General Relativistic Electrodynamics with Applications in Cosmology and Astrophysics

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Thesis Presented for the Degree of

Doctor of Philosophy

in the Department of Mathematics and Applied Mathematics
University of Cape Town

August 2005

Declaration

The work presented in this thesis is partly based on collaborations with my supervisors Peter Dunsby (Department of Mathematics and Applied Mathematics, University of Cape Town) and Mattias Marklund (Department of Physics, Umeå University, Sweden), as well as Chris Clarkson (Institute of Cosmology and Gravitation, University of Portsmouth, UK), Caroline Zunckel (Department of Mathematics and Applied Mathematics, University of Cape Town), Martin Servin (Department of Physics, Umeå University, Sweden) and Christos Tsagas (DAMTP, University of Cambridge, UK).

Part of the work has been published in the papers listed below, where a square bracket indicates the sections which are partially based on them:

- Marklund M, Dunsby P K S, Betschart G, Servin M and Tsagas C G (2003), *Class. Quantum Grav.* **20** 1823 [Section 3.1]
- Betschart G, Dunsby P K S, and Marklund M (2004), *Class. Quant. Grav.* **21** 2115 [Section 4.2, Appendix A]
- Clarkson C A, Marklund M, Betschart G and Dunsby P K S (2004), *Astroph. J.* **613** 492 [Chapter 6, Appendix B]
- Betschart G and Clarkson C A (2004), *Class. Quant. Grav.* **21** 5587 [Chapter 5]
- Betschart G, Zunckel C, P K S Dunsby and Marklund M, gr-qc/0503006 [Section 4.3, Appendix A]

Sections 6.6 and 6.7 reflect the work of Chris Clarkson, who wrote the computer code for the numerical analysis and produced the plots.

My second year of my PhD was spent at the Department of Electromagnetics, Chalmers University of Technology, Gothenburg, Sweden, where I obtained a Licentiate of Engineering degree from Chalmers in 2004 based on: a course work component, a Licentiate seminar and a technical report (*Plasma Physics on Curved Spacetimes*, 36 pages plus appended publications).

I herby declare that the presented thesis has not previously been submitted to this or any other university for a degree and that it represents my own work.

Gerold Betschart

Abstract

In this thesis, we apply general relativistic electrodynamics to the study of plasmas on curved spacetimes, taking a perturbative viewpoint. In order to facilitate a gauge-invariant analysis, we employ the covariant 1+3 and 1+1+2 decompositions of general relativity. The former decomposition is ideally suited to homogeneous spacetimes, while the latter is useful for spherically symmetric ones. After reviewing the fundamentals of these formalisms, we turn to general relativistic electrodynamics and use them to decompose Maxwell's and the Lorentz-force equations. We describe a plasma in terms of a charged multi-component fluid and derive the corresponding conservation equations. Describing the early Universe as a charge-neutral two-component fluid linearised around the Einstein-de Sitter model, we investigate primordial velocity perturbations in this plasma. A new weakly damped plasma mode is identified in addition to the standard gravitational instability picture. Vortical velocity perturbations are further found to generate magnetic fields of sufficient strength to provide a seed for the galactic dynamo. Within the standard model of cosmology, we investigate the amplification of magnetic fields due to gravitational waves. The amplification depends on the initial expansion-normalised shear anisotropy and, crucially, on the square of the initial ratio between the magnetic field (or gravity wave) scale and the Hubble scale, which makes the amplification especially effective for inflation-produced super-horizon magnetic fields interacting with inflationary relic gravity waves. We use the 1+1+2 machinery to study locally rotationally symmetric spacetimes as well as scalar field end electromagnetic perturbations living on them. These perturbations are found to be governed by a covariant master equation, which generalises the well-known Regge-Wheeler equation for Schwarzschild perturbations. Finally, we explore the interaction between gravitational radiation, stemming from the ringdown of a perturbed Schwarzschild black hole, and a static dipole magnetic field surrounding the hole. This interaction produces an electromagnetic signature of the ringdown phase, which lies in the radio frequency band for dipole field strengths comparable to magnetars. The detection of such a signal would provide indirect evidence for gravitational radiation, and would strengthen a direct observation of gravity waves by other means if made simultaneously.

Acknowledgement

It is a real pleasure for me to express my gratitude towards my supervisors, Peter and Mattias. Your subtle guidance which left me the freedom to pursue my own interests and explore my own ideas, always useful advice and continual encouragement throughout the thesis work and, above all, your friendship - all this made working with you such a joy! I have particularly fond memories of all the hikes we did together in the Cape Town area, especially the more adventurous ones with Peter. I also like to thank Alison and Pernilla, your wives, for the good times we spent all together at numerous braais and dinners over the years.

I thank all my collaborators for their valuable contributions, in particular Caroline and Chris. I tremendously enjoyed sharing an office with Chris during my first year at UCT. Thanks, Chris (and Vivian), for all the fun we had and for trying to answer my sheer endless stream of questions, which allowed me to master a lot of unfamiliar mathematics in a short time.

Many people at the Mathematics Department of UCT contributed their share in giving me a good time, be it for stimulating my thoughts with interesting discussions, for helping out at computer problems, for keeping my tutorial duties more pleasant, for giving me various lifts, or for simply hanging around with me. Among these I particularly like to mention: Teresa, Jannie, Sante, Anslyn, Jeff, Nazeem, Craig, Naureen, Uli, Bonita, Kevin, Mike, George E., Charles, Jurie, John, George J., Peter B. Thanks guys!

In 2003, I spent a nice year in Gothenburg at the Department of Electromagnetics, Chalmers University of Technology. Although Mattias and me have been the only group members working in general relativity, the group quickly made me feel at home. I like to thank in particular: Fredrik, Lukas, Ulf, Pontus, Jean-Michel, Richard, Håkan, Dan, Mietek. It was cool to be there! I further had a lot of fun visiting Marek's courses at Chalmers; one scientific anecdote told per lecture was truly amazing.

Both at UCT and Chalmers, my life was made considerably easier due to the assistance of always helpful secretaries, especially Monica (Chalmers) and Di (UCT). Thanks a lot!

I thank Brian (Linköping University) for hospitality, as well as Lennart, Michael, Krister and Gert (all Umeå University) for three joyful weeks I spent at their institute.

Being abroad and getting visitors from my home village Muotathal in Switzerland is always great. It was a hell lot of fun to travel around in Sweden with Hugo, Heinz, Geri and Bruno, as it was to travel through South Africa together with Urs, Hugo, Roman, Marco and René. I am particularly indebted to Hugo for bringing a little coffee machine to Gothenburg, surprising me in Johannesburg with a Swiss chocolate Easter egg bought by my mother, and giving me money, donated by Bruno S., to buy a good bottle of wine. A fine bottle of single malt whisky left in Gothenburg by Heinz, Geri and Bruno was also very much appreciated. Thanks go also to Isabelle and Franz, and to Helen and Bertha for visiting me in Cape Town. Travel plans to the Middle East anybody?

Of course, visiting friends in Switzerland is no less pleasant. Among these I especially like to mention: Lucia, Sandra, Judith, Christian, the organisers of the "Bergenboden-Chilbi" and the New Year parties at "Theaterstübli", and the honourable members of the "Donnerstags-Club Muotathal". I also like to include my relatives Othmar, Ernst and Edith for their much appreciated hospitality. Hope to see you soon, guys!

It is always a great joy to receive all these lovely postcards and e-mails from Szu-Chieh, my "little sister". I sincerely hope to be able to show you in person the beauty of Switzerland soon!

Last but not least, I thank my parents, Leonie and Peter, my brother Wädi, and my relatives for all their love and support they gave me throughout all these years.

Liebe Eltern, lieber Wädi: Ich danke euch von ganzem Herzen für all die Liebe und Unterstützung, die ihr mir über die ganzen Jahre habt zukommen lassen. Ich weiss, dass es euch alles andere als einfach gefallen ist, mich in die Ferne ziehen zu lassen. Trotzdem habt ihr meine Entscheidungen respektiert und seid immer zu mir gestanden. Ihr seid schlicht die besten!

Contents

Chapter 1

Introduction

1.1 Plasmas and relativity

Plasmas and electromagnetic fields are ubiquitous in our Universe and play an important role in many astrophysical and cosmological processes. In fact, the biggest part of the Universe's matter content is in the plasma state. Plasmas may be found, e.g., in stars, accretion disks of rotating black holes (BHs), the Earth's ionosphere and contribute to the inter-galactic medium. Although plasma physics can often be adequately addressed within the Newtonian or the special relativistic framework, there are occasions where Einstein's theory of general relativity (GR) has to be taken into account – namely whenever the underlying four-dimensional spacetime in which physics takes place strongly deviates from being flat, that is, when the curvature of spacetime cannot be neglected. A prominent example is our Universe, which is expanding and thus constantly changing its curvature. Near strongly gravitating compact objects such as neutron stars (NSs) or black holes the spacetime-geometry becomes immensely distorted. In the case of a BH, the distortion is so strong that not even light can resist its pull and gets trapped once it has passed the hole's boundary, the famous event horizon. Obviously, in such situations gravitational effects on a plasma can neither be discarded nor treated as small corrections within a special relativistic description, but have to be considered within the full nonlinear theory of GR.

A number of techniques can be used to analyse the equations describing general relativistic plasmas. Depending on the nature of the problem one might employ analytical, numerical and/or perturbative methods. Analytical results are usually only obtainable under severe symmetry assumptions, which unavoidably restricts their applicability. Moreover, the inherent complexity of Einstein's theory means that numerical techniques are also non-trivial to apply but remain often the only tools to study the nonlinearity of Einstein's field equations (EFEs) in their full glory. The field of numerical relativity produced stunning results in the last couple of decades, for example, the simulation of BH-BH mergers or the simulation of the growth of primordial density fluctuations into structures that resemble the observed large scale structure of our Universe. However, in regard to general relativistic plasmas the most useful method so far seems to be the perturbative one, possibly combined with numerical methods. In general, we may distinguish between two types of approach.

Non-gravitating plasmas on curved background spacetimes: This approach is perhaps the simplest way to analyse the influence of general relativistic gravity on plasmas because it neglects the back-scattering to the gravitational field. In astrophysical applications it is often a very good approximation to treat the plasma as a test field (or rather test fluid), which does not disturb the background spacetime. The only difference from the flat spacetime case is that gravitational terms may enter the equations. Within this approach there are two subcases: (a) weak gravitational fields, described by a single potential, or weak gravitational waves (GWs), which are a purely non-Newtonian effect; (b) strong gravitational fields, where one uses exact solutions to Einstein's field equations for the background model. The 'membrane paradigm' formalism (see [1] and references therein) was developed for this purpose, with the background spacetime being the Schwarzschild or Kerr black hole solution.

Self-gravitating plasmas: In this case one takes into account the plasma contribution to the total gravitational field. This approach, which is technically more demanding than cases (a) and (b) above, is applicable to early Universe studies, when most of the baryonic matter was ionised. A prominent example is the analysis of the cosmic microwave background (CMB), in which the photons are described by the Boltzmann equation and the collisions are due to Thomson scattering with the electrons (see [2] and references therein).

A considerable amount of research has been done on the interaction between plasmas and GWs and on the use of electromagnetic fields for the detection of GWs (see [3–5] and references therein). For spacetimes containing gravitational radiation, wave interactions in plasmas are expected to occur, with the gravitational wave acting as a pump. Using this heuristic picture, The authors of [6] used kinetic theory on a gravitational wave background to show that there can indeed be parametric excitation of plasma waves, with a growth rate proportional to the gravitational wave amplitude. Bingham *et al.* [7] have used the weak gravitational wave approximation in order to study the scattering of gravitational waves in supernovae (SN), and they have pointed out that the expected gravitational wave form may change due to such scattering. Furthermore, Ref. [8] used the cold fluid approximation to show the possible existence of radio waves due to the emission of weak gravitational waves from binary pulsars. Using an analogy to frequency upshifting of short laser pulses in laboratory plasmas (see, e.g., [9]), it was shown in Ref. [10] that weak gravitational waves could induce similar phenomena in magnetised multi-component plasmas. In Ref. [11] the exact plane-fronted parallel (pp) solution [12] to Einstein's field equations was used in order to gain a better understanding of nonlinear effects on plasmas due to gravitational waves; it was shown that nonlinearities can excite longitudinal electromagnetic modes as well as plasma modes. Ignat'ev has investigated the effects of such a pp-wave on a magnetohydrodynamic (MHD) plasma, and also found an exact solution to the MHD equations in this case [13].

A number of papers employ the formalism of the membrane paradigm [1], together with the appropriate fluid equations, in order to look into the properties of plasmas in the vicinity of compact astrophysical objects such as black holes or neutron stars (usually under the MHD approximation). The authors of Ref. [14] studied high-frequency electromagnetic (EM) waves in a plasma outside a spherically symmetric black hole obtaining the dispersion relation. It was further found that the propagation of electromagnetic waves is determined by an inhomogeneous wave equation. Building on this paper, the authors showed the possibility of EM-wave outburst from black holes due to mode conversion [15]. Khanna [16] obtained the MHD equations describing an inviscid, fully ionised plasma in the vicinity of a rotating black hole using a two-component plasma theory and found a generalised Ohm's law.

Work has also been done on fluid dynamics and kinetic gas theory in the context of cosmology. Notably, the book by Bernstein [17] treats the kinetics of gases in the Friedmann-Lemaître-Robertson-Walker (FLRW) model. Note though that there are relatively few relativistic cosmological studies that take into account plasma effects and the behaviour of matter in the presence of electromagnetic fields (see e.g. [18–25]). Thus, the general relativistic treatment of plasmas, both in astrophysics as well as in cosmology, look like a field open to investigation.

1.2 Perturbations in general relativity

General relativistic treatments require the rigourous setup of a self-consistent set of equations to describe the plasma dynamics. Moreover, when perturbative techniques are employed, there are extra considerations, such as those related to the gauge-invariance of the approach. The gauge group of GR is the infinite group of diffeomorphisms, mirroring the freedom of choosing a set of coordinates and transforming to another if wished. This gauge freedom, however, introduces unphysical gauge modes when slight deviations from the background spacetime (perturbations) are considered, which have to be eliminated. Several methods are in use to cure this problem. In cosmology, the most well-known approach is the one due to Bardeen [26], who showed how to extract the relevant gauge-invariant quantities in terms of the metric. In this thesis, we will employ another method due to Ellis & Bruni [27], the 1+3 approach, building on the work of Hawking [28], Olson [29] and Stewart & Walker [30], which does not use the metric directly but describes the spacetime instead by means of covariant quantities, obtained by a local foliation of spacetime into 'time' and 'space' via the introduction of a congruence of observers with 4-velocity u^a. The latter method has the advantage that its quantities have a clear physical and/or geometrical meaning, in contrast to the Bardeen variables, and that covariant and gauge-invariant perturbation variables are easily identified, hence fixing the gauge problem right from the start (see [2] for a comprehensive review). The single fluid analysis of Ellis & Bruni [27] + has been extended to multi-component systems by Dunsby, Bruni & Ellis [31], where a number of possible cosmological applications was discussed.

We emphasise that the two approaches are opposite in the following sense: The metric approach regards the full real spacetime as composed of a background (usually endowed with pleasant symmetries) together with perturbations on it,

$$\text{real spacetime} = \text{background} + \text{perturbation}\ .$$

It is a bottom-top approach, tackling the real spacetime by starting from the background metric. Moreover, Bardeen's approach is developed for a homogeneous and isotropic background. The 1+3 method is a top-bottom approach, starting from the equations which govern the real spacetime, and linearises them about a suitable background, which is thought to be very close to the real spacetime. This procedure separates the introduced variables into background (zeroth-order) quantities and perturbations, allowing also a clear distinction between first-order, second-order etc. perturbations. The approach makes it further possible to do perturbations around any background, and also to go beyond first-order in perturbation theory in a transparent way.

Yet another method in use is the tetrad approach, most notably the Newman-Penrose (NP) formalism (see, e.g., [32, 33]), which employs a full tetrad in order to obtain a description of the spacetime in terms of scalar equations. The NP-formalism is widely used to investigate perturbations of the Schwarzschild and Kerr spacetimes. Although the tetrad formalism is a powerful tool, the partial-frame methods such as the 1+3 formalism and its 1+1+2 extension by Clarkson & Barrett [34] are conceptually more convenient to handle because their quantities have a physically simpler interpretation. The 1+1+2 formalisms was recently developed in order to investigate perturbations of the Schwarzschild geometry along similar lines as does the 1+3 method for perturbed cosmological models. It is particularly well-adapted to spacetimes having a distinguished spacelike direction vector field n^a, spherically symmetric spacetimes, for example. Finally, it should be kept in mind that the partial-frame methods just described as well as the NP-formalism are extremely useful tools not only for studying properties of a given spacetime but also for investigating its perturbations.

1.3 The mystery of the origin of large scale cosmic magnetic fields

The origin of cosmological magnetic fields, with characteristic strengths between 10^{-7} to 10^{-5} G [35,36], that are prevalent throughout galaxies clusters, disk and spiral galaxies and high-redshift condensations has generated much debate in recent years, with the majority of this work being focused on providing mechanisms that generate these galactic fields on large scales (see [37, 38] and references therein). The candidate mechanisms are diverse, often depending on the required seed field strengths.

It has been suggested that the fields observed today could be a result of the amplification of a relatively large magnetic seed field through protogalactic collapse at the onset of structure formation (cf. [39], for example). Due to the high conductivity of the primordial plasma, the seed magnetic field remains frozen into the plasma. As the gas collapses to current measured densities, the flux lines of the frozen-in cosmological magnetic field get compressed, inducing adiabatic amplification.

Another popular mechanism, which requires a relatively weaker pre-existing seed field, is amplification via the galactic dynamo by means of parametric resonance [35, 40–42]. The combined effect of differential rotation across the disk and the cyclonic turbulent motions of the ionised gas is believed to lead to the exponential amplification of a smaller primordial field until the back-reaction of the plasma opposes further growth. Although the dynamo mechanism is strongly supported by the close correlation between the observed structure of the galactic fields and the spiral pattern of galaxies, there is some argument over its efficiency and hence the amount of amplification that can occur through this process. The major problem with all of these mechanisms is that they assume the presence of a pre-existing seed field whose origin is still to be established.

A further idea relies on turbulence (disrupted flow) and shocks, which occur during the stages of structure formation, inducing weaker magnetic fields via battery-type mechanisms, which operate as a result of large-scale misalignments of gradients in electron number density and pressure (or temperature) [43–45].

There have been numerous attempts to generate early, pre-recombination, magnetic fields with strengths suitable to support and maintain the dynamo by exploiting the different out-of-equilibrium epochs that are believed to have taken place between the end of the inflationary era and decoupling [46–51]. These fields are facilitated by currents that arise from local charge separation generated by vortical velocity fields prevalent in the early plasma (cf. also [52, 53]).

One problem with the above mechanisms is that they are causal in nature: the scales over which the fields are coherent cannot exceed the particle horizon during that epoch. Given that such phase transitions took place at very early times, where the comoving horizon size was small, tight constraints must be placed on the coherence length of these magnetic fields. However, pre big bang models based on string theory [54, 55], in which vacuum fluctuations of the magnetic field are amplified by the dilaton field, predict super-horizon fields.

Inflation has long been suggested as a solution to the causality problem since it naturally achieves correlations on super-horizon scales. However, adjustments to the standard inflationary models need to be made because magnetic fields surviving this epoch are small on account of the inability of vector fields to couple gravitationally to the conformally flat metric resulting from the exponentially fast expansion. A way around this obstacle is by breaking the conformal invariance of electromagnetism since this alters the way the underlying gauge fields couple to gravity. There are many ways of doing this, which explains the variety of the proposed mechanisms in the literature [56–62]. Such inflationary scenarios have not been without critique, though [63].

It has also been proposed that inflation is followed by a period of preheating, in which the parametric resonance of the causal oscillations of the inflaton field and the accompanying perturbations can lead to amplification on super-horizon scales [64–66]. Other authors have advocated the breakdown of Lorentz invariance either in the context of string theory and non-commutative varying speed of light theories, or due to the dynamics of large extra dimensions [67–69]. The success of these proposals, however, is usually achieved at the expense of simplicity.

In order for these proposed mechanisms to be viable, they must, in addition, produce seed fields that satisfy the criteria for the subsequent amplification processes to work. To be a candidate seed field for the galactic dynamo, the induced field must exceed a minimum coherence scale in order to prevent the destabilisation of the dynamo action. The time scale over which the amplification takes place also dictates a minimum field strength. In the case of a dark-energy dominated Universe the seed field can be as low as $B \sim 10^{-34}$ G on a coherence scale of 10 kpc today [70], while the seed's strength has to be risen towards $B \sim 10^{-23}$ G on a coherence scale of 10 kpc today [71] for a matter dominated Universe. Davis *et al.* [72, 73] proposed an inflationary mechanism that exploits the natural coupling between the Z-boson and the gravitational background. Unfortunately, the magnetic fields produced only just fall within dynamo limits in the case of a dark energy dominated Universe. However, if these fields are allowed to interact with inflationary relic gravitational waves for a sufficiently long time, they might be amplified to strengths relevant to the galactic dynamo mechanism [74]. Recently, the production of a magnetic seed field due to the rotational velocity of ions and electrons, caused by the nonlinear evolution of primordial density perturbations in the cosmic plasma during pre-recombination radiation and matter eras, was investigated in [75] and a rms amplitude $B \approx 10^{-23}(\lambda/\mathrm{Mpc})^{-2}\,\mathrm{G}$ at recombination on comoving scales $\lambda \gtrsim 1\,\mathrm{Mpc}$ was reported. Although there is an abundance of models addressing the origin of the observed large scale magnetic fields in our Universe, it seems still to be a long way to go before this debate can be declared closed.

1.4 Gravitational waves

A fascinating prediction of Einstein's theory of general relativity is the existence of gravitational waves and there is an enormous effort worldwide to detect gravitational radiation (see, e.g., [76–79]). Hopefully, within the next few years detectors like LIGO will be able to detect and measure the gravity waves emitted from events such as black hole mergers [80] and exploding and collapsing stars. A pressing problem for all GW detectors at present is the extraction of the actual waveform from the huge amount of noise invariably generated in the detection process [81–83]. Many of these events could be accompanied by an electromagnetic (EM) counterpart with the same waveform, yet the frequency of this generated EM radiation will be very low, generally less than about 10 kHz, and would be typically absorbed by the interstellar medium. However, photon frequency conversion [8, 10, 84] could overcome this absorbtion process by increasing the frequency to detectable levels. Although events in supernovae (SN) II and some compact binary mergers [85] are accompanied by an optical counterpart, many events such as BH-BH merger and BH ringdown are not. Thus, an independent simultaneous detection of the EM waveform mirroring that of the GW would be highly desirable for GW detection, especially when an optical signal is not available.

When a plane gravity wave passes through a magnetic field, it vibrates the magnetic field lines, thus creating EM radiation with the same frequency as the forcing GW (see, e.g., [8, 86, 87] and references therein). This would provide exactly the mechanism required: virtually all stars have a strong magnetic field threading through and surrounding them, and this field becomes immensely strong as the field lines are compressed as the star collapses to a BH or neutron star; anything up to 10^{14}G – possibly even higher – seems possible in magnetars [88]. It has been proposed that this mechanism may indeed has been observed, being partly responsible for the afterglow observed in some gamma-ray bursts (GRBs) and SN events, an argument strengthened by certain anomalous GRBs and SN light curves (see [89] and references therein for a detailed discussion). The basic idea is that these events are thought to form a BH or neutron star after the initial explosion (envelope ejection). During the formation process, a substantial fraction of the mass will be released as GWs, which might subsequently be converted into EM radiation as it passes through the surrounding thin plasma.

Studies of generation of EM radiation by GWs in astrophysical situations so far have provided order of magnitude estimates [89] and some of the extra complexities involved when a thin plasma is present [6, 8, 14, 90, 91]. In particular, a thin plasma can increase the frequency of the electromagnetic radiation, whose origin is from a plane gravity wave passing through a uniform, static, magnetic field, thus strengthening the observational potential of the EM-GW interaction still further [10, 84]. While these investigations have given a good indication of the physical processes we may expect, the effect has not yet been studied in a strong gravitational field, the most promising place we may expect such an interaction to take place. To facilitate these studies, a covariant, self-consistent description of plasmas in strong gravitational fields has to be developed. A promising possibility would be to use the covariant 1+1+2 formalism in order to formulate a covariant multifluid theory or a magnetohydrodynamic single fluid theory. The latter would be a valuable tool for describing plasmas surrounding BH or neutron stars, allowing for more accurate models of the magnetosphere or for a more reliable analysis of the GW-plasma interaction.

1.5 Thesis outline

The thesis is organised as follows:

In chapter 2, we outline the covariant 1+3 and 1+1+2 formalisms of general relativity and introduce the basic definitions and quantities used in later chapters. A covariant description of arbitrary spacetimes is thus obtained. The bearing of these formalisms on the gauge-problem of general relativity is discussed.

In chapter 3, we discuss electrodynamics on curved spacetimes. We develop the splitting of Maxwell's equations and the Lorentz-force equation both in the framework of the 1+3- and the 1+1+2 formalism. We introduce the general relativistic equations (energy, momentum and particle number conservation in particular) for dealing with charged multi-component fluids in both frameworks. This chapter thus provides the fundamental tools needed for investigating plasmas in general relativity.

Chapter 4 is devoted to large scale cosmological magnetic fields. First, we apply the 1+3 multifluid description of the previous chapter to the case of a charged two-component fluid for matter-dominated FLRW dust cosmologies and propose a mechanism for generating a primordial magnetic seed field based solely on the physics of self-gravitating plasmas. The mechanism is similar to Harrison's protogalaxy model [92, 93], and the Biermann battery effect [94], in the sense of yielding vorticity-driven magnetic fields, but we note that the battery effect in our formalism would be of second order, while the Harrison effect relies on Thomson scattering. Since velocity and density perturbations naturally occur in the early Universe, it is interesting to examine whether such perturbations can induce magnetic fields which can be sustained at appreciable levels until the onset of nonlinear gravitational collapse. All obtained modes show a typical high-frequency behaviour mirroring their plasma origin.

Second, we investigate the interaction between an on average homogeneous magnetic seed field, envisaged to have produced during inflation, and gravitational wave relics from inflation. The analysis of the interaction, which is viewed as a second-order perturbative effect, reveals that the magnetic seed might be amplified by several orders of magnitude under favourable circumstances, producing a plausible seed for the galactic dynamo mechanism.

In chapter 5, the 1+1+2 formalism is employed to describe the physically important group of so-called locally rotationally symmetric (LRS) class II spacetimes, for which the 1+1+2 equations reduce to relatively simple scalar equations. Scalar field and electromagnetic perturbations on these spacetimes, which include the spherically symmetric ones, are investigated and a generalised covariant Regge-Wheeler master equation is obtained. The findings are discussed in detail for the Schwarzschild, Vaidya, Kantowski-Sachs and Lemaître-Tolman-Bondi spacetimes.

In chapter 6, we investigate the induced electromagnetic field from the interaction of a strong magnetic dipole field surrounding a BH with GWs emitted during BH ringdown, the settling down of a BH after an initial perturbation. Shortly after a BH is perturbed, it radiates away its curvature deformations as GWs with certain characteristic frequencies which are independent of the initial perturbation, and dependent only on its mass (in the case of a Schwarzschild BH). These complex frequencies form solutions known as quasi-normal modes which govern the BH ringdown process [95, 96]. As the ringdown process is thought to be independent of the initial perturbation, this particular situation should be a primer for more complex situations, such as the late stages of BH-BH merger [80]. Substantial amplification of the EM field is found near the horizon and photon sphere, much stronger than in the case of plane GWs [8], where the amplification grows linearly with interaction distance.

Chapter 7 contains our conclusions and an outlook for further work.

Some useful material, which has been employed in several occasions in the main text, has been banned to various appendices in order to provide easier reading as well as quick reference.

Units

Unless otherwise stated we adhere to units in which $\kappa \equiv 8\pi G = 1 = c = \hbar$.

Chapter 2

Covariant and gauge-invariant formalisms

In general relativity (GR), the geometry of our Universe (spacetime) is dynamically linked to its energy-matter content via *Einstein's field equations* (EFEs)

$$G_{ab} \equiv R_{ab} - \tfrac{1}{2} R\, g_{ab} = T_{ab} - \Lambda\, g_{ab} \;, \tag{2.1}$$

where G_{ab} is the *Einstein tensor*, R_{ab} the *Ricci tensor*, R the *Ricci scalar*, g_{ab} the spacetime metric, T_{ab} the *energy-momentum tensor* and Λ the *cosmological constant.* Because of the *twice-contracted Bianchi identities*, these guarantee the conservation of total energy-momentum

$$\nabla_b G^{ab} = 0 \qquad \Rightarrow \qquad \nabla_b T^{ab} = 0 \;, \tag{2.2}$$

provided the cosmological constant Λ satisfies the relation $\nabla_a \Lambda = 0$, i.e., it is constant in time and space. Here and in the following, ∇_a denotes the *covariant derivative*, that is, the unique torsion-free derivative operator naturally associated with the spacetime metric g_{ab} such that $\nabla_a\, g_{bc} = 0$.

There are several ways to describe spacetimes (solutions of EFEs):

(I) the metric $g_{ij}(x^k)$ is expressed in coordinates x^k, with its connection given through the Christoffel symbols;

(II) the metric is described by a tetrad, with its connection given through the Ricci rotation coefficients;

(III) via covariantly defined variables with respect to a partial frame formalism such as the $1+3$ or $1+1+2$ decompositions of GR.

While methods (I) and (II) are well-suited for studying a particular spacetime, e.g. by choosing coordinates adapted to underlying symmetries, the covariant method (III) is well-apt for a consistent perturbative treatment because it describes physics and geometry by tensor quantities and relations, which remain valid in all coordinate systems. Thus, spurious gauge modes cannot appear.

The 1+3 approach slices the spacetime in 'time' and 'space' by means of a fundamental observer and is suited for investigating small deviations from homogeneity and isotropy of the standard model of cosmology. However, in order to study BH perturbations, the formalism has to be extended since the hole's attraction defines an additional distinguished direction. The extension, the 1+1+2 approach, was recently established by Clarkson & Barrett [34]. Although the 1+1+2-formalism was mainly introduced to study perturbations of the Schwarzschild geometry, which is the geometry of a non-rotating BH, it is indeed extremely useful for investigating the much larger class of so-called locally rotationally symmetric (LRS) spacetimes. A nice feature of these partial frame formalisms is that in the just mentioned examples the concomitant 'background' spacetimes are described in terms of relatively simple *scalar* equations and merely the description of the perturbations calls for vector or tensor equations. This is somewhat analogous to choosing coordinates adapted to the underlying symmetry of spacetime in the usual metric description.

2.1 The 1+3 formalism

Instead of using Einstein's field equations directly, the 1+3 formulation provides an alternative description of spacetime in terms of scalars, 3-vectors and projected symmetric trace-free (PSTF) 3-tensors and their concomitant equations, obtained by using the Ricci and Bianchi identities. These quantities have a clear physical or geometrical meaning. The $1+3$ covariant approach is based on Refs. [97–102]. A comprehensive review can be found in Ref. [2].

2.1.1 1+3 covariant variables and their properties

4-velocity of fundamental observer

A spacetime $(\mathcal{M}, \mathrm{g})$ is split into 'space' and 'time' relative to a congruence of observers with 4-velocity

$$u^a = \frac{dx^a}{d\tau} \;, \qquad u_a u^a = -1 \;, \tag{2.3}$$

where τ is proper time measured along the observers' worldlines. If these observers are thought to represent the average motion of matter, e.g. in cosmological or radiation spacetimes, they are dubbed 'fundamental observers'.

The 4-velocity u^a induces unique *projection tensors*

$$U^a{}_b = -u^a u_b \Rightarrow U^a{}_c U^c{}_b = U^a{}_b \ , \ U^a{}_a = 1 \ , \ U_{ab} u^b = u_a \ , \tag{2.4}$$

$$h_{ab} = g_{ab} + u_a u_b \Rightarrow h^a{}_c h^c{}_b = h^a{}_b \ , \ h^a{}_a = 3 \ , \ h_{ab} u^b = 0 \ . \tag{2.5}$$

The first projects parallel to the 4-velocity vector u^a, and the second determines the (orthogonal) metric properties of the instantaneous rest-spaces of observers moving with 4-velocity u^a. For the rest-spaces, there is a naturally defined *volume element*:

$$\varepsilon_{abc} \equiv u^d \eta_{dabc} \qquad \Rightarrow \qquad \varepsilon_{abc} = \varepsilon_{[abc]} \ , \ \varepsilon_{abc} u^c = 0 \ , \tag{2.6}$$

where η_{abcd} is the 4-dimensional volume element ($\eta_{abcd} = \eta_{[abcd]}$, $\eta_{0123} = \sqrt{|\det g_{ab}|}$). Hitherto, the spacetime volume element may be written as

$$\eta_{abcd} = 2\,\varepsilon_{ab[c}\, u_{d]} - 2\, u_{[a}\, \varepsilon_{b]cd} \ . \tag{2.7}$$

The spatial volume element satisfies the following useful identities:

$$\varepsilon^{abc}\varepsilon_{def} = 3!\, h^{[a}{}_d\, h^b{}_e\, h^{c]}{}_f \ , \tag{2.8}$$

$$\varepsilon^{abc}\varepsilon_{cef} = 2!\, h^{[a}{}_e\, h^{b]}{}_f \ , \tag{2.9}$$

$$\varepsilon^{abc}\varepsilon_{bcf} = 2!\, h^a{}_f \ , \tag{2.10}$$

$$\varepsilon^{abc}\varepsilon_{abc} = 3! \ . \tag{2.11}$$

The 4-velocity u^a and the spatial projection tensor h_{ab} induce *two derivatives*: the covariant time derivative (denoted with a dot) along the observers' worldlines, where for any tensor $T^{a\cdots b}{}_{c\cdots d}$

$$\dot{T}^{a\cdots b}{}_{c\cdots d} = u^e \nabla_e T^{a\cdots b}{}_{c\cdots d} \ , \tag{2.12}$$

and the fully orthogonally projected covariant derivative D, where for any tensor $T^{a\cdots b}{}_{c\cdots d}$

$$\mathrm{D}_e T^{a\cdots b}{}_{c\cdots d} = h_e{}^f h^a{}_g \cdots h^b{}_h\, h_c{}^i \cdots h_d{}^j\, \nabla_f T^{g\cdots h}{}_{i\cdots j} \ , \tag{2.13}$$

with total projection on all free indices. We stress that these derivatives do generally not commute and therefore give rise to various commutator relations, which play an integral part in all partial frame formalisms. Directly from the definitions one obtains for derivatives of the spatial projector tensor and the 3-volume element

$$\mathrm{D}_a\, h_{bc} = \mathrm{D}_a\, \varepsilon_{bcd} = 0 \;; \tag{2.14}$$

$$\dot{h}_{ab} = 2\, u_{(a}\, \dot{u}_{b)} \;, \quad \dot{\varepsilon}_{abc} = 3\, \dot{u}^d\, \varepsilon_{d[ab}\, u_{c]} \;. \tag{2.15}$$

Hence, the fully projected time derivatives of h_{ab} and ε_{abc} both identically vanish.

Moreover, we use angle brackets to denote orthogonal projections of vectors and the orthogonally projected symmetric trace-free (PSTF) part of tensors:

$$v^{\langle a\rangle} = h^a{}_b\, v^b \;, \qquad T^{\langle ab\rangle} = [\, h^{(a}{}_c\, h^{b)}{}_d - \tfrac{1}{3}\, h^{ab}\, h_{cd}\,]\, T^{cd} \;; \tag{2.16}$$

for convenience the angle brackets are also used to denote orthogonal projections of covariant time derivatives along u^a ('*Fermi derivatives*'):

$$\dot{v}^{\langle a\rangle} = h^a{}_b\, \dot{v}^b \;, \qquad \dot{T}^{\langle ab\rangle} = [\, h^{(a}{}_c\, h^{b)}{}_d - \tfrac{1}{3}\, h^{ab}\, h_{cd}\,]\, \dot{T}^{cd} \;. \tag{2.17}$$

In analogy to the standard vector analysis in three dimensions we introduce the covariant spatial div and curl operators for vectors and second-rank PSTF tensors,

$$\mathsf{div}\, V = \mathrm{D}^a V_a \quad , \quad (\mathsf{div}\, T)_a = \mathrm{D}^b T_{ab} \;; \tag{2.18}$$

$$\mathsf{curl}\, V_a = \varepsilon_{abc} \mathrm{D}^b V^c \quad , \quad \mathsf{curl}\, T_{ab} = \varepsilon_{cd\langle a} \mathrm{D}^c T_{b\rangle}{}^d \;. \tag{2.19}$$

Kinematical quantities

These are the fundamental quantities in the 1+3 formulation giving us information about the expansion, shearing and rotation of the congruence of wordlines traced out by the chosen observers. They are obtained by splitting the covariant derivative of u_a into its irreducible parts, defined by their symmetry properties:

$$\nabla_a u_b = -\, u_a\, \dot{u}_b + \mathrm{D}_a u_b = -\, u_a\, \dot{u}_b + \tfrac{1}{3}\, \Theta\, h_{ab} + \sigma_{ab} + \omega_{ab} \;, \tag{2.20}$$

where the trace $\Theta \equiv \mathrm{D}_a u^a$ is the *(volume) rate of expansion* of the congruence; $\sigma_{ab} \equiv \mathrm{D}_{\langle a} u_{b\rangle}$ is the trace-free symmetric *rate of shear* tensor ($\sigma_{ab} = \sigma_{(ab)}$, $\sigma_{ab}\, u^b = 0$, $\sigma^a{}_a = 0$), describing the rate of distortion of the congruence; and $\omega_{ab} \equiv \mathrm{D}_{[a} u_{b]}$ is the skew-symmetric *vorticity* tensor ($\omega_{ab} = \omega_{[ab]}$, $\omega_{ab}\, u^b = 0$), describing the rotation of the congruence relative to a non-rotating (Fermi-propagated) frame. It is often convenient to define the *vorticity vector* ω^a,

$$\omega^a \equiv \tfrac{1}{2}\, \varepsilon^{abc}\, \omega_{bc} = \tfrac{1}{2}\, \mathsf{curl}\, u^a \quad \Rightarrow \quad \omega_a\, u^a = \omega_{ab}\, \omega^b = 0 \;,\; \omega_{ab} = \varepsilon_{abc}\, \omega^c \;. \tag{2.21}$$

Finally $\dot{u}^a = u^b \nabla_b u^a$ is the *relativistic acceleration* vector, which represents the influence of forces other than gravity on the observer (a free-falling observer has vanishing acceleration in her rest-frame).

With the aid of the rate of expansion Θ it is possible to define an *average length scale* a via

$$\frac{\dot{a}}{a} = \frac{1}{3}\Theta = H \;, \tag{2.22}$$

where H is the Hubble parameter. Whence, volumes vary on average as a^3. Sometimes, the square *magnitudes*

$$\omega^2 = \tfrac{1}{2}\,\omega_{ab}\,\omega^{ab} = \omega_a\,\omega^a \geq 0 \;, \qquad \sigma^2 = \tfrac{1}{2}\,\sigma_{ab}\,\sigma^{ab} \geq 0 \;, \tag{2.23}$$

are useful, particularly because they are positive.

Energy-momentum tensor

The matter-energy content of the Universe (spacetime) is specified by its *energy-momentum tensor* T_{ab} which can be decomposed relative to u^a in the form

$$T_{ab} = \mu\, u_a\, u_b + q_a\, u_b + u_a\, q_b + p\, h_{ab} + \pi_{ab} \;, \tag{2.24}$$

where $\mu \equiv (T_{ab}\, u^a u^b)$ is the *relativistic energy density* relative to u^a, $q^a \equiv -T_{bc}\, u^b\, h^{ca}$ is the *relativistic momentum density* (the *energy flux* relative to u^a), $p \equiv \frac{1}{3}(T_{ab}\, h^{ab})$ is the *isotropic pressure*, and $\pi_{ab} \equiv T_{cd}\, h^c{}_{\langle a}\, h^d{}_{b\rangle}$ is the trace-free *anisotropic pressure* (stress). Its trace is $T = T_a{}^a = 3p - \mu$. Clearly, these matter variables obey

$$q_a\, u^a = 0 \;, \; \pi^a{}_a = 0 \;, \; \pi_{ab} = \pi_{(ab)} \;, \; \pi_{ab}\, u^b = 0 \;. \tag{2.25}$$

These quantities have to be related by the *equations of state* to capture the physics. In cosmology for example, one often requires

$$q^a = \pi_{ab} = 0 \qquad \Leftrightarrow \qquad T_{ab} = \mu\, u_a\, u_b + p\, h_{ab} \;, \tag{2.26}$$

characterising a 'perfect fluid' with equation of state $p = p(\mu, s)$, where s is the entropy density. The simplest possibility $p = 0$ denotes pressure-free matter ('dust' or 'Cold Dark Matter'). In general, an equation of state determines p from μ and possibly other thermodynamical variables. In addition, it is common to impose various *energy conditions* such as

$$\mu > 0 \;, \qquad \mu + p > 0 \;, \qquad \mu + 3p > 0 \;. \tag{2.27}$$

The latter requirement is typically violated in inflationary models. Furthermore, the *isentropic speed of sound* $c_s^2 = (\partial p/\partial \mu)_{s=\text{const}}$ is restricted to the range

$$0 \leq c_s^2 \leq 1 \qquad \Leftrightarrow \qquad 0 \leq \left(\frac{\partial p}{\partial \mu}\right)_{s=\text{const}} \leq 1 \tag{2.28}$$

to guarantee local stability of matter (lower bound) and causality (upper bound).

Curvature tensors and their properties

Spacetime curvature is encapsulated in the *Riemann curvature tensor* ${R_{abc}}^d$, defined by the relation (*Ricci identity*)

$$(\nabla_a \nabla_b - \nabla_b \nabla_a)\, V_c = {R_{abc}}^d\, V_d \tag{2.29}$$

for any dual vector field V_c. The Riemann tensor measures the failure of a vector to return to its initial value when parallel transported around a small closed curve and determines the relative accelerations of infinitesimally nearby geodesics by means of the *geodesic deviation equation* (see, for example, [103]). It has the following symmetry properties

$$R_{abcd} = R_{[ab][cd]} = R_{cdab}\,, \qquad {R_{[abc]}}^d = 0\,, \tag{2.30}$$

and satisfies the Bianchi identity

$$\nabla_{[a}\, {R_{bc]d}}^e = 0\,. \tag{2.31}$$

By contraction one obtains the *Ricci tensor* $R_{ab} = {R_{acb}}^c = R_{ba}$ and a further contraction yields the *Ricci scalar* $R = {R_a}^a$. Applying a twofold contraction to the Bianchi identity (2.31) gives the above stated *twice-contracted Bianchi identity*

$$\nabla_a\, {R_c}^a + \nabla_b\, {R_c}^b - \nabla_c\, R = 0 \qquad \Longleftrightarrow \qquad \nabla^a\, G_{ab} = 0\,, \tag{2.32}$$

which ensures via EFEs (2.1) the conservation of energy and momentum. By taking the trace of EFEs (2.1) and substituting for the trace of the energy-momentum tensor one obtains an expression for the Ricci scalar in terms of matter variables and the cosmological constant, namely

$$R = \mu - 3p + 4\Lambda\,. \tag{2.33}$$

Using this expression in the EFEs and replacing the energy-momentum tensor with its decomposition (2.24) yields

$$R_{ab} = \tfrac{1}{2}\,(\mu + 3p - 2\Lambda)\, u_a u_b + \tfrac{1}{2}\,(\mu - p + 2\Lambda)\, h_{ab} + 2\, u_{(a}\, q_{b)} + \pi_{ab}\,, \tag{2.34}$$

the 1+3 split of the Ricci tensor. The Ricci tensor is thus locally at each point fully determined in terms of the matter variable and the cosmological constant.

It is very useful to decompose the Riemann curvature tensor into a trace part and a trace-free part, the latter being the *Weyl tensor* $C_{abc}{}^{d}$ which is defined by the equation

$$R_{abcd} = C_{abcd} + g_{a[c}\, R_{d]b} - g_{b[c}\, R_{d]a} - \tfrac{1}{3}\, R\, g_{a[c}\, g_{d]b}\ . \tag{2.35}$$

The Weyl tensor is sometimes called the *conformal tensor* because $C_{abc}{}^{d}$ (the position of indices is important!) is invariant under conformal transformations. It has the symmetry properties (2.30) of the Riemann tensor and is trace-free on all indices. The 1+3 decomposition of the Weyl tensor $C_{abc}{}^{d}$ mimicks the reduction of the Faraday tensor into electric and magnetic components—it is split relative to u^a into *'electric'* and *'magnetic' Weyl curvature* parts according to

$$E_{ab} = C_{abcd}\, u^b\, u^d \quad \Rightarrow \quad E^a{}_a = 0\ ,\ E_{ab} = E_{(ab)}\ ,\ E_{ab}\, u^b = 0\ , \tag{2.36}$$

$$H_{ab} = \tfrac{1}{2}\, \varepsilon_{ade}\, C^{de}{}_{bc}\, u^c \quad \Rightarrow \quad H^a{}_a = 0\ ,\ H_{ab} = H_{(ab)}\ ,\ H_{ab}\, u^b = 0\ . \tag{2.37}$$

The Weyl tensor may thus be written as

$$\begin{aligned} C_{abcd} \;=\;& \left(4\, g_{a[p}\, g_{q]b}\, g_{c[r}\, g_{s]d} - \eta_{abpq}\, \eta_{cdrs}\right) u^p\, u^r\, E^{qs} \\ & +2\left(\eta_{abpq}\, g_{c[r}\, g_{s]d} + g_{a[p}\, g_{q]b}\, \eta_{cdrs}\right) u^p\, u^r\, H^{qs}\ . \end{aligned} \tag{2.38}$$

The magnetic and electric parts of the Weyl tensor represent the 'free gravitational field', enabling gravitational action at a distance (tidal forces, gravitational waves), and influence the motion of matter and radiation through the geodesic deviation equation for timelike and null vectors, respectively [104–108].

An extremely useful 1+3 decomposition of the Riemann curvature tensor $R_{abcd}{}^{d}$ can now be obtained by inserting the findings (2.38), (2.34) and (2.33) into equation (2.35) and expanding completely. The derivation gives finally

$$\begin{aligned} R^{ab}{}_{cd} &= R^{ab}_{P\,cd} + R^{ab}_{I\,cd} + R^{ab}_{E\,cd} + R^{ab}_{B\,cd}\ ; \\ R^{ab}_{P\,cd} &= \tfrac{2}{3}\,(\mu + 3p - 2\Lambda)\, u^{[a}\, u_{[c}\, h^{b]}{}_{d]} + \tfrac{2}{3}\,(\mu + \Lambda)\, h^{[a}{}_{[c}\, h^{b]}{}_{d]}\ , \\ R^{ab}_{I\,cd} &= -\,2\, u^{[a}\, h^{b]}{}_{[c}\, q_{d]} - 2\, u_{[c}\, h^{[a}{}_{d]}\, q^{b]} - 2\, u^{[a}\, u_{[c}\, \pi^{b]}{}_{d]} + 2\, h^{[a}{}_{[c}\, \pi^{b]}{}_{d]}\ , \\ R^{ab}_{E\,cd} &= 4\, u^{[a}\, u_{[c}\, E^{b]}{}_{d]} + 4\, h^{[a}{}_{[c}\, E^{b]}{}_{d]}\ , \\ R^{ab}_{B\,cd} &= 2\, \varepsilon^{abe}\, u_{[c}\, H_{d]e} + 2\, \varepsilon_{cde}\, u^{[a}\, H^{b]e}\ . \end{aligned} \tag{2.39}$$

Here P is the perfect fluid part, I the imperfect fluid part, while E and B are the parts due to the electric and magnetic Weyl tensor, respectively.

2.1.2 $1+3$ covariant propagation and constraint equations

In the 1+3 formulation, an arbitrary spacetime may be completely characterised by the irreducible set of geometrical quantities,

$$\{\Theta, \sigma_{ab}, \omega_{ab}, \dot{u}^a, E_{ab}, H_{ab}\}, \tag{2.40}$$

together with the irreducible set of matter variables,

$$\{\mu, p, q^a, \pi_{ab}, \Lambda\}, \tag{2.41}$$

provided an equation of state is prescribed. The cosmological constant Λ has been included in the set of matter variables because it acts formally like an energy density term in EFEs. From the EFEs (2.1) and its associated integrability conditions it is possible to obtain tensor equations for the above introduced covariant variables, resulting in propagation and constraint equations. These tensor equations covariantly describe the spacetime and may be viewed as an alternative formulation of EFEs.

Ricci identity

The first set arises from the Ricci identity for the fundamental timelike vector field u^a, i.e.

$$2\,\nabla_{[a}\nabla_{b]}u^c = R_{ab}{}^c{}_d\,u^d\ , \tag{2.42}$$

on substituting in from (2.20) and (2.39). The *propagation equations* are obtained by separating out the parallel projected part into trace, symmetric trace-free, and skew symmetric parts:

1. The *Raychaudhuri equation* [109]

$$\dot{\Theta} - \mathsf{div}\,\dot{u} = -\tfrac{1}{3}\,\Theta^2 + \dot{u}_a\dot{u}^a - 2\,\sigma^2 + 2\,\omega^2 - \tfrac{1}{2}\,(\mu + 3p) + \Lambda\ , \tag{2.43}$$

is the *basic equation of gravitational attraction* [97]– [101]. It demonstrates the repulsive nature of a positive cosmological constant and identifies $(\mu+3p)$ as the active gravitational mass density. The appearance of the pressure in the gravitational mass is a peculiar feature of GR. In terms of the scale factor a the Raychaudhuri equation can be rewritten as

$$3\,\frac{\ddot{a}}{a} = -2\left(\sigma^2 - \omega^2\right) + \mathsf{div}\,\dot{u} + \dot{u}_a\dot{u}^a - \frac{1}{2}\,(\mu + 3p) + \Lambda\ , \tag{2.44}$$

showing how the curvature of the curve $a(\tau)$ along each worldline (parameterised by proper time τ along that worldline) is determined by the kinematical quantities and the total energy density. Equation (2.44) plays a fundamental role for various singularity theorems.

2. The *vorticity propagation equation*

$$\dot{\omega}^{\langle a\rangle} - \tfrac{1}{2}\,\mathsf{curl}\,\dot{u}^a = -\tfrac{2}{3}\,\Theta\,\omega^a + \sigma^a{}_b\,\omega^b\ ; \qquad (2.45)$$

3. The *shear propagation equation*

$$\dot{\sigma}^{\langle ab\rangle} - \mathrm{D}^{\langle a}\dot{u}^{b\rangle} = -\tfrac{2}{3}\Theta\sigma^{ab} + \dot{u}^{\langle a}\,\dot{u}^{b\rangle} - \sigma^{\langle a}{}_c\,\sigma^{b\rangle c} - \omega^{\langle a}\,\omega^{b\rangle} - (E^{ab} - \tfrac{1}{2}\pi^{ab}) \qquad (2.46)$$

shows how the tidal gravitational field E_{ab} directly induces shear (which then feeds into the Raychaudhuri and vorticity propagation equations, thereby changing the nature of the congruence flow).

The *constraint equations* are obtained by first projecting (2.42) orthogonally and then either contracting over indices b and c or multiplying with $\varepsilon^{ab}{}_c$ or multiplying with ε^{abf} and taking the PSTF part.

1. The (0α)-*constraint*

$$0 = (C_1)^a = \mathrm{D}_b\,\sigma^{ab} - \tfrac{2}{3}\,\mathrm{D}^a\Theta + \varepsilon^{abc}\,[\,\mathrm{D}_b\,\omega_c + 2\,\dot{u}_b\,\omega_c\,] + q^a \qquad (2.47)$$

relates the heat flux to the spatial inhomogeneity of the expansion;

2. The *vorticity divergence identity*

$$0 = (C_2) = \mathsf{div}\,\omega - \dot{u}_a\,\omega^a\ ; \qquad (2.48)$$

3. The H_{ab}-*constraint*

$$0 = (C_3)^{ab} = H^{ab} + 2\,\dot{u}^{\langle a}\,\omega^{b\rangle} + \mathrm{D}^{\langle a}\omega^{b\rangle} - \mathsf{curl}\,\sigma^{ab} \qquad (2.49)$$

characterises the magnetic Weyl tensor as being constructed from the 'distortion' of the vorticity and the 'curl' of the shear.

Twice-contracted Bianchi identities

The second set of equations stems from the twice-contracted Bianchi identities (2.32). Projecting parallel to u^a gives the *energy conservation equation*

$$\dot{\mu} + \mathsf{div}\,q = -\,\Theta\,(\mu + p) - 2\,\dot{u}_a\,q^a - \sigma^a{}_b\,\pi^b{}_a\ , \qquad (2.50)$$

while projecting orthogonal to u^a gives the *momentum conservation equation*

$$\dot{q}^{\langle a\rangle} + \mathrm{D}^a p + \mathrm{D}_b\,\pi^{ab} = -\tfrac{4}{3}\,\Theta\,q^a - \sigma^a{}_b\,q^b - (\mu+p)\,\dot{u}^a - \dot{u}_b\,\pi^{ab} - \varepsilon^{abc}\,\omega_b\,q_c\ . \tag{2.51}$$

It is worth pointing out that for perfect fluids, characterised by Eq. (2.26), these reduce to

$$\dot{\mu} = -\Theta\,(\mu+p)\ , \tag{2.52}$$

$$0 = \mathrm{D}_a p + (\mu+p)\,\dot{u}_a\ . \tag{2.53}$$

This shows that $(\mu+p)$ is the inertial mass density, and also governs the conservation of energy. If this quantity is zero (an effective cosmological constant) or negative, the behaviour of matter will be anomalous.

Other Bianchi identities

The third set of equations arises from the Bianchi identities (2.31). By contracting once and using the splitting of R_{abcd} into R_{ab}, R and C_{abcd} as well as (2.32), we obtain

$$\nabla_a\,C_{bcd}{}^a + \nabla_{[b}\left(R_{c]d} - \tfrac{1}{6}\,R\,g_{c]d}\right) = 0\ . \tag{2.54}$$

The once-contracted Bianchi identities (2.54) give a further pair of propagation equations and a further pair of constraint equations when covariantly decomposed.

The propagation equations are the *$\dot{E}$-equation*,

$$\begin{aligned}\dot{E}^{\langle ab\rangle} &+ \tfrac{1}{2}\,\dot{\pi}^{\langle ab\rangle} - \mathsf{curl}\,H^{ab} + \tfrac{1}{2}\,\mathrm{D}^{\langle a}q^{b\rangle}\\ &= -\tfrac{1}{2}\,(\mu+p)\,\sigma^{ab} - \Theta\left(E^{ab} + \tfrac{1}{6}\,\pi^{ab}\right) + 3\,\sigma^{\langle a}{}_c\left(E^{b\rangle c} - \tfrac{1}{6}\,\pi^{b\rangle c}\right) - \dot{u}^{\langle a}\,q^{b\rangle}\\ &\quad + \varepsilon^{cd\langle a}\left[2\,\dot{u}_c\,H^{b\rangle}{}_d + \omega_c\left(E^{b\rangle}{}_d + \tfrac{1}{2}\,\pi^{b\rangle}{}_d\right)\right]\ ,\end{aligned} \tag{2.55}$$

and the *$\dot{H}$-equation*,

$$\begin{aligned}\dot{H}^{\langle ab\rangle} + \mathsf{curl}\,E^{ab} - \tfrac{1}{2}\,\mathsf{curl}\,\pi^{ab} &= -\Theta\,H^{ab} + 3\,\sigma^{\langle a}{}_c\,H^{b\rangle c} + \tfrac{3}{2}\,\omega^{\langle a}\,q^{b\rangle}\\ &\quad - \varepsilon^{cd\langle a}\left[2\,\dot{u}_c\,E^{b\rangle}{}_d - \tfrac{1}{2}\,\sigma^{b\rangle}{}_c\,q_d - \omega_c\,H^{b\rangle}{}_d\right]\ ,\end{aligned} \tag{2.56}$$

respectively. These equations describe gravitational radiation, e.g. they can be combined to yield a wave equation for E_{ab} as well as H_{ab}.

The constraint equations are the (div E)-*equation*,

$$0 \;=\; (C_4)^a \;=\; \mathrm{D}_b\left(E^{ab}+\tfrac{1}{2}\,\pi^{ab}\right)-\tfrac{1}{3}\,\mathrm{D}^a\mu+\tfrac{1}{3}\,\Theta\,q^a-\tfrac{1}{2}\,\sigma^a{}_b\,q^b-3\,\omega_b\,H^{ab} - \varepsilon^{abc}\left[\sigma_{bd}\,H^d{}_c-\tfrac{3}{2}\,\omega_b\,q_c\right] , \qquad (2.57)$$

wherein the the spatial gradient of the energy density acts as source, and the (div H)-*equation*,

$$0 \;=\; (C_5)^a \;=\; \mathrm{D}_b\,H^{ab}+(\mu+p)\,\omega^a+3\,\omega_b\left(E^{ab}-\tfrac{1}{6}\,\pi^{ab}\right) + \varepsilon^{abc}\left[\tfrac{1}{2}\,\mathrm{D}_b q_c+\sigma_{bd}\,(E^d{}_c+\tfrac{1}{2}\,\pi^d{}_c)\right] , \qquad (2.58)$$

wherein the vorticity acts as source. These equations show that scalar modes will result in a non-zero divergence of the electric Weyl tensor, and vector (vorticity) modes in a non-zero divergence of the magnetic Weyl tensor, respectively.

2.1.3 On the geometry of the observers' rest-spaces

From the dual formulation of Frobenius' Theorem (see, e.g., [103]) follows the handy criterion for a vector field ξ^a:

$$\xi^a \quad \text{is hypersurface-orthogonal} \qquad \Leftrightarrow \qquad \xi_{[a}\nabla_b\,\xi_{c]}=0 \, . \qquad (2.59)$$

If the criterion is fulfilled by the timelike vector field u^a, the rest-spaces of the observers mesh together to form a submanifold of the spacetime whose intrinsic curvature and metric is induced from the embedding spacetime. If u^a fails the criterion, the distribution of rest-spaces (3-vector spaces) is not integrable but it is still formally possible to define a 3-'curvature' tensor.

u^a is hypersurface-orthogonal

Since the congruence u^a is hypersurface-orthogonal, we have

$$0=u_{[a}\nabla_b\,u_{c]}=u_{[a}\mathrm{D}_b\,u_{c]}=u_{[a}\,\omega_{bc]} \qquad \Leftrightarrow \qquad \omega_{ab}=0 \, , \qquad (2.60)$$

that is, in the language of the 1+3 formalism the Frobenius condition is equivalent to the vanishing of the vorticity. Clearly, u^a fulfills the Frobenius criterion iff $u_a = -f\nabla_a\,t$ for some coordinate-dependent functions $f(x^a)$ and $t(x^a)$; hence, u^a is perpendicular to the hypersurfaces $t = const.$ The function f can be normalised to $f = 1$ if in addition the acceleration $\dot{u}^a$ also vanishes.

Since these orthogonal 3-spaces form a submanifold Σ, with the induced metric given by

h_{ab} and the covariant derivative defined by D_a, we may define its intrinsic 3-curvature tensor ${}^{(3)}R_{abc}{}^d$ by the three-dimensional version of the Ricci identity

$$2\,\mathrm{D}_{[a}\mathrm{D}_{b]}V_c = {}^{(3)}R_{abc}{}^d\,V_d \tag{2.61}$$

for any dual vector field V_a living in the 3-submanifold Σ. The 3-curvature tensor is related to the Riemann curvature tensor of the spacetime via the *Gauss equation* (cf. [103])

$${}^{(3)}R_{abcd} = (R_{abcd})_\perp - K_{ac}\,K_{bd} + K_{bc}\,K_{ad}\ , \tag{2.62}$$

where $\perp$ means projection with h_{ab} on all indices and K_{ab} denotes the extrinsic curvature (second fundamental form),

$$K_{ab} = \mathrm{D}_a\,u_b = \tfrac{1}{3}\,\Theta\,h_{ab} + \sigma_{ab} \equiv \Theta_{ab}\ . \tag{2.63}$$

The 1+3 decomposition (2.39) of the Riemann tensor yields

$$\left(R^{ab}{}_{cd}\right)_\perp = \tfrac{2}{3}\,(\mu+\Lambda)\,h^{[a}{}_{[c}\,h^{b]}{}_{d]} + 2\,h^{[a}{}_{[c}\,\pi^{b]}{}_{d]} + 4\,h^{[a}{}_{[c}\,E^{b]}{}_{d]}\ . \tag{2.64}$$

Using this in (2.62) and contracting reveals an expression for the 3-Ricci tensor,

$${}^{(3)}R_{ab} = \left[\tfrac{2}{3}\,(\mu+\Lambda) - \tfrac{2}{9}\,\Theta^2\right]h_{ab} - \tfrac{1}{3}\,\Theta\,\sigma_{ab} + E_{ab} + \tfrac{1}{2}\,\pi_{ab} + \sigma_{ac}\,\sigma^c{}_b\ , \tag{2.65}$$

which implies for the 3-Ricci scalar

$${}^{(3)}R = 2\,(\mu+\Lambda) - \tfrac{2}{3}\,\Theta^2 + 2\,\sigma^2\ . \tag{2.66}$$

This is a generalised Friedmann equation, showing how energy density, expansion and shear determine the average curvature of the 3-space.

Moreover, we mention that the *Codazzi-Mainardi equation* (see [103])

$$\mathrm{D}_a\,K^a{}_b - \mathrm{D}_b\,K^a{}_a = R_{cd}u^d h^c{}_b \tag{2.67}$$

is equivalent to the (0α)-constraint (2.47) when the vorticity vanishes.

We finally note that a spacetime which is either homogeneous or isotropic at every point is necessarily a spacetime of constant curvature (see [103], for example). Whence we have in this case, where $\tilde{K}$ is spatially constant,

$${}^{(3)}R_{abcd} = \tilde{K}\,h_{c[a}\,h_{b]d} \qquad \Rightarrow \qquad {}^{(3)}R_{ab} = \tilde{K}\,h_{ab} \qquad \Rightarrow \qquad {}^{(3)}R = 3\,\tilde{K}\ . \tag{2.68}$$

u^a is not hypersurface-orthogonal

If the congruence u^a fails to be hypersurface-orthogonal, then the congruence has non-vanishing vorticity, $\omega_{ab} \neq 0$, as follows from (2.60). Although the orthogonal 3-spaces do not mesh together to form a submanifold, it is still possible to define formally a 3-'curvature' tensor [110] by setting

$$ {}^{(3)}R_{abcd} = (R_{abcd})_{\perp} - K_{ac}\,K_{bd} + K_{bc}\,K_{ad}\ , \tag{2.69} $$

where K_{ab} is now given by

$$ K_{ab} = \mathrm{D}_a\,u_b = \tfrac{1}{3}\,\Theta\,h_{ab} + \sigma_{ab} + \omega_{ab} = \Theta_{ab} + \omega_{ab}\ . \tag{2.70} $$

The 3-'curvature' tensor has the following properties:

$$ {}^{(3)}R_{abcd} = {}^{(3)}R_{[ab][cd]}\ , \tag{2.71} $$

$$ {}^{(3)}R_{[abc]}{}^{d} = 2\,\omega_{[ab}\,K_{c]}{}^{d}\ , \tag{2.72} $$

$$ {}^{(3)}R^{ab}{}_{cd} - {}^{(3)}R_{cd}{}^{ab} = 8\,\omega^{[a}{}_{[c}\Theta^{b]}{}_{d]}\ . \tag{2.73} $$

The 3-'Ricci tensor' ${}^{(3)}R_{ab} = {}^{(3)}R_{acb}{}^{c}$ now becomes

$$ \begin{aligned} {}^{(3)}R_{ab} = {} & \left[\tfrac{2}{3}\,(\mu+\Lambda) - \tfrac{2}{9}\,\Theta^2\right] h_{ab} - \tfrac{1}{3}\,\Theta\,(\sigma_{ab}+\omega_{ab}) + E_{ab} + \tfrac{1}{2}\,\pi_{ab} \\ & + \sigma_{ac}\,\sigma^{c}{}_{b} + \sigma_{ac}\,\omega^{c}{}_{b} + \omega_{ac}\,\sigma^{c}{}_{b} + \omega_{ac}\,\omega^{c}{}_{b}\ , \end{aligned} \tag{2.74} $$

where the skew part is given by

$$ {}^{(3)}R_{[ab]} = \tfrac{1}{3}\,\omega_{ab}\,\Theta + \omega_{bc}\,\sigma^{c}{}_{a} - \omega_{ac}\,\sigma^{c}{}_{b}\ . \tag{2.75} $$

Finally, the 3-'Ricci scalar' ${}^{(3)}R = {}^{(3)}R^{a}{}_{a}$ is calculated to be

$$ {}^{(3)}R = 2\,(\mu+\Lambda) - \tfrac{2}{3}\,\Theta^2 + 2\left(\sigma^2 - \omega^2\right)\ . \tag{2.76} $$

We mention that the introduction of the 3-'curvature' tensor ${}^{(3)}R_{abc}{}^{d}$ considerably eases the calculation of various commutation relations.

2.1.4 Commutator relations

In GR, the curvature of spacetime manifests itself explicitly in the fact that the covariant derivatives associated with the given connection do in general not commute when spacetime is curved. The concomitant commutator relations are used all over the place in covariant 1+3 formalism (of course, they show up in various appearances in every formulation of GR, for example, in the Newman-Penrose formalism). Typically, our basic equations constitute a system of first-order differential equations which need to be manipulated into wave equations in order to decouple the system or to ease the physical interpretation. It is here where the commutator relations have to be used. Due to their general bearing, we display at this place the most important ones. All commutators stem from the Ricci identities for spacetime scalars f, vectors V^a and second-rank tensors W^{ab}, respectively:

$$\nabla_{[a}\nabla_{b]}f = 0\,, \tag{2.77}$$

$$2\,\nabla_{[a}\nabla_{b]}V^c = R_{ab}{}^c{}_d V^d\,, \tag{2.78}$$

$$2\,\nabla_{[a}\nabla_{b]}W^{cd} = -R_{ab}{}^{ec}\,W_e{}^d - R_{ab}{}^{ed}\,W^c_{\ e}\,. \tag{2.79}$$

The according commutator relations for the 3-spaces orthogonal to the congruence u^a follow by successively writing out the 3-commutators explicitly and then using the Ricci identities (2.77)–(2.79), the irreducible splitting (2.20) of $\nabla_a u_b$ and the 'generalised' Gauss equation (2.69).

For scalar functions f one then obtains the following relations:

$$\mathrm{D}_{[a}\mathrm{D}_{b]}f = \omega_{ab}\,\dot{f}\,, \tag{2.80}$$

$$\mathrm{D}_a\dot{f} - (\mathrm{D}_a f)^{\cdot}_{\perp} = -\dot{u}_a\dot{f} + \left(\tfrac{1}{3}\Theta\,h_a{}^b + \sigma_a{}^b + \omega_a{}^b\right)\mathrm{D}_b f\,. \tag{2.81}$$

For 3-vectors V^a living in the orthogonal 3-space ($V^a u_a = 0$) the following holds:

$$2\,\mathrm{D}_{[a}\mathrm{D}_{b]}V^c = 2\,\omega_{ab}\,\dot{X}^{\langle c\rangle} - {}^{(3)}R_{abs}{}^c\,V^s\,, \tag{2.82}$$

$$\begin{aligned}\mathrm{D}_a\dot{V}_b - (\mathrm{D}_a V_b)^{\cdot}_{\perp} &= -\dot{u}_a\dot{V}_{\langle b\rangle} + \left(\tfrac{1}{3}\Theta\,h_a{}^c + \sigma_a{}^c + \omega_a{}^c\right)(V_c\dot{u}_b + \mathrm{D}_c V_b)\\ &\quad - H_a{}^d\,\varepsilon_{dbc}\,V^c - \tfrac{1}{2}h_{ab}\,q_c V^c + \tfrac{1}{2}V_a\,q_b\,.\end{aligned} \tag{2.83}$$

Second-rank tensors W_{ab} orthogonal to the congruence u^a ($W_{ab}\,u^a = 0 = W_{ab}\,u^b$) obey the relations

$$2\,\mathrm{D}_{[a}\mathrm{D}_{b]}W^{cd} = 2\,\omega_{ab}\left(\dot{W}^{cd}\right)_{\perp} - {}^{(3)}R_{abs}{}^c\,W^{sd} - {}^{(3)}R_{abs}{}^d\,W^{cs}\,, \tag{2.84}$$

$$\begin{aligned}\mathrm{D}_a\dot{W}_{bc} - (\mathrm{D}_a W_b c)^{\cdot}_{\perp} &= \left(\tfrac{1}{3}\Theta\,h_a{}^d\sigma_a{}^d + \omega_a{}^d\right)(\dot{u}_b W_{dc} + \dot{u}_c W_{bd} + \mathrm{D}_d W_{bk})\\ &\quad + \left[h_{a[e}q_{b]} - \varepsilon_{ebd}H_a{}^d\right]W^e_{\ c} + \left[h_{a[e}q_{c]} - \varepsilon_{ecd}H_a{}^d\right]W_b{}^e\\ &\quad - \dot{u}_a\left(\dot{W}_{bc}\right)_{\perp}\,.\end{aligned} \tag{2.85}$$

2.2 The 1+1+2 formulation

In this section, we outline the key features of the 1+1+2 covariant formalism necessary for the description of the Schwarzschild spacetime, the geometry outside static, compact, spherically symmetric objects. Why then does the 1+3 approach work for FLRW but not for Schwarzschild?

In the FLRW case, all covariant variables are scalar quantities: expansion Θ, energy density μ and pressure p which are subsequently governed by simple *scalar first order* ODEs. Vectorial and tensorial equations will first appear in the description of the perturbed spacetime (cf. chapter 4).

In the case of the Schwarzschild spacetime on the other hand, relative to static observers with 4-velocity u^a aligned with the timelike Killing vector field, the only background quantities are the acceleration $\dot{u}^a$ (needed to prevent infall) and the electric Weyl tensor E_{ab} (accounting for the tidal forces). The corresponding 1+3 equations for these variables comprise a set of *tensorial* PDEs, and a description of perturbed Schwarzschild becomes untractable. However, to deal with spherical symmetry, i.e. the preferred radial direction defined by the compact object's gravitational pull, a spacelike congruence n^a has to be introduced into the observers' rest-space leading to a larger set of covariant variables with concomitant equations (evolution, propagation, constraint). The resulting Schwarzschild background equations then become *scalar first order* ODEs, true tensorial equations showing up only for the perturbed Schwarzschild spacetime. In fact, these features remain true for all spacetimes belonging to the class of so-called LRS spacetimes, of which Schwarzschild is a prominent member, suggesting that 1+1+2 is a handy tool for investigating them or the perturbed regime (cf. chapter 5).

The 1+1+2 formalism as presented here was developed only recently by Clarkson & Barrett [34]. A similar formalism was introduced in [111] and further worked out in [112–114]. It was partially studied in the context of symmetric solutions of EFEs in [112, 115, 116].

2.2.1 1+1+2 covariant variables

In the following, it is assumed that a 1+3 covariant decomposition (as described above) has been carried out already, with all tensors split into scalars, vectors and PSTF tensors with respect to the timelike threading vector field u^a.

The spacelike congruence n^a

Let us introduce a for the moment arbitrary unit vector n^a orthogonal to u^a: $n^a\, n_a = 1,\ u^a\, n_a = 0$. Then the projection tensor

$$N_a{}^b \equiv h_a{}^b - n_a\, n^b = g_a{}^b + u_a\, u^b - n_a\, n^b \tag{2.86}$$

projects vectors orthogonal to n^a and u^a onto 2-spaces, which we refer to as the *sheet*. The sheet carries a natural 2-volume element

$$\varepsilon_{ab} \equiv \varepsilon_{abc}\, n^c = u^d\, \eta_{dabc}\, n^c \qquad \Rightarrow \qquad \varepsilon_{(ab)} = 0 = \varepsilon_{ab}\, n^b\,, \tag{2.87}$$

induced by the volume element ε_{abc} of the 3-spaces. From the definition of ε_{ab} and N_{ab} it is straightforward to derive the following useful relations:

$$\begin{aligned} \varepsilon_{abc} &= n_a\, \varepsilon_{bc} + n_b\, \varepsilon_{ca} + n_c\, \varepsilon_{ab}\,, && (2.88)\\ \varepsilon_{ab}\, \varepsilon^{cd} &= N_a{}^c\, N_b{}^d - N_a{}^d\, N_b{}^c\,, && (2.89)\\ \varepsilon_a{}^c\, \varepsilon_{bc} &= N_{ab}\,, && (2.90)\\ \varepsilon^{ab}\, \varepsilon_{ab} &= 2\,. && (2.91) \end{aligned}$$

Note that for a 2-vector Ψ^a, ε_{ab} may be used to form a vector orthogonal to Ψ^a but of the same length.

Any 3-vector ψ^a can now be irreducibly split into a scalar, Ψ, which is the part of the vector parallel to n^a, and a 2-vector, Ψ^a, lying in the sheet orthogonal to n^a:

$$\psi^a = \Psi\, n^a + \Psi^a, \quad \text{with} \quad \Psi \equiv \psi_a\, n^a, \quad \text{and} \quad \Psi^a \equiv N^{ab}\, \psi_b \equiv \psi^{\bar{a}}\,, \tag{2.92}$$

where a bar over an index henceforth denotes projection with N_{ab}. Similarly, any PSTF tensor, ψ_{ab}, can now be split into scalar, 2-vector and 2-tensor parts:

$$\psi_{ab} = \psi_{\langle ab\rangle} = \Psi\left(n_a\, n_b - \tfrac{1}{2} N_{ab}\right) + 2\, \Psi_{(a}\, n_{b)} + \Psi_{ab}\,, \tag{2.93}$$

where

$$\begin{aligned} \Psi &\equiv n^a\, n^b\, \psi_{ab} = -N^{ab}\, \psi_{ab}\,,\\ \Psi_a &\equiv N_a{}^b\, n^c\, \psi_{bc} = \Psi_{\bar{a}}\,,\\ \Psi_{ab} &= \Psi_{\{ab\}} \equiv \left(N_{(a}{}^c N_{b)}{}^d - \tfrac{1}{2} N_{ab}\, N^{cd}\right) \psi_{cd}\,. && (2.94) \end{aligned}$$

Curly brackets denote the part of a tensor which is PSTF with respect to n^a. Thus, such second-rank tensors in the 1+1+2 formalism are precisely the *transverse-traceless* (TT) tensors, which, e.g., appear naturally in the description of GWs in the context of linearised Einstein gravity. For later use we note also that the projection tensors satisfy $h_{\{ab\}} = 0 = N_{\{ab\}}$, and that the tensor $M_{ab} \equiv n_a\, n_b - \frac{1}{2} N_{ab}$ obeys $M_{ab} = M_{\langle ab\rangle}$.

The congruence n^a defines two new derivatives (the splitting of the spatial D_a-derivative) for any tensor $\psi_{a\cdots b}{}^{c\cdots d}$:

$$\hat{\psi}_{a\cdots b}{}^{c\cdots d} \equiv n^e\,\mathrm{D}_e\,\psi_{a\cdots b}{}^{c\cdots d}\,, \tag{2.95}$$

$$\delta_e\,\psi_{a\cdots b}{}^{c\cdots d} \equiv N_e{}^j\,N_a{}^f\cdots N_b{}^g\,N_h{}^c\cdots N_i{}^d\,\mathrm{D}_j\,\psi_{f\cdots g}{}^{h\cdots i}\,. \tag{2.96}$$

The hat-derivative is the spatial D_a-derivative along the vector field n^a in the surfaces orthogonal to u^a, while the δ-derivative is the spatial D-derivative projected onto the sheet, with total projection on every free index. As in the 1+3 formalism, these derivatives do not commute but give rise to a variety of commutation relations among them, some of which will be addressed at a later stage.

Kinematical quantities relative to n^a

In analogy with (2.20), these are obtained by splitting the covariant spatial derivative of n_a into its irreducible parts, defined by their symmetry properties:

$$\mathrm{D}_a\,n_b = n_a\,a_b + \tfrac{1}{2}\,\phi\,N_{ab} + \xi\,\varepsilon_{ab} + \zeta_{ab}\,, \tag{2.97}$$

where $a_a \equiv n^c\,\mathrm{D}_c\,n_a = \hat{n}_a$ is the sheet's *acceleration*, $\phi \equiv \delta_a\,n^a$ its *expansion*, $\xi \equiv \frac{1}{2}\,\varepsilon^{ab}\,\delta_a n_b$ its *twisting* (the rotation of n^a) and $\zeta_{ab} \equiv \delta_{\{a}n_{b\}}$ its *shear* (distortion), respectively. The other derivative of n^a gives its change along u^a,

$$\dot{n}_a = \mathcal{A}\,u_a + \alpha_a \quad \text{where} \quad \alpha_a \equiv \dot{n}_{\bar{a}} \quad \text{and} \quad \mathcal{A} = n^a\,\dot{u}_a\,. \tag{2.98}$$

The new variables a_a, ϕ, ξ, ζ_{ab}, $\mathcal{A}$ and α_a are fundamental objects in spacetime, and their dynamics gives us information about the spacetime geometry. They are treated on the same footing as the kinematical variables of u^a in the 1+3 approach (which also appear here).

Splitting of the kinematical, Weyl and matter tensors

The splitting is done in accordance with the decompositions (2.92) and (2.93), respectively. For the 4-acceleration, vorticity and shear, one arrives at

$$\dot{u}^a = \mathcal{A}\,n^a + \mathcal{A}^a\,, \tag{2.99}$$

$$\omega^a = \Omega\,n^a + \Omega^a\,, \tag{2.100}$$

$$\sigma_{ab} = \Sigma\left(n_a\,n_b - \tfrac{1}{2}N_{ab}\right) + 2\,\Sigma_{(a}\,n_{b)} + \Sigma_{ab}\,. \tag{2.101}$$

The shear scalar, σ, may then be expressed in the form

$$\sigma^2 \equiv \tfrac{1}{2}\,\sigma_{ab}\,\sigma^{ab} = \tfrac{3}{4}\,\Sigma^2 + \Sigma_a\,\Sigma^a + \tfrac{1}{2}\,\Sigma_{ab}\,\Sigma^{ab}\ . \tag{2.102}$$

Furthermore, for the electric and magnetic Weyl tensors one gets

$$E_{ab} = \mathcal{E}\left(n_a\,n_b - \tfrac{1}{2}N_{ab}\right) + 2\,\mathcal{E}_{(a}\,n_{b)} + \mathcal{E}_{ab}\ , \tag{2.103}$$

$$H_{ab} = \mathcal{H}\left(n_a\,n_b - \tfrac{1}{2}N_{ab}\right) + 2\,\mathcal{H}_{(a}\,n_{b)} + \mathcal{H}_{ab}\ . \tag{2.104}$$

Finally, the fluid variables heat flux q^a and anisotropic pressure π_{ab} are split into

$$q^a = Q\,n^a + Q^a\ , \tag{2.105}$$

$$\pi_{ab} = \Pi\left(n_a\,n_b - \tfrac{1}{2}N_{ab}\right) + 2\,\Pi_{(a}\,n_{b)} + \Pi_{ab}\ . \tag{2.106}$$

In terms of 1+1+2 variables, the energy-momentum tensor (2.24) reads explicitly as

$$T_{ab} = \mu\,u_a\,u_b + p\,h_{ab} + 2\,u_{(a}\left[Q\,n_{b)} + Q_{a)}\right] + \Pi\left(n_a\,n_b - \tfrac{1}{2}N_{ab}\right) + 2\,\Pi_{(a}\,n_{b)} + \Pi_{ab}\ , \tag{2.107}$$

where the decompositions (2.105) and (2.106) have been used.

2.2.2 Derivatives and commutators

For later reference, we investigate helpful splittings of various kinds of derivatives involving the fundamental variables of the 1+1+2 formalism as well as the corresponding commutation relations. Let us start with the full split of the covariant derivative of the fundamental unit vectors u_a and n_a, respectively. The full covariant derivative of n^a in terms of the relevant 1+1+2 variables is

$$\nabla_a\,n_b = -\mathcal{A}\,u_a\,u_b - u_a\,\alpha_b + n_a\,u_b + \left(\tfrac{1}{3}\Theta + \Sigma\right)\left(\Sigma_a - \varepsilon_{ac}\,\Omega^c\right)u_b + n_a\,a_b + \tfrac{1}{2}\phi\,N_{ab} + \xi\,\varepsilon_{ab} + \zeta_{ab}\ , \tag{2.108}$$

while the full decomposition of the covariant derivative of u^a is

$$\begin{aligned}\nabla_a u_b = {} & -u_a\left(\mathcal{A}\,n_b + \mathcal{A}_b\right) + \left(\tfrac{1}{3}\Theta + \Sigma\right)n_a\,n_b + n_a\left(\Sigma_b + \varepsilon_{bc}\,\Omega^c\right) \\ & + \left(\Sigma_a - \varepsilon_{ac}\,\Omega^c\right)n_b + \left(\tfrac{1}{3}\Theta - \tfrac{1}{2}\Sigma\right)N_{ab} + \Omega\,\varepsilon_{ab} + \Sigma_{ab},\end{aligned} \tag{2.109}$$

which in turn implies the useful relation

$$\hat{u}_a = \left(\tfrac{1}{3}\Theta + \Sigma\right)n_a + \Sigma_a + \varepsilon_{ab}\,\Omega^b. \tag{2.110}$$

Furthermore, the spatial gradient of a scalar Ψ and a 2-vector $\Psi_a = \Psi_{\bar{a}}$, respectively, have the decomposition

$$\begin{aligned} \mathrm{D}_a\Psi &= \hat{\Psi}\, n_a + \delta_a\Psi\ , &(2.111)\\ \mathrm{D}_a\Psi_b &= -\left(\Psi_c\, a^c\right) n_a\, n_b + n_a\hat{\Psi}_{\bar{b}} - \left[\tfrac{1}{2}\phi\,\Psi_a + \left(\xi\,\varepsilon_{ac} + \zeta_{ac}\right)\Psi^c\right] n_b + \delta_a\Psi_b\ , &(2.112)\end{aligned}$$

while the spatial gradient of a PSTF 2-tensor $\Psi_{ab} = \Psi_{\{ab\}}$ splits according to

$$\mathrm{D}_a\Psi_{bc} = -2\, n_a\, n_{(b}\,\Psi_{c)d}\, a^d + n_a\,\hat{\Psi}_{bc} - 2\left[\tfrac{1}{2}\phi\,\Psi_{a(b} + \left(\xi\,\varepsilon_{ad} + \zeta_{ad}\right)\Psi_{(b}{}^{d}\right] n_{c)} + \delta_a\Psi_{bc}\ . \qquad (2.113)$$

Of particular interest are the various derivatives of the sheet projection tensor N_{ab} and the sheet volume element ε_{ab}. One obtains the following useful relations:

$$\begin{aligned} \dot{N}_{ab} &= 2\,u_{(a}\,\dot{u}_{b)} - 2\,n_{(a}\,\dot{n}_{b)} = 2\,u_{(a}\,\mathcal{A}_{b)} - 2\,n_{(a}\,\alpha_{b)}\ , &(2.114)\\ \hat{N}_{ab} &= 2\,u_{(a}\,\hat{u}_{b)} - 2\,n_{(a}\,\hat{n}_{b)} \\ &= 2\,u_{(a}\left[\left(\tfrac{1}{3}\Theta + \Sigma\right) n_{b)} + \Sigma_{b)} + \varepsilon_{b)c}\,\Omega^c\right] - 2\,n_{(a}\,a_{b)}\ , &(2.115)\\ \delta_c\, N_{ab} &= 0\ , &(2.116)\\ \dot{\varepsilon}_{ab} &= -2\,u_{[a}\,\varepsilon_{b]c}\,\mathcal{A}^c + 2\,n_{[a}\,\varepsilon_{b]c}\,\alpha^c\ , &(2.117)\\ \hat{\varepsilon}_{ab} &= 2\,n_{[a}\,\varepsilon_{b]c}\,a^c\ , &(2.118)\\ \delta_c\,\varepsilon_{ab} &= 0\ . &(2.119)\end{aligned}$$

In general, the three derivatives defined so far, namely the dot derivative ($\dot{}$), the hat-derivative ($\hat{}$) and the delta-derivative (δ_a) do not commute. Instead, when acting on a scalar ψ, they satisfy:

$$\begin{aligned} \hat{\dot{\psi}} - \dot{\hat{\psi}} &= -\mathcal{A}\dot{\psi} + \left(\tfrac{1}{3}\Theta + \Sigma\right)\hat{\psi} + \left(\Sigma_a + \varepsilon_{ab}\,\Omega^b - \alpha_a\right)\delta^a\psi\ , &(2.120)\\ \delta_a\dot{\psi} - \left(\delta_a\psi\right)^{\cdot}_{\perp} &= -\mathcal{A}_a\,\dot{\psi} + \left(\alpha_a + \Sigma_a - \varepsilon_{ab}\,\Omega^b\right)\hat{\psi} + \left(\tfrac{1}{3}\Theta - \tfrac{1}{2}\Sigma\right)\delta_a\psi \\ &\quad + \left(\Sigma_{ab} + \Omega\,\varepsilon_{ab}\right)\delta^b\psi\ , &(2.121)\\ \delta_a\hat{\psi} - \left(\delta_a\psi\right)^{\wedge}_{\perp} &= -2\,\varepsilon_{ab}\,\Omega^b\,\dot{\psi} + a_a\,\hat{\psi} + \tfrac{1}{2}\phi\,\delta_a\psi + \left(\zeta_{ab} + \xi\,\varepsilon_{ab}\right)\delta^b\psi\ , &(2.122)\\ \delta_{[a}\delta_{b]}\psi &= \varepsilon_{ab}\left(\Omega\,\dot{\psi} - \xi\,\hat{\psi}\right)\ . &(2.123)\end{aligned}$$

Here and in the following the $\perp$ denotes projection onto the sheet. (The same symbol is used in the 1+3 description to mean projection onto the observer's rest-space, alas, there won't be any opportunity to confuse them). The derivation of these commutator relations follows patterns analogous to the 1+3 case and is more or less straightforward but somewhat vexing. For example, the last two equations are the decomposition of the 1+3 commutation relation (2.80), written as

$$\mathsf{curl}\,\mathrm{D}_a\psi = 2\,\dot{\psi}\,\omega_a \;. \tag{2.124}$$

From equation (2.123), we see that our sheet will be a genuine 2-surface in the spacetime (and, in particular, that the derivative δ_a will be a true covariant derivative on this surface) if and only if $\xi = \Omega = 0$. (Recall that the 1+3 spatial metric h_{ab} corresponds to a genuine 3-surface when $\omega^a = 0$.) Otherwise, the sheet is really just a collection of tangent planes. In addition, by Frobenius' theorem (see [103]), the two vectors u^a and n^a are 2-surface forming if and only if the commutator $[u,n]^a$ in (2.120) has no component in the sheet: that is, when Greenberg's vector [111]

$$\Sigma^a + \varepsilon^{ab}\Omega_b - \alpha^a \tag{2.125}$$

vanishes (compare also [116]).

The commutation relations for 2-vectors ψ_a are

$$\begin{aligned}
\hat{\dot{\psi}}_{\bar{a}} - \dot{\hat{\psi}}_{\bar{a}} &= -\mathcal{A}\,\dot{\psi}_{\bar{a}} + \left(\tfrac{1}{3}\Theta + \Sigma\right)\hat{\psi}_{\bar{a}} + \left(\Sigma_b + \varepsilon_{bc}\,\Omega^c - \alpha_b\right)\delta^b\psi_a \\
&\quad + \mathcal{A}_a\left(\Sigma_b + \varepsilon_{bc}\,\Omega^c\right)\psi^b + \mathcal{H}\,\varepsilon_{ab}\,\psi^b \;,
\end{aligned} \tag{2.126}$$

$$\begin{aligned}
\delta_a\dot{\psi}_b - \left(\delta_a\psi_b\right)^{\cdot}_{\perp} &= -\mathcal{A}_a\,\dot{\psi}_b + \left(\alpha_a + \Sigma_a - \varepsilon_{ac}\,\Omega^c\right)\hat{\psi}_{\bar{b}} + \left(\tfrac{1}{3}\Theta - \tfrac{1}{2}\Sigma\right)\left(\delta_a\psi_b + \psi_a\,\mathcal{A}_b\right) \\
&\quad + \left(\Sigma_{ac} + \Omega\,\varepsilon_{ac}\right)\left(\delta^c\psi_b + \psi^c\,\mathcal{A}_b\right) + \tfrac{1}{2}\left(\psi_a\,Q_b - N_{ab}\,\psi^c\,Q_c\right) \\
&\quad - \left(\tfrac{1}{2}\phi\,N_{ac} + \xi\,\varepsilon_{ac} + \zeta_{ac}\right)\psi^c\,\alpha_b + H_a\,\varepsilon_{bc}\,\psi^c \;,
\end{aligned} \tag{2.127}$$

$$\begin{aligned}
\delta_a\hat{\psi}_b - \left(\delta_a\psi_b\right)^{\hat{}}_{\perp} &= -2\,\varepsilon_{ac}\,\Omega^c\,\dot{\psi}_{\bar{b}} + a_a\,\hat{\psi}_{\bar{b}} + \tfrac{1}{2}\phi\left(\delta_a\psi_b - \psi_a\,a_b\right) + \left(\zeta_{ac} + \xi\,\varepsilon_{ac}\right)\left(\delta^c\psi_b - \psi^c\,a_b\right) \\
&\quad -2\left(\Omega\,\varepsilon_{a[b} + \Sigma_{a[b}\right)\left(\Sigma_{c]} + \varepsilon_{c]d}\Omega^d\right)\psi^c \\
&\quad -\psi_a\left[\left(\tfrac{1}{2}\Sigma - \tfrac{1}{3}\Theta\right)\left(\Sigma_b + \varepsilon_{bc}\,\Omega^c\right) + \tfrac{1}{2}\Pi_b + \mathcal{E}_b\right] \\
&\quad + N_{ab}\left[\left(\tfrac{1}{2}\Sigma - \tfrac{1}{3}\Theta\right)\left(\Sigma_c + \varepsilon_{cd}\,\Omega^d\right) + \tfrac{1}{2}\Pi_c + \mathcal{E}_c\right]\psi^c \;,
\end{aligned} \tag{2.128}$$

$$\begin{aligned}
\delta_{[a}\delta_{b]}\psi^c &= \left[\left(\tfrac{1}{3}\Theta - \tfrac{1}{2}\Sigma\right)^2 - \tfrac{1}{4}\phi^2 + \tfrac{1}{2}\Pi + \mathcal{E} - \tfrac{1}{3}\left(\mu + \Lambda\right)\right]\psi_{[a}N_{b]}{}^c \\
&\quad -\psi_{[a}\left[-\left(\tfrac{1}{3}\Theta - \tfrac{1}{2}\Sigma\right)\left(\Sigma_{b]}{}^c + \Omega\,\varepsilon_{b]}{}^c\right) + \tfrac{1}{2}\phi\left(\zeta_{b]}{}^c + \xi\,\varepsilon_{b]}{}^c\right) + \tfrac{1}{2}\Pi_{b]}{}^c + \mathcal{E}_{b]}{}^c\right] \\
&\quad + N_{[a}{}^c\left[-\left(\tfrac{1}{3}\Theta - \tfrac{1}{2}\Sigma\right)\left(\Sigma_{b]d} + \Omega\,\varepsilon_{b]d}\right) + \tfrac{1}{2}\phi\left(\zeta_{b]d} + \xi\,\varepsilon_{b]d}\right) + \tfrac{1}{2}\Pi_{b]d} + \mathcal{E}_{b]d}\right]\psi^d \\
&\quad - \left[\left(\Sigma_{[a}{}^c + \Omega\,\varepsilon_{[a}{}^c\right)\left(\Sigma_{b]d} + \Omega\,\varepsilon_{b]d}\right) - \left(\zeta_{[a}{}^c + \xi\,\varepsilon_{[a}{}^c\right)\left(\zeta_{b]d} + \xi\,\varepsilon_{b]d}\right)\right]\psi^d \\
&\quad + \varepsilon_{ab}\left(\Omega\,\dot{\psi}^{\bar{c}} - \xi\,\hat{\psi}^{\bar{c}}\right) \;.
\end{aligned} \tag{2.129}$$

Analogous relations for second-rank tensors hold but are rather involved. Since tensor commutators will not be needed in the later course of the thesis, there is no need to display them here.

2.2.3 1+1+2 covariant evolution, propagation and constraint equations

It was found above that in the 1+1+2 formalism an arbitrary spacetime is completely characterised through the irreducible set of geometrical variables,

$$\{\Theta,\, \mathcal{A},\, \Omega,\, \Sigma,\, \phi,\, \xi,\, \mathcal{E};\, \mathcal{H},\, \mathcal{A}^a,\, \Omega^a,\, \Sigma^a,\, \alpha^a,\, a^a,\, \mathcal{E}^a,\, \mathcal{H}^a;\, \Sigma_{ab},\, \zeta_{ab},\, \mathcal{E}_{ab},\, \mathcal{H}_{ab}\}\ , \tag{2.130}$$

together with the irreducible set of matter variables,

$$\{\mu,\, p,\, Q,\, \Lambda,\ ,\Pi,\, Q^a,\, \Pi^{ab}\}\ , \tag{2.131}$$

once an equation of state is given.

However, the splitting of the 1+3 equations alone does not fully determine the new 1+1+2 variables. Instead, the 1+3 equations have to be augmented with the Ricci identities for n^a:

$$R_{abc} \equiv 2\nabla_{[a}\nabla_{b]}n_c - R_{abcd}\, n^d = 0\ , \tag{2.132}$$

where R_{abcd} is the Riemann curvature tensor. This 3-index tensor may be covariantly split using the two vector fields u^a and n^a, and gives dynamical equations for the covariant parts of the derivative of n^a (namely α_a, a_a, ϕ, ξ and ζ_{ab}) in the form of *evolution* equations, along u^a, and *propagation* equations, along n^a. The constraint equations arise by splitting the corresponding 1+3 constraints or by taking suitable projections of R_{abc}. It is interesting to note that there is some redundancy: not all information that R_{abc} contains is needed in order to determine the full 1+1+2 equations because part of that information is already contained in the 1+3 equations.

The full set of the 1+1+2 equations for arbitrary spacetimes is rather long and will not be displayed here. The complete set of 1+1+2 equations is available from Chris Clarkson [117]. For this thesis, only the substantially smaller set of equations governing the so-called LRS spacetimes, which include spacetimes with spherical symmetry, will be needed. These will be stated explicitly in chapter 5.

2.3 Gauge-invariant perturbation theory

An often occurring problem in practice is that one is able to solve EFEs under favourable circumstances, such as high symmetry, but is actually interested in a somewhat less symmetric, more realistic situation. A typical example provides our Universe which appears isotropic and homogeneous at large scales but becomes gradually lumpy at smaller scales. Thus, our Universe is well described in terms of a FLRW model at large scales but deviates from it at small scales due to the inhomogeneities, which can be taken into account by perturbing the FLRW model 'slightly'. Another example is the outside region of a spherically symmetric matter distribution, which is described by the Schwarzschild solution. By perturbing the Schwarzschild solution, one gains some insight in what happens during gravitational collapse of the central region.

We are thus led to consider a fictitious background spacetime $(\bar{\mathcal{M}}, \bar{g}_{ab})$ and a perturbation thereof, that is the physical, realistic spacetime $(\mathcal{M}, g_{ab})$. In order to quantify the deviation of the physical from the background spacetime, we need a map $\Phi : \bar{\mathcal{M}} \to \mathcal{M}$ which identifies points in the background $\bar{\mathcal{M}}$ with corresponding points in the realistic spacetime $\mathcal{M}$, in accord with $\bar{g}_{ab} \to g_{ab} = \bar{g}_{ab} + \delta\, g_{ab}$. If coordinates are chosen in the background manifold $\bar{\mathcal{M}}$, then the correspondence Φ introduces a coordinate system on the physical manifold $\mathcal{M}$. Given a physical quantity Q (e.g. the Ricci scalar, the energy density μ, the density contrast δ, π_{ab} etc.) on $\mathcal{M}$ and the corresponding physical quantity $\bar{Q}$ on $\bar{\mathcal{M}}$, then we define the *perturbation* δQ of Q at the point $p \in \mathcal{M}$ by

$$\delta Q(p) = Q(p) - \bar{Q}(\Phi^{-1}(p)) \,. \tag{2.133}$$

It is usually understood that the perturbation δQ is small. However, δQ can be assigned any value one likes at the point p by simply altering the correspondence Φ. Any change of the map Φ which leaves the background manifold $\bar{\mathcal{M}}$ unchanged is called a *gauge transformation*. Such a change reflects the freedom of choosing different coordinates $\{x^a\}$ in the physical manifold $\mathcal{M}$:

$$x^a \to \tilde{x}^a = x^a + \xi^a \,. \tag{2.134}$$

Thus, if we alter the initial correspondence (maybe only slightly) to obtain the new identification map $\tilde{\Phi}$, the definition of the perturbation is now

$$\delta\tilde{Q}(p) = Q(p) - \bar{Q}(\tilde{\Phi}^{-1}(p)) \,. \tag{2.135}$$

It is evident that the difference

$$\Delta Q(p) = \delta\tilde{Q}(p) - \delta Q(p) = \bar{Q}(\Phi^{-1}(p)) - \bar{Q}(\tilde{\Phi}^{-1}(p)) \tag{2.136}$$

is a pure gauge artifact and bears no physical significance. Such gauge artifacts (leading to spurious modes) need to be identified and subsequently be eliminated from the discussion. This is the essence of the *gauge problem* of GR.

It follows from (2.136) that any tensorial quantity Q is gauge-invariant if it vanishes in the background manifold $\bar{\mathcal{M}}$ because then the perturbation δQ will be the same whatsoever the correspondence Φ is. The only other possibilities for gauge-invariant quantities are: a scalar which is constant in $\bar{\mathcal{M}}$, or a tensor that is a constant linear combination of Kronecker deltas. This constitutes the fundamental *gauge-invariance Lemma* of Stewart & Walker [30]. A spin-off of the Lemma is that a tensorial quantity is gauge-invariant at a particular perturbative order if it vanishes at all lower orders than this [118, 119].

In the 1+3 formalism, and its 1+1+2 extension, the determination of gauge-invariant variables is with the aid of the Stewart-Walker Lemma conceptually very simple. Choose a background spacetime $\bar{\mathcal{M}}$ and identify all non-zero covariant quantities in the lists (2.40)–(2.41) or (2.130)–(2.131), respectively, needed for a complete description of the spacetime. All quantities, which vanish in the chosen background $\bar{\mathcal{M}}$ are then automatically gauge-invariant and are considered as being of first order. The big bonus of this procedure is that the perturbation variables have a direct physical or geometrical meaning, besides from being covariantly defined. The first-order approximation of the physical spacetime $\mathcal{M}$ is then readily found by linearising the general 1+3 (1+1+2) equations accordingly. Higher-order approximations may be obtained by keeping terms up to the required order.

Despite its simplicity of the above outlined perturbation method, most of the literature on perturbation theory focuses on the metric approach. Here, one chooses a metric in the background spacetime $\bar{\mathcal{M}}$, introduces metric fluctuations and solves the resulting linearised EFEs. The metric fluctuations are irreducibly classified and their behaviour under gauge transformations (2.134) is analysed. One then faces the choice of either fixing a particular gauge and keeping carefully track of it and the resulting gauge freedom or identifies gauge-invariant variables whose physical or geometrical meaning is often rather obscure than obvious. Moreover, metric based perturbation theory becomes rather intricate beyond the linear level. Cosmological perturbation theory is discussed, for example, in [26, 27, 120–125], while [33] is a good source for the theory of black hole perturbations.

Chapter 3

Covariant electrodynamics

The purpose of this chapter is to develop the description of Maxwell's theory of electrodynamics on curved spacetimes from the point of view of the 1+3 and 1+1+2 formalism, respectively. Maxwell's theory is covariantly displayed as

$$\nabla_b F^{ab} = \mu_0 j^a , \qquad \nabla_{[a}F_{bc]} = 0 , \tag{3.1}$$

where the Maxwell field is characterised through the antisymmetric, Lorentz-invariant Faraday tensor F_{ab} and the 4-current is denoted with j^a. The appearance of the magnetic permeability of the vacuum, μ_0, in Maxwell's field equations (3.1) indicates the usage of SI-units but we will set $c = 1$ for convenience such that the relation between permeability and permittivity of the vacuum, ϵ_0, becomes $\mu_0 \epsilon_0 = c^{-2} = 1$. The Maxwell field couples to the gravitational field via the EM energy-momentum tensor, $T^{ab}_{\rm em}$, associated with the Faraday tensor F_{ab} given by the expression

$$\mu_0 T^{ab}_{\rm em} \equiv F_{ac} F^c{}_b + \tfrac{1}{4} g_{ab} F_{cd} F^{cd} . \tag{3.2}$$

The EM energy-momentum tensor is conserved, $\nabla_b T^{ab}_{\rm em} = 0$, for free EM fields, while in the presence of sources the conservation equations follow from

$$\nabla_b T^{ab}_{\rm em} = j_b F^{ab} , \tag{3.3}$$

where j^a is the 4-current arising from charged particles.

When dealing with Maxwell's theory in GR, one immediately faces the issue of the treatment of these Maxwell fields. A possibility is to regard them as part of the 'background' spacetime, in which case one has to solve the coupled Einstein-Maxwell system (2.1) and (3.1). Well-known examples of this procedure are the celebrated charged black hole solutions: the Reissner-Nordström solution in the case of non-rotating charged black holes, and the Kerr-Newman solution for its generalisation towards the rotating analogue. Another possibility is to consider the Maxwell fields as perturbations on a given 'background' spacetime, which leads to great simplification because the spacetime dynamics decouples from the Maxwell fields. This viewpoint is fully justified in the context of cosmology or astrophysics, where the Maxwell fields are typically of very weak and local nature such that their effect on spacetime curvature becomes negligible at zeroth order.

In the following, we discuss the splitting of the Faraday tensor and the resulting Maxwell equations in the 1+3 and 1+1+2 frameworks. Further, the equation of motion for charged particles, that is, the relativistic Lorentz-force equation is investigated. Finally, the covariant description of a collection of relativistic, interacting charged fluids is addressed.

3.1 The 1+3 split of electrodynamics

3.1.1 Faraday and electromagnetic stress tensor

From the Maxwell field strength tensor F_{ab} of an electromagnetic field one deduces the *electric* and *magnetic field* parts as measured by an observer with 4-velocity u^a [101]:

$$E_a \equiv F_{ab}\, u^b \,, \tag{3.4}$$
$$B_a \equiv \tfrac{1}{2}\, \varepsilon_{abc}\, F^{bc} \,, \tag{3.5}$$

where the electromagnetic fields are purely spatial, $E_a\, u^a = 0 = B_a\, u^a$. Thus, the Maxwell field strength tensor can be decomposed as

$$F_{ab} = u_a\, E_b - u_b\, E_a + \varepsilon_{abc}\, B^c \,, \tag{3.6}$$

while the electromagnetic energy-momentum tensor, $T^{ab}_{\rm EM}$, takes the form

$$\mu_0\, T^{ab}_{\rm EM} = \tfrac{1}{2}\left(E^2 + B^2\right) u^a\, u^b + \tfrac{1}{6}\left(E^2 + B^2\right) h^{ab} + 2\, u^{(a}\, \varepsilon^{b)cd}\, E_c\, B_d - \left(E^{\langle a}\, E^{b\rangle} + B^{\langle a}\, B^{b\rangle}\right) \,. \tag{3.7}$$

Here, we have set, as usual, $E^2 = E_a\, E^a$ and $B^2 = B_a\, B^a$. Comparing the above expression with the irreducible decomposition (2.24) of an arbitrary stress tensor, we immediately deduce that fundamental observers measure the energy density $\mu_{\rm EM}$, pressure $p_{\rm EM}$, heat flux (Poynting vector) $q^a_{\rm EM}$ and anisotropic stress $\pi^{ab}_{\rm EM}$ of the EM field to be

$$\begin{aligned}
\mu_{\rm EM} &= \tfrac{1}{2}\left(\epsilon_0 E^2 + \mu_0^{-1} B^2\right) , && (3.8)\\
p_{\rm EM} &= \tfrac{1}{6}\left(\epsilon_0 E^2 + \mu_0^{-1} B^2\right) , && (3.9)\\
q^a_{\rm EM} &= \mu_0^{-1}\, \varepsilon^{abc} E_b B_c , && (3.10)\\
\pi^{ab}_{\rm EM} &= -\left(\epsilon_0 E^{\langle a} E^{b\rangle} + \mu_0^{-1} B^{\langle a} B^{b\rangle}\right) , && (3.11)
\end{aligned}$$

which are the familiar expressions known from special relativity. The remaining quantity which needs to be split is the 4-current j^a:

$$j^a = \rho_c\, u^a + j^{\langle a\rangle} , \tag{3.12}$$

where ρ_c is the charge density and $j^{\langle a\rangle}$ the 3-current (density), respectively.

3.1.2 Maxwell's field equations

As shown in [101], the 1+3 Maxwell's equations are obtained by inserting (3.6) and (3.12) into (3.1) yielding two propagation and two constraint equations for the EM fields:

$$\begin{aligned}
\dot{E}^{\langle a\rangle} - \mathrm{curl}\, B^a &= -\tfrac{2}{3}\,\Theta\, E^a + \sigma^a{}_b\, E^b + \varepsilon^{abc}\left(\dot{u}_b\, B_c + \omega_b\, E_c\right) - \mu_0 j^{\langle a\rangle} , && (3.13)\\
\dot{B}^{\langle a\rangle} + \mathrm{curl}\, E^a &= -\tfrac{2}{3}\,\Theta\, B^a + \sigma^a{}_b\, B^b - \varepsilon^{abc}\left(\dot{u}_b\, E_c - \omega_b\, B_c\right) , && (3.14)\\
0 &= \mathrm{D}_a E^a - 2\,\omega_a\, B^a - \frac{\rho_c}{\epsilon_0} , && (3.15)\\
0 &= \mathrm{D}_a B^a + 2\,\omega_a\, E^a , && (3.16)
\end{aligned}$$

where $\rho_c = -j_a\, u^a$ denotes the charge density.

It is instructive to consider for a moment Maxwell's equations as given by (3.13)–(3.16) in order to see how the curvature of spacetime affects and interacts with an EM field. In general, EM fields couple to the kinematical quantities ($\dot{u}_a$, Θ, σ_{ab}, ω_{ab}) and it is this coupling, absent in flat spacetime, which gives rise to a plethora of fascinating phenomena. The coupling with expansion is crucial for plasmas in a cosmological setup, while the coupling with shear, and hence gravitational radiation, opens the possibility for excitation of a detectable EM signal when a gravity wave passes through a plasma in a strong gravitational field. Finally, the coupling with vorticity is central for plasmas surrounding spinning compact objects such as black holes or neutron stars. This coupling, for example, might be responsible via some variant of the Blandford-Znajek-mechanism [126] for the gigantic jets observed in many galaxies.

3.1.3 Lorentz-force equation

Let us turn to the investigation of the equation of motion for a particle with mass m, charge q and 4-velocity $\tilde{u}^a$ ($\tilde{u}^a\,\tilde{u}_a = -1$) in the presence of EM fields. The equation of motion in this case is given by the relativistic Lorentz-force equation,

$$m\,\tilde{u}^b\,\nabla_b\,\tilde{u}_a = q\,\tilde{F}_{ab}\,\tilde{u}^b = F_{ab}\,\tilde{u}^b\;, \tag{3.17}$$

where the last equality follows from the Lorentz-invariance of the Faraday tensor. The 4-velocity $\tilde{u}^a$ of the particle is linked to the 4-velocity u^a of the fundamental observer by a Lorentz transformation

$$\tilde{u}^a = \gamma\,(u^a + V^a)\;; \qquad \gamma = (1 - V^a\,V_a)^{-1/2}\;, \qquad u_a\,V^a = 0\;. \tag{3.18}$$

Here, V^a is the relative velocity of the moving particle as measured by the fundamental observer. From the definition of the Lorentz factor γ one easily derives the following useful relations:

$$V^a\,V_a = \frac{\gamma^2 - 1}{\gamma^2}\;, \qquad V^a\,\dot{V}_a = \frac{\dot{\gamma}}{\gamma^3}\;, \qquad V^b\,\nabla_a\,V_b = \frac{\nabla_a\gamma}{\gamma^3}\;. \tag{3.19}$$

Employing (3.6), (3.18) and (3.19) in (3.17), one arrives at the following 1+3 decomposition of the Lorentz-force equation:

$$E_a\,V^a = \gamma\,\frac{m}{q}\left[\frac{\dot{\gamma}}{\gamma} + \frac{1}{3}\frac{\gamma^2 - 1}{\gamma^2}\,\Theta + \frac{V^a\,\mathrm{D}_a\gamma}{\gamma} + V_a\,\dot{u}^a + V^a\,V^b\,\sigma_{ab}\right]\;, \tag{3.20}$$

$$\begin{aligned} E_a + \varepsilon_{abc}\,V^b\,B^c = \gamma\,\frac{m}{q}\Bigg[&\left(\frac{\dot{\gamma}}{\gamma} + \frac{1}{3}\Theta + \frac{V^b\,\mathrm{D}_b\gamma}{\gamma}\right)V_a + \dot{u}_a + \dot{V}_{\langle a\rangle} \\ &+ (\sigma_{ba} + \omega_{ba} + \mathrm{D}_b\,V_a)\,V^b\Bigg]\;. \end{aligned} \tag{3.21}$$

We thus recognise that the first equation (3.20) describes the work done by the electric field E_a upon the charge q moving with velocity V^a. The second equation (3.21) is the general relativistic Lorentz-force equation, which demonstrates how the geometry of spacetime (through its kinematical variables) and the derivatives of the relative velocity V^a (or γ) contribute to the total, effective acceleration of the particle.

3.1.4 Charged multifluids

The multi-component fluid

We assume a family of fundamental observers moving with 4-velocity u^a and a collection of perfect fluids with individual 4-velocities given by

$$u^a_{(i)} = \gamma_{(i)} \left(u^a + V^a_{(i)}\right) , \tag{3.22}$$

where $\gamma_{(i)} \equiv \left(1 - V^2_{(i)}\right)^{-1/2}$ is the Lorentz-boost factor and $V^a_{(i)} u_a = 0$ (i is numbering each fluid). By assumption each fluid has, in its own rest frame, an energy momentum tensor of the form

$$T^{ab}_{(i)} = \left(\mu_{(i)} + p_{(i)}\right) u^a_{(i)}\, u^b_{(i)} + p_{(i)}\, g^{ab} , \tag{3.23}$$

where $\mu_{(i)}$ and $p_{(i)}$ are the fluid's energy density and pressure respectively, while g_{ab} is the spacetime metric. Note that in general each species has its own equation of state. Relative to the fundamental frame u^a, however, the above reads

$$T^{ab}_{(i)} = \breve{\mu}_{(i)}\, u^a\, u^b + \breve{p}_{(i)}\, h^{ab} + 2\, u^{(a}\, \breve{q}^{b)}_{(i)} + \breve{\pi}^{ab}_{(i)} , \tag{3.24}$$

which is the stress-energy tensor of an imperfect fluid with

$$\breve{\mu}_{(i)} \equiv \gamma^2_{(i)} \left(\mu_{(i)} + p_{(i)}\right) - p_{(i)} , \tag{3.25}$$

$$\breve{p}_{(i)} \equiv p_{(i)} + \tfrac{1}{3}\, \gamma^2_{(i)} \left(\mu_{(i)} + p_{(i)}\right) V^2_{(i)} , \tag{3.26}$$

$$\breve{q}^a_{(i)} \equiv \gamma^2_{(i)} \left(\mu_{(i)} + p_{(i)}\right) V^a_{(i)} , \tag{3.27}$$

$$\breve{\pi}^{ab}_{(i)} \equiv \gamma^2_{(i)} \left(\mu_{(i)} + p_{(i)}\right) \left(V^a_{(i)}\, V^b_{(i)} - \tfrac{1}{3}\, V^2_{(i)}\, h^{ab}\right) , \tag{3.28}$$

and $h^{ab} \equiv g^{ab} + u^a u^b$ is the projection tensor orthogonal to u^a. Note that $\breve{q}^a_{(i)}$ is the heat flow and $\breve{\pi}^{ab}_{(i)}$ is the anisotropic pressure of each fluid component relative to u^a. Clearly, both quantities depend entirely on the motion of the species relative to u^a.

The gravitational field

The dynamics of the gravitational field is determined by Einstein's equations, forming a closed system once the equation of state for the individual fluid components has been established. Of course, in the presence of other physical fields (e.g. anisotropic stresses or spinor fields) we need to supplement the system with the corresponding evolution and constraint equations (e.g. see [127, 128] and references therein). In the presence of an electromagnetic field, the conservation laws for the individual charged species are

$$\nabla_b T^{ab}_{(i)} = j^b_{(i)} F^a{}_b + J^a_{(i)} , \tag{3.29}$$

with $j^a_{(i)} = \rho_{c(i)} u^a_{(i)}$ being the 4-current and $\rho_{c(i)} \equiv -u_a j^a_{(i)}$ the charge density in the rest frame of the fluid, and μ_0 is the permeability of free space. The term $J^a_{(i)}$ represents interactions other than electromagnetic between the fluids and splits as

$$J^a_{(i)} = \varepsilon_{(i)} u^a + f^a_{(i)} , \tag{3.30}$$

where $\varepsilon_{(i)}$ is the work per unit volume due to the interaction and $f^a_{(i)}$ is the force density orthogonal to u^a. Because of overall energy-momentum conservation we require that

$$\sum_i J^a_{(i)} = 0 \tag{3.31}$$

and write the total fluid equations as

$$\sum_i \nabla_b T^{ab}_{(i)} = F^a{}_b \sum_i j^b_{(i)} . \tag{3.32}$$

Moreover, particle conservation ensures that

$$\nabla_a \left(n_{(i)} u^a_{(i)} \right) = 0 , \tag{3.33}$$

where $n_{(i)}$ is the number density of the individual species in their own rest frame. Finally, we point out that the current density in Eq. (3.29) can be written

$$j^a_{(i)} = q_{(i)} n_{(i)} u^a_{(i)} , \tag{3.34}$$

where $q_{(i)}$ is the individual charge of the particles that make up the fluid.

The nonlinear fluid equations

Charged fluids will interact with each other in the presence of EM fields. The plasma consisting of the interacting fluids is governed by Maxwell's equations on one hand and by the matter conservation equations on the other. The electrodynamics of the plasma is thus described in terms of Maxwell's equations (3.13)–(3.16), where $\rho_{\rm c}$ and $j^{(a)}$ are now

$$\begin{aligned} \rho_{\rm c} &= \sum_i \rho_{\rm c(i)}\,, && (3.35)\\ j^{(a)} &= \sum_i j^{(a)}_{(i)}\,, && (3.36) \end{aligned}$$

the total charge density and the total 3-current density, respectively.

The conservation laws of the individual fluid components, relative to the u^a frame, are obtained by inserting decompositions (3.25)–(3.28) into Eq. (3.29). In particular, by projecting (3.29) onto u^a we arrive at the *energy density conservation* equation

$$\begin{aligned} \dot{\mu}_{(i)} &= -\left(\mu_{(i)}+p_{(i)}\right)\left(\Theta+\mathrm{D}_a V^a_{(i)}\right)-\gamma^{-1}_{(i)}\left(\mu_{(i)}+p_{(i)}\right)\left(\dot{\gamma}_{(i)}+\gamma_{(i)}\,\dot{u}_a\,V^a_{(i)}+V^a_{(i)}\,\mathrm{D}_a\gamma_{(i)}\right)\\ &\quad -V^a_{(i)}\,\mathrm{D}_a\,\mu_{(i)}+\gamma^{-1}_{(i)}\,\varepsilon_{(i)}\,. \qquad (3.37) \end{aligned}$$

On the other hand, we derive the *momentum density conservation* equation

$$\begin{aligned} \left(\mu_{(i)}+p_{(i)}\right)\left(\dot{u}^a+\dot{V}^{(a)}_{(i)}\right) &= -\gamma^{-2}_{(i)}\,\mathrm{D}^a p_{(i)}-\tfrac{1}{3}\Theta\left(\mu_{(i)}+p_{(i)}\right)V^a_{(i)}-\dot{p}_{(i)}\,V^a_{(i)}\\ &\quad -\left(\mu_{(i)}+p_{(i)}\right)\left(V^b_{(i)}\,\mathrm{D}_b\,V^a_{(i)}+\sigma^a{}_b\,V^b_{(i)}+\varepsilon^{abc}\,\omega_b\,V_{(i)c}\right)\\ &\quad +\gamma^{-1}_{(i)}\left(\mu_{(i)}+p_{(i)}\right)\left(V^a_{(i)}\,\dot{\gamma}_{(i)}+V^a_{(i)}\,V^b_{(i)}\,\mathrm{D}_b\gamma_{(i)}\right) \qquad (3.38)\\ &\quad -V^a_{(i)}\,V^b_{(i)}\,\mathrm{D}_b\,p_{(i)}+\gamma^{-1}_{(i)}\,\rho_{\rm c(i)}\left(E^a+\varepsilon^{abc}\,V_{(i)b}\,B_c\right)+\gamma^{-1}_{(i)}\,f^a_{(i)}\,, \end{aligned}$$

by projecting (3.29) orthogonal to u^a. Furthermore, the *particle number conservation*, expressed by Eq. (3.33), takes the form

$$\dot{n}_{(i)}=-\Theta\,n_{(i)}-n_{(i)}\,\dot{u}_a\,V^a_{(i)}-\gamma^{-1}_{(i)}\left[\dot{\gamma}_{(i)}\,n_{(i)}+\mathrm{D}_a\left(\gamma_{(i)}\,n_{(i)}\,V^a_{(i)}\right)\right]\,. \qquad (3.39)$$

Similarly, the total fluid equations (see Eq. (3.32)) provide the *total energy density conservation*,

$$\dot{\mu}=-\Theta\,(\mu+p)-\mathrm{D}_a\,q^a-2\,\dot{u}_a\,q^a-\sigma^a{}_b\,\pi^b{}_a\,, \qquad (3.40)$$

and the *total momentum density conservation*

$$\begin{aligned} (\mu+p)\,\dot{u}^a &= -\mathrm{D}^a\,p-\tfrac{4}{3}\Theta\,q^a-\dot{q}^{(a)}-\sigma^a{}_b\,q^b-\varepsilon^{abc}\,\omega_b\,q_c-\mathrm{D}_b\,\pi^{ab}-\dot{u}_b\,\pi^{ab}\\ &\quad +\rho_{\rm c}\,E^a+\varepsilon^{abc}\,j_b\,B_c\,. \qquad (3.41) \end{aligned}$$

Here, we defined $\mu=\sum_i\breve{\mu}_{(i)}$, $p=\sum_i\breve{p}_{(i)}$, $q^a=\sum_i\breve{q}^a_{(i)}$, $\pi^{ab}=\sum_i\breve{\pi}^{ab}_{(i)}$ and the accentuated quantities are given by (3.25)–(3.28).

Maxwell's equations (3.13)–(3.16) together with the conservation equations (3.37)–(3.41) constitute the fundamental equations for the multifluid description of relativistic plasmas in curved spacetimes. One obtains a closed system of plasma equations once equations of state for each individual fluid are prescribed. In general, the second law of thermodynamics should be employed, too. The exposed formalism allows in principle for the inclusion of interactions which are not of an EM origin, e.g., thermal effects such as collisions between the fluid particles.

3.2 The 1+1+2 split of electrodynamics

3.2.1 Faraday and electromagnetic stress tensor

In accordance with (2.92), the electric, magnetic and current 3-vector fields are irreducibly decomposed into scalar and 2-vector parts as

$$E^a \equiv \mathscr{E} n^a + \mathscr{E}^a , \tag{3.42}$$

$$B^a \equiv \mathscr{B} n^a + \mathscr{B}^a , \tag{3.43}$$

$$j^{(a)} \equiv \mathscr{J} n^a + \mathscr{J}^a , \tag{3.44}$$

where all the 2-vector fields live on the sheet, e.g., $\mathscr{E}_a u^a = 0 = \mathscr{E}_a n^a$. Thus, using also (2.88), the Maxwell field strength tensor (3.6) can be decomposed as

$$F_{ab} = 2\,\mathscr{E}\, u_{[a}\, n_{a]} + 2\, u_{[a}\, \mathscr{E}_{a]} + \mathscr{B}\, \varepsilon_{ab} + 2\, n_{[a}\, \varepsilon_{b]c}\, \mathscr{B}^c , \tag{3.45}$$

while the electromagnetic energy-momentum tensor, T^{ab}_{EM}, takes the form

$$\begin{aligned} \mu_0\, T^{ab}_{\mathrm{EM}} = \; & \tfrac{1}{2}\left(\mathscr{E}^2 + \mathscr{E}_a\, \mathscr{E}^a + \mathscr{B}^2 + \mathscr{B}_a\, \mathscr{B}^a\right) u^a\, u^b + \tfrac{1}{6}\left(\mathscr{E}^2 + \mathscr{E}_a\, \mathscr{E}^a + \mathscr{B}^2 + \mathscr{B}_a\, \mathscr{B}^a\right) h^{ab} \\ & + 2\, u^{(a}\, n^{b)}\, \varepsilon^{cd}\, \mathscr{E}_c\, \mathscr{B}_d + 2\, u^{(a}\, \varepsilon^{b)c}\left(\mathscr{B}\, \mathscr{E}_c - \mathscr{E}\, \mathscr{B}_c\right) - 2\left[\mathscr{E}\, \mathscr{E}^{(a} + \mathscr{B}\, \mathscr{B}^{(a}\right] n^{b)} \\ & - \tfrac{1}{3}\left(2\,\mathscr{E}^2 - \mathscr{E}^c\, \mathscr{E}_c + 2\,\mathscr{B}^2 - \mathscr{B}^c\, \mathscr{B}_c\right)\left(n^a\, n^b - \tfrac{1}{2} N^{ab}\right) - \left(\mathscr{E}^{\{a}\, \mathscr{E}^{b\}} + \mathscr{B}^{\{a}\, \mathscr{B}^{b\}}\right) . \end{aligned} \tag{3.46}$$

Comparing the above expression with the irreducible decomposition (2.107) of an arbitrary stress tensor in terms of 1+1+2 variables, we easily see that the energy density μ_{EM}, the pressure p_{EM}, the heat flux (Poynting vector) q^a_{EM} and the anisotropic stress π^{ab}_{EM} of the EM field measured by the fundamental observer translate now into the following expressions:

$$
\begin{aligned}
\mu_{\rm EM} &= \tfrac{1}{2}\left[\epsilon_0\left(\mathscr{E}^2+\mathscr{E}_a\,\mathscr{E}^a\right)+\mu_0^{-1}\left(\mathscr{B}^2+\mathscr{B}_a\,\mathscr{B}^a\right)\right]\ ; &(3.47)\\
p_{\rm EM} &= \tfrac{1}{6}\left[\epsilon_0\left(\mathscr{E}^2+\mathscr{E}_a\,\mathscr{E}^a\right)+\mu_0^{-1}\left(\mathscr{B}^2+\mathscr{B}_a\,\mathscr{B}^a\right)\right]\ ; &(3.48)\\
Q_{\rm EM} &= \mu_0^{-1}\,\varepsilon^{ab}\,\mathscr{E}_a\,\mathscr{B}_b\ , &(3.49)\\
Q^a_{\rm EM} &= \mu_0^{-1}\,\varepsilon^a{}_b\left(\mathscr{B}\,\mathscr{E}^b-\mathscr{E}\,\mathscr{B}^b\right)\ ; &(3.50)\\
\Pi_{\rm EM} &= -\tfrac{1}{3}\left[\epsilon_0\left(2\,\mathscr{E}^2-\mathscr{E}^c\,\mathscr{E}_c\right)+\mu_0^{-1}\left(2\,\mathscr{B}^2-\mathscr{B}^c\,\mathscr{B}_c\right)\right]\ , &(3.51)\\
\Pi^a_{\rm EM} &= -\left[\epsilon_0\,\mathscr{E}\,\mathscr{E}^a+\mu_0^{-1}\,\mathscr{B}\,\mathscr{B}^a\right]\ , &(3.52)\\
\Pi^{ab}_{\rm EM} &= -\left[\epsilon_0\,\mathscr{E}^{\{a}\,\mathscr{E}^{b\}}+\mu_0^{-1}\,\mathscr{B}^{\{a}\,\mathscr{B}^{b\}}\right]\ . &(3.53)
\end{aligned}
$$

These equations may also be directly obtained from the corresponding 1+3 equations (3.8)–(3.11).

3.2.2 Maxwell's field equations

Maxwell's equations in terms of the 1+1+2 variables are most easily found by inserting the decompositions (3.42)–(3.44) into the 1+3 Maxwell's equations (3.13)–(3.16) and projecting the vector equations parallel and orthogonal to n^a. The result is the 1+1+2 form of Maxwell's equations:

There are two 2-vector equations (the sheet part)

$$
\begin{aligned}
\dot{\mathscr{E}}_{\bar a}+\varepsilon_{ab}\left(\hat{\mathscr{B}}^b-\delta^b\mathscr{B}\right) &= +\xi\,\mathscr{B}_a-\left(\tfrac{1}{2}\phi+\mathcal{A}\right)\varepsilon_{ab}\,\mathscr{B}^b-\left(\tfrac{2}{3}\Theta+\tfrac{1}{2}\Sigma\right)\mathscr{E}_a-\Omega\,\varepsilon_{ab}\,\mathscr{E}^b\\
&\quad +\mathscr{E}\left(-\alpha_a+\Sigma_a+\varepsilon_{ab}\,\Omega^b\right)+\mathscr{B}\,\varepsilon_{ab}\left(\mathcal{A}^b-a^b\right)\\
&\quad +\Sigma_{ab}\,\mathscr{E}^b-\varepsilon_{ab}\,\zeta^{bc}\,\mathscr{B}_c-\mu_0\,\mathscr{J}_a\ , \qquad (3.54)
\end{aligned}
$$

$$
\begin{aligned}
\dot{\mathscr{B}}_{\bar a}-\varepsilon_{ab}\left(\hat{\mathscr{E}}^b-\delta^b\mathscr{E}\right) &= -\xi\,\mathscr{E}_a+\left(\tfrac{1}{2}\phi+\mathcal{A}\right)\varepsilon_{ab}\,\mathscr{E}^b-\left(\tfrac{2}{3}\Theta+\tfrac{1}{2}\Sigma\right)\mathscr{B}_a-\Omega\,\varepsilon_{ab}\,\mathscr{B}^b\\
&\quad +\mathscr{B}\left(-\alpha_a+\Sigma_a+\varepsilon_{ab}\,\Omega^b\right)-\mathscr{E}\,\varepsilon_{ab}\left(\mathcal{A}^b-a^b\right)\\
&\quad +\Sigma_{ab}\,\mathscr{B}^b+\varepsilon_{ab}\,\zeta^{bc}\,\mathscr{E}_c\ , \qquad (3.55)
\end{aligned}
$$

two scalar equations stemming from the projection along n^a,

$$\dot{\mathscr{E}} - \varepsilon_{ab}\,\delta^a\mathscr{B}^b = +2\xi\,\mathscr{B} + \mathscr{E}^a\,\alpha_a - \left(\tfrac{2}{3}\Theta - \Sigma\right)\mathscr{E} + \Sigma^a\,\mathscr{E}_a - \mu_0\,\mathscr{J} + \varepsilon_{ab}\left(\mathcal{A}^a\,\mathscr{B}^b + \Omega^a\,\mathscr{E}^b\right)\,, \tag{3.56}$$

$$\dot{\mathscr{B}} + \varepsilon_{ab}\,\delta^a\mathscr{E}^b = -2\xi\,\mathscr{E} + \mathscr{B}^a\,\alpha_a - \left(\tfrac{2}{3}\Theta - \Sigma\right)\mathscr{B} + \Sigma^a\,\mathscr{B}_a - \varepsilon_{ab}\left(\mathcal{A}^a\,\mathscr{E}^b - \Omega^a\,\mathscr{B}^b\right)\,, \tag{3.57}$$

together with the standard constraint equations

$$\hat{\mathscr{E}} + \delta_a\,\mathscr{E}^a = -\phi\,\mathscr{E} + \mathscr{E}_a\,a^a + 2\,\Omega\,\mathscr{B} + 2\,\Omega^a\,\mathscr{B}_a + \frac{\rho_c}{\epsilon_0}\,, \tag{3.58}$$

$$\hat{\mathscr{B}} + \delta_a\,\mathscr{B}^a = -\phi\,\mathscr{B} + \mathscr{B}_a\,a^a - 2\,\Omega\,\mathscr{E} - 2\,\Omega^a\,\mathscr{E}_a\,. \tag{3.59}$$

We thus end up with six Maxwell equations in the 1+1+2 description. Observe that equations (3.56) and (3.57) turn into additional constraints if the EM fields are required to be stationary.

3.2.3 Lorentz-force equation

To obtain the 1+1+2 version of the 1+3 decomposition of the Lorentz force equation (3.17), all we need to do is to augment the 1+1+2 quantities with the decomposition

$$V^a = v\,n^a + v^a\,, \qquad v^a = v^{\bar{a}}\,, \tag{3.60}$$

of the relative velocity V^a, which appears in the Lorentz transformation (3.18) linking the particle frame with the frame of the fundamental observer. This transformation becomes whence explicitly

$$\tilde{u}^a = \gamma\,(u^a + v\,n^a + v^a)\;; \qquad \gamma = \left(1 - v^2 - v^a\,v_a\right)^{-1/2}\,, \qquad u_a\,v^a = 0 = n_a\,v^a\,. \tag{3.61}$$

The various relations (3.19) regarding the Lorentz factor γ can now be written as

$$v^2 + v^a\,v_a = \frac{\gamma^2 - 1}{\gamma^2}\,, \qquad v\,\dot{v} + v^a\,\dot{v}_a = \frac{\dot{\gamma}}{\gamma^3}\,, \qquad v\,\nabla_a\,v + v^b\,\nabla_a\,v_b = \frac{\nabla_a\gamma}{\gamma^3}\,. \tag{3.62}$$

Employing the above relations in the 1+3 split (3.20)–(3.21) of the Lorentz-force equation, as well as (2.112), it is straightforward to find the following 1+1+2 decomposition of the Lorentz-force equation:

$$\begin{aligned}
\mathscr{E}\, v + \mathscr{E}_a\, v^a &= \gamma\, \frac{m}{q} \left[\frac{\dot{\gamma}}{\gamma} + \frac{1}{3}\frac{\gamma^2 - 1}{\gamma^2}\, \Theta + \frac{v\, \hat{\gamma} + v^a\, \delta_a \gamma}{\gamma} + v\, \mathcal{A} + v_a\, \mathcal{A}^a \right. \\
&\qquad \left. + v^2\, \Sigma + 2\, v\, v^a\, \Sigma_a + v^a\, v^b\, \Sigma_{ab} \right] \,, \qquad (3.63)
\end{aligned}$$

$$\begin{aligned}
\mathscr{E} + \varepsilon_{ab}\, v^a\, \mathscr{B}^b &= \gamma\, \frac{m}{q} \left[\left(\frac{\dot{\gamma}}{\gamma} + \frac{1}{3}\Theta + \frac{v\, \hat{\gamma} + v^a\, \delta_a \gamma}{\gamma} \right) v + \mathcal{A} + \dot{v} - v^a\, \alpha_a \right. \\
&\qquad + \Sigma\, v + \Sigma_a\, v^a + \varepsilon_{ab}\, \Omega^a\, v^b + v\, \hat{v} - v\, v^a\, a_a + v^a\, \delta_a\, v \\
&\qquad \left. - \tfrac{1}{2} \phi\, v^a\, v_a - \zeta_{ab}\, v^b\, v^c \right] \,, \qquad (3.64)
\end{aligned}$$

$$\begin{aligned}
\mathscr{E}_a + \varepsilon_{ab} \left(\mathscr{B}\, v^b - v\, \mathscr{B}^b \right) &= \gamma\, \frac{m}{q} \left[\left(\frac{\dot{\gamma}}{\gamma} + \frac{1}{3}\Theta + \frac{v\, \hat{\gamma} + v^a\, \delta_a \gamma}{\gamma} \right) v_a + \mathcal{A}_a + v\, \alpha_a + \dot{v}_{\bar{a}} \right. \\
&\qquad - \tfrac{1}{2} \Sigma\, v_a + v\, \Sigma_a + \Sigma_{ab}\, v^b + \varepsilon_{ab} \left(v\, \Omega^b - \Omega\, v^b \right) + v^b\, \delta_b\, v_a \\
&\qquad \left. + v \left(v\, a_a + \hat{v}_{\bar{a}} + \tfrac{1}{2} \phi\, v_a - \xi\, \varepsilon_{ab}\, v^b + \zeta_{ab}\, v^b \right) \right] \,. \qquad (3.65)
\end{aligned}$$

An interpretation for the above equations is readily at hand. The first equation (3.63) describes the work done by the electric field upon the moving charge q singling out the contributions along the direction n^a and from the sheet. The second equation (3.64) is the component along n^a of the general relativistic Lorentz-force, while equation (3.65) is the part of the Lorentz-force which lies in the sheet. The RHS of equations (3.64)–(3.65) show, as observed before, in explicit detail how the geometry of spacetime (through its kinematical variables) and the derivatives of the relative velocity $V^a = v\, n^a + v^a$ (or γ) contribute to the total, effective acceleration of the particle.

3.2.4 Charged multifluids

The treatment of multifluids in the 1+1+2 formalism proceeds in analogy to the 1+3 case. In what follows, the 1+1+2 splits (3.42)–(3.44), (3.60) of the EM fields, current density and relative velocity, respectively, are presupposed.

The multi-component fluid

We assume a family of fundamental observers moving with 4-velocity u^a and a collection of perfect fluids with individual 4-velocities given by

$$u^a_{(i)} = \gamma_{(i)} \left(u^a + v\, n^a + v^a_{(i)} \right) \,, \qquad (3.66)$$

where $\gamma_{(i)} \equiv \left(1 - v_{(i)}^2 - v_{(i)}^a\, v^{(i)}\, v_a\right)^{-1/2}$ is the Lorentz-boost factor and $v_{(i)}^a\, u_a = 0 = v_{(i)}^a\, n_a$ (i is numbering each fluid). By assumption each fluid has, in its own rest frame, an energy-momentum tensor of the form

$$T_{(i)}^{ab} = \left(\mu_{(i)} + p_{(i)}\right) u_{(i)}^a\, u_{(i)}^b + p_{(i)}\, g^{ab}\,, \tag{3.67}$$

where $\mu_{(i)}$ and $p_{(i)}$ are the fluid's energy density and pressure respectively, while g_{ab} is the spacetime metric. Note that in general each species has its own equation of state. Relative to the fundamental frame u^a, however, the above reads

$$\begin{aligned} T_{(i)}^{ab} &= \breve{\mu}_{(i)}\, u^a\, u^b + \breve{p}_{(i)}\, h^{ab} + 2\, \breve{Q}_{(i)}\, u^{(a}\, n^{b)} + 2\, u^{(a}\, \breve{Q}_{(i)}^{b)} \\ &\quad + \breve{\Pi}_{(i)} \left(n^a\, n^b - \tfrac{1}{2} N^{ab}\right) + 2\, \breve{\Pi}_{(i)}^{(a}\, n^{b)} + \breve{\Pi}_{(i)}^{ab}\,, \end{aligned} \tag{3.68}$$

which is the stress-energy tensor of an imperfect fluid with

$$\begin{aligned} \breve{\mu}_{(i)} &\equiv \gamma_{(i)}^2 \left(\mu_{(i)} + p_{(i)}\right) - p_{(i)}\,, && (3.69)\\ \breve{p}_{(i)} &\equiv p_{(i)} + \tfrac{1}{3}\, \gamma_{(i)}^2 \left(\mu_{(i)} + p_{(i)}\right) \left(v^2 + v_{(i)}^2\right)\,, && (3.70)\\ \breve{Q}_{(i)} &\equiv \gamma_{(i)}^2 \left(\mu_{(i)} + p_{(i)}\right) v_{(i)}\, n^a\,, && (3.71)\\ \breve{Q}_{(i)}^a &\equiv \gamma_{(i)}^2 \left(\mu_{(i)} + p_{(i)}\right) v_{(i)}^a\,, && (3.72)\\ \breve{\Pi}_{(i)} &\equiv \tfrac{2}{3} \gamma_{(i)}^2 \left(\mu_{(i)} + p_{(i)}\right) \left(v_{(i)}^2 - \tfrac{1}{2}\, v_a^{(i)}\, v_{(i)}^a\right)\,, && (3.73)\\ \breve{\Pi}_{(i)}^a &\equiv \gamma_{(i)}^2 \left(\mu_{(i)} + p_{(i)}\right) v_{(i)}\, v_{(i)}^a\,, && (3.74)\\ \breve{\Pi}_{(i)}^{ab} &\equiv \gamma_{(i)}^2 \left(\mu_{(i)} + p_{(i)}\right) \left(v_{(i)}^{(a}\, v_{(i)}^{b)} - \tfrac{1}{2}\, v_{(i)}^c\, v_c^{(i)}\, N^{ab}\right)\,, && (3.75) \end{aligned}$$

and $h^{ab} \equiv g^{ab} + u^a\, u^b = N_{ab} + n^a\, n^b$ is the projection tensor orthogonal to u^a, whereas N^{ab} projects orthogonal to n^a. We remind the reader that (3.71)–(3.72) and (3.73)–(3.75) represent nothing but the irreducible components of the spatial heat flow $\breve{q}_{(i)}^a$ and the spatial anisotropic pressure $\breve{\pi}_{(i)}^{ab}$ of each fluid component, respectively. Clearly, all these quantities depend entirely on the motion of the species relative to u^a and n^a.

The gravitational field

The paragraph dealing with the gravitational field of multifluids in the 1+3 section can be taken over verbatim. The only quantity that needs to be decomposed is the 3-force density $f_{(i)}^a$,

$$f_{(i)}^a = F_{(i)}\, n^a + F_{(i)}^a\,, \qquad F_{(i)}^a = F_{(i)}^{\bar{a}}\,. \tag{3.76}$$

Therefore, the term $J^a_{(i)}$ representing interactions other than electromagnetic between the fluids splits now according to

$$J^a_{(i)} = \varepsilon_{(i)}\, u^a + F_{(i)}\, n^a + F^a_{(i)} \ , \tag{3.77}$$

where $\varepsilon_{(i)}$ is the work per unit volume due to the interaction, $F_{(i)}$ is the force density along the direction n^a and $F^a_{(i)}$ is the force density in the sheet.

The nonlinear fluid equations

Charged fluids will interact with each other in the presence of EM fields. The plasma consisting of the interacting fluids is governed by Maxwell's equations on one hand and by the matter conservation equations on the other. The electrodynamics of the plasma is now described in terms of Maxwell's equations (3.54)–(3.57), where the total charge density $\rho_{\rm c}$ is as usual defined by

$$\rho_{\rm c} = \sum_i \rho_{\rm c(i)} \ , \tag{3.78}$$

while the total 3-current density $j^{\langle a\rangle}$ is decomposed as

$$j^{\langle a\rangle} = \left(\sum_i \mathscr{I}_{(i)}\right) n^a + \sum_i \mathscr{I}^a_{(i)} = \mathscr{I}\, n^a + \mathscr{I}^{\,a} \ . \tag{3.79}$$

The conservation laws of the individual fluid components are readily obtained from the corresponding 1+3 equations. The *energy density conservation* equation (3.37) now becomes

$$\begin{aligned} \dot{\mu}_{(i)} \;&=\; -\left(\mu_{(i)} + p_{(i)}\right)\left(\Theta + \phi\, v_{(i)} - a_a\, v^a_{(i)} + \hat{v}_{(i)} + \delta_a\, v^a_{(i)}\right) - v_{(i)}\, \hat{\mu}_{(i)} - v^a_{(i)}\, \delta_a\, \mu_{(i)} \\ &\quad - \left(\mu_{(i)} + p_{(i)}\right)\left(\mathcal{A}\, v_{(i)} + \mathcal{A}_a\, v^a_{(i)} + \frac{\dot{\gamma}_{(i)} + v_{(i)}\,\hat{\gamma}_{(i)} + v^a_{(i)}\,\delta_a \gamma_{(i)}}{\gamma_{(i)}}\right) + \frac{\varepsilon_{(i)}}{\gamma_{(i)}} \ . \end{aligned} \tag{3.80}$$

On the other hand, from (3.39) we derive the equation for *momentum density conservation along*

n^a,

$$
\begin{aligned}
\left(\mu_{(i)} + p_{(i)}\right)\left(\mathcal{A} + \dot{v}_{(i)} - v^a_{(i)}\,\alpha_a\right) \;=\;& -\left(\mu_{(i)} + p_{(i)}\right)\Big[\tfrac{1}{3}\Theta\, v_{(i)} + v_{(i)}\,\hat{v}_{(i)} + v^b_{(i)}\,\delta_b\, v_{(i)} \\
& - \left(v_{(i)} - \tfrac{1}{2}\phi\right) v_{(i)a} v^a_{(i)} - \zeta_{ab}\, v^a_{(i)}\, v^b_{(i)} + v_{(i)}\,\Sigma + v^a_{(i)}\,\Sigma_a \\
& + \varepsilon_{ab}\,\Omega^a\, v^b_{(i)} - \frac{\dot{\gamma}_{(i)} + v_{(i)}\,\hat{\gamma}_{(i)} + v^a_{(i)}\,\delta_a \gamma_{(i)}}{\gamma_{(i)}}\, v_{(i)}\Big] \\
& - \frac{\hat{p}_{(i)}}{\gamma^2_{(i)}} - \left(\dot{p}_{(i)} + v_{(i)}\,\hat{p}_{(i)} + v^a_{(i)}\,\delta_a\, p_{(i)}\right) v_{(i)} \\
& + \frac{\rho_{\mathrm{c}(i)}}{\gamma_{(i)}}\left(\mathscr{E} + \varepsilon_{ab}\, v^a_{(i)}\, B^b\right) + \frac{F_{(i)}}{\gamma_{(i)}}\,, \qquad (3.81)
\end{aligned}
$$

by projecting parallel to n^a, and the equation for *momentum density conservation in the sheet,*

$$
\begin{aligned}
\left(\mu_{(i)}\right. \;\;+\;\; & \left.p_{(i)}\right)\left(\mathcal{A}^a + \dot{v}^{\bar{a}}_{(i)} + v_{(i)}\,\alpha^a\right) \qquad (3.82) \\
=\;& -\left(\mu_{(i)} + p_{(i)}\right)\Big[\tfrac{1}{3}\Theta\, v^a_{(i)} + v^2_{(i)}\, a^a + v_{(i)}\,\hat{v}^{\bar{a}}_{(i)} + v^b_{(i)}\,\delta_b\, v^a_{(i)} \\
& + v_{(i)}\left(\tfrac{1}{2}\phi\, v^a_{(i)} - \xi\,\varepsilon^a{}_b\, v^b_{(i)} + \zeta^a{}_b\, v^b_{(i)}\right) - \tfrac{1}{2}\Sigma\, v^a_{(i)} + \Sigma^a{}_b\, v^b_{(i)} + v_{(i)}\,\Sigma^a \\
& + \varepsilon^{ab}\left(v_{(i)}\,\Omega_b - \Omega\, v_{(i)c}\right) - \frac{\dot{\gamma}_{(i)} + v_{(i)}\,\hat{\gamma}_{(i)} + v^a_{(i)}\,\delta_a \gamma_{(i)}}{\gamma_{(i)}}\, v^a_{(i)}\Big] - \frac{\delta^a\, p_{(i)}}{\gamma^2_{(i)}} + \frac{F^a_{(i)}}{\gamma_{(i)}} \\
& - \left(\dot{p}_{(i)} + v_{(i)}\,\hat{p}_{(i)} + v^a_{(i)}\,\delta_a\, p_{(i)}\right) v^a_{(i)} + \frac{\rho_{\mathrm{c}(i)}}{\gamma_{(i)}}\left[\mathscr{E}^a + \varepsilon^{ab}\left(\mathscr{B}\, v_{(i)b} - v_{(i)}\,\mathscr{B}_b\right)\right]\,,
\end{aligned}
$$

by projecting orthogonal to n^a. Furthermore, the *particle number conservation* (3.39) takes now the form

$$
\begin{aligned}
\dot{n}_{(i)} \;=\;& -\Big[\Theta + (\phi + \mathcal{A})\, v_{(i)} + (\mathcal{A}_a - a_a)\, v^a_{(i)} + \hat{v}_{(i)} + \delta_a\, v^a_{(i)} \\
& + \frac{\dot{\gamma}_{(i)} + v_{(i)}\,\hat{\gamma}_{(i)} + v^a_{(i)}\,\delta_a \gamma_{(i)}}{\gamma_{(i)}}\Big]\, n_{(i)} - \left(v_{(i)}\,\hat{n}_{(i)} + v^a_{(i)}\,\delta_a\, n_{(i)}\right)\,. \qquad (3.83)
\end{aligned}
$$

The total fluid equations are obtained in a similar fashion. Whence, the *total energy density conservation* (3.40) reads now

$$
\begin{aligned}
\dot{\mu} \;=\;& -\Theta\,(\mu + p) - \left(\phi\, Q + \hat{Q} - Q_a\, a^a + \delta_a\, Q^a\right) - 2\,(\mathcal{A}\, Q + \mathcal{A}_a\, Q^a) \\
& - \left(\tfrac{3}{2}\Sigma\,\Pi + 2\,\Sigma_a\,\Pi^a + \Sigma_{ab}\,\Pi^{ab}\right)\,. \qquad (3.84)
\end{aligned}
$$

From the total momentum density conservation equation (3.41) follow *total momentum conser-*

vation along n^a,

$$
\begin{aligned}
(\mu + p)\,\mathcal{A} \;&=\; -\hat{p} - \tfrac{4}{3}\Theta\, Q - \left(\dot{Q} - Q_a\,\alpha^a\right) - \left(\Sigma\, Q + \Sigma_a\, Q^a\right) - \varepsilon^{ab}\,\Omega_a\, Q_b - \left(\mathcal{A}\,\Pi + \mathcal{A}_a\,\Pi^a\right) \\
&\quad - \left(\hat{\Pi} + \tfrac{3}{2}\phi\,\Pi - 2\,\Pi_a\, a^a + \delta_a\,\dot{\Pi}^a - \Pi_{ab}\,\zeta^{ab}\right) + \rho_c\,\mathscr{E} + \varepsilon^{ab}\,\mathscr{J}_a\,\mathscr{B}_b\;. \qquad (3.85)
\end{aligned}
$$

and *total momentum conservation in the sheet,*

$$
\begin{aligned}
(\mu + p)\,\mathcal{A}^a \;&=\; -\delta^a p - \tfrac{4}{3}\Theta\, Q^a - \left(\dot{Q}^{\bar{a}} + Q\,\alpha^a\right) - \left(\Sigma^a{}_b\, Q^b + Q\,\Sigma^a - \tfrac{1}{2}\Sigma\, Q^a\right) \\
&\quad -\varepsilon^{ab}\left(Q\,\Omega_b - \Omega\, Q_b\right) - \left(\mathcal{A}\,\Pi^a - \tfrac{1}{2}\Pi\,\mathcal{A}^a + \mathcal{A}_b\,\Pi^{ab}\right) \\
&\quad - \left(\hat{\Pi}^{\bar{a}} + \tfrac{3}{2}\phi\,\Pi^a + \tfrac{3}{2}\Pi\, a^a - \tfrac{1}{2}\delta^a\,\Pi - \Pi^a{}_b a^b + [\zeta^a{}_b - \xi\,\varepsilon^a{}_b]\,\Pi^b + \delta_b\,\Pi^{ab}\right) \\
&\quad +\rho_c\,\mathscr{E}^a + \varepsilon^{ab}\left(\mathscr{B}\,\mathscr{J}_b - \mathscr{J}\,\mathscr{B}_b\right)\;. \qquad (3.86)
\end{aligned}
$$

Here, we defined the total fluid variables

$$
\begin{aligned}
\mu &= \sum_i \breve{\mu}_{(i)}\;, \qquad p = \sum_i \breve{p}_{(i)}\;; \qquad (3.87) \\
Q &= \sum_i \breve{Q}_{(i)}\;, \qquad Q^a = \sum_i \breve{Q}^a_{(i)}\;; \qquad (3.88) \\
\Pi &= \sum_i \breve{\Pi}_{(i)}\;, \qquad \Pi^a = \sum_i \breve{\Pi}^a_{(i)}\;, \qquad \Pi^{ab} = \sum_i \breve{\Pi}^{ab}_{(i)}\;; \qquad (3.89)
\end{aligned}
$$

and the accentuated quantities are given by (3.69)–(3.75).

In the 1+1+2 formalism, Maxwell's equations (3.54)–(3.57) together with the conservation equations (3.80)–(3.86) constitute the fundamental equations for the multifluid description of relativistic plasmas in curved spacetimes. One obtains a closed system of plasma equations once equations of state for each individual fluid are prescribed. In general, the second law of thermodynamics should be employed, too. The presented formalism allows in principle for the inclusion of interactions which are not of an EM origin, e.g., thermal effects such as collisions between the fluid particles.

Chapter 4

Cosmic magnetic fields

In this chapter, we will pursue two alternative paths for generating large scale magnetic fields. The first path leads us to investigate velocity perturbations in the early Universe, such as observed in the CMB. In this early state, the Universe may be viewed as consisting primarily of an overall neutral plasma wherein the velocity perturbations give rise to currents and hence electromagnetic fields. Given that this process can be uphold long enough after recombination, vortical velocity perturbations might generate a magnetic field which fulfills the dynamo requirements. The second path brings us to study the nonlinear interaction between gravitational waves (as produced during inflation) and magnetic fields in a FLRW Universe. This interaction turns out to be able to amplify a pre-existing seed field by several orders of magnitude. Consequently, an inflationary magnetic seed such as proposed in [72, 73] is readily boosted towards within the standard requirements for the dynamo to work.

4.1 The FLRW background spacetime

To discuss cosmic magnetic fields on large scales, we need a model for the Universe. Astronomical observations agree in that our Universe is remarkably isotropic and homogeneous at large scales. These symmetry requirements are fulfilled by the so-called Friedman-Lemaître-Robertson-Walker (FLRW) spacetimes, the standard model(s) of cosmologists, allowing for spatial sections with either flat, closed or open geometry. Recent data from WMAP [129,130] suggest an almost flat but acceleratingly expanding Universe due to a non-zero cosmological constant Λ. However, since Λ typically only becomes dominant at late stages when linear perturbation theory breaks down anyway, we may safely discard the Λ-term in our following investigations (but keep it for completeness in the principal equations).

4.1.1 The FLRW background equations

The FLRW models are characterised by a perfect fluid matter tensor and the condition of everywhere-isotropy. Thus, relative to the congruence of fundamental observers with 4-velocity u^a ($u^a u_a = -1$), the kinematical variables have to be locally isotropic, which implies the vanishing of the 4-acceleration $\dot{u}_a \equiv u^b \nabla_b u_a$, shear $\sigma_{ab} \equiv \mathrm{D}_{\langle a} u_{b\rangle}$ and vorticity $\omega_{ab} \equiv \mathrm{D}_{[a} u_{b]}$ ($0 = \dot{u}_a = \sigma_{ab} = \omega_{ab}$). Furthermore, the models have to be not only conformally flat, that is, the electric and magnetic components of the Weyl tensor vanish ($0 = E_{ab} = H_{ab}$), but also spatially homogeneous implying the vanishing of the spatial gradients of the energy density μ, the pressure p and the expansion $\Theta \equiv \mathrm{D}_a u^a$ ($0 = \mathrm{D}_a \mu = \mathrm{D}_a \Theta = \mathrm{D}_a p$). As usual, the spatial derivative $\mathrm{D}_a \equiv h_a{}^b \nabla_b$ is obtained by projection of the spacetime covariant derivative ∇_a onto the 3-space (with metric $h_{ab} \equiv g_{ab} + u_a u_b$) orthogonal to the observer's worldline. As a consequence, the key background equations are the energy conservation equation

$$\dot{\mu} + \Theta(\mu + p) = 0 \,, \tag{4.1}$$

the Raychaudhuri equation

$$\dot{\Theta} = -\tfrac{1}{3}\Theta^2 - \tfrac{1}{2}(\mu + 3p) + \Lambda \,, \tag{4.2}$$

and the Friedmann equation

$$\mu + \Lambda = \frac{1}{3}\Theta^2 + \frac{3K}{a^2} \,, \tag{4.3}$$

where the constant $K = (-1, 0, 1)$ indicates the (open, flat, closed) geometry of the spatial sections and a denotes the scale factor.

4.1.2 Flat FLRW models with $\Lambda = 0$

An important subclass build flat FLRW Universes with $\Lambda = 0$ where the matter obeys a barotropic equation of state, $p = p(\mu)$. For our purposes here it is sufficient to assume an equation of state for the matter in the form $p = w\,\mu$, with constant barotropic index w. The background equations (4.1)–(4.3) restrained to our stated assumptions imply the following evolution equation for the scale factor:

$$\frac{\ddot{a}}{a} + \frac{1}{2}(1 + 3w)\left(\frac{\dot{a}}{a}\right)^2 = 0 \,. \tag{4.4}$$

By integrating once choosing initial conditions such that $\Theta_i \equiv \Theta(t_i) = 3\,H_i$ for some arbitrary initial time t_i with $H = \dot{a}/a$ the inverse Hubble radius, we obtain for the expansion

$$\frac{1}{3}\Theta = \frac{\dot{a}}{a} = \frac{2}{3(1+w)(t-t_i)+2/H_i} \; . \tag{4.5}$$

Integrating once more, we find for the scale factor the solution

$$a(t) = a_i \left[\tfrac{3}{2} H_i \,(1+w)\,(t-t_i) + 1\right]^{\frac{2}{3(1+w)}} \; . \tag{4.6}$$

The introduction of a dimensionless time variable, τ, defined as

$$\tau \equiv \tfrac{3}{2} H_i \,(1+w)\,(t-t_i) + 1 \; , \tag{4.7}$$

will turn out to be extremely useful. The new time variable allows for a convenient integration of almost all equations to be considered later, irrespective of the barotropic index and taking the initial conditions explicitly into account as well. For example, the scale factor evolves simply as $a = a_i\,\tau^{2/(3(1+w))}$ and the Hubble radius as $H = H_i/\tau$. Henceforth, the energy density satisfies

$$\mu = 3\,H_i^2 \left(\frac{a_i}{a}\right)^{3(1+w)} \; , \tag{4.8}$$

from which we recover the familiar results that $\mu \sim a^{-3}$ in a dust Universe and $\mu \sim a^{-4}$ in a radiation dominated Universe, respectively. Moreover, $\tau = 1$ corresponds to the initial time t_i. The significance of the variable τ lies in the fact that it not only easily reproduces the right scaling of various quantities but also includes the proper initial conditions, which will ease the integration of equations in later sections. A further useful feature is that in terms of the time variable τ the expressions for Θ (or μ) are identical for a constant barotropic index w.

During the late stages of the Universe, whose matter is well approximated as pressureless (dust), it is often convenient to describe the evolution of the Universe in terms of the redshift z. The redshift is defined through the relation

$$1 + z(t) \equiv \frac{a_0}{a(t)} = (1+z_i)\,\tau(t)^{-\frac{2}{3}} \; , \tag{4.9}$$

where $z_i = z(t_i)$ is the redshift at some initial time t_i, a_0 denotes the scale factor today and today's Hubble expansion rate H_0 is measured in proper units by

$$H_0 = 100\,h\,\mathrm{km\,s^{-1}\,Mpc^{-1}} = h \times (9.77813\,\mathrm{Gyr})^{-1} \; ; \tag{4.10}$$

here, h denotes the normalised Hubble expansion rate, with recent observational data [130] favouring a value of $h = 0.71^{+0.04}_{-0.03}$.

4.2 Cosmic magnetic fields from velocity perturbations in the early Universe

Since the early Universe, being much smaller and hotter than today, was in a plasma state, it is reasonable to model this state in terms of multifluids, at least during the radiation and early matter dominated era where the Universe predominantly consists of photons, electrons and positrons. In such a primordial plasma, velocity and density perturbations should occur in a natural way (for example, due to quantum fluctuations). Since the velocity perturbations produce currents, they also induce electromagnetic fields. An intriguing possibility is that such EM fields might serve as the long-sought seeds for the dynamo mechanism. Employing the multifluid approach discussed in the previous chapter, our mechanism is similar to Harrison's protogalaxy model [92,93], and the Biermann battery effect [94], in the sense of yielding vorticity driven magnetic fields, but we note that the battery effect in our formalism would be of second order, while the Harrison effect relies on Thomson scattering.

The issue of tackling the various first-order perturbations when linearising the exact multifluid and Maxwell equations around a FLRW model is dealt with by using a two-parameter approximation scheme characterised by two smallness parameters ε_g and ε_{em}, respectively:

- ε_g - gravitational: σ_{ab}, ω_{ab}, $\mathrm{D}_c\,\sigma_{ab}$, $\mathrm{D}_c\,\omega_{ab}$, etc.
- ε_{em} - electromagnetic: E_a, B_a.

In this way, terms which are second-order in the gravitational variables, ε_g^2, second-order in the induced electromagnetic fields, ε_{em}^2, or of 'mixed' order $\varepsilon_g\varepsilon_{em}$ (cross terms) are neglected.

4.2.1 The fluid equations linearised about FLRW models

We introduce fluids into the picture as previously discussed in the subsection 3.1.4 about charged multifluids in the 1+3 formalism. Consider a family of fundamental observers moving with 4-velocity u^a and a collection of fluids with individual 4-velocities given by

$$u^a_{(i)} = \gamma_{(i)}(u^a + V^a_{(i)}) \,, \tag{4.11}$$

where $\gamma_{(i)} \equiv (1 - V^2_{(i)})^{-1/2}$ is the Lorentz-boost factor and $V^a_{(i)}\, u_a = 0$ (i is numbering each fluid). Assuming perfect fluids, each fluid has, in its own rest frame, an energy momentum tensor of the form

$$T^{ab}_{(i)} = (\mu_{(i)} + p_{(i)})\, u^a_{(i)}\, u^b_{(i)} + p_{(i)}\, g^{ab}\,, \tag{4.12}$$

where $\mu_{(i)}$ and $p_{(i)}$ are the fluid's energy density and pressure respectively, while g_{ab} is the spacetime metric. Note that in general each species has its own equation of state.

Now, the homogeneity and the isotropy of the background FLRW spacetime demands that the shear σ^{ab}, the vorticity ω^a, the acceleration $\dot{u}^a$ as well as the EM fields vanish to zeroth order, the latter bearing the consequence of a zero background charge ρ_c by virtue of (3.15). Furthermore, due to the background symmetry all spatial gradients and velocity components orthogonal to u^a must vanish in the background implying that spatial inhomogeneities are first order quantities and that $\gamma_{(i)} = 1$ to first order for all species i. For simplicity, we will consider only collisionless plasmas, that is, we take exclusively electromagnetic interactions between the fluids into account. We chose to work in the *energy frame*, defined by the vanishing of the total heat flux,

$$q^a = \sum_i \gamma^2_{(i)} (\mu_{(i)} + p_{(i)})\, V^a_{(i)} = 0\,. \tag{4.13}$$

For each fluid, the linearised equations of energy, momentum and number density conservation then take the form

$$\begin{aligned}
\dot{\mu}_{(i)} &= -\left(\Theta + \mathrm{D}_a V^a_{(i)}\right)\left(\mu_{(i)} + p_{(i)}\right)\,, && (4.14)\\
\left(\mu_{(i)} + p_{(i)}\right)\left(\dot{u}^a + \dot{V}^a_{(i)}\right) &= -\mathrm{D}^a p_{(i)} - V^a_{(i)}\, \dot{p}_{(i)} - \tfrac{1}{3}\Theta\left(\mu_{(i)} + p_{(i)}\right) V^a_{(i)}\,, && (4.15)\\
\dot{n}_{(i)} &= -\left(\Theta + \mathrm{D}_a V^a_{(i)}\right) n_{(i)}\,, && (4.16)
\end{aligned}$$

where $n_{(i)}$ denotes the number density of species (i). In the non-relativistic limit, the total energy density is simply $\mu = \sum_i \mu_{(i)}$ and the total pressure is $p = \sum_i p_{(i)}$. The linearised total fluid equations thus read

$$\begin{aligned}
\dot{\mu} &= -\Theta(\mu + p)\,, && (4.17)\\
(\mu + p)\,\dot{u}^a &= -\mathrm{D}^a p\,, && (4.18)
\end{aligned}$$

the last equation implying that (in the energy frame) the acceleration is at least second order in a dust Universe. Finally, Maxwell's equations give the following evolution equation for the EM fields,

$$\begin{aligned}
\dot{E}^{\langle a\rangle} &= -\tfrac{2}{3}\,\Theta\, E^a + \mathsf{curl}\, B^a - \mu_0\, j^{\langle a\rangle}\,, && (4.19)\\
\dot{B}^{\langle a\rangle} &= -\tfrac{2}{3}\,\Theta\, B^a - \mathsf{curl}\, E^a\,, && (4.20)
\end{aligned}$$

together with the constraint equations

$$\mathrm{D}_a E^a = \frac{\rho_{\mathrm{c}}}{\epsilon_0}, \tag{4.21}$$

$$\mathrm{D}_a B^a = 0, \tag{4.22}$$

where we have as usual defined $\mathsf{curl}\, B^a \equiv \epsilon^{abc}\mathrm{D}_b B_c$, and analogously for $\mathsf{curl}\, E^a$.

Equations (4.14)–(4.22) are our main tools for investigating plasma effects in cosmology up to linear order in perturbation theory. For simplicity, we will restrain the analysis to pressureless (dust) Universes with $\Lambda = 0$ and model the matter content as a 2-component fluid (electrons and positrons) with vanishing total charge.

4.2.2 Electrically induced velocity perturbations

Consider an Einstein-de Sitter background and a two-fluid system, with each component having a dust-like energy-momentum tensor relative to its own frame. In the background, the only non-zero scalars are the total density $\mu = \mu_1 + \mu_2$ and the expansion Θ. Note that to zero order the total charge vanishes (i.e. $\rho_{\mathrm{c}} = -e(n_1 - n_2) = 0$), since both species have equal but opposite charges $q_1 = -e = -q_2$. It follows that ρ_{c} is a first order gauge-invariant variable [30]. Furthermore, $\mu_i = m_i n_i$ since no thermal effects are included [m_i is the mass of the particle belonging to species (i)]. In this environment, it helps to introduce the variables

$$N \equiv n_1 + n_2\,, \quad n \equiv n_1 - n_2\,, \quad V^a \equiv \tfrac{1}{2}(V_1^a + V_2^a)\,, \quad v^a \equiv \tfrac{1}{2}(V_1^a - V_2^a)\,. \tag{4.23}$$

Given our frame choice (i.e. $q^a = 0$), equation (3.27) leads to the first order result $\mu_1 v_1^a = -\mu_2 v_2^a$ and subsequently to the following relation

$$V^a = -\frac{\delta\mu}{\mu}\, v^a \tag{4.24}$$

between V^a and v^a, where $\delta\mu \equiv \mu_1 - \mu_2$ and $\delta\mu/\mu$ is independent of time. The latter is true because both μ and $\delta\mu$ scale identically with time; more explicitly,

$$\frac{\delta\mu}{\mu} = \frac{(m_1\,\bar{n}_1 - m_2\,\bar{n}_2)}{(m_1\,\bar{n}_1 + m_2\,\bar{n}_2)}\,, \tag{4.25}$$

with $\bar{n}_{(i)}$ denoting number density at some fixed time. Since we have $\mu_i = m_i n_i$, the total energy density can be written as

$$\mu = \tfrac{1}{2}(m_1 + m_2)\, N + \tfrac{1}{2}(m_2 - m_1)\, n, \tag{4.26}$$

a relation which will be used very often in the following analysis. The Raychaudhuri equation then becomes

$$\dot{\Theta} = -\tfrac{1}{3}\,\Theta^2 - \tfrac{1}{4}\,(m_1 + m_2)\,N - \tfrac{1}{4}\,(m_1 - m_2)\,n\,. \tag{4.27}$$

Then, employing equations (4.15) and (4.16) we obtain the propagation formulae for N, n and v^a

$$\dot{N} = -\left(\Theta + \mathrm{D}_a\, V^a\right) N\,, \tag{4.28}$$

$$\dot{n} = -\Theta\, n - N\, \mathrm{D}_a\, v^a\,, \tag{4.29}$$

$$\dot{v}^{\langle a\rangle} = -\frac{1}{3}\,\Theta\, v^a - \frac{e}{2}\,\frac{(m_1 + m_2)}{m_1\, m_2}\, E^a\,. \tag{4.30}$$

As expected, equations (4.28) and (4.29) show how velocity perturbations, depending on the sign of their 3-divergence, can increase or decrease the number density dilution caused by the expansion. More importantly, equation (4.30) shows that the presence of the electric field acts as a source of linear velocity perturbations in the charged plasma, even when such perturbations are originally absent (i.e. when $v_a = 0$ initially). In what follows we will see that a non-zero initial velocity perturbation can give rise to density fluctuations [cf. (4.29)], generating a perturbed non-zero charge density which through equation (4.21) may seed electric fields.

4.2.3 Velocity induced density perturbations

Consider the dimensionless, first-order, gauge-invariant variable

$$\Delta \equiv \frac{a^2}{N}\,\mathrm{D}^2 N\,, \tag{4.31}$$

where a is the background scale factor and $\mathrm{D}^2 = h^{ab}\,\mathrm{D}_a\,\mathrm{D}_b$ is the covariant Laplacian operator normal to u^a. The above describes inhomogeneities in the total number density of the particles and, consequently, it also describes inhomogeneities in the total energy density. To linear order the evolution of Δ is determined by the system

$$\dot{\Delta} = -\mathscr{X} + \frac{\delta\mu}{\mu}\, a\,\mathrm{D}^2\,\mathscr{V}\,, \tag{4.32}$$

$$\dot{\mathscr{X}} = -\tfrac{2}{3}\,\Theta\,\mathscr{X} - \tfrac{1}{4}\,N\left[(m_1 + m_2)\,\Delta + (m_1 - m_2)\,a^2\,\mathrm{D}^2\,Y\right]\,, \tag{4.33}$$

$$\dot{\mathscr{V}} = -\tfrac{1}{3}\,\Theta\,\mathscr{V} + \tfrac{3}{4}\,\alpha^2\,\mu\, a\, Y\,, \tag{4.34}$$

$$\dot{Y} = -\frac{1}{a}\,\mathscr{V}\,, \tag{4.35}$$

where $\alpha^2 \equiv 4\,e^2/(3\,\epsilon_0\,m_1\,m_2)$. In deriving the above we have employed the first order gauge-invariant variables

$$\mathscr{X} \equiv a^2\,\mathrm{D}^2\,\Theta\,, \qquad \mathscr{V} \equiv a\,\mathrm{D}_a\,v^a\,, \qquad Y \equiv \frac{n}{N}\,, \tag{4.36}$$

and used Maxwell's equation (4.21). Note that $\mathscr{X}$ and $\mathscr{V}$ describe scalar inhomogeneities in the expansion and the relative velocity of the species respectively, while Y determines the net charge of the total fluid. Given that equations (4.34) and (4.35) have decoupled from the rest of the system we, can obtain the following propagation equation for Y:

$$\ddot{Y} + \tfrac{2}{3}\,\Theta\,\dot{Y} + \tfrac{3}{4}\,\alpha^2\,\mu\,Y = 0\,, \tag{4.37}$$

The solution to equation (4.37) will act as an inhomogeneous driving term in the corresponding propagation equation for Δ:

$$\ddot{\Delta} + \frac{2}{3}\,\Theta\,\dot{\Delta} - \frac{1}{2}\,\mu\,\Delta = \left(\frac{3}{4}\,\alpha^2 + \frac{1}{2}\right)\frac{\delta\mu}{\mu}\,\mu\,a^2\,\mathrm{D}^2\,Y\,, \tag{4.38}$$

obtained by taking the derivative of equation (4.32) and using (4.33). According to equations (4.35) and (4.38), velocity inhomogeneities act as sources of density fluctuations. Note that the right hand side of (4.38) is a pure multifluid effect, where the part containing α^2 stems from the plasma description.

In order to solve equations (4.37) and (4.38) it is standard to decompose the physical (perturbed) fields into a spatial and temporal part, using as eigenfunctions $Q_{(k)}$, solutions of the scalar Helmholtz equation [131, 132]. In particular we write

$$\Delta = \Delta_{(k)}\,Q^{(k)}\,, \qquad Y = Y_{(k)}\,Q^{(k)}\,, \tag{4.39}$$

where $\mathrm{D}_a\,Y_{(k)} = 0 = \mathrm{D}_a\,\Delta_{(k)}$, $\dot{Q}_{(k)} = 0$ and $\mathrm{D}^2\,Q^{(k)} = -(k^2/a^2)\,Q^{(k)}$. For an Einstein-de Sitter background, the expansion and energy density evolve as $\Theta = 2/t = 3\,H_i/\tau$ and $\mu = 4/(3\,t^2) = 3\,H_i/\tau^2$, where the dimensionless time variable τ defined in (4.7) was used. Hence, applying the harmonic splitting given above, equations (4.38) and (4.37) become (a prime means differentiation with respect to τ)[1]

$$\Delta''_{(k)} + \frac{4}{3\,\tau}\,\Delta'_{(k)} - \frac{2}{3\,\tau^2}\,\Delta_{(k)} = -k^2\left(\alpha^2 + \frac{2}{3}\right)\frac{\delta\mu}{\mu}\,\frac{1}{\tau^2}\,Y_{(k)}\,, \tag{4.40}$$

[1]Since the dimensionful cosmic time t differs from the dimensionless time τ merely by a constant shift, dotted and primed equations relate to each other by the simple replacement (dot, t) $\longleftrightarrow$ (prime, τ).

and

$$Y''_{(k)} + \frac{4}{3\,\tau}\,Y'_{(k)} + \frac{\alpha^2}{\tau^2}\,Y_{(k)} = 0\;, \tag{4.41}$$

respectively. In order to estimate the value of the parameter α we substitute back for the gravitational constant G and write

$$\alpha^2 = \frac{4}{3}\left(\frac{m_e}{m_1}\right)\left(\frac{m_e}{m_2}\right)\left(\frac{e^2}{\epsilon_0}\right)\left(\frac{1}{8\,\pi\,G\,m_e^2}\right) \sim \left(\frac{m_e}{m_1}\right)\left(\frac{m_e}{m_2}\right)\times 10^{42}\;. \tag{4.42}$$

Since $\alpha \gg 1$ the solutions to the above equations are

$$\Delta_{(k)} = C_1\,\tau^{2/3} + C_2\,\tau^{-1} + k^2\,\frac{\delta\mu}{\mu}\,Y_{(k)}\;, \tag{4.43}$$

$$Y_{(k)} = [\mathrm{C}_1\,\cos(\alpha\,\ln\tau) + \mathrm{C}_2\,\sin(\alpha\,\ln\tau)]\,\tau^{-\frac{1}{6}}\;. \tag{4.44}$$

Hence, in addition to the usual growing and decaying modes of the standard gravitational instability picture, we have obtained a mode representing high-frequency plasma oscillations with a weak damping envelope. This mode is triggered by velocity distortions in the charged plasma and, as expected, has negligible large scale effect. However, the extra plasma modes become increasingly important as we move on to progressively smaller scales (i.e. for $k \gg 1$).

It should also be pointed out that a *finite temperature* will in general cause Landau damping of the plasma oscillations. The effect (requiring kinetic treatment) is small for wavelengths much larger than the Debye length (which is proportional to the thermal velocity of the plasma particles) and in this case the dust fluid approximation is well justified.

4.2.4 Velocity induced electromagnetic fields

For a *cold* plasma, the currents for each fluid species may be written as

$$j^a_{(i)} = q_{(i)}\,n_{(i)}\,u^a_{(i)} = q_{(i)}\,n_{(i)}\,(u^a + V^a_{(i)})\;, \tag{4.45}$$

where $q_{(i)}$ is the charge and $V^a_{(i)}$ is the velocity of the species under consideration. Since we require the plasma to be neutral on the whole, the species are of opposite charge. Hence, the total 3-current $j^{\langle a\rangle}$ appearing in Maxwell's equations reads to first order

$$j^{\langle a\rangle} = j_1^{\langle a\rangle} + j_2^{\langle a\rangle} = -e\,N\,v^a\;. \tag{4.46}$$

From Maxwell's equations (4.19)–(4.22), using (4.46) and (4.29), one can then deduce second order wave equations for the induced electromagnetic fields. They are

$$\ddot{E}_{\langle a\rangle} - \mathrm{D}^2 E_a + \tfrac{5}{3}\,\Theta\,\dot{E}_{\langle a\rangle} + \left[\tfrac{2}{9}\,\Theta^2 + \left(\tfrac{3}{4}\,\alpha^2 + \tfrac{1}{3}\right)\mu\right] E_a = 2\,\beta^2\,\mu\left(\mathrm{D}_a Y - \tfrac{1}{3}\,\Theta\, v_a\right)\ , \tag{4.47}$$

$$\ddot{B}_{\langle a\rangle} - \mathrm{D}^2 B_a + \tfrac{5}{3}\,\Theta\,\dot{B}_{\langle a\rangle} + \left[\tfrac{2}{9}\,\Theta^2 + \tfrac{1}{3}\,\mu\right] B_a = -2\,\beta^2\,\mu\,\mathsf{curl}\ v_a\ , \tag{4.48}$$

where $\beta^2 = \mu_0\, e/(m_1 + m_2)$. Observe that B_a and $\mathsf{curl}\ v_a$ are both purely solenoidal, whereas $\mathrm{D}_a Y$ has no solenoidal part. It is worthwhile to note that the magnetic field is solely sourced by inhomogeneities in the velocity in contrast to the electric field which is sourced by inhomogeneities in the number density and velocity perturbations. Both equations look strikingly similar, the differences originating either from the total current or from a gradient in the charge density (in the case of $\mathrm{D}_a Y$). The additional $3\,\alpha^2/4$-term in the electric wave equation comes from the non-stationarity of the total current and its largeness — $\alpha^2 \sim 10^{42}$ for an e^+e^-–plasma — leads directly to the high-frequency behaviour of plasma effects, as will be shown below (cf. the preceding section for a discussion of the high-frequency plasma mode in the gravitational instability picture).

It will be useful to introduce expansion normalised variables,

$$\mathscr{E}_a \equiv \frac{E_a}{\Theta}\ , \qquad \mathscr{B}_a \equiv \frac{B_a}{\Theta}\ , \qquad \mathscr{K}_a \equiv \frac{\mathsf{curl}\ v_a}{\Theta}\ . \tag{4.49}$$

Equations (4.47) and (4.48), together with equations for the driving terms, then read

$$\ddot{\mathscr{E}}_{\langle a\rangle} - \mathrm{D}^2\mathscr{E}_a + \left(\Theta - \frac{\mu}{\Theta}\right)\dot{\mathscr{E}}_{\langle a\rangle} - \left[\frac{1}{9}\,\Theta^2 - \left(\frac{3}{4}\,\alpha^2 + \frac{1}{3}\right)\mu\right]\mathscr{E}_a = 2\,\beta^2\,\frac{\mu}{\Theta}\left(\mathrm{D}_a Y - \frac{1}{3}\,\Theta v_a\right), \tag{4.50}$$

$$\ddot{\mathscr{B}}_{\langle a\rangle} - \mathrm{D}^2\mathscr{B}_a + \left(\Theta - \frac{\mu}{\Theta}\right)\dot{\mathscr{B}}_{\langle a\rangle} - \left(\frac{1}{9}\,\Theta^2 - \frac{1}{3}\,\mu\right)\mathscr{B}_a = -2\,\beta^2\,\mu\,\mathscr{K}_a\ , \tag{4.51}$$

while the equations for the driving terms are

$$\dot{v}_{\langle a\rangle} + \frac{1}{3}\,\Theta\, v_a = -\frac{3}{8}\,\frac{\alpha^2}{\beta^2}\,\Theta\,\mathscr{E}_a, \tag{4.52}$$

$$\dot{\mathscr{K}}_{\langle a\rangle} + \left(\frac{1}{3}\,\Theta - \frac{1}{2}\,\frac{\mu}{\Theta}\right)\mathscr{K}_a = \frac{3}{8}\,\frac{\alpha^2}{\beta^2}\left[\dot{\mathscr{B}}_{\langle a\rangle} + \left(\frac{1}{3}\,\Theta - \frac{1}{2}\,\frac{\mu}{\Theta}\right)\mathscr{B}_a\right]. \tag{4.53}$$

Equation (4.53) follows from (4.49) using (4.52) and Maxwell's equation (4.20). In order to find solutions to the above equations, we extract from them the scalar and solenoidal (vector) parts (cf. appendix A.3) and solve them separately.

Scalar modes

In analogy with (A.20), we set $\mathscr{V} \equiv a\,\mathrm{D}^a v_a$ and $\mathcal{E} \equiv a\,\mathrm{D}^a \mathscr{E}_a$. Equation (4.52) then transforms into

$$\dot{\mathscr{V}} + \frac{1}{3}\,\Theta\mathscr{V} = -\frac{3}{8}\,\frac{\alpha^2}{\beta^2}\,\Theta\,\mathcal{E} = \frac{3}{4}\,\alpha^2\,\mu\,a\,Y\,, \tag{4.54}$$

where the last equality is a direct consequence of Maxwell's equation (4.21). Combining $\dot{Y} = -\mathscr{V}/a$ with (4.54) and using (4.17) and (4.27) together with

$$a\,\mathrm{D}^a\,\mathrm{D}^2\,\mathscr{E}_a = \mathrm{D}^2\,\mathcal{E} + \left(-\tfrac{2}{9}\,\Theta^2 + \tfrac{2}{3}\,\mu\right)\mathcal{E}\,, \tag{4.55}$$

one can show that the scalar part of the electric wave equation (4.50) reduces to

$$\ddot{\mathcal{E}} + \left(\frac{4}{3}\,\Theta - \frac{\mu}{\Theta}\right)\dot{\mathcal{E}} + \left[\frac{2}{9}\,\Theta^2 + \left(\frac{3}{4}\,\alpha^2 - \frac{1}{2}\right)\mu\right]\mathcal{E} = 0\,. \tag{4.56}$$

It is also easy to see that equation (4.54) additionally gives rise to propagation equations for $\mathscr{V}$ and Y:

$$\ddot{\mathscr{V}} + \tfrac{1}{3}\,\Theta\,\dot{\mathscr{V}} + \left[-\tfrac{1}{9}\,\Theta^2 + \left(\tfrac{3}{4}\,\alpha^2 - \tfrac{1}{6}\right)\mu\right]\mathscr{V} = 0\,, \tag{4.57}$$

$$\ddot{Y} + \tfrac{2}{3}\,\Theta\,\dot{Y} + \tfrac{3}{4}\,\alpha^2\,\mu\,Y = 0\,. \tag{4.58}$$

Hence, equations (4.56)–(4.58) all emanate from (4.54). Note that in deriving equation (4.56), the Laplacian terms cancel, due to the Coulomb field-like nature of the scalar part of electric and the velocity field.

We now specialise our considerations to a flat FLRW background with zero cosmological constant, for which $\mu = \Theta^2/3$ and $\Theta = 2/t = 3\,H_i/\tau$ always hold. In view of (4.54), which tells us that perturbations in the number density Y feed the scalar part of the electric and velocity field, we choose to determine the solutions of above equations in terms of initial conditions for the velocity modes v_{S}: $v_i \equiv v(\tau = 1)$ and $v_i' \equiv v'(1)$ (a prime stands for ∂_τ). Employing the dimensionless time variable τ, the solutions read (we drop the subscript S which indicates the scalar modes)

$$v_{(k)} = \frac{1}{\sqrt{\tau}}\left\{v_i \cos(\omega \ln\tau) + \frac{1}{\omega}\left(\frac{1}{2}v_i + v_i'\right)\sin(\omega\ln\tau)\right\}, \tag{4.59}$$

$$\mathscr{E}_{(k)} = -\frac{4\,\beta^2}{9\,\alpha^2}\,\frac{1}{\sqrt{\tau}}\left\{\left(2\,v_i + 3\,v_i'\right)\cos(\omega\ln\tau) + \frac{\left(2 - 18\,\alpha^2\right)v_i + 3\,v_i'}{6\,\omega}\,\sin(\omega\ln\tau)\right\}, \tag{4.60}$$

$$Y_{(k)} = \frac{2}{9\,\alpha^2}\,\frac{1}{L}\,\tau^{-\frac{1}{6}}\left\{\left(2\,v_i + 3\,v_i'\right)\cos(\omega\ln\tau) + \frac{\left(2 - 18\,\alpha^2\right)v_i + 3\,v_i'}{6\,\omega}\,\sin(\omega\ln\tau)\right\}. \tag{4.61}$$

Here, we introduced the Hubble length $\lambda_H \equiv 1/H_i$ as well as the physical wavelength λ associated with the constant comoving wavenumber $k = 2\pi\, a/\lambda = 2\pi\, a_i/\lambda_i$ and defined a dimensionless scale, L, via (an index i stands for evaluation at initial time t_i)

$$\frac{k}{a_i\, H_i} = 2\pi \left(\frac{\lambda_H}{\lambda}\right)_i \equiv \frac{1}{L}\,. \tag{4.62}$$

We further used the fact that for flat models the scale factor grows like $a(\tau) = a_i\, \tau^{2/3}$. The frequency of the displayed solutions is proportional to $\omega \equiv \sqrt{\alpha^2 - 1/36}$ and grows logarithmically in time. The solutions show the same high-frequency behaviour that was obtained in the previous section. It is obvious from the solution (4.61)that the velocity induced density perturbation modes $Y_{(k)}$ are completely negligible on large scales but become important at small scales $k \gg 1 \gg L$.

Vector modes

According to (A.23), we set $\tilde{\mathscr{E}}_a \equiv a\,\mathrm{curl}\, \mathscr{E}_a$ etc., and obtain from equations (4.50)–(4.53) with the help of the relation (A.17):

$$\ddot{\tilde{\mathscr{E}}}_{\langle a\rangle} - \mathrm{D}^2\tilde{\mathscr{E}}_a + \left(\Theta - \frac{\mu}{\Theta}\right)\dot{\tilde{\mathscr{E}}}_{\langle a\rangle} + \left[-\frac{1}{9}\,\Theta^2 + \left(\frac{3}{4}\,\alpha^2 + \frac{1}{3}\right)\mu\right]\tilde{\mathscr{E}}_a = -\frac{2}{3}\,\beta^2\,\mu\,\tilde{v}_a\,, \tag{4.63}$$

$$\dot{\tilde{v}}_{\langle a\rangle} + \frac{1}{3}\,\Theta\,\tilde{v}_a = -\frac{3}{8}\,\frac{\alpha^2}{\beta^2}\,\Theta\,\tilde{\mathscr{E}}_a\,, \tag{4.64}$$

$$\ddot{\tilde{\mathscr{B}}}_{\langle a\rangle} - \mathrm{D}^2\tilde{\mathscr{B}}_a + \left(\Theta - \frac{\mu}{\Theta}\right)\dot{\tilde{\mathscr{B}}}_{\langle a\rangle} + \left(-\frac{1}{9}\,\Theta^2 + \frac{1}{3}\,\mu\right)\tilde{\mathscr{B}}_a = -2\,\beta^2\,\mu\,\tilde{\mathscr{K}}_a\,, \tag{4.65}$$

$$\dot{\tilde{\mathscr{K}}}_{\langle a\rangle} + \left(\frac{1}{3}\,\Theta - \frac{1}{2}\,\frac{\mu}{\Theta}\right)\tilde{\mathscr{K}}_a = \frac{3}{8}\,\frac{\alpha^2}{\beta^2}\left[\dot{\tilde{\mathscr{B}}}_{\langle a\rangle} + \left(\frac{1}{3}\,\Theta - \frac{1}{2}\,\frac{\mu}{\Theta}\right)\tilde{\mathscr{B}}_a\right]\,. \tag{4.66}$$

Specialising to a flat FLRW background as before and performing a harmonic decomposition (see appendix A.3), these equations[2] become in terms of the dimensionless time variable τ

$$\mathscr{E}''_{(k)} + \frac{4}{3\,\tau}\mathscr{E}'_{(k)} + \left[\frac{4}{9\,L^2}\,\frac{1}{\tau^{4/3}} + \frac{\alpha^2}{\tau^2}\right]\mathscr{E}_{(k)} \;=\; -\frac{8\,\beta^2}{9\,\tau^2}\,v_{(k)}\,, \tag{4.67}$$

$$v'_{(k)} + \frac{2}{3\,\tau}\,v_{(k)} \;=\; -\frac{3\,\alpha^2}{4\,\beta^2\,\tau}\,\mathscr{E}_{(k)}\,, \tag{4.68}$$

[2]The RHS of equation (4.69) corrects a typo in the published version [53]; the correction ensures that all induced modes (4.71)–(4.74) oscillate with the same frequency.

and

$$\mathscr{B}''_{(k)} + \frac{4}{3\,\tau}\,\mathscr{B}'_{(k)} + \frac{4}{9\,L^2}\,\frac{1}{\tau^{4/3}}\,\mathscr{B}_{(k)} = -\frac{8\,\beta^2}{3\,\tau^2}\,\mathscr{K}_{(k)}\,, \tag{4.69}$$

$$\mathscr{K}'_{(k)} + \frac{1}{3\,\tau}\,\mathscr{K}_{(k)} = \frac{3\,\alpha^2}{8\,\beta^2}\left[\mathscr{B}'_{(k)} + \frac{1}{3\,\tau}\,\mathscr{B}_{(k)}\right], \tag{4.70}$$

where we have dropped the index V [denoting that the variables in equations (4.67)–(4.70) are vector harmonic coefficients], and used the dimensionless length scale L together with (4.62) for the contribution of the Laplacian in (4.63) and (4.65), respectively. These equations imply closed third-order differential equations for each of the variables, which can be solved analytically in terms of Bessel and Lommel functions modified by weak damping envelopes. If the wavelength of the mode is much greater than the horizon, eg. $\lambda_i \gg \lambda_{H_i}$, we may neglect the terms containing L^2. However, we can neglect that term in (4.67) and the system (4.69)–(4.70) throughout because the α^2–term dominates almost always except for very late times or ultra-short wavelengths (this may be confirmed also by considering the third-order equations). It follows that the above equations can then be solved analytically and the general (real) solutions are found to be

$$v_{(k)} = C_1 + \frac{1}{\sqrt{\tau}}\left\{C_2\,\cos\left(\omega\,\ln\tau\right) + C_3\,\sin(\omega\,\ln\tau)\right\}\,, \tag{4.71}$$

$$\mathscr{E}_{(k)} = -\frac{8\,\beta^2}{9\,\alpha^2}\,C_1 - \frac{2\,\beta^2}{9\,\alpha^2}\,\frac{1}{\sqrt{\tau}}\left\{(C_2 + 6\,\omega\,C_3)\cos\left(\omega\ln\tau\right) + (C_3 - 6\,\omega\,C_2)\sin\left(\omega\ln\tau\right)\right\}, \tag{4.72}$$

$$\mathscr{K}_{(k)} = \frac{1}{\tau^{1/6}}\left\{D_1\cos(\omega\,\ln\tau) + D_2\sin(\omega\,\ln\tau)\right\}\,, \tag{4.73}$$

$$\mathscr{B}_{(k)} = D_3\frac{1}{\tau^{1/3}} + \frac{8\,\beta^2}{3\,\alpha^2}\,\frac{1}{\tau^{1/6}}\left\{D_1\,\cos(\omega\,\ln\tau) + D_2\,\sin(\omega\,\ln\tau)\right\}\,, \tag{4.74}$$

where $\omega = \sqrt{\alpha^2 - 1/36}$. The forcing term $\mathscr{K}$ has only two modes because its governing equation reduces two a second-order ODE in the long-wavelength limit $L \to \infty$ when $k = 0$. Observe that the vector modes of $\mathscr{K}_a$ are linearly related to those of v_a [according to (4.49)], thus we have $a\,\Theta\,\mathscr{K} \sim v$. Therefore consistency requires the vanishing of the integration constant C_1.[3] Since we think of the electromagnetic fields as being generated by the velocity perturbations, we choose initial conditions for v and $\mathscr{K}$, respectively: namely, $v_i \equiv v(1)$, $v'_i \equiv v'(1)$ and $\mathscr{K}_i \equiv \mathscr{K}(1)$, $\mathscr{K}'_i \equiv \mathscr{K}'(1)$. For this set of initial conditions, the integration constants C_i and D_i become

$$C_2 = v_i\,, \quad C_3 = \frac{v_i + 2\,v'_i}{2\,\omega}\,, \quad D_1 = \mathscr{K}_i\,, \quad D_2 = \frac{\mathscr{K}_i + 6\,\mathscr{K}'_i}{6\,\omega}\,; \tag{4.75}$$

[3]When the exact solutions of the third-order ODEs for $k \neq 0$ are considered, the parts containing the integration constant C_1 become time-dependent.

there is no restriction for D_3 since this part represents the standard evolution of a homogeneous magnetic field in cosmology (free solution), which is not induced by the velocity perturbation.[4] The solutions for the velocity perturbation and the (expansion normalised) electric field agree then with those found in the scalar case and show the same behaviour in time, as expected. The induced expansion normalised magnetic field attains two parts, a standard decaying part and a weakly decaying oscillatory part due to the plasma. Observe that the velocity induced electromagnetic fields decay slower than their expansion normalised homogeneous counterparts, which fall off like $\tau^{-1/3}$. The plasma thus effectively acts against the expansion's tendency to dilute the fields.

4.2.5 Generated electromagnetic fields

Observe that the magnetic field (4.74) is rather slowly decaying and therefore still could play a role in some astrophysical processes under favourable conditions. Of particular interest is the question of the magnetic field strength generated by the vortical motions of the plasma alone, e.g., without keeping a standard decaying field in the background. From the solutions (4.73) and (4.74) follows directly that the induced oscillatory magnetic field is proportional to the vortical motion, that is, velocity perturbations in the plasma always induce magnetic fields.

As an illustration, assume that at some arbitrary initial time t_i ($\tau = 1$) there exist vortical velocity perturbations of amplitude $\mathscr{K}_i$ in the plasma, which are thought to occur naturally due to gravity. In the solutions (4.73) and (4.74), the terms containing the integration constant D_2 may be dropped because we have typically $\mathscr{K}_i$, $\mathscr{K}_i' \ll \omega \approx 10^{21}$. Thus, the amplitude of the induced magnetic field can roughly be approximated as

$$|\mathscr{B}(\tau)| \lesssim \frac{8}{3}\frac{\beta^2}{\alpha^2}\frac{1}{\tau^{1/6}}\mathscr{K}_i\,. \tag{4.76}$$

Restoring SI units, we find for the physical magnetic field [5], $B = \Theta\,\mathscr{B} = 2\,\mathscr{B}/t$,

$$|B(t)| \lesssim \left(\frac{m_1}{m_e}\right)\left(\frac{m_2}{m_e}\right)\left(\frac{m_e}{m_1+m_2}\right)\mathscr{K}_i\left(\frac{t_i}{t}\right)^{1/6}\frac{1}{t}\times 2\times 10^{-7}\,\mathrm{G}\,, \tag{4.77}$$

or alternatively in terms of redshift z, using $1+z = (1+z_i)/\tau^{2/3}$,

$$|B(z)| \lesssim \mathscr{K}_i\, h\left(\frac{1+z}{1+z_i}\right)^{1/4}(1+z)^{3/2}\times 10^{-24}\,\mathrm{G}\,, \tag{4.78}$$

[4] In the inhomogeneous case $k \neq 0$, this part of the solution is clearly induced by the velocity perturbations, which is readily seen by regarding the solution of the corresponding third-order ODE.

[5] We follow common practice using Gauss (instead of Tesla) as the unit of the magnetic flux.

where we also have employed $\Theta = 3\,H = 3\,H_0\,(1+z)^{3/2}$ with $H_0 = h\,(9.8\,\text{Gyrs})^{-1}$ the Hubble expansion rate and $h \simeq 0.7$ the dimensionless Hubble parameter, and neglected the mass factor.[6] Hence, velocity curl perturbations of magnitude $\mathscr{K}_i \sim 10^{-5}$ present in an e^+e^-–plasma at the time of decoupling, when Thomson scattering becomes negligible ($z \sim 1000$), would be accompanied by a magnetic field with strength $B \sim 10^{-25}\,\text{G}$. Evolving this field towards a redshift of $z \sim 100$, a redshift well within the limits before nonlinear effects become important, the field acquires a strength of $B \sim 10^{-26}\,\text{G}$. Redshifting to $z \simeq 10$, at the onset of the dynamo mechanism [35, 40–42], reduces the field strength further to $B \sim 10^{-28}\,\text{G}$. The requirements of the galactic dynamo are thus readily satisfied. [One could repeat the analysis by starting the mechanism at decoupling, say, with initially vanishing velocity perturbation and vanishing magnetic field but with $\mathscr{K}_i'/\omega \sim 10^{-5}$. The results remain the same.][7] It should be noted that the choice $\mathscr{K}_i \sim 10^{-5}$ for the initial velocity perturbation is rather conservative: given that at decoupling $t_i \sim 300'000$ yrs, this means that $(\mathsf{curl}\,v_a)_i \sim 10^{-18}\,\text{s}^{-1}$ in proper units.

It is possible to include in the presented analysis a standard decaying magnetic field ($D_3 \neq 0$ in (4.74)); for example, the remnant of a magnetic field at last scattering [133], produced at the electroweak or GUT phase transition, which decays adiabatically after decoupling in the absence of velocity perturbations. As a consequence of our first order approximation scheme, such a field would simply add to the field generated by the velocity perturbations. From a physical point of view, however, it would be natural to expect that such a field present at last scattering would be affected by the vortical motions of the plasma as well, an effect possibly boosting the initial magnetic field to a higher strength. To study such a scenario properly means to extend our approximation scheme to *second* order in the perturbation parameters. The problem might be tractable due to recent progress in treating second order perturbations in the covariant approach [135, 136] and is left for future investigations.

The above argument is also applicable for the decaying electric field (for both scalar and vector modes). Reasoning as above, the magnitude of the velocity induced electric field is approximated by

$$|\mathscr{E}(\tau)| = \frac{4}{3}\,\frac{\beta^2}{\alpha}\,v_i\,\frac{1}{\sqrt{\tau}}\,, \tag{4.79}$$

where v_i is the magnitude of the initial scalar velocity perturbation. We resort once again to SI units and find for the physical field

$$|E(t)| = \left(\frac{m_1}{m_e}\right)\left(\frac{m_2}{m_e}\right)\left(\frac{m_e}{m_1+m_2}\right) v_i \left(\frac{t_i}{t}\right)^{1/2} \frac{1}{t} \times 2 \times 10^{-3}\,\text{Vm}^{-1}, \tag{4.80}$$

[6] The mass parameter in (4.78) and (4.80) is always of the order of one for an electron-positron– or an electron-ion–plasma.

[7] The constraint concerning the initial velocity perturbation $\mathscr{K}_i$ and v_i, respectively, stems from the standard CMB results [134].

or alternatively,

$$|E(z)| = v_i\, h\, \frac{(1+z)^{9/4}}{(1+z_i)^{3/4}} \times 2\times 10^{-23}\ \mathrm{Vm}^{-1}. \tag{4.81}$$

Hence, using the same data as above, i.e., velocity perturbations of magnitude $v_i \sim 10^{-5}$ in an e^+e^-–plasma present at decoupling ($z \sim 1000$), would lead to an induced electric field of strength $E \sim 10^{-26}\,\mathrm{Vm}^{-1}$ at $z = 100$.

Comparing the energy density of the induced electric and magnetic fields, we find that $E^2/(c^2B^2) \sim 10^{-7}$, thus showing the well known fact that the electric field contribution, due to Debye shielding, is negligible in a cosmological context.

4.3 Primordial magnetic seed field amplification by gravitational waves

In this section, we explore an alternative mechanism that looks at the interaction of a pre-existing magnetic field with a gravitational wave (GW) spectrum which accompanies most inflationary scenarios. We aim to show that this interaction can produce a sufficiently large amplification of a seed field present at the end of inflation to easily meet the requirements for the dynamo [70,71] to work.

The issue of how to deal with the coupling between gravitational waves and the seed magnetic field is rather subtle. A commonly used approximation in the literature is to assume that the magnetic field is weak and that its contribution to the energy-momentum tensor is such that it does not disturb the isotropy of the FLRW background [20–23, 25]. This is done by assuming that the energy density of the magnetic field $\tilde{B}_a$ is much less than the matter energy density: $\mu_0^{-1}\,\tilde{B}^2 \ll \mu$ and that its anisotropic pressure is negligible: $\Pi_{ab} \equiv -\mu_0^{-1}\,\tilde{B}_{\langle a}\tilde{B}_{b\rangle} \approx 0$. The problem with this approximation is that it is not gauge-invariant in a strict mathematical sense, so one can therefore not guarantee that, when calculating the magnetic field which arises through its coupling with linear perturbations of FLRW (such as gravitational waves), it leads to physically meaningful results. In order to solve this problem we develop a self-consistent framework based on second-order perturbation theory, employing the methods initiated by recent work of Clarkson [135] and Clarkson *et al.* [136]. Here the seed magnetic field is treated as a homogeneous linear perturbation of the background FLRW model and couplings to gravitational degrees of freedom that arise when perturbing the background are taken to be second order in the perturbation theory. Adopting this approach allows us to write Maxwell's equations in a way that makes them manifestly gauge-invariant to second order with interaction terms that clearly describe the modes induced by the gravity wave-magnetic field interaction.

The results show that, in the presence of gravitational radiation, the magnitude of the magnetic field is amplified proportionally to the shear distortion caused by the propagating waves. Crucially, however, the gravitational boost is also proportional to the square of the field's original scale. Notice that the same results were already reported in [74], where the weak-field approximation was used; however, this was achieved at the expense of rather contrived initial conditions on the generated magnetic field as will be pointed out below. These results immediately suggests, following the argumentation of [74], that the mechanism presented here could lead to significant amplification when dealing with large scale magnetic fields. Indeed, when applied to fields of roughly 10^{-34} G spanning a comoving scale of about 10 kpc today, the mechanism leads to an amplification of up to 13 orders of magnitude. Such magnetic fields are proposed to emerge from a period of inflation as a consequence of the amplification of the standard model Z-boson field due to its coupling with the electroweak Higgs field. At preheating, the spectrum of the Z field is transferred to the hypercharge field, which remains frozen into the cosmic medium and is converted into an ordinary magnetic field during the electroweak phase transition [72, 73]. The size of the boost can easily bring these magnetic fields well within the galactic dynamo requirements, without the need for extra amplification during reheating. In fact, the enhancement is so effective that in can bring the field within the dynamo limits even within conventional cosmological models which are not dark-energy dominated. This is more easily achieved when the extra strengthening of the field, due to the adiabatic collapse of the protogalaxy, is also taken into account.

4.3.1 Setting out the stage

If we wish to study the interaction between gravitational waves and a magnetic field in a cosmological setting, we immediately face a second-order problem in perturbation theory because both the magnetic field as well as GWs are absent in the exact FLRW background, and may thus be individually regarded as first order perturbations. Using the 1+3 covariant approach [2,97,101], we therefore develop a two-parameter expansion in two smallness parameters: $\epsilon_{\tilde{B}}$ represents the magnitude of a homogeneous magnetic field and ϵ_g represents the magnitude of the GW. The magnitude of the interaction GW $\times$ magnetic field is of order $\mathcal{O}(\epsilon_{\tilde{B}}\epsilon_g)$ as is the magnitude of the in such a manner generated electromagnetic fields. However, at second-order level, only terms of order $\mathcal{O}(\epsilon_{\tilde{B}}\,\epsilon_g)$ are kept while terms of order $\mathcal{O}(\epsilon_g^2)$ and $\mathcal{O}(\epsilon_{\tilde{B}}^2)$ are discarded.

It follows that the perturbation spacetimes may be divided up and denoted as shown below:

- $\mathcal{B}$ = Exact FLRW as background spacetime, $\mathcal{O}(\epsilon^0)$;
- $\mathcal{F}_1$ = Exact FLRW perturbed by a homogeneous magnetic field $\tilde{B}$ whose energy density and curvature are neglected, $\mathcal{O}(\epsilon_{\tilde{B}})$;
- $\mathcal{F}_2$ = Exact FLRW with gravitational perturbations $\mathcal{O}(\epsilon_g)$;

- $\mathcal{S} = \mathcal{F}_1 + \mathcal{F}_2$ allows for inclusion of interactions terms of order $\mathcal{O}(\epsilon_{\tilde{B}}\epsilon_g)$.

We will generally refer to terms of order $\mathcal{O}(\epsilon_{\tilde{B}})$ and $\mathcal{O}(\epsilon_g)$ appearing in $\mathcal{F}$ as 'first-order' and to variables of mixed order $\mathcal{O}(\epsilon_{\tilde{B}}\epsilon_g)$ appearing in $\mathcal{S}$ as 'second-order'.

It should be noticed that the absence of an electric field in $\mathcal{F}_1$ does not necessarily imply that there is no electric field at all but rather that the electric field is *perturbatively* smaller than the magnetic field. This is in accordance with the standard assumption that the very early Universe was a good conductor (see, for example, [137] for an example of how this works). The inclusion of an electric field in $\mathcal{F}_1$ is possible, in principle, but would require to alter the perturbation scheme because then interactions between gravitational waves and the electric field needed to be taken into account as well. However, a more realistic way of describing the interaction between gravitational waves and electromagnetic fields should employ a multifluid description (see chapter 3 and [52, 53]), which allows for modelling the currents, but that is beyond the scope of the present study.

Having outlined the different stages we turn to review the concomitant equations. We keep them as general as possible, which will allow us to illuminate the effects of spatial geometry, cosmological constant Λ and equation of state for the matter on the interaction. We limit ourselves to the irrotational case, that is, we require the vorticity ω_{ab} to vanish throughout.

4.3.2 First-order perturbations

The FLRW background has already been reviewed in subsection 4.1.1 and we therefore head straightforward towards the stage of first-order perturbations.

The homogeneous magnetic field $\tilde{B}_a$

We assume the magnetic field $\tilde{B}_a$ to be spatially homogeneous at first order ($\mathrm{D}_a\tilde{B}_b = 0$) and thus consider the gradient of $\tilde{B}_a$ as well as the magnetic anisotropy $\Pi_{ab} = -\tilde{B}_{\langle a}\tilde{B}_{b\rangle}$ as being of second order. We presuppose that such a field was produced by some primordial process, which left a relic field on average homogeneous over a typical coherence length. Since there are no electric fields or charges in the $\mathcal{F}_1$ perturbation spacetime, the magnetic induction equation takes the form

$$\beta_a \equiv \dot{\tilde{B}}_{\langle a\rangle} + \tfrac{2}{3}\,\Theta\,\tilde{B}_a = 0\ . \tag{4.82}$$

As a result, the magnetic field scales as

$$\tilde{B}_a = \tilde{B}_a^0\left(\frac{a_0}{a}\right)^2\ , \tag{4.83}$$

where a denotes the scale factor, e.g., $\Theta = 3\,\dot{a}/a = 3\,H$, where H denotes the inverse Hubble length.

Gravitational waves

Gravitational waves are covariantly described via transverse parts of the electric (E_{ab}) and magnetic (H_{ab}) Weyl components, which are PSTF tensors [138, 139]. The pure tensor modes are transverse, obtained by switching off scalar and vector modes ($0 = \mathrm{D}_a\,\mu = \mathrm{D}_a\,\Theta = \mathrm{D}_a\,p = \omega_a = \dot{u}_a$), which results in the constraints

$$0 = \mathrm{D}^a\,\sigma_{ab} = \mathrm{D}^a\,E_{ab} = \mathrm{D}^a\,H_{ab} = H_{ab} - \mathsf{curl}\,\sigma_{ab}\ . \tag{4.84}$$

The propagation equations for these tensor modes are simply

$$\dot{\sigma}_{\langle ab\rangle} + \tfrac{2}{3}\,\Theta\,\sigma_{ab} = -E_{ab}\ , \tag{4.85}$$
$$\dot{E}_{\langle ab\rangle} + \Theta\,E_{ab} = \mathsf{curl}\,(\mathsf{curl}\,\sigma_{ab}) - \tfrac{1}{2}\,(\mu + p)\,\sigma_{ab}\ , \tag{4.86}$$

together with the background equations for Θ and μ. Since every first-order gauge-invariant (FOGI) tensor satisfies the linearised identity

$$\mathsf{curl}\,(\mathsf{curl}\,T_{ab}) = -\mathrm{D}^2\,T_{ab} + \tfrac{3}{2}\,\mathrm{D}_{\langle a}\,\mathrm{D}^c\,T_{b\rangle c} + \left(\mu + \Lambda - \tfrac{1}{3}\,\Theta^2\right)T_{ab}\ , \tag{4.87}$$

we see that the gravitational waves are completely determined by a closed wave equation for the shear, namely

$$\ddot{\sigma}_{\langle ab\rangle} - \mathrm{D}^2\,\sigma_{ab} + \tfrac{5}{3}\,\Theta\,\dot{\sigma}_{\langle ab\rangle} + \left(\tfrac{1}{9}\,\Theta^2 + \tfrac{1}{6}\,\mu - \tfrac{3}{2}\,p + \tfrac{5}{3}\,\Lambda\right)\sigma_{ab} = 0\ . \tag{4.88}$$

Evidently, in the light of (4.84) and (4.85), once the solution for the shear is known, it also determines the first-order contribution to the Weyl tensor.

4.3.3 Gravito-magnetic interaction and gauge problem

Maxwell's equations govern the interaction between GW and magnetic fields. If we require charge neutrality and neglect currents as well as the back-reaction of induced second-order magnetic fields with the shear, we obtain the EM evolution equations

$$\dot{E}_{\langle a\rangle} + \tfrac{2}{3}\,\Theta\,E_a = \mathsf{curl}\,B_a\ , \tag{4.89}$$
$$\dot{B}_{\langle a\rangle} + \tfrac{2}{3}\,\Theta\,B_a = \sigma_{ab}\,\tilde{B}^b - \mathsf{curl}\,E_a\ , \tag{4.90}$$

and the constraint equations

$$D^a\,E_a \;=\; 0\,, \tag{4.91}$$
$$D^a\,B_a \;=\; 0\,. \tag{4.92}$$

Observe that the EM fields have to be divergence-free at all orders due to disregarding vorticity effects. Moreover, the system is not gauge-invariant because it contains a mixture of second-order (E_a, $\mathsf{curl}\,E_a$, $\mathsf{curl}\,B_a$) and first-order terms (σ_{ab}), while B_a now comprises the full magnetic field (the first-order contribution plus the induced field). The situation we are interested in is the interaction between the shear σ_{ab} and the first-order magnetic field $\tilde{B}^a$, neglecting the back-reaction with the induced magnetic field. How to disentangle the different magnetic field perturbations in a consistent way?

In special relativity, the standard procedure would be to use a power series expansion of the magnetic field,

$$B^a = \epsilon_{\tilde{B}}\tilde{B}^a_1 + \epsilon_g\,\epsilon_{\tilde{B}}\,B^a_2 + \mathcal{O}(\epsilon_g^2,\epsilon_{\tilde{B}}^2)\,, \tag{4.93}$$

where the first-order field $\tilde{B}^a_1$ satisfies the magnetic induction equation (4.82). Although the insertion of this expansion into the above system yields only second-order terms, the procedure does not work in general relativity since the commutation relations for the various differential operators (cf. appendix A) could not be consistently satisfied. To stress this important point clearly, we consider the commutation relation between the (proper) time derivative and the spatial gradient applied to the magnetic field. It is evident that the case where the commutator relation is introduced after the expansion of B^a,

$$\left(D^b\,B^a\right)^{\cdot}_{\perp} = \epsilon_g\,\epsilon_{\tilde{B}}\left(D^b\,B^a\right)^{\cdot}_{\perp} = \epsilon_g\,\epsilon_{\tilde{B}}\left[D^b\,\dot{B}^a_2 - \tfrac{1}{3}\,\Theta\,D^b\,B^a_2\right]\,, \tag{4.94}$$

does not agree with the case where the linearised identity for $(D^a\,B^b)^{\cdot}$ is substituted before using the power series expansion (4.93):

$$\begin{aligned}\left(D^b\,B^a\right)^{\cdot}_{\perp} &= D^b\,\dot{B}^a - \tfrac{1}{3}\,\Theta\,D^b\,B^a + H^{bd}\,\varepsilon_{dac}\,B^c + \sigma^d{}_c\,D^c\,B_a \\ &= \epsilon_g\,\epsilon_{\tilde{B}}\left[D^b\dot{B}^a_2 - \tfrac{1}{3}\,\Theta\,D^b\,B^a_2\right] + \epsilon_{\tilde{B}}\,H^b{}_d\,\varepsilon^{dac}\,\tilde{B}^1_c\,.\end{aligned} \tag{4.95}$$

Here, $\perp$ denotes projection onto the fundamental observer's rest space. This inconsistency can only be resolved if all interaction terms are zero. It is via the commutation relations that Weyl curvature is brought in through the back door which couples to the magnetic field and thus affects the interaction. It is this feature that renders the power series procedure faulty.

The difficulty arises because the magnetic field B^a is not gauge-invariant in $\mathcal{S}$ as it does not vanish in $\mathcal{F}_1$. We therefore need to define a new second order gauge-invariant (SOGI) variable which satisfactorily describes the effects that we wish to investigate. However, a look at Maxwell's equations above reveals that $\beta_a \equiv \dot{B}_{\langle a\rangle} + \frac{2}{3}\,\Theta\,B_a$ is the sought SOGI variable which has to be used at second order instead of the magnetic field B_a. We chose to describe the interaction in terms of the variable $I_a \equiv \sigma_{ab}\,\tilde{B}^b$. Hence, Maxwell's equations can be written in truly gauge-invariant terms at second order, namely

$$\dot{E}_{\langle a\rangle} + \tfrac{2}{3}\,\Theta\,E_a = \mathsf{curl}\,B_a\ , \tag{4.96}$$

$$\beta_a + \mathsf{curl}\,E_a = I_a\ . \tag{4.97}$$

Observe that the standard constraints $0 = \mathrm{D}^a\,B_a = \mathrm{D}^a\,E_a$ imply

$$0 = \mathrm{D}^a\,\beta_a = \mathrm{D}^a\,I_a\ , \tag{4.98}$$

where the latter is equivalent to the expression $0 = \sigma_{ab}\,\mathrm{D}^a\,\tilde{B}^b$, which is automatically satisfied since spatial gradients are regarded as second-order. Clearly, if the idealised assumption of infinite conductivity is made, in particular that all electric fields are naught, Maxwell's equations reduce to $\beta_a = I_a$. In this specific case, once the solution for I_a is known, the (not gauge-invariant) generated magnetic field measured by the fundamental observer can be obtained via a standard integration of β_a. However, it is important to stress that β_a is the fundamental variable, whose deviation from zero quantifies the evolution of the magnetic field at second order in a truly gauge-invariant manner.

4.3.4 Wave equations for the main variables

Having written the central Maxwell's equations as a system of differential equations of purely SOGI variables, we now turn to the derivation of wave equations for those. We thereby make no restrictions to the spatial geometry or to the equation of state and also keep the cosmological constant; this has the advantage of letting us draw some conclusions on how these parameters influence the interaction between GWs and magnetic fields. In particular, it will turn out that neglecting the current in Maxwell's equations and at the same time requiring a homogeneous magnetic field at first-order level leads to consistent equations in spatially flat models only.

Wave equation for the interaction variable

Let us first derive the wave equation for the interaction variable $I_a = \sigma_{ab}\,\tilde{B}^b$. Even though the shear σ_{ab} belongs to $\mathcal{F}_2$ and the magnetic field $\tilde{B}_a$ to $\mathcal{F}_1$, the commutator relations do not lead to ambiguities for I_a since they manifest themselves only at third order in this case. In order to derive an evolution equation for I_a, we need the auxiliary quantity $J_a \equiv E_{ab}\,\tilde{B}^b$. Then, using equations (4.82), (4.85), (4.86) and (4.87), we arrive at the system

$$\begin{aligned} \dot{I}_{\langle a\rangle} + \tfrac{4}{3}\,\Theta\, I_a &= -J_a\,, & (4.99)\\ \dot{J}_{\langle a\rangle} + \tfrac{5}{3}\,\Theta\, J_a &= -\mathrm{D}^2 I_a + \left[\tfrac{1}{2}\,(\mu - p) + \Lambda - \tfrac{1}{3}\,\Theta^2\right] I_a\,, & (4.100) \end{aligned}$$

where we employed that spatial gradients of the magnetic field are second-order and thus $\mathrm{D}^2 I_a = \mathrm{D}^2\left(\sigma_{ab}\,\tilde{B}^b\right) = \left(\mathrm{D}^2\,\sigma_{ab}\right)\tilde{B}^b$. Eliminating the auxiliary variable J_a , the general closed wave equation for I_a is found to be

$$\ddot{I}_{\langle a\rangle} - \mathrm{D}^2 I_a + 3\,\Theta\,\dot{I}_{\langle a\rangle} + \left[\tfrac{13}{9}\,\Theta^2 - \tfrac{1}{6}\,\mu - \tfrac{5}{2}\,p + \tfrac{7}{3}\,\Lambda\right] I_a = 0\,. \tag{4.101}$$

In the case of infinite conductivity, the solution to equation (4.101) instantly yields the solution of β_a, from which the induced magnetic field measured by the fundamental observer might be obtained by integration.

Wave equation for the electric field

To derive the wave equation for the induced electric field, we first differentiate equation (4.96) and equate the result with the second-order identity

$$(\mathsf{curl}\, B_a)^{\cdot}_{\perp} = -\Theta\,\mathsf{curl}\, B_a + \mathsf{curl}\,\beta_a - H_{ab}\,\tilde{B}^b \tag{4.102}$$

to obtain

$$\ddot{E}_{\langle a\rangle} + \tfrac{5}{3}\,\Theta\,\dot{E}_{\langle a\rangle} + \left[\tfrac{4}{9}\,\Theta^2 - \tfrac{1}{3}\,(\mu + 3p) + \tfrac{2}{3}\,\Lambda\right] E_a = \mathsf{curl}\,\beta_a - H_{ab}\,\tilde{B}^b\,. \tag{4.103}$$

Secondly, using equation (4.97) to substitute for $\mathsf{curl}\,\beta_a$ above and the expansion

$$\mathsf{curl}\ (\mathsf{curl}\, E_a) = -\mathrm{D}^2 E_a - \left[\tfrac{2}{9}\,\Theta^2 - \tfrac{2}{3}\,(\mu + \Lambda)\right] E_a\,, \tag{4.104}$$

we find a forced wave equation for the induced electric field, namely

$$\ddot{E}_{\langle a\rangle} - \mathrm{D}^2 E_a + \tfrac{5}{3}\,\Theta\,\dot{E}_{\langle a\rangle} + \left[\tfrac{2}{9}\,\Theta^2 + \tfrac{1}{3}\,(\mu - 3\,p) + \tfrac{4}{3}\,\Lambda\right] E_a = K_a\,, \tag{4.105}$$

where the forcing term $K_a \equiv \mathsf{curl}\, I_a - H_{ab}\,\tilde{B}^b = \varepsilon_{cd[a}\,\mathrm{D}\,\sigma_{b]}{}^{c}\,B^b$ has no divergence. It is possible to show that the forcing term K_a, as well as $\mathsf{curl}\, I_a$ and $H_{ab}\,\tilde{B}^b$, respectively, can be found from the wave equation

$$\ddot{K}_{\langle a\rangle} - \mathrm{D}^2 K_a + \tfrac{11}{3}\,\Theta\,\dot{K}_{\langle a\rangle} + \left[\tfrac{22}{9}\,\Theta^2 - \tfrac{1}{3}\,(\mu + 9\,p) + \tfrac{8}{3}\,\Lambda\right] K_a = 0\,. \tag{4.106}$$

For example, the wave equation for $\mathsf{curl}\, I_a$ follows by taking the curl of equation (4.101) and using the expansion (4.104), while the case $H_{ab}\,\tilde{B}^b$ is similar to the derivation of the wave equation for the interaction term I_a.

It will be useful for later purposes to consider the electric field's rotation. By taking the curl of equation (4.105), we immediately arrive at

$$(\mathsf{curl}\, E_a)^{\cdot\cdot}_{\perp} - \mathrm{D}^2\,(\mathsf{curl}\, E_a) + \tfrac{7}{3}\,\Theta\,(\mathsf{curl}\, E_a)^{\cdot}_{\perp} + \left[\tfrac{7}{9}\,\Theta^2 + \tfrac{1}{6}\,(\mu - 9\,p) + \tfrac{5}{3}\,\Lambda\right] \mathsf{curl}\, E_a = \mathsf{curl}\, K_a\ . \quad (4.107)$$

Because $\mathsf{curl}\,\left(H_{ab}\,\tilde{B}^b\right) = -\mathrm{D}^2\, I_a + \left[-\tfrac{5}{18}\,\Theta^2 + \tfrac{5}{6}\,(\mu + \Lambda)\right] I_a$ holds, we note the interesting result

$$\mathsf{curl}\, K_a = \left[\tfrac{1}{18}\,\Theta^2 - \tfrac{1}{6}\,(\mu + \Lambda)\right] I_a\ . \quad (4.108)$$

That is, for a cosmological model with flat spatial sections we have $\mathsf{curl}\, K_a = 0$ and, therefore, the electric field's rotation is not induced by the interaction between magnetic fields and GWs at second-order level – the generated electric field is curl-free. As a consequence, the interaction between magnetic fields and GWs produces in a spatially flat Universe a magnetic field as in the limit of high conductivity.

However, upon closer inspection of the forcing term K_a in equation (4.105) one discovers that this term is actually identically zero because of the identity [113]

$$0 = \varepsilon^{abc}\, V_b\left(\mathrm{D}_d\, A_c{}^d\right) - 2\, V_b\, \varepsilon^{cd[a}\left(\mathrm{D}_c\, A^{b]}{}_d\right)\ , \quad (4.109)$$

which holds for any vector V_a and tensor $A_{ab} = A_{\langle ab\rangle}$ perpendicular to the congruence u_a. Thus, equation (4.108) implies that our chosen perturbative scheme is only consistent if the cosmological model is spatially flat (cf. also footnote 8 below). In essence, we see that the requirement of having a spatially homogeneous and thus curl-free magnetic field at first-order perturbation level can only be uphold when the Universe is *spatially flat*. Furthermore, the interaction between GWs and a magnetic field generates in this particular case no electric fields (at least not at second order).

The generated magnetic field

We have already pointed out that for spatially flat models the generated magnetic field follows directly from the interaction variable since in this case we have $\beta_a = I_a$. For closed or open models, however, a wave equation for β_a is needed to determine the induced magnetic field. The sought equation may be obtained by adopting the constraint equation (4.97) to equation (4.107) and substituting for curl K_a via equation (4.108), which leads to

$$\begin{aligned} \ddot{\beta}_{\langle a\rangle} - \mathrm{D}^2\beta_a &+ \tfrac{7}{3}\,\Theta\,\dot{\beta}_{\langle a\rangle} + \left[\tfrac{7}{9}\,\Theta^2 + \tfrac{1}{6}\,(\mu - 9\,p) + \tfrac{5}{3}\,\Lambda\right]\beta_a \\ &= \ddot{I}_{\langle a\rangle} - \mathrm{D}^2\,I_a + \tfrac{7}{3}\,\Theta\,\dot{I}_{\langle a\rangle} + \left[\tfrac{13}{18}\,\Theta^2 + \tfrac{1}{3}\,\mu - \tfrac{3}{2}\,p + \tfrac{11}{6}\,\Lambda\right] I_a\,. \end{aligned} \qquad (4.110)$$

Observe that for models with flat spatial sections the LHS and RHS of the above equation become identical – in agreement with the comment following equation (4.107). A slight simplification is achieved by employing equation (4.101) yielding finally a forced wave equation for β_a:

$$\ddot{\beta}_{\langle a\rangle} - \mathrm{D}^2\,\beta_a + \tfrac{7}{3}\,\Theta\,\dot{\beta}_{\langle a\rangle} + \left[\tfrac{7}{9}\,\Theta^2 + \tfrac{1}{6}\,(\mu - 9\,p) + \tfrac{5}{3}\,\Lambda\right]\beta_a = -\tfrac{2}{3}\,\Theta\,\dot{I}_{\langle a\rangle} - \left[\tfrac{13}{18}\,\Theta^2 - \tfrac{1}{2}\,(\mu + 2\,p - \Lambda)\right] I_a\,. \qquad (4.111)$$

It is evident that the variable I_a and hence the gravitational waves source fluctuations in the magnetic field variable β_a. Another variant to derive equation (4.111) consists in differentiating Maxwell's equation (4.97) twice, using equation (4.96) to get rid of the curl E_a-term and applying the corresponding commutation relations. This clearly demonstrates the consistency of our approximation scheme.

4.3.5 Generated magnetic field

After having derived the fundamental equations governing the interaction between GWs and magnetic fields as well as the generated electromagnetic fields, we turn to the task of solving them. In view of the previous findings, it is only sensible to investigate the solutions for spatially flat models. For the sake of simplicity, we restrict ourselves to models with zero cosmological constant Λ. We assume the matter to obey a barotropic equation of state, $p = w\,\mu$, with constant barotropic index w. We will further make frequent use of the dimensionless time variable τ introduced in subsection 4.1.2.

Since we are merely considering universes with *flat* spatial geometry, the induced magnetic field can be found by integrating over β_a. To this end, it suffices to solve for the interaction variable I_a. A standard harmonic decomposition [31, 131, 132] is used to take care of the Laplacian operator. We expand the shear $\sigma_{ab} = \sum_k \sigma^{(k)}\,Q^{(k)}_{ab}$ in pure tensor harmonics, where as usual $\dot{Q}^{(k)}_{\langle ab\rangle} = 0$ and $\mathrm{D}^2\,Q^{(k)}_{ab} = -(k^2/a^2)\,Q^{(k)}_{ab}$ hold. Moreover, each gravitational wave mode is associated with the physical wave length $\lambda_{\mathrm{GW}} = 2\,\pi\,a/k$. Since the magnetic field in $\mathcal{F}_1$ obeys curl $\tilde{B}_a = 0$, it follows that $\mathrm{D}^2\,\tilde{B}_a = -\mathrm{curl}\,(\mathrm{curl}\,\tilde{B}_a) = 0$ and therefore that the expansion of the magnetic field $\tilde{B}_a = \sum_n \tilde{B}^{(n)}\,Q^{(n)}_a$ in pure vector (solenoidal) harmonics reduces to $\tilde{B}_a = \tilde{B}^{(0)}\,Q^{(0)}_a$, where $\tilde{B}^{(0)} = \tilde{B}_i\,(a_i/a)^2$ and the index i stands for evaluation at some initial time t_i. This just means that the magnetic field $\tilde{B}^a$ is spatially constant, e.g., in agreement with the assumption of homogeneity [8]. Of course, the solenoidal harmonics also obey the relations $\dot{Q}^{(n)}_{(a)} = 0$ and $\mathrm{D}^2\, Q^{(n)}_a = -(n^2/a^2)\, Q^{(n)}_a$. Perturbations in $\mathcal{S}$ are conveniently decomposed with the vector harmonics [9] $V^{(\ell)}_a \equiv Q^{(k)}_{ab}\, Q^b_{(n)}$, which are readily verified to fulfill the standard requirements $\dot{V}^{(\ell)}_{(a)} = 0$ and $\mathrm{D}^2\, V^{(\ell)}_a = -(\ell^2/a^2)\, V^{(\ell)}_a$, where the wavenumber ℓ satisfies $\ell^2 = (k_a + n_a)(k^a + n^a)$. Because the magnetic field in $\mathcal{F}_1$ has got only the zero mode in our investigation, the wavenumber ℓ coincides with the wavenumber k of the shear.

Using the unified time variable τ and the harmonics explained above, we transform the wave equation (4.101) for the interaction variable I_a into an ordinary differential equation:

$$\frac{9}{4}\,(1+w)^2\, I''_{(\ell)} + \frac{27\,(1+w)}{2\,\tau} I'_{(\ell)} + \left[\left(\frac{\ell}{a_i\, H_i}\right)^2 \tau^{-\frac{4}{3(1+w)}} + \frac{25 - 15\,w}{2\,\tau^2}\right] I_{(\ell)} = 0\,, \qquad (4.112)$$

where a prime means differentiation with respect to τ. Initial conditions are chosen as follows:

$$I_{(\ell)}(t_i) \;=\; \sigma_{(k)}(t_i)\, \tilde{B}_i\,, \qquad (4.113)$$

$$I'_{(\ell)}(\tau = 1) \;=\; \tilde{B}_i \left[\sigma'_{(k)}(1) - \frac{4}{3\,(1+w)}\sigma_{(k)}(1)\right]\;; \qquad (4.114)$$

here, $\dot{\sigma}_{(k)}(t_i) = (3/2)\, H_i\, (1+w)\, \sigma'_{(k)}(1)$ was used and $\tilde{B}_i$ is the initial amplitude of the first-order magnetic field. For every mode k we have initially $\sigma(t_i) = \sigma_i$ and $\sigma'(1) = \sigma'_i$.

Long-wavelength limit

In the long-wavelength limit ($\ell \to 0$), that is, when the wavelength of the shear perturbation tends towards infinity, the solution of equation (4.112) is easily found to be

$$I^{(0)}(\tau) = C_1\, \tau^{-\frac{10}{3(1+w)}} + C_2\, \tau^{\frac{-5+3\,w}{3(1+w)}}\,, \qquad (4.115)$$

where C_1 and C_2 are constants of integration. If the initial conditions (4.113)–(4.114) are chosen, the corresponding integration constants are

$$C_1 \;=\; \frac{(-5+3\,w)\, I_{(\ell)}(1) - 3\,(1+w)\, I'_{(\ell)}(1)}{5+3\,w}\,, \qquad (4.116)$$

$$C_2 \;=\; \frac{10\, I_{(\ell)}(1) + 3\,(1+w)\, I'_{(\ell)}(1)}{5+3\,w}\,. \qquad (4.117)$$

[8] In light of the commutator relation (4.104), which holds for $\tilde{B}^a$ in $\mathcal{F}_1$, the requirement of $\mathrm{D}_a\, \tilde{B}_b = 0$ and therefore $\mathrm{curl}\, \tilde{B}_a = 0$ is only consistent for a spatially flat Universe—in an open or closed Universe, a current is needed to uphold the magnetic field's homogeneity.

[9] It should be kept in mind that all above introduced harmonics are exclusively defined on the background FLRW spacetime.

We remark that this solution is in agreement with the result obtained by multiplying the first-order magnetic field (4.83) with the long-wavelength solution of the shear equation (4.88). Whence, the total magnetic field in the presence of long-wavelength GWs is

$$B^{(0)}(\tau) = \tilde{B}_i\, \tau^{-\frac{4}{3(1+w)}} \left[1 - \frac{C_1}{\tilde{B}_i\, H_i}\, \frac{2}{3\,(1-w)} \left(\tau^{\frac{-1+w}{1+w}} - 1\right) + \frac{C_2}{\tilde{B}_i\, H_i}\, \frac{1}{1+3\,w} \left(\tau^{\frac{2+6\,w}{3(1+w)}} - 1\right)\right] , \tag{4.118}$$

where $\tilde{B}_i$ is the magnitude of the first-order magnetic field interacting with the GWs at initial time t_i, and it is required for physical reasons that the induced magnetic field vanishes initially. We stress that the interaction always leads to an amplification of the magnetic field for any physically acceptable choice of equation of state because of the growing contribution in the second line of equation (4.118).

Let us look at some important special cases. For the sake of simplicity, we take $I'_{(\ell)}(1) = 0$ for granted. In the matter-dominated era, where the matter is accurately described as dust, $w = 0$ and $a = a_i\, \tau^{2/3}$, this yields for the magnetic field mode

$$B^{(0)}_{\mathrm{Dust}}(a) = \tilde{B}_i \left(\frac{a_i}{a}\right)^2 \left[1 + \frac{2}{3}\, \frac{\sigma_i}{H_i} \left\{\left(\frac{a_i}{a}\right)^{3/2} - 1\right\} + \frac{2\,\sigma_i}{H_i} \left\{\frac{a}{a_i} - 1\right\}\right] , \tag{4.119}$$

whereas for a radiation-dominated era, where $w = 1/3$ and $a = a_i\, \tau^{1/2}$, the magnetic field mode is

$$B^{(0)}_{\mathrm{Rad}}(a) = \tilde{B}_i \left(\frac{a_i}{a}\right)^2 \left[1 + \frac{2}{3}\, \frac{\sigma_i}{H_i} \left\{\frac{a_i}{a} - 1\right\} + \frac{5}{6}\, \frac{\sigma_i}{H_i} \left\{\left(\frac{a}{a_i}\right)^2 - 1\right\}\right] . \tag{4.120}$$

Whence, in the long-wavelength limit the amplification depends mainly on the scale factor and the size of the initial GW distortion relative to the horizon $(\sigma/H)_i$.

General case with $\ell \neq 0$

The general solution to the interaction equation (4.112) is:

$$I_{(\ell)}(\tau) = \tau^{\frac{-5+w}{2(1+w)}} \left[D_1\, J_1\,(x,\, y) + D_2\, J_2\,(x,\, y)\right] , \tag{4.121}$$

where D_1, D_2 are integration constants and $J_1(x, y)$, $J_2(x, y)$ denote Bessel functions of the first and second kind, respectively, whose arguments are determined by

$$x \equiv \frac{3\,w + 5}{2\,(1 + 3\,w)} , \qquad y \equiv \frac{\ell}{a_i\, H_i}\, \frac{2}{1 + 3\,w}\, \tau^{\frac{1+3\,w}{3(1+w)}} . \tag{4.122}$$

Observe that in the limit of long wavelengths, $\ell \to 0$, the solution (4.115) is recovered. The not gauge-invariant, generated magnetic field can be calculated from the solution (4.121) in an analytic fashion for every barotropic parameter w. We will state here only the total magnetic field solution in the case of dust and radiation, respectively. For dust, where $w = 0$ and $a = a_i \tau^{2/3}$, the full magnetic field is

$$B^{(\ell)}_{\rm Dust}(a) = \tilde{B}_i \left(\frac{a_i}{a}\right)^2 \left[1 + \frac{3}{4\pi^2}\left(\frac{\lambda_{\rm GW}}{\lambda_{\rm H}}\right)^2_i \left(\frac{\sigma_i}{H_i} + \frac{\sigma'_i}{2H_i}\right) + \mathcal{O}(a^{-1})\right] , \tag{4.123}$$

while for radiation, where $w = 1/3$ and $a = a_i \tau^{1/2}$, the total magnetic field modes obey

$$B^{(\ell)}_{\rm Rad}(a) = \tilde{B}_i \left(\frac{a_i}{a}\right)^2 \left[1 + \frac{3}{4\pi^2}\left(\frac{\lambda_{\rm GW}}{\lambda_{\rm H}}\right)^2_i \left(\frac{\sigma_i}{H_i} + \frac{2\sigma'_i}{3H_i}\right) + \mathcal{O}(a^{-1})\right] . \tag{4.124}$$

Here, we introduced the gravitational wavelength $\lambda_{\rm GW} = 2\pi a/k$ and the Hubble length $\lambda_{\rm H} = 1/H$. The undisplayed remainders $\mathcal{O}(a^{-1})$ in the expressions above contain oscillating functions which decay at least as fast as the inverse scale factor a^{-1}. Note that when the long-wavelength limit of the full solutions above is taken, the findings (4.119) and (4.120) are rediscovered. The results (4.123)–(4.124) clearly show how the generated magnetic field depends on the initial conditions and that the late time behaviour is almost identical for both dust and radiation. It should be noted that the interaction can only be effective if the wavelength of the GWs matches the size of the magnetic field region, $\lambda_{\rm GW} \sim \lambda_{\tilde{B}}$: in the case of $\lambda_{\rm GW} \gg \lambda_{\tilde{B}}$ the magnetic field cannot be physically affected by the GWs, while for $\lambda_{\rm GW} \ll \lambda_{\tilde{B}}$ the effect becomes negligible due to its quadratic dependence on $\lambda_{\rm GW}$. If we divide the findings (4.123)–(4.124) through the energy density of the background radiation, the dominant contribution can be summarised as follows,

$$\frac{B}{\mu_\gamma^{1/2}} \simeq \left[1 + \frac{1}{10}\left(\frac{\lambda_{\tilde{B}}}{\lambda_{\rm H}}\right)^2_i \left(\frac{\sigma}{H}\right)_i\right]\left(\frac{\tilde{B}}{\mu_\gamma^{1/2}}\right)_i \quad \text{if} \quad \lambda_{\rm GW} \sim \lambda_{\tilde{B}} < \infty , \tag{4.125}$$

where the wavenumber indices have been suppressed and $\sigma'_i = 0$ was assumed. A significant amplification of the original magnetic field can thus be achieved for super-horizon gravitational waves in the resonant case, $\lambda_H \ll \lambda_{\rm GW} \sim \lambda_{\tilde{B}} < \infty$, if the shear anisotropy σ/H is not too small. Note that a result almost identical to (4.125) was obtained in [74], wherein the factor 1/10 is replaced by 10 instead. However, our result holds for *any* finite gravitational wavelength, $\lambda_{\rm GW} \sim \lambda_{\tilde{B}}$, while the result in [74] assumes $\lambda_H \ll \lambda_{\rm GW}$. Moreover, [74] used somewhat contrived initial conditions, namely a non-vanishing initial generated magnetic field (see equation (19) in [74]), leading to an abrupt amplification of the field whereas we chose initial conditions such that there is no generated field when the interaction kicks in. In particular, with the mechanism presented in [72, 73] in mind, the initial field $\tilde{B}$ is thought to be produced during inflation, after which it starts to interact with the inflationary gravitational wave spectrum.

4.3.6 Application

In order to estimate the amplification of the seed field due to the interaction with GWs we reproduce in this subsection the analysis presented in [74] using the same parameter values. In this way, we can easily compare the results derived within our gauge-invariant method and the ones obtained within the weak-field approximation later on. For convenience, we use natural units in this subsection.

Given that the evolution of the (spatially flat) Universe is dominated by a dark energy component such as a cosmological constant or quintessence, the minimum seed required for the dynamo mechanism to work is of the order of $10^{-30}\,$G at the time of completed galaxy formation and coherent on a scale at least as large as the largest turbulent eddy, roughly ~ 100 pc [70]. Such a collapsed magnetic field corresponds to a field $\tilde{B}$ of $\sim 10^{-34}\,$G with coherence length $\lambda_{\tilde{B}} \sim 10\,$kpc on a comoving scale if the field remains frozen into the cosmic plasma from the epoch of radiation decoupling to galaxy formation. Its field strength compared to the energy density of the background radiation, μ_γ, gives raise to the ratio $\tilde{B}/\mu_\gamma^{1/2} \sim 10^{-29}$, which stays constant as long as the magnetic flux is conserved and the magnetic field is frozen into the cosmic medium.

During inflation, the Hubble parameter H remains constant and is taken to be $H \sim 10^{13}\,$GeV [72, 73]. The scale of the magnetic field therefore implies $\lambda_{\tilde{B}}/\lambda_H \sim 10^{20}$ at the end of inflation, at which time the magnetic field strength was $\tilde{B} \sim 10^{22}\,$G [72, 73]. A general prediction of all inflationary scenarios is the production of large scale gravitational waves whose energy density is roughly

$$\mu_{\mathrm{GW}} \simeq \left(\frac{m_{\mathrm{Pl}}}{\lambda_{\mathrm{GW}}}\right)^2 \left(\frac{H}{m_{\mathrm{Pl}}}\right)^2 . \tag{4.126}$$

Here, λ_{GW} denotes the wavelength of the GWs and $m_{\mathrm{Pl}} = G^{-1/2} = 1.22 \times 10^{19}\,$GeV the Planck mass with G Newton's constant (see, for example, [140–143]). This implies for the induced shear anisotropy

$$\left(\frac{\sigma}{H}\right)_i \simeq \left(\frac{\lambda_H}{\lambda_{\mathrm{GW}}}\right)_i \left(\frac{H}{m_{\mathrm{Pl}}}\right) , \tag{4.127}$$

where the suffix i indicates the end of the inflationary epoch. Typical inflationary models predict $H/m_{\mathrm{Pl}} \sim 10^{-6}$, which lies comfortably within the bound $H/m_{\mathrm{Pl}} \lesssim 10^{-5}$ stemming from the quadrupole anisotropy of the CMB.

The interaction of such a primordial magnetic field with GWs produced by inflation leads to a substantial amplification of the former. Resorting to our result (4.125) and applying (4.127), we find for the magnetic field

$$\frac{B}{\mu_\gamma^{1/2}} \simeq \left[1+\frac{1}{10}\left(\frac{\lambda_{\tilde{B}}}{\lambda_{\rm H}}\right)_i\left(\frac{H}{m_{\rm Pl}}\right)\right]\left(\frac{\tilde{B}}{\mu_\gamma^{1/2}}\right)_i . \qquad (4.128)$$

Substituting $(\lambda_{\tilde{B}}/\lambda_H)_i \sim 10^{20}$ and $H/m_{\rm Pl} \sim 10^{-6}$ into the above expression, we obtain that GWs amplify the original magnetic field as much as 13 orders of magnitude. The mechanism thus brings an inflationary seed such as in [72,73] up to $\sim 10^{-21}$ G, which is comfortably within the requirements of the galactic dynamo mechanism [70]. In universes with zero cosmological constant, the minimum seed for the dynamo has to be raised from $\sim 10^{-30}$ G to $\sim 10^{-23}$ G [71]. Clearly, the efficiency of the amplification process makes it possible to meet these requirements as well, in particular when the field's enhancement during preheating and protogalactic collapse are taken into account.

We stress that the efficiency of the mechanism depends crucially on the ratio between the coherence length $\lambda_{\tilde{B}}$ of the initial magnetic field and the initial size of the horizon λ_H. This ratio, however, disappears when the long-wavelength limit (for which $\lambda_{\rm GW} \gg \lambda_{\tilde{B}}$) is taken (see subsection 4.3.5). Even though the solutions (4.119), (4.120) show a growth proportional (quadratic) to the scale factor, the factor of proportionality $(\sigma/H)_i$ [$\sim 10^{-26}$ in the example] is far too small in order to achieve an effective amplification. Whence, the interaction between GW and on average homogeneous magnetic fields is completely negligible in the limit of long-wavelength gravity waves.

4.3.7 Comparison with the weak-field approximation and discussion

The interaction between GWs and magnetic fields in the cosmological setting has recently been investigated in [74], where the so-called weak-field approximation [20–23, 25] was used. Therein one allows for a weak magnetic test field $\tilde{B}_a$ in the background, whose energy density, anisotropic stress and spatial dependence have negligible impact on the background dynamics: $\tilde{B}^2 \ll \mu$ and $\pi_{ab} = -\tilde{B}_{\langle a}\tilde{B}_{b\rangle} \simeq 0 \simeq {\rm D}_a\tilde{B}_b$ to zero order. In order to isolate linear tensor perturbations, it is necessary to impose ${\rm D}_a\tilde{B}^2 = 0 = \varepsilon_{abc}\tilde{B}^b\,{\rm curl}\,\tilde{B}^c$ in addition to the standard constraints $\omega_a = 0 = {\rm D}_a\mu = {\rm D}_a p$ associated with pure perfect fluid cosmologies. In the weak-field approximation, the main equations governing the induced magnetic field arising from the interaction between a weak background magnetic field $\tilde{B}^a$ and GWs were derived in [74] for the case of a spatially flat Universe with vanishing cosmological constant Λ and a barotropic equation of state $p = w\,\mu$:

$$\ddot{B}_{(\ell)} + \frac{5}{3}\Theta\dot{B}_{(\ell)} + \left[\frac{1}{3}(1-w)\Theta^2 + \frac{\ell^2}{a^2}\right]B_{(\ell)} = 2\left(\dot{\sigma}_{(k)} + \frac{2}{3}\Theta\sigma_{(k)}\right)\tilde{B}_i^{(n)}\left(\frac{a_i}{a}\right)^2 , \qquad (4.129)$$

where the GWs are determined by the shear wave equation

$$\ddot{\sigma}_{(k)} + \frac{5}{3}\,\Theta\,\dot{\sigma}_{(k)} + \left[\frac{1}{6}\,(1-3\,w)\,\Theta^2 + \frac{k^2}{a^2}\right]\sigma_{(k)} = 0\;. \tag{4.130}$$

Here, the shear is harmonically decomposed as $\sigma_{ab} = \sigma_{(k)}\,Q^{(k)}_{ab}$, while for the induced magnetic field $B^{(\ell)}_a = B_{(\ell)}\,V^{(\ell)}_a$ with $V^{(\ell)}_a = Q^{(k)}_{ab}\,Q^b_{(n)}$ was adopted. The background magnetic field evolves as $\tilde{B}_a = \tilde{B}^i_a\,(a_i/a)^2$ and $\tilde{B}^i_a = \tilde{B}^i_{(n)}\,Q^{(n)}_a$ is assumed.

We want to compare our results with the corresponding ones in the weak-field approximation. For simplicity, we restrict ourselves here to the case of dust. As pointed out above, the only allowed magnetic wavenumber for the interacting magnetic field is $n = 0$, when $\mathrm{D}_a\,\tilde{B}_b$ and thus $\mathsf{curl}\,\tilde{B}_a$ are required to vanish in the background due to homogeneity, which leads to $\ell = k$. The solution for the generated magnetic field in the weak-field approximation, e.g., equation (21) in [74], however, is not applicable in the limit $n \to 0$. This can be traced back to the choice for the initial conditions for the generated magnetic field made by the authors of [74] when solving equations (4.129)–(4.130), see equation (19) in [74].

In what follows below, we solve equations (4.129)–(4.130) again, including the full solution for the shear instead of merely keeping the dominant part as done in [74]. We specify the initial conditions by choosing for every mode k of the shear $\sigma_{(k)}(a_i) = \sigma_i$, $\dot{\sigma}_{(k)}(a_i) = 0$ and for every mode $\ell = k$ of the generated magnetic field $B_{(\ell)}(a_i) = 0 = \dot{B}_{(\ell)}(a_i)$. The solution, including the background field, for an arbitrary wavenumber k of the shear has the structure

$$B^{(\ell)}_{\mathrm{Dust}}(a) = \tilde{B}_i\left(\frac{a_i}{a}\right)^2\left[1 + \frac{\sigma_i}{H_i}f\left(\sqrt{a};k\right) + \mathcal{O}\left(a^{-\frac{1}{2}}\right)\right]\;, \tag{4.131}$$

where the function $f\left(\sqrt{a};k\right)$ is built of several oscillatory terms with amplitude $(\lambda_{\mathrm{GW}}/\lambda_{\mathrm{H}})^2_i$ at most and the un-displayed part falls off at least as fast as $a^{-1/2}$. If this is compared with our result (4.123), one observes that it differs having another time behaviour. More strikingly, however, is that now the term $f\left(\sqrt{a};k\right)$ not only amplifies the seed field but also grows like $\sqrt{a}$ in the limit $k/(a_i\,H_i) \ll 1$. This is in clear contrast to the gauge-invariant result (4.123), where the seed undergoes amplification but then still decays adiabatically like a^{-2}. On the other hand, in the limit of long-wavelength GWs ($k \to 0$), the exact full solution is now

$$B^{(0)}_{\mathrm{Dust}}(a) = \tilde{B}_i\left(\frac{a_i}{a}\right)^2\left[1 + \frac{\sigma_i}{H_i}\left\{\frac{20}{3} - 14\left(\frac{a}{a_0}\right)^{1/2} + \frac{36}{5}\left(\frac{a}{a_i}\right) + \frac{2}{15}\left(\frac{a}{a_i}\right)^{3/2}\right\}\right]\;. \tag{4.132}$$

Again, we obtain a solution whose time behaviour differs from that found in (4.119). However, the weak-field solutions agree with our presented solutions when merely the dominant part of the solutions is considered, at least in the examples regarded above. The reason for the fact that the solutions obtained within the weak-field approximation are in general not congruent with the solutions from the presented gauge-invariant approach has to be sought in the non-gauge-invariance of the weak-field approximation, where the magnetic field $\tilde{B}_a$ interacting with the GWs is treated as a weak background field. However, gauge-invariance requires $\tilde{B}_a$ to vanish strictly in the FLRW background. We remind the reader once more that our procedure solves firstly for the gauge-invariant variable $\beta_a = \dot{B}_{\langle a \rangle} + \frac{2}{3}\,\Theta\, B_a$, from which the magnetic field B_a as measured in the frame of reverence u^a can then subsequently be found.

A further important remark concerns the issue of conductivity. We have seen earlier that, within our assumptions and for spatially flat Universes, the gravito-magnetic interaction leads to an induced magnetic field whatsoever the conductivity of the cosmic medium is. This is so because the interaction does not generate rotational electric field modes which might affect the magnetic field. In the weak-field approximation, however, the situation is completely different. If one supposes the conductivity of the cosmic medium that high that electric fields are quickly dissipated away, yielding a curl-free induced magnetic field, then equation (4.129) no longer applies and one simply has to use

$$\dot{B}_{(\ell)} + \frac{2}{3}\,\Theta B_{(\ell)} = \sigma_{(k)}\, \tilde{B}_i^{(0)} \left(\frac{a_i}{a}\right)^2 \tag{4.133}$$

instead, while the equation for the shear (4.130) is unaltered. This means that the weak-field approximation produces identical results as our gauge-invariant perturbation approach in the high conductivity limit, and only there. It is hence evident that in the weak-field approximation the conductivity of the cosmic medium has a crucial bearing on the generated magnetic field, in stark contrast to the result of our gauge-invariant approach (see also [144]).

4.4 Summary

After briefly reviewing on cosmic magnetic fields, we linearised in section 4.3 the multi-component fluid equations derived in chapter 3 about a FLRW Universe and then applied them to an Einstein-de Sitter (EdS) Universe. Our matter field is an ion-electron plasma with zero average pressure (which made the EdS model a suitable background). We showed how, when there is a residual net charge, the presence of an electric field can lead to velocity perturbations even when the latter are originally absent. We also found that velocity distortions can source inhomogeneities in the number density, and therefore in the energy density, of the fluid. In fact, our linear equations reveal the presence of an extra mode, representing high-frequency plasma oscillations, in addition to the standard growing and decaying modes. This mode is likely to be important on scales considerably smaller than the Hubble radius and therefore is of little importance as far as structure formation is concerned. It does illustrate, however, interesting small-scale physics that could play a role during the latter stages of galaxy formation.

We also applied our covariant equations to look into the generation of electromagnetic fields due to velocity perturbations in a plasma. The corresponding wave equations, with the velocity distortions playing the role of a source, were given, and they were solved for both scalar and vector perturbations. The solutions show high-frequency behaviour typical of a plasma. In the matter-dominated era, we found a net magnetic field with a magnitude $B \sim 10^{-30}\,\mathrm{G}$ today, due to the non-vanishing vorticity of the velocity perturbations in the two-component fluid. Since velocity perturbations occur naturally in the early Universe, this magnetic field thus represents a suitable candidate for a seed which could be amplified by the galactic dynamo. Moreover, the model is self-consistent and does not invoke any other physics than GR and general relativistic electrodynamics.

There are a number of ways to generalise the discussion presented in section 4.3. One possibility is to include thermal effects which occur in a photon-baryon plasma giving a non-zero acceleration to first order. This may lead to possible coupling between acoustic and plasma oscillations. The inclusion of a radiation gas and Thomson scattering would bring the model to a form more accurately describing the plasma at pertinent epochs, as well as bringing it closer to Harrison's protogalaxy model [92, 93]. In addition, one could apply the ponderomotive force concept between neutrinos and electrons (see [127, 145] and references therein) to cosmology in a covariant context. In this picture, derived from the theory of electroweak interactions, there is an effective interaction between electrons and neutrinos due to density gradients in either species. For instance, the (non-relativistic) force density exerted by neutrinos on the electrons is given by [145]

$$f^a_{(e)} = -\frac{1}{\sqrt{2}}\left(1 + 4\,\sin^2\theta_W\right) G_F\, n_e\, \mathrm{D}^a n_\nu \;, \tag{4.134}$$

where θ_W is the Weinberg angle and G_F is the Fermi constant. The expression (4.134), together with its neutrino counterpart, could act as a driving force for density fluctuations in the early Universe, possibly giving a neutrino signature in the CMB, having an alternating structure as compared to the regular CMB spectrum. The neutrino-driven instability discussed by Silva *et al.* [146] (see also Ref. [147] for the covariant relativistic form of the same equations), using kinetic theory, could in principle be transferred to a gauge-invariant covariant formalism, suitable for cosmological applications (see also [148]), but this is left for future studies.

In section 4.4, we have investigated the properties of magnetic fields in the presence of cosmological gravitational waves, using a two-parameter perturbation scheme. Using proper second-order gauge-invariant variables (SOGI), we were able to obtain results in terms of clearly defined quantities, with no ambiguity concerning the physical validity of the variables. The full set of equations determining the evolution of the gravitational waves and the generated electromagnetic fields was presented, and the integration shows growing behaviour of the induced magnetic field due to the interaction of a 'background' magnetic field with gravitational waves. The results were discussed in different fluid regimes, in particular dust and radiation, and it was established that the dominant contribution to the magnetic field is the same in both fluid regimes. Indeed, the magneto-GW interaction was found to boost super-horizon magnetic fields existing at the end of the inflationary era, like those predicted in [72,73], by more than 10 orders of magnitude bringing them easily within the required dynamo limits. Moreover, we further recalculated the induced magnetic field employing the weak-field approximation, thereby extending previous results in [74], and compared the solutions with ours derived in a gauge-invariant manner using SOGI variables. It was found that there is a significant difference in the growth behaviour of the magnetic field when SOGI variables are used as compared to the case of a weak-field approximation scheme. While the two methods agree in the limit of high conductivity, they seem to be compatible otherwise only in the limit of infinitely long-wavelength gravitational waves when the dominant part of the solution is considered.

Chapter 5

Scalar field and electromagnetic perturbations on LRS class II spacetimes

5.1 Overview

The covariant 1+3 approach, discussed in chapter 2, has proven to be a powerful tool in relativistic cosmology, especially through its application of the gauge-invariant, covariant perturbation formalism (see [2] and references therein). This perturbation formalism works extremely well in cosmological applications when the background model is homogeneous and isotropic, that is of Friedman-Lemaître-Robertson-Walker (FLRW) type. However, if the spacetime under consideration has less symmetry, the 1+3 approach is no longer ideally suited because its splitting in 'time' and 'space' relative to the fundamental observer is not sensitive to another preferred direction apart from 'time'. The description of spacetime through covariant quantities, defined in the observer's rest-space, is simply blind to a second distinguished direction. The 1+1+2 approach [34], outlined in chapter 2, remedies this by slicing the 'space' further into 'sheets' orthogonal to a second preferred direction. Analogously, the quantities of the rest-space are further covariantly split in such a way that obtained quantities still have a clear geometrical or physical meaning. The 1+1+2 approach thus naturally extends the 1+3 approach and keeps its many benefits.

In this chapter, we illustrate the 1+1+2 formalism with its application to scalar and electromagnetic perturbations on spacetimes which are *locally rotationally symmetric* (LRS) [149, 150] possessing a continuous isotropy group at each point and hence a multiply-transitive isometry group. Since LRS spacetimes exhibit locally a preferred spatial direction, the 1+1+2 formalism is ideally suited for a covariant description of these spacetimes, yielding a complete characterisation in terms of covariant scalar quantities which have either a clear physical or direct geometrical meaning. Such a covariant classification of LRS perfect fluid spacetime geometries has already been presented in [112], whereas orthonormal frame methods have been employed in [151–153]. We will include LRS imperfect fluids in our treatment but the emphasis will be mainly on LRS class II spacetimes, that is LRS spacetimes without vorticity terms such that their sheets become a genuine 2-surfaces.

Schwarzschild black hole perturbations are well understood and it has been known for a long time that they are all governed by master equations known as the Regge-Wheeler equation [154], a Schrödinger equation with a slightly different potential for scalar, electromagnetic and gravitational perturbations, respectively [33, 95, 96, 154–156], due to the differing spins of the perturbing fields. Using the 1+1+2 formalism, we find the covariant generalisation of the Regge-Wheeler equation for scalar perturbations, as described by the Klein-Gordon equation, for all LRS spacetimes, and present the generalised Regge-Wheeler equation for electromagnetic perturbations, governed by Maxwell's equations, in the case of LRS class II spacetimes. We discuss the resulting wave equations in detail for Schwarzschild and Vaidya spacetimes [12, 157], the latter being closely related to the former by having a non-increasing mass. We also describe (source-free) electromagnetic perturbations on the Schwarzschild geometry by a linear first order system of ODEs plus an algebraic constraint, once spherical and time harmonics have been introduced. This allows for a quick determination of some electromagnetic field configurations, such as the solutions describing a static magnetic dipole or a static uniform magnetic field at infinity. We also discuss the key wave equations for two classes of dust Universe models. Specifically, we have a look at the inhomogeneous Lemaître-Tolman-Bondi (LTB) spacetimes [12, 158–160], and the spatially homogeneous Kantowski-Sachs (KS) spacetime [12, 161, 162] in the spherically symmetric case.

5.2 LRS class II spacetimes

In this section, we discuss LRS class II spacetimes in terms of the 1+1+2 formalism. The discussion follows in parts van Elst & Ellis [112], but generalises their treatment of LRS class II perfect fluids towards imperfect fluids employing a new, somewhat streamlined notation.

The 1+1+2 formalism is ideally suited to study spacetimes which exhibit local rotational symmetry (LRS) because they have a unique preferred spatial direction at each point, defined covariantly, for example, by an eigendirection of a degenerate rate of the shear tensor field or by a vorticity vector field. This direction constitutes a local axis of symmetry - all observations are identical under rotations about it and are the same in all spatial directions perpendicular to it [149,150]. Hence, after a 1+1+2 split, if the spacelike unit vector field n^a ($n^a n_a = 1$, $n^a u_a = 0$) is chosen parallel to the axis of symmetry, all covariantly defined 1+1+2 vectors and tensors must vanish due to the LRS symmetry. It follows that LRS spacetimes may be covariantly characterised by scalar quantities (modulo equations of state for the matter variables), namely

$$\textbf{LRS}: \qquad \{\mathcal{A}, \Theta, \phi, \xi, \Sigma, \Omega, \mathcal{E}, \mathcal{H}, \mu, p, Q, \Pi, \Lambda\} \,. \tag{5.1}$$

The variables (5.1) thus *fully* describe LRS spacetimes and are what is solved for in the 1+1+2 approach. For LRS class II, one requires in addition that the vorticity terms vanish, $\Omega = 0 = \xi$, which further constrains the magnetic Weyl curvature $\mathcal{H}$ to vanish [117]. Thus, a substantially smaller set of scalars describes LRS class II spacetimes:

$$\textbf{LRS class II}: \qquad \{\mathcal{A}, \Theta, \phi, \Sigma, \mathcal{E}, \mu, p, Q, \Pi, \Lambda\} \,. \tag{5.2}$$

These LRS class II quantities satisfy a set of covariant evolution and/or propagation equations, obtained from the Bianchi and Ricci identities for the unit vector fields u^a and n^a, respectively:

Propagation:

$$\begin{aligned}
\hat{\phi} &= -\tfrac{1}{2}\phi^2 + \left(\tfrac{1}{3}\Theta + \Sigma\right)\left(\tfrac{2}{3}\Theta - \Sigma\right) - \tfrac{2}{3}(\mu + \Lambda) - \mathcal{E} - \tfrac{1}{2}\Pi \,, && (5.3)\\
\hat{\Sigma} - \tfrac{2}{3}\hat{\Theta} &= -\tfrac{3}{2}\phi\Sigma - Q \,, && (5.4)\\
\hat{\mathcal{E}} - \tfrac{1}{3}\hat{\mu} + \tfrac{1}{2}\hat{\Pi} &= -\tfrac{3}{2}\phi\left(\mathcal{E} + \tfrac{1}{2}\Pi\right) + \left(\tfrac{1}{2}\Sigma - \tfrac{1}{3}\Theta\right) Q \,; && (5.5)
\end{aligned}$$

Evolution:

$$\begin{aligned}
\dot{\phi} &= -\left(\Sigma - \tfrac{2}{3}\Theta\right)\left(\mathcal{A} - \tfrac{1}{2}\phi\right) + Q \,, && (5.6)\\
\dot{\Sigma} - \tfrac{2}{3}\dot{\Theta} &= -\mathcal{A}\phi + 2\left(\tfrac{1}{3}\Theta - \tfrac{1}{2}\Sigma\right)^2 + \tfrac{1}{3}(\mu + 3p - 2\Lambda) - \mathcal{E} + \tfrac{1}{2}\Pi \,, && (5.7)\\
\dot{\mathcal{E}} - \tfrac{1}{3}\dot{\mu} + \tfrac{1}{2}\dot{\Pi} &= +\left(\tfrac{3}{2}\Sigma - \Theta\right)\mathcal{E} + \tfrac{1}{4}\left(\Sigma - \tfrac{2}{3}\Theta\right)\Pi + \tfrac{1}{2}\phi Q - \tfrac{1}{2}(\mu + p)\left(\Sigma - \tfrac{2}{3}\Theta\right) \,; && (5.8)
\end{aligned}$$

Propagation/evolution:

$$\begin{aligned}
\hat{\mathcal{A}} - \dot{\Theta} &= -(\mathcal{A} + \phi)\mathcal{A} + \tfrac{1}{3}\Theta^2 + \tfrac{3}{2}\Sigma^2 + \tfrac{1}{2}(\mu + 3p - 2\Lambda) \,, && (5.9)\\
\dot{\mu} + \hat{Q} &= -\Theta(\mu + p) - (\phi + 2\mathcal{A})Q - \tfrac{3}{2}\Sigma\Pi \,, && (5.10)\\
\dot{Q} + \hat{p} + \hat{\Pi} &= -\left(\tfrac{3}{2}\phi + \mathcal{A}\right)\Pi - \left(\tfrac{4}{3}\Theta + \Sigma\right)Q - (\mu + p)\mathcal{A} \,. && (5.11)
\end{aligned}$$

This system of equations extends the corresponding system in [112] (see section 6) to imperfect fluids.

Since the vorticity Ω vanishes, the unit vector field u^a is hypersurface-orthogonal to the spacelike 3-surfaces whose intrinsic curvature can be calculated from the Gauss equation for u^a. For the intrinsic Ricci-curvature, one finds from this

$$\begin{aligned} {}^3R_{ab} &= \left[\tfrac{2}{3}(\mu+\Lambda)+\mathcal{E}+\tfrac{1}{2}\Pi+\Sigma^2-\tfrac{1}{3}\Theta\Sigma-\tfrac{2}{9}\Theta^2\right] n_a n_b \\ &+ \left[\tfrac{2}{3}(\mu+\Lambda)-\tfrac{1}{2}\mathcal{E}-\tfrac{1}{4}\Pi+\tfrac{1}{4}\Sigma^2+\tfrac{1}{6}\Theta\Sigma-\tfrac{2}{9}\Theta^2\right] N_{ab}\ . \end{aligned} \tag{5.12}$$

The intrinsic Ricci-scalar of the 3-surfaces is therefore

$$ {}^3R = 2\left[\mu+\Lambda-\tfrac{1}{3}\Theta^2+\tfrac{3}{4}\Sigma^2\right]\ . \tag{5.13}$$

This relation constitutes the generalised Friedman equation.

On the other hand, the additional vanishing of the sheet distortion, ξ, implies that the sheet is in this case a genuine 2-surface. The Gauss equation for n^a together with the 3-Ricci identities determine the 3-Ricci curvature tensor of the spacelike 3-surfaces orthogonal to u^a to be

$$ {}^3R_{ab} = -\left[\hat{\phi}+\tfrac{1}{2}\phi^2\right] n_a n_b - \left[\tfrac{1}{2}\hat{\phi}+\tfrac{1}{2}\phi^2-K\right] N_{ab}\ , \tag{5.14}$$

thus giving for the 3-Ricci scalar

$$ {}^3R = -2\left[\hat{\phi}+\tfrac{3}{4}\phi^2-K\right]\ . \tag{5.15}$$

Here, K is the Gaussian curvature of the sheet, ${}^2R_{ab} = K\,N_{ab}$. Combining the last equation with equations (5.13) and (5.3), we can express the Gaussian curvature K in the form

$$ K = \tfrac{1}{3}(\mu+\Lambda)-\mathcal{E}-\tfrac{1}{2}\Pi+\tfrac{1}{4}\phi^2-\left(\tfrac{1}{3}\Theta-\tfrac{1}{2}\Sigma\right)^2\ . \tag{5.16}$$

Evolution and propagation equations for K can be calculated from equations (5.3)–(5.8) to give

$$\begin{aligned} \dot{K} &= -\left(\tfrac{2}{3}\Theta-\Sigma\right)K\ , &(5.17)\\ \hat{K} &= -\phi K\ . &(5.18) \end{aligned}$$

By specification of (2.120) follows that every scalar ψ in LRS II has to satisfy the commutation relation

$$ \hat{\dot{\psi}}-\dot{\hat{\psi}} = -\mathcal{A}\,\dot{\psi}+\left(\tfrac{1}{3}\Theta+\Sigma\right)\hat{\psi}\ . \tag{5.19}$$

We applied this commutator relation to the Gaussian curvature K in order to check for consistency of the equations (5.3)–(5.11), which is indeed guaranteed due to equations (5.17) and (5.18).

From equation (5.17) follows the neat result that whenever the Gaussian curvature K of the sheet is constant in time but non-vanishing, that is, the sheets have either spherical or hyperbolic geometry, the shear Σ is proportional to the expansion Θ:

$$K \neq 0 \quad \text{and} \quad \dot{K} = 0 \quad \Longrightarrow \quad \Sigma = \tfrac{2}{3}\Theta \, . \tag{5.20}$$

As a matter of frame choice, it is always possible to pick an observer with $\dot{K} = 0$ in some spacetime region, corresponding to a 'static' observer.[1] Adopting such a choice leads to great simplifications in the equations, the caveat being that such a choice is often not the most natural one, e.g., in Friedman-Lemaître-Robertson-Walker (FLRW) spacetimes. For example, taking a static observer to describe spherically symmetric spacetimes, the constraint (5.16) allows for a covariant definition of the radial parameter r via

$$r^{-2} = \tfrac{1}{3}(\mu + \Lambda) - \mathcal{E} - \tfrac{1}{2}\Pi + \tfrac{1}{4}\phi^2 \, ; \qquad \dot{r} = 0 = \delta_a r \, . \tag{5.21}$$

(A non-static observer has $\dot{r} \neq 0$). In situations with spherical symmetry it is convenient to express the hat-derivative in terms of the radial parameter r. If we associate an affine parameter ρ with the hat-derivative, we find using equation (5.18)

$$\hat{X} = \frac{dX}{d\rho} = \frac{1}{2} r \, \phi \, \frac{\partial X}{\partial r} + \hat{\tau} \, \dot{X} \, , \tag{5.22}$$

where τ denotes proper time and X may be any scalar.

The description of LRS spacetimes which are also static becomes particularly simple. In this case, there exists a timelike hypersurface-orthogonal Killing vector field, which one may identify with the 4-velocity field u^a after normalisation. Consequently, all terms in (5.3)–(5.11) containing a dot-derivative vanish. It is then very easy to show [using also (5.20)] that spherically symmetric perfect fluid spacetimes must have vanishing shear and expansion, or that the spherically symmetric vacuum solution must be the Schwarzschild solution (cf. subsection 5.4.1).

5.3 Perturbations on LRS class II backgrounds

In this section, we investigate scalar and electromagnetic perturbations on LRS class II background spacetimes. The scalar end electromagnetic fields are treated as test fields, i.e., they do not affect the geometry of the background. Hence, the Stewart-Walker Lemma [30] ensures that these fields are gauge-invariant. Our goal is the derivation of the concomitant covariant and gauge-invariant wave equations which describe these perturbations. These wave equations

[1]Clearly, static observers cannot be defined globally when the spacetime possesses an event horizon.

are the generalisation of the well-known Regge-Wheeler equations from the Schwarzschild case towards LRS class II spacetimes, yet written in covariant notation.

5.3.1 Commutation relations

In order to achieve this aim, we will make frequent use of commutation relations. It is easily seen from the relations (2.120)–(2.123) that all first order scalars ψ have to satisfy the following commutation relations between the different derivatives introduced above:

$$\begin{aligned}
\hat{\dot{\psi}} - \dot{\hat{\psi}} &= -\mathcal{A}\,\dot{\psi} + \left(\tfrac{1}{3}\Theta + \Sigma\right)\hat{\psi} \,, && (5.23)\\
\delta_a \dot{\psi} - N_a{}^b\,(\delta_b \psi)^{\cdot} &= -\tfrac{1}{2}\left(\Sigma - \tfrac{2}{3}\Theta\right)\delta_a \psi \,, && (5.24)\\
\delta_a \hat{\psi} - N_a{}^b\,(\delta_b \psi)^{\hat{}} &= +\tfrac{1}{2}\phi\,\delta_a \psi \,, && (5.25)\\
\delta_{[a}\,\delta_{b]}\,\psi &= 0 \,. && (5.26)
\end{aligned}$$

Analogously, the commutation relations (2.126)–(2.129) for first order 2-vectors ψ_a reduce in the LRS II case to

$$\begin{aligned}
\hat{\dot{\psi}}_{\bar{a}} - \dot{\hat{\psi}}_{\bar{a}} &= -\mathcal{A}\,\dot{\psi}_{\bar{a}} + \left(\tfrac{1}{3}\Theta + \Sigma\right)\hat{\psi}_{\bar{a}} \,, && (5.27)\\
\delta_a \dot{\psi}_b - N_a{}^c N_b{}^d\,(\delta_c \psi_d)^{\cdot} &= -\tfrac{1}{2}\left(\Sigma - \tfrac{2}{3}\Theta\right)\delta_a \psi_b \,, && (5.28)\\
\delta_a \hat{\psi}_b - N_a{}^c N_b{}^d\,(\delta_c \psi_d)^{\hat{}} &= +\tfrac{1}{2}\phi\,\delta_a \psi_b \,, && (5.29)\\
\delta_{[a}\,\delta_{b]}\,\psi_c &= -K\,\psi_{[a}\,N_{b]c} \,. && (5.30)
\end{aligned}$$

5.3.2 Harmonics

It is useful to expand all first-order perturbations in harmonics. Note that all functions and relations below are defined in the background only; we solely expand first-order variables, so zeroth-order equations are sufficient.

In analogy to the spatial harmonics defined in [132], we introduce dimensionless sheet harmonic functions Q, defined on the background, as eigenfunctions of the 2-dimensional Laplace-Beltrami operator such that for positive, negative or vanishing curvature K

$$\delta^2 Q = -\frac{k^2}{r^2}\,Q \,, \qquad \hat{Q} = 0 = \dot{Q} \qquad (0 \le k^2 \in \mathbf{R}) \,. \tag{5.31}$$

The function r is, up to an irrelevant constant, covariantly defined by

$$\frac{\hat{r}}{r} \equiv \frac{1}{2}\phi \,, \qquad \frac{\dot{r}}{r} \equiv \frac{1}{3}\Theta - \frac{1}{2}\Sigma \,, \qquad \delta_a r \equiv 0 \,, \tag{5.32}$$

which, in the light of equations (5.17)–(5.18), is nothing but the covariant version of the common setting $K \equiv \kappa/r^2$ with $\kappa = \pm 1$ for spherical and hyperbolic 2-geometry, respectively, where the constant is fixed by the relation (5.16). However, the covariant definition (5.32) extends to the plane-symmetric case as well, where there is no natural length scale. We can now expand any first order scalar ψ in terms of these functions formally as

$$\psi = \sum_k \psi_{\mathsf{S}}^{(k)} Q^{(k)} = \psi_{\mathsf{S}} Q \,, \tag{5.33}$$

where the sum over k is implicit in the last equality. The S subscript reminds us that ψ is a scalar, and that a harmonic expansion has been made.

We also need to expand vectors in harmonics. We therefore define the *even* (electric) parity vector harmonics as

$$Q_a^{(k)} = r\,\delta_a Q^{(k)} \quad \Rightarrow \quad \hat{Q}_{\bar{a}} = 0 = \dot{Q}_{\bar{a}} \,, \quad \delta^2 Q_a = \left(1 - k^2\right) r^{-2} Q_a \,; \tag{5.34}$$

where the (k) superscript is implicit, and we define *odd* (magnetic) parity vector harmonics as

$$\bar{Q}_a^{(k)} = r\,\varepsilon_{ab}\,\delta^b Q^{(k)} \quad \Rightarrow \quad \hat{\bar{Q}}_{\bar{a}} = 0 = \dot{\bar{Q}}_{\bar{a}} \,, \quad \delta^2 \bar{Q}_a = \left(1 - k^2\right) r^{-2} \bar{Q}_a \,. \tag{5.35}$$

Note that $\bar{Q}_a = \varepsilon_{ab} Q^b \Leftrightarrow Q_a = -\varepsilon_{ab} \bar{Q}^b$, so that ε_{ab} is a parity operator. The crucial difference between these two types of vector harmonics is that $\bar{Q}_a$ is solenoidal, so

$$\delta^a \bar{Q}_a = 0 \,, \tag{5.36}$$

while

$$\delta^a Q_a = -k^2 r^{-1} Q \,. \tag{5.37}$$

Note also that

$$\varepsilon_{ab}\,\delta^a Q^b = 0 \,, \quad \text{and} \quad \varepsilon_{ab}\,\delta^a \bar{Q}^b = +k^2 r^{-1} Q \,. \tag{5.38}$$

The harmonics are orthogonal: $Q^a \bar{Q}_a = 0$ (for each k), which implies that any first-order vector ψ_a can now be written

$$\psi_a = \sum_k \psi_{\mathsf{V}}^{(k)} Q_a^{(k)} + \bar{\psi}_{\mathsf{V}}^{(k)} \bar{Q}_a^{(k)} = \psi_{\mathsf{V}} Q_a + \bar{\psi}_{\mathsf{V}} \bar{Q}_a \,. \tag{5.39}$$

Again, we implicitly assume a sum over k in the last equality, and the V subscript reminds us that ψ_a is a vector expanded in harmonics.

We like to point out that the harmonics introduced here naturally generalise the spherical harmonics used in [34]. In particular, the various formulae for scalar and vector spherical harmonics stated in [34] also hold for our generalised harmonics. More precisely, we obtain the following relations:

$$\Psi = \Psi_{\mathsf{S}}\, Q\,, \tag{5.40}$$

$$\delta_a \Psi = r^{-1}\, \Psi_{\mathsf{S}}\, Q_a\,, \tag{5.41}$$

$$\varepsilon_{ab}\, \delta^b\, \Psi = r^{-1}\, \Psi_{\mathsf{S}}\, \bar{Q}_a\,; \tag{5.42}$$

together with

$$\Psi_a = +\Psi_{\mathsf{V}}\, Q_a + \bar{\Psi}_{\mathsf{V}}\, \bar{Q}_a\,, \tag{5.43}$$

$$\varepsilon_{ab}\, \Psi^b = -\bar{\Psi}_{\mathsf{V}}\, Q_a + \Psi_{\mathsf{V}}\, \bar{Q}_a\,, \tag{5.44}$$

$$\delta^a\, \Psi_a = -k^2\, r^{-1}\, \Psi_{\mathsf{V}}\, Q\,, \tag{5.45}$$

$$\varepsilon_{ab}\, \delta^a\, \Psi^b = +k^2\, r^{-1}\, \bar{\Psi}_{\mathsf{V}}\, Q\,, \tag{5.46}$$

where the sum over harmonic indices is suppressed.

5.3.3 Scalar perturbations

Let us consider perturbations of LRS spacetimes due to a *massive* scalar field, ψ, with mass M, whose equation of motion is the Klein-Gordon equation,

$$\left(g^{ab}\, \nabla_a\, \nabla_b + M^2\right) \psi = \left(\nabla^a\, \nabla_a + M^2\right) \psi = 0\,. \tag{5.47}$$

We investigate this simple equation first in an arbitrary spacetime and specialising to LRS II spacetimes afterwards. The corresponding 1+3 equation is easily derived using $g^{ab} = h^{ab} - u^a\, u^b$ and the following expression for the spatial Laplacian of a scalar field,

$$\mathrm{D}^a\, \mathrm{D}_a\, \psi = h^{ab}\, \mathrm{D}_a\, \mathrm{D}_b\, \psi = h^{ab}\, \nabla_a\, \nabla_b\, \psi + \Theta\, \dot{\psi}\,, \tag{5.48}$$

and leads to the wave equation

$$\ddot{\psi} - \mathrm{D}^a\, \mathrm{D}_a\, \psi + \Theta\, \dot{\psi} - \dot{u}^a\, \mathrm{D}_a\, \psi + M^2\, \psi = 0\,. \tag{5.49}$$

Thus, for a general spacetime, the evolution of a scalar field is affected by effects of expansion and acceleration, as measured by the fundamental observer.

The 1+1+2 form of equation (5.49) is readily obtained using $h^{ab} = N^{ab} + n^a\, n^b$ as well as the relation

$$\delta^a\,\delta_a\,\psi = N^{ab}\,\delta_a\,\delta_b\,\psi = N^{ab}\,\mathrm{D}_a\,\mathrm{D}_b\,\psi - \phi\,\hat{\psi}\;. \tag{5.50}$$

The resulting equation reads as

$$\ddot{\psi} - \hat{\hat{\psi}} + \Theta\,\dot{\psi} - (\mathcal{A}+\phi)\,\hat{\psi} + (a^a - \mathcal{A}^a - \delta^a)\,\delta_a\,\psi + M^2\,\psi = 0\;. \tag{5.51}$$

Equation (5.51) is the fully split Klein-Gordon equation for a massive scalar field in an arbitrary spacetime, given in a covariant and gauge-invariant fashion. For LRS spacetimes, the wave equation (5.51) closes, yielding

$$\ddot{\psi} - \hat{\hat{\psi}} + \Theta\,\dot{\psi} - (\mathcal{A}+\phi)\,\hat{\psi} + \left(M^2 - \delta^2\right)\psi = 0\;. \tag{5.52}$$

We emphasise that this is true for *all* LRS spacetimes. Equation (5.52) constitutes the generalised Regge-Wheeler equation for scalar perturbations on all LRS background spacetimes. While the generalised Regge-Wheeler equation (5.52) is not affected by rotational effects at all, it is in general sensitive to the observer's acceleration $\mathcal{A}$, the spacetime expansion Θ, the sheet expansion ϕ, as well as the mass M of the perturbation field ψ.

For LRS class II spacetimes, it is sometimes convenient to rescale the scalar field ψ according to $\psi \equiv r^{-1}\Psi$, where the function r is defined via (5.32). In terms of the rescaled field Ψ, the Regge-Wheeler equation (5.52) reads

$$\ddot{\Psi} - \hat{\hat{\Psi}} + \left(\Sigma + \tfrac{1}{3}\Theta\right)\dot{\Psi} - \mathcal{A}\,\hat{\Psi} + \left[M^2 - \mathcal{E} - \tfrac{1}{6}\left(\mu - 3p + 4\Lambda\right) - \delta^2\right]\Psi = 0\;. \tag{5.53}$$

This form of the Regge-Wheeler equation has several advantages over (5.52): firstly, it allows us to introduce the Gaussian curvature K at the expense of the Weyl curvature $\mathcal{E}$, for example, and secondly, it simplifies in vacuo. We emphasise that this form is *generic*: we will show in the next section that electromagnetic perturbations are described covariantly by the same equation but having a different potential [that is, the term in square brackets in (5.53)] once appropriate harmonics have been used to get rid of the sheet Laplacian δ^2.

5.3.4 Electromagnetic perturbations

In accordance with (2.92) and (3.42)–(3.44), the electric, magnetic and current 3-vector fields are irreducibly decomposed into scalar and 2-vector parts as

$$E^a = \mathscr{E}\,n^a + \mathscr{E}^a\;, \quad B^a = \mathscr{B}\,n^a + \mathscr{B}^a\;, \quad j^{\langle a\rangle} = \mathscr{J}\,n^a + \mathscr{J}^a\;. \tag{5.54}$$

Thus, Maxwell's equations for electromagnetic test fields, sourced by the total electric charge ρ_c and the currents $\mathscr{J}$ and $\mathscr{J}_a$, respectively, on a LRS class II geometry become in the sheet approach the splitting for arbitrary spacetimes is displayed in subsection 3.2.2):

$$\begin{aligned}
\hat{\mathscr{E}} + \delta_a \mathscr{E}^a &= -\phi\, \mathscr{E} + \frac{\rho_c}{\epsilon_0}\,, && (5.55)\\
\hat{\mathscr{B}} + \delta_a \mathscr{B}^a &= -\phi\, \mathscr{B}\,, && (5.56)\\
\dot{\mathscr{E}} - \varepsilon_{ab}\, \delta^a \mathscr{B}^b &= +\left(\Sigma - \tfrac{2}{3}\Theta\right) \mathscr{E} - \mu_0\, \mathscr{J}\,, && (5.57)\\
\dot{\mathscr{B}} + \varepsilon_{ab}\, \delta^a \mathscr{E}^b &= +\left(\Sigma - \tfrac{2}{3}\Theta\right) \mathscr{E}\,, && (5.58)\\
\dot{\mathscr{E}}_{\bar{a}} + \varepsilon_{ab} \left(\hat{\mathscr{B}}^b - \delta^b \mathscr{B}\right) &= -\left(\tfrac{1}{2}\phi + \mathcal{A}\right) \varepsilon_{ab}\, \mathscr{B}^b - \left(\tfrac{1}{2}\Sigma + \tfrac{2}{3}\Theta\right) \mathscr{E}_a - \mu_0\, \mathscr{J}_a\,, && (5.59)\\
\dot{\mathscr{B}}_{\bar{a}} - \varepsilon_{ab} \left(\hat{\mathscr{E}}^b - \delta^b \mathscr{E}\right) &= +\left(\tfrac{1}{2}\phi + \mathcal{A}\right) \varepsilon_{ab}\, \mathscr{E}^b - \left(\tfrac{1}{2}\Sigma + \tfrac{2}{3}\Theta\right) \mathscr{B}_a\,. && (5.60)
\end{aligned}$$

Note that ε_{ab} is a parity operator, $\varepsilon_{ac}\, \varepsilon^c{}_b = -N_{ab}$, thus we can switch parity between 2-vectors X_a and Y_a via

$$X_a = \varepsilon_{ab}\, Y^b \Longleftrightarrow Y_a = -\varepsilon_{ab}\, X^b\,. \tag{5.61}$$

Using this relation, the parity-reversed form of equations (5.59) and (5.60) read

$$\begin{aligned}
\hat{\mathscr{E}}_{\bar{a}} + \varepsilon_{ab}\, \dot{\mathscr{B}}^b - \delta_a \mathscr{E} &= -\left(\tfrac{1}{2}\phi + \mathcal{A}\right) \mathscr{E}_a - \left(\tfrac{1}{2}\Sigma + \tfrac{2}{3}\Theta\right) \varepsilon_{ab}\, \mathscr{B}^b\,, && (5.62)\\
\hat{\mathscr{B}}_{\bar{a}} - \varepsilon_{ab}\, \dot{\mathscr{E}}^b - \delta_a \mathscr{B} &= +\left(\tfrac{1}{2}\phi + \mathcal{A}\right) \mathscr{B}_a - \left(\tfrac{1}{2}\Sigma + \tfrac{2}{3}\Theta\right) \varepsilon_{ab}\, \mathscr{E}^b + \mu_0\, \varepsilon_{ab}\, \mathscr{J}^b\,. && (5.63)
\end{aligned}$$

From Maxwell's equations, one deduces (by a somewhat tedious calculation exploiting the earlier displayed commutation relations as well as the above parity-reversed equations) the following wave equations for the electromagnetic fields along the distinguished direction n_a:

$$\begin{aligned}
\ddot{\mathscr{E}} - \hat{\hat{\mathscr{E}}} &- \left(\mathcal{A} + 2\phi\right) \hat{\mathscr{E}} - \left(\Sigma - \tfrac{5}{3}\Theta\right) \dot{\mathscr{E}}\\
&- \left[\delta^2 + \tfrac{1}{2}\phi^2 - 2\mathcal{E} + \left(\tfrac{1}{3}\Theta + \Sigma\right)\left(\tfrac{3}{2}\Sigma - \Theta\right) - \tfrac{1}{3}\left(\mu - 3p + 4\Lambda\right)\right] \mathscr{E}\\
&= \frac{\hat{\rho}_c}{\epsilon_0} + \mu_0\, \dot{\mathscr{J}} + \left(\phi + \mathcal{A}\right) \frac{\rho_c}{\epsilon_0} + \mu_0\, \Theta\, \mathscr{J}\,, && (5.64)\\
\ddot{\mathscr{B}} - \hat{\hat{\mathscr{B}}} &- \left(\mathcal{A} + 2\phi\right) \hat{\mathscr{B}} - \left(\Sigma - \tfrac{5}{3}\Theta\right) \dot{\mathscr{B}}\\
&- \left[\delta^2 + \tfrac{1}{2}\phi^2 - 2\mathcal{E} + \left(\tfrac{1}{3}\Theta + \Sigma\right)\left(\tfrac{3}{2}\Sigma - \Theta\right) - \tfrac{1}{3}\left(\mu - 3p + 4\Lambda\right)\right] \mathscr{B}\\
&= 0\,. && (5.65)
\end{aligned}$$

Equations (5.64)–(5.65) are the generalisation of the famous Regge-Wheeler equation, for electromagnetic perturbations on the Schwarzschild background, towards LRS class II spacetimes, although written in covariant manner. We emphasise that $\mathscr{E}$ and $\mathscr{B}$ decouple from each other and that in the absence of sources the equations are identical closed wave equations.

It will turn out advantageous to rescale the fields and the sources under consideration similarly as above for Klein-Gordon fields; that is, we define

$$\mathscr{E} \equiv r^{-2} E \,, \quad \mathscr{B} \equiv r^{-2} B \,, \quad \rho_{\rm c} \equiv r^{-2} \varrho_{\rm c} \,, \quad \mathscr{J} \equiv r^{-2} J \,, \tag{5.66}$$

and substitute into the wave equations (5.64)–(5.65) to obtain

$$\begin{aligned}
\ddot{E} - \hat{\hat{E}} &- \mathcal{A}\,\hat{E} + \left(\Sigma + \tfrac{1}{3}\Theta\right)\dot{E} - \left[\delta^2 + \left(\Sigma - \tfrac{2}{3}\Theta\right)\left(2\Sigma + \tfrac{1}{6}\Theta\right)\right] E \\
&= \frac{\hat{\varrho}_{\rm c}}{\epsilon_0} + \mu_0\,\dot{J} + \mathcal{A}\,\frac{\varrho_{\rm c}}{\epsilon_0} + \mu_0\left(\Sigma + \tfrac{1}{3}\Theta\right) J \,, \qquad (5.67)\\
\ddot{B} - \hat{\hat{B}} &- \mathcal{A}\,\hat{B} + \left(\Sigma + \tfrac{1}{3}\Theta\right)\dot{B} - \left[\delta^2 + \left(\Sigma - \tfrac{2}{3}\Theta\right)\left(2\Sigma + \tfrac{1}{6}\Theta\right)\right] B = 0. \qquad (5.68)
\end{aligned}$$

Again, if we neglect the source terms, the equations become identical closed wave equations. Moreover, the equations (5.67)–(5.68) are then of the same form as equation (5.53) for the Klein-Gordon field, the only difference being a slightly altered potential term.

The wave equations for the electromagnetic perturbations lying in the sheet are derived analogously and read

$$\begin{aligned}
\ddot{\mathscr{E}}_{\bar a} - \hat{\hat{\mathscr{E}}}_{\bar a} &- (3\,\mathcal{A} + \phi)\,\hat{\mathscr{E}}_{\bar a} - \left(\Sigma - \tfrac{5}{3}\Theta\right)\dot{\mathscr{E}}_{\bar a} \\
&+ \left[\tfrac{1}{4}\phi^2 - \mathcal{E} - \mathcal{A}\,(\mathcal{A} + \phi) + \tfrac{2}{9}\Theta^2 - \tfrac{2}{3}\Theta\,\Sigma - \tfrac{7}{4}\Sigma^2 + \tfrac{1}{3}\,(\mu - 3p + 4\Lambda) - \delta^2\right]\mathscr{E}_a \\
&= (\phi - 2\,\mathcal{A})\,\delta_a\,\mathscr{E} + 3\Sigma\,\varepsilon_{ab}\,\hat{\mathscr{B}}^b + \left(3\,\mathcal{A}\,\Sigma - Q + \hat{\Theta} - \dot{\mathcal{A}}\right)\varepsilon_{ab}\,\mathscr{B}^b \\
&\quad - \frac{1}{\epsilon_0}\,\delta_a\,\rho_{\rm c} - \mu_0\,\dot{\mathscr{J}}_{\bar a} - \left(\Theta - \tfrac{3}{2}\Sigma\right)\mu_0\,\mathscr{J}_a \,, \qquad (5.69)
\end{aligned}$$

$$\begin{aligned}
\ddot{\mathscr{B}}_{\bar a} - \hat{\hat{\mathscr{B}}}_{\bar a} &- (3\,\mathcal{A} + \phi)\,\hat{\mathscr{B}}_{\bar a} - \left(\Sigma - \tfrac{5}{3}\Theta\right)\dot{\mathscr{B}}_{\bar a} \\
&+ \left[\tfrac{1}{4}\phi^2 - \mathcal{E} - \mathcal{A}\,(\mathcal{A} + \phi) + \tfrac{2}{9}\Theta^2 - \tfrac{2}{3}\Theta\,\Sigma - \tfrac{7}{4}\Sigma^2 + \tfrac{1}{3}\,(\mu - 3p + 4\Lambda) - \delta^2\right]\mathscr{B}_a \\
&= (\phi - 2\,\mathcal{A})\,\delta_a\,\mathscr{B} - 3\,\Sigma\,\varepsilon_{ab}\,\hat{\mathscr{E}}^b - \left(3\,\mathcal{A}\,\Sigma - Q + \hat{\Theta} - \dot{\mathcal{A}}\right)\varepsilon_{ab}\,\mathscr{E}^b \\
&\quad + \mu_0\,\varepsilon_{ab}\left[\delta^b\mathscr{J} - \hat{\mathscr{J}}^b - \left(\tfrac{1}{2}\phi + 2\,\mathcal{A}\right)\mathscr{J}^b\right] . \qquad (5.70)
\end{aligned}$$

In contrast to equations (5.64)–(5.65), these do not decouple – not even in the absence of sources. For example, in addition to the source terms, the magnetic 2-vector field $\mathscr{B}_a$ gives rise to forcing terms for the electric 2-vector field $\mathscr{E}_a$, which is also forced by the 'radial' electric field.

However, it turns out that *in the absence of sources* knowledge of the 'radial' part $\mathscr{E}$ (or $\mathscr{B}$) of the perturbations, e.g., by solving the concomitant Regge-Wheeler equation (5.67), suffices to completely determine the electromagnetic perturbations. To see this, we expand all perturbations into harmonics and decompose the governing Maxwell's equations into their harmonic components. (For details, we refer the reader to subsection 5.3.2 and to appendix B). From the scalar equations (5.55)–(5.58) we find for each fixed harmonic index k

$$\begin{aligned}
\hat{\mathscr{E}}_{\mathsf{S}} + \phi\,\mathscr{E}_{\mathsf{S}} &= +\frac{k^2}{r}\,\mathscr{E}_{\mathsf{V}}\,, && (5.71)\\
\hat{\mathscr{B}}_{\mathsf{S}} + \phi\,\mathscr{B}_{\mathsf{S}} &= +\frac{k^2}{r}\,\mathscr{B}_{\mathsf{V}}\,, && (5.72)\\
\dot{\mathscr{E}}_{\mathsf{S}} - \left(\Sigma - \tfrac{2}{3}\Theta\right)\mathscr{E}_{\mathsf{S}} &= +\frac{k^2}{r}\,\bar{\mathscr{B}}_{\mathsf{V}}\,, && (5.73)\\
\dot{\mathscr{B}}_{\mathsf{S}} - \left(\Sigma - \tfrac{2}{3}\Theta\right)\mathscr{B}_{\mathsf{S}} &= -\frac{k^2}{r}\,\bar{\mathscr{E}}_{\mathsf{V}}\,. && (5.74)
\end{aligned}$$

It is thus obvious that a solution for the 'radial' fields $\mathscr{E}$ and $\mathscr{B}$ determines the sheet fields $\mathscr{E}_a$ and $\mathscr{B}_a$, respectively.

The 2-vector equations (5.59) and (5.60), or (5.62) and (5.63), respectively, split into two parts; the first part is the odd parity part, given by

$$\begin{aligned}
\dot{\bar{\mathscr{E}}}_{\mathsf{V}} + \hat{\mathscr{B}}_{\mathsf{V}} - \frac{1}{r}\,\mathscr{B}_{\mathsf{S}} &= -\left(\tfrac{1}{2}\phi + \mathcal{A}\right)\mathscr{B}_{\mathsf{V}} - \left(\tfrac{1}{2}\Sigma + \tfrac{2}{3}\Theta\right)\bar{\mathscr{E}}_{\mathsf{V}}\,, && (5.75)\\
\dot{\bar{\mathscr{B}}}_{\mathsf{V}} - \hat{\mathscr{E}}_{\mathsf{V}} + \frac{1}{r}\,\mathscr{E}_{\mathsf{S}} &= +\left(\tfrac{1}{2}\phi + \mathcal{A}\right)\mathscr{E}_{\mathsf{V}} - \left(\tfrac{1}{2}\Sigma + \tfrac{2}{3}\Theta\right)\bar{\mathscr{B}}_{\mathsf{V}}\,, && (5.76)
\end{aligned}$$

and the second part is the even parity part, given by

$$\begin{aligned}
\dot{\mathscr{E}}_{\mathsf{V}} - \hat{\bar{\mathscr{B}}}_{\mathsf{V}} &= +\left(\tfrac{1}{2}\phi + \mathcal{A}\right)\bar{\mathscr{B}}_{\mathsf{V}} - \left(\tfrac{1}{2}\Sigma + \tfrac{2}{3}\Theta\right)\mathscr{E}_{\mathsf{V}}\,, && (5.77)\\
\dot{\mathscr{B}}_{\mathsf{V}} + \hat{\bar{\mathscr{E}}}_{\mathsf{V}} &= -\left(\tfrac{1}{2}\phi + \mathcal{A}\right)\bar{\mathscr{E}}_{\mathsf{V}} - \left(\tfrac{1}{2}\Sigma + \tfrac{2}{3}\Theta\right)\mathscr{B}_{\mathsf{V}}\,. && (5.78)
\end{aligned}$$

We remark that the even parity equations are redundant since propagating the constraints (5.73)–(5.74) and inserting them into (5.77) and (5.78) just yields back equation (5.71) and (5.72). On the other hand, the odd parity equations are not implied by the scalar ones and have been used in the derivation of the generalised Regge-Wheeler equation (5.64). If one eliminates $\bar{\mathscr{E}}_{\mathsf{V}}$ and $\bar{\mathscr{B}}_{\mathsf{V}}$ from equations (5.77)–(5.78) using the constraints (5.73)–(5.74), it becomes obvious that the electromagnetic perturbations fall into two distinct classes whose equations decouple from each other, namely:

$$\begin{aligned}
&\textit{polar}\text{ perturbations}: && \{\mathscr{E}_{\mathsf{S}}, \mathscr{E}_{\mathsf{V}}, \bar{\mathscr{B}}_{\mathsf{V}}\}\,, && (5.79)\\
&\textit{axial}\text{ perturbations}: && \{\mathscr{B}_{\mathsf{S}}, \mathscr{B}_{\mathsf{V}}, \bar{\mathscr{E}}_{\mathsf{V}}\}\,. && (5.80)
\end{aligned}$$

Moreover, the resulting equations involving either $\mathscr{E}_{\mathsf{S}}$ and $\mathscr{E}_{\mathsf{V}}$ or $\mathscr{B}_{\mathsf{S}}$ and $\mathscr{B}_{\mathsf{V}}$ are identical.

5.4 Examples

We have shown that scalar and vector perturbations in LRS class II spacetimes are governed by simple master equations. We shall now discuss these master equations in some specific spacetimes. First, we shall recover the well known Regge-Wheeler equations for the Schwarzschild spacetime, which we then generalise to the Vaidya radiation spacetime. We then discuss solutions from the Lemaître-Tolman-Bondi and Kantowski-Sachs families.

5.4.1 The Schwarzschild spacetime

The background

Schwarzschild spacetime is fully determined by any two of three non-zero scalar functions ϕ , $\mathcal{A}$ and $\mathcal{E}$. These functions obey the background equations

$$\hat{\phi} = -\tfrac{1}{2}\phi^2 + \mathcal{A}\phi\,, \tag{5.81}$$

$$\hat{\mathcal{A}} = -\mathcal{A}\,(\phi + \mathcal{A})\;; \tag{5.82}$$

together with the constraint

$$\mathcal{E} + \mathcal{A}\phi = 0\,. \tag{5.83}$$

The parametric solutions for these variables are [34]

$$\mathcal{E} = -\frac{2m}{r^3}\,, \tag{5.84}$$

$$\phi = \frac{2}{r}\sqrt{1 - \frac{2m}{r}}\,, \tag{5.85}$$

$$\mathcal{A} = \frac{m}{r^2}\left(1 - \frac{2m}{r}\right)^{-1/2}\,. \tag{5.86}$$

These form a one-parameter family of solutions, parameterised by the constant m, which is just the Schwarzschild mass. The (exterior) Schwarzschild solution is given for $2m < r < \infty$.

The perturbations

In order to make the connection with the standard Regge-Wheeler equations [95], it is instructive to contrast the found covariant and gauge-invariant wave equations for scalar and electromagnetic perturbations with the ones governing gravitational perturbations. The fundamental object for the study of gravitational perturbations on a Schwarzschild background is the Regge-Wheeler tensor W_{ab} [34], defined by

$$W_{ab} = \tfrac{1}{2}\,\phi\, r^2\, \zeta_{ab} - \tfrac{1}{3},r^2\, \mathcal{E}^{-1}\, \delta_{\{a}\, \delta_{b\}}\, \mathcal{E}\ , \tag{5.87}$$

a dimensionless, gauge-invariant, transverse-traceless tensor which obeys the wave equation

$$\ddot{W}_{\{ab\}} - \hat{\hat{W}}_{\{ab\}} - \mathcal{A}\, \hat{W}_{\{ab\}} + \phi^2\, W_{ab} - \delta^2\, W_{ab} = 0\ . \tag{5.88}$$

If one expands equation (5.88) into spherical tensor harmonics, the odd and even parity parts of equation (5.88) both become

$$\ddot{W}_{\mathsf{T}} - \hat{\hat{W}}_{\mathsf{T}} - \mathcal{A}\, \hat{W}_{\mathsf{T}} + \left[\frac{L}{r^2} + 3\,\mathcal{E}\right] W_{\mathsf{T}} = 0\ , \tag{5.89}$$

where $W_{\mathsf{T}} = W_{\mathsf{T}}^{(\ell)}$ are the tensor harmonic components of W_{ab}, $\ell = 1, 2, \ldots$ and $L = \ell(\ell+1) = k^2$. It turns out that equation (5.89) is actually the Regge-Wheeler equation [154] when written in appropriate coordinates. Converting from ρ, the affine parameter associated with the hat-derivative, to the parameter r, $\rho \to r$, and then to the 'tortoise' coordinate of Regge and Wheeler,

$$r_* = r + 2\,m \ln\left(\frac{r}{2\,m} - 1\right)\ , \tag{5.90}$$

and also introducing the Schwarzschild time via $d\tau = \sqrt{1 - \frac{2\,m}{r}}\, dt$, we find that (6.10) becomes:

$$\left(-\frac{\partial^2}{\partial\, t^2} + \frac{\partial^2}{\partial\, r_*^2} + \mathscr{V}_T\right) W_{\mathsf{T}} = 0\ , \qquad \mathscr{V}_T = \left(1 - \frac{2,m}{r}\right)\left[\frac{L}{r^2} - \frac{6\,m}{r^3}\right]\ , \tag{5.91}$$

where $\mathscr{V}_T$ is the Regge-Wheeler potential for gravitational perturbations.

It is now straightforward to obtain the familiar Regge-Wheeler equations for the case of scalar and electromagnetic perturbations. From equation (5.53) one finds immediately for a massive scalar field ψ with mass M, $\psi = r^{-1}\Psi$, that

$$\ddot{\Psi} - \hat{\hat{\Psi}} - \mathcal{A}\,\hat{\Psi} - \left[\mathcal{E} - M^2 + \delta^2\right]\Psi = 0\ . \tag{5.92}$$

Introducing scalar spherical harmonics and performing as above one readily gets

$$\left(-\frac{\partial^2}{\partial\, t^2} + \frac{\partial^2}{\partial\, r_*^2} + \mathscr{V}_S\right)\Psi_{\mathsf{S}} = 0\ , \qquad \mathscr{V}_S = \left(1 - \frac{2\,m}{r}\right)\left[\frac{L}{r^2} + \frac{2\,m}{r^3} + M^2\right]\ , \tag{5.93}$$

where Ψ_{S} is a scalar harmonic component of Ψ and $\mathscr{V}_S$ is the Regge-Wheeler potential for scalar perturbations. This equation was originally derived in [155] employing the Newman-Penrose formalism. Notice the striking similarity between the Regge-Wheeler potential for the massless scalar field perturbations, $\mathscr{V}_S$, and the potential for gravitational perturbations, $\mathscr{V}_T$ [95].

The electromagnetic case is even simpler. For this case, the covariant Regge-Wheeler equation (5.67) for the electric field perturbation $\mathscr{E}$, $\mathscr{E} = r^{-2}E$, reduces in the Schwarzschild background to

$$\ddot{E} - \hat{\hat{E}} - \mathcal{A}\hat{E} - \delta^2 E = 0 \; . \tag{5.94}$$

Adopting once more scalar spherical harmonics and proceeding as before leads to

$$\left(-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r_*^2} + \mathscr{V}_V\right) E_{\mathsf{S}} = 0 \; , \quad \mathscr{V}_V = \left(1 - \frac{2m}{r}\right)\left[\frac{L}{r^2}\right] \; , \tag{5.95}$$

where $\mathscr{V}_V$ is the Regge-Wheeler potential for electromagnetic perturbations (the magnetic case is completely analogous). The Regge-Wheeler equation for radial electric perturbations (5.95), and its magnetic counterpart, was originally derived in [156] employing the Newman-Penrose formalism.

It is remarkable that the (source-free) electromagnetic perturbations can also be described by a linear first-order system of ODE's after expanding Maxwell's equations into spherical *and* time harmonics [34] (possible because the background is static). The time derivatives of first order quantities are decomposed into their Fourier components by assuming an $e^{i\omega\tau}$ time dependence for the first order variables; factors of $i\omega$ just represent time derivatives, $d/d\tau$. Note that

$$\hat{\omega} = -\mathcal{A}\omega \quad \Rightarrow \quad \omega = \sigma\left(1 - \frac{2m}{r}\right)^{-1/2} = \frac{2\sigma}{\phi r} \; , \tag{5.96}$$

arising from the commutation relation between the dot- and hat-derivatives. The harmonic function ω is defined with respect to proper time τ of observers moving along u^a, while σ is the *constant* harmonic index associated with time t of observers at infinity. They are related by $\omega\tau = \sigma t$. Inserting time harmonics into equations (5.71)–(5.76) one finds that equations (5.73)–(5.74) turn into constraints for the odd modes $\bar{\mathscr{E}}_{\mathsf{V}}$ and $\bar{\mathscr{B}}_{\mathsf{V}}$, respectively,

$$\bar{\mathscr{B}}_{\mathsf{V}} = \frac{i\omega r}{L}\mathscr{E}_{\mathsf{S}} \; , \qquad \bar{\mathscr{E}}_{\mathsf{V}} = -\frac{i\omega r}{L}\mathscr{B}_{\mathsf{S}} \; , \tag{5.97}$$

which may subsequently be eliminated from equations (5.75)–(5.76). The final result is a linear first order system of ODE's,

$$\hat{\mathbf{Y}} = \mathbf{A}\,\mathbf{Y} \; , \tag{5.98}$$

where the matrix $\mathbf{A}$ is given as

$$\mathbf{A} = \begin{pmatrix} -\phi & \frac{L}{r} \\ \frac{1}{r} - \frac{r\,\omega^2}{L} & -\frac{1}{2}\phi - \mathcal{A} \end{pmatrix} \tag{5.99}$$

and the perturbation vector $\mathbf{Y}$ is defined in the case of polar and axial perturbations by

$$\mathbf{Y}_{\text{polar}} = \begin{pmatrix} \mathscr{E}_{\mathsf{S}} \\ \mathscr{E}_{\mathsf{V}} \end{pmatrix} \qquad \text{and} \qquad \mathbf{Y}_{\text{axial}} = \begin{pmatrix} \mathscr{B}_{\mathsf{S}} \\ \mathscr{B}_{\mathsf{V}} \end{pmatrix} , \tag{5.100}$$

respectively. We emphasise the observation that in a stationary situation, e.g. $\omega = 0$, the constraints (5.97) imply $\bar{\mathscr{E}}_{\mathsf{V}} = \bar{\mathscr{B}}_{\mathsf{V}} = 0$ in the absence of charges, which means that electromagnetic perturbations cannot have a solenoidal part in the sheet. A further implication is that a pure magnetic field can only exist on the Schwarzschild geometry if it has no solenoidal contribution in the sheet and is stationary and/or sheet-like. In the stationary situation, one can, if desired, express the general solution for the matrix equation (5.99), for an arbitrary multipole-moment ℓ, in terms of hypergeometric functions. The solutions for the $\ell = 1$ equations are of particular importance because they represent the magnetic fields often used to model the magnetosphere of a compact object: namely the case of a static asymptotically uniform magnetic field as well as a static dipole field, which falls off like $1/r^3$ at infinity. The general solutions for the case $\ell = 1$ are, in terms of r:

$$\mathscr{B}_{\mathsf{S}} = C_1 + \frac{1}{(2\,m)^3}\left[\ln\left(1 - \frac{2\,m}{r}\right) + \frac{2\,m}{r}\left(1 + \frac{m}{r}\right)\right] C_2 \,, \tag{5.101}$$

$$\mathscr{B}_{\mathsf{V}} = \sqrt{1 - \frac{2\,m}{r}}\left\{C_1 + \frac{1}{(2\,m)^3}\left[\ln\left(1 - \frac{2\,m}{r}\right) + \frac{2\,m}{r}\frac{r - m}{r - 2\,m}\right] C_2\right\} . \tag{5.102}$$

Here, $\bar{\mathscr{B}}_{\mathsf{V}} = 0$ and the boundary conditions $\{C_1 = \mathscr{B}_\infty,\, C_2 = 0\}$ correspond to the static magnetic field with uniform field strength $\mathscr{B}_\infty$ at infinity while the setting $\{C_1 = 0,\, C_2 = -3\,\mathscr{B}_\infty\}$ gives the dipole magnetic field with $\mathscr{B}_\infty$ the magnitude of $\mathscr{B}_{\mathsf{S}}\, r^3$ as $r \to \infty$. If needed, the found solutions may be easily expressed in terms of a Schwarzschild tetrad, say. For other (somewhat more tedious) methods see [103, 163, 164].

5.4.2 Vaidya spacetime

The background

The energy-momentum tensor of Vaidya's radiating sphere spacetime is that of a radiation fluid, e.g.,

$$T_{ab} = \mu\, k_a\, k_b \,, \qquad k_a\, k^a = 0 \,. \tag{5.103}$$

A convenient form of its metric for the case of outgoing radiation is given in terms of retarded time $u = t - r_*$ by

$$g_{ab} = -\left(1 - \frac{2\,m(u)}{r}\right) du^2 - 2\,du\,dr + r^2\,d\Omega^2\,, \tag{5.104}$$

where $m(u)$ is an arbitrary non-increasing function, $m'(u) \equiv dm(u)/du \leq 0$; if $m(u)$ is constant, the metric is equivalent to the Schwarzschild metric. Thus, the Vaidya spacetime describes a collapsing star that irradiates a part of its mass, where the total power output measured at infinity is given by $-m'(u) = 4\,\pi\,r^2\,\mu$ [165–170].

A covariant description of the Vaidya spacetime is obtained as follows. Spherical symmetry requires the null-vector k_a to be of the form $k_a = u_a \pm n_a$, where a plus (minus) corresponds to outflowing (inflowing) radiation. Thus, a static observer will encounter the radiation fluid as an imperfect one, with the energy-density μ, the isotropic pressure $p = \frac{1}{3}\mu$, the radial heat-flux $Q = \pm\mu$ and the trace-free part of the anisotropic sheet pressure $\Pi = \frac{2}{3}\mu$. Working in the static frame ($\dot{r} = 0$), the non-stationary spherically symmetric Vaidya spacetime in the case of outgoing radiation is described covariantly by the following set of equations:

$$\begin{aligned}
\hat{\mathcal{A}} - \dot{\Theta} &= -\mathcal{A}\,(\mathcal{A} + \phi) + \Theta\,(\Theta - \phi)\;, && (5.105)\\
\hat{\phi} &= -\phi\left(\tfrac{1}{2}\phi - \mathcal{A} - 2\,\Theta\right)\;, && (5.106)\\
\dot{\phi} &= -\Theta\,\phi\,, && (5.107)\\
\hat{\Theta} + \dot{\Theta} &= -\Theta\left(\tfrac{1}{2}\phi + 3\,\Theta + 3\,\mathcal{A}\right)\;. && (5.108)
\end{aligned}$$

The corresponding constraints, implied by our frame choice , are

$$\begin{aligned}
K &\equiv r^{-2} = \tfrac{1}{4}\phi^2 - \mathcal{E}\;, && (5.109)\\
\Sigma &= \tfrac{2}{3}\Theta\,, && (5.110)\\
\mu &= -\Theta\,\phi = 3\,p = Q = \tfrac{3}{2}\,\Pi\;, && (5.111)\\
\mathcal{E} &= \mu - \mathcal{A}\,\phi\;. && (5.112)
\end{aligned}$$

Note that the equations governing the Vaidya spacetime reduce to the ones describing Schwarzschild spacetime [see (5.81)–(5.83)] when the expansion Θ vanishes.

In order to solve the equations, we find it useful to express our derivatives in terms of coordinates $\{r, u\}$, where r is the radial parameter and u labels time. In particular, we have $\rho = \rho(r, u)$ and $\tau = \tau(r, u)$ and the derivatives become for an arbitrary scalar X

$$\hat{X} = \frac{r\,\phi}{2}\,\frac{\partial X}{\partial r} + \hat{u}\,\frac{\partial X}{\partial u}\;, \tag{5.113}$$

and

$$\dot{X} = \dot{u}\,\frac{\partial X}{\partial u}\,, \tag{5.114}$$

respectively. One neat possibility to define the coordinate u is through the relations

$$\dot{u} = -\hat{u} = \frac{1}{\hat{r}} = \frac{2}{r\,\phi}\,, \tag{5.115}$$

which is compatible with the commutator relation (5.23) and identifies u with the u-coordinate used in the metric (5.104), as will become clear in an instant. Upon switching to the coordinates $\{r, u\}$ and putting $\mathcal{A} \equiv \mathcal{A}_{\mathsf{Sch}} - \Theta$ [as suggested by equations (5.106) and (5.108)], the system of equations transforms into

$$\begin{aligned}
\left(\tfrac{1}{4}\,r^2\,\phi^2\,\partial_r - \partial_u\right)\mathcal{A}_{\mathsf{Sch}} &= -\tfrac{1}{2}\,r\,\phi\left[\mathcal{A}_{\mathsf{Sch}}\left(\mathcal{A}_{\mathsf{Sch}} + \phi + \Theta\right) + \tfrac{1}{2}\phi\,\Theta\right]\,, && (5.116)\\
r\,\partial_r\phi &= -\phi + 2\,\mathcal{A}_{\mathsf{Sch}}\,, && (5.117)\\
2\,\partial_u\phi &= -r\,\Theta\,\phi^2\,, && (5.118)\\
r\,\phi\,\partial_r\,\Theta &= -\Theta\left(\phi + 6\,\mathcal{A}_{\mathsf{Sch}}\right)\,. && (5.119)
\end{aligned}$$

The solutions may be readily found by making an educated guess. Clearly, by spherical symmetry, the radiation has to move outward radially and the expansion ϕ of the radial congruence n^a therefore should be of the same form as for the Schwarzschild spacetime but with a decreasing mass parameter. Since equation (5.117) is identical with the analogous one for the Schwarzschild spacetime, we are led to identify $\mathcal{A}_{\mathsf{Sch}}$ with the acceleration of a static Schwarzschild observer, in particular, $\mathcal{A}_{\mathsf{Sch}}$ represents the acceleration caused by the star's instantaneous gravitational field. It is then straightforward to work out all other quantities. The result is:

$$\begin{aligned}
\phi &= \frac{2}{r}\sqrt{1 - \frac{2\,m(u)}{r}}\,, && (5.120)\\
\Theta &= \frac{m'(u)}{r}\left(1 - \frac{2\,m(u)}{r}\right)^{-3/2} = \tfrac{3}{2}\,\Sigma\,, && (5.121)\\
\mathcal{A} &= \mathcal{A}_{\mathsf{Sch}} - \Theta; \qquad \mathcal{A}_{\mathsf{Sch}} = \frac{m(u)}{r^2}\left(1 - \frac{2\,m(u)}{r}\right)^{-1/2}\,, && (5.122)\\
\mu &= -\frac{2\,m'(u)}{r^2}\left(1 - \frac{2\,m(u)}{r}\right)^{-1} = 3\,p = Q = \tfrac{3}{2}\,\Pi\,, && (5.123)\\
\mathcal{E} &= -\frac{2\,m(u)}{r^3}\,. && (5.124)
\end{aligned}$$

It is worth pointing out that these solutions can also be worked out directly from the Vaidya metric (5.104) by choosing the splitting unit vectors to be

$$u^a = \left(1 - \frac{2\,m(u)}{r}\right)^{-1/2} \left(\frac{\partial}{\partial u}\right)^a \tag{5.125}$$

and

$$n^a = -\left(1 - \frac{2\,m(u)}{r}\right)^{-1/2} \left(\frac{\partial}{\partial u}\right)^a + \left(1 - \frac{2\,m(u)}{r}\right)^{1/2} \left(\frac{\partial}{\partial r}\right)^a , \tag{5.126}$$

respectively.

The obtained solutions may be interpreted as follows. First note that from equation (5.123) follows that a physically acceptable, that is, positive energy density μ requires a non-increasing (but otherwise unrestricted) mass, $m'(u) \leq 0$, suggesting that a part of the star's mass is released as radiation during collapse, with total luminosity $-m'(u) = 4\,\pi\,r^2\,\mu$ at infinity (in proper units). Further, the expansion Θ and shear Σ are negative since $m'(u) \leq 0$, meaning that the spatial sections of the spacetime are contracting, which tends to push nearby observers together. Equation (5.122) hence tells us that an observer has to balance its always radially outward directed acceleration according to the diminishing gravitational attraction of the collapsing star and the growing contraction in order to stay at rest. Finally, the expressions for sheet expansion ϕ and tidal forces $\mathcal{E}$ are as for a star in equilibrium but with a time-dependent mass due to mass-radiation conversion during collapse.

The perturbations

With the background solutions for the Vaidya spacetime at hand, it is now an easy task to rewrite the covariant Regge-Wheeler equations for the case of scalar and electromagnetic perturbations into in terms of coordinates. From equation (5.53) one obtains immediately for a massive scalar field ψ with mass M, $\psi = r^{-1}\Psi$, that

$$\ddot{\Psi} - \hat{\hat{\Psi}} - \mathcal{A}\,\hat{\Psi} + \Theta\,\dot{\Psi} - \left[\mathcal{E} - M^2 + \delta^2\right]\Psi = 0 \;, \tag{5.127}$$

while equation (5.67) for the electromagnetic perturbations, $\mathscr{E} = r^{-2}E$, gives

$$\ddot{E} - \hat{\hat{E}} - \mathcal{A}\,\hat{E} + \Theta\,\dot{E} - \delta^2 E = 0 \;, \tag{5.128}$$

Introducing scalar spherical harmonics and adopting the coordinates $\{r, u\}$ from above, we find the corresponding Regge-Wheeler equation

$$\left[\left(1 - \frac{2\,m(u)}{r}\right)\frac{\partial^2}{\partial r^2} - 2\,\frac{\partial^2}{\partial u\,\partial r} + \frac{2\,m(u)}{r^2}\,\frac{\partial}{\partial r} + \left(\frac{L}{r^2} + \lambda\,\frac{2\,m(u)}{r^3} + \lambda\,M^2\right)\right]\mathscr{P}_{\mathrm{S}} = 0 \;, \tag{5.129}$$

where $\mathscr{P}_\mathrm{S}$ denotes the scalar harmonic component of either Ψ or E and $L = \ell\,(\ell+1)$; the parameter λ takes the value one for perturbations in a scalar field and the value zero for electromagnetic perturbations.

It is instructive to compare the equations (5.129) with the corresponding equations (5.93) and (5.95) in the Schwarzschild case. We found it most favourable to do the comparison by employing coordinates $\{t, r_*\}$ which are defined for $r > 2\,m(u)$ by

$$t = u + r_* \,, \qquad r_* = r + 2\,m(u)\left[\ln\left(\frac{r}{2\,m(u)} - 1\right)\right] . \tag{5.130}$$

In terms of the coordinates $\{t, r_*\}$, the equations (5.129) now become

$$\begin{aligned}\Bigg\{[-1 - h(u,r)]\,\frac{\partial^2}{\partial t^2} \;&+\; [1 - h(u,r)]\,\frac{\partial^2}{\partial r_*^2} - \frac{4\,m'(u)}{r - 2\,m(u)}\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial r_*}\right) - 2\,h(u,r)\frac{\partial^2}{\partial t\,\partial r_*}\\ &+\; \left(1 - \frac{2\,m(u)}{r}\right)\left[\frac{L}{r^2} + \lambda\,\frac{2\,m(u)}{r^3} + \lambda\,M^2\right]\Bigg\}\,\mathscr{P}_\mathrm{S} = 0 \,,\end{aligned} \tag{5.131}$$

where the function $h(u,r)$,

$$h(u,r) = 4\,m'(u)\left[\ln\left(\frac{r}{2\,m(u)} - 1\right) - \left(1 - \frac{2\,m(u)}{r}\right)^{-1}\right] , \tag{5.132}$$

was introduced for brevity's sake. Clearly, when the mass $m(u)$ stays constant, the equations (5.131) reduce to the familiar Regge-Wheeler equations (5.93) and (5.95), respectively. Note that all new terms in the Regge-Wheeler equations for the Vaidya spacetime (5.131) are proportional to the mass change in time, $m'(u)$, and thus proportional to the expansion Θ [cf. equation (5.121)]. This was to be expected as, contrasting the Schwarzschild case, the only new ingredient in the covariant Regge-Wheeler equations for the Vaidya spacetime [see (5.127) and (5.128)] is proportional to the expansion Θ as well. Finally, the accompanying Regge-Wheeler potentials retain the Schwarzschild form but become time-dependent in the Vaidya case.

5.4.3 More general spacetimes

It was shown in [150] that the metric of every LRS class II spacetime can be given in diagonal form if local comoving coordinates are chosen:

$$ds^2 = -A^{-2}(t,x)\,dt^2 + B^2(t,x)\,dx^2 + C^2(t,x)\,[\,dy^2 + D^2(y,k)\,dz^2\,] \,, \tag{5.133}$$

where $D(y,k) = (\sin y\ , y\ , \sinh y)$ for $k = (1\ , 0\ , -1)$ labelling the closed, flat or open geometry of the sheet. With the obvious choice $u^a = A\,(\partial_t)^a$ and $n^a = B^{-1}\,(\partial_x)^a$, respectively, the kinematical quantities are now

$$\begin{aligned}
\phi &= 2\frac{\hat{C}}{C}\,, && (5.134)\\
\mathcal{A} &= -\frac{\hat{A}}{A}\,, && (5.135)\\
\Theta &= \frac{\dot{B}}{B} + 2\frac{\dot{C}}{C}\,, && (5.136)\\
\Sigma &= \frac{2}{3}\left(\frac{\dot{B}}{B} - \frac{\dot{C}}{C}\right)\,, && (5.137)
\end{aligned}$$

and equations (5.17) and (5.18) can therefore be integrated to yield the Gaussian curvature

$$K = \frac{C_1}{C^2(t,x)}\,, \tag{5.138}$$

where the integration constant C_1 may be normalised such that $C_1 = k$. From equation (5.4) or (5.6) one immediately obtains for the heatflux

$$Q = 2\left(\frac{\dot{\hat{C}}}{C} - \frac{\dot{B}\,\hat{C}}{B\,C}\right)\,, \tag{5.139}$$

while equation (5.9) directly yields the relation

$$\tfrac{1}{2}\,(\mu + 3\,p - 2\,\Lambda) = -\left[\frac{\hat{\hat{A}}}{A} - 2\left(\frac{\hat{A}}{A}\right)^2 + \frac{\ddot{B}}{B} + 2\frac{\ddot{C}}{C} + 2\frac{\hat{A}\,\hat{C}}{A\,C}\right]\,, \tag{5.140}$$

which can be employed to constrain the metric functions such that, e.g., $\mu + 3\,p > 0$ holds. If a simple equation of state, $p = \alpha\,\mu$, say, is assumed, one finds from this relation an expression for the energy density μ. On the other hand, the field equations imply

$$\mu + \Lambda - K = \left(\frac{\dot{C}}{C}\right)^2 - \left(\frac{\hat{C}}{C}\right)^2 + 2\frac{\dot{B}\,\dot{C}}{B\,C} - 2\frac{\hat{\hat{C}}}{C} \tag{5.141}$$

which is in general different from the expression for μ obtained from equation (5.140) and therefore gives an additional constraint for the metric functions. Finally, the pressure variables and the electric Weyl curvature can be calculated similarly but the resulting expressions are somewhat long and will not be stated here.

The perturbations

When local comoving coordinates are chosen, the wave equations governing the scalar and electromagnetic perturbations in the sourcefree case are best given in form of equation (5.52) and (5.67), respectively, since there only the kinematical variables enter and these are simple expressions in terms of the metric functions [cf. equations (5.134)–(5.137)]. The corresponding wave equations read in the case of scalar field perturbations ψ as

$$\ddot{\psi} - \hat{\hat{\psi}} + \left(\frac{\dot{B}}{B} + 2\,\frac{\dot{C}}{C}\right)\dot{\psi} + \left(\frac{\hat{A}}{A} - 2\,\frac{\hat{C}}{C}\right)\hat{\psi} + (M^2 - \delta^2)\,\psi = 0 \,, \tag{5.142}$$

while in the case of (rescaled) electromagnetic perturbations E (or B) they become

$$\ddot{E} - \hat{\hat{E}} + \left(\frac{\dot{B}}{B}\right)\dot{E} + \left(\frac{\hat{A}}{A}\right)\hat{E} + \left[3\,\frac{\dot{B}\,\dot{C}}{B\,C} - 2\left(\frac{\dot{C}}{C}\right)^2 - \delta^2\right] E = 0. \tag{5.143}$$

Once the metric functions A, B and C in (5.133) are given for a known LRS class II solution, these wave equations may readily be transformed into their concomitant coordinate analogues.

Observe that the 'radial' parameter r, covariantly defined by the relations (5.32), agrees with the metric function $C(t,x)$ in the line element (5.133) in account of (5.134) and (5.136)–(5.137). Thus the physical electromagnetic perturbations $\mathscr{E}$ are obtained from the rescaled ones E by the transformation $\mathscr{E} = C^{-2}(t,x)\,E$. If the scalar field ψ is analogously rescaled as $\psi = C^{-1}(t,x)\,\Psi$, then the wave equation for the scalar field perturbation (5.142) takes the Regge-Wheeler form

$$\ddot{\Psi} - \hat{\hat{\Psi}} + \left(\frac{\dot{B}}{B}\right)\dot{\Psi} + \left(\frac{\hat{A}}{A}\right)\hat{\Psi} + \left[M^2 + \frac{\hat{\hat{C}}}{C} - \frac{\dot{\dot{C}}}{C} - \frac{\dot{B}\,\dot{C}}{B\,C} - \frac{\hat{A}\,\hat{C}}{A\,C} - \delta^2\right]\Psi = 0 \,. \tag{5.144}$$

It is thus obvious that once a harmonic decomposition has been applied to the equations (5.143) and (5.144), the resulting wave equations will only differ in their corresponding potential term.

If in addition to the LRS symmetry further symmetries are present, then the wave equations for the perturbations simplify considerably. For example, for stationary LRS class II spacetimes we have

$$\ddot{\Psi} - \hat{\hat{\Psi}} + \left(\frac{\hat{A}}{A}\right)\hat{\Psi} + \left[M^2 + \frac{\hat{\hat{C}}}{C} - \frac{\hat{A}\hat{C}}{AC} - \delta^2\right]\Psi = 0 \,, \tag{5.145}$$

$$\ddot{E} - \hat{\hat{E}} + \left(\frac{\hat{A}}{A}\right)\hat{E} - \delta^2 E = 0 \,; \tag{5.146}$$

while for spatially homogeneous spacetimes (where one may set $A = 1$) we have

$$\ddot{\Psi} - \hat{\hat{\Psi}} + \left(\frac{\dot{B}}{B}\right)\dot{\Psi} + \left[M^2 - \frac{\ddot{C}}{C} - \frac{\dot{B}\,\dot{C}}{B\,C} - \delta^2\right]\Psi = 0\,, \tag{5.147}$$

$$\ddot{E} - \hat{\hat{E}} + \left(\frac{\dot{B}}{B}\right)\dot{E} + \left[3\,\frac{\dot{B}\,\dot{C}}{B\,C} - 2\left(\frac{\dot{C}}{C}\right)^2 - \delta^2\right]E = 0\,. \tag{5.148}$$

Application – LTB dust Universe

As an example of an inhomogeneous spacetime, let us consider the Lemaître-Tolman-Bondi (LTB) [12, 158–160] dust Universe (with $\Lambda = 0$), whose metric is given by

$$ds^2 = -dt^2 + \frac{(Y')^2}{k - \varepsilon\, f^2(r)}\,dr^2 + Y^2\left[d\vartheta^2 + D^2(\vartheta, k)\,d\varphi^2\right]. \tag{5.149}$$

Here, a prime means $\partial/\partial\, r$ and a dot will denote $\partial/\partial\, t$. Moreover, $Y = Y(t, r)$ is the solution of

$$\dot{Y}^2 - 2\,m(r)/Y = -\varepsilon\, f^2(r)\,, \tag{5.150}$$

wherein $\varepsilon = (-1, 0, 1)$ corresponds to the hyperbolic, parabolic and elliptic solution, respectively. The constraint equation can be integrated completely, which yields an additional function $t_B(r)$. Therefore, there are three functions which can be prescribed at will: the 'mass' $m(r)$, the 'energy' $f(r)$ and the 'bang time' $t_B(r)$.

It follows from equations (5.134)–(5.137) that the acceleration $\mathcal{A}$ has to vanish and that the non-zero dynamical variables take the form

$$\phi = 2\,\frac{\sqrt{k - \varepsilon\, f^2}}{Y}, \tag{5.151}$$

$$\Theta = \frac{\dot{Y}'}{Y'} + 2\,\frac{\dot{Y}}{Y}\,, \tag{5.152}$$

$$\Sigma = \frac{2}{3}\left(\frac{\dot{Y}'}{Y'} - \frac{\dot{Y}}{Y}\right). \tag{5.153}$$

An expression for the energy density μ can be gained from equation (5.140) and yields, using the above mentioned constraint,

$$\mu = \frac{2\,m'}{Y'\,Y^2}\,. \tag{5.154}$$

Finally, the wave equations (5.143) and (5.144) governing the perturbations $\mathscr{P}$ can be written as

$$\ddot{\mathscr{P}} - \frac{k-\varepsilon f^2}{(Y')^2}\left[\mathscr{P}'' - \left(\frac{Y''}{Y'} + \frac{\varepsilon f f'}{k-\varepsilon f^2}\right)\mathscr{P}'\right] + \frac{\dot{Y}'}{Y'}\,\dot{\mathscr{P}} + (V-\delta^2)\mathscr{P} = 0\ ; \tag{5.155}$$

the potential V is given by

$$V = V_{EM} = 3\,\frac{\dot{Y}\,\dot{Y}'}{Y\,Y'} - 2\left(\frac{\dot{Y}}{Y}\right)^2 \tag{5.156}$$

in the case of electromagnetic perturbations, whereas

$$V = V_S = M^2 + \frac{m}{Y^3} + \left(\frac{\dot{Y}\dot{Y}' + \varepsilon f f'}{YY'}\right) \tag{5.157}$$

denotes the potential in the case of scalar field perturbations.

Application – Kantowski-Sachs dust Universe

As a further example we consider a spherically symmetric Kantowski-Sachs dust Universe with $\Lambda = 0$ [12, 161, 162]. The metric is written as

$$ds^2 = -dt^2 + B^2(t)\,dr^2 + C^2(t)\,d\Omega^2\ , \tag{5.158}$$

where the metric functions are conveniently expressed in terms of a parameter $\eta(t)$ satisfying $dt = 2\,C\,d\eta$, namely

$$B = m\,(\eta\,\tan\eta + 1) + b\,\tan\eta\ , \tag{5.159}$$
$$C = c\,\cos^2\eta\ ; \qquad m\,,b\,,c = const\ . \tag{5.160}$$

Since this spacetime is spatially homogeneous, the acceleration $\mathcal{A}$ and expansion ϕ of the fundamental observer's congruence have to vanish [cf. equations (5.135)–(5.134)]. The remaining quantities are found to be

$$\mu = \frac{m}{B\,C^2}\ , \tag{5.161}$$
$$\mathcal{E} = \frac{m\,\cos^2\eta - 3\,B}{3\,B\,C^2\,\cos^2\eta}\ , \tag{5.162}$$
$$\Theta = c\,\frac{m\left(\eta - 3\,\sin\eta\,\cos\eta - 4\,\eta\,\sin^2\eta\right) + b\left(1 - 4\,\sin^2\eta\right)}{2\,B\,C^2}\ , \tag{5.163}$$
$$\Sigma = c\,\frac{m\left(\eta + 3\,\sin\eta\,\cos\eta + 2\,\eta\,\sin^2\eta\right) + b\left(1 + 2\,\sin^2\eta\right)}{3\,B\,C^2}\ . \tag{5.164}$$

The equations for the perturbations $\mathscr{P}$ now become in terms of the parameter η

$$\left\{ \frac{\partial^2}{\partial\eta^2} - \frac{4\,C^2}{B^2}\,\frac{\partial^2}{\partial r^2} + \frac{B\left(1+2\sin^2\eta\right) - m\,\cos^2\eta}{B\,\sin\eta\,\cos\eta}\,\frac{\partial}{\partial\eta} + 4\,C^2\left(V-\delta^2\right) \right\} \mathscr{P} = 0\,, \tag{5.165}$$

where

$$V = V_S = M^2 + \frac{2\,B - m\,\cos^2\eta}{2\,B\,C^2\,\cos^2\eta} \tag{5.166}$$

in the case of scalar perturbations, and

$$V = V_{EM} = \frac{3\,m\,\cos^2\eta - B\left(3+4\,\sin^2\eta\right)}{2\,B\,C^2\,\cos^2\eta} \tag{5.167}$$

in the case of electromagnetic perturbations, respectively.

5.5 Summary

Employing the 1+1+2 formalism of Clarkson & Barrett [34], we presented a covariant description of LRS class II spacetimes in terms of scalar quantities, which all have either a clear physical or geometrical meaning. We investigated scalar and electromagnetic perturbations (test fields) on LRS class II spacetimes and found that they are governed by covariant wave equations [see equations (5.53) and (5.67)], which are the covariant generalisations of the Regge-Wheeler equation, known to describe perturbations of the Schwarzschild spacetime. In particular, it was shown that both scalar and electromagnetic perturbations (in the absence of sources) are governed by master equations of the same form, namely the covariant *generalised Regge-Wheeler equation*,

$$\ddot{\mathscr{P}} - \hat{\hat{\mathscr{P}}} - \mathcal{A}\,\hat{\mathscr{P}} + \left(\Sigma + \tfrac{1}{3}\Theta\right)\dot{\mathscr{P}} + \left(V - \delta^2\right)\mathscr{P} = 0\,, \tag{5.168}$$

where the potential V is different for the two cases considered. To arrive at this specific form, one had to rescale the perturbations with the 'radial' parameter r, which is induced from the Gaussian curvature K of the sheet and can thus only be defined covariantly for LRS class II spacetimes [cf. (5.32)]. The findings have been discussed in some detail for the particular cases of Schwarzschild and Vaidya spacetimes, and for some dust Universe models. While the master equations may be written simply as equation (5.168) in covariant form, they can become very untidy when written explicitly in coordinates, demonstrating some of the advantages of using a covariant approach.

Throughout this chapter, we excluded gravitational perturbations of LRS class II spacetimes from the discussion. While gravitational perturbations of Schwarzschild spacetime are known to obey the master equation (5.168) [see [34] and the discussion in (5.4.1)], it is not clear at all if and how these results may be generalised towards more general LRS class II spacetimes. The most interesting result of the analysis [34] was that *both* even and odd gravitational perturbations can be described by a single tensor, namely the Regge-Wheeler tensor W_{ab}. It would be particularly interesting to investigate if it is possible to generalise W_{ab} to the non-vacuum case. It has been known for a long time that scalar field, electromagnetic and gravitational perturbations are governed by master equations in the case of vacuum, algebraically special (Petrov type D) spacetimes [30]. So far, we demonstrated the existence of master equations for scalar and electromagnetic perturbations in the case of LRS II spacetimes (including matter). The study of gravitational perturbations in these cases has to be left for future research.

Chapter 6

Electromagnetic signature of black hole ringdown

6.1 Overview

A perturbed star viewed as an out-of-equilibrium fluid will radiate away its deformation in form of gravitational waves thereby restoring equilibrium. Fascinatingly, the same is true for the case of BHs although there is no fluid at all. A slightly perturbed BH will try to achieve its highly symmetrical ground state by radiating away the spacetime deformations through GWs which are carried by the spacetime itself, highlighting the fact that GR is a dynamical theory. Such a GW has the characteristics of a sound wave produced by whacking a bell: an initially complicated oscillatory pattern goes over into a ringdown phase, distinguished by a single frequency independent of initial conditions, and ultimately in a power law tail. It is this analogy that gave rise to the name BH ringdown for the settling down of a vibrating BH into equilibrium.

A typical BH is immersed in a surrounding plasma, usually accompanied by an accretion disc, which sustains a magnetosphere. If such a BH undergoes a perturbation, the produced GWs will interact with the plasma and consequently generate an EM signal that will bear all the characteristic features of the forcing GW. In particular, the EM signal will show the typical ringdown, ringing with the same frequency as the forcing GW. If such an EM signal would be detectable, it could be used for an independent confirmation of a GW signal observed simultaneously with GW detectors on Earth.

To investigate this interaction in a realistic manner, a proper description of the magnetosphere in terms of a 1+1+2 covariant fluid mechanics, preferably in the magnetohydrodynamic limit, is needed. Unfortunately, such a tool is not at hand yet, despite some recent progress in the case of the 1+3 framework [144]. Thus, we have to content ourselves for the moment to the 'simple' case where GWs interact with a strong static magnetic field, which is the goal of this chapter.

Nevertheless, it is still possible to draw a picture of what happens in the more general case. There are two types of first-order perturbations now, the first is associated with the GWs while the second is due to the magnetosphere (magnetic field, charge and current density). Therefore, we treat GWs and the plasma at the same footing, that is, we neglect contributions from their corresponding energy-momentum tensors in comparison to the curvature of the BH. The interaction of GWs and the magnetosphere is thus a second order effect meaning that the induced EM signal is second order, too. We are therefore let to divide up the perturbation 'background' spacetimes in the following manner:

- $\mathcal{B}$ = Exact Schwarzschild, $\mathcal{O}(\epsilon^0)$;
- $\mathcal{F}_1$ = Schwarzschild with gravitational perturbations $\mathcal{O}(\epsilon_g)$;
- $\mathcal{F}_2$ = Exact Schwarzschild perturbed by a plasma carrying a magnetosphere: $\mathcal{O}(\epsilon_{\mathscr{B}})$;
- $\mathcal{S} = \mathcal{F}_1 + \mathcal{F}_2$ allowing for interaction terms in the plasma equations (Maxwell's equations, fluid or magnetohydrodynamic equations, etc. — depending on the level of sophistication): the induced EM fields will be $\mathcal{O}(\epsilon_{\mathscr{B}}\epsilon_g)$.

The resulting equations containing the interaction terms will in general be a mixture of first and second-order perturbation terms and will therefore not be gauge-invariant. Upon defining appropriate second-order gauge-invariant (SOGI) variables, which vanish in $\mathcal{F}$ and encapsule the various interaction terms as well as the induced EM fields, and deriving the concomitant propagation and evolution equations, it should be possible to consistently replace the non-gauge-invariant equations by a system of gauge-invariant, purely second-order perturbation equations. The resulting system might be put in an initial value formulation or, in the static subclass, in a quasi-normal mode formulation.

The described strategy was successfully carried out in [136] for a toy model scenario where the magnetosphere is simply modelled by a static magnetic dipole field without a supporting plasma. It is exactly this scenario which is of interest for the rest of this chapter.

6.2 $\mathcal{B}$: The Schwarzschild background

The 1+1+2 covariant approach neatly addresses the problem of gauge freedom but there is still freedom of choosing a frame, that is the arbitrary orientation of the spatial congruence n^a. For the Schwarzschild geometry, a natural choice is to define n^a parallel to the acceleration $\dot{u}^a$ of the observers, the $\mathcal{A}^a = \delta^a\phi = \delta^a\mathcal{A} = 0$ frame. Thus, for a family of *static* observers, in the background there are only the zeroth-order scalars: $\mathcal{E}$, the *radial tidal force*; $\mathcal{A}$, the acceleration a static observer must apply radially outwards (to prevent infall); and ϕ, the spatial expansion of the radial vector n^a. These are determined by the radial propagation equations:

$$\hat{\phi} = -\tfrac{1}{2}\phi^2 - \mathcal{E}\,, \tag{6.1}$$
$$\hat{\mathcal{E}} = -\tfrac{3}{2}\phi\mathcal{E}\,; \tag{6.2}$$

together with the constraint

$$\mathcal{E} + \mathcal{A}\phi = 0\,. \tag{6.3}$$

Defining the affine parameter ρ by $\hat{} = d/d\rho$, and another radial parameter r by [cf (5.32)]

$$\hat{r} = \tfrac{1}{2}\phi r\,, \qquad \dot{r} = 0 = \delta_r\,, \tag{6.4}$$

the parametric solution to these equations, giving a complete description of the BH, are given by

$$\mathcal{E} = -\frac{2m}{r^3}, \tag{6.5}$$
$$\phi = \frac{2}{r}\sqrt{1-\frac{2m}{r}}, \tag{6.6}$$
$$\mathcal{A} = \frac{m}{r^2}\left(1-\frac{2m}{r}\right)^{-1/2}; \tag{6.7}$$

where

$$\rho = 2m\cosh^{-1}\left(\sqrt{\frac{r}{2m}}\right) + r\sqrt{1-\frac{2m}{r}}, \tag{6.8}$$

relates the affine parameter ρ associated with the radial vector n^a with the usual Schwarzschild coordinate $r = \left(\frac{1}{4}\phi^2 - \mathcal{E}\right)^{-1/2}$ [see also (5.21)].

In $\mathcal{F}$ and $\mathcal{S}$ we keep all powers of these variables.

We give a plot of ϕ and $\mathcal{A}$ as a function of the Schwarzschild coordinate r in Fig. 6.1. This shows how the expansion of n^a starts from zero at the horizon $r = 2m$, is largest at the photon sphere, before dropping to zero again as $r \to \infty$.

6.3 $\mathcal{F}_1$: The gravity wave perturbation

As shown in [34], GW perturbations are governed completely, in the $\mathcal{A}^a = \delta^a\phi = \delta^a\mathcal{A} = 0$ frame, by the *Regge-Wheeler tensor* W_{ab} which is a gauge- and frame-invariant TT tensor,

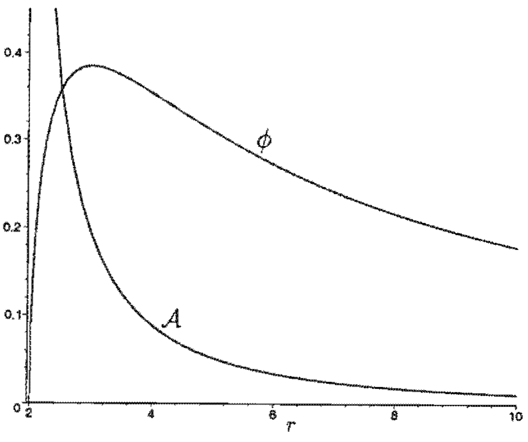

Figure 6.1: A plot of ϕ and $\mathcal{A}$ with r, showing the maxima of ϕ at the photon sphere, $r = 3m$. In contrast, $\mathcal{A}$ falls from infinity at the horizon $r = 2\,m$.

defined as [34]

$$W_{ab} = \tfrac{1}{2}\,\phi\, r^2\, \zeta_{ab} - \tfrac{1}{3}\, r^2\, \mathcal{E}^{-1}\, \delta_{\{a}\, X_{b\}}\ , \tag{6.9}$$

and $X_a = \delta_a\,\mathcal{E}$ is the gauge-invariant variable [30] describing the angular fluctuation in the radial tidal force, while ζ_{ab} describes the shear of the spatial congruence n_a. This tensor contains in compact form the curved space generalisation of the two flat-space GW polarisations h_+ and $h_\times$, and obeys the covariant tensorial form of the Regge-Wheeler equation [34, 154]

$$-\ddot{W}_{ab} + \hat{\hat{W}}_{ab} + \mathcal{A}\,\hat{W}_{ab} - \phi^2\, W_{ab} + \delta^2\, W_{ab} = 0\ . \tag{6.10}$$

Every other object in $\mathcal{F}_1$ is determined by linear combinations of $\{W_\mathrm{T}\,, \hat{W}_\mathrm{T}\,, \dot{W}_\mathrm{T}\,, \hat{\dot{W}}_\mathrm{T}\}$, once appropriate harmonics are used. Notice that equation (6.10) governs GWs of both parities; this should be compared to the metric approach where one finds that odd parity GWs are described by the Regge-Wheeler (RW) equation while even parity GWs are described by the Zerilli equation [171]. According to [34], the even perturbations may be conveniently described by the variables $\{W_\mathrm{T}, \mathcal{Z}\}$ where W_T obeys the Regge-Wheeler equation and $\mathcal{Z}$ the Zerilli equation. However, the Zerilli variable $\mathcal{Z}$ can be expressed as a linear combination of W_T and $\hat{W}_\mathrm{T}$ and is therefore not a fundamental object.

For simplicity, we will only consider the case where the GWs are of *odd parity*. For purely odd perturbations, the gravitational field is governed by W_{ab}, and the other GW variables that we require are related to this via the *covariant* gauge-invariant equations [34]

$$
\begin{aligned}
\zeta_{ab} &= \frac{2}{\phi\, r^2}\, W_{ab}\,, && (6.11)\\
\dot{\Sigma}_{\{ab\}} &= r^{-2}\, W_{ab} + \frac{2}{\phi\, r^2}\, \hat{W}_{ab}\,, && (6.12)\\
\dot{\alpha}_{\bar{a}} = -\dot{\Sigma}_{\bar{a}} = \mathcal{E}_a &= \frac{2}{\phi\, r^2}\, \delta^b\, W_{ab}\,, && (6.13)\\
\varepsilon_{ab}\, \dot{\mathcal{H}}^b &= -\frac{2}{\phi\, r^2}\, \delta^b\, \hat{W}_{bc}\,, && (6.14)\\
\dot{\mathcal{H}} &= -\frac{2}{\phi\, r^2}\, \varepsilon_{ab}\, \delta^a\, \delta^c\, W^b_{\ c}\,. && (6.15)
\end{aligned}
$$

There are similar relations for $\mathcal{E}_{ab}$ and $\mathcal{H}_{ab}$ but we will not need them (all other 1+1+2 variables are zero).
In $\mathcal{S}$ we neglect all products of these quantities.

Because the background $\mathcal{B}$ is spherically symmetric, the RW tensor W_{ab} may be expanded in spherical harmonics (cf. Appendix B),

$$
W_{ab} = \sum_{\ell_g=2}^{\infty} \overset{(\ell_g)}{W_{ab}}\,, \tag{6.16}
$$

if separable solutions are assumed. The harmonic components of the RW tensor obey then the constraint equations

$$
\begin{aligned}
\delta^2\, \overset{(\ell_g)}{W_{ab}} &= \left(\phi^2 - 3\,\mathcal{E} - L_g\, r^{-2}\right) \overset{(\ell_g)}{W_{ab}}\,, && (6.17)\\
\delta^a\, \delta^b\, \overset{(\ell_g)}{W_{ab}} &= 0\,. && (6.18)
\end{aligned}
$$

6.3.1 Time harmonics and quasi-normal modes

Because the background is static, it is often advantageous to introduce time harmonics as defined in [34]. The time derivatives of first order quantities are decomposed into their Fourier components by assuming an $e^{i\omega\tau}$ time dependence for the first order variables; factors of $i\omega$ just represent time derivatives, $d/d\tau$. Note that

$$
\hat{\omega} = -\mathcal{A}\,\omega \quad \Rightarrow \quad \omega = \sigma\left(1 - \frac{2\,m}{r}\right)^{-1/2} = \frac{2\,\sigma}{\phi\, r}\,, \tag{6.19}
$$

arising from the commutation relation between the dot- and hat-derivatives [see (2.120)]. The harmonic function ω is defined with respect to proper time τ of observers moving along u^a, while σ is the *constant* harmonic index associated with time t of observers at infinity. They are related by $\omega\,\tau = \sigma\,t$.

A GW perturbation, as described by the Regge-Wheeler and Zerilli equations, has to be restricted by boundary conditions in order to yield a physically acceptable solution. Clearly, a GW cannot be allowed to propagate out of the horizon. The relevant boundary conditions for GW detection are those that represent a GW perturbation which propagates outwards at infinity ($r \sim r_* \to \infty$) and inwards to the horizon ($r \to 2\,m$, $r_* \to -\infty$). The form of the Regge-Wheeler and Zerilli variables corresponding to this are

$$\psi \sim e^{i\sigma r_*} \quad \text{as} \quad r_* \to -\infty \quad \text{and} \quad \psi \sim e^{-i\sigma r_*} \quad \text{as} \quad r_* \to +\infty\,, \tag{6.20}$$

where $\psi = W$ or $\psi = \mathcal{Z}$ (see, e.g., [95]) and r_* is the tortoise Regge-Wheeler coordinate [154] defined by

$$d\rho = \tfrac{1}{2}\,\phi\, r dr_* = \left(\tfrac{1}{2}\,\phi\, r\right)^{-1} dr \quad \Rightarrow \quad r_* = r + 2\,m \ln\left(\frac{r}{2\,m} - 1\right)\,, \tag{6.21}$$

which shifts the horizon towards $-\infty$. It turns out that the only solutions with boundary conditions (6.20) require *discrete* values of the complex frequency parameter σ, with $\Im(\sigma) > 0$; these are referred to as *quasinormal frequencies*, and the solutions constructed from them as *quasinormal modes* (QNMs). Because of the $e^{i\omega\tau}$ time dependence, these decay exponentially in time, which corresponds to energy radiated to infinity or the horizon as GW. This damping in time is important as ψ grows exponentially as $r \to \infty$. For more details on QNMs, the reader is referred to references [95, 96, 172].

6.4 $\mathcal{F}_2$: The static magnetic field

It follows from Maxwell's equations [see subsection (3.2.2) in chapter 3] that the static ($\dot{\mathscr{B}} = \dot{\mathscr{B}}_a = 0$) magnetic field is governed by the 1+1+2 equations

$$\hat{\mathscr{B}} = -\delta_a\,\mathscr{B}^a - \phi\,\mathscr{B}\,, \tag{6.22}$$

$$\hat{\mathscr{B}}_a = \delta_a\,\mathscr{B} - \left(\tfrac{1}{2}\,\phi + \mathcal{A}\right)\,\mathscr{B}_a\,, \tag{6.23}$$

$$0 = \varepsilon_{ab}\,\delta^a\,\mathscr{B}^b\,, \tag{6.24}$$

where the last equation tells us that the field is purely of even parity. The solution to these equations for arbitrary harmonic index, when harmonically decomposed, can only be written as a complicated combination of hypergeometric functions.

In $\mathcal{S}$ we neglect all products of the magnetic field with itself.

Since the background $\mathcal{B}$ is spherically symmetric and the static magnetic field is viewed as a first-order perturbation, the magnetic field solution may be expanded in spherical harmonics. This implies that we can write

$$\mathscr{B} = \sum_{\ell_{\mathscr{B}}=1}^{\infty} \mathscr{B}^{(\ell_{\mathscr{B}})} , \tag{6.25}$$

$$\mathscr{B}_a = \sum_{\ell_{\mathscr{B}}=1}^{\infty} \mathscr{B}_a^{(\ell_{\mathscr{B}})} , \tag{6.26}$$

where the harmonic components of the static magnetic field obey the constraint equations, where $L = \ell\,(\ell+1)$,

$$\delta^2\, \mathscr{B}^{(\ell_{\mathscr{B}})} = -L_{\mathscr{B}}\, r^{-2}\, \mathscr{B}^{(\ell_{\mathscr{B}})} , \tag{6.27}$$

$$\delta^2\, \mathscr{B}_a^{(\ell_{\mathscr{B}})} = (1 - L_{\mathscr{B}})\, r^{-2}\, \mathscr{B}_a^{(\ell_{\mathscr{B}})} , \tag{6.28}$$

$$\varepsilon^{ab}\, \delta_a\, \mathscr{B}_b^{(\ell_{\mathscr{B}})} = 0 . \tag{6.29}$$

The solution for the $\ell = 1$ equations are of particular importance because they embrace fields often used to model the magnetosphere of a compact object: namely the case of a static asymptotically uniform magnetic field as well as a static dipole field, which falls off like $1/r^3$ at infinity. The general solutions for the case $\ell = 1$ were given already before in (5.101)–(5.102) in terms of r. For our purpose, what is of interest here is merely the solution for the static magnetic dipole field, which is given by

$$\mathscr{B}_{\mathsf{S}} = -\frac{3\,\mathscr{B}_\infty}{(2\,m)^3}\left[\ln\left(1 - \frac{2\,m}{r}\right) + \frac{2\,m}{r}\left(1 + \frac{m}{r}\right)\right] , \tag{6.30}$$

$$\mathscr{B}_{\mathsf{V}} = -\sqrt{1 - \frac{2\,m}{r}}\,\frac{3\,\mathscr{B}_\infty}{(2m)^3}\left[\ln\left(1 - \frac{2\,m}{r}\right) + \frac{2\,m}{r}\,\frac{r-m}{r-2\,m}\right] , \tag{6.31}$$

where $\mathscr{B}_\infty$ denotes the magnitude of $\mathscr{B}_{\mathsf{S}}\, r^3$ as $r \to \infty$.

6.5 The interaction between GWs and magnetic field

Here we will introduce a set of auxiliary variables, all of order $\mathcal{O}(\epsilon_{\mathscr{B}}\, \epsilon_g)$, which allow us to convert Maxwell's equations into a linear (in differential order) system of *gauge-invariant* ordinary differential equations (gauge-invariant because they vanish at all perturbative orders lower than this [118, 119]). We refer to these as the interaction variables. A quick glance at the right hand side of Maxwell's equations (3.54)–(3.59) reveals that we are dealing with products of tensorial spherical harmonics, which are not particularly pleasant. Instead of explicitly using tensor spherical harmonics in the GW×B products in Maxwell's equations, we shall absorb them into the following interaction variables, which makes the resulting equations considerably neater. There is no extra work involved here, although it may not appear that way; we would otherwise still require the key equations (6.37) and (6.51)–(6.56) given below. The latter in particular are crucial relations among all the coupled tensor/vector/scalar spherical harmonics which appear (these are the products of the variables (6.33)–(6.36) with the variables (6.43)–(6.46), although we have absorbed the magnetic field strength and the GW amplitude). There is another reason for defining the variables in the manner we do: while variables such as α_a appear in Maxwell's equations, our solution in $\mathcal{F}_2$ only gives us $\dot{\alpha}_{\bar{a}}$; we circumvent this problem by absorbing the time derivatives into our new variables below.

With these considerations in mind, we define the four *interaction variables*

$$\overset{(\ell_{\mathscr{B}},\ell_g)}{\chi_a} = \left(\overset{(\ell_{\mathscr{B}},\ell_g)}{\chi_{1a}}, \overset{(\ell_{\mathscr{B}},\ell_g)}{\chi_{2a}}, \overset{(\ell_{\mathscr{B}},\ell_g)}{\chi_{3a}}, \overset{(\ell_{\mathscr{B}},\ell_g)}{\chi_{4a}}\right) \tag{6.32}$$

as follows:

$$\dot{\overset{(\ell_{\mathscr{B}},\ell_g)}{\chi_{1a}}} = \left(\phi\, r^2\right)^{-1} \overset{(\ell_g)}{W_{ab}}\, \mathscr{B}^b_{(\ell_{\mathscr{B}})}, \tag{6.33}$$

$$\dot{\overset{(\ell_{\mathscr{B}},\ell_g)}{\chi_{2a}}} = \left(\phi\, r^2\right)^{-1} \overset{(\ell_g)}{\hat{W}_{ab}}\, \mathscr{B}^b_{(\ell_{\mathscr{B}})}, \tag{6.34}$$

$$\dot{\overset{(\ell_{\mathscr{B}},\ell_g)}{\chi_{3a}}} = \left(\phi\, r^2\right)^{-1} \overset{(\ell_g)}{W_{ab}}\, \delta^b \mathscr{B}^{(\ell_{\mathscr{B}})}, \tag{6.35}$$

$$\dot{\overset{(\ell_{\mathscr{B}},\ell_g)}{\chi_{4a}}} = \left(\phi\, r^2\right)^{-1} \overset{(\ell_g)}{\hat{W}_{ab}}\, \delta^b \mathscr{B}^{(\ell_{\mathscr{B}})}, \tag{6.36}$$

for each $\ell_{\mathscr{B}} \leftrightarrow \ell_g$ interaction. We use a bold font as a matrix shorthand for the '4-vector' these variables form. These variables obey the propagation equations

$$\dot{\overset{(\ell_{\mathscr{B}},\ell_g)}{\boldsymbol{\chi}_a}} = \boldsymbol{\Gamma}^{(\ell_{\mathscr{B}},\ell_g)}\, \overset{(\ell_{\mathscr{B}},\ell_g)}{\boldsymbol{\chi}_a}, \tag{6.37}$$

with the *interaction matrices* given by, for each $\ell_{\mathscr{B}}$ and ℓ_g,

$$\boldsymbol{\Gamma}^{(\ell_{\mathscr{B}},\ell_g)} = \begin{pmatrix} -(\phi+\mathcal{A}) & 1 & 1 & 0 \\ \Delta^{(\ell_g)} & -(\phi+2\,\mathcal{A}) & 0 & 1 \\ L_{\mathscr{B}}\, r^{-2} & 0 & -2\,\phi & 1 \\ 0 & L_{\mathscr{B}}\, r^{-2} & \Delta^{(\ell_g)} & -(2\,\phi+\mathcal{A}) \end{pmatrix}, \tag{6.38}$$

where

$$\Delta^{(\ell_g)} \equiv -\omega^2 + 3\,\mathcal{E} + L_g\, r^{-2}\,. \tag{6.39}$$

We have introduced the time harmonics of [34], as discussed briefly above, into these equations for notational simplicity. In order to simplify our presentation, we will define a set of auxiliary interaction variables as follows (some of which may be a little surprising, but they are all required). First, define

$$\boldsymbol{V}_a^{(\ell_g)} = \begin{pmatrix} V_{1a}^{(\ell_g)} \\ V_{2a}^{(\ell_g)} \end{pmatrix} = \begin{pmatrix} \delta^b\, W_{ab}^{(\ell_g)} \\ \delta^b\, \hat{W}_{ab}^{(\ell_g)} \end{pmatrix}\,, \tag{6.40}$$

where we use a bold font to denote the '2-vector' matrix. Similarly we define

$$\boldsymbol{\lambda}_a^{(\ell_{\mathscr{B}})} = \begin{pmatrix} \lambda_{1a}^{(\ell_{\mathscr{B}})} \\ \lambda_{2a}^{(\ell_{\mathscr{B}})} \end{pmatrix} = \begin{pmatrix} \mathscr{B}_a^{(\ell_{\mathscr{B}})} \\ \delta_a\, \mathscr{B}^{(\ell_{\mathscr{B}})} \end{pmatrix}. \tag{6.41}$$

For simplicity of presentation, we introduce the shorthand notation '$\circ$' which takes two '2-vectors' to form a '4-vector' as

$$\boldsymbol{V} \circ \boldsymbol{\lambda} = (V_1\,\lambda_1\,,\ V_2\,\lambda_1\,,\ V_1\,\lambda_2\,,\ V_2\,\lambda_2)\ . \tag{6.42}$$

We use these to define the following '4-vector' variables as follows

$$\dot{\boldsymbol{K}}^{(\ell_{\mathscr{B}},\ell_g)} = (\phi\, r)^{-1}\,\varepsilon_{ab}\boldsymbol{V}^a_{(\ell_g)} \circ \boldsymbol{\lambda}^b_{(\ell_{\mathscr{B}})}\,, \tag{6.43}$$

$$\dot{\boldsymbol{\Psi}}_a^{(\ell_{\mathscr{B}},\ell_g)} = \phi^{-1}\quad \boldsymbol{V}_a^{(\ell_g)} \circ \delta^c\,\boldsymbol{\lambda}_c^{(\ell_{\mathscr{B}})}\,, \tag{6.44}$$

$$\dot{\boldsymbol{M}}_a^{(\ell_{\mathscr{B}},\ell_g)} = \phi^{-1}\quad \boldsymbol{V}^b_{(\ell_g)} \circ \delta_{\{a}\,\boldsymbol{\lambda}_{b\}}^{(\ell_{\mathscr{B}})}\,, \tag{6.45}$$

$$\dot{\boldsymbol{J}}_a^{(\ell_{\mathscr{B}},\ell_g)} = \phi^{-1}\,\varepsilon_{cd}\,\delta^c\,\boldsymbol{V}^d_{(\ell_g)} \circ \varepsilon_{ab}\,\boldsymbol{\lambda}^b_{(\ell_{\mathscr{B}})}\,, \tag{6.46}$$

where, for example, $\boldsymbol{K} = (K_1, K_2, K_3, K_4)$ gives the shorthand for four of these sixteen new variables. These variables are all $\mathcal{O}(\epsilon_{\mathscr{B}}\epsilon_g)$. They are all constructed to obey the same propagation equation as $\boldsymbol{\chi}$, viz:

$$\hat{\boldsymbol{K}}^{(\ell_{\mathscr{B}},\ell_g)} = \boldsymbol{\Gamma}^{(\ell_{\mathscr{B}},\ell_g)}\,\boldsymbol{K}^{(\ell_{\mathscr{B}},\ell_g)}\,, \tag{6.47}$$

$$\hat{\boldsymbol{\Psi}}_a^{(\ell_{\mathscr{B}},\ell_g)} = \boldsymbol{\Gamma}^{(\ell_{\mathscr{B}},\ell_g)}\,\boldsymbol{\Psi}_a^{(\ell_{\mathscr{B}},\ell_g)}\,, \tag{6.48}$$

$$\hat{\boldsymbol{M}}_a^{(\ell_{\mathscr{B}},\ell_g)} = \boldsymbol{\Gamma}^{(\ell_{\mathscr{B}},\ell_g)}\,\boldsymbol{M}_a^{(\ell_{\mathscr{B}},\ell_g)}\,, \tag{6.49}$$

$$\hat{\boldsymbol{J}}_a^{(\ell_{\mathscr{B}},\ell_g)} = \boldsymbol{\Gamma}^{(\ell_{\mathscr{B}},\ell_g)}\,\boldsymbol{J}_a^{(\ell_{\mathscr{B}},\ell_g)}\,. \tag{6.50}$$

We have defined all these variables as the time integral of combinations of the RW tensor and the static magnetic field. This is because for some of the GW variables appearing in Maxwell's equations it is their time derivatives which are related to the RW tensor [see, e.g., equations (6.12) and (6.13)]. Defining auxiliary interaction variables this way which satisfy the propagation equations (6.47)–(6.50) removes this problem, and absorbs it into the initial (or boundary) conditions.

By taking various δ-derivatives of these variables and using the appropriate commutation relations [see [34] or subsection (2.2.2)], together with equation (6.18), we can show that they all obey the following constraints, which are crucial identities for consistency of the resulting equations later, and allow us to relate all the interaction terms to χ when we split Maxwell's equations into spherical harmonics.

$$\begin{aligned}
0 &= J_a^{(\ell_{\mathscr{B}},\ell_g)} + 2\,M_a^{(\ell_{\mathscr{B}},\ell_g)} + 2\,r\,\varepsilon_{ab}\,\delta^b K^{(\ell_{\mathscr{B}},\ell_g)} + (L_g-2)\,\chi_a^{(\ell_{\mathscr{B}},\ell_g)} - \Psi_a^{(\ell_{\mathscr{B}},\ell_g)}\,, && (6.51)\\
0 &= r\,\delta^2 K^{(\ell_{\mathscr{B}},\ell_g)} + \varepsilon_{ab}\,\delta^a\,\Psi^b_{(\ell_{\mathscr{B}},\ell_g)} - (L_g-2)\,\varepsilon_{ab}\,\delta^a\,\chi^b_{(\ell_{\mathscr{B}},\ell_g)}\,, && (6.52)\\
0 &= \left[r\,\delta^2 + (L_g - L_{\mathscr{B}})\,r^{-1}\right] K^{(\ell_{\mathscr{B}},\ell_g)} + \varepsilon_{ab}\,\delta^a\,\Psi^b_{(\ell_{\mathscr{B}},\ell_g)} - 2\,\varepsilon_{ab}\,\delta^a\,M^b_{(\ell_{\mathscr{B}},\ell_g)}\,, && (6.53)\\
0 &= L_{\mathscr{B}}\left[\delta^2 + (L_{\mathscr{B}} - L_g)\,r^{-2}\right] K^{(\ell_{\mathscr{B}},\ell_g)} - r\left[\delta^2 + (L_{\mathscr{B}} + L_g)\,r^{-2}\right]\varepsilon_{ab}\,\delta^b\,\Psi^a_{(\ell_{\mathscr{B}},\ell_g)}\,, && (6.54)\\
0 &= \left(r^2\,\delta^2 + L_{\mathscr{B}}\right)\delta^a\,\Psi_a^{(\ell_{\mathscr{B}},\ell_g)} - (L_g-2)\,L_{\mathscr{B}}\,\delta^a\,\chi_a^{(\ell_{\mathscr{B}},\ell_g)}\,, && (6.55)\\
0 &= \left(r^2\,\delta^2 + L_g\right)\delta^a\,\Psi_a^{(\ell_{\mathscr{B}},\ell_g)} - 2\,L_g\,\delta^a\,M_a^{(\ell_{\mathscr{B}},\ell_g)}\,. && (6.56)
\end{aligned}$$

These twenty-four constraint equations propagate consistently.

For each $\ell_{\mathscr{B}} \leftrightarrow \ell_g$ interaction, the system of equations describing *the gravitational wave – magnetic field interaction* are given above. Not all these variables appear explicitly in Maxwell's equations, but they couple to them through the system of propagation equations (6.47)–(6.50) and constraints (6.51)–(6.56). We now discuss how these enter Maxwell's equations. Consider, for example, the term $\zeta_{ab}\,\mathscr{B}^b$ which appears in the evolution equation for $\mathscr{E}_a$, as can be seen from equation (3.54). We can relate this to the interaction variables above as follows: using equation (6.11) we have

$$\zeta_{ab}\,\mathscr{B}^b = \frac{2}{\phi\,r^2}\left(\sum_{\ell_g} W_{ab}^{(\ell_g)}\right)\left(\sum_{\ell_{\mathscr{B}}} B^b_{(\ell_{\mathscr{B}})}\right) = 2\sum_{\ell_{\mathscr{B}},\ell_g}\dot{\chi}_{1a}^{(\ell_{\mathscr{B}},\ell_g)}\,. \qquad (6.57)$$

Similarly for the other products. We therefore use the following abbreviations:

$$K = \sum_{\ell_{\mathscr{B}},\ell_g} K^{(\ell_{\mathscr{B}},\ell_g)}, \quad \chi_a = \sum_{\ell_{\mathscr{B}},\ell_g} \chi_a^{(\ell_{\mathscr{B}},\ell_g)}, \quad M_a = \sum_{\ell_{\mathscr{B}},\ell_g} M_a^{(\ell_{\mathscr{B}},\ell_g)}, \quad J_a = \sum_{\ell_{\mathscr{B}},\ell_g} J_a^{(\ell_{\mathscr{B}},\ell_g)}, \qquad (6.58)$$

while for Ψ_a we define:

$$\Psi_a = \sum_{\ell_{\mathscr{B}},\ell_g} \left(\Psi_{1a}^{(\ell_{\mathscr{B}},\ell_g)}, \Psi_{2a}^{(\ell_{\mathscr{B}},\ell_g)}, L_{\mathscr{B}}^{-1} \Psi_{3a}^{(\ell_{\mathscr{B}},\ell_g)}, L_{\mathscr{B}}^{-1} \Psi_{4a}^{(\ell_{\mathscr{B}},\ell_g)} \right) . \tag{6.59}$$

(The definition for Ψ_a is slightly different because it is defined having a $\delta^2 \mathscr{B}^{(\ell_{\mathscr{B}})} \sim L_{\mathscr{B}} \mathscr{B}^{(\ell_{\mathscr{B}})}$ term in it.) These definitions prevent summations appearing explicitly later.

6.5.1 The gauge-invariant form of Maxwell's equations

Neglecting terms $\mathcal{O}\left(\epsilon_{\mathscr{B}} \times \text{even parity gravity waves}\right)$ and $\mathcal{O}(\epsilon_{\mathscr{B}}^2)$ and choosing the frame in $\mathcal{F}_2$ such that $\mathcal{A}^a = \delta^a \phi = \delta^a \mathcal{A} = 0$ we find that Maxwell's equations become,

$$\begin{aligned} \hat{\mathscr{E}} + \delta_a \mathscr{E}^a + \phi \mathscr{E} &= 0 \, , & (6.60) \\ \hat{\mathscr{B}} + \delta_a \mathscr{B}^a + \phi \mathscr{B} &= 0 \, , & (6.61) \\ \dot{\mathscr{E}} - \varepsilon_{ab} \delta^a \mathscr{B}^b &= 0 \, , & (6.62) \\ \dot{\mathscr{B}} + \varepsilon_{ab} \delta^a \mathscr{E}^b &= 0 \, , & (6.63) \end{aligned}$$

and

$$\begin{aligned} \dot{\mathscr{E}}_{\bar{a}} + \varepsilon_{ab} \left(\hat{\mathscr{B}}^b - \delta^b \mathscr{B} \right) + \left(\tfrac{1}{2} \phi + \mathcal{A} \right) \varepsilon_{ab} \mathscr{B}^b &= -2 \varepsilon_{ab} \dot{\chi}_1^b \, , & (6.64) \\ \dot{\mathscr{B}}_{\bar{a}} - \varepsilon_{ab} \left(\hat{\mathscr{E}}^b - \delta^b \mathscr{E} \right) - \left(\tfrac{1}{2} \phi + \mathcal{A} \right) \varepsilon_{ab} \mathscr{E}^b &= 4 \Psi_{3a} + \phi \chi_{1a} + 2 \chi_{2a} \, . & (6.65) \end{aligned}$$

The terms on the left are those which govern an EM field around a BH; those on the right are the interaction terms. Note that these equations are a mixture of first and second order quantities, and are thus not gauge-invariant, and therefore not integrable.

In order to convert Maxwell's equations into gauge-invariant form, it is not enough to define the interaction variables above; we must also do something with the magnetic field: in $\mathcal{S}$ the magnetic field appearing in Maxwell's equations has a contribution from the static background field in $\mathcal{F}_1$ which we must somehow subtract off. The standard route to do this is as a series expansion, but this does not work here. If we imagine that B_a is written as a power series,

$$B^a = \epsilon_{\mathscr{B}} \left(B_1^a + \epsilon_g B_2^a + \cdots \right) \tag{6.66}$$

where B_1^a satisfies the $\mathcal{F}_1$ equations, (6.24), then one would imagine B_1^a would cancel out of the second-order Maxwell's equations when B^a appears alone leaving just B_2^a; when it appears multiplying an $\mathcal{F}_2$ term it is only B_1^a which contributes. However, this is not the case. It is possible to show from the commutation relations for the hat and dot derivatives acting on B^a that this leads to an inconsistency, implying that the interaction terms must be zero. Consider, for example, the scalar part of the magnetic field:

$$\mathscr{B} = \epsilon_{\mathscr{B}}\, \mathscr{B}_1 + \epsilon_g\, \epsilon_{\mathscr{B}}\, \mathscr{B}_2 + \mathcal{O}(\epsilon_g^2, \epsilon_{\mathscr{B}}^2) \tag{6.67}$$

where $\mathscr{B}_1$ satisfies $\dot{\mathscr{B}}_1 = 0$ and $\hat{\mathscr{B}}_1 = F$ where $\dot{F} = 0$, representing the background solution $\mathcal{F}_1$. Now, employing the commutation relation (2.120) we obtain

$$\hat{\dot{\mathscr{B}}} = \epsilon_g\, \epsilon_{\mathscr{B}}\, \hat{\dot{\mathscr{B}}}_2 = \epsilon_g\, \epsilon_{\mathscr{B}} \left(\dot{\hat{\mathscr{B}}}_2 - \mathcal{A} \dot{\mathscr{B}}_2 \right) \tag{6.68}$$

by using the commutation relation *after* substituting from the expansion (6.67), and neglecting terms $\mathcal{O}(\epsilon_g^2)$. Alternatively,

$$\hat{\dot{\mathscr{B}}} = \dot{\hat{\mathscr{B}}} - \mathcal{A}\, \dot{\mathscr{B}} - 2\, \alpha_a\, \delta^a \mathscr{B} = \epsilon_g\, \epsilon_{\mathscr{B}} \left(\dot{\hat{\mathscr{B}}}_2 - \mathcal{A}\, \dot{\mathscr{B}}_2 \right) - 2\, \epsilon_{\mathscr{B}}\, \alpha_a\, \delta^a \mathscr{B}_1, \tag{6.69}$$

where we applied the commutator *before* using the expansion (6.67). This is clearly a contradiction if $\alpha_a\, \delta^a \mathscr{B}_1 \neq 0$, which is the case here. [The correct form of calculating this equation results in equation (6.72).] In fact, this problem usually arises when using covariant (partial-frame) methods for second-order perturbation theory (cf. subsection 4.3.3 for the analogous situation in a cosmological setting). In contrast to metric-based approaches, the solutions for perturbed derivative *operators* are never sought, so *they must always operate on quantities of the same perturbative order.* We must therefore define some gauge-invariant variables for the magnetic field.

Gauge-invariant variables for the magnetic field

Whilst in subsection 4.3.3 we previously used the gauge-invariant variable $\dot{B}_{\langle a \rangle} + \frac{2}{3}\, \Theta\, B_a$, we similarly define the variables

$$\beta \equiv \dot{\mathscr{B}}\ , \qquad \text{and} \qquad \beta_a \equiv \dot{\mathscr{B}}_{\bar{a}} = \dot{\mathscr{B}}_a + \left(\alpha^b\, \mathscr{B}_b \right) n_a\ , \tag{6.70}$$

which are gauge-invariant in $\mathcal{S}$, as they vanish in $\mathcal{F}$ [118, 119]. To convert Maxwell's equations into a gauge-invariant system of equations, we must somehow replace every occurrence of $\mathscr{B}$ with β, and $\mathscr{B}_a$ with β_a. Note first that

$$\beta = -\varepsilon_{ab}\, \delta^a\, \mathscr{E}^b, \tag{6.71}$$

which follows immediately from equation (6.63). Meanwhile, the commutation relation between hat- and dot-derivatives [see equation (2.120) in chapter 2] when applied to $\mathscr{B}$ results in the propagation equation

$$\hat{\beta} = -(\phi + \mathcal{A})\,\beta - \delta_a\,\beta^a + 4\,\delta_a\,\Psi_3^a + \phi\,\delta_a\,\chi_1^a + 2\,\delta_a\,\chi_2^a\ , \tag{6.72}$$

where we have used equation (6.61), and the appropriate commutation relation (2.127) for dot-δ derivatives on vectors. However, this equation also arises from propagating (6.71) using Maxwell's equations, as it should. Hence, because equation (6.71) is a consistent constraint, the propagation equation for β is redundant. This implies that equation (6.71) can replace equations (6.61) and (6.63). To find a propagation equation for β_a we must propagate equation (6.70) using the appropriate commutation relation for vectors [see (2.126)], giving

$$\hat{\beta}_a = \varepsilon_{ab}\,\ddot{\mathscr{E}}^b - \left(\tfrac{1}{2}\,\phi + 2\,\mathcal{A}\right)\beta_a + \delta_a\,\beta - 2\,\ddot{\chi}_{1a} - \phi\,\chi_{3a} - 2\,\chi_{4a} - 2\,r^{-2}\left(\Psi_{1a} + 2\,M_{1a} + J_{1a}\right)\ , \tag{6.73}$$

which replaces equation (6.64). It is this equation which brings Weyl curvature into Maxwell's equations through the commutation relations. A key remaining evolution equation comes from calculating $\ddot{\mathscr{E}}$ using equation (6.62):

$$\ddot{\mathscr{E}} = \varepsilon_{ab}\,\delta^a\,\beta^b + \phi\,\varepsilon_{ab}\,\delta^a\,\chi_1^b + 2\,\varepsilon_{ab}\,\delta^a\,\chi_2^b\ , \tag{6.74}$$

which propagates consistently. We will use this evolution equation, which is just the gauge-invariant form of equation (6.62), to replace equation (6.60). Therefore, *Maxwell's equations are now just the two vector propagation equations (6.73) and (6.65), together with the two scalar non-propagation equations (6.71) and (6.74).* The last two serve as definitions for β and $\mathscr{E}$ after time harmonics are used; these then become constraints.

Note how converting the gauge-dependent form of Maxwell's equations, equations (6.60)–(6.65), which contain a mixture of first and second perturbation orders, into a gauge-invariant *second order* system has introduced many more interaction terms into the equations, terms arising purely from the Ricci identities. These terms are essentially hidden in the frame derivatives (dot, hat and δ) when acting on $\mathscr{B}$ in equations (6.60)–(6.65), and in $\mathscr{B}$ itself, illustrating the importance of using a full set of gauge-invariant variables.

Although equations we have derived are gauge-invariant to order $\mathcal{O}\left(\epsilon_{\mathscr{B}}\,\epsilon_g\right)$, they are actually valid up to $\mathcal{O}\left(\epsilon_{\mathscr{B}}^2\,\epsilon_g\right)$, $\mathcal{O}\left(\epsilon_{\mathscr{B}}^3\right)$, which can be easily seen as follows. If we include terms $\mathcal{O}\left(\epsilon_{\mathscr{B}}^2\right)$ (i.e., the energy density and anisotropic pressure of the static magnetic field) in the gravity sector, then changes to W_{ab} (and other GW variables) is $\mathcal{O}\left(\epsilon_{\mathscr{B}}^2\right)$, making the change to the GW-B variables $\mathcal{O}\left(\epsilon_{\mathscr{B}}[\epsilon_g + \epsilon_{\mathscr{B}}^2]\right)$. However, the equations are not gauge-invariant at this order, because the variables are non-zero at lower perturbative order (i.e., at order $\mathcal{O}(\epsilon_{\mathscr{B}}\,\epsilon_g)$; see, e.g., [118, 119]).

The gauge-invariant form of the equations now shows exactly the terms and couplings involved in generating the EM field. Consider, for example, the covariant wave equation for $\mathscr{E}$:

$$-\ddot{\mathscr{E}} + \hat{\hat{\mathscr{E}}} + \delta^2\mathscr{E} + (2\,\phi + \mathcal{A})\,\hat{\mathscr{E}} - \left(\tfrac{1}{2}\,\phi^2 + 2\,\mathcal{E}\right)\mathscr{E} = -2\,\phi\,\varepsilon_{ab}\,\delta^a\chi_1^b - 4\,\varepsilon_{ab}\,\delta^a\chi_2^b - 4\,\varepsilon_{ab}\,\delta^a\Psi_3^b. \quad (6.75)$$

The LHS of this equation is just the contribution from the BH geometry, and can be related simply by a change of variables to the usual Regge-Wheeler equation for an electromagnetic field around a BH as was already discussed in detail in the preceding chapter, in particular (5.64) and subsection (5.4.1) [compare also with equations (6.10) and (6.80)]. The RHS, on the other hand, is the source from the interaction terms, and has contributions from the time integral and angular derivative of the dot-product between the transverse traceless RW shearing tensor and the angular (sheet) part of the magnetic field, and the 2-divergence of the RW tensor times the magnitude of the radial part of the magnetic field.

6.6 The initial value and quasi-normal mode formulations

6.6.1 Spherical harmonics

In order to numerically integrate the system of equations we must split them using spherical harmonics, which removes the tensorial nature of the equations, and turns equations (6.51)–(6.56) into *algebraic* relations. In the Appendix we have given an overview of the spherical harmonics we use, which were developed in [34]; the generalisation to open or flat 2-surfaces was discussed in the preceding chapter, subsection 5.3.2.

A spherical harmonic decomposition of all variables then implies, from equations (6.51)–(6.56), that the spherical harmonic components of each of the variables $\boldsymbol{K}^{(\ell_{\mathscr{B}},\ell_g)}$, $\boldsymbol{\Psi}_a^{(\ell_{\mathscr{B}},\ell_g)}$, $\boldsymbol{M}_a^{(\ell_{\mathscr{B}},\ell_g)}$, $\boldsymbol{J}_a^{(\ell_{\mathscr{B}},\ell_g)}$ are proportional to the harmonic components of $\boldsymbol{\chi}_a^{(\ell_{\mathscr{B}},\ell_g)}$. So, for example, for each ℓ:[1]

$$\Psi_{\mathsf{V}}^{(\ell_{\mathscr{B}},\ell_g)} = \frac{L_{\mathscr{B}}\,l_g}{L_{\mathscr{B}} - L}\,\chi_{\mathsf{V}}^{(\ell_{\mathscr{B}},\ell_g)}\,, \quad (6.76)$$

$$\bar{\Psi}_{\mathsf{V}}^{(\ell_{\mathscr{B}},\ell_g)} = \frac{L_{\mathscr{B}}\,l_g\,(L + L_g - L_{\mathscr{B}})}{(L_{\mathscr{B}} + L)\,L_g - (L - L_{\mathscr{B}})^2}\,\bar{\chi}_{\mathsf{V}}^{(\ell_{\mathscr{B}},\ell_g)}\,, \quad (6.77)$$

with similar relations for $\boldsymbol{K}^{(\ell_{\mathscr{B}},\ell_g)}, \boldsymbol{M}_a^{(\ell_{\mathscr{B}},\ell_g)}, \boldsymbol{J}_a^{(\ell_{\mathscr{B}},\ell_g)}$. Here and in the following, we make use of the abbreviations $L \equiv \ell(\ell+1)$ and $l \equiv L - 2$, and similarly for the indexed quantities such as L_g etc.

[1] Observe that expression (6.76) corrects a typo in reference [136].

Because the equations are linear in the second order variables, when we split into spherical harmonics, the equations decouple into two distinct subsets of opposing parity; the *parity mixing* which occurs between the magnetic field and the GW is contained in the interaction variables. We will call the set of equations containing $\mathscr{E}_{\mathsf{V}}$ the *even parity equations*, and those containing $\bar{\mathscr{E}}_{\mathsf{V}}$ the *odd parity equations*. Unfortunately, all the other variables are of the 'opposite' parity to E_a in each system of equations, so this may cause confusion (so, e.g., $\bar{\beta}_{\mathsf{V}}$ and $\bar{\chi}_{\mathsf{V}}$ are of even parity, etc.).

6.6.2 Initial value formulation

A useful form of writing Maxwell's equations is as wave equations. For this we use the variables

$$W_{\mathsf{S}} = r^2 \mathscr{E}_{\mathsf{S}} \qquad (\text{EVEN}) \,, \tag{6.78}$$

$$\bar{W}_{\mathsf{S}} = \tfrac{1}{2}\phi r^3 \beta_{\mathsf{S}} \quad (\text{ODD}) \,. \tag{6.79}$$

Then these variables satisfy for each ℓ wave equations of the following form :

$$-\ddot{W} + \hat{\hat{W}} + \mathcal{A}\hat{W} - \frac{L}{r^2} W = S(\chi_a) \;, \tag{6.80}$$

where $W = \{W_{\mathsf{S}}, \bar{W}_{\mathsf{S}}\}$ and the even and odd source terms are defined as

$$S_{\mathsf{S}} = -2\,L\,r \sum_{\ell_{\mathscr{B}}=1}^{\infty} \sum_{\ell_g=2}^{\infty} \left\{ \phi\, \bar{\chi}_{1\mathsf{V}}^{(\ell_{\mathscr{B}},\ell_g)} + 2\,\bar{\chi}_{2\mathsf{V}}^{(\ell_{\mathscr{B}},\ell_g)} + 2\,\frac{l_g\,(L + L_g - L_{\mathscr{B}})}{(L_{\mathscr{B}} - L)^2 - (L_{\mathscr{B}} + L)\,L_g}\, \dot{\bar{\chi}}_{3\mathsf{V}}^{(\ell_{\mathscr{B}},\ell_g)} \right\}, \tag{6.81}$$

$$\bar{S}_{\mathsf{S}} = L \sum_{\ell_{\mathscr{B}}=1}^{\infty} \sum_{\ell_g=2}^{\infty} \left\{ -2\,\phi\, r^2\, \ddot{\chi}_{1\mathsf{V}}^{(\ell_{\mathscr{B}},\ell_g)} - 2\,\phi\,\frac{l_g\,(L - 3L_{\mathscr{B}})}{L - L_{\mathscr{B}}}\,\chi_{1\mathsf{V}}^{(\ell_{\mathscr{B}},\ell_g)} \right.$$
$$\left. + \frac{(L_{\mathscr{B}} - 4\,l_g - L)\,\phi^2 r^2 + 4\,l_g}{L - L_{\mathscr{B}}}\,\chi_{3\mathsf{V}}^{(\ell_{\mathscr{B}},\ell_g)} + 2\,r^2\,\frac{L_{\mathscr{B}} + l_g - L}{L - L_{\mathscr{B}}}\,\chi_{4\mathsf{V}}^{(\ell_{\mathscr{B}},\ell_g)} \right\}. \tag{6.82}$$

These wave equations may replace Maxwell's equations and, more importantly, are in the form of an initial value problem. We then have, for each parity, one forced wave equation for the EM field, plus two evolution equations for the interaction variables (χ_1 and χ_3), plus two constraints (propagation equations); the set of four differential equations for the interaction variables may be easily turned into a set of two coupled wave equations instead by eliminating two χ_a variables (either χ_1 and χ_3 or χ_2 and χ_4). Eliminating χ_{2a} using the first equation of (6.37) and χ_{4a} using the third equation turns the remaining two into wave equations: for the odd parity equations, we find

$$
\begin{aligned}
-\ddot{\chi}_{1\mathrm{V}}^{(\ell_{\mathscr{B}},\ell_g)} + \dot{\hat{\chi}}_{1\mathrm{V}}^{(\ell_{\mathscr{B}},\ell_g)} + \mathcal{A}\,\hat{\chi}_{1\mathrm{V}}^{(\ell_{\mathscr{B}},\ell_g)} &= -2\,(\phi+\mathcal{A})\,\hat{\chi}_{1\mathrm{V}}^{(\ell_{\mathscr{B}},\ell_g)} + 2\,\hat{\chi}_{3\mathrm{V}}^{(\ell_{\mathscr{B}},\ell_g)} \\
&\quad + \left[6\,\mathcal{E} - \tfrac{1}{2}\,\phi^2 - \mathcal{A}^2 + (L_g - L_{\mathscr{B}})\,r^{-2}\right]\chi_{1\mathrm{V}}^{(\ell_{\mathscr{B}},\ell_g)} + (3\,\phi + 2\,\mathcal{A})\,\chi_{3\mathrm{V}}^{(\ell_{\mathscr{B}},\ell_g)}\,, \qquad (6.83) \\
-\ddot{\chi}_{3\mathrm{V}}^{(\ell_{\mathscr{B}},\ell_g)} + \dot{\hat{\chi}}_{3\mathrm{V}}^{(\ell_{\mathscr{B}},\ell_g)} + \mathcal{A}\hat{\chi}_{3\mathrm{V}}^{(\ell_{\mathscr{B}},\ell_g)} &= 2\,L_{\mathscr{B}}\,r^{-2}\,\hat{\chi}_{1\mathrm{V}}^{(\ell_{\mathscr{B}},\ell_g)} - 4\,\phi\hat{\chi}_{3\mathrm{V}}^{(\ell_{\mathscr{B}},\ell_g)} \\
&\quad + 2\,L_{\mathscr{B}}\,r^{-2}\,(\phi+\mathcal{A})\,\chi_{1\mathrm{V}}^{(\ell_{\mathscr{B}},\ell_g)} + \left[7\,\mathcal{E} - 3\,\phi^2 + (L_g - L_{\mathscr{B}})\,r^{-2}\right]\chi_{3\mathrm{V}}^{(\ell_{\mathscr{B}},\ell_g)}\,, \qquad (6.84)
\end{aligned}
$$

with identical equations for the even variables.

The full solution for the induced EM radiation is given by the variables W: for *even* perturbations, $\mathscr{E}_{\mathrm{V}}$ is given by equation(6.60) and $\bar{\beta}_{\mathrm{V}}$ by equation (6.74); for *odd* perturbations, $\bar{\mathscr{E}}_{\mathrm{V}}$ is given by equation (6.71) and β_{V} by equation (6.72).

6.6.3 Quasi-normal mode formulation using temporal harmonics

While the covariant equations above are given with time derivatives, allowing the problem to be put in the form suitable for solving as an initial value problem, it is often advantageous to use time harmonics [as briefly discussed in subsection (6.3.1)]. In particular, the effect of BH ringdown is conventionally studied by this method, as the ringdown phase is characterised by a set of quasi-normal frequencies [95, 96], which are independent of the initial perturbation. We achieve this by replacing all dot-derivatives by a factor of $i\omega$, with the usual understanding that subsequent equations are then for the spatial parts only [34], although formally it is significantly more complicated [95, 96, 172]. The harmonic function ω is defined with respect to the proper time, τ, of observers travelling on u^a, and satisfies equation (6.19); σ is the *constant* harmonic index associated with time, t, measured by observers at infinity. Note that they are related by $\omega\tau = \sigma t$.

The time derivative of the second-order interaction variables $\boldsymbol{K}^{(\ell_{\mathscr{B}},\ell_g)}$, $\boldsymbol{\Psi}_a^{(\ell_{\mathscr{B}},\ell_g)}$, $\boldsymbol{\chi}_a^{(\ell_{\mathscr{B}},\ell_g)}$, $\boldsymbol{M}_a^{(\ell_{\mathscr{B}},\ell_g)}$, $\boldsymbol{J}_a^{(\ell_{\mathscr{B}},\ell_g)}$ acts only on the GW part of the term, because the time derivative of the magnetic field is already second order. Therefore, when these terms are split into time harmonics, and the interaction equations (6.47)–(6.50) are solved, the usual boundary conditions on the GW variable W_{ab} will take effect – that the GW cannot propagate out of the horizon, or in from infinity. This implies that the allowed frequencies σ must be discrete with positive imaginary part [95, 96]. This represents modes which decay exponentially in time, but whose amplitudes grow exponentially with radius.

Our method presented here, which sets up the equations as a set of purely second-order, linear, gauge-invariant differential equations, means that when we solve them we do not view quadratic first-order effects as quadratic forcing terms in the second-order equations, but as second-order quantities in their own right; the first-order equations are forgotten about. Therefore the propagation equations governing the interaction variables, equations (6.37) and (6.47)–(6.50) , also must be confined to these frequencies. Hence, the coupling between the equations for the induced EM field and the interaction variables implies that the *allowed independent frequencies of the induced EM radiation must be identical to those of the forcing GW* – the quasi-normal frequencies; that is, the GW and EM radiation satisfy the same dispersion relation, and are in resonant interaction. Other frequencies correspond to EM waves which are not induced by the interaction terms with these boundary conditions (and form part of the homogeneous solution for the EM field); there is no need to consider these here. Therefore, when we split the system of equations using the time harmonics, each ℓ_g picks out a set of allowed frequencies in the interaction equations, thus *removing the summations over* ℓ_g in Maxwell's equations. For each ℓ_g there is one system of equations for each quasi-normal frequency $\omega_{(\ell_g)}$ associated with that particular ℓ_g. The complete solution for $\mathscr{E}_{\mathsf{V}}$, for example, may then be written schematically *for each* ℓ as

$$\mathscr{E}_{\mathsf{V}} = \sum_{\ell_g=2}^{\infty} \mathscr{E}_{\mathsf{V}}(\ell_g) = \sum_{\ell_g=2}^{\infty} \sum_{\omega\in\{\omega_g\}_n} \mathscr{E}_{\mathsf{V}}^{(\omega)}(\ell_g)\, e^{i\omega\tau} \,, \tag{6.85}$$

where $\{\omega_g\}_n$ denotes the set of all quasi-normal frequencies for a given ℓ_g.

From the wave equations give above, it is clear that for each parity, while there are three EM variables, there are only two degrees of freedom in the EM radiation; in the even case for example these are W_{S} and $\hat{W}_{\mathsf{S}}$, resulting in a straightforward wave equation. We can of course stick to these variables in the QNM formulations of the problem, but the system is naturally first order in the variables $\mathscr{E}_{\mathsf{V}}$ and $\bar{\beta}_{\mathsf{V}}$ (or $\mathscr{E}_{\mathsf{S}}$) in the even case once the extra degree of freedom is removed (similarly for the odd case). There doesn't seem much advantage whichever way we choose the variables so we will remove the scalar (radial) parts of the EM field from the system of equations, using equations (6.71) and (6.74). Our key equations assume then the displayed form for even and odd parity, respectively.

EVEN PARITY:

$$\begin{aligned}
\hat{\mathscr{E}}_{\mathsf{V}} &= \sum_{\ell_{\mathscr{B}}=1}^{\infty} \left\{ -\left(1 + L\,\omega^{-2} r^{-2}\right) \left[\phi\, \bar{\chi}_{1\mathsf{V}}^{(\ell_{\mathscr{B}},\ell_g)} + 2\,\bar{\chi}_{2\mathsf{V}}^{(\ell_{\mathscr{B}},\ell_g)}\right] + 4\, \frac{l_g\,(L + L_g - L_{\mathscr{B}})}{(L_{\mathscr{B}} - L)^2 - (L_{\mathscr{B}} + L)\,L_g}\, \bar{\chi}_{3\mathsf{V}}^{(\ell_{\mathscr{B}},\ell_g)} \right\} \\
&\quad - \left(\tfrac{1}{2}\phi + \mathcal{A}\right) \mathscr{E}_{\mathsf{V}} + \left(1 - L\,\omega^{-2} r^{-2}\right) \bar{\beta}_{\mathsf{V}} \,, \qquad (6.86) \\
\hat{\bar{\beta}}_{\mathsf{V}} &= -\left(\tfrac{1}{2}\phi + 2\mathcal{A}\right) \bar{\beta}_{\mathsf{V}} - \omega^2 \mathscr{E}_{\mathsf{V}} + \sum_{\ell_{\mathscr{B}}=1}^{\infty} \left[2\left(\omega^2 - l_g r^{-2}\right) \bar{\chi}_{1\mathsf{V}}^{(\ell_{\mathscr{B}},\ell_g)} - \phi\, \bar{\chi}_{3\mathsf{V}}^{(\ell_{\mathscr{B}},\ell_g)} - 2\,\bar{\chi}_{4\mathsf{V}}^{(\ell_{\mathscr{B}},\ell_g)}\right], \qquad (6.87) \\
\hat{\bar{\boldsymbol{\chi}}}_{\mathsf{V}}^{(\ell_{\mathscr{B}},\ell_g)} &= \boldsymbol{\Gamma}^{(\ell_{\mathscr{B}},\ell_g)}\, \bar{\boldsymbol{\chi}}_{\mathsf{V}}^{(\ell_{\mathscr{B}},\ell_g)}\,; \qquad (6.88)
\end{aligned}$$

ODD PARITY:

$$\begin{aligned}
\hat{\bar{\mathscr{E}}}_{\mathsf{V}} &= -\left(\tfrac{1}{2}\phi + \mathcal{A}\right)\bar{\mathscr{E}}_{\mathsf{V}} - \beta_{\mathsf{V}} + \sum_{\ell_{\mathscr{B}}=1}^{\infty}\left\{\left[\phi\,\chi_{1\mathsf{V}}^{(\ell_{\mathscr{B}},\ell_g)} + 2\,\chi_{2\mathsf{V}}^{(\ell_{\mathscr{B}},\ell_g)}\right] + 4\,\frac{l_g}{L_{\mathscr{B}} - L}\,\chi_{3\mathsf{V}}^{(\ell_{\mathscr{B}},\ell_g)}\right\}, && (6.89)\\
\hat{\beta}_{\mathsf{V}} &= -\left(\tfrac{1}{2}\phi + 2\,\mathcal{A}\right)\beta_{\mathsf{V}} - \left(-\omega^2 + L\,r^{-2}\right)\bar{\mathscr{E}}_{\mathsf{V}} \\
&\quad + \sum_{\ell_{\mathscr{B}}=1}^{\infty}\left\{2\left[\omega^2 + \frac{(L - 3\,L_{\mathscr{B}})\,l_g}{(L_{\mathscr{B}} - L)\,r^2}\right]\chi_{1\mathsf{V}}^{(\ell_{\mathscr{B}},\ell_g)} - \phi\,\chi_{3\mathsf{V}}^{(\ell_{\mathscr{B}},\ell_g)} - 2\,\chi_{4\mathsf{V}}^{(\ell_{\mathscr{B}},\ell_g)}\right\}, && (6.90)\\
\hat{\boldsymbol{\chi}}_{\mathsf{V}}^{(\ell_{\mathscr{B}},\ell_g)} &= \boldsymbol{\Gamma}^{(\ell_{\mathscr{B}},\ell_g)}\boldsymbol{\chi}_{\mathsf{V}}^{(\ell_{\mathscr{B}},\ell_g)}, && (6.91)
\end{aligned}$$

for each ℓ_g and $\omega \in \{\omega_g\}_n$. Each parity consists of a set of six coupled ordinary differential equations in the radial parameter ρ.

6.7 Numerical examples

We have now set up the equations as a gauge-invariant linear system of differential equations in purely second order variables, in two different ways. The first is as a set of three coupled wave equations (for each parity), which may be numerically integrated as an initial value problem once some initial data is specified. The second is as a six-dimensional system of first-order ordinary differential equations (for each parity) which are Fourier decomposed in time, which is suitable for integration once appropriate boundary conditions are satisfied. There are of course advantages and disadvantages to both, which we discuss presently.

While it would be desirable to be able to integrate these equations in a situation which is astrophysically accurate in some sense, this is quite a non-trivial problem as it involves specifying initial data from a fully nonlinear integration of the field equations in a situation such as, for example, BH-BH merger. This is beyond the purpose of the present discussion, as we would like to get an overall estimate of the strength and importance of the effect in this first instance.

In general, the summations over ℓ_g and $\ell_{\mathscr{B}}$ in the equations for the generated EM radiation mean that these coupled systems of equations are infinite dimensional. However, for a static magnetic field around a BH the dominant contribution to the field strength will be dipolar, and the GW emitted by a compact object will typically be dominated by the quadrupole radiation (for example, when two BHs collide head-on from an initially small separation, the emitted radiation is pure quadrupole [173]; other studies with high energy collisions support this conclusion [174–177]). Therefore, in this section we will investigate numerically the $\ell_g = 2$, $\ell_{\mathscr{B}} = 1$ interaction while ignoring the contribution from the others.

As we mentioned earlier, the case of an $\ell_{\mathscr{B}} = 1$ magnetic field has two solutions, one which is uniform at infinity, and one which falls off at large distances like $1/r^3$, a dipole [see (5.101)–

(5.102)]. Both of these are of interest astrophysically, as magnetic fields surrounding compact objects can extend considerable distances when supported by a plasma (i.e., a BH 'embedded' in an external magnetic field), but be purely dipolar close in. It is clearly important to distinguish the two cases in the $\boldsymbol{\chi}_a$ variables when we integrate the equations, in order to determine which type of field is responsible for what. For both solutions, the ratio $\mathscr{B}_{\mathsf{V}}/\mathscr{B}_{\mathsf{S}}$ is a known function of r, with no dependence on any boundary conditions — see, e.g., equations (6.30)–(6.31). This implies that

$$\frac{\chi_1}{\chi_3} = \frac{\chi_2}{\chi_4} = r\left.\frac{\mathscr{B}_{\mathsf{V}}}{\mathscr{B}_{\mathsf{S}}}\right|_{\text{uniform } or \text{ dipole solution}}, \tag{6.92}$$

where χ_i $(i = 1, \cdots, 4)$ represents either the odd or even parity part of χ_{ia}. The ratio $\mathscr{B}_{\mathsf{V}}/\mathscr{B}_{\mathsf{S}}$ is given by equations (6.30)–(6.31) for the dipolar field, while for the field which is uniform at infinity (characterised by $\hat{\mathscr{B}}_{\mathsf{S}} = 0$) it equals $\frac{1}{2}\,\phi r$ as can be inferred from the corresponding solution (5.101)–(5.102). Thus, if we desire the magnetic field to be one of these solutions, we can use equation (6.92) to constrain the boundary conditions, or simply replace χ_3, χ_4 in the equations. In Figure 6.2 we show a plot of the ratio $-\chi_3/\chi_1$ for the pure dipole field which shows how the dominant contribution to the interaction terms at large distances and close to the horizon is dominated by χ_1 (or χ_2; the figure for $-\chi_2/\chi_4$ is identical). Thus, χ_3 and χ_4, containing the radial part of the magnetic field, only contribute significantly in the vicinity of the photon sphere. We will consider only the pure dipole solution here, and hereafter remove χ_3, χ_4 using equation (6.92) (we remove these two because, as Figure 6.2 shows, replacing $\chi_1 4$ and χ_2 would make numerical solutions become unstable at small and large distances).

The induced EM radiation will of course be of much higher amplitude far from the BH if we allow for the presence of the uniform magnetic field as part of the static background, as the interaction distance will be vastly increased. For the pure dipole magnetic field, the interaction distance is effectively curtailed at large r, because the magnetic field strength falls off so fast that $\mathscr{E} \gg \chi$ by $r \sim 20\,m$ or so. In astrophysical situations where the magnetic field extends far from the source (supported by an accretion disk, or entangled in the ejected envelope of the progenitor star, for example), we would expect further, linearly growing amplification, (beyond say $r \sim 20\,m$) over the amplification we report below. To study this conclusively, however, one needs to include a plasma into the discussion, a task which is beyond the scope of the present investigation.

Hereafter we set $m = 1$ (which just defines the units of r), and we use the tortoise coordinate r_* of Regge and Wheeler introduced in (6.21). Because the system of equations we are investigating is linear, the units we use are physically irrelevant, and is tied into the physical amplitude of our initial data which we normalise at unity (so that if units are chosen for χ, say, we can immediately read off the actual amplitudes for $\mathscr{E}$).

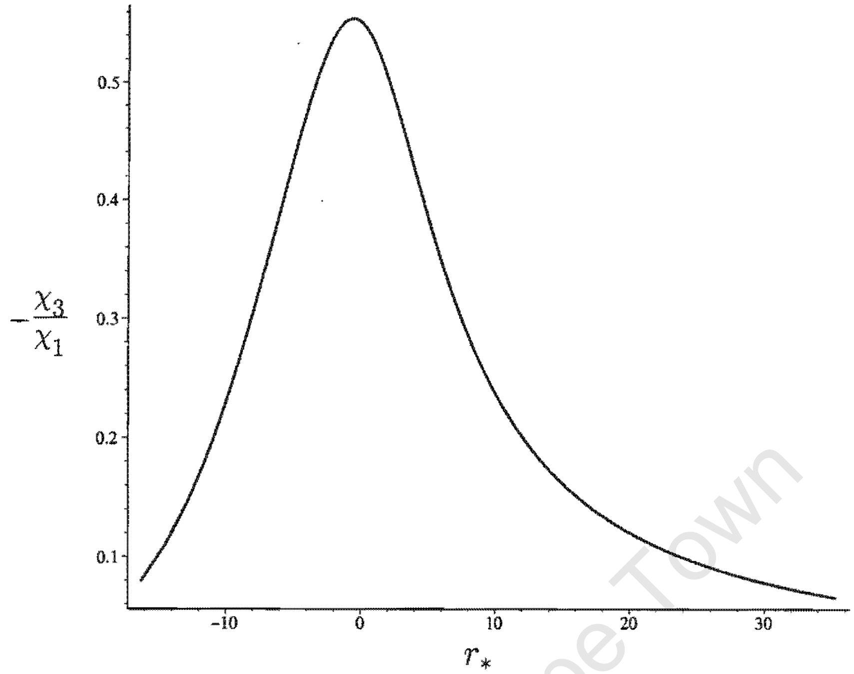

Figure 6.2: For a pure dipole background magnetic field this figure, which is a plot of equation (6.92), shows how the relative contributions from the interaction terms are dominated by χ_1 (or χ_2), except in the region just inside the photon sphere (the peak is at about $r \sim 2.4\,m$), where χ_3 becomes significant (recall that χ_3 is defined using the angular gradients of the radial part of the background magnetic field). The 'tortoise' coordinate r_* is defined by equation (6.21).

6.7.1 The initial value problem

Here we envisage the following situation: at some initial time $t = 0$ the interaction is 'turned on' with some typical initial profile for the GW [i.e., the tensor W_{ab}, which translates in this case to $\chi^a(t=0) = \chi^a_0$], at which time the induced EM field is zero, but with non-zero second time derivatives ('acceleration'). Although intuitively reasonable for modelling a situation such as BH formation or where the magnetic field becomes very strong very quickly, say, we require this switching on of the interaction because otherwise Maxwell's equations will not be consistent for a general χ^a_0.

A common way of specifying initial data for this type of problem is to consider GW scattering off a BH, with the initial data given by a static narrow Gaussian peak at some distance from the hole [172]. This then splits in two as the RW equation is evolved, with the part falling into the hole of most interest: this scatters off the photon sphere and starts the black hole vibrating (roughly speaking), with a characteristic waveform which is largely independent of the initial data, dominated by the quasi-normal modes of the BH (which only depend on its mass) [95, 172, 178, 179]. We will use this scenario with $W_0 \sim \exp\left(-(r_* - 20)^2\right)$ at $t = 0$, which we normalise so that at $t = 0$ and $r_* = 20$, $\chi_{1_V} = 1$. We will not consider a pulse originating further from the hole because the dipole field falls off so fast with distance; the qualitative results remain the same.

We then evolve our key equation (6.80) and the wave equation for χ_1 [modified by replacing χ_3 with equation (6.92) as discussed above] with this initial data. This then gives the solution for W_S, which we convert to $\mathscr{E}_V$. Results are shown in Figures (6.3) and (6.4) for $\log_{10}|\mathscr{E}_V|$.

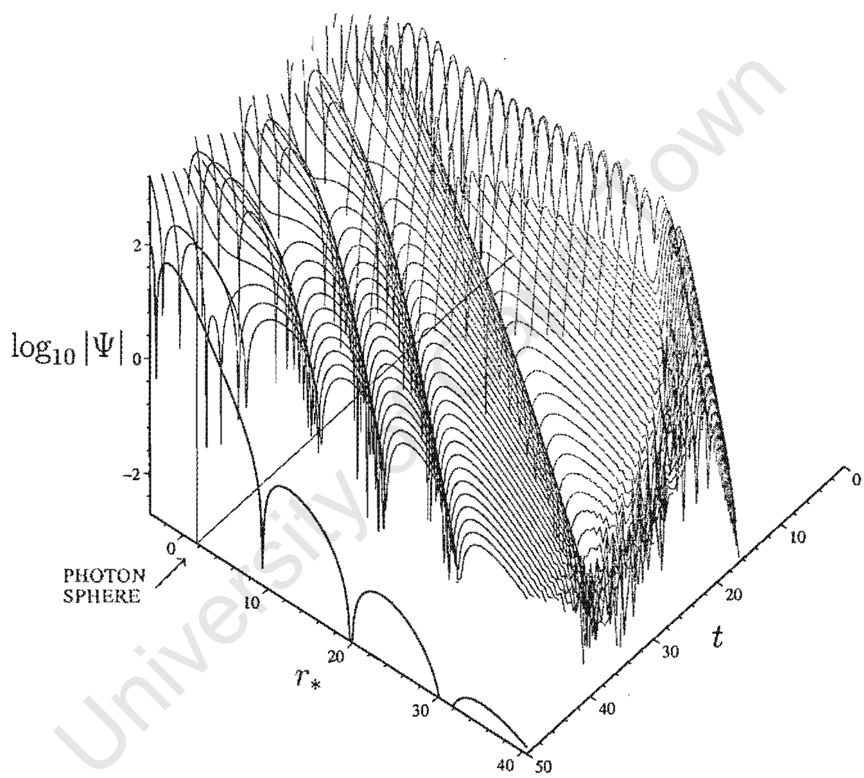

Figure 6.3: The induced EM radiation from static Gaussian initial data for the GW at $t = 0$, such that the EM field is zero. This pulse splits into two, one falling into the hole and the other propagating to infinity. The part falling into the hole is partially reflected at about $t \sim 15$ generating the 'ringing' we see later at $t = 50$ in the thick curve (modulated by the magnetic field here – this is χ_{1_V}). During this in-fall, the GW-B interaction produces substantial amounts of EM radiation which is reflected back away from the hole, and is further increased by the subsequent BH ringing.

These figures show the EM radiation generated and subsequently amplified during the scattering of the GW off the photon sphere. The ringing of the BH then generates a continuous stream of

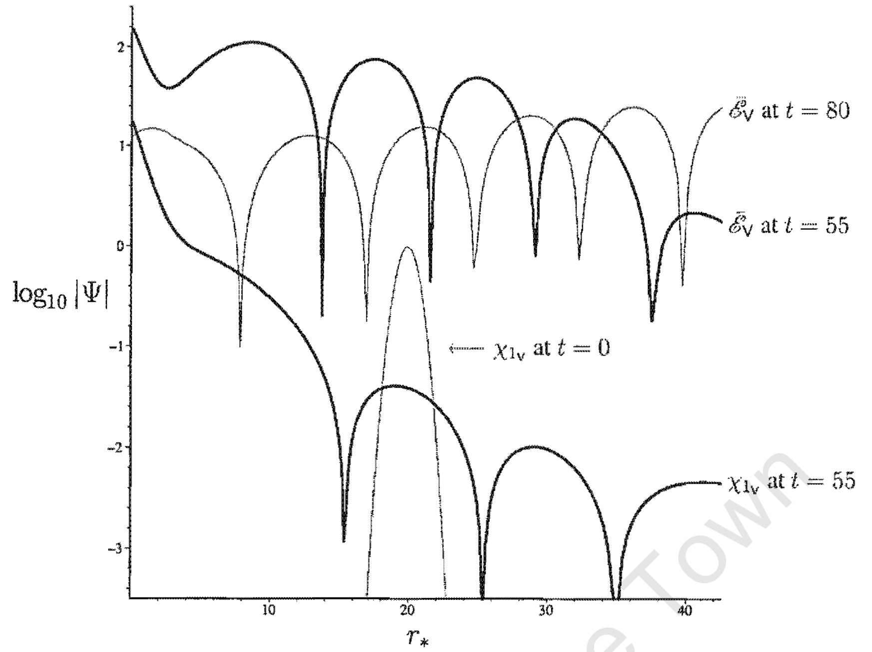

Figure 6.4: The induced EM radiation from static Gaussian initial data for the GW at $t = 0$, such that $W \sim \exp\left(-(r_* - 20)^2\right)$, normalised so that at $t = 0$ and $r_* = 20$, $\chi_{1_{\mathrm{V}}} = 1$; at $t = 0$ the EM field is zero. This pulse splits into two, one falling into the hole and the other propagating to infinity. The part falling into the hole is partially reflected generating the 'ringing' we see at $t = 55$ (modulated by the magnetic field here in $\chi_{1_{\mathrm{V}}}$), in the wave which is moving to the right . During this in-fall, the GW-B interaction produces substantial amounts of EM radiation, which is further increased by the subsequent BH ringing. Thus we see at $t = 55$ the EM field generated is two orders of magnitude larger than the initial pulse, and over three orders of magnitude larger than the interaction terms at the same distance from the hole. At $t = 80$ we see that the main amplification has taken place with the peak of the induced EM radiation moving past $r_* \sim 40$.

EM radiation, which at its peak is over *two orders of magnitude* larger than the initial pulse of radiation (by the time it is reflected back out to $r_* = 20$). This radiation mirrors very closely the GW waveform making it a suitable EM counterpart for GW emission.

6.7.2 The quasi-normal mode approach

We shall now integrate the equations in the frequency domain, summing over the QNMs of the BH, which will tell us about the strength of the interaction in the latter stages of a perturbation of a BH independently of the initial perturbation [172]. We imagine that the interaction starts at $t = 0$ at some inner radius r_0, so for $r < r_0$ we assume that $\mathscr{E}_{\mathrm{S}} = \beta_{\mathrm{S}} = 0$, while χ does its own thing; at $r = r_0$ we choose boundary conditions for each ω_n such that all EM terms and their derivatives are equal to zero; for want of accurate boundary conditions for the GW, we randomly[2] choose $\chi^{(\omega_n)} = \chi_0$. In order to compare differing amplifications for each parity, we use the same χ_0 for both parities. We then integrate equations (6.86)–(6.91) out to some $r = r_{\max}$ for each QNM frequency ω_n. Then, for each variable at $r = r_{\max}$ we can simply add up the QNMs. This then gives a good approximation to the time decay of the signal as it passes $r = r_{\max}$ after $t \gtrsim t_{\max} = r_{\max} - r_0 + 2m \ln[(r_{\max} - 2m)/(r_0 - 2m)]$ [95, 172]. We use the first twelve QNM frequencies as tabulated in [180] for $\sigma_n = \frac{1}{2}\phi r \omega_n$ – see equation (6.19).

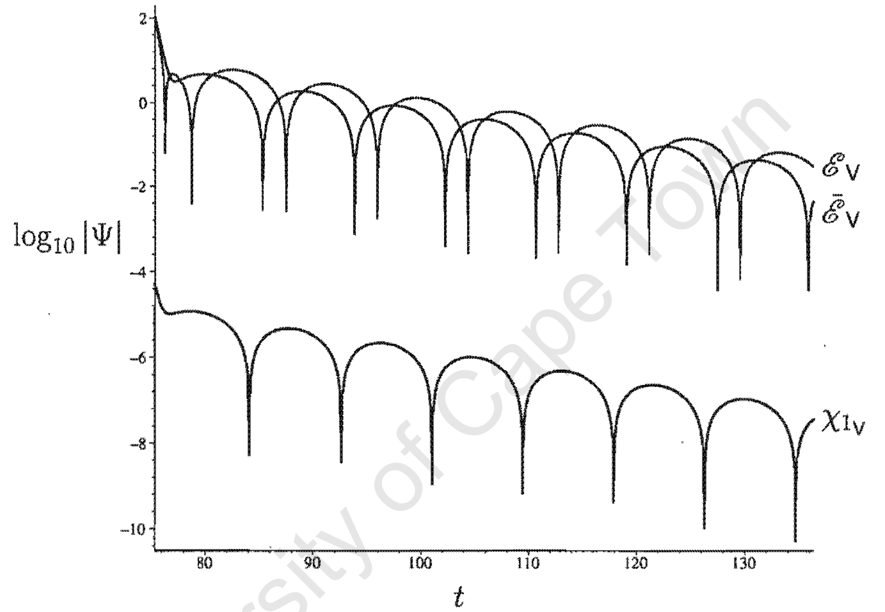

Figure 6.5: Late time behavior showing the temporal evolution of the generated EM waveforms for an observer at $r = 65m$, using a quasi-normal mode expansion. Choosing boundary conditions at $r_0 = 2.05\,m$, the QNM sum will approximate the true solution for this observer after $t \simeq 76\,m$. We see that the EM waveform is substantially larger than the interaction variables, for which we have shown the largest, $\chi_{1_{\rm V}}$. The differing amplification of the electric field for each parity is also not insubstantial, with the even parity case being larger than the odd, in this case, demonstrating polarisation of the induced EM radiation.

In Figure 6.5 we show a typical result of this integration, for an observer situated at $r \simeq 65$, with $r_0 = 2.05$. The generated electric field is shown, for both parities, as is the largest interaction variable, which is $\chi_{1_{\rm V}}$ in this case. The units of the graph are arbitrary: dividing each variable by $|\chi_0|$ (to make each variable dimensionless), say, will merely shift all the curves

[2]Although this may seem somewhat arbitrary, it is no more arbitrary than choosing a Gaussian distribution as in the last section. We have performed the numerical integration below for many different choices of χ_0, and the results are qualitatively similar.

up or down. At large distances from the source, the behaviour of the fields can be represented as an amplitude over a potential of the distance function. In the case of the gravitational wave, the fall-off scales like $1/r$, while for a spherical electromagnetic wave it behaves as $1/r$. At the same time, the background magnetic dipole field has $1/r^3$-dependence. We can therefore normalise with the respect to the fall-off of the field strengths, in order to get a scale-invariant form of the amplification. In the situation given in Figure 6.5, normalising the curve for $\chi_{1\mathrm{v}}$ raises it up by $3\log_{10}(65/2.05) \sim 4.5$. Hence, at large distances from the source *the scale-invariant amplification of the EM radiation is over two orders of magnitude larger the the magnitude of the GW times the magnetic field strength.* At this distance from the source the interaction is no longer taking place to a significant degree, implying that this level of amplification is a generic feature. Note also from Figure 6.5 that the amplification of the electric field parities is different, implying that the EM wave is polarised.

6.8 Estimates

To estimate the implications of this amplification we have found, consider the case of a compact object such as a BH or neutron star. The interaction between gravity waves and the magnetic field is quantified by the variable $\chi_i \sim h_i B_i$, where h_i is the amplitude of the gravitational wave at the onset of the interaction, $B_i \sim B_\mathrm{s}(r_\mathrm{s}/r_i)^3$ is the field strength at the distance r_i from the compact object with 'radius' (e.g., surface of a neutron star or BH horizon) r_s and surface magnetic field B_s. The interaction produces an EM signal E_out which is typically two orders of magnitude larger at $r_\text{cut-off}$ than the original perturbation χ_1 at $r_\text{cut-off}$, where the interaction effectively switches off. From Figure (6.4) we extrapolate that at $r_\text{cut-off} \sim 40\,m$ we have roughly $\chi_\text{cut-off} \sim 10^{-2.5}\chi_i$, leading to an induced electric field strength

$$E_\mathrm{out} \sim 3 \times 10^8 h_i \left(\frac{B_\mathrm{s}}{1\,\mathrm{T}}\right)\left(\frac{r_\mathrm{s}}{r_i}\right)^3 \mathrm{V\,m^{-1}}. \tag{6.93}$$

The induced signal attenuates inversely with distance D outside the interaction region, e.g., $r \gtrsim 40\,m$. At a distance D from the source, the spectral energy flux $\Phi_\omega = (\frac{1}{2}\,\epsilon_0\, c\, E^2)/\omega$ can then be calculated to be

$$\Phi_\omega \sim 1.4 \times 10^6\, h_i^2 \left(\frac{B_\mathrm{s}}{1\,\mathrm{T}}\right)^2 \left(\frac{r_\mathrm{s}}{r_i}\right)^6 \left(\frac{r_\text{cut-off}}{100\,\mathrm{km}}\right)^2 \left(\frac{10\,\mathrm{kpc}}{D}\right)^2 \left(\frac{1\,\mathrm{kHz}}{\omega}\right) \mathrm{Jy}. \tag{6.94}$$

As an example, consider a magnetar of mass $m = 1.5\,M_\odot$ with radius $r_\mathrm{s} = 9$ km, e.g. twice its Schwarzschild radius, and take $r_i = 4\,r_\mathrm{s}$ and $r_\text{cut-off} = 90$ km, assuming a magnetic field strength B_s in the range of 10^5 to 10^{10} T. An occurring instability such as a supernova explosion or a bar mode instability is likely to produce a GW with $h_i \sim 10^{-3}$ and frequency ω of about $1 - 10$ kHz [181], which is also the frequency of the induced EM wave. This leads to $E_{\rm out} \sim 5\times10^{8} - 10^{13}\,{\rm V\,m^{-1}}$. If such an event happened within our galaxy ($D \sim 10$ kpc), $\Phi_\omega \sim 10^{5} - 10^{15}$ Jy, if we assume that 10% of the signal's energy undergoes mode conversion shifting the frequency up to $30 - 300$ kHz [8]. To achieve a higher detection rate, one has to gather events from a farther distance. Events within the Virgo Cluster ($D \sim 15$ Mpc) would have flux $\Phi_\omega \sim 10^{-1} - 10^{9}$ Jy. The proposed radio telescope Astronomical Low Frequency Array [182] is expected to operate in the range from 30 kHz to 30 MHz, with minimum detection level of 1000 Jy, making such events an exciting possibility for indirect gravitational wave detection.

6.9 Conclusions

We have investigated the scenario of GWs around a Schwarzschild BH interacting with a strong, static, magnetic field in its vicinity. This interaction produces a stream of EM radiation mirroring the BH ringdown, with a stronger amplitude than one may expect from estimates of the interaction in flat space, due to nonlinear amplification in the vicinity of the photon sphere. This interaction may play an important role in GRBs and perhaps some SN events, in addition to neutron star physics, and may be a useful mechanism to aid in GW detection.

We converted the Einstein-Maxwell equations into a linear, gauge-invariant system of differential equations by utilising the 1+1+2 covariant approach to perturbations of Schwarzschild. We also introduced a set of second-order 'interaction' variables to aid in simplifying the derivation, and a new variable for the magnetic field, both of which made the system of equations manifestly gauge-invariant. It was then a simple matter to convert the system of equations into wave equations for integration as an initial value problem, or as a harmonically decomposed (in time) system of first-order ordinary differential equations, which could then be integrated using a BH quasi-normal mode expansion, an important approximation method for late time behaviour. We integrated the system of equations using both of these techniques.

A key point of this section was to set up a suitable formalism to study this GW-B interaction around a BH, and to put the equations into a suitable gauge-invariant form for numerical integration. The next step is to include a plasma, as various plasma instabilities could be induced by such a process, making detection of this sort of induced radiation a genuine possibility. This will also help model some of the relativistic effects which take place after a SN explosion. In fact, EM waves in a plasma in an *exact* Schwarzschild spacetime are pretty complicated and unexpected [14], so it is an interesting question in its own right to ask what happens when GWs are thrown into the mix.

Chapter 7

Conclusions and outlook

Plasmas are encountered on a wide range in our Universe. They are observed in the Earth's ionosphere, in stars, in accretion disks of black holes, or in the inter-galactic medium, to name just a few examples. Even more, the very early Universe being much hotter and denser than nowadays must have been in a plasma state for some time before the expansion of the Universe led to sufficient cooling such that prevalent inhomogeneities could eventually evolve under the influence of gravity into the large scale structures we see today. Although for many astrophysical applications, e.g., solar physics, the treatment of plasmas within a special relativistic or Newtonian framework is completely satisfactory, the inclusion of gravity as described by general relativity becomes compulsory when the spacetime's curvature cannot be neglected. There are two main areas where this is of interest. The first area is the cosmology of the very early Universe, the second area is the part of astrophysics which investigates the physics of strongly gravitating compact objects such as neutron stars and black holes. In addition, the electrodynamics of plasmas becomes richer on curved spacetimes than in Minkowski space. On the one hand, the electromagnetic fields become susceptible to the expansion (contraction) of the Universe, which typically leads to a diffusion (compression) of the fields, on the other hand, the EM fields couple also to gravity waves and the intrinsic rotation of spacetime inducing new EM phenomena unknown in 'ordinary' electrodynamics. These couplings manifest themselves as source terms in Maxwell's equations and allow for energy transfer from GWs and/or rotational energy of a black hole, say, towards the EM fields.

A physically reasonable way to treat plasmas in the cosmological setup or near compact objects is to view them as a perturbation on a given background spacetime, typically described by the FLRW metric in the case of cosmology and the Schwarzschild (Kerr) metric for a non-rotating (rotating) compact object. Since the mapping between the background and the perturbed spacetime is not unique (gauge freedom), special care has to be taken into account to eliminate unphysical gauge modes. The 1+3 and 1+1+2 formulations of general relativity avoid this problem by describing the spacetime in terms of covariantly defined quantities which are governed by tensorial equations. Since the background spacetimes one is interested in in applications are chosen to have a high degree of symmetry, FLRW and Schwarzschild in our case, only a few of these covariant quantities are needed to describe the background fully while the rest vanishes in it. The variables which vanish in a given background spacetime are automatically gauge-invariant as a consequence of the Stewart-Walker Lemma, and it is then straightforward to 'linearise' the general tensorial equations around the background in order to obtain approximate equations for the real, perturbed spacetime. A further advantage of these approaches is that the variables describing the spacetime have all a clear physical or geometric meaning.

Chapter 2 was devoted to the 1+3 and 1+1+2 decompositions of general relativity. The 1+3 formalism achieves a split of spacetime into 'time' and 'space' by the aid of a family of observer with a timelike 4-velocity field u^a, whilst the 1+1+2 cousin further foliates 'space' by 'sheets' introducing an additional spacelike vector field n^a orthogonal to u^a. A set of covariant variables is obtained by an irreducible decomposition of the Riemann curvature tensor, the energy-momentum tensor as well as the covariant derivative of these vector fields. First-order partial differential evolution/propagation equations and constraint equations for these covariant quantities follow from the application of the Ricci and Bianchi identities, in conjunction with Einstein's field equations, to the vector fields u^a and n^a, respectively. These fields introduce further various derivative operators, which in general do not commute with each other when acting on any tensorial quantity. Special care was given to the arising commutation relations since they are crucial when one derives wave equations for the covariant variables and also when one investigates the propagation of the constraints. Moreover, the gauge problem in general relativity and its relation to the covariant decompositions was discussed.

The electrodynamics of electromagnetic fields and plasmas on arbitrary spacetimes was developed in chapter 3 within both the 1+3 and 1+1+2 formalism. The fundamental decompositions of Maxwell's equations were derived together with the Lorentz-force equation, that is, the general relativistic equation of motion for a massive charged test particle in the presence of EM fields on a curved spacetime. We chose to describe the plasma as a multi-component, imperfect fluid, whose individual components belong each to a separate particle species forming a perfect fluid in their rest-frame. The general nonlinear fluid equations were presented, in particular, the conservation equations for the energy, momentum and particle number density were given both for each single fluid component as well as for the total fluid.

The main aim of chapter 4 was to pursue two different mechanisms for generating a primordial magnetic seed field, which might then get amplified by some variant of the galactic dynamo mechanism towards field strengths as observed in large scale cosmic magnetic fields.

The investigated mechanisms are physically self-consistent because they are only based on general relativistic plasma physics. First, the 1+3 formalism was used to study velocity and density perturbations in the Einstein-de Sitter Universe whose matter was modelled as a two-component plasma neglecting thermal effects. It was found that the velocity perturbations in the charged plasma are sourced by the electric field. Velocity perturbations were shown to induce density perturbations in the plasma whose spectrum contains beside the usual growing and decaying modes also weakly damped high-frequency plasma oscillations, which become negligible at larger scales. It was further demonstrated how linear velocity perturbations in the cold plasma generate electromagnetic fields and analytic solutions for the scalar and vector modes of those were obtained in the long-wavelength limit. Thus, expansion normalised vortical motions of order $(\mathsf{curl}\, v^a/\Theta) \sim 10^{-5}$ in an e^+e^--plasma at the time of decoupling $z \sim 1000$) are expected to be accompanied by a magnetic field of order $10^{-25}\,\mathrm{G}$. Given that the simple plasma model is valid until $z \sim 100$, this magnetic field will have a strength of $B \sim 10^{-28}\,\mathrm{G}$ at the onset of galaxy formation $z \sim 10$ and hence satisfies the galactic dynamo requirements. Second, the interaction between a primordial homogeneous magnetic field and gravity waves was studied in the case of FLRW Universes, which was done in a gauge-invariant fashion employing purely second-order perturbation variables. Wave equations for the main variables were derived and solved for FLRW models with flat spatial sections in the radiation- and matter-dominant eras. It was found that the gravito-magnetic interaction can amplify a primordial super-horizon seed by several orders of magnitude in the resonant case, that is, when the wavelengths of the GWs match the size of the seed field region. The amplification is proportional to the initial expansion normalised shear distortion and, crucially, to the squared ratio of the seed scale to the initial Hubble radius when the interaction kicks in. The results from the gauge-invariant approach have been contrasted to results obtained with the weak-field approximation and it was found that the two methods are only compatible with each other in the limit of high conductivity or in the infinite wavelength limit.

Chapter 5 saw the application of the 1+1+2 formalism to the so-called LRS class II spacetimes, for which the 'sheets' of that formalism become genuine 2-surfaces with well-defined Gaussian curvature. The full 1+1+2 equations governing these class of spacetimes were given, generalising the treatment in [112] towards such with imperfect fluid stress-energy tensors. Harmonics were defined formally and covariantly for all allowed sheet geometries in analogy to the covariantly defined spherical harmonics as given in [34]. Scalar field and electromagnetic perturbations were studied and a covariant Regge-Wheeler master equation for these was found, which generalises the covariant Regge-Wheeler equation known from the Schwarzschild case (see [34]). This master equation has been discussed in detail for the Schwarzschild and Vaidya spacetimes as well as the Kantowski-Sachs and Lemaître-Tolman-Bondi dust Universes.

The generated electromagnetic radiation resulting from the interaction of gravitational waves scattering off a perturbed compact object with a static dipole magnetic field surrounding it was studied with the help of the 1+1+2 formalism in chapter 6. This may be viewed as a simple model to simulate the expected production of EM radiation during, e.g., BH-NS mergers or gravitational collapse of a compact object carrying a magnetosphere. The study of the toy model required the introduction of a set of second-order covariant variables (both for interaction terms and the magnetic field) to guarantee the gauge-invariance of the treatment. Maxwell's equations propagate consistently when rewritten in terms of these second-order variables. The generated EM radiation could be described either in the form of wave equations, allowing for an initial value formulation, or as a system of first-order ODEs, adapted to a quasi-normal mode analysis. Numerically integrated examples demonstrated that the interaction produces an EM signal, which mirrors the characteristics of the forcing GW, but whose amplitude is typically 2-3 orders of magnitude bigger than the GW's amplitude. Clearly, the production of EM radiation through this gravito-magnetic interaction becomes the more pronounced the stronger the initial magnetic field is. Numerical estimates involving magnetars, which can have field strengths of up to 10^{10} T, indicated that such EM signals might be detectable by future space-based telescopes such as ALFA. This opens up the possibility for an indirect detection of GWs, which could give complimentary evidence to a simultaneous registration of a GW by an earth-based detector like LIGO.

In this thesis, we tried to get a handle on some effects of general relativistic plasma physics thought important in cosmology and astrophysics by employing a perturbative approach on the basis of the 1+3 and 1+1+2 decompositions of general relativity. There are many possibilities to extend the work presented here. For example, it would be desirable to obtain from the multifluid equations a single-fluid description in a covariant manner, which might be used to formulate a generalised version of magnetohydrodynamics for curved spacetimes. Of particular interest would be the case of a quasi-neutral fluid comprised of two particle species of opposite charge, for which one might derive the generalised Ohm's law in analogy to standard plasma physics. Such a magnetohydrodynamic theory would facilitate the description of plasmas near strongly gravitating compact objects and might lead to more realistic studies of the interaction between GWs and plasmas by providing approximate means for the plasma current. Evidently, such a theory might be also useful for modelling the cosmic medium in the early Universe. However, we have neglected collisions and thermal effects in the plasma throughout this thesis for reason of simplicity. Clearly, these issues need to be addressed within our formalisms in future studies if one wants to embody a more accurate plasma description.

It should be emphasised that although we mainly applied the 1+3 and 1+1+2 formalism to the standard cosmological models and the Schwarzschild black hole spacetime, respectively, their range of usefulness extends farther than these examples. In practice, the 1+3 formalism is ideally suited to spatially homogeneous spacetimes, for which the 1+3 equations reduce to 'simple' scalar equations due to the absence of spatial gradients. A similar simplification occurs in the 1+1+2 formalism when applied to spherically symmetric (more generally, LRS) spacetimes. This stays even true for extensions of general relativity as long as the field equations can be cast into the effective form $G_{ab} = T_{ab}^{eff}$, where G_{ab} denotes the usual Einstein tensor and T_{ab}^{eff} the effective energy-momentum tensor. Whilst the 1+3 formalism has been successfully used for some years now, predominantly in the field of cosmology (perturbation theory, CMB, gravitational lensing, $f(R)$-theories of gravity, etc.), the 1+1+2 formalism in its form as presented in this thesis is of a more recent nature. The author hopes to have convinced the reader of the usefulness of the latter formalism, too.

As a last remark, we direct our reader's attention to the somewhat paradoxical point that, even though general relativity is a metric theory, the covariant decompositions used at length in this thesis in general do not invoke the metric tensor in explicit terms. Like the Devil shies away from holy water, one indeed tries to avoid the use of the metric tensor (expressed in some chosen coordinates, say) whenever possible but favours instead a description based on a congruence of local observers. It is here where the actual power of these covariant decompositions of general relativity lies: yielding a characterisation of spacetime in terms of a few covariant variables with a clear geometrical or physical meaning. Having said that, one might wonder if this set of covariant quantities can be of use in the realms of quantum gravity too. At high energies one expects that the smooth spacetime picture of GR must break down and should be replaced by some fuzzy structure instead. From the point of view of observers, a natural way to introduce fuzziness of 'spacetime' would be to propagate the covariant quantities themselves to quantum fields. It seems extremely interesting to investigate this idea in close detail and to see where it leads to. However, before promoting the covariant quantities to quantum fields, one might first try to formulate quantum field theory on curved spacetimes within the 1+3 and 1+1+2 framework and contrast it with the traditional formulation of quantum field theory on curved spacetimes.

Appendix A

Some useful FLRW relations

In this appendix, we present some very useful material, which was frequently employed in chapter 4.

A.1 Commutation relations up to second order

Here we present various commutator relations which have been used in the sections 4.2. and 4.3 dealing with cosmic magnetic fields. The relations are given up to the required second order in our perturbation scheme. The vanishing of vorticity, $\omega_{ab} = 0$, is assumed throughout. All appearing tensors are PSTF, $S_{ab} = S_{\langle ab \rangle}$, and all vectors V_a, W_a are purely spatial. The objects V_a, W_a and S_{ab} all vanish in the FLRW background.

Commutators for first-order vectors V_a:

$$
\begin{aligned}
(\mathrm{D}_a V_b)^{\cdot}_{\perp} &= \mathrm{D}_a \dot{V}_b - \tfrac{1}{3}\,\Theta\,\mathrm{D}_a V_b - \sigma_a{}^c\,\mathrm{D}_c V_b + H_a{}^d\,\varepsilon_{dbc} V^c\,, &\text{(A.1)}\\
(\mathsf{curl}\, V_a)^{\cdot}_{\perp} &= \mathsf{curl}\,\dot{V}_a - \tfrac{1}{3}\,\Theta\,\mathsf{curl}\, V_a - \varepsilon_{abc}\,\sigma^{bd}\,\mathrm{D}_d V^c - H_{ab} V^b\,, &\text{(A.2)}\\
\mathrm{D}_{[a}\mathrm{D}_{b]} V_c &= \left[\tfrac{1}{9}\,\Theta^2 - \tfrac{1}{3}\,(\mu+\Lambda)\right] V_{[a} h_{b]c} + \left(\tfrac{1}{3}\,\Theta\,\sigma_{c[a} - E_{c[a}\right) V_{b]} \\
&\quad + h_{c[a}\left(E_{b]d} - \tfrac{1}{3}\,\Theta\,\sigma_{b]d}\right) V^d\,, &\text{(A.3)}\\
\mathsf{curl}\,\mathsf{curl}\, V_a &= -\mathrm{D}^2 V_a + \mathrm{D}_a(\mathsf{div}\, V) + \tfrac{2}{3}\left(\mu + \Lambda - \tfrac{1}{3}\,\Theta^2\right) V_a + \left(E_{ab} - \tfrac{1}{3}\,\Theta\,\sigma_{ab}\right) V^b\,. &\text{(A.4)}
\end{aligned}
$$

Commutators for first-order tensors S_{ab}:

$$
\begin{aligned}
(\mathrm{D}_a S_{bc})^{\cdot}_{\perp} &= \mathrm{D}_a \dot{S}_{bc} - \tfrac{1}{3}\Theta\,\mathrm{D}_a S_{bc} - \sigma_a{}^d\,\mathrm{D}_d S_{bc} + 2\,H_a{}^d\,\varepsilon_{de(b}\,S_{c)}{}^e\,, && \text{(A.5)}\\
\left(\mathrm{D}^b S_{ab}\right)^{\cdot}_{\perp} &= \mathrm{D}^b \dot{S}_{ab} - \tfrac{1}{3}\Theta\,\mathrm{D}^b S_{ab} - \sigma^{bc}\,\mathrm{D}_c S_{ab} + \varepsilon_{abc}\,H^b{}_d\,S^{cd}\,, && \text{(A.6)}\\
(\mathrm{curl}\, S_{ab})^{\cdot}_{\perp} &= \mathrm{curl}\,\dot{S}_{ab} - \tfrac{1}{3}\Theta\,\mathrm{curl}\,S_{ab} - \sigma_e{}^c\,\varepsilon_{cd(a}\,D^e S_{b)}{}^d + 3\,H_{c(a}\,S_{b)}{}^c\,, && \text{(A.7)}\\
\mathrm{curl}\,\mathrm{curl}\,S_{ab} &= -D^2 S_{ab} + \left(\mu + \Lambda - \tfrac{1}{3}\Theta^2\right) S_{ab} + \tfrac{3}{2}\,D_{(a}\,D^c S_{b)c} \\
&\quad +3\,S_{c\langle a}\left(E_{b\rangle}{}^c - \tfrac{1}{3}\Theta\,\sigma_{b\rangle}{}^c\right); && \text{(A.8)}
\end{aligned}
$$

Commutators for second-order vectors W_a:

$$
\begin{aligned}
(\mathrm{D}_a W_b)^{\cdot}_{\perp} &= \mathrm{D}_a \dot{W}_b - \tfrac{1}{3}\Theta\,\mathrm{D}_a W_b\,, && \text{(A.9)}\\
\mathrm{D}_{[a}\mathrm{D}_{b]} W_c &= \left[\tfrac{1}{9}\Theta^2 - \tfrac{1}{3}(\mu+\Lambda)\right] W_{[a}\,h_{b]c}\,, && \text{(A.10)}\\
\mathrm{curl}\,\mathrm{curl}\,W_a &= -\mathrm{D}^2 W_a + \mathrm{D}_a(\mathrm{div}\,W) + \tfrac{2}{3}\left(\mu + \Lambda - \tfrac{1}{3}\Theta^2\right) W_a\,. && \text{(A.11)}
\end{aligned}
$$

A.2 Commutator expressions containing D^2

In this section, we give some useful expressions for commuting spatial derivatives, up to first order, in the case of an irrotational dust Universe (see also [102]). We will assume in the following that $\mathrm{D}_a N$ and X_a are first-order quantities.

$$
\begin{aligned}
(\mathrm{D}^2 N)^{\cdot} &= \mathrm{D}^2 \dot{N} - \tfrac{2}{3}\Theta\,\mathrm{D}^2 N\,, && \text{(A.12)}\\
\mathrm{D}_{[a}\mathrm{D}_{b]}\mathrm{D}_c N &= \left[\tfrac{1}{9}\Theta^2 - \tfrac{1}{3}(\mu+\Lambda)\right]\mathrm{D}_{[a} N\,h_{b]c}\,, && \text{(A.13)}\\
\mathrm{D}_a \mathrm{D}^2 N &= \mathrm{D}^2\mathrm{D}_a N + \left[\tfrac{2}{9}\Theta^2 - \tfrac{2}{3}(\mu+\Lambda)\right]\mathrm{D}_a N\,, && \text{(A.14)}\\
\mathrm{D}_{[a}\mathrm{D}_{b]}\mathrm{D}^c X^d &= \left[-\tfrac{1}{9}\Theta^2 + \tfrac{1}{3}(\mu+\Lambda)\right]\left[h^c{}_{[a}\,\mathrm{D}_{b]}\,X^d + h^d{}_{[a}\,\mathrm{D}^c X_{b]}\right]\,, && \text{(A.15)}\\
\mathrm{D}_a \mathrm{D}^2 X_b &= \mathrm{D}^2\mathrm{D}_a X_b + \left[\tfrac{2}{9}\Theta^2 - \tfrac{2}{3}(\mu+\Lambda)\right]\left[2\,\mathrm{D}_{(a} X_{b)} - h_{ab}\,\mathrm{D}_c X^c\right]\,, && \text{(A.16)}\\
a\,\varepsilon^{abc}\,\mathrm{D}_b\,\mathrm{D}^2 X_c &= \mathrm{D}^2\left(a\,\varepsilon^{abc}\,\mathrm{D}_b X_c\right)\,. && \text{(A.17)}
\end{aligned}
$$

A.3 Isolating scalar and vector modes

We define our harmonics as eigenfunctions of the Laplace–Beltrami operator [131, 132]

$$
\mathrm{D}^2 Q^{(k)} = -\frac{k^2}{a^2}\,Q^{(k)}\,, \qquad \dot{Q}^{(k)} = 0\,, \tag{A.18}
$$

where $Q^{(k)}$ stands for a scalar, vector or tensor harmonic with harmonic index k. The harmonics $Q^{(k)}$ are defined in the FLRW background spacetime and are therefore of zeroth order. For example, a first-order vector field X^a may be expanded covariantly in scalar and vector (solenoidal) harmonics

$$X^a = X_{\mathsf{S}}\, Q^a_{\mathsf{S}} + X_{\mathsf{V}}\, Q^a_{\mathsf{V}} \;, \tag{A.19}$$

where an implicit summation over the harmonic index k in this expansion is understood and the modes X_{S} and X_{V} are of first order. In order to extract the purely scalar modes, X_{S}, of a first order vector field, one basically takes the divergence (multiplied by the scale factor a for convenience) and readily obtains the following relations:

$$X \equiv a\,\mathrm{D}_b\,X^b = X_{\mathsf{S}}\,(k\,Q_{\mathsf{S}}) \;, \tag{A.20}$$

$$\dot{X} = a\,\mathrm{D}_b\,\dot{X}^b = \dot{X}_{\mathsf{S}}\,(k\,Q_{\mathsf{S}}) \;, \tag{A.21}$$

$$\ddot{X} = a\,\mathrm{D}_b\,\ddot{X}^b = \ddot{X}_{\mathsf{S}}\,(k\,Q_{\mathsf{S}}) \;. \tag{A.22}$$

The solenoidal modes, X_{V}, can be obtained by applying curl and noting that the curly harmonics $\tilde{Q}^a_{\mathsf{V}} \equiv a\,\mathsf{curl}\,Q^a_{\mathsf{V}}$ also satisfy relation (A.18). The relations analogous to the scalar case are now

$$\tilde{X}^a \equiv a\,\mathsf{curl}\;X^a \;\; = X_{\mathsf{V}}\,\tilde{Q}^a_{\mathsf{V}} \;, \tag{A.23}$$

$$\dot{\tilde{X}}^{\langle a\rangle} = a\,\mathsf{curl}\;\dot{\tilde{X}}^{\langle a\rangle} = \dot{X}_{\mathsf{V}}\,\tilde{Q}^a_{\mathsf{V}} \;, \tag{A.24}$$

$$\ddot{\tilde{X}}^{\langle a\rangle} = a\,\mathsf{curl}\;\ddot{\tilde{X}}^{\langle a\rangle} = \ddot{X}_{\mathsf{V}}\,\tilde{Q}^a_{\mathsf{V}} \;. \tag{A.25}$$

Thus, for fixed comoving wave number k, X and X_{S} as well as $\tilde{X}^a$ and X_{V} will obey identical equations. We like to stress that all relations above are valid within the limits of our two-parameter approximation scheme.

Appendix B

Spherical harmonics for Schwarzschild perturbations

Harmonics have already been introduced for LRS class II spacetimes in a covariant way in chapter 5. Since the spherically symmetric Schwarzschild spacetime plays an important role in applications, we discuss spherical harmonics for this case in somewhat more detail here than was done in chapter 5. In particular, we also define tensorial spherical harmonics, which are used for investigating gravitational waves. The spherical harmonics allow us to replace the 'angular' derivative operator δ_a appearing naturally in the 1+1+2 formalism by a harmonic coefficient. The harmonics are defined in analogy with the FLRW case [132] (see also subsection 5.3.2), and we refer to [183] for details of other approaches. Note that all functions and relations below are defined in the background only; we only expand first-order perturbation variables, so zeroth-order equations are sufficient. The presentation below follows closely [34], where spherical harmonics were introduced in the literature in a covariant manner.

We introduce spherical harmonic functions $Q = Q^{(\ell,m)}$, with $m = -\ell, \cdots, \ell \in \mathbb{Z}$, defined on the background, such that

$$\delta^2 Q = -\frac{\ell(\ell+1)}{r^2} Q\,, \quad \hat{Q} = 0 = \dot{Q}\,. \tag{B.1}$$

The function r is covariantly defined by [cf (5.32)]

$$\phi = 2\frac{\hat{r}}{r}\,, \quad \dot{r} = 0 = \delta_a r\,; \tag{B.2}$$

and gives a natural length scale to the spacetime. This factor is included in our definition (B.1) so that the equation propagates; it is trivial to show that it evolves also. We have defined r so far up to an arbitrary constant, which reflects our freedom in choosing a particular normalisation of the spherical harmonic functions; we will find it most useful for our purposes to fix this freedom by covariantly defining [see also (5.21)]

$$r \equiv \left(\tfrac{1}{4}\phi^2 - \mathcal{E}\right)^{-1/2}, \tag{B.3}$$

i.e., we identify r defined here with the usual Schwarzschild parameter. We can now expand any first order scalar Ψ in terms of these functions as

$$\Psi = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{m=\ell} \Psi_{\mathsf{S}}^{(\ell,m)}\, Q^{(\ell,m)} = \Psi_{\mathsf{S}}\, Q, \tag{B.4}$$

where the sum over ℓ and m is implicit in the last equality. The S subscript reminds us that Ψ is a scalar, and that a spherical harmonic expansion has been made. Due to the spherical symmetry of the background, m never appears in any equation; we will just ignore it from now on.

We also need to expand vectors and tensors in spherical harmonics. We therefore define the *even* (electric) parity vector spherical harmonics for $\ell \geq 1$ as

$$Q_a^{(\ell)} = r\,\delta_a\, Q^{(\ell)} \quad \Rightarrow \quad \hat{Q}_a = 0 = \dot{Q}_a\ , \quad \delta^2 Q_a = \frac{1-\ell(\ell+1)}{r^2}\, Q_a\ ; \tag{B.5}$$

where the (ℓ) superscript is implicit, and we define *odd* (magnetic) parity vector spherical harmonics as

$$\bar{Q}_a^{(\ell)} = r\,\varepsilon_{ab}\,\delta^b\, Q^{(\ell)} \quad \Rightarrow \quad \hat{\bar{Q}}_a = 0 = \dot{\bar{Q}}_a\ , \quad \delta^2 \bar{Q}_a = \frac{1-\ell(\ell+1)}{r^2}\, \bar{Q}_a\ . \tag{B.6}$$

Note that $\bar{Q}_a = \varepsilon_{ab}\, Q^b \Leftrightarrow Q_a = -\varepsilon_{ab}\, \bar{Q}^b$, so that ε_{ab} is a parity operator. The crucial difference between these two types of vector spherical harmonics is that $\bar{Q}_a$ is solenoidal, so

$$\delta^a \bar{Q}_a = 0\ , \tag{B.7}$$

while

$$\delta^a Q_a = -\frac{\ell(\ell+1)}{r}\, Q\ . \tag{B.8}$$

Note also that

$$\varepsilon_{ab}\,\delta^a Q^b = 0\ , \quad \text{and} \quad \varepsilon_{ab}\,\delta^a \bar{Q}^b = +\frac{\ell(\ell+1)}{r}\, Q\ . \tag{B.9}$$

The harmonics are orthogonal: $Q^a\,\bar{Q}_a = 0$ (for each ℓ), which implies that any first-order vector

Ψ_a can now be written

$$\Psi_a = \sum_{\ell=1}^{\infty} \Psi_{\mathsf{V}}^{(\ell)}\, Q_a^{(\ell)} + \bar{\Psi}_{\mathsf{V}}^{(\ell)}\, \bar{Q}_a^{(\ell)} = \Psi_{\mathsf{V}}\, Q_a + \bar{\Psi}_{\mathsf{V}}\, \bar{Q}_a \ . \tag{B.10}$$

Again, we implicitly assume a sum over ℓ in the last equality, and the V reminds us that Ψ^a is a vector expanded in spherical harmonics.

Similarly we define even and odd tensor spherical harmonics for $\ell \geq 2$ as

$$Q_{ab} = r^2\, \delta_{\{a}\, \delta_{b\}}\, Q \ , \quad \Rightarrow \quad \hat{Q}_{ab} = 0 = \dot{Q}_{ab} \ , \quad \delta^2\, Q_{ab} = \left[\phi^2 - 3\,\mathcal{E} - \frac{\ell\,(\ell+1)}{r^2}\right] Q_{ab} \ , \tag{B.11}$$

$$\bar{Q}_{ab} = r^2\, \varepsilon_{c\{a}\, \delta^c\, \delta_{b\}}\, Q \ , \quad \Rightarrow \quad \hat{\bar{Q}}_{ab} = 0 = \dot{\bar{Q}}_{ab} \ , \quad \delta^2\, \bar{Q}_{ab} = \left[\phi^2 - 3\,\mathcal{E} - \frac{\ell\,(\ell+1)}{r^2}\right] \bar{Q}_{ab} \ , \tag{B.12}$$

which are orthogonal: $Q_{ab}\, \bar{Q}^{ab} = 0$, and are parity inversions of one another: $Q_{ab} = -\varepsilon_{c\{a}\, \bar{Q}_{b\}}{}^c \Leftrightarrow \bar{Q}_{ab} = \varepsilon_{c\{a}\, Q_{b\}}{}^c$. Any first-order tensor may be expanded

$$\Psi_{ab} = \sum_{\ell=2}^{\infty} \Psi_{\mathsf{T}}^{(\ell)}\, Q_{ab}^{(\ell)} + \bar{\Psi}_{\mathsf{T}}^{(\ell)}\, \bar{Q}_{ab}^{(\ell)} = \Psi_{\mathsf{T}}\, Q_{ab} + \bar{\Psi}_{\mathsf{T}}\, \bar{Q}_{ab} \ . \tag{B.13}$$

With the help of these spherical harmonics it is now possible to replace terms containing the angular derivative δ_a, which appears in the perturbation equations, with harmonic coefficients. Here we list all the replacements which must be made for scalars, vectors and tensors. Note that sums over ℓ and m are implicit in these equations.

Scalar:

$$\Psi = \Psi_{\mathsf{S}}\, Q \ , \tag{B.14}$$

$$\delta_a \Psi = r^{-1}\, \Psi_{\mathsf{S}}\, Q_a \ , \tag{B.15}$$

$$\varepsilon_{ab}\, \delta^b\, \Psi = r^{-1}\, \Psi_{\mathsf{S}}\, \bar{Q}_a \ . \tag{B.16}$$

Vector:

$$\Psi_a = +\Psi_{\mathsf{V}}\, Q_a + \bar{\Psi}_{\mathsf{V}}\, \bar{Q}_a \ , \tag{B.17}$$

$$\varepsilon_{ab}\, \Psi^b = -\bar{\Psi}_{\mathsf{V}}\, Q_a + \Psi_{\mathsf{V}}\, \bar{Q}_a \ , \tag{B.18}$$

$$\delta^a\, \Psi_a = -\ell\,(\ell+1)\, r^{-1}\, \Psi_{\mathsf{V}}\, Q \ , \tag{B.19}$$

$$\varepsilon_{ab}\, \delta^a\, \Psi^b = +\ell\,(\ell+1)\, r^{-1}\, \bar{\Psi}_{\mathsf{V}}\, Q \ , \tag{B.20}$$

$$\delta_{\{a}\, \Psi_{b\}} = r^{-1}\left(\Psi_{\mathsf{V}}\, Q_{ab} - \bar{\Psi}_{\mathsf{V}}\, \bar{Q}_{ab}\right) \ , \tag{B.21}$$

$$\varepsilon_{c\{a}\delta^c \Psi_{b\}} = r^{-1}\left(\bar{\Psi}_{\mathsf{V}}\, Q_{ab} + \Psi_{\mathsf{V}}\, \bar{Q}_{ab}\right) \ . \tag{B.22}$$

Tensor:

$$\begin{aligned}
\Psi_{ab} &= +\Psi_{\mathsf{T}}\, Q_{ab} + \bar{\Psi}_{\mathsf{T}}\, \bar{Q}_{ab}\ , && \text{(B.23)}\\
\varepsilon_{c\{a}\, \Psi_{b\}}{}^{c} &= -\bar{\Psi}_{\mathsf{T}}\, Q_{ab} + \Psi_{\mathsf{T}}\, \bar{Q}_{ab}\ , && \text{(B.24)}\\
\delta^{b}\, \Psi_{ab} &= +\left(1 - \tfrac{1}{2}\,\ell\,(\ell+1)\right) r^{-1} \left(\Psi_{\mathsf{T}}\, Q_{a} - \bar{\Psi}_{\mathsf{T}}\, \bar{Q}_{a}\right)\ , && \text{(B.25)}\\
\varepsilon_{c\{d}\, \delta^{d}\, \Psi_{a\}}^{\ c} &= -\left(1 - \tfrac{1}{2}\,\ell\,(\ell+1)\right) r^{-1} \left(\bar{\Psi}_{\mathsf{T}}\, Q_{a} + \Psi_{\mathsf{T}}\, \bar{Q}_{a}\right)\ . && \text{(B.26)}
\end{aligned}$$

After the harmonic expansion is introduced in the perturbation equations, each vector and tensor equation produces two harmonics equations for each ℓ, one of odd parity and one of even parity, due to the orthogonality of the vector and tensor harmonics. The importance of using spherical harmonics lies in Eqs. (B.15), (B.16), (B.19) and (B.20), which is where changes occur in the even and odd parity relations. Observe that spherical harmonics simply re-write the equations picking up some sign changes as well as some factors of ℓ's along the way.

Another benefit of these harmonic relations is that they allow us to derive various properties of the δ_a–derivative in a simple manner. For example, one readily checks that

$$2\,\delta_{\{a}\,\delta^{c}\,\Psi_{b\}c} - \delta^{2}\,\Psi_{ab} = \left(\mathcal{E} - \tfrac{1}{2}\,\phi^{2}\right)\Psi_{ab} \qquad \text{(B.27)}$$

holds, an important relation used in deriving the covariant form of the Regge-Wheeler equation for gravitational perturbations of the Schwarzschild spacetime [34].

Last but not least, we will sometimes use the following aliases

$$\begin{aligned}
\mathrm{L} &\equiv \ell\,(\ell+1)\ , && \text{(B.28)}\\
l &\equiv (\ell-1)\,(\ell+2) = \mathrm{L} - 2 && \text{(B.29)}
\end{aligned}$$

for reasons of brevity.

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