Towards a geoid consistent vertical datum in South Africa

UNIVERSITY OF CAPE TOWN

FACULTY OF ENGINEERING AND THE BUILT ENVIRONMENT

By: Matthews Siphiwe Mphuthi

MPHMAT021

Department of Architecture, Planning and Geomatics: Division of Geomatics
Faculty of Engineering and the Built Environment

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Supervisor:
Associate Professor Patroba Achola Odera

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Abstract

Traditionally, vertical datums have been realised through mean sea level (MSL) data, at one or more tide gauge station(s), followed by a precise levelling procedure to establish a network of benchmarks. Most countries around the world are still using mean sea level based vertical datums, and South Africa is not an exception. However, these vertical datums suffer from a myriad of problems such as; numerous errors from the levelling networks and tide gauge sea level measurements, high cost of maintenance and upgrade, instability due to high MSL variability, inconsistency with data acquired by satellite and space-based measurement instruments and techniques, just to mention a few. Therefore there is a need to establish a geoid-based vertical datum to mitigate the limitations of mean sea level based vertical datum and to open further frontiers in geodesy, geophysics and geodynamics research, and related applications.

Establishment of a national geoid-based vertical geodetic datum requires critical studies on the existing national height system(s) and related distortions, appropriate height system and related reference surface, offset between a local height datum and the intended reference surface, among others. The world is moving towards global unification of vertical datums to modernise the vertical positioning technique, an international height reference system (IHRS) would provide a globally unified height reference system. The horizontal positioning is already realised on the international terrestrial reference frame (ITRF) with high precision, and a similar approach for the realisation of a new vertical datum for South Africa is required. This study carries out analysis on the following aspects over South Africa: comparison between spheroidal, orthometric and normal height systems; accuracy of levelling network; vertical datum offset in relation to geoid, quasigeoid and the IHRS. It concludes by providing a unique framework for establishing a geoid-based vertical datum in South Africa.

A numerical investigation of the correlation between the South African spheroidal, orthometric and normal height systems is conducted. It is determined that the spheroidal orthometric height system is more correlated with the normal height system (~ 21.3 cm on average) than the orthometric height system (~ 40.7 cm on average). A further numerical assessment was conducted to determine the magnitude of misclosures and the empirical value for the first order levelling network on the levelling loops. It was determined that majority of the levelling loops fall within the acceptable empirical value for the first order levelling network ($c = 0.003$). However, only one levelling loop does not fall within the acceptable range of misclosure for the first order levelling network, with a misclosure from spirit levelling measurement of -10.2 cm while the estimated acceptable misclosure is 9.7 cm.

The vertical datum offset between the South African local vertical datum and global vertical datum was achieved by estimating the vertical datum offset and the geopotential values on the four fundamental benchmarks. A single-point-based geodetic boundary value problem (GBVP) approach was used following Molodensky's theory for estimating the height anomalies from the disturbing potential using Bruns's formula. The gravity potential at each tide gauge benchmark (TGBM) in South Africa deviates from the potential of the global reference surface by 0.585, -2.023, -2.597 and $2.105\ \mathrm{m^2s^{-2}}$ for Cape Town, Port Elizabeth, East London and Durban tide gauge benchmarks, respectively. The corresponding vertical datum offset between the international height reference system and the four fundamental benchmarks over South Africa are 5.973, -20.647, -26.518, and 21.496 cm for Cape Town, Port Elizabeth, East London and Durban tide gauge benchmarks, respectively. The datum offsets between the land levelling datum (LLD) and the global vertical datum has been estimated, for the first time over South Africa, in this study.

A preliminary geoid-based vertical datum in relation to the IHRS for South Africa was determined and evaluated using 138 GPS/levelling data points distributed over the country. However, since it would be difficult to identify exactly which data points are associated with a particular TGBM, the TGBM in Cape Town was held fixed for this analysis. During this analysis, the spheroidal orthometric height was unified to the IHRS (H^{SIHRS}), an existing bias between the H^{SIHRS} and the local quasigeoid is estimated to be approximately 15.8 cm on average. An adequate data coverage is required to improve the quality of the determined vertical datum offset for the South African vertical datum in relation to the global vertical datum. It is proposed in this study that a normal height system should be adopted for South Africa, with the relevant reference surface being quasigeoid model. Some considerations to be taken during the implementation and adoption of a consistent geoid-based vertical datum in South Africa are discussed.

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I owe my deepest gratitude to my lovely wife, Nokwazi P. Mphuthi and kids for all their support, patience and encouragement. I would like to expand my thanks to the Lord above for making it possible for me to make it this far.

Dedication

This thesis is dedicated to the memory of my mother, Molelekeng J. Mphuthi. She might have departed in her physical form, but her love, support, and guidance continue to inspire me every day.

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SYMBOL	DESCRIPTION	UNITS
GM_g	mass gravity constant of the geopotential model	m^3s^{-2}
a_g	semi-major axis of the geopotential model	meters
GM	mass gravity constant of the reference ellipsoid	m^3s^{-2}
a	semi-major axis of the reference ellipsoid	meters
f	flattening parameter of the reference ellipsoid	no units
b	semi-minor axis of the reference ellipsoid	meters
E	linear eccentricity	meters
e	first (numerical) eccentricity	no units
e'	second (numerical) eccentricity	no units
ω	angular velocity of the Earth's rotation	s^{-1}
λ	longitude	degrees or radians
φ	geographic latitude	degrees or radians
$\bar{\varphi}$	geocentric latitude	degrees or radians
r	local elliptic radius	meters
$\bar{g}$	mean gravity	ms^{-2}
γ	normal gravity	ms^{-2}
γ_a, γ_b	normal gravity at the equator and at the poles	ms^{-2}
$\bar{C}_{n,m}, \bar{S}_{n,m}$	fully normalized spherical harmonic coefficients	no units

	(or Stokes' coefficients) of the gravity model	no units
$P_{n,m}$	associated Legendre functions of the first kind	no units
$\bar{P}_{n,m}$	fully normalized harmonics	no units
$TGBM$	Tide Gauge Bench Mark.	no units
ITRS.	International Terrestrial Reference System	no units
$IHRS$	International Height Reference System	no units

1 INTRODUCTION

1.1 Background

Vertical positioning is required for a number of scientific and engineering applications. Due to the accelerating growth in the use of space-based technologies, such as, Global Navigation Satellite Systems (GNSS - GPS, Galileo, GLONASS and BeiDou), Very Long Baseline Interferometry (VLBI), Satellite Laser Ranging (SLR), Lunar Laser Ranging (LLR) and Satellite Altimetry, there is a need to establish a geoid-based vertical datum to facilitate effective application of space-based technologies for vertical positioning. Space-based technologies have entirely changed the vertical positioning techniques (Torge & Müller, 2012). However, height measured from space-based instruments are purely geometric heights, and they are referenced to the ellipsoid (mathematically defined reference surface). Moreover, geometric heights do not carry any physical meaning and they cannot be used to conduct surface analysis such as predicting the flow of water (Heiskanen & Moritz, 1967).

Earth's gravity field information has to be integrated in the height determination, to achieve physically meaningful heights. A differential levelling technique is conducted to determine physically meaningful heights, geometrical height differences between benchmarks are corrected using correction term computed from the gravity information. Spirit levelling technique is used to estimate the geometrical height differences between two benchmarks. However, in order to obtain absolute heights, a height datum is required. Height datum is a surface of zero height to which levelling heights can be referenced.

Traditionally, levelling based vertical datum have been realised by being fixed to the mean sea level (MSL) value at one or more tide gauge benchmark (TGBM) station/s. Heights at the tide gauge station/s may be referenced to a Chart Datum (CD), which is the point below the lowest low water observed for sufficiently long period (this is applied in South Africa). The fixed point of origin at the tide gauge station, realise a surface of zero height that is not theoretically matched to a single equipotential surface due to combined distorting effects of mean sea level measurements and Sea Surface Topography (SST). Then the height of the TGBM/s is transferred to the inland benchmarks by using terrestrial levelling techniques. A schematic depiction of this approach is given in Figure 1-1:

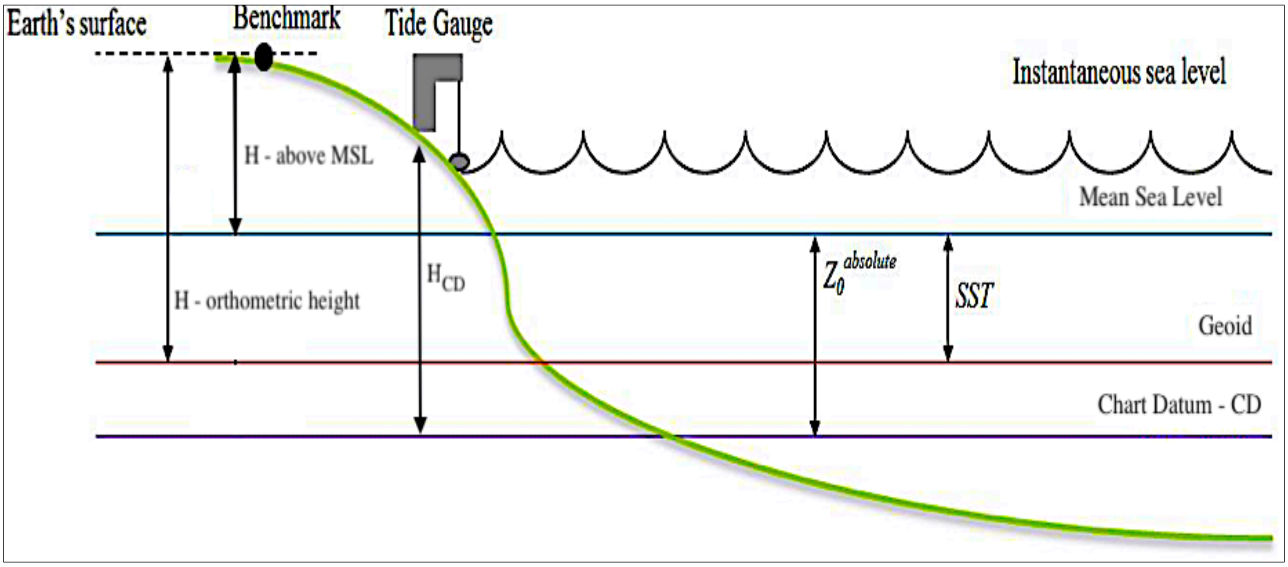

Figure 1-1: Relationship between Tide gauge and the MSL

There are a number of issues associated with the levelling based vertical datum, such as, numerous errors from the levelling networks and tide gauge sea level measurements, higher cost of maintenance and upgrade, instability due to MSL variability, inconsistency and time consuming. However, the approach to determine physically meaningful heights has changed entirely by a geoid based vertical datum. Geoid height from a gravimetric geoid model can be used to convert the geometrical heights to physical heights. This has escalated the need for consistent geoid heights that are within the obtainable accuracy of the ellipsoidal heights, obtained from space-based positioning instruments. An accurate geoid height (N) is required for transformation of the ellipsoidal height (h) to physically meaningful height (H). This geometric relationship is expressed by equation (1:1), with an assumption that the geoid and mean sea level are coincident, however, practically this is not the case in a strict sense.

$$N = h - H \tag{1:1}$$

This geometric relationship has been elaborated in a number of geodetic literatures (Rapp, 1994; Burša et al., 2001, 2004; Amos & Featherstone, 2009; Sjöberg, 2010; Kuczynska-Siehien et al., 2017; Sánchez & Sideris, 2017). This approach provides principle for height datum modernisation, there are few countries progressing towards this approach, only New Zealand and Canada have physically implemented the gravimetric quasigeoid/geoid based vertical datum. This approach overcomes some of the flaws of differential levelling – difficult access to mountainous and remote areas, expensive to conduct and labour intensive. However, it provides direct access to physical height at any point using space-based instruments. A vertical datum defined in a more rigorous sense, is defined by an equipotential surface represented by a geopotential value W_0, which provides a reference surface for physical heights.

The South African vertical datum is known as the Land Levelling Datum (LLD), it was realised over a century ago (since early 1900's) on the adopted mean sea level (MSL) at four tide gauge stations (situated in Cape Town, Port Elizabeth, East London and Durban). It was connected to the network of primary levelling benchmarks which were then adjusted in a piecemeal fashion (Wonnacott & Merry, 2011). It is classified as a spheroidal orthometric height system. However, it provides a poor approximation to both true orthometric and normal heights systems (Merry, 1985).

The spheroidal orthometric height system is somewhat more close to the normal orthometric height system. The geoidal orthometric correction was computed for only four levelling loops around Cape Town, meaning that the actual gravity measurements were taken for only those loops. The spheroidal orthometric correction was applied to all the height differences from first order levelling networks, computed from normal gravity (Merry, 1977, 1985; Wonnacott & Merry, 2011). In this system, spheropotential number is used instead of the geopotential number, which is derived from the normal gravity (Odumosu *et al.,* 2015).

The tide observations at these four tide gauges were referred to datum benchmarks. It was mentioned on the Technical Publication 17 report (Anonymous, 1965) and by Wonnacott and Merry (2012) that the mean sea level at these tide gauges were determined within a short period (1-2 years). However, there is no further information on the exact duration of the tide gauge observations. Both the datum and the levelling network are associated with significant distortions. The land levelling datum and spheroidal orthometric height system are associated with a number of obstacles, as defined by Wonnacott & Merry (2012):

- The LLD was established from the tide gauge observations, which were observed more than a century ago. There is an ambiguity as to how the datum was integrated to the levelling networks.
- The primary levelling networks were adjusted independently, and only the spheroidal orthometric correction was applied to all the levelling loops.
- A trigonometrical height technique and necessary adjustments were employed to determine heights of the trigonometrical beacons.

The purpose of this study is to make contribution towards the gravimetric geoid based national vertical datum for South Africa. This is going to be beneficial to the nation towards joining the International Height Reference System (IHRS), which is tightly linked to the geometric coordinates reference frame, International Terrestrial Reference Frame (ITRF). The International Association of Geodesy (IAG) took a resolution in July 2015, that there is an urgent need for the establishment of an integrated national and regional geodetic vertical reference system, with an aim of establishing a unified global vertical reference system (Sánchez & Sideris, 2017). This will make it possible for the space-based positioning instruments to be used directly to determine physically meaningful heights, which is the most cost effective approach today.

The issue of modernising vertical datum is a unique problem that requires a unique solution for each and every country. To mention few examples, New Zealand before the implementation of a geoid based vertical datum, had its vertical datum defined by thirteen different local vertical datums (LVDs) based on a network of twelve different tide gauge stations (Claessens *et al*., 2011). Then a unified local vertical datum was developed in September 2009, New Zealand Vertical Datum (NZVD2009) uses a New Zealand quasigeoid 2009 (NZGeoid2009) model as its reference surface. It was developed for the purpose of unifying the thirteen LVDs and to also provide a consistent vertical datum for New Zealand (Claessens *et al*., 2011).

In another case, Australian vertical datum is based on a well-established Australian Height Datum (AHD) which uses a normal-orthometric height system. It was realised by an adjustment of approximately 195,000 km of spirit levelling observations, fixed to MSL observations at multiple tide-gauges (Featherstone & Kuhn, 2006). Furthermore, in Australia a gravimetric quasigeoid model was computed, known as an AUSGeoid09 model, and it has been posteriori fitted to the AHD so as to provide a product that is practically useful for the more direct determination of AHD heights from satellite based instruments (Featherstone *et al.,* 2011). This is theoretically more compatible since both the AHD and a quasigeoid (AUSGeoid09) uses normal-orthometric height system.

Realisation of a geoid-based vertical datum in South Africa requires a unique approach just as the examples from two different countries mentioned above, they each had unique approach in revisiting the definition and realisation of their respective height systems and vertical datums. The common approach of redefining height systems and vertical datums are to modernise and unify them, and most importantly to be rigorously defined.

The continental (African) geoid modelling history is not that long, in the Southern Africa there are three types of regional geoids (namely, UCT2003, UCT2004, and UCT2006) and in 2003 a continental quasigeoid model was developed by Merry (2003), under African Geoid Project of 2003 (AGP2003), it was developed to provide support for infrastructural development across the continent. An official hybrid geoid model for South Africa was developed in 2009, namely SAGEOID10 (Merry, 2007; Chandler & Merry, 2010). A very recent geoid model for Africa (AFRgeo2019) was modelled in 2019 for the similar purpose of providing a unified reference surface for Africa (Abd-Elmotaal et al., 2020).

The South African hybrid geoid model (SAGEOID10) was developed for the purpose of transforming GPS derived ellipsoidal heights to spheroidal orthometric heights on the LLD. The SAGEOID10 hybrid geoid model is only valid within the borders of South Africa, as it was calibrated using data within the country (Chandler & Merry, 2010). However, it still does not sort out the issues associated with the LLD, such as the existing instability and inconsistency.

1.2 Problem Statement

The current South African vertical datum has a number of problems. The first order levelling networks over South Africa, defining the LLD, were not adjusted as a whole but in a piecemeal approach. This means that the LLD is inconsistent. The LLD was determined from the MSL observations at four fundamental benchmarks (TGBMs), which were measured approximately over a century ago. It is a known fact that MSLs are unstable, this affects the stability of a vertical datum defined from MSL observations, such as the LLD. It will not be economically viable to maintain or to readjust the current South African vertical datum.

National height datums are generally developed from an assumption that the geoid and mean sea level coincide at one or more tide gauge stations, and the tidal data is used as a datum. However, the two may depart from each other by as much as 200 cm (Merry, 1993). Moreover, these issues affect the definition and the unification of South African vertical datum with neighbouring countries, in order to be able to engage on cross country engineering projects. The existing levelling-based vertical datum does not meet the accuracy requirement of modern geodesy, therefore unique strategies must be implemented to reduce the problems resulting from the LLD as the vertical datum for South Africa.

An accurate gravimetric geoid model is required for the nation, not only for the reasons mentioned above but also for taking part on the IHRS, which is tightly linked to the geometric coordinates reference frame, International Terrestrial Reference Frame (ITRF). The existing datasets that are defined in terms of LLD should be converted to the gravimetric geoid based vertical datum using a consistent transformation, the geoid model and the appropriate offset. The relationship between the proposed gravimetric geoid based datum and the other height systems used in South Africa is diagrammatically illustrated in Figure 1-2:

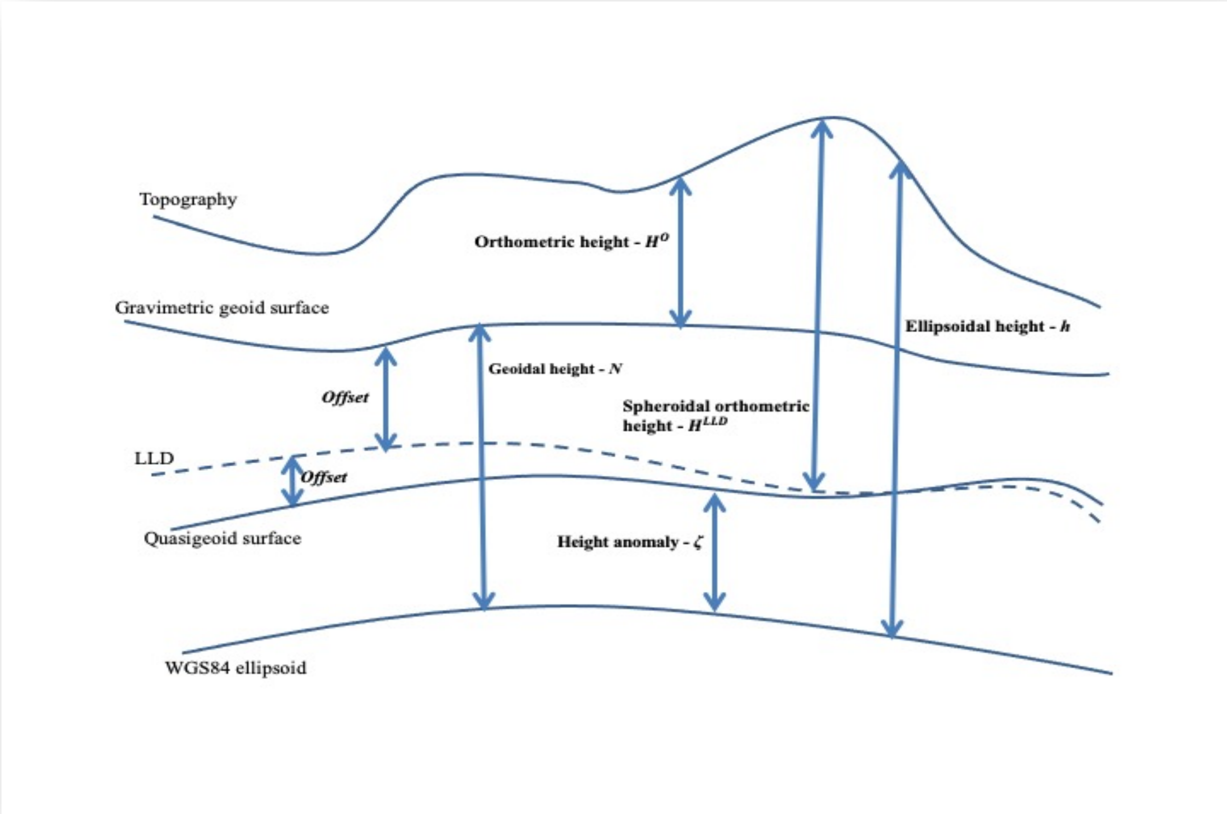

Figure 1-2: Relationships between height systems

As indicated in Figure 1-2, there is an existing datum offset among reference surfaces. The datum offset between LLD and the existing quasigeoid in South Africa was estimated to be approximately 20.0 cm on average (Chandler & Merry, 2010). However, the quasigeoid model was integrated to the GPS/levelling data using a correction surface to develop a hybrid geoid model (SAGEOID10). The datum offset is an indication that the local geoid model does not coincide with the local vertical datum as it is always assumed in most practical applications. Furthermore, this assumption ignores the existence of the sea surface topography (SST) and existing distortions in the local vertical datum (Odera & Fukuda, 2015b). The relationship between LLD and the reference surface (geoid or quasigeoid model) to be adopted has to be resolved for appropriate adjustment. The datum offset can be modelled using the relationship between the land levelling datum, local geoid model and reference ellipsoid, as given in the following expression:

$$\boldsymbol{Offset} = h - H - N \tag{1:2}$$

The modernization of a vertical datum for South Africa is necessary to make it compatible with space-based instruments. It should be replaced by a gravimetric geoid-based model that describes the vertical datum with respect to an ellipsoid. This also means that some existing levelling network points will probably be neglected and this would result to fewer ground-truth data to test geoid models over time. A geoid-based vertical datum would consist of continuously operating GNSS reference stations (CORS). Land surveyors will be able to create/install benchmarks using a space-based instrument. A modernised vertical datum is defined less by levelling observations and more by modelling, meaning that it can be updated or improved as more adequate datasets become available.

A decision has to be made whether to adopt a geoid or quasigeoid model as a reference surface, for a rigorously defined height system in South Africa. This will enable estimation of vertical datum offset for the South African vertical datum, in relation to the international height reference system. The local vertical datum offsets can be used for unification of the South African vertical datum at the four benchmarks in a manner that is consistent to the international height reference system.

The LLD was established from a set of primary levelling networks, which were not rigorously adjusted and actual gravity data was not considered on the function of the spheroidal orthometric correction. This gives an indication that LLD is not properly defined and it is inconsistent, therefore a replacement is required. A well-defined geoid-based vertical datum will provide South Africa with a consistent and reliable vertical datum. It will also provide a better way to study environmental issues such as change of sea-level, seismic activities, and volcanic and tectonic activities. Moreover, it will provide integrated spatial data infrastructure for engineering and cadastral applications (especially for 3D cadastre).

1.3 Research objectives

A geoid model is not only important in geodetic science but also in scientific and economic fields. Moreover, a geoid is an equipotential (but not the quasigeoid) surface in the Earth's gravity field that can serve both as a geodetic datum and a reference surface in geophysics (Sjöberg, 2018). However, there are many considerations surrounding a geoid based vertical datum that need to be thought through before it is fully implemented and adopted in South Africa. The main focus of this study is to make contributions towards development of a gravimetric geoid-based national vertical datum. The specific objectives of this study are listed below:

a) To determine differences between spheroidal, orthometric and normal height systems using the GPS/levelling data points over South Africa.
b) To conduct an assessment to determine the magnitude of misclosures and the empirical value for the first order levelling network on the main levelling loops in South Africa.

c) To estimate the vertical datum offset and the geopotential value over South Africa on the four fundamental benchmarks, situated on the tide gauge stations (Cape Town, Port Elizabeth, East London and Durban), in relation to International Height Reference System.
d) To conduct an analysis to determine a consistent height system with a corresponding reference surface for South Africa.
e) To conduct a preliminary investigation of the quasigeoid and geoid datum offsets on IHRS over South Africa.
f) To develop a general procedure for the implementation of a quasigeoid/geoid vertical datum in South Africa.

1.4 Thesis outline

In chapter 2, a theoretical framework for establishment of a geoid-based vertical geodetic datum is covered, the fundamental of height types and systems is presented, and the recent levelling-based vertical datum in South Africa is also presented (including the role of the local/global geoid in a solution for unification of local vertical datums). In chapter 3, the South African vertical geodetic datum and related modernisation efforts are presented. Empirical investigations for the establishment of a geoid-based vertical datum in South Africa are given in chapter 4. In chapter 5, a proposed framework for the establishment and implementation of a modernised vertical datum in South Africa is presented. Chapter 6 provides conclusion and future work with respect to the stated research objectives.

2 THEORETICAL FRAMEWORK FOR THE ESTABLISHMENT OF A GEOID-BASED VERTICAL DATUM

2.1 Height systems

Height is defined as the distance of a point above a specified reference surface of constant potential to the Earth surface, measured along the direction of gravity between the point and the reference surface (Vaníček, 1976). A reference surface is also known as a vertical datum or height datum. Height systems have different reference surfaces, such as geoid and quasigeoid. The difference between the geoid and quasigeoid is that the former implies that orthometric heights must be considered while the latter implies the use of the so-called normal heights. However, the geoid is a physically meaningful surface, it is sensitive to the density variations within the Earth. While, the quasigeoid is not a physically meaningful surface, and it requires integration over the Earth's surface (Vaníček *et al*., 2012). However, the computation of a geoid model (in contrast to the quasigeoid) requires the topographic density distribution knowledge (which is frequently not well known), this implies that the height anomaly can be computed more accurately than the geoid (Sjöberg, 2013).

Height of a point on the Earth surface is also referred to as a vertical position, the height or elevation of a point is defined as a one dimensional coordinate system. There are two different classes of heights as mentioned in section 1.1; geometric and physical heights. The geometric height systems are purely mathematical, they do not need Earth's gravity field information to be determined and they do not have any physical meaning, this also includes the ellipsoidal height system. In the contrary, the physical heights are considered to be natural as they are connected to the Earth's gravity field and they also represent potential differences that provide direction of the flow of water (e.g. orthometric, normal, and dynamic height systems).

2.1.1 Ellipsoidal height system

Ellipsoidal height, is defined by a perpendicular distance from a particular point P on the Earth's surface to the reference ellipsoid surface, as depicted in Figure 2-1. It is measured solely from space-based instruments, and it is a purely geometrical height system. This type of height system is only practical if the information of the geoid undulation is available, to determine the geometric relationship as expressed by equation (1:1). Ellipsoidal heights cannot be used for most engineering and scientific applications, physical heights are mostly preferred for these applications.

2.1.2 Orthometric height system

Orthometric height H^O, is defined as a geometrical distance from the point P on the Earth's surface along the plumbline to the point P_O on the geoid's surface, as depicted on Figure 2-1. The separation between the reference ellipsoid and the geoid is referred to as a geoid height N. It is considered natural and it is hard to realise perfectly in practice, since it requires knowledge of gravity variations or mass-density distribution inside the topography.

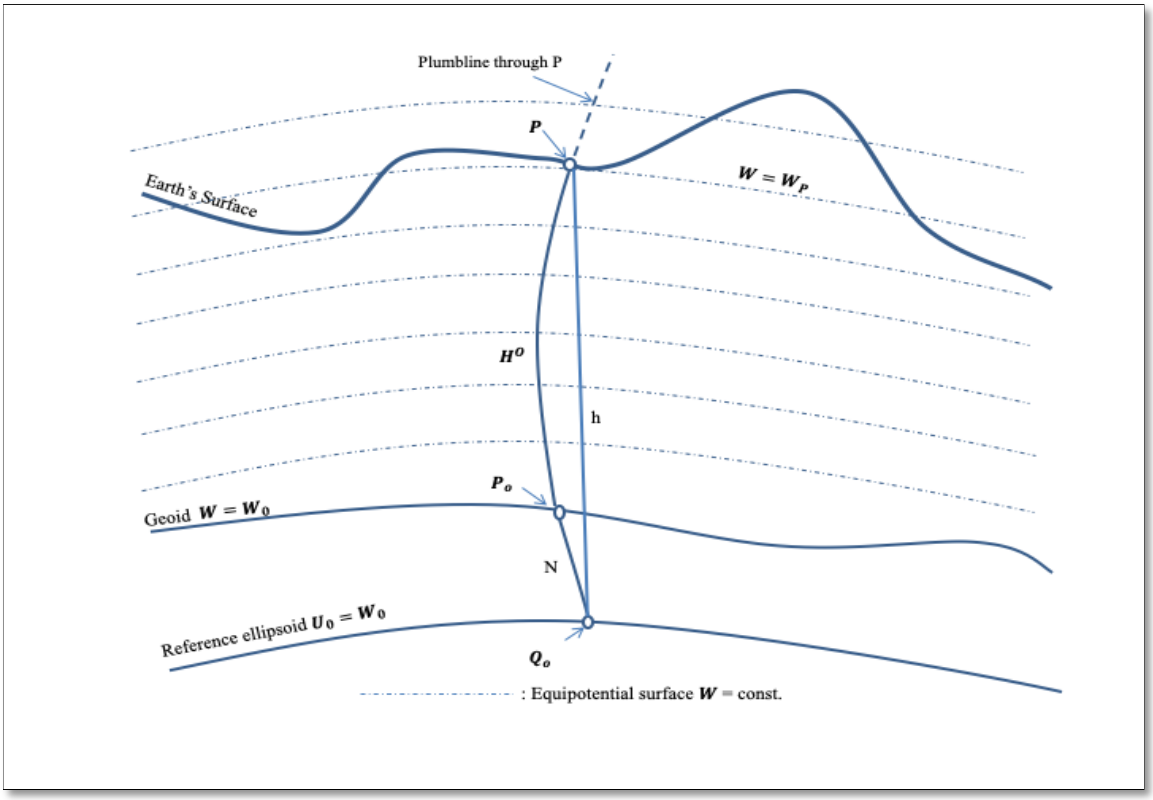

Figure 2-1: Orthometric and ellipsoidal height representation

According to Heiskanen and Moritz (1967), orthometric heights have an unequalled geometrical and physical significance. It can be determined as follows (Heiskanen & Moritz, 1967):

$$H^O = C/\bar{g} \quad (2:1)$$

where $\bar{g} = \frac{1}{H}\int_{P_0}^{P} gdn$ is the actual mean gravity along the plumb line measured in Gal (Heiskanen & Moritz, 1967), while C is the geopotential number, measured in $Gal.m$ and it represents the difference in gravity potential between the constant value at the geoid (W_0) and the potential at the point $\boldsymbol{P}$ on the surface of the Earth (W_P), it can be expressed as follows:

$$C = W_0 - W_p = -\int_{P_0}^{P} dW = \int_{P_0}^{P} g dn, \qquad (2:2)$$

where dW represents the potential difference in differential meaning, dn is the height difference in differential meaning and g is the observed gravity. A graphical representation of the orthometric height system is depicted in Figure 2-1.

2.1.2.1 Normal orthometric height system

The normal-orthometric height system, H^{N-O} (Figure 2-2), was developed for the situation where actual gravity values are not available at levelling stations. It is the slightly curved-line distance reckoned along the normal gravity plumbline from the point, P_o^{N-O} , on the surface of the quasigeoid to the point P on the surface of the Earth (Featherstone & Kuhn, 2006). In this system the geopotential number from equation (2:1) above is replaced with the spheropotential or a normal-geopotential number, C^S, derived from the normal gravity and the normal-orthometric height is determined as follows (Yilmaz, 2008):

$$H^{N-O} = C^S / \overline{\gamma} \qquad (2:3)$$

where $\overline{\gamma}$ represents a mean normal gravity. It only makes use of the normal gravity field as an approximation of the Earth's gravity field to derive all necessary gravity-field-related quantities. Normal-orthometric heights are easy to compute, because of the avoidance of making actual gravity observations but they are less likely to predict fluid flows correctly than normal heights. As depicted in Figure 2-2, the reference surface for the normal-orthometric height is the quasigeoid (this is only acceptable in practice but may not be valid theoretically).

2.1.2.2 Spheroidal orthometric height system

The spheroidal orthometric height system does not require actual gravity observations, but an approximation to the actual gravity on the levelling route based upon the normal gravity. Merry (1985) mentioned that this system corresponds somewhat more closely to the normal height system, but it is a poor approximation to the true orthometric height. In the South African spheroidal orthometric height system, gravity on the levelling routes is approximated using theoretical normal gravity as a basis. In this system the mean normal gravity ($\bar{\gamma}^*$) along the normal plumbline is computed using a Vignal's approach as follows (Merry, 1985):

$$\bar{\gamma}^* = \gamma - 0.1543H^S \quad (2:4)$$

Therefore the spheroidal orthometric height can be expressed as:

$$H^S = C^{S^*}/\bar{\gamma}^* \quad (2:5)$$

The spheropotential, C^{S^*}, is computed using an approximated normal gravity, γ^*, reduced to the elevation of the point using the free-air vertical gradient of normal gravity (Merry, 1985), it can be generally expressed as follows:

$$C^{S^*} = \int \gamma^* dn = \sum \gamma^* dn, \quad (2:6)$$

where an approximation to the actual gravity on the levelling route is expressed as:

$$\gamma^* = \gamma - 0.3086H^S \quad (2:7)$$

2.1.2.3 Helmert orthometric height system

The Helmert orthometric height system is computed in the same manner as the rigorous orthometric height system, given by equation (2:1). However, in the Helmert orthometric height system the mean gravity along the actual plumbline is estimated from the normal density factor (Vaníček, 1976),

$$\bar{g}^H = g + \frac{1}{2}FH_o - 2\pi G\rho H_o \quad (2:8)$$

where $\bar{g}^H$ is the Helmert mean gravity, g is the observed gravity at the topographical surface, F is linear vertical normal gravity gradient, G is the universal gravitational constant, H_o is the orthometric height of the surface at the same point and ρ is the topographic density constant. Therefore, the Helmert orthometric height (H^H) can be obtained as follows:

$$H^H = {}^{C}/_{\bar{g}^H} \tag{2:9}$$

This approximation is based on the Poincare–Prey relationship for mean gravity and the Bouguer shell gravity expression that accounts for the topographic mass above the geoid but neglects the terrain effects (Vaníček, 1976).

2.1.2.4 Mader orthometric height system

The Mader approximation of the orthometric height assumes that the terrain effect vary linearly between the geoid and the surface, and so uses a simple mean of the values of the effect evaluated for the Earth's surface and the geoid. In this approach, the mean gravity value ($\bar{g}^M$) along the plumbline is estimated as follows (Mader, 1954; Santos *et al.*, 2005):

$$\bar{g}^M = g + \frac{1}{2}FH_o - \frac{1}{2}(g^T - g_o^T) \tag{2:10}$$

where g^T and g_o^T are the vertical components of gravity due to topographic masses at the ground and at the geoid surfaces, respectively. Mader orthometric height (H^M) can then be obtained as follows:

$$H^M = {}^{C}/_{\bar{g}^M} \tag{2:11}$$

2.1.2.5 Niethammer orthometric height system

A practical and more accurate approximation of the orthometric height system is given by Niethammer's approach. In this approach, the mean gravity is computed considering the integral mean of terrain effects evaluated at discrete points along the plumbline. The mean gravity can be estimated as follows (Niethammer, 1932; Santos *et al.*, 2005):

$$\bar{g}^{Ntm} = g + \frac{1}{2}FH_o - g^T + g_o^T \tag{2:12}$$

Therefore, the Niethammer orthometric height can be obtained as follows:

$$H^{Ntm} = {}^{C}\!/_{\bar{g}^N} \tag{2:13}$$

Both the Mader and Niethammer orthometric height systems consider the roughness of the terrain, residual to the Bouguer plate or shell when estimating the integral-mean value of gravity along the actual plumbline. Furthermore, they also include a terrain effect term in the computation of mean gravity. However, these height system are not in wide practical use due to their computational complexity (Santos *et al*., 2005).

2.1.3 Normal height system

Due to the difficulty of determining a pure orthometric height, other forms of height systems are used as an approximation to the rigorous orthometric height system. In this case, the actual mean gravity along the actual plumbline is replaced with a mean normal gravity ($\bar{\gamma}$) computed along the normal gravity plumbline. It can be determined as follows (Vaníček, 1976):

$$H^N = C/\bar{\gamma} \tag{2:14}$$

A theoretical replacement of the Earth surface is introduced in this system, referred to as a telluroid surface. It is defined as an auxiliary surface obtained by the point-wise projection of points on the Earth's surface along the straight-line ellipsoidal normal to points that have the same normal gravity potential (U) as the actual gravity potential (W) on the earth's surface, it is as depicted in Figure 2-2.

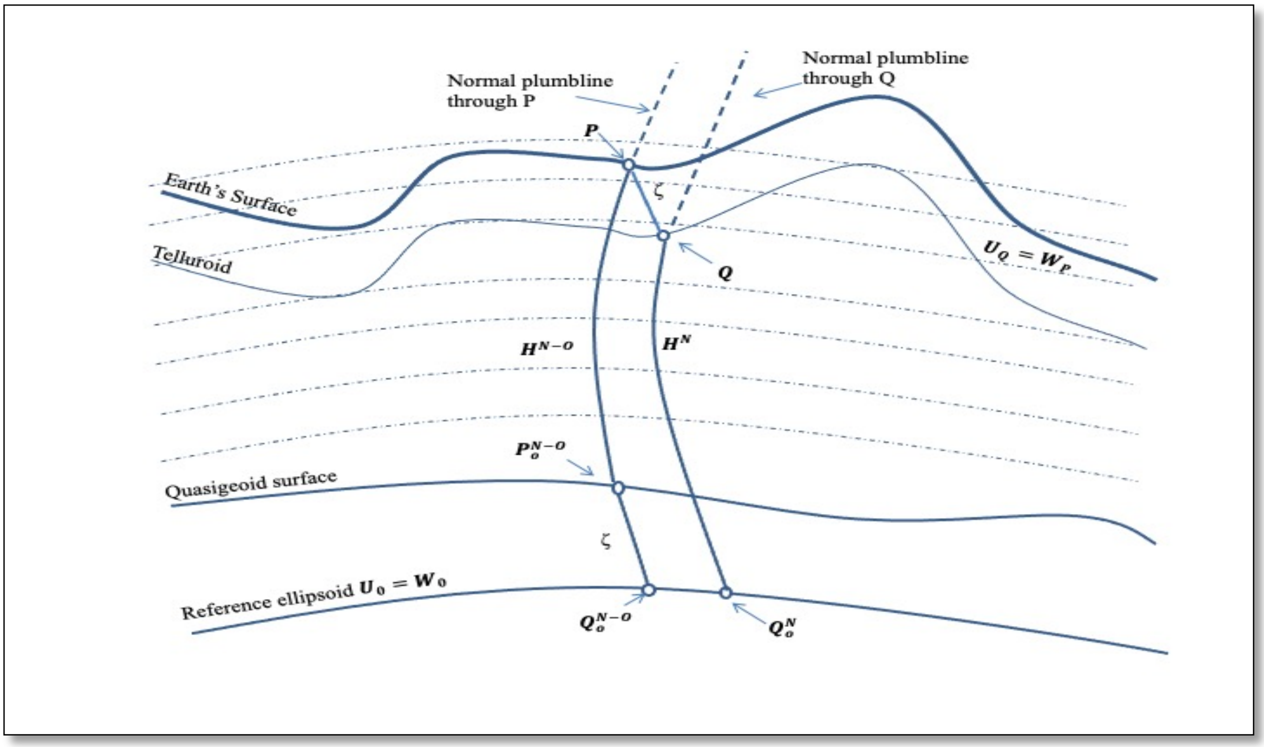

Figure 2-2: Normal and normal orthometric height representation

The quasigeoid height, ζ, is the straight-line distance reckoned along the ellipsoidal surface normal from the point, P_O^{N-O}, on the surface of the quasigeoid to the point, Q_O^{N-O}, on the surface of the reference ellipsoid. It is the same length as the height anomaly, ζ, the straight-line distance reckoned along the ellipsoidal normal from the point P on the Earth's surface to the point, Q ,on the surface of the telluroid.

2.1.4 Dynamic height system

The dynamic height system, H^D, is computed in the same way as the orthometric height, however the mean gravity value in equation (2:1) is replaced with the constant value of normal gravity at mid-latitude (γ_{45}). It can be determined as follows (Vaníček, 1976):

$$H^D = C/\gamma_{45} \qquad (2:15)$$

In the dynamic height system, the potential difference is converted to a straight distance relative to the geoid. Dynamic heights are considered to be physically meaningful, however, they are not always preferable as a practical height system because it has no geometrical meaning.

2.1.5 Relationship between height systems

The conceptual differences between the LLD, geoid and quasi-geoid are defined in the previous section and it is depicted in Figure 2-3.

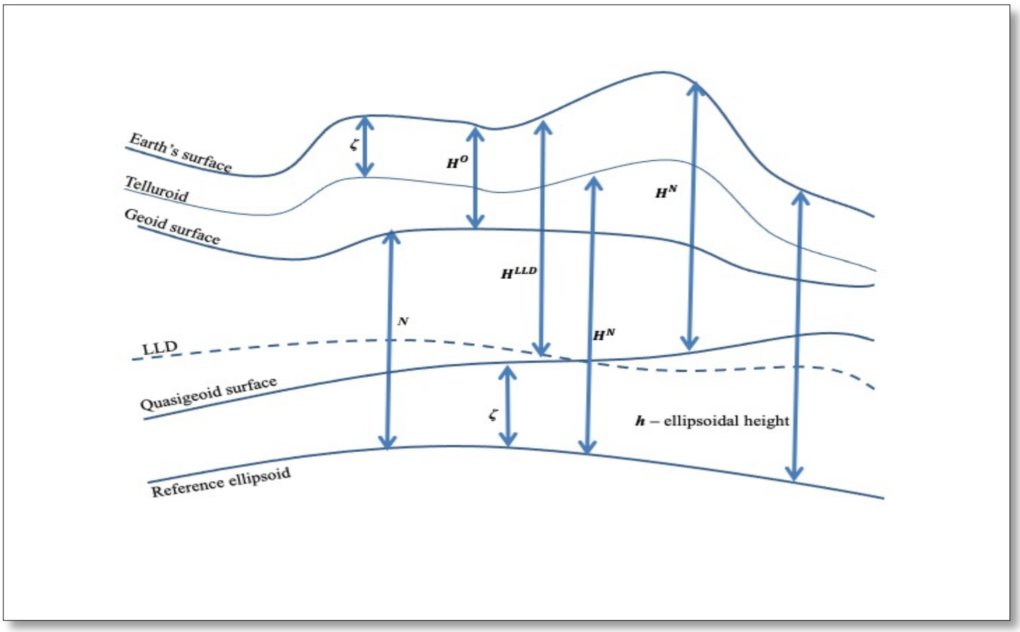

Figure 2-3: Relationship between height reference surfaces and height systems

There are a number of theoretical and practical differences in the computation and determination of different height surfaces, the key difference between the geoid and quasigeoid arise from the assumption made concerning the treatment or handling of the Earth's topographic density during the solution of the geodetic boundary value problem (Foroughi & Tenzer, 2017). The separation between the reference ellipsoid and the quasigeoid is called the height anomaly, given as follow,

$$\zeta = h - H^N \tag{2:16}$$

The telluroid is defined as an auxiliary surface obtained by the point-wise projection of points on the topographic surface along the straight-line at a distance equal to height anomaly (ζ) below the Earth surface. The distance between the ellipsoid and the telluroid along the normal plumbline is referred to as the normal height (H^N). The relationship between the ellipsoidal height (h), orthometric height (H^O) and geoidal heights (N) is expressed as follows:

$$N = h - H^O \tag{2:17}$$

Furthermore, the separation between the geoid and the quasigeoid can be rigorously defined as follows:

$$N - \zeta = H^N - H^O = C/\bar{\gamma} - C/\bar{g} \tag{2:18}$$

Therefore the difference between orthometric and normal height can be determined as:

$$H^N - H^O = H^O \frac{(\bar{g} - \bar{\gamma})}{\bar{\gamma}} \; or \; H^N \frac{(\bar{g} - \bar{\gamma})}{\bar{g}} \tag{2:19}$$

It is quite difficult to evaluate the actual mean gravity along the actual plumbline ($\bar{g}$), since it requires knowledge about topographic density distribution around the point of interest. Therefore, the differential term ($\bar{g} - \bar{\gamma}$) can be approximated by using Bouguer gravity anomaly (Δg_P^B) at the computational point P, hence equation (2:19) can be expressed as (Heiskanen & Moritz, 1967),

$$\zeta - N \approx H^N - H^O \approx H^O \frac{\Delta g_P^B}{\bar{\gamma}} \tag{2:20}$$

The above equation gives a standard practical approach for defining the geoid and quasigeoid separation term, in an approximate form. However, recent studies on determining $\zeta - N$ indicate that a more rigorous approach is required on extremely rough terrains (Flury & Rummel, 2009; Sjöberg, 2010). This approach considers the attraction of topographic masses to the mean gravity along the plumbline. The approach by Flury & Rummel (2009) and Sjöberg (2010) was not used in this study, as the South African terrain is not extremely rough or rugged. The South African topography is fairly flat (see Figure 4-2) compared to the selected test areas 'with extreme topography' such as Zugspitze and Grossglockner reported by Flury & Rummel (2009). The extension of equation (2:20) can be expressed as (Flury & Rummel, 2009).

$$\zeta - N \approx H^O \frac{\Delta g_P^B}{\bar{\gamma}} + \frac{(V_{P_0}^{TOP} - V_P^{TOP})}{\bar{\gamma}}. \tag{2:21}$$

where $V_{P_0}^{TOP} and \, V_P^{TOP}$ are the topographic potentials at the geoid and Earth's surface point, respectively. The computation of the Bouguer gravity anomalies requires determination of the Bouguer gravity reduction. During the Bouguer gravity reduction the gravitational effect of the intermediate topography is removed in two stages, as mentioned bellow:

- The simple Bouguer reduction assumes that the topography attracts as an infinitely lateral plate of thickness equal to the observation elevation,
- Then accounts for departures of the topography from this simple plate approximation, and takes into account the negative gravitational effect of the residual topography.

The Bouguer gravity anomaly (Δg^B) can be computed using the generalised formula, it uses point absolute gravity and DTM data, refer to Featherstone and Dentith (1997) for more details. The mean value of the normal gravity between the reference ellipsoid and the telluroid, $\bar{\gamma}$, is computed as follows (Heiskanen & Moritz, 1967):

$$\bar{\gamma} = \gamma \left[1 - (1 + f + m - 2f \sin^2 \varphi) \frac{H^N}{a} + \frac{H^{N^2}}{a^2}\right] \qquad (2:22)$$

where γ is the normal gravity on the surface of the reference ellipsoid, H^N is the normal height, f is the geometrical flattening, m is the Clairaut constant, a is the length of the semi-major axis, and φ is the geodetic latitude at the point of interest.

The difference between the LLD and quasigeoid reference surfaces is simply the difference between the normal and the spheroidal orthometric height system, this can be determined as follows,

$$H^N - H^S = C/\bar{\gamma} - C^{S^*}/\bar{\gamma}^* \qquad (2:23)$$

Note that the mean normal gravity for the normal and spheroidal height systems are computed using different approaches, see equations (2:4) and (2:22). Therefore the separation term can be defined as follows:

$$H^N - H^S = \frac{(\bar{\gamma}^* \sum g - \bar{\gamma} \sum \gamma^*)dn}{\bar{\gamma}.\bar{\gamma}^*} = \frac{(g\bar{\gamma}^* - \bar{\gamma}\gamma^*)H^S}{A} \qquad (2:24)$$

where A is expressed as follows:

$$A = \left[1 - (1 + f + m - 2fsin^2\varphi)\frac{H^N}{a} + \frac{(H^N)^2}{a^2}\right](\gamma^2 - 0.1543H^S\gamma) \qquad (2:25)$$

The separation between the quasigeoid and the LLD reference surface has to be carefully assessed to enable transformation from spheroidal orthometric height system to normal height system. The difference between orthometric and spheroidal height systems can be expressed as,

$$H^O - H^S = \frac{(\ \bar{\gamma}^* \sum g - \bar{g} \sum \gamma^*)dn}{\bar{g}.\bar{\gamma}^*} = \frac{(\ \bar{\gamma}^* g - \bar{g}\gamma^*)H^S}{\bar{g}.\bar{\gamma}^*} \quad (2:26)$$

The assessment of the comparison between orthometric, normal and spheroidal heights will contribute towards redefining South African vertical datum. The height differences will provide an adjustment parameter for the South African height system. Moreover, it is going to provide general knowledge of how close LLD is to the geoid and quasigeoid reference surfaces. There is still a greater need of the gravity data linked with accurate horizontal and vertical position for better assessment of these comparisons.

Several studies of comparison of different height systems have been conducted around the world by various authors, to name a few (Featherstone & Kirby, 1998; Sadiq & Ahmad, 2009; Odera & Fukuda, 2015a; Foroughi & Tenzer, 2017). In the study conducted by Featherstone & Kirby (1998), the maximum difference between the normal orthometric and normal height was estimated to be 15 cm, with a standard deviation of $\pm$ 1.8 cm in Australia. In the study conducted in Pakistan by Sadiq & Ahmad (2009), the study was conducted in two study areas (one area with high elevations and the other with low elevations), the standard deviation of the separation term between orthometric and normal height was estimated as ±7.7 cm on the area with high elevations. In contrast with the low elevation area, the standard deviation of the separation term between orthometric and normal height becomes as small as ±2 cm. In the study conducted in Japan by Odera & Fukuda (2015a), the difference between Helmert and rigorous orthometric heights varies from -30.9 to 0 cm, with a mean value of 0.4 cm and a standard deviation of ±1.7 cm.

In the study conducted in the area of Himalaya and Tibet by Foroughi & Tenzer (2017), the maximum separation between the Helmert's orthometric and normal height reaches −406 cm, when computed with a spectral resolution complete to the spherical harmonic degree of 2160. A further investigation of the dependence of the separation between the Helmert's orthometric and normal height on the spectral resolution of used gravity and topographic models, indicate that a higher degree harmonic spectrum (from 360 to 2160) modifies the separation to 200 cm or even more, especially at the foothills of Himalaya. Orthometric and normal heights are the most commonly adopted height systems for realisation of the vertical datum worldwide. Therefore, determination of the correlation between the spheroidal, orthometric and normal height systems, will contribute towards a redefinition of South African vertical datum.

2.2 Vertical geodetic datum

A vertical geodetic datum is a surface of zero elevation to which heights of various points are referenced. Heights are defined in a number of different ways, some of them are not linked to the Earth's gravitational potential. Moreover, heights are referenced from different types of vertical datums. There are two main options that are generally adopted for definition of height datums.

i. The traditional mean sea level/ levelling based vertical datum; this type of datum is defined by an average tide gauge measurements over a certain period. Before the era of space-based instruments, it was the only possible option to realise physical height over most countries. However, it is expensive to maintain and it is also difficult to realise in extremely mountainous areas.
ii. Geoid-based vertical datum, this height datum is defined by a gravimetric geoid or quasigeoid model. In this approach, a local vertical datum would be defined by a gravity potential. Meaning that, every point is given a geopotential number (C_A, C_B, etc.) with respect to this reference surface. Height reference systems could be realized solely by space-based instruments.

The second option is currently the most preferred one by the geodetic community. Any approach that South Africa will take with regards to redefining its height system must consider historical perspective of the LLD such as possible future use of the existing South Africa's extensive data sets of the primary levelling networks. The fact that they have been widely used and that they represent a large financial investment, it is enough to consider their possible future value.

2.2.1 Mean Sea Level-based vertical geodetic datum

National height datums are generally developed from an assumption that the geoid and mean sea level coincide at one or more tide gauge stations, and tidal data is used as a datum. Therefore, this means that any variability (temporal or spatial) in the mean sea level will affect the stability of the vertical datum. Variability in the mean seal level can be caused by ocean currents, winds, pressure, melting of the polar ice-caps, and density changes (Singh, 2018).

The tide gauge benchmark's height is established from the long-term average of mean sea level observations, as briefly elaborated in section 1.1. A differential levelling technique incorporated with gravity data is conducted to determine absolute, physically meaningful heights to a network of benchmarks over a region or country. This has been a standard approach to obtain physically meaningful heights over the past centuries. A schematic depiction of this approach is given in Figure 2-4:

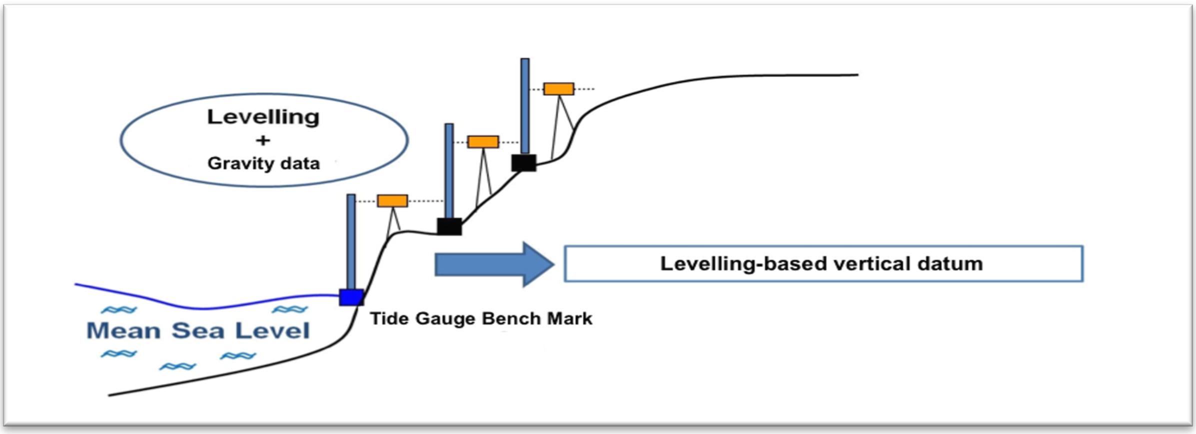

Figure 2-4: Levelling-based vertical datum (Amjadiparvar, 2015)

Then the local vertical datum is set up over a regional or continental scale using a terrestrial levelling technique. Spirit levelling is the classical method for precisely measuring height difference between two points, as indicated in Figure 2-5. Heights from the datum are then connected to the in-land benchmarks by means of levelling network. Considering Figure 2-5, height for benchmark **B** would be computed as $H_B = H_A + \delta n_{AB}$, if the height for benchmark **A** was known. The procedure to measure geometric height difference between two benchmarks is depicted in Figure 2-5:

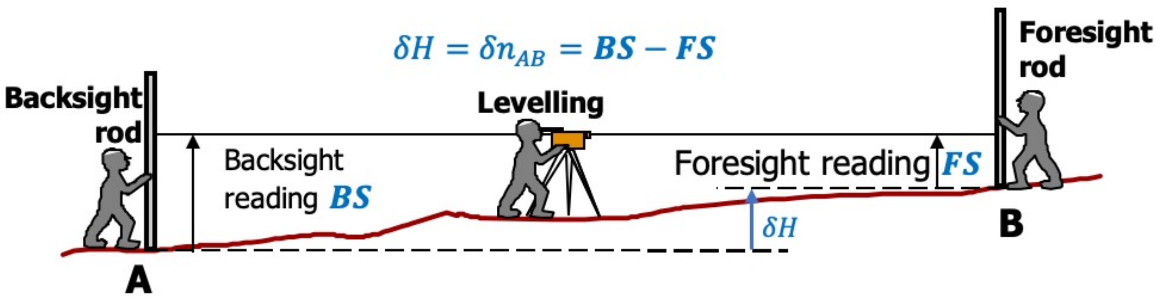

Figure 2-5: Geometric height difference between two benchmarks (Amjadiparvar, 2015)

2.2.2 Geoid-based vertical geodetic datum

The land levelling datum does not meet the accuracy requirements of modern geodesy, therefore, a modernised approach should be adopted to redefine the South African vertical datum. A modernised vertical datum has a number of benefits, such as, it provides a stable and consistent vertical datum, compatible with space-based instruments and it is not expensive to maintain. The surface of zero height on the geoid-based vertical datum is defined by an equipotential surface.

The first step into establishing a geoid consistent vertical datum depends on modelling a high-resolution gravimetric geoid model from spherical harmonic coefficients of global geopotential model (GGM) combined with satellite, airborne, terrestrial gravity data and high resolution digital topographic model. An adequate distribution of GPS/levelling data would be required to accurately determine vertical datum offset or corrector surface, to provide an adjustment between the geoid-based and the levelling-based vertical datum. Moreover, in a geoid-based vertical datum, ellipsoidal heights obtained from space-based instruments can be easily converted to physical heights.

However, a unique approach is proposed for South Africa to achieve a geoid-consistent vertical datum. A gravimetric geoid model of high-resolution must be modelled for South Africa to be able to join IHRS. This will enable estimation of the vertical datum offset between the LLD and IHRS, at the four fundamental benchmarks. The determined vertical datum offset can then be used for unification of the South African vertical datum to the global vertical datum. A few examples of geoid-based vertical datum are discussed below, as it may add some value to look at current global practice and experience with respect to adopted approaches in modernising vertical datum.

New Zealand had thirteen separate local levelling datums, each of them was based on local mean sea level observed at different tide gauges at different times. It was realized in 1949, and it was named New Zealand Geodetic Datum 1949 (NZGD49) (Amos, *et al*., 2005). These datums were unstable, and therefore they could not be connected by precise levelling. In this case each of the datums may result in equipotential zero height surfaces. Moreover, the traditional approach to establishing a national height datum has been becoming more complex, due to the emerging new technologies that offer an alternative approach to the height datum problem.

Satellite-based instruments now offer a new and inexpensive means of establishing a national 3D reference system that can be monitored in real time. The New Zealand Vertical Datum 2016 (NZVD2016) was developed to unify the 13 separate existing Local Vertical Datums (LVDs) (McCubbine et al., 2018). The NZVD2016 uses the New Zealand gravimetric quasigeoid model 2016 (NZGeoid2016) as its reference surface. It is the first country to implement the use of the geoid based vertical datum. This provided New Zealand with the unified vertical datum, and better use of the GNSS technology within the country (McCubbine et al., 2018).

The **Australian** gravimetric quasi geoid model of 2009 (AUSGeoid09) (Featherstone *et al*., 2011) was developed as an improvement from the AUSGeoid98 geoid model. The AUSGeoid09 quasi geoid model has been fitted to the Australian vertical datum to develop the AUSGeoid09 hybrid geoid model. This similar practice is employed in South Africa, in that a hybrid geoid model was developed for the purpose of transforming ellipsoidal height into LLD height system.

The Australian Height Datum 1971 (AHD71), was formally defined in 1971 when 97,230 km of two-way levelling was simultaneously adjusted, by setting the average value observed at 32 tide gauges around Australia between 1966 and 1968 to a height of zero metres. The AHD is recognized as being imperfect realization of an equipotential surface because some of the tide gauges are not well sited, mean sea level was determined over a limited period only, and sea surface topography has been ignored. It contains approximately 1 m north-south tilt and 0.5 m regional distortions with respect to the quasigeoid, meaning that GNSS-gravimetric-quasigeoid and AHD heights are inconsistent (Featherstone *et al.*, 2011). However, the AHD remains the official vertical datum in Australia.

Natural Resources **Canada** (NRCan) has released the Canadian Geodetic Vertical Datum of 2013 (CGVD2013), which uses the Canadian Gravimetric Geoid 2013 (CGG2013) as reference surface (Véronneau & Huang, 2016). This height reference system replaced the Canadian Geodetic Vertical Datum of 1928 (CGVD28). The CGVD28 was established by precise levelling between more than 80,000 stations that formed part of the primary vertical control network. The MSL used in establishing CGVD28 was determined in 1928, based upon data collected at six tide gauges. Two of these gauges are on the Pacific Ocean (Vancouver and Prince Rupert), one on the St Lawrence River (Pointe-au-Pere, near Rimouski, Quebec), two on the Atlantic Ocean (Halifax and Yarmouth) and the last at Rouses Point in southern Quebec (Véronneau & Huang, 2016). It was contaminated by systematic error due to the use of normal orthometric corrections rather than Helmert corrections, Long-term sea level change, post-glacial rebound, particularly in the Hudson Bay region, and the negligence of sea-surface topography (Kingdon *et al.*, 2005). The CGVD2013 vertical datum is defined by an equipotential surface having a gravity potential of $W_P = 62\ 636\ 856\ \mathrm{m^2s^{-2}}$. Canada is the second country in the world to officially adopt a geoid based vertical datum, New Zealand being the first (Véronneau & Huang, 2016).

The **North American** Vertical Datum of 1988 (NAVD88) provides vertical datum for the United States of America, it is based on orthometric heights system. The NAVD88 datum was established to replace the National Geodetic Vertical Datum of 1929 (NGVD29) (Roman & Weston, 2013). The NGVD29 datum was established by holding mean sea level fixed at twenty six tide gauge stations (five were situated in Canada) and an intensive levelling network of 106 724 km was adjusted, however, 31 565 km of that levelling network was located in Canada (Zilkoski *et al.*, 1992). This height datum was constrained to one tide gauge, resulting in non-zero heights for mean sea level at other tide gauges.

The orthometric corrections were not computed from the observed gravity data and thus normal orthometric heights were established. However, the geopotential numbers were computed from observed gravity data during the adjustment of the NAVD88, making it possible for all points to be computed on the Helmert orthometric height system. The same geopotential surface (geoid), represented by $W_P = 62\,636\,856\ \mathrm{m^2s^{-2}}$ (as Canada) was selected for adoption in 2022 as North American vertical datum, it provides the best fit with the tide gauges (Roman & Weston, 2013).

2.2.3 International Height Reference System

The resolution for development of an international height reference system (IHRS) was released by International Association of Geodesy (IAG) in July 2015 (IAG, 2015), also known as the global vertical datum. To establish a coordinated approach for monitoring global changes, and to also provide control information and products for prediction of several Earth's science phenomena and geohazard monitoring.

The IHRS would provide a globally unified height reference system, defined by a surface of equal potential of the Earth's gravity field, which would be realised by a conventional value, $W_O = 62{,}636{,}853.4\ \mathrm{m^2s^{-2}}$ (Burša *et al.*, 2001, 2004; Sánchez and Sideris, 2017). A number of recent researches have shown that this value may have vary by $1 - 2\ \mathrm{m^2s^{-2}}$ due to change in sea level (Rülke *et al.*, 2013; Albarici *et al.*, 2019). The value of W_O in practice depends on the realisation of the vertical datum (Amjadiparvar *et al.*, 2013).

A local vertical datum defined in terms of an equipotential surface (W_P) can be related with the potential of the global vertical datum as follows, $C_P = \Delta W = W_O - W_P$. The estimated potential difference can be used to determine vertical datum offset between the local and global vertical datum. This relationship is elaborated in detail in section 4.3.

In general, national vertical datums are defined by selecting fundamental benchmark/s at a coastal tide gauge stations and setting $H = 0$, $W_0 = W_P$, and then they are connected to the national levelling network. In this study, the vertical datum offset for South African vertical datum is computed by means of the Geodetic Boundary Value Problem approach considering the four independent tide gauge stations (located in Cape Town, Durban, East London and Port Elizabeth), using GPS/levelling and gravity data over South Africa.

The vertical datum offset is estimated to obtain the existing discrepancies between the South African local vertical datum and the global vertical datum, using the height anomalies computed from a GGM. To be able to transform from one datum to the other, it is necessary to determine geopotential differences of the four fundamental benchmarks on different equipotential surfaces (i.e. local vertical datum and global vertical datum). The global geoid is defined as the equipotential surface coinciding with the mean sea surface around the globe or worldwide. Therefore, the evaluation of the gravity potential value difference over South Africa provides us with direct link between South African local vertical datum and the potential value of global geoid.

2.2.3.1 Theoretical Background for determining vertical offset between the South African vertical datum and IHRS

The relationship between the gravity potential (W) and the corresponding normal potential (U) of the reference ellipsoid can be determined from estimating the disturbing potential (T), this can be expressed as follows (Hofmann-Wellenhof & Moritz, 2005):

$$W_P = U_P + T_P \tag{2:27}$$

The normal potential at a point P on the Earth surface is determined as follows (Hofmann-Wellenhof & Moritz, 2005):

$$U_P = U_0 + \frac{\partial U_0}{\partial h} h_P, \tag{2:28}$$

where h_p represents an ellipsoidal height at the point P, U_0 is the normal gravity potential obtained directly from the World Geodetic System 1984 (WGS84) reference ellipsoid and $\frac{\partial U_0}{\partial h}$ is the gradient of normal gravity potential. In this study, a single-point-based Geodetic Boundary Value Problem (GBVP) approach is employed to determine the vertical datum offset for height system unification. This is done by following Molodensky theory for estimating the height anomalies from the disturbing potential using Bruns's formula.

The disturbing potential at the point P is computed from the spherical harmonic coefficients of the latest Gravity field and steady-state Ocean Circulation Explorer (GOCE) based GGM (truncated at n = m = 200), TIM6 (Zingerle et al., 2019). According to Odera (2019), the GOCE-based GGM, especially timewise solution (TIM) has the best agreement with the latest gravimetric quasigeoid model over South Africa (with a mean and standard deviation of − 18.7 and ± 21.7 cm, respectively). It was integrated with the residual gravity anomalies using Stokes's integral while residual terrain model (RTM) was used to cater for the contribution of short wavelength component. This was done by evaluating Stokes's integral of the free-air gravity anomalies combined with the Molodensky G_1 term. The solution to the GBVP at point P in terms of the disturbing potential is expressed as follows (Torge & Müller, 2012):

$$T_P = T_{GGM} + \frac{R}{4\pi}\iint_{\sigma}(\Delta g - \Delta g_{GGM} + G_1).S(\psi)d\sigma + T_{RTM} \tag{2:29}$$

where, $\Delta g -$ free-air gravity anomalies, $R -$ mean radius of the Earth, $\Delta g_{GGM} -$ gravity anomalies generated by the GGM, $\psi -$ spherical distance/ geocentric angle, $d\sigma -$ an infinitesimal surface element of the unit sphere σ (corresponding to ellipsoidal coordinates), $S(\psi) -$ Stokes's function. The Stokes's Kernel function can be computed as expressed by equation (2:30) (Hofmann-Wellenhof & Moritz, 2005), the Stokes' integral in equation (2:29) was evaluated using technique described in detail by Yun (1999), (see section 3, eq. 6). A brief elaboration of the technique for computer programming was given by Bracewell (1978).

$$S(\psi) = \frac{1}{\sin(\frac{\psi}{2})} - 6\sin\frac{\psi}{2} + 1 - 5\cos\psi - 3\cos\psi.\ln\left[\sin\frac{\psi}{2} + \sin^2\frac{\psi}{2}\right]. \tag{2:30}$$

The corresponding disturbing potential from the Molodensky height anomaly (T_{RTM}) was evaluated as follows (Forsberg, 1985):

$$T_{RTM} = -\pi G\rho H_P^2 - \frac{G\rho R^2}{6}\iint_{\sigma}\frac{(H^3 - H_P^3)}{\ell^3}d\sigma \tag{2:31}$$

where $H \, and \, H_P$ are the heights of roving point and computation point, respectively, $G -$ is the Newtons gravitational constant, $\rho -$ represents the constant topographic and water mass density distribution of $2670\ kg.m^{-3}$ and $997\ kg.m^{-3}$, respectively, and $\ell -$ is the planar distance between the computational point and the roving points. The residual gravity anomalies are in principle converted into residual disturbing potential, using 2D Fourier transform with a spherical approximation of the RTM terrain correction integration (Yun, 1999). Moreover, a digital elevation model (DEM) from the Shuttle Radar Topography Mission (SRTM) at 90 metre spatial resolution was used to evaluate the Molodensky G_1 term and the indirect effect contribution, this can be expressed as follows (McCubbine et al., 2018a):

$$G_1 = \frac{\Delta\varphi\Delta\lambda}{2\pi}\left[(H.\Delta g) * \frac{1}{\ell^3} - H_P\left(\Delta g * \frac{1}{\ell^3}\right)\right] \tag{2:32}$$

where $\Delta\varphi$ and $\Delta\lambda$ are the differences in latitude and longitude, respectively. The G_1 term contribution were computed for the central $1^0 \times 1^0$ grid only of the $4^0 \times 4^0$ gravity data grid, in order to handle/reduce any edge effect. It is used as a terrain correction for regional quasigeoid computation. Furthermore, it is more significant in mountainous regions and relies heavily upon a detailed accurate DEM (McCubbine et al., 2018a). A computer program designed from python was used for this computation. Therefore substituting equations (2:28) and (2:29) into equation (2:27) to determine the gravity potential at a point P on the local vertical datum yields,

$$W_P = U_0 + \frac{\partial U_0}{\partial h} h_P + T_{GGM} + \frac{R}{4\pi} \iint_\sigma (\Delta g - \Delta g_{GGM} + G_1).S(\psi)d\sigma + T_{RTM} \quad (2:33)$$

Hence the gravity potential difference between global and local vertical reference (LLD) surfaces at a point *P* can be expressed as,

$$\delta W_P = W_0 - W_P = W_0 - (U_0 + \frac{\partial U_0}{\partial h} h_P + T_{GGM} + \frac{R}{4\pi} \iint_\sigma (\Delta g - \Delta g_{GGM} + G_1).S(\psi)d\sigma + T_{RTM}), \quad (2:34)$$

The h_P, in this case refers to the ellipsoidal height at the tide gauge benchmark (TGBM). The height anomaly at the TGBM is estimated from the Bruns's formula and the gradient of the normal potential gives an approximation of the normal gravity value $\left(\frac{\partial U_0}{\partial h} \approx \gamma\right)$. Therefore, equation (2:34) can be expressed as,

$$\delta W_P = W_0 - W_P = (W_0 - U_0) + \gamma(h_P - H_P^{LLD} - \zeta_{GGM} - \zeta_{res} - \zeta_{RTM}), \quad (2:35)$$

where ζ_{GGM} gives the contribution of the GGM, expressed as,

$$\zeta_{GGM} = \zeta_0 + \frac{GM_g}{r\gamma}. \sum_{n=2}^{n_{max}} \left(\frac{a_g}{r}\right)^n . \sum_{m=0}^{n} (\Delta\bar{C}_{n,m} \cos m\lambda + \Delta\bar{S}_{n,m} \sin m\lambda) . \bar{P}_{n,m}(\sin\bar{\varphi}) \quad (2:36)$$

where: γ is the normal gravity in ms^{-2}, a_g is the semi-major axis of the geopotential model in m, GM_g is the gravity mass constant of the geopotential model in m^3/s^2, r is the radial distance to the computational point in m, $\Delta\bar{C}_{n,m}$ is the difference between the fully normalised harmonic coefficient $\bar{C}_{n,m}$ and harmonic coefficient generated by the normal gravity field $C^*_{n,m}$, $\Delta\bar{S}_{n,m}$ is the difference between the fully normalised spherical harmonic coefficient $\bar{S}_{n,m}$ and the harmonic coefficient generated by the normal gravity field $S^*{}_{n,m}$, n and m are the degree and order for a geopotential model, $\bar{P}_{n,m}$ is fully normalised harmonics Legendre function (detailed evaluation is given in sub-section 2.2.3.2) while $\bar{\varphi}$ and λ are geocentric latitude and longitude of the computation point.

The zero-degree harmonic term to the GGM geoid undulations with respect to a specific reference ellipsoid (ζ_0) is given as $\zeta_0 = \frac{GM_g - GM_0}{R\gamma} - \frac{W_O - U_0}{\gamma}$ (Hofmann-Wellenhof & Moritz, 2005). The contribution of residual gravity anomalies ($\Delta g - \Delta g_{GGM} + G_1$) with the effect of the GGM and the terrain removed (ζ_{res}) is expressed as,

$$\zeta_{res} = \frac{R}{4\pi\gamma} \iint_{\sigma} (\Delta g - \Delta g_{GGM} + G_1).S(\psi)d\sigma \tag{2:37}$$

The contribution of the indirect effect on the height anomaly at the point P (ζ_{RTM}) is given by Amos (2007) as,

$$\zeta_{RTM} = \frac{-\pi G\rho H_P^2}{\gamma} - \frac{G\rho R^2}{6\gamma} \iint_{\sigma} \frac{(H^3 - H_P^3)}{\ell^3} d\sigma \tag{2:38}$$

After estimation of the local gravity potential value W_P, using equation (2:33), the vertical datum offsets on the South African vertical datum in relation to the IHRS was computed as,

$$\delta\zeta_P = {}^{\delta W_P}/_{\gamma_P}. \tag{2:39}$$

A unified vertical datum will provide a reference surface for engineering projects across countries, flooding control initiatives, plate tectonic movements determination and analysis, coastal hazard studies, unification of national gravity anomaly database, and improvement of the continental geoid, amongst other applications.

2.2.3.2 The Legendre polynomials

The associated Legendre functions given by $P_{n,m}(\sin\bar{\varphi})$ and the fully normalised Legendre functions are given by $\bar{P}_{n,m}(\sin\bar{\varphi})$, also known as fully normalised harmonics. The abbreviation $t = \sin\bar{\varphi}$ and $u = \cos\bar{\varphi}$ will be used for convenience. The first order of the Legendre's functions of the first kind is defined as (Vaníček, 1976),

$$P_{n,m}(t) = \frac{1}{2^n n!}(1-t^2)^{\frac{m}{2}} \frac{d^{n+m}}{dt^{n+m}}(t^2-1)^n, \tag{2:40}$$

In this analysis, recursive formulas were used to compute the associated Legendre functions (Losch & Seufer, 2003),

$$P_{n+1,0}(t) = (2n+1)tP_{n,0}(t) - nP_{n-1,0}(t)\,,$$

$$P_{n,n}(t) = (2n-1)\sin\theta\, P_{n-1,n-1}(t),$$

$$P_{n,m}(t) = P_{n-2,m}(t) + (2n-1)\sin\theta\, P_{n-1,m-1}(t). \tag{2:41}$$

with the beginning values as follow:

$$P_{0,0}(t) = 1,$$
$$P_{1,0}(t) = t,\;\; P_{1,1}(t) = u$$
$$P_{2,0}(t) = \frac{3}{2}t^2 - \frac{1}{2},$$

$$P_{2,1}(t) = 3ut, \qquad P_{2,2}(t) = 3u^2.$$

A recursive method was used to compute fully normalised associated Legendre functions. However, there are two different types of recursive method, namely, sectorial and non-sectorial method. A non-sectorial $(i.e.\ n > m)$ method for computing fully normalised associated Legendre functions $(\bar{P}_{n,m}(t))$ is expressed as follows (Holmes & Featherstone, 2001):

$$\bar{P}_{n,m}(t) = a_{n,m}\, t\, \bar{P}_{n-1,m}(t) - b_{n,m}\bar{P}_{n-2,m}(t), \qquad for\ all\ n > m \tag{2:42}$$

where:-

$$a_{n,m} = \sqrt{\frac{(2n-1)(2n+1)}{(n-m)(n+m)}}, \tag{2:43}$$

$$b_{n,m} = \sqrt{\frac{(2n+1)(n+m-1)(n-m-1)}{(n-m)(n+m)(2n-3)}}, \tag{2:44}$$

Moreover, the recursion formula for computing the $\bar{P}_{m,m}(t)$ using sectorial method $(i.e. n = m)$, is used in the recursion equation (2:42) as seed values. The initial values are computed as follows, $\bar{P}_{0,0}(t) = 1 \; and \; \bar{P}_{1,1}(t) = \sqrt{3}\,u$. The $\bar{P}_{m,m}(t)$ for maximum $m \; and \; n$ values can be computed as follows (Holmes & Featherstone, 2001),

$$\bar{P}_{m,m}(t) = u\sqrt{\frac{2m+1}{2m}}\bar{P}_{m-1,m-1}(t), \qquad for\ all\ m > 1, \tag{2:45}$$

The recursion process as indicated by equations (2:42) and (2:45) using the lower triangular matrix is demonstrated in Figure 2-6. The indicated circles in Figure *2-6* represents a value of $\bar{P}_{n,m}(t)$, that corresponds to a pair of recursive terms $(a_{n,m}, b_{n,m})$ and to a specific combination of m and n (Holmes & Featherstone, 2001).

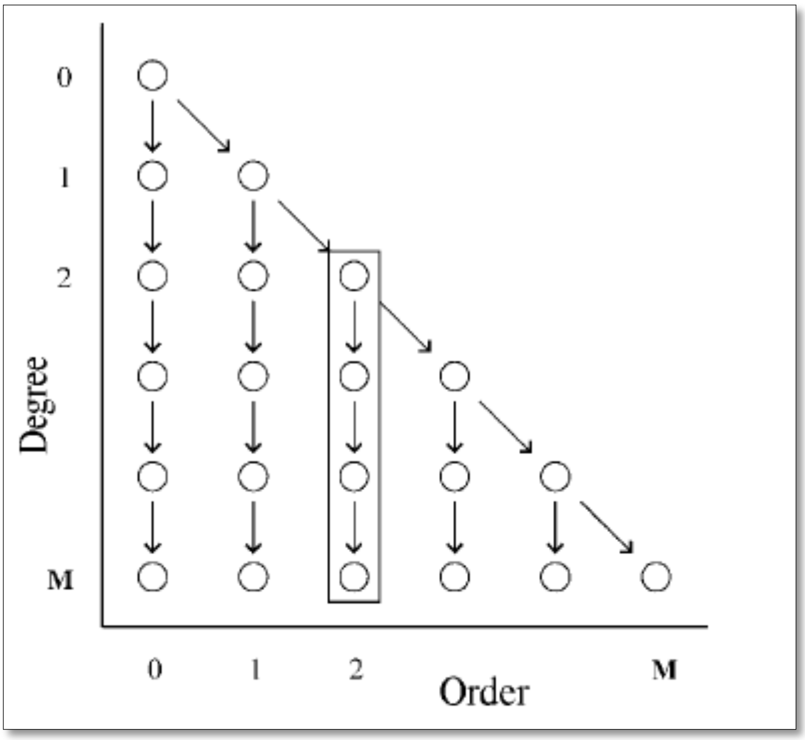

Figure 2-6*: Schematic of the recursion sequence employed in the standard, first modified and second modified forward column algorithms to compute* $\boldsymbol{\bar{P}_{n,m}(t)}$ (Holmes & Featherstone, 2001)

The recursion method provides numerically stable results for maximum degree and order, hence it is advisable to use it for computation of fully normalised Legendre functions, especially for computer analysis. The associated Legendre function can be expressed more explicitly as follows (Moritz, 1980):

$$P_{n,m}(t) = 2^{-n}(1-t^2)^{\frac{m}{2}}\sum_{k=0}^{r}(-1)^k \frac{(2n-2k)!}{k!\,(n-k)!\,(n-m-2k)!}t^{n-m-2k}, \tag{2:46}$$

The term r is the greatest integer from $\leq (n-m)/2$; whichever is an integer between $(n-m)/2$ or $(n-m-1)/2$. The polynomial functions of $\sin\overline{\varphi}$ can be used to define the Legendre functions (Moritz, 1980). Moreover, a fully normalised associated Legendre function can be expressed as follows (Losch & Seufer, 2003),

$$\bar{P}_{n,m}(t) = \sqrt{k(2n+1))\frac{(n-m)!}{(n+m)!}}P_{n,m}(t), \tag{2:47}$$

$$with\ k = \begin{cases} 1\ for\ m = 0 \\ 2\ for\ m \neq 0 \end{cases}$$

It should be noted that this method becomes numerically unstable as the degree and order increases, this is due to the factorial of a larger number. The geoid undulation is computed from geocentric geodetic coordinates, this is because the spherical harmonics models are formulated from the centre of the earth. The longitude for both the geocentric and geodetic coordinates remains the same. However, the latitude is transformed from geodetic φ to geocentric $\overline{\varphi}$ latitude as follows (Losch & Seufer, 2003):

$$\overline{\varphi} = arctan\frac{z}{\sqrt{x^2-y^2}} = \arctan\left[\left(\frac{b}{a}\right)^2 \tan\varphi\right]. \tag{2:48}$$

The normal gravity computed as the function of latitude can be expressed as follows (Heiskanen & Moritz, 1967):

$$\gamma_0(\overline{\varphi}) = \gamma_a\frac{1+\kappa sin^2\overline{\varphi}}{\sqrt{1-e^2sin^2\overline{\varphi}}}\text{ , with } \kappa = \frac{b\gamma_b - a\gamma_a}{a\gamma_a} \tag{2:49}$$

where:-

a and b are the semi-major and semi-minor axis of the reference ellipsoid, respectively,

γ_a and γ_b are the normal gravity at the equator and at the pole, respectively,

e is the first numerical eccentricity of the reference ellipsoid.

The local elliptic radius $r(\overline{\varphi})$ is computed as follows (Losch & Seufer, 2003):

$$r(\overline{\varphi}) = \sqrt{x^2 + y^2 + z^2} = a\sqrt{1 + \frac{e^2(1 - e^2)sin^2\overline{\varphi}}{1 - e^2 sin^2\overline{\varphi}}}, \quad (2:50)$$

It can also be approximated as:

$$r(\overline{\varphi}) = a(1 - f sin^2\overline{\varphi}). \quad (2:51)$$

where:-

f is the flattening parameter from the reference ellipsoid.

3 THE SOUTH AFRICAN VERTICAL GEODETIC DATUM

3.1 Historical development of South African Height datum

The South African height datum is based upon tide gauge observations taken more than a century ago, connected to the network of primary levelling benchmarks, which were then adjusted in a piecemeal fashion. However, the determination of the mean sea level at the tide gauge stations is subjected to a number of systematic distortions (Merry, 1985). Hence, there are existing distortions on both the primary levelling networks, and the datum. The land levelling datum is estimated to be 15-20 cm below the mean sea level (Merry, 1990), and the global sea levels are estimated to be rising at a rate of approximately *1.7 mm/year* during the last 2000 years (Blake, 2010). The African sea level data set is not enough in comparison to most part of the world, in respect to size and quality. The sea level rise along the Southern African coastal has not been constant, such that the rate of sea level rise in South Africa varies from Western, Southern and Eastern coastal region by approximately 0.42, 1.57 and 3.4 mm/year, respectively (Blake, 2010).

This means that the discrepancy between LLD and MSL, estimated by Merry (1990), has increased over time and it needs to be taken into account during the re-computation of the South African vertical datum, if the mean sea level approach is selected. A brief historical background about the establishment of the South African height system is given by Wonnacott & Merry (2012). Furthermore, the height of the datum reference (TGBM in Cape Town - BM 1), as depicted in Figure 3-1, was obtained from the Harbour Engineer on the 1st of April in 1925 and the mean sea level measurements referred to were obtained in 1900 and 1907 (Anonymous, 1965).

Precise levelling surveys have been carried out along all the major rail and road routes throughout South Africa, and four precise primary levelling networks were adjusted independently. It is mentioned on the technical publication report 17 by Anonymous (1965) that the entire levelling network must be adjusted concurrently, after a levelling network covering the entire country is completed. However, this adjustment has not been undertaken.

Figure 3-1: Cape Town TGBM - BM1

The national town survey marks and the trigonometrical beacons are referred to the LLD datum. The South African height system is referred to as the spheroidal orthometric height system. It was given the name because only the spheroidal orthometric correction was applied to all the primary levelling networks. The spheroidal orthometric correction is defined as a function of the latitude of the point, the gravitational constant, normal gravity, and the mean height of the two points. However, the geoidal orthometric correction was only computed for four levelling loops (Merry, 1977). This provides proof that the LLD is not rigorously defined.

During the establishment of the South African verical datum, around 1940's, mean sea level data from four tide gauge stations were transferred to the benchmarks of the primary levelling network by means of both precise levelling and trigonometric levelling. However, the sea level data was measured over a century ago, as depicted in Figure 3-2, this image was obtained from the South African National Geo-Spatial Information (NGI).

The sea level data is collected over a period of years (usually 18.6 years - complete lunar cycle) and averaged to determine the MSL reference surface (Searson, 1994; Schumann, 2012). The distribution of the South African coastline tide gauge stations (the blue triangles indicate the tide gauge stations connected to the four fundamental benchmarks) is depicted in Figure 3-3, this data was obtained from the South African Navy Hydrographic Office (SANHO) (http://www.sanho.co.za/).

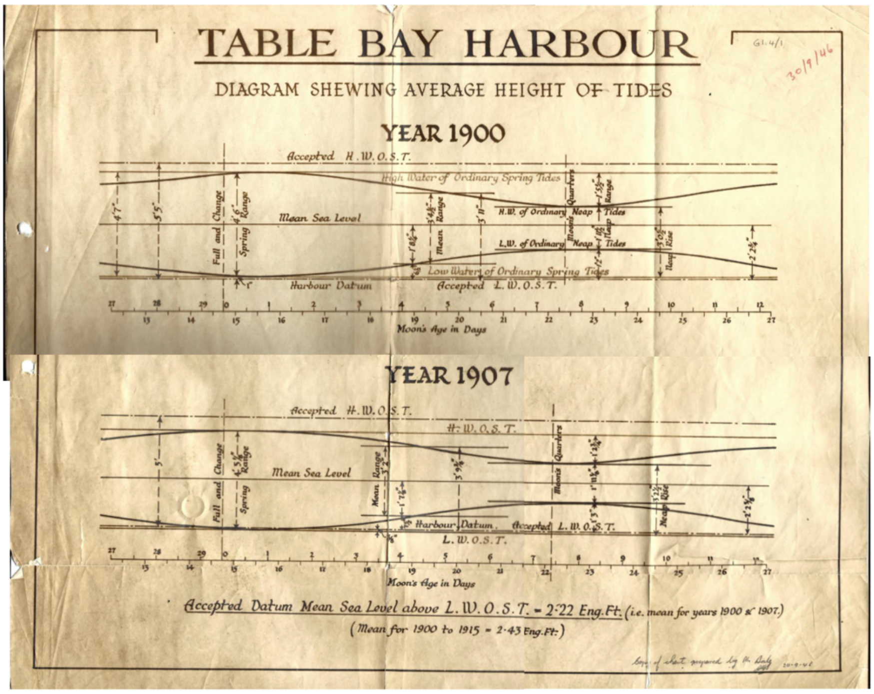

Figure 3-2: Table Bay Tidal measurement (1900 & 1907)

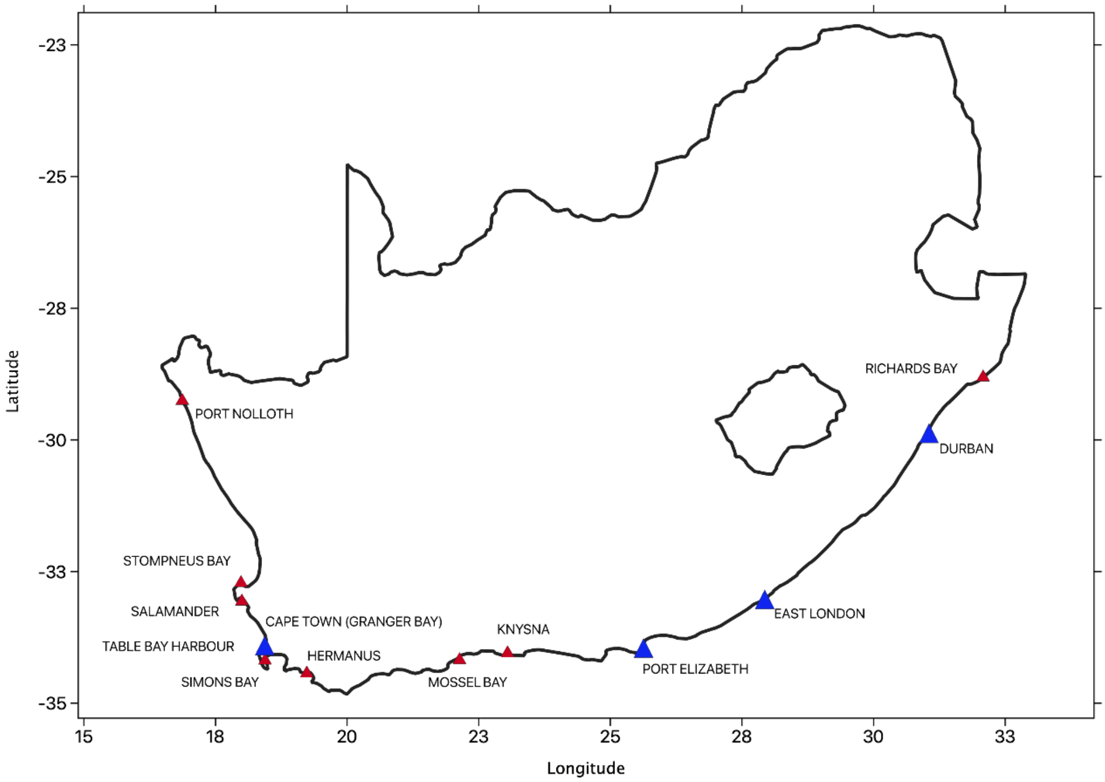

Figure 3-3: Tide gauge stations along the South African coastline (blue triangles indicate the tide gauge stations connected to the four fundamental benchmarks while red triangles represent the tide gauge stations forming part of the tide gauge network)

Chart Datum (CD) is used as a reference for ocean tide heights, which is specified in relation to the LLD. The offsets of the Chart Datum relative to the LLD at each tide gauge stations (only the four fundamental benchmarks are depicted) and the tide gauge benchmark (TGBM) was also measured in relation to the Revised Local Reference (RLR), these measurements are depicted in Figure 3-4 to Figure 3-7. These data were obtained from the Permanent Service for Mean Sea Level (PSMSL) website (https://www.psmsl.org/data/).

The RLR datum at each tide gauge station is defined to be approximately 7000 mm below MSL, in order to avoid negative numbers in the resulting RLR mean values. It is defined as a common datum in order to construct time series of sea level data at each tide gauge stations. The time series has been conduct to monitor sea level deviations over time on the four fundamental benchmarks, as depicted in Figure 3-4 to Figure 3-7. BM1 is the first height of a benchmark that was determined above MSL in South Africa, it is estimated to be 3.425 m above LLD within the 1960 – 1978 period.

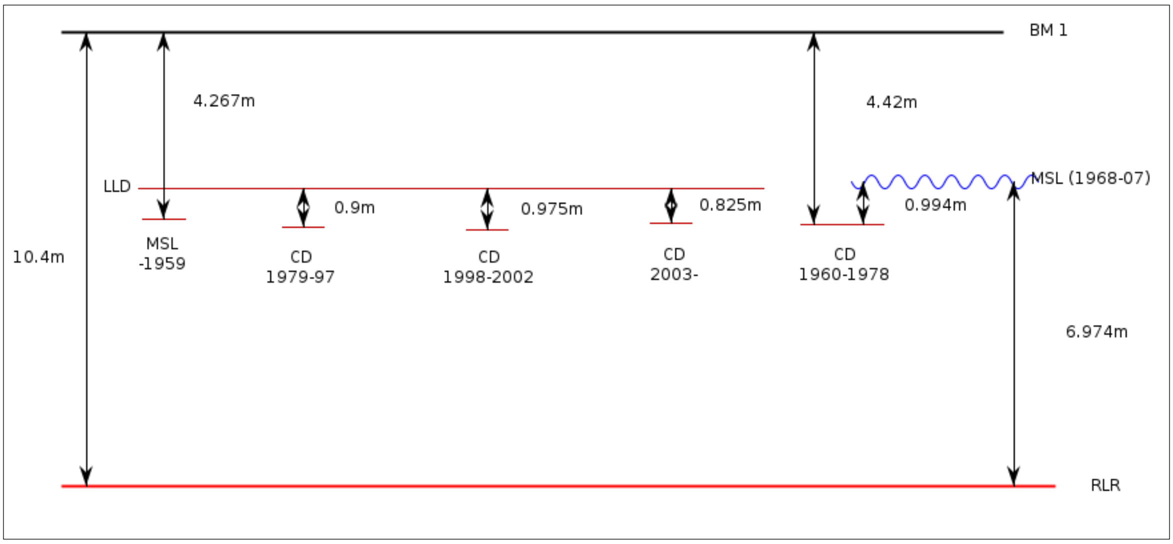

Figure 3-4: Cape Town TGBM

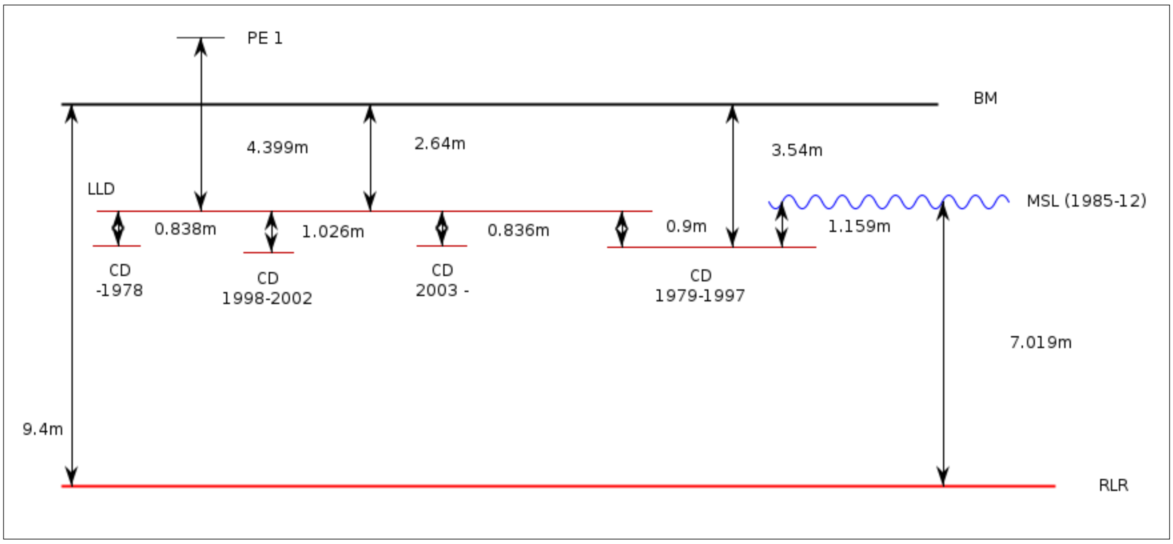

Figure 3-5: Port Elizabeth TGBM

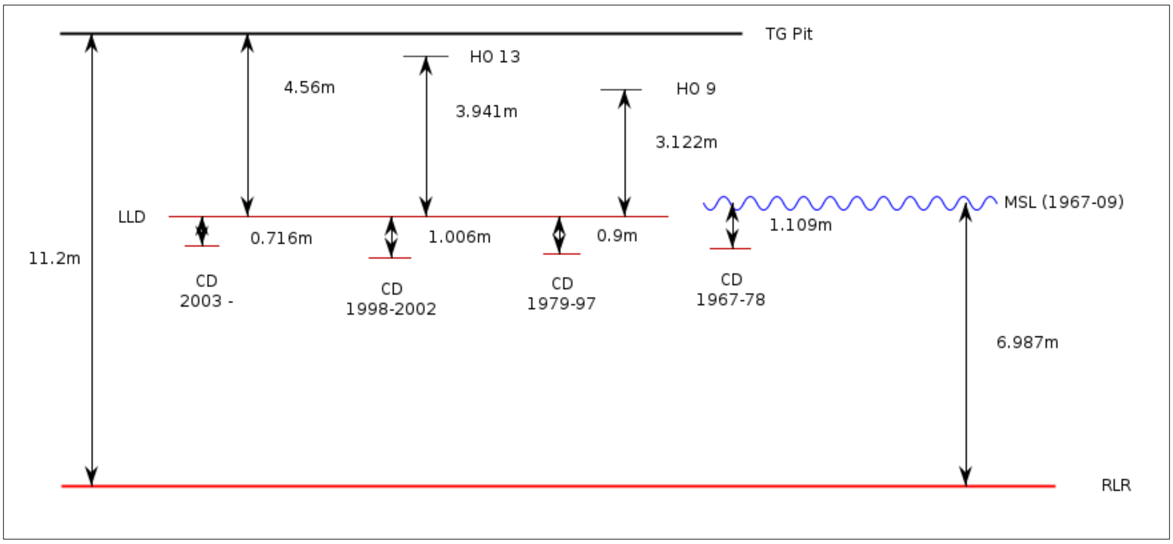

Figure 3-6: East London TGBM

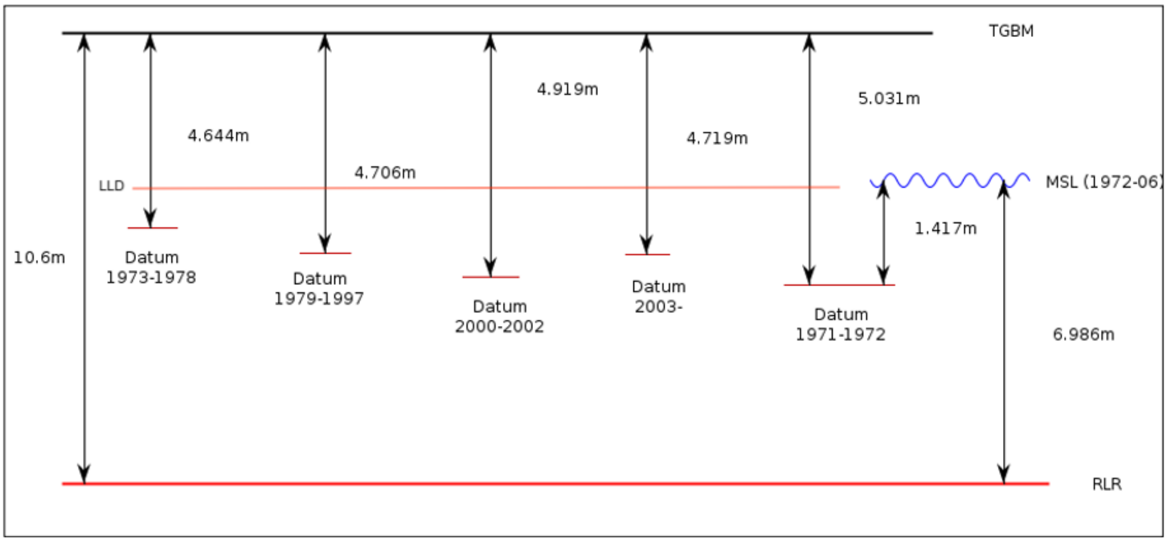

Figure 3-7: Durban TGBM

The vertical datum defined by MSL is subject to the following issues:

i. It is defined by a fixed value of MSL that has been determined over a specific time interval. However, the MSL is an unstable surface.

ii. The MSL is not an equipotential surface. Meaning that the sea surface topography, may result into differences between vertical datums based on coastal MSL observation (Amos, 2007).

3.2 Distribution of levelling network

A very extensive run of the first order levelling network has been carried out gradually over the years, i.e. from 1940's to 1960's (Merry, 1977, 1985; Wonnacott & Merry, 2011). The extensive nature of the first order levelling network that has been completed in South Africa, is depicted in Figure 3-8.

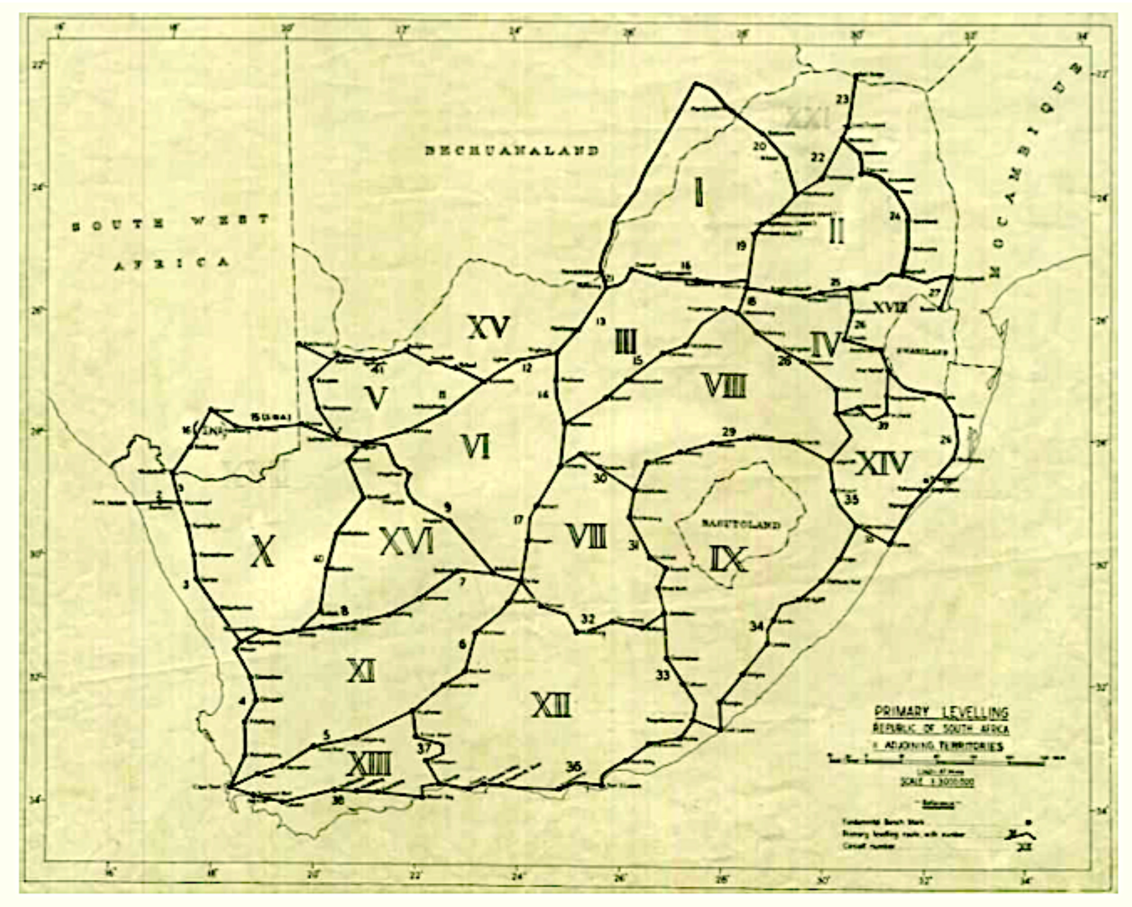

Figure 3-8: South African first order levelling network (Wonnacott & Merry, 2011)

The South African vertical datum was then defined as the spheroidal orthometric height system. The spheroidal orthometric correction was applied to all the first order levelling networks. The observed gravity data over the four levelling loops, was used to compute the geoidal orthometric correction which was then applied only on the four levelling loops, a detailed elaboration is given by Merry (1977). Gravity data is usually collected in relative form, as a difference between two benchmarks, it is used in combination with precise levelling data, to determine the potential differences from which physical heights are computed.

3.3 Problems with the South African vertical datum

There are a number of problems associated with the South African vertical datum. As mentioned in the previous sections, the current South African vertical datum (LLD) is based on tide gauge observations of MSL which were observed more than a century ago. According to the Technical Publication report 17 by Anonymous (1965), the tide gauge measurements were taken over less than a full tidal cycle and there is an ambiguity towards how the primary levelling networks were connected with the tide gauge datum. This provide an indication that the LLD has lost its significance, even the height of existing benchmarks above mean sea level does not match with the recently determined mean sea level of South African coast.

Furthermore, the South African vertical datum consists of a network of primary levelling benchmarks, distributed across the country. The primary levelling networks were developed and adjusted independently. However, as mentioned in the previous section, only the spheroid orthometric correction was applied to all the levelling loops in the network, and the correction consisting of actual gravity data (geoidal orthometric correction) was only applied to very few levelling loops. A trigonometrical heighting technique and necessary adjustment was employed to determine heights of the national trigonometrical beacons (Anonymous, 1965).

It is mentioned on the Technical Publication report 17 that the adjustments of all the other levelling networks, from the previous reports, were considered provisional. However, a unified levelling network adjustment is still to be undertaken. As a result, both the datum and the primary levelling networks consist of distortions. Therefore, a modernised vertical datum for South Africa is required.

3.4 Attempts towards a geoid based vertical datum

The preliminary stages of geoid determination in South Africa are covered by van Gysen & Kryński (1993) and Wonnacott (1999). The SAGEOID10 is the South African hybrid geoid model obtained by fitting gravimetric geoid model onto the local vertical datum through GPS/levelling data. The gravimetric geoid model was computed as follows (Chandler & Merry, 2010):

i. The long wavelength component was computed from EGM2008 (Pavlis et al., 2012) geopotential model truncated at degree 360.
ii. The contribution of the medium and short wavelength components were computed using both land and marine gravity anomalies, together with a terrain correction.

The geometric geoid model was computed using 79 benchmarks distributed throughout the country. The geoid height (*N*) of the geometric geoid model was computed from the difference between the ellipsoidal height (h), and the orthometric height (H) as expressed by equation (1:1) (Chandler & Merry, 2010).

The WGS84 ellipsoid was used as the reference ellipsoid during the computation of the SAGEOID10 hybrid geoid model. A correction surface was computed from the comparison of the geometric geoid model (obtained from GPS and precise levelling data) and the gravimetric geoid model, correction surface was derived by interpolating corrections using Kriging technique. The correction surface was applied to the gravimetric geoid model to remove biases and tilts on the gravimetric geoid model.

The gravimetric geoid model was converted to a hybrid geoid model consistent with the GPS/levelling data using the correction surface (Chandler & Merry, 2010). The 79 GPS/levelling precisely measured data points from the 118 GPS/levelling data points were used for the development of the SAGEOID10 hybrid geoid model. The remaining 39 GPS/levelling points were used for the validation of the SAGEOID10 hybrid geoid model. The standard deviation of the SAGEOID10 hybrid geoid model was determined to be 7cm at the 39 GPS/levelling validation points (Chandler & Merry, 2010).

The current vertical datum has lost its significance where even the height of existing benchmarks above mean sea level does not match with the recently determined mean sea level of South African coast. The SAGEOID10 hybrid geoid model, has not solved the issues associated with the LLD, as mentioned in the previous section. This approach constrains the local quasigeoid model to fit onto the land levelling datum. The SAGEOID10 hybrid geoid model was developed for the purpose of converting ellipsoidal height to spheroidal orthometric height system.

The biggest challenging factor towards redefining the South African vertical datum is inconsistency of the levelling networks, meaning that the land levelling datum is not uniform. Moreover, the levelling networks were adjusted independently, making it not feasible to identify benchmarks associated with a particular fundamental benchmark. The most rigorous approach to readjust the South African levelling network, would be tying the whole first order levelling network into one and employing a rigorous levelling adjustment method. However, this approach would be tedious, given that it may not be possible to trace and digitise all of the original data as it is not stored in digital format.

In this study a vertical datum offset between the LLD and global vertical datum is estimated at the four TGBMs, due to the challenges mentioned above, it is suggested that only one TGBM should be held fixed, in order for the datum offset to be applied directly to unify the LLD to IHRS. This approach has been determined to be more realistic for the establishment of a geoid consistent vertical datum by Odera & Fukuda (2015b), as it does not constrain the local geoid to the local vertical datum.

4 EMPIRICAL INVESTIGATIONS FOR THE ESTABLISHMENT OF A GEOID-BASED VERTICAL DATUM IN SOUTH AFRICA

4.1 Comparison between orthometric, normal and spheroidal orthometric heights over South Africa

4.1.1 Introduction

This study has been carried out to compare the differences between the South African spheroidal orthometric, orthometric and normal height systems. The comparison is achieved by determining the differences between the orthometric and normal, spheroidal and normal, and spheroidal and orthometric heights at 141 GPS/levelling stations over South Africa. It is mainly conducted for the purpose of establishing a height system which is more consistent with the South African spheroidal orthometric height system. Such a height system must be based on the known vertical datum reference surfaces (geoid or quasigeoid). The theoretical and practical differences in the computation and determination of these height systems are elaborated briefly in chapter 2.

4.1.2 Data and methods

The comparison between the orthometric and normal height is determined from the Bouguer gravity anomalies and the topographic heights on land (see equation (2:20)). The primary gravity data used was provided by the South African Council for Geoscience (formerly, South African Geological Survey, www.geoscience.org.za. There are approximately 105,408 existing gravity data stations over South Africa, gravity data distribution over the country is shown in Figure 4-1. The gravity network is tied to the International Gravity Standardisation Net of 1971 (IGNS71) system and the precision of the gravimetric observations range from ±0.02 to ±0.5 mGal. However, the horizontal and vertical positions of these data points were approximated from the classical topographical maps based on the Cape datum and they are not linked to the TrigNet system (TrigNet is a network of continuously operating GNSS base stations covering South Africa all managed and controlled by a single control centre situated in the offices of National Geospatial Information (NGI)), hence they are not so accurate but appropriate for determination of Bouguer gravity anomalies. The distribution of the South African terrestrial gravity data is depicted in Figure 4-1 and the statistics of free-air and Bouguer gravity anomalies are presented in Table 4.1.

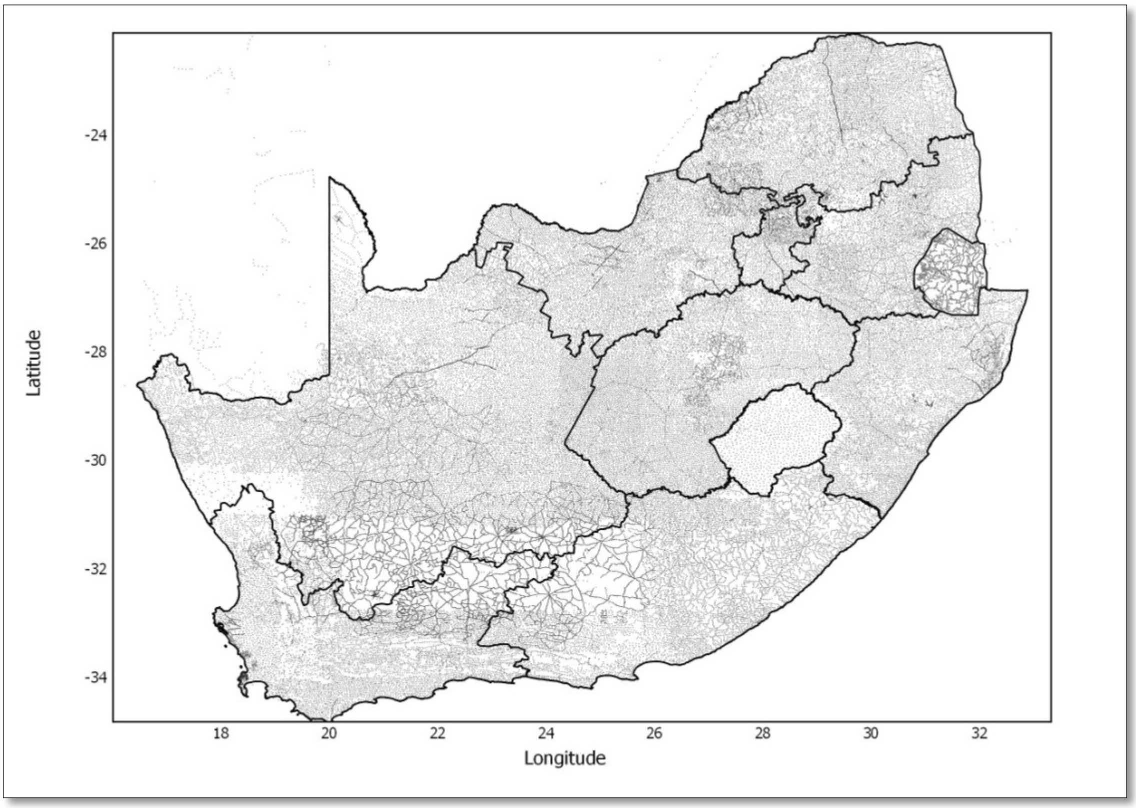

Figure 4-1: Distribution of the terrestrial gravity data over South Africa

Table 4.1: Statistics of the gravity data over South Africa

Gravity Data (mGals)	Min.	Max.	Mean	Standard dev.
Bouguer Anomaly	-194.31	81.16	97.55	45.10
Free-Air gravity anomaly	-101.29	212.90	17.30	30.88

The assessment of the separation between orthometric and normal heights has been conducted from the 141 GPS/levelling stations over South Africa. It has been done for more accurate results since the latitude and height plays a greater role on this analysis. The distribution of the 141 GPS/levelling stations over South Africa is depicted in Figure 4-2.

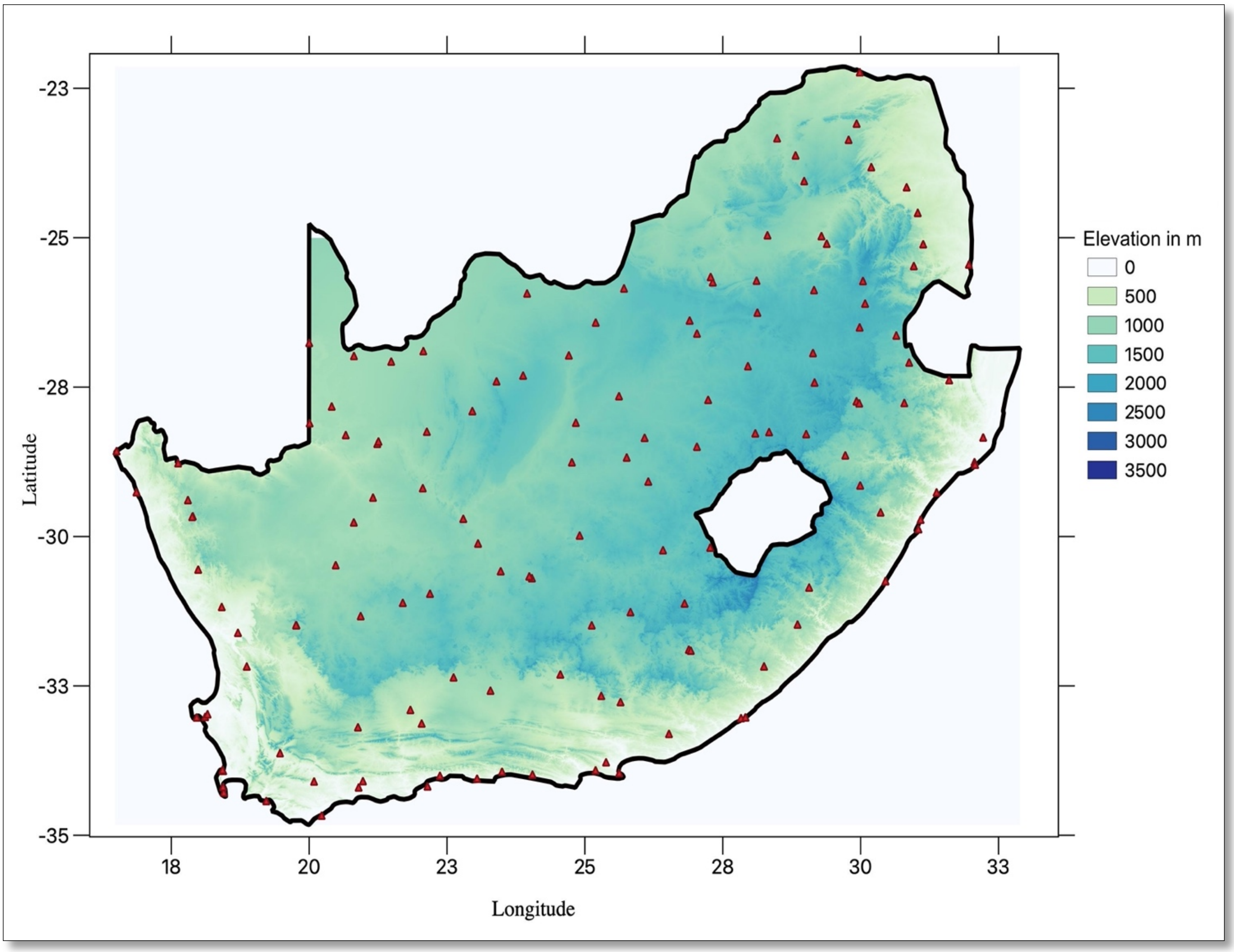

Figure 4-2: Distribution of the 141 GPS/levelling stations

The difference between the orthometric and normal height has been computed approximately as a function of the Bouguer gravity anomalies and the topographic height at the 141 GPS/levelling stations using equation (2:20). The differences between the normal and spheroidal height, and the orthometric and spheroidal height were obtained using equations (2:24) and (2:26), respectively. The actual gravity is obtained by interpolating Bouguer gravity anomalies at the GPS/levelling points, then computing free-air gravity anomaly and applying terrain correction. The mean actual gravity is estimated by the Helmert's formula (Hofmann-Wellenhof & Moritz, 2005).

4.1.3 Results and discussion

The magnitude of the orthometric-normal height separation has been computed at each and every GPS/levelling data point using equation (2:20), as depicted in a contour plot in Figure 4-3. It shows an increasing trend in absolute value over the eastern side of the country (i.e. along the Mpumalanga province), detecting a maximum value of 75.2 cm. The trend seems to be lowering when moving towards the south western part of the country, detecting a minimum absolute value of about 5.0 cm.

The mean and standard deviation of the differences between orthometric and normal heights over South Africa are 19.4 and ± 17.6 cm, respectively. The statistics of the differences between orthometric and normal heights are given in Table 4.2.

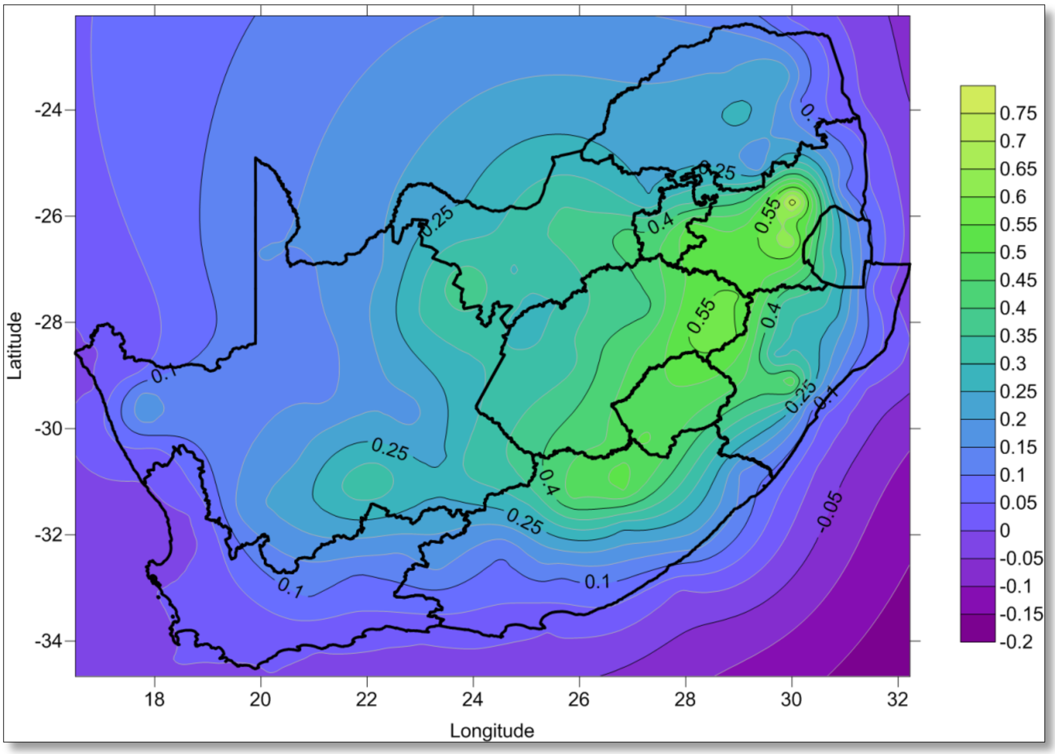

Figure 4-3: The differences between orthometric and normal heights over South Africa derived from 141 GPS/levelling stations (units are in m)

The differences between spheroidal orthometric and normal heights are evaluated at the 141 GPS/levelling stations using equation (2:24). The differences between spheroidal orthometric and normal heights over South Africa are presented in Figure 4-4.

There are a number of contributing factors towards the existing differences between the normal and spheroidal orthometric heights, such as unavoidable random errors during observations of data or systematic biases to mention a few. The mean and standard deviation of the differences between the normal and spheroidal orthometric heights over South Africa are -21.3 cm and ± 23.8 *cm*, respectively. Moreover, the separation between the normal height system and the spheroidal orthometric height system over South Africa is estimated as -40.7 cm on average, with an absolute maximum separation of about 76.2 cm and a standard deviation of ± 25.3 cm. The statistics of the differences between spheroidal orthometric and normal heights are given in Table 4.2. This provides proof that the spheroidal orthometric height system is more closer to the normal height system than orthometric height systems over South Africa as determined by Merry (1985). Although, Merry (1985) used a different approach and data to conduct this investigation and it was not conducted for the whole country.

The contour plot of the normal-spheroidal orthometric height separation from the 141 GPS/levelling stations (Figure 4-4), depicts an increase in magnitude on the western side of the country. A maximum magnitude is detected slightly above the Northern Cape Province, situated on the north western part of the country, with the maximum separation of about 65.0 cm and minimum absolute value of about 5.0 cm.

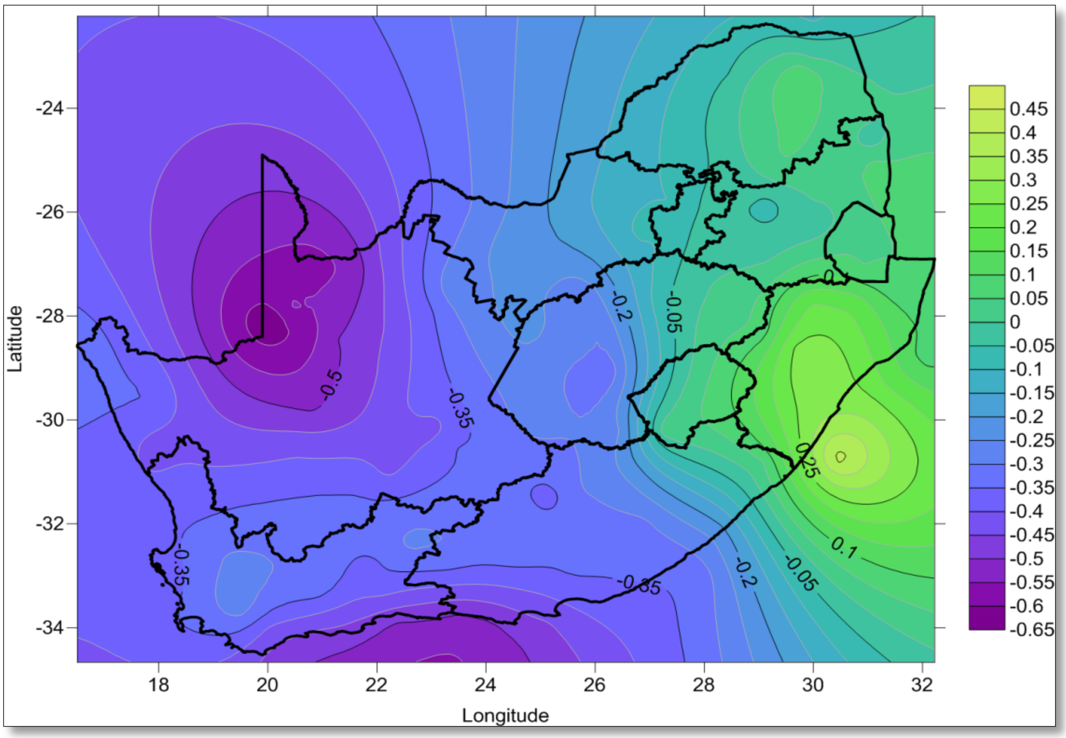

Figure 4-4: The differences between normal and spheroidal heights from 141 GPS/levelling stations (units are in m)

The contour plot of the normal-spheroidal orthometric height separation from the 141 GPS/levelling stations (Figure 4-5), depicts an increase in magnitude on the western, southern, and eastern side of the country. The maximum magnitude peak of about 80.0 cm are detected on the Northern Cape, Eastern Cape and Mpumalanga provinces and the minimum absolute values of about 45.0 cm are detected on the East coastal region of the country.

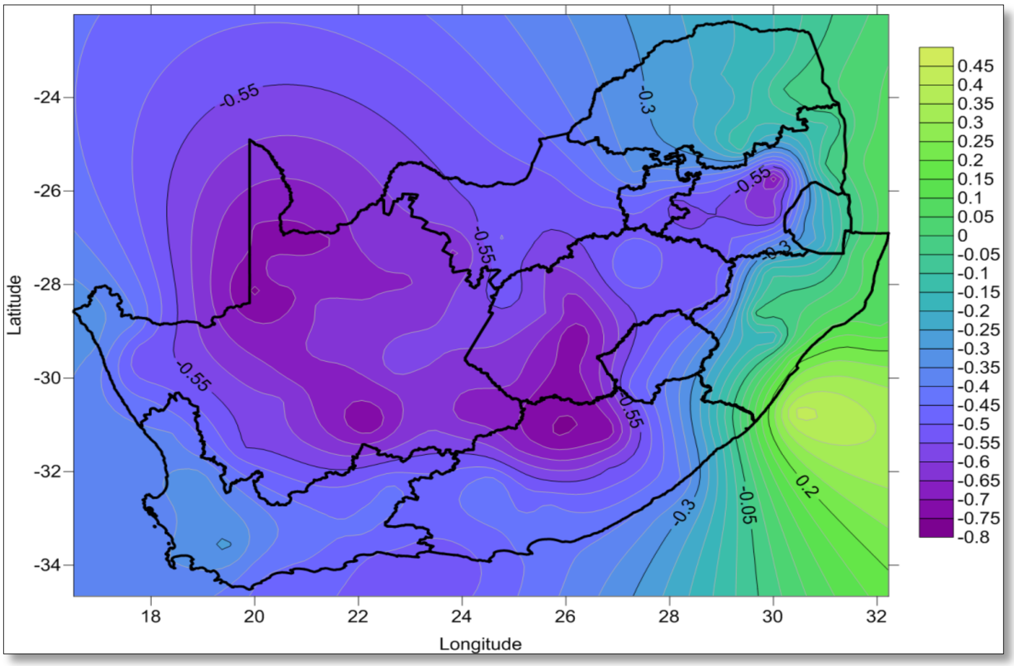

Figure 4-5: The difference between orthometric and spheroidal orthometric heights from 141 GPS/levelling stations (units are in m)

The difference between the spheroidal orthometric and normal height takes a parabolic curve trend line when plotted against elevation (Figure 4-6), the magnitude of the differences increase as the elevation increases over South Africa. Furthermore, the differences between normal-spheroidal orthometric and orthometric-spheroidal orthometric heights are not maintaining any defined pattern when compared with elevation (Figure 4-6). The statistics of the height differences are shown in Table 4.2, it can be deduced that the spheroidal orthometric height system is closer to the normal height system than the orthometric height system with a mean and standard deviation of -21.3 and ± 23.8 cm, respectively.

Table 4.2: Statistics of the differences between orthometric, normal and spheroidal orthometric heights estimated from 141 GPS/Levelling stations.

Differences	**Min. (m)**	**Max. (m)**	**Mean (m)**	**Standard dev. (m)**
$H^O - H^N$	0.000	0.752	0.194	0.176
$H^N - H^S$	-0.647	0.421	-0.213	0.238
$H^O - H^S$	-0.762	0.421	-0.407	0.253

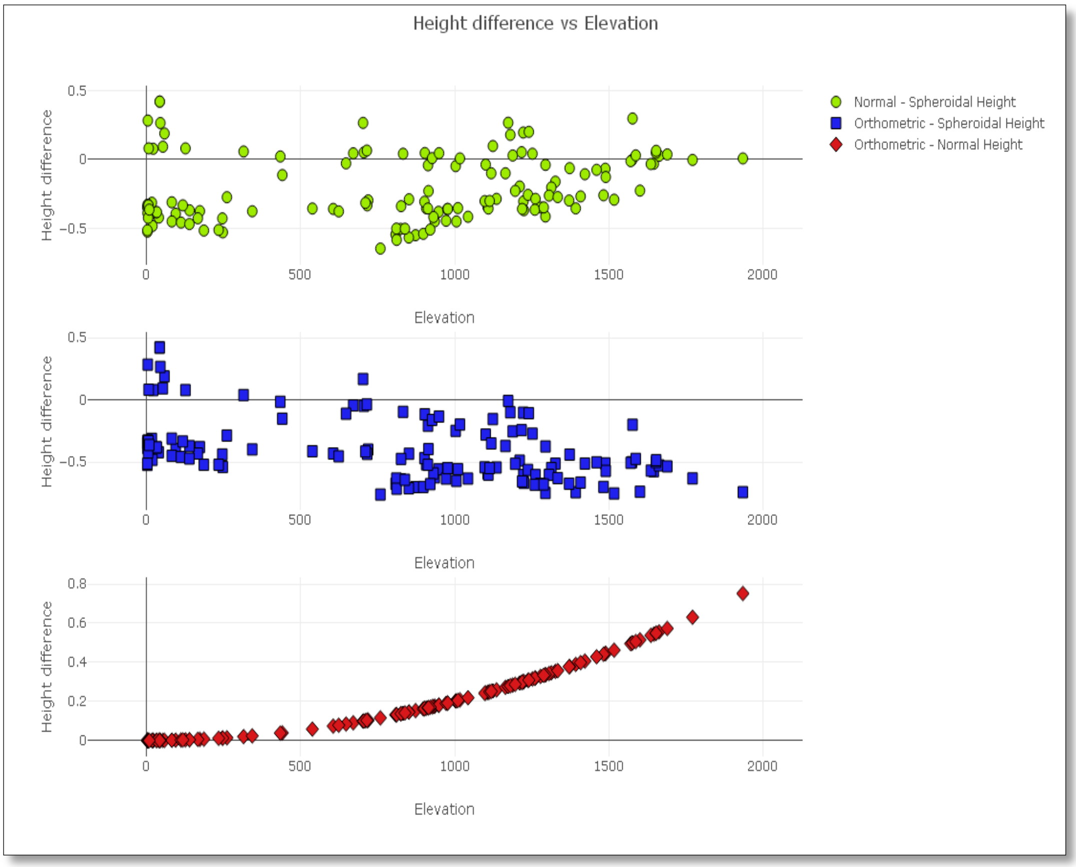

Figure 4-6: Relationship between elevation and the height differences over South Africa (units are in m)

4.2 Analysis of errors in the South African height datum

4.2.1 Introduction

There are a number of existing gross and systematic errors in the primary levelling networks defining the LLD, as mentioned on the Technical Publication report 17 of the primary levelling network of South Africa. However, gross errors in one-way levelling are more likely to be observational or field-booking errors, while gross errors found in two-way sections are more likely to be data entry errors made after the section height differences had been averaged (Filmer & Featherstone, 2009). The systematic errors in the levelling network mentioned in the Technical Publication report 17 are listed below:

i. Earth curvature, terrestrial refraction and collimation errors,
ii. Instrument and levelling staff destruction,
iii. Progressive change in refraction – when levelling uphill or downhill,
iv. Unequal heating of instrument,
v. Levelling staff index error,

vi. Expansion of invar strip of levelling staff,

vii. Levelling staff bubble out of adjustment.

4.2.2 Data and methods

The adjustment of the South African levelling network is referred to as provisional on the Technical Publication report 17, and there is no evidence of the final adjustment conducted in the country. In conducting a differential levelling, errors propagate in proportion to the square root of the total distance travelled. Furthermore, some of the errors in the levelling network are due to neglect of the actual gravity field. Misclosure assessment is undertaken to verify that forward and backward runs of a levelling traverse, including any individual bays, are within the maximum acceptable misclosure. The acceptable misclosure for different orders of levelling can be computed as (Filmer & Featherstone, 2009),

$$\mathrm{m} = c.\sqrt{k} \tag{4:1}$$

where m $-$ represent an acceptable misclosure, $c -$ an empirical value describing the outcome, and $k -$ total travelled distance in km. The value of c is chosen based on the quality of the levelling, and it is usually being defined by the expected maximum closing error. Compiled orders of levelling are given in Table 4.3 (COTO, 2013),

Table 4.3: Levelling misclosures for different levelling orders

Order	Purpose	Maximum closure (m)
Precision order	Deformation surveys	0.001 √km
First order	Major levelling control	0.003 √km
Second order	Minor levelling control	0.007 √km
Third order	Levelling for construction	0.012 √km

The primary levelling network depicting the major levelling loops connecting the four fundamental benchmark in South Africa is given in Figure 4-7.

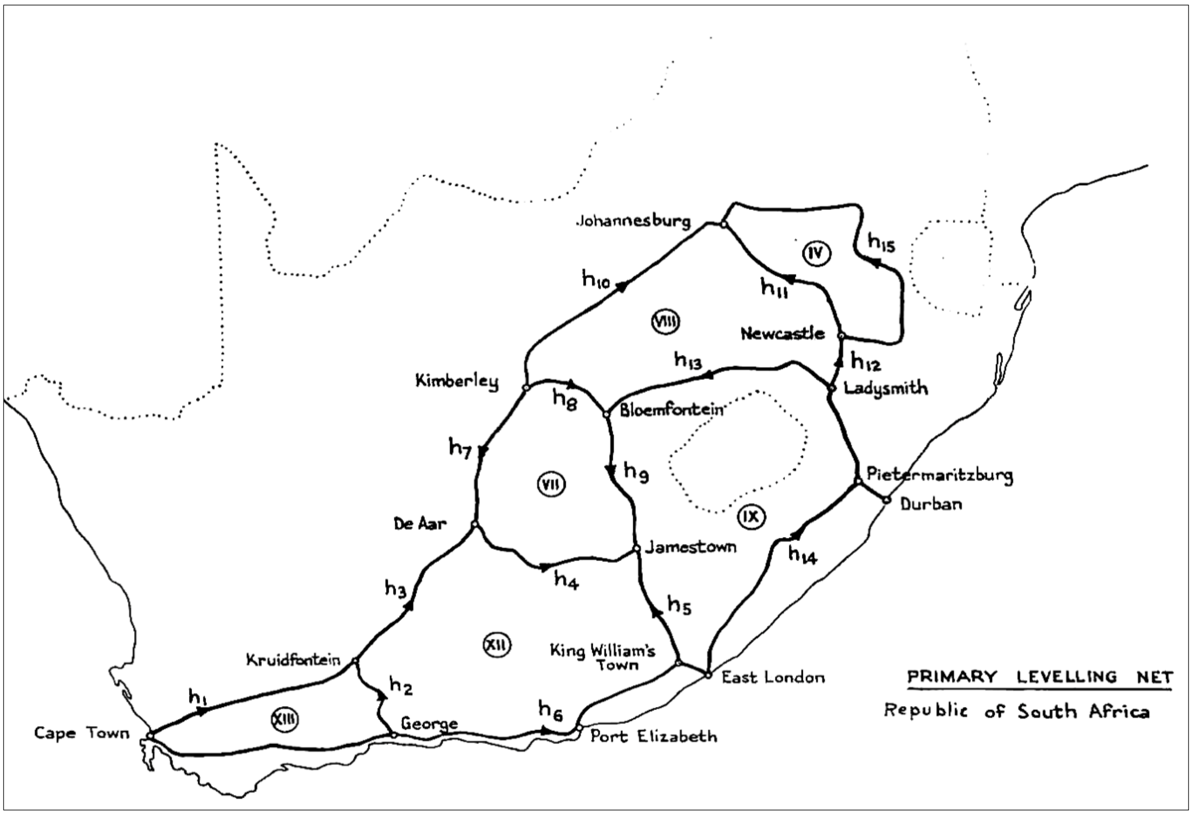

Figure 4-7: Primary levelling network in south Africa (Anonymous, 1965)

4.2.3 Results and discussion

The primary levelling network depicted in Figure 4-7 falls under the first order levelling network. The acceptable misclosure for each levelling loops (iv, vii, viii, ix, xii, and xiii) are computed using the data as given in the Technical Publication report 17, as illustrated in Table 4.4.

Table 4.4: Possible levelling loop closure error

Levelling loop	Total distance (km)	Acceptable misclosure (m)	Measured misclosure (m)	Computed c
iv	1043.303	0.097	-0.102	0.003
vii	1063.098	0.098	-0.044	0.001
viii	1540.830	0.118	0.062	0.002
ix	1828.725	0.128	0.079	0.002
xii	1862.215	0.129	0.114	0.003
xiii	1147.910	0.102	-0.028	0.001

Almost all the levelling loops fall within the acceptable range of misclosure for first order levelling network, except for levelling loop iv as indicated in Table 4.4. The determined misclosure from spirit levelling measurement for levelling loop iv was −10.2 cm while the estimated acceptable misclosure was 9.7 cm. The levelling loop xiii was the most accurate, with the misclosure of −2.8 cm from spirit levelling measurements and the estimated acceptable misclosure of 10.2 cm. Thus the computed levelling loop misclosures were distributed to each set up accordingly during the adjustment process.

A further analysis was conducted by comparing the empirical value from the first order levelling (c = 0.003), as give in Table 4.3, with the empirical value computed from the measured misclosure ($c = \frac{m}{\sqrt{k}}$). This comparison gave an indication that all the levelling loops fall within the acceptable empirical value for the first order levelling network. It is determined from this investigation that loops vii and xiii (0.001) are better than loops viii and ix (0.002) followed by loops iv and xii (0.003). This trend of accuracy variation was not investigated from the Technical Publication report 17, it is possible that this type of computational analysis was not commonly practiced in the 1950s. This analysis provides a better measure for detecting erroneous levelling loops, as it accounts for the levelling distance, than the measured misclosure.

Height differences in levelling networks are usually observed in each direction on a levelling loop, this enables detection of erroneous levelling sections and also to reduce any possible systematic and gross errors in a levelling network. However, height differences between the fundament benchmarks in the Technical Publication Report 17 by Anonymous (1965) are given only as the average of the accumulated height differences.

4.3 Estimation of vertical datum offset for the South African vertical datum, in relation to the international height reference system

4.3.1 Introduction

The South African land levelling datum (LLD) has been providing the reference frame for a variety of practical applications, such as, building of roads, development of infrastructures and variety of developmental activities in the country. In order for South Africa to meet the standards of the global vertical datum, the South African vertical datum must be defined by a gravity potential value. This will enable development of a height system on a defined accuracy that would provide transformation of ellipsoidal height to a rigorously defined height system.

To achieve this, the South African vertical datum should be defined by means of a gravimetric quasigeoid or geoid model, this approach would provide South Africa with a consistent vertical datum. The main focus of this study is to estimate the vertical datum offset for the South African vertical datum, at the four fundamental benchmarks, in relation to the IHRS. The national primary levelling networks can be adjusted using geopotential differences, instead of height differences. This is conducted by studying the relationship between gravity potential and height, in the vertical datum definition and realisation. The growing need for a global reference surface requires a unification of all existing vertical datums around the world, which is a scientific problem of high practical significance (Sánchez et al., 2018).

Unification of height systems requires determination of the adjustment parameters or datum offsets between existing vertical datums, each of which is defined with a fundamental surface of zero elevation. Vertical datum offset is an existing discrepancies between datums, it can be estimated from GPS/levelling data of benchmarks on Land, GPS/levelling data of Tide Gauge stations, Global Geopotential Models (GGM) and a precise quasigeoid or geoid model (Singh, 2018). A geoid is defined as the surface of equal potential of the Earth's gravity field which on average coincides with an undisturbed MSL, it is suitable for providing a reference surface for vertical datums. Moreover, traditionally, local vertical datums were established from a selected network of tide gauge station/s, and the spirit levelling technique was used to develop a local height network. Height differences (dH) measured during levelling are scaled by gravity (g) to determine the difference in gravity potential (dW, also known as change in gravity potential), this relationship can be expressed as follows (Hofmann-Wellenhof & Moritz, 2005):

$$dW = g.dH \tag{4:2}$$

Difference in gravity potential is known as geopotential number (C_P), defined as the difference in potential between the constant value at the global geoid (W_0) and the potential at the point P on the surface of the Earth (W_P), it can be expressed as follows (Vaníček, 1976):

$$C_P = W_0 - W_P = -\int_0^{\mathrm{P}} dW = -\int_0^{P} gdH, \tag{4:3}$$

The negative sign in equation (4:3) above indicates the fact that an increase in height invokes a decrease in gravity potential. It should also be noted that over a short or in regions of low gravitational variation, this difference will be insignificant (Vaníček, 1976).

4.3.2 Data and methods

A number of different data sets were made available for the purpose of this study. The land and marine gravity data over South Africa were provided by the South African Council for Geoscience (SACGS) and Bureau Gravimetrique Internationale (BGI). However, the marine gravity data was coarse, it was supplemented with a global marine gravity model from CryoSat-2 and Jason-1 (Sandwell *et al*., 2014). Both the land and marine gravity measurements were generated into a grid file using a Kriging method from *Golden Surfer software* (to ensure compatibility within the data), then the global marine gravity data was added to fill in the existing gaps on the marine gravity data. The global marine data grid was downloaded with the companion gridded bathymetric data set (from https://topex.ucsd.edu/cgi-bin/get_data.cgi). The marine gravity anomalies were computed from satellite altimetry data, they were reduced using fast Fourier technique, the method is described in details by Sandwell & Smith (2009). Moreover, both the horizontal and vertical coordinates associated with the gravity data from SACGS and BGI are of low accuracy, as they have been interpolated from a 1:50000 map, this will introduce distortions on the resulting gravity anomalies.

The gravity data was screened for duplications using *Golden Surfer software*, the free-air gravity anomalies on the land gravity data ranges between -101.3 and 129.3 mGal with a mean and standard deviation of 16.3 and ± 31.1 mGal, respectively. The free-air gravity anomalies from the land gravity data was compared to the set of free-air gravity anomalies generated using the GOCE-based GGM (TIM6) harmonic coefficients (truncated at degree and order 200) on zero-tide system. Thereafter, a mean difference of -2.1 mGal with a standard deviation of ± 10.7 mGal was obtained.

The free-air gravity anomalies on the marine gravity data ranges between -97.5 and 115.7 mGal with a mean and standard deviation of 2.4 mGal and ± 30.4 mGal, respectively. The free-air gravity anomalies from the marine gravity data was compared to the set of free-air gravity anomalies generated using the GOCE-based GGM (TIM6) harmonic coefficients (truncated at degree and order 200) on zero-tide system. Thereafter, a mean difference of 11.8 mGal with a standard deviation of ± 17.5 mGal was obtained. The gravity data was limited to a $4^0 \times 4^0$ grid around each TGBM to reduce computation time and considering that the delimited region had sufficient data to conduct the analysis around each TGBM, as depicted in Figure 4-8. This idea was adopted from the study literature conducted by (Sánchez et al, 2018).

The first-order gravity data have a maximum uncertainty of ± 1 mGal while the accuracy of first-order levelling network in South Africa is estimated at $1.9\sqrt{L}\,mm$, L being the distance of a levelling line in km. The GPS measurements of the TGBMs were collected by the Nation Geo-Spatial Information (NGI), South African government agency. The ellipsoidal heights were determined using differential carrier phase GPS measurements, linked to the national network of permanent GPS stations, TrigNet. The coordinates are in the ITRF2008(20016.2) reference frame and refer to the WGS84 ellipsoid. The internal accuracy of GPS coordinates is approximately ± 1 and ± 2 cm on the horizontal and vertical position, respectively (Odera, 2019). The differences between ellipsoidal and spheroidal orthometric heights are considered as height anomalies, as the South African LLD provides heights that are closer to normal height system (Merry, 1985; Odera, 2019).

The SRTM data at 3 arc second (90 m resolution) DEM was used for computation of the terrain effect (G_1 term). The DEM is uniform on the specified grid ($4^0 \times 4^0$) around each TGBM, as depicted in Figure 4-9 to Figure 4-12. The remove-compute-restore method is used to compute the height anomalies of the TGBMs. The long wavelength component of the disturbing potential was determined from the spherical harmonic coefficients of the latest GOCE-based GGM (TIM6 truncated at degree and order 200), the harmonic coefficients on zero-tide system was used for geodetic calculations. The *zero-tide* model is compliant with Stokes formula (because all of the external masses are removed). Its advantage over the *tide-free* model is that it does not require the use of an assumption regarding elasticity (in the removal of the indirect effect) and so the reduction can be done completely by potential theory (Amos, 2007). The medium wavelength component was determined from the gravity data residuals ($\Delta g - \Delta g_{GGM} + G_1$), using Stokes's integral as described in the previous sub-section. The residual terrain model (RTM) was used to cater for the contribution of the short wavelength component. A computer program designed from *python* was used for this computation. The four fundamental tide gauge benchmarks located in Cape Town (TGBM_CPT), Port Elizabeth (TGBM_PEL), East London (TGBM_ELN), and Durban (TGBM_DBN) over South Africa are shown in Figure 4-8.

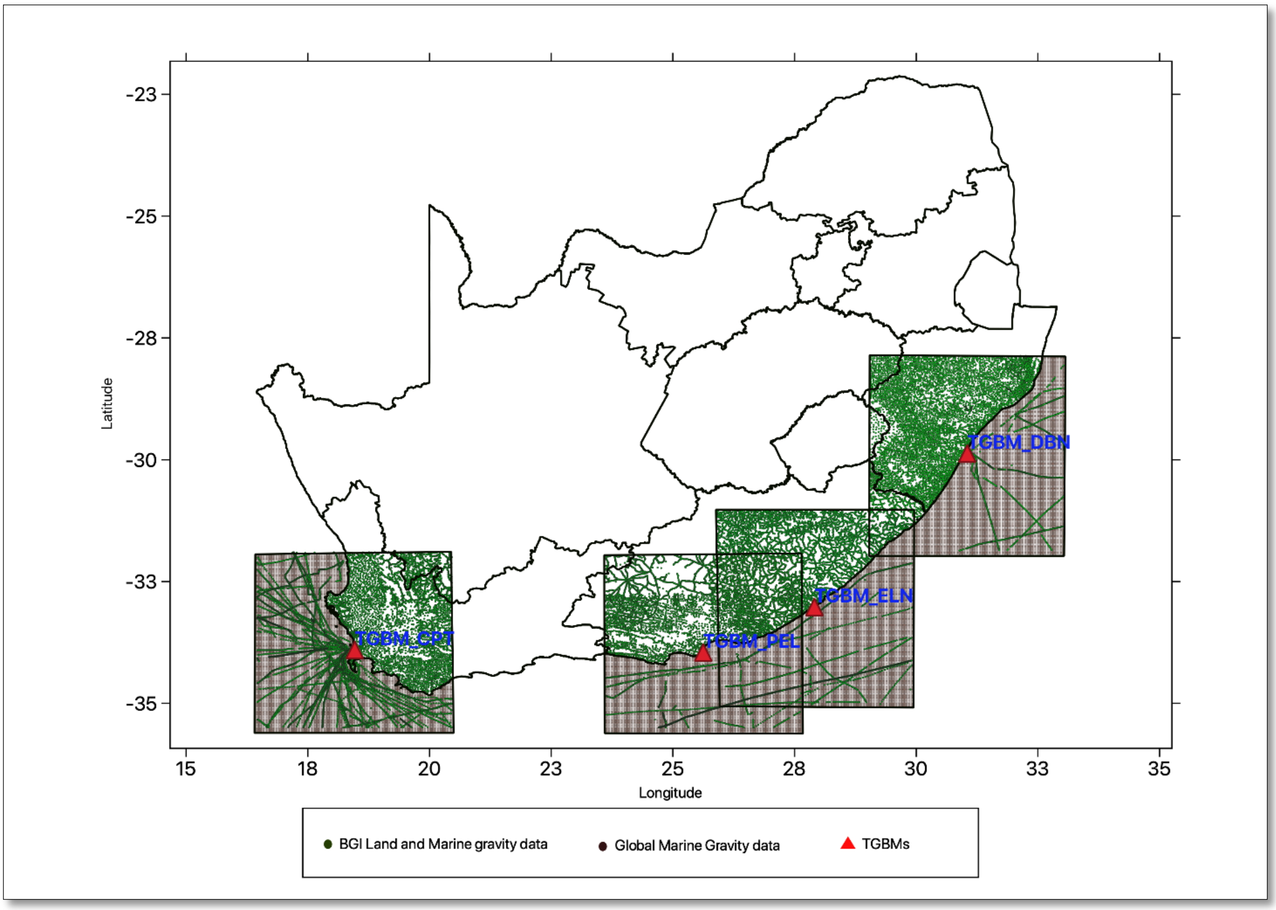

Figure 4-8: Distribution of gravity data around the four fundamental benchmarks over South Africa

The elevation map around each TGBM was generated using DEM from SRTM90 to provide a terrain visualisation, as depicted in Figure 4-9 to Figure 4-12. A kriging interpolation method was used to generate contour maps, because it is statistically more sophisticated and it allows identifying distortions in the data. Moreover, it was used to evaluate the contribution of the indirect effect on the height anomaly.

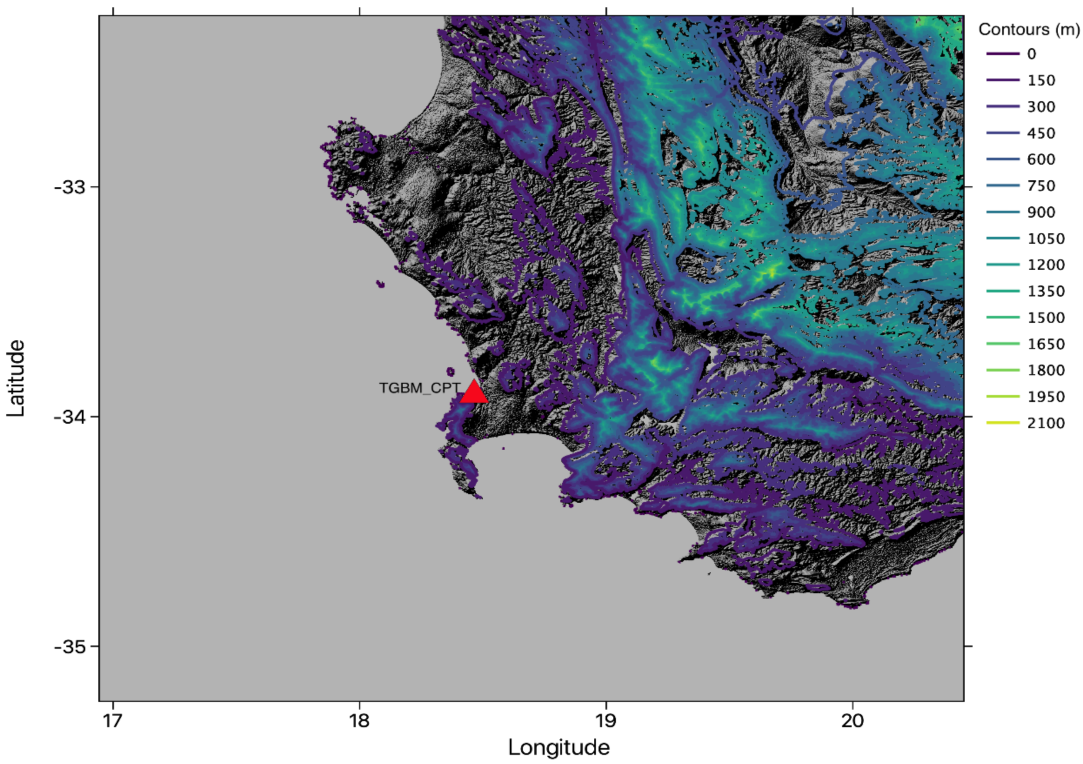

Figure 4-9: Elevation around Cape Town TGBM

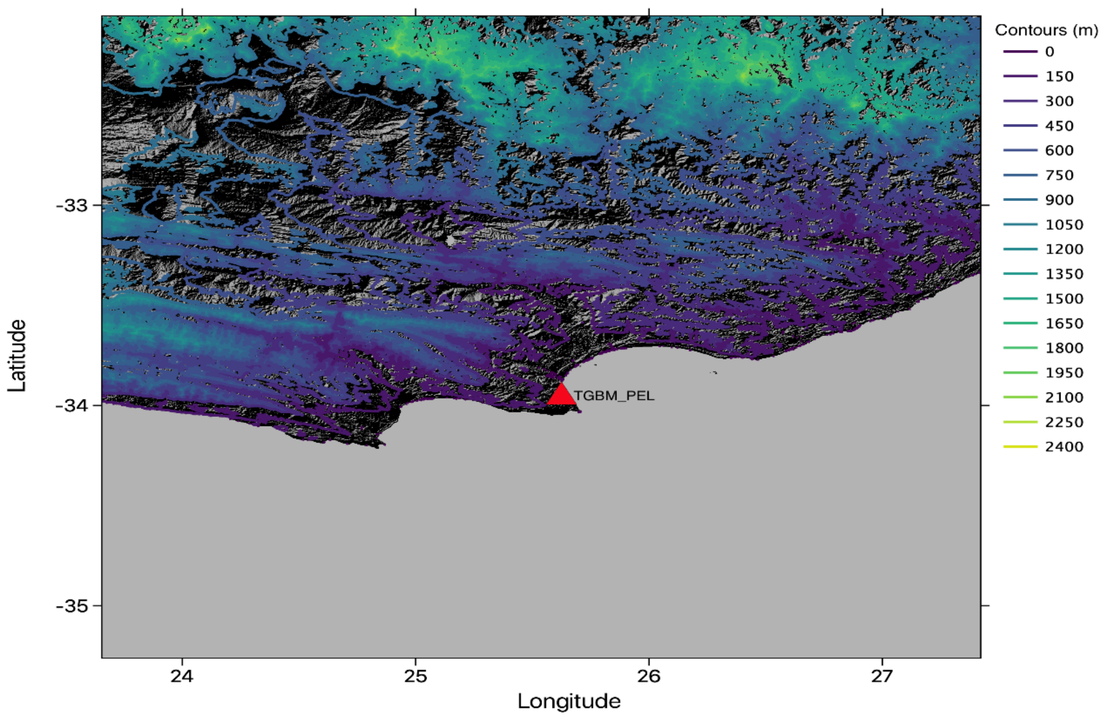

Figure 4-10: Elevation around Port Elizabeth TGBM

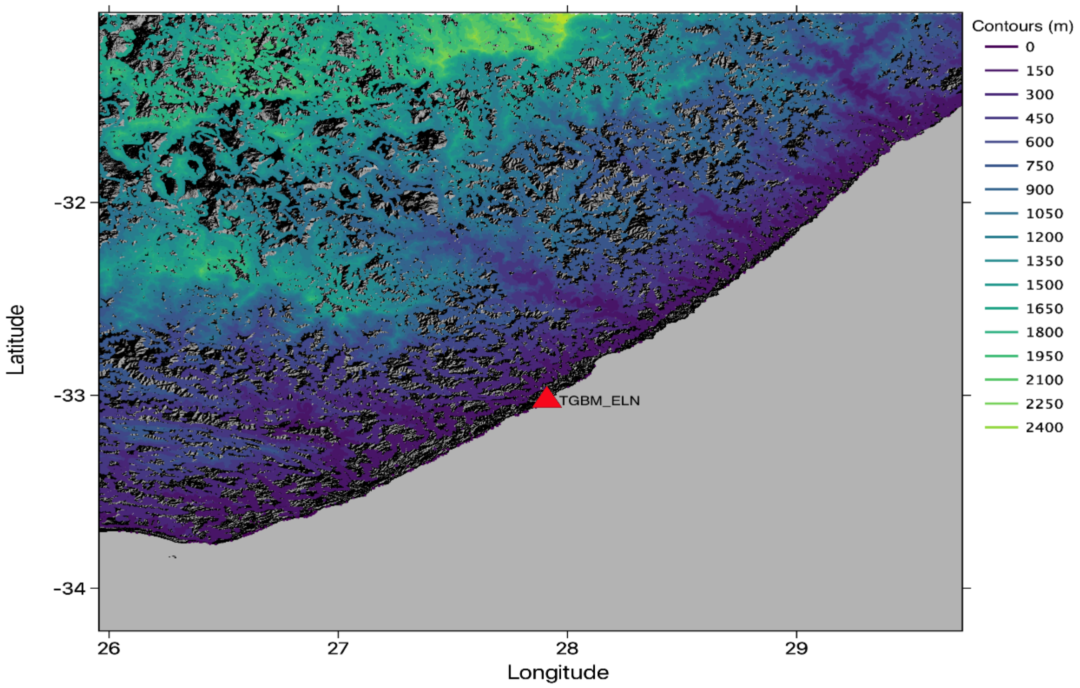

Figure 4-11: Elevation around East London TGBM

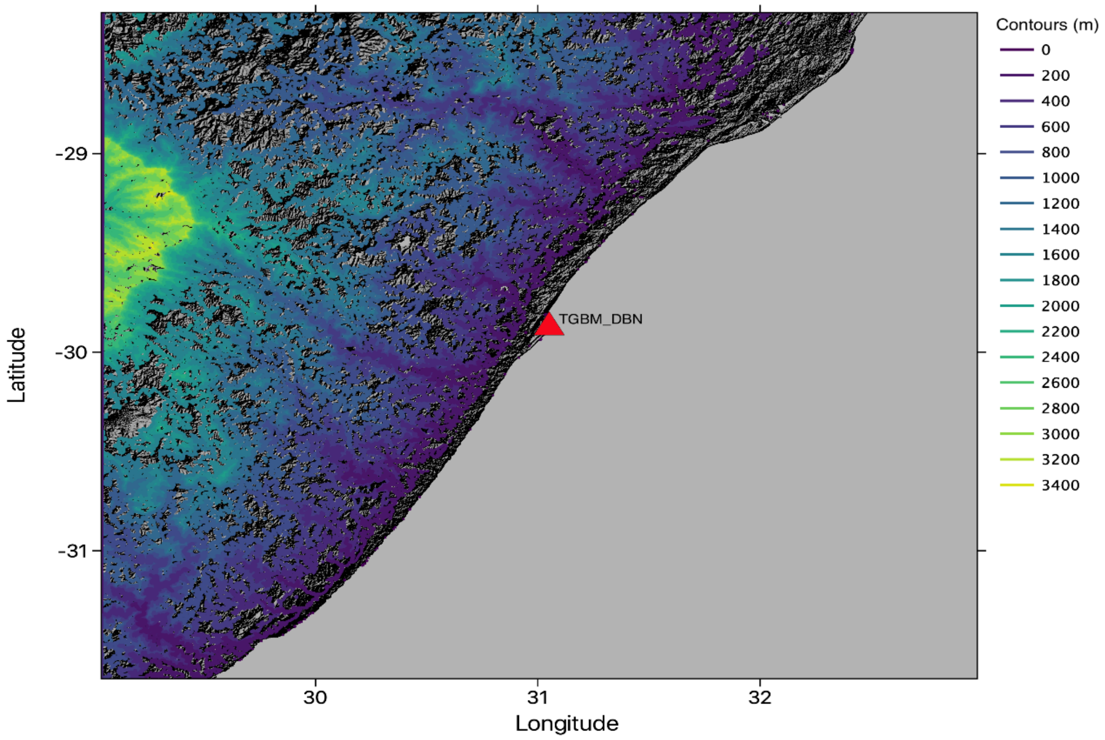

Figure 4-12: Elevation around Durban TGBM

The vertical datum offset on the fundamental benchmarks is estimated using equation (2:35) this provides an adjustment factor for datum transformation over South Africa and the gravity potential value for each fundamental benchmark can be estimated.

4.3.3 Results and discussion

The gravity residuals used on Stokes integral, as expressed in equation (2:37), was determined from the gravity anomalies computed from the observed gravity data, the gravity anomalies generated by the coefficients of the spherical harmonics, from the GOCE based GGM and the Molodensky G_1 term determined from a convolution of heights with gravity anomalies. The residual gravity anomalies around each TGBM are depicted in Figure 4-13 to Figure 4-16, to illustrate the quality of gravity data around each TGBM.

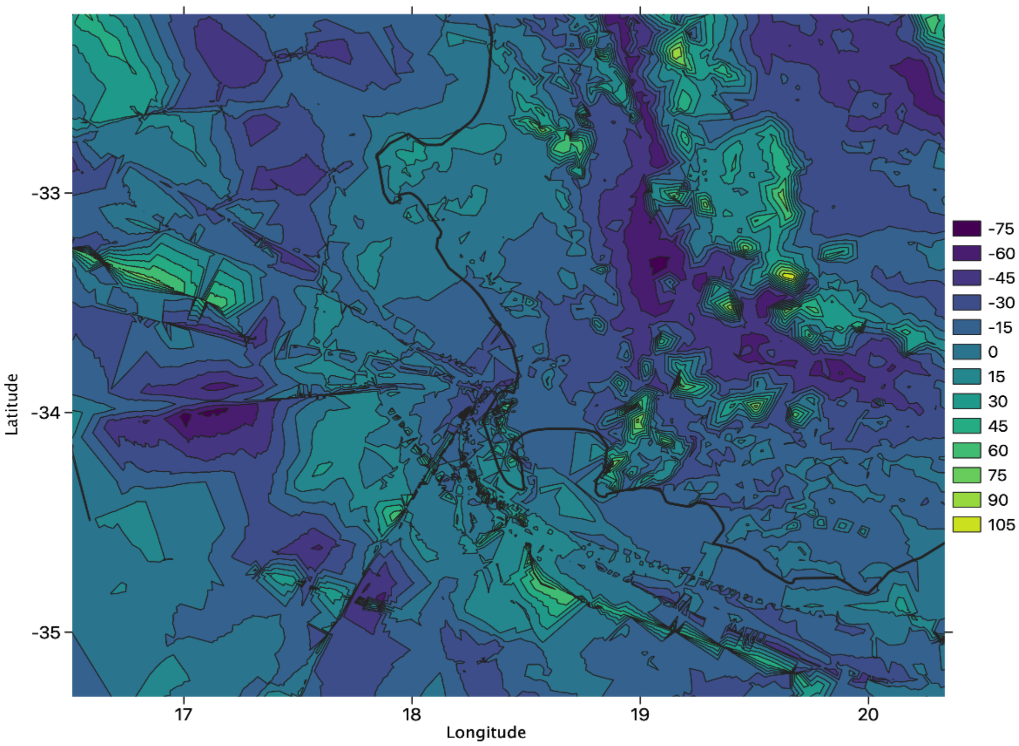

Figure 4-13: Residual gravity anomalies around Cape Town TGBM (units are in mGal)

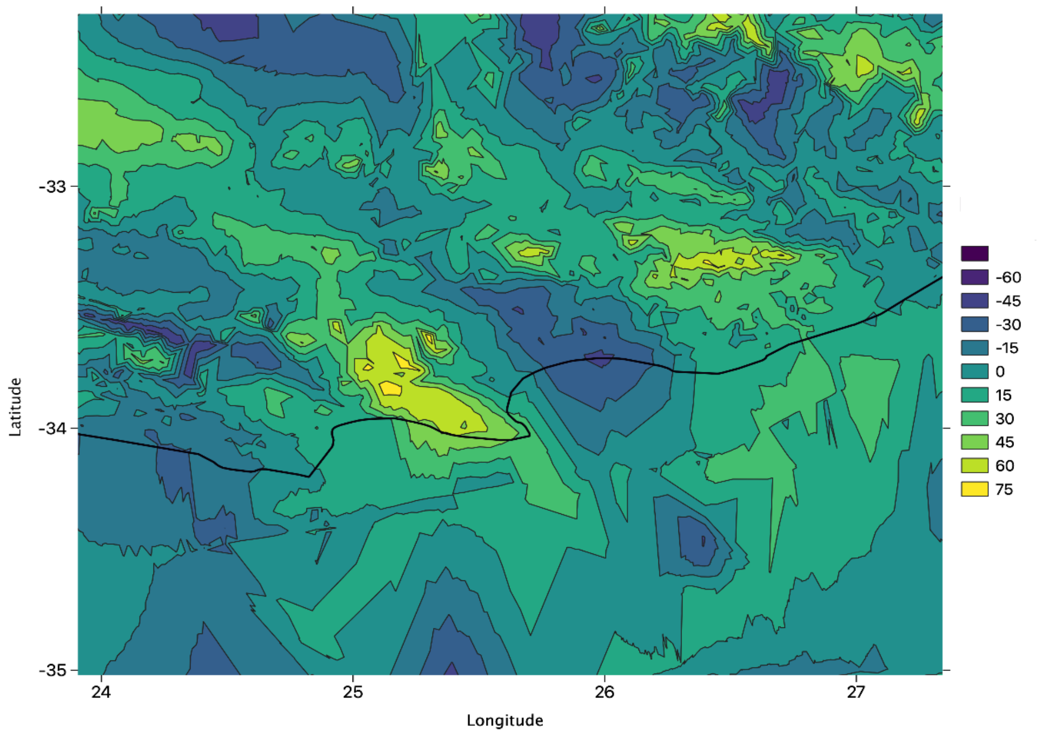

Figure 4-14: Residual gravity anomalies around Port Elizabeth TGBM (units are in mGal)

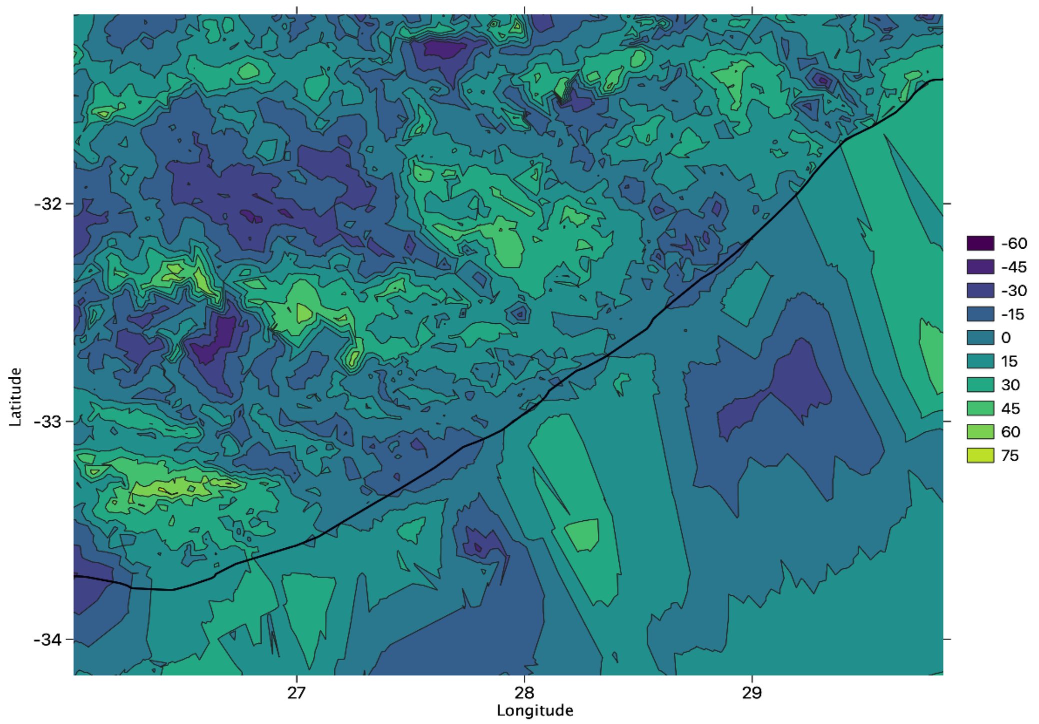

Figure 4-15: Residual gravity anomalies around East London TGBM (units are in mGal)

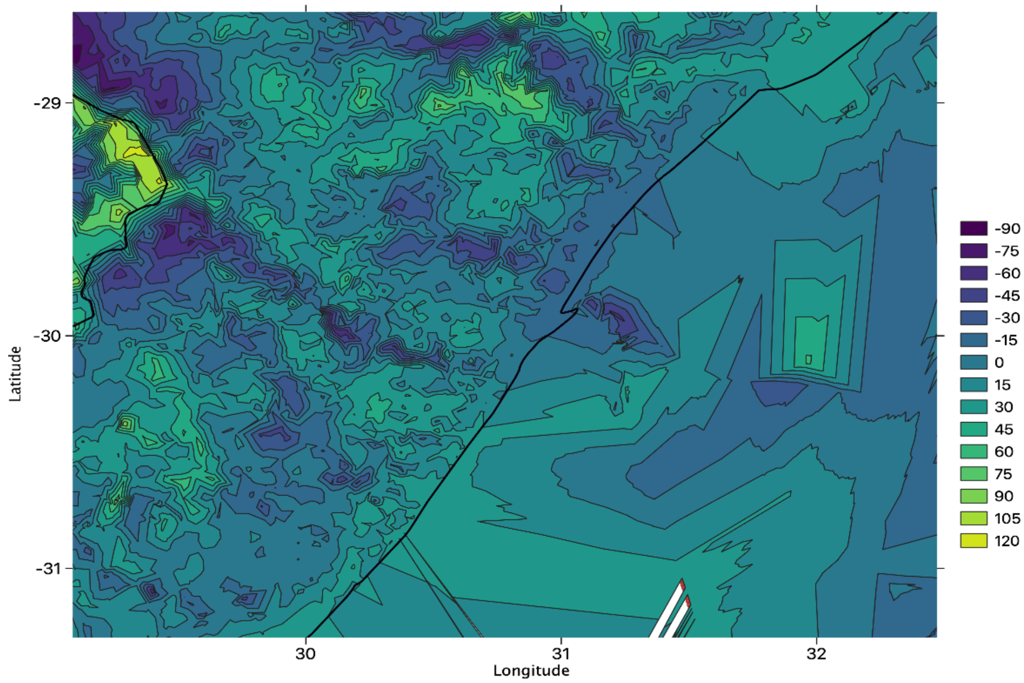

Figure 4-16: Residual gravity anomalies around Durban TGBM (units are in mGal)

The vertical datum offset is determined on the four fundamental benchmarks over South Africa in relation to the IHRS, estimated potential discrepancies are depicted in Figure 4-17. The South African vertical datum is defined by four primary levelling networks initiated from these fundamental benchmarks, and they were adjusted in a piecemeal fashion. Therefore, these discrepancies give an indication of the existing inconsistencies within the four primary levelling networks, and it also indicates an adjustment shift that corresponds to a particular network.

The vertical datum offset at each TGBM was evaluated using equation (2:39), and the potential difference between the local and the global reference surface was evaluated using equation (2:35). The components involved in the computation of the vertical datum offset, at each TGBM, are as illustrated in Table 4.5.

Table 4.5: Vertical datum offset parameters and estimated offset at each TGBM.

TGBM	h_P(m)	H_P^{LLD}(m)	ζ_{GGM}(m)	ζ_{res}(m)	ζ_{RTM}(m)	$W_P(\mathrm{m^2s^{-2}})$	$\delta W_P(\mathrm{m^2s^{-2}})$
CPT	34.423	3.6281	31.996	0.085	-1.518	62636852.815	0.585
PEL	31.487	4.2233	29.276	0.016	-1.994	62636855.423	-2.023
ELN	33.823	4.4153	30.642	0.018	-1.159	62636855.997	-2.597
DBN	32.678	4.3076	28.465	-0.010	-0.472	62636851.295	2.105

The gravity potential at each TGBM in South Africa deviates from the potential of the global reference surface by $0.585, -2.023, -2.597$ and $2.105\ \mathrm{m^2s^{-2}}$ for Cape Town, Port Elizabeth, East London and Durban, respectively. These deviations are depicted in Figure 4-17.

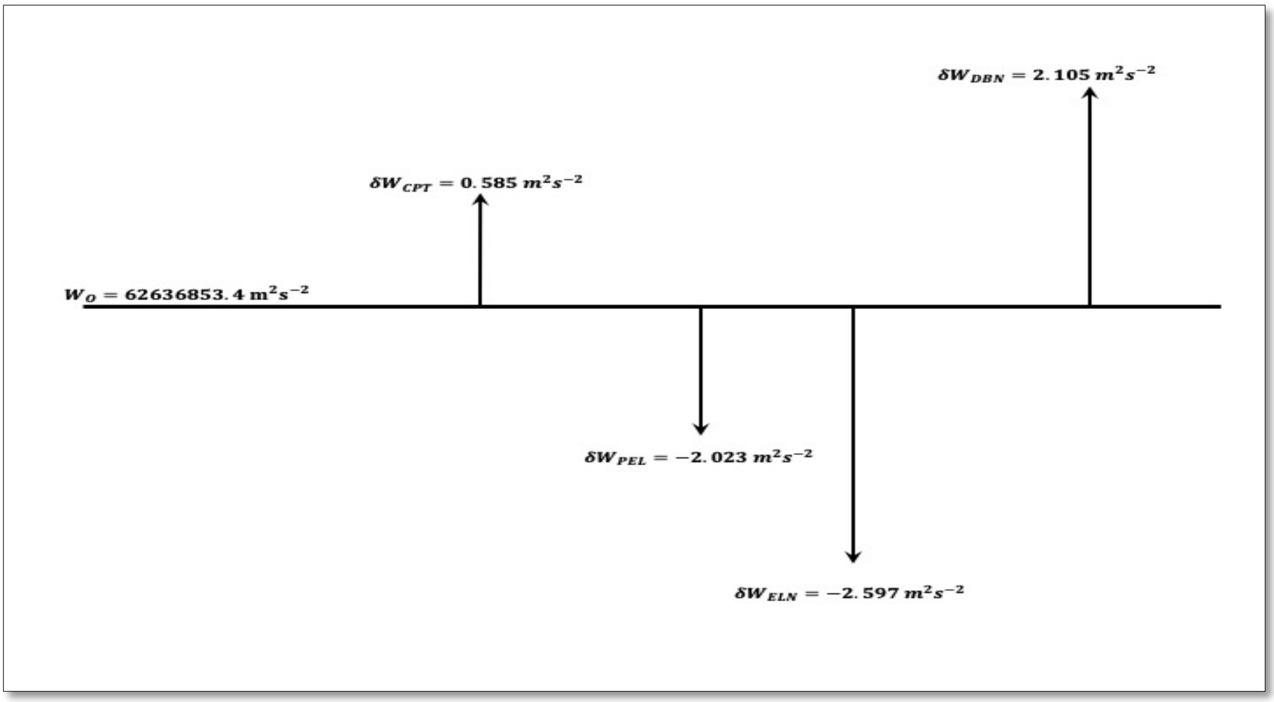

Figure 4-17: Vertical datum offset on the four fundamental BM in relation to the global vertical datum

The corresponding linear vertical datum offsets between the international height reference system and the four fundamental benchmarks over South Africa are 5.973 , -20.647, -26.518, and 21.496 cm for Cape Town, Port Elizabeth, East London and Durban, respectively. These offsets can be used for unification of the South African vertical datum at the four benchmarks in a manner that is consistent to the international height reference system. The estimated gravity potential on the four fundamental benchmarks are given in Table 4.5.

These datum offsets should be as reliable as possible, because they form part of the datum transformation parameters. The quality of the fundamental benchmarks can be improved by being connected to the gravity data networks. The desired physical heights system can be deduced from geopotential values, as elaborated in chapter 2. The advantages of using geopotential value for height determination is that there is no need to compute orthometric or normal corrections to the measured height differences, and it is very easy to convert between height systems, as one does not have to compute a new set of corrections.

4.4 Towards a local geoid-based vertical datum over South Africa

4.4.1 Introduction

A gravimetric geoid model is the commonly preferred option to provide for a consistent vertical datum around the world. The process of transformation from one reference surface to the other becomes much more complex. However, no matter how complex it is, the relationship between the current datum and new datum can be resolved. A decision has to be made whether to adopt a geoid or quasigeoid model as a reference surface, for a rigorously defined height system in South Africa. Even though they have their underlying issues, uncertainty in the topographic density distribution for geoid modelling and convergence issues with the quasigeoid determination (Amos & Featherstone, 2009).

The GPS/levelling data is used to assess an existing vertical datum offset between the local geoid model and local vertical datum. The geometric ellipsoidal height (h) combined with the physical height (H) can be related to the reference surface as, $N = h - H$. Furthermore, a vertical datum offset (δ) can be determined from this relationship as, $N - (h - H) = \delta$. In an ideal case, the datum offset determined above would be equal to zero '0', where the origin of the vertical datum coincides with the geoid or quasigeoid surface. However, this is not the case in reality, due to approximate modelling of gravity effects on the precise levelling network, adjustment techniques used for the precise levelling network, systematic rise/fall in mean sea level, and errors in the tide gauge observation used to define the datum, to name a few. Therefore, the origin of the local vertical datum would not necessarily coincide with the geoid or quasigeoid surface.

The computed vertical datum offset from the GPS/levelling data would contain unavoidable random errors and systematic biases (Amos & Featherstone, 2009). The vertical datum offset generally defines how the geoid or quasigeoid model fits the GPS/levelling data, in some cases it is referred to as a corrector surface, created to remove existing biases (Chandler & Merry, 2010; Younis, 2017) and also provide an accuracy evaluation of a reference surface (geoid or quasigeoid model).

In the South African context, for the purpose of adequately redefining the vertical datum, each primary network must have their own datum offset. Since the South African local vertical datum is based on four primary levelling networks, which were adjusted in a piecemeal fashion. Therefore, as the current existing local vertical datum is inconsistent it would be necessary for the primary levelling networks to be unified, by using the offsets at the four fundamental benchmarks or in the locations where they overlap, before redefining the South African local vertical datum. A datum offset can be used for establishing a geoid consistent vertical datum and to unify the land levelling datum using a precise local geoid model.

4.4.2 Data and method

This assessment will assist in the decision making regarding an adequate vertical reference surface and the best corresponding height system for South Africa. This would be carried out by determining how well the spheroidal orthometric (H^S), normal (H^N) and orthometric (H^O) height systems fit the existing quasigeoid and geoid model over South Africa. The existing quasigeoid model (SAGEOID10) and the geoid model converted from the SAGEOID10, are used for evaluating the corresponding height systems.

One hundred and thirty eight (138) GPS/levelling data points (Figure 4-18) with their corresponding spheroidal orthometric, normal and orthometric heights over South Africa have been used for height system evaluation, 3 GPS/levelling data points were identified as outliers hence they were discarded for this analysis (hence the use of 138 instead of 141 GPS/levelling points).

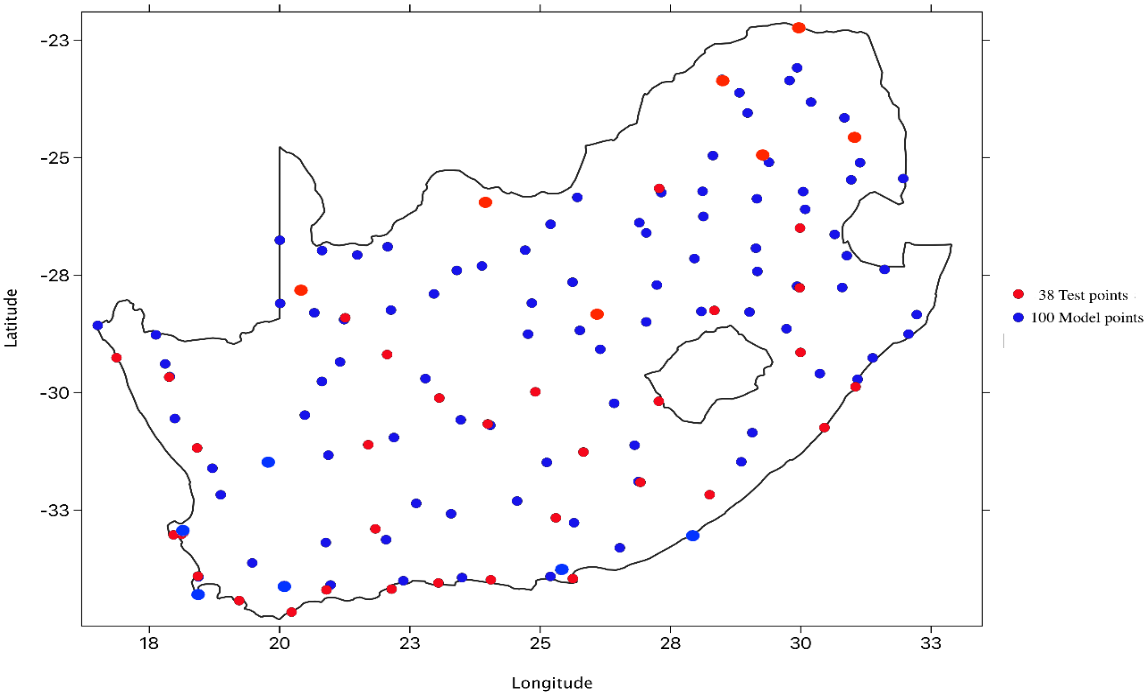

Figure 4-18: Distribution of the 138 GPS/levelling data points over South Africa

A total of 100 GPS/levelling data points were used to create a correction surface, while 38 GPS/levelling points were used for cross validation (Figure 4-18). The accuracy of a reference surface has been assessed using the datum offset from the 138 GPS/levelling data points, with the corresponding physical height or height system, as illustrated in Table 4.6 and Table 4.7.

Table 4.6: Quasigeoid offset on the 138 GPS/levelling stations over South Africa

Datum offset	Min. (m)	Max. (m)	Mean (m)	Standard dev. (m)
$\zeta - (h - H^S)$	-0.267	0.647	0.195	0.227
$\zeta - (h - H^N)$	-0.247	0.783	0.208	0.229
$\zeta - (h - H^O)$	-0.256	0.667	0.425	0.268

Table 4.7: Geoid offset on the 138 GPS/levelling stations over South Africa

Datum offset	Min. (m)	Max. (m)	Mean (m)	Standard dev. (m)
$N - (h - H^S)$	-0.265	0.763	0.413	0.249
$N - (h - H^N)$	-0.246	1.502	0.643	0.360
$N - (h - H^O)$	-0.247	0.783	0.425	0.251

Results provide more proof that the spheroidal orthometric height is more compatible with the quasigeoid model than the geoid. It would make sense for South Africa to adopt the quasigeoid model as its reference surface and its corresponding height system, i.e. the normal height. The South African local height system is more comparable with both the quasigeoidal surface and the normal height system (as determined in the previous section). Availability of more high-quality data, especially the gravity data, would improve the quality of the local quasigeoid model to enable it to be adopted as a reference surface. However, the geoid, is a smooth, physically meaningful surface, convex everywhere and describable by a simple mathematical expression while the quasigeoid, has no physical meaning and contains folds, making it quite difficult to describe mathematically, as defined in the previous section.

The existing residuals in Table 4.6 and Table 4.7 give an illustration of how much each height system fits with a certain reference surface (geoid or quasigeoid model), this is probably due to the existing biases in the South African local vertical datum and quasigeoid (i.e. the SAGEOID10 model before it was converted into a hybrid model). The smaller the standard deviation of the residuals the better the agreement. In order to reduce effects of the mentioned random and systematic errors, least squares parametric model can be used.

The datum offset can be modelled using a least squares parametric case model, a geometric geoidal ($N^{GPS/levelling}$) and gravimetric geoidal height (N^{GM}) is used for this analysis. A corrector surface is modelled using a four parametric mathematical model, to reduce existing distortions on the computed datum offset, it can be expressed as follow (Kiamehr, 2006):

$$\begin{aligned} \boldsymbol{N^{GPS/levelling}} - \boldsymbol{N^{GM}} &= \boldsymbol{h} - \boldsymbol{H} - \boldsymbol{N^{GM}} \\ &= x_0 + x_1 \cos\varphi \cos\lambda + x_2 \cos\varphi \sin\lambda + x_3 \sin\varphi, \end{aligned} \tag{4:4}$$

where:-

$N^{GPS/levelling}\ and\ N^{GM}$ – Geometric and gravimetric geoidal height,

$x_0, x_1, x_2, and\ x_3$ – Unknown parameters.

Therefore, least squares adjustment for the above mathematical model (Kiamehr, 2006):

$$\Delta\boldsymbol{N} = \boldsymbol{N^{GPS/levelling}} - \boldsymbol{N^{GM}} = \boldsymbol{Ax} + \boldsymbol{V}, \tag{4:5}$$

where:-

A- represents a Jacobian or design matrix consisting of the coefficients of the unknown parameters, with a size of $(n \times u)$ matix.

x- Solution vector or a vector of unknown parameters, with a size of $(u \times 1)$ matrix.

V- Residuals to be minimised, with a size of $(n \times 1)$ matrix.

The values for *n* and *u* in this study are 100 (observations) and 4 (unknown parameters), respectively. The solution vector for the unknown parameters (u) is determined using the following expression:

$$\boldsymbol{x} = \left(\boldsymbol{A^T P A}\right)^{-1} \boldsymbol{A^T P} \Delta \boldsymbol{N}, \qquad (4:6)$$

The letter $\boldsymbol{P}$ in the equation above represents the weight matrix for the geoidal height differences from the geometric geoidal height and a gravimetric geoidal height. However, in this case an identity matrix is used as the weight matrix to avoid complications on the results. The residuals of the adjustment are then determined from the system of observation equations by the following expression:

$$\boldsymbol{V_i} = \Delta \boldsymbol{N} - \boldsymbol{Ax}, \qquad (4:7)$$

A computer program was designed to perform the whole process of the least squares adjustment described above for modelling correction surfaces.

4.4.3 Results and discussion

A cross validation approach was used for evaluation of the local quasigeoid and geoid model, only 100 GPS/levelling data points were used for the creation of the correction surface as describe in the previous sub-section and the remaining 38 GPS points were used for validation. The statistics results for this evaluation at 38 GPS/levelling validation points before parametric model fitting are illustrated in Table 4.8 and Table 4.9.

Table 4.8: Quasigeoid offset at 38 GPS/levelling validation points before parametric model fitting

Datum offset	Min. (m)	Max. (m)	Mean (m)	Standard dev. (m)
$\zeta - (h - H^S)$	-0.196	0.647	0.168	0.222
$\zeta - (h - H^N)$	-0.088	0.783	0.183	0.218
$\zeta - (h - H^O)$	-0.190	0.667	0.436	0.245

Table 4.9: Geoid offset at 38 GPS/levelling validation points before parametric model fitting

Datum offset	Min. (m)	Max. (m)	Mean (m)	Standard dev. (m)
$N-(h-H^S)$	-0.078	0.763	0.421	0.245
$N-(h-H^N)$	-0.085	1.502	0.689	0.361
$N-(h-H^O)$	-0.088	0.783	0.436	0.241

The statistics of the results in Table 4.8 and Table 4.9 indicate that the normal and orthometric heights provided a best fit when compared with the quasigeoid and geoid model, with a standards deviation of ± 21.8 and ± 24.1 cm, respectively. This was expected since they were compared to their corresponding reference surfaces. However, the spheroidal height provided a best fit with the quasigeoid model (±22.2 cm) than the geoid model (± 24.5 cm). The validation process is conducted for independent testing of the generated model. The statistics of the results at 38 GPS/levelling validation points after parametric model fitting are illustrated in Table 4.10 and Table 4.11.

Table 4.10: Quasigeoid datum offset at 38 GPS/levelling validation points after parametric model fitting

Datum offset	Min. (m)	Max. (m)	Mean (m)	Standard dev. (m)
$\zeta-(h-H^S)$	-0.016	0.029	0.047	0.063
$\zeta-(h-H^N)$	-0.037	0.036	0.003	0.051
$\zeta-(h-H^O)$	-0.020	0.067	0.066	0.076

Table 4.11: Geoid datum offset at 38 GPS/levelling validation points after parametric model fitting

Datum offset	Min. (m)	Max. (m)	Mean (m)	Standard dev. (m)
$N-(h-H^S)$	-0.033	0.041	0.058	0.076
$N-(h-H^N)$	-0.054	0.093	-0.080	0.083
$N-(h-H^O)$	-0.075	0.075	0.003	0.039

The quasigeoid and the geoid datum offsets at the validation data points provided similar conclusion as before parametric model fitting. However, with improved results, the normal and orthometric height provided a best fit when compared with the quasigeoid and geoid model, with a standards deviation of $\pm 5.1$and ± 3.9 cm, respectively. Moreover, the spheroidal height provided a best fit with the quasigeoid model (±6.3 cm) than the geoid model (±7.6 cm). The distribution of the validation data points are depicted in Figure 4-19 to Figure 4-24.

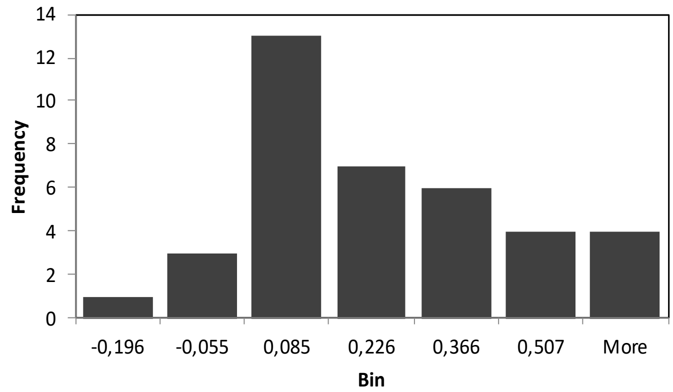

Figure 4-19: Histogram of the difference between $\boldsymbol{H^S}$, GPS/levelling and quasigeoidal surface after parametric model fitting (Bin range in m)

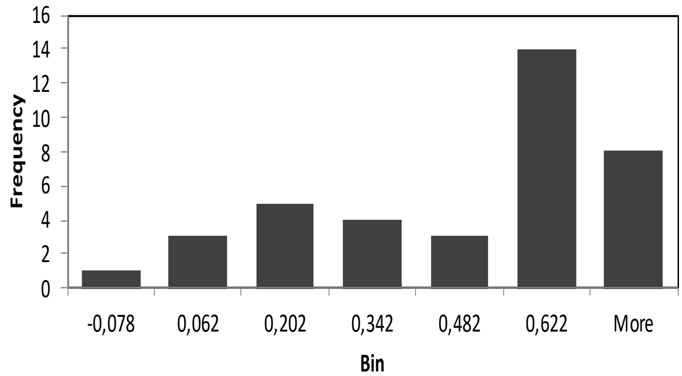

Figure 4-20: Histogram of the difference between $\boldsymbol{H^S}$, GPS/levelling and geoidal surface after parametric model fitting (Bin range in m)

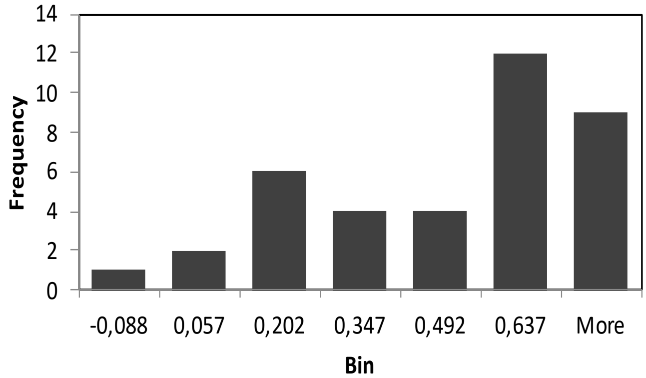

Figure 4-21: Histogram of the difference between $\boldsymbol{H^N}$, GPS/levelling and quasigeoidal surface after parametric model fitting (Bin range in m)

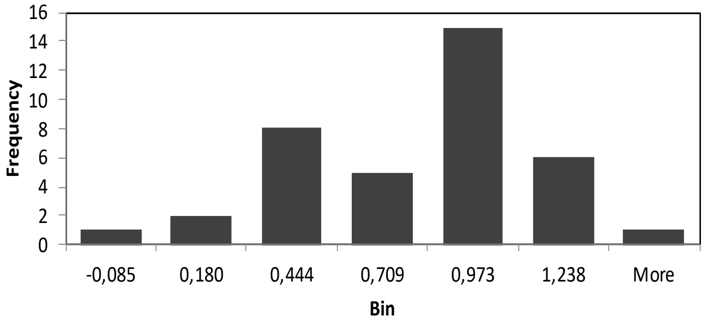

Figure 4-22: Histogram of the difference between $\boldsymbol{H^N}$, GPS/levelling and geoidal surface after parametric model fitting (Bin range in m)

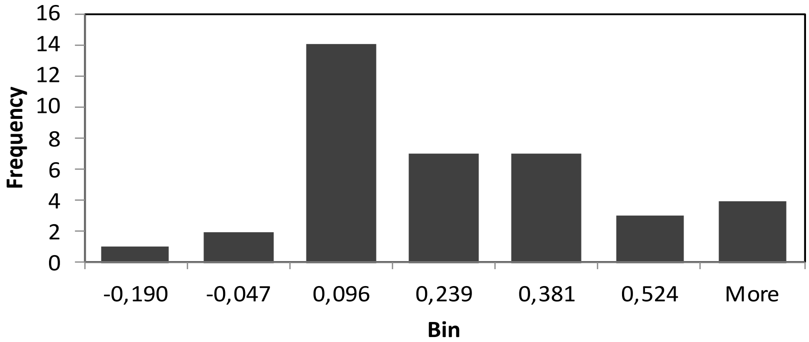

Figure 4-23: Histogram of the difference between $\boldsymbol{H^O}$, GPS/levelling and quasigeoidal surface after parametric model fitting (Bin range in m)

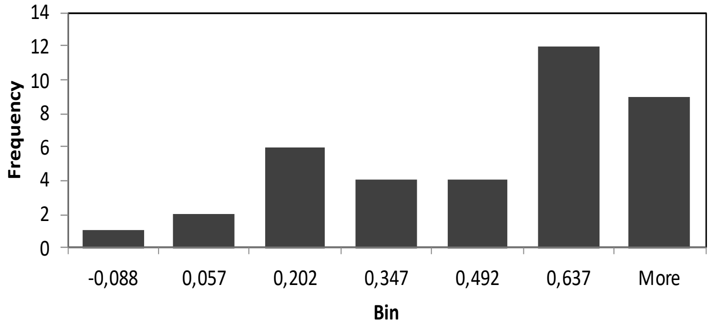

Figure 4-24: Histogram of the difference between $\boldsymbol{H^O}$, GPS/levelling and geoidal surface after parametric model fitting (Bin range in m)

The plotted histograms depicted in Figure 4-19 to Figure 4-24 indicate that the distribution of the validation data is slightly skewed and their distribution is approximately symmetric.

4.5 Towards a global geoid-based vertical datum over South Africa

4.5.1 Introduction

The world is moving towards global unification of vertical datums to modernise the vertical positioning technique. The modernization of a vertical datum is necessary to make it compatible with space-based instruments, and it would be easily accessible anywhere across the country. The horizontal positioning is already realised on the ITRF with high precision, and a similar approach for the realisation of a new vertical datum for South Africa is required. In order for this to be achieved in this study, the relationship between the LLD and IHRS has been determined for appropriate adjustment, in the previous section.

The vertical datum offset between the LLD and IHRS was modelled, using equation (2:39), on the four fundamental benchmarks. The estimated linear offset between the four TGBMs on LLD and the global vertical datum are; 5.973, -20.647, -26.518, and 21.496 cm for Cape Town, Port Elizabeth, East London and Durban, respectively. Due to the inconsistencies on the establishment of the LLD (as mentioned in sub-section 4.4), it will impose similar strain during the unification of the LLD to the IHRS. However, for the purpose of unification, it is proposed in this study that TGBM at Cape Town to be held fixed. An attempt is also made to unify the LLD to the IHRS over South Africa using few benchmarks observed around each of the four TGBMs.

4.5.2 Data and methods

A preliminary unification of the LLD to the IHRS over South Africa was conducted using the 138 GPS/levelling data points (Figure 4-18). They were collected randomly across the country by the Chief Directorate: National Geo-spatial Information (CD:NGI) of South Africa. However, for the purpose of unifying LLD to the IHRS, the linear offset at each TGBM should be applied to all benchmarks connected to it. It was impractical to identify exactly which data points are associated with a particular TGBM.

Therefore, for the purpose of this analysis, the TGBM in Cape Town was held fixed, as it was mentioned on the Technical Publication Report 17 that the Cape Town TGBM was given more weight than the other TGBMs. It should be noted that the South African vertical datum (LLD) is not based only on the TGBM in Cape Town, this was conducted as an analysis during the process of developing the LLD (as an attempt to fundamentally fix LLD to the Cape Town TGBM only).

A further preliminary unification of the LLD to the IHRS over South Africa was conducted on the four TGBMs using few benchmarks which were observed around each TGBM. The GPS/levelling data points around each TGBM were also collected by the CD:NGI as part of an attempt to readjust the LLD. Local-scaled precise levelling networks were conducted around each fundamental benchmark for this purpose. The distribution of the benchmarks around each TGBM is depicted in Figure 4-25 to Figure 4-28.

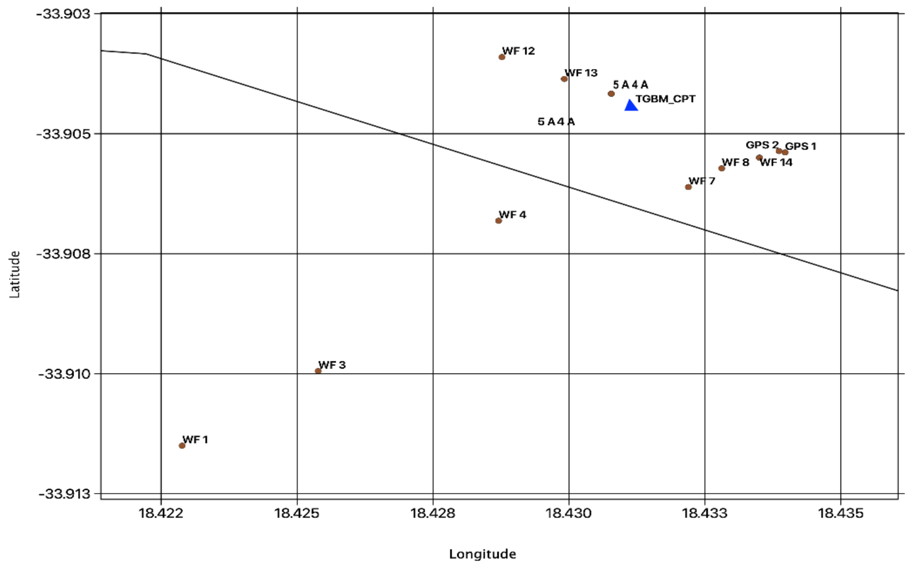

Figure 4-25: Distribution of benchmarks around TGBM_CPT

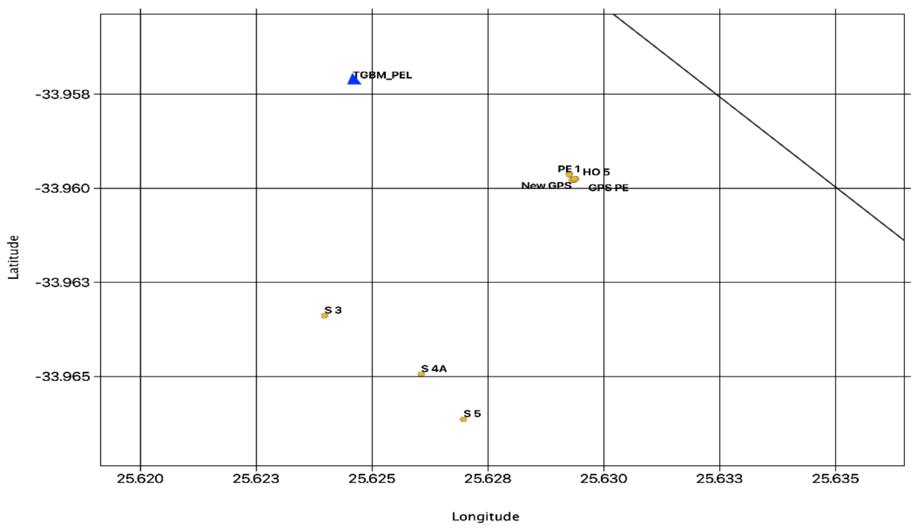

Figure 4-26: Distribution of benchmarks around TGBM_PEL

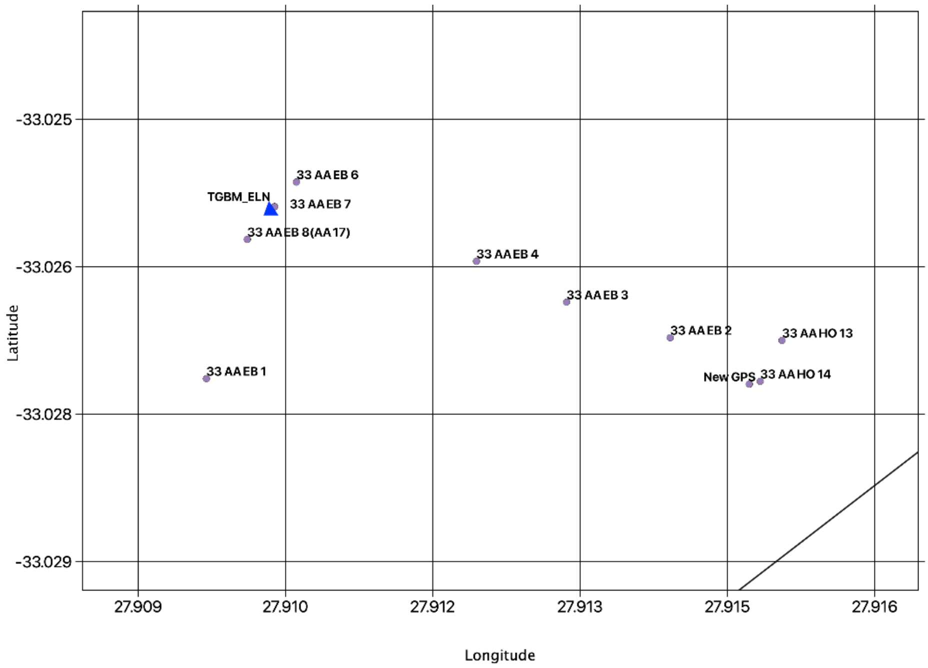

Figure 4-27: Distribution of benchmarks around TGBM_ELN

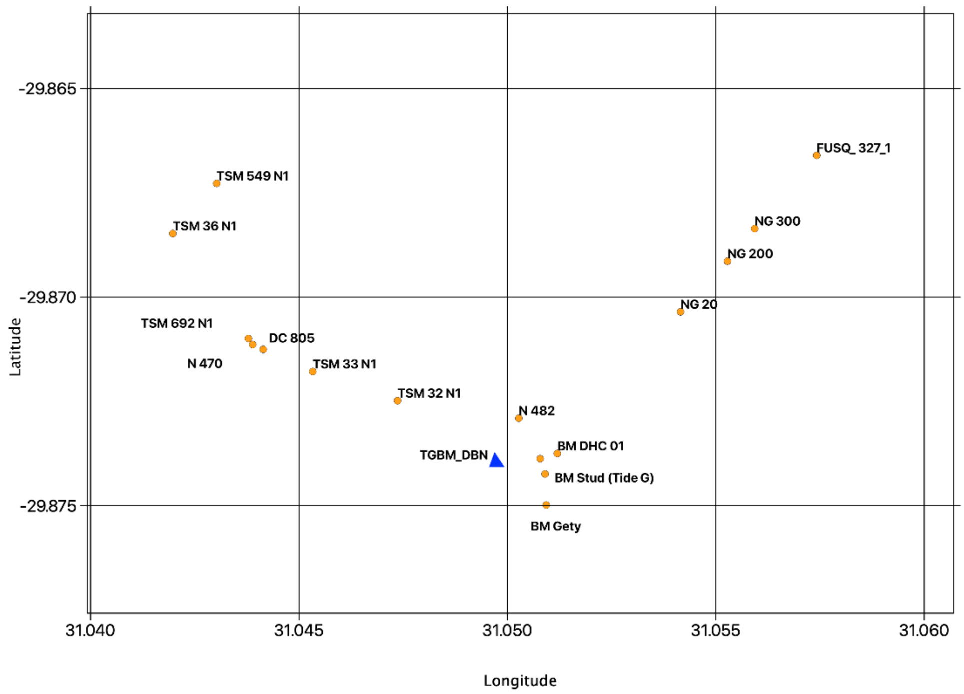

Figure 4-28: Distribution of benchmarks around TGBM_DBN

An analysis of variance (ANOVA) was conducted to determine the amount of variability within data used to estimate the datum offsets on both the quasigeoid and geoid reference surfaces. The statistical F-test is used as it is an omnibus test, meaning that it tests for an overall difference among means.

4.5.3 Results and discussion

The possibility of unifying LLD to the global geoid-based vertical datum is investigated using 138 GPS/levelling data points over South Africa. The linear offsets from the TGBM in Cape Town between LLD and the global vertical datum, local quasigeoid and the global vertical datum, and local geoid and the global vertical datum are $5.973, 40.573,$ and 38.574 cm, respectively. They were applied to all the spheroidal orthometric (H^S), normal (H^N), and orthometric (H^O) heights of the 138 data points.

The linear offsets were computed using the approach described in the previous section. This was conducted for unification of LLD to the global vertical datum. Moreover, the datum offset analysis was conducted to determine how well the adjusted spheroidal orthometric (H^{SIHRS}), normal (H^{NIHRS}), and orthometric (H^{OIHRS}) heights based on the IHRS fit with the existing quasigeoid and geoid model over South Africa. The descriptive statistics of this analysis are given in Table 4.12 and Table 4.13.

Table 4.12: *Quasigeoid offset on the 138 GPS/levelling stations based on IHRS over South Africa*

Datum offset	Min.(m)	Max.(m)	Mean(m)	Standard dev.(m)
$\zeta - (h - H^{SIHRS})$	-0.481	0.587	0.158	0.238
$\zeta - (h - H^{NIHRS})$	-0.818	0.376	0.020	0.233
$\zeta - (h - H^{OIHRS})$	-0.798	0.281	-0.160	0.242

Table 4.13: *Geoid offset on the 138 GPS/levelling stations based on IHRS over South Africa*

Datum offset	Min. (m)	Max. (m)	Mean (m)	Standard dev. (m)
$\mathrm{N} - (h - H^{SIHRS})$	-0.481	0.702	0.357	0.243
$N - (h - H^{NIHRS})$	-0.817	1.096	0.220	0.360
$N - (h - H^{OIHRS})$	-0.798	0.396	0.040	0.250

The statistic results indicate that the datum offsets on the 138 GSP stations based on IHRS when compared to the local quasigeoid and geoid models before fitting produced similar conclusion as attained in the previous section. The existing bias (mean) on the 138 GPS station between the adjusted height systems ($H^{SIHRS}, H^{NIHRS}, and\ H^{OIHRS}$) and the reference surfaces (quasigeoid and geoid model) has improved (see mean values in Tables 4.6, 4.7, 4.12 and 4.13).

The normal and orthometric height systems indicated a great improvement mainly when compared with their corresponding reference surface, with an average of 2.0 and 4.0 cm on the quasigeoid and geoid surfaces, respectively (Tables 4.12 and 4.13). The spheroidal height system produced a smaller mean bias when compared with the quasigeoid than when compared with the geoid, 15.8 and 35.7 cm, respectively. However, the standard deviation remained the same as expected since it is the measure of goodness of fit. Moreover, an ANOVA test was conducted to determine the amount of variability within data, the results are given in Table 4.14 and Table 4.15.

Table 4.14: ANOVA results from quasigeoid offsets

Anova: Single Factor

SUMMARY

Groups	***Count***	***Sum***	***Average***	***Variance***
$\zeta - (h - H^{SIHRS})$	138	21.598	0.158	0.057
$\zeta - (h - H^{NIHRS})$	138	2.758	0.020	0.055
$\zeta - (h - H^{OIHRS})$	138	-21.875	-0.160	0.059

ANOVA

Source of Variation	***SS***	***df***	***MS***	***F***	***P-value***	***F crit***
Between Groups	6.938	2	3.469	59.965	0.000	3.018
Within Groups	23.604	411	0.058			
Total	30.542	413				

Table 4.15: ANOVA results from geoid offsets

ANOVA: Single Factor

SUMMARY

Groups	*Count*	*Sum*	*Average*	*Variance*
$\mathbf{N}-\left(\boldsymbol{h}-\boldsymbol{H}^{SIHRS}\right)$	138	48.971	0.357	0.059
$\boldsymbol{N}-\left(\boldsymbol{h}-\boldsymbol{H}^{NIHRS}\right)$	138	30.132	0.220	0.130
$\boldsymbol{N}-\left(\boldsymbol{h}-\boldsymbol{H}^{OIHRS}\right)$	138	5.498	0.040	0.062

ANOVA

Source of Variation	*SS*	*df*	*MS*	*F*	*P-value*	*F crit*
Between Groups	6.938	2	3.469	41.403	0.000	3.018
Within Groups	34.187	411	0.084			
Total	41.125	413				

where:

- Sum – sum of all variables in each group,
- SS – sum of squares
- df – degree of freedom
- MS – mean sums of squares
- F - F ratio is simply a ratio of the mean sums of squares due to between-group differences to the mean sums of squares due to within-group differences.
- P-value - The level of significance of the F value
- F crit - The value needed to reject the null hypothesis

The ANOVA output values from both the quasigeoid and geoid offsets are $F_{(2.411)} = 59.965$ and $F_{(2.411)} = 41.403$, respectively. The critical value at the 0.05 level of significance (also known as the significance level or type I error) with 2 degrees of freedom in the numerator and 411 degrees of freedom in the denominator, for both the quasigeoid and geoid offsets is the same (3.018). The F-value for geoid offsets (41.403) is smaller than the F-value for the quasigeoid offsets (59.965), this denotes that the group means cluster together more tightly than the within-group variability. The distance between the means is small relative to the random error within each group.

This means that the value needed for rejection of the null hypothesis is 3.018. The null hypothesis in this case, states that there is no difference among the means of the three different groups ($H_o\colon \mu_1 =$

$\mu_2 = \mu_3$). The F-test checks for an overall difference among groups. The obtained F values for both the quasigeoid and geoid offset are extremely larger than the critical value. This means that the null hypothesis cannot be accepted. Nonetheless, their P-values are approximately equal to zero, this implies that there is a significant difference among the three sets of datum offsets from both the quasigeoid and geoid data group.

A further datum offset analysis was carried out on the four TGBMs using few benchmarks which were observed around each TGBM (Figure 4-25 to Figure 4-28). The linear offsets at each TGBM, between each height system and the global vertical datum are given in Table 4.16. The relationship between linear offsets and the TGBM latitude is depicted in Figure 4-29.

Table 4.16: Linear offsets at each TGBM

TGBM	Linear Offsets (m)		
	LLD	**Quasigeoid**	**Geoid**
CPT	0.060	0.406	0.386
PEL	-0.206	0.219	0.199
ELN	-0.265	0.064	0.044
DBN	0.215	-0.068	-0.088

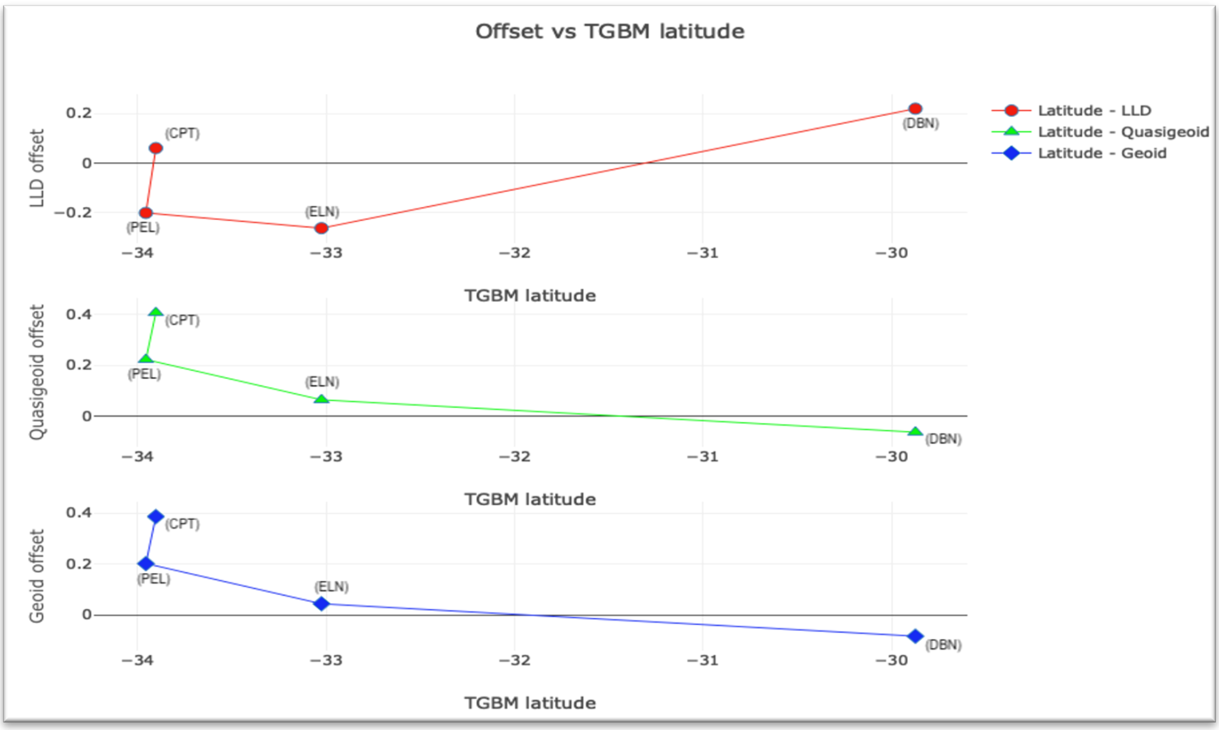

Figure 4-29: Relationship between the linear offsets and latitudes of the TGBMs (units are in m)

The linear offsets from the quasigeoid and geoid maintained a similar trend, as indicated in Figure 4-29. However, the linear offset from LLD differed. It should be noted that the LLD is not well defined theoretically and practically. While the quasigeoid and geoid reference surfaces are well defined both in theory and practice. The linear offsets, as given in Table 4.16 were applied accordingly to transform the H^S, H^N and H^O of the benchmarks around each TGBM to the IHRS. The descriptive statistics of this analysis is given in Table 4.17 to Table 4.30.

Table 4.17: Quasigeoid offset on the benchmarks around the TGBM_CPT

Datum offset	Min.(m)	Max.(m)	Mean(m)	Standard dev.(m)
$\zeta - (h - H^{SIHRS})$	0.287	0.288	0.287	0.000
$\zeta - (h - H^{NIHRS})$	-0.059	-0.058	-0.059	0.000
$\zeta - (h - H^{OIHRS})$	-0.039	-0.038	-0.039	0.000

Table 4.18: Geoid offset on the benchmarks around the TGBM_CPT

Datum offset	Min. (m)	Max. (m)	Mean (m)	Standard dev. (m)
$N - (h - H^{SIHRS})$	0.279	0.282	0.281	0.001
$N - (h - H^{NIHRS})$	-0.067	-0.064	-0.065	0.001
$N - (h - H^{OIHRS})$	-0.047	-0.044	-0.045	0.001

An ANOVA was conducted to determine the amount of variability within data around the TGBM_CPT. The ANOVA results are as given in Table 4.19 and Table 4.20.

Table 4.19: ANOVA results from quasigeoid offsets around the TGBM_CPT

Anova: Single Factor

SUMMARY

Groups	*Count*	*Sum*	*Average*	*Variance*
$\zeta - (h - H^{SIHRS})$	13	3.732	0.287	0.000
$\zeta - (h - H^{NIHRS})$	13	-0.766	-0.059	0.000
$\zeta - (h - H^{OIHRS})$	13	-0.506	-0.039	0.000

ANOVA

Source of Variation	*SS*	*df*	*MS*	*F*	*P-value*	*F crit*
Between Groups	0.981	2	0.491	2550683.2	0.000	3.259
Within Groups	0.000	36	0.000			
Total	0.981	38				

Table 4.20: ANOVA results from geoid offsets around the TGBM_CPT

Anova: Single Factor

SUMMARY

Groups	*Count*	*Sum*	*Average*	*Variance*
$\mathrm{N} - (h - H^{SIHRS})$	13	3.648	0.281	0.000
$N - (h - H^{NIHRS})$	13	-0.850	-0.065	0.000
$N - (h - H^{OIHRS})$	13	-0.590	-0.045	0.000

ANOVA

Source of Variation	*SS*	*df*	*MS*	*F*	*P-value*	*F crit*
Between Groups	0.981	2	0.491	411400.5	0.000	3.259
Within Groups	0.000	36	0.000			
Total	0.981	38				

Table 4.21: Quasigeoid offset on the benchmarks around the TGBM_PEL

Datum offset	Min.(m)	Max.(m)	Mean(m)	Standard dev.(m)
$\zeta - (h - H^{SIHRS})$	0.627	0.628	0.628	0.001
$\zeta - (h - H^{Nihrs})$	0.202	0.203	0.203	0.001
$\zeta - (h - H^{Oihrs})$	0.222	0.223	0.223	0.001

Table 4.22: *Geoid offset on the benchmarks around the TGBM_PEL*

Datum offset	Min. (m)	Max. (m)	Mean (m)	Standard dev. (m)
$\mathrm{N} - (h - H^{SIHRS})$	0.707	0.709	0.708	0.001
$N - (h - H^{NIHRS})$	0.282	0.284	0.283	0.001
$N - (h - H^{OIHRS})$	0.302	0.304	0.253	0.001

An ANOVA was conducted to determine the amount of variability within data around the TGBM_PEL. The ANOVA results are given in Table 4.23 and Table 4.24.

Table 4.23: ANOVA results from quasigeoid offsets around the TGBM_PEL

Anova: Single Factor

SUMMARY

Groups	*Count*	*Sum*	*Average*	*Variance*
$\zeta - (h - H^{SIHRS})$	11	6.906	0.628	0.000
$\zeta - (h - H^{NIHRS})$	11	2.231	0.203	0.000
$\zeta - (h - H^{OIHRS})$	11	2.451	0.223	0.000

ANOVA

Source of Variation	*SS*	*df*	*MS*	*F*	*P-value*	*F crit*
Between Groups	1.265	2	0.633	2485181.5	0.000	3.316
Within Groups	0.000	30	0.000			
Total	1.265	32				

Table 4.24: ANOVA results from geoid offsets around the TGBM_PEL

Anova: Single Factor

SUMMARY

Groups	*Count*	*Sum*	*Average*	*Variance*
$N-(h-H^{SIHRS})$	11	7.786	0.708	0.000
$N-(h-H^{NIHRS})$	11	3.111	0.283	0.000
$N-(h-H^{OIHRS})$	11	3.331	0.253	0.000

ANOVA

Source of Variation	*SS*	*df*	*MS*	*F*	*P-value*	*F crit*
Between Groups	1.265	2	0.633	1391701.7	0.000	3.316
Within Groups	0.000	30	0.000			
Total	1.265	32				

Table 4.25: Quasigeoid offset on the benchmarks around the TGBM_ELN

Datum offset	Min.(m)	Max.(m)	Mean(m)	Standard dev.(m)
$\zeta-(h-H^{SIHRS})$	0.594	0.594	0.594	0.000
$\zeta-(h-H^{Nihrs})$	0.265	0.265	0.265	0.000
$\zeta-(h-H^{Oihrs})$	0.285	0.285	0.285	0.000

Table 4.26: Geoid offset on the benchmarks around the TGBM_ELN

Datum offset	Min. (m)	Max. (m)	Mean (m)	Standard dev. (m)
$N-(h-H^{SIHRS})$	0.537	0.538	0.537	0.001
$N-(h-H^{NIHRS})$	0.208	0.209	0.208	0.001
$N-(h-H^{OIHRS})$	0.228	0.229	0.228	0.001

An ANOVA was conducted to determine the amount of variability within data around the TGBM_ELN. The ANOVA results are given in Table 4.27 and Table 4.28.

Table 4.27: ANOVA results from quasigeoid offsets around the TGBM_ELN

Anova: Single Factor

SUMMARY

Groups	*Count*	*Sum*	*Average*	*Variance*
$\zeta - (h - H^{SIHRS})$	11	6.532	0.594	0.000
$\zeta - (h - H^{NIHRS})$	11	2.913	0.265	0.000
$\zeta - (h - H^{OIHRS})$	11	3.133	0.285	0.000

ANOVA

Source of Variation	*SS*	*df*	*MS*	*F*	*P-value*	*F crit*
Between Groups	0.748	2	0.374	2.341E+28	0.000	3.316
Within Groups	0.000	30	0.000			
Total	0.748	32				

Table 4.28: ANOVA results from geoid offsets around the TGBM_ELN

Anova: Single Factor

SUMMARY

Groups	*Count*	*Sum*	*Average*	*Variance*
$N - (h - H^{SIHRS})$	11	5.910	0.537	0.000
$N - (h - H^{NIHRS})$	11	2.291	0.208	0.000
$N - (h - H^{OIHRS})$	11	2.511	0.228	0.000

ANOVA

Source of Variation	*SS*	*df*	*MS*	*F*	*P-value*	*F crit*
Between Groups	0.748	2	0.374	1372153.4	0.000	3.316
Within Groups	0.000	30	0.000			
Total	0.748	32				

Table 4.29: Quasigeoid offset on the benchmarks around the TGBM_DBN

Datum offset	Min.(m)	Max.(m)	Mean(m)	Standard dev.(m)
$\zeta - (h - H^{SIHRS})$	-0.503	-0.501	-0.503	0.001
$\zeta - (h - H^{Nihrs})$	-0.220	-0.218	-0.220	0.001
$\zeta - (h - H^{Oihrs})$	-0.200	-0.198	-0.200	0.001

Table 4.30: *Geoid offset on the benchmarks around the TGBM_DBN*

Datum offset	Min. (m)	Max. (m)	Mean (m)	Standard dev. (m)
$\mathrm{N} - (h - H^{SIHRS})$	-0.202	-0.200	-0.201	0.001
$N - (h - H^{NIHRS})$	0.081	0.083	0.082	0.000
$N - (h - H^{OIHRS})$	0.101	0.103	0.102	0.000

An ANOVA was conducted to determine the amount of variability within data around the TGBM_DBN. The ANOVA results are given in Table 4.31 and Table 4.32.

Table 4.31: ANOVA results from quasigeoid offsets around the TGBM_DBN

Anova: Single Factor

SUMMARY

Groups	*Count*	*Sum*	*Average*	*Variance*
$\zeta - (h - H^{SIHRS})$	16	-8.042	-0.503	0.000
$\zeta - (h - H^{NIHRS})$	16	-3.514	-0.220	0.000
$\zeta - (h - H^{OIHRS})$	16	-3.194	-0.200	0.000

ANOVA

Source of Variation	*SS*	*df*	*MS*	*F*	*P-value*	*F crit*
Between Groups	0.919	2	0.459	926644.7	0.000	3.204
Within Groups	0.000	45	0.000			
Total	0.919	47				

Table 4.32: ANOVA results from geoid offsets around the TGBM_DBN

Anova: Single Factor						
SUMMARY						
Groups	***Count***	***Sum***	***Average***	***Variance***		
$\mathbf{N}-\left(\boldsymbol{h}-\boldsymbol{H}^{\boldsymbol{SIHRS}}\right)$	16	-3.217	-0.201	0.000		
$\boldsymbol{N}-\left(\boldsymbol{h}-\boldsymbol{H}^{\boldsymbol{NIHRS}}\right)$	16	1.311	0.082	0.000		
$\boldsymbol{N}-\left(\boldsymbol{h}-\boldsymbol{H}^{\boldsymbol{OIHRS}}\right)$	16	1.631	0.102	0.000		
ANOVA						
Source of Variation	***SS***	***df***	***MS***	***F***	***P-value***	***F crit***
Between Groups	0.919	2	0.459	1837845.3	0.000	3.204
Within Groups	0.000	45	0.000			
Total	0.919	47				

The ANOVA outputs provided in Table 4.19, Table 4.20, Table 4.23, Table 4.24, Table 4.27, Table 4.28, Table 4.31, and Table 4.32, from both the quasigeoid and geoid linear offsets indicate that their null hypothesis cannot be accepted on the basis that their obtained *F* value are extremely larger than their critical value at the 0.05 level. Moreover, their *P*-values are approximately equal to zero, meaning that there is a significant difference among the three sets of linear offsets from both the quasigeoid and geoid on all the TGBMs.

The datum offset from the benchmarks around the TGBM in Cape Town provided a smaller offset. The orthometric height gave a smaller bias offset when compared to the quasigeoid than the normal height, as given in Table 4.17, this is unusual as the quasigeoid model is a reference surface for normal height system. Nonetheless, the orthometric height gave a smaller bias offset when compared to the geoid model, as given Table 4.18, this was expected as the geoid model is a reference surface for orthometric height system.

The datum offsets trend is assessed using the mean values of the quasigeoid and geoid datum offsets from the benchmarks around each of the four TGBMs, this is illustrated in Figure 4-30 and Figure 4-31.

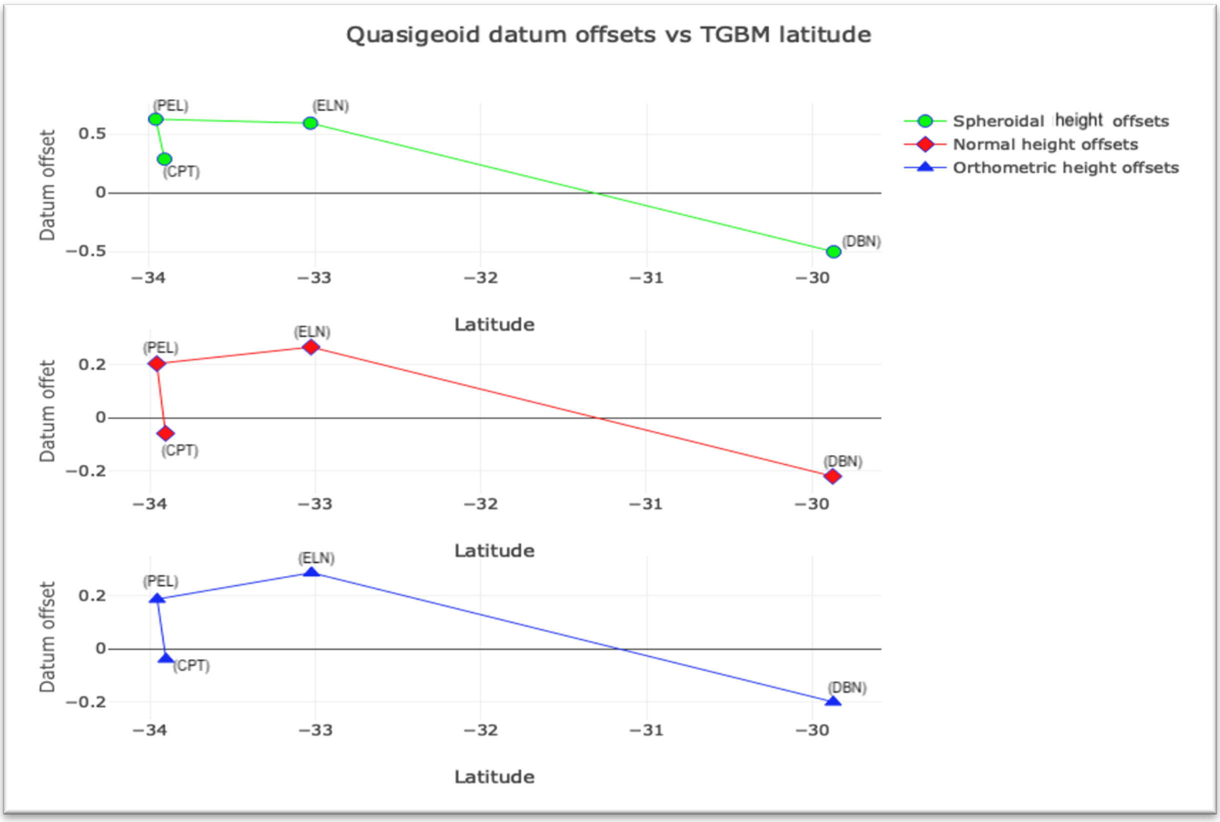

Figure 4-30: Quasigeoid datum offset trends from benchmarks around each of the four TGBMs (in m)

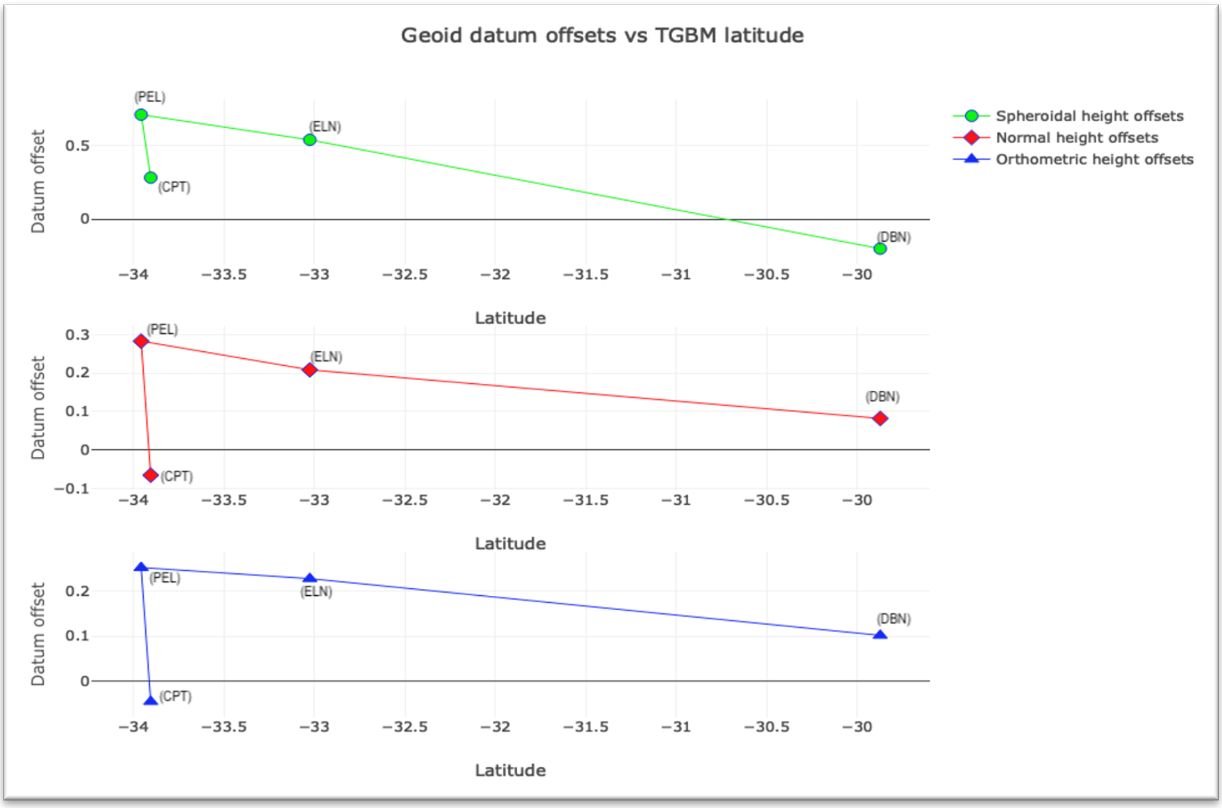

Figure 4-31: Geoid datum offset trends from the benchmarks around each of the four TGBMs (in m)

The quasigeoid and geoid datum offset on the spheroidal height has maintained a similar trend. However, on the normal and orthometric height, there is a major change on the datum offsets, major change emerge on the TGBM in Port Elizabeth and Durban. The datum offsets on the TGBM in Cape Town has maintained a similar trend on both the quasigeoid and geoid datum offsets. This indicates that the behaviour of the observed data in Cape Town is stable.

5 PROPOSED FRAMEWORK FOR THE ESTABLISHMENT OF A GEOID-BASED VERTICAL GEODETIC DATUM IN SOUTH AFRICA

5.1 Introduction

There are some considerations to be taken surrounding the implementation and adoption of a geoid consistent vertical datum in South Africa. The primary factors to be considered include data quality and availability, politics, and difficulties that maybe encountered by the stakeholders. Implementation and adoption of a geoid based vertical datum will bring about a different approach towards height determination.

The level of accuracy of the existing local vertical datum (i.e. LLD) is not reliable as the horizontal reference system, realised from the ITRS. The main objective of this study is to contribute towards the mandate by International Association of Geodesy (IAG) to refer all existing local vertical datums to one global vertical datum.

The IAG has adopted a conventional geopotential value (W_0) for global vertical datum, which would be related to the local vertical datums geopotential values, this would provide precise coordinates on the ITRS and heights defined by potential quantities for South Africa. In principle unification of height, requires determination of the adjustment parameters or datum offsets between existing vertical datums, each of which is defined with a fundamental surface of zero elevation. The computation of the gravity potential on the local vertical datum (W_P) can be carried out using the methodology based on Molodensky theory.

It has been outlined in the studies conducted on the previous sections, that the South African vertical datum is unstable and inconsistent. Moreover, a hybrid geoid model was developed in South Africa, SAGEOID10 (Chandler & Merry, 2010), for the purpose of integrating observations from space-based instruments with the LLD. However, the existing inconsistencies have not been handled or resolved.

A reliable reference frame is required for consistent analysis, and a combination of physical and geometric heights in order to explore the advantages of satellite geodesy in South Africa. To also develop a vertical datum to a defined accuracy that enables the generation of rigorously defined heights from ellipsoidal heights.

5.2 Prospects for a geoid-based vertical geodetic datum in South Africa and related challenges

The existing limitations on the current South African vertical datum were discussed in chapter 3. The main objective of this study is to provide contribution towards developing a geoid consistent vertical datum for South Africa in relation to the IHRS. The primary component of any vertical reference system is the fundamental surface of zero elevation. The vertical datum offset between the LLD and global vertical datum was estimated using a single-point based GBVP approach, the vertical datum offset was estimated on the four fundamental benchmarks (presented in sub-section 4.3).

A preliminary investigation of unifying LLD to the global vertical datum in South Africa was conducted (presented in sub-section 4.4), for the sake of successful implementation of a geoid consistent vertical datum. The LLD was established from a set of levelling networks that were adjusted independently and there is no further information indicating benchmarks or regions of the levelling networks that are associated with a particular TGBM. Therefore, for the purpose of this assessment, the TGBM in Cape Town is held fixed as it was given more weight than the other TGBMs during the establishment of the LLD.

It has been determined in this study that the LLD provides a best fit with the quasigeoid model than the geoid model. Therefore, a quasigeoid model was used to estimate gravity potential values at each TGBM in relation to the IHRS (presented in sub-section 4.3). A further investigation of the geometrical relationship between the LLD and quasigeoid based vertical datum was conducted, refer to sub-section 4.4. The preliminary adjustment of the LLD to the IHRS was accomplished in the following order:

- The vertical datum offsets in relation to the global equipotential surface (W_0) at each TGBM was estimated. A single-point-based geodetic boundary value problem (GBVP) approach was used following Molodensky theory for estimating the height anomalies from the disturbing potential using Bruns's formula.
- The height anomalies (gravimetric quasigeoid heights) for each TGBM station were computed from the global geopotential model (GOCE, TIM6), and set of gravity data (marine and land gravity data). The residual gravity anomalies were converted into residual height anomalies, using 2D Fourier transform with a spherical approximation of the RTM terrain correction integration.
- The datum offset at the TGBM in Cape Town was used to unify the LLD within a global vertical reference system. Only the TGBM in Cape Town was held fixed, since it was given more weight than the other three TGBMs (East London, Port Elizabeth, and Durban) during the establishment of the LLD.

In order for this adjustment procedure to be implemented successfully, the LLD should be uniformed or if it is feasible the regional shifts can be applied to all the benchmarks associated with particular TGBM as presented in sub-section 4.5. The vertical datum offset between LLD and the global vertical datum can be determined more accurately in future with the availability of more adequate data.

The upcoming international satellite gravity missions, such as the concepts based on laser ranging between low-flying pairs of satellites that offer the most improved solution for measuring the time-variable gravity field, combined with theoretical progress in geoid modelling, will contribute in improving the accuracy of the South African geoid model.

The advantage of using the geoid defining the global mean sea level is the direct compatibility with a global standard, to be able to integrate the geoid undulations with ellipsoidal heights obtained from space-based instruments. It will also provide the determination of the absolute sea surface topography from satellite radar altimetry. The global vertical datum provides reference frame with long-term stability and homogeneous consistency worldwide. Moreover, very small changes such as the sea level rise/fall can be detected (no matter how small it is) if a globally stable reference frame is established. This height reference system will contribute greatly to the study of the Earth system, such as, mapping, monitoring, determining and understanding changes in the Earth's mass distribution, rotation, and shape.

Adopting a global vertical reference system for vertical positioning will assist the South African economy to become more efficient and competitive, and contribute greatly towards economic activities and applications, such as engineering, improve scientific research and mapping. The South African horizontal positioning is already realized by the International Terrestrial Reference Frame, this provide South Africa with a unified global horizontal datum with higher level of accuracy, however, the existing vertical datum is not reliable at this level of accuracy. Moreover, attempting to readjust the primary levelling network would be a tedious task and it is not advisable, because these data are not digitally available – they are kept on paper-based records at the National Geo-Spatial Information (NGI).

There are a number of challenges associated with adopting a modernised vertical datum for South Africa. The availability of adequate gravity data and budget (cost associated with acquiring gravity data) are the most contributing technical challenges in realising a new height system for South Africa. The available gravity data is not that much accurate as their horizontal and vertical position are not linked to the TrigNet system and they were approximated from the classical topographical maps based on the Cape datum. This pose a negative impact on the accuracy of the geoid model for South Africa. However, there are a number of existing satellite missions dedicated to provide measurements of the Earth gravity field and global gravity data.

Accuracy of the position (horizontal and vertical position) can be easily affected if there is any signal interference to the space-based instrument (can be electromagnetic interference and even radio signal). Long-occupation GPS technique (static GPS survey) might be recommended on other areas, to avoid the effect of accuracy from multipath and satellite receiver geometry. The perceived challenge also include the legal traceability of heights determined by a different technique. To minimise confusion, proper digital documentation should be prepared to maintain traceability and avoid fragmentation. As in every transition, there would be a resistance stage as most geodetic datum stakeholders and users are already accustomed to the current mean sea level based vertical datum in South Africa.

Therefore, a stakeholder consultation has to be conducted, to develop necessary recommendation to be suitable for stakeholders. Stakeholder consultation can be done by gathering their insights, perceptions, and awareness of barriers; this can be conducted through questionnaire-led interviews. Moreover, implementation policies and guidelines would need to be set to have the desired effect, and review any legislation that involves heights.

A geoid-based vertical datum can be refined more frequently than the levelling-based vertical datum overtime as new data and improved computation technique become available. To refine a geoid-based vertical datum requires less effort than refining a levelling-based vertical datum, for instance the LLD has never been readjusted or refined since it was established. Geoid-based vertical datum should be refined reasonably frequently over time to allow the required temporal stability by the stakeholders and to avoid confusions.

If a geoid-based vertical datum is adopted for South Africa, the historical perspective of the LLD should be considered, there is high possibility that it might be neglected. It will play a huge role in verifying future geoid models, however, the possibility of using other options should be investigated. It is recommended that NGI should continue with the physical maintenance and monitoring of the national benchmarks (i.e. trig-beacons and town survey marks), to ensure datum stability and observe the data required to derive the mathematical transformation for users.

Politics in South Africa plays a big role, it would be required to ensure that this is done competently, within clear policy frameworks, and following procedures that protect the public from attempts to make personal gains. Involvement of international relations department may be required to collect gravity data beyond national boundaries, this data is required to avoid edge effect during geoid modelling.

6 CONCLUSION AND FUTURE WORK

6.1 Conclusion

The current South African vertical datum is constrained to four tide gauge stations, it is inconsistent and unstable. It is defined by an outdated sea level data, which was measured approximately a century ago. The first order levelling networks were adjusted in a piece meal fashion. A vertical datum defined by MSL data have a limited life-span and it requires to be replaced or upgraded after a certain period. However, to readjust it would be a tedious task and relatively expensive for the country, therefore it is not advisable. The current South African vertical datum is not compatible with space-based instruments.

A modernised vertical datum for South Africa is required to update the existing vertical datum and to modernise the vertical positioning technique as it is not advance as that of modern space-based horizontal positioning. The horizontal positioning is already realised on the ITRF with high precision, and a similar approach for the realisation of a new vertical datum for South Africa is required. One of the main objectives of the IAG is to promote the implementation of an integrated global geodetic reference frame that provides a reliable frame for consistent analysis and modelling of global phenomena and processes affecting the Earth's gravity field, the Earth's surface geometry and the Earth's rotation.

In order for South Africa to meet the standards of the global vertical datum, for the purpose of unification of vertical datums, a precise gravimetric geoid model is required, a single-point-based GBVP approach was used for this study. This approach provided South Africa with the datum offset and the geopotential values on the four fundamental benchmarks in relation to the global vertical datum, this is presented in sub-section 4.3.

National height systems are generally developed from an assumption that the geoid and mean sea level coincide at one or more tide gauge stations, and tidal data is used as a datum. However, the mean sea level is subjected to both temporal and spatial variability, this affect the stability of the vertical datum of this nature. The sea level rise along the Southern African coastal are estimated to be rising in order of approximately 0.42, 1.57 and 3.4 mm/year on the western, southern and eastern coastal regions, respectively. This indicates a great need for a stable and consistent vertical datum for Southern Africa. An investigation was conducted in sub-section 4.2, to determine the magnitude of misclosures and the empirical value for the first order levelling network on the levelling loops. It was discovered that almost all the levelling loops fall within the acceptable range of misclosure for first order levelling network, except for levelling loop iv, with a misclosure from spirit levelling measurement of -10.2 cm while the estimated acceptable misclosure is 9.7 cm. Although, the levelling loop xiii was the most accurate, with the misclosure of -2.8 cm from spirit levelling measurements and the estimated acceptable misclosure is 10.2 cm. Furthermore, it was determined from an investigation using empirical value (c) for the first order levelling network that loops vii and xiii (c = 0.001) are more accurate than loops viii and ix (c = 0.002) followed by loops iv and xii (c = 0.003). This analysis provides a better measure for detecting erroneous levelling loops, as it accounts for the levelling distance.

A comparison between various height systems (orthometric, normal and spheroidal orthometric) was conducted over South Africa at 141 GPS/levelling stations (sub-section 4.1). In order to determine the height that is closer to the spheroidal orthometric height system. The magnitudes of the orthometric-normal separation in South Africa are relatively smaller; however, the South African spheroidal orthometric height system is more consistent with the normal height system than the orthometric height system. The estimated magnitude of the orthometric-normal separation on average is 19.4 cm over South Africa, with a standard deviation of ±17.6 cm. The separation between the normal height system and the South African spheroidal orthometric height system is approximately -21.3 cm on average, with a standard deviation of ±23.8 cm. Moreover, the separation between the orthometric height system and the spheroidal orthometric height system is estimated as -40.7 cm on average, with a standard deviation of ±25.3 cm. The role of the correlation between the spheroidal, orthometric and normal height systems will need to be revaluated, using gravity data linked with accurate horizontal and vertical position for the purpose of redefining South African vertical datum and better assessment.

The vertical datum offset on the South African vertical datum in relation to the global vertical datum, has been estimated using a single-point-based GBVP solution at four TGBMs. The gravity data on a $4^0 \times 4^0$ grid around each fundamental benchmark was selected, for the purpose of estimating their disturbing potential, this was performed in combination with the spherical harmonics coefficients from the GOCE based GGM, TIM-R6 (truncated at degree and order 200). The results (sub-section 4.3) suggest that the gravity potential at each TGBM in South Africa deviates from the potential of the global reference surface by $0.585, -2.023, -2.597$ and $2.105\ \mathrm{m^2s^{-2}}$ for Cape Town, Port Elizabeth, East London and Durban, respectively. The corresponding vertical datum offset between the international height reference system and the four fundamental benchmarks over South Africa are 5.973, -20.647, -26.518, and 21.496 cm for Cape Town, Port Elizabeth, East London and Durban, respectively. This evaluation provides us with direct link to the IHRS, and a positive step towards the South African vertical datum realisation and unification in a manner which is consistent with the IHRS.

An analysis was conducted in sub-section 4.4 to determine a consistent height system with a corresponding reference surface for South Africa. This assessment was carried out using a vertical datum offset computed on the 138 GPS/levelling data points distributed over South Africa. Based on the statistical measures, it is concluded that the South African local height system is more compatible with the quasigeoid surface, than the geoid surface. A least squares adjustment technique was used for estimation of an absolute accuracy of the quasigeoid and geoid based on different height systems. The normal and orthometric height systems provided a best fit when compared with the quasigeoid and geoid model, with standards deviations of ± 21.8 and ± 24.1 cm, respectively. This was expected since they were compared to their corresponding reference surfaces.

However, the spheroidal height provided a best fit with the quasigeoid than the geoid model, with a standard deviation of ± 22.2 cm. This evaluation was further carried out using a cross validation approach. The normal and orthometric height provided a best fit when compared with the quasigeoid and geoid model, with a standards deviation of ± 5.1 cm and ± 3.9 cm, respectively. Moreover, the spheroidal height provided a best fit with the quasigeoid than the geoid model, with a standard deviation of ±6.3 cm. This means that the LLD is more compatible with the quasigeoid than the geoid model.

The possibility of unification of the South African vertical datum within geopotential space has been investigated in sub-section 4.5, the TGBM in Cape Town was held fix for this analysis. An average vertical datum offset between the LLD and global vertical datum was estimated. The linear offsets from the TGBM in Cape Town are, 5.973, 40.573, and 38.574 cm, between the spheroidal orthometric (H^S) and the global vertical datum, the normal (H^N) and the global vertical datum, and orthometric (H^O) and the global vertical datum, respectively.

The bias between the adjusted height systems ($H^{SIHRS}, H^{NIHRS}, and\ H^{OIHRS}$) and the reference surfaces (quasigeoid and geoid) at 138 GPS/levelling stations improved with the same amount as their linear offsets. The normal and orthometric height systems indicated a great improvement mainly when compared with their corresponding reference surfaces (quasigeoid and geoid), with an average of 2.0 and 4.0 cm, respectively. The spheroidal height system produced a smaller bias when compared with the quasigeoid (15.8 cm) than when compared with the geoid (35.7 cm).

An analysis of variance (ANOVA) was conducted to determine the amount of variability within data used to estimate the datum offsets on both the quasigeoid and geoid reference surfaces. The ANOVA output from both the quasigeoid and geoid offsets are $F_{(2,411)} = 59{,}965\ and\ F_{(2,411)} = 41.403$, respectively. The critical value at the 0.05 level of significance with 2 degrees of freedom in the numerator and 411 degrees of freedom in the denominator, for both the quasigeoid and geoid offsets is 3.018. The null hypothesis was not accepted and it was determined that there is a significant difference among the three sets of datum offsets from both the quasigeoid and geoid data groups.

The datum offset from the benchmarks around the TGBM in Cape Town provided a smaller offset. The orthometric height gave a smaller bias offset when compared to the quasigeoid than the normal height, this is unusual as the quasigeoid model is a reference surface for normal height system. Nonetheless, the orthometric height gave a smaller bias offset when compared to the geoid model, this was expected as the geoid model is a reference surface for orthometric height system. Due to the numerical investigations conducted in this study, see sub-section 4.4, it is proposed that South Africa should adopt the normal height system and its relevant reference surface being the quasigeoid

Considerations to be taken for implementation of a consistent quasigeoid based vertical datum in South Africa are presented in chapter 5. In an attempt to modernise the South African vertical datum, for reduction of existing inconsistencies within the current vertical datum and to also reduce on cost for maintenance and upgrade. A reliable reference frame is required for consistent analysis, and combination of physical and geometric heights in order to explore the advantages of satellite geodesy. This development will provide an easy transformation of the ellipsoidal height from space-based instrument to a more physically meaningful height system.

It is recommended that stakeholder consultations and review of relevant implementation policy in the country would be required, before the implementation of a consistent geoid based vertical datum in South Africa. Political influence would be required to ensure that this is done competently within clear policy frameworks and following procedures that protect the public from attempts to make personal gains. There may also be a need for a datum conversion surface, proper documentation should be prepared to maintain traceability and avoid fragmentation.

In conclusion, the realisation of a new national vertical datum for South Africa by a global consistent vertical datum is strongly proposed. This will provide South Africa with a vertical datum defined by an equipotential surface in relation to the global vertical datum. Precise height anomalies are required at the TGBMs, for determination of the vertical datum offset for the South African vertical datum in relation to the global vertical datum. It is suggested in this study that the TGBM in Cape Town should be held fixed as it was given more weight during the establishment of the LLD. This will encourage major cross country engineering projects to enable economic growth for the country and the continent as a whole, and this will also play a big role in scientific applications.

6.2 Future work

In order to modernise the South African vertical datum, an adequate land and marine gravity data coverage, with accurate horizontal position, over South Africa is required in combination with a best fitting global geopotential models for modelling a gravimetric quasigeoid model of higher quality. Since gravity, levelling and ellipsoidal height measurements over South Africa were done in different epochs, a proper transformation of gravity and heights data available in South Africa to a specific reference epoch would be required for linking the South African vertical datum (i.e. the Land Levelling Datum). Densification and updating/re-adjusting the South African gravity data coverage will provide higher precision quasigeoid model. Establishing a state-of-the art absolute gravity network over South Africa is recommended for future works.

Both the airborne and terrestrial gravity data acquisition techniques should be employed for densification of gravity data coverage. Terrestrial gravimetry can be used to collect gravity data on all the benchmarks forming part of the set of first order levelling networks defining the LLD. This is necessary for a rigorous adjustment of the levelling datum, as it is still required for validation purposes. A more precise DEM can also assist to achieve better results for establishing a geoid-based vertical datum, such as a DEM produced from airborne laser scanning technique. Moreover, refining/reprocessing GNSS data using up to date products (i.e. precise orbits and clocks, ionospheric corrections, etc.) and scientific GNSS software (e.g. Bernese 5.2, GAMIT/GLOBK) and ITRF2014 or the released ITRF2020 to determine ellipsoidal heights.

A modern vertical datum should satisfy a number of primary requirements, such as, it should be defined by a gravity potential, be consistent, stable and reliable, compatible with space based instruments, it also should be dynamic in nature, suitable for scientific research and be integrated into global height datum with ease, and satisfy a large number of economic activities. The information about GGMs, gravity observations, physical heights (e.g. normal heights), and ellipsoidal heights at the TGBM/s are required to compute the geopotential for the local vertical datum. A vertical datum of this nature can be easily related with the global vertical datum, using the GBVP approach. Furthermore, adopting this vertical positioning technique will assist the South African economy to become more efficient and competitive.

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