Covariant Perturbations in $f(R)$ - Gravity of Multi-component Fluid Cosmologies

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"...Whether these ideas [modifications of General Relativity] are leading physicists down blind alleys or are steps toward the master theory of the universe only time will tell. But they are wild ideas that could be crazy enough to be right."

-Heinz R. Pagels, *Perfect Symmetry*

To Getch, who has always been there for me :
from 0, 1, 2,..., A, B, C... to $\pm$, $\neg$, $\{\}$, $\sim$, i, ∂, $\int$, T^{ab}, Δ, Ω.

Declaration

I hereby declare that this thesis has not been submitted, either in the same or different form, to this or any other university for a degree and that it represents my own work.

Abstract

We study the evolution of scalar cosmological perturbations in the $1 + 3$ Covariant Gauge-Invariant formalism for generic f(R) theories of gravity. Working in the energy frame of the total matter, we give a complete set of equations describing the evolution of matter and curvature fluctuations for a multi-fluid cosmological medium. We then specialize to a radiation-dust fluid described by barotropic equations of state. We apply the perturbation equations around a background solution of R^n gravity and look at exact solutions for scales much smaller and much larger than the Hubble radius.

Keywords: general relativity, modified gravity, multi-component fluids

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Chapter 1

INTRODUCTION

There are at least three areas where the standard General Relativity theory faces serious challenges from other competing fundamental theories. Attempts to unite quantum mechanics with general relativity have so far been unsuccessful, and with the remarkable success the former has achieved for the last hundred years, many are questioning the unique status of General Relativity and suggest that a fundamental modification of the theory could lead to quantum-gravity. In standard particle physics, all the forces of nature save gravity have been shown to be manifestations of an underlying unified theory of nature, and hopes are there that modification of Einstein's General Relativity could lead to a grand unification of all the known forces as a single theory of everything. And, most important to us here is the missing link between the observed universe and the theory invented a priori to explain it.

The recent discovery of the accelerated expansion of the Universe has shed new light on the not-for-so-long established standard model of cosmology based on Einstein's General Theory of Relativity. Observational analyses show that only a tiny fraction ($\sim 4\%$) of the energy content of the Universe is known to exist in normal matter form, whereas $\sim 23\%$ of it exists in a little understood *dark matter* form. Dubbed as *Dark Energy*, the remaining ($> 72\%$) of the energy budget of the Universe is often believed to be the cause of the accelerated cosmic expansion.

Many candidates have been put forward as an explanation for Dark Energy [1–4], but most of them fall under one of these three forms: the cosmological constant Λ, exotic scalar fields (such as *Quintessence*) and geometrical dark energy (modified gravity, i.e., Scalar-Tensor and $f(R)$models). In this work we will only concentrate on the interpretation of Dark Energy as geometrical interpretation of a more fundamental theory of gravity, $f(R) - gravity$.

Whereas it is well known that dynamical evolution of large scale structures is always present in the Universe, it is not very well understood whether there is a dynamical mechanism to generate the seeds of these structures or whether the seeds are laid down as part of the initial conditions of he Universe [5–10, 12–14]. From a physical point of view, the dynamical picture is more attractive because it is more consistent with cosmological observations and the Big Bang theory which is our best model of the Universe. Most cosmologists now believe that large scale structures have formed through the growth of density perturbations via gravitational instability. Thus a cosmological perturbation theory has been developed to understand the formation of large scale structures, based on the linearity assumption of the structures, which seems to hold in the early and the late time evolution stages of the Universe [15].

An excellent framework to study cosmological perturbations is the 1+3 covariant approach which has been developed to analyze the evolution of linear perturbations of FRW models in General Relativity [16–20]. In recent years higher order theories of gravity have attracted lots of attention, where a detailed analysis of FLRW models using the dynamical system techniques shows that there exist classes of fourth order theories which admit a transient decelerated expansion phase during which structure formation can take place, followed by a DE-like era which drives the present cosmological acceleration. In [21] the evolution of scalar perturbations of FLRW models in fourth order gravity has been studied for single barotropic fluids using the 1+3 covariant approach; and the solutions of the perturbation equations show that a decelerated phase is not necessarily required to form large scale structures. This divergence from the standard GR can provide us with a distinguishable signature of the fourth order theories, which can be tested against observations. But, since the Universe consists of a mixture of interacting fluids, a complete treatment of perturbations in fourth order theories requires taking this fact into account. The aim of this thesis is to present a general framework to study the multi-fluid cosmological perturbation theories with a variable equation of state for a general f(R)-gravity theory, using the 1+3 covariant approach.

The outline of this thesis will be as follows: In chapter 2, we will give an overview of the standard models and principles of cosmology. The third chapter is dedicated to a discussion of the covariant and gauge invariant formalisms of cosmological perturbations. In Chapter 4, we discuss the shortcomings and pitfalls of the Standard Model and the need for generalized gravitational theories. Then we give a review of the covariant approach of cosmological perturbations as applied to higher order theories of gravity. Chapters 5 and 6 focus on the analysis of perturbations in single and multi-fluid cosmological media, respectively. As applications, we present in chapter 7 solutions of the multi-fluid perturbation equations in the short wavelength and long wavelength regimes for a universe dominated by a radiation-dust mixture. We conclude the main body of the thesis by pointing out the implications of this work and the open problems that need to be addressed in subsequent works.

The traditional sign and natural unit conventions are in use. Thus the natural units ($\hbar = c = k_B = 8\pi G = 1$) i.e., the normalized reduced Planck's constant, speed of light, Boltzmann constant and Newton's gravitational constant (also denoted by κ), in that order are assumed unless otherwise purposefully retained, and Latin indices of tensors run from 0 to 3. The symbols ∇ and ; represent the usual covariant derivative, ∂ and , correspond to partial differentiation and an over dot shows differentiation with respect to proper time.

We use the $(-,+,+,+)$ signature and the Riemann tensor is defined by

$$R^a{}_{bcd} = \Gamma^a{}_{bd,c} - \Gamma^a{}_{bc,d} + \Gamma^e{}_{bd}\Gamma^a{}_{ce} - \Gamma^f{}_{bc}\Gamma^a{}_{df}, \tag{1.1}$$

where the $\Gamma^a{}_{bd}$, etc. are the usual affine connection symbols defined by

$$\Gamma^a{}_{bd} = \frac{1}{2}g^{ae}(g_{be,d} + g_{ed,b} - g_{bd,e}). \tag{1.2}$$

The Ricci tensor is obtained by contracting the first and the third indices:

$$R_{ab} = g^{cd}R_{acbd}, \tag{1.3}$$

and the scalar curvature (Ricci scalar) is defined to be the trace of the Ricci tensor:

$$R = g^{ab}R_{ab} = R^a{}_a. \tag{1.4}$$

Moreover the following are standard notations used in the thesis:

g :	the determinant of the metric g_{ab}	(1.5)
(ab) :	symmetrization over the indices a and b	(1.6)
$[ab]$:	anti-symmetrization over the indices a and b	(1.7)

Chapter 2

THE CONCORDANCE COSMOLOGY

2.1 The Cosmological Principle

Cosmology is the study of the Universe, and the Universe, by definition, is the entirety of all physically existing things, including ourselves. We are no privileged citizens of a privileged place in the Universe, unlike the fast held creeds until as recently as a few hundred years ago used to argue. This assumption that the Earth occupies no unique position in the Universe is known as the *Copernican Principle.* A more encompassing hypothesis upon which most modern cosmological models are built is known as the *Cosmological Principle.* This principle states that *the large-scale structure of the Universe is homogeneous everywhere and isotropic about every point in the Universe.*

A fundamental observer in a *homogeneous* Universe sees the same general structure of the Universe in time. As a consequence these fundamental observers see the sequence of events by which all the observers can synchronize their clocks to measure a universally defined *cosmic time t.* Our fundamental thermodynamical quantities such as the energy density μ and pressure p are then functions of this time, i.e., $\mu = \mu(t)$, $p = p(t)$, etc.

The *isotropic* version of the Universe asserts that the Universe, at large scales, looks the same in whichever direction the fundamental observer chooses to look. State-of-the-art cosmological observations attest to this hypothesis within an accuracy of 3 % and 0.001% for the distribution of matter in the Universe and the cosmic microwave background (CMB), respectively [22].

In short, the homogeneity and isotropy hypotheses are only simplifying assumptions that claim that, on large enough scales, the Universe looks the same at every location and in every direction, respectively.

2.2 The FLRW Universe

The standard way of studying the Universe is by assuming that the large-scale evolution of spacetime can be determined by applying Einstein's Field Equations of Gravitation everywhere [23]: global evolution will follow from local physics. The standard models of cosmology are based on the assumption that if averaged over a large enough physical scale, the Universe is isotropic to all fundamental observers (the preferred family of observers associated with the average motion of matter in the Universe). If the isotropy is exact, then it follows that the Universe is spatially homogeneous as well. In such (homogeneous and isotropic) a universe matter moves along irrotational and shear-free geodesic curves with tangent vector u^a. This means that there exists a canonical time-variable t for which $u_a = -t_{,a}$. These symmetry properties are embodied exactly in the Friedmann - Lemaître - Robertson - Walker (FLRW, for short) geometries, geometries used to describe the large-scale, isotropic and homogeneous structure of the Universe. Consequently the FLRW geometries are conformally flat, i.e., the Weyl tensor is zero:

$$C_{ijkl} = R_{ijkl} + \tfrac{1}{2}(R_{ikgjl} + R_{jlgik} - R_{ilgjk} - R_{jkgil}) - \tfrac{1}{6}R(g_{ikgjl} - g_{il}g_{jk}) = 0. \qquad (2.1)$$

This tensor represents the free gravitational field, enabling non-local effects such as tidal forces and gravitational waves which do not occur in the exact FLRW geometries. Comoving coordinates in this geometry are usually chosen such that the metric takes the form:

$$ds^2 = -dt^2 + a^2(t)d\sigma^2, \quad u_a = \delta^a{}_0(a = 0, 1, 2, 3), \qquad (2.2)$$

where $a(t)$ is the scale factor. That the scale factor is a time-dependent parameter tells us the fact that the Universe expands. The worldlines with tangent vector $u^a = \frac{dx^a}{dt}$ represent the histories of fundamental observers. The (t = constant) space sections are surfaces of homogeneity and have maximal symmetry: they are 3-spaces of constant curvature $K = \frac{k}{a^2(t)}$ where k is the sign of K. The normalized metric $d\sigma^2$ characterizes a 3-space of normalized constant curvature k; coordinates (r, θ, ϕ) can be chosen such that

$$d\sigma^2 = d^2r + f^2(r)(d^2\theta + sin^2\theta d^2\phi), \qquad (2.3)$$

where

$$f(r) = \begin{cases} sin(r) & \text{for } k = +1, \\ r & \text{for } k = 0, \\ sinh(r) & \text{for } k = -1. \end{cases}$$

The rate of expansion at any time t is characterized by the Hubble parameter $H(t) = \frac{\dot{a}}{a}$. The time evolution of the metric can be determined from the EFEs, showing the effect of matter on space-time curvature, to the metric. As a consequence of local isotropy, the energy-momentum tensor (EMT) T_{ab} necessarily takes a perfect fluid form relative to the preferred worldlines with tangent vector u^a:

$$T_{ab} = (\mu + p)u_a u_b + pg_{ab}. \qquad (2.4)$$

The energy density $\mu(t)$ and pressure term $p(t)$ are the time-like and space-like eigenvalues of T_{ab}, respectively. The energy-density conservation equation determines the integrability conditions for the EFEs:

$$T^{ab}{}_{;b} = 0 \Leftrightarrow \dot{\mu} + 3(\mu + p)\frac{\dot{a}}{a} = 0. \tag{2.5}$$

Upon prescribing a suitable equation of state function such that $p = w\mu$ where w isa function of energy density (μ) and temperature (T), the integrability conditions become determinate. Baryonic matter(*dust*) is pressureless and hence $w_d = 0$ whereas radiation has $p_r = \mu_r/3 \Leftrightarrow w = 1/3, \mu_r = aT_r^4$, which by (2.5) imply

$$\mu_d \propto a^{-3}, \quad \mu_r \propto a^{-4}, \quad T_r \propto a^{-1}. \tag{2.6}$$

The scale factor $a(t)$ evolves according to

$$3\frac{\ddot{a}}{a} = -\tfrac{1}{2}\kappa(\mu + 3p) + \Lambda, \tag{2.7}$$

where κ is the gravitational constant and Λ the *cosmological constant.*[1] This equations is known as the *Raychaudhuri equation* and is the basic equation of gravitational interactions. The equation shows that the active gravitational mass density of the matter and fields present is $\mu_{grav} = \mu + 3p$. Ordinary matter has a positive gravitational mass density:

$$\mu + 3p > 0 \Leftrightarrow w > -1/3 \tag{2.8}$$

(the *Strong Energy Condition*, SEC for short), so ordinary matter will tend to cause the Universe to decelerate ($\ddot{a} < 0$). A positive cosmological constant causes an accelerating expansion ($\ddot{a} > 0$), whereas when matter and a cosmological constant are both present, either result (i.e., an accelerated or decelerated expansion) may occur depending on which effect is dominant. The first integral of equations (2.5, 2.7) when $\dot{a} \neq 0$ is the *Friedmann equation*

$$\frac{\dot{a}^2}{a^2} = \frac{k\mu}{3} + \frac{\Lambda}{3} - \frac{k}{a^2}. \tag{2.9}$$

This is just the Gauss equation relating the 3-space curvature to the 4-space curvature, showing how matter directly causes a curvature of 3-spaces. Because of the spacetime symmetries, the ten EFEs are equivalent to the two Friedmann equations.[2] Models of this kind, that is with a Robertson-Walker (RW) geometry with metric (Eqns. 2.2,2.3) and dynamics governed by equations (2.5, 2.7, 2.9), are called Friedmann-Lemaître-Robertson-Walker universes (FLRW). The Friedmann equation (2.9) controls the expansion of the Universe, and the conservation equation (2.5) controls the density of matter as the Universe expands; when $\dot{a} \neq 0$, Eqn. (2.7) will necessarily hold if (2.5, 2.9) are both satisfied [23].

[1]A cosmological constant can also be regarded as a fluid with pressure p related to the energy density μ by $(p = -\mu, w = -1)$.

[2]Strictly speaking both of these equations (2.7 & 2.9) are known as *Friedmann equations*, but we will use the aforementioned names throughout this thesis.

Once an integrable matter description (specifying the equation of state $w = w(\mu, T)$ explicitly or implicitly) for each matter component is given, the existence and uniqueness of solutions follows both for a single matter component and for a combination of different kinds of matter, for example $\mu = \mu_{bar} + \mu_{rad} + \mu_{CDM} + \mu_{\nu}$ where we include cold dark matter (CDM) and neutrinos (ν). Initial data for such solutions at an arbitrary time t_0 (eg. today) consists of:

* The Hubble constant $H_0 = (\frac{\dot{a}}{a})|_{t=0} = 100h$km/sec/Mpc;

* A dimensionless density parameter $\Omega_{i0} = \kappa\frac{\mu_{i0}}{3H^2}$ for each type of matter present (labelled by i);

* If $\Lambda \neq 0$, either $\Omega_{\Lambda 0} = \frac{\Lambda}{3H_0^2}$, or the dimensionless *deceleration parameter* $q_0 = -(\frac{\ddot{a}}{a})|_{t=0}H_0^{-2}$.

If the equations of state for the matter are specified, a unique solution for $(a(t), \mu(t))$ is determined, i.e., a unique corresponding cosmic history. The total energy density is the sum of the terms Ω_{i0} for each type of matter present. Thus

$$\Omega_{m0} = \Omega_{bar0} + \Omega_{r0} + \Omega_{CDM0} + \Omega_{\nu 0}, \tag{2.10}$$

$$\Omega_0 = \Omega_{m0} + \Omega_{\Lambda 0}, \tag{2.11}$$

give us the total matter energy density and the total energy density(the cosmological constant included).

If the pressure term p is negligible relative to the matter term μ in (2.7), then we get

$$q_0 = \frac{1}{2}\Omega_{m0} - \Omega_{\Lambda 0}, \tag{2.12}$$

thus showing that a cosmological constant Λ can cause an accelerated cosmic expansion ($q_0 < 0$). A vanishing Λ ($q_0 = \frac{1}{2}\Omega_{m0}$) on the other hand shows that matter can cause deceleration of the expansion. The Friedmann Eqn.(2.9) evaluated at the present time t_0 gives the spatial curvature

$$K_0 = \frac{k}{a_0^2} = H_0^2(\Omega_0 - 1). \tag{2.13}$$

We speak of *open*, *flat* or *closed* universes depending on whether $K_0 < 0$ ($\Omega_0 < 1$), $K_0 = 0$ ($\Omega_0 = 1$), or $K_0 > 0$ ($\Omega_0 > 1$). To completely define the geometry of the homogeneous and isotropic Universe, we define the density parameter Ω_k to measure the curvature of space, in such a way that

$$\Omega_m + \Omega_\Lambda + \Omega_k = 1. \tag{2.14}$$

Owing to their extreme geometrical simplicity the FLRW models are the standard models of modern cosmology [23]. They have been used extensively in the analysis of the gravitational effect of matter (*dust* and radiation) on the global evolutionary dynamics of the Universe and the local background physics of the evolution of matter itself.

2.2.1 Solutions of the EFEs

For dust and non-interacting radiation (2.9) becomes

$$3\frac{\dot{a}^2}{a^2} = \frac{D}{a^3} + \frac{R}{a^4} + \frac{\Lambda}{3} - 3\frac{k}{a^2}, \tag{2.15}$$

where $D \equiv \kappa\mu_{(d)0}a_0^3$ and $R \equiv \kappa\mu_{(r)0}a_0^4$ with $\dot{D} = 0$ and $\dot{R} = 0$. The behaviour of the equation depends on the value of Λ.

For $\Lambda = 0$ the Universe starts off at a very dense initial state, where its energy density and curvature tend to infinity. Its future fate depends on the value of the spatial curvature, or equivalently the density parameter Ω_0. The Universe expands forever if ($k = 0 \Leftrightarrow \Omega_0 = 1$) or ($k < 0 \Leftrightarrow \Omega_0 < 1$), but collapses to a future singularity if ($k > 0 \Leftrightarrow \Omega_0 > 1$). Thus $\Omega_0 = 1$ corresponds to the critical density μ_{crit} separating $\Lambda = 0$ FLRW models that recollapse in the future from those that expand forever, and Ω_0 is just the ratio of the matter density to this critical density:

$$\Omega_{crit} = 1 \Leftrightarrow \kappa\mu_{crit} = 3H_0^2 \Rightarrow \Omega_0 = \frac{\kappa\mu_0}{3H_0^2} = \frac{\mu_0}{\mu_{crit}}. \tag{2.16}$$

When $\Lambda < 0$, all solutions start at a singularity and recollapse.

When Λ is positive, there are some interesting possible scenarios:

* If $k = 0$ or $k = -1$, all solutions start at a singularity and expand forever.

* If $k = +1$, there can again be models with a singular start, either expanding forever or collapsing to a future singularity. However in this case a static solution (the Einstein static universe) is also possible, as well as models asymptotic to this static state in either the future or the past.

* Models with $k = +1$ can bounce (collapsing from infinity to a minimum radius and re-expanding).

In the standard cosmological models, the Universe contains a realistic mixture of matter components (baryons, radiation, neutrinos, cold dark matter, a scalar field, and perhaps a cosmological constant). Here are some very specific models with simple expanding solutions [23]:

* The Einstein-de Sitter model, for which ($p = 0, \Lambda = 0, k = 0$) $\Rightarrow \Omega_0 = 1$. This is the simplest expanding non-empty solution:

  $$a(t) = Ct^{2/3}, \tag{2.17}$$

  starting from a singular state at time $t = 0$ (C is an arbitrary constant). Its age (the proper time since the start of the Universe) when the Hubble constant takes the value H_0 is $\tau_0 = \frac{2}{3H_0}$. This is a good model of the expansion of the universe since radiation domination ended until the recent times when a cosmological constant started to dominate the expansion. It is also a good model of the far future universe if $k = 0$ and $\Lambda = 0$.

* The Milne model, for which $(\mu = p = 0, \Lambda = 0, k = -1) \Rightarrow \Omega_0 = 0$, giving a linearly expanding empty solution:

$$a(t) = Ct. \tag{2.18}$$

This is just a flat spacetime as seen by a uniformly expanding set of observers, singular at $t = 0$. Its age is $\tau_0 = \frac{1}{H_0}$. It is a good model of the far future universe if $k < 0$ and $\Lambda = 0$.

* The de Sitter universe, for which $(\mu = p = 0, \Lambda \neq 0, k = 0) \Rightarrow \Omega_0 = 0$, giving the steady state expanding empty solution:

$$a(t) = Ce^{Ht}, \tag{2.19}$$

where C and H are constants. In this model, the Universe expands at a constant rate, and hence there is no start and its age is infinite. It is a good model of the far future universe for those cases which expand forever with $\Lambda > 0$. It can alternatively be understood as a solution with $\Lambda = 0$ and containing matter with the exceptional equation of state $\mu + p = 0$.

Other FLRW forms of the de Sitter Universe include: a geodesically complete form with $k = +1$, $a(t) = a_0 cosh(Ht)$ (a regular bounce), and another geodesically incomplete form with $k = -1$, $a(t) = a_0 sinh(Ht)$ (a singular start). The fact that there are no preferred time-like directions or space sections in this spacetime of constant curvature has led to there being no uniqueness in the solutions of such a universe . In general, however, the Raychaudhuri equation and the SEC lead to the following theorem [23]:

Friedmann-Lemaître Universe Singularity Theorem 1 *In a FL universe with* $\Lambda \leq 0$ *and* $\mu + 3p > 0$ *at all times, at any instant* t_0 *when* $H_0 \equiv (\frac{\dot{a}}{a})_0 > 0$ *there is a finite time* $t_* : t_0 - (\frac{1}{H_0}) < t_* < t_0$*, such that* $a(t) \rightarrow 0$ *as* $t \rightarrow t_*$*; the Universe starts at a spacetime singularity there, with* $\mu \rightarrow \infty$ *and* $T \rightarrow \infty$ *if* $\mu + p > 0$.

This state of singularity, famously known as the **Big Bang**, is generally taken to be not only the start to matter and spacetime, but to physics itself.

2.2.2 Distance Scales and Ages in Cosmology

Cosmography, measurement of the 'distance' between two observed cosmological objects or events, is one of the most challenging tasks in cosmology. This is due to the expansion of the Universe, which makes the comoving distances between any two objects to constantly change. We often use such directly observable quantities as the luminosity of a quasar,the redshift of a galaxy, or the angular size of the CMB power spectrum acoustic peaks to indirectly measure another quantity not directly observable, but mathematically calculable, such as the comoving coordinates of the quasar or the galaxy.

All distance measures in the FLRW geometry somehow make use of the fact that light travels on radial null geodesics $x^a(\lambda)$ in spacetime [23, 25], the tangent of which is $k^a = \frac{dx^a}{d\lambda}$ such that $k^a{}_{;b}k^b = 0, k^a k_a = 0$. Thus using Eqn. 2.2 for a photon emitted at a time t_e the comoving radial distance $r(t_e, t_o)$ it travels before it is received by an observer (receiver) at a later time t_0 is given by

$$ds^2 = 0, \tag{2.20}$$

with

$$d\theta = 0 = d\phi, \tag{2.21}$$

and hence

$$r(t_e, t_o) = \int_{t_e}^{t_o} \frac{dt}{a} = \int_{a_e}^{a_o} \frac{da}{a\dot{a}}. \tag{2.22}$$

The Hubble constant H_0 (the 0 referring to the present epoch) is a key parameter in cosmological distance measures since the age and the size of the observable region of the Universe scales with its present rate of expansion. The proportionality between the recession speed v and distance d between two cosmological objects in the expanding Universe is given by

$$v = H_o d. \tag{2.23}$$

The exact value of H_0 is yet to be determined, but present estimates show that it lies somewhere around [22, 25, 26]

$$H_0 = 72 \pm 5\text{kms}^{-1}Mpc^{-1} \quad \text{or} \quad H_0 = 100h\text{kms}^{-1}Mpc^{-1}, \tag{2.24}$$

h being a normalized dimensionless number parameterizing our uncertainty in H_0. The corresponding Hubble time given by $\tau_H = \frac{1}{H_0}$ is the time taken for light to traverse a Hubble distance $D_H = \frac{c}{H_0}$. Thus these quantities give us a rough estimate of the scales of the Universe: $\tau_H \simeq 1.2 - 1.5 \times 10^{10}$ years, $D_H \simeq 1.2 - 1.5 \times 10^{26}$m $\simeq 3700 - 4700$Mpc.

Redshift

The redshift z of an object emitting a wavelength λ_e (and a frequency ν_e) and observed with wavelength λ_o (and a corresponding frequency ν_o) is defined to be the fractional Doppler shift of its emitted light (photons) due to its radial motion [23, 25]:

$$z \equiv \frac{\lambda_o}{\lambda_e} - 1 = \frac{\nu_e}{\nu_o} - 1. \tag{2.25}$$

In general,

$$1 + z = (1 + z_c)(1 + z_v), \tag{2.26}$$

where z_v is the redshift due to the local peculiar motion of the object whereas z_c is the cosmological redshift due to the expansion of the Universe given in terms of the scale factor as

$$1 + z_c = \frac{a(t_0)}{a(t_e)}, \tag{2.27}$$

For comoving objects, we note that $z_v = 0$ and hence $z_c = z$.

Distance Types

The **Proper Distance** D_p is defined by [24]:

$$D_p = \int dt = \int_0^z \frac{dz'}{[(1+z')H(z')]}. \tag{2.28}$$

The **Line-of-Sight Comoving Distance** D_c between two nearby objects in the Universe is the constant distance between them at any epoch if the two objects are moving with the Hubble flow, and is defined by [24, 25]

$$D_c = \int \frac{cdt}{a} = D_H \int_0^z \frac{dz'}{\sqrt{\Omega_M(1+z')^3 + \Omega_k(1+z')^2 + \Omega_\Lambda}}. \tag{2.29}$$

The **Transverse Comoving Distance** D_M measures the distance between two events at the same redshift but separated by a certain angle in the sky and is mathematically defined by

$$D_M = \begin{cases} D_H \frac{1}{\Omega_k} sinh[\sqrt{\Omega_k} D_c/D_H] & \text{for } \Omega_k > 0, \\ D_c & \text{for } \Omega_k = 0, \\ D_H \frac{1}{|\Omega_k|} sinh[\sqrt{|\Omega_k|} D_c/D_H] & \text{for } \Omega_k < 0. \end{cases}$$

The **Angular Diameter Distance** D_A is one of the most important and widely used distances in cosmology. It defines the ration of an object's transverse physical size to its radian angular size:

$$D_A = \frac{\ell}{\Delta\theta}. \tag{2.30}$$

It is also related to the comoving distance through

$$D_M = (1+z)D_A. \tag{2.31}$$

Another very important distance measure of astronomical importance is the Luminosity Distance D_L

$$D_L \equiv \sqrt{\frac{L}{4\pi f}}, \tag{2.32}$$

which relates two bolometric quantities, the Luminosity L and the flux f of a distant object such as a supernova. Here also there is a simple mathematical relation between this quantity and the comoving and angular diameter distances:

$$D_L = (1+z)D_M = (1+z)^2 D_A. \tag{2.33}$$

Sometimes we might be interested in predicting the evolutionary properties of objects at high redshift. This can be achieved by taking the difference between the present age of the Universe t_0 and the age t_e of the Universe when the photons were emitted as measured by a hypothetical observer attached to the object. This difference in time gives us what is known as the **Lookback Time** t_L of the object. Mathematically this is given as

$$t_L = t_H \int_0^z \frac{dz'}{(1+z')\sqrt{\Omega_M(1+z')^3 + \Omega_k(1+z')^2 + \Omega_\Lambda}}. \tag{2.34}$$

The **Light Travel Distance LTD** is the distance the emitted photons travel during the lookback time:

$$LTD = ct_L. \tag{2.35}$$

2.2.3 Causal and Visual Horizons

Due to the finite speed of light, and the fact that causal effects cannot propagate faster than light, the feature of cosmological structure formation and our observational knowledge of the cosmos is highly constrained [23, 27]. We can neither influence nor be influenced by regions outside our past null cone; in short, there are regions of the Universe beyond which we have no access. The boundary separating the accessible part of the Universe from the inaccessible is called a *horizon*. We have two main types of horizons in cosmology:

The **Particle Horizon**

$$\chi_{ph} = \int_0^{t_0} \frac{dt'}{a(t')} \tag{2.36}$$

is the maximum distance particles could move to an observer during the Universe's period of existence. In other words, this is the largest region of spacetime we could have probed so far. The physical distance to the matter comprising this horizon is

$$D_{ph} = a(t_0)\chi_{ph}. \tag{2.37}$$

This horizon exists for all FLRW spacetimes for all ordinary matter (dust) and radiation, since (2.36) converges in those cases. This horizon always grows, and once matter enters the horizon, it never leaves [23]. Apart from limiting causality (and hence cosmic structure), the particle horizon can also set limits on what is testable in the Universe. It has been found that in a perturbed FLRW Universe, once a causal contact has taken place, it remains forever (until the end of the Universe) [23].

The other commonly discussed horizon in cosmology is the **Event Horizon**. It represents the largest comoving distance that light emitted now can ever reach at an observer any time in the future:

$$\chi_{eh} = \int_{t_0}^{t_{\inf}} \frac{dt'}{a(t')}. \tag{2.38}$$

It sets the maximum extent to the particle horizon, and is said to exist if the integral in (2.38) does not diverge. One common application of event horizons is in the general relativistic description of blackholes, where the escape velocity of the blackhole inside the horizon is superluminal. Light emitted from beyond the horizon can never reach the observer and time itself stops at the boundary. Within the horizon, all light-like paths, and hence all paths in the forward light cones of particles within the horizon, are warped so as to fall farther into the hole.

It is worth mentioning that the comoving Hubble radius $\lambda_H = 1/aH$ determines the relevant physical scales for local causal influences in an expanding universe. This radius increases during any standard evolutionary history of the Universe (such as the radiation-dominated and dust-dominated epochs).

2.3 Inflation

Despite the impressive successes the Big Bang model of the Universe has enjoyed, there are many serious puzzles that it also leaves unanswered.

2.3.1 The Horizon Problem

As we have seen earlier, (2.36) converges since $a \propto t^{\frac{1}{2}}$ in the early radiation-dominated epoch of the Universe and at later times the Universe enters the dust-dominated phase, in which case [28]

$$D_{ph} \simeq \frac{6000}{\sqrt{\Omega(z)}} h^{-1} Mpc. \tag{2.39}$$

This implies that at last scattering the particle horizon was only ~100 Mpc in size and subtending an angle of ~ 1 degree in the sky. But this is in contradiction with the large number of causally disconnected patches we see on the CMB sky, all at the same temperature.

2.3.2 The Flatness Problem

The evolution of the curvature density parameter

$$\begin{aligned}\Omega_k \equiv -\frac{c^2 k}{a^2 H^2} = 1 - \Omega_m - \Omega_\Lambda = \left[\frac{(1+z)H_0}{H(z)}\right]^2 \\ = \frac{\Omega_{k,0}}{(1+z)\Omega_{d,0} + (1+z)^2\Omega_{r,0} + (1+z)^{-2}\Omega_{\Lambda,0} + \Omega_{k,0}},\end{aligned} \tag{2.40}$$

where the matter content of the Universe comprises of only dust and radiation can be shown to be [31]

$$\dot{\Omega}_k = 2\Omega_k H q = \Omega_k H(\Omega_d + 2\Omega_r - 2\Omega_\Lambda). \tag{2.41}$$

In the absence of a cosmological constant, we see that $\Omega_d + 2\Omega_r > 0$. If at high enough redshift (i.e., at an early cosmic time) Ω_k were slightly different from zero, then the spatial curvature would rapidly evolve away from the spatially flat case, i.e., Ω_k would either approach 1 in the open case or diverge to $-\infty$ in the closed universe case. But a positive Λ term would dominate the dust and radiation terms at some finite cosmic time such that Ω_k is *fine-tuned* to 0. The above relations show that no matter how much $\Omega_{k,0}$ is different from zero, Ω_z at high z could not have been significantly far from zero, the deviation being only 1 in 10^{60} at the Planck epoch, for example. The flatness problem is then: why the fine tuning?

Another problem from particle physics is the so-called **Anti-matter problem** [28]. At $kT \geq m_p c^2$, m_p being the proton mass, there are roughly equal numbers of photons(γ), protons (p) and antiprotons($\bar{p}$) in equilibrium, whereas the ratios today stand at $N_p/N_\gamma \sim 10^{-9}$ and $N_{\bar{p}}/N_p \sim 0$. Since baryon number is a conserved quantity, it would then imply the necessity that $N_p/N_{\bar{p}} = 1 + O(10^{-9})$ during baryogenesis. What, then, is the source of this initial asymmetry?

2.3.3 The Structure Problem

The Universe is not absolutely homogeneous. What brought about the matter clumping that finally led to cosmic structures like galaxies and clusters? It is generally believed that inhomogeneities must have existed in the primordial matter to account for the structures we observe today. But according to perturbation theory, any small inhomogeneities in the primordial matter rapidly grow into large ones through gravitational self-interaction. This implies that an extreme smoothness in the primordial matter must have existed for these inhomogeneities of galactic scale to exist at present. If we extrapolate further to 10^{-45}s after the Big Bang, then an almost perfect smoothness, but not quite an absolute smoothness, must be assumed to have existed. Why the primordial matter must have been so smooth is not accounted for in the standard Big Bang FLRW model. This problem is variously called the **smoothness problem**, the **homogeneity problem** or the **structure problem** all for obvious reasons [28, 29].

2.3.4 The Magnetic Monopole Problem

This problem suggests that if the Universe were very hot at early times (this amounts to the Universe kicking off with a Hot Big Bang), a large number of heavy, stable magnetic monopoles would be produced, and should be detected observationally.

To tackle these problems,a special epoch of exponential expansion, termed as *inflation*, was proposed in the early 1980s. This epoch is characterized by a decreasing comoving Hubble radius:

$$\frac{d(1/aH)}{dt} < 0 \Leftrightarrow \ddot{a} > 0, \tag{2.42}$$

leading to the Hubble radius being swallowed by the outpacing accelerated growth of the expansion. As a consequence,all physical conditions become correlated on scales much larger than the Hubble radius [30], *smoothening* the primordial matter fluctuations along the way.

The inflationary theory of cosmic evolution starts by assuming that at some point in the early Universe, the matter energy density was dominated by some form of matter, called *scalar field* ϕ with a negative pressure. In the absence of the cosmological constant, the Raychaudhuri equation (2.7) reduces to

$$\ddot{a} = -\frac{1}{6}(\mu + 3p)a \tag{2.43}$$

which means that

$$p < -\frac{1}{3}\mu \tag{2.44}$$

in order for inflation to occur. Thus the Friedmann equation (2.9) reads

$$\frac{\dot{a}^2}{a^2} = \frac{1}{3}\mu - \frac{k}{a^2}. \tag{2.45}$$

Since the scale factor must increase faster than $a(t) \propto t$, the curvature term becomes negligible.

The scalar field is assumed to have the energy - momentum tensor

$$T_{ab} = (\partial_a \partial_b \phi)(\partial_b \phi) - g_{ab}\left[\frac{1}{2}(\partial_c \phi)(\partial^c \phi) - V(\phi)\right] \tag{2.46}$$

and the Lagrangian

$$\mathcal{L} = \frac{1}{2} g^{ab}(\partial_a \phi)(\partial_b \phi) - V(\phi) \tag{2.47}$$

and is governed by the Euler-Lagrange Equation

$$\Box^2 \phi + \frac{dV}{d\phi} = 0, \tag{2.48}$$

where $\Box^2 \equiv \nabla^a \nabla_a = g^{ab}\nabla_a \nabla_b$ is the covariant d'Alembertian operator. Treating ϕ as a perfect fluid with negligible spatial variations [31] we get the energy and the pressure associated with it to be

$$\mu_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi),$$
$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi). \tag{2.49}$$

The equation of motion for the scalar field will then be

$$\dot{\mu}_\phi + 3(\mu_\phi + p_\phi)H = 0 \Rightarrow \ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0. \tag{2.50}$$

This, coupled with the Friedmann equation in the scalar field-dominated universe ,

$$H^2 = \frac{1}{3}\left[\frac{1}{2}\dot{\phi}^2 + V(\phi)\right] \tag{2.51}$$

completely describe the evolution of the scalar field and the expansion during this epoch. The condition (2.44) and Eqn.(2.49) will constrain ϕ and V such that for inflation to occur $\dot{\phi}^2 < V(\phi)$.

Making a further approximation, called the *slow-roll approximation* $\dot{\phi}^2 \ll V(\phi)$ makes Eqns (2.50, 2.51) analytically solvable. This approximation will simplify our inflation equations into

$$3H\dot{\phi} = -\frac{dV}{d\phi}, \tag{2.52}$$

$$H^2 = \frac{1}{3}V(\phi), \tag{2.53}$$

$$\dot{H} = -\frac{1}{2}\dot{\phi}^2. \tag{2.54}$$

Eqn.(2.54) is not an independent equation on its own, but tells us how the rates of change of the Hubble parameter and the scalar field are related. These equations further tell us that the scale factor grows exponentially, keeping the Hubble parameter *constant*:

$$a(t) \propto e^{Ht} \Rightarrow a(t) \propto e^{\sqrt{\frac{1}{3}V(\phi)}t}. \tag{2.55}$$

If the Universe followed the standard radiation-dominated epoch after inflation, then

$$a(t) \propto t^{\frac{1}{2}} \propto \frac{1}{T}, \tag{2.56}$$

where T is a measure of the typical particle energy: $E \sim k_B T$. This means that

$$\frac{a_{inf}}{a_0} \sim \left(\frac{t_{inf}}{t_0}\right)^{\frac{1}{2}} \sim \frac{T_0}{T_{inf}}. \tag{2.57}$$

CMB analyses show that at present $T_0 \sim 3$K and we have seen earlier that $t_0 \sim \frac{1}{H_0} \sim 10^{18}$s. Using (2.40) we can write

$$\frac{\Omega_{k,inf}}{\Omega_{k,0}} = \left(\frac{H_0}{H_{inf}}\right)^2 \left(\frac{a_0}{a_{inf}}\right)^2 \sim \frac{t_{inf}}{t_0}. \tag{2.58}$$

Since inflation is generally thought to have occurred somewhere between the Planck era and the period of GUT phase transition, particle physics has it that the ratio (2.58) lies somewhere between 10^{-60} and 10^{-54}, thus showing that an extreme fine-tuning must have occurred for Ω_k to attain its present value ($-0.5 < \Omega_{k,0} < 0.5$). The implication of this is that the scale factor must have grown by a factor of about $10^{27} - 10^{30}$ or about 60-70 e-foldings during inflation, thus solving the flatness problem. Once again, if the Universe underwent a radiation-dominated like expansion during its earliest stages, the particle horizon at inflation would be

$$D_{ph,inf} = 2ct_{inf}, \tag{2.59}$$

resulting in a size of a causally connected region about $10^{-33} - 10^{-27}$m across, in sharp contrast to the $10^{-3}m - 1m$ wide region obtainable from (2.58) [31]. This goes in accord with the previous discussion that the scale factor must have grown by about 60-70 e-foldings, and hence solves the horizon problem.

As far as the monopole problem is concerned, supercooling of the Universe occurs at the inflationary phase transition-suppressing the production of the monopoles.

Inflation is an extremely short period scenario ($\sim 10^{-36}s - \sim 10^{-32}s$) after the Big Bang and it ends when the scalar field is converted into radiation, a process called *reheating*. Reheating lead the Universe to the hot Big Bang epoch.

2.4 Dark Matter and Dark Energy

Thanks to the advances made in observational cosmology, there are now different techniques of determining the energy and matter content of the Universe, such as temperature fluctuations in the CMB, distance-luminosity relations analyses in supernovae, large scale structure and big bang nucleosynthesis. In the Concordance Model, the presently accepted energy budget breakdown of the Universe shows that [26, 32–35]

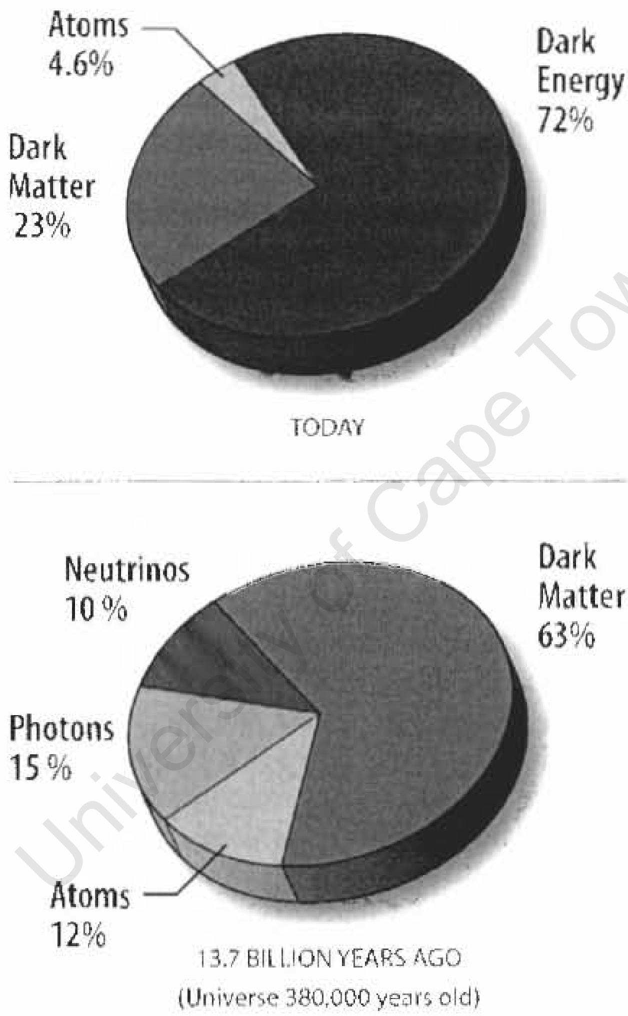


Figure 2.1: The energy content of the Universe then and now. Credit: NASA/WMAP Science Team (2008).

* $\Omega_m \sim 0.28$, i.e., $\sim 28\%$ of the total energy of the Universe exists in the form of non-relativistic matter, of which only a tiny fraction ($\Omega_b \sim 0.046$) is known to exist in baryonic matter form, whereas the remaining matter is not as yet properly understood. Known as as *Dark Matter*, this missing matter is believed

to exist in two forms: non-relativistic cold dark matter ($\Omega_{CDM} \sim 0.23$) and non-relativistic hot dark matter ($\Omega_{HDM} < 0.0152$).

* $\Omega_\Lambda \sim 0.72$, i.e., the largest portion ($\sim 72\%$) of the Universe's energy budget goes into the unknown! This unknown form of energy is termed *Dark Energy*. It was discovered when, in 1998, observational data from supernovae hinted at an accelerating expansion of the present Universe. According to Eqns.(2.7) and (2.9), this would be possible if the Universe is dominated by a component "fluid" with negative pressure. This called for the reinstatement of the cosmological constant Λ as the new component fluid with $w_\Lambda = -1$. Ironically, the same cosmological constant that Einstein once used to produce a static universe is now held responsible for speeding up cosmic expansion.

2.5 The Anthropic Principle(s)

One of the most fundamental philosophical questions in cosmology is why the Universe appears the way it does, with all the different parameters (and their numerical values). A proposed answer to this question lies in the *Anthropic Principle*: We see the universe the way it is because we exist" [36, 37]. But why do we (intelligent life forms) exist, and how did we come into existence in the first place? There are two main versions of the Anthropic Principle that try to answer these questions.

A. The Weak Anthropic Principle (WAP)

According to this principle, in a universe that is large or infinite in spacetime, the conditions necessary for the development of intelligent life will be met only in certain regions that are limited in space and time. The intelligent beings in these regions should therefore not be surprised if they observe that their locality in the Universe satisfies the conditions that are necessary for their existence [36]. Thus the various physical relations between fundamental quantities and the initial/boundary conditions in nature are such that intelligent observers should develop to observe them, and hence restrict the times and spaces from which the Universe can be observed. Such seemingly coincidental phenomena that raise questions such as: Why did the Big Bang occur 13.7 billion years ago? Why is the Universe so smooth (homogeneous and isotropic)? Why does the Universe's expansion accelerate now, and not much earlier (since the Big Bang)? all resort to the WAP for an explanation.

B. The Strong Anthropic Principle(SAP)

An even bolder principle of anthropic nature is the Strong Anthropic Principle, which claims that an intelligent existence is a necessary condition for any universe model to make sense [23]. This principle claims that there are many different regions with varying initial conditions (this could either mean there being multi-universes or many different regions of our Universe with different initial conditions)where the laws of physics may not necessarily be the same. In only a few of these regions such as this

Universe would there be just the right conditions for intelligent observers to develop and wonder why the Universe is the way it is, for which the answer would be: Had it been slightly different, we would not have been here to ask and wonder.

There is also a much more controversial version of the Anthropic Principle, namely the Final Anthropic Principle (FAP). This 'principle' states that intelligent observers must necessarily evolve in the Universe and once they come into existence, they will never die out (or never so until the end of the Universe itself) [37]. The 'principle' has been deemed too unscientific to be taken seriously that at times it has been mocked at as a Completely Ridiculous Anthropic Principle (CRAP) [23].

2.6 Open Issues

As we mentioned earlier, the Concordance Model is built upon a universe where more than 95% of its matter/energy content is yet to be known. The inflationary scenario is still a controversial issue (mainly because what triggered it is not precisely known) and the Big Bang scenario by itself is a problem, as symmetry (and hence all the laws of physics) break at the Big Bang: no one knows what happened exactly at the Big Bang and "before". This means that either the [Concordance] Model has to undergo a rigorous scrutiny or a brand new cosmological theory has to come up.

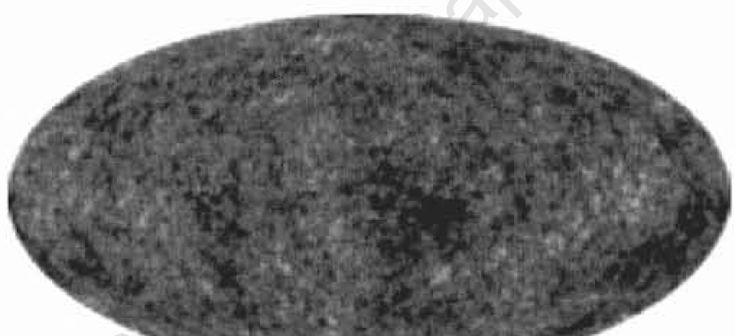


Figure 2.2: The Cosmic Microwave Background Radiation.

One often forwarded suggestion is that the FLRW metric may not be the right geometry for all scales, i.e., isotropy and homogeneity of spacetime could be broken as we go to higher and higher redshifts. In such a scenario, there could be two potential ways out of the dark energy conundrum:

* The Universe is inhomogeneous and hence only inhomogeneous cosmological models, such as the Lemaître-Tolman-Bondi (LTB) models can best describe it. We note here that Cosmological Principle is no longer valid ;

* Back reaction, where inhomogeneities around a FLRW background are treated but the cosmological dynamics is described by averaging the nonlinear EFEs.

Another suggestion is, as we mentioned in the Introduction, that cosmology based on the standard General Relativity theory may not be the right description and hence that it should be studied through modified-gravity approach, to which the present thesis is dedicated, with a particular emphasis on structure growth in $f(R)$ gravity models.

Chapter 3

THE 1+3 COVARIANT FORMALISM

We have made a passing mention earlier that the Universe is not perfectly smooth: primordial fluctuations seeded the formation of structures at large scales and hence the real Universe is slightly lumpy. Theories of cosmological perturbations tell us how these small fluctuations grow and form the large scale structures (galaxies, clusters and superclusters) in the real Universe.

There are basically two approaches to cosmological perturbations: the standard *Gauge-Invariant Perturbation Theory* and the *Covariant Perturbation Theory*.

Pioneered by Lifshitz (1946) and later developments made by Bardeen (1980) and Kodama and Sasaki (1984) the standard gauge-invariant approach is based on the foliation of the background spacetime with hypersurfaces and perturbing away from it. It is a non-local, linear theory applicable once we have fixed our metric, i.e., it is coordinate-dependent. Since it is a linear theory, nonlinear effects are not accounted for. The main shortcoming of this approach, however, is in handling the unphysical gauge modes that are inherent to the theory [5–10, 12–14, 16–19, 38–41].

The covariant formalism is a way of describing spacetime via covariantly defined variables with respect to a partial frame formalism such as $1+3$ or $1+1+2$. It is a suitable method to describe physics and geometry by tensor quantities and relations valid in all coordinate systems. Although earlier attempts had been made by Hawking, Ehlers and Olson, the covariant and gauge-invariant method was first formulated by Ellis and Bruni and further developed by Bruni, Dunsby and Ellis [16–19]. It is a local, covariant theory based on threading spacetimes with frames. This approach differs from the standard one in that it starts from the theory and reduces to linearities in a particular background. Nonlinearities can be accommodated, but the main advantage of this approach is that no unphysical gauge modes appear here.

3.1 The 1+3 Covariant Approach

In this approach, a fundamental observer slices spacetime into time and space (hence $1+3$ to indicate the number of dimensions involved in each slice) to investigate deviations from homogeneity and isotropy of the Universe. It provides an alternative description of spacetime in terms of scalars, 3-vectors and projected symmetric trace-free(PSTF) 3-tensors and their corresponding equations using the Ricci and Bianchi identities [20, 42, 43].

3.1.1 Covariant Variables

The 4-velocity The average motion of a cosmological fluid at a point can always be represented by a family of preferred worldlines in spacetime, with a uniquely defined average 4-velocity with respect to fundamental observers associated with the worldlines given as

$$u^a = \frac{dx^a}{d\tau}, \qquad u_a u^a = -1, \tag{3.1}$$

where τ is proper time measured along the worldlines [42]. For any u^a, there exist unique projection tensors

$$U^a{}_b = -u^a u_b \Rightarrow U^a{}_c U^c{}_b = U^a{}_b,\ U^a{}_a = 1,\ U_{ab}u^b = u_a \tag{3.2}$$

$$h_{ab} = g_{ab} + u_a u_b \Rightarrow h^a{}_c h^c{}_b = h^a{}_b,\ h^a{}_a = 3, h_{ab}u^b = 0. \tag{3.3}$$

$U^a{}_b$ projects *along* the 4-velocity vector u^a whereas h_{ab} projects the metric properties of the instantaneous restspaces of observers *orthogonal* to u^a. The volume element for the 3-restspaces is defined by

$$\eta_{abc} = u^d \eta_{dabc} \Rightarrow \eta_{abc} = \eta_{[abc]},\ \eta_{abc}u^c = 0, \tag{3.4}$$

where η_{abcd} is the 4-dimensional volume element such that

$$\eta_{abcd} = \eta_{[abcd]} = 2\eta_{ab[c}u_{d]} - 2u_{[a}\eta_{b]cd}. \tag{3.5}$$

In particular, $\eta_{0123} = \sqrt{|det\ g_{ab}|}$.
η_{abc} satisfies the following identities [43]:

$$\eta^{abc}\eta_{def} = 3! h^{[a}{}_d h^b{}_e h^{c]}{}_f \tag{3.6}$$

$$\eta^{abc}\eta_{cef} = 2! h^{[a}{}_e h^{b]}{}_f \tag{3.7}$$

$$\eta^{abc}\eta_{bcf} = 2! h^a{}_f \tag{3.8}$$

$$\eta^{abc}\eta_{abc} = 3! \tag{3.9}$$

Since it is a time-space split formalism, we define a covariant time derivative *along* the fundamental worldlines

$$\dot{T}^{a...b}{}_{c...d} \equiv u^e \nabla_e T^{a...b}{}_{c...d} \tag{3.10}$$

and a fully *orthogonally* projected covariant derivative

$$\tilde{\nabla}_e T^{a...b}{}_{c...d} \equiv h^f{}_e h^a{}_g ... h^b{}_i h^t{}_c ... h^m{}_d \nabla_f T^{g...i}{}_{t...m} \tag{3.11}$$

for any tensor $T^{a...b}{}_{c...d}$. It is worth mentioning here that $\tilde{\nabla}$ is not the same as the 3-dimensional (spatial) covariant derivative ${}^3\nabla$ unless u^a is vorticity-free. Orthogonal projections of vectors, the orthogonally PSTF part of tensors, and orthogonal projections of covariant time derivatives along u^a (known as *'Fermi derivatives'*) are denoted by angular brackets as follows:

$$v^{\langle a \rangle} = h^a{}_b v^b, \qquad T^{\langle ab \rangle} = \left[h^{(a}{}_c h^{b)}{}_d - \frac{1}{3} h^{ab} h_{cd} \right] T^{cd}, \tag{3.12}$$

$$\dot{v}^{\langle a \rangle} = h^a{}_b \dot{v}^b, \qquad \dot{T}^{\langle ab \rangle} = \left[h^{(a}{}_c h^{b)}{}_d - \frac{1}{3} h^{ab} h_{cd} \right] \dot{T}^{cd}. \tag{3.13}$$

3.1.2 Kinematic Quantities

These are the quantities that tell us about the overall spacetime kinematics, i.e., the expansion, shear and vorticity of the fundamental worldlines, obtained by splitting $\nabla_a u_b$ into its irreducible parts defined by their symmetry properties:

$$\nabla_a u_b = \tilde{\nabla}_a u_b - u_a \dot{u}_b = \frac{1}{3}\Theta h_{ab} + \sigma_{ab} + \omega_{ab} - u_a \dot{u}_b, \tag{3.14}$$

$\Theta \equiv \tilde{\nabla}_a u^a$ being the volume rate of expansion of the fluid with $\Theta = 3H$. $\sigma_{ab} \equiv \tilde{\nabla}_{\langle a} u_{b \rangle}$ is the symmetric, trace-free rate of shear tensor ($\sigma_{ab} = \sigma_{(ab)}, \sigma_{ab} u^b = 0, \sigma^a{}_a = 0$) and describes the rate of distortion of the fluid flow; and $\omega_{ab} \equiv \tilde{\nabla}_{[a} u_{b]}$ is the skew-symmetric vorticity tensor ($\omega_{ab} = \omega_{[ab]}, \omega_{ab} u^b = 0$), describing the rotation of the fluid relative to a non-rotating(Fermi-propagated)frame. The vorticity vector ω^a is defined to be

$$\omega^a \equiv \frac{1}{2} \eta^{abc} \omega_{bc} \Rightarrow \omega_a u^a = 0, \ \omega_{ab} = \eta_{abc} \omega^c. \tag{3.15}$$

The following definitions are frequently used in the literature:

$$\sigma^2 \equiv \frac{1}{2} \sigma_{ab} \sigma^{ab} \geq 0, \qquad \omega^2 \equiv \frac{1}{2} \omega_{ab} \omega^{ab} = \omega_a \omega^a \geq 0. \tag{3.16}$$

The relativistic acceleration vector

$$a_a \equiv \dot{u}_a = u_{a;b} u^b \tag{3.17}$$

represents the effects of non-gravitational forces (such as pressure) and vanishes for a particle moving only under gravitational or inertial forces [20].

The representative length scale is the cosmological scale factor $a(\tau)$ defined in terms of the expansion Θ and the Hubble parameter $H(\tau)$ as

$$\frac{\dot{a}}{a} = \frac{1}{3} \Theta = H. \tag{3.18}$$

Matter Description

The EMT specifies the matter-energy content of the Universe and can be decomposed along the fluid flow lines

$$T_{ab} = \mu u_a u_b + q_a u_b + u_a q_b + p h_{ab} + \pi_{ab}, \tag{3.19}$$

where

$$\mu = T_{ab}u^a u^b, \qquad q^a = -T_{bc}u^b h^{ca}, \qquad p = \tfrac{1}{3}(T_{ab}h^{ab}), \qquad \pi_{ab} = T_{cd}h^c{}_{\langle a}h^d{}_{b\rangle} \tag{3.20}$$

are the relativistic energy density with respect to u^a, the relativistic momentum density (energy flux), the relativistic isotropic pressure and the trace-free anisotropic pressure of the fluid. The trace of the EMT above is given by

$$T = T^a{}_a = 3p - \mu. \tag{3.21}$$

We note that

$$q_a u^a = 0 \qquad \pi^a{}_a = 0 \qquad \pi_{ab} = \pi_{(ab)}, \qquad \pi_{ab}u^b = 0. \tag{3.22}$$

In a perfect cosmological fluid both q^a abd π_{ab} vanish, and the equation of state $p = p(\mu, s)$, where s is the entropy density of the fluid, characterizes the thermodynamics of the fluid.

The following energy conditions generally put constraints on μ and p in any cosmological model [44]:

* Null Energy Condition (NEC): $\mu + p \geq 0$;
* Weak Energy Condition (WEC): $\mu \geq 0, \mu + p \geq 0$;
* Strong Energy Condition (SEC): $\mu + p \geq 0, \mu + 3p \geq 0$;
* Dominant Energy Condition (DEC): $\mu \geq |p|$.

Inflationary models typically violate the SEC.

The isentropic speed of sound of the fluid is defined by

$$c_s^2 = (\partial p/\partial \mu)_{s=const}. \tag{3.23}$$

In order for matter stability and causality to be preserved the acoustic speed should be constrained

$$0 \leq c_s^2 \leq 1. \tag{3.24}$$

Causality breaks beyond $c_s^2 = 1$ and matter with $c_s^2 < 0$ is unstable.

3.1.3 Geometry of Spacetime: the Ricci and Bianchi Identities

The field equations

$$R_{ab} - \tfrac{1}{2}Rg_{ab} + \Lambda g_{ab} = \kappa T_{ab} \tag{3.25}$$

determine the trace part of the gravitational field at each point in spacetime from the matter at that point. They are obtained using the Ricci and the Bianchi identities.

The Ricci identities

The curvature of spacetime is represented by the Riemann tensor R_{abcd} via the Ricci identities:

$$(\nabla_a\nabla_b - \nabla_b\nabla_a)u_c = R_{abc}{}^d u_d. \tag{3.26}$$

These identities tell us about the non-commutativity of the second covariant derivatives of a dual vector(the 4-velocity in this particular case). The Riemann tensor has the following symmetry properties :

$$R_{abcd} = R_{[ab][cd]} = R_{cdab}, \qquad R_{a[bcd]} = 0, \tag{3.27}$$

which result in its 20 independent components. The tensor can be split into its trace part given by the Ricci tensor $R_{ab} = R^c{}_{acb}$ (10 components) and the trace-free part define as the Weyl tensor or conformal curvature tensor C_{abcd} (10 components). The Weyl tensor is given by

$$C^{ab}{}_{cd} = R^{ab}{}_{cd} - 2g^{[a}{}_{[c}R^{b]}{}_{d]} + \frac{R}{3}g^{[a}{}_{[c}g^{b]}{}_{d]}. \tag{3.28}$$

Because it is trace-free,

$$C^a{}_{bad} = 0 \tag{3.29}$$

and can be split into its "electric" and "magnetic" parts, E_{ab} and H_{ab} respectively given by

$$E_{ab} \equiv C_{agbh}u^g u^h, \qquad H_{ab} = \tfrac{1}{2}\eta_{ae}{}^{gh}C_{ghbd}u^e u^d. \tag{3.30}$$

These tensors are each symmetric and trace-free in the local rest frame (LRF) of u^a:

$$\begin{aligned} E_{ab} &= E_{(ab)}, & H_{ab} &= H_{(ab)}, \\ E^a{}_a &= 0, & H^a{}_a &= 0, \\ E_{ab}u^b &= 0, & H_{ab}u^b &= 0. \end{aligned} \tag{3.31}$$

Using these tensors the Weyl tensor can be rewritten as

$$C_{abcd} = (\eta_{abpq}\eta_{cdrs} + g_{abpq}g_{cdrs})u^p u^r E^{qs} + (\eta_{abpq}g_{cdrs} + g_{abpq}\eta_{cdrs})u^p u^r H^{qs}, \tag{3.32}$$

where

$$g_{abcd} = g_{ac}g_{bd} - g_{ad}g_{bc}. \tag{3.33}$$

H_{ab} and E_{ab} represent the free gravitational field, enabling gravitational action at a distance (tidal forces, gravitational waves), and influence the motion of matter and radiation through the geodesic deviation for timelike and null vectors respectively [43]. Whereas the magnetic part does not have a Newtonian analogue, the electric part can be clarified by its Newtonian analogue

$$E_{\alpha\beta} = \phi_{,\alpha\beta} - \tfrac{1}{3}h_{\alpha\beta}\phi^{\delta}{}_{,\delta} \tag{3.34}$$

ϕ being the usual Newtonian gravitational potential [20].

The Bianchi Identities

The Bianchi identities read

$$\nabla_{[a}R_{bc]d}{}^{e} = 0. \tag{3.35}$$

The twice-contracted Riemann tensor yields the Ricci scalar $R \equiv R^a{}_a$. Using this and contracting twice (3.35) we obtain

$$\nabla_a R_c{}^a + \nabla_b R_c{}^b - \nabla_c R = 0 \Leftrightarrow \nabla^a G_{ab} = 0. \tag{3.36}$$

The *trace equation*

$$R = \mu - 3p + 4\Lambda \tag{3.37}$$

can be obtained by substituting the trace of the EMT into the trace of the EFEs (3.25). Using this result we can write out the $1+3$-split of the Ricci tensor

$$R_{ab} = \tfrac{1}{2}(\mu + 3p - 2\Lambda)u_a u_b + \tfrac{1}{2}(\mu - p + 2\Lambda)h_{ab} + 2u_{(a}g_{b)} + \pi_{ab}. \tag{3.38}$$

Spatial Gradients

The spatial gradient (orthogonal to u^a) of any scalar function f in the LRF of fundamental observers O_u is defined as

$$f_a = \tilde{\nabla}_a f. \tag{3.39}$$

In particular we define here the gradients of the thermodynamical quantities μ, p, Θ [20]

$$X_a = \tilde{\nabla}_a \mu, \qquad Y_a = \tilde{\nabla}_a p, \qquad Z_a = \tilde{\nabla}_a \Theta \tag{3.40}$$

and the divergence of the acceleration vector and its spatial gradient are given by

$$A = a^a{}_{;a}, \qquad A_a = \tilde{\nabla}_a A \tag{3.41}$$

3.1.4 Propagation Equations

A complete description of any arbitrary spacetime using the covariant approach requires the irreducible sets of the geometrical and thermodynamical quantities

$$\begin{aligned}&\{\Theta, \sigma_{ab}, \omega_{ab}, \dot{u}^a, E_{ab}, H_{ab}\},\\ &\{\mu, p, q_a, \pi_{ab}, \Lambda\}\end{aligned} \tag{3.42}$$

with a prescribed equation of state of the cosmological fluid. The evolution equations for the kinematic quantities are obtained by separating the parallel projected part of the Ricci identity for the fundamental timelike 4-velocity u^a into trace, symmetric trace-free and skew-symmetric parts.

The Raychaudhuri equation

This is the basic equation of gravitational attraction and gives the propagation equation for the expansion

$$\dot{\Theta} + \tfrac{1}{3}\Theta^2 - A + 2(\sigma^2 - \omega^2) + \tfrac{1}{2}\kappa(\mu + 3p) - \Lambda = 0, \qquad (3.43)$$

which, in terms of the scale factor, can also be given by

$$\frac{3\ddot{a}}{a} = 2(\omega^2 - \sigma^2) + A - \tfrac{1}{2}\kappa(\mu + 3p) + \Lambda. \qquad (3.44)$$

The $(\mu + 3p)$-term in this equation represents the active gravitational mass density; the pressure term appears as a general-relativistic effect. A non-negative Λ acts as a repulsive force and hence tends to speed up the expansion; the vorticity tends to hold the matter apart while the shear tends to cause contraction. The divergence of the acceleration represents spatial pressure gradients and affects the average distance of the worldlines through its divergence.

The shear propagation equation

The twice-projected symmetric part of the Ricci identities(3.26) yields the evolution equation for the shear:

$$\dot{\sigma}^{\langle ab\rangle} - \tilde{\nabla}^{\langle a}\dot{u}^{b\rangle} = \tfrac{2}{3}\Theta\sigma^{ab} + \dot{u}^{\langle a}\dot{u}^{b\rangle} - \sigma^{\langle a}{}_c\sigma^{b\rangle c} - \omega^{\langle a}\omega^{b\rangle} - (E^{ab} - \tfrac{1}{2}\pi^{ab}). \qquad (3.45)$$

It shows how the anisotropic pressure π_{ab} and the "electric" part of the Weyl tensor E_{ab} induce distortion in the surrounding fluid flow.

The vorticity propagation equation

This equation is obtained from the twice-projected skew-symmetric part of the Ricci identities and is given by

$$\dot{\omega}^{\langle a\rangle} - \tfrac{1}{2}\eta^{abc}\tilde{\nabla}_b\dot{u}_c = \tfrac{2}{3}\Theta\omega^a + \sigma^a{}_b\omega^b. \qquad (3.46)$$

3.1.5 Constraint Equations

These equations are also obtained from the Ricci identity, through orthogonal projection ,indices contraction and by taking the PSTF parts. These equations do not involve time derivatives of the kinematic quantities.

The shear constraint

This equation shows how the energy flux vector q^a controls the spatial gradients of Θ, ω_{ab} and σ_{ab}:

$$\tilde{\nabla}_b\sigma^{ab} - \tfrac{2}{3}\tilde{\nabla}^a\Theta + \eta^{abc}[\tilde{\nabla}_b\omega_c + 2\dot{u}_b\omega_c] + \kappa q^a = 0. \qquad (3.47)$$

The vorticity constraint equation

$$\tilde{\nabla}_a\omega^a - (\dot{u}_a\omega^a) = 0. \qquad (3.48)$$

This equation is also known as the *vorticity divergence identity* [42]. As will be discussed in the next section and Chapters 4 and 5, the second term on the LHS is second order, and hence the equation shows the decoupling of vector and tensor modes from the scalar modes in the linearized background.

The H_{ab} constraint equation

$$H^{ab} + 2\dot{u}^{\langle a}\omega^{b\rangle} + \tilde{\nabla}^{\langle a}\omega^{b\rangle} - \eta^{cd\langle a}\tilde{\nabla}_c\sigma^{b\rangle}{}_d = 0. \qquad (3.49)$$

This constraint characterizes the "magnetic" Weyl tensor as being constructed from the distortion of the vorticity and the curl of the shear, the "curl" being defined by the expression $(curl\ \sigma)^{ab} = \eta^{cd\langle a}\tilde{\nabla}_c\sigma^{b\rangle}{}_d$.

The twice-contracted Bianchi identities

The twice-contracted Bianchi identities (3.36) give us two important conservation equations. Projecting the identity along u^a results in the energy conservation equation

$$\dot{\mu} + \tilde{\nabla}^a q_a = -\Theta(\mu + p) - 2\dot{u}_a q^a - \sigma^a{}_b\pi^b{}_a, \qquad (3.50)$$

whereas orthogonal projection with respect to u^a leads to the momentum conservation equation

$$\dot{q}^{\langle a\rangle} + \tilde{\nabla}^a p + \tilde{\nabla}_b\pi^{ab} = -\frac{4}{3}\Theta q^a - \sigma^a{}_b q^b - (\mu + p)\dot{u}^a - \dot{u}_b\pi^{ab} - \eta^{abc}\omega_b q_c. \qquad (3.51)$$

3.1.6 Maxwell-like gravitational equations

Using the Weyl tensor and the Bianchi identities we can write

$$\nabla_a C_{bcd}{}^a + \nabla_{[}(R_{c]d} - \tfrac{1}{6}Rg_{c]d}) = 0. \qquad (3.52)$$

Covariant decomposition of these identities will produce the following evolution and constraint equations :

The $\dot{E}$-equation

$$\begin{aligned}\dot{E}^{\langle ab\rangle} &+ \tfrac{1}{2}\dot{\pi}^{\langle ab\rangle} - curl H^{ab} + \tfrac{1}{2}\tilde{\nabla}^{\langle a}q^{b\rangle} \\ &= -\tfrac{1}{2}(\mu + p)\sigma^{ab} - \Theta(E^{ab} + \tfrac{1}{6}\pi^{ab}) + 3\sigma^{\langle a}{}_c(E^{b\rangle c} - \tfrac{1}{6}\pi^{b\rangle c}) - \dot{u}^{\langle a}q^{b\rangle} \\ &\quad + \eta^{cd\langle}\left[2\dot{u}_c H^{b\rangle}{}_d + \omega_c(E^{b\rangle}{}_d + \tfrac{1}{2}\pi^{b\rangle}{}_d)\right]. \end{aligned} \qquad (3.53)$$

The $\dot{H}$-equation

$$\dot{H}^{\langle ab\rangle} + curl E^{ab} - \tfrac{1}{2} curl \pi^{ab} = -\Theta H^{ab} + 3\sigma^{\langle a}{}_{c}H^{b\rangle c} + \tfrac{3}{2}\omega^{\langle a}q^{b\rangle}$$
$$- \eta^{cd\langle a}\left[2\dot{u}_c E^{b\rangle}{}_{d} - \tfrac{1}{2}\sigma^{b\rangle}{}_{c}q_d - \omega_c H^{b\rangle}{}_{d}\right]. \quad (3.54)$$

These propagation equations describe gravitational radiation and are analogues to the Maxwell's equations for electromagnetic radiation [20].

The Divergence of E equation

$$\tilde{\nabla}_a(E^{ab} + \tfrac{1}{2}\pi^{ab}) - \tfrac{1}{3}\tilde{\nabla}^a\mu + \tfrac{1}{3}\Theta q^a - \tfrac{1}{2}\sigma^a{}_b q^b - 3\omega H^{ab} - \eta^{abc}[\sigma_{bd}H^d{}_c - \tfrac{3}{2}\omega_b q_c] = 0. \quad (3.55)$$

The Divergence of H equation

$$\tilde{\nabla}_b H^{ab} + (\mu + p)\omega^a + 3\omega_b(E^{ab} - \tfrac{1}{6}\pi^{ab}) + \eta^{abc}\left[\tfrac{1}{2}\tilde{\nabla}_b q_c + \sigma_{bd}(E^d{}_c + \tfrac{1}{2}\pi^d{}_c)\right]. \quad (3.56)$$

Eqns.(3.55) and (3.56) are constraint equations sourced by the spatial gradient of the energy density and the vorticity, respectively. The first equation shows how scalar modes are coupled to the divergence of the electric Weyl tensor while the second one shows how vector (vorticity) modes are coupled to the divergence of the magnetic Weyl tensor [43].

Commutation relations in the covariant formalism will be presented in the *APPENDIX.*

3.2 Gauge-invariant Perturbation Theory

The Universe at large scales is almost homogeneous and isotropic, and is well described by the FLRW background model. But deviations from symmetry (homogeneity and isotropy) start to become more and more apparent as we go to smaller and smaller scales. Such deviations can be accounted for by perturbing the FLRW slightly [20,43].

Let's consider a background spacetime $(\bar{\mathcal{M}}, \bar{g}_{ab})$ in the homogeneous, isotropic Universe and the real, physical spacetime $(\mathcal{M}, g_{ab})$. Quantifying the deviation of the physical from the background spacetime requires a mapping $\Phi : \bar{\mathcal{M}} \to \mathcal{M}$ which identifies points in the background $\bar{\mathcal{M}}$ with their corresponding points in $\mathcal{M}$ such that $\bar{g}_{ab} \to g_{ab} = \bar{g}_{ab} + \delta g_{ab}$. Given a physical quantity Q on $\mathcal{M}$ and the corresponding quantity $\bar{Q}$ on $\bar{\mathcal{M}}$, the perturbation δQ of Q at a point $p \in \mathcal{M}$ is defined as

$$\delta Q(p) = Q(p) - \bar{Q}(\Phi^{-1}(p)). \quad (3.57)$$

The perturbation δQ is usually taken to be small, but can be assigned to take any value at the point p by altering the mapping function Φ: there is no *a priori* reason for choosing a particular mapping over another.The freedom of choice of the mapping between $\bar{\mathcal{M}}$ and $\mathcal{M}$ is called the *gauge freedom.* Any change of the mapping Φ which leaves the background manifold $\bar{\mathcal{M}}$ unchanged is called a *gauge transformation.*

Gauge transformations reflect the freedom of choosing different coordinates $\{x^a\}$ on the manifold $\mathcal{M}$, i.e.,

$$x^a \to \tilde{x}^a = x^a + \epsilon^a(x) \tag{3.58}$$

for an arbitrary infinitesimal vector field $\epsilon^a(x)$. Thus in the new correspondence $\tilde{\Phi}$,the perturbation becomes

$$\delta\tilde{Q}(p) = Q(p) - \bar{Q}(\tilde{\Phi}^{-1}(p)). \tag{3.59}$$

Taking the difference of (3.57) and (3.59) we get

$$\Delta Q(p) \equiv \delta\tilde{Q}(p) - \delta\bar{Q}(p) = \bar{Q}(\bar{\Phi}^{-1}(p)) - \bar{Q}(\tilde{\Phi}^{-1}(p)). \tag{3.60}$$

This difference is a pure gauge artifact and leads to unphysical modes-and hence need to be identified and eliminated when they arise. In fact this is a standard *gauge problem* of General Relativity.

The Stewart-Walker Lemma

We can notice from Eqn.(3.60) that Q is *gauge-invariant* if it vanishes in the background spacetime $\bar{\mathcal{M}}$. In general, the Stewart-Walker Lemma states that the linear perturbation δQ of a tensorial quantity $\bar{Q}$ on the background spacetime $(\bar{\mathcal{M}}, \bar{g})$ is gauge-invariant (GI) if and only if one of the following holds:

1. $\bar{Q}$ vanishes;
2. $\bar{Q}$ is a constant scalar;
3. $\bar{Q}$ is a constant linear combination of products of Kronecker deltas.

In this thesis we will use this Lemma to define the complete set of covariant GI quantities, which, in our context, are the quantities that vanish in the background.

Chapter 4

$f(R)$ THEORIES OF GRAVITY

4.1 Introduction

We have discussed in Chapter 2 that the late time accelerated expansion of the Universe does not have a well established explanation in the Concordance Model of cosmology. If GR is the correct theory of gravitation that controls the expansion of the Universe, then most of its energy content exists neither in the luminous nor in the dark, invisible forms of matter but instead in the form of an exotic, unclustered, invisible dark energy. There has been a surge of attempts to explain the observed discrepancy between theory and observation recently, most of which fall into one of the following classes [44–52]:

* a non-zero cosmological constant Λ
* dark energy
* modification of GR

The Cosmological Constant

One of the earliest explanations put forward as a dynamical origin of cosmic acceleration is a non-zero cosmological constant with equation of state $w_\Lambda = -1$. A cosmological constant coming into dominance at late times can cause cosmic acceleration and make the Universe enter an irreversible de Sitter phase. But this model has two main problems: the *cosmological constant problem* and the *coincidence problem* [53, 54].

The cosmological constant problem refers to the huge ($\sim 10^{120}$ orders of magnitude) discrepancy between the "observed" value of Λ responsible for cosmic acceleration and that predicted by quantum field theoretic arguments for the energy of the quantum vacuum at Planckian scales. There are even some supersymmetric theories that require a cosmological constant that is exactly zero. Apparently an extreme or infinite fine-tuning must have occurred at the very early Universe if these energy scale discrepancies are to reconcile. The total fractional energy density (Eqn. 2.14) is close to 1.0 (or is precisely 1.0 if the Universe is taken to be perfectly flat) at the present time when we are here to observe it, after about 13.7 billion years of expansion when it was always greater than 1.0. Since Ω_Λ is the only constant component, it is natural to be curious about why Λ is so finely tuned as to be dominant only now. This is the essence of the so called coincidence problem.

Dark energy is probably the most extensively speculated candidate in recent years. However, since it is a merely phenomenological explanation with no prediction of its existence from the Big Bang or inflationary cosmology, its dynamical nature remains dark, and none of the variant models put forward (such as the *decaying* Λ, *quintessence*, *k-essence*, etc.) have been convincingly viable so far.

This takes us to the third idea: the search for a new (*modified*) gravitational physics that could be responsible for the observed cosmic acceleration. Historically, there have been several proposed "modified" gravity models such as braneworld cosmologies, and Gauss-Bonnet gravity theories. In this thesis we will consider models that involve infrared modifications of the standard GR that become significant only at low curvatures in the matter dominated era: the *f(R) gravity models.* These models are purely phenomenological and are built by including higher order curvature invariants in the Einstein-Hilbert action [21, 44, 46, 47, 51, 52, 55]

$$\mathcal{A} = \int d^4x\sqrt{-g}\left[\frac{1}{2}R + \mathcal{L}_m\right], \tag{4.1}$$

and rewriting the generalized higher-order-gravity action

$$\mathcal{A}_{f(R)} = \int d^4x\sqrt{-g}\,[f(R) + \mathcal{L}_m], \tag{4.2}$$

where the $\mathcal{L}_m$ and R terms show that the Lagrangian contains matter and curvature contributions,respectively. Throughout this thesis "matter" refers to all component contributions of the cosmic fluid except curvature. Thus it includes, for example, baryonic matter and CDM(collectively referred to as *dust*) and radiation.

A generalization of the EFEs can be derived in three ways [47]: in the *metric*, *Palatini* or *metric-affine* formalisms.

The metric (second order) formalism

In this formalism, the metric g_{ab} is the only independent variable with respect to which the action (4.2) is varied to derive the field equations

$$f'G_{ab} = f'(R_{ab} - \frac{1}{2}g_{ab}R) = T^m_{ab} + \frac{1}{2}g_{ab}(R - Rf') + \nabla_b\nabla_a f' - g_{ab}\nabla_c\nabla^c f', \tag{4.3}$$

where

$$f \equiv f(R), \qquad f' \equiv \frac{df}{dR}, \qquad T^m_{ab} \equiv \frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}L_m)}{\delta g_{ab}}. \tag{4.4}$$

Due to the introduction of fourth order derivatives of the metric in the last two terms of the RHS of Eqn. (4.3), this formalism is sometimes referred to as a *fourth order theory of gravity* (FOG).

The Palatini (first order) formalism

The Palatini formalism treats both the metric and the affine connection $\Gamma^{a}{}_{bc}$ as independent variables. Hence the field equations are derived by varying the action with respect to both the metric and the affine connections. In GR it results in the same field equations as those obtained via the metric formalism, but this no longer holds for $f(R)$ theories whose Lagrangians are no longer linear:

$$G_{ab} = \frac{T_{ab}}{f'} - \frac{1}{2}(R - \frac{f}{f'})g_{ab} + \frac{1}{f'}(\nabla_a\nabla_b - g_{ab}\Box)f' - \frac{3}{2(f')^2}\left[\nabla_a f'\nabla_b f' - \frac{1}{2}g_{ab}\nabla_c f'\nabla^c f'\right]. \tag{4.5}$$

We note that there are no second order covariant derivatives of f',and hence the alias *first order* formalism.

The metric-affine formalism

The matter part of the action (4.2) depends explicitly on the affine connection and hence introduces a torsion associated with matter. The theory is not yet a well-explored one.

In this thesis we will only focus on $f(R)$ theories based on the metric formalism. Hence let's go back to Eqn. (4.3) and rewrite it in a more convenient and compact form

$$G_{ab} = \tilde{T}^{m}_{ab} + T^{R}_{ab} = T_{ab}, \tag{4.6}$$

where

$$\tilde{T}^{m}_{ab} = \frac{T^{m}_{ab}}{f'}, \quad \text{and} \quad T^{R}_{ab} = \frac{1}{f'}\left[\frac{1}{2}g_{ab}(f(R) - Rf') + \nabla_b\nabla_a f' - g_{ab}\nabla_c\nabla^c f'\right] \tag{4.7}$$

are the effective EMTs for standard matter and *curvature* treated as a *fluid*, respectively. Assuming that the momentum conservation of standard matter $T^{m;b}_{ab} = 0$ still holds prompts us to conclude that T_{ab} is divergence-free, i.e., $T^{;b}_{ab} = 0$, and that $\tilde{T}^{m}_{ab}$ and T^{R}_{ab} are not individually conserved [21]:

$$\tilde{T}^{m;b}_{ab} = \frac{T^{m;b}_{ab}}{f'} - \frac{f''}{f'^2}T^{m}_{ab}R^{;b}, \tag{4.8}$$

$$T^{R;b}_{ab} = \frac{f''}{f'^2}\tilde{T}^{m}_{ab}R^{;b}. \tag{4.9}$$

4.2 Covariant Decomposition and Projection

Using the 1 + 3 covariant formalism, we can decompose the effective total energy-momentum tensor,

$$T_{ab} = \mu u_a u_b + p h_{ab} + 2q_{(a}u_{b)} + \pi_{ab} \tag{4.10}$$

into the following dynamical quantities:

$$\mu = T_{ab}u^a u^b = \tilde{\mu}^m + \mu^R, \tag{4.11}$$

$$p = \frac{1}{3}T_{ab}h^{ab} = \tilde{p}^m + p^R, \tag{4.12}$$

$$q_a = -T_{bc}h^b_a u^c = \tilde{q}^m_a + q^R_a, \tag{4.13}$$

$$\pi_{ab} = T_{cd}h^c_{\langle a}h^d_{b\rangle} = \tilde{\pi}^m_{ab} + \pi^R_{ab}, \tag{4.14}$$

with

$$\tilde{\mu}^m = \frac{\mu^m}{f'}, \quad \tilde{p}^m = \frac{p^m}{f'}, \quad \tilde{q}^m_a = \frac{q_a}{f'}, \quad \tilde{\pi}^m_{ab} = \frac{\pi^m_{ab}}{f'}. \tag{4.15}$$

In a perfect cosmological medium, the quantities q^m_a and π^m_{ab} both vanish.

The effective thermodynamical quantities for the curvature fluid are given by:

$$\mu^R = T^R_{ab}u^a u^b = \frac{1}{f'}\left[\frac{1}{2}(Rf' - f) - \Theta f''\dot{R} + f''\tilde{\nabla}^2 R + f''\dot{u}_b\tilde{\nabla}^b R\right], \tag{4.16}$$

$$p^R = \frac{1}{3}T^R_{ab}h^{ab} = \left[\frac{1}{2}(f - Rf') + f''\ddot{R} + f'''\dot{R}^2 + \frac{2}{3}\left(\Theta f''\dot{R} - f''\tilde{\nabla}^2 R - f'''\tilde{\nabla}^a R\tilde{\nabla}_a R\right) + f''\dot{u}_b\tilde{\nabla}^b R\right], \tag{4.17}$$

$$q^R_a = -T^R_{bc}h^b_a u^c = -\frac{1}{f'}\left[f'''\dot{R}\tilde{\nabla}_a R + f''\tilde{\nabla}_a\dot{R} - \frac{1}{3}f''\Theta\tilde{\nabla}_a R\right], \tag{4.18}$$

$$\pi^R_{ab} = T^R_{cd}h^c_{\langle a}h^d_{b\rangle} = \frac{1}{f'}\left[f''\tilde{\nabla}_{\langle a}\tilde{\nabla}_{b\rangle}R + f'''\tilde{\nabla}_{\langle a}R\tilde{\nabla}_{b\rangle}R + \sigma_{ab}\dot{R}\right]. \tag{4.19}$$

The parallel and orthogonal projection of the twice-contracted Bianchi identities (3.36) with respect to the flow lines give us the energy and momentum equations equations

$$\dot{\mu}^m = -\Theta(\mu^m + p^m), \tag{4.20}$$

$$\dot{\mu}^R + \tilde{\nabla}^a q^R_a = -\Theta(\mu^R + p^R) - 2\dot{u}^a q^R_a - \sigma^{ab}\pi^R{}_{ba} + \mu^m\frac{f''\dot{R}}{f'^2}, \tag{4.21}$$

$$\dot{q}^R_{\langle a\rangle} + \tilde{\nabla}^a p^R + \tilde{\nabla}^b\pi^R_{ab} = -\frac{4}{3}\Theta q^R_a - \sigma^b{}_a q^R_b - (\mu^R + p^R)\dot{u}_a - \dot{u}^b\pi^R_{ab} - \eta^{bc}{}_a\omega_b q^R_c + \mu^m\frac{f''\tilde{\nabla}_a R}{f'^2}. \tag{4.22}$$

The curvature fluid and effective matter exchange energy and momentum; and this is exhibited by the last terms on the RHS of the last two equations, respectively. The conservation of momentum of standard matter relates the energy density and the isotropic pressure to the relativistic acceleration:

$$\tilde{\nabla}^a p^m = -(\mu^m + p^m)\dot{u}^a. \tag{4.23}$$

4.3 Propagation and Constraint Equations

The propagation and constraint equations we saw in the previous chapter take the following form in higher order theories of gravity [21]:

The generalized propagation equation

$$\dot{\Theta} + \tfrac{1}{3}\Theta^2 + \sigma_{ab}\sigma^{ab} - 2\omega_a\omega^a - \tilde{\nabla}^a\dot{u}_a + \dot{u}_a\dot{u}^a + \tfrac{1}{2}(\tilde{\mu}^m + 3\tilde{p}^m) = -\tfrac{1}{2}(\mu^R + 3p^R). \quad (4.24)$$

The vorticity equation

$$\dot{\omega}_{\langle a\rangle} + \tfrac{2}{3}\Theta\omega_a + \tfrac{1}{2}curl\ \dot{u}_a - \sigma_{ab}\omega^b = 0. \quad (4.25)$$

The shear propagation equation

$$\dot{\sigma}_{\langle ab\rangle} + \tfrac{2}{3}\Theta\sigma_{ab} + E_{ab} - \tilde{\nabla}_{\langle a}\dot{u}_{b\rangle} + \sigma_{c\langle a}\sigma^c_{b\rangle} + \omega_{\langle a}\omega_{b\rangle} - \dot{u}_{\langle a}\dot{u}_{b\rangle} = \tfrac{1}{2}\pi^R_{ab}. \quad (4.26)$$

Gravito-electric propagation equation

$$\dot{E}_{\langle ab\rangle} + \Theta E_{ab} - curl\ H_{ab} + \tfrac{1}{2}(\tilde{\mu}^m + \tilde{p}^m)\sigma_{ab} - 2\dot{u}^c\eta_{cd(a}H^d_{b)} - 3\sigma_{c\langle a}E^c_{b\rangle} + \omega^c\eta_{cd(a}E^d_{b)}$$
$$= -\tfrac{1}{2}(\mu^R + p^R)\sigma_{ab} - \tfrac{1}{2}\dot{\pi}^R_{\langle ab\rangle} - \tfrac{1}{2}\tilde{\nabla}_{\langle a}q^R_{b\rangle} - \tfrac{1}{6}\Theta\pi^R_{ab} - \tfrac{1}{2}\sigma^c_{\langle a}\pi^R_{b\rangle c} - \tfrac{1}{2}\omega^c\eta^d_{c(a}\pi^R_{b)d}. \quad (4.27)$$

Gravito-magnetic propagation equation

$$\dot{H}_{ab} + \Theta H_{ab} + curl E_{ab} - 3\sigma_{c\langle a}H^c_{b\rangle} + \omega^c\eta_{cd(a}H^d_{b)} + 2\dot{u}^c\eta_{cd(a}E_{b)}{}^d$$
$$= \tfrac{1}{2}curl\ \pi^R_{ab} - \tfrac{3}{2}\omega_{\langle a}q^R_{b\rangle} + \tfrac{1}{2}\sigma^c_{(a}\eta_{b)c}{}^d q^R_d. \quad (4.28)$$

The vorticity constraint equation

$$\tilde{\nabla}^a\omega_a - \dot{u}^a\omega_a = 0. \quad (4.29)$$

The Shear constraint equation

$$\tilde{\nabla}^a\sigma_{ab} - curl\ \omega_a - \tfrac{2}{3}\tilde{\nabla}_a\Theta + 2[\omega, \dot{u}]_a = -q^R_a. \quad (4.30)$$

The Gravito-magnetic constraint equation

$$curl\ \sigma_{ab} + \tilde{\nabla}_{\langle a}\omega_{b\rangle} - H_{ab} + 2\dot{u}_{\langle a}\omega_{b\rangle} = 0. \quad (4.31)$$

The Gravito-electric divergence equation

$$\tilde{\nabla}^b E_{ab} - \tfrac{1}{3}\tilde{\nabla}_a\tilde{\mu}^m - [\sigma, H]_a + 3H_{ab}\omega^b = \tfrac{1}{2}\sigma^b{}_a q^R_b - \tfrac{3}{2}[\omega, q^R]_a - \tfrac{1}{2}\tilde{\nabla}^b\pi^R_{ab} + \tfrac{1}{3}\tilde{\nabla}_a\mu^R - \tfrac{1}{3}\Theta q^R_a. \quad (4.32)$$

The Gravito-magnetic divergence equation

$$\tilde{\nabla}^b H_{ab} - (\tilde{\mu}^m + \tilde{p}^m)\omega_a + [\sigma, E]_a - 3E_{ab}\omega^b = -\tfrac{1}{2}curl\ q^R_a + (\mu^R + p^R)\omega_a - \tfrac{1}{2}[\sigma, \pi^R]_a - \tfrac{1}{2}\pi^R_{ab}\omega^b. \quad (4.33)$$

It is worth noting that the corresponding GR equations we saw earlier can be recovered from their $f(R)$ counterparts upon setting $f(R) = R$, in which case the RHS of the above equations all vanish.

4.4 Background Dynamics of $f(R)$ Gravity

Since our perturbation equations should be given relative to a background metric, we have to carefully choose the background that best describes the observed universe. In light of this, we choose the FLRW background for two main reasons. Firstly, this metric has a remarkable symmetry (CMB analyses show that this background is isotropic to within 1 in 10^5) and is, therefore, efficient in handling the perturbation equations more easily than any other metric. Secondly, most of the perturbations in GR are done around this background [21], and this makes comparison of our work with already existing ones easier.

The Friedmann, Raychaudhuri and energy conservation equations of the background are

$$\Theta^2 = 3(\tilde{\mu}^m + \mu^R) - \tfrac{3}{2}\tilde{R}, \tag{4.34}$$
$$\dot{\Theta} + \tfrac{1}{3}\Theta^2 + \tfrac{1}{2}(\tilde{\mu}^m + 3\tilde{p}^m) + \tfrac{1}{2}(\mu^R + 3p^R) = 0, \tag{4.35}$$
$$\dot{\mu}^m + \Theta(\mu^m + p^m) = 0, \tag{4.36}$$

where $\tilde{R}$ is the 3-Ricci scalar defined by $\tilde{R} = 6K/a^2$, K being the spatial curvature (taking values of 0 or ± 1).

The simplest and widely studied form of higher order $f(R)$ gravitational theories is R^n gravity where $f(R) = \chi R^n$ and for which (4.2) becomes [21,52]

$$\mathcal{A} = \int d^4x\sqrt{-g}\,[\chi R^n + \mathcal{L}_m]. \tag{4.37}$$

$\chi = \chi(n)$ is a coupling running constant chosen in such away that it has the right dimensions and reduces to unity for GR. In this toy model of gravity, the modified EFEs take the form

$$nR^{n-1}G_{ab} = \chi^{-1}\tilde{T}^m_{ab} + \tfrac{1}{2}g_{ab}(n-1)R^n + \left[n(n-1)R^{n-2}R^{;ab} + n(n-1)(n-2)R^{n-3}R^{;a}R^{;b}\right](g_{ac}g_{bd} - g_{ab}g_{cd}). \tag{4.38}$$

For comparison with Eqn. (4.6), we can rewrite this equation (for $R \neq 0$) as

$$\begin{aligned} G_{ab} &= \frac{\tilde{T}^m_{ab}}{n\chi R^{n-1}} + \tfrac{1}{2n}g_{ab}(1-n)R + \left[(n-1)\frac{R^{;ab}}{R} + (n-1)(n-2)\frac{R^{;a}R^{;b}}{R^2}\right](g_{ac}g_{bd} - g_{ab}g_{cd}) \\ &\equiv T^m_{ab} + T^R_{ab}, \end{aligned} \tag{4.39}$$

where the non-Einsteinian part of the EMT is taken as an effective fluid with a different thermodynamical behaviour from standard matter. The corresponding Friedmann system of equations for a barotropic equation of state $p = w\mu$ are:

$$2n\frac{\ddot{a}}{a} + \tfrac{1}{3}n(n-1)\Theta\frac{\dot{R}}{R} + n(n-1)\frac{\dot{R}}{R} + n(n-1)(n-2)\frac{\dot{R}^2}{R^2} + (n-1)\frac{R}{3} + \frac{\mu}{3n\chi R^{n-1}}(1+3w) = 0, \tag{4.40}$$

$$\Theta^2 + \frac{9k}{a^2} + 3(n-1)\Theta\frac{\dot{R}}{R} + \frac{3R}{2n}(n-1) - \frac{\mu}{3n\chi R^{n-1}} = 0, \tag{4.41}$$

where R is given by

$$R = -6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right). \tag{4.42}$$

We then need the energy conservation equation (4.36) to complete the system.

A dynamical systems analysis of R^n gravity was presented in [52] where it was shown that the model has a Friedmann-like transient solution

$$a = a_0 t^{\frac{2n}{3(1+w)}} \tag{4.43}$$

before it enters the de Sitter-like, accelerated expansion phase. In this case, we obtain the following expressions for the expansion, the Ricci scalar, the curvature fluid energy density, the curvature fluid pressure and the effective matter energy density respectively:

$$\Theta = \frac{2n}{(1+w)t}, \tag{4.44}$$

$$R = \frac{4n\left[4n - 3(1+w)\right]}{3(1+w)^2t^2}, \tag{4.45}$$

$$\mu_R = \frac{2(n-1)\left[2n(3w+5) - 3(1+w)\right]}{3(1+w)^2t^2}, \tag{4.46}$$

$$p_R = \frac{2(n-1)\left[n(6w^2+8w-2) - 3w(1+w)\right]}{3(1+w)^2t^2}, \tag{4.47}$$

$$\mu_m = \left(\frac{3}{4}\right)^{1-n} n\chi \left(\frac{n(4n-3(1+w))}{(1+w)^2t^2}\right)^{n-1} \frac{4n^2 - 2(n-1)\left[2n(3w+5) - 3(1+w)\right]}{3(1+w)^2t^2}. \tag{4.48}$$

In the following chapters these background quantities will be used to find solutions to our perturbation equations in both the long and short wavelength limits.

Chapter 5

PERTURBATIONS IN SINGLE FLUIDS

5.1 Introduction

Single fluid covariant perturbation equations in $f(R)$ gravity have been derived by Carloni, Dunsby and Troisi [21]. These equations describe the evolution of the perturbations in a barotropic fluid with a constant equation of state around a FLRW background where the local rest frame is chosen to coincide with the energy frame of standard matter. In this chapter we present a review of these mathematically well defined equations and the solutions thereof.

5.2 Linearized Equations

The evolution - and - conservation equations we presented in Chapter 4 (Eqns 4.24 - 4.33) are exact non - linear covariant equations valid for any $f(R)$ cosmology in any spacetime where our fundamental observer is comoving with standard matter. In this section we present the corresponding linearized equations around a FLRW background metric. In the linearization procedure, the Stewart-Walker Lemma defines which terms are GI and linear. Thus all inhomogeneous and anisotropic quantities that vanish in this background, e.g. q_a^R and π_{ab}^R, are taken to be of linear order.

Making use of the background cosmological equations (4.34, 4.35, 4.36), we can write the linearized equations of propagation and constraint in the following form:

$$\dot{\Theta} + \tfrac{1}{3}\Theta^2 - \tilde{\nabla}^a \dot{u}_a + \tfrac{1}{2}(\tilde{\mu}^m + 3\tilde{p}^m) = -\tfrac{1}{2}(\mu^R + 3p^R), \tag{5.1}$$

$$\dot{\omega}_a + \tfrac{2}{3}\Theta\omega_a + \tfrac{1}{2}curl\ \dot{u}_a = 0, \tag{5.2}$$

$$\dot{\sigma}_{ab} + \tfrac{2}{3}\Theta\sigma_{ab} + E_{ab} - \tilde{\nabla}_{\langle a}\dot{u}_{b\rangle} = -q_a^R, \tag{5.3}$$

$$\dot{E}_{\langle ab\rangle} + \Theta E_{ab} - curl\ H_{ab} + \tfrac{1}{2}(\tilde{\mu}^m + \tilde{p}^m)\sigma_{ab} = -\tfrac{1}{2}(\mu^R + p^R)\sigma_{ab} - \tfrac{1}{2}\dot{\pi}^R_{\langle ab\rangle} - \tfrac{1}{2}\tilde{\nabla}_{\langle a}q^R_{b\rangle} - \tfrac{1}{6}\Theta\pi^R_{ab}, \tag{5.4}$$

$$\dot{H}_{ab} + \Theta H_{ab} + curl E_{ab} = \tfrac{1}{2}curl\ \pi^R_{ab}, \tag{5.5}$$

$$\tilde{\nabla}^a \omega_a = 0, \tag{5.6}$$
$$\tilde{\nabla}^a \sigma_{ab} - curl\ \omega_a - \tfrac{2}{3}\tilde{\nabla}_a \Theta = -q^R_a, \tag{5.7}$$
$$curl\ \sigma_{ab} + \tilde{\nabla}_{\langle a}\omega_{b\rangle} - H_{ab} = 0, \tag{5.8}$$
$$\tilde{\nabla}^b E_{ab} - \tfrac{1}{3}\tilde{\nabla}_a \tilde{\mu}^m = -\tfrac{1}{2}\tilde{\nabla}^b \pi^R_{ab} + \tfrac{1}{3}\tilde{\nabla}_a \mu^R - \tfrac{1}{3}\Theta q^R_a, \tag{5.9}$$
$$\tilde{\nabla}^b H_{ab} - (\tilde{\mu}^m + \tilde{p}^m)\omega_a = -\tfrac{1}{2} curl\ q^R_a + (\mu^R + p^R)\omega_a. \tag{5.10}$$

We need the linearized conservation equations and the acceleration equation to close the above system:

$$\dot{\mu}^m = -\Theta(\mu^m + p^m), \tag{5.11}$$
$$\dot{\mu}^R + \tilde{\nabla}^a q^R_a = -\Theta(\mu^R + p^R) + \mu^m \frac{f''\dot{R}}{f'^2}, \tag{5.12}$$
$$\dot{q}^R_{\langle a\rangle} + \tilde{\nabla}^a p^R + \tilde{\nabla}^b \pi^R_{ab} = -\tfrac{4}{3}\Theta q^R_a - (\mu^R + p^R)\dot{u}_a + \mu^m \frac{f''\tilde{\nabla}_a R}{f'^2}, \tag{5.13}$$
$$\tilde{\nabla}^a p^m = -(\mu^m + p^m)\dot{u}^a. \tag{5.14}$$

5.3 Dynamics of Scalar Perturbations

We can uniquely decompose the different dynamical quantities we saw earlier into their scalar, vector and tensor components [21]:

$$V_a = \bar{V}_a + \hat{V}_a = \eta^{abc}\tilde{\nabla}_b \bar{V}_c + \tilde{\nabla}_a \hat{V}, \tag{5.15}$$
$$W_{ab} = \bar{W}_{ab} + \hat{W}_{ab} + W^*_{ab} = \bar{W}_{ab} + \tilde{\nabla}_a \hat{W}_b + \tilde{\nabla}_a \tilde{\nabla}_b W^*, \tag{5.16}$$

with

$$\tilde{\nabla}^a \bar{V}_a = 0, \qquad \eta^{abc}\tilde{\nabla}_b \hat{V}_c = 0, \tag{5.17}$$
$$\tilde{\nabla}^a \bar{W}_{ab} = 0, \qquad (curl\hat{W})_{ab} = 0, \qquad (curl W^*)_{ab} = 0. \tag{5.18}$$

Since density fluctuations are associated with the evolution of scalar perturbations, we will discard all the non-scalar quantities by setting

$$V_a = \tilde{\nabla}_a V, \qquad W_{ab} = \tilde{\nabla}_{\langle a}\tilde{\nabla}_{b\rangle} W, \tag{5.19}$$

where

$$curl V_a = 0 = curl W_{ab}, \quad \tilde{\nabla}^a W_{ab} = \tfrac{2}{3}\tilde{\nabla}^2(\tilde{\nabla}_a W), \qquad \omega_a = 0 = H_{ab} \tag{5.20}$$

in linear perturbations of the FLRW background.

The spatial gradients defined in (3.40) describe fluid inhomogeneities, but to give a physically relevant characterization of spatial variation of the matter inhomogeneities in the lumpy universe, we need to define the comoving fractional density and expansion gradient variables

$$D_a^m = \frac{a}{\mu_m}\tilde{\nabla}_a \mu_m, \qquad Z_a = a\tilde{\nabla}_a\Theta,$$
$$C_a = a\tilde{\nabla}_a\tilde{R}. \tag{5.21}$$

Quantities with index m refer to *matter* (such as radiation, dust). The quantities D_a^m and Z_a define comoving fractional density gradient and comoving gradient of the expansion, respectively and can, in principle, be measured observationally [17]. C_a defines inhomogeneities in the 3-curvature scalar and is related to other gradient quantities via a constraint equation derived from the Gauss relation (Eqn. B-13).

To describe inhomogeneities in the curvature fluid, two additional variables need to be defined: the dimensionless gradients describing inhomogeneities in the Ricci scalar

$$\mathcal{R}_a = a\tilde{\nabla}_a R, \ \Re_a = a\tilde{\nabla}_a \dot{R}. \tag{5.22}$$

The velocity of the curvature fluid relative to u_a^m given by

$$V_a^R = -\frac{\tilde{\nabla}_a R}{\dot{R}}. \tag{5.23}$$

To obtain the evolution equations for the above defined variables, we differentiate the gradients with respect to cosmic time and make use of the commutation relations from the Appendix and the linearized forms of the propagation and constraint equations presented in the previous section. Thus the propagation equation for the matter density gradient of a vorticity-free, perfect matter fluid with a constant barotropic equation of state w is given by

$$\dot{D}_a^m - w\Theta D_a^m + (1+w)Z_a = 0. \tag{5.24}$$

This equation depicts how the expansion hinders the growth of the density perturbations.

The evolution equation of Z_a is obtained by taking the time derivative of the comoving spatial gradient of the expansion defined in (5.21), and is given by

$$\dot{Z}_a = (\dot{R}\frac{f''}{f'} - \frac{2}{3}\Theta)Z_a + \left[\frac{w-1}{1+w}\frac{\mu_m}{f'} + \frac{w}{1+w}\left(\frac{R}{2} - \frac{f}{f'} - 2\dot{R}\Theta\frac{f''}{f'} - \frac{K}{a^2}\right)\right]D_a^m$$
$$+ \Theta\frac{f''}{f'}\Re_a + \left[\frac{1}{2} - \frac{1}{2}\frac{ff''}{f'^2} + \frac{f''\mu_m}{f'^2} - \dot{R}\Theta(\frac{f''}{f'})^2 + \dot{R}\Theta\frac{f'''}{f'} + \frac{2K}{a^2}\frac{f''}{f'}\right]\mathcal{R}_a$$
$$- \frac{f''}{f'}\tilde{\nabla}^2\mathcal{R}_a - \frac{w}{1+w}\tilde{\nabla}^2 D_a^m. \tag{5.25}$$

In deriving this equation, we have used the relation of the 4-acceleration in terms of the pressure gradient $\tilde{\nabla}_b p$,

$$\dot{u}_b = -\frac{\tilde{\nabla}_b p_m}{\mu_m + p_m}, \tag{5.26}$$

taking the divergence of which we obtain

$$\tilde{\nabla}^b \dot{u}_b = -\frac{\tilde{\nabla}^b \tilde{\nabla}_b p_m}{\mu_m + p_m}. \tag{5.27}$$

Hence using the identities [19, 56]

$$\tilde{\nabla}_b \tilde{\nabla}^b \tilde{\nabla}_a p = \tilde{\nabla}_b \tilde{\nabla}_a \tilde{\nabla}^b p = \tilde{\nabla}_b \tilde{\nabla}_a Y^b, \qquad \tilde{R}^b{}_a = \frac{1}{3} h^b{}_a \tilde{R} \tag{5.28}$$

and taking terms upto linear order we can write

$$A_a = \frac{1}{\mu_m + p_m}(\frac{2K}{a^2} - \tilde{\nabla}^2) Y_a. \tag{5.29}$$

Applying Eqns. (5.1, 5.29) and the *trace equation* (B-13) we find the evolution equation for Z_a as given in (5.25).

The derivation of the equation for $\mathcal{R}_a$ is straightforward and yields

$$\dot{\mathcal{R}}_a = \Re_a - \frac{w}{1+w} \dot{R} D^m_a. \tag{5.30}$$

Finally the evolution equation for $\Re_a$ is obtained by taking the spatial gradient of the trace equation and is given by

$$\begin{aligned} \dot{\Re}_a = &-\left(\Theta + 2\dot{R}\frac{f'''}{f''}\right)\Re_a - \dot{R} Z_a + \left(\frac{(1-3w)\mu_m}{3f''} - \frac{w}{1+w}\ddot{R}\right) D^m_a \\ &- \left[\ddot{R}\frac{f'''}{f''} + \dot{R}^2 \frac{f^{(4)}}{f''} + \Theta \dot{R}\frac{f'''}{f''} + \frac{f'}{3f''} - \frac{R}{3} + \frac{2K}{a^2}\right]\mathcal{R}_a + \tilde{\nabla}^2 \mathcal{R}_a. \end{aligned} \tag{5.31}$$

These 4 evolution equations give a closed system of equations describing inhomogeneities in the matter for a general $f(R)$ theory of gravity. In addition, from the Friedmann constraint (B-15), we can obtain an equation connecting C_a with other gradient variables

$$\begin{aligned} &\frac{C_a}{a^2} + \left[\tfrac{4}{3}\Theta + 2\frac{\dot{R} f''}{f'}\right] Z_a - 2\frac{\mu_m}{f'} D^m_a + \left[2\Theta \dot{R}\frac{f'''}{f'} - \frac{f''}{f'}(\frac{f}{f'} - 2\frac{\mu_m}{f'} + 2\dot{R}\Theta\frac{f''}{f'} + \frac{4K}{a^2})\right]\mathcal{R}_a \\ &+ 2\Theta\frac{f''}{f'}\Re_a - 2\frac{f''}{f'}\tilde{\nabla}^2\mathcal{R}_a = 0. \end{aligned} \tag{5.32}$$

This equation does not add any useful information on its own; it rather imposes constraint on the other dynamical variables.

The evolution equation for the above constraint is given by

$$
\begin{aligned}
\dot{C}_a = 12K^2 & \left[\frac{3f'^2 + 2a^2(\Theta f'' - 3\ddot{R}f''')}{a^2 f'^2} \frac{f''}{2\Theta f' + 3\dot{R}f''}\right] \mathcal{R}_a + K \left\{ \frac{6f'C_a}{a^2(2\Theta f' + 3\dot{R}f'')} \right. \\
& + D^m_a \left(\frac{16w\Theta}{3(1+w)} - \frac{12\mu_m}{2\Theta f' + 3\dot{R}f''}\right) - \frac{12f''}{2\Theta f' + 3\dot{R}f''}\tilde{\nabla}^2\mathcal{R}_a \\
& - \left[\frac{2a^2(\Theta f'' - 3\dot{R}f''')}{3f'} \frac{12\dot{R}\Theta f' f''' - 6f''\left(f - 2\mu_m + 2\dot{R}\Theta f''\right)}{(2\Theta f' + 3\dot{R}f'')f'}\right] \mathcal{R}_a \\
& \left. + \left(\frac{12\Theta f''}{2\Theta f' + 3\dot{R}f''} + \frac{2f''}{f'}\right)\Re_a \right\} + \tilde{\nabla}^2 \left[\frac{4wa^2\Theta}{3(1+w)}D^m_a + \frac{2a^2 f''}{f'}\Re_a - \frac{2a^2(\Theta f'' - 3\dot{R}f'')}{3f'}\mathcal{R}_a\right].
\end{aligned}
\tag{5.33}
$$

This equation will be used in the long wavelength analysis of the perturbations where C_a is known to be a conserved quantity.

5.4 Scalar Equations

The gradient equations we just described are general evolution equations of the perturbations. But only the scalar part of the gradient variables are understood to play a role in matter clustering, and hence in structure formation. Thus we locally decompose the gradient quantities into their corresponding scalar and vector components by taking the divergence of the vector quantities in analogy with the one we have already seen in (3.14):

$$
a\tilde{\nabla}_b X_a = X_{ab} = \tfrac{1}{3}h_{ab}X + \Sigma^X_{ab} + X_{[ab]}, \tag{5.34}
$$

where $\Sigma^X_{ab} = X_{(ab)} - \frac{1}{3}h_{ab}X$ describes shear whereas $X_{[ab]}$ describes the vorticity. Vorticity and shear describe the rotation and distortion of the density gradient field, respectively.

This decomposition extracts the scalar part of the perturbation gradients, which are responsible for spherically symmetric clustering of matter. Thus we have

$$
\begin{aligned}
&\Delta_m = a\tilde{\nabla}^a D^m_a \qquad Z = a\tilde{\nabla}^a Z_a \qquad C = a\tilde{\nabla}^a C_a \\
&\mathcal{R} = a\tilde{\nabla}^a \mathcal{R}_a \qquad \Re = a\tilde{\nabla}^a \Re_a.
\end{aligned}
\tag{5.35}
$$

These variables characterize the spherically symmetric part (trace) of the gradients. It follows that the scalar dynamical equations are obtained by taking the divergence of their corresponding gradient equations:

$$
\dot{\Delta}_m = w\Theta\Delta_m - (1+w)Z, \tag{5.36}
$$

$$\dot{Z} = (\dot{R}\frac{f''}{f'} - \frac{2}{3}\Theta)Z + \left[\frac{w-1}{1+w}\frac{\mu_m}{f'} + \frac{w}{1+w}\left(\frac{R}{2} - \frac{f}{f'} - 2\dot{R}\Theta\frac{f''}{f'} - \frac{3K}{a^2}\right)\right]\Delta_m,$$
$$+\Theta\frac{f''}{f'}\Re + \left[\frac{1}{2} - \frac{1}{2}\frac{ff''}{f'^2} + \frac{f''\mu_m}{f'^2} - \dot{R}\Theta(\frac{f''}{f'})^2 + \dot{R}\Theta\frac{f'''}{f'}\right]\mathcal{R} - \frac{f''}{f'}\tilde{\nabla}^2\mathcal{R} - \frac{w}{1+w}\tilde{\nabla}^2\Delta_m, \tag{5.37}$$

$$\dot{\mathcal{R}} = \Re - \frac{w}{1+w}\dot{R}\Delta_m, \tag{5.38}$$

$$\dot{\Re} = -\left(\Theta + 2\dot{R}\frac{f'''}{f''}\right)\Re - \dot{R}Z + \left(\frac{(1-3w)\mu_m}{3f''} - \frac{w}{1+w}\ddot{R}\right)\Delta_m$$
$$-\left[\ddot{R}\frac{f'''}{f''} + \dot{R}^2\frac{f^{(4)}}{f''} + \Theta\dot{R}\frac{f'''}{f''} + \frac{f'}{3f''} - \frac{R}{3}\right]\mathcal{R} + \tilde{\nabla}^2\mathcal{R}, \tag{5.39}$$

$$\frac{C}{a^2} + \left[\frac{4}{3}\Theta + 2\frac{\dot{R}f''}{f'}\right]Z - 2\frac{\mu_m}{f'}\Delta_m + \left[2\Theta\dot{R}\frac{f'''}{f'} - \frac{f''}{f'}(\frac{f}{f'} - 2\frac{\mu_m}{f'} + 2\dot{R}\Theta\frac{f''}{f'})\right]\mathcal{R}$$
$$+2\Theta\frac{f''}{f'}\Re - 2\frac{f''}{f'}\tilde{\nabla}^2\mathcal{R} = 0. \tag{5.40}$$

The evolution of the scalar constrain is given by

$$\begin{aligned}\dot{C} = 12K^2&\left[\frac{f'^2 + 2a^2(\Theta f'' - 3\ddot{R}f''')}{a^2f'^2}\frac{f''}{2\Theta f' + 3\dot{R}f''}\right]\mathcal{R} + K\left\{\frac{6f'C}{a^2(2\Theta f' + 3\dot{R}f'')}\right.\\
&+\Delta_m\left(\frac{16w\Theta}{3(1+w)} - \frac{12\mu_m}{2\Theta f' + 3\dot{R}f''}\right) - \frac{12f''}{2\Theta f' + 3\dot{R}f''}\tilde{\nabla}^2\mathcal{R}\\
&-\left[\frac{2a^2(\Theta f'' - 3\dot{R}f''')}{3f'}\frac{12\dot{R}\Theta f'f''' - 6f''\left(f - 2\mu_m + 2\dot{R}\Theta f''\right)}{(2\Theta f' + 3\dot{R}f'')f'}\right]\mathcal{R}\\
&\left.+\left(\frac{12\Theta f''}{2\Theta f' + 3\dot{R}f''} + \frac{2f''}{f'}\right)\Re\right\} + \tilde{\nabla}^2\left[\frac{4wa^2\Theta}{3(1+w)}\Delta_m + \frac{2a^2f''}{f'}\Re - \frac{2a^2(\Theta f'' - 3\dot{R}f'')}{3f'}\mathcal{R}\right].\end{aligned} \tag{5.41}$$

5.5 Harmonic Analysis

The above evolution equations can be thought of as a coupled system of harmonic oscillators of the form

$$\ddot{X} + A\dot{X} + BX = C(Y, \dot{Y}), \tag{5.42}$$

where the second term from the left represents the friction (damping) term, the third one, the restoring force term while C represents the source forcing term. A key assumption in the analysis of the equation here is that we can apply the separation of variables technique

$$X(x,t) = X(\vec{x})X(t),$$
$$Y(x,t) = Y(\vec{x})Y(t) \tag{5.43}$$

and write

$$X = \sum_k X^k(t) Q_k(\vec{x}),$$
$$Y = \sum_k Y^k(t) Q_k(\vec{x}), \tag{5.44}$$

where $Q_k(x)$ are the eignefunctions of the covariantly defined Laplace-Beltrami operator on an almost FLRW spacetime:

$$\tilde{\nabla}^2 Q = -\frac{k^2}{a^2} Q. \tag{5.45}$$

$k = \frac{2\pi a}{\lambda}$ is the order of the harmonic and $\dot{Q}_k(\vec{x}) = 0$ (Q is covariantly constant). In this way the the evolution equations and the constraint equation can be converted into ordinary differential equations-thus rendering them easier to solve. After harmonic decomposition Eqns. (5.36 -5.40) and can be rewritten in the following form:

$$\dot{\Delta}^k_m = w\Theta\Delta^k_m - (1+w)Z^k, \tag{5.46}$$

$$\dot{Z}^k = (\dot{R}\frac{f''}{f'} - \frac{2}{3}\Theta)Z^k + \left[\frac{w-1}{1+w}\frac{\mu_m}{f'} + \frac{w}{1+w}\left(\frac{R}{2} - \frac{f}{f'} - 2\dot{R}\Theta\frac{f''}{f'} - \frac{3K}{a^2}\right) + \frac{w}{1+w}\frac{k^2}{a^2}\right]\Delta^k_m$$
$$+\Theta\frac{f''}{f'}\Re^k + \left[\frac{1}{2} + \frac{k^2}{a^2}\frac{f''}{f'} - \frac{1}{2}\frac{ff''}{f'^2} + \frac{f''\mu_m}{f'^2} - \dot{R}\Theta(\frac{f''}{f'})^2 + \dot{R}\Theta\frac{f'''}{f'}\right]\mathcal{R}^k, \tag{5.47}$$

$$\dot{\mathcal{R}}^k = \Re^k - \frac{w}{1+w}\dot{R}\Delta^k_m, \tag{5.48}$$

$$\dot{\Re}^k = -\left(\Theta + 2\dot{R}\frac{f'''}{f''}\right)\Re^k - \dot{R}Z^k + \left(\frac{(1-3w)\mu_m}{3f''} - \frac{w}{1+w}\ddot{R}\right)\Delta^k_m$$
$$-\left[\frac{k^2}{a^2} + \ddot{R}\frac{f'''}{f''} + \dot{R}^2\frac{f^{(4)}}{f''} + \Theta\dot{R}\frac{f'''}{f''} + \frac{f'}{3f''} - \frac{R}{3}\right]\mathcal{R}^k, \tag{5.49}$$

$$\frac{C^k}{a^2} + \left[\frac{4\Theta}{3} + \frac{2\dot{R}f''}{f'}\right]Z^k - 2\frac{\mu_m}{f'}\Delta^k_m + \left[2\Theta\dot{R}\frac{f'''}{f'} - \frac{f''}{f'}(\frac{f}{f'} - \frac{2\mu_m}{f'} + 2\dot{R}\Theta\frac{f''}{f'} - 2\frac{k^2}{a^2})\right]\mathcal{R} + \frac{2\Theta f''}{f'}\Re^k$$
$$= 0. \tag{5.50}$$

along with

$$\dot{C}^k = 12K^2\left[\frac{f'^2 + 2a^2(\Theta f'' - 3\ddot{R}f''')}{a^2 f'^2}\frac{f''}{2\Theta f' + 3\dot{R}f''}\right]\mathcal{R}^k + K\left\{\frac{6f'C^k}{a^2(2\Theta f' + 3\dot{R}f'')}\right.$$
$$+\Delta^k_m\left(\frac{16w\Theta}{3(1+w)} - \frac{12\mu_m}{2\Theta f' + 3\dot{R}f''}\right) + \frac{12f''}{2\Theta f' + 3\dot{R}f''}\frac{k^2}{a^2}\mathcal{R}^k$$
$$-\left[\frac{2a^2(\Theta f'' - 3\dot{R}f''')}{3f'}\frac{12\dot{R}\Theta f'f''' - 6f''\left(f - 2\mu_m + 2\dot{R}\Theta f''\right)}{(2\Theta f' + 3\dot{R}f'')f'}\right]\mathcal{R}^k$$
$$\left.+\left(\frac{12\Theta f''}{2\Theta f' + 3\dot{R}f''} + \frac{2f''}{f'}\right)\Re\right\} - \frac{k^2}{a^2}\left[\frac{4wa^2\Theta}{3(1+w)}\Delta_m + \frac{2a^2f''}{f'}\Re^k - \frac{2a^2(\Theta f'' - 3\dot{R}f'')}{3f'}\mathcal{R}^k\right]. \tag{5.51}$$

The first four first order equations can be reduced to a coupled system of second-order differential equations:

$$\ddot{\Delta}^k_m - \left[(w-\frac{2}{3})\Theta + \frac{\dot{R}f''}{f'}\right]\dot{\Delta}^k_m + \left[w\frac{k^2}{a^2} + (w-1)\frac{\mu_m}{f'} - w\frac{f}{f'}\right]\Delta^k_m$$
$$= \frac{1+w}{2}\left[-1 - \frac{2k^2}{a^2}\frac{f''}{f'} + (f - 2\mu_m + 2\dot{R}\Theta f'')\frac{f''}{f'^2} - 2\dot{R}\Theta\frac{f'''}{f'}\right]\mathcal{R} - \frac{1+w}{f'}\Theta f''\dot{\mathcal{R}}^k, \tag{5.52}$$

$$\ddot{\mathcal{R}}^k + (2\dot{R}\frac{f'''}{f''} + \Theta)\dot{\mathcal{R}}^k + \left[\frac{k^2}{a^2} + \ddot{R}\frac{f'''}{f''} + \dot{R}^2\frac{f^{(4)}}{f''} + \Theta\dot{R}\frac{f'''}{f''} + \frac{f'}{3f''} - \frac{R}{3}\right]\mathcal{R}^k$$
$$= -\left[\frac{1}{3}(3w-1)\frac{\mu_m}{f''} + \frac{w}{1+w}\left(2\ddot{R} + 2\dot{R}^2\frac{f'''}{f''} + 2\dot{R}\Theta\right)\right]\Delta^k_m + \frac{1-w}{1+w}\dot{R}\dot{\Delta}^k_m. \tag{5.53}$$

The corresponding equations in GR can be retrieved by setting $f(R) = R$. This yields

$$\ddot{\Delta}^k_m - (w-\frac{2}{3})\Theta\dot{\Delta}^k_m + \left[w\frac{k^2}{a^2} + (\frac{1}{2} + w - \frac{3}{2}w^2)\mu_m\right]\Delta^k_m = 0, \tag{5.54}$$

$$\mathcal{R}^k = (1-3w)\mu_m\Delta^k_m. \tag{5.55}$$

A phase space analysis of the FLRW background dynamics shows that for specific intervals of n, the R^n model gives a set of initial conditions for the Universe to undergo a transition of decelerated expansion which then evolves into an accelerated phase at late times [52]. It is argued that this decelerated phase could have been a suitable era for structure formation to take place.

On scales much larger than the Hubble radius, a close look at Eqn. (5.51) in a flat background shows that C will be conserved, i.e., $\dot{C} = 0$, thus leaving us with 3 independent variables to deal with. Making use of (4.44-4.48) our first order equations will then reduce to

$$\dot{\Delta}_m = \left[\frac{1+w-2n}{1+w} - \frac{6(n-1)n}{n+3(n-1)w-3}\right]\frac{\Delta_m}{t} - \frac{3(1+w)^2}{4a_0^2\left[n+3(n-1)w-3\right]\left[4n-3(1+w)\right]}t^{1-\frac{4n}{3(1+w)}}C_0$$
$$- \frac{9(n-1)(1+w)^3t^2}{4\left[n+3(n-1)w-3\right]\left[4n-3(1+w)\right]}t^2\Re + \left[\frac{3(n-1)(1+w)^2\left[n(6w+8)-15(1+w)\right]}{4\left[n+3(n-1)w-3\right]\left[4n-3(1+w)\right]}\right]t\mathcal{R}, \tag{5.56}$$

$$\dot{\mathcal{R}} = \Re + \frac{8nw\left[4n-3(1+w)\right]}{3(1+w)^3}\frac{\Delta_m}{t^3}, \tag{5.57}$$

$$\dot{\Re} = -2\left[\frac{(n-4)+2(n-2)w}{(1+w)t} - \frac{3n(n-1)}{n+3w(n-1)-3}\right]\Re + \frac{2n(4n-3w-3)}{(1+w)\left[n+3(n-1)w-3\right]}\frac{C_0}{a_0^2}t^{-\frac{4n}{3(1+w)}-2}$$
$$-2\left[\frac{9n(n-2)(n-1)}{n+3(n-1)w-3} + 2n^2 - 7n - \frac{3n^2(9n-26)+57n}{9(1+w)(n-1)} - \frac{8n^2(n-2)}{9(1+w)^2(n-1)} + 6\right]\frac{\mathcal{R}}{t^2} +$$
$$\frac{16n\left[4n-3(1+w)\right]\left[4n+3(n-1)w-3\right]\left[(9w(1+w)+8)\,n^2 - (3w(9w+8)+13)\,n + 3(1+w)(1+6w)\right]}{27(n-1)(1+w)^4\left[n+3(n-1)w-3\right]}\frac{\Delta_m}{t^4}, \tag{5.58}$$

where C_0 is the value of the conserved quantity C. These equations can be combined to give a decoupled third order perturbation equation [21]

$$(n-1)\dddot{\Delta}_m-(n-1)\left(\frac{4nw}{1+w}-5\right)\frac{\ddot{\Delta}_m}{t}+\mathcal{D}_1(n,w)\frac{\dot{\Delta}_m}{t^2}+\mathcal{D}_2(n,w)\frac{\Delta_m}{t^3}+\mathcal{D}_3(n,w)C_0t^{-(\frac{4n}{3(1+w)})-1}=0, \tag{5.59}$$

where the $\mathcal{D}$s are constant coefficients for prescribed n and w values defined as

$$\mathcal{D}_1(n,w) = -\frac{2\left[-9\left(2(n-1)n+1\right)w^2+6n\left(n(4n-7)+1\right)w+18w+n\left(4n(8n-19)+33\right)+9\right]}{9(1+w)^2}, \tag{5.60}$$

$$\mathcal{D}_2(n,w) = \frac{\left[(2n-1)w-1\right]\left[4n-3(1+w)\right]\left[3(1+w)+n\left(n(6w+8)-9w-13\right)\right]}{9(1+w)^3}, \tag{5.61}$$

$$\mathcal{D}_3(n,w) = -\frac{n\left[21w-6n(2+w)+31\right]-18(1+w)}{6a_0^2}. \tag{5.62}$$

The general solution of (5.59) was obtained in [21]:

$$\Delta_m(t) = A_1t^{(\frac{2nw}{1+w})-1}+A_2t^{\alpha}+A_3t^{\beta}-A_4\frac{C_0}{a_0^2}t^{2-(\frac{4n}{3(1+w)})}, \tag{5.63}$$

A_1,A_2 and A_3 being arbitrary integration constants to be determined from initial conditions and with

$$\alpha = -\frac{1}{2}+\frac{nw}{1+w}+\frac{\sqrt{(n-1)\left[4(3w+8)^2n^3-4\left(3w(18w+55)+152\right)n^2+3(1+w)(87w+139)n-81(1+w)\right]}}{6(n-1)(1+w)^2} \tag{5.64}$$

$$\beta = -\frac{1}{2}+\frac{nw}{1+w}-\frac{\sqrt{(n-1)\left[4(3w+8)^2n^3-4\left(3w(18w+55)+152\right)n^2+3(1+w)(87w+139)n-81(1+w)\right]}}{6(n-1)(1+w)^2} \tag{5.65}$$

$$A_4 = \frac{9(1+w)^3\left[18(1+w)+\left(6n(2+w)-21w-31\right)\right]}{8\left[n(6w+4)-9(1+w)\right]\left[6(2+w)n^3-(19+9w)n^2-3(1+w)(1+3w)n+9(1+w)^2\right]}. \tag{5.66}$$

It has been shown in [21] that for $n \in (0.33, 0.71)$ and $n \in (1, 1.32)$, the C_2 and C_3 modes show damped oscillatory signatures that become sub-dominant at late times and that only for $n \in [1.32, 1.43)$, the modes grow slower than $t^{2/3}$. Another important distinction of R^n in this limit is that structure formation can take place even in accelerated backgrounds (see fig 5.1) which is not a feature of the Λ-dominated expansion phase in GR.

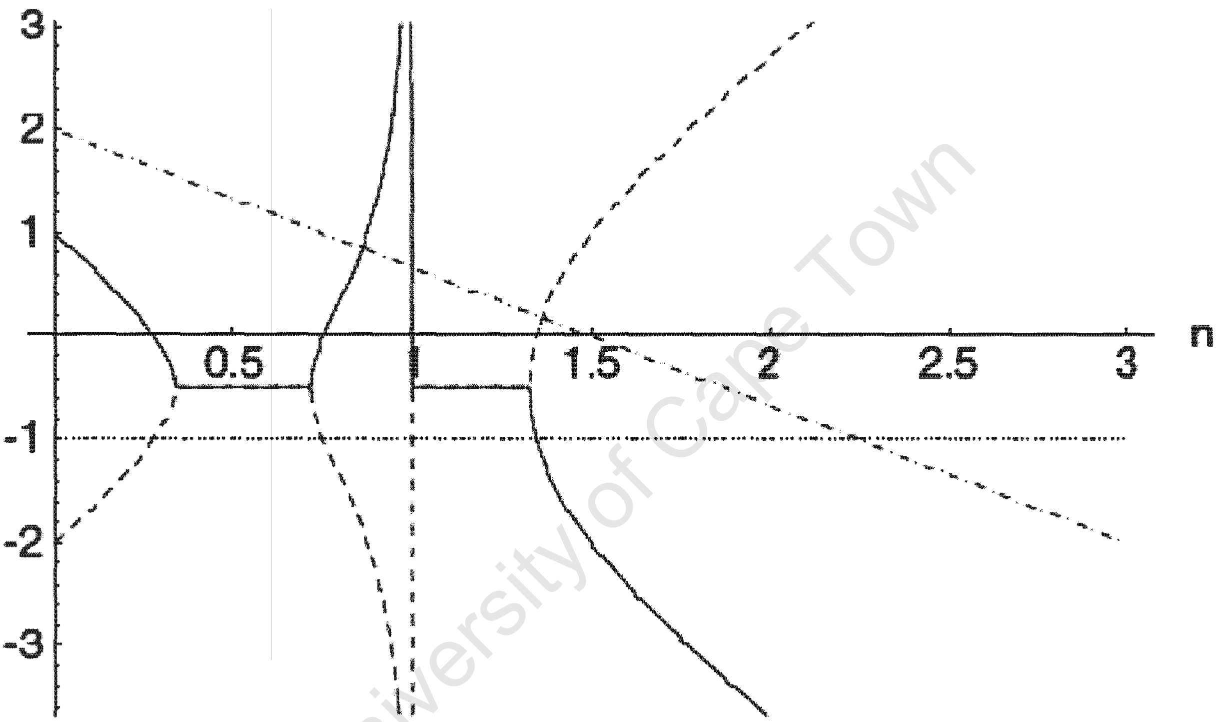


Figure 5.1: Plot of the real part of the exponents of each mode of the solution (5.63) against n in the case of dust. The continuous and dashed lines show the t^{α} and t^{β} modes, respectively, whereas the dashed - dot and the dotted lines represent the $t^{\frac{6-4n}{3}}$ and t^{-1} modes, respectively [21].

Chapter 6

PERTURBATIONS IN MULTI-FLUIDS

6.1 Preliminaries on Multi-component Fluids

In the previous chapter, we have seen perturbation equations that describe a universe filled with one kind of matter with a constant equation of state. However, the real universe consists of different components of matter fluids such as radiation, cold dark matter and baryons. Thus in analyzing the formation of structures in a universe containing such fluids, where each fluid interacts with other fluids besides having its own dynamical behavior, it is imperative that we consider the governing perturbation equations for each component.

A detailed analysis of multi-component covariant cosmological perturbations in General Relativity is given in [18–20]. The main objective of this work is two-fold: first, we want to extend the single fluid $f(R)$ perturbations into multi-component ones. Second, it is important to extend the general relativistic multi-fluid perturbations into those of a generic $f(R)$ gravity to account for structure formation in the late time accelerated universe.

The thermodynamical description of a relativistic matter fluid is dictated by the energy momentum tensor T^{m}_{ab}, the particle flux N^{a} and the entropy flux S^{a} of the system. Whereas T^{m}_{ab} and S^{a} always satisfy the conservation of 4-momentum and the second law of thermodynamics, namely

$$T^{m;b}_{ab} = 0, \quad S^{a}{}_{;a} \geq 0, \tag{6.1}$$

particle flux conservation, i.e., the condition $N^{a}{}_{;b} = 0$ may at times be violated.

The total energy-momentum tensor of the standard matter in a general frame is sourced by the energy density μ^{m}, isotropic pressure p^{m}, the energy flux $q^{m}_{a} (= q^{m}_{\langle a \rangle})$ and the anisotropic pressure $\pi^{m}_{ab} (= \pi^{m}_{\langle ab \rangle})$:

$$T^{m}_{ab} = \mu^{m} u^{m}_{a} u^{m}_{b} + p^{m} h^{m}_{ab} + 2 q^{m}_{(a} u^{m}_{b)} + \pi^{m}_{ab}, \tag{6.2}$$

and defines our thermodynamical quantities. In this case the effective energy-momentum tensor of standard matter is defined by Eqn.(6.6). If we impose the SEC, $T^{m}_{ab} V^{a} V^{b} \geq$

0, for all time-like vectors V^a, then T^m_{ab} will have a unique unit time-like vector $u^a_{E(m)}$ ($u^a_{E(m)}u^{E(m)}_a = -1$). Another time-like vector u^a_N can be defined along the flux N^a_N, i.e., $u^a_N = \frac{N^a}{\sqrt{-N b N^b}}$.

For a perfect fluid (or an unperturbed fluid in the background space), $u^a_{E(m)}$, u^a_N and S^a are all parallel [18] and a unique hydrodynamic 4-velocity u^a_m can be defined for the fluid flow, in which case

$$T^m_{ab} = \mu^m u^m_a u^m_b + p^m h^m_{ab}, \qquad N^a = n u^a_m, \qquad S^a = s u^a_m, \tag{6.3}$$

where μ^m and p^m are related by the equation of state $p^m = p^m(\mu^m, s)$. $n = -N^a u^m_a$ and $s = -S_a u^a_m$ define the particle and entropy densities, respectively, in the local rest frame of an observer attached to u^a_m.

We can also decompose the EMT with respect to another frame, say n^a, but in this case we need to introduce a particle drift $\tilde{j}^a = \tilde{h}^a{}_{b(m)} N^a$ [17, 18, 57].

If the matter fluid is imperfect, fluid hydrodynamic four-velocity is no longer unique and our EMT will take the more general expression given above (Eqn. 6.2) and the particle flux includes a *drift* term:

$$N^a = n u^a_m + j^a. \tag{6.4}$$

Choosing the relevant frame is a very crucial component of multi-fluid perturbation theories. u^a_m is the velocity of fundamental observers in the universe.[1] In the particle frame $u^a_m = u^a_N$, called the Eckart choice (Gauge), an observer $O_{u=u_N}$ sees no particle drift and hence $j^a = j^a_N = 0$. If, on the other hand, we consider the energy frame $u^a_m = u^a_E$, also known as the Landau Gauge, an observer $O_{u=u^m_E}$ measures no energy flux ($q^m_a = q^E_a = 0$) along the flow line and the EMT takes the form (6.3). In this work, our gauge choice will be the Landau Gauge, with the energy frame coinciding with the energy frame of the total standard matter fluid. Therefore all decompositions will be with respect to the 4-velocity vector $u^a = u^a_m$.

For multi-component matter fluids, the decomposition yields

$$T^m_{ab} = \sum_i T^i_{ab}, \tag{6.6}$$

where

$$T^i_{ab} = \mu_i u^i_a u^i_b + p_i h^i_{ab} + q^i_a u^i_b + q^i_b u^i_a + \pi^i_{ab}, \tag{6.7}$$

$$h^i_{ab} = g_{ab} + u^i_a u^i_b, \tag{6.8}$$

$$N^a_i = n_i u^a_i + j^a_i, \tag{6.9}$$

[1] Fluid flow vector u^a is uniquely defined as the future directed time-like eigenvector of the Ricci tensor:

$$u^a = \frac{dx^a}{d\tau}, \tag{6.5}$$

where $x^a(\tau)$ describes the worldline of the fluid in terms of the proper time τ. In our multi-fluid picture, it corresponds to the normal to the surface of homogeneity.

u_a^i being the normalized fluid 4-velocity vector for the i^{th} component, $u_i^a u_a^i = -1$ which we can fix by either choosing the energy frame $u_i^a = u_{Ei}^a$ thereby setting $q_i^a = q_{Ei}^a = 0$, or the particle frame $u_i^a = u_{Ni}^a$ for which $j_i^a = j_{Ni}^a = 0$ for that component. The velocity of the i^{th} fluid component relative to the fundamental observer O_u is defined to be

$$V_i^a \equiv u_i^a - u_m^a. \tag{6.10}$$

$V_i^a \neq 0$ for *tilted*, inhomogeneous cosmological media whereas the special case where u_i^a coincides with u_m^a describes an *untilted* homogeneous cosmological medium.

In the energy frame of standard matter, for which $u^a = u_m^a$, $h_m^{ab} = u_m^a u_m^b$, decomposition of the matter stress-energy tensor with respect to the 4-velocity u_m^a gives the following thermodynamical quantities:

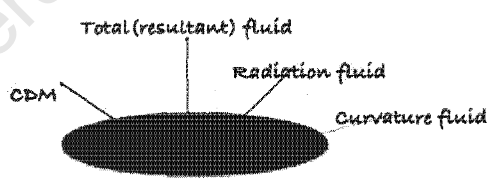
$$\mu^m = T_{ab}^m u_m^a u_m^b = \sum_{i=1}^{N} \mu_i, \tag{6.11}$$

$$p^m = \frac{1}{3} T_{ab}^m h_m^{ab} = \sum_{i=1}^{N} p_i, \tag{6.12}$$

$$q_a^m = -T_{bc}^m h^b{}_{a(m)} u^c = \sum_{i=1}^{N} (\mu_i + p_i) V_a^i, \tag{6.13}$$

$$\pi_{ab}^m = T_{cd}^m h_{\langle a(m)}^c h_{b(m)\rangle}^d = 0 \quad \text{(to \quad linear order)} \tag{6.14}$$

In the linearization procedure, we take into account the fact that, since V_a^i is small, all quadratic and higher order terms of this quantity are dropped.



$$T_{ab} = \mu u_a u_b + p h_{ab} + q_a u_b + q_b u_a + \pi_{ab}$$

$$T_{ab}^i = \mu_i u_a^i u_b^i + p_i h_{ab}^i + q_a^i u_b^i + q_b^i u_a^i + \pi_{ab}^i$$

Figure 6.1: The Multi-fluid diagram

Having fixed the frame to the energy frame of the standard matter u_m^a, we now decompose the energy momentum tensor of the curvature "fluid" relative to u_m^a. Consequently, the corresponding thermodynamical quantities in the curvature "fluid" are given by

$$\mu^R = T_{ab}^R u_m^a u_m^b = \frac{1}{f'} \left[\frac{1}{2}(Rf' - f) - \Theta f'' \dot{R} + f'' \tilde{\nabla}^2 R + f'' \dot{u}_b \tilde{\nabla}^b R \right], \tag{6.15}$$

$$p^R = \frac{1}{3}T^R_{ab}h^{ab}_m = \left[\frac{1}{2}(f - Rf') + f''\ddot{R} + f'''\dot{R}^2 + \frac{2}{3}\left(\Theta f''\dot{R} - f''\tilde{\nabla}^2 R \right.\right.$$
$$\left.\left. - f'''\tilde{\nabla}^a R\tilde{\nabla}_a R\right) + f''\dot{u}_b\tilde{\nabla}^b R\right], \tag{6.16}$$

$$q^R_a = -T^R_{bc}h^b{}_{a(m)}u^c_m = -\frac{1}{f'}\left[f'''\dot{R}\tilde{\nabla}_a R + f''\tilde{\nabla}_a\dot{R} - \frac{1}{3}f''\Theta\tilde{\nabla}_a R\right], \tag{6.17}$$

$$\pi^R_{ab} = T^R_{cd}h^c_{\langle a(m)}h^d_{b(m)\rangle} = \frac{1}{f'}\left[f''\tilde{\nabla}_{\langle a}\tilde{\nabla}_{b\rangle}R + f'''\tilde{\nabla}_{\langle a}R\tilde{\nabla}_{b\rangle}R + \sigma_{ab}\dot{R}\right]. \tag{6.18}$$

In the background FLRW universe, $V_i^a = 0$ and all perfect fluid components have the same velocity. Applying the Stewart-Walker Lemma shows that V_i^a is a first-order gauge-invariant (GI) quantity. If we choose our fluid flow vector u^a to coincide with the energy frame of the total matter, i.e., $u^a_E = u^a_m$, then exact FLRW models will be characterized by vanishing shear and vorticity of u^a_m and all spatial gradients orthogonal to u^a_m of any scalar quantity [17]:

$$\sigma_{ab} = \omega_{ab} = 0, \qquad \tilde{\nabla}_a f = 0. \tag{6.19}$$

It follows that since

$$X_a = \tilde{\nabla}_a\mu_m = 0, \quad Y_a = \tilde{\nabla}_a p_m, \quad Z_a = \tilde{\nabla}_a\Theta = 0 \tag{6.20}$$

in the background metric, then $\mu^m = \mu^m(t)$, $p^m = p^m(t)$ and $\Theta = \Theta(t)$. This necessitates the EMT having the perfect fluid form, and hence the vanishing of the anisotropic pressure π^m_{ab} and the energy flux q^m_a. We emphasize that from now on, we will fix our frame to be along u^a_m, and hence that, by *energy frame*, we interchangeably mean the *energy frame of the total matter.*

There are two aspects of dealing with perturbations in multi-fluid systems. The first one is where we are interested in the total fluctuation dynamics of the entire fluid. But this alone does not give us a complete picture of the fluid until a characterization of how each component of the fluid also plays a role in the overall fluctuation and how it relates with the rest of the components is known. Thus in the following sections we will investigate these differing yet complementary aspects of a multi-component cosmic fluid.

6.2 Total Fluid Perturbations

The dynamics of the total "fluid" obeys the Friedmann, Raychaudhuri and energy conservation equations,

$$\Theta^2 = 3(\tilde{\mu}^m + \mu^R) - \tfrac{3}{2}\tilde{R}, \tag{6.21}$$
$$\dot{\Theta} + \tfrac{1}{3}\Theta^2 + \tfrac{1}{2}(\tilde{\mu}^m + 3\tilde{p}^m) + \tfrac{1}{2}(\mu^R + 3p^R) = 0, \tag{6.22}$$
$$\dot{\mu}^m + \Theta(\mu^m + p^m) = 0, \tag{6.23}$$

in the homogeneous and isotropic FLRW background.

6.2.1 The Inhomogeneity Variables

In addition to the gradient variables defined in (5.21), which define inhomogeneities of the total fluid quantities in this case, we define the following gradient variables to characterize inhomogeneities that arise when more than one fluid components are involved:

$$D_a^i = a\frac{\tilde{\nabla}_a \mu_i}{\mu_i}, \qquad Y_a^i = \tilde{\nabla}_a p_i = \frac{1}{a}\left[c_{si}^2 \mu^i D_a^i + p^i \varepsilon_a^i\right],$$
$$\varepsilon_a = \frac{a}{p_m}\left(\frac{\partial p}{\partial s}\right)\tilde{\nabla}_a s, \qquad \varepsilon_a^i = \frac{a}{p_i}\left(\frac{\partial p^i}{\partial s}\right)\tilde{\nabla}_a s, \tag{6.24}$$

where ε_a is defined through the relation, [2]

$$p_m \varepsilon_a = \sum_i p_i \varepsilon_a^i + \frac{1}{2}\sum_{i,j}\frac{h_i h_j}{h}(c_{si}^2 - c_{sj}^2) S_a^{ij}, \tag{6.25}$$

and defines the dimensionless variable ε_a that quantifies entropy perturbations in the total fluid. The index i refers to the i^{th}-component fluid. In situations where each component is well described by a perfect fluid, $\varepsilon_a^i \simeq 0$. Thus in subsequent discussions all terms containing this quantity are dropped.

The total pressure gradient of the fluid is given by

$$Y_a^m = \tilde{\nabla}_a p_m = \frac{1}{a}\left[c_s^2 \mu_m D_a^m + p_m \varepsilon_a\right]. \tag{6.26}$$

6.2.2 Linear Conservation Equations for Matter

We know that μ, p and Θ are zeroth order (or background) quantities. Here we assume that D_a, Z_a, σ_{ab}, ω_{ab}, a_a, π_{ab}, q_a and their first derivatives are first order quantities. The linearized forms of the evolution equations of the background quantities can thus be given as

$$\dot{\mu}_m + (\mu_m + p_m)\Theta + Q = 0, \tag{6.27}$$
$$(\mu_m + p_m)a_a + Y_a + \tilde{\nabla}\pi_a^b + \dot{q}_a^m + \tfrac{4}{3}\Theta q_a^m = 0, \tag{6.28}$$
$$\dot{\Theta} + \tfrac{1}{3}\Theta^2 - A + \tfrac{1}{2}(\mu + 3p) = 0, \tag{6.29}$$

where

$$Q^m = q^a_{m;a} \simeq \tilde{\nabla}_a q_m^a \quad \text{to linear order.} \tag{6.30}$$

In a more compact form we can re-write the conservation equations as [19]

$$\dot{\mu}_m + 3hH\tilde{\nabla}_a \Psi_m^a = 0, \tag{6.31}$$
$$h a_a + Y_a^m + h(F_a^m + \Pi_a^m) = 0, \tag{6.32}$$

[2] We will discuss this under the section *Relative Equations*

where

$$h = \mu_m + p_m, \qquad \Psi_a^m = \frac{q_a^m}{h},$$
$$\Pi_a^m = \frac{1}{h}\tilde{\nabla}^b \pi_{ab}^m, \qquad F_a^m = \dot{\Psi}_a - (3c_s^2 - 1)H\Psi_a^m. \tag{6.33}$$

But since we are working in the energy frame of the total matter,

$$q_a^m = 0 \Rightarrow \Psi_a^m = 0. \tag{6.34}$$

6.2.3 Linear Gradient Equations

Using the above linearized equations, we can now write the evolution equations and the constraint equation of the gradient inhomogeneity variables.

The evolution of the total matter fluctuations is given by

$$\dot{D}_a^m + (1+w)Z_a - w\Theta D_a^m = 0. \tag{6.35}$$

The comoving expansion evolves according to

$$\dot{Z}_a = (\dot{R}\frac{f''}{f'} - \frac{2}{3}\Theta)Z_a + \left[\frac{(2c_s^2 - w - 1)}{(1+w)}\frac{\mu_m}{f'} + \frac{c_s^2}{(1+w)}\left(\frac{R}{2} - \frac{f}{f'} - 2\dot{R}\Theta\frac{f''}{f'} - \frac{K}{a^2}\right)\right] D_a^m$$
$$+ \frac{w}{(1+w)}\left[2\frac{\mu_m}{f'} + \frac{R}{2} - \frac{f}{f'} - 2\dot{R}\Theta\frac{f''}{f'} - \frac{K}{a^2}\right]\varepsilon_a + \Theta\frac{f''}{f'}\Re_a$$
$$+ \left[\frac{1}{2} - \frac{1}{2}\frac{ff''}{f'^2} + \frac{f''\mu_m}{f'^2} - \dot{R}\Theta(\frac{f''}{f'})^2 + \dot{R}\Theta\frac{f'''}{f'} + \frac{2K}{a^2}\frac{f''}{f'}\right]\mathcal{R}_a - \frac{f''}{f'}\tilde{\nabla}^2\mathcal{R}_a - \frac{c_s^2\tilde{\nabla}^2 D_a^m}{1+w} - \frac{w\tilde{\nabla}^2\varepsilon_a}{1+w}, \tag{6.36}$$

whereas the equations for the curvature inhomogeneities and the constraint are as follows:

$$\dot{\mathcal{R}}_a = \Re_a - \dot{R}\left[\frac{c_s^2}{1+w}D_a^m + \frac{w}{1+w}\varepsilon_a\right], \tag{6.37}$$

$$\dot{\Re}_a = -\left(2\dot{R}\frac{f'''}{f''} + \Theta\right)\Re_a - \dot{R}Z_a + \left[\frac{(1-3c_s^2)\mu_m}{3f''} - \frac{c_s^2}{1+w}\ddot{R}\right]D_a^m - \left[\frac{w\mu_m}{f''} + \frac{w}{1+w}\ddot{R}\right]\varepsilon_a$$
$$- \left(\frac{2K}{a^2} + \ddot{R}\frac{f'''}{f''} + \dot{R}^2\frac{f^{(4)}}{f''} + \dot{R}\Theta\frac{f'''}{f''} + \frac{1}{3}\frac{f'}{f''} - \frac{R}{3}\right)\mathcal{R}_a + \tilde{\nabla}^2\mathcal{R}_a, \tag{6.38}$$

$$\frac{C_a}{a^2} + \left(\frac{4}{3}\Theta + 2\frac{\dot{R}f''}{f'}\right)Z_a - 2\frac{\mu_m}{f'}D_a^m + \left[2\dot{R}\Theta\frac{f'''}{f'} - \frac{f''}{f'}\left(\frac{f}{f'} - 2\frac{\mu_m}{f'} - \frac{4K}{a^2}\right)\right]\mathcal{R}_a$$
$$+ 2\Theta\frac{f''}{f'}\Re_a - 2\frac{f''}{f'}\tilde{\nabla}^2\mathcal{R}_a = 0. \tag{6.39}$$

The constraint evolves as

$$\dot{C}_a = 12K^2\left[\frac{3f'^2 + 2a^2(\Theta f'' - 3\ddot{R}f''')}{a^2 f'^2}\frac{f''}{2\Theta f' + 3\dot{R}f''}\right]\mathcal{R}_a + K\left\{\frac{6f'C_a}{a^2(2\Theta f' + 3\dot{R}f'')}\right.$$

$$
+ D^m_a \left(\frac{16w\Theta}{3(1+w)} - \frac{12\mu_m}{2\Theta f' + 3\dot{R} f''}\right) - \frac{12 f''}{2\Theta f' + 3\dot{R} f''}\tilde{\nabla}^2 \mathcal{R}_a
$$
$$
- \left[\frac{2a^2(\Theta f'' - 3\dot{R} f''')}{3f'} \frac{12\dot{R}\Theta f' f''' - 6f''\left(f - 2\mu_m + 2\dot{R}\Theta f''\right)}{(2\Theta f' + 3\dot{R} f'')f'}\right]\mathcal{R}_a
$$
$$
+ \left(\frac{12\Theta f''}{2\Theta f' + 3\dot{R} f''} + \frac{2f''}{f'}\right)\Re_a \Bigg\} + \tilde{\nabla}^2\left[\frac{4wa^2\Theta}{3(1+w)}D^m_a + \frac{2a^2 f''}{f'}\Re_a - \frac{2a^2(\Theta f'' - 3\dot{R} f'')}{3f'}\mathcal{R}_a\right]. \tag{6.40}
$$

where the isentropic speed of sound $c_s^2 = \frac{\dot{p}_m}{\dot{\mu}_m}$ and the equation of state $w = \frac{p_m}{\mu_m}$ are related via $\dot{w} = (1+w)(w - c_s^2)\Theta$ to linear order. It is interesting to see that because of our frame choice being the energy frame of the total matter u^a_m, the propagation equation for the total density gradient (Eqn.6.35) is a simplified one whereas this is no longer the case for the component fluids (*cf.* Eqn. 6.65).

6.2.4 Scalar Equations

As in the single fluid case we define

$$
\Delta_m = a\tilde{\nabla}^a D^m_a, \qquad Z = a\tilde{\nabla}^a Z_a,
$$
$$
C = a\tilde{\nabla}^a C_a, \qquad \mathcal{R} = a\tilde{\nabla}^a \mathcal{R}_a,
$$
$$
\Re = a\tilde{\nabla}^a \Re_a \qquad \varepsilon = a\tilde{\nabla}^a \varepsilon_a, \tag{6.41}
$$

and hence the corresponding scalar evolution and constraint equations become

$$
\dot{\Delta}_m = w\Theta\Delta_m - (1+w)Z, \tag{6.42}
$$

$$
\dot{Z} = (\dot{R}\frac{f''}{f'} - \frac{2}{3}\Theta)Z + \left[\frac{(2c_s^2 - w - 1)}{(1+w)}\frac{\mu_m}{f'} + \frac{c_s^2}{(1+w)}\left(\frac{R}{2} - \frac{f}{f'} - 2\dot{R}\Theta\frac{f''}{f'} - \frac{3K}{a^2}\right)\right]\Delta_m
$$
$$
+ \frac{w}{(1+w)}\left[2\frac{\mu_m}{f'} + \frac{R}{2} - \frac{f}{f'} - 2\dot{R}\Theta\frac{f''}{f'} - \frac{3K}{a^2}\right]\varepsilon + \Theta\frac{f''}{f'}\Re
$$
$$
+ \left[\frac{1}{2} - \frac{1}{2}\frac{f f''}{f'^2} + \frac{f''\mu_m}{f'^2} - \dot{R}\Theta(\frac{f''}{f'})^2 + \dot{R}\Theta\frac{f'''}{f'}\right]\mathcal{R} - \frac{f''}{f'}\tilde{\nabla}^2\mathcal{R} - \frac{c_s^2\tilde{\nabla}^2\Delta_m}{1+w} - \frac{w\tilde{\nabla}^2\varepsilon}{1+w} \tag{6.43}
$$

$$
\dot{\mathcal{R}} = \Re - \dot{R}\left[\frac{c_s^2}{1+w}\Delta_m + \frac{w}{1+w}\varepsilon\right], \tag{6.44}
$$

$$
\dot{\Re} = -(2\dot{R}\frac{f'''}{f''} + \Theta)\Re - \dot{R}Z - \left[\ddot{R}\frac{f'''}{f''} + \dot{R}^2\frac{f^{(4)}}{f''} + \dot{R}\Theta\frac{f'''}{f''} + \frac{1}{3}\frac{f'}{f''} - \frac{R}{3}\right]\mathcal{R}
$$
$$
+ \left(\frac{(1-3c_s^2)\mu_m}{3f''} - \frac{c_s^2}{1+w}\ddot{R}\right)\Delta_m - \left(\frac{w\mu_m}{f''} + \frac{w}{1+w}\ddot{R}\right)\varepsilon + \tilde{\nabla}^2\mathcal{R}, \tag{6.45}
$$

$$
\frac{C}{a^2} + \left[\frac{4}{3}\Theta + 2\frac{\dot{R}f''}{f'}\right]Z - 2\frac{\mu_m}{f'}\Delta_m + \left[2\Theta\dot{R}\frac{f'''}{f'} - \frac{f''}{f'}(\frac{f}{f'} - 2\frac{\mu_m}{f'} + 2\dot{R}\Theta\frac{f''}{f'})\right]\mathcal{R} + 2\Theta\frac{f''}{f'}\Re - 2\frac{f''}{f'}\tilde{\nabla}
$$

$$= 0, \tag{6.46}$$

$$\dot{C} = 12K^2\left[\frac{f'^2 + 2a^2(\Theta f'' - 3\ddot{R}f''')}{a^2 f'^2}\frac{f''}{2\Theta f' + 3\dot{R}f''}\right]\mathcal{R} + K\left\{\frac{6f'C}{a^2(2\Theta f' + 3\dot{R}f'')}\right.$$
$$+ \Delta_m\left(\frac{16w\Theta}{3(1+w)} - \frac{12\mu_m}{2\Theta f' + 3\dot{R}f''}\right) - \frac{12f''}{2\Theta f' + 3\dot{R}f''}\tilde{\nabla}^2\mathcal{R}$$
$$- \left[\frac{2a^2(\Theta f'' - 3\dot{R}f''')}{3f'}\frac{12\dot{R}\Theta f' f''' - 6f''\left(f - 2\mu_m + 2\dot{R}\Theta f''\right)}{(2\Theta f' + 3\dot{R}f'')f'}\right]\mathcal{R}$$
$$\left. + \left(\frac{12\Theta f''}{2\Theta f' + 3\dot{R}f''} + \frac{2f''}{f'}\right)\Re\right\} + \tilde{\nabla}^2\left[\frac{4wa^2\Theta}{3(1+w)}\Delta_m + \frac{2a^2f''}{f'}\Re - \frac{2a^2(\Theta f'' - 3\dot{R}f'')}{3f'}\mathcal{R}\right]. \tag{6.47}$$

6.2.5 Harmonic Analysis

Following similar harmonic decomposition procedure as in Section 5.3, we write the above scalar equations as

$$\dot{\Delta}^k_m + (1+w)Z^k - w\Theta\Delta^k_m = 0, \tag{6.48}$$

$$\dot{Z}^k = (\dot{R}\frac{f''}{f'} - \frac{2}{3}\Theta)Z^k + \left[\frac{(1+3w)c_s^2 - 2(1+w)}{2(1+w)}\frac{\mu_m}{f'} + \frac{2c_s^2\Theta^2 + 3c_s^2(\mu_R + 3p_R)}{6(1+w)} + \frac{c_s^2}{1+w}\frac{k^2}{a^2}\right]\Delta^k_m$$
$$+ \left[\frac{2f'\Theta^2 + 3(1+3w)\mu_m + 3f'(\mu_R + 3p_R)}{6f'(1+w)} + \frac{1}{1+w}\frac{k^2}{a^2}\right]w\varepsilon^k + \Theta\frac{f''}{f'}\Re^k$$
$$+ \left[\frac{1}{2} + \frac{k^2}{a^2}\frac{f''}{f'} - \frac{1}{2}\frac{ff''}{f'^2} + \frac{f''\mu_m}{f'^2} - \dot{R}\Theta(\frac{f''}{f'})^2 + \dot{R}\Theta\frac{f'''}{f'}\right]\mathcal{R}^k, \tag{6.49}$$

$$\dot{\mathcal{R}}^k = \Re^k - \dot{R}\left[\frac{c_s^2}{1+w}\Delta^k_m + \frac{w}{1+w}\varepsilon^k\right], \tag{6.50}$$

$$\dot{\Re}^k = -\left(2\dot{R}\frac{f'''}{f''} + \Theta\right)\Re^k - \dot{R}Z^k + \left[\frac{(1-3c_s^2)\mu_m}{3f''} - \frac{c_s^2}{1+w}\ddot{R}\right]\Delta^k_m - \left[\frac{w\mu_m}{f''} + \frac{w}{1+w}\ddot{R}\right]\varepsilon^k$$
$$- \left(\frac{k^2}{a^2} + \ddot{R}\frac{f'''}{f''} + \dot{R}^2\frac{f^{(4)}}{f''} + \dot{R}\Theta\frac{f'''}{f''} + \frac{1}{3}\frac{f'}{f''} - \frac{R}{3}\right)\mathcal{R}^k, \tag{6.51}$$

$$\frac{C^k}{a^2} + \left[\frac{4}{3}\Theta + 2\frac{\dot{R}f''}{f'}\right]Z^k - 2\frac{\mu_m}{f'}\Delta^k_m + \left[2\Theta\dot{R}\frac{f'''}{f'} - \frac{f''}{f'}(\frac{f}{f'} - 2\frac{\mu_m}{f'} + 2\dot{R}\Theta\frac{f''}{f'} - 2\frac{k^2}{a^2})\right]\mathcal{R} + 2\Theta\frac{f''}{f'}\Re$$
$$= 0. \tag{6.52}$$

In the same manner, the evolution of the constraint equation takes the form

$$\dot{C}^k = 12K^2\left[\frac{f'^2 + 2a^2(\Theta f'' - 3\ddot{R}f''')}{a^2f'^2}\frac{f''}{2\Theta f' + 3\dot{R}f''}\right]\mathcal{R}^k + K\left\{\frac{6f'C^k}{a^2(2\Theta f' + 3\dot{R}f'')}\right.$$
$$+ \Delta^k_m\left(\frac{16w\Theta}{3(1+w)} - \frac{12\mu_m}{2\Theta f' + 3\dot{R}f''}\right) + \frac{12f''}{2\Theta f' + 3\dot{R}f''}\frac{k^2}{a^2}\mathcal{R}^k$$

$$- \left[\frac{2a^2(\Theta f'' - 3\dot{R} f''')}{3f'} \frac{12\dot{R}\Theta f' f''' - 6f''\left(f - 2\mu_m + 2\dot{R}\Theta f''\right)}{(2\Theta f' + 3\dot{R} f'')f'} \right] \mathcal{R}^k$$
$$+ \left(\frac{12\Theta f''}{2\Theta f' + 3\dot{R} f''} + \frac{2f''}{f'}\right)\Re \Big\} - \frac{k^2}{a^2}\left[\frac{4wa^2\Theta}{3(1+w)}\Delta_m + \frac{2a^2 f''}{f'}\Re^k - \frac{2a^2(\Theta f'' - 3\dot{R} f'')}{3f'}\mathcal{R}^k\right]. \quad (6.53)$$

As in the single fluid case, this equation will be used when we discuss the long wavelength limits of our perturbations in the radiation and dust backgrounds.

6.2.6 Second-order Equations

The first order equations (6.48, 6.49, 6.50, 6.51) combine to give the second order pair of equations

$$\ddot{\Delta}^k_m + \left[(c_s^2 + \frac{2}{3} - 2w)\Theta - \dot{R}\frac{f''}{f'}\right]\dot{\Delta}^k_m + \left[\left(\frac{3}{2}w^2 + 5c_s^2 - 4w - 1\right)\frac{\mu_m}{f'} + \frac{1}{2}(3w - 5c_s^2)\frac{f}{f'}\right.$$
$$\left. + (c_s^2 - w)\left(2R - 4\dot{R}\Theta\frac{f''}{f'} - \frac{12K}{a^2}\right) + c_s^2\frac{k^2}{a^2}\right]\Delta^k_m + \left[2\frac{\mu_m}{f'} + \frac{R}{2} - \frac{f}{f'} - \dot{R}\Theta\frac{f''}{f'} - \frac{3K}{a^2} + \frac{k^2}{a^2}\right]w\varepsilon^k$$
$$= \frac{1+w}{2}\left[-1 - \frac{2k^2}{a^2}\frac{f''}{f'} + (f - 2\mu_m + 2\dot{R}\Theta f'')\frac{f''}{f'^2} - 2\dot{R}\Theta(\frac{f''}{f'})^2 - 2\dot{R}\Theta\frac{f'''}{f'}\right]\mathcal{R}^k - (1+w)\Theta\frac{f''}{f'}\dot{\mathcal{R}}^k \quad (6.54)$$

$$\ddot{\mathcal{R}}^k + (2\dot{R}\frac{f'''}{f''} + \Theta)\dot{\mathcal{R}}^k + \left[\frac{k^2}{a^2} + \ddot{R}\frac{f'''}{f''} + \dot{R}^2\frac{f^{(4)}}{f''} + \dot{R}\Theta\frac{f'''}{f''} + \frac{1}{3}\frac{f'}{f''} - \frac{R}{3}\right]\mathcal{R}^k + \frac{c_s^2 - 1}{1+w}\dot{R}\dot{\Delta}^k_m$$
$$+ \left[\frac{(3c_s^2 - 1)\mu_m}{3f''} + \frac{w + c_s^2}{1+w}\dot{R}\Theta + \frac{c_s^2}{1+w}\left(2\ddot{R} + 2\dot{R}^2\frac{f'''}{f''}\right) + \frac{\dot{R}}{1+w}\left(\dot{c_s^2} + c_s^2(c_s^2 - w)\Theta\right)\right]\Delta^k_m$$
$$+ \frac{w}{1+w}\dot{R}\dot{\varepsilon}^k + \left[\frac{w\mu_m}{f''} + \frac{2w - c_s^2}{1+w}\dot{R}\Theta + \frac{w}{1+w}\left(2\ddot{R} + 2\dot{R}^2\frac{f'''}{f''}\right)\right]\varepsilon^k = 0. \quad (6.55)$$

The corresponding equations of these in the general relativistic limit reduce to:

$$\mathcal{R}^k = (1 - 3c_s^2)\mu_m\Delta^k_m - 3w\mu_m\varepsilon^k, \quad (6.56)$$
$$\ddot{\Delta}^k_m + (c_s^2 + \frac{2}{3} - 2w)\Theta\dot{\Delta}^k_m + \left[\frac{3w(w + c_s^2) + (c_s^2 - w) - 2}{2}\mu_m + c_s^2\frac{k^2}{a^2} - \frac{4}{3}(w - c_s^2)\Theta^2\right]\Delta^k_m$$
$$+ \left[\frac{2\Theta^2 + 3(1 + 3w)\mu_m}{6} + \frac{k^2}{a^2}\right]w\varepsilon^k + \frac{1+w}{2}\left((1 - 3c_s^2)\mu_m\Delta^k_m - 3w\mu_m\varepsilon\right) = 0$$
$$\Rightarrow \ddot{\Delta}^k_m + (c_s^2 + \frac{2}{3} - 2w)\Theta\dot{\Delta}^k_m + \left[(\frac{3}{2}w^2 + 3c_s^2 - 4w - \frac{1}{2})\mu_m + (w - c_s^2)\frac{12K}{a^2} + c_s^2\frac{k^2}{a^2}\right]\Delta^k_m$$
$$+ w\left(\frac{k^2}{a^2} - \frac{3K}{a^2}\right)\varepsilon^k = 0. \quad (6.57)$$

This last equation is the standard equation presented in the same form as in [18].

6.3 The Component Equations

The component background equations in the FLRW spacetime are given by

$$\Theta^2 = 3(\tilde{\mu}^m + \mu^R) - \tfrac{3}{2}\tilde{R}, \tag{6.58}$$

$$\dot{\Theta} + \tfrac{1}{3}\Theta^2 + \tfrac{1}{2}(\tilde{\mu}^m + 3\tilde{p}^m) + \tfrac{1}{2}(\mu^R + p^R) = 0, \tag{6.59}$$

$$\dot{\mu}_i + \Theta(\mu_i + p_i) = 0, \tag{6.60}$$

It is worth noting that the Friedmann and Raychaudhuri equations are described by the sum total of all the matter and curvature components of the energy density and pressure, rather than by the individual components.

The speed of sound $c_{si}^2 = \frac{\dot{p}_i}{\dot{\mu}_i}$ and the barotropic equation of state $w_i = \frac{p_i}{\mu_i}$ of the i^{th}-component fluid are related by the familiar expression

$$\dot{w}_i = -(1 + w_i)(c_{si}^2 - w_i)\Theta. \tag{6.61}$$

As in the total fluid case, the linearized energy and momentum conservation equations for the component fluids can be written (neglecting all the interaction terms) as

$$\dot{\mu}^i + 3h_i H\tilde{\nabla}_a \Psi_i^a = 0, \tag{6.62}$$

$$h_i a_a + Y_a^i + h_i(F_a^i + \Pi_a^i) = 0, \tag{6.63}$$

with

$$h_i = \mu_i + p_i, \qquad \Psi_a^i = \frac{\tilde{q}_a^i}{h_i} = \frac{q_a^i}{h_i} + V_a^i,$$
$$\Pi_a^i = \frac{1}{h_i}\tilde{\nabla}^b \pi_{ab}^i, \qquad F_a^i = \dot{\Psi^i}_a - (3c_{si}^2 - 1)H\Psi_a^i. \tag{6.64}$$

6.3.1 Linear Gradient Equations

Deriving the evolution equation for D_a^i in terms of the above quantities (6.64), we obtain

$$\dot{D}_a^i - 3H(w_i - c_{si}^2)D_a^i + (1 + w_i)Z_a = 3aH(1 + w_i)(F_a^m + \Pi_a^m) + \frac{1}{\mu^i h}3Hh_i(c_s^2\mu_m D_a^m + p_m\varepsilon_a)$$
$$- a(1 + w_i)\tilde{\nabla}_a\tilde{\nabla}^b\Psi_b^i - 3Hw_i\varepsilon_a^i. \tag{6.65}$$

For matter fluids in the energy frame, $\Psi_a^m = 0,\ F_a^m = 0$ and Π_a^m is a negligible second-order quantity. But we note the dependence of this equation on the velocity ψ_a^i of the i^{th}-fluid.

Upon using the relation (6.25), the above equation yields

$$\dot{D}_a^i - (w_i - c_{si}^2)\Theta D_a^i + (1 + w_i)Z_a = a\Theta(1 + w_i)(F_a^m + \Pi_a^m)$$
$$+ \frac{1 + w_i}{1 + w}c_s^2\Theta D_a^m + \frac{1}{h}(1 + w_i)\Theta\left\{\sum_i p_i\varepsilon_a^i + \frac{1}{2}\sum_{i,j}\frac{h_i h_j}{h}(c_{si}^2 - c_{sj}^2)S_a^{ij}\right\}$$
$$- a(1 + w_i)\tilde{\nabla}_a\tilde{\nabla}^b\Psi_b^i - w_i\Theta\varepsilon_a^i. \tag{6.66}$$

The momentum equations for the total and component fluids

$$h a_a + Y_a + h(F^m_a + \Pi^m_a) = 0 \Rightarrow a_a = -\frac{Y^m_a}{h} - F^m_a - \Pi^m_a,$$
$$h_i a_a + Y^i_a + h_i(F^i_a + \Pi^i_a) = 0 \Rightarrow h_i a_a + Y^i_a + h_i\left[\dot{\Psi}^i_a - (3c^2_{si} - 1)H\Psi^i_a + \Pi^i_a\right] = 0, \tag{6.67}$$

can be combined and rewritten as

$$h_i\left[-\frac{Y^m_a}{h} - F^m_a - \Pi^m_a\right] + Y^i_a + h_i\left[\dot{\Psi}^i_a - (3c^2_{si} - 1)H\Psi^i_a + \Pi^i_a\right] = 0. \tag{6.68}$$

This yields the equation of motion for Ψ^i_a:

$$\dot{\Psi}^i_a - (3c^2_{si} - 1)H\Psi^i_a = F^m_a + \Pi^m_a - \Pi^i_a + \frac{1}{ahh_i}\left[h_i c^2_s \mu_m D^m_a + h_i p_m \varepsilon_a - h c^2_{si}\mu^i D^i_a + h p^i \varepsilon^i_a\right]. \tag{6.69}$$

The propagation equations for the expansion and the curvature inhomogeneity variables are the same as in the total fluid since these quantities are global dynamical properties of the total fluid and not of the components.

6.3.2 Relative Evolution Equations

The relative variables defined below relate the differences in the fluctuations of pairs of the different components of the fluid. The derivation of their governing evolution equations follow similar lines of argument as those for the total and component equations. These variables depend only on the choice of the individual velocities, not on the choice of the overall frame [18].

$$S^{ij}_a = \frac{\mu_i D^i_a}{h_i} - \frac{\mu_j D^j_a}{h_j} = \frac{1}{1+w_i}D^i_a - \frac{1}{1+w_j}D^j_a,$$
$$\varepsilon^{ij}_a = \frac{p_i \varepsilon^i_a}{h_i} - \frac{p_j \varepsilon^j_a}{h_j} = \frac{w_i}{1+w_i}\varepsilon^i_a - \frac{w_j}{1+w_j}\varepsilon^j_a,$$
$$\Pi^{ij}_a = \Pi^i_a - \Pi^j_a,\ V^{ij}_a = V^i_a - V^j_a,\ \Psi^{ij}_a = \Psi^i_a - \Psi^j_a. \tag{6.70}$$

The evolution equations for these variables are therefore given by

$$\dot{S}^{ij}_a + a\tilde{\nabla}_a\tilde{\nabla}^b\Psi^{ij}_a + \Theta\varepsilon^{ij}_a = 0, \tag{6.71}$$
$$\dot{\Psi}^{ij}_a - (3c^2_{sj} - 1)H\Psi^{ij}_a - 3(c^2_{si} - c^2_{sj})H\Psi^i_a = -\frac{1}{ah_i}(c^2_{si} - c^2_{sj})\mu_i D^i_i - \frac{1}{a}c^2_{sj}S^{ij}_a + \frac{1}{a}\varepsilon^{ij}_a - \Pi^{ij}_a. \tag{6.72}$$

Using the relations

$$\sum_l h_l \Psi^{il}_a = \sum_l h_l\left(\Psi^i_a - \Psi^l_a\right) = h(\Psi^i_a - \Psi^m_a), \tag{6.73}$$

and

$$\sum_l h_l S^{il}_a = \sum_l h_l\left(\frac{\mu_i}{h_i}D^i_a - \frac{\mu_l}{h_l}D^l_a\right) = h\frac{\mu_i}{h_i}D^i_a - \mu_m D^m_a, \tag{6.74}$$

we can rewrite the relative density perturbation equation and the relative velocity equation in terms of the total matter and relative variables:

$$\dot{S}_a^{ij} + a\tilde{\nabla}_a\tilde{\nabla}^b\Psi_a^{ij} + \Theta\varepsilon_a^{ij} = 0, \tag{6.75}$$

$$\dot{\Psi}_a^{ij} - [3(c_{si}^2 + c_{sj}^2) - 1]H\Psi_a^{ij} + \frac{\Theta}{h}\sum_l h_l(c_{sj}^2\Psi_a^{il} - c_{si}^2\Psi_a^{jl}) = \Theta(c_{si}^2 - c_{sj}^2)\Psi_a$$
$$- \frac{1}{ah}(c_{si}^2 - c_{sj}^2)\mu_m D_a^m - \frac{1}{a}(c_{si}^2 - c_{sj}^2)S_a^{ij} - \frac{1}{a}\varepsilon_a^{ij} + \frac{1}{ah}\sum_l h_l(c_{sj}^2 S_a^{il} - c_i^2 S^{jl}) - \Pi_a^{ij}. \tag{6.76}$$

These equations show the coupling between isothermal and adiabatic perturbations, and the generation thereof, of one from the other.

6.3.3 The Case of Two Fluids

Let's now consider a two-fluid case, where the relative inhomogeneities in fluids 1 and 2 are given as

$$\Psi_a^1 = \Psi_a + \frac{h_2}{h}\Psi_a^{12}, \tag{6.77}$$

$$\mu_1 D_a^1 = \frac{h_1}{h}(\mu_m D_a^m + h_2 S_a^{12}). \tag{6.78}$$

The evolution equations for the relative variables are given by

$$\dot{S}_a^{12} + a\tilde{\nabla}_a\tilde{\nabla}^b\Psi_a^{12} + \Theta\varepsilon_a^{12} = 0, \tag{6.79}$$

$$\dot{\Psi}_a^{12} - \left[\frac{3}{h}(h_2 c_{s1}^2 + h_1 c_{s2}^2) - 1\right] H\Psi_a^{12} - 3H(c_{s1}^2 - c_{s2}^2)\Psi_a - \frac{1}{ah}(c_{s1}^2 - c_{s2}^2)\mu_m D_a^m$$
$$= -\frac{1}{ah}(h_2 c_{s1}^2 + h_1 c_{s2}^2)S_a^{12} - \Pi_a^{12} - \frac{1}{a}\varepsilon_a^{12}. \tag{6.80}$$

Following the definition in [18]

$$c_z^2 \equiv \frac{1}{h}(h_2 c_{s1}^2 + h_1 c_{s2}^2), \tag{6.81}$$

we can rewrite (6.80) in a more compact form :

$$\dot{\Psi}_a^{12} - (3c_z^2 - 1)H\Psi_a^{12} - 3(c_{s1}^2 - c_{s2}^2)H\Psi_a^m = -\frac{1}{ah}(c_{s1}^2 - c_{s2}^2)\mu_m D_a^m - \frac{1}{a}c_z^2 S_a^{12} - \Pi_a^{12} - \frac{1}{a}\varepsilon_a^{12}. \tag{6.82}$$

Reducing further to perfect fluid situations, we have $\Pi_a^{12} = \Psi_a^m = 0$ and $\Psi_a^{12} = V_a^{12}$ and this will lead the equations to

$$\dot{S}_a^{12} + a\tilde{\nabla}_a\tilde{\nabla}^b V_b^{12} + 3H\varepsilon_a^{12} = 0, \tag{6.83}$$

$$\dot{V}_a^{12} - (3c_z^2 - 1)HV_a^{12} = -\frac{1}{ah}(c_{s1}^2 - c_{s2}^2)\mu_m D_a^m - \frac{1}{a}c_z^2 S_a^{12} - \frac{1}{a}\varepsilon_a^{12}. \tag{6.84}$$

A more detailed treatment of radiation-dust perturbations as an example of a two-fluid system will be given in the next chapter.

6.3.4 Scalar Equations

Our scalar variables for the component fluids are given as

$$\Delta^i_m = a\tilde{\nabla}^a D^i_a, \qquad V_i = a\tilde{\nabla}^a V^i_a, \qquad \Psi_i = a\tilde{\nabla}^a \Psi^i_a,$$
$$\varepsilon_i = a\tilde{\nabla}^a \varepsilon^i_a, \qquad F_i = a\tilde{\nabla}^a F^i_a, \qquad \Pi_i = a\tilde{\nabla}^a \Pi^i_a, \tag{6.85}$$

and for the relative scalar variables:

$$S_{ij} = a\tilde{\nabla}^a S^{ij}_a = \frac{\Delta_{mi}}{1+w_i} - \frac{\Delta_{mj}}{1+w_j}, \tag{6.86}$$
$$V_{ij} = a\tilde{\nabla}^a V^{ij}_a, \qquad \Psi_{ij} = a\tilde{\nabla}^a \Psi^{ij}_a,$$
$$\varepsilon_{ij} = a\tilde{\nabla}^a \varepsilon^{ij}_a, \qquad \Pi_{ij} = a\tilde{\nabla}^a \Pi^{ij}_a. \tag{6.87}$$

Using these variables, we can write the corresponding propagation equations as follows:

$$\dot{\Delta}^i_m - \Theta(w_i - c^2_{si})\Delta^i_m + (1+w_i)Z = \frac{1+w_i}{1+w}c^2_s\Theta\Delta_m + \frac{1+w_i}{1+w}w\Theta\varepsilon - w_i\Theta\varepsilon_i$$
$$+ a\Theta(1+w_i)(F^m + \Pi^m) - a(1+w_i)\tilde{\nabla}^2\Psi_i, \tag{6.88}$$
$$\dot{\Psi}_i - (3c^2_{si} - 1)H\Psi_i = F^m + \Pi^m - \Pi_i + \frac{1}{ahh_i}\left[h_i c^2_s \mu_m \Delta_m + h_i p_m \varepsilon - h c^2_{si}\mu^i\Delta^i_m + hp^i\varepsilon_i\right], \tag{6.89}$$
$$\dot{S}^{ij} + a\tilde{\nabla}^2\Psi_{ij} + \Theta\varepsilon_{ij} = 0, \tag{6.90}$$
$$\dot{\Psi}^{ij} - [3(c^2_{si} + c^2_{sj}) - 1]H\Psi^{ij} + \frac{\Theta}{h}\sum_l h_l(c^2_{sj}\Psi^{il} - c^2_{si}\Psi^{jl}) = \Theta(c^2_{si} - c^2_{sj})\Psi - \frac{1}{ah}(c^2_{si} - c^2_{sj})\mu_m\Delta_m$$
$$- \frac{1}{a}(c^2_{si} - c^2_{sj})S^{ij} - \frac{1}{a}\varepsilon^{ij} + \frac{1}{ah}\sum_l h_l(c^2_{sj}S^{il} - c^2_i S^{jl}) - \Pi^{ij}. \tag{6.91}$$

Specializing these for the two-component fluid system, we get

$$\dot{S}^{12} + a\tilde{\nabla}^2\Psi_{12} + \Theta\varepsilon_{12} = 0, \tag{6.92}$$
$$\dot{\Psi}_{12} - (3c^2_z - 1)H\Psi_{12} - 3(c^2_{si} - c^2_{s2})H\Psi^m = -\frac{1}{ah}(c^2_{s1} - c^2_{s2})\mu_m\Delta_m - \frac{1}{a}c^2_z S_{12}. - \Pi_{12} - \frac{1}{a}\varepsilon_{12}, \tag{6.93}$$
$$\tag{6.94}$$

which, for the perfect-fluid case, will further simplify to

$$\dot{S}_{12} + a\tilde{\nabla}^2 V_{12} + \Theta\varepsilon_{12} = 0, \tag{6.95}$$
$$\dot{V}_{12} - (3c^2_z - 1)HV_{12} = -\frac{1}{ah}(c^2_{s1} - c^2_{s2})\mu_m\Delta_m - \frac{1}{a}c^2_z S_{12} - \frac{1}{a}\varepsilon_{12}. \tag{6.96}$$

6.3.5 Harmonic Analysis

In the harmonic space, the perfect-fluid component (where $\Psi_i \simeq V_i$) equations will take the form

$$\dot{\Delta}_i^k - (w_i - c_{si}^2)\Theta\Delta_i^k + (1+w_i)Z^k = \frac{1+w_i}{1+w}c_s^2\Theta\Delta_m^k + \frac{1+w_i}{1+w}w\Theta\varepsilon^k + (1+w_i)\frac{k^2}{a}V_i, \quad (6.97)$$

$$\dot{V}_i^k - (c_{si}^2 - \frac{1}{3})\Theta V_i^k = \frac{1}{ahh_i}\left[h_i c_s^2\mu\Delta_m + h_i p\varepsilon^k - hc_{si}^2\mu^i\Delta_i^k\right], \quad (6.98)$$

$$\dot{V}_{ij}^k - (c_{sj}^2 - \frac{1}{3})\Theta V_{ij}^k - (c_{si}^2 - c_{sj}^2)\Theta V_i^k = -\frac{1}{ah_i}(c_{si}^2 - c_{sj}^2)\mu_i\Delta_i^k - \frac{1}{a}c_{sj}^2 S_{ij}^k, \quad (6.99)$$

$$\dot{S}_{ij}^k - \frac{k^2}{a}V_{ij}^k = 0. \quad (6.100)$$

6.3.6 Second-order Equations

The set of second-order differential equations involving Δ_m^i, $\mathcal{R}$, $\Re$, S_{ij} and V_{ij}, in the barotropic two-perfect fluid case are given by

$$\begin{aligned}
&\ddot{\Delta}_i^k + (\frac{2}{3} - w_i)\Theta\dot{\Delta}_i^k - \frac{1+w_i}{1+w}\left(\dot{R}\frac{f''}{f'} + (c_s^2 - c_{si}^2)\Theta\right)\dot{\Delta}^k + c_{si}^2\frac{k^2}{a^2}\Delta_i^k \\
&- \frac{1+w_i}{1+w}\left[(1+w)\frac{\mu_m}{f'} - \left(2\frac{\mu_m}{f'} - \frac{f}{f'} - \Theta\dot{R}\frac{f''}{f'}\right)c_s^2 + \dot{c}_s^2\Theta\right. \\
&\left.+(c_s^2 - c_{si}^2)(c_s^2 - w)\left(3\frac{\mu_m}{f'} + \frac{3R}{2} - \frac{3f}{2f'} - 3\dot{R}\Theta\frac{f''}{f'} - \frac{9K}{a^2}\right) - w\dot{R}\Theta\frac{f''}{f'}\right]\Delta^k - \frac{1+w_i}{1+w}w\Theta\dot{\varepsilon}^k \\
&- \frac{1+w_i}{1+w}\left[\left(w - 3(c_s^2 + wc_{si}^2)\right)\frac{\mu_m}{f'} - \frac{1}{2}\left(w + 3(c_s^2 + wc_{si}^2)\right)\frac{f}{f'} - \left(2w - 3(c_s^2 + wc_{si}^2)\right)\dot{R}\Theta\frac{f''}{f'}\right. \\
&\left.+(w - c_s^2 - c_{si}^2 w)\left(\frac{3R}{2} - \frac{9K}{a^2}\right)\right]\varepsilon^k \\
&+ (1+w_i)\Theta\frac{f''}{f'}\dot{\mathcal{R}}^k + (1+w_i)\left[\frac{1}{2} + \frac{k^2}{a^2}\frac{f''}{f'} - \frac{1}{2}\frac{ff''}{f'^2} + \frac{f''\mu_m}{f'^2} - \dot{R}\Theta(\frac{f''}{f'})^2 + \dot{R}\Theta\frac{f'''}{f'}\right]\mathcal{R}^k = 0,
\end{aligned} \quad (6.101)$$

$$\begin{aligned}
&\ddot{\mathcal{R}}^k + (2\dot{R}\frac{f'''}{f''} + \Theta)\dot{\mathcal{R}}^k + \left[\frac{k^2}{a^2} + \ddot{R}\frac{f'''}{f''} + \dot{R}^2\frac{f^{(4)}}{f''} + \dot{R}\Theta\frac{f'''}{f''} + \frac{1}{3}\frac{f'}{f''} - \frac{R}{3}\right]\mathcal{R}^k + \frac{c_s^2 - 1}{1+w}\dot{R}\dot{\Delta}_m^k \\
&+ \left[\frac{(3c_s^2 - 1)\mu_m}{3f''} + \dot{c}_s^2\dot{R} + \frac{w + c_s^2}{1+w}\dot{R}\Theta + \frac{c_s^2}{1+w}\left(2\ddot{R} + 2\dot{R}^2\frac{f'''}{f''}\right) + c_s^2(c_s^2 - w)\dot{R}\Theta\right]\Delta_m^k \\
&+ \frac{w}{1+w}\dot{R}\dot{\varepsilon}^k + \left[\frac{w\mu_m}{f''} + \frac{2w - c_s^2}{1+w}\dot{R}\Theta + \frac{w}{1+w}\left(2\ddot{R} + 2\dot{R}^2\frac{f'''}{f''}\right)\right]\varepsilon^k = 0,
\end{aligned} \quad (6.102)$$

$$\ddot{S}_{ij}^k = \frac{k^2}{a}\dot{V}_{ij} - \frac{k^2}{3a}\Theta V_{ij}, \quad (6.103)$$

$$\begin{aligned}
\ddot{V}_{ij}^k = {} & (c_z^2 - \frac{1}{3})\Theta\dot{V}_{ij} + \left[\dot{c}_z^2\Theta - (c_z^2 - \frac{1}{3})\left(\frac{1}{3}\Theta^2 + \frac{1}{2}(1+3w)\frac{\mu_m}{f'} + \frac{1}{2}(\mu_R + 3p_R)\right)\right]V_{ij} \\
& - \frac{c_{si}^2 - c_{sj}^2}{a(1+w)}\dot{\Delta}_m + \frac{c_{si}^2 - c_{sj}^2}{a(1+w)}(\frac{1}{3} + w - c_s^2)\Theta\Delta_m - \frac{c_z^2}{a}\dot{S}_{ij} + \frac{c_z^2\Theta - 3\dot{c}_z^2}{3a}S_{ij}.
\end{aligned} \quad (6.104)$$

Since Eqns. (6.103) and (6.104) are linearly dependent equations, we can pick up either one of them and, in combination with the other two equations, close our system of second order equations.

Chapter 7

RADIATION-DUST PERTURBATIONS

7.1 Background Setup

We now are in a position to write the perturbation equations for a universe dominated with radiation and pressureless matter or dust mixture. General relativistic covariant perturbations in such a universe have have been investigated by Dunsby [19,20], and Dunsby, Bruni, and Ellis [18].

Since the component fluids satisfy the conservation equations of the background separately, we can write,

$$\dot{\mu}_d + \Theta\mu_d = 0, \tag{7.1}$$

$$\dot{\mu}_r + \tfrac{4}{3}\Theta\mu_r = 0. \tag{7.2}$$

The speed of sound and the equation of state parameter for dust are both zero, and for radiation, they are $\frac{1}{3}$. Equations (6.58) and (6.59) close the system of equations that completely describe background.

For the total fluid we have

$$w = \frac{p}{\mu_m} = \frac{p_d + p_r}{\mu_d + \mu_r} = \frac{\mu_r w_r}{\mu_d + \mu_r} = \frac{1}{3}\frac{\mu_r}{\mu_d + \mu_r}, \tag{7.3}$$

$$c_s^2 = \frac{1}{(1+w)(\mu_m)}\left(\mu_d c_{sd}^2 + \frac{4}{3}\mu_r c_{sr}^2\right) = \frac{\frac{4}{3}\mu_r c_{sr}^2}{(1+w)(\mu_d+\mu_r)} = \frac{4\mu_r}{3(3\mu_d + 4\mu_r)}, \tag{7.4}$$

$$c_z^2 = \frac{1}{h}(h_r c_{sd}^2 + h_d c_{sr}^2) = \frac{3\mu_d c_{sr}^2}{3\mu_d + 4\mu_r} = \frac{\mu_d}{3\mu_d + 4\mu_r}. \tag{7.5}$$

And their variations in time are given by

$$\dot{w} = (1+w)(w - c_s^2)\Theta = -\frac{\mu_d\mu_r}{9(\mu_d+\mu_r)^2}, \tag{7.6}$$

$$\dot{c}_s^2 = -\frac{4\mu_d\mu_r\Theta}{3(3\mu_d+4\mu_r)^2} = -c_z^2 c_s^2\Theta, \tag{7.7}$$

$$\dot{c}_z^2 = \frac{4\mu_d\mu_r\Theta}{3(3\mu_d+4\mu_r)^2} = c_z^2 c_s^2\Theta = -\dot{c}_s^2. \tag{7.8}$$

7.2 Total Fluid Equations

Upon expanding Eqn.(6.25) for a radiation - dust mixture we get

$$p_m\varepsilon = -\frac{4\mu_d\mu_r}{3(3\mu_d+4\mu_r)}S_{dr}, \tag{7.9}$$

and hence

$$\varepsilon = -\frac{4\mu_d}{3\mu_d+4\mu_r}S_{dr}. \tag{7.10}$$

We can thus readily derive the evolution equation for ε; and it is given by

$$\dot\varepsilon = -\frac{16}{3}\frac{\mu_d\mu_r}{(3\mu_d+4\mu_r)^2}\Theta S_{dr} - \frac{4\mu_d}{3\mu_d+4\mu_r}\dot S_{dr} = -4c_z^2c_s^2\Theta S_{dr} - 4c_z^2\dot S_{dr}. \tag{7.11}$$

Using these relations and applying the general total fluid second order equations to the radiation-dust mixture yields

$$\begin{aligned}
&\ddot\Delta^k_m + \left[(c_s^2+\frac{2}{3}-2w)\Theta - \dot R\frac{f''}{f'}\right]\dot\Delta^k_m + \left[\left(\frac{3}{2}w^2+5c_s^2-4w-1\right)\frac{\mu_m}{f'} + \frac{1}{2}(3w-5c_s^2)\frac{f}{f'}\right.\\
&\left.+(c_s^2-w)\left(2R-4\dot R\Theta\frac{f''}{f'}-\frac{12K}{a^2}\right)+c_s^2\frac{k^2}{a^2}\right]\Delta^k_m - 4wc_z^2\left[2\frac{\mu_m}{f'}+\frac{R}{2}-\frac{f}{f'}-\dot R\Theta\frac{f''}{f'}-\frac{3K}{a^2}+\frac{k^2}{a^2}\right]S^k_{dr}\\
&=\frac{1+w}{2}\left[-1-\frac{2k^2}{a^2}\frac{f''}{f'}+(f-2\mu_m+2\dot R\Theta f'')\frac{f''}{f'^2}-2\dot R\Theta(\frac{f''}{f'})^2-2\dot R\Theta\frac{f'''}{f'}\right]\mathcal{R}^k-(1+w)\Theta\frac{f''}{f'}\dot{\mathcal{R}}^k,
\end{aligned} \tag{7.12}$$

$$\begin{aligned}
&\ddot{\mathcal{R}}^k + (2\dot R\frac{f'''}{f''}+\Theta)\dot{\mathcal{R}}^k + \left[\frac{k^2}{a^2}+\ddot R\frac{f'''}{f''}+\dot R^2\frac{f^{(4)}}{f''}+\dot R\Theta\frac{f'''}{f''}+\frac{1}{3}\frac{f'}{f''}-\frac{R}{3}\right]\mathcal{R}^k+\frac{c_s^2-1}{1+w}\dot R\dot\Delta^k_m\\
&+\left[\frac{(3c_s^2-1)\mu_m}{3f''}+\frac{w+c_s^2}{1+w}\dot R\Theta+\frac{c_s^2}{1+w}\left(2\ddot R+2\dot R^2\frac{f'''}{f''}\right)+\frac{\dot R}{1+w}\left(\dot{c_s^2}+c_s^2(c_s^2-w)\Theta\right)\right]\Delta^k_m\\
&-\frac{4w}{1+w}c_z^2\dot R\dot S^k_{dr}-4wc_z^2\left[\frac{\mu_m}{f''}+\frac{2}{1+w}\left(\ddot R+\dot R\Theta+\dot R^2\frac{f'''}{f''}\right)\right]S^k_{dr}=0,
\end{aligned} \tag{7.13}$$

$$\ddot S^k_{dr} + (c_s^2+\frac{1}{3})\Theta\dot S^k_{dr} + \frac{k^2}{a^2}c_z^2S^k_{dr} - \frac{k^2}{a^2}(c_z^2+\frac{3}{4}c_s^2)\Delta^k_m = 0, \tag{7.14}$$

where Δ_m and S_{dr} can be expanded as

$$\Delta_m = \frac{\mu_d\Delta_d+\mu_r\Delta_r}{\mu_d+\mu_r}, \tag{7.15}$$

$$S_{dr} = \Delta_d - \tfrac{3}{4}\Delta_r. \tag{7.16}$$

These three equations form a closed system of equations characterizing the density, curvature and entropy fluctuations of the total mixture.

7.3 Component Equations

Applying our component perturbation equations(6.97-6.100) to a radiation/dust fluid background we obtain

$$\dot\Delta^k_d + Z^k = c_s^2\Theta\Delta^k_d - \frac{4c_s^2c_z^2\mu_r}{3(\mu_d+\mu_r)}\Theta S^k_{dr} + \frac{k^2}{a}V^k_d, \tag{7.17}$$

$$\dot{\Delta}^k_r + \tfrac{4}{3}Z^k = \tfrac{4}{3}c_s^2\Theta\Delta^k_r - \frac{16c_s^2c_z^2\mu_r}{9(\mu_d+\mu_r)}\Theta S^k_{dr} + \frac{4k^2}{3a}V^k_r, \tag{7.18}$$

$$\dot{S}^k_{dr} - \frac{k^2}{a}V^k_{dr} = 0, \tag{7.19}$$

$$\dot{V}^k_{dr} + c_s^2\Theta V^k_{dr} = \frac{1}{a}\frac{\mu_d+\mu_r}{3\mu_d+4\mu_r}\Delta^k_m - \frac{1}{a}c_z^2 S^k_{dr}. \tag{7.20}$$

It can also be shown that

$$\dot{V}^k_{dr} = -\frac{a}{k^2}c_s^2\Theta\dot{S}_{dr} + \frac{1}{a}(c_z^2 + \frac{3}{4}c_s^2)\Delta_m - \frac{1}{a}c_z^2 S_{dr}. \tag{7.21}$$

The perturbations of the density gradients of the radiation component of the fluid is dictated by the second order equation

$$\begin{aligned}
&\ddot{\Delta}^k_r + \left[\left(\frac{1}{3} - \frac{4}{3(1+w)}(\frac{3w\mu_d}{3\mu_d+4\mu_r} + \frac{(c_s^2-\frac{1}{3})\mu_r}{\mu_d+\mu_r})\right)\Theta - \frac{4\mu_r}{3(1+w)(\mu_d+\mu_r)}\dot{R}\frac{f''}{f'}\right]\dot{\Delta}^k_r \\
&+ \frac{4\mu_d}{3(1+w)}\left[\left(\frac{4w}{3\mu_d+4\mu_r} - \frac{c_s^2-\frac{1}{3}}{\mu_d+\mu_r}\right)\Theta - \frac{\dot{R}f''}{(\mu_d+\mu_r)f'}\right]\dot{\Delta}^k_d \\
&+ \frac{4}{3(1+w)}\left[\frac{k^2}{3a^2} + \left(\frac{(w-c_s^2)\mu_r}{\mu_d+\mu_r} - \frac{3w\mu_d}{3\mu_d+4\mu_r} + \frac{\mu_d\mu_r}{3(\mu_d+\mu_r)^2}\right)\dot{R}\Theta\frac{f''}{f'}\right. \\
&- \left(\frac{4\mu_d\mu_r}{(3\mu_d+4\mu_r)^2}\frac{3w\mu_d+(3w-1)\mu_r}{3(\mu_d+\mu_r)} - \frac{3(c_s^2-\frac{2w}{3})\mu_d}{3\mu_d+4\mu_r} + \frac{(c_s^2-\frac{1}{3})(c_s^2-w)\mu_r}{\mu_d+\mu_r} - \frac{(c_s^2-\frac{1}{3})\mu_d\mu_r}{3(\mu_d+\mu_r)^2}\right)\Theta^2 \\
&\left.- \left(\frac{(1+w-2c_s^2)\,\mu_r}{\mu_d+\mu_r} - \frac{6w\mu_d}{3\mu_d+4\mu_r}\right)\frac{\mu_d+\mu_r}{f'} - \left(\frac{3w\mu_d}{3\mu_d+4\mu_r} + \frac{c_s^2\mu_r}{\mu_d+\mu_r}\right)\frac{f}{f'}\right]\Delta^k_r \\
&+ \frac{4}{3(1+w)}\left[\left(\frac{(w-c_s^2)\mu_d}{\mu_d+\mu_r} + \frac{4w\mu_d}{3\mu_d+4\mu_r} - \frac{\mu_d\mu_r}{3(\mu_d+\mu_r)^2}\right)\dot{R}\Theta\frac{f''}{f'}\right. \\
&+ \left(\frac{4\mu_d\mu_r}{3(3\mu_d+4\mu_r)^2}\frac{4w\mu_r+(4w+1)\mu_d}{\mu_d+\mu_r} - \frac{(c_s^2-\frac{1}{3})\mu_d\mu_r}{3(\mu_d+\mu_r)^2} - \frac{4(c_s^2-\frac{2w}{3})\mu_d}{3\mu_d+4\mu_r} - \frac{(c_s^2-\frac{1}{3})(c_s^2-w)\mu_d}{\mu_d+\mu_r}\right)\Theta^2 \\
&\left.- \left(\frac{(1+w-2c_s^2)\,\mu_d}{\mu_d+\mu_r} + \frac{8w\mu_d}{3\mu_d+4\mu_r}\right)\frac{\mu_d+\mu_r}{f'} + \left(\frac{4w\mu_d}{3\mu_d+4\mu_r} - \frac{c_s^2\mu_d}{\mu_d+\mu_r}\right)\frac{f}{f'}\right]\Delta^k_d \\
&+ \frac{4}{3}\Theta\frac{f''}{f'}\dot{\mathcal{R}}^k + \frac{4}{3}\left[\frac{1}{2} + \frac{k^2}{a^2}\frac{f''}{f'} - \frac{1}{2}\frac{ff''}{f'^2} + \frac{f''(\mu_r+\mu_d)}{f'^2} - \dot{R}\Theta(\frac{f''}{f'})^2 + \dot{R}\Theta\frac{f'''}{f'}\right]\mathcal{R}^k = 0.
\end{aligned} \tag{7.22}$$

Similarly the evolution equation of the dust density gradient is given by

$$\ddot{\Delta}_d^k + \left[\left(\frac{2}{3} + \frac{\mu_d}{1+w}(\frac{4w}{3\mu_d+4\mu_r} - \frac{c_s^2}{\mu_d+\mu_r})\right)\Theta - \frac{\mu_d}{(1+w)(\mu_d+\mu_r)}\dot{R}\frac{f''}{f'}\right]\dot{\Delta}_d^k$$
$$-\frac{1}{1+w}\left[\left(\frac{3w\mu_d}{3\mu_d+4\mu_r} + \frac{c_s^2\mu_r}{\mu_d+\mu_r}\right)\Theta + \frac{\mu_r}{\mu_d+\mu_r}\dot{R}\frac{f''}{f'}\right]\dot{\Delta}_r^k$$
$$+\frac{\mu_d}{1+w}\left[\left(\frac{w-c_s^2}{\mu_d+\mu_r} + \frac{4w}{3\mu_d+4\mu_r} - \frac{\mu_r}{3(\mu_d+\mu_r)^2}\right)\dot{R}\Theta\frac{f''}{f'}\right.$$
$$+\left(\frac{4\mu_r}{3(3\mu_d+4\mu_r)^2}\frac{4w\mu_r+(4w+1)\mu_d}{\mu_d+\mu_r} - \frac{4(c_s^2-w)}{3\mu_d+4\mu_r} - \frac{(c_s^2-w)c_s^2}{\mu_d+\mu_r} - \frac{c_s^2\mu_r}{3(\mu_d+\mu_r)^2}\right)\Theta^2$$
$$\left.-\left(\frac{1+w-2c_s^2}{\mu_d+\mu_r} + \frac{8w}{3\mu_d+4\mu_r}\right)\frac{\mu_d+\mu_r}{f'} + \left(\frac{4w}{3\mu_d+4\mu_r} - \frac{c_s^2}{\mu_d+\mu_r}\right)\frac{f}{f'}\right]\Delta_d^k$$
$$+\frac{1}{1+w}\left[\left(\frac{(w-c_s^2)\mu_r}{\mu_d+\mu_r} - \frac{3w\mu_d}{3\mu_d+4\mu_r} - \frac{\mu_d\mu_r}{3(\mu_d+\mu_r)^2}\right)\dot{R}\Theta\frac{f''}{f'} + \left(\frac{c_s^2\mu_d\mu_r}{3(\mu_d+\mu_r)^2} + \frac{3(c_s^2-w)\mu_d}{3\mu_d+4\mu_r}\right.\right.$$
$$\left.-\frac{(c_s^2-w)c_s^2\mu_r}{\mu_d+\mu_r} + \frac{4\mu_d\mu_r}{(3\mu_d+4\mu_r)^2}\frac{(1-3w)\mu_r-3w\mu_d}{3(\mu_d+\mu_r)}\right)\Theta^2$$
$$\left.-\left(\frac{3w\mu_d}{3\mu_d+4\mu_r} + \frac{c_s^2\mu_r}{\mu_d+\mu_r}\right)\frac{f}{f'} - \left(\frac{(1+w-2c_s^2)\mu_r}{\mu_d+\mu_r} - \frac{6w\mu_d}{3\mu_d+4\mu_r}\right)\frac{\mu_d+\mu_r}{f'}\right]\Delta_r^k$$
$$+\Theta\frac{f''}{f'}\dot{\mathcal{R}}^k + \left[\frac{1}{2} + \frac{k^2}{a^2}\frac{f''}{f'} - \frac{1}{2}\frac{ff''}{f'^2} + \frac{f''(\mu_r+\mu_d)}{f'^2} - \dot{R}\Theta(\frac{f''}{f'})^2 + \dot{R}\Theta\frac{f'''}{f'}\right]\mathcal{R}^k = 0. \quad (7.23)$$

In terms of the component perturbation variables we can write the evolution equation for the curvature gradient as

$$\ddot{\mathcal{R}}^k + (2\dot{R}\frac{f'''}{f''} + \Theta)\dot{\mathcal{R}}^k + \left[\frac{k^2}{a^2} + \ddot{R}\frac{f'''}{f''} + \dot{R}^2\frac{f^{(4)}}{f''} + \dot{R}\Theta\frac{f'''}{f''} + \frac{1}{3}\frac{f'}{f''} - \frac{R}{3}\right]\mathcal{R}^k$$
$$+\left(\frac{c_s^2-1}{1+w}\frac{\mu_d}{\mu_d+\mu_r} - \frac{4wc_z^2}{1+w}\right)\dot{R}\dot{\Delta}_d^k + \left(\frac{c_s^2-1}{1+w}\frac{\mu_r}{\mu_d+\mu_r} + \frac{3wc_z^2}{1+w}\right)\dot{R}\dot{\Delta}_r^k$$
$$+\left\{\left[(3c_s^2-1)\frac{\mu_d+\mu_r}{3f''} + \frac{w+c_s^2}{1+w}\dot{R}\Theta + \frac{c_s^2}{1+w}\left(2\ddot{R} + 2\dot{R}^2\frac{f'''}{f''}\right) + \frac{\dot{R}}{1+w}\left(\dot{c_s^2} + c_s^2(c_s^2-w)\Theta\right)\right]\frac{\mu_d}{\mu_d+\mu_r}\right.$$
$$\left.-4wc_z^2\left[\frac{\mu_d+\mu_r}{f''} + \frac{2}{1+w}\left(\ddot{R} + \dot{R}\Theta + \dot{R}^2\frac{f'''}{f''}\right)\right] + \frac{c_s^2-1}{3(1+w)}\frac{\mu_d\mu_r}{(\mu_d+\mu_r)^2}\dot{R}\Theta\right\}\Delta_d^k$$
$$+\left\{\left[(3c_s^2-1)\frac{\mu_d+\mu_r}{3f''} + \frac{w+c_s^2}{1+w}\dot{R}\Theta + \frac{c_s^2}{1+w}\left(2\ddot{R} + 2\dot{R}^2\frac{f'''}{f''}\right) + \frac{\dot{R}}{1+w}\left(\dot{c_s^2} + c_s^2(c_s^2-w)\Theta\right)\right]\frac{\mu_r}{\mu_d+\mu_r}\right.$$
$$\left.+3wc_z^2\left[\frac{\mu_d+\mu_r}{f''} + \frac{2}{1+w}\left(\ddot{R} + \dot{R}\Theta + \dot{R}^2\frac{f'''}{f''}\right)\right] - \frac{c_s^2-1}{3(1+w)}\frac{\mu_d\mu_r}{(\mu_d+\mu_r)^2}\dot{R}\Theta\right\}\Delta_r^k = 0.$$
$$(7.24)$$

The entropy perturbations follow the evolution equation

$$\ddot{S}_{dr}^k = \frac{k^2}{a}\dot{V}_{dr} - \frac{k^2}{3a}\Theta V_{dr}, \quad (7.25)$$

and this can be further expanded as :

$$\ddot{S}^k_{dr} + (c_s^2 + \frac{1}{3})\Theta\dot{S}_{dr} + \frac{k^2}{a^2}c_z^2 S_{dr} - \frac{k^2}{a^2}(c_z^2 + \frac{3}{4}c_s^2)\Delta_m = 0. \tag{7.26}$$

In terms of component gradients (7.26) can also be given alternatively by

$$\ddot{S}^k_{dr} + (c_z^2 + 2c_s^2)\Theta\dot{\Delta}_d - \frac{3}{4}(c_z^2 + 2c_s^2)\Theta\dot{\Delta}_r - \frac{3k^2}{4a^2}(c_z^2 + c_s^2)\Delta_r = 0. \tag{7.27}$$

In the coming sections, we will consider the short- and long wavelength limits of our radiation/dust perturbation equations in a flat ($K = 0$) FLRW spacetime and look for their exact solutions.

We observe that (7.1) and (7.2) integrate to give

$$\mu_d = \mu_{(d)0}(\frac{a_0}{a})^3, \tag{7.28}$$

$$\mu_r = \mu_{(r)0}(\frac{a_0}{a})^4. \tag{7.29}$$

and thus the total energy density of the mixture is

$$\mu_m = \mu_d + \mu_r = \mu_{(r)0}a_0^4(1 + \frac{\mu_{(d)0}}{\mu_{(r)0}})a^{-4} = \mu_{(r)0}a_0^4(1 + \alpha a)a^{-4}, \tag{7.30}$$

where $\alpha \equiv \frac{\mu_{(d)0}}{a\mu_{(r)0}}$. Defining $\beta \equiv \frac{1}{a_0^2}\sqrt{\frac{3}{\mu_{(r)0}}}$ we can rewrite (7.30) as

$$\mu_m = \frac{3}{\beta^2}a^{-4}(1 + \alpha a), \tag{7.31}$$

and the pressure $p = p_d + p_r = p_r = \frac{1}{3}\mu_r$ as

$$p = \frac{1}{\beta^2}a^{-4}. \tag{7.32}$$

Since we do not have an explicit expression of the Hubble parameter H and the curvature scalar R as functions of the scale factor a in $f(R)$, an exact multi-fluid background solution is not available. This, however, is an important issue and will be investigated in a future work.

In this chapter we will look at solutions in the short wavelength and long wavelength approximations for perturbations both in the radiation and dust dominated epochs where we can use the exact single fluid background solution given by $a = a_0 t^{\frac{2n}{3(1+w)}}$ [21].

7.4 SHORT WAVELENGTH SOLUTIONS

7.4.1 Perturbations in the Radiation-dominated Epoch

Let us now look at the case where the characteristic size of the fluid inhomogeneities is much less than the Jeans length for the radiation fluid but is still larger than the mean free path of the photon. Similar investigation has been made by [19] for the case of General

Relativity.[1] Here we assume that we can neglect the interaction between the component fluids and that the radiation energy density can be taken as *almost* homogeneous, meaning $\Delta_r \approx 0$.
This amounts to studying dust and curvature fluctuations on a homogeneous radiation background, whereby radiation affects the growth of the dust fluctuations by speeding up the cosmic expansion [18]. In this epoch, the component and relative equations we have seen in the previous chapter will take the form

$$\dot{\Delta}_d^k + Z^k = \frac{\Theta}{h}(c_s^2\mu_m\Delta_m^k + p_m\varepsilon^k) + a(\frac{k^2}{a^2})V_d^k, \tag{7.33}$$

$$\begin{aligned}
&\dot{Z}^k - (\dot{R}\frac{f''}{f'} - \frac{2}{3}\Theta)Z^k - \left[\frac{(2c_s^2 - w - 1)}{(1+w)}\frac{\mu_m}{f'} - \frac{c_s^2}{(1+w)}\left(\frac{R}{2} - \frac{f}{f'} - 2\dot{R}\Theta\frac{f''}{f'}\right)\right]\Delta_m^k \\
&- \frac{w}{(1+w)}\left[2\frac{\mu_m}{f'} + \frac{R}{2} - \frac{f}{f'} - 2\dot{R}\Theta\frac{f''}{f'}\right]\varepsilon^k - \frac{1}{h}(\frac{k^2}{a^2})\left[c_s^2\mu_m\Delta_m^k + p_m\varepsilon^k\right] + \Theta\frac{f''}{f'}\Re^k \\
&- \left[\frac{1}{2} + \frac{k^2}{a^2}\frac{f''}{f'} - \frac{1}{2}\frac{ff''}{f'^2} + \frac{f''\mu_m}{f'^2} - \dot{R}\Theta(\frac{f''}{f'})^2 + \dot{R}\Theta\frac{f'''}{f'}\right]\mathcal{R}^k = 0,
\end{aligned} \tag{7.34}$$

$$\dot{V}_d^k + \frac{1}{3}\Theta V_d^k = \frac{1}{ah}\left[c_s^2\mu_m\Delta_m^k + p_m\varepsilon^k\right], \tag{7.35}$$

$$\dot{V}_{dr}^k - (c_z^2 - \frac{1}{3})\Theta V_{dr}^k = \frac{1}{3ah}\mu_m\Delta_m^k - \frac{1}{a}c_z^2 S_{dr}^k, \tag{7.36}$$

$$\dot{\mathcal{R}}^k = \Re^k - \dot{R}\frac{1}{h}\left[c_s^2\mu_m\Delta_m^k + p_m\varepsilon^k\right], \tag{7.37}$$

$$\begin{aligned}
\dot{\Re}^k = &-\left(2\dot{R}\frac{f'''}{f''} + \Theta\right)\Re^k - \dot{R}Z^k + \frac{\mu_m}{3f''}\Delta_m^k - \frac{1}{f''}\left[c_s^2\mu_m\Delta_m^k + p_m\varepsilon^k\right] \\
&- \left(\frac{k^2}{a^2} + \ddot{R}\frac{f'''}{f''} + \dot{R}^2\frac{f^{(4)}}{f''} + \dot{R}\Theta\frac{f'''}{f''} + \frac{1}{3}\frac{f'}{f''} - \frac{R}{3}\right)\mathcal{R}^k.
\end{aligned} \tag{7.38}$$

Since $\Delta_r << \Delta_d$ we have

$$c_s^2\mu_m\Delta_m^k + p_m\varepsilon^k = \frac{1}{3}\mu_r\Delta_r^k \approx 0, \tag{7.39}$$

whence

$$S_{dr}^k \approx \Delta_d^k. \tag{7.40}$$

Using these approximations the above set of equations can be rewritten as

$$\dot{\Delta}_d^k + Z^k - a(\frac{k^2}{a^2})V_d^k = 0, \tag{7.41}$$

$$\begin{aligned}
&\dot{Z}^k - (\dot{R}\frac{f''}{f'} - 2H)Z^k + \frac{\mu_d}{f'}\Delta_d^k - \Theta\frac{f''}{f'}\Re^k \\
&- \left[\frac{1}{2} + \frac{k^2}{a^2}\frac{f''}{f'} - \frac{1}{2}\frac{ff''}{f'^2} + \frac{f''\mu_r}{f'^2} - 3H\dot{R}(\frac{f''}{f'})^2 + 3H\dot{R}\frac{f'''}{f'}\right]\mathcal{R}^k = 0,
\end{aligned} \tag{7.42}$$

$$\dot{V}_d^k + HV_d^k = 0, \tag{7.43}$$

[1] Long wavelength General Relativity solutions for such a universe have also been studied by [19] and [6].

$$\dot{V}^k_{dr} + \frac{4}{3}\frac{\mu_r}{h} H V^k_{dr} = 0, \tag{7.44}$$
$$\dot{\mathcal{R}}^k = \Re^k, \tag{7.45}$$
$$\begin{aligned}\dot{\Re}^k = &-\left(2\dot{R}\frac{f'''}{f''}+3H\right)\Re^k - \dot{R}Z^k + \frac{\mu_d}{3f''}\Delta^k_d \\ &-\left(\frac{k^2}{a^2}+\ddot{R}\frac{f'''}{f''}+\dot{R}^2\frac{f^{(4)}}{f''}+3H\dot{R}\frac{f'''}{f''}+\frac{1}{3}\frac{f'}{f''}-\frac{R}{3}\right)\mathcal{R}^k.\end{aligned} \tag{7.46}$$

We can write Eqns. (7.41-7.46) as the pair of second order differential equations given below:

$$\begin{aligned}&\ddot{\Delta}^k_d + \left(2H - \frac{3\dot{R}f''}{4f'}\frac{\mu_d}{\mu_r}\right)\dot{\Delta}^k_d - \frac{\mu_d}{f'}\Delta^k_d + 3H\frac{f''}{f'}\dot{\mathcal{R}}^k \\ &+\left[\frac{1}{2}+\frac{k^2}{a^2}\frac{f''}{f'}-\frac{1}{2}\frac{ff''}{f'^2}+\frac{f''\mu_r}{f'^2}-3H\dot{R}(\frac{f''}{f'})^2+3H\dot{R}\frac{f'''}{f'}\right]\mathcal{R}^k = 0,\end{aligned} \tag{7.47}$$
$$\begin{aligned}&\ddot{\mathcal{R}}^k + \left(2\dot{R}\frac{f'''}{f''}+3H\right)\dot{\mathcal{R}}^k - \frac{3\dot{R}\mu_d}{4\mu_r}\dot{\Delta}^k_d - \frac{\mu_d}{3f''}\Delta^k_d \\ &+\left(\frac{k^2}{a^2}+\ddot{R}\frac{f'''}{f''}+\dot{R}^2\frac{f^{(4)}}{f''}+3H\dot{R}\frac{f'''}{f''}+\frac{f'}{3f''}-\frac{R}{3}\right)\mathcal{R}^k = 0.\end{aligned} \tag{7.48}$$

It can be shown that H and $\dot{R}\frac{f''}{f'}$ are of the same order whereas $\frac{\mu_d}{\mu_r} << 1$, implying that curvature and radiation fluids effectively dominate the fluctuation dynamics. In effect, terms like $\mu_d\Delta^k_d$ merely sub-dominate in the curvature-radiation-dust "mixture". Hence we can safely approximate the above equation by

$$\ddot{\Delta}^k_d + 2H\dot{\Delta}^k_d + 3H\frac{f''}{f'}\dot{\mathcal{R}}^k + \left[\frac{1}{2}+\frac{k^2}{a^2}\frac{f''}{f'}-\frac{1}{2}\frac{ff''}{f'^2}+\frac{f''\mu_r}{f'^2}-3H\dot{R}(\frac{f''}{f'})^2+3H\dot{R}\frac{f'''}{f'}\right]\mathcal{R}^k = 0, \tag{7.49}$$
$$\ddot{\mathcal{R}}^k + \left(2\dot{R}\frac{f'''}{f''}+3H\right)\dot{\mathcal{R}}^k + \left(\frac{k^2}{a^2}+\ddot{R}\frac{f'''}{f''}+\dot{R}^2\frac{f^{(4)}}{f''}+3H\dot{R}\frac{f'''}{f''}+\frac{f'}{3f''}-\frac{R}{3}\right)\mathcal{R}^k = 0. \tag{7.50}$$

These fluid equations tell us that, deep in the radiation-dominated era, the matter and curvature fluctuations decouple.

In the R^n-gravity toy model, the simplest form of higher order $f(R)$ gravitational theories discussed earlier, the corresponding expressions to Eqns. (4.44 - 4.48) in the radiation-dominated epoch become

$$\Theta = \frac{3n}{2t} \Rightarrow H = \frac{n}{2t}, \tag{7.51}$$
$$R = \frac{3n\,(n-1)}{t^2}, \tag{7.52}$$
$$\mu_R = \frac{3(n-1)\,(3n-1)}{2t^2}, \tag{7.53}$$

$$p_R = \frac{(n-1)(n-1)}{2t^2}, \tag{7.54}$$

$$\mu_m = \left(\frac{3}{4}\right) n\chi \left(\frac{3n(n-1)}{t^2}\right)^{n-1} \frac{8n-5n^2-2}{t^2}. \tag{7.55}$$

Since $n = 1$ is a singular limit in R^n models of $f(R)$-gravity theories, we need to analyze our general relativistic (GR) limit and the non-GR equations separately.

The General Relativistic Case

General Relativity is a subclass of the generalized R^n models of gravity where $n = 1$. In this limit, Eqns. (7.49) and (7.50) reduce to

$$\ddot{\Delta}^k_d + 2H\dot{\Delta}^k_d + \frac{1}{2}\mathcal{R}^k = 0, \tag{7.56}$$

$$\mathcal{R}^k = 0, \tag{7.57}$$

thus yielding the standard *Meszaros equation* for the density contrast of dust in a radiation background, valid on small scales,

$$\ddot{\Delta}^k_d + \frac{1}{t}\dot{\Delta}^k_d = 0. \tag{7.58}$$

The general solution of this equation is given by

$$\Delta^k_d(t) = C_1 + C_2 \ln t. \tag{7.59}$$

Dust density perturbations grow logarithmically deep in the radiation epoch.

For $n \neq 1$, with $w = \frac{1}{3}$ in the radiation-dominated epoch, Eqns. (7.49) and (7.50) takes the following form:

$$\ddot{\Delta}^k_d + \frac{n}{t}\dot{\Delta}^k_d + \frac{t}{2}\dot{\mathcal{R}}^k + \left[\frac{12-5n}{4} + \frac{n}{12}\left(\frac{\lambda_H}{\lambda}\right)^2_{eq} t^{2-n}\right]\mathcal{R}^k = 0, \tag{7.60}$$

$$\ddot{\mathcal{R}}^k - \left(\frac{5n-16}{2t}\right)\dot{\mathcal{R}}^k + \left[\frac{n^2}{4}\left(\frac{\lambda_H}{\lambda}\right)^2_{eq} t^{-n} - \frac{6(n-2)}{t^2}\right]\mathcal{R}^k = 0, \tag{7.61}$$

where we have used the fact that $\frac{k^2}{a^2} = \frac{n^2}{a^2}\left(\frac{\lambda_H}{\lambda}\right)^2_{eq} t^{-n}$ with normalized time $t_{eq} = 1$ at the time of *radiation-matter equality*.

Quasi-static Analysis

The system of equations (7.60)-(7.61) yields a very complicated pair of Bessel - hypergeometric type analytic solutions. Analyses of these solutions show that the temporal variations of $\mathcal{R}^k$ can be neglected, i.e., $\ddot{\mathcal{R}}^k \simeq 0$ and $\dot{\mathcal{R}}^k \simeq 0$, because they quickly damp away. It can shown that the solutions of the resulting equations is a good approximation to the actual solution of the exact equations. This is depicted by the plots below where the quasi - static solutions, as the solutions obtained using this approximation are called, and those obtained by numerically integrating the full equations are plotted against time. In this scheme, the density perturbations lead to the decoupled, simple equation

$$\ddot{\Delta}_d^k + \frac{n}{t}\dot{\Delta}_d^k = 0. \tag{7.62}$$

This equation admits the general solution

$$\Delta_d^k(t) = C_1 + C_2 t^{1-n}. \tag{7.63}$$

We choose the initial conditions $\Delta^k_{(r)eq} = \dot{\Delta}^k_{(r)eq} = 10^{-5}$. The initial conditions determine the values of the integration constants C_1 and C_2.

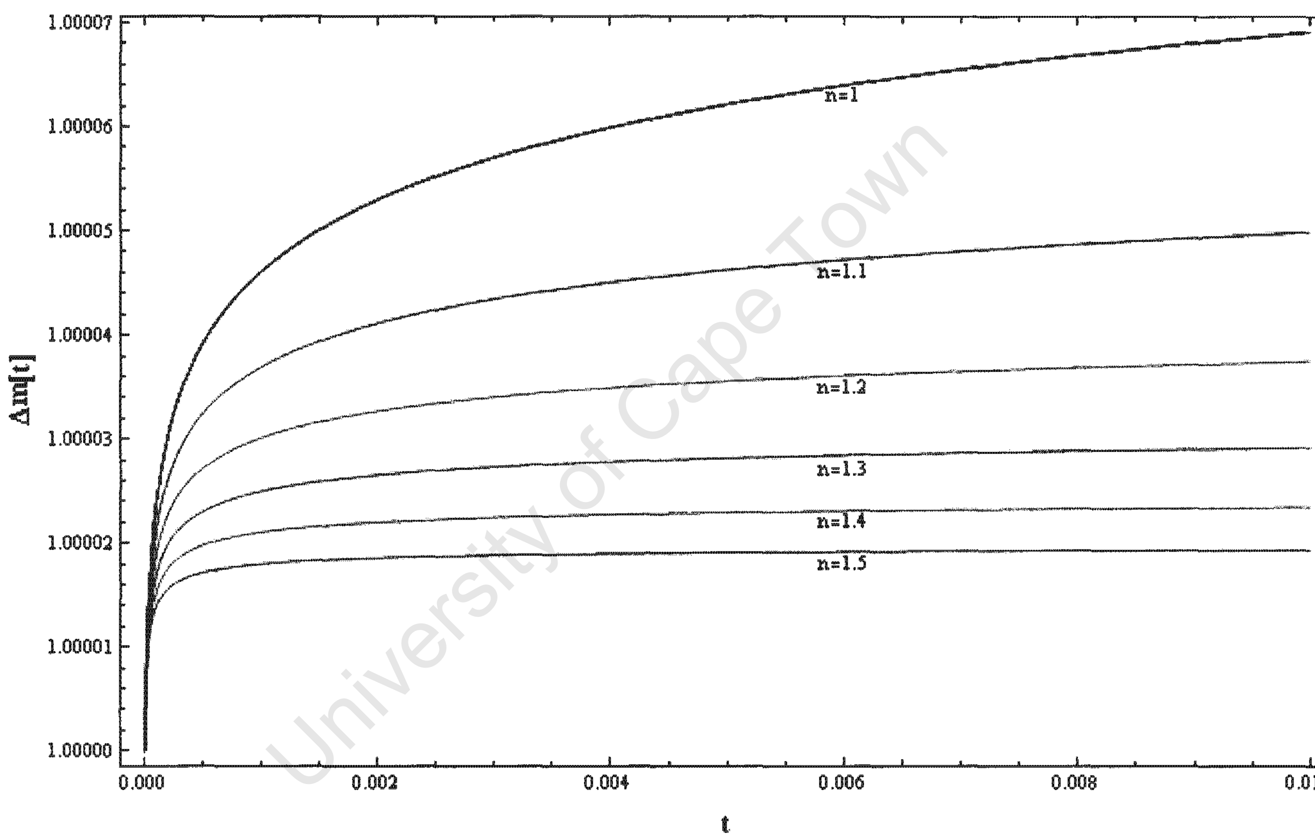


Figure 7.1: Analytic solutions in the radiation-dominated epoch. The horizontal axis stands for the normalized cosmic time.

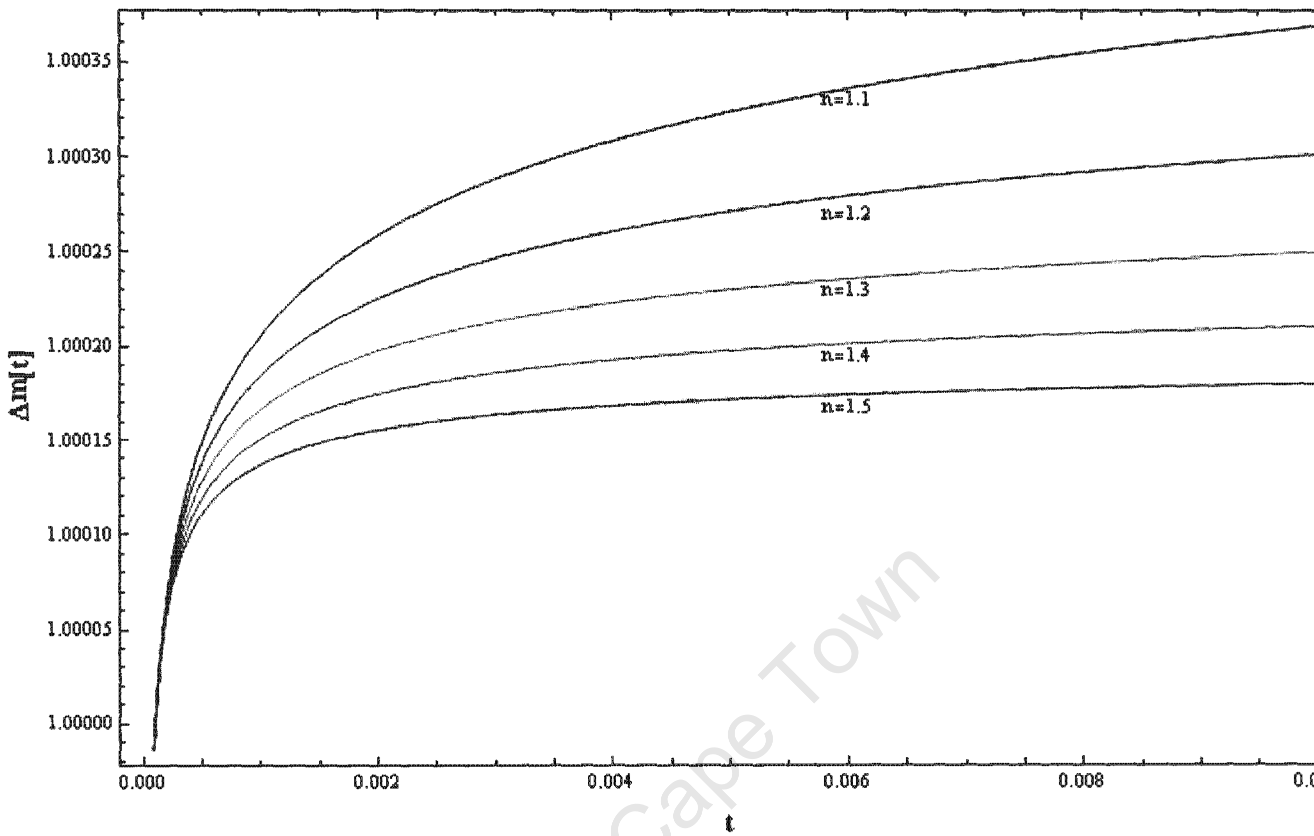


Figure 7.2: Numerical solutions in the radiation-dominated epoch, showing that they match the exact solutions in the quasi-static approximation in Fig. 7.1.

On these scales, radiation suppresses the growth of fluctuations as they enter the horizon before radiation-dust equality, and dust (baryon) self-gravitation is not yet strong enough to offset the cosmic expansion. This is because the expansion scale factor grows faster than the perturbation amplitudes do. The phenomenon is known in the literature as the *Meszaros effect.*

It is clear from the above analysis that the Meszaros effect puts a constraint on the value of n in R^n gravity. To do so, all we need do is to determine the allowed values of n for which the perturbation amplitudes grow slower than the expansion in the radiation dominated era, i.e.,

$$\frac{d}{dt}\left[\frac{\Delta_d^k(t)}{a(t)}\right] \propto \frac{d}{dt}\left[\frac{t^{1-n}}{t^{\frac{n}{2}}}\right] < 0 \Rightarrow 1-\frac{3n}{2} < 0 \tag{7.64}$$

This means that only values of $n > \frac{2}{3}$ give a growth rate compatible with the Meszaros effect.

7.4.2 Perturbations in the Dust-dominated Epoch

During this epoch of the Universe the dust energy density dominates in the two-fluid dynamics and all order-of-magnitude approximations go in line with the assumption that $\mu_d >> \mu_r$. The first order evolution equations for this epoch are then given by

$$\dot{\Delta}_d^k + Z^k = \frac{\Theta}{h}(c_s^2\mu_m\Delta_m^k + p_m\varepsilon^k) + a(\frac{k^2}{a^2})V_d^k, \tag{7.65}$$

$$\begin{aligned}&\dot{Z}^k - (\dot{R}\frac{f''}{f'} - \frac{2}{3}\Theta)Z^k - \left[\frac{(2c_s^2-w-1)}{(1+w)}\frac{\mu_m}{f'} - \frac{c_s^2}{(1+w)}\left(\frac{R}{2}-\frac{f}{f'}-2\dot{R}\Theta\frac{f''}{f'}\right)\right]\Delta_m^k\\ &-\frac{w}{(1+w)}\left[2\frac{\mu_m}{f'}+\frac{R}{2}-\frac{f}{f'}-2\dot{R}\Theta\frac{f''}{f'}\right]\varepsilon^k - \frac{1}{h}(\frac{k^2}{a^2})\left[c_s^2\mu_m\Delta_m^k + p\varepsilon^k\right] + \Theta\frac{f''}{f'}\Re^k\\ &-\left[\frac{1}{2}+\frac{k^2}{a^2}\frac{f''}{f'}-\frac{1}{2}\frac{ff''}{f'^2}+\frac{f''\mu_m}{f'^2}-\dot{R}\Theta(\frac{f''}{f'})^2+\dot{R}\Theta\frac{f'''}{f'}\right]\mathcal{R}^k = 0,\end{aligned} \tag{7.66}$$

$$\dot{V}_d^k + \frac{1}{3}\Theta V_d^k = \frac{1}{ah}\left[c_s^2\mu_m\Delta_m + p_m\varepsilon^k\right], \tag{7.67}$$

$$\dot{V}_{dr}^k - (c_z^2 - \frac{1}{3})\Theta V_{dr}^k = \frac{1}{3ah}\mu_m\Delta_m^k - \frac{1}{a}c_z^2 S_{dr}^k, \tag{7.68}$$

$$\dot{\mathcal{R}}^k = \Re^k - \dot{R}\frac{1}{h}\left[c_s^2\mu_m\Delta_m^k + p_m\varepsilon^k\right], \tag{7.69}$$

$$\begin{aligned}\dot{\Re}^k = &-\left(2\dot{R}\frac{f'''}{f''}+\Theta\right)\Re^k - \dot{R}Z^k + \frac{\mu_m}{3f''}\Delta_m^k - \frac{1}{f''}\left[c_s^2\mu_m\Delta_m^k + p_m\varepsilon^k\right]\\ &-\left(\frac{k^2}{a^2}+\ddot{R}\frac{f'''}{f''}+\dot{R}^2\frac{f^{(4)}}{f''}+\dot{R}\Theta\frac{f'''}{f''}+\frac{1}{3}\frac{f'}{f''}-\frac{R}{3}\right)\mathcal{R}^k.\end{aligned} \tag{7.70}$$

Imposing the short-wavelength assumptions (as in 7.39)

$$c_s^2\mu_m\Delta_m^k + p_m\varepsilon^k \approx \frac{1}{3}\mu_r\Delta_r^k \approx 0, \tag{7.71}$$

$$S_{dr}^k \approx \Delta_d^k, \tag{7.72}$$

will lead our equations for this regime to taking the simpler expressions

$$\dot{\Delta}^k_d + Z^k + a\tilde{\nabla}^2 V^k_d = 0, \tag{7.73}$$

$$\dot{Z}^k - (\dot{R}\frac{f''}{f'} - 2H)Z^k + \frac{\mu_d}{f'}\Delta^k_d - \Theta\frac{f''}{f'}\Re^k - \left[\frac{1}{2} + \frac{k^2}{a^2}\frac{f''}{f'} - \frac{1}{2}\frac{ff''}{f'^2} + \frac{f''\mu_d}{f'^2} - 3H\dot{R}(\frac{f''}{f'})^2 + 3H\dot{R}\frac{f'''}{f'}\right]\mathcal{R}^k = 0, \tag{7.74}$$

$$\dot{V}^k_d + HV^k_d = 0, \tag{7.75}$$

$$\dot{V}^k_{dr} + \frac{4}{3}\frac{\mu_r}{h}HV^k_{dr} = 0 \Rightarrow \dot{V}^k_{dr} + \frac{4\mu_r}{3\mu_d}HV^k_{dr} = 0, \tag{7.76}$$

$$\dot{\mathcal{R}}^k = \Re^k, \tag{7.77}$$

$$\dot{\Re}^k = -\left(2\dot{R}\frac{f'''}{f''} + 3H\right)\Re^k - \dot{R}Z^k + \frac{\mu_d}{3f''}\Delta^k_d - \left(\frac{k^2}{a^2} + \ddot{R}\frac{f'''}{f''} + \dot{R}^2\frac{f^{(4)}}{f''} + 3H\dot{R}\frac{f'''}{f''} + \frac{1}{3}\frac{f'}{f''} - \frac{R}{3}\right)\mathcal{R}^k. \tag{7.78}$$

The resulting second order equations is then

$$\ddot{\Delta}^k_d + \left(2H - \dot{R}\frac{f''}{f'}\right)\dot{\Delta}^k_d - \frac{\mu_d}{f'}\Delta^k_d + 3H\frac{f''}{f'}\dot{\mathcal{R}}^k + \left[\frac{1}{2} + \frac{k^2}{a^2}\frac{f''}{f'} - \frac{1}{2}\frac{ff''}{f'^2} + \frac{f''\mu_d}{f'^2} - 3H\dot{R}(\frac{f''}{f'})^2 + 3H\dot{R}\frac{f'''}{f'}\right]\mathcal{R}^k = 0, \tag{7.79}$$

$$\ddot{\mathcal{R}}^k + \left(2\dot{R}\frac{f'''}{f''} + 3H\right)\dot{\mathcal{R}}^k - \dot{R}\dot{\Delta}^k_d - \frac{\mu_d}{3f''}\Delta^k_d + \left(\frac{k^2}{a^2} + \ddot{R}\frac{f'''}{f''} + \dot{R}^2\frac{f^{(4)}}{f''} + 3H\dot{R}\frac{f'''}{f''} + \frac{f'}{3f''} - \frac{R}{3}\right)\mathcal{R}^k = 0. \tag{7.80}$$

As can be seen, these two equations differ from their counterparts in the radiation-dominated epoch in that they form a *coupled* system of equations. In this epoch the expressions given in (4.44-4.48), upon setting $w = 0$, reduce to

$$\Theta = \frac{2n}{t} \Rightarrow H = \frac{2n}{3t}, \tag{7.81}$$

$$R = \frac{4n\,(4n-3)}{3t^2}, \tag{7.82}$$

$$\mu_R = \frac{2(n-1)\,(10n-3)}{3t^2}, \tag{7.83}$$

$$p_R = -\frac{4n(n-1)}{3t^2}, \tag{7.84}$$

$$\mu_m \simeq \mu_d = \left(\frac{4n}{3t^2}\right)^n \chi\,(4n-3)^{n-1}\left(\frac{13n - 8n^2 - 3}{2}\right). \tag{7.85}$$

The General Relativistic Limit

Equations (7.79) and (7.80) when reduced to the case of General Relativity in this epoch are given by

$$\ddot{\Delta}^k_d + 2H\dot{\Delta}^k_d - \mu_d\Delta^k_d, + \frac{1}{2}\mathcal{R}^k = 0, \tag{7.86}$$

$$-\frac{\mu_d}{3}\Delta_d^k + \frac{1}{3}\mathcal{R}^k = 0. \tag{7.87}$$

These then lead to the decoupled equation

$$\ddot{\Delta}_d^k + \frac{4}{3t}\dot{\Delta}_d^k - \frac{2}{3t^2}\Delta_d^k = 0. \tag{7.88}$$

This equation admits a general solution of the form

$$\Delta_d^k(t) = C_1 t^{-1} + C_2 t^{\frac{2}{3}}, \tag{7.89}$$

giving a standard solution of the perturbations in the dust dominated epoch.

For $n \neq 1$, the equations take the form

$$\ddot{\Delta}_d^k + \left(\frac{10n-6}{3t}\right)\dot{\Delta}_d^k + \left(\frac{2(8n^2-13n+3)}{3t^2}\right)\Delta_d^k + \frac{3(n-1)}{2(4n-3)}t\dot{\mathcal{R}}^k + \left[\frac{n(n-1)}{3(4n-3)}\left(\frac{\lambda_H}{\lambda}\right)^2_{eq} t^{2-\frac{4n}{3}} + \frac{27n^2-8n^3-18n}{2n(4n-3)}\right]\mathcal{R}^k = 0, \tag{7.90}$$

$$\ddot{\mathcal{R}}^k + \left[\frac{8n\left([n(8n-13)+3](4n-3)\right)}{27(n-1)t^4}\right]\Delta_d^k + \frac{8n(4n-3)}{3t^3}\dot{\Delta}_d^k + \frac{8-2n}{t}\dot{\mathcal{R}}^k + \left[\frac{4n^2}{9}\left(\frac{\lambda_H}{\lambda}\right)^2_{eq} t^{-\frac{4n}{3}} - \frac{2[n(8n+5)-69]+54}{9(n-1)t^2}\right]\mathcal{R}^k = 0. \tag{7.91}$$

where $\frac{k^2}{a^2} = \frac{4n^2}{9}\left(\frac{\lambda_H}{\lambda}\right)^2_{eq} t^{-\frac{4n}{3}}$ during this epoch.

Quasi-static Analysis

In the quasi-static limit with $\left(\frac{\lambda_H}{\lambda}\right)^2_{eq} >> 1$, we get a single second order equation

$$\ddot{\Delta}_d^k + \frac{4n}{3t}\dot{\Delta}_d^k + \left[\frac{4(8n^2-13n+3)}{9t^2}\right]\Delta_d^k = 0, \tag{7.92}$$

the solution of which is given by

$$\Delta_d^k(t) = C_1 t^{-\frac{2n}{3}+\frac{1}{2}+\frac{\sqrt{-112n^2+184n-39}}{6}} + C_2 t^{-\frac{2n}{3}+\frac{1}{2}-\frac{\sqrt{-112n^2+184n-39}}{6}}. \tag{7.93}$$

C_1 and C_2 are integration constants to be determined from initial conditions.

At $t = t_{eq} = 1$ we have

$$\Delta_{(d)eq}^k(t) = C_1 + C_2, \tag{7.94}$$

and differentiating (7.93) gives

$$\dot{\Delta}_d^k(t) = \left[-\frac{2n}{3} + \frac{1}{2} + \frac{\sqrt{-112n^2+184n-39}}{6}\right] C_1 t^{-\frac{2n}{3}-\frac{1}{2}+\frac{\sqrt{-112n^2+184n-39}}{6}} + \left[-\frac{2n}{3} + \frac{1}{2} - \frac{\sqrt{-112n^2+184n-39}}{6}\right] C_2 t^{-\frac{2n}{3}-\frac{1}{2}-\frac{\sqrt{-112n^2+184n-39}}{6}}, \tag{7.95}$$

which, at equality, will give

$$\dot{\Delta}^k_{(d)eq}(t) = \left[-\frac{2n}{3} + \frac{1}{2} + \frac{\sqrt{-112n^2+184n-39}}{6}\right] C_1 + \left[-\frac{2n}{3} + \frac{1}{2} - \frac{\sqrt{-112n^2+184n-39}}{6}\right] C_2. \tag{7.96}$$

Solving (7.94) and (7.96) simultaneously yields

$$C_1 = \frac{3\left[\dot{\Delta}^k_{(d)eq} - \left(-\frac{2n}{3} + \frac{1}{2} - \frac{\sqrt{-112n^2+184n-39}}{6}\right)\Delta^k_{(d)eq}\right]}{\sqrt{-112n^2+184n-39}}, \tag{7.97}$$

$$C_2 = \frac{3\left[-\dot{\Delta}^k_{(d)eq} + \left(-\frac{2n}{3} + \frac{1}{2} + \frac{\sqrt{-112n^2+184n-39}}{6}\right)\Delta^k_{(d)eq}\right]}{\sqrt{-112n^2+184n-39}}. \tag{7.98}$$

The following plots show the evolution of the density perturbations $\left(\frac{\Delta^k_d}{\Delta^k_{(d)eq}}\right)(t)$ in time (t from 1 to infinity, where $t = t_{eq} = 1$ is the normalized time at equality)for the above linearly independent solutions, C_1 and C_2 having been obtained by setting $\Delta^k_{(d)eq} = 10^{-5}$ and $\dot{\Delta}^k_{(d)eq} = 10^{-5}$.

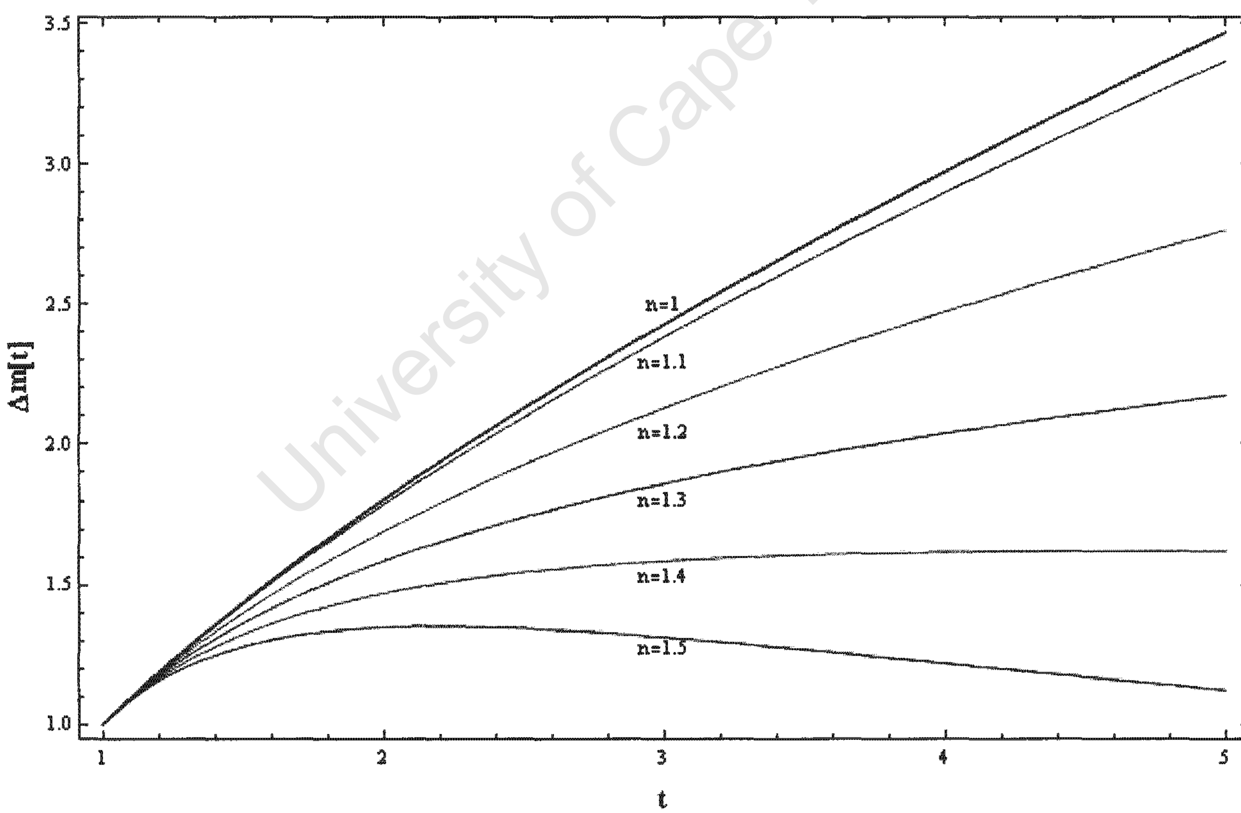


Figure 7.3: Analytic solutions in the dust-dominated epoch. The horizontal axis stands for the normalized cosmic time.

These plots are consistent with the drop in power observed in the matter power spectrum in [21].

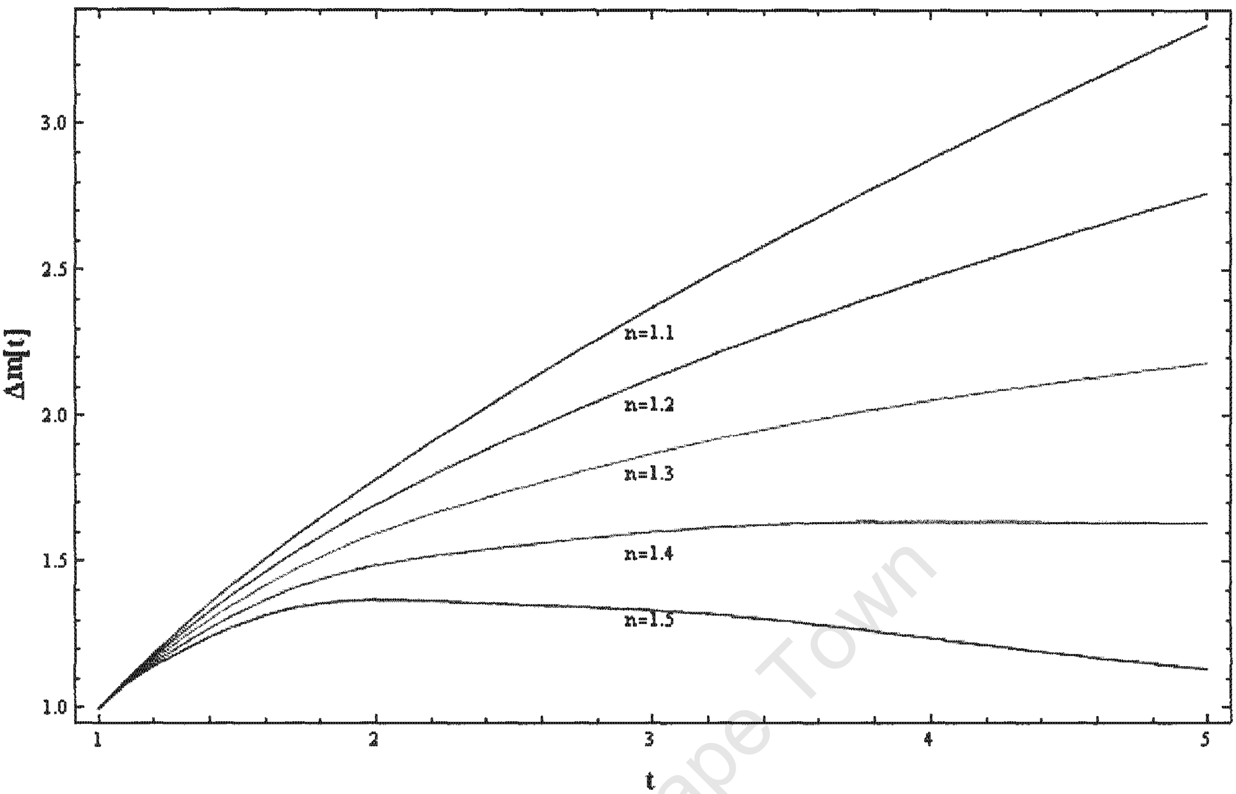


Figure 7.4: Numerical solutions in the dust-dominated epoch, showing a good match with the quasi - static solutions in Fig. 7.3.

7.5 LONG WAVELENGTH SOLUTIONS

7.5.1 Background Model

For specific intervals of n a set of initial conditions provides with non-zero measures for which the cosmic histories include a transient decelerated phase which evolves towards an accelerated phase . Structure formation is believed to have taken place during the transient regime [21]. Here we analyze the evolution of scalar perturbations during this phase,in the *long wavelength limit.* In this limit the wavenumber k is so small that $\lambda = \frac{2\pi a}{k} >> \lambda_H$, i.e., $\frac{k^2}{a^2H^2} << 1$. All Laplacian terms can therefore be neglected and spatially flat ($K = 0$) backgrounds guarantee the conservation of C : $\dot{C} = 0$. This follows from an earlier discussion we saw, Eqn.(5.33).

We consider only adiabatic perturbations ($\Rightarrow S_{ij} = 0$) in this thesis and hence, for a radiation-dust mixture, the equation for the evolution of entropy perturbations

$$\dot{S}_{dr} + a\tilde{\nabla}^2 V_{dr} = 0 \tag{7.99}$$

implies that

$$V_{dr} = 0. \tag{7.100}$$

Combining this with the equation

$$\dot{V}_{dr} - (c_z^2 - \frac{1}{3})\Theta V_{dr} = -\frac{1}{ah}(c_{sd}^2 - c_{sr}^2)\mu\Delta_m - \frac{1}{a}c_z^2 S_{dr}, \tag{7.101}$$

we obtain

$$(c_{sd}^2 - c_{sr}^2)\mu_m\Delta_m = 0. \tag{7.102}$$

We thus have the following system of equations:

$$\dot{\Delta}_m + (1+w)Z - w\Theta\Delta_m = 0, \tag{7.103}$$

$$\dot{Z} = (\dot{R}\frac{f''}{f'} - \frac{2}{3}\Theta)Z + \left[\frac{(2c_s^2 - w - 1)}{(1+w)}\frac{\mu_m}{f'} + \frac{c_s^2}{(1+w)}\left(\frac{R}{2} - \frac{f}{f'} - 2\dot{R}\Theta\frac{f''}{f'}\right)\right]\Delta_m + \Theta\frac{f''}{f'}\Re + \left[\frac{1}{2} - \frac{1}{2}\frac{ff''}{f'^2} + \frac{f''\mu_m}{f'^2} - \dot{R}\Theta(\frac{f''}{f'})^2 + \dot{R}\Theta\frac{f'''}{f'}\right]\mathcal{R}, \tag{7.104}$$

$$\dot{\mathcal{R}} = \Re - \frac{c_s^2}{1+w}\dot{R}\Delta_m, \tag{7.105}$$

$$\dot{\Re} = -\left(2\dot{R}\frac{f'''}{f''} + \Theta\right)\Re - \dot{R}Z - \frac{(3c_s^2 - 1)\mu_m}{3f''}\Delta_m - \left(\ddot{R}\frac{f'''}{f''} + \dot{R}^2\frac{f^{(4)}}{f''} + \dot{R}\Theta\frac{f'''}{f''} + \frac{1}{3}\frac{f'}{f''} - \frac{R}{3}\right)\mathcal{R}, \tag{7.106}$$

$$\frac{C_0}{a^2} + \left(\frac{4}{3}\Theta + 2\frac{\dot{R}f''}{f'}\right)Z - 2\frac{\mu_m}{f'}\Delta_m + \left[2\dot{R}\Theta\frac{f'''}{f'} - \frac{f''}{f'}\left(\frac{f}{f'} - 2\frac{\mu_m}{f'} + 2\dot{R}\Theta\frac{f''}{f'}\right)\right]\mathcal{R} + 2\Theta\frac{f''}{f'}\Re = 0. \tag{7.107}$$

where C_0 is the conserved value for the quantity C.

In terms of the background R^n solutions and making use of the conservation of C the above equations can be rewritten as

$$\dot{\Delta}_m = \left[\frac{1+w-2n}{1+w} - \frac{6(n-1)n}{n+3(n-1)w-3}\right]\frac{\Delta_m}{t} - \frac{3(1+w)^2}{4a_0^2\left[n+3(n-1)w-3\right]\left[4n-3(1+w)\right]}t^{1-\frac{4n}{3(1+w)}}C_0 - \frac{9(n-1)(1+w)^3t^2}{4\left[n+3(n-1)w-3\right]\left[4n-3(1+w)\right]}t^2\Re + \left[\frac{3(n-1)(1+w)^2\left[n(6w+8)-15(1+w)\right]}{4\left[n+3(n-1)w-3\right]\left[4n-3(1+w)\right]}\right]t\mathcal{R}, \tag{7.108}$$

$$\dot{\mathcal{R}} = \Re + \frac{8nc_s^2\left[4n-3(1+w)\right]}{3(1+w)^3}\frac{\Delta_m}{t^3}, \tag{7.109}$$

$$\dot{\Re} = -2\left[\frac{(n-4)+2(n-2)w}{(1+w)t} - \frac{3n(n-1)}{n+3w(n-1)-3}\right]\Re + \frac{2n(4n-3w-3)}{(1+w)\left[n+3(n-1)w-3\right]}\frac{C_0}{a_0^2}t^{-\frac{4n}{3(1+w)}-2} - 2\left[\frac{9n(n-2)(n-1)}{n+3(n-1)w-3} + 2n^2 - 7n - \frac{3n^2(9n-26)+57n}{9(1+w)(n-1)} - \frac{8n^2(n-2)}{9(1+w)^2(n-1)} + 6\right]\frac{\mathcal{R}}{t^2} + \frac{16n\left[4n-3(1+w)\right]\left[4n+3(n-1)w-3\right]\left[(9w(1+w)+8)n^2-(3w(9w+8)+13)n+3(1+w)(1+6w)\right]}{27(n-1)(1+w)^4\left[n+3(n-1)w-3\right]}\frac{\Delta_m}{t^4}. \tag{7.110}$$

7.5.2 Perturbations in the Radiation-dominated Epoch

The second order set of equations governing the dynamics of density perturbations in the radiation-dominated epoch is given by

$$\ddot{\Delta}^k_m + \frac{n(9n-14)+4}{2(n-2)t}\dot{\Delta}^k_m + \frac{n\left(n(n(19n-54)+58)-32\right)+8}{2(n-2)^2t^2}\Delta^k_m$$
$$+\frac{2\left(n(3n-4)+2\right)}{3(n-2)^2}t\dot{\mathcal{R}}^k - \frac{\left(n(15n-22)+14\right)}{3(n-2)}\mathcal{R}^k + \frac{4(n^2-1)}{3(n-2)^2a_0^2}t^{-n}C_0 = 0, \quad (7.111)$$
$$\ddot{\mathcal{R}}^k - \frac{n(11n-32)+32}{2(n-2)t}\dot{\mathcal{R}}^k + \frac{3\left(n(5n-9)+8\right)}{2t^2}\mathcal{R}^k - \frac{3n\left(n(n-3)+2\right)}{2(n-2)t^3}\dot{\Delta}^k_m$$
$$-\frac{3n(n-1)\left(n(19n-28)+4\right)}{4(n-2)t^4}\Delta^k_m - \frac{3n(n-1)}{(n-2)a_0^2}t^{-(n+2)}C_0 = 0. \quad (7.112)$$

Making use of the conservation of C we can eliminate our $\dot{\mathcal{R}}$ and $\mathcal{R}$ quantities in favour of Δ_m (and its derivatives) and C_o. This way we can get a decoupled third order equation for Δ_m:

$$\dddot{\Delta}^k_m - \frac{n-5}{t}\ddot{\Delta}^k_m + \frac{\left(n(24-19n)+8\right)}{4t^2}\dot{\Delta}^k_m + \frac{(n-2)\left(n(5n-8)+2\right)}{2t^3}\Delta^k_m - \frac{12-7n}{3a_0^2}t^{-(n+1)}C_0. \quad (7.113)$$

The general solution of this equation is give by

$$\Delta^k_m(t) = \frac{2\left(24-14n\right)}{9(7n^3-18n^2+16)}t^{2-n}C_0 + C_1t^{\frac{n}{2}-1} + C_2t^{-\frac{1}{2}+\frac{n}{4}+\frac{\sqrt{3(81n^2-44n+12)}}{4}}$$
$$+C_3t^{-\frac{1}{2}+\frac{n}{4}-\frac{\sqrt{3(81n^2-44n+12)}}{4}}, \quad (7.114)$$

where C_1, C_2 and C_3 are arbitrary integration constants to be evaluated from initial conditions whereas C_0 is the conserved quantity defined earlier.

7.5.3 Perturbations in the Dust-dominated Epoch

Proceeding in a similar fashion as for the radiation - dominated regime, we get the second order equations of the dust - dominated case given by

$$\ddot{\Delta}^k_m + \frac{n(8n-13)+3}{(n-3)t}\dot{\Delta}^k_m + \frac{\left(n(8n-13)+3\right)\left(n(16n-15)+9\right)}{3(n-3)^2t^2}\Delta^k_m + \frac{3(n-1)\left(n(16n-15)+9\right)}{4(n-3)^2(4n-3)}t\dot{\mathcal{R}}^k$$
$$-\frac{n\left[\left(n(16n(8n-31)+711)-540\right]+189}{4(n-3)^2(4n-3)}\mathcal{R}^k - \frac{n\left(27+54n-56n^2\right)-27)}{4(n-3)^2(4n-3)a_0^2}t^{-\frac{4n}{3}}C_0 = 0, \quad (7.115)$$
$$\ddot{\mathcal{R}}^k - \frac{4(n-1)\left(n(2n-5)+6\right)}{\left(n(n-4)+3\right)t}\dot{\mathcal{R}}^k + \frac{4\left[n\left(n(2n(16n-65)+213)-198\right)+81\right]}{9\left(n(n-4)+3\right)t^2}\mathcal{R}^k$$
$$-\frac{16n(3-4n)^2\left(n(8n-13)+3\right)}{27\left((n-4)n+3\right)t^4}\Delta^k_m - \frac{2n\left(n(4n-7)+3\right)}{\left(n(n-4)+3\right)a_0^2}t^{-(n+2)}C_0 = 0. \quad (7.116)$$

These equations decouple at the third order to give the equation

$$\dddot{\Delta}^k_m + \frac{5}{t}\ddot{\Delta}^k_m - \frac{2\left[n\left(4n(8n-19)+33\right)+9\right]}{9(n-1)t^2}\dot{\Delta}^k_m - \frac{2(4n-3)\left(n(8n-13)+3\right)}{9(n-1)t^3}\Delta^k_m$$
$$-\frac{\left(n(12n-31)+18\right)}{6(n-1)a_0^2}t^{-(n+1)}C_0 = 0, \quad (7.117)$$

the general solution of which is given by

$$\Delta^k_m(t) = \frac{9\left[12n^2-31n+18\right]}{8(48n^4-184n^3+159n^2+63n-81)}t^{2-\frac{4n}{3}}C_0 + C_1t^{-1} + C_2t^{-\frac{3n-3+\sqrt{(n-1)(256n^3-608n^2+417n-81)}}{6(n-1)}}$$

$$+ C_3 t^{-\frac{3n-3-\sqrt{(n-1)(256n^3-608n^2+417n-81)}}{6(n-1)}} . \tag{7.118}$$

This solution is the same as that obtained for the single-fluid equation (5.63) with $w = 0$, except that here Δ_m is the total density gradient of the mixture (dominated by dust), and not of the dust alone. Nonetheless, the discussions on the solutions of the single fluid *dust* case still hold here for the multi-fluid perturbations in the *dust-dominated* era.

Chapter 8

DISCUSSION AND CONCLUSION

In this thesis we have extended the $1+3$ Covariant and Gauge-invariant cosmological perturbations to a multi-component fluid universe with a general equation of state parameter for generic $f(R)$ theories of gravity. The linearized evolution equations of the density and curvature perturbations of such a universe have been derived in a frame that coincides with the four-velocity vector of the total matter. We then have applied background transient solutions of R^n gravity to these equations for a two-fluid system dominated by radiation and dust and looked for solutions in both the short and long wavelength approximations.

In the short wavelength limit, the full integration of the equations analytically proved to be complicated and a quasi-static approximation technique is used instead. For the values of n considered in the vicinity of 1, the quasi-static approximation is reasonably valid, i.e., it is in a good fit with the solutions found by numerically integrating the full equations for certain initial conditions. It is also worth mentioning that the equations in this limit generate standard solutions when $n = 1$, for which GR is recovered.

The two-fluid perturbations in the multi-fluid approach are shown to reduce to their respective single fluid cases when radiation and dust-dominated cases are considered in the long wavelength limit.

Investigating a way of solving the equations analytically or exploring better approximation techniques is left for future work. Finding signatures from the plots of the power spectrum by solving the field equations and the background for solutions that connect the Friedmann-like era with the late time accelerated de-Sitter-like phase for specific $f(R)$ models is also left for future work.

APPENDIX

A: Covariant Identities

For any scalar f,vector V^a and second-rank tensor W^{ab},the following commutation relations are valid identities in a FLRW geometry:

$$(\tilde{\nabla}_a f)^{\cdot} = \tilde{\nabla}_a \dot{f} - \tfrac{1}{3}\Theta\tilde{\nabla}_a f + \dot{f}\dot{u}_a \tag{A-1}$$

$$\tilde{\nabla}^2(\nabla_a f) = \tilde{\nabla}_a(\nabla^2 f) - \frac{2K}{a^2}\tilde{\nabla}_a f - 2\dot{f}\omega_a \tag{A-2}$$

$$(\tilde{\nabla}^2 f)^{\cdot} = \tilde{\nabla}^2 \dot{f} - \frac{2}{3}\Theta\tilde{\nabla}^2 f + \dot{f}\tilde{\nabla}^a \dot{u}_a \tag{A-3}$$

$$(\tilde{\nabla}_a V_b)^{\cdot} = \tilde{\nabla}_a \dot{V}_b - \tfrac{1}{3}\Theta\tilde{\nabla}_a V_b \tag{A-4}$$

$$\tilde{\nabla}_{[a}\tilde{\nabla}_{b]}V_c = -\frac{K}{a^2}V_{[a}h_{b]c} \tag{A-5}$$

$$\tilde{\nabla}^b(\tilde{\nabla}_{\langle a}V_{b\rangle}) = \tfrac{1}{2}\tilde{\nabla}^2 V_a + \tfrac{1}{6}\tilde{\nabla}_a(\tilde{\nabla}^b V_b) + \frac{K}{a^2}V_a \tag{A-6}$$

$$(\tilde{\nabla}_a W_{cd})^{\cdot} = \tilde{\nabla}_a \dot{W}_{cd} - \tfrac{1}{3}\Theta\tilde{\nabla}_a W_{cd} \tag{A-7}$$

where $V_a = V_{\langle a\rangle}$ and $W_{ab} = W_{\langle ab\rangle}$ are first order quantities.

B: Some Useful Relations in $f(R)$

The following $f(R)$ relations are generally used throughout this thesis:

$$\mu = \frac{\mu_m}{f'} + \mu_R \tag{B-8}$$

$$\mu_R = \frac{1}{f'}\left[\frac{1}{2}(Rf' - f) - \Theta f''\dot{R} + f''\tilde{\nabla}^2 R\right] \tag{B-9}$$

$$p = \frac{p_m}{f'} + p_R \tag{B-10}$$

$$p_R = \frac{1}{f'}\left[\frac{1}{2}(f - Rf') + f''\ddot{R} + f'''\dot{R}^2 + \frac{2}{3}\Theta f''\dot{R} - \frac{2}{3}f''\tilde{\nabla}^2 R\right] \tag{B-11}$$

$$R = \mu - 3p = \tilde{\mu}_m + \mu_R - 3\tilde{p}_m - 3p_R \tag{B-12}$$

$$3\ddot{R}f'' + 3\dot{R}^2 f''' + 3\Theta\dot{R}f'' - 3\tilde{\nabla}^2 f' = \mu_m + Rf' - 2f - 3p_m \tag{B-13}$$

$$\mu_R + 3p_R = (1 - 3w)\frac{\mu_m}{f'} - \frac{f}{f'} - 2\Theta\dot{R}\frac{f''}{f'} + 2\frac{f''}{f'}\tilde{\nabla}^2 R \tag{B-14}$$

$$\Theta^2 = 3\frac{\mu_m}{f'} + \frac{3R}{2} - \frac{3f}{2f'} - 3\dot{R}\Theta\frac{f''}{f'} + 3\frac{f''}{f'}\tilde{\nabla}^2 R - \frac{9K}{a^2} \tag{B-15}$$

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