Flaws, Approximations and Uncertainties in the Estimation of the Exposed-to-Risk

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Abstract

This research analyses the theoretical basis of exposed-to-risk estimation. It defends the conventional actuarial approach against criticisms raised by Hoem (1984), and, in so doing, examines in detail the development of the actuarial profession's estimation techniques. Maximum likelihood estimates are shown to be closely related to the estimates of decremental probabilities derived using the conventional actuarial approach.

The correct treatment of deaths when estimating the initial exposed-to-risk is considered and contrasted with what is often used in practice. The relationship between the initial and central exposed-to-risk is considered for a single decrement, two decrements and for select rates. The implications of alternative assumptions and approximations are considered. Some inaccuracies in tuition material of the Faculty and Institute of Actuaries and articles written about exposed-to-risk are highlighted. Other problem areas, such as the bias of calculated rates and estimation under policy and calendar year rate intervals, are also considered.

Contents

Chapter 1

Introduction

The objective of this research is to examine the rationale of exposed-to-risk theory, the nature of the assumptions that are commonly made in deriving estimates of rates of mortality and the implications for the estimates of choosing particular assumptions.

The method of exposed-to-risk estimation traditionally favoured by the actuarial profession has its origins in research conducted in the early nineteenth century. A substantial body of literature on the subject has been produced since then. The commonly accepted theory at the present time can be found in texts of the various professional actuarial bodies on the subject of exposed-to-risk, for example Batten (1978), Gershenson (1961) and Benjamin and Pollard (1980). Hoem (1984) asserts that this approach is flawed, that is, that the theory underlying the conventional actuarial method of estimating mortality rates is based on an incorrect assumption. Although the magnitude of the error introduced into our estimates by such an incorrect assumption would be small, the assertion, if correct, would imply that much of the profession's understanding of exposed-to-risk theory has been incomplete. These arguments against the conventional approach are therefore considered in detail.

The actuarial approach requires an a priori assumption about the form of the force of decrement. Estimates are then derived on the assumption that the observed decrements resulted from the assumed underlying force of decrement. The dependence of the estimates obtained on the assumption chosen is, perhaps, not always fully understood. This dependence and its consequences needs to be emphasised. For example, the notion of giving 'full exposure' to deaths and the treatment of deaths in general is often interpreted incorrectly. The implications of making a given assumption are analysed for particular circumstances.

The actuarial field is not the only one in which estimates are made of decremental probabilities. More general estimation techniques can be found in statistical theory. For example, it is possible to estimate the form of the survival function instead of assuming that it follows a particular form. Using statistical theory, estimates can be derived which have properties deemed to be desirable, for example, estimates that maximise the likelihood. The properties of these latter estimates are considered relative to the properties of the estimates generally used in actuarial work.

There are many areas of the theory that continue to be considered problematic. Examples include the relationship between the assumption made about the underlying force of decrement, and that made regarding the distribution of deaths over the rate interval and the relationship between initial and central exposure for a single decrement, for multiple decrements and for select rates. Each of these and other areas is considered and clarified in this research.

Chapter 2

Background

2.1 Definitions and Notation

Throughout this research, unless otherwise specified, the following notation will be used

$\overset{\circ}{l}_{x+r}$ = number of lives attaining exact age $x+r$ during the period of investigation.

$\overset{\circ}{l}_{x+r}$ is a step-function with respect to $r \in [0,1)$.

b_{x+r}, e_{x+r} = number of lives at the start and end, respectively, of the period of investigation, then aged $x+r$ exact.

n_{x+r}, w_{x+r} = number of new entrants and withdrawals, respectively, during the period of investigation, aged $x+r$ exact at date of entry or exit.

θ_{x+r} = number of deaths during the period of investigation aged $x+r$ exact at death.

b_{x+r}, e_{x+r}, n_{x+r}, w_{x+r} and θ_{x+r} are discrete functions with respect to $r \in [0,1)$.

$\theta_{(x)}$ = total number of deaths during the period of investigation aged x last birthday at date of death.

$$= \sum_r \theta_{x+r} ,$$

where the summation is taken over all values of $r \in [0, 1)$ at which θ_{x+r} is non-zero.

We will make the assumption that the age of entry to and, provided they survive, exit from the investigation is pre-determined for each individual. In other words, a given individual will exit from observation at a defined age, or exit from observation at an earlier age if they die. This assumption is considered further in Chapter 3. Alternatively, we could state that the distributions used are conditional on the ages at entry to, and, provided they survive, ages at exit from the investigation of the lives observed.

If a different pattern of deaths were to be observed, the same number of beginners and new entrants at each age $x + r$ would be observed. Hence

$$E[b_{x+r}] = b_{x+r} \quad , \quad \text{and}$$
$$E[n_{x+r}] = n_{x+r}$$

for all $r \in [0, 1)$.

If a different pattern of deaths were to be observed, the functions w_{x+r}, e_{x+r} and $\overset{\circ}{l}_{x+r}$ would take on different values. The values we observe for these functions are, therefore, observed values of random functions. The expected value of $\overset{\circ}{l}_{x+r}$ will be a function which is discontinuous at a finite number of points in the interval $[0,1)$.

Unless otherwise stated, all expressions are applicable to the life year rate interval commencing at age x exact, although the results can be generalised for other life year rate intervals by taking r to be the time since the start of the specific rate year. The extension of the results to policy or calendar year rate intervals is not as straightforward. These rate intervals are considered further in section 5.2.

For each individual, i, observed, let $x + s_i$ be the age at which observation of individual i commences, whether due to attainment of age x, new entry, or the start of the period of investigation; and let $x + t_i$ be the age at which observation ceases, whether due to death, withdrawal, end of the period of investigation or attainment of age $x + 1$. Thus

$$\begin{aligned} 0 \leq \; & s_i < 1, \text{ and} \\ 0 < \; & t_i \leq 1 \end{aligned}$$

Let D_i be a function such that D_i equals 1 if the individual dies, and 0 otherwise. If N lives are assumed to be observed in total it is obvious that

$$\begin{aligned} N &= \sum_i 1 \text{ , and} \\ \theta_{(x)} &= \sum_{i:D_i=1} 1 \\ &= \sum_i 1 - \sum_{i:D_i=0} 1 \end{aligned}$$

Let $f(s,t,q_x)$ be a function of q_x such that

$$f(s,t,q_x) = {}_{t-s}q_{x+s}$$

The form of this function under various assumptions will be considered in later sections.

2.2 Historical Development

Perhaps the first exposition of the commonly used actuarial approach was that of Wittstein in 1862 (Seal, 1977). He considered an example in which A lives enter observation aged x exact, B lives enter observation aged x last birthday at entry, and C lives exit from observation, other than by death, aged x last birthday at exit. He assumed that movements B and C occur uniformly over the year of age so that the net movement at age $x+t$ could be assumed to be constant for all t and equal to $(B-C)\delta t$. He then derived the approximate relationship

$$\theta_{(x)} \approx A\ q_x + (B-C)\int_0^1 {}_{1-t}q_{x+t}\ dt$$

Wittstein considered two possible assumptions about the form of ${}_{1-t}q_{x+t}$

$${}_{1-t}q_{x+t} = \frac{(1-t)q_x}{1-tq_x} \quad \text{, and}$$

$${}_{1-t}q_{x+t} = 1-(1-q_x)^{1-t}$$

The first, the 'uniform assumption', implied a uniform distribution of deaths over the year of age x to $x+1$, and the second, the 'constant force of mortality assumption', implied a constant force of mortality over the same year of age.

Using these assumptions and expanding the logarithmic series, Wittstein, taking the first term in this expansion, derived the estimate

$$q_x \approx \frac{\theta_{(x)}}{A + \frac{1}{2}(B - C)} \tag{2.1}$$

He showed that under the uniform assumption the next term was

$$-\frac{1}{6}(B - C)\frac{(\theta_{(x)})^2}{(A + \frac{1}{2}(B - C))^3} \tag{2.2}$$

Under the constant force of mortality assumption the next term was half of (2.2). Since the value of this term and those following it are, in practice, very small, the choice of assumption makes little difference to the magnitude of the estimate derived.

The denominator of the estimate in equation (2.1) has come to be termed the initial exposed-to-risk. The concept of the exposed-to-risk as the time period of exposure giving rise to the observed number of deaths has been traced back to Woolhouse in 1867 by Seal (1977). The length of time taken into account for the deaths when calculating the initial exposed-to-risk is an area of particular uncertainty, and is considered further in Chapter 4.

According to Seal (1977), Wittstein was also the first to consider the assumption most commonly used in actuarial work on exposed-to-risk, the so-called 'Balducci' assumption, that is

$$_{1-t}q_{x+t} = (1-t)q_x$$

Seal (Hoem, 1984) has established that, in fact, Balducci did not make any substantial contribution to the theory based on this assumption, but was mainly concerned with a related assumption. A summary of the properties of the three assumptions, uniform, Balducci and constant force of mortality, is given in Appendix B.

The line of reasoning expressed by Wittstein has become the accepted approach in much of the actuarial literature on this subject. The American and British texts covering exposed-to-risk theory, for example Batten (1978) and Benjamin and Pollard (1980), suggest that the estimate of q_x, $\overset{\circ}{q}_x$, given by

$$\theta_{(x)} = \overset{\circ}{l}_x \, \overset{\circ}{q}_x + \sum_r (b_{x+r} + n_{x+r} - w_{x+r} - e_{x+r}) f(r, 1, \overset{\circ}{q}_x) \qquad (2.3)$$

should be used. This is the same as Wittstein's approach but without the assumption of a uniform distribution of movements to and from, other than by death, the observed population. This approach will be referred to in this research as the 'conventional' approach to the calculation of the exposed-to-risk.

With this line of reasoning accepted as correct, much of the actuarial literature published in the nineteenth and twentieth centuries focussed on the more practical aspects such as the collection of data, and the development of methods to deal satisfactorily with data of a particular form. Whitall in 1893 (Bailey and Haycocks, 1947) was the first to distinguish between the three rate years, the so-called life year, policy year and calendar year rate intervals, used to describe the rate year appropriate to the form in which data are classified.

In order to calculate the exposed-to-risk, methods have been developed to suit both the form of the data and the degree of accuracy required. Three methods are generally employed in practice.

1. The 'continuous method', used when data are available grouped by numbers entering or exiting observation and according to nature of entry or exit.

2. The 'census method', used when census data of the population are available at intervals of time.

3. The 'direct method', used when detailed data are available for each life observed so that an exact exposure can be calculated for each life.

For a given method, the formula used to estimate the central exposed-to-risk is the same, irrespective of the assumed underlying force of mortality. The initial exposed-to-risk calculated, however, will depend on the assumption made. The differences between the central and initial exposed-to-risk under different assumptions are investigated in Chapter 4.

Insured lives and pension fund members are generally exposed to more than one decrement, for example, death, withdrawal, and, in the case of pension fund members, ill-health and age retirement as well. Analysis of multiple decrements appears to have commenced with Bernoulli's study of smallpox sufferers in 1766 (Seal, 1977) and was continued in the actuarial field with the study of pension fund financing. Statisticians refer to the subject of multiple decrements as 'the theory of competing risks', for example Nelson (1982), and have substantially extended this research in the context of Markov processes.

The statistical and actuarial fields have often developed independently of one another. For example, Chiang's proportionality assumption presented in 1961 to describe the relationship between independent and dependent rates was given explicitly by Greville in 1948 (Seal, 1977). Also, the Kaplan-Meier estimator was suggested by Böhmer to the International Congress of Actuaries in 1912 (Hoem, 1984) for use in actuarial applications, but has not been incorporated into the standard actuarial practice.

Biostatistics concentrates on the estimation of the survival function or force of mortality and hence uses methods which do not make prior assumptions about the form of the force of mortality. These methods involve plotting functions of the observed data, and comparing the results with those that would be obtained if alternative assumptions held true. The most appropriate assumption is then chosen. The methods focus on either the survival function or the 'hazard function', that is, the instantaneous failure rate.

The latter function corresponds with the force of decrement referred to in actuarial theory. Possible methods which are commonly used include the Herd-Johnson method, the Kaplan-Meier or product-limit method, the Nelson-Aalen hazard approach and various methods termed somewhat inappropriately 'actuarial methods'. Chiang (Seal, 1977) describes these methods and their biomedical applications.

Chapter 3

Estimation of q_x

3.1 An Attack on the Conventional Approach

In his paper 'A Flaw in Actuarial Exposed-to-Risk Theory', Hoem (1984) argues that the conventional approach to calculating the exposed-to-risk is flawed. He suggests, as others have before him (for example Seal (1977)), that the 'correct' approach requires knowledge of the 'maximum age' to which each individual was observed in the specified rate interval, or would have been observed had they not died. If we define this 'maximum age' to be, for the ith individual, $x + \alpha_i$, the expected number of deaths is given by

$$E[\theta_{(x)}] = \sum_i {}_{\alpha_i - s_i}q_{x+s_i} \tag{3.1}$$

The 'maximum age' is the age of exit of lives who do not die, but would generally be unknown for lives who die. Note that the estimate of q_x derived from equation (3.1), $\tilde{q}_x$, is obtained by substituting $\theta_{(x)}$ for $E[\theta_{(x)}]$ in this equation, that is

$$\theta_{(x)} = \sum_i f(s_i, \alpha_i, \tilde{q}_x)$$

This compares with the estimate obtained under the conventional approach, $\mathring{q}_x$, as given by equation (2.3). This equation can be written as

$$\theta_{(x)} = \sum_i f(s_i, 1, \mathring{q}_x) - \sum_{i:D_i=0} f(t_i, 1, \mathring{q}_x) \tag{3.2}$$

$$= \sum_i \Big(f(s_i, 1, \mathring{q}_x) - (1 - D_i) f(t_i, 1, \mathring{q}_x) \Big) \tag{3.3}$$

Hoem considers the development of moment relations using the three common assumptions, that is, the uniform, Balducci and constant force of mortality assumptions.

Hoem then analyses the conventional approach as described in actuarial texts, specifically Greville (1978). From this, he argues that the conventional approach is based on 'a faulty argument which involves the erroneously symmetric treatment of entrants to and exits from the study population during the period of investigation'. He suggests that the conventional approach is based on the notion that the 'general term' on the right hand side of equation (3.3), namely

$$f(s_i, 1, \mathring{q}_x) - (1 - D_i) f(t_i, 1, \mathring{q}_x)$$

represents the 'expected number of deaths' for the ith individual. He asserts that this expression is not the expected deaths for the ith individual.

Hoem goes on to describe what we have defined as the 'maximum age' of the individuals observed. Although equation (3.1) is obviously correct, he concedes that it requires knowledge of the 'maximum age' of the deaths. In the absence of this information, he suggests that an approximate method for deriving moments can be used to obtain estimates of q_x, namely that, 'for simplicity', α_i be taken as 1 for the deaths. In effect, this is equivalent to assuming that all the deaths would have been observed to age $x + 1$ had they not died.

He then proceeds to calculate 'wholly correct moment relations', using equation (3.1) under the uniform and Balducci assumptions. He shows that the resulting moments are different from those obtained using the conventional approach. However, he acknowledges that the use of $\alpha_i = 1$ for the deaths is a 'practical approximation', which is required in order to derive useful results.

Finally, Hoem derives maximum likelihood estimates of q_x which do not require any assumption about the maximum age of the deaths. He shows that the expressions derived under the uniform and Balducci assumptions are complicated and must be solved by numerical iteration. The maximum likelihood estimate found under the constant force of mortality assumption is, however, easily calculated and, in general, the asymptotic variance of the estimate is easy to estimate. He concludes that this maximum likelihood estimate is superior, for all practical purposes, to those obtained using the conventional approach, his simplified approach, and the maximum likelihood estimates obtained under the other two assumptions.

3.2 A Defence of the Conventional Approach

Hoem's assertion that the conventional approach is based on a sum of the 'expected number of deaths' for the ith individual over all individuals is fallacious. The actual form of the expected deaths under the conventional approach is more involved, as is shown below.

Hoem argues that the probability of survival from age of entry to age of exit has not been taken into account in the conventional approach. However, the conventional approach does allow for this probability since it uses the observed number of exits, which will be less than those that would have been observed had no mortality occurred. The quantity deducted is an expression applicable only to lives that we know have not died. Effectively, the conventional approach is conditional on what we have actually observed in the investigation.

To show that the approach of equation (3.1) is equivalent to the conventional approach we proceed as follows. Treating D_i as a random variable, the expected deaths of the ith individual is given by

$$E[D_i] = {}_{\alpha_i - s_i}q_{x+s_i} \tag{3.4}$$

and the expected total number of deaths is just the sum of equation (3.4) over all individuals, which is obviously the same as equation (3.1).

Suppose, however, that the N lives could be divided into M sub-groups, with all individuals in the jth group ($j = 1, \ldots, M$) having a common age at entry to observation, $x + s_j$, and a common 'maximum age', $x + \alpha_j$. Let N^j represent the number of lives in the jth group, and $\theta^j_{(x)}$ represent the observed number of deaths from the jth group. Thus

$$N = \sum_{j=1}^{M} N^j$$

and

$$\theta_{(x)} = \sum_{j=1}^{M} \theta^j_{(x)}$$

The expected number of deaths from the jth group is

$$E[\theta^j_{(x)}] = N^j{}_{\alpha_j - s_j}q_{x+s_j} \tag{3.5}$$

The expected total number of deaths is

$$\begin{aligned} E[\theta_{(x)}] &= \sum_{j=1}^{M} N^j{}_{\alpha_j - s_j}q_{x+s_j} \\ &= \sum_{j=1}^{M} N^j({}_{1-s_j}q_{x+s_j} - {}_{\alpha_j - s_j}p_{x+s_j}\,{}_{1-\alpha_j}q_{x+\alpha_j}) \end{aligned}$$

Since, from equation (3.5)

$$N^j{}_{\alpha_j - s_j}p_{x+s_j} = N^j - E[\theta^j_{(x)}]$$

we have that

$$E[\theta_{(x)}] = \sum_{j=1}^{M}(N^j{}_{1-s_j}q_{x+s_j} - (N^j - E[\theta^j_{(x)}])_{1-\alpha_j}q_{x+\alpha_j}) \tag{3.6}$$

By taking the expectation over the whole of the second summation we get

$$E[\theta_{(x)}] = \sum_{j=1}^{M} N^j{}_{1-s_j}q_{x+s_j} - E\left[\sum_{j=1}^{M}(N^j - \theta^j_{(x)})_{1-\alpha_j}q_{x+\alpha_j})\right]$$

Now $(N^j - \theta^j_{(x)})$ is the number of lives out of the N^j lives in the jth group who did not die, that is, those for which $D_i = 0$ so

$$E[\theta_{(x)}] = \sum_{j=1}^{M}\sum_{k=1}^{N^j} {}_{1-s_j}q_{x+s_j} - E\left[\sum_{j=1}^{M}\sum_{(i\in j:D_i=0)} {}_{1-\alpha_j}q_{x+\alpha_j}\right]$$

where the expression $(i \in j : D_i = 0)$ indicates those lives in group j who did not die. Thus

$$E[\theta_{(x)}] = \sum_i {}_{1-s_i}q_{x+s_i} - E\left[\sum_{i:D_i=0} {}_{1-\alpha_i}q_{x+\alpha_i}\right] \tag{3.7}$$

Since, for the lives who do not die, the 'maximum age' is just the actual age of exit, that is $\alpha_i = t_i$, we can express equation (3.7) as

$$E[\theta_{(x)}] = \sum_i {}_{1-s_i}q_{x+s_i} - E\left[\sum_{i:D_i=0} {}_{1-t_i}q_{x+t_i}\right] \tag{3.8}$$

The second term on the right-hand-side of equation (3.8) is the expected value of a function of the random variables D_i. Note that equation (3.7) can also be expressed as

$$E\left[\sum_i D_i\right] = \sum_i {}_{1-s_i}q_{x+s_i} - E\left[\sum_i (1-D_i)_{1-\alpha_i}q_{x+\alpha_i}\right]$$

which implies that

$$\sum_i E[D_i] = \sum_i {}_{1-s_i}q_{x+s_i} - \sum_i (1 - E[D_i])_{1-\alpha_i}q_{x+\alpha_i} \tag{3.9}$$

If, in equation (3.9), we substitute D_i for $E[D_i]$ $(i = 1, \ldots, N)$ we get

$$\begin{aligned}\sum_i D_i &= \sum_i f(s_i, 1, \overset{\circ}{q}_x) - \sum_i (1 - D_i) f(\alpha_i, 1, \overset{\circ}{q}_x) \\ &= \sum_i f(s_i, 1, \overset{\circ}{q}_x) - \sum_{i:D_i=0} f(t_i, 1, \overset{\circ}{q}_x)\end{aligned}$$

which is the conventional method of estimation as expressed in equation (3.2). Thus the approach of equation (3.1) and the conventional approach are logically consistent.

The conventional approach only requires knowledge of the α_i of lives who do not die. It is thus a conditional approach in that it uses information that we have observed, and, in so doing, avoids having to make assumptions about information that is not known. The expression on the right hand side of equation (3.8) is the correct interpretation for the expected deaths under the conventional approach.

Thus it is clear that Hoem's assertion that the conventional approach is flawed is incorrect. Notwithstanding this, the approach that he suggests will not in general produce useful results since the 'maximum age', $x + \alpha_i$, will usually be unknown in the case of the deaths. To make equation (3.1) useful in practice, an assumption has to be made about the value to be substituted for α_i in the case of the deaths before any estimates can be derived.

In the above analysis, we have assumed that the N lives could be placed into M groups, each group having a common 'maximum age'. Although the conventional approach does not require knowledge of the 'maximum age' for deaths it is however based on the underlying assumption that, had the deaths not died, they would have been observed on average to a similar 'maximum age' as those who did not die. Thus, the conventional approach is an acceptable practical approach, which allows direct calculation of the required estimates without requiring an explicit assumption about the 'maximum age' of the deaths.

3.3 Equivalent Average Maximum Age of Deaths

As discussed above, Hoem used $\alpha_i = 1$ as a 'practical approximation' in calculating 'wholly correct moment relations'. However, as is shown below, in order for equation (3.1) to yield correct results, the deaths would have to be assumed, on average, to have a common 'maximum age' which is less than $x + 1$. The only exception would be where all lives exiting from observation other than by death, exit at age $x + 1$ in which case the common 'maximum age' would be $x + 1$. Hoem's use of $\alpha_i = 1$ for the general situation thus explains why his 'correct' moments are substantially different from those calculated using the conventional approach.

In order to see this, suppose that there is a function β_i of D_i defined by

$$\beta_i = t_i \quad \text{if} \quad D_i = 0 \quad \text{and} \quad \beta_i = \beta \quad \text{if} \quad D_i = 1$$

such that

$$E[\theta_{(x)}] = \sum_i {}_{\beta_i - s_i}q_{x+s_i} \tag{3.10}$$

This can be expressed as

$$E\left[\sum_{i:D_i=1} 1\right] = E\left[\sum_{i:D_i=1} {}_{\beta-s_i}q_{x+s_i}\right] + E\left[\sum_{i:D_i=0} {}_{t_i-s_i}q_{x+s_i}\right]$$

The expectations must be introduced because the summation has been expressed as a function of the random variables, D_i. By rearranging, we see that

$$E\left[\sum_{i:D_i=1} {}_{\beta-s_i}p_{x+s_i}\right] = E\left[\sum_{i:D_i=0} {}_{t_i-s_i}q_{x+s_i}\right]$$

and therefore that

$$E\left[\sum_{i:D_i=1} \frac{{}_{\beta}p_x}{{}_{s_i}p_x}\right] = E\left[\sum_{i:D_i=0} {}_{t_i-s_i}q_{x+s_i}\right] \tag{3.11}$$

From the fact that

$$\sum_i \frac{1}{{}_{s_i}p_x} = E\left[\sum_{i:D_i=0} \frac{1}{{}_{t_i}p_x}\right]$$

(see Appendix A)

and rearranging, we get that

$$E\left[\sum_{i:D_i=1} \frac{1}{{}_{s_i}p_x}\right] = E\left[\sum_{i:D_i=0} \left(\frac{1}{{}_{t_i}p_x} - \frac{1}{{}_{s_i}p_x}\right)\right]$$

Substituting this in equation (3.11) we get

$${}_{\beta}p_x\, E\left[\sum_{i:D_i=0} \left(\frac{1}{{}_{t_i}p_x} - \frac{1}{{}_{s_i}p_x}\right)\right] = E\left[\sum_{i:D_i=0} {}_{t_i-s_i}q_{x+s_i}\right]$$

Dividing each side of this equation by

$$_{\beta}p_x E\left[\sum_{i:D_i=0} {}_{t_i-s_i}q_{x+s_i}\right]$$

gives

$$\frac{1}{_{\beta}p_x} = \frac{E\left[\sum_{i:D_i=0}\left(\frac{1}{_{t_i}p_x} - \frac{1}{_{s_i}p_x}\right)\right]}{E\left[\sum_{i:D_i=0}\left({}_{t_i-s_i}q_{x+s_i}\right)\right]}$$

However since

$$\frac{1}{_{t_i}p_x} - \frac{1}{_{s_i}p_x} = \frac{_{t_i-s_i}q_{x+s_i}}{_{t_i}p_x}$$

we have that

$$\frac{1}{_{\beta}p_x} = \frac{E\left[\sum_{i:D_i=0}\left({}_{t_i-s_i}q_{x+s_i}\left[\frac{1}{_{t_i}p_x}\right]\right)\right]}{E\left[\sum_{i:D_i=0}\left({}_{t_i-s_i}q_{x+s_i}\right)\right]} \tag{3.12}$$

Thus, it can be seen that $\frac{1}{_{\beta}p_x}$ is a weighted average of the $\frac{1}{_{t_i}p_x}$ for lives that are expected not to die and not, as might be expected, of those expected to die. This weighted average increases our understanding of the underlying treatment of the deaths. Effectively, it is assumed that the deaths, had they not died, would, on average, have been exposed to a similar 'maximum age' as the lives who did not die. Effectively, the above analysis derives an explicit expression of the assumption underlying the conventional approach.

3.4 Maximum Likelihood Estimation

The likelihood function for the distribution of deaths can be derived by applying survival analysis. The following is a brief explanation of the principles involved. A more detailed analysis can be found in Nelson (1982).

Suppose that the time from age 0 to failure is a random variable T, with probability density function $f(t)$ and that units are observed from age γ. Data are said to be complete if each unit is observed until it fails. If units are removed from observation before failure or are still running at the end of the period of investigation, their failure times are known only to be longer than their observed running times. This data is said to be right censored. If units come under observation at an age greater than γ so that, had the unit failed before coming under observation it would not have been recorded as a failure, the data are said to be left truncated.

A unit, observed from age $u \geq \gamma$, which fails at age $v > u$ contributes a term $f(v)/\mathcal{F}(u)$ to the likelihood, where

$$\mathcal{F}(u) = \int_u^\infty f(t)dt$$

A unit, observed from age $u \geq \gamma$, which is censored at age $v > u$ contributes a term $\mathcal{F}(v)/\mathcal{F}(u)$ to the likelihood. The full likelihood for N independent units is

$$\Lambda = \prod_F \frac{f(v_i)}{\mathcal{F}(u_i)} \prod_C \frac{\mathcal{F}(v_i)}{\mathcal{F}(u_i)}$$

where the products are taken over failed units and censored units respectively.

Now, in our investigation, lives commence the rate interval at age x, the ith life is observed from age $x + s_i$ to age $x + t_i$ and the following relations hold

$$f(v_i) = f(x+t_i) = {}_{x+t_i}p_0\mu_{x+t_i}$$
$$\mathcal{F}(u_i) = \mathcal{F}(x+s_i) = {}_{x+s_i}p_0$$
$$\mathcal{F}(v_i) = \mathcal{F}(x+t_i) = {}_{x+t_i}p_0$$

and

$$\Lambda = \prod_{i:D_i=1} {}_{t_i-s_i}p_{x+s_i}\mu_{x+t_i} \prod_{i:D_i=0} {}_{t_i-s_i}p_{x+s_i} \tag{3.13}$$

By making an assumption about the underlying force of mortality, differentiating equation (3.13) with respect to the parameter being estimated, that is, q_x, and equating the differential to zero, a maximum likelihood estimate of q_x can be derived. These maximum likelihood estimates of q_x are closely related to the estimates derived using the conventional approach. This is demonstrated in Appendix C for the three common assumptions.

1. The uniform assumption.

   The maximum likelihood estimator of q_x, $\hat{q}_x$, is given by

   $$\theta_{(x)} = \hat{q}_x \left[\sum_{i:D_i=0} \frac{t_i}{1-t_i\hat{q}_x} - \sum_i \frac{s_i}{1-s_i\hat{q}_x} \right]$$

   It is shown in Appendix C.1 that this maximum likelihood estimate is asymptotically equal to the estimate obtained under the conventional approach using equation (3.2).

2. The Balducci assumption.

   The maximum likelihood estimator of q_x, $\hat{q}_x$ is given by

   $$\theta_{(x)} = \hat{q}_x \left[\sum_i \frac{1-s_i}{1-(1-s_i)\hat{q}_x} - \sum_{i:D_i=0} \frac{1-t_i}{1-(1-t_i)\hat{q}_x} \right]$$

$$- \quad 2\hat{q}_x \sum_{i:D_i=1} \frac{1-t_i}{1-(1-t_i)\hat{q}_x}$$

It is shown in Appendix C.2 that this maximum likelihood estimate and the conventional estimate obtained from equation (3.2) under the Balducci assumption are asymptotically equal.

3. The constant force of mortality assumption.

   The maximum likelihood estimator of μ, $\hat{\mu}$, is given by

$$\theta_{(x)} = \hat{\mu} \sum_i (t_i - s_i)$$

   It is shown in Appendix C.3 that the maximum likelihood estimate and the conventional estimate of μ, as given by equation (A.7), are the same. The estimate of q_x obtained from this estimate is a maximum likelihood estimate of q_x.

Finally, since the Gompertz assumption can be used to describe the shape of the mortality curve over a large range of ages, the maximum likelihood estimates of B and c have also been derived and compared to those obtained under the conventional aproach. The maximum likelihood estimates of B and c, $\hat{B}$ and $\hat{c}$, are given by the simultaneous equations

$$\begin{aligned}
\theta_{(x)} &= \frac{\hat{B}}{\ln \hat{c}} \sum_i \hat{c}^x(\hat{c}^{t_i} - \hat{c}^{s_i}) \\
\sum_{i:D_i=1} (x+t_i) &= \frac{\hat{c}\hat{B}}{\ln \hat{c}} \sum_i \hat{c}^{x+s_i-1}((x+t_i)\hat{c}^{t_i-s_i} - (x+s_i)) \\
&- \frac{\hat{c}\hat{B}}{(\ln \hat{c})^2} \sum_i \hat{c}^{x+s_i-1}(\hat{c}^{t_i-s_i} - 1)
\end{aligned}$$

(A summary of the properties of this assumption is included in Appendix B.)

It is shown in Appendix C.4 that the maximum likelihood estimates are one possible pair of solutions from the set of solutions satisfying the conventional approach.

Chapter 4

The Relationship Between the Central and Initial Exposed-to-Risk

4.1 'Exposing to the End of the Rate Interval'

Under equation (3.1), the expected number of deaths to the ith individual is ${}_{\alpha_i - s_i}q_{x+s_i}$, that is, the probability of dying between age $x + s_i$ and $x + \alpha_i$, irrespective of whether or not the life died. This probability must be considered separately from the length of time taken into account for the ith individual when calculating the initial exposed-to-risk. The latter amount is dependent upon the assumption made about the force of mortality as is shown below.

It is often said that for the conventional approach the deaths should be 'exposed to the end of the rate interval'. For example, in an Institute of Actuaries's examiner's report (Institute of Actuaries, 1990; Subject 6, Question 1) this expression is used in the definition of the initial exposed-to-risk. As we have seen, under the Balducci assumption, the conventional estimate of q_x is given by, from equation (3.2),

$$\theta_{(x)} = \mathring{q}_x \left(\sum_i (1 - s_i) - \sum_{i:D_i=0} (1 - t_i) \right) \tag{4.1}$$

The initial exposed-to-risk can thus be expressed as

$$E_x = \left(\sum_{i:D_i=0} (t_i - s_i) + \sum_{i:D_i=1} (1 - s_i) \right)$$

For lives who do not die, the length of time taken into account when calculating the initial exposed-to-risk coincides with the period for which they were exposed to the risk of dying. For the deaths, the length of time taken into account in the initial exposed-to-risk is the period from age of entry to observation up until attainment of age $x + 1$. The latter age will coincide with or be greater than the individual's 'maximum age'. In particular, for a death whose 'maximum age' is determined by the end of the period of investigation, the period taken into account will extend beyond the end of the period of investigation. Another way of expressing this is that under the conventional approach in equation (4.1), no deduction is made for the deaths, but only for lives who withdraw or reach the end of the period of investigation.

As Batten (1978) points out, this result follows purely from the imposition of the Balducci assumption and a logical interpretation is not therefore necessarily possible, and is not in fact required. It is just a mathematical result. The statement that 'deaths should be exposed to the end of the rate year' is only applicable where the Balducci assumption has been made, and must be interpreted as meaning that the period taken into account for the deaths when estimating the initial exposed-to-risk should extend to the end of the rate interval.

It must be emphasised that deaths are not actually 'exposed' or even assumed to have been exposed to the end of the rate interval since they obviously cannot give rise to further deaths after they have died. In addition, if the deaths had not died they would only have been exposed to their 'maximum age'. 'Exposing to the end of the rate interval' must not be taken as being the same as assuming that $\alpha_i = 1$.

A fairly common variation of the above error is the assertion that all deaths should be given a full year's exposure in the initial exposed-to-risk, for example Benjamin and Pollard (1980, p. 38). This is patently incorrect, even under the Balducci assumption since if $s_i > 0$ for a life who dies, his or her contribution to the initial exposed-to-risk will be less than 1.

If an alternative assumption is made regarding the shape of the force of mortality, a different form for the contribution to the initial exposed-to-risk of the deaths will result (Dorrington, 1989). For example, under the uniform assumption, the conventional approach, becomes, from equation

(3.8),

$$E[\theta_{(x)}] = \sum_i \frac{(1-s_i)q_x}{1-s_iq_x} - E\left[\sum_{i:D_i=0} \frac{(1-t_i)q_x}{1-t_iq_x}\right] \tag{4.2}$$

Since

$$\frac{(1-r)q_x}{1-rq_x} = q_x\left[(1-r) + \frac{r(1-r)q_x}{1-rq_x}\right]$$

equation (4.2) can be re-written as

$$\begin{aligned} E[\theta_{(x)}] &= q_x\left(\sum_i\left((1-s_i) + \frac{s_i(1-s_i)q_x}{1-s_iq_x}\right)\right) \\ &- q_x\left(E\left[\sum_{i:D_i=0}\left((1-t_i) + \frac{t_i(1-t_i)q_x}{1-t_iq_x}\right)\right]\right) \\ &= q_x\left(\sum_i(1-s_i) - E\left[\sum_{i:D_i=0}(1-t_i)\right]\right) \\ &+ q_x\left(\sum_i\frac{s_i(1-s_i)q_x}{1-s_iq_x} - E\left[\sum_{i:D_i=0}\frac{t_i(1-t_i)q_x}{1-t_iq_x}\right]\right) \end{aligned} \tag{4.3}$$

The expression shown in equation (4.3) indicates that the period taken into account for the initial exposed-to-risk for each of the lives is the same as that under the Balducci assumption, with an adjustment for those expected to die between the age of entry and age $x+1$, and those expected to die between the age at exit and age $x+1$ for those who exit other than by death.

4.2 The Form of $E_x - E_x^c$ for a Single Decrement

We indicated in Chapter 2 that the difference between the estimates obtained for a given population using different assumptions about the underlying force of mortality will, for most practical purposes, be negligible. This is due to the fact that the force of mortality under the different assumptions changes little over a single year of age. The expected deaths between age $x+r$ and $x+r+\delta r$ are given by

$$E[\theta_{x+r}]\delta r = E[\overset{\circ}{l}_{x+r}]\mu_{x+r}\delta r$$

Thus, if μ_{x+r} varies little for $0 \leq r < 1$, the expected distribution of deaths over the rate interval is virtually exclusively determined by the shape of $E[\overset{\circ}{l}_{x+r}]$.

The central exposed-to-risk is given by

$$E_x^c = \sum_i (t_i - s_i)$$

Commonly it is assumed that the underlying force of mortality is Balducci. In this instance, the initial exposed-to-risk is given by

$$E_x = \sum_i (1 - s_i) - \sum_{i:D_i=0} (1 - t_i)$$

and, therefore,

$$E_x - E_x^c = \sum_{i:D_i=1} (1 - t_i)$$

The expected value of this difference can be expressed as

$$\begin{aligned} E\left[\sum_{i:D_i=1}(1-t_i)\right] &= \int_0^1 E[\theta_{x+r}](1-r)\,dr \\ &= \int_0^1 E[\overset{\circ}{l}_{x+r}]\mu_{x+r}(1-r)\,dr \end{aligned} \tag{4.4}$$

where the integral can be expressed as the sum of several integrals over each interval in which $E[\overset{\circ}{l}_{x+r}]$ is continuous.

If we assume that expected deaths occur uniformly over the rate interval, that is that

$$E[\theta_{x+r}]\delta r = E[\theta_{(x)}]\delta r$$

then equation (4.4) becomes

$$\begin{aligned} E[E_x - E_x^c] &= E[\theta_{(x)}]\int_0^1 (1-r)\,dr \\ &= \frac{1}{2}E[\theta_{(x)}] \end{aligned}$$

The difference $E_x - E_x^c$ can thus be approximated by $\frac{1}{2}\theta_{(x)}$. This is the most commonly used approximation and is correct on the assumption that the Balducci assumption holds and

$$E[\overset{\circ}{l}_{x+r}]\mu_{x+r}\delta r \quad = \quad E[\theta_{(x)}]\delta r$$

which, under the Balducci assumption, implies

$$\begin{aligned} E[\overset{\circ}{l}_{x+r}]\delta r &= E[\theta_{(x)}]\frac{(p_x + rq_x)}{q_x}\delta r \\ &= \left(E[\theta_{(x)}]\frac{p_x}{q_x} + E[\theta_{(x)}]r\right)\delta r \end{aligned} \tag{4.5}$$

If the observed population between age $x+r$ and $x+r+\delta r$ does not vary approximately linearly with r as defined by equation (4.5) above, the assumption that

$$E_x = E_x^c + \frac{1}{2}\theta_{(x)} \tag{4.6}$$

may not be reasonable.

In order for the assumption that

$$E_x = E_x^c + \frac{1}{2}\theta_{(x)}$$

to be reasonable under the uniform, constant force of mortality or Gompertz assumption in the underlying life table, together with the assumption of a uniform distribution of deaths over the rate interval, it can be shown that the required shape of $E[\mathring{l}_{x+r}]$ is

$$\begin{aligned}
E[\mathring{l}_{x+r}] &= E[\theta_{(x)}]\frac{(1-rq_x)}{q_x} \\
&= \frac{E[\theta_{(x)}]}{q_x} - E[\theta_{(x)}]r, \\
E[\mathring{l}_{x+r}] &= \frac{E[\theta_{(x)}]}{\mu}, \text{ and} \\
E[\mathring{l}_{x+r}] &= E[\theta_{(x)}]Bc^{-(x+r)}
\end{aligned}$$

respectively. For most practical purposes, these expressions for $E[\mathring{l}_{x+r}]$ will differ little from a constant value.

Since in most investigations, one will have little idea about the shape of $\mathring{l}_{x+r}$, it is more useful to consider the implications of the above for the expected movements. Since

$$\overset{\circ}{l}_{x+r+\delta r}=\overset{\circ}{l}_{x+r}+\sum_{s=r}^{r+\delta r}(b_{x+s}+n_{x+s}-w_{x+s}-e_{x+s}-\theta_{x+s})$$

the expected value of the change in $\overset{\circ}{l}_{x+r}$ over the interval of age $x+r$ to $x+r+\delta r$ can be expressed as

$$\begin{aligned} E[\overset{\circ}{l}_{x+r+\delta r}-\overset{\circ}{l}_{x+r}] &= \sum_{s=r}^{r+\delta r}(b_{x+s}+n_{x+s}) \\ &- E\left[\sum_{s=r}^{r+\delta r}(\theta_{x+r}+w_{x+r}+e_{x+r})\right] \end{aligned} \tag{4.7}$$

Suppose it is assumed that the underlying force of mortality is Balducci and that deaths are distributed uniformly over the rate interval. Then, from equation (4.5)

$$E[\overset{\circ}{l}_{x+r+\delta r}]-E[\overset{\circ}{l}_{x+r}]=E[\theta_{(x)}]\delta r \tag{4.8}$$

Equations (4.7) and (4.8) together imply that

$$\sum_{s=r}^{r+\delta r}(b_{x+s}+n_{x+s})-E\left[\sum_{s=r}^{r+\delta r}(w_{x+r}+e_{x+r})\right]=2E[\theta_{(x)}]\delta r$$

since, if deaths are uniformly distributed over the rate interval,

$$E\left[\sum_{s=r}^{r+\delta r}\theta_{x+r}\right]=\theta_{(x)}\delta r$$

Thus in order for the assumptions to be reasonable, expected net movements other than death in any interval of age $x+r$ to $x+r+\delta r$ should be approximately twice the number of deaths in that interval. Similarily it can be shown for the other assumptions about the underlying force of mortality, that with an assumption of a uniform distribution of deaths over the rate interval the required relationship is :

1. The Uniform Distribution.

$$\sum_{s=r}^{r+\delta r}(b_{x+s}+n_{x+s})-E\left[\sum_{s=r}^{r+\delta r}(w_{x+r}+e_{x+r})\right]=0$$

that is, that expected net movements other than death in the interval of age $x+r$ to $x+r+\delta r$ should approximately cancel each other out in that interval.

2. The Constant Force of Mortality Assumption.

$$\sum_{s=r}^{r+\delta r}(b_{x+s}+n_{x+s})-E\left[\sum_{s=r}^{r+\delta r}(w_{x+r}+e_{x+r})\right]=E[\theta_{(x)}]\delta r$$

that is, that expected net movements other than death in the interval of age $x+r$ to $x+r+\delta r$ should be approximately equal to the expected number of deaths in that interval.

3. The Gompertz Assumption.

$$\begin{aligned}&\sum_{s=r}^{r+\delta r}(b_{x+s}+n_{x+s})-E\left[\sum_{s=r}^{r+\delta r}(w_{x+r}+e_{x+r})\right]\\ =\ &E[\theta_{(x)}]\delta r+E[\theta_{(x)}]Bc^{-(x+r)}(c^{-\delta r}-1)\end{aligned}$$

4.3 The Form of $E_x - E_x^c$ for Two Decrements

Suppose that we consider two decrements, death and withdrawal. We are now assuming that the lives are exposed to two decrements. As in the single decrement case, b_{x+r} and n_{x+r} would not take on different values if a different pattern of deaths and withdrawals were to be observed. The values of

$\mathring{l}_{x+r}$ and e_{x+r} would vary with different patterns of death and withdrawal, so that the values that we observe are in fact observed values of random functions with respect to the decrements of death and withdrawal.

Let $w_{(x)}$ be the total number of withdrawals during the period of investigation aged x last birthday at date of exit. Let the independent initial exposed-to-risk for death and withdrawal respectively, be E_x^θ and E_x^w, and let the dependent initial exposed-to-risk be E_x. In the following, subscripts or superscripts θ and w will be used to denote functions relating to mortality and withdrawal respectively. Let the relationship between the independent initial and the central exposed-to-risk for the two decrements considered separately as single decrements be expressed as

$$\begin{aligned} E_x^\theta &= E_x^c + f_\theta, \quad \text{and} \\ E_x^w &= E_x^c + f_w \end{aligned}$$

The dependent overall decrement is

$$(aq)_x = q_x^\theta + q_x^w - q_x^\theta q_x^w$$

This is estimated by

$$\begin{aligned} \frac{\theta_{(x)} + w_{(x)}}{E_x} &= \frac{\theta_{(x)}}{E_x^c + f_\theta} + \frac{w_{(x)}}{E_x^c + f_w} - \frac{\theta_{(x)} w_{(x)}}{(E_x^c)^2 + E_x^c f_\theta + E_x^c f_w + f_\theta f_w} \\ &= \frac{\theta_{(x)} + w_{(x)} + \big(\theta_{(x)} f_w + w_{(x)} f_\theta - \theta_{(x)} w_{(x)}\big)/E_x^c}{E_x^c + f_\theta + f_w + f_\theta f_w / E_x^c} \end{aligned} \tag{4.9}$$

Suppose that the assumptions made about the underlying forces of mortality and withdrawal in the underlying single decrement tables, and the assumptions made about the distribution of deaths and withdrawals over the rate intervals were such that we could assume that

$$f_\theta = \frac{1}{2}\theta_{(x)} \quad \text{and} \quad f_w = \frac{1}{2}w_{(x)}$$

Equation (4.9) then becomes

$$\frac{\theta_{(x)} + w_{(x)}}{E_x} = \frac{\theta_{(x)} + w_{(x)}}{E_x^c + \frac{1}{2}\theta_{(x)} + \frac{1}{2}w_{(x)} + \frac{\frac{1}{4}\theta_{(x)}w_{(x)}}{E_x^c}}$$

This is the relationship usually presented (for example Puzey (1987)). However, as is shown below, this result may, in some circumstances, be unreasonable on the chosen assumptions.

If we assume that the underlying force of mortality in the single decrement table is Balducci and that deaths occur uniformly over the rate interval in the double decrement table, that is, that

$$E[\theta_{x+r}]\delta r = E[\theta_{(x)}]\delta r$$

then, similarily to the single decrement case, this implies that

$$E[\mathring{l}_{x+r}] = E[\theta_{(x)}]\frac{p_x^\theta}{q_x^\theta} + E[\theta_{(x)}]r \tag{4.10}$$

Thus, the population attaining age $x+r$ to $x+r+\delta r$ should vary approximately linearly with $r \in [0,1)$ and with a positive gradient.

If we assume that the underlying force of withdrawal in the single decrement table is Balducci or Uniform, and that withdrawals occur uniformly over the rate interval in the double decrement table, then the expected population attaining between age $x+r$ and $x+r+\delta r$ will be, respectively,

$$E[\overset{\circ}{l}_{x+r}]\delta r = \left(E[w_{(x)}]\frac{p_x^w}{q_x^w} + E[w_{(x)}]r\right)\delta r \ , \text{ and} \tag{4.11}$$

$$E[\overset{\circ}{l}_{x+r}]\delta r = \left(\frac{E[w_{(x)}]}{q_x^w} - E[w_{(x)}]r\right)\delta r \tag{4.12}$$

theta is, that $\overset{\circ}{l}_{x+r}$ should vary approximately linearly with $r \in [0,1)$ and with a positive, or negative gradient respectively. In order for both the assumptions underlying equations (4.10) and (4.11) to apply, we would require that

$$\begin{aligned} E[\theta_{(x)}] &= E[w_{(x)}] \quad , \text{ and} \\ q_x^\theta &= q_x^w \end{aligned}$$

which is unlikely to be the case in practice. Suppose that the assumptions of a Balducci underlying force of mortality and force of withdrawal, and a uniform distribution of deaths over the rate interval are considered to be reasonable. The expected distribution of withdrawals is then constrained to, from equation (4.10),

$$\begin{aligned} E[w_{x+r}] &= E[\overset{\circ}{l}_{x+r}]\mu_{x+r}^w \\ &= \left(E[\theta_{(x)}]\frac{p_x^\theta}{q_x^\theta} + E[\theta_{(x)}]r\right)\mu_{x+r}^w \\ &= \left(E[\theta_{(x)}]\frac{p_x^\theta}{q_x^\theta} + E[\theta_{(x)}]r\right)\frac{q_x^w}{(p_x^w + rq_x^w)} \end{aligned}$$

which is clearly not uniform with repect to r. However, if q_x^θ and w_x^w are of similar magnitude, then w_{x+r} will be approximately uniform for $r \in [0,1)$.

Suppose now that we used the assumption underlying equations (4.10) and (4.12). Clearly the expressions underlying these equations are not consistent since one assumes an increasing population aged $x+r$, while the other assumes a decreasing population aged $x+r$. In practice, the magnitude of the difference between a positive gradient of $E[\theta_{(x)}]$ and a negative gradient of $E[w_{(x)}]$ would be small, but the results would not be theoretically sound.

The assumptions made about the underlying forces of decrement in the single decrement tables and the distributions of the decrements over the rate intervals can not be considered in isolation from each other; the relationships between the assumptions must be recognised. The most significant assumptions should be chosen first, with the other assumptions constrained by those already chosen. The latter assumptions should, however still be checked to ensure that they are reasonable.

The form of the dependent initial exposed-to-risk is, in fact, totally dependent on the assumptions made. If we assumed that the dependent overall rate of decrement was Balducci, that is that

$$_{1-r}(aq)_{x+r} = (1-r)(aq)_x$$

then the dependent initial exposed to risk will be of the form

$$E_x \quad = \quad \overset{\circ}{l}_x + \sum_r (b_{x+r} + n_{x+r} - e_{x+r})(1-r)$$

$$= E_x^c + \sum_r \theta_{x+r}(1-r) + \sum_r w_{x+r}(1-r)$$

In this case the expected value of the dependent initial exposed-to-risk is

$$E[E_x] = E[E_x^c] + \int_0^1 E[\theta_{x+r} + w_{x+r}](1-r)dr$$

If it is expected that the combined decrement of death and withdrawal is uniformly distributed over the rate interval, then

$$E[E_x] = E[E_x^c] + \frac{1}{2}E[\theta_x + w_x]$$

and the estimate of E_x will be

$$E_x^c + \frac{1}{2}(\theta_{(x)} + w_{(x)}) \qquad (4.13)$$

The form of f_θ and f_w will depend on the form assumed for the underlying forces of mortality and withdrawal and the assumed distributions of deaths and withdrawals over the rate interval. The combined effect of the two decrements must result in a Balducci distribution for the overall decrement. It is not possible for the underlying distributions, in the single decrement tables, of both death and withdrawal to be Balducci since it can be shown that if both underlying decrements are Balducci then

$$_{1-r}(aq)_{x+r} = (1-r)(aq)_x + (1-r)rq_{(x)}^w q_x^\theta$$

If the assumption of a Balducci overall rate of decrement is considered reasonable, then the most simple approach will be to assume that the initial independent exposed-to-risk has the form defined by equation (4.13).

4.4 The Form of $E_{x,t} - E^c_{x,t}$ for Select Rates

Define the following notation

$q_{x,t}$ = probability that a life aged x exact with duration t exact will die before attaining age $x+1$ exact with duration $t+1$ exact.

$\overset{\circ}{l}_{x+r,t+s}$ = number of lives attaining exact age $x+r$ with duration $t+s$ exact during the period of investigation.

$b_{x+r,t+s}, e_{x+r,t+s}$ = number of lives at the start and end, respectively, of the period of investigation, then aged $x+r$ exact with duration $t+s$ exact.

$w_{x+r,t+s}$ = number of withdrawals during the period of investigation aged $x+r$ exact with duration $t+s$ exact at the date of withdrawal.

$\theta_{x+r,t+s}$ = number of deaths during the period of investigation aged $x+r$ exact with duration $t+s$ exact at the date of death.

$\theta_{(x,t)}$ = number of deaths during the period of investigation aged x last birthday with curtate duration t at the date of death.

$=\sum_r \sum_s \theta_{x+r,t+s}$

By analogy with the ultimate functions, $b_{x+r,t+s}$, $e_{x+r,t+s}$, $w_{x+r,t+s}$ and $\theta_{x+r,t+s}$ will be discrete functions of $(r,s) \in [(0,0),(1,1))$, while $\overset{\circ}{l}_{x+r,t+s}$ will be a function of (r,s) with a finite number of discontinuities on the interval $[(0,0),(1,1))$.

For a life aged $x+r$ exact with duration $t+s$ exact, $0 \leq r, s < 1$, let the probability of dying aged x last birthday and curtate duration t be defined by

$$_{1-max(r,s)}q_{x+r,t+s} = \left(1 - max(r,s)\right)q_{x,t} \tag{4.14}$$

that is, the probability of dying before attaining the sooner of age $x+1$ exact and duration $t+1$ exact, is proportional to the outstanding period during which the life will be aged x last birthday and curtate duration t. This is similar to the Balducci assumption made for ultimate rates.

In section 4.2 it was explained that the assumption made about the underlying force of mortality has a negligible effect on the magnitude of the estimate of the rate of decrement. Although the assumption expressed in equation (4.14) may seem unreasonable, its effect will, in the same way, be small. This assumption was chosen because it greatly simplifies and yet does not materially affect the magnitude of the estimate derived below.

Using the conventional approach, the expected number of deaths aged x last birthday and with duration t curtate is

$$\begin{aligned} E[\theta_{(x,t)}] &= \int_0^1 E[\overset{\circ}{l}_{x,t+s}]_{1-s}q_{x,t+s}ds \ + \int_0^1 E[\overset{\circ}{l}_{x+r,t}]_{1-r}q_{x+r,t}dr \\ &+ \sum_r \sum_{s \leq r} \left(b_{x+r,t+s} - E\left[w_{x+r,t+s} + e_{x+r,t+s}\right]\right) {}_{1-r}q_{x+r,t+s} \\ &+ \sum_r \sum_{s > r} \left(b_{x+r,t+s} - E\left[w_{x+r,t+s} + e_{x+r,t+s}\right]\right) {}_{1-s}q_{x+r,t+s} \end{aligned}$$

Under the assumption in equation (4.14), the conventional estimate of $q_{x,t}$, denoted $\mathring{q}_{x,t}$, is given by

$$
\begin{aligned}
\theta_{(x,t)} &= \int_0^1 \mathring{l}_{x,t+s}\,(1-s)\,\mathring{q}_{x,t}\,ds + \int_0^1 \mathring{l}_{x+r,t}\,(1-r)\,\mathring{q}_{x,t}\,dr \\
&+ \sum_r \sum_{s\le r} (b_{x+r,t+s} - w_{x+r,t+s} - e_{x+r,t+s})(1-r)\,\mathring{q}_{x,t} \\
&+ \sum_r \sum_{s>r} (b_{x+r,t+s} - w_{x+r,t+s} - e_{x+r,t+s})(1-s)\,\mathring{q}_{x,t}
\end{aligned}
$$

The initial exposed-to-risk is, therefore,

$$
\begin{aligned}
E_{x,t} &= \int_0^1 \mathring{l}_{x,t+s}\,(1-s)ds + \int_0^1 \mathring{l}_{x+r,t}\,(1-r)dr \\
&+ \sum_r \sum_{s\le r} (b_{x+r,t+s} - w_{x+r,t+s} - e_{x+r,t+s})(1-r) \\
&+ \sum_r \sum_{s>r} (b_{x+r,t+s} - w_{x+r,t+s} - e_{x+r,t+s})(1-s)
\end{aligned}
$$

From the form of this equation, specifically the treatment of exits other than by death, it is obvious that the relationship between the initial and central exposed-to-risk is

$$
E_{x,t} - E^c_{x,t} = \sum_r \sum_{s\le r} (1-r)\theta_{x+r,t+s} + \sum_r \sum_{s>r} (1-s)\theta_{x+r,t+s} \quad (4.15)
$$

On the assumption that expected deaths are independently uniformly distributed over the age and duration rate intervals, that is, that

$$
E[\theta_{x+r,t+s}]\delta r \delta s = E[\theta_{(x,t)}]\delta r \delta s
$$

the expected value of the difference in equation (4.15) becomes

$$\begin{aligned}
E[E_{x,t} - E^c_{x,t}] &= \int_0^1 \int_0^r (1-r)E[\theta_{x+r,t+s}]dsdr + \int_0^1 \int_r^1 (1-s)E[\theta_{x+r,t+s}]dsdr \\
&= E[\theta_{(x,t)}] \left(\int_0^1 (r - r^2)dr + \int_0^1 (\frac{1}{2} - r + \frac{1}{2}r^2)dr \right) \\
&= \frac{1}{3}E[\theta_{(x,t)}] \qquad (4.16)
\end{aligned}$$

and therefore $E[E_{x,t} - E^c_{x,t}]$ will be estimated by $\frac{1}{3}\theta_{(x,t)}$. Thus, using assumptions similar to those used for ultimate rates we have derived a relationship between the initial and central exposed-to-risk for select rates. The relationship derived is not, as is often suggested, (for example Institute of Actuaries, (1988), Subject 6, Question 4; and Puzey (1987)) $\frac{1}{2}\theta_{(x,t)}$. In order to see that the fraction should intuitively be less than $\frac{1}{2}$, consider the following explanation.

For ultimate rates, in the absence of beginners, new entrants, withdrawals and enders, each life will be observed for a full year from age x to $x+1$, unless they die in that interval. For select rates, in the absence of beginners, withdrawals and enders, each life will only be observed for a full year if their birthday and policy anniversary coincide. It is unlikely that birthdays and policy anniversaries will occur on the same day for all lives. We often make the assumption that birthdays and policy anniversaries are independent of each other although in practice we would probably observe some relationship between them. If an individual's birthday and policy anniversary do not coincide, they will enter a new rate interval at each birthday and policy anniversary. The maximum possible period that the individual could be observed in each rate interval is thus dependent on the interval between their birthday and policy anniversary.

Now, for a death to be included in $\theta_{(x,t)}$ the life, at the time of death, must be both aged x last birthday and duration t curtate. At any time, a life will continue to be classified as 'aged x' with 'duration t' for a maximum period up until the sooner of the next birthday or policy anniversary. At the time of this event, the life changes classification, that is, enters a new rate interval. For ultimate rates, at a given time the life would continue to be classified as 'aged x' for a maximum period up to the next birthday. This maximum period for ultimate rates would be expected to be longer than the maximum period under select rates given that policy anniversaries and birthdays do not coincide for all the lives.

A reasonable estimate of the relationship between the central and initial exposed-to-risk under ultimate rates is $\frac{1}{2}\theta_{(x)}$. It is to be expected that the adjustment to the central exposed-to-risk to get the initial exposed-to-risk under select rates would be less than the comparable adjustment under ultimate rates since at any time lives will continue to be exposed in the same rate interval for a shorter time under select rates than under ultimate rates. Thus, we would expect the adjustment under select rates to be less than $\frac{1}{2}\theta_{(x,t)}$. If our assumption of deaths distributed independently uniformly over the rate years is reasonable, the derivation above indicates that the appropriate adjustment would be of the order of $\frac{1}{3}\theta_{(x,t)}$.

Chapter 5

Areas of Uncertainty

5.1 Bias of Estimates

Equation (3.1) and the conventional approach both yield unbiased estimates of the expected number of deaths. Seal (1954) states that the estimates of q_x found using equation (3.1) will be unbiased only when a 'rather dubious' assumption is made about the form of the force of decrement, namely

$$_{\alpha_i - s_i}q_{x+s_i} = (\alpha_i - s_i)q_x$$

Since this will only be true if $q_x = 0$[1] , the assumption can only be considered to be approximate. Hence, for all practical purposes, equation (3.1) will yield biased estimates of q_x.

[1]This assumption can be expressed, for some $s_i \leq u_i \leq \alpha_i$, as

$$\begin{aligned} _{\alpha_i - s_i}q_{x+s_i} &= {}_{u_i - s_i}q_{x+s_i} + {}_{u_i - s_i}p_{x+s_i}\ {}_{\alpha_i - u_i}q_{x+u_i} \\ &= (u_i - s_i)q_x + (1 - (u_i - s_i)q_x)(\alpha_i - u_i)q_x \\ &= (\alpha_i - s_i)q_x - (u_i - s_i)(\alpha_i - u_i)(q_x)^2 \end{aligned}$$

which is only true for all u_i if $q_x = 0$.

Breslow and Crowley (1974) show that the estimate

$$\frac{\theta_{(x)}}{N}$$

is an unbiased estimator of q_x when $s_i = 0$ and $\alpha_i = 1$ for all lives under investigation. For any observed population, we could base our estimate for q_x only on the sub-group of lives for whom $s_i = 0$ and $\alpha_i = 1$. This estimate, which Kaplan and Meier refer to as the 'reduced sample estimate' (Breslow and Crowley, 1974) is unbiased. Breslow and Crowley consider only the situation where $s_i = 0$ for all lives, there are no enders and the estimate calculated is

$$\frac{\theta_{(x)}}{N - \frac{1}{2}w_{(x)}}$$

They show that observing lives with $s_i > 0$ and/or $\alpha_i < 1$ can introduce bias into this estimate. Their reasoning can be extended to the conventional approach under the Balducci assumption as follows.

Suppose that the $\overset{\circ}{l}_x$ lives could be split into $\overset{\circ}{l}{}^1_x$ lives for whom $\alpha_i = 1$ and the balance,

$$\overset{\circ}{l}{}^2_x = \overset{\circ}{l}_x - \overset{\circ}{l}{}^1_x$$

for whom $\alpha_i < 1$. Similarily, let $\theta^1_{(x)}$ be all those deaths during the period of investigation for whom $s_i = 0$ and $\alpha_i = 1$ and let

$$\theta^2_{(x)} = \theta_{(x)} - \theta^1_{(x)}$$

be the balance of the deaths at age x last birthday. Under the assumption of a Balducci underlying force of mortality the conventional estimate of q_x

is given by

$$\overset{\circ}{q}_x = \frac{\theta_{(x)}}{\overset{\circ}{l}_x + \sum_r (b_{x+r} + n_{x+r} - w_{x+r} - e_{x+r})(1-r)} = \frac{\theta_{(x)}}{E_x}$$

This can be expressed as

$$\begin{aligned} \overset{\circ}{q}_x &= \frac{\theta^1_{(x)} + \theta^2_{(x)}}{\overset{\circ}{l}{}^1_x + \overset{\circ}{l}{}^2_x + \sum_r (b_{x+r} + n_{x+r} - w_{x+r} - e_{x+r})(1-r)} \\ &= \left(\frac{\overset{\circ}{l}{}^1_x}{E_x}\right)\left(\frac{\theta^1_{(x)}}{\overset{\circ}{l}{}^1_x}\right) + \left(\frac{E_x - \overset{\circ}{l}{}^1_x}{E_x}\right)\left(\frac{\theta^2_{(x)}}{E_x - \overset{\circ}{l}{}^1_x}\right) \end{aligned}$$

Now

$$\frac{\theta^1_{(x)}}{\overset{\circ}{l}{}^1_x}$$

is the reduced sample estimate. Thus, the conventional estimate is a weighted average of the unbiased reduced sample estimate and an estimate based only on lives who could be observed for a maximum period less than one year. Any bias in our conventional estimates therefore results from the fact than some individuals would be observed for less than the full life year even if they did not die.

Roberts (1987) attempted to investigate the bias in the estimates of q_x and m_x under different assumptions about the force of mortality. He defined "expected instantaneous m- and q-type rates' at age $x + r$ as

$$E[\overset{\circ}{m}(r)] = \frac{\mu_{x+r}\, \overset{\circ}{l}_{x+r}\, dr}{\overset{\circ}{l}_{x+r}\, dr} = \mu_{x+r} \tag{5.1}$$

$$E[\overset{\circ}{q}(r)] = \frac{\mu_{x+r}\, \overset{\circ}{l}_{x+r}\, dr}{\overset{\circ}{l}_{x+r}\, dr + \mu_{x+r}\, \overset{\circ}{l}_{x+r}\, (1-r)dr} \tag{5.2}$$

$$= \frac{\mu_{x+r}}{1 + \mu_{x+r}(1-r)} \tag{5.3}$$

It is obvious that the denominator of equation (5.2) is, in fact, derived on the basis of the Balducci assumption yet Roberts continues his analysis by considering the form of equations (5.1) and (5.2) under the constant force of mortality, Balducci, uniform and other assumptions. He is thus guilty of the error described in section 4.1, namely adding exposure time for deaths to the end of the rate interval in the general case, although this is only applicable under the Balducci assumption. The denominator of equation (5.2) should be an instantaneous initial exposure whose form depends, as shown in section 4.1, on the assumption made about the underlying force of mortality.

Notwithstanding this, Roberts assumes the exposure at age $x+r$ to be non-random and, further that the expectation operator can, without much error, be assumed to act separately on the numerator and denominator of equations (5.1) and (5.2). These two assumptions effectively assume away bias in the estimate of q_x which makes Roberts' conclusion that the Balducci estimate of q_x is unbiased erroneous. This is demonstrated below.

The estimate of q_x under the conventional approach with the Balducci assumption is given by

$$\begin{aligned} \overset{\circ}{q}_x &= \frac{\theta_{(x)}}{\left(\sum_i (1-s_i) - \sum_{i:D_i=0} (1-t_i)\right)} \quad \text{or} \qquad (5.4) \\ \theta_{(x)} &= \left(\sum_i (1-s_i) - \sum_{i:D_i=0} (1-t_i)\right) \overset{\circ}{q}_x \end{aligned}$$

$$= E_x \mathring{q}_x$$

Now, since E_x and $\mathring{q}_x$ are obviously not independent, the expected deaths must be expressed as

$$E[\theta_{(x)}] = E\left[\left(\sum_i (1-s_i) - \sum_{i:D_i=0}(1-t_i)\right)\mathring{q}_x\right] \tag{5.5}$$

If the expectation operator could be taken over numerator and denominator separately, as Roberts indicates, the expected value of equation (5.4) would become

$$\begin{aligned} E[\mathring{q}_x] &= \frac{E[\theta_{(x)}]}{E\left[\sum_i(1-s_i) - \sum_{i:D_i=0}(1-t_i)\right]} \quad \text{or} \\ E[\theta_{(x)}] &= E[\mathring{q}_x]E\left[\sum_i(1-s_i) - \sum_{i:D_i=0}(1-t_i)\right] \end{aligned} \tag{5.6}$$

Comparing equation (5.6) to equation (5.5) shows that the implication of taking the expectation over numerator and denominator separately is that

$$E\left[\mathring{q}_x\left(\sum_i(1-s_i) - \sum_{i:D_i=0}(1-t_i)\right)\right] = E[\mathring{q}_x]E\left[\sum_i(1-s_i) - \sum_{i:D_i=0}(1-t_i)\right]$$

that is, that $\mathring{q}_x$ and E_x are independent. The correct conclusion from Roberts' analysis is that the error introduced by assuming that the expectation can be taken separately over numerator and denominator is small, and therefore that the bias in the Balducci estimate is small.

It is more useful to consider the following analysis. Using the approach of equation (3.1) with the Balducci assumption, we have that

$$E[\theta_{(x)}] = \sum_i \frac{(\alpha_i - s_i)q_x}{1-(1-\alpha_i)q_x}$$

It can be seen that if $\alpha_i = 1$ for all i, the estimate of q_x obtained will be unbiased. Therefore, under this approach, if there are any members who are withdrawals or enders, the estimate of q_x obtained will be biased.

Using the approach of equation (3.1) under the uniform assumption gives

$$E[\theta_{(x)}] = \sum_i \frac{(\alpha_i - s_i)q_x}{1-s_i q_x}$$

Thus, if $s_i = 0$ for all i, the estimate of q_x obtained will be unbiased. However, in this case, the estimate is a function of the α_i of all lives. If, as is usual, α_i is unknown for the deaths, this estimate cannot be calculated.

Using the conventional approach as expressed in equation (3.2) with the Balducci and uniform assumptions respectively, we have that

$$\begin{aligned} E[\theta_{(x)}] &= \left(\sum_i (1-s_i) - E\left[\sum_{i:D_i=0}(1-t_i)\right]\right) q_x, \text{ and} \\ E[\theta_{(x)}] &= \left(\sum_i \frac{(1-s_i)}{1-s_i q_x} - E\left[\sum_{i:D_i=0}\frac{(1-t_i)}{1-t_i q_x}\right]\right) q_x \text{ respectively} \end{aligned}$$

Under the Balducci assumption, the estimate will again be unbiased if $\alpha_i = 1$ for all i. Note that it is not sufficient in this case to have $\alpha_i = 1$ only for lives who do not die since the expected value is a function of the random variables D_i; α_i must be one for any life who could have died, and so must be one for all lives. Under the uniform assumption, it can be seen that the estimate of q_x will be biased if there are any lives with $s_i > 0$ or $\alpha_i < 1$.

The estimate of μ under the conventional approach as expressed in equation (A.7) with the constant force of mortality assumption is obtained from

$$\begin{aligned} E[\theta_{(x)}] &= \mu E\left[\sum_i (t_i - s_i)\right] \quad , \text{ that is} \\ \overset{\circ}{\mu} &= \frac{\theta_{(x)}}{E_x^c} \end{aligned}$$

$\overset{\circ}{\mu}$ will only be an unbiased estimate of μ if E_x^c is non-random. This will only occur in the trivial case where no deaths are expected. Deriving the estimate of q_x from the estimate of μ, that is,

$$\overset{\circ}{q}_x = 1 - e^{-\overset{\circ}{\mu}}$$

introduces further positive bias.

5.2 Other Rate Intervals

The results presented in this research can be extended to policy and calendar year rate intervals if information observed and assumptions used are appropriately re-defined. An additional difficulty is, however, introduced in that we have to decide the age to which calculated rates apply. This problem does not arise with a life year rate interval since all lives commence the rate interval at the same age.

Consider the rate interval commencing on the policy anniversary on which lives are aged x last birthday. Suppose that the ith life is aged $x+u_i$ exact at the start of the rate interval where $0 \le u_i < 1$. The ith life thus enters investigation at age $x+u_i+s_i$, exits from observation at age $x+u_i+t_i$ and has a 'maximum age' $x+u_i+\alpha_i$. $\theta_{(x)}$ is now defined to be the number of deaths observed during the period of investigation who were 'aged x last birthday on the policy anniversary prior to the date of death'. Let the 'correct' age to which calculated rates apply be $x+u$.

The expression in equation (3.8) for the expected deaths under the conventional approach becomes, under this policy year rate interval,

$$E[\theta_{(x)}] = \sum_i {}_{1-s_i}q_{x+u_i+s_i} - E\left[\sum_{i:D_i=0} {}_{1-t_i}q_{x+u_i+t_i}\right] \tag{5.7}$$

Commonly, the average age at the start of the rate interval for all lives observed is considered, that is,

$$x + \frac{1}{N}\sum_i u_i$$

and this is the age to which calculated rates are assumed to apply. If birthdays are assumed to be uniformly distributed over the policy year, then

$$E[u_i] = \frac{1}{2}$$

and rates are assumed to apply to age $x+\frac{1}{2}$ exact.

Alternatively, the expected average age at death, namely,

$$x + E\left[\frac{\sum_{i:D_i=1}(u_i + t_i)}{\theta_{(x)}}\right]$$

is considered. If we can assume that the bias introduced by taking the expectation over numerator and denominator separately is small, this becomes

$$x + \frac{E[\sum_{i:D_i=1} u_i] + E[\sum_{i:D_i=1} t_i]}{E[\theta_{(x)}]}$$

On the assumption that birthdays are uniformly distributed over the policy year,

$$E[\sum_{i:D_i=1} u_i] = \frac{1}{2}E[\theta_{(x)}]$$

On the assumption that deaths are uniformly distributed over the rate interval

$$E[\sum_{i:D_i=1} t_i] = \frac{1}{2}E[\theta_{(x)}]$$

Thus, deducting $\frac{1}{2}$ from the expected average age at death will give the expected average age at the start of the rate interval of those that died. Calculated rates are then assumed to apply to this age, that is $x + \frac{1}{2}$.

Obviously, both of these methods give only approximate estimates of $x + u$, and take no account of the age ranges over which individual lives will be observed. If the ith life was observed from age $x + u_i$ to $x + u_i + 1$, the expected number of deaths would be

$$E[\theta_{(x)}] = \sum_i q_{x+u_i}$$

$q_{x+u} = \frac{1}{N}E[\theta_{(x)}]$ would thus be an arithmetic average of the q_{x+u_i}. $\frac{1}{N}\theta_{(x)}$ would be an unbiased estimate of q_{x+u}. If birthdays were assumed to be uniformly distributed over the policy year, then the expected value of q_{x+u} would be

$$E[q_{x+u}] = \int_0^1 q_{x+r}\, dr$$

This is the theoretically correct approach underlying the approximate methods considered above. However, it is only appropriate where all lives are observed for the full rate year. If we calculated the 'reduced sample estimate', that is, the estimate based only on lives for whom $s_i = 0$ and $\alpha_i = 1$, this would be an unbiased estimate of q_{x+u}. Where, as is usual, the estimate is based on all lives observed however, allowance must be made for the fact that some of the lives would be observed for less than the full rate interval.

If we could assume that, for the ith individual,

$$_{1-r}q_{x+u_i+r} = (1-r)q_{x+u_i} \tag{5.8}$$

which is, effectively, a Balducci assumption for each individual over the year of age $x + u_i$ to $x + u_i + 1$, equation (5.7) becomes

$$E[\theta_{(x)}] = \sum_i (1 - s_i)q_{x+u_i} - E\left[\sum_{i:D_i=0} (1 - t_i)q_{x+u_i}\right] \tag{5.9}$$

Equation (5.9) can be expressed as

$$E[\theta_{(x)}] = \sum_i (1 - s_i)q_{x+u_i} - E\left[\sum_i (1 - D_i)(1 - \alpha_i)q_{x+u_i}\right]$$

Note that α_i must be used instead of t_i in the second summation because the expectation has been expressed as a sum over all lives, and not just over lives who do not die. This equation becomes

$$E[\theta_{(x)}] = \sum_i ((1-s_i) - E[(1-D_i)(1-\alpha_i)])\, q_{x+u_i}$$

which, since $(1-D_i)(1-\alpha_i) = (1-D_i)(1-t_i)$ can be expressed as

$$E[\theta_{(x)}] = \sum_i ((1-s_i) - E[(1-D_i)(1-t_i)])\, q_{x+u_i}$$

The expected value of the initial exposed-to-risk is, as for the life year rate interval,

$$\begin{aligned} E[E_x] &= \sum_i (1-s_i) - E\left[\sum_{i:D_i=0} (1-t_i)\right] \\ &= \sum_i ((1-s_i) - E[(1-D_i)(1-t_i)]) \end{aligned}$$

If we can assume that the bias introduced by taking the expectation over numerator and denominator separately is small, then the expected value of our estimate of q_{x+u}, namely

$$\mathring{q}_{x+u} = \frac{\theta_{(x)}}{E_x}$$

can be expressed as

$$\frac{E[\theta_{(x)}]}{E[E_x]} = \frac{\sum_i ((1-s_i) - E[(1-D_i)(1-t_i)])\, q_{x+u_i}}{\sum_i ((1-s_i) - E[(1-D_i)(1-t_i)])} \qquad (5.10)$$

that is, it is a weighted average of the q_{x+u_i}. Thus, under the assumption expressed in equation (5.8), the conventional approach estimates a rate determined by the weighted average expressed in equation (5.10). For all practical purposes, it is usually considered sufficient to consider the usual estimate based on expected average age at the start of the rate interval, although it must be understood that this method is only an approximate approach.

5.3 Age to Which Select Rates Apply

When deriving select rates for an investigation in which deaths are classified by age and duration at date of death, care must be taken in determining the age to which calculated rates apply. Consider the example where $\theta_{(x,t)}$ deaths are observed to occur at age x last birthday with curtate duration t at date of death. Let $E[\theta_{x+r,t+s}]\delta r\delta s$ be the expected number of deaths aged between $x+r$ and $x+r+\delta r$ with duration between $t+s$ and $t+s+\delta s$ at the date of death. Thus

$$\begin{aligned} E[\theta_{(x,t)}] &= \int_0^1 \int_0^1 E[\theta_{x+r,t+s}] dr ds \\ &= \int_0^1 \int_0^1 E[\mathring{l}_{x+r,t+s}]\mu_{x+r,t+s} dr ds \end{aligned}$$

where the integrals can be expressed as the sum of several integrals over each interval over which $E[\mathring{l}_{x+r,t+s}]$ is continuous.

The expected average age at entry of the deaths can be estimated, and calculated rates assumed to apply to this age at entry, over the $(t+1)$th policy year. Determining the correct age to which rates apply is, in fact a fairly complex problem, analogous to the problems considered in the previous section. For all practical purposes an arithmetic average of the age at entry of the deaths may give a sufficiently accurate estimate of the 'correct' age at entry. However, it must again be realised that it is an approximation.

The average age at entry is given by

$$\frac{\sum_s \sum_r (x+r-t-s)\theta_{x+r,t+s}}{\theta_{(x,t)}}$$

On the assumption that the expectation can be taken separately over numerator and denominator without significant effect, the expected average age at entry is given by

$$\frac{\int_0^1 \int_0^1 (x+r-t-s)E[\theta_{x+r,t+s}]drds}{E[\theta_{(x,t)}]}$$

On the assumption that expected deaths are independently uniformly distributed over the year of age x to $x+1$ and the $(t+1)$th policy year, the average age at entry would be approximated by

$$\begin{aligned} & \int_0^1 \int_0^1 (x+r-t-s)drds \\ = \; & x-t+\int_0^1 (\frac{1}{2}-s)ds \\ = \; & x-t \end{aligned}$$

The calculated rates would thus be assumed to apply to lives aged $x-t$ exact at entry between duration t and $t+1$. This is the conclusion drawn by Puzey (1987), but he states that 'no assumptions are required'. It is obvious that, even if the above approach can only be taken as being approximately correct, we have had to assume that the expectation can be taken over numerator and denominator separately and that deaths are independently uniformly distributed over the age and duration rate intervals. Puzey's statement is somewhat inaccurate.

Chapter 6

Conclusion

In this research, several areas of uncertainty that I and my colleagues have experienced when working with exposed-to-risk theory have been investigated. Problem areas have been placed on a solid theoretical basis so that the implications of making particular assumptions and approximations can be more fully understood.

Although, in many instances, the magnitude of the difference between possible estimates or the bias of a particular estimate would, for all practical purposes, be negligible, it is important that our estimates be theoretically sound. Situations where the difference may be more significant have been indicated so that the magnitude of any error can be minimised. In addition, the theory established in the actuarial field is not only limited to estimation of mortality or withdrawal rates, it can also be extended to non-life applications. The effect of the assumptions highlighted in this research, although immaterial when estimating mortality rates, may become more significant when the theory is applied to other circumstances. Thus, it is necessary to quantify and understand the implications of the assumptions that are made.

Notation and expressions commonly used have been analysed and rigorously redefined so that they produce a consistent whole. This means, in many cases, that we can continue to use current notation or estimates, provided that the correct assumptions are stated. In other cases, the commonly used approaches have been shown to be incorrect, or correct only in limited circumstances. This has been highlighted by referring to inaccuracies in published material. In such cases, it is hoped that this research will increase understanding of the principles involved so that theoretically correct results can be presented in future.

The conventional actuarial estimates have been shown to be theoretically sound, justifying the actuarial techniques established since the early nineteenth century. They are closely related to maximum likelihood estimates, the properties of which statisticians advocate as desirable. Although, under the uniform and Balducci assumptions, the commonly used conventional estimates are only asymptotically equal to the maximum likelihood estimates and are, in general, biased, they avoid the difficulty which would be encountered with other estimates, for example those which require the 'maximum age' of deaths.

This research has focussed on particular areas of interest in the context of exposed-to-risk theory. The topic is, however, diverse, presenting several aspects worthy of further fruitful research. It is hoped that the results presented stimulate further understanding of and interest in the subject.

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Appendix A

Useful Expressions and Their Derivations

The information presented in this appendix summarises several of the trains of thought underlying this research. It is convenient to group the results into three parts. Derivations, where required, are given in Appendices A.1 to A.4.

1. The following expressions will be used in various derivations and should be intuitively obvious

$$\sum_i \frac{1}{{}_{s_i}p_x} = E\left[\sum_{i:D_i=0} \frac{1}{{}_{t_i}p_x}\right] \tag{A.1}$$

$$E[\theta_{(x)}] = E\left[\sum_{i:D_i=0} \left(\frac{{}_{t_i}q_x}{{}_{t_i}p_x}\right)\right] - \sum_i \left(\frac{{}_{s_i}q_x}{{}_{s_i}p_x}\right) \tag{A.2}$$

Derivations of these relations are given below in Appendices A.1 and A.2.

2. It should be obvious that

$$\begin{aligned}\mathring{l}_{x+r} &= \sum_{i:s_i\le r} 1 - \sum_{i:t_i<r} 1 \\ &= \sum_{i:s_i\le r} 1 - \sum_{i:D_i=0\,;\,t_i<r} 1 - \sum_{i:D_i=1\,;\,t_i<r} 1 \end{aligned} \tag{A.3}$$

It is shown in Appendix A.4 that the expected value of equation (A.3) can be expressed as

$$E[\mathring{l}_{x+r}] = \sum_{i:s_i\le r} {}_{r-s_i}p_{x+s_i} - E\left[\sum_{i:D_i=0\ ;\ t_i<r} {}_{r-t_i}p_{x+t_i}\right] \tag{A.4}$$

3. Since

$$E[\theta_{x+r}]\delta r = E[\mathring{l}_{x+r}]\mu_{x+r}\delta r \tag{A.5}$$

we can express the expected number of deaths in the alternative form

$$E[\theta_{(x)}] = \int_0^1 E[\mathring{l}_{x+r}]\mu_{x+r}dr \tag{A.6}$$

which is equivalent to

$$E[\theta_{(x)}] = E\left[\sum_i \int_{s_i}^{t_i} \mu_{x+r}dr\right] \tag{A.7}$$

It is shown below in Appendix A.3 that equation (A.6) is equivalent to the conventional approach.

A.1 Derivation of Equation (A.1) from the Conventional Approach

We can rearrange equation (3.6) in the following way

$$\begin{aligned}\sum_{j=1}^{M} N^j &- \sum_{j=1}^{M}(N^j - E[\theta^j_{(x)}]) \\ &= \sum_{j=1}^{M} N^j{}_{1-s_j}q_{x+s_j} - \sum_{j=1}^{M}(N^j - E[\theta^j_{(x)}])_{1-\alpha_j}q_{x+\alpha_j}\end{aligned}$$

and, therefore

$$\sum_{j=1}^{M} N^j{}_{1-s_j}p_{x+s_j} = \sum_{j=1}^{M}(N^j - E[\theta^j_{(x)}])_{1-\alpha_j}p_{x+\alpha_j}$$

Dividing through on both sides by p_x gives

$$\sum_{j=1}^{M} N^j \frac{1}{{}_{s_j}p_x} = \sum_{j=1}^{M}(N^j - E[\theta^j_{(x)}])\frac{1}{{}_{\alpha_j}p_x} \tag{A.8}$$

This relation can be expressed as

$$\begin{aligned}\sum_{j=1}^{M}\sum_{k=1}^{N^j}\frac{1}{{}_{s_j}p_x} &= E\left[\sum_{j=1}^{M}(N^j - \theta^j_{(x)})\frac{1}{{}_{\alpha_j}p_x}\right] \\ \sum_{i}\frac{1}{{}_{s_i}p_x} &= E\left[\sum_{j=1}^{M}\sum_{(i\in j:D_i=0)}\frac{1}{{}_{\alpha_j}p_x}\right] \\ &= E\left[\sum_{i:D_i=0}\frac{1}{{}_{t_i}p_x}\right]\end{aligned}$$

which is equation (A.1).

A.2 Derivation of Equation (A.2) from the Conventional Approach

Rearranging equation (3.6) gives

$$
\begin{aligned}
E[\theta_{(x)}] &= \sum_{j=1}^{M} N^j \left(\frac{q_x - s_j q_x}{s_j p_x}\right) - \sum_{j=1}^{M}(N^j - E[\theta^j_{(x)}]) \left(\frac{q_x - \alpha_j q_x}{\alpha_j p_x}\right) \\
&= \sum_{j=1}^{M}(N^j - E[\theta^j_{(x)}]) \left(\frac{\alpha_j q_x}{\alpha_j p_x}\right) - \sum_{j=1}^{M} N^j \left(\frac{s_j q_x}{s_j p_x}\right) \\
&+ \quad q_x \left(\sum_{j=1}^{M} N^j \frac{1}{s_j p_x} - \sum_{j=1}^{M}(N^j - E[\theta^j_{(x)}]) \frac{1}{\alpha_j p_x}\right) \\
&= \sum_{j=1}^{M}(N^j - E[\theta^j_{(x)}]) \left(\frac{\alpha_j q_x}{\alpha_j p_x}\right) - \sum_{j=1}^{M} N^j \left(\frac{s_j q_x}{s_j p_x}\right)
\end{aligned}
$$

from equation (A.8).

This can be expressed as

$$
\begin{aligned}
E[\theta_{(x)}] &= E\left[\sum_{j=1}^{M}(N^j - \theta^j_{(x)}) \left(\frac{\alpha_j q_x}{\alpha_j p_x}\right)\right] - \sum_{j=1}^{M} \sum_{k=1}^{N^j} \left(\frac{s_j q_x}{s_j p_x}\right) \\
&= E\left[\sum_{j=1}^{M} \sum_{(i \in j : D_i = 0)} \left(\frac{\alpha_j q_x}{\alpha_j p_x}\right)\right] - \sum_{i} \left(\frac{s_i q_x}{s_i p_x}\right) \\
&= E\left[\sum_{i : D_i = 0} \left(\frac{t_i q_x}{t_i p_x}\right)\right] - \sum_{i} \left(\frac{s_i q_x}{s_i p_x}\right)
\end{aligned}
$$

which is equation (A.2).

A.3 Proof of Equivalence of Equation (A.6) and the Conventional Approach

Since, from equation (A.4)

$$E[\mathring{l}_{x+r}] = \sum_{i:s_i \le r} {}_{r-s_i}p_{x+s_i} - E\left[\sum_{i:D_i=0\,;\,t_i<r} {}_{r-t_i}p_{x+t_i}\right]$$

equation (A.6) becomes

$$\begin{aligned}
E[\theta_{(x)}] &= \int_0^1 E[\mathring{l}_{x+r}]\mu_{x+r}dr \\
&= \int_0^1 \left(\sum_{i:s_i \le r} {}_{r-s_i}p_{x+s_i} - E\left[\sum_{i:D_i=0\,;\,t_i<r} {}_{r-t_i}p_{x+t_i}\right]\right)\mu_{x+r}dr \\
&= \sum_i \int_{s_i}^1 {}_{r-s_i}p_{x+s_i}\mu_{x+r}dr - E\left[\sum_{i:D_i=0} \int_{t_i}^1 {}_{r-t_i}p_{x+t_i}\mu_{x+r}dr\right] \\
&= \sum_i {}_{1-s_i}q_{x+s_i} - E\left[\sum_{i:D_i=0} {}_{1-t_i}q_{x+t_i}\right]
\end{aligned}$$

which is the conventional approach as expressed in equation (3.8).

A.4 Proof of Equation (A.4)

It obvious that

$$\begin{aligned}
E\left[\sum_{i:D_i=1\,;\,t_i<r} 1\right] &= \sum_{i:s_i \le r\,;\,\alpha_i \ge r} {}_{r-s_i}q_{x+s_i} + \sum_{i:s_i \le r\,;\,\alpha_i<r} {}_{\alpha_i-s_i}q_{x+s_i} \\
&= \sum_{i:s_i \le r} {}_{r-s_i}q_{x+s_i} - \sum_{i:s_i \le r\,;\,\alpha_i<r} {}_{\alpha_i-s_i}p_{x+s_i}\,{}_{r-\alpha_i}q_{x+\alpha_i}
\end{aligned}$$

This can be expressed as

$$E\left[\sum_{i:D_i=1\,;\,t_i<r} 1\right] = \sum_{i:s_i\leq r} {}_{r-s_i}q_{x+s_i} - E\left[\sum_{i:D_i=0\,;\,t_i<r} {}_{r-t_i}q_{x+t_i}\right]$$

Thus, the expected value of equation (A.3) can be expressed as

$$\begin{aligned} E[\mathring{l}_{x+r}] &= \sum_{i:s_i\leq r} 1 - E\left[\sum_{i:D_i=0\,;\,t_i<r} 1\right] - \sum_{i:s_i\leq r} {}_{r-s_i}q_{x+s_i} \\ &+ \quad E\left[\sum_{i:D_i=0\,;\,t_i<r} {}_{r-t_i}q_{x+t_i}\right] \\ &= \sum_{i:s_i\leq r} {}_{r-s_i}p_{x+s_i} - E\left[\sum_{i:D_i=0\;;\;t_i<r} {}_{r-t_i}p_{x+t_i}\right] \end{aligned}$$

which is equation (A.4)

Appendix B

Properties of Some Common Assumptions

Assumption	Parameter				
	μ_{x+t}	${}_{t-s}q_{x+s}$	${}_{1-s}q_{x+s}$	${}_{t}q_{x}$	l_{x+t}
Uniform	$\frac{q_x}{1-tq_x}$	$\frac{(t-s)q_x}{1-sq_x}$	$\frac{(1-s)q_x}{1-sq_x}$	tq_x	$l_x - td_x$
Balducci	$\frac{q_x}{p_x+tq_x}$	$\frac{(t-s)q_x}{p_x+tq_x}$	$(1-s)q_x$	$\frac{tq_x}{p_x+tq_x}$	$\frac{l_{x+1}}{p_x+tq_x}$
Constant μ	μ	$1-p_x^{t-s}$	$1-p_x^{1-s}$	$1-p_x^t$	$l_x p_x^t$
Gompertz	Bc^{x+t}	$1-g^{c^x(c^t-c^s)}$	$1-g^{c^x(c-c^s)}$	$1-\frac{g^{c^{x+1}}}{g^{c^{x+s}}}$	$l_0 g^{c^{x+t}-1}$

Appendix C

Maximum Likelihood Estimation

C.1 The Uniform Assumption

Under the uniform assumption, the likelihood in equation (3.13) becomes

$$\Lambda = \prod_i \frac{1 - t_i q_x}{1 - s_i q_x} \prod_{i:D_i=1} \frac{q_x}{1 - t_i q_x}$$

and hence

$$\ln \Lambda = \sum_{i:D_i=0} \ln(1 - t_i q_x) - \sum_i \ln(1 - s_i q_x) + \theta_{(x)} \ln q_x$$

Differentiating with respect to q_x gives

$$\frac{\mathrm{d} \ln \Lambda}{\mathrm{d} q_x} = - \sum_{i:D_i=0} \frac{t_i}{1 - t_i q_x} + \sum_i \frac{s_i}{1 - s_i q_x} + \frac{\theta_{(x)}}{q_x}$$

The maximum likelihood estimator of q_x, $\hat{q}_x$, is derived by equating this to 0. Rearranging gives

$$\theta_{(x)} = \hat{q}_x \left[\sum_{i:D_i=0} \frac{t_i}{1 - t_i \hat{q}_x} - \sum_i \frac{s_i}{1 - s_i \hat{q}_x} \right] \tag{C.1}$$

As N tends toward infinity, equation (C.1) tends toward

$$E[\theta_{(x)}] = q_x \left(E\left[\sum_{i:D_i=0} \frac{t_i}{1 - t_i q_x} \right] - \sum_i \frac{s_i}{1 - s_i q_x} \right) \tag{C.2}$$

Now, as can be shown using Appendix A.2, this is the same as the conventional approach of equation (3.8) under the uniform assumption, that is,

$$E[\theta_{(x)}] = \sum_i \frac{(1 - s_i)q_x}{1 - s_i q_x} - E\left[\sum_{i:D_i=0} \frac{(1 - t_i)q_x}{1 - t_i q_x} \right] \tag{C.3}$$

Thus the estimate derived using maximum likelihood estimation is asymptotically equal to the estimate derived using the conventional approach as expressed in equation (3.8).

C.2 The Balducci Assumption

Under the Balducci assumption the likelihood in equation (3.13) becomes

$$\Lambda = \prod_i \frac{1 - (1 - s_i)q_x}{1 - (1 - t_i)q_x} \prod_{i:D_i=1} \frac{q_x}{1 - (1 - t_i)q_x}$$

and hence

$$\begin{aligned} \ln \Lambda &= \sum_i \ln(1 - (1 - s_i)q_x) - \sum_{i:D_i=0} \ln(1 - (1 - t_i)q_x) \\ &+ \quad \theta_{(x)} \ln q_x \quad - \quad 2 \sum_{i:D_i=1} \ln(1 - (1 - t_i)q_x) \end{aligned}$$

Differentiating with respect to q_x gives

$$\begin{aligned} \frac{d \ln \Lambda}{dq_x} &= -\sum_i \frac{1-s_i}{1-(1-s_i)q_x} + \sum_{i:D_i=0} \frac{1-t_i}{1-(1-t_i)q_x} \\ &+ \frac{\theta_{(x)}}{q_x} + 2\sum_{i:D_i=1} \frac{1-t_i}{1-(1-t_i)q_x} \end{aligned}$$

The maximum likelihood estimator of q_x, $\hat{q}_x$ is derived by equating this to 0. Rearranging gives

$$\begin{aligned} \theta_{(x)} &= \hat{q}_x \left[\sum_i \frac{1-s_i}{1-(1-s_i)\hat{q}_x} - \sum_{i:D_i=0} \frac{1-t_i}{1-(1-t_i)\hat{q}_x} \right] \\ &- 2\hat{q}_x \sum_{i:D_i=1} \frac{1-t_i}{1-(1-t_i)\hat{q}_x} \end{aligned} \tag{C.4}$$

As N tends toward infinity, equation (C.4) tends toward

$$\begin{aligned} E[\theta_{(x)}] &= q_x \left(\sum_i \frac{1-s_i}{1-(1-s_i)q_x} - E\left[\sum_{i:D_i=0} \frac{1-t_i}{1-(1-t_i)q_x} \right] \right) \\ &- 2q_x \int_0^1 \frac{E[\theta_{x+r}](1-r)}{1-(1-r)q_x} dr \end{aligned} \tag{C.5}$$

Under the Balducci assumption, equation (A.4) becomes

$$E[\mathring{l}_{x+r}] = \sum_{i:s_i \leq r} \frac{1-(1-s_i)q_x}{1-(1-r)q_x} - E\left[\sum_{i:D_i=0\,;\,t_i<r} \frac{1-(1-t_i)q_x}{1-(1-r)q_x} \right]$$

and equation (A.5) becomes

$$\begin{aligned} E[\theta_{x+r}] &= q_x \sum_{i:s_i \leq r} \frac{1-(1-s_i)q_x}{(1-(1-r)q_x)^2} \\ &- q_x E\left[\sum_{i:D_i=0\,;\,t_i<r} \frac{1-(1-t_i)q_x}{(1-(1-r)q_x)^2} \right] \end{aligned} \tag{C.6}$$

Thus

$$
\begin{aligned}
2q_x \int_0^1 \frac{E[\theta_{x+r}](1-r)}{1-(1-r)q_x} dr &= 2(q_x)^2 \int_0^1 \left(\sum_{i:s_i \le r} \frac{(1-(1-s_i)q_x)(1-r)}{(1-(1-r)q_x)^3} \right) dr \\
- & \quad 2(q_x)^2 E\left[\int_0^1 \sum_{i:D_i=0\,;\,t_i<r} \frac{(1-(1-t_i)q_x)(1-r)}{(1-(1-r)q_x)^3} dr \right] \\
= & \sum_i 2(q_x)^2(1-(1-s_i)q_x) \int_{s_i}^1 \frac{(1-r)}{(1-(1-r)q_x)^3} dr \\
- & \quad E\left[\sum_{i:D_i=0} 2(q_x)^2(1-(1-t_i)q_x) \int_{t_i}^1 \frac{(1-r)}{(1-(1-r)q_x)^3} dr \right] \qquad (C.7)
\end{aligned}
$$

Now

$$
\begin{aligned}
\int_u^1 \frac{(1-r)}{(1-(1-r)q_x)^3} dr &= \left[-\frac{(1-r)}{2q_x(1-(1-r)q_x)^2} \right]_u^1 \\
&- \frac{1}{2q_x} \int_u^1 \frac{1}{(1-(1-r)q_x)^2} dr \\
&= \frac{1-u}{2q_x(1-(1-u)q_x)^2} \\
&+ \frac{1}{2(q_x)^2} - \frac{1}{2(q_x)^2(1-(1-u)q_x)} \qquad (C.8)
\end{aligned}
$$

Substituting (C.8) into equation (C.7) gives

$$
2q_x \int_0^1 \frac{E[\theta_{x+r}](1-r)}{1-(1-r)q_x} dr = \sum_i \left(\frac{(1-s_i)q_x}{1-(1-s_i)q_x} + (1-(1-s_i)q_x) - 1 \right)
$$

$$
-E\left[\sum_{i:D_i=0} \left(\frac{(1-t_i)q_x}{1-(1-t_i)q_x} + (1-(1-t_i)q_x) - 1 \right) \right]
$$

By substituting this into equation (C.5) we see that

$$
E[\theta_{(x)}] = \sum_i (1-s_i)q_x - E\left[\sum_{i:D_i=0} (1-t_i)q_x \right] \qquad (C.9)
$$

which is the conventional approach of equation (3.8) under the Balducci assumption. Thus, the estimate derived using maximum likelihood estimation is asymptotically equal to the estimate derived using the conventional approach as expressed in equation (3.8).

C.3 The Constant Force of Mortality Assumption

Under the constant force of mortality assumption, the likelihood in equation (3.13) becomes

$$\Lambda = \prod_i e^{-\mu(t_i - s_i)} \prod_{i:D_i=1} \mu$$

and hence

$$\ln \Lambda = -\mu \sum_i (t_i - s_i) + \theta_{(x)} \ln \mu$$

Differentiating with respect to μ gives

$$\frac{\mathrm{d} \ln \Lambda}{\mathrm{d}\mu} = -\sum_i (t_i - s_i) + \frac{\theta_{(x)}}{\mu}$$

The maximum likelihood estimator of μ, $\hat{\mu}$, is derived by equating this to 0. Rearranging gives

$$\theta_{(x)} = \hat{\mu} \sum_i (t_i - s_i)$$

The conventional approach as expressed in equation (A.7) becomes, under the constant force of mortality assumption,

$$E[\theta_{(x)}] = \mu E\left[\sum_i (t_i - s_i)\right]$$

Substituting $\theta_{(x)}$ for $E[\theta_{(x)}]$ and the sum over i for its expected value gives the conventional estimate of μ. It can be seen that this estimate is same as the maximum likelihood estimate, $\hat{\mu}$.

C.4 The Gompertz Assumption

Under the Gompertz assumption the likelihood in equation (3.13) becomes

$$\Lambda = \prod_i g^{c^x(c^{t_i}-c^{s_i})} \prod_{i:D_i=1} Bc^{x+t_i}$$

and hence

$$\ln\Lambda = -\frac{B}{\ln c}\sum_i (c^x(c^{t_i}-c^{s_i})) + \theta_{(x)}\ln B + \ln c \sum_{i:D_i=1}(x+t_i)$$

since

$$\ln g = -\frac{B}{\ln c}$$

Differentiating with respect to B and c respectively gives

$$\begin{aligned}
\frac{\mathrm{d}\ln\Lambda}{\mathrm{d}B} &= -\sum_i \frac{c^x(c^{t_i}-c^{s_i})}{\ln c} + \frac{\theta_{(x)}}{B} \\
\frac{\mathrm{d}\ln\Lambda}{\mathrm{d}c} &= -B\sum_i\left[\frac{c^{x+s_i-1}}{\ln c}[(x+t_i)c^{t_i-s_i} - (x+s_i)]\right] \\
&+ B\sum_i\left[\frac{c^{x+s_i-1}}{(\ln c)^2}(c^{t_i-s_i}-1)\right] + \sum_{i:D_i=1}\frac{(x+t_i)}{c}
\end{aligned}$$

Equating each of these to 0 will give simultaneous equations in the maximum likelihood estimates of B and c, $\hat{B}$ and $\hat{c}$

$$\theta_{(x)} = \frac{\hat{B}}{\ln \hat{c}} \sum_i \hat{c}^x(\hat{c}^{t_i} - \hat{c}^{s_i}) \tag{C.10}$$

$$\begin{aligned} \sum_{i:D_i=1} (x+t_i) &= \frac{\hat{c}\hat{B}}{\ln \hat{c}} \sum_i \hat{c}^{x+s_i-1}((x+t_i)\hat{c}^{t_i-s_i} - (x+s_i)) \\ &- \frac{\hat{c}\hat{B}}{(\ln \hat{c})^2} \sum_i \hat{c}^{x+s_i-1}(\hat{c}^{t_i-s_i} - 1) \end{aligned}$$

Using the Gompertz assumption in the conventional approach as expressed in equation (A.7) gives

$$\begin{aligned} E[\theta_{(x)}] &= Bc^x E\left[\sum_i \left[\frac{c^r}{\ln c}\right]_{s_i}^{t_i}\right] \\ &= \frac{B}{\ln c} E\left[\sum_i c^x(c^{t_i} - c^{s_i})\right] \end{aligned}$$

Substituting $\theta_{(x)}$ for $E[\theta_{(x)}]$ and the sum over i for its expected value in this equation gives the relationship between B and c under the conventional approach. Since only one equation is obtained for two variables, there are an infinite number of possible solutions. The most reasonable combination will be found by graphical plotting and trial and error as described in Benjamin and Pollard (1980). The maximum likelihood estimates will, as shown in equation (C.10), satisfy the relationship of the conventional approach. They provide a single possible combination from the set of solutions satisfying the conventional requirements.