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Pricing Methods for American Options

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Synopsis

This thesis is about the comparison of Pricing models for the valuation of American Options.

Three classes of numerical approaches are considered. These are Lattice Methods, Analytic Approximations and Monte Carlo Simulation. Methods will be contrasted in terms of accuracy and speed of the computed American option price.

One particular method utilises regression when estimating the American option price. For this approach the impact of outliers and multicollinearity is examined and alternative regression models fitted.

Monte Carlo Simulation is implemented to calculate early exercise probabilities of American options in the South African market. Results are compared for both call and put options.

A test set of 3550 options is simulated with parameters mirroring the South African economy. On this set, the accuracy of all methods is assessed relative to a benchmark price, which is computed by a convergent lattice approach.

Finally, American Symmetry is used to evaluate both put and call options.

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Notation and Definition of Symbols

The following symbols correspond to the given definitions and are taken to hold throughout the remainder of the thesis.

S is the asset price at time t ;
K the strike price of the option
r the annual continuously compounded constant riskless interest rate
 δ the annual continuously compounded constant dividend yield
 σ the instantaneous annualised standard deviation of returns;
T the time to maturity of the option measured in years;

P the price of an American Put
p the price of a European put
C the price of an American Call
c the price of a European call

BS Black-Scholes Option Price
GBM Geometric Brownian Motion

Lattice Methods

AB Accelerated Binomial method
BA Binomial Average Method
BBS Binomial with Black Scholes modification
BBSR BBS with Richardson Extrapolation
CRR Cox, Ross and Rubinstein model
SBSR SR with Black Scholes modification
SMO Leisen's model
SR SMO model with extrapolation
TRI Trinomial model

Analytic Approximations

JU Multipiece Exponential Approach with Extrapolation
GJ The Geske-Johnson Approach
LBA Lower Bound Approximation
LUBA Lower and Upper Bound Approximation
ML The Analytic Method of Lines
SUL Gaussian Quadrature and Chebyshev Function Approximation

MCS Monte Carlo Simulation

Chapter 0: Introduction and Organisation

0.1 Introduction

The overwhelming majority of traded options are of American type. Yet their valuation, even in the standard case of a lognormal process for the underlying asset, remains unsolved. The focus of the dissertation is on the pricing of American options, based on a single underlying that pays a continuous dividend yield.

In 1973 the first option-pricing model with a closed form solution appeared in the literature. Fisher Black and Myron Scholes published the Black Scholes formula for pricing European options. This model subsequently became the standard pricing model for financial options.

Over the last two decades, the volumes of derivatives traded on the stock exchanges all over the world, has exceeded the volume traded in the underlying shares by an order of magnitude. Investors are increasingly demanding more specific portfolio exposure as well as tailored payoffs based on contingent events.

With increasing demand for sophisticated products, the need to price them accurately became apparent. In contrast to European options, where exercise is possible only at expiry, American options permit exercise at any time before the option expires. This single additional exercise feature has complicated the question of pricing American options to the extent that no closed form solution exists for the American option problem to date.

Consequently, researchers have resorted to numerical techniques to approximate the American option price. Numerical approaches can be categorised into one of three classes: Lattice Methods, Analytic Approximations and Monte Carlo Simulations. Each class and the particular method in it, has its unique strengths and weaknesses.

In this thesis methods will be contrasted in terms of their accuracy and speed with which the American option price is computed.

0.2 Organisation of Thesis

In **Chapter 1** a review of the literature on the various approaches is provided. In an Appendix, a summary of the derivation of the Black Scholes formula for valuing European options is given. In particular the Black Scholes Partial Differential Equation (PDE) is stated as some approximations rely on a modification of this PDE to obtain an approximate solution for the American Price.

The European option-pricing problem is defined and the American option-pricing problem stated. Following this, two further important aspects for American option pricing, namely Richardson Extrapolation and some Symmetry Properties are described.

A literature review of lattice methods, analytic approximations as well as Monte Carlo Simulations follows.

Chapter 2 is devoted to Lattice approaches. In particular, the mechanics of a number of lattice approaches are discussed. Initially, an important basic approach, the CRR model (Cox, Ross and Rubinstein) is summarized and its convergence demonstrated numerically. Subsequently, various attempts to improve the CRR speed-accuracy trade off are considered. Finally, a table of put values is calculated.

Chapter 3 describes a number of analytic approximations. In particular a number of the most recent and best (in terms of speed and accuracy) methods are considered. Each approach is summarised and its computational implementation described. Again, a table of American put values is computed.

Chapter 4 investigates a regression approach utilised in one of the techniques in Chapter 3. In particular, outliers and multicollinearity are investigated and alternative regression models produced.

Chapter 5 describes three Monte Carlo approaches for American options. Subsequently, two variance reduction techniques commonly used in practice, are summarised. Finally, an approach that assumes the optimal exercise boundary known is implemented to calculate early exercise probabilities in the SA context.

In **Chapter 6** a large number of option input parameters are simulated and specific variations considered, e.g. short term, highly volatile, etc. On this set of options, all methods are compared with respect to the accuracy of the price, as well as the time taken for its computation.

Finally, conclusions are drawn and recommendations made in **Chapter 7** while references are given in **Chapter 8**.

Chapter 1: Literature Review

This literature review presents the theoretical developments necessary for an investigation into the pricing of American options on a single underlying, which pays a continuous dividend yield.

1.1 Stating the pricing problem

To begin, however, the derivation of the Black-Scholes formula is given. In 1973, Fisher Black and Myron Scholes presented the first equilibrium option-pricing model. In the same year, Robert Merton extended their model in several important ways. The Black-Scholes pricing formula for European options is an analytic formula that solves the price of the expected (under risk-neutral probabilities) discounted payoff at maturity with exercise possible only at maturity. The most standard assumption concerning asset price behaviour is that of Geometric Brownian Motion (GBM). In **Appendix A**, the implied, by GBM, stochastic differential equation for the return of a single underlying, is transformed into the Black Scholes partial differential equation, utilising Ito's lemma and the construction of an arbitrage-free (and therefore risk-preference independent/neutral) portfolio. This partial differential equation is:

$$\frac{\partial C}{\partial t} + rS \frac{\partial C}{\partial S} + \sigma^2 S^2 \frac{\partial^2 C}{2\partial S^2} - rC = 0$$

For a European call, c , this PDE is solved subject to the following boundary condition:

$$c(S_T, T) = \text{Max}(0, X - S_T),$$

In **Appendix A** the well-known BS pricing formula is then obtained.

For an American call, C , however, one additional condition is required:

$$C(S_t, T) \geq \text{Max}(0, X - S_T), \text{ for all } t < T.$$

This condition means that prior to expiration the American call value must be at least its exercise value, since it can be exercised at any time. An American call will be exercised if the spot price exceeds a critical stock price and these critical stock prices together form an early exercise boundary B_t .

The above problem is a free boundary problem, and to date no explicit easily computable solution has been derived. However, the solution of the problem is of great economic interest as the overwhelming majority of options traded are of American type (Ait-Sahalia & Carr, 1997).

1.1.1 Comment1: Richardson Extrapolation

Richardson Extrapolation is a technique that has been introduced to option pricing by Geske & Johnson (1984) and is now applied even in Monte Carlo Simulations.

For a sequence of N convergent prices, Richardson extrapolation can be used to “predict” the theoretical limit of the solution.

From Marchuk & Shaidurov (1983), an N point Richardson extrapolation is the following weighted average of N converging values:

$$\hat{P} = \sum_{n=1}^N \frac{(-1)^{N-n} n^N}{n!(N-n)!} P^{(n)}$$

In the case where N=2 this gives: $P = -P^{(1)} + 2P^{(2)}$, while if N=3 one gets: $P = 1/2 P^{(1)} - 4P^{(2)} + (9/2)P^{(3)}$.

Extrapolation is usually based on the number of exercise opportunities or the number of time steps. However, Carr & Faquet (1994) apply extrapolation to the number of time derivative discretisations and Ju (1998) applies it to the number of multipiece-exponential functions for approximating the exercise boundary.

Alternative extrapolation technique have also be used in option pricing. Sullivan (2000) uses geometric (rather than arithmetic) extrapolation. Ho, Stapleton & Subrahmanyam (1994) propose exponential extrapolation.

1.1.2 Comment 2: Symmetry Properties for American Options

Just like Put-Call parity for European options, there exists a parity result for the American Option price as well as for the critical stock price determining the boundary for early exercise (Carr & Faquet (1994), McDonald & Schroeder (1998) and Detemple (2001)).

For call and put prices:

$$C(S,K; \delta,r) = P(K,S; r,\delta)$$

Furthermore, the optimal exercise boundaries B^P and B^C are connected by the following symmetry relation for all times to maturity:

$$B_t^P(K,r,\delta) = \frac{S_0 K}{B_t^C(S_0,\delta,r)}$$

These symmetry properties allow the calculation of a Call (or Put) price or boundary, by interchanging the inputs in a model that is based a Put (or Call) respectively.

1.2 Introduction to the Literature Review

Over the last two decades, a number of pricing approaches have been developed. These can be broadly classified into Lattice methods, Analytic approximations and Monte Carlo methods. Within these three classes, every single pricing approach has its own unique strengths and weaknesses, to suit specific investor needs. Approaches differ on dimensions such as speed, accuracy, flexibility to incorporate different types of dividend or stochastic processes, ease of the implementation, etc.

The chapter now proceeds by reviewing the developments in each of the three main pricing classes in turn.

1.2.1 A Review of Lattice Methods

Cox, Ross & Rubinstein (1979) introduced the binomial method for valuation of both European and American options. A similar formulation was independently developed by Rendleman & Bartter (1979). The method discretises both the time and state space (i.e. stock price process), in order to approximate the option price. If riskless profitable arbitrage is not possible, the binomial method explicitly establishes a trading strategy that replicates the price of the option. The value of the call calculated in this way, is a function of only one random variable, which is the stock price. In particular, the call price is independent of the risk preference of the investor, and therefore investors can be assumed risk neutral.

The binomial-pricing method has three independent parameters, subject to only two constraints. These are, that the mean and variance converge in the limit to those of the appropriate lognormal diffusion process. Consequently the selection of parameters has one degree of freedom. Several possible parameter sets have appeared in the literature. These include those of Cox, Ross & Rubinstein (**CRR**) (1979), Rendleman & Bartter (1979), Jarrow & Rudd (1983), Tian (1993), etc. A summary of the first three approaches is given in Omberg (1987).

When pricing American call where the underlying does not pay dividends, the BS can be used to calculate the price. When no dividends are paid, it is never optimal to exercise the call early. However, if the underlying pays dividends, the CRR can be adjusted to deal with a continuously payable dividend yield, a single known dividend yield and a single known dividend payment. See Hull (1998) for a summary.

The binomial method for European options converges to the BS-equation, as the discrete time intervals are made arbitrarily small (Cox, Ross & Rubinstein (1979), Rendleman & Bartter (1979), Bingham & Kiesel (1998), etc.). Convergence of the binomial method for pricing American options is proved in Amin & Khanna (1994). This fact is important because it allows the price to be calculated to any degree of accuracy. Hence the Binomial method is frequently used to calculate the benchmark price in numerical comparisons. A modified version of the binomial method, which has a faster rate of convergence, is implemented in this thesis.

The binomial method is a very flexible method and is now widely used to value options, particularly where no analytic (closed form) solution exists, e.g. Barrier or Lookback options. However convergence and flexibility comes at a price: the accuracy and speed relative to other recent methods is weak (Broadie & Detemple (1996)). A precise definition of speed and accuracy will be given in Chapter Six.

In order to improve the speed-accuracy trade-off several modifications have been proposed in the literature over the last two decades. For American options, because a Binomial tree with n time steps has the number of nodes of the order n^2 , the computation time as n gets large, increases with the order n^2 . This makes acceleration or extrapolation an attractive alternative amongst others.

Omberg (1987) applies Richardson extrapolation to the Binomial method, by varying the number of time steps in the model. However, convergence in the number of time steps is oscillatory rather than uniform. Furthermore, these oscillations are not symmetric around the true value for in- and out- the money options. This is also confirmed by Kamrad & Ritchken (1991) and Leisen & Reimer (1996). Omberg showed that it was impossible to select the parameters of the binomial model in such a way as to ensure uniform convergence (without mismatching the mean or variance of the binomial process).

Given oscillations, of whatever nature, practitioners have applied a binomial average method (**BA**). There is no particular inventor to this method. The average of the n^{th} and $(n+1)^{\text{st}}$ step binomial prices is calculated, which should, due to oscillation (mostly!) around the true value, lie closer to the true value. This method should improve speed, because a tree with n and $n+1$ time steps is faster to compute than a tree with $2n$ time steps.

Breen (1991) describes an application of Richardson extrapolation to the CRR, called the accelerated binomial method (**AB**). For a fixed number of time steps n in the binomial model, Breen calculates prices allowing exercise at times n only, at $n/2$ and n , at $n/3$ and $2n/3$ and n , (etc). He argues, that convergence should be uniform from below as an option with more exercise opportunities should be more valuable than an option with less exercise opportunities. In other words, extrapolation is applied to the number of exercise opportunities rather than the number of time steps.

However, as Omberg (1987) pointed out, care should be taken that convergence is indeed uniform. For example, the number of exercise opportunities should increase in a geometric fashion (1,2,4,...) and not in an arithmetic fashion (1,2,3,...), as at least the same set of exercise opportunities should be used, as the number of exercise opportunities increases. However, Breen (1991) maintained his original arithmetic formulation and his final version of the AB method was implemented in this thesis. On a small sample of options it was found that the arithmetic formulation was more accurate than the geometric formulation, which may perhaps suggest that geometric exercise opportunities, although uniform convergent, perhaps converge too fast given the type of extrapolation, i.e. Richardson.

The AB method converges to the three-point Geske and Johnson (1984) (**GJ**) approximation, not to the true American option value. To have convergence to the American option value, not only the number of time-steps, but also the number of exercise opportunities must increase to infinity.

Another variation of the binomial method is the trinomial method (**TRI**). As for the binomial approach, the continuous distribution of the stock price is replaced by a suitable discrete process, where the discrete distribution tends to the appropriate limit. In particular, the discrete stock process is now allowed to take on one of three (hence trinomial) instead of two (for the binomial) values at the end of a time step. In this way, accuracy is improved as a more detailed description of the stock price results.

Parkinson (1977) employed a three-jump model to value the American put, but his approach seems difficult to generalise to situations involving more than one state variable. Stapleton and Subrahmanyam (1984) discuss a three-jump model, but they did not examine its numerical efficiency or solve for the jump probabilities.

Boyle (1988) uses a multiperiod trinomial procedure, where the lattice jump probabilities are obtained by equating the first two moments of the underlying lognormal distribution to those of the approximating distribution. To ensure all jump probabilities were nonnegative, Boyle introduced a stretch parameter, λ , (to adjust the size of the up and down jumps), which has to be constrained.

Kamrad & Ritchken (1991) also develop a three state model. The mean and variance of the approximating distribution are chosen to equal the mean and variance of the logarithmic returns process $\ln\{S(t+dt)/S(t)\}$, which is a normal random variable with mean μdt and variance $\sigma^2 dt$. As with Boyle (1988), a stretch parameter λ was introduced.

However, unlike Boyle's trinomial procedure, any value for $\lambda, \lambda \geq 1$, yields a feasible set of probabilities in the case where the underlying pays no dividend. For this reason the Kamrad & Ritchken's approach was implemented in this thesis. However, even for this approach, where a positive continuously compounded dividend is paid, a negative probability may still result.

Kamrad & Ritchken also show that a trinomial method with n steps requires less computational effort (measured in terms of multiplications and additions) than a binomial model with $2n$ steps. Furthermore, in their numerical study the trinomial method with 20 steps was almost always more accurate than the binomial method with 40 steps. It seems therefore that the binomial method is "beaten". However, storage requirements of both methods still need to be considered. Further, today a practical number of time steps falls in the region of 200-300.

In a numerical comparison, Broadie & Detemple (1996) show that for standard American options, the trinomial method's increased accuracy (compared to CRR) is very nearly offset by its increased computational time. This issue will be further investigated in Chapter Two (as well as in Chapter Six).

Yet another approach, aimed at improving the speed/accuracy trade-off of the CRR, is suggested by Broadie & Detemple (1996). The binomial with Black Scholes

modification (**BBS**) and the Binomial with Black Scholes modification and Richardson extrapolation (**BBSR**).

In the first modification, the BS formula replaces the usual 'continuation value' at the time step just before option maturity. Subsequent analysis shows that the error, compared to the CRR, is substantially reduced for the same number of time steps and that convergence to the true value is smoother.

The smoother convergence suggests that Richardson extrapolation to the BBS method may be useful, resulting in the BBSR method. Two-point Richardson extrapolation is preferred to three-point extrapolation. Unlike the AB, which extrapolates based on the number of exercise opportunities, BBSR bases its extrapolation directly on the number of time steps. Broadie & Detemple (1996) show that the BBSR is a significant improvement over the CRR.

The smoother convergence and its benefit when extrapolating, is demonstrated graphically in Chapter Two.

In their publication, Leisen & Reimer (1996) technically analyse convergence behaviour as well as convergence speed. The authors explain the non-symmetric oscillatory convergence behaviour for in-and-out the money options. When plotting the asset price against an increasing number of time steps, they observe that asset prices fall progressively closer to the spot price. Stated differently, as n increases the distance between the first (second, etc.) asset price and the spot becomes smaller. Hence, when fixing a strike at a position away from the spot, the distances to surrounding tree nodes is changing for different values of n , and this causes the computed option- price to oscillate in a non-symmetric fashion.

Based on the above insight, the authors propose an approximation of normally distributed terms by inverting normal approximations to the binomial distribution. The authors suggest an improvement to the European part by inverting three normal approximations to the binomial including two approximations by Peizer & Pratt (1968).

Leisen (1998) shows in an analysis that when extrapolated, the models based on the inversion of normal approximations, perform very similar (slightly worse) to the extrapolated **SMO** model described below.

Having diagnosed the source of the irregular oscillation patterns as described above, the SMO model directly modifies the CRR tree parameters such that the strike lies at the centre of the tree at maturity. Smoother convergence results and this is also demonstrated in Chapter Two.

Finally, Leisen & Reimer (1996) measure convergence speed by the order of convergence of the price approximation. If e_n is the absolute difference between the discrete and continuous price then they define a sequence of lattices to converge with order $\rho > 0$, if $e_n \leq K/n^\rho$ for large n and for some constant K . The larger ρ is, the quicker the convergence.

Using simulation an order of convergence one is suggested for the European CRR. They then show this analytically. Finally, they suggest that the CRR for American options has an order of convergence one, and Leisen (1998) confirms this analytically.

The final lattice approach investigated in this thesis combines two existing improvement features. The “new” model, which combines the Black-Scholes modification from Broadie & Detemple and the extrapolated SMO model, we call **SBSR**. Superior performance is expected as two improvement features are combined.

Further Comments

- The Binomial Method is very easy to implement and quite flexible in that it can often easily be adapted to price non-standard or exotic options (See Hull (1998)).
- The greeks (i.e. Delta, Gamma, etc.) can be calculated. See for example Hull (1998), Pelsler & Vorst (1994) and Willmott (1999).
- Several limiting cases, other than the BS exists. For example a Jump Stock Price Movement model (Bingham & Kiesel (1998)).
- Generalisations of the binomial approach to the multinomial method, is given in Boyle (1988), Omberg (1988), Boyle, Evnine & Gibbs (1989) amongst others. This generalisation involves the pricing of options on the maximum of more than a single underlying asset. These applications are very time-consuming in the binomial method

The **finite difference method** is a numerical method related to lattice methods. It was introduced by Brennan and Schwartz (1977) and Schwartz (1977). See also Willmott, Howison & Dewyne (1997) for a recent discussion of this method. Convergence of the Brennan and Schwartz method is proved in Jaillet, Lamberton & Lapeyre (1990).

This method approximates the PDE on which the option pricing formula is based. In particular, the partial derivatives occurring in partial differential equations are replaced by approximations based on the Taylor series expansion of functions near the point or points of interest.

Once the diffusion equation is obtained from the BS partial differential equation, forward differencing results in the explicit finite difference method, while backward differencing results in the implicit finite difference scheme. If central differencing is used the Crank-Nicolson method arises.

Geske & Shastri (1985) find that the explicit finite difference method with a logarithmic transformation is more efficient than the binomial method for a larger number of options, and more efficient than the implicit finite difference method. Brennan & Schwartz (1977) demonstrate that binomial and trinomial trees with an infinite number of time steps, is the same as the explicit finite difference method.

The main disadvantage of the explicit finite difference methods is that there exists a small probability that no solution is found for the option price. Although the implicit finite difference method may also prove unstable under certain scenarios, the solution is stable, where the explicit finite difference method is unstable. However, due to these complications, finite difference methods are not considered further here.

This concludes the literature review on Lattice approaches. In the next section analytic approximations are discussed.

1.2.2 A Review of Analytic Approximations

Analytic approximations have developed along several distinct strains, depending on how the pricing problem is modified through making one or more simplifying assumptions. The discussion in this section proceeds by describing some of the most current and prominent methods.

1.2.2.1 The Compound Option Approach

The compound option model was developed in Geske (1979) in the context of a call option where the underlying is a call option on the assets of the firm with the strike price the liabilities. The Black –Scholes equation is inadequate for pricing this option, as the volatility of the underlying option is changing. In particular, it is not constant (as the BS model assumes) due to leverage effects, but is instead a function of the level of the underlying. From this, the compound option solution technique was developed.

Early work for the pricing of American options focused on the case of discrete dividends. If a single dividend is paid during the life of the option, Geske (1981), Roll (1977), and Whaley (1981) obtain a closed form solution, which is based on the compound option model.

Utilising the solution technique of a compound option given in Geske (1979), Geske and Johnson (1984) obtain an exact solution for the American option-pricing problem. Their formula involves an infinite series of multivariate cumulative normal functions. At every instant there is a positive probability of premature exercise, and this situation is equivalent to an infinite sequence of options on options. At each instant, the option will be exercised, if it hasn't been exercised at any previous instant and the payoff exceeds the option value if not exercised. The solution is an n-fold compound option. Unlike other authors who approximate the PDE (e.g. Brennan & Schwartz (1977)) or the stock price process (e.g. CRR), the GJ formula is evaluated as the exact solution to the PDE subject to a discrete exercise boundary.

Geske & Johnson obtain the price of a put that can only be exercised at T (using BS), plus the price of another put that can be exercised at T/2 and T (which requires bivariate normal integration), plus the price of another put that can only be exercised at T/3, 2T/3 and T (which requires trivariate normal integration), etc. Finally, Richardson extrapolation based on the number of exercise opportunities is suggested.

Because multivariate cumulative normal functions require the computation of multidimensional integrals, the possibility of including many terms in the approximation in order to gain accuracy is very limited. This is because multivariate normal integration is very time consuming as the number of dimension increases.

Consequently, Bunch and Johnson (1992) implement a modified two-point Geske-Johnson approach, in an attempt to avoid the computation of the multidimensional integrals. The Geske-Johnson approach is modified by optimally placing the exercise points (as opposed to equal spacing), and mostly only the computation of the cumulative bivariate normal density is required.

Their “maximum method” uses two-point Richardson extrapolation after maximising the option price, by choosing the first of two possible exercise times from 8 equally spaced exercise times, as the one that maximises the price of the option. Finally, the modified two-point GJ approach (MGJ) is defined as the “maximum method” with an added modification for deep in-the-money options, which are evaluated by the three-point GJ approach.

Since the compound option model approximates the exercise-opportunity set by restricting it to a sub-optimal set of discrete times, the values generated are lower bounds. MGJ method is expected to perform better, because the computed put values are generally higher than for GJ, and hence are expected to lie closer to (if not overshoot) the limit of the sequence.

Broadie & Detemple (1996) show that although the GJ2 method is faster but less accurate than the modified GJ2 approach, both are dominated in both speed and accuracy by even the CRR. In other words, for the same accuracy, we can find a faster binomial method (with a small n), that is more accurate.

However, the GJ2 method is implemented along with a very efficient bivariate normal integration program published by Genz (1992). In his paper, various multivariate normal integration methods are compared, including the one by Schervish (1984). On Genz’s homepage, further improvements to the method in his (1992) publication are discussed.

1.2.2.2 The Method of Lines

In the method of lines (Carr & Faquet (1994)), the time derivative is replaced by a finite difference approximation. Mathematically, the method of lines approach is equivalent to the randomisation approach discussed in Carr (1998).

The free boundary problem given by the Black-Scholes PDE relates the value of an American option to the underlying stock price at time t . In this PDE the time is discretised, while the stock price is left continuous. Then the time derivative is replaced with a finite difference of the change in stock price over a discrete time interval. The free boundary problem involving a PDE is replaced with a recurrent sequence of free boundary problems involving ODE’s, each of which is solved analytically. As the number of discrete time intervals gets arbitrarily large, the authors conjecture that the solution of the modified PDE converges to the solution of the exact PDE. Numerical analysis support convergence, but no formal proof is provided. An explicit solution for the normalised (division by the strike) American put value is obtained. Both the stock price and the early exercise boundary are generated.

From Numerical Simulations, the authors find that the ML slightly undervalues the American put option. Hence a modified three-point Richardson extrapolation is used. The authors suggest that this modified extrapolation approach with only 3 option prices, represents a satisfactory trade-off between speed and accuracy.

The approach is entirely analytic, besides numerically solving for the critical stock-price in the case where the underlying pays a continuous proportional dividend. Errors arise in finite difference methods due to the truncation of the spatial domain and in lattices due to space discretisation. The analytic nature of the ML avoids these errors. In addition computation is speeded up, because the valuation formula does not

use distribution functions or any other special functions besides powers and natural logarithms. Arbitrary accuracy can be achieved at the expense of lower speed, simply by adding more points to the extrapolation.

1.2.2.3 An Approximation based on Bounds

Broadie & Detemple (1996) derive a lower bound for the American call option price based on a capped option with an appropriately chosen constant cap. Furthermore, they provide a procedure, based on capped call options, to compute a uniform lower bound to the optimal exercise boundary of the American call option. Finally, an integral representation of the American option price is used in conjunction with the lower bound on the optimal exercise boundary, to obtain an upper bound for the theoretical price of the option. The integral representation splits the American option price into its European value and the early exercise premium. It is discussed in more detail below. A final value for the American option is then obtained using regression. This method is not convergent, but very accurate.

In this thesis, outliers and collinearity of this regression approach are analysed and alternative biased regression approaches considered.

1.2.2.4 Gaussian Quadrature and Function Approximation

Sullivan (2000) proposes an approach that combines Gaussian quadrature with function approximation using Chebyshev polynomials. The technique builds on the risk-neutral pricing relation introduced by the binomial method. In particular, Gaussian quadrature is used to evaluate the risk neutral expectation, while the stock price process is left log normal (i.e. continuous). The option value function is approximated by Chebyshev polynomials rather than it being restricted to a discrete set of points. An approximation of the option value function in two dimensions, across stock prices and over time to maturity is produced.

The Quadratic Approximation

A further technique called the quadratic approximation, uses a technique originally developed by MacMillan(1986), Barone-Adesi & Whaley (1987) and splits the American option price into the European value and the early exercise premium. Together with some simplifying assumptions, the solution to the PDE for the early exercise premium is approximated by a quadratic equation. Ju (1998) finds that these approximations are very fast and many times faster than most other method. However, a serious shortcoming is that they are not very accurate, especially for long maturity options. In addition, the method is not convergent, because there is no control parameter to change to improve the accuracy. For this reason, this method was not considered any further.

Integral Representation Method

As for the quadratic approximation, the integral formula expresses the value of the American option as the value of the corresponding European option augmented by the present value of the gains from early exercise (See for example Kim (1990)). The gains from exercise, in turn, depend parametrically on the optimal exercise boundary, which is the solution of a non-linear integral equation subject to a boundary condition. While the option price has an explicit solution, the exercise boundary is implicitly defined by the integral equation so that a recursive numerical procedure is required to solve for the exercise boundary and option price

Huang, Subrahmanyam & Yu (1996) implement a four-point Richardson extrapolation scheme to the integral representation method. To implement this method, only six points on an approximation to the early exercise boundary are employed. This integral representation method involves only the univariate cumulative normal function and is therefore very fast. However, it is not very accurate, especially for moderate and long maturity options (Ju (1998), Sullivan (2000)), therefore it is not included in the later analysis.

1.2.2.5 The Multipiece Exponential Function approach

Another approximation based on the integral representation method, is proposed in Ju (1998). Since the accuracy of the integral does not critically depend on the optimal exercise boundary (it is an argument to the logarithmic function), Ju approximates the boundary by a multi-piece exponential function and evaluates the resulting integral in closed form. This method is fast and accurate, as the most of the risk-neutral integration is done analytically. The method is convergent and they demonstrate numerically, that a three-point extrapolation scheme attains penny-accuracy. Furthermore, the price is exact as time to maturity goes either to zero or to infinity. Hence, as opposed to many other methods, accuracy for longer-term options is maintained. This method is included in the analysis in this thesis.

Johnson's regression formula (1983) and Omberg's exponential formula(1987) are two further approaches. However, both are not defined for positive dividends and are therefore excluded from further discussion.

1.2.3 A Review of Monte Carlo Methods

Simulation is an attractive method for asset pricing because of the generality in the types of assets (e.g. stochastic volatility and interest rates, default risk, etc.) it can handle and the ease with which various stochastic processes (e.g. GBM, extreme value, etc.) can be applied. Furthermore, multiple state variables and path dependencies can be easily taken into account.

Monte Carlo methods were introduced to option pricing by Boyle (1977). However, applications have been restricted to European type instruments. A number of Monte Carlo approaches for American Options have appeared in the literature. For a review see Boyle, Broadie and Glasserman (1997). The authors identify three important approaches for American option that have recently appeared in the literature. First, Tilly's (1993) bundling algorithm, with an extensions discussed in Carriere (1996). Second, Barraquand and Martineau's (1995) stratified state aggregation algorithm, which is extended in Raymar & Zwecher (1997). Finally, Broadie and Glasserman's (1997) biased estimates, one high and one low.

A number of acceleration and variance reduction techniques are applied in these simulation approaches. Boyle, (1977) and Hull & White (1990) provide an introduction to the control variate technique and antithetic sampling in option pricing. For a recent review and other methods see Boyle et al. (1997).

As stated above the major difficulty in valuing early-exercise features is the need to estimate the optimal exercise boundary. Implicit in the calculation of the American call is the decision to exercise at the first time the payoff exceeds a certain continuation value. In simulation, this lack of knowledge of the early exercise boundary has resulted in estimates that are biased. Broadie & Glasserman (1997) show that using simulation, there can be no unbiased general estimator of the American option price.

1.2.3.1 An Unbiased Forward Simulation approach

Recently, Basso, Pardon and Pianca (2002 a,b,c), have implemented a two-step forward simulation procedure for valuing American calls. The first step of the valuation generates the optimal exercise boundary, while the second part does the actual simulation.

A number of approaches for computing the early exercise boundary have been suggested in the recent literature. For a review, see Basso et al. (2002a,b). They note that while the accuracy of the optimal exercise boundary is not essential for pricing options, it is crucial when it comes to evaluating early exercise features such as average exercise times, the probability of early exercise, the overall probability of the option being exercised, etc.

In their publication, they investigate numerically the accuracy of three techniques that seem to be among the most adapt to obtain an accurate boundary. The first is the binomial approach with an interpolated boundary and omission of the first few steps. Second, the numerical algorithms proposed by Bunch & Johnson (2000) and finally the randomisation approach by Carr (1998). See also Carr & Faquet (1994).

Numerically they show that Carr's approach is the most accurate, robust and computationally efficient. The Bunch & Johnson's approach proved to be unstable under certain conditions, while the binomial approach is computationally expensive. In this thesis we will implement the forward simulation approach with Carr's approach for estimating the boundary. In doing so, exercise probabilities are calculated in the South African context for both Put and Call options.

Finally, although MCS is computationally ineffective in its basic form, convergence is typically independent of the number of state variables / underlying assets. Hence, MCS should become increasingly attractive compared to lattice methods, when for example, valuing American options on the maximum of a number of underlying assets.

University of Cape Town

APPENDIX A: The Black-Scholes Option Pricing Formula

A very frequent assumption concerning the asset price behaviour is that of Geometric Brownian Motion (GBM). In the continuous time framework, GBM for the asset price, implies the following stochastic differential equation for the return of a single non-dividend paying underlying.

Equation 1

$$\frac{dS}{S} = \mu dt + \sigma dz$$

The Wiener process dz , is normally distributed with mean zero and variance dt and the dz 's for non-overlapping increments of time are independent. If, as is true for Equation 1, μ and σ depend only on S and time t , then the stochastic differential equation is called an Ito process. Furthermore, the Markov property is assumed. Hence, future asset price depends only on the current level.

Now, using Ito's Lemma in stochastic calculus, the stochastic process for the value of the derivative security C , where S obeys GBM is given by:

Equation 2

$$dC = \left(\frac{\partial C}{\partial S} \mu S + \frac{\partial C}{\partial t} + \frac{\partial^2 C}{2 \partial S^2} \sigma^2 S^2 \right) dt + \left(\frac{\partial C}{\partial S} \sigma S \right) dz$$

Both, the process for the asset price as well as the process for the derivative, have the same source of uncertainty. Therefore the two securities can be combined to construct a risk-less portfolio.

Consider a portfolio P , which is short one unit in the option and long $\partial C / \partial S$ units of the underlying asset. The change in value of this portfolio over period dt is:

Equation 3

$$dP = -dC + \frac{\partial C}{\partial S} dS$$

Substituting for dS from Equation 1 and for dC from equation 2 into 3 and simplifying gives:

Equation 4

$$dP = - \left(\frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} \right) dt$$

Since the portfolio is risk-less, in equilibrium its return should be the riskless rate r . Rearranging $dP/P = rdt$ leads to the Black-Scholes partial differential equation:

Equation 5

$$\frac{\partial C}{\partial t} + rS \frac{\partial C}{\partial S} + \sigma^2 S^2 \frac{\partial^2 C}{2 \partial S^2} - rC = 0$$

Remarks on the Black-Scholes PDE:

- 1) The equation holds for any derivative whose payoff depends only on S and t . For example, the analytic Black-Scholes formula for European options is the solution to the PDE subject to the boundary conditions at maturity T : $C_T = \max(0, S_T - K)$ for a call and $C_T = \max(0, K - S_T)$ for a put.
- 2) Implicit Assumptions include that the short selling of securities with full use of the proceeds is permitted, there are no transaction costs, all securities are infinitely divisible, no dividends are paid during the life of the derivative, the risk-less rate of interest is constant, and that securities trading is continuous.
- 3) Since μ does not appear in the PDE, the value of the derivative must be independent of investor risk preferences. In particular, if all investors are risk neutral, then $\mu = r$. If in addition the asset pays a continuous dividend yield at rate δ , then $\mu = r - \delta$.

The implication of risk-neutral pricing is that the present value of any future random cash flow is given as the expected value (using risk-neutral probabilities) of the random future value discounted at the constant risk-less rate.

Equation 6

$$C = e^{-r(T-t)} E_t[\max(0, S_T - K)]$$

Applying Ito's lemma to $\ln S$, where S follows GBM, and with $\mu = r - \delta$,

Equation 7

$$d \ln S = \left(r - \delta - \frac{1}{2} \sigma^2 \right) dt + \sigma dz$$

Since dz is a Wiener process,

$$\ln S_T \sim \text{Normal} \left(\ln S_t + (r - \delta - \frac{1}{2} \sigma^2)(T - t), \sigma \sqrt{T - t} \right)$$

Equation 8

The Black-Scholes equation for a European call now follows:¹

$$C = S_0 e^{-\delta(T-t)} N(d_1) - X e^{-r(T-t)} N(d_2)$$

$$d_1 = \frac{\ln(S_0 / X) + (r - \delta + \sigma^2 / 2)(T - t)}{\sigma \sqrt{T - t}}$$

where

$$d_2 = \frac{\ln(S_0 / X) + (r - \delta - \sigma^2 / 2)(T - t)}{\sigma \sqrt{T - t}}$$

¹ See Clewlow & Strickland (1998) as well as Hull (1998) for another key result

Chapter 2: Lattice Methods

The purpose of the thesis is to compare methods for calculating American option prices with respect to speed and accuracy of the approximating methods. With this in mind, this chapter serves to describe a number of lattice methods.

The Binomial method is a type of lattice approach. It's usefulness stems from the fact that it is easy to implement and that it can be applied to a variety of pricing problems, (e.g. barrier options). Very importantly, however, it converges to the true price for both American options and European options.

This chapter starts by summarizing the mathematics of the binomial model and deriving the parameterisation of the Cox-Ross and Rubinstein (CRR) model. Given theoretical convergence of the CRR, this is graphically demonstrated. Subsequently, the other convergence improvements discussed in the literature review are also graphically illustrated. Then, the computational implementation of the selected lattice methods is described and a table of values produced. Finally, the pseudo code of our CRR implementation is given in the Appendix.

2.1 The Multiplicative Binomial method

When pricing American options or options with complex payoffs such as Barrier Lookback and Asian options, often no closed-form analytic solution exists. Therefore numerical procedures must be used to solve the Black-Scholes partial differential equation. The binomial model provides a simple and intuitive alternative.

The Binomial model consists of one risk-less asset B and a single risky asset S (that does not pay dividends for the moment). A total of N + 1 trading dates are assumed to fall in equally spaced intervals between time 0 and the trading horizon T. The interval between trading dates is of length Δt , with $N = T/\Delta t$. Let $S_n = S(n\Delta t)$ be the price at the nth date and $S(t)$ the price at time t. Let R be the continuously compounded return, and r be the effective rate per period, then $(1+r) = e^{R\Delta t}$.

During each period Δt the security price either goes up by a factor u ($u > 1$) with probability p, or it goes down by a factor d ($0 < d < 1$) with a probability 1-p. These moves are independent of each other. Under this assumption the asset price has a binomial distribution over the N periods.

Consider a call option on the single asset, which matures at the end of a single period Δt . In the same way as with the Black-Scholes model, a risk-less portfolio can be constructed. Consider a long position of Δ units of the underlying asset and a short position of one call option. Regardless of whether the asset price goes up or down, the value of the portfolio must be equal at the end of Δt :

Equation 1

$$-C_u + \Delta u S = -C_d + \Delta d S$$

Rearranging

Equation 2

$$\Delta = \frac{C_u - C_d}{(u - d)S}$$

Since the portfolio is risk-less it must earn r :

Equation 3

$$(-C_u + \Delta uS) = e^{r\Delta t}(-C + \Delta S)$$

Substituting for Δ and rearranging for the call price at the start of the period C :

Equation 4

$$C = e^{-r\Delta t} \left(\frac{e^{r\Delta t} - d}{u - d} C_u + \frac{u - e^{r\Delta t}}{u - d} C_d \right)$$

Now, define the risk-neutral probabilities as follows:

Equation 5

$$p = \frac{e^{r\Delta t} - d}{u - d}$$

Remarks

- 1) As for the Black-Scholes model the payoff of the derivative has not been restricted to any particular form yet;
- 2) The option price is independent of the expected return of the stock. Therefore the option price is independent of the risk preference of investors, which allows us to interpret p as risk-neutral probabilities.
- 3) Hence, Equation 4 can be interpreted as taking discounted expectations of future payoffs under risk-neutral probabilities.

Implicitly, it was assumed that the option payoff can be replicated. A market where all option payoffs can be replicated is called a complete market. If we further impose the assumption that the market is arbitrage-free, i.e. there is no possibility of a risk-less profit, this translates mathematically into the existence of a unique risk-neutral probability, and vice versa.

If now the collection of all single periods is considered, and over this period the market is assumed to be complete and arbitrage-free, then a unique risk-neutral probability can be constructed.

Approximating the Continuous-time Process

So, in a risk neutral world, the expected return from all securities is the risk-free interest rate. Also, future cash flows can be valued by discounting their expected values at the risk-free interest rate r .

As for the asset price in continuous-time, the drift and volatility are two key properties of an asset price. Naturally, therefore, the binomial tree should be constructed to be consistent with the Black-Scholes model for European options and therefore to choose u , d and p to match the risk-neutral mean and variance of the GBM.

Since, $E(S) = Se^{r\Delta t}$, where S is the price at the beginning of Δt , it follows that:

Equation 6

$$Se^{r\Delta t} = pSu + (1-p)Sd, \text{ or } e^{r\Delta t} = pu + (1-p)d.$$

Note also that the variance of the proportional change in the stock price during Δt is $\sigma^2\Delta t$ under the assumption of GBM. Equating variances gives:

Equation 7

$$pu^2 + (1-p)d^2 - [pu + (1-p)d]^2 = \sigma^2\Delta t$$

Substituting from Equation 6 gives:

Equation 8

$$e^{r\Delta t}(u+d) - ud - e^{2r\Delta t} = \sigma^2\Delta t$$

Equation 6 and 8 impose two conditions on p , u and d . A third condition, which leads to the Cox, Ross and Rubinstein model is $u = 1/d$.

When terms of higher order than Δt are ignored,² the CRR model is given by the following three conditions:

$$p = \frac{e^{r\Delta t} - d}{u - d}; u = e^{\sigma\sqrt{\Delta t}}; d = e^{-\sigma\sqrt{\Delta t}}$$

To see this, consider the Taylor series expansion of the exponential function. The choices for the CRR model guarantee, that in the continuous time limit the binomial tree converges (weakly) to a continuous time Brownian motion. The CRR results in the same mean for any step size, but the same variance only in the limit.

Now a tree of prices can be constructed and at maturity the payoff calculated. For European options, this payoff is discounted backwards through the tree, by taking the discounted expected value at every node of the tree. For American options, this discounted expected value is additionally compared to the immediate payoff at every time-step and the maximum taken. The value at the very first node obtained in this way is the option price.

A continuous dividend at rate δ can be included if r is replaced by $r-\delta$ in the calculation of p . In theory negative probabilities can result, if $(r-\delta)\Delta t < -\sigma(\Delta t)^{1/2}$. However, negative probabilities are impossible to occur, given the ranges of distribution parameters considered in this thesis, even with American symmetry.

² See Wilmott, Howison and Dewyne (1997) for a solution that does not ignore higher order terms.

2.2 Convergence and Improvement

As stated in the literature review, the binomial model for American options converges to the true value, as the number of time steps tends to infinity. A number of possibilities exist for accelerating convergence and these are discussed below.

It will be demonstrated below, that under particular parameter scenarios some acceleration techniques may not perform as intended. Further an exact analysis of all parameter combinations where ‘unintended’ performance results, is a tedious and complex task and instead, examples of ‘unintended’ performance are used to illustrate the point under consideration.

Convergence of the CRR for American Options

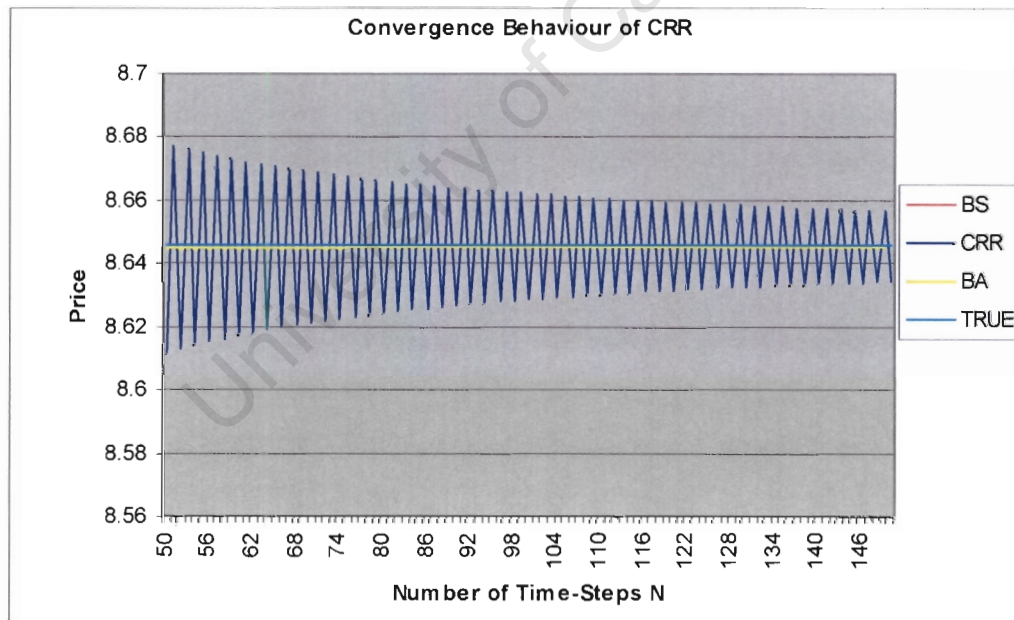
The convergence behaviour of the methods is analysed using time-steps between $N = 50$ and $N = 150$ inclusive. The “true” value of options considered in this chapter is approximated by an $N = 5000$ time-step CRR.

Consider the following at-the-money call option-pricing problem:

$S = 100$, $K = 100$, $r = 0.1$, $\delta = 0.03$, $\sigma = 0.25$ and $T = 0.5$.

The graph below demonstrates the convergence of the CRR to its “true” value. Note that convergence is oscillatory and more or less symmetric around the true price.

Graph 1



Given symmetric oscillations, the **BA** method is proposed: this method calculates the average of two consecutive prices, which should lie closer to the true price. Two prices calculated with n and $n + 1$ time steps require less time than one price calculated with $n+1$ prices. However, the average may in fact not lie closer to the true value when convergence is not perfect, as shown in the Graph 2 and Graph 3 below.

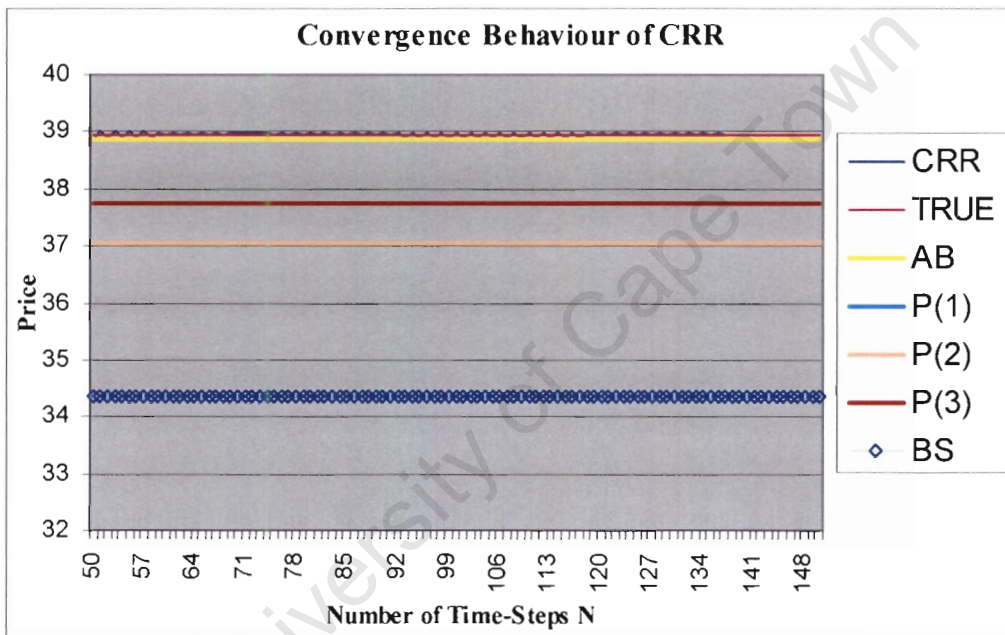
Note that, for GRAPH 1, the BS, CRR, BA and the TRUE value lie very close together. This is not generally true.

Based on the symmetric convergence, Omberg's attempt to extrapolate based on the number of time steps seemed promising. However, as stated in the literature review and as will be demonstrated by the graphs below, this approach turns out not to be successful, since oscillations is non-symmetric for in- and out-the-money options.

Given unstable oscillations for in- and out-the-money options, Breen (1991) makes a second attempt to improve convergence behaviour through the **AB** method: call prices with only 1, only 2 and only 3 exercise opportunities are extrapolated in a three-point extrapolation. Extrapolation in this way circumvents the problem of non-symmetric oscillatory convergence in N .

The graph below demonstrates the convergence of the option with the following parameters: $S = 125$, $K = 100$, $r = 0.01$, $\delta = 0.06$, $\sigma = 0.45$ and $T = 3$.

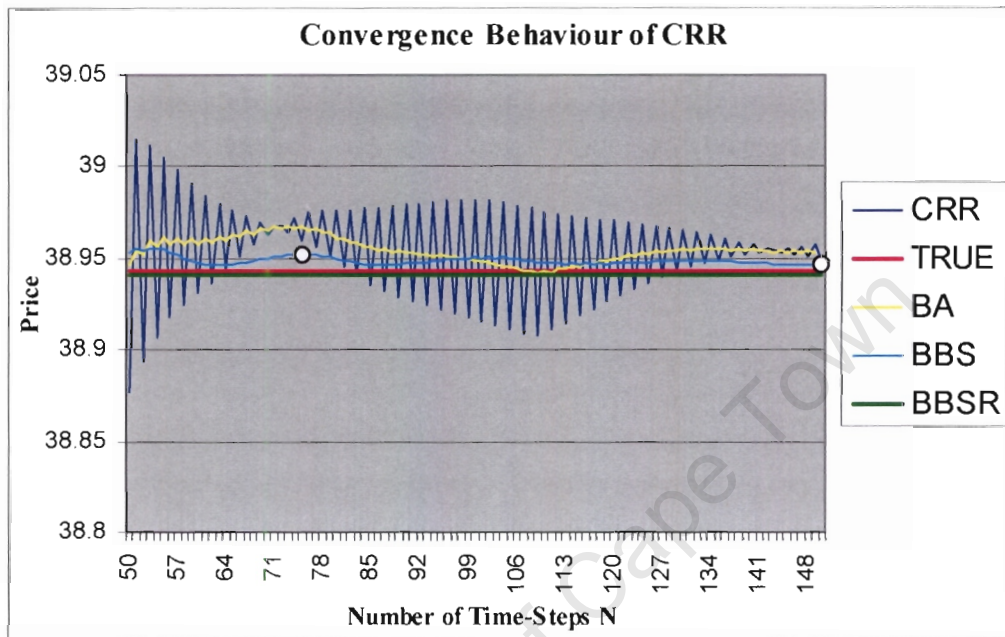
Graph 2



The graph demonstrates extrapolation based on the number of exercise opportunities of the AB method. The price $P(1)$ is close to the BS, $P(2)$ at about 37 and $P(3)$ just below 38. When extrapolated, AB lies close to the CRR. Note that the BS price is differs by more than four currency units from the true price.

In order to make the oscillations visible the scale is adjusted. Graph 3 below demonstrates imperfect oscillations. For some n , consecutive oscillations lie entirely above the true value and hence the rationale for the BA method breaks down. Furthermore, the **BBS** method is demonstrated. It was invented to smooth out the convergence in the number of time-steps by utilising the BS model in the penultimate time-step of the binomial tree.

Graph 3

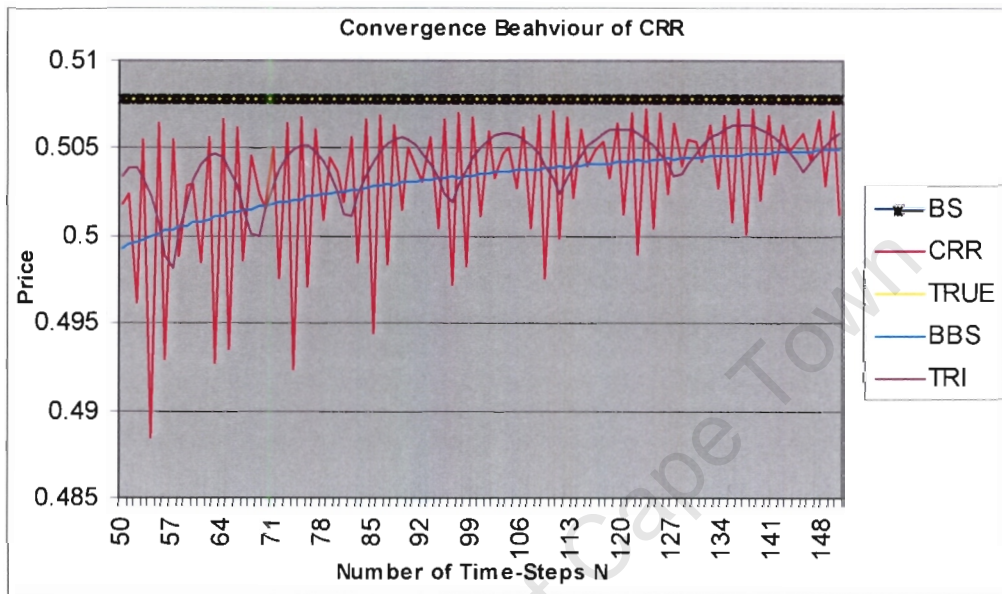


The BBS shows a great reduction in oscillations. Given smoother convergence, two-point Richardson extrapolation is now used to accelerate convergence, i.e. **BBSR**. BBSR results from extrapolating the two white dots corresponding to $N = 75$ and $N = 148$ on the BBS line. The BBSR method is considered to be a very fast and accurate lattice approach.

Another improvement attempts to reduce the error introduced when approximating the continuous time asset process with a binomial lattice. Instead the **TRI** proposes a finer lattice for approximating the asset price process.

For the sake of demonstrating non-symmetric convergence for out-of-the money options the following parameter input was selected: $S = 75$, $K = 100$, $r = 0.1$, $\delta = 0.03$, $\sigma = 0.25$ and $T = 0.5$.

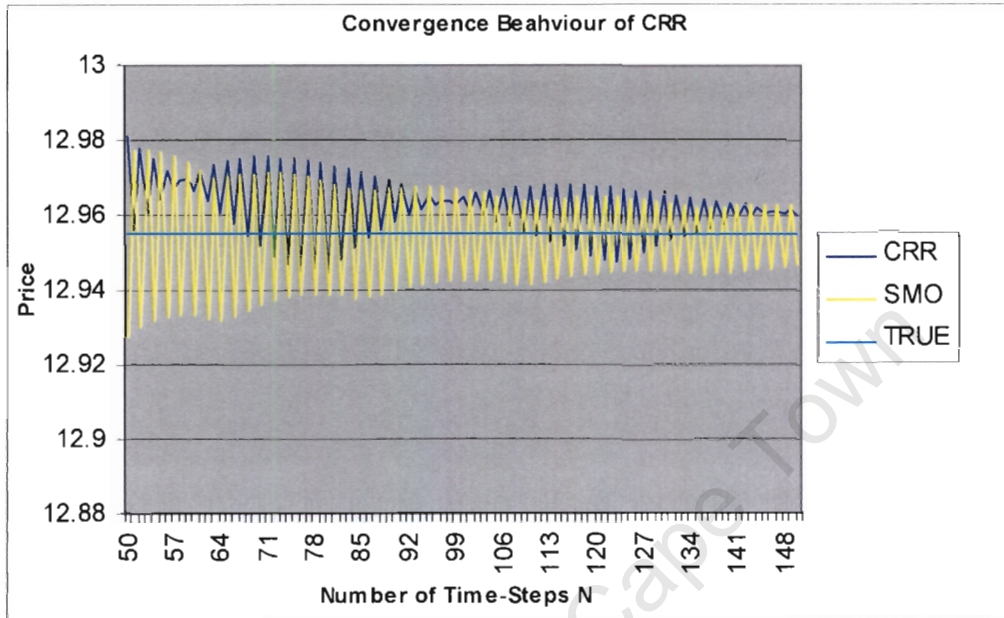
Graph 4



As stated in the literature review, the TRI mostly improves accuracy over the CRR. From the graph, it is evident that accuracy of the TRI relative to the CRR is very much dependent on the number of time-steps. For some N the CRR dominates the TRI and vice versa.

The SMO approach will be considered next. Improvement stems from locating K at the centre of the binomial tree at maturity, by modifying the CRR parameters directly. This represents a mismatching of the variance term of the continuous process, but yields much smoother convergence. Therefore extrapolation yields very good results.

The following example is used: $S = 100$, $K = 110$, $r = 0.07$, $\delta = 0$, $\sigma = 0.3$ and $T = \frac{1}{2}$.



It is evident that the alternating amplitude of the CRR oscillations is greatly reduced and is instead decreasing in a more linear fashion and again extrapolation suggests itself.

Finally, having highlighted various improvement features, one “new” model suggests itself simply by putting two of the improvement features together. SBSR model combines the BS modification with the SMO approach.

Conclusion

The above section demonstrated graphically the various ways through which lattice methods sought to improve convergence. It was demonstrated how in some cases superior accuracy was attained. However, in order to show superiority of a particular method in general, both speed and accuracy need to be considered. This issue is investigated in Chapter Six.

2.3 Lattice Computer Implementation

Numerical accuracy of any particular method can be established from applying a number of error measures (See Chapter Six). **Speed** is measured by the time taken to compute a number of option prices. The results will be sensitive to the particular hardware and/or software employed, as already noted by Geske & Shastri (1985).

In VBA-for Excel, the declaration of variables as certain data types specifies the precision of the variable and hence the amount of memory utilised. In all the programs variables were declared for double precision. Furthermore, the efficiency of the individual procedure may be improved by considering the following:

- Minimizing storage requirements, by using a single vector instead of a matrix of prices, for both the stock prices as well as during discounting;
- Incorporating the discount factor in the risk-neutral probabilities to avoid one extra multiplication when valuing recursively;
- Perhaps peculiar to VBA is the fact that the function, which takes the maximum of two numbers, requires a lot of time, as this function has to be called from Excel. This is especially important when pricing American options using the Binomial method. Speed can be improved substantially, by replacing the MAX-function with an IF-statement;
- Once-off annualising of the option parameters, e.g. thereby saving one multiplication when discounting;
- The use of power functions was considered as an alternative to multiplication at times. The time for the computation of a small sample of options was very similar;
- Writing a double precision univariate normal integration program, and comparing the speed with that of the built-in program in Excel. On a small sample of values the programs were similar on both the accuracy and speed dimension. Multivariate normal integration will be further discussed in chapter Three.

The implementation of the **CRR** program is based on the pseudo code given in Broadie & Detemple (1996). As such a single vector of prices instead of the entire tree is stored at any one time. The risk-neutral probability is adjusted by incorporating the discount rate in it. Finally, the stock prices need only be calculated once, thereafter values are accessed through looping, which at times requires accessing every second position in a vector. Given these modifications, it was found that speed greatly improved. Pseudo code for the CRR given in **Appendix A** of this chapter.

Given the CRR program, the **BA** program is easily constructed by running the CRR program for N and $N+1$ time-steps and taking the average.

The **AB** model is also based on the pseudocode given in Broadie & Detemple (1996). The authors emphasise the importance of using a recursive formula in the computation of the terms in the binomial formula. In particular, if a binomial term b_j has already been computed, then b_{j+1} can be derived from:

$$b_{j+1} = b_j \frac{n-j+1}{j} * \frac{q}{p}, \text{ where } : b_j = \binom{n}{j} p^{n-j} q^j$$

For any given number of exercise opportunities, these terms need only be computed once. Three-point extrapolation is employed and hence it is assumed the method is employed with the number of time-steps divisible by 6.

The **BBS** program is easily obtained from the CRR program by computing the penultimate option value using BS. **BBSR** (two-point extrapolation) is then implemented by computing two values using the BBS.

The **TRI** is again based on the pseudocode given in Broadie & Detemple (1996). As with the CRR, the discount factor is included in the risk-neutral probability. Furthermore, the “stretch parameter” in Kamrad & Ritchken’s (1991) model is set to $(3/2)^{1/2}$, which results in risk neutral probabilities of 1/3 each and has been shown in literature to result in the fastest convergence. (Boyle (1988); Kamrad & Ritchken (1991)).

As with the CRR, a single vector is used for discounting. Also, a single price vector is calculated and looping utilised thereafter.

The **SMO** model is a binomial model that adjusts the CRR approach in two ways: First in the calculation of the input parameters. Secondly in the calculation of the price vector, which no longer need only be calculated once. This is because the size of the up- and down movement is no longer reciprocal.

As already mentioned above, although the programs are made as efficient as possible, relative performance may still vary when another Software is used. This is because functions defined using different Software ultimately require varying amounts of time.

2.4 A Numerical Implementation of Methods

In this section the Lattice methods are implemented with $N = 300$ and a range of values for the AMERICAN PUT calculated. The table below is reproduced and extended from Aitsahlia & Carr (1997) for the CRR, AB, BBS and TRI. Two additional decimal places were added. Furthermore The BS, AB, BBSR and SMO are shown on the next page.

K	S	d	r	vol	T	BS	CRR	BA	AB
100	80	0.02	0.06	0.4	3	25.845288	29.260868	29.263358	29.322791
100	85	0.02	0.06	0.4	3	23.940445	26.913337	26.921641	26.895555
100	90	0.02	0.06	0.4	3	22.188813	24.810536	24.806316	24.77116
100	95	0.02	0.06	0.4	3	20.57805	22.887642	22.883663	22.800911
100	100	0.02	0.06	0.4	3	19.09658	21.118519	21.130391	21.00057
100	105	0.02	0.06	0.4	3	17.733629	19.549335	19.540923	19.458681
100	110	0.02	0.06	0.4	3	16.479226	18.085127	18.089651	17.974934
100	115	0.02	0.06	0.4	3	15.324186	16.764463	16.762892	16.66325
100	120	0.02	0.06	0.4	3	14.260095	15.548499	15.548275	15.457504
100	100	0.02	0.02	0.4	3	25.518559	25.869988	25.888858	25.875732
100	100	0.02	0.04	0.4	3	22.122477	23.285814	23.300718	23.179733
100	100	0.02	0.06	0.4	3	19.09658	21.118519	21.130391	21.00057
100	100	0.02	0.08	0.4	3	16.412338	19.261011	19.270526	19.206274
100	100	0.02	0.1	0.4	3	14.041977	17.64977	17.657349	17.700175
100	100	0	0.06	0.4	3	17.427221	19.844465	19.855024	19.765615
100	100	0.02	0.06	0.4	3	19.09658	21.118519	21.130391	21.00057
100	100	0.04	0.06	0.4	3	20.834164	22.474821	22.488116	22.39991
100	100	0.02	0.06	0.3	3	13.36087	15.16119	15.169383	15.09338
100	100	0.02	0.06	0.35	3	16.23935	18.149533	18.159595	18.052864
100	100	0.02	0.06	0.4	3	19.09658	21.118519	21.130391	21.00057
100	100	0.02	0.06	0.45	3	21.922024	24.058095	24.071724	23.931564
100	100	0.02	0.06	0.5	3	24.707618	26.960605	26.975901	26.821074
100	100	0.02	0.06	0.4	0.5	10.071305	10.267695	10.275329	10.258481
100	100	0.02	0.06	0.4	1	13.386799	13.86943	13.879085	13.843379
100	100	0.02	0.06	0.4	1.5	15.539816	16.359243	16.369988	16.278281
100	100	0.02	0.06	0.4	2	17.080888	18.274268	18.285636	18.218992
100	100	0.02	0.06	0.4	2.5	18.228339	19.82454	19.836253	19.719538
100	100	0.02	0.06	0.4	3	19.09658	21.118519	21.130391	21.00057
100	100	0.02	0.06	0.4	3.5	19.754943	22.221062	22.232968	22.11101
100	100	0.02	0.06	0.4	4	20.249432	23.174869	23.186724	23.080831
100	100	0.02	0.06	0.4	4.5	20.612532	24.008988	24.020686	23.937902
100	100	0.02	0.06	0.4	5	20.868239	24.745471	24.757035	24.687534

In addition to the option price, the table demonstrates the difference between the American and European option price. This difference can be quite substantial and increases the more the option is in-the-money, the lower (for calls) / higher (for puts) the interest rate, the higher (for calls) / lower (for puts) the dividend yield, the higher the volatility and the longer the term of the option.

And,

K	S	d	r	vol	T	TRI	BBS	BBSR	SMO
100	80	0.02	0.06	0.4	3	29.254487	29.261293	29.260712	29.246518
100	85	0.02	0.06	0.4	3	26.915758	26.92255	26.920961	26.909185
100	90	0.02	0.06	0.4	3	24.79471	24.804233	24.801961	24.790696
100	95	0.02	0.06	0.4	3	22.868799	22.881595	22.87875	22.867436
100	100	0.02	0.06	0.4	3	21.118317	21.132916	21.129695	21.118519
100	105	0.02	0.06	0.4	3	19.525555	19.540114	19.536376	19.525323
100	110	0.02	0.06	0.4	3	18.07436	18.087082	18.083855	18.071706
100	115	0.02	0.06	0.4	3	16.749905	16.759366	16.755593	16.743548
100	120	0.02	0.06	0.4	3	15.538582	15.545157	15.542148	15.528355
100	100	0.02	0.02	0.4	3	25.879882	25.894836	25.888197	25.869988
100	100	0.02	0.04	0.4	3	23.290008	23.304434	23.299185	23.285814
100	100	0.02	0.06	0.4	3	21.118317	21.132916	21.129695	21.118519
100	100	0.02	0.08	0.4	3	19.257134	19.272365	19.272182	19.261011
100	100	0.02	0.1	0.4	3	17.642129	17.658723	17.661019	17.64977
100	100	0	0.06	0.4	3	19.840745	19.857101	19.854696	19.844465
100	100	0.02	0.06	0.4	3	21.118317	21.132916	21.129695	21.118519
100	100	0.04	0.06	0.4	3	22.478132	22.491423	22.487599	22.474821
100	100	0.02	0.06	0.3	3	15.161539	15.171041	15.169089	15.16119
100	100	0.02	0.06	0.35	3	18.149668	18.161619	18.158986	18.149533
100	100	0.02	0.06	0.4	3	21.118317	21.132916	21.129695	21.118519
100	100	0.02	0.06	0.45	3	24.056784	24.074683	24.071225	24.058095
100	100	0.02	0.06	0.5	3	26.957545	26.979319	26.975777	26.960605
100	100	0.02	0.06	0.4	0.5	10.27226	10.277339	10.273907	10.267695
100	100	0.02	0.06	0.4	1	13.874235	13.88144	13.877225	13.86943
100	100	0.02	0.06	0.4	1.5	16.363453	16.372538	16.368114	16.359243
100	100	0.02	0.06	0.4	2	18.277305	18.288196	18.284012	18.274268
100	100	0.02	0.06	0.4	2.5	19.825987	19.838771	19.835005	19.82454
100	100	0.02	0.06	0.4	3	21.118317	21.132916	21.129695	21.118519
100	100	0.02	0.06	0.4	3.5	22.218728	22.23541	22.233169	22.221062
100	100	0.02	0.06	0.4	4	23.170512	23.189139	23.187426	23.174869
100	100	0.02	0.06	0.4	4.5	24.001875	24.023108	24.022611	24.008988
100	100	0.02	0.06	0.4	5	24.736678	24.759399	24.760893	24.745471

2.5 Conclusion

This chapter has focused on lattice methods for approximating the American option price. Initially the inputs into the CRR were summarised. Then the particular mechanisms by which various methods aim to achieve improvement over the CRR were highlighted. Finally, a table of prices was calculated.

We will now proceed with an analysis of analytic approximations in Chapter Three.

Appendix A: CRR Pseudo Code

All variables are declared for double precision and the inputs read into the program.
Then:

```
For z = (w - 1) * h + 1 To w * h
  r(z) = r(z) * t(z)
  vol(z) = vol(z) * (t(z) ^ (1 / 2))
  sdy(z) = sdy(z) * t(z)
  dt(1) = 1 / n

'CRR
u(1) = Exp(vol(z) * (dt(1) ^ (1 / 2)))
d(1) = 1 / u(1)
q(1) = (Exp((r(z) - sdy(z)) * dt(1)) - d(1)) / (u(1) - d(1))
qq(1) = q(1) * Exp(-r(z) * dt(1))
pq(1) = (1 - q(1)) * Exp(-r(z) * dt(1))

s(n + 1) = so(z)
For j = 1 To n
  s(n + 1 - j) = s(n + 1 - j + 1) * u(1)
  s(n + 1 + j) = s(n + 1 + j - 1) * d(1)
Next

For j = 1 To 2 * n + 1 Step 2
  If s(j) > k(z) Then
    v(j) = s(j) - k(z)
  Else
    v(j) = 0
  End If
Next

i = n - 1
Do While i >= 0
  For j = -i To i Step 2
    v(n + 1 + j) = (pq(1) * v(n + 1 + j + 1)) + (qq(1) * v(n + 1 + j - 1))
    If (s(n + 1 + j) - k(z)) > v(n + 1 + j) Then
      v(n + 1 + j) = s(n + 1 + j) - k(z)
    End If
  Next
  i = i - 1
Loop

Range("i3").Offset(z - 1, 0).Value = v(n + 1)
Next
```

Chapter 3: Analytic Approximations

In the previous chapter, lattice methods for pricing American options were discussed and convergence improvement features analysed. The focus of this chapter is on analytic approximations and five methods will be discussed in this chapter. Among them, they constitute the most efficient methods for pricing American call options on a single, dividend paying underlying.

The first method implemented is GJ2, the two-point Geske-Johnson approximation. Second and third are the three-point Method of Lines (ML) and approximations based on Lower and Upper bounds (LBA and LUBA). Two further very recent methods are the Gaussian Quadrature method (SUL) and the Multipiece Exponential method (JU).

3.1 The Two-Point Geske-Johnson Method

A compound option is an option on an option. Geske (1979) originally shows how to value compound options. The evaluation of the compound option approach to American Options, requires the evaluation of an infinite series of multivariate normal probabilities. In particular, the **GJ2** approach requires Bivariate normal integration.

3.1.1 Bivariate Normal Intergation

Given the standard bivariate normal distribution function, Sheppard (1900) transformed the bivariate expression to a one-dimensional integral. Drezner and Wesolowsky (1990) (DW) differentiated this expression with respect to ρ (the correlation coefficient) and integrated from 0 to ρ . In Genz (1992) (G) an additional substitution of $r = \sin(\theta)$ is made and a transformed expression for the Bivariate distribution function obtained. In both methods, integration is done numerically using Gaussian Quadrature.

In a detailed simulation analysis in single precision, the accuracy of G's method was superior to that of DW.

In both methods there exists a singularity when $\rho = 1$, and the methods loose accuracy when ρ is close to one. Genz (1992) provides an additional improvement when the correlation coefficient is close to one.

Now, for Geometric Brownian motion the correlation between two consecutive time periods is $0.5^{1/2}$ ($=0.7071$), which is the correlation coefficient utilised in the GJ2 method. For this correlation, Genz (1992) recommends the use of 12 point Gaussian Quadrature, and this is implemented in double precision. As with all other methods, univariate normal integration is performed using the function provided by Excel.

3.1.2 Computer Implementation

The GJ2 method is implemented to calculate the value of the American PUT. In order to proceed with the bivariate integration, the GJ2 method requires the critical stock price at $T/2$. Barone-Adesi & Whaley (1987) provide an initial guess to the stock price for an American option. This provides a lower bound to the price required by GJ2 and hence the Bisection method is implemented, with the strike as the upper bound. The critical price is determined to be correct within 0.001. Finally, two-point Richardson Extrapolation is used.

3.2 Gaussian Quadrature and Function approximation

Sullivan (2000) develops a technique for approximating the option value function using a combination of numerical integration and function approximation. The integral for the American put option is divided into two parts: one part gives the integral below the early exercise boundary, which has a closed form solution U . The other part, W , gives the risk-neutral expectation above the early exercise boundary. Gaussian quadrature is used to evaluate it.

As one moves farther back from maturity, Gaussian quadrature on its own would represent an exponential effort to calculate option prices. Hence, an approximating function is proposed to summarize the option price information by means of a Chebyshev polynomial at each time step. This implies that at each time step, the Gaussian rule need only be employed for one-dimensional integrals.

3.2.1 Computer implementation

The Gaussian quadrature method is implemented for the American put. At one time step away from maturity, the Black-Scholes formula is used to calculate the exercise boundary ($lowB$) using the bisection method. A lower guess is obtained from Barone-Adesi & Whaley (1987) approximation of the critical stock price for an American option. At the first time step the upper guess of the exercise boundary in the bisection method is given by the strike price. As with all remaining time steps, the exercise boundary is calculated such that the option is equal to the immediate exercise value within 0.0001.³

Still needed is an upper boundary for the Chebyshev function approximation. At time step 1, the initial upper guess is given by $uppb(1) = strike + 2 * (strike - exerciseb(1))$. This is incremented by a fixed 15% of this $uppb(1)$ until the option value is less than 0.005. Then bisection is applied until the option value is greater than 0.005 and less than 0.001.

Then 8-point Chebyshev function approximation is fitted to the log-option prices. Clenshaw recurrence is used to evaluate the function.⁴ Then 16-point Gaussian quadrature is used to find a lower (exercise) boundary in the similar way as for time step 1. The upper boundary on the other hand, is found using the BS equation, rather than quadrature in order to speed up the search.⁵ Again a Chebyshev polynomial is fitted, new upper and lower bounds found at the next time step, etc.

At $t=2$, a lower guess for the lower (exercise) boundary is given by the Barone-Adesi & Whaley (1987) approximation.

For $t > 2$, it is given as $lowb(t) = 2 * lowb(t-1) - lowb(t-2)$.

On the other hand, a initial guess of the upper boundary is given by $uppb(t) = uppb(t-1) * dx$, where $dx = uppb(t-1) / uppb(t-2)$. This guess is incremented by a fixed 10% of this initial upper guess for as long as the option value is less than 0.005. If this loop is only run once, bisection is applied with $uppb(t-1)$ as the lower guess, else $uppb(t) - increment$.

³ Recommended by Michael Sullivan per Email

⁴ See Press, et al (1992) for a reference on Gaussian Quadrature as well as Chebyshev function approximation.

⁵ Recommended by Michael Sullivan per Email

The replacement of the BS formula for the gaussian quadrature and chebyshev approximation in the search for the upper bound calculation does not only improve the speed, but also avoids errors in the function approximation.

Recall that the upper boundary at the previous time step is likely to be less than the new upper boundary. Therefore, irrespective of rescaling the new upper boundary to the previous chebyshev interval, the chebyshev function would be evaluated at a stock price that falls outside the range over which the chebyshev function was originally defined. Then, during gaussian quadrature, stock prices are adjusted even further away. This extrapolation may represent a source of “instability”, as the function approximation may no longer be accurate.

From the above then, the method is implemented in two ways. First is the geometric four-point extrapolation **SUL4** as is recommended by Sullivan (2000). Here four option prices are found allowing for 1,3,9 and 27 opportunities for exercise and then extrapolated using the following weights: (-1, 39, -351, 729) / 416.

Given that it takes about $\frac{1}{4}$ of a second to compute a option price using the CRR with 300 time steps, an attempt to improve the speed in this approach, is the two-point extrapolation based on 6 and 12 exercise opportunities respectively **SUL2**. The weights used in the extrapolation are the usual (-1,2).

3.3 The Analytic Method of lines (ML)

Carr & Faquet (1994) approximate the partial differential equation by replacing the derivative of the option price with respect to time, with a finite difference of the change in stock price over a discrete time interval.

Rearranging this PDE implies that the free boundary problem is replaced with a recurrent sequence of free boundary problems involving ODE's, each of which is solved analytically.

The approach is entirely analytic, besides numerically solving for the critical stock-price in the case where the underlying pays a continuous proportional dividend. In particular, no special functions besides powers and the natural logarithm is needed. Finally, arbitrary accuracy can be achieved at the expense of lower speed, simply by adding more points to the extrapolation.

3.3.1 Computer Implementation

This program is implemented for the American put. The three-point method of lines was programmed directly from the equations in the Appendix of the paper. Since options based on an underlying that pays a constant proportional dividend are considered, the critical stock price need to be solved numerically. Using Newton's Method the critical stock price is obtained accurate to 0.0001. Their modified three-point extrapolation is used, with the middle weight adjusted by a factor of $[1 - 0.0002*(5-T)]$, where T is the time to maturity.

3.4 The Multipiece Exponential Function Approach

In this approach American options are priced by approximating the early exercise boundary as a multipiece exponential function. This exponential boundary allows the premium integral to be evaluated analytically. Two-point (**JU2**) as well as three-point (**JU3**) Richardson extrapolation, based on the number of exponential functions used to approximate the exercise boundary, is applied.

3.4.1 Computer Implementation

The Program is implemented for the American put⁶. The exercise boundary is approximated by one or more functions of the form:

$$\text{boundary} = B * \exp(b * \text{time}).$$

Once analytically integrated, the price of an American option with a one, two, three, etc. piece exponential boundary can be obtained. What remains is to numerically solve for the B's and b's respectively. By applying the two-dimensional Newton's method to the "value match" and "high contact" condition, the B's and b's are obtained. The "value match" condition equates the price to the exercise premium, while the "high contact" condition is the derivative of the "value match" condition with respect to the boundary.

For the one-piece exponential function, an initial guess for B is obtained by the approximation given by Baroni-Adesi & Whaley (1987). Thereafter, the Baroni-Adesi & Whaley (1987) equation for the critical stock price is implemented and solved by a variation of the Bisection method. The process continues until the adjustment to the boundary is less than 0.01 for a maximum of 7 iterations.

Initially, b is set to zero. Using Kramer's rule the "value match" and "high contact" conditions are solved simultaneously. The process continues until both adjustments to B and b are less than 0.001 for a maximum of 20 iterations. Thereafter, previously calculated B's and b's are used as initial estimates for the next B's and b's respectively.

The above is the standard method, however, Ju has included a further adjustment. If the difference between the exercise boundary at maturity ($B_T = \min(K, Kr/\delta)$), and the initial guess by Baroni-Adesi & Whaley (1987) is less than 5% of B_T , then the multipiece exponential function becomes a multipiece step function. In other words, all b's are set to zero and not adjusted.

At this point the procedure was speeded up, by eliminating redundant calculations in Kramer's rule as well as derivative calculations relating to the b's, as the solution is now a one-dimensional problem.

Finally, extrapolation based on the two-piece exponential **JU2** as well as the three-piece exponential **JU3** boundary approximation are calculated.

⁶ Professor Ju provided me with Fortran code, which was recoded in VBA.

3.5 LBA and LUBA methods

This approach is implemented for the American CALL. Broadie & Detemple (1996) propose two approximation methods based on lower and upper bounds for the American option price.

The authors propose a lower bound approximation to the American option price labeled LBA as well as an approximation, based on a weighted average of both the lower and upper bound, labeled LUBA.

3.5.1 A Lower Bound on the American Call Option Value

The lower bound is based on the value of a 'capped call' option. Let $C_t(S_t)$ [$c_t(S_t)$] denote the value of the American [European] call option, and $C_t(S_t, L)$ denote the value of the capped call option with automatic exercise when S hits L and with constant cap L and payoff $\max(\min(S, L) - K, 0)$. The analytic formula for these capped call options is derived in Broadie & Detemple (1995).

This formula for $C_t(S_t, L)$ gives an immediate lower bound for $C_t(S_t)$ for any L . Hence, $C_t(S_t, L)$ can be maximised with respect to L to give a lower bound on the American call option value $C_t^l(S_t)$. This maximisation is a univariate differentiable unconstrained optimisation problem (with a unique maximum).

3.5.2 Implementation of the lower bound

Consider the bisection method on the derivative of $C_t(S_t, L)$ with respect to L : $C_t(S_t, L)'$. To begin, an initial guess equal to the maximum of the strike and spot price is specified for call options. If at the outset $C_t(S_t, L)'$ is negative then the initial guess is reduced in steps of 5.

An initial guess for the upper bound was obtained from incrementing the initial guess by 250 for as long as $C_t(S_t, L)'$ was positive. Then, using bisection, the maximum cap L^* was calculated to be correct within 0.05. Finally, $C_t(S_t, L)$ is evaluated at L^* .

Over a small sample of 20 options the answers differed in the 6th decimal place when instead the cap was calculated to be within 0.005, and this took about 0.55 instead of 0.49 seconds in total. The more accurate version was implemented, as accuracy in high decimal places is what subsequent regression is hoping to utilise.

When American symmetry was used to evaluate the put price, the initial guess is the strike, instead of the maximum of the strike and spot price, as for calls. Further, increments were in steps of 20, instead of 250. This is because interest rates in SA are much higher relative to dividend yields, so that when a high dividend yield is used in the model, L is closer to the strike price. It also means, that in the program for Put options, the initial evaluation of $C_t(S_t, L)'$ is never negative.

However, for call options there is a possibility that the initial evaluation of $C_t(S_t, L)'$ is negative, in which case one could directly calculate the payoff and hence the option price. However, the high interest rates and low dividend yields in SA, make this possibility remote.

3.5.3 An Upper Bound on the American Call Option Value

Let $D(L,t)$ ⁷ be defined as the derivative of $C_t(S_t,L)$ with respect to L , evaluated as S_t approaches L from below. Then the solution of $D(L,t) = 0$ gives a lower bound to the optimal exercise boundary at a certain time to expiry. Solving for L at various discretisations in the time to maturity interval, a lower bound for the boundary is constructed. This implies an upper bound for the American option price.

The integral representation (see for example Kim (1990)), separates the American price into the European part and an integral, representing the early exercise premium. Some methods discussed earlier solve for the optimal exercise boundary by recursively solving the “value match” condition, i.e. solving for the stock price that equates the option value to its payoff on exercise. In contrast, Broadie et al. (1996) obtain the boundary via the capped call option route as described in the paragraph above. To obtain the upper bound for the American Call option $C_t^u(S_t)$, knowledge of this approximate boundary is utilised in conjunction with numerical integration to calculate $C_t^u(S_t)$.

3.5.4 Implementation of the upper bound

Newton’s method is used to solve $D(L,t) = 0$ for L^* at $n = 8$ discretisations during the time interval. An initial guess is the strike price. Iterations proceed until the boundary is correct within 0.1. Finally, Simpson’s rule also with $n = 8$ is used to evaluate the integral. When calculating the 20 values in the table given in the author’s publication, it took only just over 1 second using Newton’s method.

Tested also, was a variation of the Bisection method and it required about 2.5 seconds for the 20 values. Here the first iteration uses Newton and for subsequent iterations a lower guess is obtained by subtracting e.g. 50 from the previous lower bound of the decreasing optimal exercise boundary. This method is not only slower, but does also not converge if the amount subtracted does not turn out to be a lower guess.

3.5.5 Regression for LBA and LUBA

The approximations are computed as follows:

$$\text{LBA} = \lambda_1 C_t^l(S_t), \text{ and}$$

$$\text{LUBA} = \lambda_2 C_t^l(S_t) + (1-\lambda_2) C_t^u(S_t) \text{ for weights } \lambda_1 \geq 1 \text{ and } 1 \geq \lambda_2 \geq 0.$$

The coefficients λ_1 and λ_2 are predicted using regression and weighted regression respectively. The predictor variables in the regression X_i , are a function of the six option price input parameters. These predictor variables are given in the **Appendix A** to this chapter and a minor modification to the Broadie & Detemple (1996) publication described.

The coefficients in the regression are estimated from a specifically sampled test set of options: A total of 1792 combinations of option input parameters have been simulated. The data represents all combinations ($7*4*4*4*4$) of a number of uniformly distributed discrete subsets, where these subsets cover a practical range of each option input parameter in the South African context. A detailed discussion of the input distributions is provided in conjunction with the overall numerical analysis in Chapter Six.

⁷ The equation for $D(L,t)$ is derived in Broadie & Detemple (1996).

LBA

A total of 10 predictor variables X_1 to X_{10} are collected in a matrix X . A least squares regression model is fitted to predict λ_1 . Let the predicted value be Y_1 .

Finally,

$$\lambda_1 = 1 \quad \text{if } C_t^l(S_t) = c_t(S_t) \text{ or if the option falls in the early exercise region;}$$
$$\lambda_1 = \max(\min(Y_1, 1.005656943), 1) \quad \text{otherwise, where 1.005656943 is the maximum } Y_1 \text{ in the regression.}$$

LUBA

Here a total of 15 X_i 's are defined in order to predict λ_2 . However, solving for λ_2 in the defining equation of LUBA, results in a division by zero, if $C_t^l(S_t) = C_t^u(S_t)$. The authors suggest therefore to weight each case by $(C_t^u(S_t) - C_t^l(S_t)) / C_t^l(S_t)$. This has the effect of emphasising cases in the regression where there is a substantial difference between $C_t^l(S_t)$ and $C_t^u(S_t)$ and is therefore an intuitively useful improvement. The λ_2 are directly calculated from $\lambda_2 = ((C_t^u(S_t) - C_t^l(S_t)) / C_t^l(S_t))$. Cases where $C_t^l(S_t) = C_t^u(S_t)$ have zero weight. Out of the 1792 cases, 160 had a zero weight.

Finally,

$$\lambda_2 = 1 \quad \text{if } C_t^l(S_t) = c_t(S_t) \text{ or if } C_t^l(S_t) \text{ falls in the early exercise region;}$$
$$\lambda_2 = \max(\min(Y_2, 1), 0) \quad \text{otherwise, where } Y_2 \text{ is the predicted value.}$$

A much more detailed discussion on these regressions will follow in the next chapter.

3.6 A Numerical Implementation of Methods

The table recalculates and extends the American PUT prices given in AitSahlia & Carr (1997). As for the Lattice approaches, two decimal places were added.

K	S	d	r	vol	T	GJ2	SUL 2	SUL 4	ML
100	80	0.02	0.06	0.4	3	29.8107044	29.264711	29.2452006	29.2323055
100	85	0.02	0.06	0.4	3	27.3733825	26.9229042	26.9099348	26.8922711
100	90	0.02	0.06	0.4	3	25.156666	24.8024945	24.7942453	24.7692461
100	95	0.02	0.06	0.4	3	23.1414016	22.8782284	22.8734704	22.8387583
100	100	0.02	0.06	0.4	3	21.3092753	21.1285599	21.1264392	21.0834988
100	105	0.02	0.06	0.4	3	19.6431058	19.534895	19.5348011	19.4899264
100	110	0.02	0.06	0.4	3	18.1269936	18.0810475	18.0825313	18.0369189
100	115	0.02	0.06	0.4	3	16.7463734	16.7528332	16.7555556	16.7070241
100	120	0.02	0.06	0.4	3	15.4880136	15.5377569	15.5414553	15.4873188
100	100	0.02	0.02	0.4	3	25.8268891	25.8902881	25.8873004	25.8788845
100	100	0.02	0.04	0.4	3	23.3248437	23.2995101	23.29778	23.2504296
100	100	0.02	0.06	0.4	3	21.3092753	21.1285599	21.1264392	21.0834988
100	100	0.02	0.08	0.4	3	19.5721942	19.2691999	19.2655801	19.2402798
100	100	0.02	0.1	0.4	3	18.0146592	17.6588661	17.6505879	17.646689
100	100	0	0.06	0.4	3	20.0641177	19.8511058	19.8505279	19.8145379
100	100	0.02	0.06	0.4	3	21.3092753	21.1285599	21.1264392	21.0834988
100	100	0.04	0.06	0.4	3	22.6305101	22.4897247	22.4857416	22.4418956
100	100	0.02	0.06	0.3	3	15.3314367	15.1668477	15.1668841	15.1374798
100	100	0.02	0.06	0.35	3	18.32989	18.1571813	18.1564099	18.1196861
100	100	0.02	0.06	0.4	3	21.3092753	21.1285599	21.1264392	21.0834988
100	100	0.02	0.06	0.45	3	24.2608866	24.070729	24.0673686	24.0189122
100	100	0.02	0.06	0.5	3	27.1775651	26.9758202	26.9713115	26.9180269
100	100	0.02	0.06	0.4	0.5	10.2310661	10.2726604	10.2735786	10.2758901
100	100	0.02	0.06	0.4	1	13.8387853	13.8754798	13.8767593	13.8669515
100	100	0.02	0.06	0.4	1.5	16.3622313	16.366387	16.3673725	16.3469437
100	100	0.02	0.06	0.4	2	18.3291195	18.2822583	18.2827754	18.2533227
100	100	0.02	0.06	0.4	2.5	19.9434087	19.8335298	19.8328185	19.7960482
100	100	0.02	0.06	0.4	3	21.3092753	21.1285599	21.1264392	21.0834988
100	100	0.02	0.06	0.4	3.5	22.4882766	22.2324328	22.2287686	22.1803577
100	100	0.02	0.06	0.4	4	23.5202763	23.1873605	23.1813514	23.1288968
100	100	0.02	0.06	0.4	4.5	24.4327833	24.0230347	24.0144282	23.9585199
100	100	0.02	0.06	0.4	5	25.2456923	24.7611524	24.7493975	24.6906347

And the next four methods:

K	S	d	r	vol	T	JU2	JU3	LOW B	UPP B
100	80	0.02	0.06	0.4	3	29.2960875	29.2662661	29.1070713	29.3741033
100	85	0.02	0.06	0.4	3	26.9605869	26.9270379	26.767353	27.0251452
100	90	0.02	0.06	0.4	3	24.8440134	24.8079289	24.6502973	24.8992926
100	95	0.02	0.06	0.4	3	22.9219254	22.8843213	22.7305436	22.9706293
100	100	0.02	0.06	0.4	3	21.1732776	21.1351138	20.9863106	21.2171218
100	105	0.02	0.06	0.4	3	19.5797813	19.5418702	19.3987417	19.619906
100	110	0.02	0.06	0.4	3	18.1254637	18.0884596	17.9514062	18.1626728
100	115	0.02	0.06	0.4	3	16.7962982	16.7607065	16.629913	16.8311946
100	120	0.02	0.06	0.4	3	15.5799315	15.5461243	15.4216229	15.6129882
100	100	0.02	0.02	0.4	3	25.9083851	25.8908609	25.8188607	25.9020728
100	100	0.02	0.04	0.4	3	23.3367499	23.3032054	23.1667215	23.3569843
100	100	0.02	0.06	0.4	3	21.1732776	21.1351138	20.9863106	21.2171218
100	100	0.02	0.08	0.4	3	19.3145684	19.2774049	19.1383511	19.3769805
100	100	0.02	0.1	0.4	3	17.6999528	17.666072	17.5452601	17.7773314
100	100	0	0.06	0.4	3	19.8980957	19.8607043	19.7079585	19.9645992
100	100	0.02	0.06	0.4	3	21.1732776	21.1351138	20.9863106	21.2171218
100	100	0.04	0.06	0.4	3	22.5299439	22.4918255	22.3574338	22.5492016
100	100	0.02	0.06	0.3	3	15.2013343	15.1744592	15.0585568	15.247172
100	100	0.02	0.06	0.35	3	18.1974157	18.1639148	18.0311261	18.2422554
100	100	0.02	0.06	0.4	3	21.1732776	21.1351138	20.9863106	21.2171218
100	100	0.02	0.06	0.45	3	24.1196312	24.077011	23.9139583	24.1620356
100	100	0.02	0.06	0.5	3	27.0286944	26.9818045	26.8060759	27.0693022
100	100	0.02	0.06	0.4	0.5	10.2830901	10.2752818	10.2257628	10.2914516
100	100	0.02	0.06	0.4	1	13.8956382	13.8793956	13.7959165	13.9118602
100	100	0.02	0.06	0.4	1.5	16.3945302	16.3711232	16.2629381	16.4184254
100	100	0.02	0.06	0.4	2	18.3171459	18.2878111	18.1615024	18.348275
100	100	0.02	0.06	0.4	2.5	19.9405341	19.8062542	19.7001375	19.9116023
100	100	0.02	0.06	0.4	3	21.1732776	21.1351138	20.9863106	21.2171218
100	100	0.02	0.06	0.4	3.5	22.2805664	22.2391766	22.0840702	22.3298615
100	100	0.02	0.06	0.4	4	23.2382189	23.1942459	23.0352234	23.2924227
100	100	0.02	0.06	0.4	4.5	24.0758763	24.0298375	23.8688246	24.1345096
100	100	0.02	0.06	0.4	5	24.8151241	24.7674698	24.6060114	24.8778002

LBA and LUBA are not included, as our SA test set does not include all ranges of input parameters given in the table.

3.7 Conclusion

This chapter has detailed various analytic approximations as well as their computer implementation. Many of these analytic approximations are likely to be faster and more accurate than most lattice approaches, often the reason being that much of the risk-neutral integration is done analytically.

The next chapter, analyses the standard regression approach for LBA and LUBA.

Appendix A: LBA and LUBA predictor variables

1) LBA Variables

$$X_1 = T$$

$$X_2 = T^{1/2}$$

$$X_3 = S / K$$

$$X_4 = r$$

$$X_5 = \delta$$

$$X_6 = \min(r/(\max(\delta, 0.00001)), 5)$$

$$X_7 = X_6^2$$

$$X_8 = (C_t^l(S_t) - c_t(S_t)) / K$$

$$X_9 = X_8^2$$

$$X_{10} = C_t^l(S_t) / c_t(S_t)$$

2) LUBA Variables

$$X_1 = T$$

$$X_2 = T^{1/2}$$

$$X_3 = r$$

$$X_4 = \delta$$

$$X_5 = \min(r/(\max(\delta, 0.00001)), 5)$$

$$X_6 = X_5^2$$

$$X_7 = \text{partial derivative of } C_t(S_t, L) \text{ with respect to } S$$

$$X_8 = X_7^2$$

$$X_9 = (C_t^l(S_t) - c_t(S_t)) / K$$

$$X_{10} = X_9^2$$

$$X_{11} = C_t^l(S_t) / c_t(S_t)$$

$$X_{12} = (C_t^u(S_t) - C_t^l(S_t)) / K$$

$$X_{13} = C_t^u(S_t) / C_t^l(S_t)$$

$$X_{14} = S / L_{t=0}^*$$

$$X_{15} = X_{14}^2$$

Comment: For Variable X_{14} , L^* is the approximate lower bound to the exercise boundary obtained in the calculation of the upper bound of the American option price. In our implementation, L^{\wedge} , the constant cap in the lower bound calculation was used instead of L^* .

This has a minimal impact on the correlation structure of the above X variables, especially so after reweighting for the weighted regression.

Chapter 4: Regression for LBA and LUBA

The aim of this chapter is to evaluate the impact that outliers, influential observations and collinearity, have on fitting regression models to the upper and lower bounds of the American option price.⁸

4.1.1 Motivating Model Selection

First, presence of collinearity in a data set, would not cause problems, if the predictions using the model are made from the same X-space, that the model was estimated from, i.e. we value only call options, since the model was fitted to call option data. However, LBA and LUBA should also be applicable to value put options, even when the model was based on call option data.

This creates a problem in the South African environment when using American put-call symmetry as the model that values calls, is based on a dividend yield of about 2% and an interest rate of about 17%. If symmetry is used, this would imply an interest rate of about 2% and a dividend yield of about 17%. As a result, predictions are based on X-values that fall outside that range that the model was originally estimated on, and it is here that collinearity should cause a problem.

In addition to stepwise and weighted regression, two biased regression approaches, ridge regression and principal component regression, will be adopted to counter the problem of collinearity.

Furthermore, the absence of collinearity in a model, would allow the pricing of puts and calls in the SA environment, with only one regression model. For put estimation, different models are necessary if the least squares estimate of a model based on put data, differs substantially from the biased regression approaches based on call data. This will be tested in Chapter Six.

A **second** argument we will explore is the presence of outliers in the dependent variable (as indicated by large studentised residuals) and influential observations in the X-space (as indicated by a high leverage through Cook's distances). Outliers and influential observations can have negative consequences for predictions.

4.1.2 The Search Process

The data set that is used to estimate the regression models, is based on all combinations of seven moneyness classes, 4 interest rate categories, 4 dividend yield categories, 4 volatility categories as well as 4 maturity categories, i.e. $7*4*4*4*4 = 1792$ combinations of input parameters. See Chapter Six for a detailed discussion. The X-variables are given in the **Appendix A** of the previous chapter. Our approach is identical to that of Brodie & Detemple (1996), with a minor modification as explained in the abovementioned **Appendix A**.

The aim of this chapter is to explore models over this X-space. In this sense, this X-space forms the "population" space of independent variables. In other words, if we would resample our 1792 categories, we would obtain a very similar sample to the

⁸ As a reference to this Chapter: Lecture Notes in Regression at Statistics Honours Level at the University of Cape Town

one we already have, as the sampling is sufficiently stratified. Put another way, our sample systematically covers all corners of the X-space.

This allows us to estimate and test models on the same data set. The process of finding good models is therefore akin to optimisation.

A different much larger data set will be used in the final application of these models. In Chapter Six, all models will be compared on a dataset, where 50 observations are sampled within a few selected combinations of “extreme” input parameter classes. This restriction is necessary, as it is practically very challenging to sample a number of options from each of the 1792 combinations and interpret the results.

The decision criteria as to which is the best model, is the RMS or Root Mean Squared (RMS) relative error. It is fully described in Chapter Six. Models that produce lower RMS's are considered better.

The least squares model will be used to identify outliers and influential observations. Once these are identified they are removed from the dataset and new models fitted. We will show that the assumption of normality does not hold. However, even without normality, we utilise least squares model to estimate residuals, since the least squares model is robust to departure from normality.

Furthermore, autocorrelations are artificial and due to the structured way the sample was calculated. Heteroskedasticity in the errors is absent. Neither of the above two, should invite a time series interpretation.

Our regression coefficients turn out to be stable once outliers are removed and also similar to those of biased regression models.

These arguments allow us to identify outliers and influential observations using least squares.

Alternatively, the identification of outliers can be based on a biased regression model in the absence of normality.

4.2 Regression for LBA

The regression for LBA predicts a constant that multiplies the lower bound of the American option price. We will first investigate collinearity and then move on to outliers.

4.2.1 Collinearity for LBA

In order to assess collinearity, we applied three tests to our data set: the condition number, the regression coefficient variance decomposition and variance inflation factors.

4.2.1.1 The Condition Number

The Singular-Value-Decomposition of the matrix of column-centred X variables was computed and the square root of the eigenvalues, or singular values obtained. The condition index is defined as the ratio of the largest singular value over each of the 10 singular values respectively.⁹ These are given below:

Condition Indices

1.0
5.5
54.9
67.0
94.5
266.7
1024.5
1135.3
9147.0
342221.6

Although condition indices between 30 and 100, indicate moderate to strong dependencies, the last 5 indices are greater than 100.

4.2.1.2 Regression Coefficient Variance Decomposition

A second measure to assess collinearity is regression coefficient variance decomposition.

Table of Regression coefficient variance decomposition

Sing Val	Var(b1)	Var(b2)	Var(b3)	Var(b4)	Var(b5)	Var(b6)	Var(b7)	Var(b8)	Var(b9)	Var(b10)
372.1598	0.0000	0.2891	0.0000	0.0006	0.7079	0.0000	0.0002	0.0022	0.0000	0.0000
67.6408	0.0000	0.0082	0.0000	0.0007	0.9898	0.0000	0.0001	0.0011	0.0000	0.0000
6.7749	0.0000	0.0000	0.9966	0.0028	0.0000	0.0000	0.0000	0.0003	0.0003	0.0000
5.5509	0.0000	0.0000	0.0000	0.0014	0.0000	0.9070	0.0848	0.0067	0.0000	0.0000
3.9380	0.0000	0.0000	0.0000	0.0000	0.0000	0.0066	0.8698	0.1234	0.0002	0.0000
1.3955	0.0003	0.0000	0.0018	0.9669	0.0014	0.0238	0.0018	0.0032	0.0007	0.0000
0.3633	0.4293	0.0000	0.0010	0.5522	0.0009	0.0127	0.0006	0.0029	0.0004	0.0000
0.3278	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0014	0.0137	0.9411	0.0438
0.0407	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
0.0011	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0499	0.4334	0.5140	0.0027

⁹ If the SVD would be computed on $X'X$, then the condition number is defined from the ratio of the eigenvalues, or squared singular values.

Let v_{ij} be the j^{th} element of the i^{th} eigenvector and w_i our i^{th} singular value. Define the ij^{th} variance-decomposition component, of the variance of the j^{th} regression coefficient associated with the i^{th} principal component as v_{ij}^2/w_i . The ratio of this component, over the sum of components over all principal axes, gives the regression coefficient variance decomposition.

Here a variance-decomposition proportion is considered high if it is greater than 0.5 for two or more variances of estimated regression coefficients. The 7th and 10th principal axis simultaneously explain most of the variance of two betas respectively.

4.2.1.3 Variance inflation factors

A third and final assessment is given by variance inflation factors. When X is centred and standardised (i.e. in correlation form), the VIF's are given by $1/(1-R_i^2)$, where R_i^2 is the coefficient of determination from the regression of X_i on the other dependent variables. Hence, a number of regressions were run on the scaled X matrix.

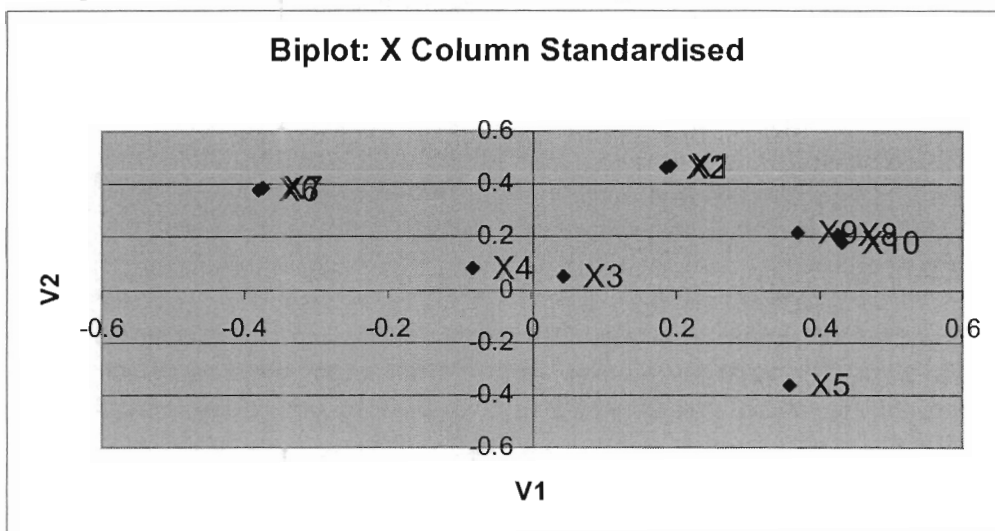
Dep Var	R ²	VIF's
SX1	0.970952	34.42561
SX2	0.970275	33.64187
SX3	0.060444	1.064332
SX4	0.333153	1.499595
SX5	0.881429	8.433757
SX6	0.990794	108.6246
SX7	0.99087	109.5336
SX8	0.966007	29.41745
SX9	0.890626	9.142981
SX10	0.927877	13.86529

Five out of the ten VIF's are greater than 10 and hence indicate strong collinearity.

From the condition indices, the regression coefficient variance decomposition and the variance inflation factors, we can conclude that serious collinearity is present in the data.

4.2.2 Displaying the Correlations in Two Dimensions

The biplot of the column standardised matrix is given below.



The quality of approximation is 64.08%. It is calculated as the sum of the ratio of the first two squared singular values, over the squared norm of the vector of singular values. The norm is just the square root of the sum of all squared elements of a vector.

The cosine of the angle between two points subtended at the origin gives the approximate correlation between the two variables. The high correlation, between X_1 and X_2 , X_6 and X_7 , as well as among X_8 , X_9 and X_{10} is apparent. These correlations are confirmed by the VIF's above. Returning to the data, we find that $X_2 = X_1^{1/2}$, $X_7 = X_6^2$ and $X_9 = X_8^2$. When one variable is defined as a function of another, this is a potential source of collinearity. Other sources of collinearity are described below.

4.2.3 Sources of Collinearity

The above diagnosis has shown that collinearity is present. In this section possible sources of collinearity are suggested. The collinearity may have arisen due to the following:

- 1) Generating new variables as transformations of another variable, e.g. ratios or powers of variables frequently may be nearly collinear with the original variables.
- 2) The trimming of data ranges may induce correlation. For example, when attempting to avoid large numbers, when dividing by e.g. a small dividend yield.
- 3) Outliers can induce artificial collinearities among the predictor variables.

A factor that could not have caused the collinearity in our data is the following:

- 1) Inadequate sampling, where sampling was done from a lower –dimensional subspace. This is unlikely as the complete X-space was categorised and therefore sampled sufficiently stratified.

4.2.4 Effects of Collinearity

In the least squares calculation of the betas, when a variable is involved in a multicollinearity, the estimate of its coefficient usually deviates far from the true value. Furthermore, the variance of the betas corresponding to small singular values is inflated. Also, small perturbations on the X's or Y's can cause significant changes in regression coefficients.

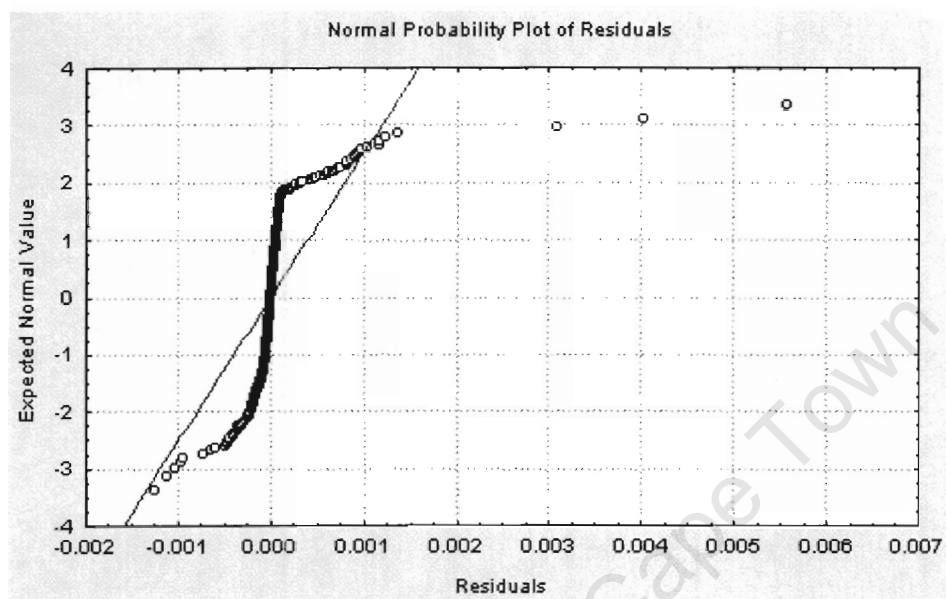
However, collinearity does not always imply a poor prediction as the whole vectors of betas is used. Only, when the model is used outside the original space of the predictor variables, predictions may be poor.

4.2.5 Testing for Normality

Least squares estimation is robust to non-normality.

4.2.5.1 The Normal Probability Plot

A normal probability plot, orders the estimated residuals and plots them against the expected $N(0,1)$ deviates.



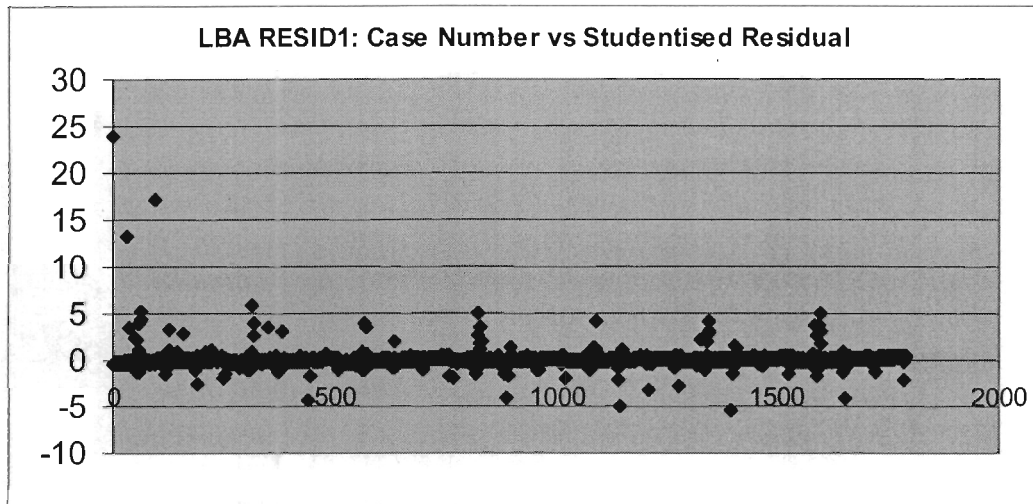
This S-shaped pattern of the residuals indicates a heavy- or long tailed error distribution. The Shapiro–Wilk test for normality calculates the correlation between the expected and observed order statistics taking the correlation structure of the ordered residuals into account. A low correlation will indicate departure from normality. The observed value is $W = 0.34491$, which is significant at less than 0.1%. Hence, we can conclude that the data is not normal. This is also supported by a skewness of 12.363 and a kurtosis of 254,167 of the residuals. Three outliers can be easily detected. These three relate to options that had a true value of less than 0.00001, i.e. the options were very out-money, had moderate to low interest rates and dividend yields, low volatilities and were short term.

4.2.5.2 Autocorrelation and Heteroskedasticity

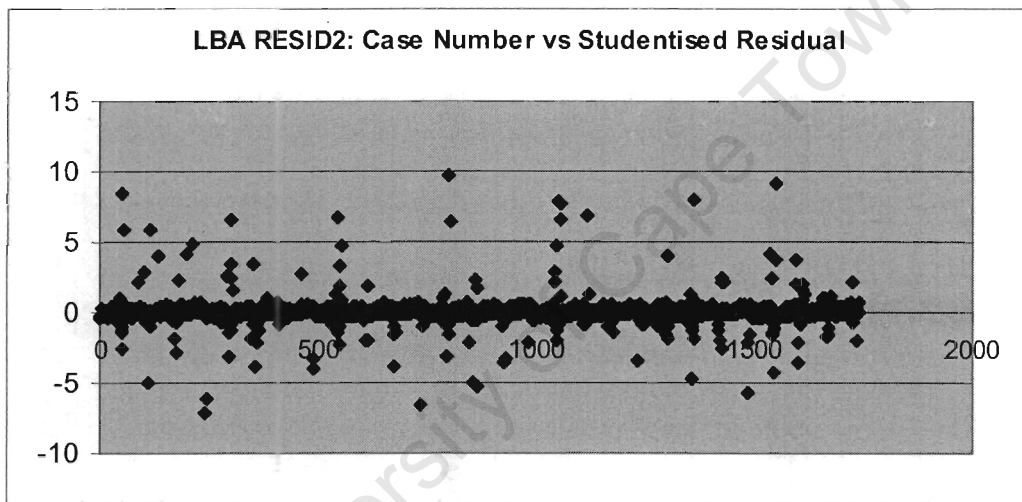
The graphs on the next page plot the case number against the studentised residual for each of the first three iterations of deleting outliers. Iterations and criteria for deleting outliers, is discussed in the next section.

Generally, at each iteration the residuals seem to be evenly scattered on either side of the x-axis. However, there does appear to be an occasional vertical stacking of outliers along the x-axis. This should not invite a time-series interpretation, but is due to the structured way the sample is simulated. In other words the order of the residuals doesn't matter. Observations with large residuals are truly outliers, probably corresponding to short term, low volatility, out-the-money, etc. options.

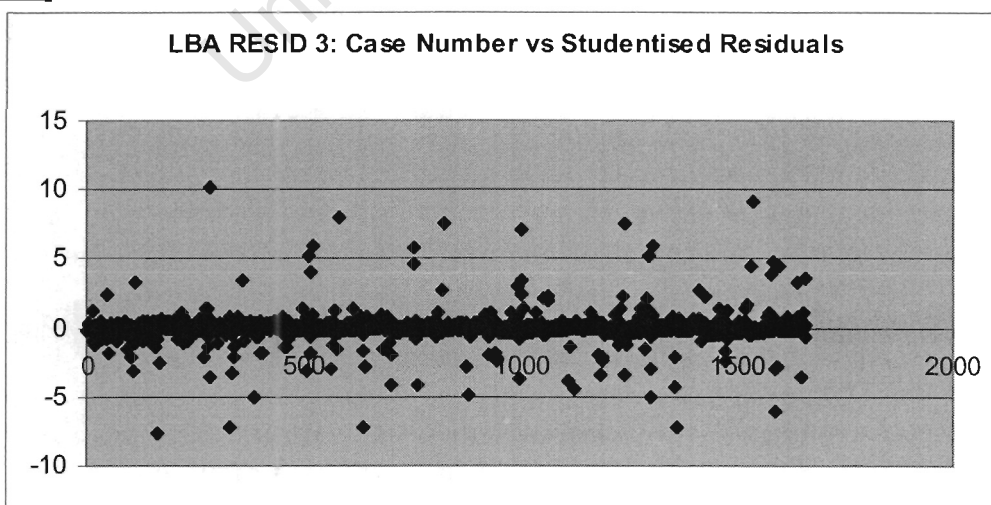
Graph 1



Graph 2



Graph 3



We will now proceed to identifying outliers in the data.

4.2.6 Outliers and Influential Observations

The least squares model was fitted to all 10 variables and 1792 observations, and the Studentised residuals (Stud Res) and Modified Cook (Mod Cook) statistic calculated. All cases that had an absolute value of the studentised residual greater than 2 and a Mod Cook greater than 3, were removed from the data set.

Forward stepwise regression is also implemented with an F-to-enter of 4.00 and an F-to-remove of 3.75.

The process of deleting outliers and influential observations was iterated and at each iterations, least squares (LS), forward stepwise, ridge and principal component (PC) models fitted. Furthermore, the following descriptive statistics were obtained.

	Skewness	Kurtosis	Shapiro-Wilk W	Num
Resid1	12.36	254.17	0.34	1792
Resid2	2.50	32.42	0.55	1740
Resid3	1.32	32.46	0.54	1660
Resid4	-0.43	15.97	0.69	1588
Resid5	-0.54	11.06	0.79	1492

Although the Aikake Information Criteria was decreasing with the iterations of removing outliers and influential observations, it was not included as it is based on the Multivariate normal likelihood and normality of the residuals is absent.

The skewness and kurtosis improves in the direction of normality, although even at the last iteration, the kurtosis of 11.056 is still somewhat greater than 3.

4.2.7 Model Selection for LBA

At each iteration of removing outliers, the RMS was calculated on the full population data set of 1792 options, even though models were derived from one of the reduced data sets. The table below provides the RMS of a number of models at each iteration:

All Variables	Least Squares	Stepwise	PC	Fact.	Ridge	Constant
Full Data	6.3458E-05	6.479E-05	6.42195E-05	-2	6.30952E-05	0.0020004
Resid 1	4.89282E-05	4.87485E-05	4.83027E-05	-2	4.82857E-05	0.0056678
Resid 2	4.57947E-05	4.5849E-05	4.57815E-05	-2	4.56598E-05	0.0043342
Resid 3	4.58547E-05	4.59134E-05	4.58612E-05	-1	4.58547E-05	LW
Resid 4	5.65804E-05	5.69464E-05				

The value of the ridge constant at Resid 3 of 1.66E-13, corresponds to the Lawless & Wang estimate of the ridge constant. From the table, the models with the lowest RMS's are obtained after the second iteration of deleting residuals. Here the final four models perform about the same, with ridge first, PC second, followed by LS and finally the stepwise model.

We can conclude that outliers have an impact on the quality of predictions.

We select the highlighted five models, to be carried over into the analysis in Chapter Six. Their Regression coefficients are given below.

	LS1	LS2	STEP	PC	RIDGE
B0	0.952167	0.949247	0.948728	0.948861	0.951955
B1	-2.6E-05	-5.9E-06	-5.3E-06	-5.02E-06	-5.2E-06
B2	2.71E-05	1.22E-05	1.03E-05	9.77E-06	9.77E-06
B3	-0.0001	9.11E-06	9.32E-06	9.29E-06	6.87E-06
B4	-0.00068	-3.5E-05	-4.8E-05	-4.22E-05	-4.3E-05
B5	-0.00039	-3E-05	-4.4E-05	-4.88E-05	-6.8E-05
B6	-0.00034	-2.2E-05	0	-7.23E-06	-1.5E-05
B7	4.52E-05	3.05E-06	0	9.65E-07	1.9E-06
B8	-0.04785	-0.07214	-0.07383	-0.07344	-0.06291
B9	-0.54046	0.276246	0.311288	0.304397	0.072577
B10	0.048662	0.05078	0.051267	0.051146	0.048069

The coefficients in the last four models are very similar. This supports the robustness of non-normality of least squares.

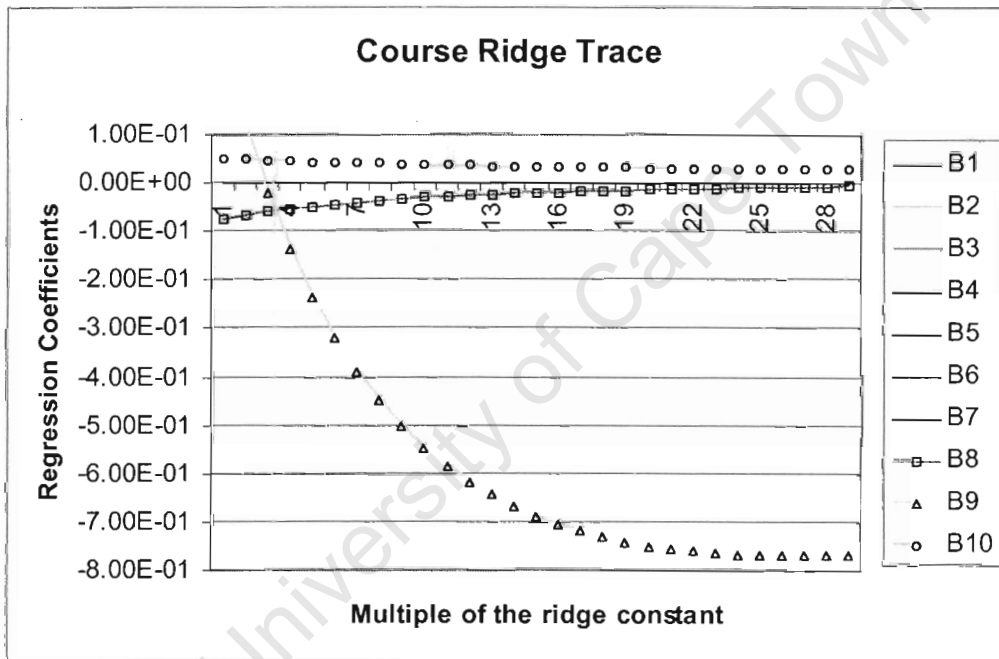
Before we describe alternative avenues that have been explored in modelling the data, let us discuss ridge regression.

4.2.8 Ridge regression

The aim of ridge regression is to choose a minimum ridge constant, such that the regression coefficients appear stable. The constant should be minimal so as to keep the bias in predictions low.

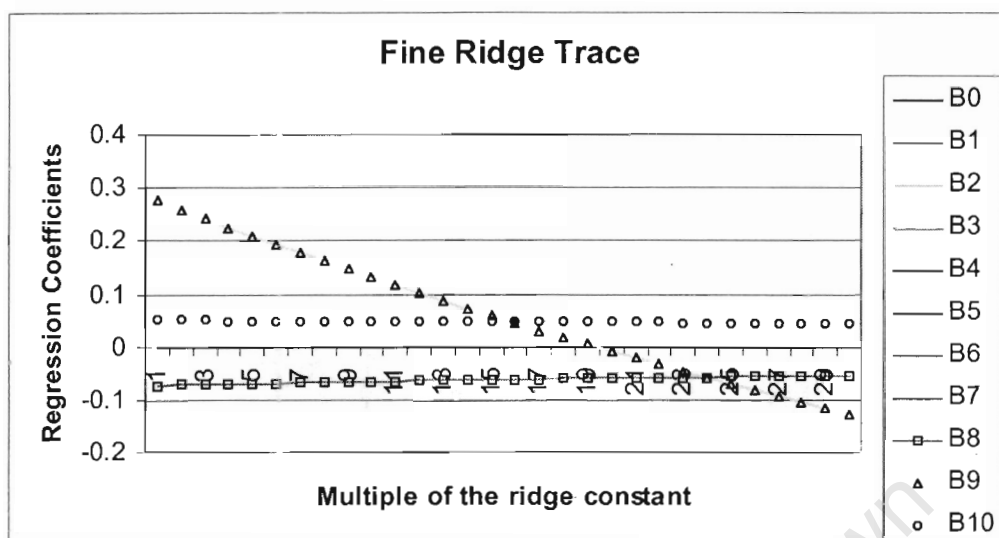
The graph below depicts the ridge trace, which is calculated from incrementing the main diagonal of the correlation matrix by 0.0003334 a total of 30 times. This initial constant is arbitrary. A certain small number is modified until we obtained a ridge model with a minimum RMS, that lies somewhere in the middle along the ridge trace created by the 30 additions to the correlation matrix. The minimum model at the RESID2 iteration occurred when the ridge constant was $14 * 0.0003334$.

The first graph gives the ridge trace calculated with an increment of ten times the one implemented for the graph on the next page. It is included to illustrate stability of coefficients and perhaps bias in ridge regression.



Regression coefficient B_9 appears to achieve stability only towards the right of the graph. Yet, the RMS is a minimum of 0.000045659, where coefficient B_9 is near the x-axis. The RMS at the right end of the graph is 0.000049192. It appears that, even though B_9 does not seem stable early on, the bias introduced by a higher ridge constant more than offsets the gain in stability of the coefficients.

The graph below depicts finer increments of the ridge constant. The minimum is located at 14 additions along the x-axis.



4.2.9 Alternative approaches to model fitting

The biplot for the initial exploration of collinearity showed that in particular 3 groups of variables were highly correlated. These were X_1 and X_2 , X_6 and X_7 as well as X_8 , X_9 and X_{10} . Because of the high VIF's, for the first two groups we deleted X_2 and X_7 , and restarted the analysis.

Subsequently, X_8 was also deleted.

The tables below show the RMS's obtained.

X2, X7 out	Least Squares	Stepwise	PC	Fact.	Ridge	Constant
Full Data	6.4787E-05	6.479E-05	6.4787E-05	all	6.4787E-05	0
Resid 1	4.84429E-05	4.8245E-05	4.84429E-05	all	4.84078E-05	0.0020004
Resid 2	4.56965E-05	4.59622E-05	4.56965E-05	all	4.56837E-05	0.0003334
Resid 3	4.60213E-05	4.58096E-05	4.60213E-05	all	4.60213E-05	0
Resid 4	5.91423E-05	5.9328E-05				
X2, X7, X8 out	Least Squares	Stepwise	PC	Fact.	Ridge	Constant
Full Data	6.77659E-05	0.002920555	6.77031E-05	-1	6.76589E-05	HK
Resid 1	5.1392E-05	5.12517E-05	5.12292E-05	-1	5.12666E-05	HK
Resid 2	4.87611E-05	4.87875E-05	4.86475E-05	-1	4.86496E-05	TC
Resid 3	4.90138E-05	4.98487E-05	4.90138E-05	all	4.90055E-05	HK
Resid 4	7.75407E-05	7.75241E-05				

TC stands for the Troskie-Charlton estimate of the ridge trace and HK for the Hoerl-Kennard estimate.

The models with all variables included perform the best.

In summary, we have isolated five models, four of which have been adjusted for outliers and some of which are robust to multicollinearity. We will proceed with the approximation based on both the lower and upper bound.

4.3 Regression for LUBA

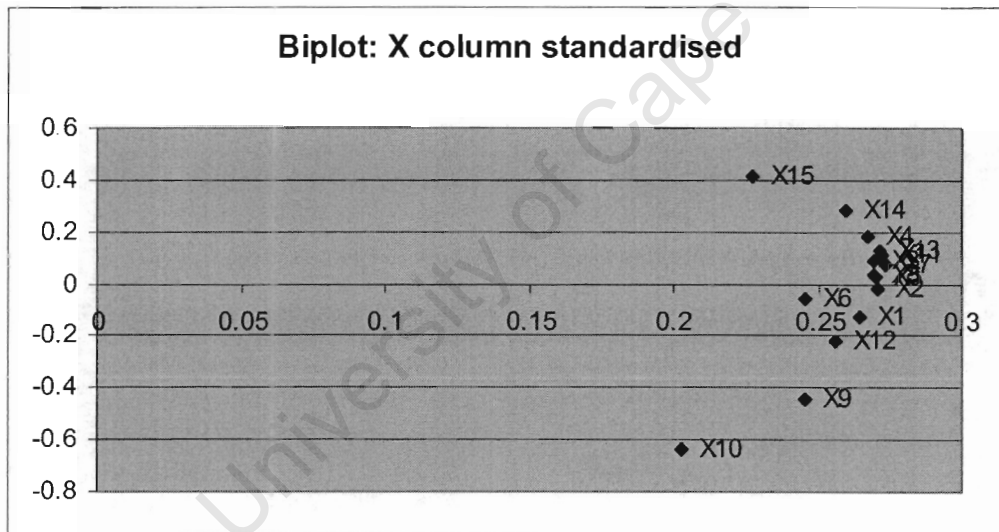
In the previous section, the regression model was used to predict a multiple of the lower bound of the American option price. This multiple is therefore in the vicinity of one and just greater than one. To utilise information provided by both an upper and a lower bound, the model fitted is one that predicts the difference between the upper and lower bound instead:

$$C = \lambda_2 * C^l + (1 - \lambda_2) C^u.$$

However, this necessitates a reweighting of the independent and dependent variables, in order to avoid division by the difference between the upper and lower bound, which is potentially very small. Put in another way, the regression model is a weighted least squares model on the residuals, with the weight equal to the square root of $(C^u - C^l) / C$, the latter being the weight given by Broadie & Detemple (1996).

The implication of applying the same weight to all dependent and independent variables relating to one option price, is that it introduces correlation into our predictor variables.

The graph below plots the first two principal components of the column-standardised X matrix. The quality of the approximation of the display is 94.45%.



The graph shows that all variables are highly correlated. From the correlation matrix, all correlations are significant, with most correlations over 70% in magnitude. The lowest correlation is 43%, which is, as can be seen from the graph, between variables X₁₀ and X₁₅. Given this high correlation, it was decided not to proceed with variable selection, as high correlation will remain even after deleting variables.

4.3.1 Collinearity for LUBA

A formal investigation of collinearity, now follows. The condition indices of the column-centred matrix are:

1
4.231206
9.460655
36.85362
53.91942
103.8223
162.7857
330.4069
502.8934
556.8406
1191.83
2791.388
4279.981
79101.46
123783.3

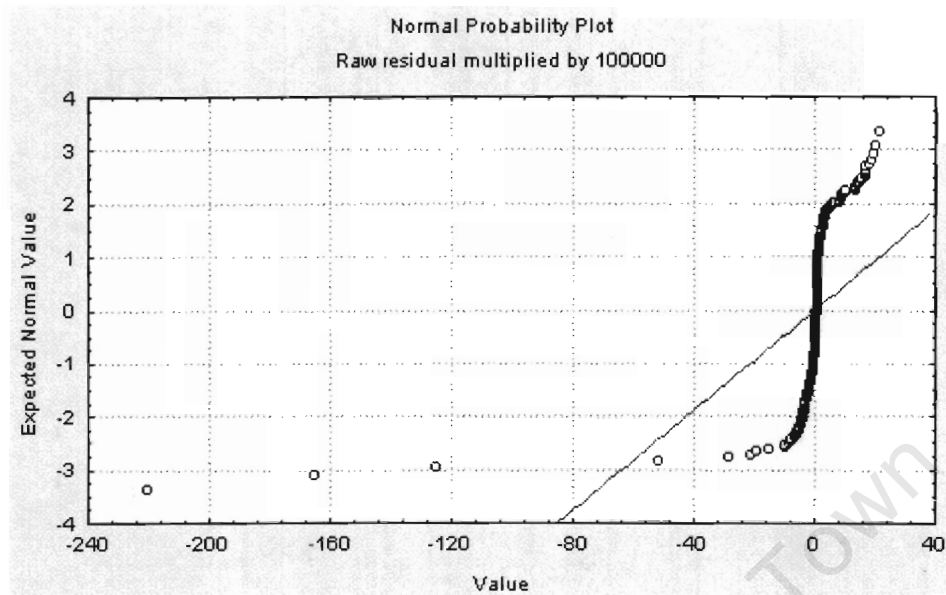
And the variance inflation factors:

Dep Var	R ²	VIF's
SX1	0.999252	1336.505
Sx2	0.999701	3342.805
SX3	0.99926	1351.899
SX4	0.999602	2509.914
SX5	0.999813	5335.325
SX6	0.996925	325.2498
SX7	0.999597	2479.544
SX8	0.999112	1126.443
SX9	0.997384	382.2338
SX10	0.985808	70.46462
SX11	0.999989	90579.71
SX12	0.992459	132.6056
SX13	0.999989	87565.67
SX14	0.998815	844.0884
SX15	0.995809	238.6088

The table of variance decomposition is given in **Appendix A** to this Chapter. Although the variance-decomposition does not indicate serious collinearity (perhaps one type of collinearity), a lot of evidence for collinearity (perhaps of another type) is provided by the condition number and the variance inflation factors. Tests may not always agree as dependencies among predictor variables may differ, for example in the number of variables involved in the correlation. Given collinearity, this should create a problem for least squares, when predicting put options.

4.3.2 Normality and Outliers in LUBA

A more immediate impact on the regression model is manifested through outliers.



The lowest three outliers correspond to out-the-money options, with a low volatility and short maturity.

We now proceed as in the previous section and delete outlying and influential observations iteratively. A summary of the indicators for normality is given below:

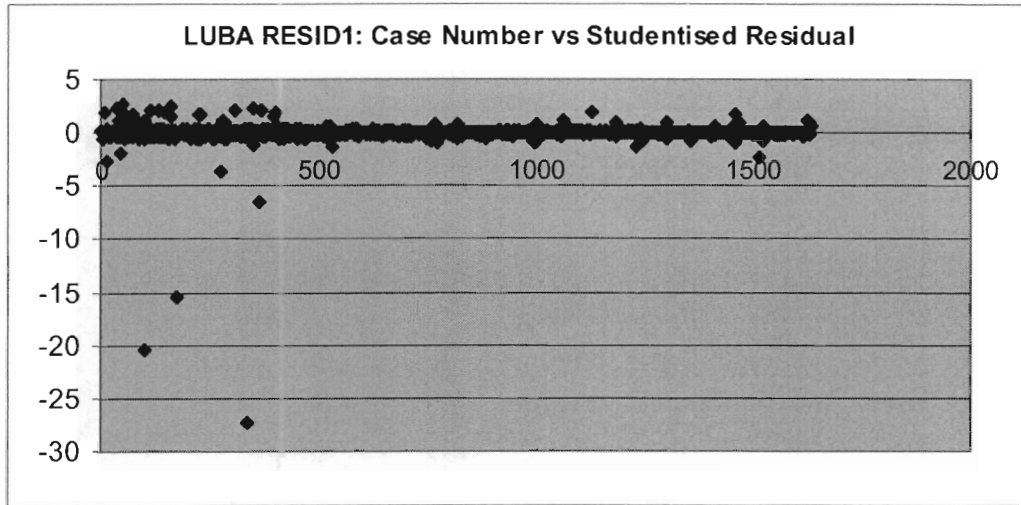
	Skewness	Kurtosis	Shapiro-Wilk W	Number
RESID 1	-20.28	487.93	0.15	1632
RESID 2	2.71	24.47	0.69	1594
RESID 3	-0.07	3.26	0.92	1493
RESID 4	-0.27	1.25	0.97	1351

The data is not normal. Note that the data set has initially been reduced to 1632 observations, as 160 observations had the lower bound equal to the upper bound and were removed from the data in order to avoid a division by zero in the weight. From the table, outliers probably caused most of the negative skewness and high kurtosis. The low kurtosis in RESID 4 may indicate that we may be removing too many observations in the tails by then.

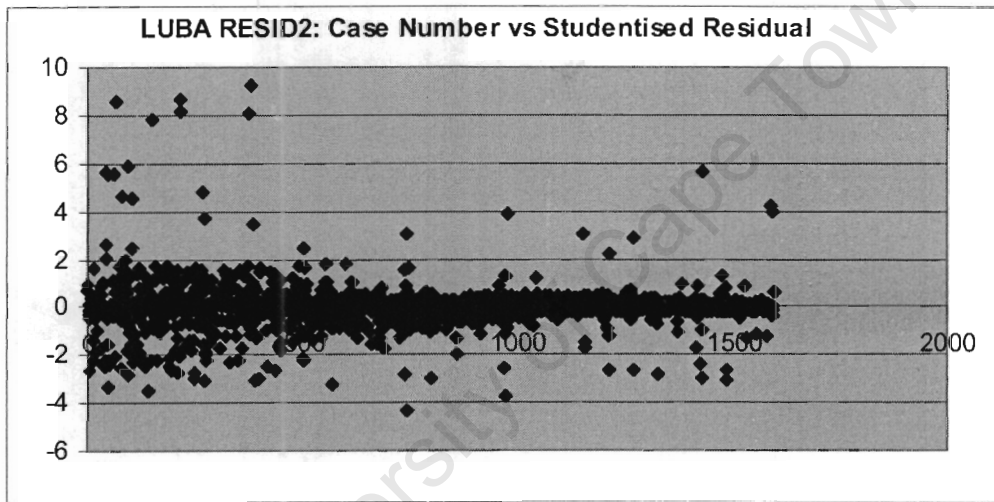
The graphs on the next page plot the case number against the studentised residual of the first 3 iterations.

It is evident that the residuals are larger for about the first 3rd of the observations. These residuals relate to out-the-money options and should not invite a time-series interpretation.

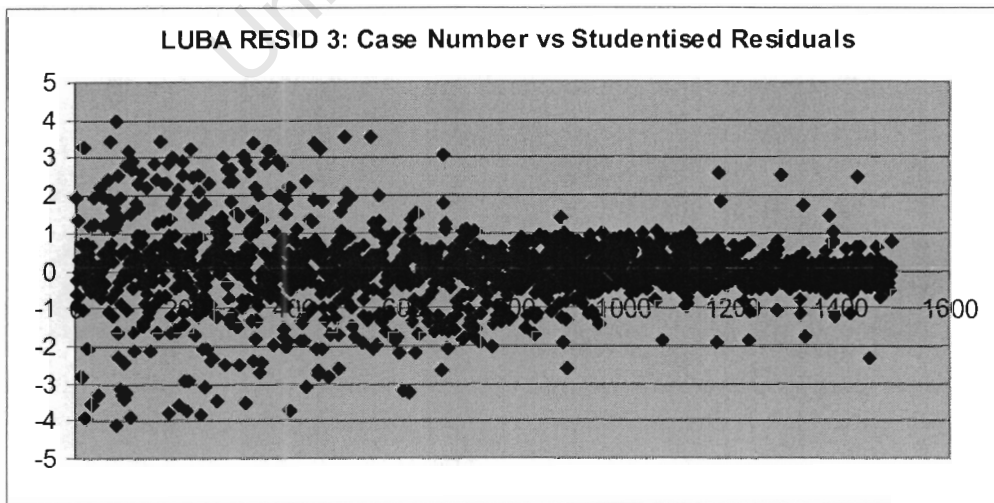
Graph 1



Graph 2



Graph 3



We will now proceed with model selection.

4.3.3 Model Selection for LUBA

At each iteration 5 types of regression models were fitted. These were least squares, forward stepwise regression, principal component regression, ridge regression as well as weighted regression. The weights for the weighted regression are

- 1 if the modified Cook's distance is less than 3,
- 3/ Mod Cook if Mod Cook is greater than 3.

The Table below summarises the RMS for the best models at each iteration:

All Variables	Least Squares	Stepwise	Weighted			
Full Data	1.05016E-05	1.0488E-05	1.0462E-05			
Resid 1	1.04627E-05	1.0461E-05	1.0482E-05			
Resid 2	1.06294E-05	1.0557E-05	1.1478E-05			
Resid 3	1.13287E-05	1.1068E-05	1.186E-05			
Resid 4	1.1739E-05	1.1603E-05				
	PC	Factors	Ridge	Konstant		
Full Data	1.0511E-05	-2	1.0502E-05	0		
Resid 1	1.04627E-05	0	1.0463E-05	0		
Resid 2	1.04869E-05	-1	1.0503E-05	0.000333	1.049E-05	TC
Resid 3	1.08793E-05	-2	1.0807E-05	0.001334	1.072E-05	TC
Resid 4						

At step one, the weighted regression model outperforms all other. This indicates that the down-weighting of residuals is important. The stepwise model is a close second. The principal component model slightly outperforms least squares, however, no ridge model does. At the second time-step, the stepwise model is the best, and the PC, ridge and least squares models are almost identical. To summarise, RESID 1 provided models with the lowest RMS's.

It was decided to carry over the six highlighted models to the final analysis in Chapter six. Their betas are given below.

	LS1	LS2	STEP	WLS	PC-1	R-TC
B0	-0.00000039	2.30E-08	0.00000002	-0.00000017	0.00000006	0.00000005
B1	-0.00222969	-0.002288099	-0.00273649	-0.00196102	-0.00777008	-0.00574595
B2	0.04914549	0.03902596	0.04093373	0.04531629	0.05523280	0.05145608
B3	0.18495828	0.242603088	0.22744764	0.19322358	0.14488383	0.11276675
B4	0.24828685	-0.032893489	0.00000000	0.15002183	0.13193919	0.27338573
B5	-0.01957464	-0.03018192	-0.02909905	-0.02193606	-0.03152990	-0.02747760
B6	0.00209432	0.003287611	0.00313382	0.00251704	0.00384327	0.00327002
B7	0.09278997	0.187355941	0.18654125	0.14166366	0.09299295	0.05207917
B8	0.05977200	-0.057441571	-0.05661493	-0.01355751	0.00656602	0.04909490
B9	-0.31216768	2.22215369	2.25900975	1.23953934	3.26186832	2.54744256
B10	-12.43448713	-55.73710778	-55.72816611	-36.64595249	-67.11221091	-46.09720287
B11	0.41367984	0.20926256	0.19320648	0.23880422	0.01420393	0.01719828
B12	24.93830935	14.8287853	14.63121002	20.31993445	8.88506829	12.72950065
B13	-0.45147659	-0.229835678	-0.21432869	-0.27729151	0.00035717	-0.00064569
B14	0.02203150	0.053710412	0.03989294	0.05736900	0.00249624	0.00086635
B15	0.01993958	-0.016013239	0.00000000	-0.01058490	0.03980735	0.03383877

The betas of the final five models differ slightly.

4.4 Conclusion

A number of regression models were applied and five models selected for LBA as well as six for LUBA. Superior models to the original least squares model were found for both LBA and LUBA when testing was done on the complete X-space. In summary, the consideration of outliers in the independent variables is important. Although the impact of collinearity could be assessed by comparing Put models on Put data with Call models on Put data in this section, it was decided to delay this analysis until Chapter Six and conduct it on a different sample. In Chapter Six, we will compare these models along with Lattice Methods and other Analytic Approximations.

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Appendix A: LUBA -Coefficient Variance Decomposition

	Var(B1)	Var(B2)	Var(B3)	Var(B4)	Var(B5)	Var(B6)	Var(B7)
Sin Val	V1	V2	V3	V4	V5	V6	V7
0.090915	0.028346	0.08226	0.129815	0.05058	4.14E-07	0.027521	0.62076
0.021487	0.000908	0.001715	0.002976	0.019366	0.002618	0.028732	0.914457
0.00961	3.73E-07	4.46E-09	1.24E-05	4.75E-05	1.22E-05	0.001298	2.82E-05
0.002467	4.07E-08	6.83E-08	3.66E-06	7.01E-06	1.46E-05	1.48E-07	5.73E-05
0.001686	0.001051	0.000547	0.014303	0.020811	0.103689	0.662622	0.003728
0.000876	0.040709	0.373979	0.043277	0.009943	0.152597	0.323564	0.001104
0.000558	3.38E-05	2.21E-06	0.001946	0.016139	0.029656	0.024613	0.000307
0.000275	1.31E-05	1.59E-06	0.000772	0.029543	0.035902	0.012842	0.001851
0.000181	2.63E-09	6.11E-09	5.2E-08	7.95E-06	3.46E-05	1.26E-05	3.05E-05
0.000163	2.12E-14	4.78E-14	1.84E-12	1.68E-10	6.12E-10	7.64E-11	2.96E-09
7.63E-05	3.53E-05	3.7E-05	0.00181	0.00157	0.00081	0.010965	0.02357
3.26E-05	1.76E-13	1.03E-12	1.74E-12	1.01E-10	2.03E-09	1.3E-08	1.44E-11
2.12E-05	3.08E-05	2.68E-05	0.001794	0.001852	0.001366	0.005994	0.016058
1.15E-06	1.71E-05	3.4E-05	0.003366	0.025998	0.04762	0.024596	0.003532
7.34E-07	2.21E-06	6.35E-06	0.001061	0.023038	0.026621	0.015999	0.000762
Var(B8)	Var(B9)	Var(B10)	Var(B11)	Var(B12)	Var(B13)	Var(B14)	Var(B15)
V8	V9	V10	V11	V12	V13	V14	V15
0.001485	0.011129	0.006525	0.030746	0.002981	0.001509	0.00043	2.93E-06
0.002065	0.015457	0.004072	0.007696	6.46E-06	2.68E-05	0.000728	5.69E-05
0.220185	0.007262	0.003497	0.072258	0.145893	0.548161	0.000864	0.000432
0.015372	0.000276	0.000429	0.004387	0.051732	0.924824	0.000637	0.00227
0.033516	0.009233	0.001282	0.049917	0.060223	0.036457	0.001817	0.00055
0.008567	0.007394	0.000266	0.005254	0.010732	0.002595	0.000347	0.000109
0.007019	0.131247	0.711854	0.040925	0.024398	0.011852	2.41E-05	1.14E-08
0.001906	0.096781	0.640124	0.14997	0.024663	0.004777	0.001047	4.27E-05
0.000579	0.000761	0.014435	0.188297	0.558019	0.100123	0.003968	0.133755
1.6E-09	2.09E-08	4.57E-07	6.74E-06	3.2E-05	2.31E-07	0.063649	0.93631
0.009327	0.000366	0.016731	0.258412	0.507531	0.167726	0.000161	0.00088
3.2E-10	4.69E-11	2.98E-07	2.61E-06	1.3E-05	5.44E-06	0.858246	0.141733
0.013764	0.006667	0.001787	0.301354	0.578996	0.067733	0.000483	0.002227
0.026079	0.736406	0.097438	0.014838	0.016716	0.002529	0.00074	0.000156
0.021417	0.703677	0.127745	0.066272	0.012553	0.000143	7.04E-05	5.76E-05

Chapter 5: Monte Carlo Simulation

A key feature of derivative instruments is that the payoff can be tailored to suit very specific needs. For derivatives, where no closed form solution exists, and where lattice approaches or analytical approximations aren't available, Monte Carlo Simulation (MCS) may still provide a way to obtain a price. MCS can deal with multiple underlying assets, random volatility, random interest rates, jumps in asset price processes, markets that allow discrete price fixings of exotic path-dependent options, etc. With regard to credit risk, in particular, it is a powerful tool for quantifying risk exposure of a credit portfolio over time. In a nutshell, the method is flexible enough to deal with a wide variety of pricing situations.

This flexibility comes at a price. MCS is relatively time consuming and a number of basic tools for improving efficiency of MCS have been developed. The aim of this chapter is to review three important methods in MCS for American options and to discuss two variance-reduction techniques in MCS.

Finally, a forward simulation approach that assumes the optimal exercise boundary known is implemented. With this approach the probabilities of option exercise will be compared for puts and for calls in the SA context.

5.1 Implementing Monte Carlo Simulation

The value of a derivative is the risk-neutral expectation of its discounted payoff. In MCS, an estimate of this expectation can be obtained by computing the average of a large number of discounted payoffs, which are calculated by evaluating a simulated price path on the payoff date. The variability of this sample mean is given by its variance.

5.1.1 Simulating GBM

Recall Equation 7 from Chapter 1, where dz is a Wiener process. The Wiener process dz , is normally distributed with mean zero and variance dt and the dz 's for non-overlapping increments of time are independent.

$$d \ln S = \left(r - \delta - \frac{1}{2} \sigma^2 \right) dt + \sigma dz$$

Discretising, we have

$$S_{t+\Delta t} = S_t \exp \left(\left\{ r - \delta - \frac{1}{2} \sigma^2 \right\} \Delta t + \sigma \sqrt{\Delta t} * e_i \right)$$

where e_i follows a standard normal distribution.¹⁰ This discretisation method has an error of $O(\Delta t)$.

¹⁰ This discrete way of simulating the time series for S is called the Euler method.

5.1.2 Two Variance Reduction Techniques

Boyle, Broadie & Glasserman (1997) propose a definition of efficiency that would facilitate the comparison of estimators taking computational effort into account.

Consider two unbiased estimates (θ_1 and θ_2) of a parameter θ , for example the price of a derivative security. The basis for comparison of these two estimates should take account of both the variance (σ^2_1 and σ^2_2) as well as the computational effort required (say b_1 and b_2). Specifically, θ_1 can be preferred to θ_2 if the efficiency of θ_1 ($=\sigma^2_1*b_1$) is less than the efficiency of θ_2 ($=\sigma^2_2*b_2$). With these definitions, Boyle, et al. (1997) describe two variance reduction techniques.

Antithetic Sampling

Given a set of random numbers Z_i , one generates a price path. Another price path is generated using $-Z_i$. For both price paths, calculate the payoffs and present values θ_1 and θ_2 . Note that $\sigma_1 = \sigma_2 = \sigma$.

An unbiased estimate of the option value is the average $\theta_a = \frac{1}{2}*(\theta_1 + \theta_2)$. Then $\sigma_a = \frac{1}{2}*(\sigma + \text{cov}(\theta_1, \theta_2))$. Thus θ_a has lower variance if $\text{cov}(\theta_1, \theta_2) \leq \sigma$.

If we assume that generating the Z_i represents a negligible fraction of the work per replication, then the work to generate θ_a is roughly double the work to generate θ_1 or θ_2 . Hence θ_a is more efficient if $2*\sigma_a \leq \sigma$, or if $\text{cov}(\theta_1, \theta_2) \leq 0$.

However, $\text{cov}(\theta_1, \theta_2) = E(\theta_1, \theta_2) - E(\theta_1)E(\theta_2) \leq 0$ by the Cauchy-Schwarz inequality. Hence improvement will result, unless θ_1 and θ_2 are perfectly positively correlated.

Control Variates

Suppose, that for two similar derivatives 1 and 2, that 2 has an explicit formula for its price V_2 . Then both derivatives are valued based on the same set of random numbers to get prices V_1' and V_2' . The estimate of V_1 , the correct value of derivative 1 is given by $V_1 = V_1' - V_2' + V_2$. The argument behind this method is that the error in V_1' will be the same as the error in V_2' and the latter is known.

Furthermore, assuming that most of the computational effort goes to generating paths of the underlying asset and not the analytical computation of V_2 , then the work required for pricing V_1' and V_2' is similar.

Comparing variances: $\sigma V_1 = \sigma V_1' + \sigma V_2' - 2\text{cov}(V_1', V_2')$, since V_2 is known. Hence, this method is effective if $\text{cov}(V_1', V_2')$ is large.

As an example, consider the price of a European option and an American option. These are positively correlated and the Price of a European option is given by the Black Scholes formula.

The above relationship can be exploited even further. Consider

$V_1^\beta = V_1' + \beta(V_2 - V_2')$, where β is a scalar.

Then $\sigma V_1^\beta = \sigma V_1' + \beta^2 \sigma V_2' - 2\beta \text{cov}(V_1', V_2')$, and σV_1^β is minimised when $\beta^* = \text{cov}(V_1', V_2') / \sigma V_2'$. The β can be estimated as the slope in a simple linear regression of an observed sample for V_1 and V_2 . V_1^β will have lower variance, whenever V_1' and V_2' are correlated.

Other variance reduction techniques, such as Moment matching or low-discrepancy sequences are discussed in Boyle, et al. (1997).

5.2 Monte Carlo Simulation for American options

An early and important attempt to simulate American options was proposed by Tilly (1993). Carriere (1996) gives an analysis and extension of Tilly's approach.

At each time period, the simulated paths are ordered by asset price and bundled into groups. Then an optimal exercise decision is estimated for each group.

Boyle et al. (1997) note the following shortcomings:

- 1) No proof of convergence is given, and the estimator is biased;
- 2) The storage requirements of the bundling and sorting is substantial for large number of simulations;
- 3) The algorithm has not been extended to additional state variables yet.

Another approach is given by Barraquand & Martineau (1995). They propose a stratification method for pricing high-dimensional American securities. They partition the payoff space instead of the state space and thus for any number of state variables a one-dimensional problem results. Transition probabilities are estimated from relative path frequencies in consecutive cells, the immediate exercise value is approximated by the average payoff in each cell and a continuation value is approximated as the discounted weighted (using transition probabilities) average of the next periods option prices. However, Boyle, et al. (1997) provide an example where their algorithm does not converge. In addition, they state that there is currently no way to bound the error in this algorithm. Boyle et al. (1997) also suggest a modification, which results in an option price that is biased low, but the average direction of the error known. The Barraquand & Martineau approach is extended in Raymar & Zwecher (1997) to two state variables rather than one.

Broadie & Glasserman (1997) propose two estimates. One is biased high due to perfect foresight in calculations, as always the larger of a continuation value (based on perfect knowledge) and an immediate exercise value is chosen.

The other is biased low, but is asymptotically unbiased. The bias results, because the estimate is an average of some unbiased estimates (based on the correct decision, e.g. exercising when immediate payoff exceeds the European value) and some biased estimates (based on the incorrect decision, e.g. continuing when continuation value exceeds immediate exercise, but with immediate exercise value exceeding the European value). These high and low estimates provide conservative bounds for the American option price.

Along with the control-variate technique (using the European value), antithetic sampling, stratified sampling and pseudo-random numbers, pruning of branches in the tree is used to reduce the number of nodes in the tree. Firstly, the European option value reduces the need for branching at the penultimate step. Secondly, at any intermediate node at which the immediate exercise value is zero, there is no need to branch, as it is optimal to continue, and just one successor node generated. In the case where the payoff is positive, consider the European values with maturity at each of the remaining time steps. All of these represent sub-optimal exercise policies and thus provide a lower bound for the continuation value. Hence, if the payoff is less than the European value, there is no need to branch.

Finally, extrapolation of the high and low estimators is used to accelerate convergence.

The simulation of exercise features of American options is discussed next.

5.3 A Forward Simulation approach

The purpose of this thesis is to value American options on a continuously dividend paying underlying. Recently, Basso, Nardon & Pianca (2002 a,b,c) have proposed a two-step forward simulation procedure for valuing American calls. The optimal exercise boundary is determined by using the ML, prior to running the simulations. Without knowing the boundary, this forward approach would not be possible. In particular, then, it applies very specifically to American options on a single underlying based on GBM and not, for example, to options that are based on a different stochastic process.

5.3.1 The optimal Exercise Boundary and its Computation

The optimal exercise Boundary for American Puts (B_t) can be characterised in number of equivalent ways:

- 1) B_t is the asset price below which it is optimal to exercise the American put;
- 2) B_t is the asset price at which one is indifferent between exercising and continuing with the option;
- 3) B_t is the highest value of the asset price for which the put value is equal to the exercise price less the stock price;
- 4) B_t is the highest value of the underlying asset price at which the put value does not depend on time to maturity.

Looking at the optimal exercise boundary from a different perspective, creates different avenues for approaching the pricing problem mathematically.

Basso et al. (2002a,b), show that that ML is the most accurate, robust and computationally efficient for obtaining B_t . In particular, they recommend that extrapolation of the critical exercise price should be based on two discretisations only, as it becomes unstable when more discretisations are extrapolated when the dividend yield is positive. Furthermore, the Bunch & Johnson (2000) approach is unstable under certain situations.

5.3.1.1 The Binomial approximation to the optimal exercise boundary

Basso et al.(2002a) note three problems that can arise, when applying a lattice approach in order to approximate the optimal exercise boundary of an option written on an asset with continuous time dynamics:

- 1) At the first time step all nodes may lie above or below the continuous time boundary i.e. the existence of a stopping node is not guaranteed.
- 2) Generally, the approximation to the optimal exercise price may be too coarse in a tree with a low number of time steps.
- 3) In a lattice approach the continuous time critical prices are not independent of the initial stock prices, as the nodes in the tree are based on the spot price.

They propose an interpolated boundary computed with a 25000-step binomial tree, with the omission of the first 5000 steps. The weights in the interpolation make use of the fact that on the optimal exercise boundary the continuation value is equal to the exercise price less the stock price. The approach has proved very robust with respect to all three problems.

Although potentially very accurate this method is time consuming.

5.4 Simulation and Exercise Probabilities

In the presence of dividends, the American Call differs from the European Call. This difference can be quite substantial and increases the more the option is in-the-money, the lower the interest rate, the higher the dividend yield, the higher the volatility and the longer the term of the option. In this section, three probabilities of exercise will be compared for American Call and Put options in the SA context.

5.4.1 Implementing the forward simulation

The forward MCS approach is implemented by employing the two-point extrapolated ML to estimate the exercise boundary. Then 5000 simulations, with 100 possible exercise opportunities are run on 50 options in each of the $7 + 4 + 4 + 4 + 4 = 23$ batches. More detail on the construction of these 23 batches is given in Chapter Six. Briefly, however, there are 7 moneyness classes, 4 interest rate classes, etc. In total, there are 1150 put options and 1150 call options.

The program then kept track of the number of times that the price path hit the boundary prior to the penultimate time step ($X < T-1$) and during the penultimate time step ($T-1 < X < T$). With this information, three probabilities / relative frequencies are calculated:

- $P(X < T-1)$ - the probability of exercise before time $T-1$;
- $P(T-1 < X < T)$ - the probability of exercise between $T-1$ and T ;
- $P(X < T)$ - total probability of early exercise.

In each case the denominator is 5000.

The slope of the exercise boundary may be different for puts and calls prior to expiry. In particular, one would expect a lot of exercise during the last time step and less exercise before it, when the slope is steep, and vice versa. The South African context adds another interesting dimension.

The table on the next page summarises the results from the simulation.

- 1) For call options, most of the exercise occurs during the penultimate time-step prior to maturity, while for puts much of the exercise occurs prior to the penultimate time-step. It follows that for call options the difference between boundary and strike is larger and the slope of the boundary steeper prior to expiry than for puts. However, in the SA context, in total more calls are exercised (though late) than puts!
- 2) Call options are only exercised prior to the penultimate time-step when the interest rate, volatility or maturity is high. Note that, for call option, high interest rates imply that the underlying should be bought as late as possible.
- 3) For calls exercise probabilities increases the more the spot exceeds the strike, the higher the dividend yield, the lower the interest and the longer maturity, and vice versa for puts.
- 4) For calls, the higher the volatility the lower the probability of early exercise and vice versa for puts, which is intuitively correct.

5.4.2 Result: Put and Call Exercise Probabilities

	Pe(<t-1)		Pe(t-1<x<T)		Pe(<T)		
CALL	Average	Stdev	Average	Stdev	Average	Stdev	CALL
OUT	0	0	0.407952	0.039021617	0.407952	0.039021617	OUT
m	0	0	0.517492	0.034464254	0.517492	0.034464254	m
o	0	0	0.627828	0.026211479	0.627828	0.026211479	o
n	0	0	0.673284	0.009229154	0.673284	0.009229154	n
e	0	0	0.714592	0.026608514	0.714592	0.026608514	e
y	0	0	0.799392	0.021063401	0.799392	0.021063401	y
IN	0	0	0.85992	0.016919835	0.85992	0.016919835	IN
d-low	0	0	0.630316	0.010202688	0.630316	0.010202688	d-low
d	0	0	0.661868	0.011388258	0.661868	0.011388258	d
d	0	0	0.68832	0.008752749	0.68832	0.008752749	d
d-high	0	0	0.713456	0.009938672	0.713456	0.009938672	d-high
r-low	0	0	0.68762	0.007147227	0.68762	0.007147227	r-low
r	0	0	0.680404	0.006024506	0.680404	0.006024506	r
r	0.00016	0.000618095	0.660896	0.012286993	0.661056	0.012108424	r
r-high	0.022928	0.021449543	0.604664	0.027401756	0.627592	0.010628031	r-high
v-low	0	0	0.741524	0.023701094	0.741524	0.023701094	v-low
v	0	0	0.688176	0.012146631	0.688176	0.012146631	v
v	0	0	0.666068	0.005455663	0.666068	0.005455663	v
v-high	0.000036	0.00013211	0.6621	0.006531462	0.662136	0.006503806	v-high
t-short	0	0	0.595784	0.019759321	0.595784	0.019759321	t-short
t	0	0	0.647288	0.015837809	0.647288	0.015837809	t
t	0.00122	0.002809968	0.734828	0.034578394	0.736048	0.0362115	t
t-long	0.035396	0.018987923	0.809288	0.015976573	0.844684	0.01868681	t-long
PUT	Pe(<t-1)		Pe(t-1<x<T)		Pe(<T)		PUT
IN	Average	Stdev	Average	Stdev	Average	Stdev	IN
IN	0.527388	0.240786485	0.226096	0.153602159	0.753484	0.099231308	IN
m	0.40744	0.207246465	0.223516	0.148563066	0.630956	0.074982907	m
o	0.293036	0.177650603	0.214784	0.132538024	0.50782	0.062096216	o
n	0.246632	0.138951375	0.1964	0.115555447	0.443032	0.031146248	n
e	0.193128	0.113801157	0.203524	0.093526894	0.396652	0.040146158	e
y	0.139084	0.088362076	0.158768	0.078418837	0.297852	0.027674399	y
OUT	0.0873	0.06044868	0.140552	0.056408113	0.227852	0.021812927	OUT
d-low	0.255644	0.134646387	0.221452	0.121689258	0.477096	0.021034054	d-low
d	0.238404	0.128599854	0.210732	0.110856146	0.449136	0.024446174	d
d	0.217284	0.130882662	0.210348	0.109493497	0.427632	0.029723415	d
d-high	0.218424	0.137523167	0.184376	0.100878036	0.4028	0.044702299	d-high
r-low	0.209756	0.128586801	0.214144	0.106569001	0.4239	0.027454073	r-low
r	0.236608	0.141395615	0.205432	0.119566573	0.44204	0.027711104	r
r	0.23392	0.125110335	0.21524	0.104108295	0.44916	0.028657784	r
r-high	0.25686	0.122598034	0.228732	0.11299563	0.485592	0.020420157	r-high
v-low	0.126716	0.154819057	0.208088	0.097688452	0.334804	0.071305282	v-low
v	0.19994	0.153216695	0.211772	0.115238132	0.411712	0.046412774	v
v	0.2996	0.116712401	0.170744	0.104353763	0.470344	0.024139392	v
v-high	0.32034	0.104003895	0.179132	0.093255354	0.499472	0.016784529	v-high
t-short	0.158984	0.138756623	0.302524	0.12861603	0.461508	0.017129052	t-short
t	0.205008	0.144128474	0.241668	0.119223232	0.446676	0.031807171	t
t	0.267096	0.129415277	0.161448	0.098873661	0.428544	0.041082953	t
t-long	0.317924	0.117065214	0.096236	0.054677111	0.41416	0.067455527	t-long

5.5 Conclusion

This chapter described three biased simulation methods for American options and two variance reduction techniques to accelerate convergence. Finally, a forward simulation approach is implemented, and the probabilities of exercise calculated for both the American call and American put. We will now turn to the final chapter of this thesis on the numerical comparison of all pricing methods.

Chapter 6: Numerical Comparison of Pricing Methods

In this chapter all methods described previously in this thesis are compared with respect to speed and accuracy.

6.1.1 Possible Comparison Dimensions

Every numerical solution procedure has its own strengths and weaknesses. In particular, procedures can be compared along the following dimensions:

- Numerical accuracy, estimated by error measures;
- Computation speed, depending on whether demand is real-time or not;
- Algorithm complexity and flexibility, the pricing of new products should not be too complex;
- Availability of price derivatives (the Greeks);
- Memory/ Storage requirements, etc.

Methods superior on one dimension may do less well on another. However, for purposes of this thesis, the dimensions of interest are accuracy and speed.

6.1.2 Accuracy and Speed Definitions

Numerical accuracy can be established using a number of error measures. Broadie & Detemple (1996) suggest the **Root Mean Squared (RMS) relative error** defined by

$$RMS = \sqrt{\frac{1}{m} \sum_{i=1}^m e_i^2}$$

where $e_i = |C_i^* - C_i| / C_i$ is the relative error, C_i is the 'true' option value, C_i^* is the estimated option value, and the index i refers to the i th option in the test set. The summation is taken over all options in the dataset satisfying $C_i \geq 0.5$. The benchmark or true price is calculated using the BBSR method with 8000 and 16000 time steps.

Another accuracy measure is the **Maximum Relative Error (MRE)** and is defined as observation e_i such that e.g. 99.5% of the sample observations are below e_i . This punishes large errors quite heavily.

Furthermore, the **Mean Absolute Error** is also not preferred because it does not penalise large errors enough.

Speed¹¹ is measured by the number of option prices computed per second, or equivalently by the time taken to compute a number of option prices. When comparing methods the exact Hardware is inconsequential, since only relative speeds matter. However, care should be taken to make the software as efficient as possible. In Chapter 2 we have specified ways through which the speed in Lattice methods can be improved.

Higher accuracy is always desirable, but not if economically insignificant price improvements are obtained at an unacceptable cost in terms of computation time. For example, a trader wishing to price a single option requires a computation speed on the order of 1 second. Generally option prices are of the order of R10, so accuracy of the order of 0,1%, (1 cent in R10) is desirable.

¹¹ The Computations were implemented on a 200MMX Proline machine (32 Ram, 200 Mhz CPU).

6.2 Construction of the Option Test Set

We assume that of interest to academics and practitioners in South Africa, are parameters assumed to have a uniform distribution with the following ranges.

For a random strike between 75 and 125, the spot is varied between 30% above and below strike as follows. Seven moneyness categories were defined: (-30% to -20%), (-20% to -10%), (-10% to 0%), (0%), (0% to 10%), (10% to 20%), (20% to 30%).

The remaining parameters were partitioned as follows:

Interest	9% to 12%	12% to 15%	15% to 18%	18% to 21%
Dividend	0.1% to 0.5%	0.5% to 2%	2% to 5%	5% to 8%
Volatility	15% to 25%	25% to 35%	35% to 45%	45% to 55%
Maturity	0.1 to 0.5	0.5 to 1	1 to 3	3 to 5

Initially a “standard” analysis was conducted with only one input parameter varying from the “average”. Seven moneyness classes, with at-the-money options considered the average, plus four interest classes with 15% considered the average, etc. In other words, unidimensional variations are considered relative to at-the-money options with $r = 0.15$, $sd_y = 0.02$, $vol = 0.35$ and $T = 1$. This gives a total of $7 + 4 + 4 + 4 + 4 = 23$ combinations.

Secondly, all combinations of “extreme” categories were considered. For deep in-, at- and out-the-money options all combinations of very high and very low interest, dividend, volatility and maturity were considered. This results in an additional $3 * 2 * 2 * 2 * 2 = 48$ combinations.

A summary of the structure of the sample is given in **Appendix A** to this chapter.

Fifty options were calculated for each combination of input parameters. In total, therefore, 3550 options were considered.

Along with Broadie & Detemple (1996), all options with a true price less than 0.5 were excluded from the analysis. This directly came to 200 options being deleted for puts and 200 options being deleted for calls. However, the eliminated options differ for Call and Put options, as now different options are out-the-money, low volatility and short-term. For call options, batches 40 to 43 inclusive have true prices less than 0.5 and for call options batches 56 to 59 inclusive have a true price less than 0.5.

6.3 Analysis of Lattice Methods

In order to enhance the interpretation of the results, a benchmark pricing method is described. The CRR with $n = 300$ time steps is chosen and all other methods are calibrated as follows: n is adjusted, until their speed is very close to that of the CRR. "Very close" means that for call options, all lattice methods took within 31 seconds of each other to compute all 3350 options in the test set. The total time taken for each method was just under 15 minutes. As benchmark, the CRR with $n = 300$ time step is reasonably fast (0.25 seconds per price) and accurate (RMS = 0.0005). The table below gives the respective number of time steps for each method:

Method	n
AB	456
BA	214,215
BBS	240
BBSR	98,196
CRR	300
SBS	212
SBSR	88,176
SMO	260
SR	116,232
TRI	182

Given the chosen 15 minutes of computation time, a certain ranking of the methods in terms of accuracy will result. Now, should a different amount of total time be selected, the relative rankings of the methods may change. However, it was decided that a more comprehensive analysis over a number of total computation times should fall beyond the scope of this thesis. In addition, there are a number of other variations that should then be explored, including perhaps different types of software, different extrapolation methods, etc.

The RMS's for all methods and for Call and Put options computed over the test set, are given in **Appendix B** and **Appendix C** respectively.

6.3.1 Summary and Interpretation of Lattice Results

The Excel sheets on the next two pages summarise Call and Put RMS's and times respectively. The "rel RMS" (to BBSR) and "rel Time" (to BBSR) are the ratios of the BBSR-RMS and BBSR-Time over those of the other methods respectively. Once calculated, the performance ratio is calculated as the product of "rel RMS" and "rel Time".

A measure of sensitivity of the methods to high vs low interest (dividend, volatility and time) is assessed by taking the ratio of the RMS's. For a parameter, a ratio of 1 means that a method is robust with respect to changes in that parameter.

- 1) Overall, SR seems to be the method of choice. It is only beaten by the BBSR on batches 24 to 71 for call options.

Summary for Calls

Batches 1 to 23	CRR	BA	BBS	BBSR	TRI	AB	SMO	SBS	SR	SBSR
RMS	0.0006483	0.0002137	0.0001734	7.272E-06	0.0005985	0.0004512	0.000797	0.0002009	1.082E-06	1.804E-06
rel RMS (to BBSR)	0.0112181	0.0340374	0.0419396	1	0.0121505	0.0161182	0.0091245	0.036204	6.7207022	4.031528
TIME	816.0313	846.8901	840.4141	835.0313	819	838.875	836.79	839.6563	825.9102	837.5625
rel TIME (to BBSR)	1.0232834	0.9859972	0.9935951	1	1.0195742	0.995418	0.9978983	0.9944918	1.0110437	0.9969779
Performance Ratio	0.011	0.034	0.042	1.000	0.012	0.016	0.009	0.036	6.795	4.019
Batches 24 to 71	CRR	BA	BBS	BBSR	TRI	AB	SMO	SBS	SR	SBSR
RMS	0.0012142	0.0011321	0.0009212	2.914E-05	0.0019061	0.0019912	0.0029245	0.0022013	3.977E-05	5.376E-05
rel RMS (to BBSR)	0.0239968	0.0257358	0.0316296	1	0.0152861	0.0146329	0.009963	0.0132358	0.7326075	0.5419251
TIME	816.0313	846.8901	840.4141	835.0313	819	838.875	836.79	839.6563	825.9102	837.5625
rel TIME (to BBSR)	1.0232834	0.9859972	0.9935951	1	1.0195742	0.995418	0.9978983	0.9944918	1.0110437	0.9969779
Performance Ratio	0.025	0.025	0.031	1	0.016	0.015	0.010	0.013	0.741	0.540
Moneyess	CRR	BA	BBS	BBSR	TRI	AB	SMO	SBS	SR	SBSR
AT	0.0006536	0.0002448	0.0002713	8.907E-06	0.0011681	0.001764	0.0007552	0.0003071	4.971E-06	8.542E-06
OUT	0.0032625	0.0036072	0.0028621	6.993E-05	0.0043398	0.0031423	0.0090518	0.0072533	0.000104	0.000123
IN	0.0002385	0.0001632	0.0001153	1.877E-05	0.0008188	0.001355	0.0004981	0.0003067	2.642E-05	4.705E-05
Interest	CRR	BA	BBS	BBSR	TRI	AB	SMO	SBS	SR	SBSR
high	0.001071	0.001032	0.0008898	2.026E-05	0.0021599	0.0015818	0.0025118	0.0018922	2.625E-05	3.373E-05
low	0.0013574	0.0012323	0.0009525	3.802E-05	0.0016522	0.0024005	0.0033371	0.0025104	5.329E-05	7.38E-05
Ratio	0.7889769	0.8374156	0.9341686	0.5328228	1.3072705	0.6589572	0.7526979	0.753735	0.4926753	0.4569973
Dividend	CRR	BA	BBS	BBSR	TRI	AB	SMO	SBS	SR	SBSR
high	0.0014231	0.0012858	0.0010426	3.912E-05	0.0017327	0.0032514	0.0034322	0.0026615	6.155E-05	8.687E-05
low	0.0010053	0.0009785	0.0007998	1.916E-05	0.0020795	0.0007309	0.0024167	0.0017412	1.799E-05	2.066E-05
Ratio	1.4156322	1.3140373	1.3036485	2.0421565	0.8332258	4.4486029	1.4202078	1.5285891	3.4210256	4.2049145
Volatility	CRR	BA	BBS	BBSR	TRI	AB	SMO	SBS	SR	SBSR
High	0.0016504	0.0015397	0.001219	3.896E-05	0.0018735	0.0024901	0.0046951	0.0037252	6.58E-05	8.671E-05
Low	0.0006907	0.0006431	0.0005638	1.734E-05	0.0019452	0.0013925	0.0007997	0.0003727	8.534E-06	1.423E-05
Ratio	2.3892852	2.3942826	2.1619232	2.2467473	0.9631186	1.7882619	5.8710759	9.9961076	7.7103828	6.0945143
Time	CRR	BA	BBS	BBSR	TRI	AB	SMO	SBS	SR	SBSR
High	0.0007814	0.0007062	0.0005494	2.927E-05	0.0022537	0.0026929	0.0008893	0.0003814	2.095E-05	3.992E-05
Low	0.0017335	0.0016433	0.0013673	2.898E-05	0.001489	0.0011491	0.0053666	0.0043853	6.236E-05	7.037E-05
Ratio	0.4507518	0.429721	0.4017815	1.0100032	1.5135555	2.3434477	0.1657121	0.0869621	0.3358822	0.5673354

- 2) The overall accuracy of puts is worse than that of calls. For example, the BBSR call is about 36 times more accurate than the BBSR put. All lattice methods have been written for call options, and then symmetry is applied to calculate the put price. Perhaps the skewness of interest and dividend yield in SA has an effect. For example, when discounting occurs at a rate of 2% for puts rather than 17% for calls. Alternatively, through symmetry a different weight may be attached to the discretisation error.

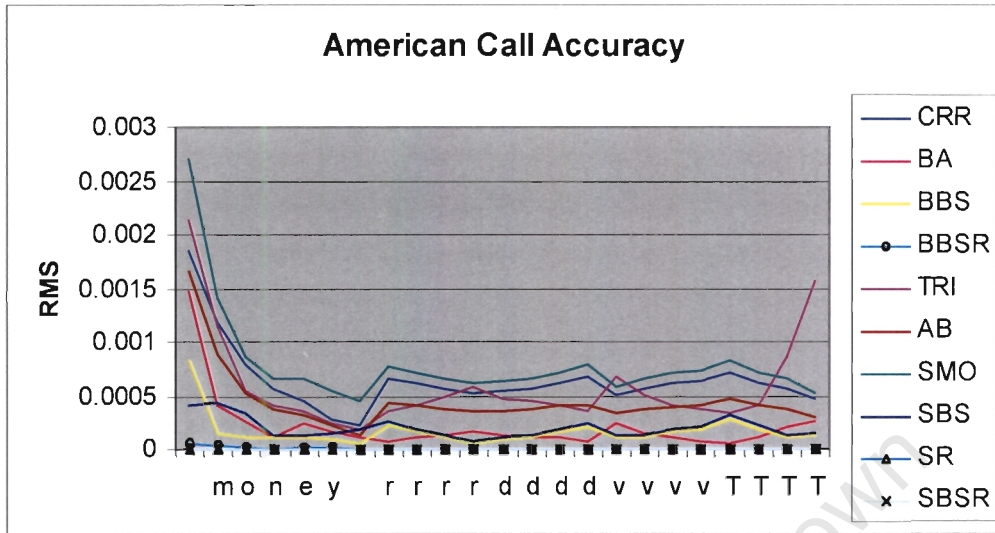
Summary for Puts

Batches 1 to 23	CRR	BA	BBS	BBSR	TRI	AB	SMO	SBS	SR	SBSR
RMS	0.0005259	0.0002707	0.0002445	0.0001822	0.0011175	0.0082672	0.0007844	0.0003507	0.0001461	0.000234
rel RMS (to BBSR)	0.3465642	0.6731931	0.7453509	1	0.1630856	0.0220439	0.232346	0.5196452	1.2475711	0.7788437
TIME	890.5703	919.3359	885.2266	855.625	878.5299	821.4063	883.3101	867.4922	888.97	853.7109
rel TIME (to BBSR)	0.9607608	0.930699	0.9665604	1	0.9739281	1.0416587	0.9686576	0.9863201	0.9624903	1.0022421
Performance Ratio	0.333	0.627	0.720	1.000	0.159	0.023	0.225	0.513	1.201	0.781
Batches 24 to 71	CRR	BA	BBS	BBSR	TRI	AB	SMO	SBS	SR	SBSR
RMS	0.0017962	0.0022353	0.0019719	0.0010888	0.0052241	0.0241868	0.0035032	0.0036821	0.0004244	0.0006476
rel RMS (to BBSR)	0.6061696	0.487113	0.5521634	1	0.2084213	0.0450171	0.3108118	0.2957094	2.5658083	1.6813914
TIME	890.5703	919.3359	885.2266	855.625	878.5299	821.4063	883.3101	867.4922	888.97	853.7109
rel TIME (to BBSR)	0.9607608	0.930699	0.9665604	1	0.9739281	1.0416587	0.9686576	0.9863201	0.9624903	1.0022421
Performance Ratio	0.582	0.453	0.534	1	0.203	0.047	0.301	0.292	2.470	1.685
Moneyiness	CRR	BA	BBS	BBSR	TRI	AB	SMO	SBS	SR	SBSR
AT	0.0013624	0.0014968	0.001396	0.0010918	0.0036351	0.0250567	0.0015741	0.0015712	0.0008504	0.0012175
OUT	0.0001641	0.0002193	0.0001984	0.0003031	0.0007074	0.0236115	0.0002891	0.0002684	0.0001723	0.0002381
IN	0.0045509	0.0059077	0.0051045	0.0021325	0.0133653	0.0237941	0.0103606	0.0110481	0.0001925	0.0004336
Interest	CRR	BA	BBS	BBSR	TRI	AB	SMO	SBS	SR	SBSR
high	0.0028887	0.003787	0.0033056	0.0018078	0.0085275	0.0363321	0.0052798	0.0059297	0.0006252	0.0009354
low	0.0007038	0.0006835	0.0006383	0.0003698	0.0019208	0.0120416	0.0017265	0.0014344	0.0002235	0.0003597
Ratio	4.1046956	5.5408894	5.1789549	4.8882038	4.439597	3.0172034	3.058033	4.1339284	2.7976688	2.6001824
Dividend	CRR	BA	BBS	BBSR	TRI	AB	SMO	SBS	SR	SBSR
high	0.0011025	0.0012905	0.0011677	0.0007094	0.0030628	0.0184676	0.0024266	0.0023237	0.0003284	0.0004174
low	0.00249	0.00318	0.0027761	0.0014683	0.0073855	0.029906	0.0045797	0.0050405	0.0005203	0.0008777
Ratio	0.4427605	0.4058056	0.4206397	0.4831281	0.4147082	0.6175219	0.5298497	0.4609993	0.6312435	0.4756121
Volatility	CRR	BA	BBS	BBSR	TRI	AB	SMO	SBS	SR	SBSR
high	0.0007159	0.0006356	0.0005586	0.0004174	0.0020643	0.021646	0.0016534	0.0011608	0.0002069	0.0003483
low	0.0030926	0.0041548	0.0036679	0.0018946	0.0090159	0.0272359	0.0057229	0.0067076	0.0006853	0.0010067
Ratio	0.2315004	0.1529858	0.1523022	0.220295	0.2289615	0.7947606	0.288906	0.1730579	0.3018571	0.3459864
Time	CRR	BA	BBS	BBSR	TRI	AB	SMO	SBS	SR	SBSR
high	0.0028707	0.0038478	0.0033807	0.0019234	0.009254	0.0412135	0.0052384	0.005991	0.0007519	0.0011049
low	0.0005069	0.0003002	0.0002814	8.732E-05	0.0003883	0.0037548	0.0014209	0.0009114	3.128E-05	9.883E-05
Ratio	5.663678	12.815888	12.014354	22.026112	23.831818	10.976179	3.6867263	6.573342	24.035443	11.179299

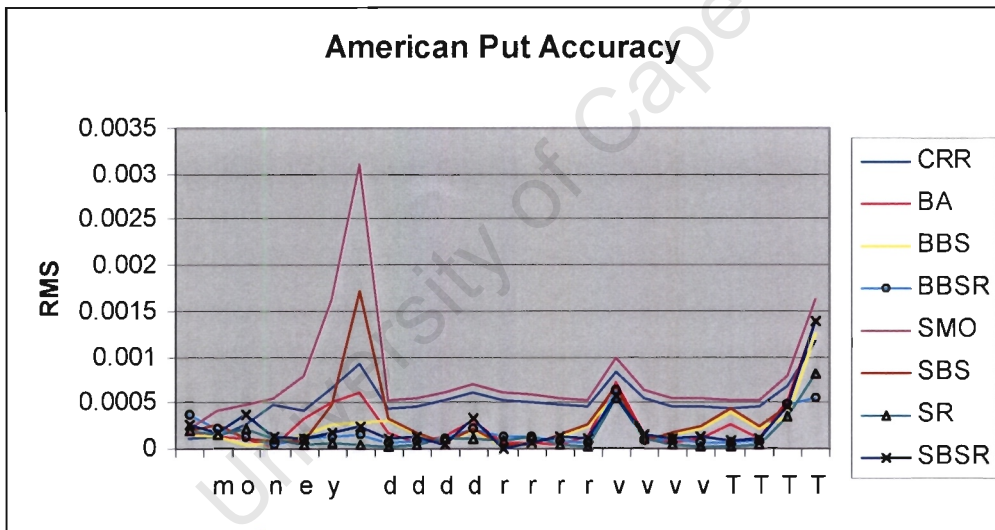
- 3) All methods appear less accurate for out-the-money options. This may be because the discretisation error arising from discretising the stock process is not sufficiently averaged, as for out-the-money options the calculated price relies heavily on only a few payoffs at maturity.
- 4) The SR and SBSR method are relatively sensitive to changes in the interest rate and dividend yield for call options, and vice versa for put options.
- 5) For calls, the accuracy of all methods is worse for high volatility options. For puts, accuracy is high for high volatility options.
- 6) For calls, the accuracy of all methods is worse for short-term option. For Puts the accuracy is better for short-term options.
- 7) On the first 23 batches, for call options the BBSR and SR are about 90 times and 600 times more accurate than the CRR for comparable speed respectively.

- 8) On the first 23 batches, for put options, the BBSR and SR are about 2.9 and 3.6 times more accurate than the CRR respectively.

The following two graphs summarise much of our previous observations.



For the American Put, the prices were obtained using American Put-Call symmetry.



The TRI and AB were excluded in order to focus on a smaller range of the RMS in the remaining methods.

This concludes our description of lattice method accuracy. The next section analyses the accuracy of the analytic approximations.

6.4 Analysis of Analytic Approximations

For the Lattice methods above, it was possible to tune all methods, so that their speed is approximately equal. For analytic approximations, this is not always possible. Changing the accuracy and hence speed, requires the addition of at least trivariate normal integration in the case of GJ2, four or more time discretisations in the method of lines and at least a four-piece exponential boundary approximation for Ju. In the case of LBA and LUBA convergence can't be improved.

Only Sullivan's method allows the inclusion of additional exercise opportunities without much additional effort. However, SUL2 and SUL4 are unstable for high dividend yields and therefore the method is unsuitable for pricing American Call options in the SA market. It also turns out, that SUL4 is twice as slow as the Lattice methods discussed above. For these reasons, SUL2 and SUL4 were run on only batches 1 to 23 for Put options.

Since the speed of the methods can't easily be made comparable directly, the RMS's were standardized by speed through the performance ratio in the same way, as done for lattice methods.

Three further least squares LBA models were fitted to put data: PLLS1, PLLS2 and PLLS3. Also, three more LUBA models were fitted to the put data: PLULS1, PLULS2 and PLULS3. In addition, a simple Luba model that averages the lower and the upper bound is included LUBA-half.

6.4.1 Summary and Interpretation of Analytical Methods

The two Excel sheets on the next two pages summarise the RMS's of all methods over the first 23 batches and the remaining 44 batches separately.

6.4.1.1 Summary for Calls

	1	2	3	4	5	6	7	8	9	10
Batches 1 to 23	upper B	LUBA-LS1	LUBA-WLS	LUBA-PC	LUBA-R	lower B	LUBA-STEP	LUBA-LS2	JU 2	LBA - R
RMS	5.345E-07	4.152E-07	4.16E-07	4.162E-07	4.163E-07	1.426E-06	4.177E-07	4.181E-07	1.066E-06	1.914E-06
rel RMS (to BBSR)	7.0557132	9.0814762	9.0649857	9.0598517	9.0586013	2.6449588	9.026814	9.0200784	3.5359204	1.9699469
TIME	208.668	327.8516	327.8516	327.8516	327.8516	95.83594	327.8516	327.8516	172.457	99.47266
rel TIME (to BBSR)	3.917755	2.493537	2.493537	2.493537	2.493537	8.5303081	2.493537	2.493537	4.7403706	8.2184401
Performance Ratio	27.643	22.645	22.604	22.591	22.588	22.562	22.509	22.492	16.762	16.190
	11	12	13	14	15	16	17	18	19	
	LBA-PC	LBA - Step	LBA-LS2	LUBA-half	JU 3	GJ2	LBA-LS1	BBSR	ML	
	2.071E-06	2.076E-06	2.645E-06	8.402E-07	1.006E-06	2.358E-05	2.7E-05	3.771E-06	0.0029195	
	1.8207889	1.8163639	1.4255264	4.4879672	3.7485235	0.1599123	0.13966	1	0.0012916	
	99.47266	99.47266	99.47266	327.8516	334.3906	31.42188	99.47266	817.5101	15.55078	
	8.2184401	8.2184401	8.2184401	2.493537	2.444776	26.017224	8.2184401	1	52.57036	
	14.964	14.928	11.716	11.191	9.164	4.160	1.148	1.000	0.068	
	1	2	3	4	5	6	7	8	9	10
Batches 24 to 71	LUBA-LS1	LUBA-WLS	LUBA-R	LUBA-STEP	LUBA-LS2	LUBA-PC	BBSR	JU 3	LBA - R	LBA-LS2
RMS	9.244E-06	1.1E-05	1.11E-05	1.297E-05	1.3E-05	1.404E-05	1.037E-05	3.685E-05	0.0001292	0.0001311
rel RMS (to BBSR)	1.1220441	0.9432047	0.9341346	0.7998234	0.7981156	0.738855	1	0.2814849	0.0802997	0.0790897
TIME	327.8516	327.8516	327.8516	327.8516	327.8516	327.8516	817.5101	334.3906	99.47266	99.47266
rel TIME (to BBSR)	2.493537	2.493537	2.493537	2.493537	2.493537	2.493537	1	2.444776	8.2184401	8.2184401
Performance Ratio	2.798	2.352	2.329	1.994	1.990	1.842	1.000	0.688	0.660	0.650
	11	12	13	14	15	16	17	18	19	
	LBA-PC	LBA - Step	upper B	LBA-LS1	lower B	LUBA-half	JU 2	ML	GJ2	
	0.0001321	0.0001324	6.707E-05	0.0001463	0.0002919	0.0001132	0.000219	0.003488	0.0021146	
	0.0785396	0.0783524	0.1546427	0.0708998	0.0355334	0.091607	0.0473619	0.0029737	0.0049052	
	99.47266	99.47266	208.668	99.47266	95.83594	327.8516	172.457	15.55078	31.42188	
	8.2184401	8.2184401	3.917755	8.2184401	8.5303081	2.493537	4.7403706	52.57036	26.017224	
	0.645	0.644	0.606	0.583	0.303	0.228	0.225	0.156	0.128	

Discussion of Call Data

- 1) On the first 23 batches, the upper bound turned out to have the best performance ratio. It was less accurate but faster than the LUBA methods. For call options, one would expect the LUBA models with deleted outliers to outperform the Least Squares model fitted to the whole data (LS1). In the LUBA regression this did not materialise, however, the performance of all LUBA methods was very close to the performance of LUBA - LS1. See **GRAPH 1** on page 74.
- 2) The performance of the LUBA methods is way better than that of BBSR in the case of Call options and vice versa for Put options.
- 3) For calls, JU2 outperformed LBA models for batches 1-23, while JU3 outperformed the LBA models for batches 24 – 71.
- 4) For Call options, the biased LBA models and LBA models with outliers deleted performed better than the LBA Least squares model fitted on the original data. This outperformance is quite large and perhaps supports the deletion of outliers prior to model fitting. See **GRAPH 2** on page 74.

6.4.1.2 Summary for Puts

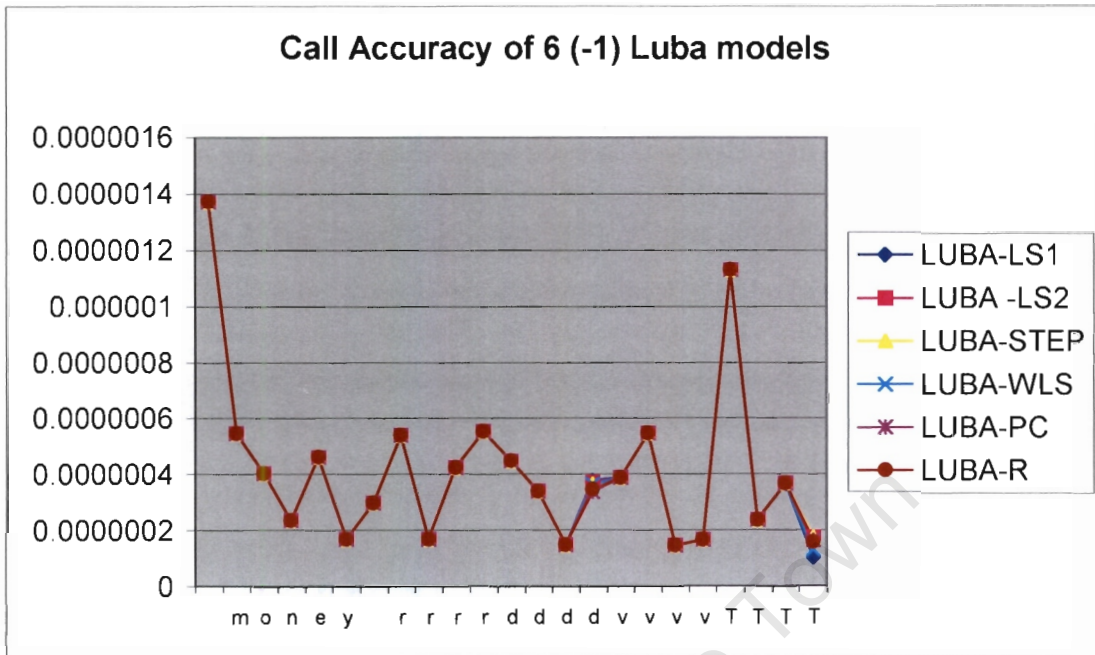
Batches 1 to 23	ML	BBSR	LBA-LS1	LBA-PC	LBA - Step	LBA-LS2	JU 3	LBA - R	PLLS1
RMS	0.0016027	0.0001822	0.0023867	0.0024549	0.002455	0.0024557	0.0005247	0.0024928	0.0033465
rel RMS (to BBSR)	0.1137108	1	0.0763577	0.0742347	0.0742328	0.0742105	0.3473058	0.0731057	0.0544576
TIME	14.83203	855.625	82.10938	82.10938	82.10938	82.10938	385.0313	82.10938	82.10938
rel TIME (to BBSR)	57.687653	1	10.420551	10.420551	10.420551	10.420551	2.2222219	10.420551	10.420551
Performance Ratio	6.560	1.000	0.796	0.774	0.774	0.773	0.772	0.762	0.567
	10	11	12	13	14	15	16	17	18
	PLLS2	LUBA-half	PLLS3	GJ2	JU 2	lower B	LUBA-LS1	LUBA-WLS	LUBA-R
	0.0036732	0.0011468	0.0045994	0.0132743	0.002315	0.00706	0.004003	0.004524	0.0047186
	0.0496135	0.1589184	0.0396226	0.0137289	0.0787236	0.0258131	0.0455264	0.0402835	0.0386218
	82.10938	269.4609	82.10938	32.46094	191.4102	77.66406	269.4609	269.4609	269.4609
	10.420551	3.1753215	10.420551	26.358602	4.4701118	11.017001	3.1753215	3.1753215	3.1753215
	0.517	0.505	0.413	0.362	0.352	0.284	0.145	0.128	0.123
	19	20	21	22	23	24	25	26	27
	LUBA-STEP	LUBA-LS2	upper B	LUBA-PC	PLULS1	SUL 2	PLULS2	PLULS3	SUL 4
	0.0047692	0.0047849	0.0078678	0.0048845	0.0065915	0.0011837	0.0070591	0.0070598	0.000689
	0.0382121	0.0380866	0.0231629	0.0373102	0.027648	0.1539558	0.0258166	0.0258138	0.2645073
	269.4609	269.4609	166.0898	269.4609	269.4609	1506.66	269.4609	269.4609	3769.262
	3.1753215	3.1753215	5.1515807	3.1753215	3.1753215	0.5678952	3.1753215	3.1753215	0.2270007
	0.121	0.121	0.119	0.118	0.088	0.087	0.082	0.082	0.060
	1	2	3	4	5	6	7	8	9
Batches 24 to 71	ML	LBA-LS1	LBA-LS2	LBA-PC	LBA - Step	LBA - R	PLLS2	PLLS1	PLLS3
RMS	0.0032813	0.0025032	0.0026583	0.0026613	0.0026628	0.0026742	0.0027249	0.0028037	0.002886
rel RMS (to BBSR)	0.2514763	0.3296514	0.3104118	0.3100673	0.3098937	0.3085695	0.3028271	0.2943164	0.2859235
TIME	14.83203	82.10938	82.10938	82.10938	82.10938	82.10938	82.10938	82.10938	82.10938
rel TIME (to BBSR)	57.687653	10.420551	10.420551	10.420551	10.420551	10.420551	10.420551	10.420551	10.420551
Performance Ratio	14.507	3.435	3.235	3.231	3.229	3.215	3.156	3.067	2.979
	10	11	12	13	14	15	16	17	18
	Lower B	LUBA-LS1	BBSR	LUBA-WLS	PLULS1	LUBA-LS2	LUBA-STEP	PLULS2	PLULS3
	0.0037279	0.0025077	0.0008252	0.0032801	0.0032927	0.0035935	0.0036312	0.0036845	0.0037158
	0.2213509	0.3290581	1	0.2515703	0.2506123	0.2296303	0.2272464	0.223958	0.2220725
	77.66406	269.4609	855.625	269.4609	269.4609	269.4609	269.4609	269.4609	269.4609
	11.017001	3.1753215	1	3.1753215	3.1753215	3.1753215	3.1753215	3.1753215	3.1753215
	2.439	1.045	1.000	0.799	0.796	0.729	0.722	0.711	0.705
	19	20	21	22	23	24	25		
	LUBA-R	LUBA-PC	LUBA-half	upper B	GJ2	JU 2	JU 3		
	0.0042546	0.0044658	0.0060316	0.014091	0.0795496	0.2559381	0.2520296		
	0.1939482	0.1847763	0.1368099	0.0585608	0.0103732	0.0032241	0.0032741		
	269.4609	269.4609	269.4609	166.0898	32.46094	191.4102	385.0313		
	3.1753215	3.1753215	3.1753215	5.1515807	26.358602	4.4701118	2.2222219		
	0.616	0.587	0.434	0.302	0.273	0.014	0.007		

Discussion of Put Data

- 5) For put options, the ML performs the best under our definition of performance. This is primarily due to the high speed of the ML. Perhaps a larger number of discretisations would have a different impact on the speed accuracy trade-off.
- 6) Second are the LBA models. Again the LS model fitted to the whole of the data, performed best, although as is the case for LUBA, the performance of the LBA models is about the same. In addition, these LBA models outperform LBA least squares models fitted to put data instead of Call data. It follows that even though collinearity was tested high, the gains from fitting biased regression models did not materialise. See **GRAPH 3** on page 75.
- 7) Similarly, LUBA models fitted to put data, did not outperform the models fitted to call data and performance is about the same.
- 8) Furthermore, the LUBA-half model outperformed all other LUBA models on batch 24-72. See **GRAPH 4** on page 75.

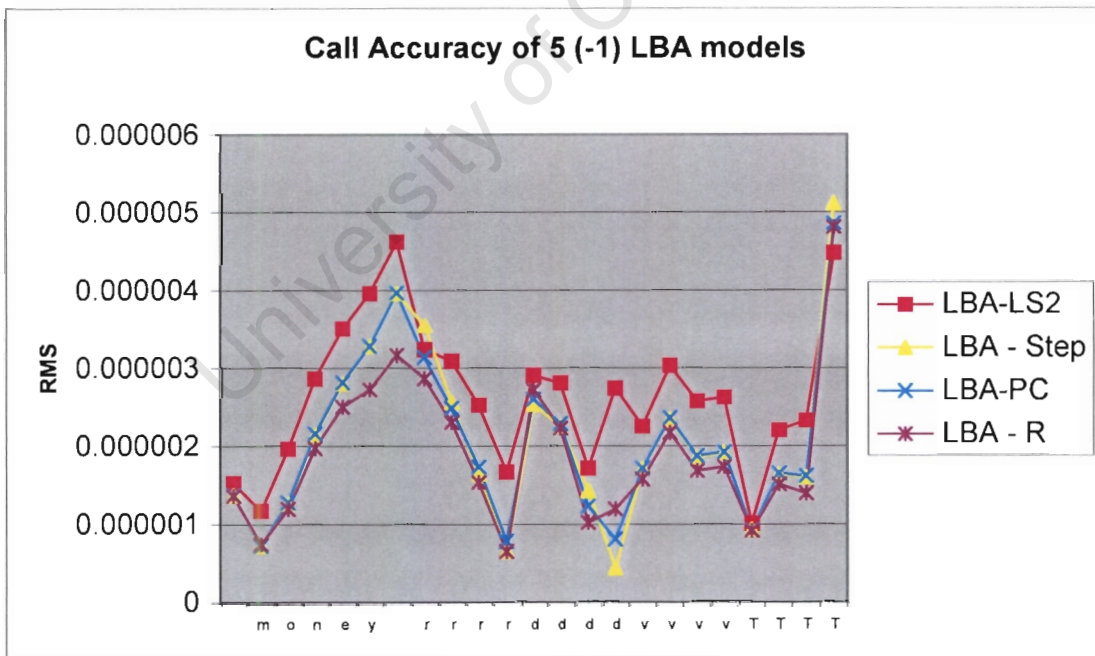
6.4.2 Graphs for Batches 1 to 23

GRAPH 1



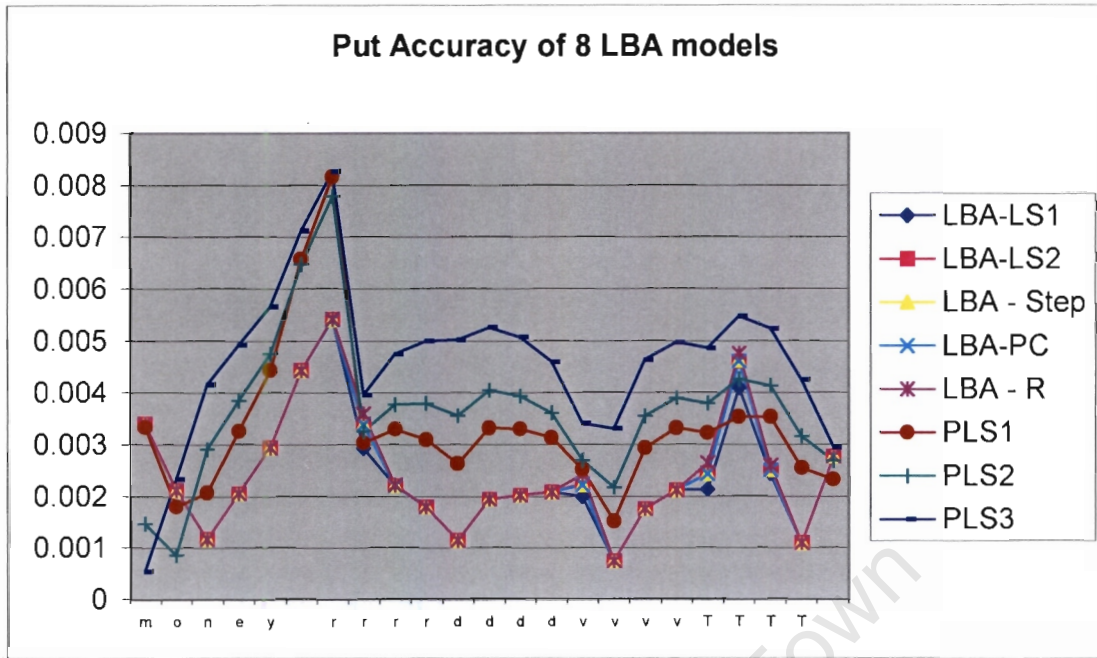
The LUBA-half model was excluded because of its poor accuracy here.

GRAPH 2

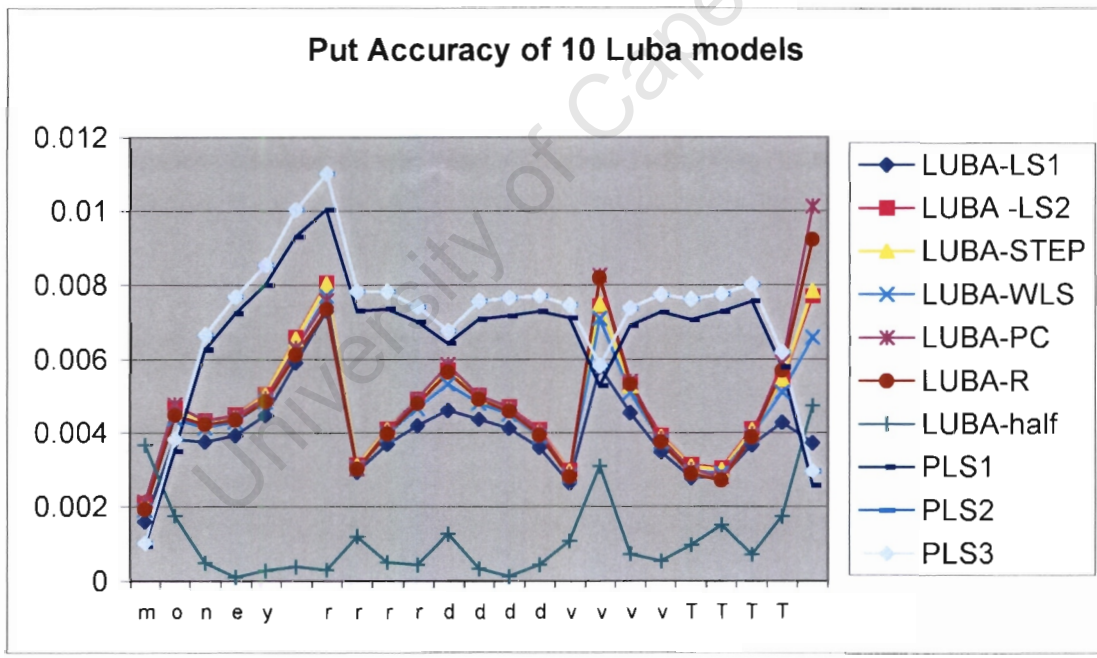


The LBA-LS1 model was excluded, because of its poor performance here.

GRAPH 3



GRAPH 4



The LUBA-half model outperforms the other LUBA models on Batches 1 to 23. The same is not true on batches 24 –71.

6.4.4 Results on Sensitivity of the Methods

Appendix F and **G** summarise the sensitivity on Batches 24 to 71 for call and put options respectively. The following can be observed:

Call Options:

- 1) LUBA models are sensitive to the moneyness, interest rate and dividend yield.
- 2) LBA models are sensitive to the dividend yield, volatility and time.
- 3) The lower and upper bound are sensitive to the interest rate, dividend yield, volatility and time.
- 4) JU's models are very sensitive to the dividend yield, volatility and time.
- 5) The ML is stable overall.

Put Options:

- 1) Ju's models are very sensitive to the moneyness, interest rate and dividend yield and volatility (but not time) of the option.
- 2) The ML, LUBA and LBA models appear more stable.

6.5 Conclusion

This chapter provided a description of the accuracy and speed of a number of lattice methods and analytic approximations.

- The results indicate that no single approach performs best overall.
- The application of American symmetry does not transfer quality of performance for neither lattice methods nor analytic approximations. Perhaps this is emphasised by the skewness of interest rates in SA.
- The best lattice approach is the SR method ahead of the BBSR method.
- The LUBA approaches achieve extremely high accuracy for call options, while the LBA approaches achieve high accuracy for put options.
- For LBA, adjustment for outliers in the regression is justified, since the performance is either very similar or improved through adjustment.
- For LUBA, the adjustment for outliers does not seem justified.
- Adjusting for collinearity through biased regression does not prove useful for either LBA or LUBA. In fact, a simple average of the upper and lower bound improves over more complicated LUBA models for Put options over batches 1-23.

The conclusions stated in the above three points, are perhaps mitigated by the fact that extreme sample categories were chosen, instead of a broad sample with sampling from each of the 1792 categories.

- Sullivan's model is sensitive to high dividend yields and hence unsuitable for pricing Call options in SA using American symmetry. The models perform poorer relative to other analytic approximations considered.
- The three point ML is very fast and seems stable overall. Although the authors of the method suggest that a satisfactory trade-off between speed and accuracy, is given by the three-point ML, one should probably add more extrapolation points in order to make the accuracy more comparable directly with those of the LUBA and LBA methods. A similar point holds for the GJ2 method.
- JU's method perform's average for call option, but less well when pricing put options.

We will now turn to conclusions and recommendations in the final chapter.

Appendix A: Sample Structure

Batch	Mon	Int	Div	Vol	Tim	Batch
1	out	stan	stan	stan	stan	1
2		stan	stan	stan	stan	2
3		stan	stan	stan	stan	3
4	at	stan	stan	stan	stan	4
5		stan	stan	stan	stan	5
6		stan	stan	stan	stan	6
7	in	stan	stan	stan	stan	7
8	at	low	stan	stan	stan	8
9	at		stan	stan	stan	9
10	at		stan	stan	stan	10
11	at	high	stan	stan	stan	11
12	at	stan	low	stan	stan	12
13	at	stan		stan	stan	13
14	at	stan		stan	stan	14
15	at	stan	high	stan	stan	15
16	at	stan	stan	low	stan	16
17	at	stan	stan		stan	17
18	at	stan	stan		stan	18
19	at	stan	stan	high	stan	19
20	at	stan	stan	stan	low	20
21	at	stan	stan	stan		21
22	at	stan	stan	stan		22
23	at	stan	stan	stan	high	23
24	at	low	low	low	low	24
25	at	low	high	low	low	25
26	at	high	low	low	low	26
27	at	high	high	low	low	27
28	at	low	low	high	low	28
29	at	low	high	high	low	29
30	at	high	low	high	low	30
31	at	high	high	high	low	31
32	at	low	low	low	high	32
33	at	low	high	low	high	33
34	at	high	low	low	high	34
35	at	high	high	low	high	35
36	at	low	low	high	high	36
37	at	low	high	high	high	37
38	at	high	low	high	high	38
39	at	high	high	high	high	39
40	out	low	low	low	low	40
41	out	low	high	low	low	41
42	out	high	low	low	low	42
43	out	high	high	low	low	43
44	out	low	low	high	low	44
45	out	low	high	high	low	45
46	out	high	low	high	low	46
47	out	high	high	high	low	47
48	out	low	low	low	high	48
49	out	low	high	low	high	49
50	out	high	low	low	high	50
51	out	high	high	low	high	51
52	out	low	low	high	high	52
53	out	low	high	high	high	53
54	out	high	low	high	high	54
55	out	high	high	high	high	55
56	in	low	low	low	low	56
57	in	low	high	low	low	57
58	in	high	low	low	low	58
59	in	high	high	low	low	59
60	in	low	low	high	low	60
61	in	low	high	high	low	61
62	in	high	low	high	low	62
63	in	high	high	high	low	63
64	in	low	low	low	high	64
65	in	low	high	low	high	65
66	in	high	low	low	high	66
67	in	high	high	low	high	67
68	in	low	low	high	high	68
69	in	low	high	high	high	69
70	in	high	low	high	high	70
71	in	high	high	high	high	71

Appendix B: Lattice Results Calls

BATCH	CRR	BA	BBS	BBSR	TRI	AB	SMO	SBS	SR	SBSR	OUT
1	0.0018443	0.0014765	0.0008241	4.989E-05	0.0021371	0.00166	0.0027064	0.000413	4.042E-06	3.638E-06	OUT
2	0.0011889	0.0004206	0.0001427	3.767E-05	0.0011509	0.0008868	0.0014086	0.0004301	2.073E-06	4.308E-06	m
3	0.0007859	0.000273	0.0001192	2.746E-05	0.0005435	0.0005208	0.0008741	0.0003399	1.285E-06	3.419E-06	o
4	0.0005746	0.000105	0.0001192	8.574E-07	0.0004185	0.000378	0.0006629	0.0001349	8.093E-07	1.059E-06	n
5	0.0004554	0.0002389	0.000119	1.272E-05	0.0003492	0.0003224	0.0006597	0.0001306	9.565E-07	1.083E-06	e
6	0.0002894	0.0001618	9.459E-05	1.23E-05	0.0002441	0.000232	0.000543	0.0001433	7.662E-07	1.218E-06	y
7	0.0002225	0.0001121	6.285E-05	6.134E-06	0.0001859	0.0001229	0.000459	0.0001916	8.113E-07	1.495E-06	IN
8	0.0006658	7.371E-05	0.0002276	1.85E-06	0.0003595	0.0004381	0.0007682	0.0002576	9.427E-07	2.289E-06	r-low
9	0.0006166	0.0001045	0.0001741	1.355E-06	0.0004172	0.0004057	0.0007114	0.000197	8.709E-07	1.676E-06	r
10	0.0005745	0.0001365	0.0001191	8.564E-07	0.0004958	0.000378	0.0006628	0.0001348	8.092E-07	1.058E-06	r
11	0.0005376	0.0001663	6.465E-05	3.81E-07	0.0005864	0.0003537	0.0006202	7.317E-05	7.554E-07	4.688E-07	r-high
12	0.0005559	0.0001251	9.231E-05	6.196E-07	0.0004656	0.0003658	0.0006414	0.0001045	7.82E-07	7.644E-07	d-low
13	0.0005739	0.0001207	0.0001183	8.489E-07	0.0004546	0.0003776	0.0006621	0.0001338	8.083E-07	1.049E-06	d
14	0.0006144	0.000105	0.0001704	1.291E-06	0.0004183	0.0004062	0.0007089	0.0001929	8.894E-07	1.594E-06	d
15	0.00068	7.299E-05	0.0002116	1.895E-06	0.0003576	0.0004123	0.0007846	0.0002395	1.016E-06	2.518E-06	d-high
16	0.000515	0.0002452	0.0001167	9.107E-07	0.0006882	0.0003389	0.0005942	0.0001321	7.255E-07	1.131E-06	v-low
17	0.0005742	0.0001565	0.0001187	8.531E-07	0.000518	0.0003778	0.0006625	0.0001344	8.088E-07	1.054E-06	v
18	0.0006135	0.0001045	0.0001659	1.271E-06	0.0004174	0.0004037	0.0007078	0.0001877	8.66E-07	1.571E-06	v
19	0.0006416	7.967E-05	0.0001907	1.483E-06	0.0003726	0.0004225	0.0007403	0.0002158	9.095E-07	1.834E-06	v-high
20	0.0007209	6.205E-05	0.0002851	2.394E-06	0.0003413	0.0004743	0.0008318	0.0003226	1.021E-06	2.963E-06	T-low
21	0.0006295	0.0001045	0.0001931	1.537E-06	0.0004171	0.0004142	0.0007263	0.0002186	8.903E-07	1.901E-06	T
22	0.0005729	0.0002005	0.0001168	8.358E-07	0.0008692	0.000377	0.000661	0.0001322	8.069E-07	1.032E-06	T
23	0.0004627	0.0002684	0.0001414	1.84E-06	0.0015575	0.0003083	0.0005339	0.0001601	1.243E-06	2.363E-06	T-high
24	0.0006998	0.000137	0.0002686	2.243E-06	0.0004216	0.0004604	0.0008074	0.0003039	9.913E-07	2.776E-06	AT
25	0.0007913	5.354E-05	0.0003254	3.154E-06	0.0003342	0.0005141	0.000913	0.0003682	1.111E-06	3.745E-06	AT
26	0.0006213	0.0002483	0.0001867	1.483E-06	0.0006378	0.0004088	0.0007168	0.0002113	8.789E-07	1.834E-06	AT
27	0.0006795	0.0001784	0.0002501	2.071E-06	0.0004868	0.0004471	0.000784	0.000283	9.625E-07	2.562E-06	AT
28	0.000766	3.251E-05	0.0003146	2.662E-06	0.0003259	0.000504	0.0008837	0.0003561	1.085E-06	3.295E-06	AT
29	0.0008157	2.189E-05	0.0003462	6.267E-06	0.0003571	0.0005314	0.0009411	0.0003916	1.767E-06	6.675E-06	AT
30	0.0007211	8.196E-05	0.0002836	2.379E-06	0.0003724	0.0004745	0.000832	0.0003209	1.022E-06	2.945E-06	AT
31	0.0007532	4.474E-05	0.0003047	2.571E-06	0.0003264	0.0004956	0.000869	0.0003449	1.067E-06	3.182E-06	AT
32	0.0004606	0.0002615	0.0001769	1.363E-06	0.001895	0.0003031	0.0005314	0.0002002	1.415E-06	1.687E-06	AT
33	0.0007279	0.0004006	0.0002062	1.964E-05	0.001003	0.0052072	0.0008404	0.0002329	3.877E-06	2.191E-05	AT
34	0.0002558	0.0002358	0.0001727	1.888E-06	0.0037564	0.0001684	0.0002951	0.0001954	2.05E-06	2.339E-06	AT
35	0.0005847	0.0006518	0.0005326	1.073E-05	0.0027327	0.0029305	0.0006741	0.0006025	6.43E-06	1.052E-05	AT
36	0.000598	0.0001818	0.0001138	6.88E-07	0.0009245	0.0003949	0.0006899	0.0001288	8.498E-07	8.436E-07	AT
37	0.0007456	0.0004564	0.0002309	5.406E-05	0.0011195	0.0074752	0.0008697	0.0002602	3.868E-05	4.723E-05	AT
38	0.0004512	0.0002421	0.0001442	1.265E-06	0.0025417	0.0002969	0.0005206	0.0001632	6.185E-07	1.581E-06	AT
39	0.0007866	0.0006884	0.0004843	3.006E-05	0.001455	0.0076119	0.0009156	0.0005498	1.674E-05	2.353E-05	AT
40											OUT
41											OUT
42											OUT
43											OUT
44	0.0078052	0.0062771	0.0046856	0.0001433	0.0064411	0.0054651	0.0200344	0.0163078	0.0001695	0.0001899	OUT
45	0.0075413	0.0115149	0.0083397	0.0001448	0.0077398	0.0065078	0.0345036	0.0317323	0.000649	0.0007191	OUT
46	0.0037298	0.0075443	0.0063629	0.0001054	0.0062964	0.0029547	0.0213341	0.017661	0.0001993	0.0002209	OUT
47	0.0084317	0.0060047	0.005202	0.0001162	0.0051252	0.0026829	0.0214859	0.0178195	0.000202	0.0002238	OUT
48	0.0019846	0.0033539	0.0027242	5.964E-05	0.0051893	0.0019188	0.0012751	0.0004353	1.258E-06	4.206E-06	OUT
49	0.0028471	0.0024221	0.00216	9.06E-05	0.0042466	0.0055565	0.0046814	0.0015504	6.36E-06	5.309E-05	OUT
50	0.0009729	0.0013248	0.0011827	1.589E-05	0.0052171	0.0006648	0.0005422	0.0002005	1.993E-06	2.217E-06	OUT
51	0.0020753	0.0026863	0.0023772	2.948E-05	0.0052622	0.001585	0.0008665	0.0002765	1.818E-06	3.587E-06	OUT
52	0.000835	0.0003516	0.0001247	2.521E-05	0.0011819	0.0006171	0.0009626	0.000289	1.353E-06	2.779E-06	OUT
53	0.0011441	0.0004637	0.0002103	6.239E-05	0.0012111	0.0063978	0.0012372	0.0003022	9.987E-06	4.655E-05	OUT
54	0.0008212	0.000515	0.0003642	1.404E-05	0.0025381	0.0004471	0.0007205	0.0001568	1.1E-06	1.216E-06	OUT
55	0.0009618	0.0008279	0.0006116	3.22E-05	0.0016286	0.0029106	0.0009785	0.0003078	3.966E-06	8.849E-06	OUT
56	5.523E-05	1.875E-05	1.604E-05	2.053E-06	5.161E-05	5.979E-05	0.0002902	0.0001969	2.101E-06	2.391E-06	IN
57	5.929E-05	4.424E-05	3.169E-05	2.666E-06	5.222E-05	0.0004092	0.0003251	0.0002427	2.603E-06	3.356E-06	IN
58	2.516E-05	1.23E-05	9.411E-06	1.507E-06	9.105E-05	3.196E-05	0.001852	0.0001531	2.092E-06	2.394E-06	IN
59	4.487E-05	1.605E-05	1.293E-05	2.405E-06	6.459E-05	2.217E-05	0.0002651	0.0001901	2.101E-06	2.393E-06	IN
60	0.0002797	0.0001415	0.0001079	1.022E-05	0.0001905	0.0001716	0.0005454	0.0002014	1.344E-06	1.651E-06	IN
61	0.0003169	0.0001777	0.0001082	1.188E-05	0.0001766	0.0004351	0.0005742	0.0002187	5.646E-06	1.121E-05	IN
62	0.000246	0.0001352	8.917E-05	7.441E-06	0.0001334	0.0001488	0.0005024	0.0002012	1.203E-06	1.607E-06	IN
63	0.0002878	0.000181	0.0001013	8.858E-06	0.0001551	0.0002572	0.0005399	0.0002014	1.448E-06	1.714E-06	IN
64	0.0001167	7.129E-05	5.317E-05	2.762E-06	0.0013597	0.0001061	0.000332	0.0001982	2.027E-06	2.271E-06	IN
65	0.0004438	0.000257	0.0001632	8.504E-05	0.0006354	0.00043743	0.0008071	0.0007094	0.0001084	0.0001368	IN
66	6.057E-05	5.911E-05	4.96E-05	9.09E-07	0.0033638	4.639E-05	0.0001539	0.0001321	2.006E-06	2.285E-06	IN
67	0.0003086	0.0004288	0.0003775	1.134E-05	0.0021031	0.0026347	0.000708	0.0007707	2.019E-05	2.222E-05	IN
68	0.0003634	0.0001752	8.353E-05	1.193E-05	0.0006855	0.0002646	0.0005726	0.0001093	8.474E-07	1.423E-06	IN
69	0.000466	0.0002968	0.0001649	9.374E-05	0.0008071	0.0051372	0.0007984	0.0004942	0.0001621	0.0003608	IN
70	0.0002469	0.000126	8.03E-05	7.133E-06	0.0021339	0.0001719	0.0004398	0.0001829	7.812E-07	1.907E-06	IN
71	0.0004952	0.0004704	0.0003963	4.043E-05	0.0010965	0.0074083	0.0009309	0.0007042	0.0001078	0.0001984	IN

Appendix C: Lattice Results Puts

Batch	CRR	BA	BBS	BBSR	TRI	AB	SMO	SBS	SR	SBSR	
1	0.0001093	0.0001829	0.0001469	0.000367	0.0003994	0.0100326	0.0002132	0.0002022	0.0001874	0.0002711	IN
2	0.0001328	0.0001285	0.0001244	0.0002186	0.0005475	0.0404219	0.0004102	0.000263	0.0001607	0.0001805	m
3	0.0002567	0.0001167	4.711E-05	0.0001237	0.0005881	0.0051473	0.0004902	0.0001096	0.00022	0.0003699	o
4	0.0004824	1.461E-05	7.593E-05	3.634E-05	0.0009763	0.0028983	0.0005544	8.524E-05	0.0001177	0.0001227	n
5	0.0004127	0.0003272	0.0001639	0.000111	0.0006418	0.0065302	0.0007815	8.895E-05	4.581E-05	0.0001144	e
6	0.0006691	0.0005117	0.0002704	0.0001329	0.0007422	0.0107666	0.0016275	0.0004781	5.987E-05	0.0001783	y
7	0.0009145	0.0006116	0.0002794	0.0001451	0.0011463	0.0093836	0.0031068	0.0017258	5.11E-05	0.000241	OUT
8	0.0004507	0.0001796	0.0003013	7.502E-05	0.0005413	0.0054351	0.000522	0.0003349	1.902E-05	0.0001043	r-low
9	0.0004713	7.17E-05	0.0001643	8.754E-05	0.0008249	0.0038231	0.0005449	0.0001851	4.169E-05	0.0001244	r
10	0.0005301	0.0001221	6.28E-05	0.0001104	0.0011623	0.0014925	0.0006124	7.608E-05	0.0001107	4.688E-05	r
11	0.0006177	0.0002994	0.0001962	0.0002105	0.0016297	0.0050598	0.0007135	0.0002276	0.0001077	0.0003284	r-high
12	0.0005183	6.67E-05	2.813E-05	0.000133	0.0012173	0.0001642	0.0006133	3.852E-05	8.515E-05	9.594E-06	d-low
13	0.0005091	6.203E-05	7.064E-05	0.0001408	0.0011052	0.0017585	0.0005885	4.934E-05	0.0001258	5.938E-05	d
14	0.0004764	4.712E-05	0.000143	8.927E-05	0.0008373	0.0033577	0.0005509	0.0001602	4.29E-05	0.0001273	d
15	0.0004634	0.0001257	0.0002476	6.634E-05	0.0005707	0.0047022	0.0005369	0.0002747	2.57E-05	0.0001068	d-high
16	0.0008414	0.000737	0.0005995	0.0006441	0.0021244	0.01488	0.0009817	0.0006867	0.0005499	0.0005668	v-low
17	0.0005438	0.0001625	8.868E-05	9.436E-05	0.0012142	0.0018915	0.0006351	9.6E-05	0.0001191	0.0001636	v
18	0.000473	5.873E-05	0.0001572	0.0001111	0.0008787	0.0108817	0.0005467	0.0001747	3.068E-05	0.0001002	v
19	0.0004685	0.0001097	0.0002246	7.011E-05	0.0007901	0.0049511	0.0005411	0.0002495	3.007E-05	0.0001285	v-high
20	0.0004489	0.0002612	0.0003951	7.683E-05	0.0003284	0.0049859	0.0005199	0.0004412	1.468E-05	8.639E-05	T-low
21	0.000459	0.0001175	0.000217	9.664E-05	0.0007161	0.0045801	0.0005304	0.0002396	4.004E-05	0.0001108	T
22	0.0006788	0.00053	0.0003541	0.0004923	0.0021016	0.0106881	0.0007904	0.0004688	0.0003442	0.0004598	T
23	0.0011667	0.001382	0.0012654	0.0005586	0.0046179	0.0331336	0.0016286	0.0014105	0.0008249	0.0013808	T-high
24	0.0004414	0.0002345	0.0003481	9.073E-05	0.0004173	0.0051152	0.0005109	0.0003884	3.299E-05	9.186E-05	AT
25	0.000523	0.0002676	0.0004519	6.72E-05	0.0001832	0.0028623	0.0006042	0.0005067	1.272E-05	7.755E-05	AT
26	0.0006542	0.0004336	0.0003481	0.0003207	0.0013848	0.01078617	0.0007467	0.000392	0.0002013	0.0003187	AT
27	0.0004856	0.0001964	0.0002557	0.0001193	0.0006945	0.0039912	0.0005652	0.0002921	6.398E-05	0.0001247	AT
28	0.0005238	0.0002746	0.0004584	6.834E-05	0.0002201	0.0023029	0.0006049	0.0005138	1.055E-05	7.889E-05	AT
29	0.0006329	0.0001885	0.0004344	4.544E-05	0.0002287	0.0007414	0.0007305	0.0004885	5.344E-06	5.257E-05	AT
30	0.0004561	0.000263	0.000399	7.644E-05	0.0003653	0.0044394	0.0005274	0.0004466	1.806E-05	8.524E-05	AT
31	0.0004863	0.0002582	0.0004179	7.016E-05	0.0002847	0.0038197	0.0005621	0.000468	1.648E-05	8.315E-05	AT
32	0.0020008	0.0028132	0.0026336	0.0025546	0.0063324	0.0301892	0.0023322	0.0027481	0.0026912	0.00362	AT
33	0.0007595	0.0005876	0.0005427	0.0005151	0.0017641	0.0168602	0.0008934	0.0005583	0.000446	0.0006539	AT
34	0.0076538	0.0094558	0.0084393	0.0053817	0.0202967	0.1067863	0.008358	0.0096159	0.0040922	0.0068068	AT
35	0.0033173	0.0050067	0.004489	0.0052327	0.0100608	0.0473186	0.0043273	0.0050134	0.0039266	0.0044608	AT
36	0.0006769	0.0005283	0.0003684	0.0003373	0.0030894	0.0103207	0.000814	0.0004517	0.0001764	0.0005317	AT
37	0.000643	0.0003631	0.0002408	0.0001378	0.0018188	0.0040284	0.0007411	0.0002709	0.0001175	0.0001299	AT
38	0.0014864	0.0019759	0.0014782	0.0015761	0.006732	0.0395713	0.0017547	0.001931	0.0008373	0.0015371	AT
39	0.0010575	0.0011022	0.0010303	0.0008757	0.0042883	0.0346993	0.0011135	0.0010538	0.0009574	0.0008267	AT
40	0	0	0	0	0	0	1.605E-15	7.471E-16	1.509E-15	1.783E-15	IN
41	0	0	0	0	0	0	3.48E-15	2.543E-15	5.241E-15	5.743E-15	IN
42	0	0	0	0	0	0	2.891E-15	3.136E-15	5.753E-15	4.702E-15	IN
43	0	0	0	0	0	0	2.853E-15	2.409E-15	5.936E-15	5.926E-15	IN
44	5.908E-05	4.627E-05	3.028E-05	3.064E-05	6.545E-05	0.0082226	0.0002613	0.0001844	2.208E-05	3.148E-05	IN
45	8.045E-05	6.662E-05	4.195E-05	1.519E-05	5.097E-05	0.0034112	0.0003257	0.0002012	1.213E-05	1.978E-05	IN
46	5.038E-05	7.356E-05	6.837E-05	0.0001408	0.0001649	0.0118	0.0001868	0.0001608	8.236E-05	0.0001066	IN
47	5.635E-05	4.675E-05	3.591E-05	0.0001013	0.0001041	0.0127194	0.0002412	0.0001893	3.589E-05	6.674E-05	IN
48	3.392E-05	0.0001093	7.896E-05	0.0002593	0.0004244	0.0005412	4.383E-05	7.325E-05	0.0001798	0.0002478	IN
49	0.0001869	0.0002751	0.0002595	0.0005495	0.0005728	0.0162429	0.0002963	0.0003289	0.0003332	0.000448	IN
50	0	0	0	0	0	0	3.053E-15	2.88E-15	7.193E-15	5.345E-15	IN
51	0	0	0	0	0	0	2.885E-15	3.077E-15	6.54E-15	5.371E-15	IN
52	0.0003238	0.0004218	0.000394	0.0006686	0.0021731	0.0649475	0.0006461	0.0005323	0.0003258	0.0004143	IN
53	0.0002178	0.0002425	0.0002177	0.0001788	0.0011184	0.0223991	0.0006612	0.0004608	0.0002111	0.0003628	IN
54	0.000965	0.0013247	0.001201	0.0021992	0.0039497	0.1092403	0.0010798	0.0012082	0.0010114	0.0016245	IN
55	0.0006515	0.0009031	0.0008473	0.000706	0.0026941	0.1282595	0.0008837	0.0009558	0.0005426	0.0004881	IN
56											OUT
57											OUT
58											OUT
59											OUT
60	0.0013762	0.0008759	0.000566	0.0001428	0.001	0.0018031	0.005321	0.0031398	2.259E-05	0.0002143	OUT
61	0.0015599	0.0010434	0.000729	0.0001101	0.0010121	0.0016908	0.0061052	0.0037954	2.221E-05	0.0001227	OUT
62	0.0013781	0.0008462	0.0004494	0.0001822	0.0007446	0.0023022	0.0053665	0.0033367	3.907E-05	0.0002726	OUT
63	0.0013737	0.0008896	0.0005933	0.0001652	0.0008452	0.0020132	0.005758	0.0037244	2.794E-05	0.0002296	OUT
64	0.0038134	0.0054144	0.0049227	0.001665	0.0140996	0.0381718	0.0114298	0.013096	0.0001488	0.0003909	OUT
65	0.0007745	0.0007679	0.0007994	0.0003609	0.00263	0.0196773	0.0035998	0.00316	7.005E-05	0.0002183	OUT
66	0.0010859	0.0427669	0.0367979	0.015186	0.0894125	0.1073742	0.0572855	0.069841	0.0012406	0.00022156	OUT
67	0.0101218	0.0147673	0.0129906	0.0055887	0.0320458	0.0617253	0.0234648	0.0281377	0.0002676	0.0004589	OUT
68	0.0004449	0.0003085	0.0003336	0.0002142	0.003395	0.0067739	0.0011393	0.0005042	5.441E-05	0.0001077	OUT
69	0.0004107	0.0002073	0.0001904	0.0001248	0.0016614	0.0086143	0.000923	0.0001541	2.186E-05	9.975E-05	OUT
70	0.001356	0.0017945	0.0017589	0.0012075	0.008213	0.0201694	0.0023456	0.0023264	0.0002595	0.0006131	OUT
71	0.0009158	0.0012105	0.0011224	0.0006423	0.0053239	0.0152141	0.0015884	0.0013611	0.0001349	0.0002598	OUT

Appendix D: Call Analytic Approximation results

BATCH	BBSR	GJ2	ML	JU 2	JU 3	lower B	upper B	LBA-LS1	LBA-LS2	LBA - Step	
1	2.649E-05	1.371E-06	0.0007592	1.371E-06	1.371E-06	1.371E-06	1.371E-06	5.363E-05	1.528E-06	1.37E-06	OUT
2	1.406E-05	5.464E-07	0.0016713	5.464E-07	5.464E-07	5.464E-07	5.464E-07	4.238E-05	1.173E-06	7.191E-07	m
3	1.346E-05	4.021E-07	0.0032522	4.021E-07	4.021E-07	4.021E-07	4.021E-07	3.205E-05	1.969E-06	1.268E-06	o
4	8.574E-07	2.359E-07	0.0032165	2.359E-07	2.359E-07	2.359E-07	2.359E-07	2.747E-05	2.865E-06	2.152E-06	n
5	7.055E-06	4.596E-07	0.0032455	4.596E-07	4.596E-07	4.596E-07	4.596E-07	2.299E-05	3.504E-06	2.804E-06	e
6	4.973E-06	1.678E-07	0.0028385	1.677E-07	1.677E-07	1.677E-07	1.676E-07	1.176E-05	3.957E-06	3.277E-06	y
7	3.843E-06	2.973E-07	0.0024942	2.968E-07	2.967E-07	2.968E-07	2.967E-07	2.895E-06	4.617E-06	3.96E-06	IN
8	1.611E-06	5.448E-07	0.0032231	5.384E-07	5.382E-07	5.388E-07	5.381E-07	5.127E-05	3.242E-06	3.549E-06	r-low
9	1.085E-06	1.683E-07	0.0032175	1.682E-07	1.682E-07	1.682E-07	1.682E-07	3.665E-05	3.089E-06	2.56E-06	r
10	6.373E-07	4.237E-07	0.0032163	4.237E-07	4.237E-07	4.237E-07	4.237E-07	1.847E-05	2.53E-06	1.645E-06	r
11	2.318E-07	5.531E-07	0.0032162	5.531E-07	5.53E-07	5.531E-07	5.531E-07	2.692E-06	1.671E-06	6.509E-07	r-high
12	5.918E-07	4.475E-07	0.0032507	4.475E-07	4.475E-07	4.475E-07	4.475E-07	2.783E-05	2.913E-06	2.536E-06	d-low
13	7.493E-07	3.374E-07	0.003229	3.374E-07	3.374E-07	3.374E-07	3.374E-07	2.679E-05	2.806E-06	2.253E-06	d
14	1.073E-06	1.978E-06	0.0031753	1.214E-07	1.229E-07	2.199E-07	1.414E-07	1.638E-05	1.717E-06	1.434E-06	d
15	1.625E-06	0.0002588	0.0029854	9.413E-06	2.315E-06	1.728E-05	2.007E-06	6.925E-05	2.741E-06	4.391E-07	d-high
16	4.763E-07	3.862E-07	0.0033926	3.862E-07	3.862E-07	3.862E-07	3.862E-07	2.129E-05	2.257E-06	1.712E-06	v-low
17	5.405E-07	5.457E-07	0.0032565	5.457E-07	5.457E-07	5.458E-07	5.458E-07	2.662E-05	3.031E-06	2.349E-06	v
18	1.103E-06	1.978E-07	0.0031895	1.46E-07	1.458E-07	1.463E-07	1.458E-07	2.717E-05	2.575E-06	1.863E-06	v
19	1.394E-06	2.189E-07	0.0031445	1.694E-07	1.706E-07	1.566E-07	1.671E-07	2.722E-05	2.628E-06	1.917E-06	v-high
20	1.899E-06	1.13E-06	0.003802	1.13E-06	1.13E-06	1.13E-06	1.13E-06	2.491E-05	1.006E-06	9.364E-07	T-low
21	1.201E-06	2.381E-07	0.0034447	2.381E-07	2.381E-07	2.381E-07	2.381E-07	2.768E-05	2.204E-06	1.638E-06	T
22	4.233E-07	5.025E-06	0.0023723	4.539E-07	4.843E-07	3.399E-07	3.755E-07	1.722E-05	2.334E-06	1.6E-06	T
23	1.349E-06	0.000268	0.001555	5.978E-06	1.165E-05	6.397E-06	1.208E-06	6.397E-06	4.482E-06	5.119E-06	T-high
24	1.554E-06	8.69E-07	0.0038781	8.69E-07	8.69E-07	8.69E-07	8.69E-07	6.296E-05	2.663E-06	3.118E-06	AT
25	2.498E-06	8.811E-05	0.0038055	1.65E-05	6.236E-06	2.256E-05	2.556E-06	0.00019	1.319E-05	1.382E-05	AT
26	5.869E-07	5.766E-07	0.0039734	5.766E-07	5.766E-07	5.766E-07	5.766E-07	1.016E-05	8.09E-07	6.681E-07	AT
27	1.204E-06	7.956E-07	0.0038607	7.956E-07	7.956E-07	7.956E-07	7.956E-07	2.164E-06	7.956E-07	7.956E-07	AT
28	2.431E-06	3.01E-07	0.0038404	3.01E-07	3.01E-07	3.01E-07	3.01E-07	5.991E-05	2.692E-06	3.205E-06	AT
29	3.036E-06	0.0003455	0.0037952	8.014E-05	1.628E-05	0.0001766	1.83E-05	0.0001844	0.0001448	0.000149	AT
30	1.938E-06	1.007E-06	0.0038426	1.007E-06	1.007E-06	1.007E-06	1.007E-06	6.76E-06	9.673E-07	9.967E-07	AT
31	2.24E-06	5.079E-06	0.0037667	5.442E-07	5.43E-07	6.546E-07	5.48E-07	3.594E-06	6.546E-07	6.546E-07	AT
32	1.072E-06	2.135E-07	0.0008734	2.135E-07	2.135E-07	2.134E-07	2.135E-07	1.753E-05	6.333E-07	6.859E-07	AT
33	4.731E-06	0.0040678	0.0009103	0.0007478	6.405E-05	0.0008882	0.0001977	0.0003973	0.0002679	0.0002671	AT
34	1.613E-06	1.099E-07	0.0026158	1.099E-07	1.099E-07	6.273E-06	1.099E-07	6.273E-06	6.273E-06	6.273E-06	AT
35	4.409E-06	0.0015342	0.0048216	2.757E-05	5.195E-05	3.54E-05	6.83E-06	8.008E-05	9.67E-05	9.695E-05	AT
36	2.622E-07	1.658E-05	0.0006621	6.762E-07	3.806E-07	6.227E-07	1.822E-07	1.647E-05	9.428E-07	9.46E-07	AT
37	2.039E-05	0.0135518	0.0014799	0.0016265	0.0001443	0.0024992	0.000634	0.0005021	0.0006531	0.0006552	AT
38	1.032E-06	4.762E-06	0.0014549	3.734E-07	1.65E-07	1.558E-07	5.194E-08	1.558E-07	1.558E-07	1.558E-07	AT
39	8.419E-06	0.0078497	0.0007869	0.0008325	0.000201	0.0005467	0.0001253	0.0003117	0.0003761	0.0003821	AT
40											OUT
41											OUT
42											OUT
43											OUT
44	4.734E-05	3.059E-06	0.0069738	3.059E-06	3.059E-06	3.059E-06	3.059E-06	8.545E-05	2.821E-06	2.686E-06	OUT
45	4.513E-05	0.0004693	0.0054676	5.245E-05	3.829E-05	0.0001763	9.794E-06	0.0002011	0.0001534	0.0001571	OUT
46	4.385E-05	2.735E-06	0.0095811	2.735E-06	2.735E-06	2.735E-06	2.735E-06	3.166E-05	2.735E-06	2.735E-06	OUT
47	4.292E-05	3.964E-06	0.0065064	2.749E-06	2.755E-06	2.745E-06	2.749E-06	1.26E-05	2.745E-06	2.745E-06	OUT
48	1.601E-05	3.729E-07	0.0026847	3.729E-07	3.729E-07	3.715E-07	3.729E-07	3.947E-05	3.715E-07	3.715E-07	OUT
49	2.884E-05	0.0044437	0.0079016	0.0008051	0.0001051	0.0009127	0.000137	0.0002731	0.0002772	0.0002768	OUT
50	8.758E-06	1.854E-07	0.0100489	1.854E-07	1.854E-07	1.97E-05	1.854E-07	1.972E-05	1.97E-05	1.97E-05	OUT
51	1.202E-05	0.000788	0.0151467	2.345E-05	2.447E-05	1.291E-05	3.18E-06	1.635E-05	2.414E-05	2.463E-05	OUT
52	1.548E-05	1.267E-05	0.0008301	5.73E-07	3.945E-07	4.345E-07	2.298E-07	3.692E-05	4.345E-07	4.345E-07	OUT
53	2.727E-05	0.0032755	0.0022997	0.0017226	0.0001594	0.0024716	0.0005123	0.0006201	0.0008048	0.0008029	OUT
54	8.629E-06	4.472E-06	0.0024118	3.401E-07	2.172E-07	1.773E-07	1.374E-07	1.062E-06	1.773E-07	1.773E-07	OUT
55	1.281E-05	0.0024951	0.0026059	0.0007292	0.0002208	0.000525	0.0001021	0.0004437	0.0004361	0.0004415	OUT
56	5.599E-07	9.824E-08	0.0038102	9.824E-08	9.824E-08	9.824E-08	9.824E-08	2.47E-05	3.011E-06	3.339E-06	IN
57	7.27E-07	0.0001393	0.0039069	2.178E-06	9.61E-07	0.0002954	3.077E-06	0.0003011	0.0002936	0.0002948	IN
58	2.855E-07	6.777E-08	0.0037734	6.777E-08	6.777E-08	6.777E-08	6.777E-08	6.777E-08	1.425E-06	1.085E-06	IN
59	4.619E-07	7.823E-08	0.0038039	7.888E-08	7.89E-08	7.884E-08	7.89E-08	7.884E-08	7.884E-08	1.373E-07	IN
60	4.532E-06	2.43E-07	0.0031455	2.43E-07	2.43E-07	2.43E-07	2.43E-07	3.521E-05	4.363E-06	4.82E-06	IN
61	5.377E-06	0.0002797	0.0030617	8.441E-05	2.513E-05	0.0002514	4.031E-05	0.0001467	0.0001989	0.0002043	IN
62	3.252E-06	2.455E-07	0.0032003	2.455E-07	2.455E-07	2.455E-07	2.455E-07	2.455E-07	1.748E-06	1.327E-06	IN
63	4.234E-06	3.069E-05	0.0031393	1.212E-06	4.914E-07	2.183E-06	3.505E-07	2.069E-06	1.112E-06	1.121E-06	IN
64	9.419E-07	1.145E-07	0.0005508	1.145E-07	1.145E-07	1.161E-07	1.145E-07	1.544E-06	1.757E-06	1.846E-06	IN
65	1.576E-05	0.0163157	0.000953	0.000581	5.32E-05	0.0009302	0.0002843	0.0008717	0.0008501	0.000858	IN
66	4.114E-07	7.182E-08	0.0018028	7.182E-08	7.182E-08	8.204E-06	7.182E-08	8.204E-06	8.168E-06	8.204E-06	IN
67	3.377E-06	0.0013096	0.0024946	6.09E-05	4.451E-05	5.217E-05	9.717E-06	9.785E-05	9.829E-05	9.73E-05	IN
68	7.719E-06	2.395E-05	0.0006752	9.242E-07	3.478E-07	9.815E-07	1.35E-07	3.21E-06	2.154E-06	2.215E-06	IN
69	2.525E-05	0.0208446	0.001003	0.0013147	0.0001496	0.0023937	0.0007029	0.0010605	0.0007201	0.0007242	IN
70	4.91E-06	9.998E-06	0.0008738	5.259E-07	4.196E-07	2.444E-07	8.25E-08	2.444E-07	4.975E-07	2.579E-07	IN
71	1.209E-05	0.0151198	0.0004525	0.0009101	0.0002991	0.0006006	0.0001463	0.0002508	0.0002919	0.0003083	IN

And,

BATCH	LBA-PC	LBA - R	LUBA-LS1	LUBA -LS2	LUBA-STEP	LUBA-WLS	LUBA-PC	LUBA-R	LUBA-half	
1	1.369E-06	1.361E-06	1.371E-06	1.371E-06	1.371E-06	1.371E-06	1.371E-06	1.371E-06	1.371E-06	OUT
2	7.237E-07	7.46E-07	5.464E-07	5.464E-07	5.464E-07	5.464E-07	5.464E-07	5.464E-07	5.464E-07	m
3	1.281E-06	1.198E-06	4.021E-07	4.021E-07	4.021E-07	4.021E-07	4.021E-07	4.021E-07	4.021E-07	o
4	2.164E-06	1.973E-06	2.359E-07	2.359E-07	2.359E-07	2.359E-07	2.359E-07	2.359E-07	2.359E-07	n
5	2.815E-06	2.506E-06	4.596E-07	4.596E-07	4.596E-07	4.596E-07	4.596E-07	4.596E-07	4.596E-07	e
6	3.285E-06	2.724E-06	1.676E-07	1.676E-07	1.676E-07	1.676E-07	1.676E-07	1.676E-07	1.677E-07	y
7	3.964E-06	3.166E-06	2.967E-07	2.967E-07	2.967E-07	2.967E-07	2.967E-07	2.967E-07	2.967E-07	IN
8	3.137E-06	2.867E-06	5.382E-07	5.382E-07	5.382E-07	5.382E-07	5.382E-07	5.382E-07	5.385E-07	r-low
9	2.489E-06	2.306E-06	1.682E-07	1.682E-07	1.682E-07	1.682E-07	1.682E-07	1.682E-07	1.682E-07	r
10	1.735E-06	1.54E-06	4.237E-07	4.237E-07	4.237E-07	4.237E-07	4.237E-07	4.237E-07	4.237E-07	r
11	7.813E-07	6.555E-07	5.531E-07	5.531E-07	5.531E-07	5.531E-07	5.531E-07	5.531E-07	5.531E-07	r-high
12	2.607E-06	2.718E-06	4.475E-07	4.475E-07	4.475E-07	4.475E-07	4.475E-07	4.475E-07	4.475E-07	d-low
13	2.29E-06	2.231E-06	3.374E-07	3.374E-07	3.374E-07	3.374E-07	3.374E-07	3.374E-07	3.374E-07	d
14	1.231E-06	1.024E-06	1.481E-07	1.48E-07	1.48E-07	1.479E-07	1.477E-07	1.477E-07	1.778E-07	d
15	8.101E-07	1.197E-06	3.777E-07	3.707E-07	3.632E-07	3.634E-07	3.364E-07	3.446E-07	7.652E-06	d-high
16	1.722E-06	1.576E-06	3.862E-07	3.862E-07	3.862E-07	3.862E-07	3.862E-07	3.862E-07	3.862E-07	v-low
17	2.361E-06	2.178E-06	5.458E-07	5.458E-07	5.458E-07	5.458E-07	5.458E-07	5.458E-07	5.458E-07	v
18	1.875E-06	1.684E-06	1.458E-07	1.458E-07	1.458E-07	1.458E-07	1.458E-07	1.458E-07	1.46E-07	v
19	1.929E-06	1.738E-06	1.661E-07	1.661E-07	1.661E-07	1.661E-07	1.66E-07	1.661E-07	1.617E-07	v-high
20	9.247E-07	9.211E-07	1.13E-06	1.13E-06	1.13E-06	1.13E-06	1.13E-06	1.13E-06	1.13E-06	T-low
21	1.656E-06	1.511E-06	2.381E-07	2.381E-07	2.381E-07	2.381E-07	2.381E-07	2.381E-07	2.381E-07	T
22	1.622E-06	1.394E-06	3.643E-07	3.653E-07	3.654E-07	3.651E-07	3.651E-07	3.652E-07	3.43E-07	T
23	4.863E-06	4.812E-06	1.006E-07	1.718E-07	1.721E-07	1.317E-07	1.644E-07	1.573E-07	2.596E-06	T-high
24	3.012E-06	3.236E-06	8.69E-07	8.69E-07	8.69E-07	8.69E-07	8.69E-07	8.69E-07	8.69E-07	AT
25	1.32E-05	1.352E-05	1.158E-06	1.181E-06	1.194E-06	1.173E-06	1.266E-06	1.24E-06	1.018E-05	AT
26	7.816E-07	8.815E-07	5.766E-07	5.766E-07	5.766E-07	5.766E-07	5.766E-07	5.766E-07	5.766E-07	AT
27	7.956E-07	7.956E-07	7.956E-07	7.956E-07	7.956E-07	7.956E-07	7.956E-07	7.956E-07	7.956E-07	AT
28	3.109E-06	3.336E-06	3.01E-07	3.01E-07	3.01E-07	3.01E-07	3.01E-07	3.01E-07	3.01E-07	AT
29	0.0001476	0.000147	1.742E-06	2.604E-06	2.609E-06	2.133E-06	2.985E-06	2.918E-06	7.918E-05	AT
30	9.846E-07	9.684E-07	1.007E-06	1.007E-06	1.007E-06	1.007E-06	1.007E-06	1.007E-06	1.007E-06	AT
31	6.546E-07	6.546E-07	5.526E-07	5.528E-07	5.527E-07	5.526E-07	5.523E-07	5.524E-07	5.858E-07	AT
32	6.483E-07	5.742E-07	2.135E-07	2.135E-07	2.135E-07	2.135E-07	2.135E-07	2.135E-07	2.135E-07	AT
33	0.0002678	0.0002432	3.92E-06	5.759E-06	5.556E-06	3.815E-06	6.111E-06	7.017E-06	0.0003458	AT
34	6.273E-06	6.273E-06	1.259E-06	1.045E-06	1.038E-06	1.093E-06	1.027E-06	1.05E-06	3.139E-06	AT
35	9.694E-05	9.63E-05	1.586E-06	1.37E-06	1.374E-06	1.427E-06	1.391E-06	1.396E-06	1.442E-05	AT
36	9.305E-07	8.767E-07	1.3E-07	1.337E-07	1.334E-07	1.324E-07	1.323E-07	1.322E-07	2.719E-07	AT
37	0.0006543	0.0006817	1.621E-05	3.372E-05	3.401E-05	2.356E-05	4.336E-05	2.445E-05	0.0009344	AT
38	1.558E-07	1.558E-07	5.507E-08	5.369E-08	5.369E-08	5.39E-08	5.367E-08	5.372E-08	8.647E-08	AT
39	0.0003808	0.0003427	5.101E-06	5.115E-06	5.106E-06	4.914E-06	5.732E-06	3.871E-06	0.0002112	AT
40										OUT
41										OUT
42										OUT
43										OUT
44	2.69E-06	2.592E-06	3.059E-06	3.059E-06	3.059E-06	3.059E-06	3.059E-06	3.059E-06	3.059E-06	OUT
45	0.0001559	0.0001552	3.586E-06	4.101E-06	4.064E-06	4.047E-06	6.182E-06	6.619E-06	8.371E-05	OUT
46	2.735E-06	2.735E-06	2.735E-06	2.735E-06	2.735E-06	2.735E-06	2.735E-06	2.735E-06	2.735E-06	OUT
47	2.745E-06	2.745E-06	2.748E-06	2.748E-06	2.748E-06	2.748E-06	2.748E-06	2.748E-06	2.745E-06	OUT
48	3.715E-07	3.692E-07	3.727E-07	3.728E-07	3.728E-07	3.728E-07	3.728E-07	3.728E-07	3.722E-07	OUT
49	0.0002767	0.0002996	8.903E-06	5.691E-06	5.619E-06	6.363E-06	5.151E-06	7.906E-06	0.0003881	OUT
50	1.97E-05	1.97E-05	3.866E-06	3.267E-06	3.235E-06	3.395E-06	3.163E-06	3.212E-06	9.843E-06	OUT
51	2.454E-05	2.288E-05	1.006E-06	1.135E-06	1.135E-06	1.064E-06	1.183E-06	1.146E-06	4.955E-06	OUT
52	4.345E-07	4.345E-07	1.792E-07	1.848E-07	1.843E-07	1.829E-07	1.825E-07	1.826E-07	1.925E-07	OUT
53	0.0008028	0.0008578	2.67E-05	3.015E-05	3.027E-05	2.851E-05	3.15E-05	2.898E-05	0.0009834	OUT
54	1.773E-07	1.773E-07	1.352E-07	1.352E-07	1.352E-07	1.352E-07	1.352E-07	1.352E-07	1.421E-07	OUT
55	0.0004401	0.0004117	4.595E-06	2.447E-06	2.509E-06	3.021E-06	2.02E-06	1.997E-06	0.0002118	OUT
56	3.272E-06	3.099E-06	9.824E-08	9.824E-08	9.824E-08	9.824E-08	9.824E-08	9.824E-08	9.824E-08	IN
57	0.0002944	0.000294	0.0001829	0.0001815	0.0001814	0.0001817	0.0001814	0.000182	0.0001468	IN
58	1.291E-06	1.091E-06	6.777E-08	6.777E-08	6.777E-08	6.777E-08	6.777E-08	6.777E-08	6.777E-08	IN
59	7.884E-08	7.884E-08	7.889E-08	7.889E-08	7.889E-08	7.889E-08	7.889E-08	7.889E-08	7.887E-08	IN
60	4.714E-06	4.444E-06	2.43E-07	2.43E-07	2.43E-07	2.43E-07	2.43E-07	2.43E-07	2.43E-07	IN
61	0.0002025	0.0002005	5.006E-06	2.826E-06	2.777E-06	2.774E-06	2.626E-06	3.046E-06	0.0001057	IN
62	1.616E-06	1.336E-06	2.455E-07	2.455E-07	2.455E-07	2.455E-07	2.455E-07	2.455E-07	2.455E-07	IN
63	1.133E-06	1.303E-06	2.561E-07	2.455E-07	2.449E-07	2.469E-07	2.418E-07	2.441E-07	9.854E-07	IN
64	1.842E-06	1.293E-06	1.145E-07	1.145E-07	1.145E-07	1.145E-07	1.145E-07	1.145E-07	1.145E-07	IN
65	0.000857	0.0007599	4.386E-05	6.242E-05	6.183E-05	5.405E-05	6.589E-05	5.586E-05	0.0003307	IN
66	8.204E-06	8.204E-06	1.638E-06	1.366E-06	1.354E-06	1.424E-06	1.341E-06	1.369E-06	4.115E-06	IN
67	9.754E-05	0.0001048	4.521E-06	3.782E-06	3.788E-06	4.029E-06	3.693E-06	3.828E-06	2.196E-05	IN
68	2.201E-06	1.719E-06	1.439E-07	1.279E-07	1.284E-07	1.321E-07	1.315E-07	1.329E-07	4.64E-07	IN
69	0.0007227	0.0007891	6.735E-05	0.0001931	0.0001925	0.000131	0.0002211	0.0001306	0.000849	IN
70	3.057E-07	2.448E-07	6.388E-08	6.422E-08	6.422E-08	6.406E-08	6.425E-08	6.418E-08	1.12E-07	IN
71	0.0003053	0.0001944	6.758E-06	1.43E-05	1.421E-05	9.346E-06	1.551E-05	5.018E-06	0.0002272	IN

Appendix E: Put Analytical Approximation Results

Batch	BBSR	GJ2	SUL2	SUL4	ML	JU 2	JU 3	lower B	upper B	LBA-LS1	
1	0.000367	0.0199143	0.0121493	0.0090104	0.0027481	0.0040628	0.003991	0.0010371	0.0075441	0.0034079	IN
2	0.0002186	0.012729	0.0025294	0.0006906	0.0017881	0.0006034	0.0001694	0.0038401	0.0072189	0.0021258	m
3	0.0001237	0.0167343	0.0003034	0.0004949	0.0011494	0.0017493	0.0003247	0.0066533	0.0073631	0.0011708	o
4	3.634E-05	0.0128769	0.0004081	0.0001461	4.234E-05	0.0022905	0.000374	0.0076781	0.007485	0.0020646	n
5	0.000111	0.007734	0.0007316	0.000117	0.0008728	0.0027726	0.0004113	0.0085285	0.0079728	0.0029472	e
6	0.0001329	0.011747	0.0012803	0.0002079	0.0048987	0.0034924	0.0005097	0.0100275	0.009251	0.0044346	y
7	0.0001451	0.026858	0.0017451	9.865E-05	0.011254	0.0036933	0.0006778	0.0110063	0.0104461	0.0054149	OUT
8	7.502E-05	0.0058517	0.0002604	2.502E-05	0.0009344	0.0021038	0.0003111	0.0078284	0.0054651	0.0029415	r-low
9	8.754E-05	0.0111531	0.0003855	0.000108	0.0003643	0.0022638	0.0003599	0.0078253	0.0069299	0.0022136	r
10	0.0001104	0.0140455	0.0004189	0.0002432	0.0005335	0.0022883	0.0003829	0.007406	0.0080988	0.0017977	r
11	0.0002105	0.0141918	0.0004111	0.0004181	0.0013406	0.0022373	0.0003911	0.0067555	0.0092334	0.0011538	r-high
12	0.000133	0.0133294	0.0004446	0.0002265	0.0003907	0.0022875	0.0003832	0.007561	0.0082169	0.0019468	d-low
13	0.0001408	0.0131136	0.000426	0.0001759	0.0002019	0.0022907	0.0003783	0.0076372	0.0077901	0.0020236	d
14	8.927E-05	0.0121491	0.0003643	0.0001179	0.0002425	0.0022789	0.0003637	0.007697	0.0069181	0.0020836	d
15	6.634E-05	0.0092464	0.0001814	3.254E-05	0.0006903	0.0021892	0.0003252	0.0074491	0.0053118	0.0019995	d-high
16	0.0006441	0.0094334	0.0004366	0.0006583	0.0026649	0.0020592	0.0003732	0.005823	0.0117229	0.0007528	v-low
17	9.436E-05	0.0135175	0.0004935	0.0002506	0.0007382	0.0022772	0.0003871	0.0073608	0.0085602	0.0017568	v
18	0.0001111	0.0114841	0.0003422	0.0001227	0.0003382	0.0022664	0.0003579	0.0077334	0.0067508	0.0012503	v
19	7.011E-05	0.0091056	0.0002137	0.0001027	0.0006636	0.0021981	0.000331	0.0076153	0.0056991	0.0021364	v-high
20	7.683E-05	0.0027181	0.0002965	3.308E-05	0.0006845	0.0018431	0.0002722	0.007744	0.0048052	0.0040979	T-low
21	9.664E-05	0.0093063	0.00039	7.253E-05	0.0003827	0.0022254	0.0003495	0.0080179	0.0067299	0.0024394	T
22	0.0004923	0.0137409	0.0003615	0.0007055	0.0012838	0.0021746	0.000374	0.0061904	0.0091171	0.0010953	T
23	0.0005586	0.0343287	0.0026522	0.0017884	0.0026538	0.001596	0.0002706	0.002966	0.012329	0.002769	T-high
24	9.073E-05	0.0044059			0.0005958	0.0019945	0.0003088	0.0082812	0.0061342	0.0037418	AT
25	6.72E-05	0.0035566			0.0005037	0.0012838	0.0002026	0.0058869	0.0027876	0.0040239	AT
26	0.0003207	0.0094452			0.0018849	0.0021794	0.0003774	0.0074278	0.0101774	0.0016448	AT
27	0.0001193	0.0086161			0.000737	0.0021475	0.0003482	0.0080113	0.0074587	0.0030848	AT
28	6.834E-05	0.0039529			0.0004292	0.0011951	0.0001945	0.0057934	0.0025864	0.0040257	AT
29	4.544E-05	0.0034242			0.0013533	0.0007067	0.0001429	0.0034741	0.0011511	0.002574	AT
30	7.644E-05	0.0031357			0.0006185	0.0017939	0.0002651	0.0075709	0.0046038	0.0041645	AT
31	7.016E-05	0.0025993			0.00052	0.0016034	0.0002321	0.0066913	0.0034905	0.0041629	AT
32	0.0025546	0.1230997			0.0034768	0.0013844	0.0002278	0.0026702	0.0154475	0.003382	AT
33	0.0005151	0.0230938			0.0010881	0.0021105	0.0003545	0.0049805	0.0080704	0.0011096	AT
34	0.0053817	0.5440405			0.004288	0.0003645	0.0001221	0.0002948	0.0212836	0.0054582	AT
35	0.0052327	0.2917701			0.0046519	0.0008055	0.0001306	0.0009419	0.0139288	0.0048761	AT
36	0.0003373	0.0222431			0.0005448	0.0021656	0.0003636	0.0053647	0.0072541	0.0006258	AT
37	0.0001378	0.0304575			0.0012936	0.002209	0.0003393	0.0048797	0.003908	0.0008713	AT
38	0.0015761	0.0571688			0.0032939	0.0014551	0.0002429	0.0024525	0.0124921	0.0033398	AT
39	0.0008757	0.0152726			0.0023656	0.0017424	0.0002981	0.0030939	0.008546	0.002662	AT
40	1.294E-10	0.0055032			0.0037502	0.6345157	0.6193184	0	0.0054566	0	IN
41	1.14E-10	0.0019419			0.0037654	0.1424316	0.1400519	0	0.0027644	0	IN
42	1.183E-10	0.0153431			0.003751	2.6240414	2.5534573	0	0.0105362	0	IN
43	1.075E-10	0.0100801			0.0037594	0.9680828	0.9504558	0	0.0076687	0	IN
44	3.064E-05	0.0071263			0.0028043	0.0008308	0.0007859	0.0022827	0.0024026	0.0009108	IN
45	1.519E-05	0.0057859			0.0025645	0.0003046	7.975E-05	0.0022205	0.001375	0.0010949	IN
46	0.0001408	0.0067437			0.0032719	0.0067662	0.0066374	0.001548	0.0032062	0.0023192	IN
47	0.0001013	0.0082305			0.0029874	0.0016239	0.0015847	0.0018568	0.0028833	0.0015444	IN
48	0.0002593	0.2807686			0.0009443	0.6364106	0.632674	4.739E-05	0.0649737	0.0007524	IN
49	0.0005495	0.0923213			0.0008262	0.1027524	0.1017994	0.0007231	0.025254	0.0030949	IN
50	1.177E-10	0.6385661			0.0009196	4.1536407	4.1496132	0	0.1361125	0	IN
51	1.274E-10	0.4658618			0.0009157	1.9272767	1.9218211	0	0.0990955	0	IN
52	0.0006686	0.0153149			0.000848	0.0008563	0.0002039	0.0031679	0.0078597	0.0023119	IN
53	0.0001788	0.0311651			0.0003192	0.0011076	0.0002379	0.0034532	0.0044819	0.0015351	IN
54	0.0021992	0.1281507			0.0022158	0.0002346	6.298E-05	0.0009111	0.0082863	0.0048368	IN
55	0.000706	0.0563655			0.0021342	0.0004453	0.0001137	0.0014293	0.0063996	0.0043272	IN
56											OUT
57											OUT
58											OUT
59											OUT
60	0.0001412	0.0140762			0.0080359	0.0007399	0.0005199	0.0064047	0.0038398	0.0049466	OUT
61	0.0001082	0.0079254			0.0067438	0.0004158	0.0003455	0.0038453	0.0018302	0.0030064	OUT
62	0.0001822	0.0238209			0.0107946	0.0016363	0.0007465	0.0092708	0.0068781	0.0066994	OUT
63	0.0001629	0.0181122			0.0089576	0.0013731	0.0005987	0.0076927	0.0049155	0.0057629	OUT
64	0.001651	0.068203			0.0073147	0.0056184	0.0007039	0.0061047	0.0174366	0.0015123	OUT
65	0.0003609	0.0222091			0.0114881	0.0047029	0.0006706	0.0085245	0.0105894	0.0030072	OUT
66	0.0055674	0.2457279			0.0116376	0.0034748	0.0004398	0.0015821	0.0137537	0.0042438	OUT
67	0.0036268	0.116284			0.0058932	0.0044794	0.0005435	0.0034782	0.0130137	0.0026677	OUT
68	0.0002142	0.0142643			0.0017589	0.0032536	0.0004477	0.0070142	0.0076244	0.0014859	OUT
69	0.0001248	0.0174596			0.0025936	0.0029886	0.0003858	0.0060287	0.0041967	0.0006555	OUT
70	0.0012075	0.0213055			0.003427	0.0030663	0.0004227	0.0042775	0.0104092	0.0017462	OUT
71	0.0006423	0.0152414			0.002312	0.0030641	0.0004289	0.0043504	0.0074407	0.0015348	OUT

And,

Batch	LBA-LS2	LBA - Step	LBA-PC	LBA - R	PLS1	PLS2	PLS3	LUBA-LS1	LUBA -LS2	LUBA-STEP	
1	0.0034079	0.0034079	0.0034079	0.0034079	0.0033409	0.001473	0.0005499	0.0015938	0.0021092	0.0020947	IN
2	0.0021258	0.0021258	0.0021258	0.0021258	0.0017949	0.0008601	0.0023346	0.0038294	0.004684	0.0046515	m
3	0.0011708	0.0011708	0.0011708	0.0011708	0.0020768	0.0029078	0.0041662	0.003764	0.0043316	0.0043032	o
4	0.0020646	0.0020646	0.0020646	0.0020646	0.003326	0.0038438	0.0049234	0.0039431	0.0044914	0.0044641	n
5	0.0029472	0.0029472	0.0029472	0.0029472	0.0044341	0.0047479	0.0056569	0.0044786	0.0050571	0.0050296	e
6	0.0044346	0.0044346	0.0044346	0.0044346	0.0065778	0.0064776	0.0071198	0.005907	0.0065783	0.0065486	y
7	0.0054149	0.0054149	0.0054149	0.0054253	0.0081617	0.0077923	0.0082601	0.0073018	0.0080554	0.0080222	OUT
8	0.0033938	0.0033901	0.0033894	0.0035962	0.0030363	0.0032636	0.0039617	0.0029486	0.0031293	0.0031116	r-low
9	0.0022136	0.0022136	0.0022136	0.0022228	0.0033029	0.0037803	0.0047475	0.0036839	0.0041111	0.0040859	r
10	0.0017977	0.0017977	0.0017977	0.0017977	0.0030948	0.0037977	0.0050061	0.0041997	0.0049185	0.0048902	r
11	0.0011538	0.0011538	0.0011538	0.0011538	0.0026334	0.0035656	0.0050201	0.0046131	0.0057012	0.005674	r-high
12	0.0019468	0.0019468	0.0019468	0.0019468	0.0033259	0.0040536	0.0052638	0.004365	0.0050092	0.0049842	d-low
13	0.0020236	0.0020236	0.0020236	0.0020236	0.0032944	0.0039379	0.0050718	0.0041203	0.0047073	0.0046809	d
14	0.0020836	0.0020836	0.0020836	0.0020836	0.0031406	0.0036149	0.0045959	0.0036087	0.0040942	0.0040662	d
15	0.0022356	0.0022246	0.0022266	0.0024243	0.0025286	0.0027025	0.0034134	0.0026466	0.0029842	0.0029573	d-high
16	0.0007528	0.0007528	0.0007528	0.0007528	0.0015306	0.0021723	0.0033079	0.0057757	0.0074422	0.0074674	v-low
17	0.0017568	0.0017568	0.0017568	0.0017568	0.0029407	0.0035456	0.0046412	0.0045533	0.0053052	0.005283	v
18	0.0021203	0.0021203	0.0021203	0.0021203	0.0033247	0.0038988	0.0049706	0.0034827	0.0039312	0.0039043	v
19	0.002445	0.0024392	0.0024394	0.0026486	0.0032277	0.0037967	0.0048631	0.0027945	0.0031285	0.0031044	v-high
20	0.0046127	0.0046189	0.0046157	0.0047675	0.0035361	0.0042602	0.0054722	0.0027802	0.0030476	0.0030024	T-low
21	0.0025159	0.0025133	0.0025136	0.0026003	0.0035338	0.004136	0.0052396	0.0036676	0.0041109	0.0040746	T
22	0.0010953	0.0010953	0.0010953	0.0010953	0.0025475	0.0031655	0.0042469	0.0042891	0.0054345	0.0054478	T
23	0.002769	0.002769	0.002769	0.002769	0.0023249	0.0026906	0.002954	0.0037224	0.0076913	0.0078438	T-high
24	0.0041385	0.0041397	0.0041376	0.004286	0.0042395	0.004938	0.0060226	0.0039409	0.0041769	0.0041356	AT
25	0.0044104	0.0044202	0.0044162	0.0044876	0.00414875	0.0017579	0.0023651	0.001665	0.0017438	0.0017355	AT
26	0.002084	0.0020829	0.002083	0.0021161	0.0036685	0.0047978	0.0064575	0.0057088	0.0070772	0.0070294	AT
27	0.0032484	0.0032463	0.0032462	0.0033456	0.003806	0.0045185	0.0057893	0.0042345	0.005061	0.0049961	AT
28	0.0046589	0.0046837	0.0046753	0.0047168	0.0017423	0.002352	0.0033885	0.001472	0.0014975	0.0014817	AT
29	0.0029889	0.0030072	0.0030011	0.0030153	0.001558	0.0012593	0.0008462	0.0005831	0.0005967	0.0005898	AT
30	0.0047247	0.0047374	0.0047328	0.0048595	0.0036663	0.0047113	0.0063288	0.0024951	0.0028907	0.0028333	AT
31	0.0046171	0.00463	0.0046257	0.0047188	0.0023965	0.003002	0.0041948	0.0017348	0.0020329	0.001992	AT
32	0.003382	0.003382	0.003382	0.003382	0.0013384	0.0016241	0.0021176	0.0032523	0.0068631	0.0071498	AT
33	0.0011096	0.0011096	0.0011096	0.0011096	0.0023736	0.0018048	0.0018467	0.0031438	0.0043653	0.0044187	AT
34	0.0054582	0.0054582	0.0054582	0.0054582	0.0002573	0.0002948	0.0002948	0.0002948	0.0004489	0.0005807	AT
35	0.0048761	0.0048761	0.0048761	0.0048761	0.0005854	0.0006843	0.0008953	0.0009045	0.0031058	0.0034601	AT
36	0.0006258	0.0006258	0.0006258	0.0006258	0.0045237	0.0047325	0.0051071	0.0033182	0.0049294	0.0049567	AT
37	0.0008745	0.0008743	0.0008743	0.0008758	0.0034888	0.0032988	0.0035532	0.0015465	0.0023119	0.0023207	AT
38	0.0033398	0.0033398	0.0033398	0.0033398	0.0020258	0.0024414	0.0024525	0.0034028	0.0099356	0.0101088	AT
39	0.002662	0.002662	0.002662	0.002662	0.0024036	0.0028706	0.0030905	0.0032613	0.0071784	0.0072455	AT
40	0	0	0	0	1.294E-10	1.294E-10	1.294E-10	0	0	0	IN
41	0	0	0	0	1.14E-10	1.14E-10	1.14E-10	0	0	0	IN
42	0	0	0	0	1.183E-10	1.183E-10	1.183E-10	0	0	0	IN
43	0	0	0	0	1.075E-10	1.075E-10	1.075E-10	0	0	0	IN
44	0.0010768	0.0010889	0.0010848	0.0011034	0.0033135	0.0028188	0.0012244	0.0012806	0.0014037	0.0013848	IN
45	0.0014951	0.0015134	0.0015076	0.0015113	0.003406	0.003406	0.0030245	0.0006071	0.0006975	0.0006855	IN
46	0.0023321	0.0023371	0.0023368	0.0021908	0.0028558	0.0012161	0.0010699	0.0013898	0.0016516	0.0016252	IN
47	0.0016104	0.0016222	0.0016209	0.0014247	0.0033741	0.0024428	0.0008023	0.0014018	0.0016637	0.0016355	IN
48	0.0007524	0.0007524	0.0007524	0.0007524	0.0004789	0.0003734	0.0002175	0.0015244	0.0021343	0.0021343	IN
49	0.0030949	0.0030949	0.0030949	0.0030949	0.0024953	0.002004	0.0014316	0.0026728	0.003508	0.0035069	IN
50	0	0	0	0	1.177E-10	1.177E-10	1.177E-10	0	0	0	IN
51	0	0	0	0	1.274E-10	1.274E-10	1.274E-10	0	0	0	IN
52	0.0025961	0.0025961	0.0025961	0.0025961	0.0017006	0.0016828	0.0018823	0.0054341	0.0078587	0.0078586	IN
53	0.002239	0.0022397	0.0022396	0.0021869	0.0016575	0.0011055	0.0010079	0.002818	0.0043485	0.0043468	IN
54	0.0048368	0.0048368	0.0048368	0.0048368	0.0047583	0.0047375	0.0045843	0.0074487	0.0082863	0.0082863	IN
55	0.0043286	0.0043286	0.0043286	0.0043286	0.0041786	0.0041209	0.0039759	0.0049497	0.0063996	0.0063996	IN
56											OUT
57											OUT
58											OUT
59											OUT
60	0.0055736	0.0055994	0.0055908	0.005624	0.0044089	0.0044406	0.0050867	0.0031408	0.0031556	0.0031435	OUT
61	0.0034375	0.0034563	0.0034502	0.0034636	0.0014559	0.0012634	0.0015256	0.001437	0.0014441	0.0014396	OUT
62	0.0072628	0.0072798	0.007274	0.0073765	0.0073004	0.0076536	0.0087704	0.0053284	0.0058341	0.0057697	OUT
63	0.0062075	0.0062232	0.0062181	0.0062927	0.0053611	0.0053585	0.006154	0.0036115	0.0039697	0.0039267	OUT
64	0.0015123	0.0015123	0.0015123	0.0015123	0.0054514	0.0043311	0.0041416	0.0061523	0.0103527	0.0106259	OUT
65	0.0030072	0.0030072	0.0030072	0.0030072	0.0070693	0.0052194	0.0044516	0.0057476	0.0071307	0.007181	OUT
66	0.0042438	0.0042438	0.0042438	0.0042438	0.0011806	0.0013721	0.001576	0.0015573	0.0021119	0.0023573	OUT
67	0.0026677	0.0026677	0.0026677	0.0026677	0.0025315	0.0019726	0.0022501	0.0022434	0.0056229	0.0059134	OUT
68	0.0014859	0.0014859	0.0014859	0.0014859	0.0068309	0.0063979	0.0064304	0.0035188	0.0042292	0.0042578	OUT
69	0.0007267	0.0007207	0.0007217	0.0008098	0.0054933	0.0044816	0.0040655	0.0017058	0.0020137	0.0020232	OUT
70	0.0017462	0.0017462	0.0017462	0.0017462	0.0042225	0.0042156	0.0042775	0.0030646	0.0058481	0.0059654	OUT
71	0.0015348	0.0015348	0.0015348	0.0015348	0.0042779	0.0041941	0.0042845	0.002346	0.0042279	0.0042921	OUT

And finally,

Batch	LUBA-WLS	LUBA-PC	LUBA-R	LUBA-half	PLS1	PLS2	PLS3	
1	0.0019179	0.002149	0.001926	0.0036702	0.0009325	0.0010233	0.0010327	IN
2	0.0043836	0.0047663	0.0044803	0.0017506	0.003509	0.0038329	0.0038398	m
3	0.0041393	0.0043364	0.0042252	0.00048	0.0062495	0.0066533	0.0066533	o
4	0.0042958	0.004452	0.0043517	9.656E-05	0.0072257	0.0076781	0.0076781	n
5	0.0048351	0.0049549	0.0048351	0.0002859	0.0080018	0.0085285	0.0085285	e
6	0.0062874	0.006302	0.0061159	0.0003888	0.0092972	0.0100275	0.0100275	y
7	0.0077034	0.0075848	0.0073278	0.0002886	0.0100375	0.0110063	0.0110063	OUT
8	0.0030631	0.0030377	0.0030142	0.0012021	0.0072926	0.0078279	0.0078284	r-low
9	0.0039564	0.0040467	0.0039709	0.000508	0.0073411	0.0078253	0.0078253	r
10	0.004668	0.00493	0.0047962	0.0004356	0.0069945	0.007406	0.007406	r
11	0.0053374	0.0058466	0.0056446	0.0012678	0.0064279	0.0067555	0.0067555	r-high
12	0.0047986	0.0050127	0.0049076	0.0003294	0.0070644	0.007561	0.007561	d-low
13	0.0045056	0.0046847	0.0045824	0.0001288	0.0071668	0.0076372	0.0076372	d
14	0.0039097	0.0040328	0.0039362	0.0004336	0.0072768	0.007697	0.007697	d
15	0.0028311	0.0028899	0.0028049	0.0010827	0.0071105	0.0074491	0.0074491	d-high
16	0.0070824	0.0082603	0.008192	0.0031047	0.0052801	0.005823	0.005823	v-low
17	0.0050805	0.0053945	0.0053079	0.0007244	0.0069009	0.0073608	0.0073608	v
18	0.0037512	0.0038482	0.003738	0.0005331	0.007255	0.0077334	0.0077334	v
19	0.0029682	0.0030114	0.0028858	0.0009669	0.0070473	0.0076153	0.0076153	v-high
20	0.0029149	0.002808	0.0027212	0.0015083	0.0072837	0.0077433	0.007744	T-low
21	0.0039337	0.0039763	0.00388	0.0007086	0.0075518	0.0080179	0.0080179	T
22	0.005102	0.005896	0.0056794	0.0017497	0.0057821	0.0061904	0.0061904	T
23	0.0065859	0.010122	0.0092048	0.0047312	0.0025756	0.002966	0.002966	T-high
24	0.0041291	0.004002	0.0040141	0.0012389	0.007519	0.0082156	0.0082812	AT
25	0.0017083	0.0016086	0.0016047	0.0015881	0.0055484	0.005821	0.0058684	AT
26	0.0066638	0.007261	0.0070776	0.0019324	0.0067593	0.0074259	0.0074278	AT
27	0.004743	0.0049303	0.0047667	0.001002	0.0075405	0.0079919	0.0080113	AT
28	0.0014833	0.0013702	0.0013461	0.0016127	0.0053637	0.0057927	0.0057934	AT
29	0.0005868	0.0005493	0.0005351	0.0011655	0.003284	0.0034649	0.0034741	AT
30	0.0026653	0.0025725	0.0023948	0.0015393	0.0071768	0.0075709	0.0075709	AT
31	0.0018564	0.001818	0.0016767	0.001626	0.0064129	0.0066913	0.0066913	AT
32	0.0060951	0.0111684	0.0109639	0.0065854	0.0021208	0.0026702	0.0026702	AT
33	0.0040959	0.0052548	0.0051843	0.002027	0.00465	0.0049485	0.0049728	AT
34	0.0002466	0.0063407	0.0053581	0.0106072	0.0002381	0.0002948	0.0002948	AT
35	0.0021655	0.0099608	0.0095108	0.0065461	0.0008	0.0009419	0.0009419	AT
36	0.0043123	0.0055549	0.0048301	0.0010837	0.0043519	0.005242	0.005356	AT
37	0.001995	0.0025824	0.0021891	0.0006077	0.0040871	0.0046939	0.0048189	AT
38	0.0077265	0.0121853	0.0111175	0.0051093	0.0019931	0.0024525	0.0024525	AT
39	0.005766	0.0082057	0.0073384	0.0028158	0.0026738	0.0030926	0.0030939	AT
40	0	0	0	0.0027283	1.294E-10	1.294E-10	1.294E-10	IN
41	0	0	0	0.0013822	1.14E-10	1.14E-10	1.14E-10	IN
42	0	0	0	0.0052681	1.183E-10	1.183E-10	1.183E-10	IN
43	0	0	0	0.0038343	1.075E-10	1.075E-10	1.075E-10	IN
44	0.0013681	0.0013077	0.0012709	0.0004072	0.0020349	0.0022322	0.0022608	IN
45	0.0006684	0.0006456	0.0006211	0.0005058	0.00208	0.0021829	0.0022002	IN
46	0.0015405	0.0015447	0.0014343	0.0011132	0.0013946	0.0015339	0.0015479	IN
47	0.0015498	0.0015494	0.0014408	0.0007337	0.0017186	0.0018273	0.0018489	IN
48	0.0021343	0.0021343	0.0021343	0.0324877	0.0001625	8.267E-05	7.128E-05	IN
49	0.0034427	0.0035493	0.0034783	0.0126247	0.0007138	0.0007182	0.0007162	IN
50	0	0	0	0.0680562	1.177E-10	1.177E-10	1.177E-10	IN
51	0	0	0	0.0495477	1.274E-10	1.274E-10	1.274E-10	IN
52	0.0077741	0.0078597	0.0078217	0.0024333	0.0025626	0.0030335	0.0030977	IN
53	0.004102	0.0044106	0.0041905	0.0007479	0.0030861	0.0033783	0.003419	IN
54	0.0082863	0.0082863	0.0082863	0.0040807	0.001077	0.0010139	0.0009621	IN
55	0.0063996	0.0063996	0.0063996	0.002534	0.0011619	0.0012934	0.0013404	IN
56								OUT
57								OUT
58								OUT
59								OUT
60	0.0031215	0.0028422	0.0027742	0.0013032	0.0056985	0.0063954	0.0064047	OUT
61	0.0014239	0.0013127	0.0012773	0.0010213	0.0034731	0.0038086	0.0038452	OUT
62	0.0054959	0.005112	0.0047944	0.0012208	0.0085533	0.0092708	0.0092708	OUT
63	0.0037239	0.003500	0.0032701	0.0014094	0.0071615	0.0076927	0.0076927	OUT
64	0.0094443	0.013487	0.0132379	0.0058125	0.0049897	0.0061042	0.0061047	OUT
65	0.0067151	0.0076229	0.0074645	0.0017137	0.0076887	0.0084547	0.0085222	OUT
66	0.0016041	0.010305	0.0098064	0.0061666	0.0011827	0.0015821	0.0015821	OUT
67	0.0045795	0.0098115	0.0094268	0.004861	0.0030788	0.0034782	0.0034782	OUT
68	0.0040191	0.0045514	0.0043452	0.0005139	0.0051928	0.0066285	0.0069276	OUT
69	0.0019065	0.002139	0.0020272	0.0009827	0.0046511	0.0056212	0.0058563	OUT
70	0.0051074	0.0075333	0.0070143	0.0031864	0.0032631	0.0042471	0.0042775	OUT
71	0.0036792	0.0052274	0.0047806	0.0016258	0.0034322	0.0042289	0.00435	OUT

Appendix F: Call Sensitivity for Analytic Approximations

	1	2	3	4	5	6	7	8	9	10
Money	BBSR	ML	LUBA-LS1	LUBA-WLS	LUBA-R	LBA - R	JU 3	LBA-LS2	LBA-PC	LBA - Step
AT	3.588E-06	0.002773	2.217E-06	2.664E-06	2.903E-06	9.639E-05	3.055E-05	9.803E-05	9.862E-05	9.886E-05
OUT	2.576E-05	0.0060382	4.824E-06	4.636E-06	4.924E-06	0.000148	4.649E-05	0.0001437	0.0001441	0.0001443
IN	5.619E-06	0.0022904	1.959E-05	2.41E-05	2.394E-05	0.0001478	3.592E-05	0.0001548	0.0001565	0.000157
AT/OUT	0.139	0.459	0.460	0.575	0.589	0.651	0.657	0.682	0.685	0.685
	11	12	13	14	15	16	17	18	19	
	LUBA-half	LUBA -LS2	LUBA-STEP	JU 2	lower B	LBA-LS1	LUBA-PC	upper B	GJ2	
	0.0001002	3.456E-06	3.462E-06	0.0002085	0.0002613	0.0001157	4.148E-06	6.183E-05	0.0017167	
	0.0001409	4.669E-06	4.672E-06	0.0002786	0.000344	0.0001484	4.869E-06	6.449E-05	0.0009582	
	0.0001055	2.878E-05	2.87E-05	0.0001848	0.0002835	0.0001753	3.081E-05	7.425E-05	0.0033796	
	0.711	0.740	0.741	0.749	0.760	0.780	0.852	0.959	1.792	
	1	2	3	4	5	6	7	8	9	10
Money	LUBA-WLS	LUBA-LS1	LUBA -LS2	LUBA-STEP	LUBA-R	LUBA-PC	GJ2	LBA - Step	LBA-PC	LBA-LS2
AT	2.664E-06	2.217E-06	3.456E-06	3.462E-06	2.903E-06	4.148E-06	0.0017167	9.886E-05	9.862E-05	9.803E-05
OUT	4.636E-06	4.824E-06	4.669E-06	4.672E-06	4.924E-06	4.869E-06	0.0009582	0.0001443	0.0001441	0.0001437
IN	2.41E-05	1.959E-05	2.878E-05	2.87E-05	2.394E-05	3.081E-05	0.0033796	0.000157	0.0001565	0.0001548
AT/IN	0.111	0.113	0.120	0.121	0.121	0.135	0.508	0.630	0.630	0.633
	11	12	13	14	15	16	17	18	19	
	BBSR	LBA - R	LBA-LS1	upper B	JU 3	lower B	LUBA-half	JU 2	ML	
	3.588E-06	9.639E-05	0.0001157	6.183E-05	3.055E-05	0.0002613	0.0001002	0.0002085	0.002773	
	2.576E-05	0.000148	0.0001484	6.449E-05	4.649E-05	0.000344	0.0001409	0.0002786	0.0060382	
	5.619E-06	0.0001478	0.0001753	7.425E-05	3.592E-05	0.0002835	0.0001055	0.0001848	0.0022904	
	0.639	0.652	0.660	0.833	0.851	0.922	0.950	1.128	1.211	
	1	2	3	4	5	6	7	8	9	10
Interest	LUBA-R	LUBA-PC	LUBA -LS2	LUBA-STEP	LUBA-WLS	LUBA-LS1	upper B	lower B	LUBA-half	LBA-LS1
high	1.463E-06	2.017E-06	1.96E-06	1.957E-06	1.774E-06	1.802E-06	1.833E-05	8.267E-05	3.267E-05	5.934E-05
low	2.074E-05	2.606E-05	2.403E-05	2.398E-05	2.022E-05	1.669E-05	0.0001158	0.0005011	0.0001938	0.0002333
Ratio	0.071	0.077	0.082	0.082	0.088	0.108	0.158	0.165	0.169	0.254
	11	12	13	14	15	16	17	18	19	
	LBA - R	LBA-LS2	LBA-PC	LBA - Step	JU 2	GJ2	BBSR	JU 3	ML	
	5.546E-05	6.234E-05	6.331E-05	6.354E-05	0.000118	0.0013255	8.157E-06	3.874E-05	0.0041347	
	0.0002029	0.0002	0.0002008	0.0002012	0.00032	0.0029036	1.259E-05	3.495E-05	0.0028413	
	0.273	0.312	0.315	0.316	0.369	0.457	0.648	1.108	1.455	
	1	2	3	4	5	6	7	8	9	10
Dividend	ML	BBSR	LBA-LS1	LUBA-LS1	LUBA-WLS	LUBA-R	LUBA -LS2	LUBA-STEP	LUBA-PC	LBA-PC
high	0.0037259	1.287E-05	0.0002713	1.77E-05	2.124E-05	2.147E-05	2.525E-05	2.52E-05	2.734E-05	0.0002612
low	0.0032501	7.871E-06	2.127E-05	7.896E-07	7.507E-07	7.379E-07	7.4E-07	7.377E-07	7.333E-07	2.975E-06
Ratio	1.146	1.635	12.757	22.414	28.298	29.095	34.125	34.158	37.286	87.781
	11	12	13	14	15	16	17	18	19	
	LBA - R	LBA - Step	LBA-LS2	JU 3	LUBA-half	upper B	lower B	JU 2	GJ2	
	0.0002554	0.0002618	0.0002594	7.314E-05	0.0002252	0.0001336	0.0005817	0.0004374	0.0042254	
	2.898E-06	2.966E-06	2.936E-06	5.543E-07	1.285E-06	5.04E-07	2.123E-06	6.219E-07	3.759E-06	
	88.159	88.265	88.336	131.957	175.239	265.147	274.025	703.255	1124.024	
	1	2	3	4	5	6	7	8	9	10
Volatility	LUBA-LS1	LUBA-R	LUBA-WLS	ML	LUBA -LS2	LUBA-STEP	LUBA-PC	LBA-LS1	LBA-LS2	LBA-PC
high	6.205E-06	9.14E-06	9.215E-06	0.002994	1.251E-05	1.249E-05	1.429E-05	0.0001674	0.0001585	0.0001599
low	1.289E-05	1.346E-05	1.314E-05	0.0040808	1.358E-05	1.354E-05	1.374E-05	0.000121	9.835E-05	9.872E-05
Ratio	0.481	0.679	0.702	0.734	0.921	0.923	1.039	1.383	1.611	1.619
	11	12	13	14	15	16	17	18	19	
	LBA - Step	LBA - R	GJ2	LUBA-half	lower B	JU 2	BBSR	upper B	JU 3	
	0.0001604	0.0001585	0.0026813	0.0001541	0.0004024	0.000307	1.461E-05	9.597E-05	5.28E-05	
	9.878E-05	9.398E-05	0.0014345	6.416E-05	0.0001593	0.0001134	5.291E-06	3.239E-05	1.77E-05	
	1.624	1.686	1.869	2.402	2.525	2.707	2.760	2.963	2.982	
	1	2	3	4	5	6	7	8	9	10
Time	ML	LUBA-LS1	BBSR	LUBA-R	LUBA-WLS	LUBA-STEP	LUBA -LS2	LUBA-PC	LBA-LS1	LBA - R
high	0.0027642	8.279E-06	1.009E-05	1.163E-05	1.16E-05	1.52E-05	1.525E-05	1.707E-05	0.0002115	0.0002018
low	0.0043566	1.04E-05	1.071E-05	1.047E-05	1.027E-05	1.029E-05	1.029E-05	1.041E-05	6.805E-05	4.198E-05
Ratio	0.634	0.796	0.942	1.111	1.129	1.478	1.482	1.640	3.108	4.808
	11	12	13	14	15	16	17	18	19	
	LBA - Step	LBA-PC	LBA-LS2	LUBA-half	lower B	JU 3	upper B	JU 2	GJ2	
	0.0002073	0.000207	0.0002057	0.0001893	0.0004961	6.336E-05	0.0001193	0.0003911	0.0038195	
	4.242E-05	4.215E-05	4.163E-05	2.2E-05	4.69E-05	5.038E-06	4.388E-06	1.251E-05	6.859E-05	
	4.888	4.970	4.942	8.603	10.578	12.576	27.189	31.254	55.686	

Appendix G: Put Sensitivity for Analytic Approximations

PART 1

Money	ML	JU 3	PLS1	JU 2	PLS2	LUBA-LS1	PLS3	LUBA-WLS	lower B
AT	0.0017278	0.0002594	0.0024726	0.0015713	0.002818	0.0025599	0.0034219	0.0035149	0.0048634
IN	0.0022361	0.6924311	0.0017637	0.7000826	0.0014942	0.0018454	0.0012013	0.0023291	0.0011025
AT/OUT	0.256	0.498	0.534	0.542	0.664	0.771	0.775	0.830	0.851
	10	11	12	13	14	15	16	17	18
	PLS3	LUBA-R	LUBA-STEP	PLS2	LUBA-LS2	LUBA-PC	PLS1	BBSR	upper B
	0.0048575	0.0049942	0.0040634	0.0048319	0.0040134	0.0053353	0.0044075	0.0010918	0.0080825
	0.0010915	0.0023174	0.0023665	0.001081	0.002372	0.0023555	0.0009995	0.0003031	0.0242973
	0.853	0.833	0.857	0.859	0.861	0.872	0.906	0.937	0.952
	19	20	21	22	23	24	25		
	LBA-LS1	LBA-Step	LBA-PC	LBA-LS2	LBA-R	LUBA-half	GJ2		
	0.003134	0.0033297	0.0033279	0.0033249	0.0033672	0.002943	0.0716426		
	0.0014205	0.0015256	0.0015249	0.0015226	0.0015016	0.0117803	0.1105793		
	1.009	1.012	1.012	1.013	1.016	1.184	1.471		
	1	2	3	4	5	6	7	8	9
Money	JU 3	JU 2	LUBA-half	upper B	GJ2	ML	LUBA-LS1	PLS1	LUBA-WLS
AT	0.0002594	0.0015713	0.002943	0.0080825	0.0716426	0.0017278	0.0025599	0.0024726	0.0035149
IN	0.6924311	0.7000826	0.0117803	0.0242973	0.1105793	0.0022361	0.0018454	0.0017637	0.0023291
OUT	0.0005211	0.0029011	0.0024848	0.008494	0.0487191	0.0067464	0.0033211	0.004632	0.004235
AT?IN	0.000	0.002	0.250	0.333	0.648	0.773	1.387	1.402	1.509
	10	11	12	13	14	15	16	17	18
	LUBA-LS2	LUBA-STEP	PLS2	LUBA-R	LBA-PC	LBA-Step	LBA-LS2	LBA-LS1	LBA-R
	0.0040134	0.0040634	0.002818	0.0049942	0.0033279	0.0033297	0.0033249	0.003134	0.0033672
	0.002372	0.0023665	0.0014942	0.0023174	0.0015249	0.0015256	0.0015226	0.0014205	0.0015016
	0.0046623	0.0047413	0.0042417	0.0058516	0.0032877	0.0032898	0.0032838	0.0031057	0.0033137
	1.692	1.717	1.886	2.155	2.182	2.183	2.184	2.206	2.242
	19	20	21	22	23	24	25		
	LUBA-PC	PLS3	BBSR	PLS1	lower B	PLS3	PLS2		
	0.0053353	0.0034219	0.0010918	0.0044075	0.0048634	0.0048575	0.0048319		
	0.0023555	0.0012013	0.0003031	0.0009995	0.0011025	0.0010915	0.001081		
	0.0061204	0.0044178	0.0011658	0.0048638	0.0057145	0.0056927	0.005626		
	2.265	2.849	3.602	4.410	4.411	4.450	4.470		
	1	2	3	4	5	6	7	8	9
Interest	lower B	PLS3	PLS2	PLS1	PLS1	LUBA-LS1	PLS2	LUBA-WLS	LUBA-LS2
high	0.0033128	0.0033107	0.0033014	0.0029826	0.002675	0.0025172	0.0027548	0.0033545	0.0037888
low	0.0041431	0.0041209	0.0040677	0.0036027	0.0029324	0.0024982	0.0026951	0.0032057	0.0033983
Ratio	0.800	0.803	0.812	0.828	0.912	1.008	1.022	1.046	1.115
	10	11	12	13	14	15	16	17	18
	LUBA-STEP	PLS3	ML	LUBA-R	LUBA-PC	LBA-R	LBA-Step	LBA-PC	LBA-LS2
	0.0038371	0.0030567	0.0036971	0.0048134	0.0051156	0.0030918	0.0030842	0.0030832	0.003081
	0.0034253	0.0027153	0.0028656	0.0036959	0.0038161	0.0022567	0.0022414	0.0022393	0.0022357
	1.120	1.126	1.290	1.302	1.341	1.370	1.376	1.377	1.378
	19	20	21	22	23	24	25		
	LBA-LS1	upper B	LUBA-half	GJ2	BBSR	JU 2	JU 3		
	0.002976	0.0187536	0.0084007	0.1228128	0.0012813	0.4414226	0.435861		
	0.0020304	0.0094284	0.0036624	0.0362863	0.000369	0.0704536	0.0681981		
	1.466	1.989	2.294	3.385	3.472	6.265	6.391		

PART 2

	1	2	3	4	5	6	7	8	9
Dividend	JU 3	JU 2	GJ2	BBSR	LUBA-half	upper B	LUBA-R	LUBA-PC	LUBA-LS1
high	0.1418712	0.1442572	0.056717	0.00062	0.0045865	0.0109659	0.0035083	0.0036854	0.0021188
low	0.362188	0.367619	0.1023821	0.0010304	0.0074767	0.0172161	0.005001	0.0052463	0.0028966
Ratio	0.392	0.392	0.554	0.602	0.613	0.637	0.702	0.702	0.731
	10	11	12	13	14	15	16	17	18
	LUBA-WLS	LUBA-STEP	LUBA-LS2	PLS3	PLS2	LBA-LS1	ML	LBA - R	LBA-LS2
	0.0027776	0.0030949	0.0030646	0.0025252	0.0024893	0.0023453	0.0030806	0.0025188	0.0025062
	0.0037826	0.0041675	0.0041224	0.0032468	0.0029605	0.0026611	0.0034821	0.0028297	0.0028105
	0.734	0.743	0.743	0.778	0.841	0.881	0.885	0.890	0.892
	19	20	21	22	23	24	25		
	LBA-PC	LBA - Step	PLS1	PLS2	PLS3	lower B	PLS1		
	0.0025092	0.0025107	0.0027	0.0036514	0.0036883	0.0037074	0.0033292		
	0.0028134	0.0028149	0.0029074	0.0037177	0.0037434	0.0037485	0.0032561		
	0.892	0.892	0.929	0.982	0.985	0.989	1.022		
	1	2	3	4	5	6	7	8	9
Volatility	JU 3	JU 2	LUBA-half	GJ2	upper B	BBSR	ML	LUBA-R	LUBA-PC
high	0.0006688	0.0017084	0.0017074	0.0227728	0.0052999	0.0004171	0.002836	0.0039117	0.0041667
low	0.5033904	0.5101678	0.0103558	0.1363263	0.0228821	0.0013148	0.0037267	0.0045976	0.004765
Ratio	0.001	0.003	0.165	0.167	0.232	0.317	0.761	0.851	0.874
	10	11	12	13	14	15	16	17	18
	LUBA-STEP	LUBA-LS1	LUBA-WLS	LUBA -LS2	LBA-LS1	PLS2	PLS3	lower B	PLS1
	0.0039622	0.0027562	0.0036099	0.0039555	0.0027596	0.0041075	0.0041464	0.0041662	0.0036815
	0.0033003	0.0022592	0.0029504	0.0032316	0.0022468	0.0032615	0.0032852	0.0032897	0.0029038
	1.201	1.220	1.224	1.224	1.228	1.259	1.262	1.266	1.268
	19	20	21	22	23	24	25		
	LBA - R	LBA-LS2	LBA-PC	LBA - Step	PLS3	PLS1	PLS2		
	0.0030098	0.0029965	0.0030015	0.0030038	0.0034368	0.0034164	0.0033348		
	0.0023386	0.0023202	0.0023211	0.0023218	0.0023352	0.002191	0.0021151		
	1.287	1.291	1.293	1.294	1.472	1.559	1.577		
	1	2	3	4	5	6	7	8	9
Time	PLS1	PLS2	PLS3	lower B	LBA - R	LBA - Step	LBA-PC	LBA-LS2	PLS3
high	0.0026366	0.0030916	0.0031423	0.0031645	0.002494	0.0024923	0.0024923	0.0024925	0.002708
low	0.0039487	0.0042775	0.0042893	0.0042914	0.0028544	0.0028333	0.0028303	0.0028242	0.0030641
Ratio	0.668	0.723	0.733	0.737	0.874	0.880	0.881	0.883	0.884
	10	11	12	13	14	15	16	17	18
	ML	LBA-LS1	PLS2	PLS1	JU 2	JU 3	LUBA-LS1	LUBA-WLS	LUBA -LS2
	0.0031042	0.0024442	0.0026934	0.0028815	0.28289	0.2808759	0.0028469	0.0040918	0.0045787
	0.0034585	0.0025622	0.0027564	0.0027259	0.2289862	0.2231832	0.0021685	0.0024685	0.0026084
	0.898	0.954	0.977	1.057	1.235	1.258	1.313	1.658	1.755
	19	20	21	22	23	24	25		
	LUBA-STEP	LUBA-R	LUBA-PC	upper B	LUBA-half	GJ2	BBSR		
	0.004665	0.0059788	0.0063159	0.0202476	0.008839	0.1357902	0.0014403		
	0.0025975	0.0025305	0.0026158	0.0079344	0.0032242	0.0233089	8.703E-05		
	1.796	2.333	2.415	2.552	2.741	5.826	16.549		

Chapter 7: Conclusions and Recommendations

Lattice methods, Analytic approximations and Monte Carlo Simulation approaches provide possible avenues for pricing derivatives. Each class has its own strengths and weaknesses. For example, lattice methods are easy to program but are quite time consuming. Analytic approximations are more difficult to program, but are often very fast and accurate. Monte Carlo Simulation is the most flexible of all classes, but also the most time consuming.

We provided a description of all methods as well as their computer implementation.

For the regression of the LBA and LUBA models, biased and weighted regression alternatives were developed. Although partial support for the deletion of outliers is found, little evidence is found in favour of adjusting for multicollinearity. Perhaps predictions for puts are made from an X-space that is not sufficiently different than the X-space that the model was build from, using calls options.

A forward MCS approach was implemented to calculate exercise probabilities in the SA market and results interpreted.

All lattice approaches and analytic approximations were compared in a final analysis in the last chapter. Although the SR model is the recommended lattice approach, no single analytic model dominates all others. This is perhaps due to the high interest rate and low dividend yield in SA.

It is unlikely that a solution to the American option-pricing problem will be found. However, given the high speed and accuracy of the methods discussed above, from a numerical viewpoint, the single asset American option problem is essentially solved.

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