRW models, which fall into the point $(0,0)$, *i.e.* the forever expanding universes, from the closed $\Omega > 1$ FLRW models which all have trajectories that fall into the turnover point (∞, ∞) , $H=0$. Notice that the plane variables are degenerate at the turnover.

model. Hence their work should be generalized to include this possibility which has not been done in such a way that provides for unique evolution curves. Madsen et al [10] suggest that one may write down the form of $\gamma(S)$, and use this in $q = 1/2\Omega(3\gamma - 2)$ to get the transformation from S to q for the mixed fluids. This suggestion only allows one to project a specific subset of trajectories in the $(\Omega_1, \Omega_2, \dots, q)$ space down into some non-unique sub-manifold. In general, such a procedure most probably filters out some phase plane structure, but retains the critical point structure, and hence can be viewed as generic in the sense that the resulting plane will be structurally stable but not unique.

An alternative path to follow is that of Janet Jones [12], she considers a 2 fluid model of the form of dust ($\mu_m, p = 0, \gamma = 1$) and an effective energy density $\mu^* = \alpha = \alpha(H, \mu_m) = mp$, (i.e. $\gamma = m + 1$). She started looking at modified single fluids, naturally the most general form of this would be to look at a fluid with viscosity, vorticity, shear, etc.. She was able to make some rather general statements about the resulting phase space for an arbitrary and well behaved singularity free α . This seems similar to the addition of some sort of bulk viscosity-like term. It may be interesting to see if this behaves like the velocity field ($\dot{\phi}$) of a subclass of scalar fields, although this is uncertain, it does seem unlikely as one would expect to require a different scalar field at every instant such that its energy momentum equation is appropriately satisfied. But it may be useful to link bulk viscous models to a subclass of scalar field models with regard to the general critical point structure of those planes. Jones wrote down the equation defining the (H, μ_m) plane from (2.49) in the form of

$$(1 + \alpha_m) \left(\frac{d\mu_m}{dH} \right) = \frac{K}{L}, \quad (2.115)$$

where :

$$K = JH - L\alpha_m, \quad (2.116)$$

$$J = (\mu_m + \gamma\alpha(H, \mu_m)), \quad (2.117)$$

$$L = (1/6)(\mu_m + (3\gamma - 2)\alpha(H, \mu_m)) + H^2, \quad (2.118)$$

with constants α_m and α_H defined, such that

$$\dot{\alpha} = \alpha_m \dot{\mu}_m + \alpha_H \dot{H}. \quad (2.119)$$

Using this she hunted down the singularities by finding the intersection of the curves C_0 where $dH/d\mu_m = 0$ and C_∞ where $dH/d\mu_m = \infty$ to find that there were only two cases to investigate : (1) when $L = 0$ and $J = 0$ and (2) when $L = 0$ and $H = 0$. Using

these cases she then derived the loci of steady states for both these cases. The restriction of such loci of the steady states ($\dot{\mu} = 0, \dot{H} = 0$) is a fundamental property of GR, the absence of which is a demonstration of the Hawking-Penrose singularity theorem. For the first case there are no periodically evolving modes allowed and all nodes are stable in an expanding universe, which is the case $H \neq 0$, while for the case $H = 0$ all forms of singularities are allowed. For the latter there are periodically evolving modes of behavior in the neighborhood of static singular points. This allows for the case of bouncing universes evolving from contracting to expanding scenarios. It was also noted that periodic behavior is associated with the violation of the energy conditions that she works with. The bottom line of her investigation was that the behavior of this system in the neighborhood of singularities is largely independent of $\alpha(H, \mu_m)$. Here if one is interested only in the generic behaviour as claimed by Ellis *i.e.* the behaviour in the immediate neighbourhood of singular points, then the single fluid and modified single fluid models of Madsen et al [10] would fully describe bulk viscous models in the sense of the work done by Jones [12].

2.7.1 Effectively 2 fluid models, $\gamma_i = \text{constant}$.

If we consider equation (2.79), and have that

$$\sum_i \gamma_i^2 \Omega_i = f(\Omega_1, \Omega_2), \quad (2.120)$$

we have what I call effectively two fluid models. This is equivalent in a well defined sense to

$$\sum_i \gamma_i^2 \Omega_i = g(q, \Omega). \quad (2.121)$$

There is an equivalence between the (Ω_0, Ω_1) and the (Ω, q) planes. Hence we consider both (2.120) and (2.121) conditions defining effectively 2 fluid models. Clearly, using (2.121), (2.79) becomes

$$\frac{dq}{d\Omega} = \frac{q}{\Omega} + \frac{1}{q(\Omega - 1)} \left(\frac{(q + \Omega)^2}{\Omega} - \frac{9}{4}g(\Omega, q) \right). \quad (2.122)$$

This is in fact what Stabell and Refsdal look at for the two fluid case. Their formulation takes a less transparent path towards the derivation of this result, and uses a very specific variable choice ⁵, hence cannot be easily generalized, while my formulation can be. For

⁵From $q = \frac{1}{2}\Omega(3\gamma - 2)$ it can be easily shown, for a 1 fluid and cosmological constant model with the substitution $\sigma = \Omega_1$, that

$$q = \sigma(3\gamma_1 - 2) - \Omega_\Lambda. \quad (2.123)$$

From this the (q, σ) planes of Stabell and Refsdal can be constructed. These are basically (q, Ω_1) planes, which must be considered in terms of slices of the full evolution space (q, Ω_1, Ω_2) . While the (q, Ω) planes

the two fluid case this can be trivially accomplished as there is a well defined map from (Ω_0, Ω_1) to (Ω, q) . Consider equation (2.15) :

$$q = \frac{1}{2}\Omega(3\gamma - 2) = \frac{1}{2}\Omega\left(3\frac{\sum_i \gamma_i \Omega_i}{\Omega} - 2\right) = \sum_i \frac{1}{2}\Omega_i(3\gamma_i - 2) = \sum_i q_i. \quad (2.124)$$

Using this we can eliminate the explicit dependence on the individual fluid components for two fluids. More generally we can eliminate any two fluids from a multi fluid model by the following two transformations derived from the above equation and using $\Omega = \sum \Omega_i$. For two fluids it is quite simple. Using (2.124) we have that

$$\Omega_2 = \frac{q - \frac{1}{2}\Omega(3\gamma_1 - 2)}{\frac{3}{2}(\gamma_2 - \gamma_1)}. \quad (2.125)$$

From the definition of the total density parameter,

$$\Omega_1 = \Omega - \left(\frac{q - \frac{1}{2}\Omega(3\gamma_1 - 2)}{\frac{3}{2}(\gamma_2 - \gamma_1)} \right). \quad (2.126)$$

That is, (2.121) becomes

$$\sum \gamma_i^2 \Omega_i = \gamma_1^2 \Omega_1(q, \Omega) + \gamma_2^2 \Omega_2(q, \Omega) = g(q, \Omega). \quad (2.127)$$

Here we have found $g(q, \Omega)$ to give us the phase plane equation

$$\begin{aligned} \frac{dq}{d\Omega} = \frac{q}{\Omega} + \frac{1}{q(\Omega - 1)} \left(\frac{(q + \Omega)^2}{\Omega} \right. \\ \left. - \frac{9}{4} \left(\gamma_1^2 \left(\Omega - \left(\frac{q - \frac{1}{2}\Omega(3\gamma_1 - 2)}{\frac{3}{2}(\gamma_2 - \gamma_1)} \right) \right) + \gamma_2^2 \left(\frac{q - \frac{1}{2}\Omega(3\gamma_1 - 2)}{\frac{3}{2}(\gamma_2 - \gamma_1)} \right) \right) \right) \end{aligned} \quad (2.128)$$

This is the transformation of the (Ω_1, Ω_2) plane which is given from (2.83) by

$$\frac{d\Omega_1}{d\Omega_2} = \frac{\Omega_1(2 - 3\gamma_1) - 2\Omega_1(\Omega_1 + \Omega_2) + 3\Omega_1(\gamma_1\Omega_1 + \gamma_2\Omega_2)}{\Omega_2(2 - 3\gamma_2) - 2\Omega_2(\Omega_1 + \Omega_2) + 3\Omega_2(\gamma_1\Omega_1 + \gamma_2\Omega_2)}. \quad (2.129)$$

For the case of a fluid plus false vacuum, that is $\gamma_1 = 0$ (2.129) trivially becomes

$$\frac{dq}{d\Omega} = \frac{q}{\Omega} + \frac{1}{q(\Omega - 1)} \left(\frac{(q + \Omega)^2}{\Omega} - \frac{3}{4}\gamma_2(\Omega(3\gamma_2 - 2) - 2q) \right). \quad (2.130)$$

Here γ_2 defines the first fluid component (traditionally either radiation or dust). This is the plane considered by Stabel and Refsdal [8]. It may be useful in the case of two fluid models to consider the (Ω, q, S) plane instead of the (Ω_1, Ω_2, S) plane as these are equivalent.

fully encode the evolution of the two fluid models.

2.8 3-component fluid, $\gamma_i = \text{constant}$.

Consider :

- **pressure-free matter / dust** : $p_1 = 0$ that is $\gamma_1 = 1$ and $\mu_1 \propto S^{-3}$,
- **radiation** : $p_2 = \mu_2/3$ that is $\gamma_2 = \frac{4}{3}$ and $\mu_2 \propto S^{-4}$,
- **false vacuum** : $p_3 = -\mu_3$ that is $\gamma_3 = 0$ and $\mu_3 = \text{constant}$.

This is slightly more general than the previous mixture and includes the false vacuum. This has been dealt with at length by Ehlers and Rindler [9] who give a thorough look at this case, the $(\Omega_m, \Omega_r, \Omega_\Lambda)$ phase plane, *i.e.* its fixed points and stability.

Consider the $(\Omega_m, \Omega_r, \Omega_\Lambda)$ phase plane, and that Ω_r is fairly well known from experiment. In this sense it is perhaps satisfactory to use its well known current parameter value to pin down its behavior and project from the full phase space into some sub-manifold that reflects the possible evolutions with regard to the less well known values. That is, where most of the uncertainty lies, for example in the Ω_m and Ω_Λ values. That is, we could map $(\Omega_m, \Omega_\Lambda) \rightarrow (\Omega, q)$ as we have discussed before in section 2.5.4. This would give us the (Ω, q, Ω_r) plane, say. Here the Ω_r behavior is well known. We could then project this into (Ω, q) in a well defined sense. There would exist no ambiguity with regards to the multivalued nature of the mapping given that Ω_r is well known at some time, such as the current time. This unique sub-manifold could then be time evolved backwards or forwards, giving an indication of the possible early and late conditions of such universes.

The uncertainties in Ω and q can also naturally be used to place constraints on the inherent uncertainties in Ω_m and Ω_Λ , for such universes, using the transformation equations (2.90) and (2.91). The (Ω, q) would be constructed in the sense of some effectively two fluid model.

The Three fluid equations for the $(\Omega_m, \Omega_r, \Omega_\Lambda)$ plane for instance are given by :

$$\left(\frac{d\Omega_r}{d\Omega_m}\right) = \left(\frac{\Omega_r}{\Omega_m}\right) \left(\frac{A-1}{A}\right), \quad (2.131)$$

$$\left(\frac{d\Omega_r}{d\Omega_\Lambda}\right) = \left(\frac{\Omega_r}{\Omega_\Lambda}\right) \left(\frac{A-1}{A+3}\right), \quad (2.132)$$

$$\left(\frac{d\Omega_m}{d\Omega_\Lambda}\right) = \left(\frac{\Omega_m}{\Omega_\Lambda}\right) \left(\frac{A}{A+3}\right). \quad (2.133)$$

Here $A = \Omega_m + 2\Omega_r - 2\Omega_\Lambda - 1$. This would be the minimal subspace that can be used to fully describe a three fluid model *eg* dust, radiation and false vacuum (non-scalar field) for all FLRW models, this generalizes Ehlers choice [9].

2.9 N-component fluids

It may be possible to find a better approximation for the effectively single fluid models by considering a large number of noninteracting fluids, that is for N large. There is the possibility of writing the average γ as an integral over some distribution function $\Gamma(\gamma, S)$

:

$$\sum_{i=0}^N \gamma_i \Omega_i(S) \longrightarrow \int_a^b \Gamma(\gamma', S) \Omega(S) d\gamma'. \quad (2.134)$$

That is, the equation $\gamma(S)\Omega(S) = \sum \gamma_i \Omega_i(S)$ becomes

$$\gamma(S)\Omega(S) = \int_a^b \Gamma(\gamma', S) \Omega(S) d\gamma' \Rightarrow \gamma(S) = \int_a^b \Gamma(\gamma', S) d\gamma'. \quad (2.135)$$

It can then be shown that there are two functions $h_0(S)$ and $h_1(S)$ such that

$$\sum_i \gamma_i \Omega_i(S) = \Omega(S) \int \Gamma(\gamma', S) d\gamma' = h_0(S) \Omega(S), \quad (2.136)$$

$$\sum_i \gamma_i^2 \Omega_i(S) = \Omega(S) \int \Gamma^2(\gamma', S) d\gamma' = h_1(S) \Omega(S), \quad (2.137)$$

where for instance $(d\gamma/dS)$, from (2.94), becomes

$$\frac{d\gamma}{dS} = \left(\frac{3}{S}\right) (h_0(S)\Omega(S) - h_1(S)). \quad (2.138)$$

Hence the extremum at $(d\gamma/dS)=0$ would occur when $\Omega(S) = h_1(S)/h_0(S)$. Similar conditions could be found for the second derivative in S , and the extremum could be classified in the usual fashion.

It is possible to specify a distribution function that would produce the current matter dominance today, but at some earlier time peak about the value appropriate for radiation dominance such that the average γ is $\frac{4}{3}$ for instance. To ensure that the fluid is causal one would require that $\Gamma = 0$ for all $\gamma' < 0$ and $\gamma' > 2$, *i.e.* the distribution has finite values only for γ' between 0 and 2. This could be interesting if one would like to see how fluids that smoothly deform from one sort of predominance to another, with regard to the (Ω, S) , (Ω, q) and (Ω, H) planes. It may be possible to find some good distribution function such that the N fluid case may be treated as an effectively single fluid case.

An interesting question would be whether or not a multi-fluid model characterized by a distribution function in γ may be a useful approximation to interacting fluids. In that, perhaps, one could use the velocity distribution functions of the fluid elements via a Maxwell velocity distribution to find some sort of $\Gamma(\gamma')$ that represents the statistical state of the fluid. If a single fluid particle interacts with another fluid particle from a

different fluid, we could possibly treat the interacting particles as a new fluid component. In this way a distribution function in γ if well chosen may be a useful trick in handling scenarios where the fluids interact in some weak sense other than gravity. Such an N interacting effectively single fluid type model with a stochastic perturbation, similar to Brownian motion with a carefully chosen diffusion parameter, may be a useful way of treating quantum fluids in the context of cosmology. The dynamics of such a classical fluid would be irreversible.

This leads one to the question of whether the limits in the integral (a,b) may be extended from (0,2), which ensures causality to $(-\infty, +\infty)$, which would perhaps include non-local effects in a very crude manner. For example, we could consider the arbitrary distribution function $\Gamma(\gamma', S) = \frac{\delta(\gamma' - \beta)^{4/3}}{1 + \alpha S^4}$ to find $\gamma(S) = \frac{4/3}{1 + \alpha S^4}$, and claim that this is some non-local distribution function.

One would like to know how sensitive the evolution curves of various scenarios are to small contaminations of extra fluid components. This seems to be clear for 2-component and 3-component fluids, in that small amounts of initial energy density of an extra fluid component seem to have very little effect some time later, so in good faith it could be ignored when studying evolution curves for non-interacting matter. If the energy density is back evolved the situation is very different as μ would be an increasing function *i.e.* small uncertainties and errors in μ would grow.

On a simpler level we also note : using 2 component (matter + radiation) or 3 component (matter + radiation + false vacuum) it should be realized that for cases where the universe initially had very much larger radiation energy density than matter energy density, that is $\mu_r \gg \mu_m$ initially, at some time later the matter component will naturally start to dominate. While for $\mu_m \gg \mu_r$, it would appear that one may fine tune the ratio μ_m/μ_r to nearly any number greater than 1 at some time in the late history of the universe. It provides a smooth transition from radiation dominated to matter dominated scenarios without having to use the baggage of epochs [10]. The false vacuum energy density in the form of a cosmological constant is constant for all time, hence will dominate the latter part of the universe's history, but would not interact with the matter or radiation but only affect the global structure of the universe through its influence on the expansion parameter. So by the time that matter domination takes over from the massively radiation dominated era the false vacuum would contribute to most of the energy density of the universe, but would be undetectable by gross interactions with matter or radiation although detectable by its effect on expansion. Perhaps perturbations of the false vacuum (however small)

would have huge observable effects in the late universe (due to the dominance of this initially small value). (The switch in μ_r dominance to μ_m dominance is easily seen in the transformed (H, μ_m, μ_r) and (μ_m, μ_r) phase plane equations (2.93) and (2.83)).

2.10 Conclusion

If one wishes to investigate the evolution of the current solution volume with regard to FLRW models with simple fluids, one should basically look at the $(\Omega_0, \Omega_1, \dots, \Omega_N)$ planes. It is these planes, rather than the (Ω, S) or $(\Omega_0, \Omega_1, \dots, \Omega_N, S)$ planes, that give a general description of the evolution of this class of possible universes *i.e.* the FLRW, perfect fluid models.

Using $\gamma(S)$ in the phase plane equation for single fluid models as obtained from multi-component fluids should be viewed with some caution. This is because one is projecting onto a sub-manifold of the full phase space, hence the resulting planes cannot be regarded as general *i.e.* generic and unique, with respect to the full phase space. This has been done by [10] with regards to the (Ω, S) plane using $\gamma(S) = \frac{4/3}{1+\alpha S^4}$. This is effectively mapping some unique trajectory from the full (Ω_1, Ω_2, S) phase space into a sub-manifold given by (Ω, S) . It has been claimed that looking at the (Ω, S) plane for general multi fluid models is in fact generic in some sense to the fluid's behavior, on the premise that this behaviour is obtained by solving the conservation equations for a mixture of perfect fluids (such as matter or radiation), giving the form of γ , as a function of S with arbitrary constants M_i representing the amount of matter and radiation present. Thus allowing one to then plot the (Ω, S) phase planes for these solutions *i.e.* a set of solutions with specified M_i , the different trajectories corresponding to different choices for H_0 at some initial time t_0 , for the given constants M_i characterizing the matter. The point then is that the particular values of the M_i are not critical: the shape of the phase plane curves is the same for all (positive) choices of M_i , so in fact one is seeing the behavior of "generic solutions", the resulting plane is in fact structurally stable. One would have something to worry about if there were critical values M_i where the nature of the solution changed. But it would seem that this is not the case.

One can use the measured values of q and Ω to get at the individual density parameter within the framework of 2 fluid FLRW perfect fluid models, or, if one knew the density parameter for a specific fluid component (as one does for the radiation), one could use q and Ω to find the other 2 fluids density parameters for 3 fluid FLRW perfect fluid models using the limits on Ω_0 , H_0 and q_0 .

2.11 Scalar fields

These are discussed for completeness. It should be understood that the horizon problem could turn out to be no problem at all. As Quantum Mechanics and Quantum Field Theory are intrinsically non-local, it is experimentally well known that Bell's inequalities are violated, this non-locality doesn't violate causality in terms of the observables. A good place to see these arguments is in [67]. The point being that using an additional scalar field to drive inflation may not be necessary to solve the problems in structure formation theories. The understanding of scalar fields in cosmology is, however, important not only with regard to inflation but also as an alternative approach to fluid models and it provides a good introduction to the handling of fields in cosmology.

Consider the field equations for an isotropic spatially homogeneous cosmological model which has a fluid matter content and a scalar field. The scalar field, the inflaton, which is also spatially homogeneous is described by a Lagrangian of the form,

$$\mathcal{L} = \frac{1}{2}(\phi_{,a}\phi^{,a} + V(\phi)). \quad (2.139)$$

Using the fully relativistic Euler-Lagrange equations and recalling that $\Gamma_{a0}^a = 3H$ we find the equation of motion for the inflaton is given by

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\partial V}{\partial \phi} = 0. \quad (2.140)$$

Multiplying both sides by $\dot{\phi}$ one finds that this becomes

$$\frac{d}{dt} \left(\frac{\dot{\phi}^2}{2} \right) + 6H \left(\frac{\dot{\phi}^2}{2} \right) = -\frac{dV(\phi)}{dt}. \quad (2.141)$$

From the usual field formulation the energy momentum tensor is obtained from the Lagrangian (2.139)

$$T_{ab} = \phi_{,a}\phi_{,b} - g_{ab} \left(\frac{1}{2}\phi_{,c}\phi^{,c} + V(\phi) \right). \quad (2.142)$$

Using the spatial homogeneity of the field [1],

$$\phi_{,a} = -\dot{\phi}u_a \quad (2.143)$$

The energy-momentum tensor becomes

$$T_{ab} = \dot{\phi}^2 u_a u_b - g_{ab} \left(-\frac{1}{2}\dot{\phi}^2 + V(\phi) \right) \quad (2.144)$$

This can be rewritten to look like

$$T_{ab} = (\mu_\phi + p_\phi)u_a u_b + g_{ab}p_\phi, \quad (2.145)$$

which is the energy momentum tensor for a perfect fluid with the energy density and pressure given respectively :

$$\mu_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi), \quad (2.146)$$

$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi). \quad (2.147)$$

One needs only to modify the resulting fields equations obtained for the perfect fluid case by adding in the scalar fields as addition pressure and energy density contributions. The equation of motion (2.140) is redundant as it is naturally included in the conservation equations for the mixed fluid case given in the previous section with the additional pressure and energy density given in (2.146) and (2.147). Hence we have from (2.6) through (2.8):

$$3\dot{H} + 3H^2 = (V(\phi) - \dot{\phi}^2 - \frac{1}{2}(\mu + 3p)), \quad (2.148)$$

$$\dot{\mu} + (\mu + p)3H = 0, \quad (2.149)$$

$$3H^2 + 3K = \left(\frac{\dot{\phi}^2}{2} + V(\phi) + \mu \right). \quad (2.150)$$

And included in (2.149) one would have along with the separate energy momentum conservation equations, equation (2.140).

Notice that the perfect fluid pressure and energy density form of the spatially homogeneous inflaton suggests 3 possibilities in terms of equations of state in the form of

$$p_\phi = (\gamma_\phi - 1)\mu_\phi. \quad (2.151)$$

- (1) $\phi_{,a}\phi^{,a} = 0$ that is $\dot{\phi}^2 = 0$: This is the traditional cosmological constant case as one then finds that $\mu_\phi = V(\phi) = -(-V(\phi)) = -p_\phi$. That is the case of $\gamma = 0$, the false vacuum.
- (2) $\phi_{,a}\phi^{,a} > 0$ that is $\dot{\phi}^2 > 0$: the space-like case.
- (3) $\phi_{,a}\phi^{,a} < 0$ that is $\dot{\phi}^2 < 0$: the time-like case.

Notice that the equation of state yields,

$$\gamma_\phi = \frac{\dot{\phi}^2}{\frac{1}{2}\dot{\phi}^2 + V(\phi)}. \quad (2.152)$$

One can see that γ is the ratio of kinetic energy over the total energy. From (2.152) one can show that

$$\dot{\phi} = \left(\frac{2\gamma_\phi V(\phi)}{2 - \gamma_\phi} \right)^{\frac{1}{2}}, \quad (2.153)$$

$$\ddot{\phi} = \frac{1}{2} \left(2\gamma_\phi V(\phi)(2 - \gamma_\phi)^3 \right)^{-\frac{1}{2}} \left(2\dot{\gamma}_\phi V(\phi) + \frac{\partial V}{\partial \phi} \dot{\phi} \gamma_\phi (2 - \gamma_\phi) \right). \quad (2.154)$$

From the conservation equation, that is the equation of motion for the scalar field, we find that by using (2.153) and (2.154) in (2.140)

$$\dot{\gamma}_\phi = \left(\frac{\partial V}{\partial \phi} \right) \frac{\gamma_\phi(\gamma_\phi - 2)}{V(\phi)} \left(\frac{2\gamma_\phi V(\phi)}{2 - \gamma_\phi} \right)^{\frac{1}{2}} - 3H\gamma_\phi(2 - \gamma_\phi). \quad (2.155)$$

Hence we have found equations for $\dot{\gamma}_\phi$ and $\dot{\phi}$, in terms of γ_ϕ , ϕ and H . To eliminate the implicit dependence on H , H must be found in terms of other useful variables such as S or Ω , in a general fashion.

2.11.1 Phase plane equations

Single scalar field type fluid

For the single fluid case, that is just a scalar field, we have that

$$\dot{\Omega}_\phi = H\Omega_\phi(1 - \Omega_\phi)(2 - 3\gamma_\phi), \quad (2.156)$$

$$\dot{H} = -H^2 \frac{1}{2} \Omega_\phi (3(\gamma_\phi - 2) + 1), \quad (2.157)$$

$$\dot{\gamma}_\phi = \left(\frac{\partial V}{\partial \phi} \right) \frac{\gamma_\phi(\gamma_\phi - 2)}{V(\phi)} \left(\frac{2\gamma_\phi V(\phi)}{2 - \gamma_\phi} \right)^{\frac{1}{2}} - 3H\gamma_\phi(2 - \gamma_\phi), \quad (2.158)$$

$$\dot{\phi} = \left(\frac{2\gamma_\phi V(\phi)}{\gamma_\phi - 2} \right)^{\frac{1}{2}}, \quad (2.159)$$

$$\dot{q} = \frac{1}{2} (\dot{\Omega}_\phi (3\gamma_\phi - 2) + 3\Omega_\phi \dot{\gamma}_\phi), \quad (2.160)$$

$$\dot{S} = HS. \quad (2.161)$$

Hence unless we can write down a general form for γ_ϕ we cannot look uniquely at the (Ω, H) , (Ω, S) planes in that these would then not reflect the full structure of the complete phase space. To be fully generic we have to look at the $(\Omega_\phi, H, \gamma_\phi, \phi)$ phase plane. There is no general way in which we can get $\gamma_\phi(S)$ without assuming some constraint of sorts to find the mapping into the lower dimensional sub-manifold. One particular trick to accomplish this is to restrict the behavior of the potential in such a way that $\partial V / \partial \phi = 0$, that is $V = \epsilon$ for some constant ϵ . This would put the $\dot{\gamma}_\phi$ equation in such a form that one could easily obtain the (Ω, S, ϕ) planes. One can restrict this even further by setting $\dot{\phi} = \beta$, where β is some constant. This case would reduce to the case of constant energy

density and constant pressure : $\mu_\phi = \frac{1}{2}\beta - \epsilon$ and $p_\phi = \frac{1}{2}\beta + \epsilon$. Here $\gamma_\phi = \frac{2\beta}{\beta - \epsilon}$; ϵ and β are both arbitrary constants. One obvious choice of β would be $\beta = 0$, giving the false vacuum case. The $(q, H, \phi, \gamma_\phi)$ plane for scalar field models will in general not be trivial due to the involved form of γ_ϕ . Clearly the most natural plane to investigate for the scalar field models is the $(H, \dot{\phi}, \phi)$ plane or the (H, γ_ϕ, ϕ) plane which can be found using the equations for \dot{H} , $\dot{\phi}$ and $\dot{\gamma}_\phi$, with the definition for the density parameter given in terms of μ_ϕ and H . These planes would be difficult to connect with observations via the dimension-less parameters.

If one were interested in the $(\phi, V(\phi))$ or $(\dot{\phi}, V(\phi))$ planes one would need a particular choice of $V(\phi)$ along with either a slow roll constraint to motivate that $\dot{\phi} = 0$ or a suitable projection to eliminate the dependence on H . Clearly in this sort of scenario one must specify $V(\phi)$ *a priori*, unless one wishes to treat $V(\phi)$ in a manner similar to the treatment by Janet Jones of an arbitrary effective energy density (section 2.7).

The plane researched by Belinskii [17] and [18], is the $(H, \dot{\phi}, \phi)$ plane rescaled using the variables :

$$x = \left(\frac{4}{3\pi G} \right)^{\frac{1}{2}} \phi, \quad (2.162)$$

$$y = \left(\frac{2}{3\pi G} \right)^{\frac{1}{2}} \dot{\phi}, \quad (2.163)$$

$$z = \frac{\dot{a}}{ma}, \quad (2.164)$$

$$\eta = mt. \quad (2.165)$$

Belinskii looked at the case $V(\phi) = m^2\phi^2$ using $m_p^2 = \frac{8\pi}{\kappa} = \frac{1}{G}$, we have chosen $\kappa = 1$ which means that in natural units $m_p^2 = 8\pi$ and $a = S$, to find the equations

$$\dot{H} = -H^2 + \frac{4\pi}{3}m_p^2(m^2\phi^2 - \dot{\phi}^2), \quad (2.166)$$

$$\ddot{\phi} = -3H\dot{\phi} - m^2\phi. \quad (2.167)$$

These can then be rewritten as three equations by using a substitution say $\beta = \dot{\phi}$ to get :

$$\dot{H} = -H^2 + \frac{4}{3}\pi m_p^2(m^2\phi^2 - \beta), \quad (2.168)$$

$$\dot{\beta} = -3H\beta - m^2\phi, \quad (2.169)$$

$$\dot{\phi} = \beta. \quad (2.170)$$

These are consistent with the equations for a general potential :

$$\dot{H} = -H^2 + \frac{1}{3}(V(\phi) - \dot{\phi}^2), \quad (2.171)$$

$$\dot{\beta} = -3H\beta - \frac{\partial V(\phi)}{\partial \phi}, \quad (2.172)$$

$$\dot{\phi} = \beta. \quad (2.173)$$

Multi fluid models with one or more scalar field contributions

The multi fluid models with many scalar fields have defining equations that look like

$$\dot{\Omega}_i = H\Omega_i(1 - \Omega)((2 - 3\gamma_i) - \Omega(2 - 3\gamma)), \quad (2.174)$$

$$\dot{H} = -H^2\left(\frac{1}{2}\Omega(3\gamma - 2) + 1\right), \quad (2.175)$$

$$\dot{\gamma}_{\phi_j} = \left(\frac{\partial V_j}{\partial \phi_j}\right) \frac{\gamma_{\phi_j}(\gamma_{\phi_j} - 2)}{V_j} \left(\frac{2\gamma_{\phi_j} V_j}{2 - \gamma_{\phi_j}}\right)^{\frac{1}{2}} - 3H\gamma_{\phi_j}(2 - \gamma_{\phi_j}), \quad (2.176)$$

$$\dot{\phi}_j = \left(\frac{2\gamma_{\phi_j} V_j}{\gamma_{\phi_j} - 2}\right)^{\frac{1}{2}}, \quad (2.177)$$

$$\dot{q} = \frac{1}{2}(\dot{\Omega}(3\gamma - 2) + 3\Omega\dot{\gamma}), \quad (2.178)$$

$$\dot{S} = HS. \quad (2.179)$$

Here constraint and definitional equations have the form

$$V_j = V_j(\phi_j), \quad (2.180)$$

$$\gamma = \sum_i \frac{\gamma_i \Omega_i}{\Omega}, \quad (2.181)$$

$$\Omega = \sum_i \Omega_i. \quad (2.182)$$

Here we use $j = 1, 2, \dots, n$, $i = 1, 2, \dots, N$ and $n \leq N$.

In principle the only natural plane to look at for general scenarios would be the $(\Omega_i, H, \gamma_{\phi_j}, \phi_j)$ plane which would be rather complex in nature. As usual any Ω_k and Ω_l could be replaced by Ω and q in the usual manner. The only way of getting this plane into the usual invariant phase plane form of the $(\Omega_1, \Omega_2, \dots, \Omega_N)$ plane, would be to strongly enforce the condition that $\dot{\gamma}_i = 0$. This would merely bring the dynamics back to the form of the perfect fluid case, as shown in the single scalar field case. This suggests that the existence of such an invariant sub-manifold is primarily due to the use of perfect fluids.

One cannot eliminate the γ_{ϕ_j} and ϕ_j dependence on H , and thence eliminate the explicit dependence on H , without such a simplifying assumption. *i.e.* one cannot discuss the $(\Omega_1, \Omega_2, \dots, \Omega_N)$ or $(\Omega_1, \Omega_2, \dots, \Omega_N, S)$ planes and expect any generality without carefully constructing N constraint equations.

2.11.2 Scalar field with a flat potential

Consider:

$$V(\phi) = V_0. \quad (2.183)$$

$V_0 = \text{constant}$; the scalar field contribution reduces using (2.152) to

- (i) *stiff matter* case, that is $\gamma_\phi = 2$, if $V(\phi) = 0$ this is a mass-less scalar field.
- (ii) *false vacuum* or cosmological constant case, that is $\gamma_\phi = 0$, when $\dot{\phi}^2 = 0$. That is the $\phi = \text{constant}$ case, the traditional cosmological constant case.

$V(\phi) = \text{Constant}$ and $\dot{\phi} = 0$ is the case considered by Stabell and Refsdal in [8], while the Madsen and Ellis [11] paper look at the $V(\phi) = \text{constant}$ case for freely evolving $\dot{\phi}$.

Consider a **single scalar field model with a flat potential** as suggested above. The evolution equations reduce to the set :

$$\dot{H} = \frac{1}{3(V_0 - \pi^2)} - H^2, \quad (2.184)$$

$$\dot{\pi} = -3H\pi, \quad (2.185)$$

$$\dot{\phi} = \pi, \quad (2.186)$$

$$\dot{S} = HS. \quad (2.187)$$

Dividing (2.185) by (2.187) it is easy to show that

$$\pi(S) = \pi_0 S^{-3}. \quad (2.188)$$

Using this in the Friedmann equations one finds

$$H(S) = \left(\frac{-k}{S^2} + \frac{1}{3} \left(\frac{1}{2} \pi^2 S^{-6} + V_0 \right) \right)^{\frac{1}{2}}. \quad (2.189)$$

This gives, on using (2.187), an equation for the scale factor $S(t)$:

$$\frac{dS}{dt}(t) = \left(-k + \frac{1}{3} \left(\frac{1}{2} \pi^2 S^{-4} + V_0 S^2 \right) \right)^{\frac{1}{2}}. \quad (2.190)$$

Hence one can find the functional form of $S(t)$; this can be used to find $H(t)$ and $\dot{\phi}(t)$. By integrating (2.186) one can get the form of ϕ . Once one has $\dot{\phi}$, one can write down $\gamma_\phi(S)$

$$\gamma_\phi(S) = \frac{2}{1 + 2 \frac{V_0 S^6}{\dot{\phi}_0^2}}. \quad (2.191)$$

This has been investigated by Madsen et al [11]. Notice that when one couples in another fluid component, we once again have the problem of not being able to write down an explicit generic expression of $\gamma(S)$. The appropriate phase plane equations can be obtained by dividing the equations (2.184) - (2.187) by combinations of each other to get the (H, π, ϕ) planes. To get the (Ω, S) plane one need merely use the functional form of $\gamma_\phi(S)$ in equation (2.36).

2.11.3 Some other inflationary potentials and comments

Massive scalar field

$$V(\phi) = \frac{\mu^2 \phi^2}{2} - \frac{\lambda \phi^4}{4} \quad (2.192)$$

This is conformally coupled via the field equations. ϕ^4 theories are particularly interesting from the particle physics point of view as these are the only renormalizable interacting field theories. The ϕ^3 have no local minimum and drops off to infinitely negative energy, while the ϕ^n theories with $n > 4$ are non-renormalizable [6] (in the sense of the Dyson series). These interaction terms allow spontaneous symmetry breaking. Recent results in string theories have indicated that perhaps renormalizability is not a good criterion for determining whether a theory is physically plausible, for instance Gross-Neveu or NJL models (Although physically useful, are not renormalizable in the traditional sense). The point here is that it is such models that offer the most hope with regard to using inflation to solve the horizon problem, but in turn the choice of even the most "physical" potentials, as motivated by particle physics, are apparently arbitrary. This is a difficult issue that must be treated with much care.

Comments

There is an extremely large number of possible inflationary potentials; it would seem that the concept of inflation is here to stay, although the exact nature of inflation is in flux. A good place to experience the full plethora of possible scenarios is in Linde's book [5].

- There are inflationary models that lead to present day values of Ω other than 1. This can be seen by looking at the (Ω, S) plane for single fluids, as an example. One can

merely evolve the FE's backwards from any present Ω_0 to find the initial conditions near the initial singularity. It is argued that inflation implies that Ω tends towards 1, which means that Ω_0 would tend to 1 if inflation lasted long enough.

- There is no proof that inflation took place and there is no good candidate for the “inflaton” yet.
- If $\Omega_0 = 1$, then there is no need for inflation. This is because for a critical $k = 0$ universe, at all times in the universe history $\Omega = 1$ remains unchanged; inflation would make no difference.
- The inflationary ideas never really ever claimed that $\Omega = 1$, but rather claimed that $\Omega = 1 \pm 10^{-4}$.
- Most Ω_0 estimates are model dependent which places a question on how we use the theory to find observational verification for the theory.
- Inflation is needed to solve the horizon problem.

2.11.4 Temperature in photon gas and multi-fluid systems

Another important physical parameter that could be investigated using a phase plane approach as demonstrated in this thesis for FLRW universes is the measure of the radiation temperature. As we shall elaborate further in the Kinetic theory part of the thesis, a Boltzmannian photon gas has an equilibrium temperature well described by the Planck distribution (3.118). This may be useful in demonstrating the evolution of the photon background temperature in FLRW universe in the context of a phase plane analysis.

That is, using

$$\mu_2 = aT^4, \quad (2.193)$$

means that (setting $a=1$)

$$\Omega_2 = \frac{T^4}{3H^2}. \quad (2.194)$$

giving the relation

$$\frac{\dot{\Omega}_2}{\Omega_2} = 4\frac{\dot{T}}{T} - \frac{2\dot{H}}{3H}. \quad (2.195)$$

Here μ_1 is taken to be dust, say. We could further include some μ_3 , a scalar field or fluid false vacuum. For single fluid models we can use (2.193) and (2.7) to show that $(TS) = \text{const}$. One could in principle write the phase plane equation out for multi-fluid

universes in terms of temperature. But one would need to include the evolution equations of H , as these cannot be eliminated in the usual fashion. That is, one could at best look at the (Ω_1, T, H) and (Ω, T, H, S) planes for example.

2.11.5 The Measure problem

One of the most popular theories at present is inflation. It purports to solve the horizon and flatness problems. The first is probably true, but the second is much more subtle. It is true that inflation drives Ω closer to 1 than non-inflationary evolution (see the (Ω, S) plane). There is, however, no reason to believe that Ω came out of the Planck era with a value which inflation was able to drive close to 1. In other words, what was the measure on the space of FLRW models at the Planck time? (Note that this is not a measure on the full set of universe models). This problem has been investigated in [20] and [21] for example. The main aim of these papers is to put a natural measure on the space, in the context of the Hamiltonian systems approach. Hawking and Page indicate that both non-inflationary and inflationary solutions have divergent total measures, so the ratio depends only on how it is evaluated, so that rather natural methods of evaluating the probability of inflation give results near unity, while others give results near zero. They claim that one cannot unambiguously conclude that inflation is highly probable. There have been suggestions of alternative measures based on the metric of the space of gravity fields by DeWitt to construct the so called unique kinematic measure on the space of FLRW simple fluid models [45].

The result of the Gibbons, Hawking, Stewart approach appears to be that inflation is not needed to solve the measure problem. They claim that the classical universe should be peaked about 1 anyway. This has also been discussed by Cole and Ellis [19], who approach the problem of inflation from a more observationally based angle. They conclude that none of the theoretical and observational arguments which have been offered to support the claim of a critical density universe are compelling, and suggest why in fact a low density universe may be favored by observations. They correctly compel cosmologists to keep an open mind about this question, until it is properly settled by empirical evidence. This questions the whole apparent need to motivate inflation from a possibly unfounded desire to have an $\Omega = 1$ universe.

Ellis concludes that Ω is nearly 1 only at very restricted periods in the evolution of the universe. The probability that Ω will be measured to be nearly 1 is strongly dependent on the time of the observations. The set of initial conditions leading to Ω not being close

to 1 at any given time, is non-zero in at least some measure in a universe in which the physics has been prescribed. The question of $\Omega=1$ and the situation of current data is still an open question at this time.

Cho and Kantowski [45] have in fact derived a unique measure for FLRW fluid models and concluded that the measure is singular at $\Omega = 1$; that this singularity, combined with the time evolution of Ω , distorts the distribution of Ω to be concentrated near 1 at early times. This concentration is misleading as the casual observer may be misled to conclude that Ω should be exactly 1. This appears to be unlikely from an observational perspective.

2.12 Conclusion, and discussion of cosmological data

The basic point of the phase plane approach is to attempt to construct a formal, qualitative as well as quantitative, framework with which to represent observational limitations directly on the phase planes describing the evolution of the universe. I have generalized previous work in the field and provided the equations for basically all the possible planes in the context FLRW universes.

In the preceding section I have generalized all the major work forming the basis of the phase-plane approach in FLRW cosmology, that is I have generalized the work of Madsen et al [10] to include the work of Stabell et al [8] (dust + Λ) and have generalised their use of the (Ω, q) planes to all multi-fluid models, as well as reproducing the specific model that they discussed which is a special case in my formulation. I have, however not, considered the use of epochs *i.e.* is the matching of single fluid epochs as an approximation to multifluid models, which is dealt with by Madsen et al [11]. I have also demonstrated the link between $(\Omega_1, \Omega_2, \dots, \Omega_N)$ planes and $(\Omega_1, \dots, \Omega_{l-1}, q, \Omega_{l+1}, \dots, \Omega_N)$ planes, *i.e.* these are in fact equivalent planes as one can be easily transformed into the other through simple transformations. This not only explains why the technique used by Stabell et al [8] worked but also leads to the work of Ehlers et al [9] on the $(\Omega_1, \Omega_2, \Omega_3)$ plane in the form of the (q, Ω_r, Ω_b) planes and the $(\Omega_r, \Omega_b, \Omega_\Lambda)$ planes that they discuss. The basic idea is to use the limits on q_0 , $\Omega_{r|0}$ and $\Omega_{b|0}$ to construct a volume of possible applicable current values on the phase-plane to determine the subset of consistent and unique evolutionary histories that could be possible in this model. Ehlers and Rindler[9] seem to conclude that the existence of the bigbang, in the model they discuss, is inevitable.

The basic results of this approach as outlined here is that two problems need to be addressed before any serious use of phase planes can be made beyond single fluids. How to usefully represent 3 and higher dimensional planes (adding an extra dimension in as

a colour sequence ?), and how and when the multi-fluid evolution equations can be used to map the behaviour into the invariant submanifold $(\Omega_1, \dots, \Omega_N)$ and then from this down into some slicing using the choice of $\gamma(S)$, as suggested and demonstrated by Madsen et al [10]. This leads to the issue of how to preserve the uniqueness of the individual trajectories in the phase phase and whether or not a generic (singular point structure is retained under the projection into a lower dimensional plane) phase-plane is sufficient with regard to comparing data to the evolutionary curves [10] (see Sections 2.4 2.5.8 and 2.10 as put forward by Ellis).

The best phase-plane like representation of data is that given by Gott et al [13] in the parameters H and Ω that is, possible values of Ω_0 and H_0 are plotted, the idea being to superimpose a phase plane (Ω, H) on this data set, which then introduces the possibility of discussing the evolution of such a parameter space. I have reproduce such (Ω, H) planes (Sections 2.2.2 2.5.3 and 2.6), it was concluded that although the (Ω, H) planes could be reproduced, a comparison with the model dependent limits in the (Ω_0, H_0) is difficult and possible ambiguous as they have explicitly used models such a single fluid with cosmological constant in the the reduction of some of the data, which means that only such models phase planes can be used, while the data is given as a (Ω, H) like plane which is only applicable for single fluids. In general one must deal with the $(\Omega_1, \Omega_2, \dots, \Omega_N, H)$ planes which cannot be simply mapped down into (Ω, H) in a general fashion unless one is willing to forgo uniqueness. This can be done in the context of the concept of effectively single fluids (see Section 2.6.1), a concept I use both to reproduce the work of Stabel et al [8] and as a technique with which to open the way for the consistent application of the (Ω_0, H_0) plane of Gott et al [13] in the FLRW context. The key-point is that the data is strongly model dependent in an explicit sense and hence must be dealt with extreme care, a project that would be far too extensive to be carried out here.

I conclude that although the dynamical systems approach is a motion towards understanding the relationship between data and the models used in cosmology, it is still far from being useful.

- (1) The reduction of data in a model independent fashion needs to be addressed.
- (2) A consistent presentation of higher dimensional phase planes (or the consistent projection into lower dimensional data representations) needs to be addressed. This seems to be well addressed in the context of the structurally stable planes projected using $\gamma = \gamma(S, M_i, \gamma_i)$.

- (3) A clear understanding of the questions of cosmology, the questions cosmology is trying to answer and the importance of the data with regards to these questions. This all boils down to the question of whether or not *the standard cosmological model for the recent universe is consistently supported by observations* [22]. This leads one to the issue of what the range of viable parameters values are *i.e.* the observational values of q_0 , Ω_0 , $\Omega_r|_0$, $\Omega_b|_0$, Λ and H_0 , this is still considered, at this time, to be an open question but a brief summary of the current status of the observational data significant to cosmology is presented below.

2.12.1 Cosmology data

This summary should not be considered definitive but rather as a brief survey of the current situation with regard to the available data sets (this is based primarily on [22] [19] [23] [13]).

- Ages
 - *White Dwarf*, minimum age estimated from *cooling ages* to be 12 - 8 Gyrs using a minimum stellar mass of 0.6 solar mass.
 - *Globular cluster*, using *evolutionary age* estimates. The errors in physics and chemistry of this technique are of the order of 15%, hence the big problem with this technique is the distance calibration. There are two procedures available, which give two independent globular cluster ages :
 - * Fit nearby Main sequence stars
 - * Use RR Lyrae stars : RR Lyrae stars are calibrated locally (i) given faint values (Galactic stars) (18 Gyrs) and (ii) brighter values (extra-Galactic) using Cepheids in the Large Magellanic Cloud (LMC) (14 Gyrs). Which is correct ? (i) The Galactic technique is consistent with both *Statistical parallaxes* and *OH line spectra* distance determinations to the galactic bulge. (ii) The problem here is either *RR Lyrae stars* in the LMC are brighter than the galactic ones, which is (unlikely) or it may be that the problem is with the RR Lyrae star classification (likely) or that the *Cepheids* are misleading us. The problem with the extra galactic method is that the RR Lyrae galactic distance scale is consistent and tied to the trigonometric parallaxes. But there does seem to be a problem with the RR Lyrae population gradients along the horizontal branch in the HR diagram while

the Cepheid scale is in good accord with the distance determination using SuperNova (SNe) geometry and expanding photosphere techniques (eg in SN1987).

- **H_0 , the Hubble parameter**

- Distance to the *Virgo Cluster* : 4 independent measurements of Cepheids in the Virgo clusters give the same distance within the individual errors. The ratio of the Virgo and Coma distances are well known hence the redshift distance to Coma is known; this gives $H_0 = 80 \pm 8$ km/s/Mpc. The problem with this is that Cepheids have been observed in the spirals of the clusters which might all be in the front of the core giving a systematic error.

The LMC distance which is used to calibrate the Cepheids can be checked :

- * Ring around SN1987A (ring delay times gives the distance of about 47 to 57 Kpc and is consistent with Cepheid distances),
- * C-type RR Lyrae stars distance determined using star pulsation theory are also consistent with Cepheid distances,
- * Binary Cepheids in LMC,
- * Expanding photosphere method for type II SNe.

The Cepheid scale seems to be more or less correct.

- *Tully-Fisher* : $H_0 = 84 \pm 8$ km/s/Mpc
- *Planetary nebulae* : $H_0 = 86 \pm 18$ km/s/Mpc
- *Fornax cluster* : $H_0 = 75 \pm 8$ km/s/Mpc
- *Surface Brightness fluctuations of elliptical galaxies* : $H_0 = 80 \pm 12$ km/s/Mpc

These measurements are all inconsistent with those of the Sandage-Tammann values. Hence one is basically left to accept the values as $66 < H_0 < 82$ km/s/Mpc at a 95% CI.

- **q_0 , the deceleration parameter.** This program has been basically unsuccessful, although hopeful.

- Magnitude-Redshift evolution of galaxies is problematic due to unknown evolutionary effects (unsuccessful).
- Angular-Diameter measurements have no standard size objects to calibrate with to use where the evolution is well understood (unsuccessful).

- Mass-Redshift curves for SNe : these are less vulnerable to evolutionary effects as the local physics is determined by specific nova events. This method is in use but the statistics are poor (hopeful).

- Λ , **Cosmological constant**

- Lensing statistics : $\Omega_\Lambda < 0.7$ 95% CI, the calculations are weakened when not using $\Omega_m + \Omega_\Lambda = 1$.
- Direct q_0 measurements, if q_0 is measured to be negative then one could expect the cosmological constant to be non-zero.

Most Λ determining techniques other than the direct q_0 measurements are strongly model dependent, and hence problematic. Nonetheless Λ may be important to solve the age problem.

- $\Omega_{r|0}$, **Radiation Density** is the only well known cosmological parameter, from CMBR observation one has that the radiation is at a temperature of $2.73 (\pm 10^{-5})$ K, and using The Stefan Boltzmann law $\mu = aT^4$ and the definition of the density parameter $\Omega = \kappa\mu/3H$ along with the Hubble parameter ($H_0 = 100h$ km/s/Mpc for $0.6 < h < 0.9$), one can (for instance) find Ω_r .

- $\Omega_{m|0}$, **Matter Density,**

- *Field galaxies* imply $\Omega_m > 0.02$ and $\Omega_m > 0.03$ if nearby galaxies have typical mass-to-light ratios.
- Measures of *large scale flows* and *CBR anisotropies* give $\Omega_m \geq 0.3$.
- Current estimates from the *primordial abundances* place the relative abundances of Deuterium (2H), 3He , 4He and Li in accordance with nucleo-synthesis predictions only if the baryonic material has a density parameter in the range $0.01 < \Omega_b h^2 < 0.015$. A higher value produces too much 4He and 7Li , a value lower produces too much deuterium and 3He . From the dynamics the total density parameter is in the range $0.1 < \Omega_0 < 0.3$. Hence there is not enough baryonic matter to satisfy the dynamical constraints. In fact one needs at least $\Omega_m h^2 \approx 0.003$ to account for star and gas absorbers in stellar matter and HI regions, and at least $\Omega_m h^2 \approx 0.03$ in the halos of galaxies to account for the rotation curves. It appears that more non-baryonic matter is needed to match the $\Omega_m \simeq 0.2$ estimate we need more than Ω_b has to offer. Even if the overall

lowest value Ω_m is taken as $\Omega_m \approx 0.03$ the problem is still there, although it could be that the dust content estimates of galaxies may be at fault, some estimates place this out by a factor of 10 in galactic disks, but this would still probably not correct for the halo mass requirements..

- Ω_0 , **Total Density** The Observationally preferred value of the total density parameter is set at $\Omega_0 \simeq 0.2 - 0.3$.

Basically if one has the bounds on H_0 , q_0 , Ω_0 , Ω_m and Ω_m in a model independent fashion, the phase-plane approach as outlined in Part A of this thesis may be very useful. I conclude with the observational limits as an indication of where the FLRW universe model is at this current time with regard to the data. The H_0 values have large discrepancies, the q_0 value is basically unknown, Ω_m has bounds but there is still no model independent consensus, the only value well known is that of the radiation temperature. Until the data issue is resolved in a *model independent fashion*, it will be very difficult to make any practical use of the phase plane methodology, beyond the basic single fluid approach of Madsen et al [10] and the theory outlined in Part A of this thesis.

I now have a close look at radiation models in the sense of Relativistic Kinetic Theory (RKT) and the Gauge Invariant Covariant (GIC) perturbation theory, remembering that the best cosmological data available, other than nucleosynthesis, is that of the CMBR. The significance of a good physical understanding of the radiation content of the universe should be obvious. It may be helpful at this stage for the reader to have a quick look at Section 3.5.8.

Chapter 3

Kinetic theory and CBR

3.1 Imperfect fluids

Once again consider that any symmetric tensor T_{ab} can be decomposed relative to a unit time-like vector u^a :

$$T_{ab} = \mu u_a u_b + p h_{ab} + q_a u_b + u_a q_b + \pi_{ab} \quad (3.1)$$

where $\pi_{[ab]} = 0 = \pi_{ab} u^b = \pi_a^a = q_a u^a$. That is, we now consider an energy-momentum tensor of this general form, where we include a flux term q_a and an anisotropic pressure term π_{ab} . This time, we explicitly find the equations of motion for such a general symmetric energy momentum tensor using the requirement of energy conservation *i.e.* using $T^{ab}_{;b} = 0$. Projecting orthogonal to u^a (onto the spatial planes¹) with the usual projection tensor h_{ab} , and parallel using u_a , we can find the momentum and energy conservation equations using the following relations for a generally decomposed covariant derivative:

$$\begin{aligned} u_{a;b} &= v_{ab} - \dot{u}_a u_b, \\ v_{ab} &= \theta_{ab} + \omega_{ab}, \quad \theta_{ab} = \theta_{(ab)}, \quad \omega_{ab} = \omega_{[ab]}, \\ \theta_{ab} &= \sigma_{ab} + \frac{1}{3} \theta h_{ab}. \end{aligned} \quad (3.2)$$

Here ω_{ab} , σ_{ab} and $\theta = 3H$ are the usual vorticity, shear, and volume expansion. The conservation equations are,

$$\dot{\mu} + (\mu + p)\theta + \pi_{ab} \sigma^{ab} + q^a_{;a} + \dot{u}_a q^a = 0, \quad (3.3)$$

$$(\mu + p)\dot{u}_a + h_a^c (p_{;c} + \pi^b_{c;b} + \dot{q}_c) + (\omega_a^b + \sigma_a^b + \frac{4}{3} \theta h_a^b) q_b = 0, \quad (3.4)$$

¹Strictly speaking, there are no well defined spatial surfaces for $\omega \neq 0$ as one may move off the fundamental observer's worldline at some time t and although remaining restricted to the spatial slice by keeping $u^a h_{ab} = 0$ can return to the fundamental observer's original worldline at some different time t^* .

the Raychaudhuri equation is

$$3\dot{H} + 3H^2 + 2(\sigma^2 - \omega^2) + \frac{1}{2}(\mu + 3p) - \dot{u}^a{}_{;a} = 0, \quad (3.5)$$

and from the generalized Gauss-Codacci² relations,

$$3H^2 - \mu - \sigma^2 + \omega^2 = -3K. \quad (3.6)$$

The pressure term may be modified to include the effects of bulk viscosity: $p \rightarrow p_t - \xi\theta$, where ξ is the coefficient of bulk viscosity. This has been discussed to some degree in section 2.7 where we considered equation of state modifications possible in perfect fluid models.

Often the *viscous assumption* is made, that is $\pi_{ab} = -2\eta\sigma_{ab}$. This is valid for $|\pi_{ab}|/p \ll 1$ and in the stationary regime. In this limit the anisotropic pressure is very much smaller than the isotropic pressure. Here η is the coefficient of shear viscosity. The energy-momentum tensor can then be reduced to the form

$$T_{ab} = \mu u_a u_b + (p - \zeta\theta)h_{ab} - 2\eta\sigma_{ab} + q_{(a}u_{b)}. \quad (3.7)$$

This is discussed in [39], but with regard to the FLRW case they considered only the situation of a fluid source where there is no shear. Hence these are models with bulk viscosity, and the singularity structure of the evolution curves of such models are characterized by the perfect fluid model planes following Jones et al [12]. A crucial point here is that, strictly speaking, bulk viscous models are irreversible if they are motivated from a thermodynamic and hence physical basis. Such models are discussed by [59] [46] in the context of dissipative cosmological models.

3.2 Gas models

An idealized fluid model of the matter is often inappropriate. The self-consistent microscopic model of relativistic kinetic theory gives a far more detailed physical description [40]. The fluid model involves information loss. There are non-equilibrium situations such as the evolution of perturbed FLRW universes into inhomogeneous or anisotropic universes, where the kinetic theory approach is well suited in handling the interaction terms. It also provides a consistent basis for the thermodynamics of fluid phenomena *eg* the theory of Isreal and Stewart is stable and causal [41]; it is a natural generalization that

²Gauss-Codacci relations have $\omega = 0$.

emerges from kinetic theory. One defines the observable tensors in terms of the distribution function $f(x^i, p^b)$, where the variables are not in 4 dimensional space-time any more (following worldlines), but are rather in 8 dimensional Liouville phase space on the mass shell, given as $P_m \{x^i, p^a | p^a p_a = -m^2\}$. What this means is that the Boltzmann equation is propagating the distribution function down a trajectory in phase space parameterized by some affine parameter. Thus all the coordinates are independent in this space. One obtains

$$p^i = \frac{dx^i}{d\lambda}, \quad (3.8)$$

$$\frac{dp^i}{d\lambda} = -\Gamma^i_{jk} p^j p^k, \quad (3.9)$$

if the particles move freely between collisions. We must of course reject the possibility that we can solve simultaneously the equations of motion of a large number of interacting particles; even if we could, the results would be of no practical value. A statistical description involving a reduction in the amount of information to be handled is required. Relativistic kinetic theory provides such a theory in the context of general relativity.

dN is the number of particles seen by an observer u^a in a volume dV in the rest space, with the 4-momentum in a range dP around p^a . The one-particle distribution function $f(x^a, p_a)$ determines the number density of the particles about the event x^a in the rest space with momentum p^a in the tangent space T_x of event x^a :

$$dN = f(x^i, p^a) (-u_a p^a) dV dP. \quad (3.10)$$

Specifically as particle worldlines are not directed normally to the space-like volume element dV , we have projected p_a onto u_a , the observer's 4-velocity, to get the correctly projected volume element in the observer's rest frame. Physically we only need to deal with :

(1) Particles on mass-shell,

$$p_a p^a = -m^2. \quad (3.11)$$

This means that $dP \rightarrow \delta(p_a p^a + m^2) dP$.

(2) Particles on the forward light cone,

$$-u_a p^a > 0. \quad (3.12)$$

Hence $dP \rightarrow 2\theta(-u_a p^a) \delta(p_a p^a + m^2) dP$.

From $f(x^a, p_a)$ we define the following useful integrals:

- The *Energy momentum tensor*:

$$T^{ab} = \int f p^a p^b dP. \quad (3.13)$$

- The *Particle 4-current*:

$$N^a = \int p^a f dP. \quad (3.14)$$

- The *Entropy 4-current*:

$$S^a = - \int p^a f (\ln(f) - 1) dP. \quad (3.15)$$

The number density n , and the entropy density s define the associated number and entropy flux densities by

$$N^a = n u^a + k^a, \quad k^a u_a = 0, \quad (3.16)$$

$$S^a = s u^a + r^a, \quad r^a u_a = 0. \quad (3.17)$$

3.2.1 Volume elements

The volume element at event x^a is spanned by $d_1 x^a$, $d_2 x^b$, $d_3 x^c$ and $d_4 x^d$ and is given by

$$dV = \eta_{ijkl} d_1 x^i d_2 x^j d_3 x^k d_4 x^l \quad (3.18)$$

where η_{ijkl} is the fully antisymmetric tensor with $\eta_{0123} = \sqrt{-g}$ and $g = \det(g_{ij})$. The volume in the tangent space is given by :

$$dP = \eta_{abcd} d_1 p^a d_2 p^b d_3 p^c d_4 p^d. \quad (3.19)$$

We can decompose the particle 4-momentum as :

$$p^a = E u^a + \lambda e^a, \quad (3.20)$$

where $e^a e_a = 1$, $e^a u_a = 0$. Here $p^a = (-u^b p_b) u^a + (h^a_b p^b)$ which corresponds to $(\lambda e^a = h^a_b p^b)$ and $(E = -u^b p_b)$, where $e^a de_a = 0 = u^a de_a$. We find that $dp^a = dE u^a + d\lambda e^a + \lambda de^a$. Using the form for η_{0123} given above and (3.19) we have that

$$dP = (+1) dE \wedge d\lambda \wedge \lambda de^{(1)} \wedge \lambda de^{(2)}. \quad (3.21)$$

Noticing that $de^{(1)} \wedge de^{(2)} \equiv d\Omega$, using the mass-shell condition $-E^2 + \lambda^2 + m^2 = 0$ we can transform coordinates *i.e.* we can change from $\{E, \lambda, e^a\} \rightarrow \{E, m, e^a\}$ or $\{\lambda, m, e^a\}$. We also wish to keep positive energies only, $E > 0$. Hence we have that $E = +(\lambda^2 + m^2)^{\frac{1}{2}}$

and $\lambda d\lambda = EdE - m dm$. We have not assumed that m is constant, we have only changed coordinates. *i.e.* we have used

$$-u_a p^a = E, \quad (3.22)$$

$$p_a p^a = -E^2 + \lambda^2. \quad (3.23)$$

Consider:

- For $\{E, \lambda, e^a\} \rightarrow \{E, m, e^a\}$, use $\lambda = +(E^2 - m^2)^{\frac{1}{2}}$. The distribution function then becomes

$$f(x^i, p^a) = f(x^i, m, E, e^a). \quad (3.24)$$

The measure becomes

$$dP^+ = (+1)dE \wedge d((E^2 - m^2)^{\frac{1}{2}}) \wedge \lambda de^{(1)} \wedge \lambda de^{(2)} \quad (3.25)$$

$$= dE \wedge \left(\frac{EdE - m dm}{(E^2 - m^2)^{\frac{1}{2}}} \right) \wedge (E^2 - m^2) d\Omega \quad (3.26)$$

$$= m dm \wedge (E^2 - m^2)^{\frac{1}{2}} dE \wedge d\Omega \equiv m dm \cdot \lambda dE \cdot d\Omega. \quad (3.27)$$

- For $\{E, \lambda, e^a\} \rightarrow \{\lambda, m, e^a\}$, use $E = +(\lambda^2 + m^2)^{\frac{1}{2}}$. The distribution function then becomes

$$f(x^i, p^a) = f(x^i, \lambda, m, e^a). \quad (3.28)$$

The measure becomes

$$dP^+ = (+1)(\lambda d\lambda + m dm) \wedge d\lambda \wedge \lambda^2 d\Omega \quad (3.29)$$

$$= m dm \wedge \frac{\lambda^2 d\lambda}{(\lambda^2 + m^2)^{\frac{1}{2}}} \wedge d\Omega. \quad (3.30)$$

The direction vectors e^a define a direction and are described in terms of angles, so $de_{(1)}^a$ and $de_{(2)}^b$ subtend a solid angle $d\Omega$.

To find the on mass-shell volume element in the tangent space, *i.e.* the volume in tangent space for a constant mass value, we must integrate out the mass dependence using a delta function to isolate the particular constant mass-value desired. For m constant we have from the mass-shell condition (3.11) that $\lambda d\lambda = EdE$. So we have from (3.11), (3.12) and (3.19) that

$$dP^+ = 2\theta(-u_a p^a) \delta(p_a p^a + m^2) dP$$

$$\begin{aligned}
&= 2\theta(+E)\delta(-E^2 + \lambda^2 + m^2)dP \\
&= \frac{\delta((\lambda^2 + m^2)^{\frac{1}{2}} - E)}{(\lambda^2 + m^2)^{\frac{1}{2}}}dE \wedge d\lambda \wedge \lambda^2 d\Omega.
\end{aligned} \tag{3.31}$$

We have used the usual relations for delta functions ³ and the form for the determinant of the tetrad metric $g = -1$ ⁴. If we integrate over E to pull out the mass-shell volume,

$$dP_m^+ = \frac{d\lambda \wedge \lambda^2 d\Omega}{(\lambda^2 + m^2)^{\frac{1}{2}}}. \tag{3.32}$$

Using $\lambda d\lambda = E dE$ and $E = +(\lambda^2 + m^2)^{\frac{1}{2}}$ we find that

$$dP_m^+ = \lambda dE \wedge d\Omega \equiv \lambda dE \cdot d\Omega, \tag{3.33}$$

as we are in orthonormal coordinates.

3.2.2 Harmonic decomposition

It is convenient to write the one-particle distribution function in the orthonormal tetrad basis as

$$f = F + F_a e^a + F_{ab} e^a e^b + F_{abc} e^a e^c e^d + \dots, \tag{3.34}$$

where F_{A_i} are Projected, Symmetric and Trace-Free (PSTF) ⁵ tensors *i.e.*

$$\begin{aligned}
F_{ab} &= F_{(ab)}, \\
F_{ab} &= F_{\langle ab \rangle} = h_a^c h_b^d F_{(cd)} - \frac{1}{3} F_{cd} h^{cd} h_{ab}, \\
F_{A_i bc} h^{bc} &= 0, \\
F_{ab} u^b &= 0.
\end{aligned} \tag{3.35}$$

This is useful as we can now write out the form of the energy momentum tensor, the particle 4-current and the entropy 4-current, in terms of these variables. We would expect

³ $\delta(x^2 - a^2) = \frac{1}{2|a|}(\delta(x - a) + \delta(x + a))$ and $\delta(ax) = \frac{\delta(x)}{a}$

⁴ $g_{ab} = g_{ij} E_a^i E_b^j = \mathbf{E}_a \cdot \mathbf{E}_b = \text{diag}(-1, +1, +1, +1)$.

⁵Projected Symmetric Trace-Free tensor (PSTF)

$$\begin{aligned}
A_{\langle ab \rangle} &= h_a^c h_b^d A_{(cd)} - \frac{1}{3} A_{cd} h^{cd} h_{ab} \\
A_{\langle ab \dots c \rangle} &= A_{\langle (ab \dots c) \rangle} \\
A_{\langle ab \dots c \rangle} u^c &= 0 = A_{\langle ab \dots c \rangle} h^{ab}
\end{aligned}$$

the pressure and energy density to be due to the scalar moment, the energy-momentum flux to be due to the vector moment projected along the specific orthonormal tetrad basis in the 3 space, and the anisotropic pressure to be due to the tensor moment.

• *Energy momentum*

$$T^{ab} = \int_m^\infty p^a p^b f dP_m^+ \quad (3.36)$$

$$= \int p^a p^b (F + F_c e^c + F_{cd} e^c e^d + F_{cde} e^c e^d e^e + \dots) dP_m^+. \quad (3.37)$$

Using

$$\begin{aligned} p^a p^b &= (Eu^a + \lambda e^a)(Eu^b + \lambda e^b) \\ &= E^2 u^a u^b + E\lambda(e^a u^b + e^b u^a) + \lambda^2 e^a e^b \end{aligned} \quad (3.38)$$

this becomes:

$$\begin{aligned} T^{ab} &= 4\pi \int (E^2)(\lambda dEd\Omega) (F + F_{dc} e^d e^c + F_{dcfg} e^d e^c e^f e^g + \dots) u^a u^b \\ &\quad + \int (E\lambda)(\lambda dEd\Omega) (F_d e^d e^{(a} u^{b)} + F_{dcf} e^d e^c e^f e^{(a} u^{b)} + \dots) \\ &\quad + \int (\lambda^2)(\lambda dEd\Omega) (F e^a e^b + F_{dc} e^d e^c e^a e^b + \dots). \end{aligned} \quad (3.39)$$

Now,

$$\int \tilde{e}^{A_l} d\Omega = \int e^{a_1} e^{a_2} \dots e^{a_l} d\Omega = \begin{cases} 0 & : \text{ odd } l \\ \frac{4\pi}{l+1} h^{(a_1 a_2} h^{a_3 a_4} \dots h^{a_{(l-1)} a_l)} & : \text{ even } l, \end{cases} \quad (3.40)$$

which is proven ⁶ in [24]. This is shown by noticing that e^a is orthogonal to u^a and remembering that F_{A_l} is a PSTF tensor. Using the PSTF conditions (3.35) it is found that

$$\begin{aligned} T^{ab} &= 4\pi \int E^2 \lambda dE \left(F + \frac{1}{5} F_{dcfg} (h^{dc} h^{fg} + h^{fc} h^{dg} + h^{gc} h^{fd} + \dots) + \dots \right) u^a u^b \\ &\quad + 4\pi \int E \lambda^2 dE \left(\frac{1}{3} F_d \frac{1}{2} (h^{(da)} u^b + h^{(db)} u^a) + \frac{1}{4} F_{dcf} \frac{1}{2} (h^{(dc} h^{fa)} u^b + h^{(dc} h^{fb)} u^a) + \dots \right) \\ &\quad + 4\pi \int \lambda^3 dE \left(\frac{1}{3} F h^{(ab)} + \frac{1}{5} F_{dc} h^{(ab} h^{dc)} + \dots \right) \\ &= 4\pi \int E^2 \lambda dE F u^a u^b + 4\pi \int E \lambda^2 dE \frac{1}{3} F_d h^{d(a} u^{b)} \\ &\quad + 4\pi \int \lambda^3 dE \frac{1}{3} F h^{ab} + 4\pi \int \lambda^3 dE \frac{1}{5} F_{dc} h^{(ab} h^{dc)}. \end{aligned} \quad (3.41)$$

⁶ $\int_0^{2\pi} d\phi \int_0^\pi (\cos \theta)^r \sin \theta d\theta = \frac{2\pi}{r+1} (\cos \theta)^{r+1} \Big|_0^\pi = \frac{4\pi}{r+1}$

Notice that due to the PSTF nature of F_{A_l} , $F_{dcfg}h^{(dc}h^{fh)} = 0$ and so on. Recall the most general form of a symmetric tensor $T_{ab} = \mu u^a u^b + q^{(a}u^{b)} + \Pi^{ab}$ where $\Pi^{ab} = p h^{ab} + \pi^{ab}$. Hence we can uniquely identify from (3.41) the following:

$$\mu = 4\pi \int E^2 \lambda dEF, \quad (3.42)$$

$$q_a = \frac{4\pi}{3} \int E \lambda^2 dEF_a, \quad (3.43)$$

$$\Pi^{ab} = \left(\frac{4\pi}{3} \int \lambda^3 dEF \right) h^{ab} + \left(\frac{4\pi}{5} \int \lambda^3 dEF_{dc} h^{(ab} h^{cd)} \right). \quad (3.44)$$

From (3.44) we are able to identify the isotropic and anisotropic pressure contributions, respectively :

$$p = \frac{4\pi}{3} \int \lambda^3 dEF, \quad (3.45)$$

$$\pi^{ab} = \frac{4\pi}{5} \int \lambda^3 dEF_{dc} h^{(ab} h^{cd)}. \quad (3.46)$$

Notice that for photons, *i.e.* radiation, we can derive the usual equation of state, as $E = \lambda$ because $m = 0$ (using $p^a p_a = -m^2$). Using (3.42) and (3.45) we find that $p = \frac{1}{3}\mu$. It can be shown ⁷ that $F_{dc}h^{(ab}h^{cd)} = \frac{2}{3}F^{ab}$. Hence we can identify the form of the anisotropic pressure using this and equation (3.44);

$$\pi_{ab} = \frac{8\pi}{15} \int \lambda^3 dEF_{ab}. \quad (3.49)$$

All higher harmonics in the integral expansion cancel since the coefficients of f , the distribution function, are PSTF tensors, *i.e.* $F_{A_l-2bc}h^{bc} = 0$. For example $F_{dcfg}h^{(dc}h^{fg)} = 0$ will always be an orthogonal projection, *i.e.* it will always be a projected tensor contracting indices such that it will vanish. Similarly $F_{dcfg}h^{(dc}h^{fg}h^{ab)} = 0$ and $F_{dcf}h^{(dc}h^{fa)} = 0$.

The kinematics and dynamics are determined directly by the first three harmonics. The first order or *vector* anisotropy F_a determines the fluxes. The second order or *tensor* anisotropies F_{ab} determines the anisotropic stresses.

7

$$h^{(ab}h^{cd)} = \frac{1}{4!} (h^{ab}h^{cd} + \dots), \quad (3.47)$$

i.e. there are 24 terms, and 8 of these will have h^{cd} contributions which when contracted with F_{cd} would vanish as F_{cd} is a PSTF tensor. Notice that $h^{ab} = h^{(ab)}$ and that $F_{dc}h^c h^d = F_{ab}$ hence

$$F_{dc}h^{(ab}h^{cd)} = \frac{1}{4!} 16 F^{ab} = \frac{2}{3} F^{ab}. \quad (3.48)$$

- *particle 4-current*

$$\begin{aligned} N^a &= \int_m^\infty \int_\Omega p^a f dP_m^+ \\ &= \int (Eu^a + \lambda e^a)(F + F_c e^c + F_{cd} e^c e^d + \dots) dP_m^+. \end{aligned} \quad (3.50)$$

Expanding, we find

$$\begin{aligned} N^a &= \int E \lambda dE d\Omega (F + F_c e^c + F_{cd} e^c e^d + \dots) u^a \\ &\quad + \int \lambda^2 dE d\Omega (F + F_c e^c + F_{cdf} e^c e^d e^f + \dots) e^a \\ &= 4\pi \int \lambda E dE \left(F + \frac{1}{3} F_{cd} h^{cd} + F_{cdfg} h^{cd} h^{fg} + \dots \right) u^a \\ &\quad + 4\pi \int \lambda^2 dE \left(\frac{1}{2} h^{ca} F_c + \frac{1}{4} F_{cdf} h^{cd} h^{fa} + \dots \right) \\ &= \left(4\pi \int \lambda E dE F \right) u^a + \left(4\pi \int \lambda^2 dE F_c \right) h^{ca}. \end{aligned} \quad (3.51)$$

We can use $N^a = nu^a + k_c h^{ac}$ and identify

$$n = 4\pi \int E \lambda dE F \quad (3.52)$$

$$k_c = \frac{4\pi}{3} \int \lambda^2 dE F_c. \quad (3.53)$$

- *entropy 4-current*

$$S^a = - \int p^a f (\ln(f) - 1) dP. \quad (3.54)$$

Clearly the entropy current is non-linear and includes all higher order moments.

3.2.3 The harmonics expansion related to Spherical harmonics

It may be useful to be able to get the usual spherical harmonic type expansion converted into the harmonic expansion. This can be done by choosing a particular tetrad basis such as

$$e^a = \{0; \cos \theta \sin \phi; \sin \theta \sin \phi; \cos \phi\}. \quad (3.55)$$

One can link the usual spherical harmonic expansion to the harmonic expansion,

$$f = \sum a_{lm} Y_{lm}(\theta, \phi) = \sum F_{A_l} e^{A_l}(\theta, \phi). \quad (3.56)$$

Some spherical harmonic relations are given in the Appendix (Section A.5).

3.2.4 Kinetic theory

Mean free path

The mean free path, λ_f , is the average distance a particle must free stream until it collides with another gas particle.

Characteristic length

The characteristic length, λ_c , is the average distance between particles in the gas, or some scale length indicative of the underlying spacing or structure in the gas or fluid. This measure has no definitive definition and is discussed in De Groot's book [29]. The ratio of these lengths can be used to separate out different physical regimes.

Define the parameter ϵ by

$$\epsilon = \frac{\lambda_f}{\lambda_c}. \quad (3.57)$$

- $\epsilon \gg 1$ *Free-flow regime*

Here the fluid description is totally inapplicable; in the limit $\epsilon \rightarrow \infty$ we have collision-free equilibrium.

- $\epsilon \approx 1$ *Transition regime*

A fluid description is only valid for $\epsilon < 1$.

- $\epsilon \ll 1$ *Hydrodynamic regime*

Collisions dominate in the limit $\epsilon \rightarrow 0$, describing detailed balancing. Here a fluid description would naturally appear.

A fluid model is clearly appropriate if the collision dominance is assumed to imply that the gas is sufficiently close to equilibrium to allow the definition of a unique smoothly varying 4-velocity and energy density. Kinetic theory is only applicable for particles whose de Broglie wavelengths are smaller than their mean free path. For collision-free matter this is clearly the case. The de Broglie wavelength $2\pi/p$ has to be smaller than the Hubble length, and the temperature must be below the Plank temperature [27]. Thermal equilibrium can only be maintained in an expanding universe for (i) the case of massless particles in spacetimes which have a conformal time-like Killing vector, such as FLRW models and (ii) the non-relativistic regime [40].

Non-equilibrium general relativistic kinetic theory uses a kinetic theory approach which is more fundamental and consistent than a phenomenological fluid picture and its associated thermodynamics. A gas model is assumed where, as before, $f = f(x^i, p^a)$ is some

single particle distribution function (3.10), describing in the usual way the distribution of particle near an event x^i with momentum p^a . Here coordinate components with late roman and greek are used, $x^i = (x^0, x^\rho)$; for general or tetrad bases, early letters are used, $p^a = (p^0, p^\alpha)$. The evolution of the distribution function is given by the Liouville equation,

$$L(f) = C[f]. \quad (3.58)$$

Here $C[f]$ is some functional giving the rate-of-change of f due to particle collisions, *i.e.* $C[f]$ is the collision term. $L(f)$ is the Liouville operator, which gives the derivative of f along the paths of motion of the particles. When collision are ignored, *i.e.* $C[f] = 0$, this reduces to the Vlasov or collision free Liouville equation,

$$L(f) = 0. \quad (3.59)$$

Between collisions the particles are considered to move freely on their paths: $x^i(\tau)$ are geodesics, here $\frac{dx^i}{d\lambda} = p^i$. From (3.8) and (3.9), *i.e.*

$$p^a{}_{;b} p^b = 0, \quad (3.60)$$

$$\frac{dp^a}{d\lambda} + \Gamma^a{}_{ab} p^a p^b = 0. \quad (3.61)$$

Interestingly enough these equations can also be easily used to give the redshift of a photon in full generality in terms of dynamic variables in a tetrad frame⁸. The rate of change of the distribution function down a worldline is

$$L_\lambda(f) = \left(\frac{\partial f}{\partial x^i} \right)_p \frac{dx^a}{d\lambda} + \left(\frac{\partial f}{\partial p^a} \right)_x \frac{dp^a}{d\lambda}. \quad (3.62)$$

i.e. The Liouville operator L is given by $L = \frac{d}{d\lambda}$ for any affine parameter λ parameterizing a trajectory in phase space (with measure $d\Gamma = dx \wedge dp$).

Using the previous definitions for the 4-current N^a , energy momentum tensor T^{ab} and the entropy current S^a , we can modify the usual source-free conservation equations (*i.e.* $C[f] = 0$ below) in such a way as to take collisions into account:

$$N^a{}_{;a} = \int C[f] dP_m^+ = N, \quad (3.63)$$

$$T^{ab}{}_{;a} = \int p^b C[f] dP_m^+ = 4\pi J^b, \quad (3.64)$$

$$S^a{}_{;a} = - \int \ln(f) C[f] dP_m^+ = S. \quad (3.65)$$

⁸See Appendix (Section A.7)

By definition, in *equilibrium* the entropy current must vanish. Mathematically expressed, in equilibrium

$$S^a_{;a} = 0; \quad (3.66)$$

no entropy is generated⁹. The H-theorem requires that $S^a_{;a} \geq 0$.

$\int \Phi C[f] dP_m^+$ is the rate of production per unit volume of some as-yet unspecified property Φ via collisions. If this rate is zero then Φ is a collisional invariant; eg $N^a_{;a}$ is the particle number creation rate and $T^{ab}_{;a}$ is the 4-momentum production rate. As 1 and p^a are collisional invariants, one attains conservation of energy and particle number ($N^a_{;a} = 0$ and $T^{ab}_{;a} = 0$).

The collision term can be decomposed into a harmonic expansion,

$$C = b + b_a e^a + b_{ab} e^a e^b + \dots = \sum b_{A_l} e^{A_l} \quad (3.70)$$

where $b_{A_l} = b_{A_l}(f, x^i, m, E)$ are also PSTF tensors. The scattering cross-section is also defined in the local tetrad basis,

$$\Sigma = \sum_{l, l'=0}^{\infty} \sigma(m, E; m', E')_{A_l B_l} e^{A_l} e^{B_l}. \quad (3.71)$$

On using (3.61), the covariant Liouville operator (3.62) is

$$L = p^i \frac{\partial}{\partial x^i} - \Gamma^i_{jk} p^j p^k \frac{\partial}{\partial p^i}, \quad (3.72)$$

and the covariant Einstein-Liouville equation (3.58) is given by:

$$\left(p^i \frac{\partial}{\partial x^i} - \Gamma^i_{jk} p^j p^k \frac{\partial}{\partial p^i} \right) f = C[f]. \quad (3.73)$$

- For the exact FLRW model, using the form of the FLRW connections (Appendix section A.2), the Liouville equation (3.73) takes on the form

$$E \frac{\partial f}{\partial t} - E^2 \frac{\dot{S}}{S} \frac{\partial f}{\partial E} = C[f]. \quad (3.74)$$

⁹ eg Maxwell-Boltzmann distribution

$$f = \exp[\alpha(x) + \beta_a(x) p^a] \quad (3.67)$$

$$S^a_{;a}(f) = -\alpha N^a_{;a} - \beta_a T^{ab}_{;b} \quad (3.68)$$

hence the equilibrium condition would be

$$N^a_{;a} = -\frac{\beta_a}{\alpha} T^{ab}_{;b}. \quad (3.69)$$

- In general the covariant Einstein-Boltzmann equation (3.73),

$$p^a \partial_a f - \Gamma_{ab}^c p^a p^b \frac{\partial f}{\partial p^c} = C(f), \quad (3.75)$$

can be rewritten using $p^a = Eu^a + \lambda e^a$, *i.e.* a (3+1) decomposition, and results (A.120), (A.121), (A.122) and (A.123) in the Appendix, (section A.7), to become¹⁰

$$\begin{aligned} L(f) &= Eu^a \partial_a f + \lambda e^a \partial_a f + u_a \Gamma_{bc}^a (Eu^b + \lambda e^b)(Eu^c + \lambda e^c) \frac{\partial f}{\partial E} - (u_a \Gamma_{bc}^a \frac{E^2}{\lambda} \\ &\quad + \frac{1}{\lambda} h_a^d \Gamma_{bc}^a) (Eu^b + \lambda e^b)(Eu^c + \lambda e^c) \frac{\partial f}{\partial e^d} \\ &= C[f]. \end{aligned} \quad (3.76)$$

Expanding the covariant Boltzmann equation equation from (3.76) in an orthonormal tetrad basis¹¹, *i.e.* using the tetrad form of the connections (A.96), (A.97), (A.98) and (A.99) in terms of the covariant derivatives of the velocity (3.2)¹², the covariant Liouville operator becomes,

$$\begin{aligned} L &= E \partial_0 + \lambda e^\alpha \partial_\alpha - \lambda \left[Eu_\alpha e^\alpha - \frac{\lambda}{3} \theta - \lambda \sigma_{\alpha\beta} e^\alpha e^\beta \right] \frac{\partial}{\partial E} \\ &\quad + \left[\frac{E^2}{\lambda} (-u^\alpha + \dot{u}_\beta e^\beta e^\alpha) - E(\sigma_\beta^\alpha + \omega_\beta^\alpha + \epsilon_{\beta\gamma}^\alpha \Omega^\gamma) e^\beta \right. \\ &\quad \left. + E(\sigma_{\beta\gamma} e^\alpha e^\beta e^\gamma) + \lambda(\dot{u}_\beta e^\alpha e^\beta - \epsilon_{\beta\delta}^\alpha n^\delta_\gamma e^\beta e^\gamma - a^\alpha) \right] \frac{\partial}{\partial e^\alpha}, \end{aligned} \quad (3.77)$$

where $a_\alpha = \gamma^\beta_{\alpha\beta}$ and $n^{\alpha\beta} = \gamma^{\alpha\beta}_{\gamma\delta} \gamma^\gamma \gamma^\delta$ (See Appendix A.6). This gives us the Einstein-Boltzmann moment equations when using a harmonically decomposed distribution function :

$$L(f) = L(F + F_a e^a + F_{ab} e^a e^b + \dots) = C\left[\sum_l F_{A_l} e^{A_l}\right] \quad (3.78)$$

$$L(F) + L(F_a e^a) + L(F_{ab} e^a e^b) \dots = \sum_l b_{A_l} e^{A_l} \quad (3.79)$$

$$\sum_l F_{A_l} L(e^{A_l}) + \sum_l L(F_{A_l}) e^{A_l} = \sum_l b_{A_l} e^{A_l} \quad (3.80)$$

$$\sum_l L'_{A_l} e^{A_l} = \sum_l b_{A_l} e^{A_l}, \quad (3.81)$$

where L'_{A_l} is a tensor coefficient of e^{A_l} arising from the action of the Liouville operator.

¹⁰This is not explicitly in a tetrad basis yet, we have only carried out a (3+1) split using independent direction angles e^α on the null cone.

¹¹See Section A.6 in the Appendix

¹²See Section A.7 in the Appendix

3.2.5 Boltzmann moment equations

Using the Boltzmann equation (3.81) we can break the equation up into the individual moment equations. That is, we obtain a hierarchy of equations classified according to the harmonic order, *i.e.* according to l , in $L'_{A_l} = b_{A_l}$. To do this one needs to remember the definition of the covariant derivatives of F_{A_l} ¹³,

$$F_{;b}^{A_l} = \partial_b F^{A_l} + \sum_k \Gamma_{cb}^{a_k} F^{a_1 a_2 \dots a_{k-1} c a_{k+1} \dots a_l}. \quad (3.83)$$

For $l = 0$, $l = 1$ and $l = 2$ this becomes respectively:

$$F_{;d}^a = \partial_d F^a + \Gamma_{de}^a F^e, \quad (3.84)$$

$$F_{;d}^{ab} = \partial_d F^{ab} + \Gamma_{de}^a F^{eb} + \Gamma_{de}^b F^{ae}, \quad (3.85)$$

$$F^{abc}_{;d} = \partial_d F^{abc} + \Gamma_{de}^a F^{ebc} + \Gamma_{de}^b F^{aec} + \Gamma_{de}^c F^{abe}. \quad (3.86)$$

These can be split up into the the spatially projected covariant derivative, and the covariant derivative orthogonal to the spatial slices, by projecting with h^{ab} or u^b respectively. Here $\partial_a F_{A_l} = (\partial F_{A_l} / \partial x^a)_{p^\alpha}$ is the partial derivative of F_{A_l} along a worldline. The form of the connections is found in the Appendix (A.6). These are used to rewrite the partial derivatives in the Liouville operator (3.77) in terms of covariant derivatives and connections. This is done following [25], in a covariant (3+1) sense.

The $l = 0$ part of (3.81) is

$$\begin{aligned} -\frac{2}{15} \lambda^{-1} \frac{\partial(\lambda^3 \sigma^{ef} F_{ef})}{\partial E} + \frac{1}{3} \lambda (F_{e,f} h^{ef}) - \frac{1}{3} E \lambda^{-1} \partial(\lambda^2 \dot{u}^d F_d) / \partial E \\ + E \dot{F} - \frac{1}{3} \lambda^2 \theta \frac{\partial F}{\partial E} = b; \end{aligned} \quad (3.87)$$

the $l = 1$ part of (3.81) is

$$\begin{aligned} -\frac{6}{35} \lambda^{-2} \frac{\partial(\lambda^4 \sigma^{ef} F_{efa})}{\partial E} + \frac{2}{5} \lambda (h_a^b F_{bc;d} h^{cd}) - \frac{2}{5} E \lambda^{-2} \frac{\partial(\lambda^2 \dot{u}^d F_{da})}{\partial E} + E h_a^d (\dot{F}_d) \\ - \theta \frac{1}{3} \lambda^2 \frac{\partial F_a}{\partial E} - \frac{2}{5} \lambda^{\frac{3}{2}} \frac{\partial(\lambda^{\frac{3}{2}} F_d \sigma^d a)}{\partial E} - E F_d \omega_a^d + \lambda h_a^b F_{,b} - \lambda E \frac{\partial F}{\partial E} u_a = b_a; \end{aligned} \quad (3.88)$$

¹³The general form for the covariant derivatives expressed in terms of the tetrad connections are given in [25]

$$\begin{aligned} \hat{\nabla}^c F_{A_l c} &= h^{cd} \partial_d F_{c A_l} + 2l F_{bc(A_{l-1}} \epsilon_{a_l)}^{bd} \bar{n}_d^c + 2l F_{b(A_{l-1}}^b a_{a_l)} - 2(l+1) a^b F_{b A_l}, \\ (F_{A_l})' &= \partial_0 F_{A_l} - l F_{d(A_{l-1}} \epsilon_{a_l)}^d \Omega^c. \end{aligned} \quad (3.82)$$

the $l = 2$ part of (3.81) is

$$\begin{aligned}
& - \frac{12}{63} \lambda^{-3} \frac{\partial(\lambda^3 \sigma^{ef} F_{efab})}{\partial E} + \frac{3}{7} \lambda (h_a^e h_b^f F_{efc;d} h^{cd}) - \frac{3}{7} E \lambda^{-3} \frac{\partial(\lambda^4 u^d F_{dab})}{\partial E} \\
& + E h_a^e h_b^f (F_{ef}) - \frac{\lambda^2}{3} \theta \frac{\partial F_{ab}}{\partial E} - \frac{4}{7} \lambda^{\frac{1}{2}} \frac{\partial \left((\lambda^{\frac{3}{2}} \{F_{d(a} \sigma_{b)}^d - \frac{1}{3} \sigma^{ef} F_{ef} h_{ab}\}) \right)}{\partial E} \\
& - 2E F_{d(a} \omega_{b)}^d + \lambda h_a^c h_b^d \{F_{(c;d)} - \frac{1}{3} F_{e;f} h^{ef} h_{cd}\} \\
& - E \lambda^2 \frac{\partial(\lambda^{-1} (F_{(a} u_{b)} - \frac{1}{3} F_c \dot{u}^c h_{ab}))}{\partial E} - \lambda^2 \frac{\partial F}{\partial E} \sigma_{ab} = b_{ab}. \tag{3.89}
\end{aligned}$$

3.2.6 Kinetic theory and Imperfect fluids

The natural decomposition of any symmetric tensor T_{ab} is given as before by equation (3.1). Using the kinetic theory approach we can relate the variables in the fluid approximation to derived thermodynamic-like quantities.

If the distribution function is known, it can be easily coupled to the geometry via the field equations in the usual manner. We have previously found the fluid variables in terms of the distribution function moments (3.42) -(3.46). The interesting thing is that we require the distribution moments only up to second order in the tetrad expansion:

$$R_{ab} u^a u^b = \frac{1}{2} (\mu + 3p), \tag{3.90}$$

$$R_{ab} u^a h_c^b = -q_c, \tag{3.91}$$

$$R_{ab} h_c^a h_d^b = \frac{1}{2} (\mu + p) h_{cd} + \pi_{cd}. \tag{3.92}$$

Using the relation $T_{ab} = \int p^a p^b f dP$ and the usual definition for p^a one finds that

$$R_{ab} u^a u^b = \frac{1}{6} \int_m^\infty (3E^2 + \lambda^2) f \lambda dE d\Omega, \tag{3.93}$$

$$R_{ab} u^a h_c^b = - \int_m^\infty E \lambda f e_c \lambda dE d\Omega, \tag{3.94}$$

$$R_{ab} h_c^a h_d^b = \frac{1}{2} \int_m^\infty (m^2 h_{cd} + \lambda^2 e_c e_d) f \lambda dE d\Omega. \tag{3.95}$$

Thus (using the PSTF tensor conditions on f):

$$\frac{1}{2} (\mu + 3p) = R_{ab} u^a u^b = \frac{1}{2} \left[4\pi \int_m^\infty E^2 \lambda f dE + 4\pi \int_m^\infty \lambda^3 f dE \right], \tag{3.96}$$

$$-q_c = R_{ab} u^a h_c^b = -\frac{4\pi}{3} \int_m^\infty E \lambda^2 F_c dE, \tag{3.97}$$

$$\begin{aligned} \frac{1}{2}(\mu + p)h_{cd} + \pi_{cd} = R_{ab}h_c^a h_d^b &= \frac{4\pi}{2} \left[\int_m^\infty E^2 \lambda F dE + \frac{1}{3} \int_m^\infty \lambda^3 F dE \right] h_{cd} \\ &+ \frac{8\pi}{15} \int_m^\infty \lambda^3 F_{cd} dE. \end{aligned} \quad (3.98)$$

The on-mass shell relations, using $p_a p^a = -m^2$ and $\lambda = +(E^2 - m^2)^{\frac{1}{2}}$, are

$$R_{ab}u^a u^b = \frac{4\pi}{2} \left[\int_m^\infty E^2 (E^2 - m^2)^{\frac{1}{2}} F dE + \int_m^\infty (E^2 - m^2)^{\frac{3}{2}} F dE \right], \quad (3.99)$$

$$R_{ab}u^a h_c^b = -\frac{4\pi}{3} \int_m^\infty E (E^2 - m^2) F_c dE, \quad (3.100)$$

$$\begin{aligned} R_{ab}h_c^a h_d^b &= \frac{4\pi}{2} \left[\int_m^\infty E^2 (E^2 - m^2)^{\frac{1}{2}} F dE + \frac{1}{3} \int_m^\infty (E^2 - m^2)^{\frac{3}{2}} F dE \right] h_{cd} \\ &+ \frac{8\pi}{15} \int_m^\infty (E^2 - m^2)^{\frac{3}{2}} F_{cd} dE. \end{aligned} \quad (3.101)$$

3.3 Gauge Invariant Covariant (GIC) Perturbation Theory

There are two basic points of view with regard to gauge invariance [48] [28] :

- **Non-local point of view:** Points in a space-time manifold are not physically distinguishable, and thus it is only gauge invariant quantities that can be observed, exactly as in any other gauge theory.
- **Local point of view:** Assume that points, of which the space-time manifold consists, are physically distinguishable - the choice of gauge corresponds to the particular reference system(frame) in which the observables are measured.

The situation is that we are given to study the real (lumpy) universe S ; this is all we can measure. We proceed to define perturbed quantities and their evolution by the way we specify a mapping Φ of idealized (fictional) space-time \bar{S} into S . This is the fitting problem.

The essential point is that by considering the lumpy universe model S , not knowing how the model \bar{S} has been used to make the idealized constructions, we will be unable to uniquely recover \bar{S} from S .

To define perturbations we have to choose a correspondence between a fictitious background space-time and the physical, real inhomogeneous universe. A change in this correspondence, keeping the background space-time fixed, is a gauge transformation¹⁴. This

¹⁴This should not be confused with Gauge Transformations in Gauge Field Theories, where the gauges discussed are generalized connections on a particular manifold structure, not mappings between "models".

transformation essentially changes the point in the background, idealized space-time corresponding to a point in the physical space-time. If the gauge condition imposed to simplify the (idealized space-time) metric leaves a residual gauge freedom, the perturbation equations will have spurious gauge modes, solutions which can be completely removed by appropriate gauge transformations and therefore have no physical reality.

Essentially the arbitrariness in defining $\delta\mu$, for instance, is because $\delta\mu$ is not gauge invariant. It can be assigned any value, at any event, by an appropriate gauge choice; it is not observable in principle, unless the gauge is fully specified by an observational procedure (for otherwise $\bar{\mu}$ is not observable). Hence to use observational data and relate it in a satisfactory fashion to the idealized model we would either want to find (1) a unique gauge choice, or we must use (2) gauge invariant variables.

The fundamental requirement for a gauge invariant quantity is that it be invariant under the choice of mappings Φ . The simplest case is a scalar \bar{f} that is constant in the unperturbed space-time \bar{S} ($\bar{f} = \text{constant}$), or any tensor \bar{f}^{ab}_{cd} that vanishes in \bar{S} : $\bar{f}^{ab}_{cd} = 0$. The reason is that in each case the mapped quantity \bar{f} in S will also be constant. The choice of corresponding Φ does not matter; they will all define the same perturbation $\delta f = f - \bar{f}$. The other possibility is a tensor that is a linear combination of the products of kronecker delta functions [31] [33].

In an *Almost-FLRW* universe all the gauge invariant variables are those that vanish in the FLRW background (eg (3.110 - 3.112) and (3.103 - 3.109)).

3.3.1 Fluid models

A FLRW universe allows only a perfect fluid space-time with

$$\sigma_{ab} = \omega_{ab} = \dot{u}_a = 0. \quad (3.102)$$

Here the FLRW fluid flow vector is $u_a = -t_{,a}$, and $E_{ab} = 0$ and $H_{ab} = 0$. The space-time is conformally flat. The almost-FLRW universe admits the following gauge invariant variables, which all vanish in the background FLRW space-time.

- The *vorticity*, *shear* and the *acceleration*:

$$\omega_{ab} \equiv h_a^c h_b^d u_{[c;d]}, \quad (3.103)$$

$$\sigma_{ab} \equiv h_a^c h_b^d u_{(c;d)} - \frac{1}{3} u_{;c}^c h_{ab}, \quad (3.104)$$

$$\dot{u}^a \equiv u^a_{;b} u^b. \quad (3.105)$$

- The *Electric* and *Magnetic* parts of the Weyl tensor¹⁵:

$$E_{ab} \equiv C_{abcd}u^c u^d, \quad (3.106)$$

$$H_{ab} \equiv \frac{1}{2}C_{acst}u^c \eta^{st}{}_{bd}u^d. \quad (3.107)$$

- *Matter tensor* components:

$$q_a \equiv -h_a^c T_{cd}u^d, \quad (3.108)$$

$$\pi_{ab} \equiv h_a^c h_b^d T_{cd} - \frac{1}{3}(h^{cd}T_{cd})h_{ab}. \quad (3.109)$$

All these variables vanish identically in the FLRW background. But we do not, so far, have gauge invariant quantities characterizing the variations of the zeroth order variables $(\mu(t), p(t), \theta(t))$ which are in general non-zero in expanding FLRW universes, and so are not gauge invariant (do not vanish in the FLRW background). We are able to construct gauge invariant quantities from these in a fairly natural manner. We define the *orthogonal spatial gradients* of these variables as follows:

$$X_a \equiv h_a^b \mu_{,b}, \quad (3.110)$$

$$Y_a \equiv h_a^b p_{,b}, \quad (3.111)$$

$$Z_a \equiv h_a^b \theta_{,b}. \quad (3.112)$$

Each of these is gauge invariant to all orders, as they all vanish¹⁶ in the FLRW background model¹⁷

The vector X_a , the *spatial projection of the energy density gradient*, leads naturally to the *fractional density gradient*,

$$D_a \equiv \frac{X_a}{\mu} = h_a^b \left(\frac{\mu_{,b}}{\mu} \right). \quad (3.113)$$

This is also gauge invariant (to 1st order) and represents the relative importance of the density gradient. The problem with D_a and X_a is that they both represent the change in density to a fixed distance whereas it is useful to consider fluctuations at a fixed scale. We define the *Comoving fractional density gradient*

$$\mathcal{D}_a \equiv S D_a. \quad (3.114)$$

¹⁵These are associated with gravitational radiation, hence the names.

¹⁶This can be seen by recalling that $\mu = \mu(t)$, $p = p(t)$, $\theta = \theta(t)$ and $h_a^b u_b = 0$; the spatial gradients of variables dependent only on proper time will obviously vanish.

¹⁷Isocurvature variations can be defined in terms of zero vorticity perturbation for $K_a = 0$ where $K_a \equiv h_a^b K_{,b} = -\frac{3}{4}\theta Z_a + 2X_a + 2(\sigma^2)_{,c} h_a^c$.

This is gauge invariant (to 1st order) and dimensionless. This can also be defined in terms of a unit direction vector e^a , which separates out its magnitude and direction:

$$\mathcal{D}_a = \mathcal{D}e_a, \quad \mathcal{D} = (\mathcal{D}_a \mathcal{D}^a)^{\frac{1}{2}}. \quad (3.115)$$

The Gauge Invariant variables chosen here are covariant, they are not dependent on the choice of coordinates (or frames). Compare these for instance with the Bardeen approach¹⁸ where a perturbed metric must first be constructed, a line metric of the form (3.180) is generally used, from this gauge invariant variables are constructed that are independent under the gauge transformations [28].

3.3.2 Gas models

Gas models are dealt with in depth in the following chapters in the context of a relativistic photon gas. The variable of central importance in the gas model is the temperature, defined in the context of kinetic theory.

The principal variable of interest is the covariant gauge invariant temperature perturbation δT , which vanishes in the FLRW background and hence is gauge invariant. This is dealt with in terms of a harmonic expansion.

Temperature and temperature anisotropies

Considering $m = 0$, *i.e.* photons, we can define the equilibrium temperature in the background in terms of the Boltzmann distribution. We use the Stefan Boltzmann law [56]

$$aT(x^i)^4 = \mu(x^i) = 4\pi \int E^3 F(E, x^i) dE, \quad (3.118)$$

where the zeroth moment is given as the Boltzmann distribution

$$F(E, x^i) = \frac{2}{\exp(E/kT(x^i)) - 1}, \quad (3.119)$$

the factor 2 takes the two different polarizations into account.

¹⁸The *Matter perturbations* used in the Bardeen approach are $\delta\mu$, δp , q_a , π_{ab} where the unperturbed energy-momentum tensor is that of a perfect fluid,

$$T_{ab} = \mu_0(t)u_{0a}u_{0b} + p_0(t)(u_{0a}u_{0b} + g_{ab}). \quad (3.116)$$

The perturbed energy momentum tensor is given by,

$$\delta T^{ab} = \mu_0 \delta\mu u_0^a u_0^b + 2aq^{(a} u_0^{b)} + p_0 \delta p (u_0^a u_0^b + g^{ab}) + p_0 a^2 \pi^{ab}. \quad (3.117)$$

These matter perturbations are gauge dependent, and hence gauge invariant potentials Φ_μ , Φ_p , Φ_q and Φ_π are constructed from the matter perturbations [28]. The metric perturbations are then related to these in a gauge independent fashion, but the gauge transformations and hence the perturbations are all constructed in coordinate terms [28] *i.e.* are NOT covariant.

This leads to a natural definition of gauge invariant temperature anisotropies. (The temperature anisotropies vanish in the background, hence they are gauge invariant [56], [31], [33].) We write

$$a(T(x^i) + \delta T(x^i, e^a))^4 \doteq 4\pi \int E^3 dE f(x^i, e^a, E). \quad (3.120)$$

Using (3.118) with (3.120) one can show that

$$\left(1 + \frac{\delta T(x^i, e^a)}{T(x^i)}\right)^4 \int_0^\infty E^3 F(x^i, E) dE = \int_0^\infty E^3 f(x^i, e^a, E) dE. \quad (3.121)$$

Using a binomial expansion, under the assumption that $\frac{\delta T}{T} \ll 1$, one can truncate the expansion. *i.e.* using $(1+x)^4 \approx 1 + 4x + O(x^2)$ one finds that (3.121) can be reduced to

$$\frac{\delta T(x^i, e^a)}{T(x^i)} \int E^3 F dE \approx \frac{1}{4} \int E^2 (F_a e^a + F_{ab} e^a e^b + \dots) dE. \quad (3.122)$$

This can be used to define the temperature anisotropies τ :

$$\frac{\delta T(x^i, e^a)}{T(x^i)} \equiv \tau = \tau_a e^a + \tau_{ab} e^a e^b + \dots = \sum_l \tau_{A_l} e^{A_l}. \quad (3.123)$$

From (3.122) and (3.118) we then have that, for small temperature anisotropies,

$$\sum_l \tau_{A_l} e^{A_l} \approx \frac{\pi}{\mu} \int E^3 (F_a e^a + F_{ab} e^a e^b + \dots) dE = \frac{\pi}{\mu} \sum_l \int E^3 F_{A_l} e^{A_l} dE. \quad (3.124)$$

This means that for small temperature anisotropies the individual moment coefficients can be found:

$$\tau_{A_l} \approx \frac{\pi}{\mu(x^i)} \int E^3 F_{A_l} dE. \quad (3.125)$$

Hence, unless we have all the moments for the distribution function expansion explicitly, we cannot find the temperature anisotropies. Hence we have that in natural units (*i.e.* $a=1$)

$$\tau_a \approx \frac{\pi}{aT^4} \int E^3 dE F_a = \frac{3q_a}{4T^4}, \quad (3.126)$$

$$\tau_{ab} \approx \frac{\pi}{aT^4} \int E^3 dE F_{ab} = \frac{15\pi_{ab}}{8T^4}, \quad (3.127)$$

$$\tau_{abc} \approx \frac{\pi}{aT^4} \int E^3 dE F_{abc} = \frac{35\xi_{abc}}{8T^4}. \quad (3.128)$$

To get these moments, one would have to solve the complete hierarchy of Boltzmann equations. What we can do is to integrate out the energy dependence of the Boltzmann

equations for the distribution functions and try to identify terms with temperature terms. *i.e.* we attempt to find, effectively, a Boltzmann-like equation for the brightness. This is done in Section 3.5 in a manner similar to that of Kodama and Sasaki [35]. But first it is important to be reminded that any practical links between the theory of temperature perturbations and observations are presented in terms of the two-point angular correlation functions.

Two point correlations

The two-point correlation function completely characterizes Gaussian noise. Its Fourier transform is the power spectrum of Gaussian noise. The function $C(\alpha)$ describing the correlation between two points in the sky an angle α apart is given by

$$C(\alpha) = \langle \tau(x^i, \theta, \phi) \tau(x^i, \theta', \phi') \rangle \quad (3.129)$$

$$= \left\langle \sum_l \tau_{A_l} e^{A_l} \sum_{l'} \tau_{A_{l'}} e'^{A_{l'}} \right\rangle \quad (3.130)$$

where the angle α is given by $\cos(\alpha) = e^a e'_a = h_{ab} e^a e'^b$, and the average is some average over angles θ and ϕ of the tetrad's coordinate realization. The two point function is the critical experimental parameter, and a covariant formulation of this is important to link the observational data to the theory. A clear covariant formulation of the two-point correlations does not exist at this time and is the topic of on-going research.

3.4 Almost FLRW conditions in the covariant formalism

3.4.1 Basic assumptions

$O[n]$ indicates order of magnitude; that is $O[n] \equiv O(\epsilon^n)$ for a small ϵ . The $O[0]$ model is taken to be the high symmetry case of perfect isotropy indicating an exact FLRW model.

- (1) Assume (for example)¹⁹ a two fluid model ; *dust* and *radiation*, described by respective energy momentum tensors. The matter/dust is described as a perfect pressure free fluid, the radiation in terms of relativistic kinetic theory variables.

- Energy momentum tensor : $T^{ab} = T_M^{ab} + T_R^{ab}$

- Conservation equation : $T_M^{ab}{}_{;b} = 0$ and $T_R^{ab}{}_{;b} = 0$

- matter/dust and radiation : $T_M^{ab} = \mu_M u^a u^b$ and $T_R^{ab} = \int f p^a p^b dP$

¹⁹Clearly this can be done for any number of fluids and gases, by adding in another energy momentum tensor contribution T_i^{ab} (say) onto the total energy momentum tensor.

– energy density : $\mu = \mu_M + \mu_R = O[0]$

- (2) One requires that momentum conservation for the dust holds (dust is geodesic), and that the universe is expanding (else a counter example exists following the EGS treatment in [56])

$$a^a \equiv \dot{u}^a = 0, \quad (3.131)$$

$$\theta = O[0] > 0. \quad (3.132)$$

It may be necessary that this condition be relaxed with regard to the tight-coupling era, as the comoving frame choice $u^a = \delta_0^a$ will most probably have to be replaced by something such as the Ekhart frame, such that the relative velocities needed in the perturbative expansion in the scattering time at decoupling may be found by iterating, following [36].

- (3) Radiation distribution function is described by the relativistic Vlasov equation. That is, there is no collision term; no entropy production. This ensures that the flux and the anisotropic pressure are not dissipative (this is a true measure of f 's deviation from equilibrium) *i.e.* we consider the post decoupling or free-streaming era,

$$L(f) = 0. \quad (3.133)$$

- (4) Relative to u^a , for all comoving matter(dust) observers, f is almost isotropic for all times from decoupling to the present day. That is, in collision free regions following the collision dominated state prior to decoupling.
- $F = F(x^i, E) = O[0]$, $\dot{F} = O[0]$,
- $F_{A_l} = F_{A_l}(x^i, E)$ and \dot{F}_{A_l} are at most $O[1]$ for all $l > 0$.

For radiation (photons, $m = 0$) from (3.42), (3.43) and (3.49) respectively it is found that

$$\mu_R = 4\pi \int_0^\infty E^3 F(x^i, E) dE (= 3p_R), \quad (3.134)$$

$$q_a = \frac{4\pi}{3} \int_0^\infty E^3 F_a(x^i, E) dE, \quad (3.135)$$

$$\pi_{ab} = \frac{8\pi}{15} \int_0^\infty E^3 F_{ab}(x^i, E) dE. \quad (3.136)$$

Hence it can be easily motivated that

$$\mu_R = O[0], \quad (3.137)$$

$$q_a = O[1], \quad (3.138)$$

$$\pi_{ab} = O[1], \quad (3.139)$$

$$\omega_{ab} = O[1], \quad (3.140)$$

$$\sigma_{ab} = O[1] \quad (3.141)$$

and

$$\dot{q}_a = O[1], \quad \hat{\nabla}_c q_a = O[1], \quad (3.142)$$

$$\dot{\pi}_{ab} = O[1], \quad \hat{\nabla}_c \pi_{ab} = O[1]. \quad (3.143)$$

There are no assumptions with regard to geometry at this stage; the Almost EGS theorem ²⁰ demonstrated that the perturbations are small, that is that the universe is almost isotropic. We proceed by looking at the general evolution equations and isolating out higher order contributions in such a manner as to retain terms only up to first order. Here the ordering is done in the sense of being *Almost FLRW*: thus

$$\dot{\mu}_R + \frac{4}{3}\mu_R\theta + \pi^{ab}\sigma_{ab} + q^a{}_{;a} = 0 \quad (3.144)$$

becomes

$$\dot{\mu}_R + \frac{4}{3}\mu_R\theta + q^a{}_{;a} \simeq 0. \quad (3.145)$$

In orders of magnitude (3.144) is $O[0]+O[0]+O[2]+O[1] = 0$ which when linearized in the Almost FLRW sense becomes $O[0]+O[0]+O[1] \simeq 0$ as in (3.145). One proceeds similarly for the momentum flux conservation equation. We have an equation in orders of magnitude that term by term $O[1]+O[1]+O[1]+O[2]=0$, which when linearized to $O[1]$ in the Almost FLRW sense gives an equation with only terms up to and including at least $O[1]$. Thus

$$\dot{q}_a + \frac{4}{3}\theta q_a + h_a^b \left(\frac{1}{3}\mu_{R;b} + \pi_b^c{}_{;c} \right) + (\sigma_{ab} + \omega_{ab})q^b = 0 \quad (3.146)$$

²⁰Almost Ehlers-Geren-Sachs Theorem demonstrates that if one is able to observe almost isotropy of the CBR about all worldlines then the geometry would be Almost FLRW (in the linearized sense discussed above). The basis for this result lies in the first order evolution equations obtained by order of magnitude arguments with regards to the non-FLRW contributions. The statement is indeed model independent within the bounds of a strict set of assumptions. This is a major motivation behind linearized perturbation theory.

becomes

$$\dot{q}_a + \frac{4}{3}\theta q_a + h_a^b \left(\frac{1}{3}\mu_{R,b} + \pi_b^c{}_{;c} \right) \simeq 0. \quad (3.147)$$

3.5 Almost FLRW temperature Boltzmann equations

3.5.1 Linearized, “Almost FLRW Boltzmann” moment equations

Using the Almost FLRW conditions (3.137)-(3.143)²¹, the linearized gauge invariant covariant Boltzmann moment equations are found from (3.87 - 3.89), for $l = 0, 1, 2$ respectively. These are,

$$\frac{1}{3}\lambda(\hat{\nabla}^e F_e) + E\dot{F} - \frac{1}{3}\lambda^2\theta\frac{\partial F}{\partial E} \simeq b, \quad (3.148)$$

$$\frac{2}{5}\lambda(\nabla_d F_{bc} h_a^b h^{cd}) + E\dot{F}_a - \frac{1}{3}\theta\lambda^2\frac{\partial F_a}{\partial E} + \lambda F_{,a} - \lambda E\frac{\partial F}{\partial E}u_a \simeq b_a, \quad (3.149)$$

$$\frac{3}{7}\lambda\hat{\nabla}^c F_{abc} + E(\dot{F}_{ab}) - \frac{1}{3}\lambda^2\theta\frac{\partial F_{ab}}{\partial E} + \lambda\hat{\nabla}_{<b}F_{a>} - \lambda^2\frac{\partial F}{\partial E}\sigma_{ab} \simeq b_{ab}. \quad (3.150)$$

Using $u_a = 0$ and $m = 0$ *i.e.* for photons,

$$E\dot{F} - \frac{1}{3}E^2\theta\frac{\partial F}{\partial E} + \frac{1}{3}E\hat{\nabla}_a F^a \simeq b, \quad (3.151)$$

$$E\dot{F}_a - \frac{1}{3}E^2\theta\frac{\partial F_a}{\partial E} + E\hat{\nabla}_a F + \frac{2}{5}E\hat{\nabla}_b F_a^b \simeq b_a, \quad (3.152)$$

$$E\dot{F}_{ab} - \frac{1}{3}E^2\theta\frac{\partial F_{ab}}{\partial E} - E^2\sigma_{ab}\frac{\partial F}{\partial E} + E\hat{\nabla}_{<a}F_{b>} + \frac{3}{7}E\hat{\nabla}_c F_{ab}^c \simeq b_{ab}. \quad (3.153)$$

Here \simeq denotes linearization of perturbed equations *i.e.* to $O[1]$ in the “Almost FLRW” sense, while remembering that \approx denotes the small temperature anisotropy approximation; in this treatment \simeq and \approx are considered as being conceptually different in that the variables in which the concept of “Almost FLRW” geometry (F_{A_l} , π_{ab} , q_a *etc*) is expressed (Section 3.4, following [55]) can only be linked to the temperature perturbations τ_{A_l} without harmonic mixing *i.e.* in the linear fashion of (3.125), if (as we know from observations) $\tau_{A_l} \ll 1$. Whether or not \simeq is equivalent to \approx is not dealt with in this thesis, although if the τ_{A_l} 's are small one naturally expects the F_{A_l} 's to be small and hence q_a and π_{ab} would be expected to be small in the sense linearization to $O[1]$ [55].

3.5.2 Temperature and Temperature anisotropies

The Boltzmann moment equation can be integrated over energy to find the Almost FLRW conservation equations, *i.e.* multiply (3.151 - 3.153) by E^2 and integrate over energy,

²¹ *i.e.* Using $u_a F^a = O[2]$, $\sigma^{ef} F_{ef} = O[2]$

using $E_{,b} = 0$ and $f = F + F_a e^a + F_{ab} e^a e^b + \dots = \sum_l F_{A_l} e^{A_l}$.

One would expect, for physically useful distributions, that they fall off exponentially at $E = \infty$ and are at least constant or exponentially falling-off at $E = 0$, *i.e.*

$$E^n F_{A_l}|_0 = 0 = E^n F_{A_l}|_\infty. \quad (3.154)$$

These are used to eliminate the terms that arise from the integrations by parts that result from eliminating the $\frac{\partial}{\partial E}$ terms. The first three are shown below; $l = 0$, $l = 1$ and $l = 2$.

For $l = 0$:

$$\int_0^\infty E^3 dE \dot{F} - \frac{\theta}{3} \int_0^\infty E^4 dE \frac{\partial F}{\partial E} + \frac{1}{3} \int_0^\infty E^3 \hat{\nabla}_a F^a \simeq \int E^2 dE b \quad (3.155)$$

or

$$\left(\int E^3 dE F \right) + \frac{4\theta}{3} \left(\int E^3 dE F \right) + \frac{1}{3} \hat{\nabla}_a \left(\int E^3 dE F^a \right) \simeq \int E^2 dE b. \quad (3.156)$$

Using equations (3.134) and (3.135) we find:

$$\left(\frac{\mu}{4\pi} \right) + \frac{4\theta}{3} \left(\frac{\mu}{4\pi} \right) + \frac{1}{3} \hat{\nabla}_a \left(\frac{3q^a}{4\pi} \right) \simeq \int E^2 dE b, \quad (3.157)$$

that is

$$\dot{\mu} + \frac{4}{3}\theta\mu + \hat{\nabla}_a q^a \simeq 4\pi \int_0^\infty E^2 dE b. \quad (3.158)$$

This is consistent, for example, with the linearized energy-density conservation equation (3.145) in the free-streaming case ($b = 0$).

For $l = 1$:

$$\begin{aligned} \int_0^\infty E^3 dE \dot{F}_a - \frac{\theta}{3} \int_0^\infty E^4 dE \frac{\partial F_a}{\partial E} + \int_0^\infty E^3 \hat{\nabla}_a F dE \\ + \frac{2}{5} \int_0^\infty E^3 \hat{\nabla}_b F_a^b \simeq \int_0^\infty E^2 dE b_a \end{aligned} \quad (3.159)$$

that is

$$\begin{aligned} \left(\int E^3 dE F_a \right) + \frac{4}{3}\theta \left(\int E^3 dE F_a \right) + \hat{\nabla}_a \left(\int E^3 dE F \right) \\ + \frac{2}{5} \hat{\nabla}_b \left(\int E^3 dE F_a^b \right) \simeq \int E^2 dE b_a. \end{aligned} \quad (3.160)$$

From equations (3.134), (3.135) and (3.136) it is found that:

$$\begin{aligned} \left(\frac{3q_a}{4\pi} \right) + \frac{4}{3}\theta \left(\frac{3q_a}{4\pi} \right) + \hat{\nabla}_a \left(\frac{\mu}{4\pi} \right) \\ + \frac{2}{5} \hat{\nabla}_b \left(\frac{15\pi^b_a}{8\pi} \right) \simeq \int E^2 dE b_a, \end{aligned} \quad (3.161)$$

that is

$$\dot{q}_a + \frac{4}{3}\theta q_a + \frac{1}{3}\hat{\nabla}_a\mu + \hat{\nabla}_b\pi_a^b \simeq \frac{4\pi}{3} \int_0^\infty E^2 dE b_a. \quad (3.162)$$

This is consistent with the linearized momentum-flux conservation equation (3.147) in the free-streaming case ($b_a = 0$).

For $l = 2$:

$$\begin{aligned} \int_0^\infty E^3 dE \dot{F}_{ab} &- \frac{1}{3}\theta \int_0^\infty E^4 dE \frac{\partial F_{ab}}{\partial E} - \sigma_{ab} \int_0^\infty E^4 dE \frac{\partial F}{\partial E} + \int_0^\infty E^3 dE \hat{\nabla}_{(a} F_{b)} \\ &- \frac{1}{3}h_{ab} \int_0^\infty E^3 dE \hat{\nabla}_c F^c + \frac{3}{7} \int_0^\infty E^3 dE \hat{\nabla}_c F_{ab}^c \simeq \int_0^\infty E^2 dE b_{ab} \end{aligned} \quad (3.163)$$

or

$$\begin{aligned} \left(\int E^3 dE F_{ab} \right) &+ \frac{4}{3}\theta \left(\int E^3 dE F_{ab} \right) + 4\sigma_{ab} \left(\int E^3 dE F \right) + \hat{\nabla}_{(a} \left(\int E^3 dE F_{b)} \right) \\ &- \frac{1}{3}h_{ab} \hat{\nabla}_c \left(\int E^3 dE F^c \right) + \frac{3}{7} \hat{\nabla}_c \left(\int E^3 dE F_{ab}^c \right) \simeq \int E^2 dE b_{ab}. \end{aligned} \quad (3.164)$$

From equations (3.134), (3.135) and (3.136) it is found that:

$$\begin{aligned} \left(\frac{15\pi_{ab}}{8\pi} \right) &+ \frac{4\theta}{3} \left(\frac{15\pi_{ab}}{8\pi} \right) + 4\sigma_{ab} \left(\frac{\mu}{4\pi} \right) + \hat{\nabla}_{(a} \left(\frac{3q_b}{4\pi} \right) \\ &- \frac{1}{3}h_{ab} \hat{\nabla}_c \left(\frac{3q_a}{4\pi} \right) + \frac{3}{7} \hat{\nabla}_c \left(\frac{35\xi_{ab}^c}{8\pi} \right) \simeq \int E^2 dE b_{ab}, \end{aligned} \quad (3.165)$$

that is

$$\pi_{ab} + \frac{4}{3}\theta\pi_{ab} + \frac{8}{15}\mu\sigma_{ab} + \frac{2}{5}\hat{\nabla}_{\langle a}q_{b\rangle} + \hat{\nabla}_c\xi_{ab}^c \simeq \frac{8\pi}{15} \int_0^\infty E^2 dE b_{ab}. \quad (3.166)$$

This in fact disagrees with the equations derived in [55] (instead of $2\hat{\nabla}_{\langle a}q_{b\rangle}$, I find $\frac{2}{5}\hat{\nabla}_{\langle a}q_{b\rangle}$). This is a problematic error and can only arise from the integration above or the derivation of the Boltzmann equations, it is most probably a typographical error in [25].

Using the small temperature anisotropy approximation (3.126) - (3.128) *i.e.*

$$\tau_a \approx \frac{3q_a}{4\mu}, \quad \tau_{ab} \approx \frac{15\pi_{ab}}{8\mu}, \quad \tau_{abc} \approx \frac{35\xi_{abc}}{8\mu}, \quad (3.167)$$

along with (3.118), equations (3.157), (3.161) and (3.165) can be rewritten to obtain the Almost FLRW temperature anisotropy gauge invariant and covariant perturbed Boltzmann

equations²². We find for $(l = 0)$, $(l = 1)$ and $(l = 2)$ respectively:

$$\left(\frac{\dot{T}}{T} + \frac{\dot{S}}{S}\right) + \frac{1}{3}\hat{\nabla}_a\tau^a \simeq \frac{\pi}{T^4} \int E^2 dEb, \quad (3.168)$$

$$\dot{\tau}_a + 4\left(\frac{\dot{T}}{T} + \frac{\dot{S}}{S}\right)\tau_a + \frac{\hat{\nabla}_a T}{T} + \frac{2}{5}\hat{\nabla}_b\tau_a^b \simeq \frac{\pi}{T^4} \int E^2 dEb_a, \quad (3.169)$$

$$\dot{\tau}_{ab} + 4\left(\frac{\dot{T}}{T} + \frac{\dot{S}}{S}\right)\tau_{ab} + \sigma_{ab} + \hat{\nabla}_{\langle a}\tau_{b\rangle} + \frac{3}{7}\hat{\nabla}_c\tau_{ab}^c \simeq \frac{\pi}{T^4} \int E^2 dEb_{ab}. \quad (3.170)$$

The hierarchy does not close. The key issues is how to handle this. An attempt to deal with this is made in the next section.

The spatial gradients need to be removed from these equations. In principle this could be done by solving the Helmholtz equation [61] [63] in the Almost FLRW space-time and using the resulting eigenfunctions to perform a Fourier like transformation, leaving only time derivatives. Notice that if the gradient $\hat{\nabla}_a\tau^a = 0$ from (3.168) remembering that $\frac{1}{3}\theta = \dot{S}/S$, one recovers the usual $\dot{T}/T = -\dot{S}/S$ for the free-streaming FLRW case. That is, $TS = \text{constant}$, as $(\dot{TS}) = \dot{S}T + S\dot{T} = 0$; multiply this by $1/ST$ to recover the free-streaming form of (3.168). This is shown for the free-flow or free-streaming regime where the fluid approximation is totally inapplicable [40]. For the case of tight-coupling where collisions dominate *i.e.* when some scattering term $C[f]$ is non-zero, one would expect to be able to recover the fluid limit, if this implies the existence of a unique 4-velocity.

When discussing the fluid limit or the tight-coupling limit the gas is tightly coupled to the fluid in the sense that the gas has a unique 4-velocity coincident with that of the matter fluid as the scattering cross-section goes to infinity (hence the scattering time goes to zero). This has not been well clarified in the covariant treatment, and is a problem which may be due to the comoving frame choice that I have used here. Perhaps a tilted frame may be more applicable when the tight-coupling limit is discussed. This is important if, for instance, the assumption of adiabatic perturbations is to be well understood [36] [50] in a fully GIC framework.

²²These look very much like the fluid conservation equations, but here there is a selfconsistent motivation for the collisionals, something that must be added on "by hand" in the fluid approach. The fluid approach is not entirely applicable in the free-streaming case; to move from a gas model to a fluid model one in fact takes a tight-coupling limit. The fluid approach is more likely to be considered valid in the collisionally dominated limit where the scattering cross-sections diverge.

3.5.3 Tilted observer

Consider an observer with 4-velocity \tilde{u}^a such that :

$$u_a \tilde{u}^a = -\cosh\phi, \quad (3.171)$$

u^a being the gas 4-velocity, and ϕ the tilt angle. Hence we can find all the kinematic variables in terms of the tilted frame.

It is then useful to define two distinct frames :

- *Eckart frame*: Consider $\tilde{u}^a = u_E^a$,

$$N^a = n_E u_E^a \quad (k_E = 0) \quad (3.172)$$

That is, u_E^a defines the kinematically averaged frame.

- *Laundau-Lifshitz frame*: Consider $\tilde{u}^a = u_L^a$,

$$T_b^a u_L^b = -\mu_L u_L^a \quad (q_L^a = 0) \quad (3.173)$$

This is included as a reminder that, although we have picked the comoving acceleration free frame, we may move into the more complicated and possibly more useful Eckart frame choice. This may be critical in understanding the tight coupling regime in terms of a covariant perturbative iterative expansion in the scattering time at decoupling, which should be carried out following [36] if the tight coupling regime is to be well understood in the covariant gauge invariant formalism.

In this treatment the frame issue is considered not to be critical; hence we continue the investigation into the Boltzmann moment equations in the comoving frame by investigating how to, if it is possible, decouple the $l = 0$ and $l = 2$ Boltzmann moment equation from the rest of the hierarchy of Boltzmann equations without truncating the moment expansion of τ_{A_l} .

3.5.4 Decoupling the $l = 0$ and $l = 1$ Almost FLRW Boltzmann equations from the complete Hierarchy

From equation (84) in Bruni et al [30] the form of the anisotropic pressure is given as

$$\pi_{\alpha\beta} = a^2 (\nabla_{\alpha\beta} \pi + \pi_{(\alpha|\beta)}^S + \pi_{\alpha\beta}^{TT}), \quad (3.174)$$

where $\nabla_{\alpha\beta} = \nabla_\alpha \nabla_\beta - \frac{1}{3} \nabla^2$, and ∇^2 is the laplacian. $\pi_{(\alpha|\beta)}^S$ is the vector contribution, S denotes solenoidal. $\pi_{\alpha\beta}^{TT}$ is the transverse trace-less contribution *i.e.* the tensor contribution. The stroke (|) denotes the part of the covariant derivative projected into the spatial

sections *i.e.* $\hat{\nabla}_\alpha T \equiv T_{|\alpha}$. In equation (136) from [30] we also have

$$\begin{aligned} \kappa\pi_{\alpha\beta} = & -\nabla_{\alpha\beta}(\Phi_H + \Phi_A) + \left(\Psi_{(\alpha|\beta)} + 2\frac{a'}{a}\Psi_{(\alpha|\beta)} \right) \\ & + \left[H_{T\alpha\beta}^{TT''} + 2\frac{a'}{a}H_{T\alpha\beta}^{TT'} + (2k - \nabla^2)H_{T\alpha\beta}^{TT} \right]. \end{aligned} \quad (3.175)$$

The apostrophe (') denotes the co-ordinate time derivative, Ψ_α is the frame dragging potential and $H_{T\alpha\beta}^{TT}$ is the Gauge invariant transverse trace-free part of the metric *i.e.* is associated with the magnetic part of the Weyl tensor [30]. The scale factor is denoted by a , which is equivalent to S in the FLRW case, around which we are perturbing. To try to interpret the physical meaning of the scalar potentials Φ_H and Φ_A it is useful to define the *stress potential* (see later) $\Phi_\pi = \frac{1}{2}(\Phi_H + \Phi_A)$ and the *newtonian gravitational potential* $\Phi_N = \frac{1}{2}(\Phi_A - \Phi_H)$. The latter follows from the scalar part of $E_{\alpha\beta}$ where, as in the newtonian theory, $E_{\alpha\beta} = \nabla_{\alpha\beta}\Phi_N$ independently of a gauge choice. Φ_N represents the part of $E_{\alpha\beta}$ that is purely tidal.

From (3.174) and (3.175), considering only scalar perturbations²³ *i.e.* we ignore all vector (solenoidal modes) and all tensorial modes, (3.174) becomes

$$\pi_{\alpha\beta} = S^2(\nabla_{\alpha\beta}\pi), \quad (3.176)$$

and from (3.175) we find, using natural units,

$$\pi_{\alpha\beta} = -\nabla_{\alpha\beta}(\Phi_H + \Phi_A). \quad (3.177)$$

This is claimed in [30] to be general for scalar perturbations. But notice that (137) in [30] holds only in the longitudinal gauge; compare this to (5.11) in Stewart [28].

Furthermore, using the gauge invariant potentials chosen from combinations of the scalar potentials that generate the metric perturbations in the Bardeen formalism, in particular equations (85-86) and (87-88) from [30], we obtain

$$\Phi_A = A - (B' + \frac{a'}{a}B) - (H_T'' + \frac{a'}{a}H_T'), \quad (3.178)$$

$$\Phi_H = H_L - \frac{1}{3}\nabla^2 H_T - \frac{a'}{a}(B + H_T'). \quad (3.179)$$

As before (') denotes conformal time derivatives. To understand the physical meaning of A, B, H_T and H_L , we digress and have a look at the Bardeen formulation of metric and matter perturbations following [28].

²³Here one should in fact be quite careful, as, in the GIC theory no consensus has been reached on how to correctly characterize scalar perturbations. It has been suggested that $H_{ab} = 0$ is the GIC condition ensuring scalar perturbations (Private communication G. F. R. Ellis).

In Bardeen notation, following [30] and [28], the perturbed line element is

$$ds^2 = a^2(\eta) \left[-(1 + 2A)d\eta^2 - 2B_\alpha dx^\alpha d\eta + ((1 + 2H_L)\gamma_{\alpha\beta} + 2H_{T\alpha\beta}) dx^\alpha dx^\beta \right]. \quad (3.180)$$

A, B_α are perturbations in the lapse function and in the shift vector respectively, while $2H_L\gamma_{\alpha\beta}$ and $2H_{T\alpha\beta}$ are perturbations in the 3 surface orthogonal to the conformal time (η) surface, *i.e.* Σ_η the level surfaces of constant time that foliate space-time (following the 3+1 formalism). This means that $N = a(1 + A)$ is the lapse function, which measures the ratio between the coordinate time and the proper time²⁴.

Comparing (3.180) to equation (4.6) in [28], *i.e.* comparing notation we find that $H_{T\alpha\beta} \equiv E_{\alpha\beta}$ and $C = H_L$, while A and B_α remain unchanged (there is a difference in notation between [30] and [28]).

For scalar perturbations $B_\alpha = D_\alpha B$; the 3 vector $B_{|a}$ must be a gradient [28], the trace-free tensor $E_{\alpha\beta}$ must be a second derivative of a scalar *i.e.* $E_{\alpha\beta} = \Delta_{\alpha\beta}$ following the notation of [28] where $\Delta_{\alpha\beta} = D_\alpha D_\beta - \frac{1}{3}\gamma_{\alpha\beta}\Delta \equiv \nabla_\alpha \nabla_\beta - \frac{1}{3}\gamma_{\alpha\beta}\nabla^2$, while in the notation of [30] $\Delta_{\alpha\beta} \equiv \nabla_{\alpha\beta}$. From the perturbation equations (4.9) in [28], under gauge transformations (generated by L and T) we are able to derive the gauge conditions on A, B, C and E in the notation of [28] (in the notation of [30] the conditions on A, B, H_T and H_L),

$$A \rightarrow A + T' + hT, \quad (3.181)$$

$$B \rightarrow B - T + L', \quad (3.182)$$

$$C \rightarrow C + \frac{1}{3}\Delta L + hT, \quad (3.183)$$

$$E \rightarrow E + L. \quad (3.184)$$

Here the gauge functions are given by L and T respectively²⁵, for $h = a'/a = aH$ following the notation of [28]. Bardeen looked for linear combinations of A, B, C and E which were gauge independent; one such choice is

$$\phi_A = A + (B' + hB) - (E'' + hE'), \quad (3.185)$$

$$\phi_C = C - \frac{1}{3}\Delta E + h(B - E'). \quad (3.186)$$

²⁴remember that we are only working in the linear theory hence $N^2 \simeq a^2(1 + 2A)$

²⁵It is interesting to note that the gravitational field has 2 degrees of freedom, while here there are four scalar potentials A, B, C and E and two gauge functions L and T .

These are invariant under the gauge transformations given in (3.181) - (3.184), these are given in a slightly different notation in ([30]) *i.e.* equations (3.178) and (3.179) above.

Longitudinal Gauge

The gauge conditions that define the longitudinal gauge are given from [28] by letting the gauge potentials L and T take on the form

$$L = -E, \quad (3.187)$$

$$T = B - E'. \quad (3.188)$$

In equations (3.182) and (3.184) this means that

$$E \rightarrow 0, \quad (3.189)$$

$$B \rightarrow 0. \quad (3.190)$$

From (3.185) and (3.186) in the longitudinal gauge the gauge invariant scalar potentials reduce to

$$\phi_A \rightarrow A, \quad (3.191)$$

$$\phi_C \rightarrow C. \quad (3.192)$$

This choice of L and T means that, in the longitudinal gauge, any calculations can be made gauge invariant by replacing ϕ_A by A and ϕ_C by C (in the covariant formalism of [30], Φ_A by A and Φ_H by H_L); then

$$\Phi_A = A, \quad (3.193)$$

$$\Phi_H = H_L. \quad (3.194)$$

Restrictions setting $\pi_{\alpha\beta} = 0$

Notice that in the longitudinal gauge if we further restrict the scalar fields by the condition ($\Phi_A = -\Phi_H$) we have limited ourselves to the perturbed line metric of the form

$$ds^2 = a^2(t) \left[-(1 + 2\Phi_A)dt^2 + (1 - 2\Phi_A)\gamma_{\alpha\beta}dx^\alpha dx^\beta \right]. \quad (3.195)$$

Here the unperturbed line metric is given by

$$ds^2 = a^2(t) \left[-dt^2 + \gamma_{\alpha\beta}dx^\alpha dx^\beta \right]. \quad (3.196)$$

For this choice (following the $\Phi = -\Psi$ choice of [50]), we have that from (3.176)

$$\pi_{\alpha\beta} = -\nabla_{\alpha\beta}(\Phi_H + \Phi_A) = -\nabla_{\alpha\beta}(\Phi_A - \Phi_A) = 0. \quad (3.197)$$

Here $\pi_{\alpha\beta} = \Delta_{\alpha\beta}\pi = -2\Delta_{\alpha\beta}\Phi_\pi = -\Delta_{\alpha\beta}(\Phi_A + \Phi_H)$ following [30]. We have obtained sufficient conditions to reduce $\pi_{\alpha\beta}$ to zero without explicitly making the perfect fluid choice. But as a result we have limited the form of the flux contributions through the metric potentials Φ_A and Φ_H to

$$q_\alpha = -S \left[hV_{S|\alpha} - \frac{2}{S^2}(\Phi'_{H|\alpha} - \frac{S'}{S}\Phi_{A|\alpha}) + hV_{C\alpha} + \frac{1}{2S^2}(\nabla^2 + 2k)\Phi_\alpha \right], \quad (3.198)$$

from equation (127) in [30]. Although this approach is physically limiting, it is followed to reproduce analogous results to [54]. In the formalism of the covariant gauge invariant theory [31] $h = \mu + p$. This is not the same h as previously introduced following the formalism of [28], which uses $h = a'/a$. Φ_α is the frame dragging potential, and can be redefined as $\Phi_\alpha = B_\alpha^S + H_{T\alpha}^S$ which is a purely solenoidal contribution to the flux, hence vanishes when we consider only scalar perturbations. The same goes for $V_{C\alpha} = v_\alpha^S - B_\alpha^S$. The flux becomes

$$q_\alpha = -ShV_{S|\alpha} - \frac{2}{S}(\Phi'_{H|\alpha} - \frac{S'}{S}\Phi_{A|\alpha}). \quad (3.199)$$

With the condition $\Phi_A = -\Phi_H$ the flux becomes

$$q_\alpha = \hat{\nabla}_\alpha \left(-S'V_S + \frac{2}{S}\Phi'_A + \frac{S'}{S}\Phi_A \right). \quad (3.200)$$

We know that $S(t) = a(t)$ following the notation used in Appendix A.2, [28] and [30]. What we have achieved here is that for (i) *Scalar perturbations* in the (ii) *Longitudinal gauge* we have along with the condition (iii) $\Phi_A = -\Phi_H$ obtained conditions setting $\pi_{\alpha\beta} = 0$, by restricting ourselves to a perturbed line metric of the form of (3.195), with the end result that $\Pi_\alpha = \hat{\nabla}^a \pi_{ab} = 0 \approx \hat{\nabla}^a \tau_{ab}$. This means that we have, by choosing this perturbed line metric, decoupled the $l = 0$ and $l = 1$ Boltzmann equations from the rest of the infinite hierarchy. That is we have decoupled equations (3.168) and (3.169). This is not general in that it is restricted to the perturbed line metric (3.195) and the interpretation thereof. (The most accurate manner in which to consider the choice of $\Phi_A = -\Phi_H$ is as a physical restriction on the GIC variable π_{ab}).

The choice of gauge in the Bardeen approach is not a problem, as not only is the theory gauge invariant, but due to the non-covariant nature of the formulation the gauge choices are not a threat to meaning of the variables used, in that by picking a gauge we

could in a sense be picking a frame, particularly a frame that is not physically defined, such as the frame defined by the matter 4-velocity (which is frame independent). The important issue here is that in the GIC formulation all the variables used are naturally frame independent *i.e.* covariant by definition, hence one must take care that in practice the form of these naturally covariant variables, when they are constructed from physically motivated arguments, are not physically defined in a frame dependent fashion. This is not a real problem but one should be careful when one makes gauge choices. Having set up the covariant formalism, based on a physical 4-velocity, one will have to transform to a new arbitrary 4 velocity (introducing a 4 - acceleration). One will then have to choose amongst all such arbitrary 4-velocities, the one which is characterized by the vanishing of certain quantities to arrive at a gauge choice based on the covariant viewpoint rather than from the usual Bardeen viewpoint, *i.e.* as expected covariance comes first, the gauge choice is secondary, not the other way around.

Conclusion

Clearly the question we should be asking is whether or not we could pick an L and T such that under the transformations (3.181), (3.182), (3.183) and (3.184) we could have that

$$\Phi_A \rightarrow -\Phi_H. \quad (3.201)$$

It would appear to be tempting to assume we would only need to choose scalar perturbations, and some gauge giving the condition above, in order to effectively have gauged out the anisotropic pressure. Clearly we *cannot* do this as we have expressed the GIC $\pi_{\alpha\beta}$ in terms of purely *gauge invariant* potentials Φ_A and Φ_H . Hence the best we can do at this stage, in an attempt to decouple the $l = 0$ and $l = 1$ Boltzmann equations from the rest of the $l > 1$ equations, is to do one of the following :

- Pick a metric that decouples the GIC Boltzmann equations as we have achieved above by using :-
 - *Scalar Perturbations*
 - *Longitudinal Gauge*
 - $\Phi_A = -\Phi_H$

This is not acceptable as we have in return :-

- Restricted the form of the heat flux without a clear physical motivation

– Restricted the form of the Almost FLRW metric without a clear physical motivation

- (i) Assume matter domination at late times, or at least that π_{ab} is zero at late times. Clearly matter domination would mean that the entire radiation energy-momentum tensor should be ignored and is problematic. An assumption of $\pi_{ab} = 0$ without a clear physical motivation is unacceptable. Hence in full generality one must conclude that in the free-streaming era it is unlikely that we would be able to decouple the $l = 0$ and $l = 1$ Boltzmann equations from the entire set as alluded to in [50], [51], [52] and [53]. A similar situation may exist if we wish to validate the multi-fluid equations in full generality for cosmological perturbations evolving during the free-streaming era [31] as we would have to either be able to gauge ξ^{abc} away (which would be unlikely as it is expressed in a gauge invariant fashion) or somehow motivate from physical grounds that $\xi^{abc} = 0$ or $\hat{\nabla}_a \xi^{abc} = 0$.

(ii) Use the conditions of thermodynamic equilibrium at decoupling *i.e.* the electron and photon temperatures coincide at decoupling hence the matter should have temperature and its temperature should be the same as that of the photons. This would place constraints on the form of q_a and hence τ_{ab} , and hence could be used to find the form of $\hat{\nabla}^a \tau_{ab}$ that follows from strict thermodynamic equilibrium. Furthermore in the strict form of the tight-coupling era as the scattering cross-section diverge the anisotropic pressure would die off due to the equilibrium conditions. This is demonstrated in section 3.5.6, following the notation set up in section 3.5.5.

3.5.5 Scattering terms

The *Transport form* of the *Collision term* can be decomposed in full generality [40]:

$$C[f] = N(x) \left[Q(x^i, p^a) - A(x^i, p^a) f(x^i, p^a) + \int_{T_x} \Sigma(p^a, p'^a) f(x^i, p'^a) dP_m^+ \right] \quad (3.202)$$

where $N(x^i)$ is the number density of the medium, $A(x^i, p^a)$ represents absorption and scattering by the medium and $Q(x^i, p^a)$ is the contribution due to spontaneous particle emission. The integral over σ refers to the rate at which transporting particles are removed or injected from p^a states by interaction with other transport particles in p'^a states (which is vanishing for photons).

For isotropic scattering by the medium (electrons in our case of interest), $Q = Q(x^i, E)$ and $A = A(x^i, E)$, the medium (comoving matter) is treated as an ideal fluid with four

velocity u^a . This ensure that the matter does not contribute to the collisional integrals, the matter is not treated microscopically. This means that

$$C[f] = N(x^i)Q(x^i, E) + N(x^i)A(x^i, E)f(x^i, p^a). \quad (3.203)$$

It is more convenient to adopt the standard radiative transfer form for the isotropic scattering term,

$$L(f) = C[f] = \frac{E}{t_c} (\bar{f} - f), \quad (3.204)$$

this, the BGK scattering term, describes how a distribution f relaxes towards or close to some distribution \bar{f} . The relaxation time, $t_c = t_c(x^i)$, is also known as the effective collision time.

As $L(f)$ describes the rate of change of f along a phase flow; $(E/t_c)\bar{f}$ can be seen as the rate of injection of photons due to scattering or emission by the matter, and $(E/t_c)f$ gives the rate of removal of photons due to scattering and absorption.

For Thompson scattering the effective collision time is

$$t_c = \frac{1}{\sigma_T n_e}. \quad (3.205)$$

Relating the relaxation form of the scattering term for Thompson scattering to the transport form, one finds that; $NQ = (E/t_c)\bar{f}$, $NA = (E/t_c)$, and $\sigma = 0$. Hence adopting the BGK form [25] [40], *i.e.*

$$b_{A_l} = t_c^{-1}(\bar{F}_{A_l} - F_{A_l}). \quad (3.206)$$

This follows from isotropic scattering, in practice we are dealing with the case of almost isotropic scattering, hence if we consider $F_{A_l} \ll 1$ and on this basis ignore harmonic mixing, and taking \bar{f} to be isotropic and representing the equilibrium distribution, we can use the BGK form for isotropic scattering and set $\bar{F}_{A_l} = 0$, for our almost isotropic case.

$$b = \gamma(x^i, E)(\bar{F} - F) \quad l = 0 \quad (3.207)$$

$$b_{A_l} = -\gamma(x^i, E)F_{A_l} \quad \forall l > 0. \quad (3.208)$$

To reiterate, the important simplicity of the BGK model is that there is no harmonic mixing, something that will most likely occur in the more general transport form of the scattering term (3.202). The BGK model, however, does not guarantee that the conservation equations ($N^a_{;a} = 0$ and $T^{ab}_{;b} = 0$) and the H-theorem ($S^a_{;a} \geq 0$) are satisfied [40].

Additional constraints are required, in fact [40] point out that the BGK scattering form generally means that $\int \gamma(f - \bar{f})dP = 0 = \int p^a(f - \bar{f})dP$, placing limits on the local rest frame. In the BGK form (3.207), we have chosen

$$\gamma(x^i, E) = \frac{E}{t_c}. \quad (3.209)$$

This means that we can in good faith adopt the BGK hierarchy as a good approximation to (3.202) if all the non-isotropic or inhomogeneous contributions to the distribution function are small.

It is important to realize the difference between the two models; the BGK collisional term models the relaxation of a distribution function towards a new configuration at some later time while the transport form describes the physical modification of a distribution function due to the actual processes of spontaneous emission, absorption and scattering. $C[f]$ is in general considered to represent the self-interactions of the gas (in the case of decoupling the modification of the photon gas through interactions with electrons). This is discussed in [40]. It has also been shown [36] that the BGK model is a good approximation to Thompson scattering, in fact [38] demonstrated that such an approximation may also be used for free-free Bremsstrahlung.

Assuming that the temperature perturbations are small (3.125) *i.e.* $\delta T/T \ll 0$ which effectively allows the use of equations (3.126), (3.127) and (3.128), we use the scattering terms of the BGK form (3.207 and 3.208) with the relaxation time chosen in terms of the scattering time t_c (3.209).

Furthermore, we assume that $(\bar{F} - F) \approx 0$, this follows from the premise that the photons will be relaxing to a Plankian. This is a subtle point ²⁶ strictly speaking this is only true for photons on there own.

3.5.6 Derivation of a second order averaged bolometric temperature equation from the $l = 0$ and $l = 1$ Boltzmann moment equations

Consider the $l = 0$ and the $l = 1$ Boltzmann equations:

$$\left(\frac{\dot{T}}{T} + \frac{\dot{S}}{S} \right) + \frac{1}{3} \hat{\nabla}_a \tau^a \simeq \frac{\pi}{T^4} B = C, \quad (3.210)$$

$$\dot{\tau}_a + 4 \left(\frac{\dot{T}}{T} + \frac{\dot{S}}{S} \right) \tau_a + \frac{\hat{\nabla}_a T}{T} + \frac{2}{5} \hat{\nabla}_b \tau_a^b \simeq \frac{\pi}{T^4} B_a = C_a. \quad (3.211)$$

²⁶Private communication with Roy Maartens

The scattering terms C and C_a are defined in terms of B and B_a , which are defined as

$$B = \int E^2 dE b, \quad (3.212)$$

$$B_a = \int E^2 dE b_a. \quad (3.213)$$

The scattering terms

Using (3.207) and (3.208),

$$b = \frac{E}{t_c} (\bar{F} - F) \quad (3.214)$$

$$b_{A_l} = -\frac{E}{t_c} F_{A_l} \quad l > 0 \quad (3.215)$$

i.e. assuming that scattering is independent of angles (usually $\cos^4(\phi)$) [36] and thus that the scattering is isotropic, and that there are no other mechanisms that cause deviations from thermodynamic equilibrium other than Thompson scattering (we neglect Raleigh scattering and so forth). Here t_c is the mean collision time or scattering time, σ_T is the Thompson scattering cross-section and n_e the electron number density²⁷. Assuming particle conservation, we have that $\dot{n}_e = -n_e\theta$. Using this, (3.209) and (3.205) one finds

$$\dot{t}_c = +t_c\theta = +3t_c \frac{\dot{S}}{S}. \quad (3.216)$$

From equations (3.214) and (3.215) using (3.209), (3.212) and (3.213) along with $(\bar{F} - F) \approx 0$:

$$B = \int E^2 \left(\frac{+E(\bar{F} - F)}{t_c} \right) dE = +\frac{1}{t_c} \int E^3 dE (\bar{F} - F) \approx 0, \quad (3.217)$$

$$B_{A_l} = \int E^2 \left(\frac{-E F_{A_l}}{t_c} \right) dE = -\frac{1}{t_c} \int E^3 F_{A_l} dE = -\frac{\tau_{A_l}}{t_c} \left(\frac{T^4}{\pi} \right). \quad (3.218)$$

From (3.210) and (3.211) we are using

$$C \approx 0, \quad (3.219)$$

$$C_a = -\frac{\tau_a}{t_c}. \quad (3.220)$$

Furthermore from (3.219) and (3.220),

$$[t_c^{-1}]' = -3t_c^{-1} \left(\frac{\dot{S}}{S} \right), \quad (3.221)$$

²⁷For a more complete treatment see Appendix E of [35], [27] or [36].

$$\hat{\nabla}^a[t_c^{-1}] = t_c^{-1} \left(\frac{\hat{\nabla}^a n_e}{n_e} \right), \quad (3.222)$$

$$\hat{\nabla}^a C_a = \hat{\nabla} \left(\frac{-\tau_a}{t_c} \right) = -t_c^{-1} \hat{\nabla}^a \tau_a + \tau_a (-\hat{\nabla}^a[t_c^{-1}]) \simeq -t_c^{-1} \hat{\nabla}^a \tau_a. \quad (3.223)$$

GIC evolution equation for τ_a

Take (3.210), multiply this by $4\tau_a$, and then subtract this from equation (3.211) to find

$$\dot{\tau}_a + \left(\frac{\hat{\nabla}_a T}{T} \right) + \frac{2}{5} \hat{\nabla}_b \tau_a^b \simeq C_a - 4C\tau_a \approx C_a. \quad (3.224)$$

This gives us a GIC evolution equation for τ_a . Notice that as $T(x^a)$ is Gauge Invariant (GI) it can then be ensured that equations (3.210) and (3.211) are GI [28] [33]. We can ensure that the above equation (3.224) is GIC as it has been constructed out of GIC variables. We can use (3.224) and (3.220) to find

$$\dot{\tau}_a + t_c^{-1} \tau_a \simeq -\frac{\hat{\nabla}_a T}{T} - \frac{2}{5} \hat{\nabla}_b \tau_a^b. \quad (3.225)$$

If the form of π_{ab} was known then the evolution of τ_a in terms of the behaviour of the averaged bolometric temperature could be investigated. As before in 3.5.1, we could limit our investigation to scalar perturbations, and adopt the longitudinal gauge with $\Phi_A = -\Phi_H$ thus setting $\tau_{ab} = 0$ and limiting the form of τ_a through the limitations place on the radiation flux q_a by (3.200).

- Notice that for $t_c \rightarrow \infty$, as $\sigma_T \rightarrow 0$ *i.e.* in the case of *Free-Streaming* as the photons decouple from the matter fluid,

$$\dot{\tau}_a \simeq -\frac{\hat{\nabla}_a T}{T}. \quad (3.226)$$

- For $t_c \rightarrow 0$, as $\sigma_T \rightarrow \infty$ *i.e.* in the case of *Tight-Coupling* as the photons are tightly coupled to the matter fluid, with very large scattering cross-sections and no mean free scattering time as they are *always* scattering, one finds

$$\tau_a = 0. \quad (3.227)$$

This is obtained by multiplying (3.225) through by t_c^{-1} and then taking the limit as t_c^{-1} becomes large. If we are considering only scalar perturbations in the longitudinal gauge with only one scalar potential then we have already set $\pi_{ab} = 0$, hence by taking the tight-coupling limit we have in fact forced the evolution to be that of a perfect fluid.

It is important to notice that in general the *Free-Streaming limit* gives us the condition

$$\dot{\tau}_a \simeq -\frac{\hat{\nabla}_a T}{T} - \frac{2}{5}\hat{\nabla}_b \tau_a^b. \quad (3.228)$$

While in general we may be tempted to try the same game with the *Tight-Coupling limit*, it must be remembered that we have assumed *a priori* that $\tau_{ab} \ll 1$ (we have adopted small anisotropies). Due to the manner in which we have constructed the scattering terms, all the higher orders in C (*i.e.* t_c^{-1}) will enter through τ_{ab} , hence unless we have set this to zero or are *very* sure that this term doesn't diverge in the limit as the C_{A_l} scattering term in the l -th Boltzmann moment equation starts to dominate, for l large. This is important; if τ_{ab} does diverge in this limit then we have a consistency problem. We know that the perturbations τ_{A_l} are small, but in the tight-coupling limit this may in fact not be the case. This means either that the tight coupling limit is unphysical or that I have applied it incorrectly. This is the motivation for (i) trying to set $\pi_{ab} = 0$ through metric and gauge restrictions (ii) suggesting that the GIC generalization of the tight-coupling limit as used in [52] [53] [36] [50] be carefully investigated.

At any rate I continue the quest for the GIC generalization of the second order time equation of the averaged bolometric temperature [54] [50] (an equation in their θ_0 and in our $T^{(k)}(x^a)$).

Second order evolution equation of $T(x^a)$

Once again considering equations (3.210) and (3.211) respectively we have from these

$$\hat{\nabla}^a \tau_a \simeq 3C - 3 \left(\frac{\dot{T}}{T} + \frac{\dot{S}}{S} \right), \quad (3.229)$$

$$\dot{\tau}_a \simeq C_a - 4 \left(\frac{\dot{T}}{T} + \frac{\dot{S}}{S} \right) \tau_a - \frac{\hat{\nabla}_a T}{T} - \frac{2}{5}\hat{\nabla}_b \tau_a^b. \quad (3.230)$$

To proceed further it is useful to first notice the form of the linearized Ricci identities *i.e.* $u^a_{;dc} - u^a_{;cd} = R^a_{bcd}u^b$ and generalizations of this [30] [31]. To $O[1]$ for $O[1]$ vector V_b and $O[1]$ tensor T_{bc} terms,

$$\begin{aligned} S\hat{\nabla}^a \dot{V}_b &\simeq (S\hat{\nabla}^a V_b) \dot{}, \\ S\hat{\nabla}^a \dot{T}_{bc} &\simeq (S\hat{\nabla}^a T_{bc}) \dot{}. \end{aligned} \quad (3.231)$$

Following [31], this can be shown by noting that $\nabla_c \nabla_e X_d - \nabla_e \nabla_c X_d = R_{dgc e} X^g$ for X_d being at least $O[1]$. It is then noticed that if we used any part of the Riemann tensor

greater than $O[0]$ in this then the product of $R_{dgce}X^g$ would be at least $O[2]$. We are in the linear theory hence we are discarding any contribution that is at least $O[2]$. This means that in the Almost FLRW theory as adopted in 3.4 we need only consider the FLRW, *i.e.* the $O[0]$, part of the full inhomogeneous Riemann tensor R_{dgce} . It is then straight-forward to show that $\hat{\nabla}_a(\dot{X}_d) - (\hat{\nabla}_a X_b)^\cdot \simeq \frac{1}{3}\theta\hat{\nabla}X_b$. From this, and using $\frac{1}{3}\theta = \dot{S}/S$, the vector relation in (3.231) can be obtained. The tensor relation follows along similar lines of argument.

The vector relation in (3.231) will be used to simplify (3.229) and (3.230), but first we must take the covariant time derivative of (3.229) and then the spatial divergence of (3.230) both of which will be linearized by noticing that $\hat{\nabla}_a S$, $\hat{\nabla}_a \dot{S}$, τ_a and $\hat{\nabla}_a T/T$ are all at least $O[1]$; meaning that the product of any two of these is at least $O[2]$, and can then be ignored in the linear theory which we are considering *i.e.* an Almost FLRW spacetime with small temperature anisotropies.

Taking the (\cdot) of (3.229) after it is multiplied through by the scale factor S , we find

$$\begin{aligned} (S\hat{\nabla}^a\tau_a)^\cdot &\simeq \left[3SC - 3S\left(\frac{\dot{T}}{T} + \frac{\dot{S}}{S}\right) \right]^\cdot \\ &\simeq 3\dot{S}C + 3S\dot{C} - 3\dot{S}\left(\frac{\dot{T}}{T} + \frac{\dot{S}}{S}\right) \\ &\quad - 3S\left(\frac{\ddot{T}}{T} + \frac{\ddot{S}}{S} - \left(\frac{\dot{T}}{T}\right)^2 - \left(\frac{\dot{S}}{S}\right)^2\right), \end{aligned} \quad (3.232)$$

while taking the spatial divergence of (3.230), linearizing and substituting for $\hat{\nabla}^a\tau_a$ from (3.229), and then multiplying through by the scale factor, one obtains

$$\begin{aligned} S\hat{\nabla}^a\dot{\tau}_a &\simeq S\left[\hat{\nabla}^a C_a - 4\left(\frac{\dot{T}}{T} + \frac{\dot{S}}{S}\right)\left(3C - 3\left(\frac{\dot{T}}{T} + \frac{\dot{S}}{S}\right)\right) \right. \\ &\quad \left. - \frac{\hat{\nabla}^2 T}{T} - \frac{2}{5}\hat{\nabla}^a\hat{\nabla}^b\tau_{ba}\right]. \end{aligned} \quad (3.233)$$

Using the linearized Ricci Identities from (3.231) *i.e.*

$$(S\hat{\nabla}^a\tau_a)^\cdot \simeq S\hat{\nabla}^a\dot{\tau}_a, \quad (3.234)$$

one finds substituting from (3.232) and (3.233),

$$\begin{aligned} S\left[\hat{\nabla}^a C_a - 4\left(\frac{\dot{T}}{T} + \frac{\dot{S}}{S}\right)\left(3C - 3\left(\frac{\dot{T}}{T} + \frac{\dot{S}}{S}\right)\right) - \frac{\hat{\nabla}^2 T}{T} - \frac{2}{5}\hat{\nabla}^a\hat{\nabla}^b\tau_{ba}\right] \\ \simeq 3\dot{S}C + 3S\dot{C} - 3\dot{S}\left(\frac{\dot{T}}{T} + \frac{\dot{S}}{S}\right) - 3S\left(\frac{\ddot{T}}{T} + \frac{\ddot{S}}{S} - \left(\frac{\dot{T}}{T}\right)^2 - \left(\frac{\dot{S}}{S}\right)^2\right). \end{aligned} \quad (3.235)$$

Here $\hat{\nabla}^2 T$ is equivalent to $\hat{\nabla}^a \hat{\nabla}_a T$. Multiplying through by $1/(3S)$, along with some algebra, (3.235) is reduced to

$$\begin{aligned} \frac{2}{15} \hat{\nabla}^b \hat{\nabla}^a \tau_{ba} \simeq & + \frac{1}{3} \hat{\nabla}^a C_a - 4C \left(\frac{\dot{T}}{T} + \frac{\dot{S}}{S} \right) + 4 \left(\frac{\dot{T}}{T} + \frac{\dot{S}}{S} \right)^2 \\ & - \frac{1}{3} \frac{\hat{\nabla}^2 T}{T} - \frac{\dot{S}}{S} C - \dot{C} + \frac{\dot{S}}{S} \left(\frac{\dot{T}}{T} + \frac{\dot{S}}{S} \right) \\ & + \frac{\ddot{T}}{T} + \frac{\ddot{S}}{S} - \left(\frac{\dot{T}}{T} \right)^2 - \left(\frac{\dot{S}}{S} \right)^2. \end{aligned} \quad (3.236)$$

Notice that $\hat{\nabla}^a \hat{\nabla}^b \tau_{ba}$ is GIC, which means that as the LHS (left hand side) of (3.236) is GIC then the RHS (right hand side) of (3.236) is GIC too. One may be tempted to think that $T(x^a)$ is not GIC, as in the Bardeen case when one is dealing with $T_0 + \delta T$ and δT can be scaled out leaving T_0 dependent on the gauge choice. In the GIC formalism, however, $T(x^a) = (\mu(x^a)/a)^{\frac{1}{3}}$; the energy density $\mu(x^a)$ is measured in the matter-observer's frame, hence $T(x^a)$ is GIC. Similarly for S , if we define it in the usual fashion as $\dot{S}/S = \frac{1}{3}\theta$. It is interesting to note that in the O[1] equations, S may be replaced by its background value S_0 in most cases.

The important result here is that the RHS of (3.236) is GIC as τ_{ba} is a GIC variable to O[1] in the Almost FLRW theory. We could now enforce the choice of scalar perturbations, in the longitudinal gauge along with $\Phi_\pi = \frac{1}{2}(\Phi_A + \Phi_H) = 0$, to thus eliminate the LHS of (3.236) giving us a second order equation of the average bolometric temperature $T(x^a)$, but it is convenient to retain the anisotropic pressure term until a later stage. We now continue, by trying to rearrange (3.236) into a more useful form. From (3.236) we find after some algebra,

$$\begin{aligned} \frac{\ddot{T}}{T} + \frac{\ddot{S}}{S} + 3 \left(\frac{\dot{T}}{T} \right)^2 + 4 \left(\frac{\dot{S}}{S} \right)^2 + 9 \left(\frac{\dot{T}}{T} \right) \left(\frac{\dot{S}}{S} \right) - \dot{C} - \left(\frac{\dot{S}}{S} \right) C - 4C \left(\frac{\dot{T}}{T} \right) \\ - 4C \left(\frac{\dot{S}}{S} \right) - \frac{1}{3} \frac{\hat{\nabla}^2 T}{T} + \frac{1}{3} \hat{\nabla}^a C_a \simeq \frac{2}{15} \hat{\nabla}^a \hat{\nabla}^b \tau_{bc}. \end{aligned} \quad (3.237)$$

Using the form of the scattering terms in (3.221) and (3.223) *i.e.* $\hat{\nabla}^a C_a \simeq -t_c^{-1} \hat{\nabla}^a \tau_a$ and $C \simeq 0$ from $(\bar{F} - F) \simeq 0$, along with (3.210) we can reduce (3.237) to

$$\begin{aligned} \frac{\ddot{T}}{T} + \frac{\ddot{S}}{S} + 3 \left(\frac{\dot{T}}{T} \right)^2 + 4 \left(\frac{\dot{S}}{S} \right)^2 + 9 \left(\frac{\dot{T}}{T} \right) \left(\frac{\dot{S}}{S} \right) \\ - \frac{1}{3} \frac{\hat{\nabla}^2 T}{T} + \frac{1}{3t_c} \left(\frac{\dot{T}}{T} + \frac{\dot{S}}{S} \right) \simeq \frac{2}{15} \hat{\nabla}^a \hat{\nabla}^b \tau_{ba}. \end{aligned} \quad (3.238)$$

Rearranging terms in terms of orders in C , notice that we have not added or subtracted any terms from the original GIC form of this equation and have therefore maintained its GIC nature. We finally get

$$\left[\frac{\ddot{T}}{T} + \frac{\ddot{S}}{S} + 3 \left(\frac{\dot{T}}{T} \right)^2 + 4 \left(\frac{\dot{S}}{S} \right)^2 + 9 \left(\frac{\dot{S}}{S} \right) \left(\frac{\dot{T}}{T} \right) - \frac{1}{3} \frac{\hat{\nabla}^2 T}{T} \right] + \frac{1}{3t_c} \left[\frac{\dot{S}}{S} + \frac{\dot{T}}{T} \right] \simeq \frac{2}{15} \hat{\nabla}^a \hat{\nabla}^b \tau_{ba}. \quad (3.239)$$

Some links to the usual GIC variables

In this section I briefly try and make some important links with the variables I use and the traditional GIC variables [31] and Section 3.3. Using $D^a = X^a/\mu \equiv h^{ab}\mu_{;b}/\mu$ we know that we can write $(4\hat{\nabla}_a T/T) \equiv D_a^{(r)}$. This gives that

$$\hat{\nabla}^a (SD_a^{(r)}) \simeq 4S \frac{\hat{\nabla}^2 T}{T}, \quad (3.240)$$

$$\Delta_{(r)} \simeq 4S^2 \frac{\hat{\nabla}^2 T}{T}. \quad (3.241)$$

These are all to $O[1]$ and follow from

$$\mathcal{D}_a^{(r)} \equiv 4S \frac{\hat{\nabla}_a T}{T}, \quad (3.242)$$

$$\Delta^{(r)} \equiv \Delta_a^{(r)} \equiv S \hat{\nabla}^a \mathcal{D}_a^{(r)}. \quad (3.243)$$

Where to $O[1]$,

$$\Delta^{(r)} = S \hat{\nabla}_a \left(\frac{4S \hat{\nabla}^a T}{T} \right) = 4S^2 \left(\frac{\hat{\nabla}^a S \hat{\nabla}_a T}{S} + \frac{\hat{\nabla}^a \hat{\nabla}_a T}{T} - \frac{\hat{\nabla}_a T \hat{\nabla}^a T}{T} \right) \simeq 4S^2 \frac{\hat{\nabla}^2 T}{T} \quad (3.244)$$

The perturbation equations in Δ and D_a as given in the GIC perturbation papers [30] [31] and [33] could be included using these relations into the Boltzmann moment equations; the key relation here is clearly (3.241).

From (3.239) for the case of scalar perturbations, in the longitudinal gauge with the $\Phi_A = -\Phi_H$ and in the free-streaming limit *i.e.* $t_c^{-1} = 0$, we have that

$$\frac{\ddot{T}}{T} + \frac{\ddot{S}}{S} + 3 \left(\frac{\dot{T}}{T} \right)^2 + 4 \left(\frac{\dot{S}}{S} \right)^2 + 9 \frac{\dot{T} \dot{S}}{T S} \simeq \frac{1}{3} \frac{\hat{\nabla}^2 T}{T} = \frac{1}{12} \frac{\Delta^{(r)}}{S^2}. \quad (3.245)$$

This is basically the divergence of (3.225) for scalar perturbations in the longitudinal gauge with the anisotropic pressure potential $\Phi_\pi = \Phi_A + \Phi_H = 0$. Either we can make the above

substitution (3.241) to remove the spatial gradients of the average bolometric temperature from my evolution equations *eg* (3.245) above and then use the perturbation equations for $\Delta^{(\tau)}$ given by [30] [31]. Or we may remove the spatial divergences in such a manner as to not directly require the explicit form of the multi-fluid perturbation equations; we wish to make a further harmonic expansion. This time we expand $T(x^a)$, the average bolometric temperature at a space-time point x^a in terms of natural harmonics in the spatial slices.

Harmonic Expansion (HE)

Following [61] [63] and [30], we are dealing primarily with scalar perturbations, hence need only worry about scalar harmonics. We use scalar harmonics which are eigenfunctions of the covariantly defined Laplace-Beltrami operator,

$$\hat{\nabla}_a \hat{\nabla}^a Q^{(k)} = -\frac{k^2}{S^2} Q^{(k)}. \quad (3.246)$$

Here $k^2 = \nu^2 - K$ is used; with the wavelength given as $\lambda = S/\nu$. [34] using $\nabla^2 Y^{(k)} = -k^2 Y^{(k)}$, for the Laplacian in the 3-spaces with metric $\gamma_{\alpha\beta}$. Hence we can expand some scalar T,

$$T = \sum_k T^{(k)} Q^{(k)}, \quad (3.247)$$

and

$$\hat{\nabla}_a T = \hat{\nabla}_a \sum_k T^{(k)} Q^{(k)} = \sum_k T^{(k)} \hat{\nabla}_a Q^{(k)}. \quad (3.248)$$

[30] stress that (3.247) and (3.248) are valid only for GI first-order quantities, and $T_{(k)}$ are first order components of T. In fact $T_{(k)}$ are claimed to be unambiguously zero in the background, and are hence GI. Furthermore we have that from (3.246) and (3.248),

$$\hat{\nabla}_a \hat{\nabla}^a T = -\sum_k \frac{k^2}{S^2} T^{(k)} Q^{(k)}. \quad (3.249)$$

The continued quest for the second order T equation

Before we can apply the HE to the equations above, we must first (i) find a way in which to deal with the non-linear terms $(\dot{T}/T)^2$ as these cannot be usefully expanded in terms of the HE *i.e.* the spatial harmonic $Q^{(k)}$ may be factored out along with the sum over k such that these can be dropped from the equations on both sides (see later), (ii) find a useful way with which to decouple the $l = 0$ and $l = 1$ Boltzmann equations from the

$l > 1$ equations by either (a) postulating or (b) physically motivating a restricted form of the anisotropic pressure π_{ab} . To facilitate these two issues *i.e.* (i) and (ii), we make the following substitutions:

$$\begin{aligned} G_1(x^a) &= \frac{\dot{T}}{T}, \\ G_2'(x^a) &= \frac{2}{15} \hat{\nabla}^a \hat{\nabla}^b \tau_{ba}, \\ G_2(x^a) &= [G_2'(x^a)]T(x^a). \end{aligned} \quad (3.250)$$

We multiply (3.239) through by T , to find

$$\begin{aligned} \left[\ddot{T} + T \left(\frac{\ddot{S}}{S} \right) + 3\dot{T} \left(\frac{\dot{T}}{T} \right) + 4T \left(\frac{\dot{S}}{S} \right)^2 + 9\dot{T} \left(\frac{\dot{S}}{S} \right) - \frac{1}{3} \hat{\nabla}^2 T \right] \\ + \frac{T}{3t_c} \left(\frac{\dot{S}}{S} \right) - \frac{1}{3t_c} \dot{T} \simeq G_2(x^a). \end{aligned} \quad (3.251)$$

After making the substitutions from (3.250), this can be rewritten as

$$\begin{aligned} \ddot{T} + \dot{T} \left(3G_1(x^a) + 9\frac{\dot{S}}{S} + \frac{1}{3t_c} \right) \\ + T \left(4 \left(\frac{\dot{S}}{S} \right)^2 + \left(\frac{\ddot{S}}{S} \right) - \frac{1}{3t_c} \frac{\dot{S}}{S} \right) - \frac{1}{3} \hat{\nabla}^2 T \simeq G_2(x^a). \end{aligned} \quad (3.252)$$

Notice that $G_2(x^a)$ includes all orders of $1/t_c$ through the coupling of τ_{ba} to all orders in l *i.e.* couples this equation into the rest of the $l > 1$ Boltzmann moment equations. To get any further with equation (3.252) we must address the issue of the form of $G_1(x^a)$ and $G_2(x^a)$.

Finding the form of $G_1(x^a)$

The total energy density μ , is given in terms of the individual energy density components; $\mu = \sum \mu_i$. For instance we use $\mu = \mu_R + \mu_M$ here for convenience, following 3.4, but it is straight-forward to generalize to more fluids by just including their perturbation equations and contributions to the total energy-momentum tensor.

One can construct \dot{T}/T in a covariant manner in terms of K , S and μ_i (in our case we consider K , S and μ_M). Here K is the intrinsic curvature, S the scale factor and μ_M the matter energy density. From the Stefan-Boltzmann law we have that

$$\frac{4\dot{T}}{T} = \frac{\dot{\mu}_R}{\mu_R}. \quad (3.253)$$

From the energy constraints (or generalized Gauss-Codacci equations) it is known that

$$\frac{1}{2} {}^{(3)}R = -3H^2 + \sigma^2 + \omega^2 + \mu. \quad (3.254)$$

Here to at least $O[1]$ the energy constraint becomes

$$\frac{1}{2} {}^{(3)}R \simeq -3H^2 + \mu. \quad (3.255)$$

(We are using $HS = \dot{S}$). The curvature is defined by

$$K = \frac{1}{6} {}^{(3)}R. \quad (3.256)$$

We Combine (3.255) with (3.256) by substituting for ${}^{(3)}R$, to get

$$\mu_R + \mu_M = \mu \simeq 3(K + H^2). \quad (3.257)$$

It then follows from (3.253) and (3.257) that

$$\frac{\dot{T}}{T} \simeq \frac{3(K + H^2) - \mu_M}{3(K + H^2) - \mu_M}. \quad (3.258)$$

The perfect fluid energy conservation equation (2.7) for pressure-free dust,

$$\mu_M = -3H\mu_M, \quad (3.259)$$

is then substituted into (3.258) to finally get

$$\frac{\dot{T}}{T} \simeq \frac{1}{4} \left(\frac{3\dot{K} + 3H(2\dot{H} + \mu_M)}{3K + 3H^2 - \mu_M} \right). \quad (3.260)$$

This means that we have found $G_1(x^a)$,

$$G_1(x^a) = \frac{1}{4} \left(\frac{3\dot{K} + 3H(2\dot{H} + \mu_M)}{3K + 3H^2 - \mu_M} \right). \quad (3.261)$$

This along with $HS = \dot{S}$ and using $k^2 = \nu^2 - K$ from 3.5.6 is then used in (3.239). Before this is done we must first try to address the issues of $G'_2(x^a)$ and $G_2(x^a)$ in (3.239).

Finding the form of $G_2(x^a)$

In the past section and some of the proceeding sections I have alluded to and sometimes discussed basically the three approaches that I could see, with regard to tackling the issue of $G_2(x^a)$, which is essentially the issue of trying to usefully deal with N differential equations, for N large. I have adopted the approach of trying the decouple the first two of

these equations *i.e.* the $l = 0$ and the $l = 1$ equations, primarily because these equations are analogous to the fluid equations [31] [30] [55]. This can be seen by comparing the linearized equations (3.145) with (3.157) and (3.147) with (3.161). Obtaining the fluid equivalent of the $l = 2$ Boltzmann equation was considered to be a non-trivial exercise and I wished to see how the Boltzmann equations could be reduced consistently to the fluid equations. The other motivation was that in [50] [52] [54] Π_a was *shown* to vanish, thus allowing the GI $l = 0$ and $l = 1$ equations to decouple from the rest of their Boltzmann moment equations²⁸, where their approach is coordinate dependent and hence non-covariant. Furthermore their approach cannot be easily generalize to O[2], that is second order processes. It should be clear that the GIC theory can easily be adapted to be a O[2] theory instead of the O[1] theory that I have used. This can be achieved merely by redefining the linearization process, that is we would merely retain terms up to at least O[2] instead of to at least O[1] and drop all term that are at least O[3]. This is something that is not trivial in the Bardeen approach, if it can be consistently done at all.

- (a) **Metric/Gauge limitations** : I have demonstrated that a gauge choice alone is not sufficient to decouple the $l = 0$ and $l = 1$ equations as π_{ab} is GI and hence cannot be gauged out of the observer frame. But with the mixture of a gauge choice and a restriction on the perturbations this can be achieved as discussed in some depth in 3.5.4 and 3.5.4 *i.e.* using the Longitudinal gauge, for scalar perturbations with $\Phi_A = -\Phi_H$; this combination is sufficient to ensure that $\pi_{ab} = 0$. The net result of

²⁸The Bardeen GI Boltzmann equation, for $\theta = \delta\rho\gamma/\rho\gamma$ (section 2.2.4) [54]

$$\dot{\theta} + \gamma^i \frac{\partial}{\partial x^i} (\theta + \Phi) + \gamma^i \frac{\partial}{\partial \gamma^i} \theta + \dot{\Phi} = \dot{\tau} (\theta_0 - \theta - \gamma_i v_b^i + \frac{1}{16} \gamma_i \gamma_j \Pi_{ij}^{\gamma}). \quad (3.262)$$

This is broken up into a moment expansion using a multipole decomposition of the anisotropies $\theta(\eta, x, \gamma) = \sum_{l=0}^{\infty} \theta_l(\eta) M_l^{-\frac{1}{2}} G_l(x, \gamma)$ (pg 66 [54])

$$\theta_0 = -\frac{k}{3} \theta_1 - \dot{\Phi}, \quad (3.263)$$

$$\theta_1 = k \left[\theta_0 + \Psi - \frac{2}{5} K_2^{\frac{1}{2}} \theta_2 \right] - \dot{\tau} (\theta_1 - V_b), \quad (3.264)$$

$$\theta_2 = k \left[\frac{2}{3} K_2^{\frac{1}{2}} \theta_1 - \frac{3}{7} K_3^{\frac{1}{2}} \theta_3 \right] - \frac{9}{10} \dot{\tau} \theta_2, \quad (3.265)$$

$$\theta_l = k \left[\frac{l}{2l-1} K_l^{\frac{1}{2}} \theta_{l-1} - \frac{l+1}{2l+3} K_{l+1}^{\frac{1}{2}} \theta_{l+1} \right] - \dot{\tau} \theta_l. \quad (3.266)$$

Notice that this is very different from the combination of the Harmonic expansions in f and $T(x^a)$ that are used in the GIC treatment. It is also worthwhile to note that $4\theta = \frac{\pi^2}{\rho\gamma} \int p^3 dp f - 1 = \delta\rho\gamma/\rho\gamma$ with f given as $f(p) = [\exp(p/(T + \delta T)) - 1]^{-1}$, while in the GIC treatment τ is used where $4\tau \approx \frac{\pi}{\mu(x^i)} \int E^3 f dE = \frac{\pi}{\mu(x^i)} \sum_l (\int E^3 F_{A_l} dE) e^{A_l}$. These are clearly two very different approaches.

such a choice is to set

$$G'_2(x^a) = 0 \Rightarrow G_2(x^a) = 0. \quad (3.267)$$

The problem with this approach is that we have most probably compromised covariance, we have in fact restricted the space-time to that of a metric given by (3.195) and have hence restricted the form of the momentum flux of the radiation to that of the form of (3.200).

- (b) **Thermodynamic limitations** : This approach is a lot trickier than making metric or gauge restrictions, as such limitations are *direct* assumptions with regard to the actual physics, and in this approach I am forced to make some very idealistic assumptions about the physics at (I) decoupling and (II) the free-streaming era, but I feel that these idealistic assumptions highlight some interesting points.

- (I) *Collisionally Dominated era* : If tight-coupling is taken to be the case strictly when the mean scattering time tends to zero, then the problem is effectively reduced to that of scattering in a FLRW space-time, *i.e.* an isotropic, spatially homogeneous geometry. Although we assume no-perturbations in the $O[0]$ contribution to the photon distribution function F , we do not kill off the perturbations entirely. It is also assumed that tight-coupling implies thermodynamic equilibrium, hence that detailed balancing holds where the photons and electrons are at the same temperature. This means that restrictions on the photon collision term can be used to restrict that of the matter. (In reality there is not an exact equilibrium, but the limit is considered here).

$$T^{ab}_{;b} = 4\pi J^a = 0 \quad (3.268)$$

$$\Rightarrow T^{ab}_{;b} = T^{ab}_{R;b} + T^{ab}_{M;b} = 0 \quad (3.269)$$

$$\Rightarrow T^{ab}_{R;b} = -T^{ab}_{M;b}, \quad (3.270)$$

where the energy-momentum tensors are given as usual. Notice that if the radiation and the dust are in thermal equilibrium, *i.e.* if the baryonic matter and the radiation prior to decoupling are in thermal equilibrium, this would mean that both the baryonic matter and the radiation must have the same temperature²⁹. This would seem to imply that the baryonic matter should in

²⁹See Appendix A.4 : This also shows that if we are in thermodynamic equilibrium for an electron - photon system in a FLRW space-time then $T_{e;b}^{ab} = 0 \Rightarrow T_{\gamma;b}^{ab} = 0$ which would mean that in the tight coupling limit $B_a = 0 = B$. This is also shown in Appendix E of [35].

fact have a temperature to make this statement meaningful. If that were the case, however, it would mean that the baryonic fluid would have a pressure prior to decoupling, that is away from the free streaming limit. In the free-streaming limit the dust could still be considered to be pressure free, as it is not in thermal equilibrium with the radiation, hence a vanishing pressure would not make the notion of temperature problematic. Hence before decoupling there would be a pressure term $n_m T_m$ assuming some non-relativistic ideal gas form with n_m the number density of the baryonic matter and T_m its temperature. This pressure contribution is generally ignored and following the usual treatment we drop the term, but list it below for completeness [36]. The other motivation for dropping the pressure term is that if we chose this as our pressure we would then in fact be choosing our energy density to be $\mu_M = n_m m + \frac{3}{2} n_m T_m$. There could also be a little confusion over which number density to use in the collision term, as we would be scattering off the baryonic fluid (we have used a fluid + gas model). We know, however, that the photons would interact with any particle that carries charge, hence before recombination the photons would scatter not only off the electrons but off the protons, increasing scattering, while after recombination we assume that photons would scatter primarily off bound electrons, a process which is considered to have a very small effect upon the mean-free path of the photons. After decoupling the matter and radiation would free-stream. We assume that the pre-recombination scattering is mainly off free electrons and that the scattering between the recombination and decoupling is very small, so that we can free-stream immediately after recombination has occurred. The number density which is then of importance is only that of the electrons. We like to think of the matter as made up of hydrogen, that is bound electron-proton pairs. Now

$$T_{R;b}^{ab} = \left(\int p^a p^b f dP_m^+ \right)_{;b} = \int p^a C[f] dP_m^+, \quad (3.271)$$

$$T_{M;b}^{ab} = \left(\mu_M u^a u^b + n_m T_m h^{ab} \right)_{;b}. \quad (3.272)$$

We once again use the photon collisional of the form (3.214) and (3.215), to write down the full collisional flux where, as before, $C[f] = \sum b_{A_i} e^{A_i}$, *i.e.*

$$\begin{aligned} \int p^b C[f] dP_m^+ &= \frac{1}{t_c} \int_m^\infty (u^b + e^b) E^3 dE ((\bar{F} - F) + F_a e^a + F_{ac} e^a e^c + \dots) d\Omega \\ &= \sigma_T n_e \left\{ \int E^3 dE \left[(\bar{F} - F) u^b + F_a e^a e^b + F_{acd} e^a e^c e^d e^b + \dots \right] d\Omega \right\} \end{aligned}$$

$$\begin{aligned}
&= \sigma_T n_e \left\{ 4\pi \left(\int E^3 dE (\bar{F} - F) \right) u^b + \frac{4\pi}{3} \left(\int E^3 dE F_a \right) h^{ab} \right\} \\
&= \sigma_T n_e q_a h^{ab} \\
&\approx \sigma_T n_e \left(\frac{4}{3} T^4 \right) \tau_a h^{ab}. \tag{3.273}
\end{aligned}$$

Using (3.272), (3.273) and (3.270) one finds, using $T_M \equiv T\gamma = T$ and $n_e \equiv n_m$, that

$$\sigma_T n_e \left(\frac{4}{3} T^4 \right) \tau_b h^{ba} \approx - \left(\mu_M u^a u^b + n_e T h^{ab} \right)_{;b}. \tag{3.274}$$

Here as discussed before we use $\mu_M = \frac{3}{2} T n_e + n_e m$ and $\dot{n}_e = -n_e \theta$ (there is no particle production i.e. particle number density conservation). We find that

$$\sigma_T n_e \left(\frac{4}{3} T^4 \right) \tau^a \approx -n_e \left[\left(\frac{3}{2} T + m \right) \dot{u}^a + \frac{3}{2} \dot{T} u^a + \frac{1}{n_e} \hat{\nabla}^a (n_e T) \right]. \tag{3.275}$$

The $u^a \theta$ terms cancel out in the RHS of (3.274). This gives us a possible form of τ_a ,

$$\tau^a \approx \frac{-1}{\sigma_T T^4} \left(\frac{3}{4} \right) \left[\left(\frac{3}{2} T + m \right) \dot{u}^a + \frac{3}{2} \dot{T} u^a + \left(\frac{\hat{\nabla}^a n_e}{n_e} \right) T + \hat{\nabla}^a T \right]. \tag{3.276}$$

The divergence of this can be taken

$$\begin{aligned}
\hat{\nabla}_a \tau^a &\approx \left(\frac{3}{T^4} \right) \left[\left(\frac{3}{2} T + m \right) \dot{u}^a + \frac{3}{2} \dot{T} u^a + \left(\frac{\hat{\nabla}^a n_e}{n_e} \right) T + \hat{\nabla}^a T \right] \\
&\quad - \left(\frac{-3}{4T^4} \right) \left[\left(\frac{3}{2} \hat{\nabla}^a T \right) \dot{u}^a + \left(\frac{3}{2} T + m \right) \hat{\nabla}^a \dot{u}_a + \frac{3}{2} (\hat{\nabla}_a \dot{T}) u^a + \frac{3}{2} \dot{T} \hat{\nabla}^a u_a \right. \\
&\quad \left. + \left(\frac{\nabla^2 n_e}{n_e} \right) T - \left(\frac{\hat{\nabla}^a n_e}{n_e} \right)^2 T + \left(\frac{\hat{\nabla}_a n_e \hat{\nabla}^a T}{n_e} \right) + \hat{\nabla}^2 T \right]. \tag{3.277}
\end{aligned}$$

Linearizing (3.277), i.e. using $\hat{\nabla}^a T = O[1]$, $\hat{\nabla}^a n_e = O[1]$, $\hat{\nabla}^a u_a = \theta$ and $\dot{u}^a = O[1]$, products of $O[1]$ variables are at least $O[2]$. From these relations, and using $u^a \hat{\nabla}_a = u^a h_{ab} \nabla^b = 0$ it is found that

$$\hat{\nabla}_a \tau^a \simeq \left(\frac{3}{4\sigma_T T^4} \right) \left[\left(\frac{3}{2} T + m \right) \hat{\nabla}^a \dot{u}_a - \frac{3}{2} \dot{T} \theta - \left(\frac{\hat{\nabla}^2 n_e}{n_e} \right) T - \hat{\nabla}^2 T \right]. \tag{3.278}$$

With $(S \hat{\nabla}_a u^a) \simeq S \hat{\nabla}_a \dot{u}^a$ [33] and using $\theta = 3\dot{S}/S$ we could replace the spatial divergence of the acceleration by using $\hat{\nabla}_a \dot{u}^a \simeq 3\dot{S}/S$. But here we choose $\dot{u}^a = 0$ such that (3.278) becomes

$$\hat{\nabla}_a \tau^a \simeq \left(\frac{-3}{4\sigma_T T^4} \right) \left[\frac{3}{2} \dot{T} \theta + \left(\frac{\hat{\nabla}^2 n_e}{n_e} \right) T + \hat{\nabla}^2 T \right]. \tag{3.279}$$

Substituting this into the zeroth order ($l = 0$) Boltzmann moment equation (3.168) for $\hat{\nabla}_a \tau^a$ we are able to decouple the $l = 0$ equation from the rest of the hierarchy of Boltzmann moment equations to find a first order equation in the average bolometric temperature

$$\left(\frac{\dot{S}}{S} + \frac{\dot{T}}{T}\right) \simeq \left(\frac{1}{4\sigma T}\right) \left(\frac{1}{T^4}\right) \left[\frac{9}{2}\dot{T}\frac{\dot{S}}{S} + \left(\frac{\hat{\nabla}^2 n_e}{n_e}\right) T + \hat{\nabla}^2 T\right] - \frac{\sigma T n_e}{4}. \quad (3.280)$$

What we have achieved is to find a covariant equation in the average bolometric temperature for the era of tight coupling where the matter fluid with which the photons are coupled also has a pressure. Hence the matter and photons can be in thermal equilibrium *i.e.* have the same temperatures, as is expected for a photon gas that has a Plankian spectrum. The interesting thing about this treatment is that we have not used *any* restrictions on the fluid and gas components other than relaxing the matter's pressure free nature *i.e.* we have not placed any restrictions on the temperature coefficients τ_{A_l} , yet have been able to decouple the average temperature behaviour in the linearized regime from all higher order moments. In the above treatment no explicit restrictions have been placed on the form of the perturbations *eg* by considering only scalar perturbations, and we have not truncated τ_{A_l} .

The key issue that this treatment is used to highlight is that of the validity of using a pressure-free fluid in tight coupling [50]. We have also, however, been able to demonstrate the existence of an alternative approach through which to find a functional form of the divergence of the anisotropic pressure, $\Pi_a = \hat{\nabla}_b \pi_a^b$. This is achieved by substituting for $\dot{\tau}_a$ and τ_a in the $l = 1$ Boltzmann equation (3.211)³⁰ or substituting for $\hat{\nabla}^a \tau_a$ directly into the $l = 0$ Boltzmann equation (3.210). This approach clearly assumes some sort of *tight-coupling* limit to ensure exact equilibrium and in this sense is unphysical but may in fact be acceptable as an approach to decoupling the $l = 0$ and $l = 1$ Boltzmann equations, as it does not rely on metric and gauge restrictions and hence we can be certain that it retains the full GIC form of the theory.

- (II) *Free-Streaming era*: At later times the gas particles only interact via gravity through the geometry. Traditionally one could write down the energy momentum conservation equations for matter and radiation, where we have a pressure free dust fluid interacting via gravity with a radiation gas. Not only is

³⁰ $\Pi_a = \frac{5}{2} \left[C_a - \dot{\tau}_a - 4 \left(\frac{\dot{T}}{T} + \frac{\dot{S}}{S} \right) \tau_a - \frac{\hat{\nabla}_a T}{T} \right].$

energy-momentum conserved by the system as a whole, but the individual components are not in thermal equilibrium and do not interact through mechanisms other than gravity, *i.e.* the geometry. We then have that

$$T_{;b}^{ab} = T_{R;b}^{ab} + T_{M;b}^{ab} = 0, \quad (3.281)$$

$$T_{R;b}^{ab} = \left(\int p^a p^b f dP_m^+ \right)_{;b} = 0, \quad (3.282)$$

$$T_{M;b}^{ab} = (\mu_M u^a u^a)_{;b} = 0 \quad (3.283)$$

hold for $B, B_a = 0$. We do not have the same restriction as for the collisionally dominated case, but we could use the form of Π_a as obtained for the tight-coupling limit as the initial conditions on the last scattering surface *i.e.* use the free-streaming $l = 0$ and $l = 1$ Boltzmann equations, but instead of using the form of τ_a as obtained in the tight coupling, one can use the form of Π_a as given in terms of (3.276) as an initial condition at the decoupling time.

- (c) **Perturbation restrictions** : It may be that the best way in which to decouple the $l = 0$ and $l = 1$ Boltzmann equations would be through the physical motivation of limitations on the forms of the perturbations *eg* limiting the investigation to either adiabatic³¹ or isocurvature perturbations. But this is problematic, as to motivate these choices, as has been done for adiabatic perturbations in [36], we need to be able to consistently carry out an iterative perturbative analysis near to the tight-coupled limit at decoupling in a fully GIC framework. How to approach this is not clear to me. The work in this area is generally carried out in the Bardeen sense following [36], hence a metric choice is made rather than placing restrictions either at $O[1]$ or $O[2]$ directly on the GIC variables. To follow [36] it seems that a metric choice needs to be made but this is unacceptable and not in line with with the GIC approach. If such a method is adopted it would probably make it very difficult to generalize the theory to $O[2]$ *i.e.* second order effects. This approach is not investigated in this thesis.

³¹These are particularly interesting as this allows one to define the surface of last scattering as a surface of constant energy density (at decoupling) and hence a constant temperature surface which can in turn be related to a surface of constant electron density; for non-adiabatic perturbations this would probably not be possible. Adiabatic perturbations in this form suggest that there are no temperature perturbations at decoupling, but that the on-sky fluctuations are generated as the light propagates down the null cone towards us. This may be problematic.

Applying the Harmonic expansion

Applying the Harmonic Expansion, assuming we know the form of $G_1(x^a)$ and $G_2(x^a)$, we use the following identities:

$$\ddot{T} = \sum_k \ddot{T}^{(k)} Q^{(k)}, \quad (3.284)$$

$$\dot{T} = \sum_k \dot{T}^{(k)} Q^{(k)}, \quad (3.285)$$

$$T = \sum_k T^{(k)} Q^{(k)}, \quad (3.286)$$

$$G_2 = \sum_k G_2^{(k)} Q^{(k)}, \quad (3.287)$$

$$\hat{\nabla}^2 T = - \sum_k \frac{k^2}{S^2} T^{(k)} Q^{(k)}. \quad (3.288)$$

Substituting these into (3.252) we find, after the sums over k and the harmonics Q have been factored out, that we have

$$\begin{aligned} \ddot{T}^{(k)} + \dot{T}^{(k)} \left(3G_1(x^a) + 9\frac{\dot{S}}{S} - 3C \right) \\ + T^{(k)} \left(4 \left(\frac{\dot{S}}{S} \right)^2 + \frac{\ddot{S}}{S} - 2C\frac{\dot{S}}{S} + \frac{1}{3} \frac{k^2}{S^2} \right) \simeq G_2^{(k)}. \end{aligned} \quad (3.289)$$

Using the substitution

$$x = ye^{-\frac{1}{2} \int A d\tau} \quad (3.290)$$

in equations of the form

$$\ddot{x} + A\dot{x} + Bx = f \quad (3.291)$$

we are able to rewrite (3.289) in the reduced form of

$$\ddot{y} + F(A, B)y = f. \quad (3.292)$$

Hence we are able to reduce the behaviour of the average bolometric temperature to that of a harmonic oscillator³². (3.252) is GIC, it should be considered that (i) the use of the Harmonic expansion and (ii) the transformation (3.290) may destroy the GIC nature of (3.289). However according to Harrison [61] and Hawking [63] the $T^{(k)}$ are claimed to

³²From the Phd thesis of Wayne Hu (pg 92 [54]), his intermediate scale acoustic solutions are constructed

be GIC, and the transformation involves an integration which never involves any gauge restrictions or frame restrictions, hence equation (3.289) is expected to be GIC. This should be carefully checked following [28]. The key point made here is that the linearized behaviour of the averaged bolometric temperature in an Almost FLRW universe in the general GIC framework can be fully described by a driven harmonic oscillator-like equation (3.289). This is considered a mirror treatment of the GI Bardeen analysis undertaken by [54] [50] [52] [53] but is covariant, and can be easily generalized to $O[2]$ effects as has been discussed before.

The point of this analysis was that as the perturbations enter the horizon, the system is no longer a single fluid. Above the photon diffusion scale, the photons and baryons are still tightly coupled by Compton scattering but the diffusion scale is less than the horizon scale. The harmonic oscillator equation above shows that the photon pressure resists the gravitational compression of the photon-matter fluid, leading to the second order equation [54].

Moment Conditions on F_{A_l}

It is crucial to notice that any explicit truncation technique in the Boltzmann moment equations is problematic, as one risks setting the shear to zero (for example). As a fix in the tight-coupling limit in the Bardeen GI formulation using the conditions,

$$\dot{\Delta}\gamma = \frac{4}{3}\dot{\Delta}b, \quad (3.293)$$

$$\theta_l \equiv V_\gamma = V_b, \quad (3.294)$$

$$\theta_l = 0 \quad l \geq 2. \quad (3.295)$$

i.e. the radiation is isotropic in the baryon rest frame and density fluctuations in the photons grow adiabatically with the baryons. The $l = 0$ and $l = 1$ Boltzmann moment equations are then found to have the form

$$\dot{\theta}_0 = -\frac{k}{2}\theta_1 - \dot{\Phi}, \quad (3.296)$$

$$\dot{\theta}_1 = -\frac{\dot{R}}{1+R}\theta_1 + \frac{1}{1+R}k\theta_0 + k\Psi. \quad (3.297)$$

Here $\dot{R} = (\dot{a}/a)R$, k is the wavenumber and Ψ and Φ are equivalent to my Φ_A and Φ_H respectively. Hu reduces the $l = 0$ and $l = 1$ Boltzmann equations to

$$\ddot{\theta}_0 + \frac{\dot{R}}{1+R}\dot{\theta}_0 + \frac{1}{3}\frac{k}{1+R}\theta_0 = F. \quad (3.298)$$

It is useful to note that Hu uses $\Psi \simeq -\Phi$ on the basis of matter domination (when pressure can be ignored pg 18,84). This is equivalent to the metric restriction that I use *i.e.* $\Phi_A = -\Phi_H$ in the GIC formulation. In [54] all the higher temperature moments are set to zero in the tight coupling limit, hence there is no shear [24]. I have attempted to decouple the $l = 1$ equation from the $l = 2$ equation without being so restrictive. Clearly much work must still be done in the GIC formulation of [54].

nal word on the Boltzmann equations, following [24], I include a list of the truncation conditions that should worry anybody who wishes to simplify gas calculations by naively truncating the harmonic expansion of f , and hence the harmonic expansion of the temperature perturbations τ . I hope that the reader now realizes that my attempt at trying to find a physical argument for setting τ_{ab} to zero, and the future attempts at restricting the behaviour of ξ_{abc} in order to allow the consistent application of the multi-fluid GIC perturbation equations, may seem unphysical and esoteric but in fact are crucial if any useful calculations are to be obtained from the GIC gas theory. What I have attempted to do here is pave the way towards the useful application of the GIC RK as a calculational tool and as a demonstration of the problems involved when using the GIC RK theory as part of a perturbative analysis of temperature anisotropies. I feel that I have succeeded in providing two useful methods of consistently decoupling the Boltzmann equations just before the electron recombination occurs at the $l = 2$ level in the Boltzmann moment expansion. Furthermore, without some sort of physically or geometrically motivated decoupling somewhere in the Boltzmann hierarchy it would be impossible to do anything useful in the sense of calculations and hence physics in the linear or second order GIC perturbation theory. My attempts at decoupling the $l = 0$ and $l = 1$ Boltzmann equations should be viewed as an opening move in this regard. It should also be impressed on the reader that at decoupling, to use $\Pi_a = 0$ directly rather than using $\pi_{ab} = 0$ to set $\Pi_a = 0$ would be far less restrictive and thus more acceptable. I was, however, unable to find a good motivation to directly set $\Pi_a = 0$.

"The crucial feature of (23)[my equation (3.299)] is that the different harmonic components are independent solutions of the Liouville equations - they do not interfere with each other".

The homogeneous Liouville solutions are given in terms of

$$f = F(m, E) + F_a(m, E)e^a + F_{ab}(m, E)e^a e^b + \dots \quad (3.299)$$

The key point here is that it is obviously true for this case, that the different harmonic solutions are independent, which means that for instance a truncation in the harmonic expansion would have no effect on the harmonic components through the Boltzmann equations. For the general case of inhomogeneous solutions, *i.e.* for

$$f = F(x^i, m, E) + F_a(x^i, m, E)e^a + F_{ab}(x^i, m, E)e^a e^b + \dots, \quad (3.300)$$

it is not that simple, as pointed out in [24] and [25].

“In looking for exact solutions of the Einstein-Boltzmann equations, an obvious procedure is to place restrictions on the harmonic components F_{A_l} of f by setting particular harmonics zero; for example, one may truncate the expansion (3)[my equation (3.300)] after some finite value L of l so that f has a finite number of harmonic terms”

The conditions that such truncations place on the harmonic moments are given by [25]. The key ones are listed below.

- **A** If the distribution function f satisfies the Liouville equation, or the Boltzmann equation, with a generalized BGK collisional, and if there is a velocity field \mathbf{u} such that relative to \mathbf{u} :
 - (i) f has a finite number of harmonic co-efficients
 - (ii) f has just four consecutive harmonic components zero
 - (iii) f has its first, second and third harmonic components zero

then the shear of \mathbf{u} vanishes.

- **B** If f satisfies the Einstein-Liouville equations and there is a velocity field \mathbf{u} such that either
 - (1) f has vanishing first, second, and third harmonic co-efficients
 - (2) the first and second harmonic co-efficients are zero and there are a finite number of moments,

then the space-time is either stationary or Robertson-Walker.

- **C** If f satisfies the Liouville, Krook and generalized Krook equations in a space-time admitting a hypersurface-orthogonal velocity field \mathbf{u} such that
 - (1) $F_{ab} = 0$ everywhere and in addition
 - (2) $F_a = F_{abc} = 0$ on an initial surface S orthogonal to \mathbf{u} ,

the shear of \mathbf{u} vanishes on the initial surface S .

Hence it should be clear that by setting $\pi_{ab} = 0$, as I have done, to decouple the $l = 0$ and $l = 1$ Boltzmann equations : (1) if $F_a = 0$ then we will have that $q_a = 0$ from (3.43) which means that we are considering a perfect fluid. $\pi_{ab} = 0$ implies from (3.49) that

$$\int E^3 F_{ab} dE = 0. \quad (3.301)$$

If this follows from $F_{ab} = 0$ then (2) If $F_a = 0 = F_{abc}$ on an initial surface S orthogonal to u , then the shear of u vanishes on S too. Without (1) or (2) there will be no apparent conditions on the other harmonic moments through the Boltzmann moment equations by making this choice. This is an important point, if the choice of setting either $\pi_{ab} = 0$ or $\xi_{abc} = 0$ is made to simplify calculations one must be extremely careful that this decision is both well motivated physically and does not place unphysical constraints through the Boltzmann equations on the other harmonic moments. If for instance the multi-fluid³³ perturbation equations in [31] are to be used, or at least claimed to be physically representative of the post decoupling development of perturbations, it is important that $\hat{\nabla}^a \xi_{abc} = 0$ so that the $l = 0$, $l = 1$ and $l = 2$ Boltzmann equations can decouple from the rest of the moment equations. This leaves one with the multi-fluid evolution equations of [31]. If however, we set $\xi_{abc} = 0$ and hence set $\tau_{abc} = 0$ one must be very certain that this does not eliminate variables of interest through the Boltzmann equations. For instance on the initial surface S if $\tau_a = 0$ or $\tau_{ab} = 0$ the shear on S would be eliminated *i.e.* $\sigma_{ab} = 0$. Non-vanishing shear corresponds to the case of non-trivial perturbations³⁴.

3.5.7 Almost FLRW temperature energy-momentum tensor

Using the small anisotropy limit one may immediately rewrite the energy momentum tensor in terms of the temperature anisotropies up to second order in l . All orders of l would contribute for the general case, *i.e.* the case for non-small temperature anisotropies. The linearized perturbed gauge invariant covariant energy momentum tensor for the photon gas and matter fluid would be given by

$$T_{ab} \approx (\mu_M + \mu)u_a u_b + \frac{1}{3}\mu h_{ab} + \frac{8\mu}{15}\tau_{ab} + \frac{8\mu}{3}u_{(a}\tau_{b)}, \quad (3.303)$$

where

$$\mu = \int_0^\infty E^3 F(x^i, E) dE = \int \frac{E^3 dE}{(\exp(E/kT) - 1)} = aT^4. \quad (3.304)$$

Hence the energy momentum tensor becomes

$$T_{ab} = (\rho + T^4)u_a u_b + \frac{1}{3}T^4 h_{ab} + \frac{8T^4}{15}\tau_{ab} + \frac{8T^4}{3}u_{(a}\tau_{b)}. \quad (3.305)$$

³³ *C'est magnifique, mais non, ce n'est pas la physique.*

³⁴ It may be useful to be reminded about the evolution of σ_{ab} ,

$$\sigma_{ab} = -\frac{2}{3}\theta\sigma_{ab} - E_{ab}, \quad (3.302)$$

and that $H^{ab} = \text{curl}\sigma_{ab}$, $E_{;b}^{ab} = \frac{1}{2}\kappa X^a = 0 \Rightarrow X^a \equiv \hat{\nabla}^a \mu = 0$ and $H_{;b}^{ab} = (\mu + p)\omega^a = 0 \Rightarrow \omega^a \equiv \frac{1}{2}\eta^{abcd}u_b\omega_{cd} = 0$ for $\mu \neq p$. (Communication with G. F. R. Ellis).

Coupling (3.305) to the geometry (Almost FLRW) via the field equations, it is found that

$$R_{ab} = \frac{1}{2}(2T^4 + \mu_M)u_a u_b + \frac{8}{3}T^4(u_a \tau_b + u_b \tau_a) + \frac{8T^4}{15}\tau_{ab} + \frac{1}{3}T^4 h_{ab} + \frac{1}{2}\rho h_{ab}. \quad (3.306)$$

For a photon gas coupled to a dust fluid, assuming small gauge invariant covariant temperature anisotropies, one has that

$$R_{ab}u^a u^b = \frac{1}{2}\mu_M + T^4, \quad (3.307)$$

$$R_{ab}u^a h_c^b = -\frac{4}{3}T^4 \tau_c, \quad (3.308)$$

$$R_{ab}h_c^a h_d^b = \frac{1}{2}\left(\frac{4}{3}T^4 + \mu_M\right)h_{cd} + \frac{8}{15}T^4 \tau_{cd}. \quad (3.309)$$

The Boltzmann equations have not been used (other than the fact that the integrated Boltzmann equations over energy lead to the conservation equations). It must be noted that these equations are in space time. Using quantities derived and defined in Liouville space, we have used the Stefan-Boltzmann law and the field equation in natural units.

It should be emphasized that these would be appropriate for large scale structure, in the sense that small scale effects such as the Doppler peaks [50] sometimes called Sakharov oscillations, would seem to be more easily accessible using Newtonian perturbation theory, for very large l . It is not clear at this stage on how to generalize the covariant gauge invariant Boltzmann treatment to extremely high l values in a useful manner. Another point is that the Kinetic Theory treatment, that is the propagation via Boltzmann equation in phase space, is valid on any simply-connected manifold. What this means is that changes in topology could be problematic if they occurred in the early universe as the Boltzmann treatment would then become invalid. The general assumption of free-streaming from decoupling though workable may not in fact be correct if re-ionization (as indicated by the Gunn-Peterson test) occurred at some time after decoupling, as the free streaming approximation would then only be valid in the pre re-ionization era.

3.5.8 Finale

Recombination occurs at the time in the early universe when the electron temperature drops such that $e + p \rightarrow H + \gamma$. If Raleigh scattering is small, there is a reduction in $e + \gamma \rightarrow e + \gamma$ scattering processes. The universe has decoupled; due to expansion the matter content (electrons, protons) have recombined and decoupled from the radiation (photon gas).

This era of decoupling produces an effective photosphere, the effective photosphere of the early universe; after sufficient recombination has occurred and the matter and radiation are considered to have decoupled the matter content then free-streams. Beyond this photosphere the universe is not transparent to photons, this is the surface of last scattering. If there is a reasonably sharp transition from opacity to transparency, there will be a surface of emission that is independent of the observer's position [60].

Ionization due to stellar and galactic ignition and possible ionization of the intergalactic medium pushes the last scattering "surface" to smaller redshifts.

The radiation horizon³⁵ in radians on the sky from last scattering, (l_s), is $\theta_{l_s} \approx \Omega^{\frac{1}{2}} z_{l_s}^{-\frac{1}{2}}$, where z_{l_s} is the last scattering redshift. Thus $2^\circ \geq \theta_{l_s} \geq 10^\circ$ are the angles subtended on the sky that are boundaries of horizons on the last scattering surface. Considering this it seems that the CBR must be arising from causally disconnected regions. Structure formation is probably due to fluctuations on scales less than 10° . The status of these fluctuations is somewhat confused. From Smoot et al (1992) [37]

$$COBE : \quad \left. \frac{\delta T}{T} \right|_{rms:scale > 10^\circ} \approx 1.10(\pm 0.18) \times 10^{-5}. \quad (3.310)$$

Structure (in the dust/matter content \equiv Galaxies) formed due to gravitational instabilities in an initial spectrum of primordial density perturbations, as imprinted on the last scattering surface *i.e.* on-sky temperature fluctuations measured by the COBE satellite.

The important limit is the black body nature of the CBR; the spectral distortions are surprisingly small, and hence contribute little after last scattering. The key point is that the temperature fluctuations in the radiation at decoupling are somehow reflected in the matter density fluctuations as the universe evolves, then as matter density fluctuations come inside the horizons (unless pressure effects intervene), gravitational instabilities occur.

An object will oscillate if it can be transversed by sound waves in the collapse timescale ($\tau \approx (G\mu)^{-\frac{1}{2}} < r/v_s$), where v_s is the sound speed. The Jeans length and mass, λ_J and M_J , are the scales at which the pressure effects are subdominant *i.e.* structure becomes unstable and gravitationally collapse occurs. This is the basic picture of the early universe with regard to the formation of gravitationally bound objects that have collapsed out due to matter fluctuation instabilities.

The problems with this picture are:

- **Horizons** : The CMBR is the same in causally disconnected regions, same within 10^{-4} . The Standard Model (SM) - FLRW universe accounts for this by the unnatural

³⁵ $d_H = S(t) \int \frac{dt'}{S(t')}$ gives the horizon size.

assumption that the early universe was highly homogeneous and isotropic on scales much greater than the causal horizons.

- **Density Fluctuations** : It is generally believed that galaxies and clusters of galaxies evolved by gravitational instabilities, from small density fluctuations in the early universe. At this time these fluctuations are postulated, *but* where in fact do these fluctuations come from ?
- **FLRW** models imply a $t = 0$ state of infinite energy density which signals the break down of GR.
- **Flatness** : Is the present universe critical $\Omega \approx 1$? deviations in Ω grow in time (for a universe which is under dense, $\Omega < 1$ [19]). But the universe is still locally flat ($k = 0$) on small scales (although not too small !).

In the light of the model outlined above, I have attempted to understand two things: the evolution of Ω in FLRW universes using and introducing a generalized phase plane approach, and more importantly and relevant from the side of real physics, to understand the temperature fluctuations at last scattering, in a fully GIC framework, that may seed matter fluctuations. Basically I have adopted a matter-radiation model as a descriptive basis in a formulation that can be easily extended to include a neutrino gas (say) at some later time. The radiation has been described in a fully GIC gas theory and the matter in terms of a GIC fluid description in the sense of an Almost FLRW universe model. The perturbations have been in the radiation variables. The key problems with the issue of the radiation perturbations was how they related to the on-sky measured temperatures. The innovation of my approach is that I have attempted to try to find a consistent method of validating the GIC fluid perturbation equations using a physical, metric or gauge truncation in the infinite hierarchy of Boltzmann equations.

Comments of Temperature

This is an important point as the on-sky temperature anisotropies have first to be related to the temperature anisotropies at decoupling, then related to the temperature differences in the current spatial slice, in order to link the on sky temperature anisotropies with the spatial temperature perturbations as characterized by Δ in the covariant gauge invariant linear theory. The two different variables are clearly related but one should not expect this to be a trivial relation (as is generally assumed). That is, the relationship between

the on-sky temperature perturbations τ and the spatial temperature perturbation at last-scattering Δ are given by the Boltzmann equations.³⁶

What we have learned here is that we have a Covariant Gauge Invariant formulation of the brightness temperature Boltzmann equations for Almost FLRW universes, for which (1) an arbitrary gauge choice cannot be taken as it may compromise covariance (2) The rest of the moment equations form a set of constraints that should be checked for consistency and (3) one must be very clear and careful about which and what temperature perturbations one is dealing with and how these have been theoretically defined when making comparisons between the Bardeen GI and GIC formalisms (4) we cannot easily decouple the Boltzmann equations from the complete moment hierarchy, which places big questions on the validity of the multi-fluid perturbation equations during both free-streaming and the decoupling eras and (5) we cannot set any three temperature anisotropies to zero without restricting the shear. There are no non-trivial perturbations for which the shear is zero, if we are to use the multi-fluid perturbation equations, it should be made clear how these can be physically motivated without truncating the Boltzmann moment equations.

Quidquid recipitur, ad modum recipientis recipitur

³⁶the keypoint here is that τ 's are not proportional to $\delta\mu \sim \Delta$, hence to expand $\frac{\delta T}{T}$ in terms of spherical harmonics and then relate the same $\frac{\delta T}{T}$ to a plane wave expansion in the spatial slices assuming De-Sitter geometry is very misleading, and is an indication of the confusion surrounding the understanding of the temperature perturbations.

Appendix A

Appendix

A.1 Basic Identities in GR

The curvature is characterized by the Ricci and Bianchi identities.

- The *Ricci Identity*:

$$u^a{}_{;dc} - u^a{}_{;cd} = R^a{}_{bcd}u^b, \quad (\text{A.1})$$

where

$$\begin{aligned} - R_{abcd} &= R_{[ab][cd]} = R_{cdab}, \\ - R_{a[bcd]} &= 0, \\ - R^a{}_{cd} &= -C^a{}_{cd} - 2g^{[a}{}_{[c}R^b]_{d]} + \frac{1}{3}Rg^{[a}{}_{[c}g^b]_{d]}. \end{aligned}$$

The Riemann curvature tensor can be algebraically separated into the trace-free component, the *Weyl tensor* C_{abcd} (10 independent components, indicative of conformal curvature), and the *Ricci tensor* R_{ab} (10 independent components).

$$C^a{}_{bad} = 0 \quad (\text{A.2})$$

Using the analogy of Maxwell equations, one can decompose this into the *Electric* and *Magnetic* part (E_{ab} and H_{ab} respectively):

$$E_{ab} = C_{agbh}u^g u^h, \quad (\text{A.3})$$

$$H_{ab} = \frac{1}{2}\eta_{ae}{}^{gh}C_{ghbd}u^e u^d. \quad (\text{A.4})$$

The Weyl tensor can be written out in terms of the metric, the electric and magnetic parts:

$$C_{abcd} = (\eta_{abpq}\eta_{cdrs} + g_{abpq}g_{cdrs})u^p u^r E^{qs} + (\eta_{abpq}g_{cdrs} + g_{abpq}\eta_{cdrs})u^p u^r H^{qs}, \quad (\text{A.5})$$

here $g_{abcd} = g_{ac}g_{bd} - g_{ab}g_{cd}$ ([4]).

- The *Bianchi Identities* :

$$R_{ab[cd;e]} = 0 \rightarrow R_{abcd;e} + R_{abec;d} + R_{abde;c} = 0, \quad (\text{A.6})$$

where we have that

$$R_{bcd;a}^a = R_{bd;c} - R_{bc;d}, \quad (\text{A.7})$$

$$R_{c;a}^a = \frac{1}{2}R_{;c}. \quad (\text{A.8})$$

Hence we have constraints on the geometry, from the Ricci and Bianchi identities. The energy-momentum should be conserved, along with making sure that the entropy is either static or increasing (H-theorem). These constraints and consistency arguments, combined with the assumed relation between the energy density and the geometry, determines the evolution:

$$T^{ab}{}_{;b} = 0, \quad (\text{A.9})$$

$$S_{;a}^a \geq 0, \quad (\text{A.10})$$

$$T^{ab} = G^{ab}. \quad (\text{A.11})$$

The Basic equations, and how to derive them:

- Conservation equations

- (I) $T^{ab}{}_{;b}u_a \rightarrow$ energy conservation equation.
- (II) $T^{ab}{}_{;b}h_{ac} \rightarrow$ momentum flux conservation equation.

- Field Equations (1) (A.11) (Assumption !)

- Constraint equations (Ricci \rightarrow 3 sets of constraints)

- (I) Shear (contract on **a,c** in the Ricci identity to get an equation in terms of the Ricci tensor and its projection along the comoving observers worldline, $R_{ab}u^b$; use the definition of covariant derivative).
- (II) Use $R_{a[bcd]} = 0 \rightarrow u_{[b;cd]} = 0 \times \eta^{acde}u_e$.
- (III) Multiply the Ricci identities by $\eta^{cdef}u_e$, symmetrize on **a** and **d** to find in the magnetic part of Weyl tensor.

- Propagation equations (again from Ricci identities).
 - (I) Raychaudhuri : Project the Ricci tensor and energy momentum tensor via the field equation to get Raychaudhuri (onto worldline) and Friedmann (into spatial section) equations.
 - (II) Shear propagation : Project \mathbf{a} and \mathbf{c} of Ricci identity and take the *symmetric trace-free* part of this.
 - (III) Vorticity propagation : Project \mathbf{a} and \mathbf{c} of Ricci identity and take *anti-symmetric* part.

A.2 FLRW space-time

Spatial homogeneity implies continuous symmetries which implies Killing vector fields, as does isotropy. We look for a space-time that is isotropic (all directions are equivalent) and homogeneous (isotropic about all points, all points are equivalent).

Cosmological Principle

- The hyper-surfaces with constant cosmic time are maximally symmetric subspaces of the whole space-time.
- Not only the metric $g_{\mu\nu}$ but all cosmic tensors such as $T_{\mu\nu}$ are invariant with respect to the isometries of these subspaces.

This can be used to motivate the FLRW metric (Weinberg 395 - 404)

$$ds^2 = dt^2 - S^2(t) \left(\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right) \quad (\text{A.12})$$

We find The Ricci tensor using the usual definition for the Christoffel coefficients, the the Ricci identity and contracting the resulting Riemann curvature tensor. That is, from the covariant derivatives $u^a_{;b} = u^a_{,b} + \Gamma^a_{bc} u^c$ enforcing the choice $g_{ij;k} = 0$ and taking permutations, one gets the usual:

$$\Gamma^l_{mn} = \frac{1}{2} g^{li} (g_{im,n} + g_{ni,m} - g_{nm,i}). \quad (\text{A.13})$$

For the FLRW metric

$$\Gamma^i_{jk} = \frac{1}{2} h^{il} \left(\frac{\partial h_{lj}}{\partial x^k} + \frac{\partial h_{lk}}{\partial x^j} - \frac{\partial h_{jk}}{\partial x^l} \right), \quad (\text{A.14})$$

$$\Gamma^0_{ij} = \frac{\dot{S}}{S} h_{ij}, \quad (\text{A.15})$$

$$\Gamma^i_{0j} = \frac{\dot{S}}{S} \delta^i_j. \quad (\text{A.16})$$

Using the Ricci identity $\alpha_{i;jk} - \alpha_{i;kj} = R^m_{ijk} \alpha_m$ the usual invariant measure of curvature R^m_{ijk} is given as

$$R^m_{ijk} = -\Gamma^m_{ij;k} + \Gamma^m_{ik;j} + \Gamma^l_{ik} \Gamma^m_{lj} - \Gamma^l_{ij} \Gamma^m_{lk}. \quad (\text{A.17})$$

We then find the usual $\left(\frac{2}{0}\right)$ tensor $R_{ab} = R^c_{abc}$ for the FLRW space-time:

$$R_{ij} : R_{00} = -3 \frac{\ddot{S}}{S}, \quad (\text{A.18})$$

$$R_{\nu\mu} = -\left(\frac{\ddot{S}}{S} + 2\frac{\dot{S}^2}{S^2} + \frac{2k}{S^2}\right)g_{\nu\mu}, \quad (\text{A.19})$$

$$R = -6\left(\frac{\ddot{S}}{S} + \frac{\dot{S}^2}{S^2} + \frac{k}{S^2}\right). \quad (\text{A.20})$$

A.3 Standard Candles and Measuring rods

Consider an object of *proper size* D and *intrinsic luminosity* L at *coordinate distance* r^1 subtending and *observer angle* of θ . The proper size is given as

$$D = Sr\theta, \quad (\text{A.21})$$

where

$$S(t) = S_0/(1+z). \quad (\text{A.22})$$

With redshift z defined by the relationship $z = (\lambda_o - \lambda_e)/\lambda_e$. The proper size is given by

$$D = \frac{S_0 r \theta}{(1+z)}. \quad (\text{A.23})$$

An area A at the observer subtends a solid angle $\Omega(\phi, \theta)$ at some source:

$$A = (S_0 r)^2 \Omega(\phi, \theta). \quad (\text{A.24})$$

The observed energy flux F (energy per unit time per area) from the source, at the observer is

$$F = \frac{L \Omega(\phi, \theta) S^2}{A 4\pi S_0^2}, \quad (\text{A.25})$$

so

$$F = \frac{L}{4\pi(S_0(1+z)r)^2}, \quad (\text{A.26})$$

where L is the *absolute luminosity* (energy per unit time produced by the source in its rest frame).

Traditionally what people have done is to take a Taylor expansion of $S(t)$ *i.e.* $S(t) = S_0(1 + H_0(t - t_0) + \frac{1}{2}q_0 H_0^2(t - t_0)^2 + \dots)$, for the usual current values of H , S and q . This relation is valid locally.

It should be noted that from (A.22) that $\dot{S} = -S_0 \dot{z}/(1+z)^2$, this along with the Hubble parameter $\dot{S} = HS$ gives us

$$\dot{z} = -H(1+z). \quad (\text{A.27})$$

We are now able to get the phase planes in redshift space, these can be used with the usual evolution equation for Ω , S and H to get the behavior of the standard candles and standard measuring rods. From the FLRW metric the observer has that

$$\dot{r} = \left(\frac{\sqrt{(1 - kr^2)}}{S} \right). \quad (\text{A.28})$$

¹We have use $\frac{dr}{\sqrt{1-kr^2}}$

Using equation (2.13) to find

$$\dot{r} = \frac{1}{S} \left(1 - H^2 S^2 (\Omega - 1) r^2\right)^{\frac{1}{2}}, \quad (\text{A.29})$$

this, for instance, gives

$$\left(\frac{dr}{dS}\right) = -H(1+z), \quad (\text{A.30})$$

$$\left(\frac{dz}{dS}\right) = -\frac{(1+z)}{S}, \quad (\text{A.31})$$

$$\left(\frac{\partial z}{\partial r}\right) = \frac{-H(1+z)S}{(1 - H^2 S^2 (\Omega - 1) r^2)}. \quad (\text{A.32})$$

These can be solved in conjunction with the evolution planes equations given previously for $\dot{\Omega}$, \dot{H} etc.. This approach could be problematic as $z = z(t, r(t))$, but the equations can in principle still be integrated. These in turn can be used to look at:

- Standard candles $F = (L/4\pi S_0^2)(r(1+z))^{-2}$,
- Standard measuring rod $D = S_0 \theta_0 (r/(1+z))$.

A.3.1 (r, z, H, Ω) plane

This plane would give an indication of the radial distance function as related to the cosmological redshift. This is unfortunately a plane with dimension greater than 2, which is problematic in terms of representation.

A.3.2 (D, z) and (F, z) planes

The standard candle, redshift and standard measuring rod, redshift planes; (D, z) and (F, z) phase planes respectively can be found from above using the same set of equations needed to obtain the (r, z, H, Ω) for single fluid or $(r, z, H, \Omega_1, \Omega_2, \dots, \Omega_N)$ planes. That is, in principle, the relation between radial distance, cosmological redshift and the standard measuring rods and candles may be found for single fluid, and effective or many fluid models. Bulk viscosity can be added in later by modifying the appropriate equation of state in the perfect model scenarios considered, or an appropriate scalar field contribution could be used to add such contributions to the behavior. These planes require a current S value that is S_0 , this can be handled in the usual manner by using the dimensionless $y = 1/(1+z)$ variable, assuming then that y is currently unity.

A.4 Pre-recombination era : and exact FLRW treatment

The Cosmic Microwave Background is considered to be black body to 1 in 10000 [36]. We wish to consider an e^- , p and γ gas which are expanding as a relativistic and non-relativistic interacting particles. After electron-positron annihilation one would expect a temperature in the order of m_e , that is, greater than $4000K \approx 0.3eV$, the electron-proton recombination temperature. Such a plasma cannot strictly be considered to be in an equilibrium distribution at a single temperature. One would also expect a different temperature for the electron gas related to the proton gas. What one can try do is model a matter fluid with an electron gas, with the usual $T_{;b}^{ab} = 0$ enforcing the conservation of energy and momenta. The following treatment is in essence that given in [27]. The fact that we observe such a highly accurate black body spectrum implies that the temperature difference between the electrons and photons is very small; this is an attempt to demonstrate that even with electron-photon scattering occurring, in a radiation dominated era this has no great effect upon the photon temperature. The energy momentum tensor for a electron gas is given by

$$T_e^{ab} = \int \frac{dp^3}{(2\pi)^3 p^0} p^a p^b f_e(p), \quad (\text{A.33})$$

where $f_e(p)$ is the Boltzmann distribution function for the electron. The collision term is of the form

$$T_{e;b}^{0b} = \frac{1}{m} \int \frac{dp^3}{(2\pi)^3} C(E) \frac{p^2}{2m}, \quad (\text{A.34})$$

where the distribution $f_e(p)$ is not in equilibrium. Hence we need to define an electron temperature. The electron gas is considered to be non-relativistic hence $E(p) = p^2/2m$, and using $E = \frac{3}{2}kT + m$,

$$\frac{3}{2}T_e n_e + n_e m = \int \frac{dp^3}{(2\pi)^3} f_e(p) \frac{p^2}{2m}. \quad (\text{A.35})$$

Here T_e and n_e are electron the *gas temperature* and electron *number density* respectively. We thus have that:

$$T_e^{00} = n_e m + \frac{3}{2}T_e n_e, \quad (\text{A.36})$$

$$T_e^{ij} = g^{ij} T_e n_e. \quad (\text{A.37})$$

These can be coupled to the geometry via the field equations. That is, one gets the usual set of equations:

$$T_{e;b}^{0b} = \frac{3}{2}n_e \left(\dot{T}_e + 2\frac{\dot{S}}{S}T_e \right). \quad (\text{A.38})$$

If T_e^{ab} is a separately conserved quantity i.e. $T_{e;b}^{ab} = 0$ then the $T_e \approx \frac{1}{3^2}$. To construct a collision term one must decide which processes contribute. The available options are:

- electron-photon scattering (Compton scattering)
- Bremsstrahlung (electron-electron and electron-proton)

Elastic electron-photon scattering is considered to be the dominant process ($n_\gamma > n_e$) i.e. Thompson scattering, this is a special case of Compton scattering. The electron-photon scattering amplitude for Thompson scattering, σ_T , is independent of energies and angles and can be factored out of the integral. The collision term is thus given by [36] [27]. We denote the electron-photon scattering amplitude by $|T_e\gamma|^2$. The Thompson scattering cross-section, $\sigma_T = (8\pi\alpha^2)/(3m^2)$,

$$C(E) = \int \frac{dk^3}{(2\pi)^3 2k} \int \frac{dk'^3}{(2\pi)^3 2k'} \int \frac{dk''^3}{(2\pi)^3 2k''} \frac{1}{2m} (2\pi)^4 |T_e\gamma|^2 \delta^4(p' + k' - p - k) \\ ((1 + f_\gamma(k))f_\gamma(k')f_e(p') - (1 + f_\gamma(k'))f_\gamma(k)f_e(p)). \quad (\text{A.39})$$

Take a Fokker-Planck expansion of the Dirac delta function:

$$\delta^{(3)}(p' + k' - p - k) \approx \delta^{(3)}(p' - p) + (k' - k) \frac{\partial}{\partial p'} \delta^{(3)}(p' - p) \\ + \frac{1}{2} (k' - k) \frac{\partial}{\partial p'} \delta^{(3)}(k' - k) \frac{\partial}{\partial p'} \delta^{(3)}(p' - p) + \dots \quad (\text{A.40})$$

This can be done because the electron is non-relativistic: $|p| = \sqrt{2mE(p)} \approx \sqrt{2mk} \gg k$. The particle is on mass-shell, the electron is non-relativistic, hence using $E = p^2/2m$, we can replace $f_e(p)$ by $f_e(E)$. This can only be done in the isotropic FLRW case, because of spatial isotropy; the distribution function is a function of $p = |p|$, the norm of the 3-momenta. The collision term then becomes

$$C(E) = A_0 f_e(E) + A_1 \frac{\partial f_e(E)}{\partial E} + E \left(\frac{2}{3} A_0 \frac{\partial f_e(E)}{\partial E} + \frac{2}{3} A_1 \frac{\partial^2 f_e(E)}{\partial E^2} \right). \quad (\text{A.41})$$

Here the coefficients A_0 and A_1 are

$$A_0 = \frac{2}{m^2} \frac{|T_e\gamma|^2 1}{2\pi} \int \frac{dk^3}{(2\pi)^3} \frac{k}{2} f_\gamma(k), \quad (\text{A.42})$$

$$A_1 = \frac{1}{2m^2} \frac{|T_e\gamma|^2 1}{2\pi} \int \frac{dk^3}{(2\pi)^3} \frac{k^2}{2} f_\gamma(k) (1 + f_\gamma(k)). \quad (\text{A.43})$$

The photon temperature T_γ is defined in terms of the photon equilibrium distribution function

$$f_{\gamma 0}(k) = \frac{1}{e^{\beta k} - 1}, \quad (\text{A.44})$$

taking derivatives,

$$\frac{d}{dk} f_{\gamma_0}(k) = -\beta f_{\gamma_0}(k)(1 + f_{\gamma_0}(k)). \quad (\text{A.45})$$

This gives

$$\int \frac{dk^3}{(2\pi)^3} k^2 f_{\gamma_0}(k)(1 + f_{\gamma_0}(k)) = \frac{4}{\beta} \int \frac{dk^3}{(2\pi)^3} k f_{\gamma_0}(k). \quad (\text{A.46})$$

In general we can define the photon gas temperature by, using $\beta = 1/T_\gamma$, the Boltzmann constant has been set to 1. The photon energy density is given by μ_γ , and counts both polarizations,

$$\int \frac{dk^3}{(2\pi)^3} k^2 f_\gamma(k)(1 + f_\gamma(k)) = \frac{4}{\beta} \int \frac{dk^3}{(2\pi)^3} k f_\gamma(k) = 2T_\gamma \mu_\gamma. \quad (\text{A.47})$$

Notice that the scattering amplitude is given by $|T_{e\gamma}|^2 = 8\pi m^2 \sigma_T$. Using this the collision term becomes :

$$C(E) = \frac{2}{3} \left(\frac{\sigma_T \mu_\gamma}{\sqrt{E}} \right) \frac{\partial}{\partial E} \left(E^{\frac{3}{2}} (f_e(E) + T_\gamma \frac{\partial f_e(E)}{\partial E}) \right). \quad (\text{A.48})$$

Using the definition of the electron gas temperature T_e and the electron number density, we can write the collision term into the energy momentum conservation equation :

$$T_{e;b}^{0b} = -2\sigma_T \mu_\gamma n_e \frac{(T_e - T_\gamma)}{m}. \quad (\text{A.49})$$

Using this and equation (A.38) we find that

$$\dot{T}_e + 2 \frac{\dot{S}}{S} T_e = -\frac{1}{\tau_e} (T_e - T_\gamma), \quad (\text{A.50})$$

where

$$\tau_e(\mu_\gamma) = \frac{4}{3} \frac{\sigma_T \mu_\gamma}{m} = \frac{32\pi}{3} \frac{\alpha^2}{m^3} \mu_\gamma, \quad (\text{A.51})$$

for the radiation dominated era

$$\mu_\gamma \approx \frac{\pi^2}{15} T_\gamma^4. \quad (\text{A.52})$$

This gives

$$\tau_e = \frac{32\pi^3 \alpha^2 T_\gamma^4}{45m^3} \approx \left(\frac{T_\gamma}{m} \right)^4 \times 10^{18}, \quad (\text{A.53})$$

radiation domination implies the expansion $H = \dot{S}/S$ goes as $\sqrt{\mu_\gamma}$, that is

$$H = \frac{1}{M_{pl}} \frac{8\pi \mu_\gamma^{\frac{1}{2}}}{3} = \frac{8\pi}{45M_{pl}^2} T_\gamma^2 \approx \left(\frac{T_\gamma}{m} \right)^2. \quad (\text{A.54})$$

We thus find that $H\tau_e \ll 1$. Using this and (A.53),

$$\left| \frac{T_\gamma - T_e}{T_e} \right| = -\tau_e \frac{\dot{T}_e}{T_e} - 2\tau_e H, \quad (\text{A.55})$$

hence we finally find that

$$\left| \frac{T_\gamma - T_e}{T_e} \right| \approx H\tau_e \ll 1. \quad (\text{A.56})$$

What we are trying to show is that the photons obey a black-body distribution prior to the recombination regime at decoupling *i.e.* the presence of electrons do not cause any great effect upon the photon gas temperature, assuming that the photon number density is very much greater than the electron number density; we are in a radiation dominated regime in which the difference between the electron and photon temperatures is very small just prior to decoupling. The electron distribution function is given by a Maxwell-Boltzmann, where scattering between photons and non-relativistic electrons dominate. During recombination photons are produced via a capture reaction,



Hence we can conclude that just before decoupling $T_e \simeq T_\gamma$ to $O[0]$, in the exact FLRW model.

A.5 Some Spherical Harmonics relations

The usual form of Spherical Harmonics [62] is

$$(m \geq 0) \quad Y_{l,m}(\theta, \phi) = (-1)^m \left[\frac{(2l+1)(l-m)!}{4\pi(l+m)!} \right] P_l(\cos(\theta)) e^{im\phi}, \quad (\text{A.58})$$

$$(m < 0) \quad Y_{lm}(\theta, \phi) = (-1)^m Y_{l,-m}^*(\theta, \phi). \quad (\text{A.59})$$

These obey the following relations :

- **Orthonormality**

$$\int_{\Omega} Y_{l',m'}^*(\theta, \phi) Y_{l,m}(\theta, \phi) d\Omega = \delta_{ll'} \delta_{mm'} \quad (\text{A.60})$$

- **Closure Relation**

$$\sum_{l=0}^{\infty} \sum_{m=-l}^{+l} Y_{l,m}^*(\theta', \phi') Y_{l,m}(\theta, \phi) = \delta(\Omega - \Omega') = \frac{\delta(\theta - \theta') \delta(\phi - \phi')}{\sin \theta}, \quad (\text{A.61})$$

- **Addition Theorem**

$$\sum_{m=-l}^{+l} Y_{l,m}(\theta', \phi') Y_{l,-m}(\theta, \phi) = \frac{(2l+1)}{4\pi} P_l(\cos \theta_{12}). \quad (\text{A.62})$$

The Legendre Polynomials are P_l and θ_{12} is the angular distance between the directions given by (θ', ϕ') and (θ, ϕ) .

Notice further that,

$$Y_{l,l} = (-1)^l \left[\frac{(2l+1)(2l)!}{4\pi 2^{2l}(l!)^2} \right]^{\frac{1}{2}}, \quad (\text{A.63})$$

$$Y_{l,0} = \left(\frac{(2l+1)}{4\pi} \right)^{\frac{1}{2}} P_l(\cos \theta), \quad (\text{A.64})$$

hence we have that up to $l = 3$:

$$l = 0 : Y_{0,0} = \left(\frac{1}{4\pi} \right)^{\frac{1}{2}}, \quad (\text{A.65})$$

$$l = 1 : \begin{cases} Y_{1,0} = \left(\frac{3}{4\pi} \right)^{\frac{1}{2}} \cos \theta \\ Y_{1,\pm 1} = \pm (-1) \left(\frac{3}{8\pi} \right)^{\frac{1}{2}} \sin \theta e^{\pm i\phi} \end{cases}, \quad (\text{A.66})$$

$$l = 2 : \begin{cases} Y_{2,0} = \left(\frac{5}{16\pi}\right)^{\frac{1}{2}} (3 \cos^2 \theta - 1) \\ Y_{2,\pm 1} = \pm(-1) \left(\frac{15}{8\pi}\right)^{\frac{1}{2}} \sin \theta \cos \theta e^{\pm i\phi} \\ Y_{2,\pm 2} = \left(\frac{15}{32\pi}\right)^{\frac{1}{2}} \sin^2 \theta e^{\pm 2i\phi} \end{cases}, \quad (\text{A.67})$$

$$l = 3 : \begin{cases} Y_{3,0} = \left(\frac{7}{16\pi}\right)^{\frac{1}{2}} \left(\frac{5}{\cos^3 \theta} - 3 \cos \theta\right) Y_{3,\pm 1} = \pm(-1) \frac{1}{4} \left(\frac{21}{4\pi}\right)^{\frac{1}{2}} \sin \theta (5 \cos^2 \theta - 1) e^{\pm i\phi} \\ Y_{3,\pm 2} = \left(\frac{105}{32\pi}\right)^{\frac{1}{2}} \sin^2 \theta \cos \theta e^{\pm 2i\phi} \\ Y_{3,\pm 3} = \pm(-1) \left(\frac{35}{4\pi}\right)^{\frac{1}{2}} \sin^3 \theta e^{\pm 3i\phi} \end{cases} \quad (\text{A.68})$$

With the usual form for the Spherical Harmonic expansion for the temperature anisotropies [50], here :

$$\frac{\delta T(x^a, e^b)}{T(x^a)} \equiv \tau = \sum \tau_{A_l} e^{A_l} = \sum_{l=1; m=-l}^{l=\infty; m=l} a_{lm} Y_{lm}. \quad (\text{A.69})$$

The primitive procedure is to identify terms with the tetrad basis and to then read of the corresponding coefficients *i.e.* using the usual $e^{i\phi} = \cos \phi + i \sin \phi$ and $e^{-i\phi} = \cos \phi - i \sin \phi$ along with functional form of the Spherical Harmonics :

$$l = 1 : \begin{cases} Y_{1,1} = -\left(\frac{3}{8\pi}\right)^{\frac{1}{2}} (\sin \theta \cos \phi + i \sin \theta \sin \phi) \\ Y_{1,0} = \left(\frac{3}{4\pi}\right)^{\frac{1}{2}} \cos \theta \\ Y_{1,-1} = +\left(\frac{3}{8\pi}\right)^{\frac{1}{2}} (\sin \theta \cos \phi - i \sin \theta \sin \phi) \end{cases}, \quad (\text{A.70})$$

$$l = 2 : \begin{cases} Y_{2,2} = \left(\frac{15}{32\pi}\right)^{\frac{1}{2}} (\sin^2 \theta \cos^2 \phi - \sin^2 \theta \sin^2 \phi + 2i \sin^2 \theta \sin \phi \cos \phi) \\ Y_{2,1} = -\left(\frac{15}{8\pi}\right)^{\frac{1}{2}} (\sin \theta \cos \theta \cos \phi + i \sin \phi \sin \theta \cos \theta) \\ Y_{2,0} = \left(\frac{5}{16\pi}\right)^{\frac{1}{2}} (3 \cos^2 \theta - 1) \\ Y_{2,-1} = +\left(\frac{15}{8\pi}\right)^{\frac{1}{2}} (\sin \theta \cos \theta \cos \phi - i \sin \phi \sin \theta \cos \theta) \\ Y_{2,-2} = \left(\frac{15}{32\pi}\right)^{\frac{1}{2}} (\sin^2 \theta \cos^2 \phi - \sin^2 \theta \sin^2 \phi - 2i \sin^2 \theta \sin \phi \cos \phi) \end{cases}, \quad (\text{A.71})$$

$$l = 3 : \left\{ \begin{array}{l} Y_{3,3} = - \left(\frac{35}{4\pi} \right)^{\frac{1}{2}} (\sin^3 \theta \cos^3 \phi - 3 \sin^3 \theta \cos \phi \sin^2 \phi \\ \quad + 3i \sin^3 \theta \sin \phi \cos^2 \phi - i \sin^3 \theta \sin^3 \phi) \\ Y_{3,2} = \left(\frac{105}{32\pi} \right)^{\frac{1}{2}} (\sin^2 \theta \cos \theta \cos^2 \phi - \sin^2 \theta \cos \theta \sin^2 \phi + 2i \cos \phi \sin \phi \sin^2 \theta \cos \theta) \\ Y_{3,1} = - \left(\frac{21}{64\pi} \right)^{\frac{1}{2}} (5 \sin \theta \cos^2 \theta \cos \phi - \sin \theta \cos \phi + 5i \sin \theta \cos^2 \theta \sin \phi - i \sin \theta \sin \phi) \\ Y_{3,0} = \left(\frac{7}{16\pi} \right)^{\frac{1}{2}} (5 \cos^3 \theta - 3 \cos \theta) \\ Y_{3,-1} = \frac{1}{4} \left(\frac{21}{4\pi} \right)^{\frac{1}{2}} (5 \sin \theta \cos^2 \theta \cos \phi - \sin \theta \cos \phi - 5i \sin \theta \cos^2 \theta \sin \phi + i \sin \theta \sin \phi) \\ Y_{3,-2} = \left(\frac{105}{32\pi} \right)^{\frac{1}{2}} (\sin^2 \theta \cos \theta \cos^2 \phi + \sin^2 \theta \cos \theta \sin^2 \phi - 2i \cos \phi \sin \phi \sin^2 \theta \cos \theta) \\ Y_{3,-3} = + \left(\frac{35}{4\pi} \right)^{\frac{1}{2}} (\sin^3 \theta \cos^3 \phi - 3 \sin^3 \theta \cos \phi \sin^2 \phi \\ \quad - 3i \sin^3 \theta \sin \phi \cos^2 \phi - i \sin^3 \theta \sin^3 \phi) \end{array} \right.$$

With the tetrad basis choice of

$$e^a = \{0, \sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta\}. \quad (\text{A.72})$$

The monopole contribution would be zero ($a_{00} = 0$), the monopole represents the averaged bolometric temperature and is not a perturbation. Where we use

$$\tau(\theta, \phi) = \sum_{l=1}^{\infty} \tau_{A_l} e^{A_l} = \tau_a e^a + \tau_{ab} e^a e^b + \dots \quad (\text{A.73})$$

Alternatively we could write the basis directly in terms of the Spherical Harmonics :

$$e^1 = \left(\frac{3}{8\pi} \right)^{-\frac{1}{2}} \left(\frac{-1}{2} \right) [Y_{1,1} - Y_{1,-1}] = \sin \theta \cos \phi, \quad (\text{A.74})$$

$$e^2 = \left(\frac{3}{8\pi} \right)^{-\frac{1}{2}} \left(\frac{-i}{2} \right) [Y_{1,1} + Y_{1,-1}] = \sin \theta \sin \phi, \quad (\text{A.75})$$

$$e^3 = \left(\frac{3}{4\pi} \right)^{-\frac{1}{2}} [Y_{1,0}] = \cos \theta. \quad (\text{A.76})$$

$$(\text{A.77})$$

To find the relationship between the τ 's and the a 's in the respective expansions we note that for instance :

$$\begin{aligned} \tau_a e^a &= \tau_1 e^1 + \tau_2 e^2 + \tau_3 e^3 \quad (\text{A.78}) \\ &= \tau_1 \left(\frac{8\pi}{3} \right)^{\frac{1}{2}} \left(\frac{-1}{2} \right) [Y_{1,1} - Y_{1,-1}] \\ &\quad + \tau_2 \left(\frac{8\pi}{3} \right)^{\frac{1}{2}} \left(\frac{-i}{2} \right) [Y_{1,1} + Y_{1,-1}] \end{aligned}$$

$$+ \tau_3 \left(\frac{4\pi}{3} \right)^{\frac{1}{2}} [Y_{1,0}] \quad (\text{A.79})$$

$$= \tau_1 \sin \theta \cos \phi + \tau_2 \sin \theta \sin \phi + \tau_3 \cos \theta. \quad (\text{A.80})$$

From (A.69) i.e. $\tau_a e^a + \tau_{ab} e^a e^b + \dots = a_{1,0} Y_{1,0} + a_{1,1} Y_{1,1} + a_{1,-1} Y_{1,-1} + \dots$ it can be shown by identifying coefficients that :

$$\left(\frac{8\pi}{12} \right)^{\frac{1}{2}} (-\tau_1 - i\tau_2) Y_{1,1} = a_{1,1} Y_{1,1}, \quad (\text{A.81})$$

$$\left(\frac{8\pi}{12} \right)^{\frac{1}{2}} (\tau_1 - i\tau_2) Y_{1,-1} = a_{1,-1} Y_{1,-1}, \quad (\text{A.82})$$

$$\left(\frac{4\pi}{3} \right)^{\frac{1}{2}} \tau_3 Y_{1,0} = a_{1,0} Y_{1,0}. \quad (\text{A.83})$$

It can then be deduced (for example) that,

$$\tau_1 = \left(\frac{3}{8\pi} \right)^{\frac{1}{2}} (a_{1,-1} - a_{1,1}), \quad (\text{A.84})$$

$$\tau_2 = i \left(\frac{3}{8\pi} \right)^{\frac{1}{2}} (a_{1,1} + a_{1,-1}), \quad (\text{A.85})$$

$$\tau_3 = \left(\frac{3}{4\pi} \right)^{\frac{1}{2}} a_{1,0}. \quad (\text{A.86})$$

A.6 Tetrads

[See [24]] Consider a general tetrad basis E_a : indices a,b,c... that is early letters are used for the tetrad basis letter i,j,k,... that is late letters are used for the coordinate basis. The tetrad components are $X^a = E_i^a X^i$, where $E_i^a E_a^j = \delta_i^j$ and $E_i^a E_b^i = \delta_b^a$. Tetrad indices are raised and lowered using the tetrad components of the metric $g_{ab} = g_{ij} E_a^i E_b^j$ and $g^{ab} g_{bc} = \delta_c^a$.

Notice that $p^a = E_i^a p^i$ and that $p^2 = g_{ab} p^a p^b = -m^2$ where

$$\frac{\partial p^2}{\partial p^a} = -2g_{ab} p^b. \quad (\text{A.87})$$

That is p^a is normal to P_m at each point p^a in the tangent space in T_x .

The tetrad components and connections are related by

$$\nabla_a E_b = \Gamma_{ab}^c E_c, \quad (\text{A.88})$$

$$[E_a, E_b] = \gamma_{ab}^c E_c. \quad (\text{A.89})$$

The differential operator $\partial_a = E_a^i \partial_i$ and the Ricci rotation coefficients $\Gamma_{abc} = E_a^j E_b^k E_{c;j;k}$, are related by

$$\partial_b g_{ac} = \Gamma_{abc} + \Gamma_{cba}, \quad (\text{A.90})$$

$$\Gamma_{abc} = \frac{1}{2}(\partial_b g_{ca} + \partial_c g_{ab} - \partial_a g_{bc}) + \frac{1}{2}(\gamma_{abc} + \gamma_{cab} - \gamma_{bca}). \quad (\text{A.91})$$

where $\gamma_{bc}^a = \Gamma_{bc}^a - \Gamma_{cb}^a$. These give the Riemann curvature tensor and Ricci scalar $R_{bc} = R_{b^a ca}$ from

$$R_{bcd}^a = \partial_c \Gamma_{db}^a - \partial_b \Gamma_{cb}^a + \Gamma_{ce}^a \Gamma_{db}^e - \Gamma_{de}^a \Gamma_{cb}^e + \Gamma_{eb}^a \gamma_{dc}^e. \quad (\text{A.92})$$

For a comoving observer with 4-velocity u^a , there is a preferred family of orthonormal tetrads associated with u^a i.e. a frame for which the time-like tetrad basis E_0 is parallel to the velocity u^a . We denote the orthonormal tetrad by E_a , in the tetrad basis

$$u^a = \delta_0^a \quad (\text{A.93})$$

$$h_{ab} = \text{diag}(0, +1, +1, +1) \quad (\text{A.94})$$

$$g_{ab} = \text{diag}(-1, +1, +1, +1). \quad (\text{A.95})$$

One finds that from (A.95) and using the Ricci rotation coefficients and the form of $u_{a;b}$ from (3.2) and using $\Gamma_{abc} = e_a \cdot \nabla_b e_c$ that

$$\Gamma_{00\nu} = \dot{u}_\nu, \quad (\text{A.96})$$

$$\Gamma_{\nu\mu 0} = \sigma_{\nu\mu} + \omega_{\nu\mu} + \frac{1}{3}\theta\delta_{\nu\mu}, \quad (\text{A.97})$$

$$\Gamma_{\nu 0\mu} = \epsilon_{\mu\nu\sigma}\Omega^\sigma. \quad (\text{A.98})$$

Indices ν, μ, σ, \dots denote values ranging over 1,2,3. Ω^σ is a 3-vector that describes the rate of rotation of the tetrad basis vectors e_ν . Notice that Γ_{bc}^a do not transform as tensors on c ; it follows that γ_{bc}^a is not tensorial on either b or c . The rotation coefficients are skew in a and c in orthonormal basis.

An alternative manner in which to express the connection in the spatial slice is as

$$\Gamma_{\alpha\beta\delta} = \frac{1}{2} \left(\epsilon_{\beta\delta\gamma} n_\alpha^\gamma + \epsilon_{\alpha\beta\gamma} n_\delta^\gamma - \epsilon_{\delta\alpha\gamma} n_\beta^\gamma \right) + (\delta_{\delta\beta} a_\alpha - \delta_{\beta\alpha} a_\delta), \quad (\text{A.99})$$

where $n_{\alpha\beta} = n_{(\alpha\beta)}$ is the symmetric part, and $\epsilon^{\alpha\beta\gamma} a_\gamma$ is the antisymmetric part of the connection.

It should be kept in mind that \dot{u}_α , $\delta_{\alpha\beta}$ and $\omega_{\alpha\beta}$ are projections of 4-tensors that transform as 3-tensors. The quantities Ω_α , $n_{\alpha\beta}$ and a_α do not transform as tensors as Γ_{bc}^a and γ_{bc}^a do not transform as tensors. **Some useful relations:**

$$\omega^a = \frac{1}{2} \eta^{abcd} u_b \omega_{cd} \quad (\text{A.100})$$

$$\omega_{ab} = \eta_{abcd} \omega^c u^d \quad (\text{A.101})$$

$$\omega_{ab} = \omega_{[ab]} \quad (\text{A.102})$$

$$\dot{u}_a = u_{a;b} u^b \quad (\text{A.103})$$

$$\dot{u}^a u_a = 0 \quad (\text{A.104})$$

$$\theta = u^a_{;a} \quad (\text{A.105})$$

$$\sigma_{ab} u^b = 0 \quad (\text{A.106})$$

$$\sigma_{ab} = \sigma_{(ab)} \quad (\text{A.107})$$

$$\sigma_a^a = 0 \quad (\text{A.108})$$

$$\Omega^\alpha = \frac{1}{2} \epsilon^{\alpha\beta\delta} E_\beta \dot{E}_\delta \quad (\text{A.109})$$

$$a_\alpha = \gamma_{\alpha\beta}^\beta \quad (\text{A.110})$$

$$n^{\alpha\beta} = \gamma_{\gamma\delta}^{(\alpha} \epsilon^{\beta)\gamma\delta} \quad (\text{A.111})$$

$$n^a = \delta_0^a \quad (\text{A.112})$$

$$h_{\alpha\beta} = \delta_{\alpha\beta} \quad (\text{A.113})$$

$$h_{00} = 0 \quad (\text{A.114})$$

A.7 Derivation of the Boltzmann equation in a local tetrad formulation

This Appendix was compiled with the help of the notes on Relativistic Kinetic Theory of G. F. R. Ellis.

(1) Consider the equation,

$$L(f) = p^a \left(\frac{\partial f}{\partial x^a} \right)_{p^a} - \Gamma_{bc}^a p^b p^c \left(\frac{\partial f}{\partial p^a} \right)_{x^a}, \quad (\text{A.115})$$

using the usual $p^a = Eu^a + \lambda e^a$ and hence $p^0 \equiv E$ and $p^\nu = \lambda e^\nu$,

(2) we can reduce (A.115) to

$$L(f) = E \left(\frac{\partial f}{\partial x^a} \right)_{p^a} u^a + \lambda \left(\frac{\partial f}{\partial x^a} \right)_{p^a} e^a - \Gamma_{bc}^0 p^b p^c \left(\frac{\partial f}{\partial E} \right)_{x^a, p^a} - \Gamma_{bc}^\beta p^b p^c \left(\frac{\partial f}{\partial p^\beta} \right)_{x^a, E} \quad (\text{A.116})$$

(3) That is, we have used the form $f(x^a, e^b) \rightarrow f(x^a, E, p^\beta)$. Using the mass-shell relation we note that

- $E = E$ and $p^\beta = (E^2 - m^2)^{\frac{1}{2}} e^\beta = \lambda e^\beta$ or $e^\beta = p^\beta / (E^2 - m^2)^{\frac{1}{2}}$,
- $m^2 = E^2 - p^\beta \delta_{\beta\alpha} p^\alpha = E^2 - \lambda^2$ (the mass-shell relation).

Using these relations the following can be derived:

$$\left(\frac{\partial E}{\partial E} \right)_{p^\alpha} = 1, \quad (\text{A.117})$$

$$\left(\frac{\partial E}{\partial p^\alpha} \right)_E = 0,$$

$$\left(\frac{\partial(m^2)}{\partial E} \right)_{p^\alpha} = 2, E$$

$$\left(\frac{\partial(m^2)}{\partial p^\alpha} \right)_E = -2p_\alpha,$$

$$\left(\frac{\partial e^\alpha}{\partial E} \right)_{p^\alpha} = \frac{+\frac{1}{2}p^\alpha(2E)}{(E^2 - m^2)^{\frac{1}{2}}} = +\frac{Ee^\alpha}{(E^2 - m^2)^{\frac{1}{2}}} = +\frac{E}{\lambda^2} e^\alpha, \quad (\text{A.118})$$

$$\left(\frac{\partial e^\beta}{\partial p^\alpha} \right)_E = \frac{1}{(E^2 - m^2)^{\frac{1}{2}}} \delta_\alpha^\beta = \frac{1}{\lambda} \delta_\alpha^\beta. \quad (\text{A.119})$$

(4) Further more, using that $f(x^a, E, p^\alpha) \equiv g(x^a, E, m, e^\alpha)$,

$$\left(\frac{\partial f}{\partial E} \right)_{p^\alpha} = \left(\frac{\partial g}{\partial E} \right)_{m, e^\alpha} \left(\frac{\partial E}{\partial E} \right)_{p^\alpha} + \left(\frac{\partial g}{\partial(m^2)} \right)_{E, e^\alpha} \left(\frac{\partial(m^2)}{\partial E} \right)_{p^\alpha}$$

$$+ \left(\frac{\partial g}{\partial e^\beta} \right)_E \left(\frac{\partial e^\beta}{\partial E} \right)_{p^\alpha}, \quad (\text{A.120})$$

$$\begin{aligned} \left(\frac{\partial f}{\partial p^\alpha} \right)_E &= \left(\frac{\partial g}{\partial E} \right)_{m, e^\alpha} \left(\frac{\partial E}{\partial p^\alpha} \right)_E + \left(\frac{\partial g}{\partial(m^2)} \right)_{E, e^\beta} \left(\frac{\partial(m^2)}{\partial E} \right)_E \\ &+ \left(\frac{\partial g}{\partial e^\beta} \right)_{E, m} \left(\frac{\partial e^\beta}{\partial p^\alpha} \right)_E. \end{aligned} \quad (\text{A.121})$$

(5) From equations (A.120 and (A.121) using the relations (A.117) - (A.119) to respectively find :

$$\left(\frac{\partial f}{\partial E} \right)_{p^\alpha} = \left(\frac{\partial g}{\partial E} \right)_{m, e^\alpha} + \left(\frac{\partial g}{\partial(m^2)} \right)_{E, e^\alpha} 2E + \left(\frac{\partial g}{\partial e^\beta} \right)_E \frac{E}{\lambda^2} e^\beta, \quad (\text{A.122})$$

$$\left(\frac{\partial f}{\partial p^\alpha} \right)_E = - \left(\frac{\partial g}{\partial(m^2)} \right)_{E, e^\alpha} 2p^\beta + \left(\frac{\partial g}{\partial e^\beta} \right)_{E, e^\alpha} \frac{1}{\lambda}. \quad (\text{A.123})$$

(6) Notice that:

$$\begin{aligned} & -\Gamma_{bc}^0 p^b p^c 2E + \Gamma_{bc}^\alpha p^b p^c 2p_\alpha \\ &= +2 \left(\Gamma_{0bc} p^b p^c p^0 + \Gamma_{abc} p^b p^c p^\alpha \right) \\ &= +2 \Gamma_{abc} p^a p^b p^c = 0. \end{aligned} \quad (\text{A.124})$$

In the tetrad frame we find that $\Gamma_{abc} = -\Gamma_{cba}$, *i.e.*, the $(\partial g / \partial(m^2))_{E, e^\alpha}$ terms cancel out in (A.116).

(7) It then follows using (A.121), (A.123) and (A.124) that we are able to rewrite (A.116):

$$\begin{aligned} L(f) &= Eu^a \frac{\partial f}{\partial x^a} + \lambda e^\alpha \frac{\partial f}{\partial x^a} - \Gamma_{bc}^0 p^b p^c \frac{\partial f}{\partial E} \\ &\quad - \frac{E}{\lambda^2} \Gamma_{bc}^0 p^b p^c e^\alpha \frac{\partial f}{\partial e^\alpha} - \frac{1}{\lambda} \Gamma_{bc}^\alpha p^b p^c \frac{\partial f}{\partial e^\alpha}. \end{aligned} \quad (\text{A.125})$$

We have, as usual, being using $u^a = \delta_0^a$ and $u_a = -\delta_0^a$.

(8) Expanding this out,

$$\begin{aligned} L(f) &= Eu^a \frac{\partial f}{\partial x^a} + \lambda e^\alpha \frac{\partial f}{\partial x^a} + \left[(-\Gamma_{0c}^0 - \Gamma_{c0}^0) E \lambda e^c - \Gamma_{\alpha\beta}^0 \lambda^2 e^\alpha e^\beta \right] \left(\frac{\partial f}{\partial E} \right) \\ &\quad - \left[\Gamma_{00}^\alpha E^2 + (\Gamma_{0\beta}^\alpha - \Gamma_{\beta 0}^\alpha) e^\beta E \lambda + \Gamma_{\beta\delta}^\alpha e^\beta e^\delta \lambda^2 \right] \frac{1}{\lambda} \frac{\partial f}{\partial e^\alpha} \\ &\quad + \left[(-\Gamma_{0c}^0 - \Gamma_{c0}^0) E \lambda e^c + (-\Gamma_{\beta\delta}^0) \lambda^2 e^\delta e^\beta \right] \frac{E}{\lambda^2} \frac{\partial f}{\partial e^\alpha}. \end{aligned} \quad (\text{A.126})$$

(9) We define the following quantities:

$$b_a \equiv (-\Gamma_{0c}^0 - \Gamma_{c0}^0)h_a^c \quad (\text{A.127})$$

$$b_{ab} \equiv -\Gamma_{(de)}^0 h_a^d h_b^e \quad (\text{A.128})$$

$$A_a \equiv -\Gamma_{a00} \quad (\text{A.129})$$

$$B_b^a \equiv (\Gamma_{0d}^c + \Gamma_{d0}^c)h_c^a h_b^d \quad (\text{A.130})$$

$$A_{ab}^d \equiv \Gamma_{(ef)}^c h_c^d h_a^e h_b^f. \quad (\text{A.131})$$

From these definitions, (A.127) - (A.131), we are able to rewrite (A.126),

$$\begin{aligned} L(f) &= Eu^a \frac{\partial f}{\partial x^a} + \lambda e^a \frac{\partial f}{\partial x^a} \\ &+ [\lambda E b_c e^c + \lambda^2 b_{ab} e^a e^b] \frac{\partial f}{\partial E} \\ &- [A^b E^2 + B_a^b \lambda e^a E + A_{de}^b e^d e^e \lambda^2] \frac{1}{\lambda} \frac{\partial f}{\partial e^b} \\ &+ [\lambda E b_c e^c + \lambda^2 b_{ab} e^a e^b] \frac{E}{\lambda^2} e^d \frac{\partial f}{\partial e^d}. \end{aligned} \quad (\text{A.132})$$

(10) Using the usual relations:

- $u_{a;b} = \omega_{ab} + \sigma_{ab} + \frac{1}{3}\theta h_{ab} - \dot{u}_a u_b$
- $u^c_{;b} = u^c_{,b} + \Gamma_{ab}^c u^a$
- $\dot{u}_a = E_a \cdot \nabla_0 E_0 = -\Gamma_{a00} = +\Gamma_{00a} = -\Gamma_{0a}^0$ (as $g^{00} = -1$)
- $\Gamma_{ab}^c u_c u^a u^b = 0$ which follows from $\Gamma_{acb} + \Gamma_{bca} = 0 \Rightarrow \Gamma_{00}^0 = 0$,

we can rewrite the definitions (A.127) - (A.131):

$$\begin{aligned} b_\alpha &= (\Gamma_{db}^c + \Gamma_{bd}^c)u_c u^b h_\alpha^d \\ &= -(\Gamma_{\alpha 0}^0 + \Gamma_{0\alpha}^0) \\ &= +\Gamma_{0\alpha 0} - \Gamma_{0\alpha}^0 \\ &= \dot{u}_\alpha, \end{aligned} \quad (\text{A.133})$$

$$b_{\alpha\beta} = \frac{1}{2}(\Gamma_{ab}^c + \Gamma_{ba}^c)u_c h_\alpha^a h_\beta^b$$

$$\begin{aligned}
&= -\frac{1}{2}(\Gamma_{\alpha\beta}^0 + \Gamma_{\alpha\beta}^0) \\
&= \frac{1}{2}(\Gamma_{0\alpha\beta} + \Gamma_{0\beta\alpha}) \\
&= \frac{1}{2}(\theta_{\alpha\beta} + \omega_{\alpha\beta} + \theta_{\beta\alpha} + \omega_{\beta\alpha}) \\
&= \theta_{\alpha\beta},
\end{aligned} \tag{A.134}$$

$$A^\beta = \dot{u}^\beta, \tag{A.135}$$

$$\begin{aligned}
B_\beta^\alpha &= \Gamma_{0\beta}^\alpha + \Gamma_{\beta 0}^\alpha \\
&= \epsilon_{\beta\delta}^\alpha \Omega^\delta + \theta_\beta^\alpha + \omega_\beta^\alpha \\
&= \theta_\beta^\alpha + \epsilon_{\beta\delta}^\alpha (\Omega^\delta + \omega^\delta),
\end{aligned} \tag{A.136}$$

$$\begin{aligned}
A_{\beta\delta}^\alpha &= \Gamma_{\beta\delta}^\alpha \\
&= \frac{1}{2} \left(\epsilon_{\beta\delta\gamma}^\alpha n^{\alpha\gamma} + \epsilon_{\beta\gamma}^\alpha n_\delta^\gamma - \epsilon_\delta^\alpha \gamma n_\beta^\gamma \right) + \left(\delta_{\delta\beta} a^\alpha - \delta_\beta^\alpha a_\delta \right).
\end{aligned} \tag{A.137}$$

(11) From The Einstein-Boltzmann equation, (A.132), we can derive a form of the Boltzmann equation in terms of variables x^a, E, m and e^a using the tetrad quantities, (A.133) - (A.137),

$$\begin{aligned}
L(f) &= Eu^a \frac{\partial f}{\partial x^a} + \lambda e^a \frac{\partial f}{\partial x^a} \\
&+ \left[\lambda E \dot{u}_\alpha e^\alpha + \lambda^2 \theta_{\alpha\beta} e^\alpha e^\beta \right] \frac{\partial f}{\partial E} \\
&- \left[E^2 \dot{u}^\alpha + \theta_\beta^\alpha + \epsilon_{\beta\delta}^\alpha (\Omega^\delta + \omega^\delta) \lambda e^\beta E + \Gamma_{\beta\delta}^\alpha e^\beta e^\delta \lambda^2 \right] \frac{1}{\lambda} \frac{\partial f}{\partial e^b} \\
&+ \left[\lambda E \dot{u}_\alpha e^\alpha + \lambda^2 \theta_{\alpha\beta} e^\alpha e^\beta \right] \frac{E}{\lambda^2} e^\alpha \frac{\partial f}{\partial e^\alpha}.
\end{aligned} \tag{A.138}$$

Here we then use the Harmonic expansion form of f ,

$$f = F + F_a e^a + F_{ab} e^a e^b + \dots = \sum_l F_{A_l} e^{A_l}, \tag{A.139}$$

along with the PSTF tensor relations,

$$F_{A_l} \equiv F_{ab\dots c} \tag{A.140}$$

$$F_{\langle ab\dots c \rangle} u^c = 0 = F_{\langle ab\dots c \rangle} h^{ab} \tag{A.141}$$

$$F_{\langle ab \rangle} = h_a^c h_b^d F_{cd} - \frac{1}{3} F_{cd} h^{cd} h_{ab}. \tag{A.142}$$

Notice that $\pi_{ab} = \pi_{(ab)}$ for example.

(12) We know that

$$p^a = Eu^a + k_c h^{ca}, \quad (\text{A.143})$$

$$p^a = Eu^a + \lambda e^a. \quad (\text{A.144})$$

For photons we know that $\lambda = E$, hence $k_c h^{ca} = \lambda e^a$ and the energy and frequency are given by

$$\nu = -u^a k_a, \quad (\text{A.145})$$

$$E = -u_a p^a. \quad (\text{A.146})$$

Hence

$$\frac{E_e}{E_r} = \frac{\nu_e}{\nu_r} = (1 + z), \quad (\text{A.147})$$

z is the redshift from the time of emission to the time of reception.

It is also helpful to note that from (3.42) and (3.52) we have that for $E = \lambda$, *i.e.* for photons

- (i) *energy density* : $\mu = 4\pi \int_0^\infty E^3 dEF$,
- (ii) *number density* : $n = 4\pi \int_0^\infty E^2 dEF$,
- (iii) *bose gas* : $F = \frac{2}{e^{E/kT} - 1}$.

The average energy $\langle E \rangle$ is given from (i), (ii) and (iii) above

$$\langle E \rangle \equiv \mu/n = (a/b)T, \quad (\text{A.148})$$

where $a = \int_0^\infty (x^3/e^x - 1)dx$ and $b = \int_0^\infty (x^2/e^x - 1)dx$. Such that from (A.147), it is found that

$$(1 + z) = \frac{E_e}{E_r} \equiv \frac{\langle E_e \rangle}{\langle E_r \rangle} = \frac{T_e}{T_r}. \quad (\text{A.149})$$

This means, for example, that as we know the recombination temperature to be about 4000 K (0.3 eV) *i.e.* $T_e \approx 4000$ K and the radiation temperature is now, on average, found to be about 2.7 K *i.e.* $T_0 \approx 2.7$ K, then from $T_e/T_0 = (1 + z)$ it is found that $z = 1300$. This is the redshift of the surface of last scattering.

A.8 The distribution function and the fractional comoving density gradient

Using a usual distribution function (+ fermions, - bosons) with the chemical potential μ_E , the scalar contribution to the distribution function is

$$F = \frac{2}{\exp((E - \mu_E)/kT) \pm 1}. \quad (\text{A.150})$$

We can set $\mu_E = 0$. Considering fermions, for example, one can find the spatial divergence of F ,²

$$\hat{\nabla}_c F = \left(\frac{\hat{\nabla}_c T}{T^2} \right) \left(\frac{E}{k} \right) \frac{1}{\cosh(E/kT) - 1}, \quad (\text{A.151})$$

$$\dot{F} = - \left(\frac{\dot{T}}{T} \right) \left(\frac{E}{kT} \right) \frac{1}{\cosh(E/kT) - 1}, \quad (\text{A.152})$$

$$\frac{\partial F}{\partial E} = \left(\frac{1}{kT} \right) \frac{1}{\cosh(E/kT) - 1}. \quad (\text{A.153})$$

Using (3.151) and (A.150) to get

$$\hat{\nabla}_c F^c \simeq \left(\frac{3E}{kT} \right) \left(\frac{\dot{T}}{T} + \frac{1}{3}\theta \right) \frac{1}{\cosh(E/kT) - 1} - \frac{3b}{E}. \quad (\text{A.154})$$

We could substitute $\frac{1}{3}\theta \simeq \frac{\dot{S}}{S}$ to first order above, to find that for free-streaming ($b=0$) and $\hat{\nabla}_c F^c \approx 0$ that $\frac{\dot{T}}{T} \approx -\frac{\dot{S}}{S}$. Continuing in this fashion one may find $\hat{\nabla}_a \hat{\nabla}_b F^{ab}$ by taking the spatial gradient of (3.152) using $(\hat{\nabla}_a F^b) \simeq \hat{\nabla}_a \dot{F}^b - (\dot{S}/S)\hat{\nabla}_a F^a$ following [33], $\hat{\nabla}_a (\partial F^b / \partial E) = \partial / \partial E (\hat{\nabla}_a F^b)$. This would require the functional form of $\partial F_a / \partial E$, unless one has that $\hat{\nabla}_c \theta = 0$.

For a gas one can easily find the fractional comoving spatial gradients:

$$D_a = S \frac{\hat{\nabla}_c \int_m^\infty E^2 (E^2 - m^2)^{\frac{1}{2}} F dE}{\int_m^\infty E^2 (E^2 - m^2)^{\frac{1}{2}} F dE} \quad (\text{A.155})$$

$$= S \frac{\int_m^\infty E^2 (E^2 - m^2)^{\frac{1}{2}} \hat{\nabla}_c F dE}{\int_m^\infty E^2 (E^2 - m^2)^{\frac{1}{2}} F dE}. \quad (\text{A.156})$$

From the Stefan-Boltzmann law, $\mu(x^a) = 4\pi \int E^3 F dE = aT^4(x^a)$, one finds directly from the definition of the fractional comoving spatial gradients (rather than substituting (from A.151) in (A.156) and integrating),

$$D_c = S \frac{\hat{\nabla}_c \mu}{\mu} = S \frac{4\hat{\nabla}_c T}{T}. \quad (\text{A.157})$$

²Notice that $\frac{\exp(E/kT)}{(\exp(E/kT)-1)^2} = \frac{1}{2} \frac{1}{\cosh(E/kT)-1}$

Here derivatives are orthogonal to the comoving matter frame velocity. Notice that $T(x^a)$ is the isotropic temperature measured at a point x^a , by averaging over the complete sky; it is the true averaged bolometric temperature at this point. There are also temperature anisotropies associated with this particular point x^a on the observers sky, which are given by $\delta T(x^a, e^b)$. But the averaged temperature may be different at some other point x'^a in the universe in the spatial slice given by the projection h^{ab} at some proper time. The quantity \mathcal{D}_a gives an indication of the difference in the averaged bolometric temperature between different regions in the spatial slice. That is the difference in the monopole temperature at different spatial positions, and should not be confused with temperature anisotropies associate with each space-time co-ordinate.

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