

Composition Operators on Banach Function Spaces

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Abstract

The aim of this thesis is to provide a survey of the topic of composition operators on spaces of (equivalence classes of) measurable functions and attempt to unify some of the most important results contained in the literature. A large class of these spaces can be equipped with norms turning them into Banach lattices. These spaces are called Banach function spaces and examples include the Lebesgue, Lorentz, Orlicz and Orlicz-Lorentz spaces. A composition operator can be defined between spaces of measurable functions as follows. Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces, where Ω_i is a set, Σ_i is a σ -algebra of subsets of Ω_i and μ_i is a positive measure on Σ_i for $i = 1, 2$. Let $F_i(\Omega_i)$ ($i = 1, 2$) be vector spaces of equivalence classes, modulo almost everywhere equality, of real or complex-valued functions on Ω_i ($i = 1, 2$) and let $\tau : \Omega_2 \rightarrow \Omega_1$ be a map, which will be referred to as the *symbol function*. For $g \in [f] \in F_1(\Omega_1)$, put

$$\begin{aligned} (\tilde{C}_\tau g)(x) &:= g(\tau(x)) & x \in \Omega_2 \text{ and} \\ C_\tau[f] &:= [\tilde{C}_\tau g] \end{aligned}$$

If C_τ maps $F_1(\Omega_1)$ into $F_2(\Omega_2)$ then C_τ is called a *composition operator* from $F_1(\Omega_1)$ into $F_2(\Omega_2)$. The first two questions which arise naturally from the definition above are, “When does \tilde{C}_τ map Σ_1 -measurable functions onto Σ_2 -measurable functions?” and “When is C_τ well-defined as a mapping of equivalence classes?”. Apart from these preliminary questions, we investigate three main problems. The first is determining the conditions on the symbol function τ such that C_τ is a continuous composition operator and providing estimates for the norm of the composition operator (in the settings where C_τ is acting on normed spaces). The second is characterizing the conditions under which the composition operator is compact. The properties of the underlying measure space will play a significant role in this part of the study. The third is characterizing those bounded linear operators on a Banach function space which are composition operators. The simplifying assumptions that the measure spaces are the same and that the spaces of measurable functions are the same will be made for the majority of results to be presented in this thesis.

There are scattered results in the literature ([37], [20], [5], [22] and [21]) regarding these three issues in the contexts of Lebesgue, Lorentz, Orlicz, Orlicz-Lorentz and Banach function spaces. The aim of this paper is to investigate composition operators in the most general of these contexts, namely Banach function spaces, to facilitate a more unified approach to the study of composition operators on spaces of measurable functions. It will also be shown which of the results in the literature for specific examples of Banach function spaces follow from this more general framework. Furthermore, we point out what appear to be errors in proofs in the literature and give proofs for similar (and often weaker) versions of the stated results. In addition to studying composition operators on Banach function spaces, we also investigate composition operators on spaces of measurable functions equipped with a (generally) non-normable linear topology and detail an important application of composition operators, namely the characterization of isometries between Lebesgue spaces.

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0.1. Introduction

The aim of this thesis is to provide a survey of the topic of composition operators on spaces of (equivalence classes of) measurable functions and attempt to unify some of the most important results contained in the literature. A large class of these spaces can be equipped with norms turning them into Banach lattices. These spaces are called Banach function spaces and examples include the Lebesgue, Lorentz, Orlicz and Orlicz-Lorentz spaces. We define the general notion of a composition operator. Let Ω_1 and Ω_2 be sets, $F_i(\Omega_i)$ ($i = 1, 2$) vector spaces of real or complex-valued functions on Ω_i ($i = 1, 2$) and let $\tau : \Omega_2 \rightarrow \Omega_1$ be a map, which will be referred to as the *symbol function*. We define the composition transformation C_τ on $F_1(\Omega_1)$ by

$$(C_\tau f)(x) := f(\tau(x)) \quad x \in \Omega_2, f \in F_1(\Omega_1)$$

If C_τ maps $F_1(\Omega_1)$ into $F_2(\Omega_2)$ then C_τ is called a *composition operator* from $F_1(\Omega_1)$ into $F_2(\Omega_2)$. It is possible to classify composition operators into three main classes based on the nature of the spaces Ω_1 and Ω_2 and the type of functions considered on these spaces. Composition operators induced by continuous symbol functions appear in the context of spaces of bounded continuous functions on topological spaces Ω_1 and Ω_2 . If spaces of analytic functions on subsets Ω_1 and Ω_2 of the complex plane are considered, then composition operators induced by analytic symbol functions arise (see [32], for example). Finally, composition operators induced by measurable symbol functions appear in the context of spaces of measurable functions over measure spaces Ω_1 and Ω_2 . The focus of this thesis will be on this third class of composition operators. Furthermore, the majority of results will be presented under the simplifying assumptions that the measure spaces are the same and the spaces of measurable functions are the same. There are three main problems to be investigated. The first is determining the conditions on the symbol function τ such that C_τ is a continuous composition operator and providing estimates for the norm of the composition operator (in the setting where C_τ is acting on normed spaces). It will be shown that this is equivalent to evaluating the conditions under which the symbol function induces a composition operator. The second is characterizing the conditions under which the composition operator is compact. The properties of the underlying measure space will play a significant role in this part of the study. The third is characterizing those linear operators on a Banach function space which are composition operators.

There are scattered results in the literature ([37], [20], [5], [22] and [21]) regarding these three issues in the contexts of Lebesgue, Lorentz, Orlicz, Orlicz-Lorentz and Banach function spaces. The aim of this paper is to investigate composition operators in the most general of these contexts, namely Banach function spaces, to facilitate a more unified approach to the study of composition operators on spaces of measurable functions. It will also be shown which of the results in the literature for specific examples of Banach function spaces follow from this more general framework. Furthermore, we point out what appear to be errors in proofs in the literature and give proofs for similar (and often weaker) versions of the stated results. In addition to studying composition operators on Banach function spaces, we also investigate composition operators on spaces of measurable functions equipped with a (generally) non-normable linear topology and detail an important application of composition operators, namely the characterization of isometries between Lebesgue spaces.

In Chapter 1 we provide the requisite background information regarding measure spaces, measurable functions, decreasing rearrangements and operators. In Chapter 2 we define linear topologies on a vector space of (equivalence classes of) real or complex-valued measurable functions, and on two of its vector subspaces. These topologies are generally non-normable. Order structures are also defined on these spaces in the case where they are real vector spaces and the interactions with the topological structures are noted. Furthermore the structure and properties of the Lorentz spaces, Orlicz spaces and Orlicz-Lorentz spaces are discussed. The salient features of Banach function spaces and an important class of Banach function spaces, the rearrangement invariant ones, are detailed in Chapters 3 and 4. In Chapter 5 the sum and intersection of Banach spaces are described and the interpolation theory to be used in the sequel is given. In Chapter 6 the connection between multiplication

operators and composition operators is demonstrated and the boundedness and compactness of multiplication operators on L^p spaces are investigated. The main results concerning the boundedness, compactness and characterization of composition operators are contained in Chapters 7,8 and 9 respectively. Finally, in Chapter 10 the application of composition operators in the characterization of isometries between L^p spaces is demonstrated.

Preliminaries

In this chapter we present the requisite background information regarding measure spaces, measurable functions, decreasing rearrangements, topological vector spaces and operators. In providing background information and useful results, our general strategy will be to provide references, but to omit proofs of results which are contained in the literature unless a particular proof serves to illuminate some important aspect of the theory. Where a particular result is not explicitly stated or proven in the literature, proofs will usually be furnished. We have tried to give some motivation for the results to be given in this chapter and to connect, where possible the concepts to be explored. There are however sections where we have merely listed those results we would like to refer to at a later stage.

We will assume familiarity with the basic concepts of functional analysis, measure theory and complex analysis as presented in [3], [6] and [31] respectively.

We start by making some comments about notation and terminology. We will adopt the convention of calling a function positive (rather than nonnegative) if $f \geq 0$ and strictly positive if $f > 0$. Similarly we will distinguish between monotone and strictly monotone. If X and Y are normed spaces, we denote the unit ball of X using B_X and the set of all bounded linear operators from X into Y using $\mathcal{B}(X, Y)$. \mathbb{F} will be used to denote an arbitrary field of scalars. If $F(\Omega)$ is a vector space of real or complex-valued functions on a set Ω , then $F_{\mathbb{R}}(\Omega)$ will be used to denote the set of all real-valued functions in $F(\Omega)$ and $F_+(\Omega)$ will be used to denote the set of all positive real-valued functions in $F(\Omega)$. We will use $L^0(\mu) = L^0(\Omega, \Sigma, \mu)$ to denote the set of all equivalence classes of complex-valued (finite almost everywhere) measurable functions on Ω where any two functions which are equal μ -a.e. have been identified. For simplicity we will use f to denote the equivalence class containing f .

1.1. Measure spaces and measurable transformations

There is often an interplay between the underlying measure space and the function space structure. This connection will be developed in the sections on resonant measure spaces and rearrangement invariant Banach function spaces. For the moment we present some results which will be useful in the sequel. Throughout this paper we will use (Ω, Σ, μ) to represent a measure space, where Ω is a set, Σ is a σ -algebra of subsets of Ω and $\mu : \Sigma \rightarrow [0, \infty]$ is a positive measure. We will assume that Ω is σ -finite, i.e. that there exists an increasing sequence of measurable sets of finite measure whose union is Ω , and that the completion process has been carried out on (Ω, Σ, μ) , i.e. Σ contains all subsets of sets of zero measure. We will use Σ_f to denote the family of measurable sets with finite measure, Σ_0 to denote the family of measurable sets with measure zero and $\Sigma_{f,0}$ to denote the family of all measurable sets with finite nonzero measure. A nonempty set $A \in \Sigma$ is called an *atom* if $B \subset A$ implies $\mu(B) = \mu(A)$ or $\mu(B) = 0$. If Σ does not contain any atoms then μ (or Ω) is called *nonatomic* or *continuous*. It can be shown that if μ is nonatomic and $A \in \Sigma$ is such that $\mu(A) = t_0$, then for every $0 \leq t \leq t_0$, there exists a measurable subset A_t of A with $\mu(A_t) = t$. Let \mathbb{R} denote the real line, \mathbb{R}^+ the positive part of the real line and m the Lebesgue measure on \mathbb{R} (or \mathbb{R}^+). As examples of nonatomic measure spaces we have $(\mathbb{R}, \mathcal{B}, m)$ and $(\mathbb{R}^+, \mathcal{M}, m)$, where \mathcal{B} is the Borel subsets of the real line and \mathcal{M} is the Lebesgue measurable subsets of the positive part of the real line. If there exists a partition of Ω consisting of countably many elements from Σ such that each element in this partition is either an atom or a null set, then (Ω, Σ, μ) is called *purely atomic*. Let \mathbb{E} denote the natural numbers \mathbb{N} (including zero) or the integers \mathbb{Z} and Let $\mathbf{w} = (w_i)_{i \in \mathbb{E}}$

be a sequence of positive real numbers. The measure space $(\mathbb{E}, \mathcal{P}(\mathbb{E}), \mu_w)$ where $\mathcal{P}(\mathbb{E})$ is the powerset of \mathbb{E} and $\mu_w(\{i\}) := w_i$ will be denoted by \mathbb{E}_w and is an example of an atomic measure space. If $\mu_w(\{i\}) = 1$ for all $i \in \mathbb{E}$, then μ_w is called the counting measure and will be denoted μ_c . The following result shows that any σ -finite measure space can be decomposed into atomic and nonatomic parts.

PROPOSITION 1.1.1. [40, p.67] *A σ -finite measure space Ω is uniquely decomposed as follows*

$$\Omega = \Omega_c \cup \Omega_a$$

where Ω_c is nonatomic and $\Omega_a = \bigcup_{n=1}^{\infty} A_n$ (where $(A_n)_{n=1}^{\infty}$ is a countable collection of disjoint atoms of strictly positive measure).

An important example of a positive measure is a Haar measure. A Borel measure μ in a locally compact topological group Ω is called a *Haar measure* if $\mu(G) > 0$ for every non-empty Borel open set G and $\mu(xA) = \mu(A)$ for every $x \in \Omega$ and for every Borel set A . The following is an important existence result for Haar measures.

THEOREM 1.1.2. [23, p.254] *In every locally compact topological group there exists at least one regular Haar measure.*

We list some properties of positive measures.

THEOREM 1.1.3. (cf. [31, p.16]) *Let μ be a positive measure on a σ -algebra Σ and let $(A_n)_{n=1}^{\infty} \subset \Sigma$. Then*

- (1) $\lim_{n \rightarrow \infty} \mu(A_n) = \mu(A)$ if $A = \bigcup_{n=1}^{\infty} A_n$ and $A_n \subset A_{n+1}$ for all $n \in \mathbb{N}^+$
- (2) $\lim_{n \rightarrow \infty} \mu(A_n) = \mu(A)$ if $A = \bigcap_{n=1}^{\infty} A_n$, $A_{n+1} \subset A_n$ for all $n \in \mathbb{N}^+$ and $\mu(A_k) < \infty$ for some $k \in \mathbb{N}^+$.

When characterizing those linear operators from a Banach function space into itself which are composition operators, it will be useful to know the relationship between set mappings and point mappings. We present some preliminaries on the algebraic structure of the underlying measure space and include Sikorski's Theorem which specifies the conditions under which a set mapping is induced by a point mapping.

PROPOSITION 1.1.4. [37, p.22] *The σ -algebra Σ of measurable sets is a Boolean algebra under the binary operations of union and intersection as join and meet respectively and the unary operation of set-complementation as Boolean complementation.*

REMARK 1.1.5. In general, we can define an order relation on a Boolean algebra by

$$x \leq y \Leftrightarrow x \vee y = y$$

In the case above, the Boolean order relation defined in this way coincides with set-inclusion. We will use the notation $A \setminus B$ to denote $A \cap B^c$

We wish to form a quotient algebra using $\Sigma_0 := \{A \in \Sigma : \mu(A) = 0\}$. We therefore require the following result.

PROPOSITION 1.1.6. (cf. [37, p.22]) *Σ_0 is a σ -ideal of the algebra Σ , i.e. Σ_0 is closed under countable union and if $B \in \Sigma_0$ and $A \subset B$, then $A \in \Sigma_0$.*

PROOF. Let $(A_n)_{n=1}^{\infty} \subset \Sigma_0$. Then $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu(A_n) = 0$ since $\mu(A_n) = 0$ for all $n \in \mathbb{N}^+$ and so $\bigcup_{n=1}^{\infty} A_n \in \Sigma_0$. Furthermore, if $A \in \Sigma_0$ and $B \subset A$, then $\mu(B) \leq \mu(A) = 0$ and so $B \in \Sigma_0$. \square

We form the quotient ring Σ/Σ_0 on which we define the Boolean operations

$$\begin{aligned} [A] \cup [B] &:= [A \cup B] \\ [A] \cap [B] &:= [A \cap B] \\ [A]^c &:= [A^c] \end{aligned}$$

PROPOSITION 1.1.7. [37, p.22] Σ/Σ_0 is a Boolean algebra when equipped with the operations defined above.

REMARK 1.1.8. Since $[A] := \{B \in \Sigma : A \sim B\}$ where $A \sim B \Leftrightarrow A \Delta B \in \Sigma_0$, we have that $[A] = [B]$ if and only if $\mu(A \Delta B) = 0$ and so if we define $[A] \subset [B]$ if and only if $[A] \cap [B] = [A]$, then

$$\begin{aligned} [A] \subset [B] &\Leftrightarrow [A \cap B] = [A] \\ &\Leftrightarrow \mu((A \cap B) \Delta A) = 0 \\ &\Leftrightarrow \mu(A \setminus B) = 0 \end{aligned}$$

Furthermore if we define $[A] \setminus [B] := [A] \cap [B]^c$ then $[A] \setminus [B] = [A] \cap [B^c] = [A \cap B^c] = [A \setminus B]$

The following result follows easily from the above remark.

PROPOSITION 1.1.9. If $A, B \in \Sigma$, then

- (1) $[\chi_A] = [\chi_B]$ if and only if $[A] = [B]$
- (2) $[\chi_A] \leq [\chi_B]$ if and only if $[A] \subset [B]$

We now present Sikorski's Theorem, which holds for a particular class of spaces, namely absolute Borel spaces, and for a particular class of set mappings, namely σ -homomorphisms. A topological space is said to be an absolute Borel space if it is homeomorphic to a Borel subset of the Hilbert cube. A map $\phi : \mathcal{A} \rightarrow \mathcal{B}$ between two Boolean algebras is called a (Boolean) homomorphism if it preserves joins, meets and complements. If ϕ preserves countable joins and countable meets then ϕ is called a σ -homomorphism.

THEOREM 1.1.10 (Sikorski). [33, p.110] Let Ω be an absolute Borel space, Σ the σ -algebra of all Borel subsets of Ω and Σ_i a σ -ideal of Σ . If $\tilde{\Sigma}$ is a σ -algebra of subsets of a set $\tilde{\Omega}$ and $\tilde{\Sigma}_i$ a σ -ideal of $\tilde{\Sigma}$ then every σ -homomorphism $\phi : \Sigma/\Sigma_i \rightarrow \tilde{\Sigma}/\tilde{\Sigma}_i$ is induced by a point mapping, i.e. there exists a map $\tau : \tilde{\Omega} \rightarrow \Omega$ such that

$$\phi([A]) = [\tau^{-1}(A)] \quad \forall [A] \in \Sigma/\Sigma_i$$

Next we define separability of a measure. Let $[\Sigma_f] := \Sigma_f/\Sigma_0$ and define $d : [\Sigma_f] \rightarrow [0, \infty)$ by

$$d([A], [B]) := \int_{\Omega} |\chi_A - \chi_B| d\mu \quad [A], [B] \in [\Sigma_f]$$

It can be shown (cf. [2, p.27]) that $([\Sigma_f], d)$ is a complete metric space. A measure μ is called *separable* if the corresponding metric space is separable. Let \mathcal{A} be a family of subsets of a set Ω . $\mathcal{M}(\mathcal{A})$ and $\mathcal{S}(\mathcal{A})$ will be used to represent the monotone class generated by \mathcal{A} and the σ -ring generated by \mathcal{A} respectively. We present one result regarding the separability of a measure.

THEOREM 1.1.11. [23, p.168] Let (Ω, Σ, μ) be a σ -finite measure space. Then μ is separable if there exists a sequence of $(A_n)_{n=1}^{\infty} \subset \Sigma$ such that $\Sigma = \mathcal{S}((A_n)_{n=1}^{\infty})$.

REMARK 1.1.12. Since $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu_w)$ is a σ -finite measure space and $\mathcal{P}(\mathbb{N}) = \mathcal{S}(\{\{n\}\}_{n \in \mathbb{N}})$, μ_w is separable for any sequence of positive real numbers $w = (w_n)_{n \in \mathbb{N}}$.

In the process of characterizing those rational functions which induce bounded composition operators on Banach function spaces over $(\mathbb{R}, \mathcal{B}, m)$ it will be useful to show that if a property holds for a particular class of sets then it holds for all Borel sets. We present one result in this direction.

THEOREM 1.1.13. [23, p.27] *Let \mathcal{R} be a ring. Then $\mathcal{M}(\mathcal{R}) = \mathcal{S}(\mathcal{R})$.*

REMARK 1.1.14. If E is a topological space, then Rudin ([31, p.12]) defines the Borel sets of E to be the smallest σ -algebra \mathcal{B} containing the open sets of E . Halmos ([23, p.62]) defines the Borel sets of the real line to be the σ -ring generated by the class of all bounded semiclosed intervals of the form $[a, b)$. Theorem 15B ([23, p.63]) shows that these two definitions are equivalent.

Next we consider some measure theoretic properties of the maps $\tau : \Omega \rightarrow \Omega$ which will be used to define composition transformations.

Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be σ -finite measure spaces. A map $\tau : \Omega_2 \rightarrow \Omega_1$ is called a measurable transformation if $\tau^{-1}(A) \in \Sigma_2$ for all $A \in \Sigma_1$. If τ is a measurable transformation such that $\mu_2(\tau^{-1}(A)) = 0$ whenever $\mu_1(A) = 0$, then τ is called nonsingular. A measurable transformation τ can be used to define a positive measure $\mu_2 \circ \tau^{-1}$ on Σ in the following natural way:

$$\mu_2 \circ \tau^{-1}(A) := \mu_2(\tau^{-1}(A)) \quad \forall A \in \Sigma_1$$

We show that, under certain conditions, τ maps any atom essentially into an atom.

PROPOSITION 1.1.15. (cf. [38, p.507]) *Let $\tau : \Omega \rightarrow \Omega$ be a measurable transformation such that*

$$(1.1.1) \quad \mu \circ \tau^{-1}(A) \leq k\mu(A) \quad \forall A \in \Sigma_f$$

for some $k > 0$. Then for any atom A with $\mu(A) > 0$ there exists an atom B such that $\mu(A \cap \tau^{-1}(B)) = \mu(A)$.

PROOF. (Ω, Σ, μ) is σ -finite, so we can write Ω_a , the atomic part of Ω , as a union of atoms A_n ($n \in \mathbb{N}^+$) of finite strictly positive measure. Assume that there exists an atom A such that $\mu(A \cap \tau^{-1}(A_n)) \neq \mu(A)$ for every atom A_n . Since A is an atom this implies that

$$(1.1.2) \quad \begin{aligned} \mu(A \cap \tau^{-1}(A_n)) &= 0 \quad \forall n \in \mathbb{N}^+ \\ \Rightarrow \mu(A \cap \tau^{-1}(\Omega_a)) &= \mu\left(A \cap \tau^{-1}\left(\bigcup_{n=1}^{\infty} A_n\right)\right) \\ &= \mu\left(\bigcup_{n=1}^{\infty} A \cap \tau^{-1}(A_n)\right) \\ &= \sum_{n=1}^{\infty} \mu(A \cap \tau^{-1}(A_n)) \\ &= 0 \end{aligned}$$

Furthermore

$$(1.1.3) \quad \begin{aligned} \Omega &= \tau^{-1}(\Omega_c) \cup \tau^{-1}(\Omega_a) \\ \Rightarrow \mu(A) &= \mu\left((A \cap \tau^{-1}(\Omega_c)) \cup (A \cap \tau^{-1}(\Omega_a))\right) \\ &= \mu(A \cap \tau^{-1}(\Omega_c)) + \mu(A \cap \tau^{-1}(\Omega_a)) \\ &= \mu(A \cap \tau^{-1}(\Omega_c)) \quad \text{by (1.1.2)} \end{aligned}$$

Let $\epsilon > 0$. Since Ω_c is nonatomic and σ -finite we can find a sequence $(B_n)_{n=1}^\infty$ of pairwise disjoint measurable subsets such that $\mu(B_n) \leq \frac{\epsilon}{k}$ for each n and $\Omega_c = \bigcup_{n=1}^\infty B_n$. Therefore

$$\begin{aligned}
 \mu(A) &= \mu\left(A \cap \tau^{-1}\left(\bigcup_{n=1}^\infty B_n\right)\right) && \text{by 1.1.3} \\
 &= \mu\left(\bigcup_{n=1}^\infty A \cap \tau^{-1}(B_n)\right) \\
 &= \sum_{n=1}^\infty \mu\left(A \cap \tau^{-1}(B_n)\right) && \text{since the } \tau^{-1}(B_n)\text{'s are pairwise disjoint} \\
 (1.1.4) \quad &= \mu\left(A \cap \tau^{-1}(B_k)\right) && \text{for some } k \in \mathbb{N}^+ \\
 &\leq \mu \circ \tau^{-1}(B_k) \\
 &\leq k\mu(B_k) && \text{by (1.1.1)} \\
 (1.1.5) \quad &\leq \epsilon
 \end{aligned}$$

Line (1.1.4) follows since $A \cap \tau^{-1}(B_n)$ is a subset of an atom and hence for each n , $\mu(A \cap \tau^{-1}(B_n)) = 0$ or $\mu(A \cap \tau^{-1}(B_n)) = \mu(A)$. Since (1.1.5) holds for any $\epsilon > 0$ we have that $\mu(A) = 0$, which is a contradiction. \square

1.2. Measurable functions

In this section we outline some of the important features of measurable functions and state some results concerning the convergence of sequences of measurable functions and the relationships between some of the different modes of convergence. Furthermore, we provide some information regarding the Radon-Nikodym derivative, rational functions and convex functions. First we describe some notation to be used. Let (Ω, Σ, μ) be a σ -finite measure space. The vector space of equivalence classes, modulo almost everywhere equality of real or complex-valued measurable functions, finite almost everywhere on Ω , will be denoted $L^0(\Omega, \Sigma, \mu)$ or $L^0(\mu)$. Unless specified otherwise we will identify functions which are equal almost everywhere and denote the equivalence class $[f]$ of a function f using f . The set of all functions in $L^0(\mu)$ which are bounded, except possibly on a set of finite measure, will be denoted $L^{00}(\mu)$.

PROPOSITION 1.2.1. *If $f \in L^0(\Omega, \Sigma, \mu)$ such that $f \neq 0$, then there exists an $\epsilon > 0$ and a $B \in \Sigma$ with $\mu(B) = \delta > 0$ such that $|f(x)| \geq \epsilon$ for all x in B .*

PROOF. Let $A = \{x \in \Omega : |f(x)| > 0\}$. $f \neq 0$ implies that $\mu(A) > 0$. For each $n \in \mathbb{N}^+$, define $A_n = \{x \in \Omega : |f(x)| > \frac{1}{n}\}$. Then $A = \bigcup_{n=1}^\infty A_n$ and so

$$\begin{aligned}
 \mu(A) &= \mu\left(\bigcup_{n=1}^\infty A_n\right) \\
 &\leq \sum_{n=1}^\infty \mu(A_n)
 \end{aligned}$$

A simple proof by contradiction shows that this implies that $\mu(A_k) > 0$ for some $k \in \mathbb{N}^+$. Let $\epsilon = \frac{1}{k}$ and $B = A_k$. \square

PROPOSITION 1.2.2. *Let $f, g \in L^0(\Omega, \Sigma, \mu)$ such that $\int_A f d\mu$ and $\int_A g d\mu$ exist for every $A \in \Sigma_f$. If*

$$(1.2.1) \quad \int_A f d\mu = \int_A g d\mu \quad \forall A \in \Sigma_f$$

then $f = g$.

PROOF. Let $f = f_1 + if_2$ and $g = g_1 + ig_2$ where $f_1, f_2, g_1, g_2 \in L^0_{\mathbb{R}}(\Omega, \Sigma, \mu)$. If (1.2.1) holds then

$$(1.2.2) \quad \begin{aligned} \int_A f_1 d\mu &= \int_A g_1 d\mu \quad \forall A \in \Sigma_f \quad \text{and} \\ \int_A f_2 d\mu &= \int_A g_2 d\mu \quad \forall A \in \Sigma_f \end{aligned}$$

Assume that $f \neq g$. Then either $f_1 \neq g_1$ or $f_2 \neq g_2$. If $f_1 \neq g_1$ then by Proposition 1.2.1 there exists an $\epsilon > 0$ and an $A \in \Sigma_f$ such that $|f_1(x) - g_1(x)| \geq \epsilon$ for all $x \in A$. A simple proof by contradiction shows that there exists a $B \subset A$ with $\mu(B) > 0$ such that $f_1(x) \geq g_1(x) + \frac{\epsilon}{3}$ for all $x \in B$ or $f_1(x) + \frac{\epsilon}{3} \leq g_1(x)$ for all $x \in B$. It follows that

$$\int_B f_1 d\mu \neq \int_B g_1 d\mu$$

which contradicts (1.2.2). Similarly, a contradiction can be reached if $f_2 \neq g_2$. It follows that $f = g$. \square

LEMMA 1.2.3. [39, p.325] *Let $1 \leq p < q < \infty$. If a measurable set A is nonatomic and $\mu(A) > 0$, then there exists an $f \in L^p(\mu)$ such that*

$$\int_A |f|^q d\mu = \infty$$

PROOF. Let $0 < \alpha < \mu(A)$. Since A is nonatomic one can find a sequence $(A_n)_{n=1}^{\infty}$ of pairwise disjoint measurable subsets of A such that for each $n \in \mathbb{N}^+$

$$\mu(A_n) = \frac{\alpha}{2^n}.$$

Define a function f on Ω as the pointwise limit

$$f = \sum_{k=1}^{\infty} \frac{1}{\mu(A_k)^{1/q}} \chi_{A_k}$$

Then the pointwise limit is well-defined since the A'_k s are pairwise disjoint. Furthermore, $f \in L^p(\mu)$ since

$$\begin{aligned} \int_{\Omega} |f|^p d\mu &= \int_{\Omega} \left(\sum_{k=1}^{\infty} \frac{1}{\mu(A_k)^{1/q}} \chi_{A_k} \right)^p d\mu \\ &= \int_{\Omega} \sum_{k=1}^{\infty} \frac{1}{\mu(A_k)^{p/q}} \chi_{A_k} d\mu \\ &= \sum_{k=1}^{\infty} \mu(A_k)^{1-p/q} \\ &= \alpha^{1-p/q} \sum_{k=1}^{\infty} \left(\frac{1}{2} \right)^{(1-p/q)k} \\ &< \infty, \end{aligned}$$

since $1 - p/q > 0$ implies that $(\frac{1}{2})^{(1-p/q)} < 1$. We show that $\int_A |f|^q d\mu = \infty$:

$$\begin{aligned} \int_{\Omega} |f|^q d\mu &= \int_{\Omega} \left(\sum_{k=1}^{\infty} \frac{1}{\mu(A_k)^{1/q}} \chi_{A_k} \right)^q d\mu \\ &= \int_{\Omega} \sum_{k=1}^{\infty} \frac{1}{\mu(A_k)} \chi_{A_k} d\mu \\ &= \sum_{k=1}^{\infty} 1 \\ &= \infty \end{aligned}$$

\square

We list some important convergence properties. f is called a *simple function* if there exists $(A_i)_{i=1}^k \subset \Sigma$ and $(\alpha_i)_{i=1}^k \subset \mathbb{F}$ such that $f = \sum_{i=1}^k \alpha_i \chi_{A_i}$. If $(A_i)_{i=1}^k \subset \Sigma_f$, then f is called a *simple function with support of finite measure*.

THEOREM 1.2.4. [31, p.15] *Let f be a positive measurable function. Then there exists a sequence $(f_n)_{n=1}^\infty$ of simple functions such that $0 \leq f_n \uparrow f$ pointwise.*

THEOREM 1.2.5. [23, p.93] *If $(f_n)_{n=1}^\infty$ is a sequence of measurable functions which is Cauchy in measure, then there exists a measurable function f such that $f_n \rightarrow f$ in measure.*

THEOREM 1.2.6 (Monotone Convergence Theorem). [31, p.21] *Let $(f_n)_{n=1}^\infty$ be a sequence of measurable functions on Ω and suppose that $0 \leq f_n \uparrow f$ μ -a.e.. Then f is measurable and*

$$\int_{\Omega} f_n d\mu \rightarrow \int_{\Omega} f d\mu \quad \text{as } n \rightarrow \infty$$

$L^0(\mu)$ and some of its subspaces will become topological vector spaces when equipped with various topologies related to convergence in measure. The relationships between different modes of convergence are therefore significant. Two such relationships are listed below.

PROPOSITION 1.2.7. [31, p.74] *Let (Ω, Σ, μ) be a finite measure space. If f is a complex valued measurable function and $(f_n)_{n=1}^\infty$ is a sequence of complex valued measurable functions such that $f_n \rightarrow f$ in measure, then there exists a subsequence $(f_{n_k})_{k=1}^\infty$ such that $f_{n_k} \rightarrow f$ pointwise μ -a.e.*

THEOREM 1.2.8. [31, p.68] *Let $1 \leq p \leq \infty$ and $(f_n)_{n=1}^\infty \subset L^p(\mu)$. If $f_n \rightarrow f$ in $L^p(\mu)$, then $(f_n)_{n=1}^\infty$ has a subsequence which converges pointwise almost everywhere to f .*

THEOREM 1.2.9. [3, p.75] *Let (Ω, Σ, μ) be a measure space, $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$ and $g \in L^{p'}(\mu)$. Define $\phi_g : L^p(\mu) \rightarrow \mathbb{F}$ by*

$$\phi_g(f) := \int f g d\mu$$

Then $\phi_g \in L^p(\mu)^$ and the map $g \mapsto \phi_g$ is an isometric isomorphism of $L^q(\mu)$ onto $L^p(\mu)^*$.*

We will require the following three results to show that monotone functions on the real line induce composition operators on certain function spaces over the real line.

THEOREM 1.2.10. [12, p.99] *Let $f : [a, b] \rightarrow \mathbb{R}$ be an integrable function. Then the function $F(x) := \int_a^x f(t) dt$ is absolutely continuous on $[a, b]$.*

LEMMA 1.2.11. (cf. [30]) *If $\phi : [a, b] \rightarrow \mathbb{R}$ is increasing, then*

- (1) *there exists $f : [a, b] \rightarrow \mathbb{R}$ such that f is left continuous, increasing and the set $\{x \in [a, b] : f(x) \neq \phi(x)\}$ is at most countable;*
- (2) *there exists a positive Borel measure μ on $[a, b]$ such that*

$$f(x) - f(a) = \mu([a, x)) \quad a \leq x \leq b$$

- (3) *f' exists m-a.e., $f \in L^1([a, b], \mathcal{B}, m)$ and*

$$f(x) - f(a) = \int_a^x f'(t) dt + s(x) \quad a \leq x \leq b$$

for some s increasing such that $s'(x) = 0$ m-a.e.

- (4) *$\phi'(x) = f'(x)$ m-a.e.*

COROLLARY 1.2.12. *If $\phi : \mathbb{R} \rightarrow \mathbb{R}$ is monotone, then ϕ' exists m-a.e.*

1.2.1. The Radon-Nikodym derivative. We state the Radon-Nikodym theorem and some related results.

THEOREM 1.2.13. [28, p.154] *Let (Ω, Σ, μ) be a σ -finite measure space and let $\tau : \Omega \rightarrow \Omega$ be a nonsingular measurable transformation. Then for any $f \in L^0(\mu)$*

$$(1.2.3) \quad \int_{\Omega} f \circ \tau d\mu = \int_{\Omega} f d\mu \circ \tau^{-1}$$

in the sense that the existence of the integral on either side implies the existence of the other and equality holds.

PROOF. It is easily checked that $\mu \circ \tau^{-1}$ is a positive measure on Σ . Let $A \in \Sigma$, then

$$\begin{aligned} \int_{\Omega} \chi_A \circ \tau d\mu &= \int_{\Omega} \chi_{\tau^{-1}(A)} d\mu \\ &= \mu \circ \tau^{-1}(A) \\ &= \int_{\Omega} \chi_A d\mu \circ \tau^{-1} \end{aligned}$$

By the linearity of integration (1.2.3) holds for all positive simple functions. Let $f \in L^0(\mu)$ such that $f \geq 0$. Then there exists a sequence of positive simple functions $(f_n)_{n=1}^{\infty}$ such that $f_n \uparrow f$ pointwise μ -a.e. Since τ is nonsingular this implies that $f_n \uparrow f$ pointwise $\mu \circ \tau^{-1}$ -a.e. For each n , we have

$$(1.2.4) \quad \int_{\Omega} f_n d\mu \circ \tau^{-1} = \int_{\Omega} f_n \circ \tau d\mu$$

Furthermore, we also have that $f_n \circ \tau \uparrow f \circ \tau$ pointwise μ -a.e. By the Monotone Convergence theorem, it therefore follows that

$$\begin{aligned} \int_{\Omega} f d\mu \circ \tau^{-1} &= \lim_{n \rightarrow \infty} \int_{\Omega} f_n d\mu \circ \tau^{-1} \\ &= \lim_{n \rightarrow \infty} \int_{\Omega} f_n \circ \tau d\mu \quad \text{since (1.2.4) holds for each } n \\ &= \int_{\Omega} f \circ \tau d\mu \end{aligned}$$

Let $f \in L^0(\mu)$, then

$$f = (g^+ - g^-) + i(h^+ - h^-)$$

where $g^+, g^-, h^+, h^- \geq 0$. Using the linearity of the integral and the fact that (1.2.3) holds for all positive measurable functions we have that (1.2.3) holds for all $f \in L^0(\mu)$. \square

THEOREM 1.2.14. [31, p.23] *Let $f : \Omega \rightarrow [0, \infty]$ be measurable and define $\nu : \Sigma \rightarrow [0, \infty]$ by*

$$\nu(A) := \int_A f d\mu \quad A \in \Sigma$$

Then ν is a measure on Σ and

$$\int_{\Omega} g d\nu = \int_{\Omega} gf d\mu$$

for every measurable function g with range in $[0, \infty]$.

THEOREM 1.2.15 (Radon-Nikodym theorem). [6, p.132] *Let (Ω, Σ, μ) be a σ -finite measure space and let ν be a positive measure on Σ . If ν is absolutely continuous with respect to μ (written $\nu \ll \mu$), then there exists a Σ -measurable function $f : \Omega \rightarrow [0, \infty)$ such that*

$$\nu(A) = \int_A f d\mu \quad \forall A \in \Sigma$$

Furthermore, the function f is unique up to μ -a.e. equality.

The function f in the theorem above is called the Radon-Nikodym derivative of ν with respect to μ and is sometimes written $\frac{d\nu}{d\mu}$.

Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be σ -finite measure spaces and let $\tau : \Omega_2 \rightarrow \Omega_1$ be a measurable transformation. Recall that we can define a measure $\mu_2 \circ \tau^{-1}$ on Σ_1 by setting

$$\mu_2 \circ \tau^{-1}(A) = \mu_2(\tau^{-1}(A)) \quad \forall A \in \Sigma_1$$

If τ is nonsingular then $\mu_2 \circ \tau^{-1}$ is absolutely continuous with respect to μ_1 and so we can apply the Radon-Nikodym theorem to obtain the following result.

COROLLARY 1.2.16. *Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be σ -finite measure spaces. If $\tau : \Omega_2 \rightarrow \Omega_1$ is a nonsingular measurable transformation, then there exists a Σ_1 -measurable function $f_\tau : \Omega_1 \rightarrow [0, \infty)$ such that*

$$\mu_2 \circ \tau^{-1}(A) = \int_A f_\tau d\mu_1 \quad \forall A \in \Sigma_1$$

Furthermore, the function f_τ is unique up to μ_1 -a.e. equality.

Let $\Omega = \Omega_c \cup \Omega_a$ where Ω_c is the nonatomic part of Ω and Ω_a is the purely atomic part of Ω . Let A_1, A_2, \dots be the atoms of Ω .

PROPOSITION 1.2.17. *If $\tau : \Omega \rightarrow \Omega$ is a nonsingular measurable transformation such that $\mu \circ \tau^{-1}(\Omega_c) = 0$, then $f_\tau = 0$ μ -a.e. on Ω_c and $f_\tau(A_n) = \frac{\mu \circ \tau^{-1}(A_n)}{\mu(A_n)}$ for each A_n .*

PROOF. We have

$$0 = \mu \circ \tau^{-1}(\Omega_c) = \int_{\Omega_c} f_\tau d\mu$$

and so $f_\tau = 0$ μ -a.e on Ω_c since f_τ non-negative. Furthermore

$$\begin{aligned} \mu \circ \tau^{-1}(A_n) &= \int_{A_n} f_\tau d\mu \\ &= f_\tau(A_n) \mu(A_n) \\ \Rightarrow f_\tau(A_n) &= \frac{\mu \circ \tau^{-1}(A_n)}{\mu(A_n)} \end{aligned}$$

□

THEOREM 1.2.18. (cf. [23, p.164]) *Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be σ -finite measure spaces. If $\tau : \Omega_2 \rightarrow \Omega_1$ is a measurable transformation such that $\mu_2 \circ \tau^{-1}$ is absolutely continuous with respect to μ_1 , then there exists a non-negative measurable function f_τ on Ω_1 such that*

$$\int_{\Omega_2} g(\tau(x)) d\mu_2(x) = \int_{\Omega_1} g(y) \cdot f_\tau(y) d\mu_1(y)$$

for every measurable function g , in the sense that if either integral exists, then so does the other and the two are equal.

REMARK 1.2.19. The function in the theorem corresponds to the Radon-Nikodym derivative since the Radon-Nikodym derivative is unique up to almost everywhere equality and if $A \in \Sigma$, then

$$\begin{aligned} \mu_2 \circ \tau^{-1}(A) &= \int_{\tau^{-1}(A)} d\mu_2 \\ &= \int_{\Omega_2} \chi_A(\tau(x)) d\mu_2(x) \\ &= \int_{\Omega_1} \chi_A(y) \cdot f_\tau(y) d\mu_1(y) \quad \text{by Theorem 1.2.18} \\ &= \int_A f_\tau(y) d\mu_1(y) \end{aligned}$$

1.2.2. Rational functions. We state three properties of rational functions and then provide an explicit description of the Radon-Nikodym derivative “induced” by a rational function.

LEMMA 1.2.20. *Let $\tau = \frac{\phi}{\psi}$ be a rational function where ψ is a polynomial with distinct complex zeros x_1, \dots, x_n of order m_1, \dots, m_n respectively. Then $|\tau'(x)| \rightarrow \infty$ as $x \rightarrow x_k$.*

LEMMA 1.2.21. *Let $\tau = \frac{\phi}{\psi}$ be a rational function and let $\deg \phi = n$, $\deg \psi = m$.*

- (1) *If $n > m + 1$, then $|\tau'(x)| \rightarrow \infty$ as $|x| \rightarrow \infty$*
- (2) *If $n = m + 1$, and the leading coefficients of ϕ and ψ are both 1, then $|\tau'(x)| \rightarrow 1$ as $|x| \rightarrow \infty$*

Let the number of zeros and number of poles of the rational function τ be denoted by $\#(Z)$ and $\#(P)$ respectively.

LEMMA 1.2.22. [35, p.332] *Let τ be a discontinuous rational function. If $\deg \phi > \deg \psi$ and $\tau'(x) \neq 0$ for all $x \in \mathbb{R}$, then $\#(Z) = \#(P)$ and the zeros and poles occur alternately on \mathbb{R} .*

It can be shown that if $\tau : \mathbb{R} \rightarrow \mathbb{R}$ is a rational function, then τ is a nonsingular measurable transformation. It follows that the Radon-Nikodym derivative $f_\tau = \frac{dm \circ \tau^{-1}}{dm}$ exists. We first calculate the Radon-Nikodym derivative “induced” by a strictly increasing function.

PROPOSITION 1.2.23. *Let $\tau : (a, b) \rightarrow \mathbb{R}$ be a strictly increasing function which is measurable with respect to the σ -algebra of Borel sets on \mathbb{R} and such that the measure $m \circ \tau^{-1}$ is nonsingular with respect to the Lebesgue measure on \mathbb{R} . Then*

$$f_\tau(x) = \frac{1}{\tau'(\tau^{-1}(x))} \quad m\text{-a.e. } x \in (a, b)$$

PROOF. We note first that τ strictly increasing implies that τ^{-1} exists and τ' exists m -a.e. by Lemma 1.2.11. Furthermore, $\tau'(x) \neq 0$ for all $x \in (a, b)$ since τ is strictly increasing. It is easily shown that

$$(\tau^{-1})'(x) = \frac{1}{\tau'(\tau^{-1}(x))}$$

Let $x, y \in (a, b)$ such that $x < y$, then

$$\begin{aligned} m(\tau^{-1}[x, y]) &= m([\tau^{-1}(x), \tau^{-1}(y)]) && \text{since } \tau \text{ is strictly increasing} \\ &= \tau^{-1}(y) - \tau^{-1}(x) \\ &= \int_x^y (\tau^{-1})'(s) dm(s) \\ (1.2.5) \quad &= \int_x^y \frac{1}{\tau'(\tau^{-1}(s))} dm(s) \end{aligned}$$

Since (1.2.5) holds for all semi-closed intervals in (a, b) , we have for any Borel set $A \subset (a, b)$

$$\begin{aligned} \int_A \frac{1}{\tau'(\tau^{-1}(s))} dm(s) &= m(\tau^{-1}(A)) \\ (1.2.6) \quad &= \int_A f_\tau(s) dm(s) \quad \text{by Corollary 1.2.16} \end{aligned}$$

Since m and $m \circ \tau^{-1}$ are positive measures we have that $\frac{1}{\tau'(\tau^{-1}(s))}, f_\tau(s) \geq 0$ m -a.e. and so (1.2.6) implies that

$$\frac{1}{\tau'(\tau^{-1}(s))} = f_\tau(s) \quad m\text{-a.e.}$$

□

To prove a sufficient condition for a rational function to induce a bounded composition operator we will use the following explicit description of the Radon-Nikodym derivative.

LEMMA 1.2.24. Let $\tau = \frac{\phi}{\psi}$ be a rational function with $\deg \phi > \deg \psi$ and $\tau'(x) \neq 0$ for all $x \in \mathbb{R}$ such that τ' exists. Let x_1, \dots, x_n be the distinct real zeros of ψ arranged in increasing order and define

$$\begin{aligned}\tau_1 &:= \tau \upharpoonright (-\infty, x_1), \\ \tau_k &:= \tau \upharpoonright (x_{k-1}, x_k) \quad \text{for } (2 \leq k \leq n) \text{ and} \\ \tau_{n+1} &:= \tau \upharpoonright (x_n, \infty)\end{aligned}$$

Then

$$f_\tau(x) = \sum_{k=1}^{n+1} \frac{1}{\tau'_k(\tau_k^{-1}(x))} \quad m\text{-a.e.}$$

PROOF. Since

$$\tau' = \frac{\phi'\psi - \phi\psi'}{(\psi)^2}$$

τ' has the same poles as τ and is continuous on the interval between consecutive poles. It therefore follows from $\tau'(x) \neq 0$ for all $x \in \mathbb{R}$ where τ' exists that τ is strictly monotone on the intervals between consecutive poles, i.e. τ_k is strictly monotone for each k . It follows by Proposition 1.2.23 that

$$f_{\tau_k}(x) = \frac{1}{\tau'_k(\tau_k^{-1}(x))} \quad m\text{-a.e. } x \in (x_{k-1}, x_k)$$

Furthermore since the Radon-Nikodym derivative is unique modulo almost everywhere equality, we have

$$\begin{aligned}f_\tau \upharpoonright (-\infty, x_1) &= f_{\tau_1} \\ f_\tau \upharpoonright (x_{k-1}, x_k) &= f_{\tau_k} \quad 2 \leq k \leq n \\ f_\tau \upharpoonright (x_n, \infty) &= f_{\tau_{n+1}}\end{aligned}$$

where equality holds m -a.e. So

$$\begin{aligned}f_\tau(x) &= \left(\sum_{k=1}^{n+1} f_{\tau_k} \right)(x) \quad m\text{-a.e.} \\ &= \sum_{k=1}^{n+1} \frac{1}{\tau'_k(\tau_k^{-1}(x))}\end{aligned}$$

□

1.2.3. Convex functions. Convex functions will be used in the definition of Orlicz spaces and Orlicz-Lorentz spaces. We recall the definition of a convex function and give two results concerning convex functions. A function $\phi : \mathbb{R} \rightarrow \mathbb{R}$ is called convex if

$$\phi(\lambda t + (1 - \lambda)s) \leq \lambda\phi(t) + (1 - \lambda)\phi(s) \quad \forall s, t \in \mathbb{R}$$

for any $\lambda \in [0, 1]$. We can characterize convex functions as follows.

THEOREM 1.2.25. [29, p.7] A function $\phi : (a, b) \rightarrow \mathbb{R}$ is convex if and only if there exists a monotone nondecreasing and left continuous function $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$\phi(t) = \phi(c) + \int_c^t \varphi(s) ds \quad t \in [c, d]$$

for every closed subinterval $[c, d] \subset (a, b)$.

PROPOSITION 1.2.26. Let ϕ be a convex function such that $\phi(0) = 0$ and let $0 \leq \alpha \leq 1$. Then $\phi(\alpha t) \leq \alpha\phi(t)$ for all $0 \leq t < \infty$.

PROOF. Let $t \in [0, \infty)$, then

$$(1.2.7) \quad \begin{aligned} \phi(\alpha t) &= \phi(\alpha t + (1 - \alpha)0) \\ &\leq \alpha\phi(t) + (1 - \alpha)\phi(0) \quad \text{since } \phi \text{ is convex and } \alpha \in [0, 1] \end{aligned}$$

$$(1.2.8) \quad = \alpha\phi(t) \quad \text{since } \phi(0) = 0$$

□

1.3. Distribution functions, decreasing rearrangements and maximal functions

Decreasing rearrangements feature prominently in the definition of Lorentz spaces and Orlicz-Lorentz spaces. Furthermore, under suitable conditions the norms of rearrangement invariant Banach function spaces can be expressed in terms of decreasing rearrangements. We therefore explore the salient features of decreasing rearrangements and maximal functions, which can be defined using decreasing rearrangements. This will require some background information regarding distribution functions.

1.3.1. Distribution functions. We define distribution functions and the notion of equimeasurability. The distribution functions of some functions which will frequently be encountered in the sequel are then given and the general properties of distribution functions listed.

Let $f \in L^0(\mu)$. The *distribution function*, μ_f of f is given by

$$\mu_f(\lambda) := \mu(\{x \in \Omega : |f(x)| > \lambda\}) \quad \text{for any } \lambda \geq 0$$

If $f \in L^0(\Omega_1, \Sigma_1, \mu_1)$ and $g \in L^0(\Omega_2, \Sigma_2, \mu_2)$, then f and g are said to be *equimeasurable* if they have the same distribution functions, i.e. $\mu_{1,f}(\lambda) = \mu_{2,g}(\lambda)$ for all $\lambda \geq 0$. If this is the case we will write $f \sim g$.

REMARK 1.3.1. It will often be convenient to let $A_{f,\lambda} = \{x \in \Omega : |f(x)| > \lambda\}$; then $\mu_f(\lambda) = \mu(A_{f,\lambda})$

The distribution functions of the zero function, characteristic functions and simple functions are given in the following example.

EXAMPLE 1.3.2. [2, p.38]

- (1) If $f = 0$ μ -a.e., then $\mu_f = 0$
- (2) If $A \in \Sigma$, then

$$\mu_{\chi_A}(\lambda) = \begin{cases} \mu(A) & \text{if } 0 \leq \lambda < 1 \\ 0 & \text{if } \lambda \geq 1 \end{cases}$$

- (3) If $f = \sum_{i=1}^n a_i \chi_{A_i}$, where $a_i > a_{i+1}$ for each i , $A_i \cap A_j = \emptyset$ for $i \neq j$ and $A_i \in \Sigma_f$ for each i , then

$$\mu_f(\lambda) = \sum_{j=1}^n m_j \chi_{[a_{j+1}, a_j)}$$

where $m_j = \sum_{i=1}^j \mu(A_i)$ and $a_{n+1} := 0$.

The following proposition highlights the properties of distribution functions.

PROPOSITION 1.3.3. (cf. [2, p.37]) Let $f, g \in L^0(\mu)$, $(f_n)_{n=1}^\infty \subset L^0(\mu)$ and $\alpha \in \mathbb{F}$. The distribution function μ_f is positive, decreasing and right continuous on $[0, \infty)$. Furthermore,

- (1) $|f| \leq |g|$ μ -a.e. implies that $\mu_f \leq \mu_g$
- (2) $\mu_{\alpha f}(\lambda) = \mu_f(\lambda/|\alpha|)$ for all $\lambda \geq 0$
- (3) $\mu_{f+g}(\lambda_1 + \lambda_2) \leq \mu_f(\lambda_1) + \mu_g(\lambda_2)$ for all $\lambda_1, \lambda_2 \geq 0$

- (4) $|f| \leq \liminf_{n \rightarrow \infty} |f_n|$ μ -a.e. implies that $\mu_f \leq \liminf_{n \rightarrow \infty} \mu_{f_n}$
(5) $|f_n| \uparrow |f|$ μ -a.e. implies that $\mu_{f_n} \uparrow \mu_f$
(6) If f is bounded except possibly on a set of finite measure, then $\mu_f(\lambda) \rightarrow 0$ as $\lambda \rightarrow \infty$.

PROOF. (6) will be shown; the remainder of the proofs can be found in [2, p.37]. Assume that $\mu_f(\lambda) \not\rightarrow 0$ as $\lambda \rightarrow \infty$, i.e. there exists an $\epsilon > 0$ such that for all $\lambda \in \mathbb{R}^+$ there exists an $\alpha_\lambda \geq \lambda$ such that $\mu_f(\alpha_\lambda) \geq \epsilon$. Therefore

$$(1.3.1) \quad \mu_f(\lambda) \geq \mu_f(\alpha_\lambda) \geq \epsilon \quad \forall \lambda \in \mathbb{R}^+$$

since μ_f is decreasing. Since f is bounded, except perhaps on a set of finite measure, we can find an $M > 0$ and an $A \in \Sigma_f$ such that $|f(x)| \leq M$ for all $x \in A^c$. For $n \in \mathbb{N}$, let $A_n = \{x \in \Omega : |f(x)| > M + n\}$, then $\mu(A_0) < \infty$, since $A_0 \subset A$, and furthermore $A_{n+1} \subset A_n$ for all $n \in \mathbb{N}$. So

$$\begin{aligned} \mu(\{x \in \Omega : |f(x)| = \infty\}) &= \mu\left(\bigcap_{n=0}^{\infty} A_n\right) \\ &= \lim_{n \rightarrow \infty} \mu(A_n) \quad \text{by Theorem 1.1.3} \\ &\geq \epsilon \quad \text{since } \mu(A_n) \geq \epsilon \forall n \in \mathbb{N} \text{ by (1.3.1)} \end{aligned}$$

Therefore $f \notin L^0(\mu)$, which is a contradiction. \square

1.3.2. Decreasing rearrangements. We start by defining the decreasing rearrangement of a function and giving explicit descriptions for the decreasing rearrangements of functions frequently encountered in the sequel. Important properties of decreasing rearrangements are then listed and the norms of L^p spaces are expressed in terms of decreasing rearrangements. Subsequently, pointwise convergence of a sequence of decreasing rearrangements will be characterized and resonant measure spaces introduced.

The *decreasing rearrangement* of a function $f \in L^0(\mu)$ is the function f^* given by

$$f^*(t) := \inf\{\lambda > 0 : \mu_f(\lambda) \leq t\} \quad \text{for any } t \geq 0$$

The decreasing rearrangements of the zero function, characteristic functions and simple functions are given in the following example.

EXAMPLE 1.3.4. [2, p.39]

- (1) If $f = 0$, then $f^* = 0$
(2) If $A \in \Sigma$, then $(\chi_A)^* = \chi_{[0, \mu(A))}$
(3) If $f = \sum_{i=1}^n \alpha_i \chi_{A_i}$, where the A_i 's are pairwise disjoint. Then

$$f^* = \sum_{i=1}^n \alpha_i \chi_{[k_{i-1}, k_i)}$$

where $k_j = \sum_{i=1}^j \mu(A_i)$ and $k_0 = 0$.

The following proposition details some of the important properties of decreasing rearrangements.

PROPOSITION 1.3.5. Let $f, g \in L^0(\mu)$, $(f_n)_{n=1}^{\infty} \subset L^0(\mu)$ and $\alpha \in \mathbb{F}$. The decreasing rearrangement f^* is positive, decreasing and right continuous on $[0, \infty)$. Furthermore,

- (1) $|f| \leq |g|$ μ -a.e. implies that $f^* \leq g^*$
(2) $(\alpha f)^* = |\alpha| f^*$
(3) $(f + g)^*(t_1 + t_2) \leq f^*(t_1) + g^*(t_2)$ for all $t_1, t_2 \geq 0$
(4) $|f| \leq \liminf_{n \rightarrow \infty} |f_n|$ μ -a.e. implies that $f^* \leq \liminf_{n \rightarrow \infty} f_n^*$

- (5) $|f_n| \uparrow |f|$ μ -a.e. implies that $f_n^* \uparrow f^*$
(6) If g is such that $\mu_g(\lambda) < \infty$ for all $\lambda > 0$, $|f_n| \leq |g|$ μ -a.e. for all n and $f_n(x) \rightarrow f(x)$ μ -a.e., then $(f - f_n)^*(t) \rightarrow 0$ for all $t > 0$
(7) f and f^* are equimeasurable
(8) f and g equimeasurable implies that $f^* = g^*$
(9) $f^*(\mu_f(\lambda)) \leq \lambda$ for $\mu_f(\lambda) < \infty$ and $\mu_f(f^*(t)) \leq t$ for $f^*(t) < \infty$
(10) If f is a decreasing positive function on $(0, \infty)$, then $f^* = f$.
(11) If there exists an $M > 0$ such that $\mu_g(\lambda) \leq M\mu_f(\lambda)$ for all $\lambda > 0$, then $g^*(t) \leq f^*(t/M)$ for all $t \geq 0$.
(12) $f^*(t) < \infty$ for all $t > 0$ if and only if f is bounded except possibly on a set of finite measure
(13) $f^*(t) \rightarrow 0$ as $t \rightarrow \infty$ if and only if $\mu_f(\lambda) < \infty$ for all $\lambda > 0$
(14) Let $\epsilon, \delta > 0$, then $f^*(\delta) \leq \epsilon$ if and only if $\mu_f(\epsilon) \leq \delta$.

PROOF. The proofs of (1) to (10) are given in [2, p.41,42] and [18, p.63-68]. The remainder of the proofs will be shown here.

(11): Since $\mu_g(\lambda) \leq M\mu_f(\lambda)$ for all $\lambda > 0$, we have that

$$\{\lambda > 0 : M\mu_f(\lambda) \leq t\} \subset \{\lambda > 0 : \mu_g(\lambda) \leq t\} \quad \forall t \geq 0$$

and so

$$\begin{aligned} g^*(t) &= \inf \{\lambda > 0 : \mu_g(\lambda) \leq t\} \\ &\leq \inf \{\lambda > 0 : M\mu_f(\lambda) \leq t\} \\ &= \inf \left\{ \lambda > 0 : \mu_f(\lambda) \leq \frac{t}{M} \right\} \\ &= f^*(t/M) \end{aligned}$$

(12): (Sufficiency): Let $t > 0$. If f is bounded, except possibly on a set of finite measure, then $\mu_f(\lambda) \rightarrow 0$ as $\lambda \rightarrow \infty$ by Proposition 1.3.3, and so there exists a $\lambda_t \in \mathbb{R}^+$ such that $\mu_f(\lambda_t) < t$. So

$$\begin{aligned} f^*(t) &= \inf \{\lambda \in \mathbb{R}^+ : \mu_f(\lambda) \leq t\} \\ &\leq \lambda_t \\ &< \infty \end{aligned}$$

(Necessity): Let $f^*(t) < \infty$ for all $t > 0$ and let $A = \{\lambda \in \mathbb{R}^+ : \mu_f(\lambda) \leq 1\}$, then $\inf A = f^*(1) < \infty$ and so $A \neq \emptyset$. Let $\lambda_1 \in A$ and $B = \{x \in \Omega : |f(x)| > \lambda_1\}$. Then $\mu(B) = \mu_f(\lambda_1) \leq 1$ and so $|f|$ is bounded by λ_1 , except possibly on B , which is a set of finite measure.

(13): (Sufficiency): Let $\epsilon > 0$ and let $\mu_f(\lambda) < \infty$ for all $\lambda > 0$. Let $n_\epsilon \in \mathbb{N}^+$ such that $\frac{1}{n_\epsilon} < \epsilon$ and let $\alpha_{n_\epsilon} = \mu_f(1/n_\epsilon)$. Since μ_f is finite-valued, $\alpha_{n_\epsilon} < \infty$ and so there exists a $t_\epsilon \in \mathbb{R}^+$ such that $t_\epsilon > \alpha_{n_\epsilon}$. Then

$$\begin{aligned} f^*(t_\epsilon) &= \inf \{\lambda \in \mathbb{R}^+ : \mu_f(\lambda) \leq t_\epsilon\} \\ &\leq \frac{1}{n_\epsilon} \quad \text{since } \mu_f(1/n_\epsilon) < t_\epsilon \\ &< \epsilon \end{aligned}$$

And so $f^*(t) < \epsilon$ for all $t \geq t_\epsilon$ since f^* is decreasing.

(Necessity): We show the contrapositive. Assume there exists some $\lambda_\infty > 0$ such that $\mu_f(\lambda_\infty) = \infty$ and hence $\mu_f(\lambda) = \infty$ for all $\lambda \leq \lambda_\infty$, since μ_f is decreasing. Then for any $t > 0$

$$\begin{aligned} f^*(t) &= \inf \{\lambda > 0 : \mu_f(\lambda) \leq t\} \\ &\geq \lambda_\infty \quad \text{since } \{\lambda > 0 : \mu_f(\lambda) \leq t\} \subset (\lambda_\infty, \infty) \end{aligned}$$

and hence $f^*(t) \rightarrow 0$ as $t \rightarrow \infty$.

(14): (Sufficiency):

$$\begin{aligned} \mu_f(\epsilon) &\leq \delta \\ \Rightarrow \inf\{\lambda > 0 : \mu_f(\lambda) \leq \delta\} &\leq \epsilon \\ \Rightarrow f^*(\delta) &\leq \epsilon \end{aligned}$$

(Necessity):

$$\begin{aligned} f^*(\delta) &\leq \epsilon \\ \Rightarrow \mu_f(\epsilon) &\leq \mu_f(f^*(\delta)) \quad \text{since } \mu_f \text{ is decreasing} \\ &\leq \delta \quad \text{by Proposition 1.3.3} \end{aligned}$$

□

REMARK 1.3.6. We have by the Proposition above that $f^*(t) < \infty$ for all $t > 0$ if and only if f is bounded, except possibly on a set of finite measure. It follows that

$$L^{00}(\mu) = \{f \in L^0(\Omega, \Sigma, \mu) : f^*(t) < \infty \forall t > 0\}$$

We will often use this implicitly when dealing with functions in $L^{00}(\mu)$.

The following proposition provides a description of the L^p norm of a function in terms of its decreasing rearrangement.

PROPOSITION 1.3.7. [2, p.43] Let $f \in L^0(\mu)$

(1) If $0 < p < \infty$, then

$$\|f\|_p = \left(\int_0^\infty (f^*(t))^p dt \right)^{1/p}$$

(2) If $p = \infty$, then

$$\|f\|_p = f^*(0)$$

The following result relates the convergence in measure of a sequence of functions to the pointwise convergence of the sequence of decreasing rearrangements.

PROPOSITION 1.3.8. Let $(f_n)_{n=1}^\infty \subset L^{00}(\mu)$. $f_n \rightarrow 0$ in measure if and only if $f_n^*(t) \rightarrow 0$ as $n \rightarrow \infty$ for all $t > 0$.

PROOF. Let $f_n^*(t) \rightarrow 0$ for all $t > 0$ and let $\epsilon, \delta > 0$. $f_n^*(\frac{\delta}{2}) \rightarrow 0$ implies that there exists a $n_{\epsilon, \delta} \in \mathbb{N}^+$ such that $n \geq n_{\epsilon, \delta}$ implies

$$\begin{aligned} f_n^*\left(\frac{\delta}{2}\right) &< \epsilon \\ \Rightarrow \mu_{f_n}(\epsilon) &\leq \frac{\delta}{2} \quad \text{by Proposition 1.3.5} \\ \Rightarrow \mu(\{x \in \Omega : |f_n(x) - 0| > \epsilon\}) &< \delta \end{aligned}$$

Since this holds for all $n \geq n_{\epsilon, \delta}$ and $\epsilon, \delta > 0$ were arbitrary, it follows that $f_n \rightarrow 0$ in measure.

Conversely, if $f_n \rightarrow 0$ in measure and $t, \epsilon > 0$, then there exists a $n_{\epsilon, t} \in \mathbb{N}^+$ such that $n \geq n_{\epsilon, t}$ implies

$$\begin{aligned} \mu \left(\left\{ x \in \Omega : |f_n(x) - 0| > \frac{\epsilon}{2} \right\} \right) &\leq t \\ \Rightarrow \mu_{f_n} \left(\frac{\epsilon}{2} \right) &\leq t \\ \Rightarrow f_n^*(t) &\leq \frac{\epsilon}{2} < \epsilon \quad \text{by Proposition 1.3.5} \\ \Rightarrow f_n^*(t) &\rightarrow 0 \text{ as } n \rightarrow \infty \end{aligned}$$

□

Hardy and Littlewood showed ([2, p.44]) that if $f, g \in L^0(\mu)$, then

$$(1.3.2) \quad \int_{\Omega} |fg| d\mu \leq \int_0^{\infty} f^*(s)g^*(s) ds$$

It follows from this that if \tilde{g} and g are equimeasurable, then

$$\int_{\Omega} |f\tilde{g}| d\mu \leq \int_0^{\infty} f^*(s)g^*(s) ds$$

since $(\tilde{g})^* = g^*$ by 1.3.5. It turns out that spaces where

$$\sup \left\{ \int_{\Omega} |f\tilde{g}| d\mu : \tilde{g} \sim g \right\} = \int_0^{\infty} f^*(s)g^*(s) ds \quad \forall f, g \in L^0(\mu)$$

have a richer structure than those spaces not having this property. These spaces are called *resonant spaces*. If the supremum on the left is attained, then the space is called *strongly resonant*. The significance of resonant measure spaces will be discussed in Section 4.1. We provide a characterization of resonant measure spaces.

THEOREM 1.3.9. [2, p.51] *A σ -finite measure space (Ω, Σ, μ) is resonant if and only if it is either nonatomic or purely atomic with all atoms having the same measure.*

1.3.3. The maximal function. The operation $f \mapsto f^*$ introduced in the previous section is not necessarily subadditive (i.e. we do not necessarily have $(f+g)^* \leq f^* + g^*$) as can be seen by considering $f = \chi_{[0,1]}$ and $g = \chi_{[1,2]}$. However, the operator which takes a function to its maximal function is subadditive. We define the maximal function and list some of its properties.

Let $A \in \Sigma$ be such that $\mu(A) = t > 0$. Applying the Hardy-Littlewood inequality (1.3.2), we obtain

$$(1.3.3) \quad \begin{aligned} \int_{\Omega} |f\chi_A| d\mu &\leq \int_0^{\infty} f^*(s)(\chi_A)^*(s) ds \\ &= \int_0^{\mu(A)} f^*(s) ds \quad \text{by Example 1.3.4} \end{aligned}$$

We therefore have that $\frac{1}{\mu(A)} \int_A |f| d\mu \leq \frac{1}{t} \int_0^t f^*(s) ds$. This shows that the average of $|f|$ over any set of measure t is dominated by the corresponding average of f^* over $(0, t)$. Furthermore, since f^* is decreasing, the latter average is maximal among all averages of f^* taken over sets of measure t . We define the *maximal function* of f , denoted f^{**} , by

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(s) ds \quad t > 0$$

EXAMPLE 1.3.10. Let $A \in \Sigma$ with $\mu(A) = t > 0$, then

$$(\chi_A)^{**}(s) = \begin{cases} 1 & \text{if } s \leq \mu(A) \\ \mu(A)/s & \text{if } s > \mu(A) \end{cases}$$

The maximal operator $f \mapsto f^{**}$ has the following properties:

PROPOSITION 1.3.11. [2, p.52] Let $f, g \in L^0(\mu)$, $(f_n)_{n=1}^\infty \subset L^0(\mu)$ and $\alpha \in \mathbb{F}$. Then f^{**} is positive, decreasing and continuous on $(0, \infty)$. Furthermore,

- (1) $f^{**} \equiv 0$ if and only if $f = 0$ μ -a.e.
- (2) $f^* \leq f^{**}$
- (3) $|f| \leq |g|$ μ -a.e. implies that $f^{**} \leq g^{**}$
- (4) $(\alpha f)^{**} = |\alpha| f^{**}$
- (5) $|f_n| \uparrow |f|$ μ -a.e. implies that $f_n^{**} \uparrow f^{**}$
- (6) $(f + g)^{**}(t) \leq f^{**}(t) + g^{**}(t)$ for all $t > 0$
- (7) If (Ω, Σ, μ) is resonant, then

$$f^{**}(t) = \frac{1}{t} \sup \left\{ \int_E |f| d\mu : \mu(E) = t \right\} \quad \forall t \in \text{ran } \mu$$

- (8) If $f^*(0) < \infty$, then $f^{**}(t) \rightarrow f^*(0)$ as $t \rightarrow 0$.
- (9) if f and g are decreasing positive functions on $(0, \infty)$, then $(f + g)^{**} = f^{**} + g^{**}$

PROOF. The proofs of (1) to (7) are given in [2, p.52-54]. We show (8) and (9).

(8): Since f^{**} is decreasing, $(f^{**}(1/n))_{n=1}^\infty$ is an increasing sequence. Furthermore, since f^* is decreasing, it is easily checked that $f^*(0) \geq f^{**}(1/n)$ for each $n \in \mathbb{N}^+$, i.e. $(f^{**}(1/n))_{n=1}^\infty$ is bounded above. Let $\epsilon > 0$. f^* is right-continuous on $[0, \infty)$ and thus there exists a $\delta > 0$ such that $0 \leq t < \delta$ implies that $f^*(0) - f(t) < \epsilon$. It follows that if we let $n_\epsilon \in \mathbb{N}^+$ such that $\frac{1}{n_\epsilon} < \delta$, then $n \geq n_\epsilon$ implies that

$$f^{**}(1/n) = n \int_0^{1/n} f^*(s) ds \geq n \left(f^*(1/n) \cdot \frac{1}{n} \right) > f^*(0) - \epsilon$$

Therefore $f^{**}(1/n) \rightarrow f^*(0)$. It follows by the continuity of f^{**} on $(0, \infty)$ that $f^{**}(t) \rightarrow f^*(0)$ as $t \rightarrow 0$.

(9): If f and g are positive decreasing functions then so is $f + g$. By Proposition 1.3.5, $(f + g)^* = (f + g)$. Thus for any $t > 0$,

$$\begin{aligned} (f + g)^{**}(t) &= \frac{1}{t} \int_0^t (f + g)^*(s) ds \\ &= \frac{1}{t} \int_0^t (f + g)(s) ds \\ &= \frac{1}{t} \int_0^t f(s) ds + \frac{1}{t} \int_0^t g(s) ds \\ &= f^{**}(t) + g^{**}(t) \quad \text{by Proposition 1.3.5} \end{aligned}$$

□

The order relationships of maximal functions will be significant. We will therefore write $f \prec g$ if $f^{**} \leq g^{**}$. The relation \prec is called the *Hardy-Littlewood-Piȓelya relation*. We will give two results in terms of the Hardy-Littlewood-Piȓelya relation, but first we state Hardy's lemma.

PROPOSITION 1.3.12 (Hardy's lemma). [2, p.56] Let f, g and h be positive measurable functions on $(0, \infty)$. If h is decreasing and

$$\int_0^t f(s) ds \leq \int_0^t g(s) ds \quad \forall t > 0$$

then

$$\int_0^\infty f(s)h(s) ds \leq \int_0^\infty g(s)h(s) ds$$

COROLLARY 1.3.13. Let $f, g, h \in L^0(\mu)$. If $f \prec g$, then

$$\int_0^\infty f^*(s)h^*(s) ds \leq \int_0^\infty g^*(s)h^*(s) ds$$

PROOF. f^*, g^* and h^* are positive measurable functions on $(0, \infty)$. Furthermore h^* is decreasing and

$$\int_0^t f^*(s) ds \leq \int_0^t g^*(s) ds \quad \forall t > 0 \text{ since } f \prec g$$

The result therefore follows by Hardy's lemma. \square

LEMMA 1.3.14. (cf. [2, p.76]) *Let f and g be nonnegative decreasing functions on $(0, \infty)$, then*

$$\int_0^\infty fg ds \leq \left(\int_0^1 g ds \right) \times \left(\sup_{0 < t < \infty} \frac{1}{\min\{1, t\}} \int_0^t f(s) ds \right)$$

PROPOSITION 1.3.15. [2, p.57] *Let (Ω, Σ, μ) be a resonant measure space and let $(A_n)_{n \in \mathbb{N}}$ be a collection of pairwise disjoint subsets of Ω , each with finite positive measure and let $A = \Omega \setminus \bigcup_{n \in \mathbb{N}} A_n$. If $f \in L^0(\mu)$ such that f is integrable on each A_n and*

$$Tf := f\chi_A + \sum_{n \in \mathbb{N}} \left(\frac{1}{\mu(A_n)} \int_{A_n} f d\mu \right) \chi_{A_n}$$

then $Tf \prec f$.

1.4. Topological vector spaces

We describe some of the basic features of topological vector spaces. These will be used when defining topologies on $L^0(\mu)$ and other related spaces. We also include Alaoglu's theorem and a result regarding the weak-star metrizable of the unit ball in a dual space, which will be used in the characterization of the conditions under which a composition operator is compact on an Orlicz space.

A topology on a vector space is called a *linear* or *vector space topology* if it is compatible with the underlying vector space structure in the sense that the operations of addition and scalar multiplication are continuous with respect to this topology. Let $A, B \subset E$ where E is a vector space. If there exists an $\alpha > 0$ such that $B \subset \lambda A$ for all $\lambda \in \mathbb{F}$ such that $|\lambda| \geq \alpha$ then A is said to *absorb* B . If A absorbs points then A is called *absorbing*. If A is a subset of a topological vector space E , and A is absorbed by every neighbourhood (or equivalently every basic neighbourhood) of the origin, then A is called *bounded*.

We give some basic results regarding topological vector spaces.

THEOREM 1.4.1. [13, p.81] *If E is a topological vector space, then there exists a basis \mathcal{U} for the neighbourhoods of the origin such that:*

- (1) *every $N \in \mathcal{U}$ is absorbing;*
- (2) *every $N \in \mathcal{U}$ is balanced;*
- (3) *for every $N \in \mathcal{U}$ there exists an $M \in \mathcal{U}$ such that $M + M \subset N$.*

Conversely, if E is a vector space over \mathbb{F} and \mathcal{U} is a filter basis on E satisfying the three conditions above, then there is a unique vector space topology on E such that \mathcal{U} is a basis for the neighbourhoods of the origin.

PROPOSITION 1.4.2. [13, p.84] *In a topological vector space E every neighbourhood of the origin contains a closed neighbourhood of the origin.*

COROLLARY 1.4.3. [13, 84] *If a topological vector space is Hausdorff, then it is regular.*

Next we state Alaoglu's theorem and the result concerning the weak-star metrizable of the unit ball in a dual space.

THEOREM 1.4.4 (Alaoglu). [3, p.130] *If E is a normed space, then B_{E^*} is weak-star compact.*

THEOREM 1.4.5. [3, p.134] *If E is a Banach space, then B_{E^*} is weak-star metrizable if and only if E is separable.*

1.5. Operators

1.5.1. Positive operators. Let $F(\Omega)$ be a vector space of real or complex-valued functions on a set Ω . We want to show that if $F(\Omega)$ is equipped with a suitable topology and $\tau : \Omega \rightarrow \Omega$ is such that $C_\tau(F(\Omega)) \subset F(\Omega)$, then C_τ is continuous with respect to the topology on $F(\Omega)$. The required result does not appear in the literature in the form we desire. Several results appearing in the literature are therefore quoted (without proof) and it is shown how these may be manipulated to obtain the desired result. We start by providing some basic definitions and results for Riesz spaces.

Let $A \subset E$, where (E, \leq) is a partially ordered set. If A has both upper bounds and lower bounds then A is called *order-bounded*. Let $(E, +, \cdot)$ be a linear space over \mathbb{R} and \leq a partial ordering on E . $(E, +, \cdot, \leq)$ is called a *partially ordered linear space* if for every $x, y, z \in E$ and $\alpha \in \mathbb{R}$ such that $\alpha \geq 0$ the following conditions hold:

- (1) $x \leq y \Rightarrow x + z \leq y + z$
- (2) $0 \leq x \Rightarrow 0 \leq \alpha x$

If $(E, +, \cdot, \leq)$ is a partially ordered linear space such that (E, \leq) is a lattice, then E is called a *Riesz space* or *vector lattice*. We write $x \vee y$ and $x \wedge y$ for $\sup\{x, y\}$ and $\inf\{x, y\}$ respectively. Furthermore, we define

$$x^+ := x \vee 0, \quad x^- := (-x) \vee 0, \quad |x| := x \vee (-x)$$

Let E be a Riesz space. A set $A \subset E$ is *solid* if $|y| \leq |x|$ and $x \in A$ implies that $y \in A$. It is easily checked that if A is a solid set, then so is λA for any $\lambda \in \mathbb{F}$. If A is a vector subspace of E , then A is called an *order ideal* if A is solid and a *Riesz subspace* if A is a sublattice of E . A vector space topology \mathcal{T} on a Riesz space is called *locally solid* if there is a neighbourhood basis of the origin consisting of solid sets. \mathcal{T} is *compatible* with the order structure on E if $E^+ := \{x \in E : 0 \leq x\}$ is closed in this topology. Finally, let $T : E_1 \rightarrow E_2$ be a linear map between Riesz spaces. If $0 \leq Tx$ whenever $0 \leq x$ then T is called a *positive* linear map. We are now in a position to state the result which will form the basis for showing that a composition operator is automatically continuous.

PROPOSITION 1.5.1. [1, p.111] *Let $T : E_1 \rightarrow E_2$ be a linear map where (E_1, \mathcal{T}_1) is a Riesz space with a compatible complete metrizable vector space topology and (E_2, \mathcal{T}_2) is a Riesz space with a vector topology such that order-bounded sets are bounded. If T is positive, then T is continuous with respect to the topologies on E_1 and E_2 .*

The desired modifications are captured by the following two results.

PROPOSITION 1.5.2. [8, p.41] *Let (E, \mathcal{T}) be a Riesz space equipped with a locally solid Hausdorff vector space topology. Then \mathcal{T} is compatible with the order structure of E .*

PROPOSITION 1.5.3. [8, p.38] *Let E be a locally solid Riesz space. Then order-bounded sets are bounded.*

COROLLARY 1.5.4. *Let $T : E \rightarrow E$ be a linear map where (E, \mathcal{T}) is a complete Hausdorff metrizable locally solid Riesz space. If T is positive, then T is continuous with respect to the topology \mathcal{T} .*

Since it is easily shown that composition operators are positive, we apply this result to composition operators and show that if τ induces a composition operator on an appropriate function space, then the induced composition operator is automatically continuous. Note first that if we let $F(\Omega)$ be a vector space of real or complex-valued functions and $F_{\mathbb{R}}(\Omega) := \{f \in F(\Omega) : \text{ran} f \subset \mathbb{R}\}$, then $F_{\mathbb{R}}(\Omega)$ is not generally a vector subspace of $F(\Omega)$, but can be viewed as a vector space over \mathbb{R} . As a result some care is required to show that if C_τ is a composition operator then it is automatically continuous. Specifically,

PROPOSITION 1.5.5. *Let $F(\Omega)$ be a vector space of real or complex-valued functions equipped with a metrizable linear topology \mathcal{T} such that the conjugation map $f \mapsto \bar{f}$ is continuous and such that the real vector space $F_{\mathbb{R}}(\Omega)$,*

equipped with the subspace topology $(\mathcal{T} \cap F_{\mathbb{R}}(\Omega))$, and a partial order \leq , is a complete metrizable locally solid Riesz space. If $\tau : \Omega \rightarrow \Omega$ is a map such that $C_{\tau}(F(\Omega)) \subset F(\Omega)$, then C_{τ} is continuous with respect to \mathcal{T} .

PROOF. Let $(f_n)_{n=1}^{\infty} \subset F(\Omega)$ be such that $f_n \rightarrow f$ in $F(\Omega)$. For each $n \in \mathbb{N}^+$ we can write $f_{1,n} = \frac{f_n + \overline{f_n}}{2}$ and $f_{2,n} = \frac{f_n - \overline{f_n}}{2i}$; then $f_n = f_{1,n} + if_{2,n}$ and $f_{1,n}, f_{2,n} \in F_{\mathbb{R}}(\Omega)$. Similarly, let $f_1 = \frac{f + \overline{f}}{2}$ and $f_2 = \frac{f - \overline{f}}{2i}$. Since the operations of conjugation, addition and scalar multiplication are continuous, we have that $f_{j,n} \rightarrow f_j$ for $j = 1, 2$. Since $C_{\tau}(F(\Omega)) \subset F(\Omega)$ and C_{τ} maps real-valued functions onto real-valued functions, we have that $C_{\tau}(F_{\mathbb{R}}(\Omega)) \subset F_{\mathbb{R}}(\Omega)$. We can therefore think of C_{τ} as a positive real linear map on $F_{\mathbb{R}}(\Omega)$. $(F_{\mathbb{R}}(\Omega), +, \cdot, \leq, \mathcal{T} \cap F_{\mathbb{R}}(\Omega))$ is a complete metrizable locally solid Riesz space. It follows by Corollary 1.5.4 that C_{τ} is continuous on $F_{\mathbb{R}}(\Omega)$. Therefore

$$\begin{aligned} C_{\tau}(f_{j,n}) &\rightarrow C_{\tau}f_j && \text{for } j = 1, 2 \\ \Rightarrow C_{\tau}(f_n) &= C_{\tau}(f_{1,n}) + iC_{\tau}(f_{2,n}) && \text{since } C_{\tau} \text{ is linear} \\ &\rightarrow C_{\tau}(f_1) + iC_{\tau}(f_2) && \text{since } C_{\tau}(f_{j,n}) \rightarrow C_{\tau}(f_j) \text{ and } F(\Omega) \text{ is equipped with a vector topology} \\ &= C_{\tau}(f) \end{aligned}$$

□

In the sequel we will be focusing on composition operators on Banach spaces of measurable functions. We show briefly that the topology induced by the norm is suitable for transferring the result above to composition operators on these spaces.

Let (E, \leq) be a Riesz space. If $\|\cdot\|$ is a norm on E such that $x, y \in E$, $|x| \leq |y| \Rightarrow \|x\| \leq \|y\|$, then $\|\cdot\|$ is called a lattice norm and $(E, \leq, \|\cdot\|)$ is called a *normed Riesz space*. If in addition $(E, \|\cdot\|)$ is complete, then $(E, \leq, \|\cdot\|)$ is called a *Banach lattice*.

REMARK 1.5.6. If $(E, \leq, \|\cdot\|)$ is a normed Riesz space, then E is a Hausdorff metrizable locally solid Riesz space. This follows since any norm topology is Hausdorff and if we let $\mathcal{N} := \{B(0, \frac{1}{n}) : n \in \mathbb{N}^+\}$, then \mathcal{N} forms a countable basis for the neighbourhoods of zero. Furthermore, it is easily checked that each of these sets is solid and hence E is a metrizable locally solid Riesz space.

1.5.2. Compact operators. The compactness of composition operators on Banach function spaces will be studied. We therefore present a few results that can be used to evaluate the compactness of an operator. We start by recalling the definition of a compact operator. Let E and F be Banach spaces and $T : E \rightarrow F$ a linear transformation. If $\overline{T(B_E)}$ is compact in F , then T is called a *compact operator*.

REMARK 1.5.7. It is easily shown that if T is a compact operator then $T \in \mathcal{B}(E, F)$. Furthermore if $T : E \rightarrow E$ is a compact operator and A is a closed subspace of E , invariant under T , then $T \upharpoonright A$ is compact.

Recall that in a normed space, a set A is compact if every sequence in A has a subsequence which converges to a point in A . Furthermore, any scalar multiple of a compact set is again a compact set. We therefore have the following simple way of showing that an operator is not compact.

PROPOSITION 1.5.8. *Let E and F be Banach spaces, $\alpha > 0$ and $T : E \rightarrow F$ a linear operator. If there exists a sequence $(x_n)_{n=1}^{\infty} \subset \alpha B_E$ such that $(Tx_n)_{n=1}^{\infty}$ does not contain a Cauchy subsequence, then T is not compact.*

If $T \in \mathcal{B}(E, F)$ and $\|Tx_n - Tx\|_F \rightarrow 0$ for every sequence $(x_n)_{n=1}^{\infty} \subset E$ such that $x_n \rightarrow x$ weakly for some $x \in E$, then T is called *completely continuous*. The relationship between compact operators and completely continuous operators is given by the following result.

PROPOSITION 1.5.9. [3, p.173] *Let E and F be Banach spaces and let $T \in \mathcal{B}(E, F)$. If T is a compact operator then T is completely continuous. Conversely, if E is reflexive and T is completely continuous, then T is compact.*

REMARK 1.5.10. In general, completely continuous operators are not necessarily compact.

A linear map $T : E \rightarrow F$ between Banach spaces is called a *finite rank operator* if the dimension of the range of T is finite. Let $\mathcal{B}_0(E, F)$ denote the set of all compact operators from E into F and let $\mathcal{B}_{00}(E, F)$ denote the set of all finite rank operators. The following result shows that $\overline{\mathcal{B}_{00}(E, F)} \subset \mathcal{B}_0(E, F)$.

PROPOSITION 1.5.11. [3, p.174] *Let E and F be Banach spaces. If $(T_n)_{n=1}^{\infty}$ is a sequence of finite rank operators from E into F such that $T_n \rightarrow T \in \mathcal{B}(E, F)$ in the operator norm, then T is a compact operator.*

PROPOSITION 1.5.12. [3, p.177] *Let E be a normed space and $T : E \rightarrow E$ a compact operator. If T has a bounded inverse, then E is finite dimensional.*

PROOF. T^{-1} is linear and bounded, therefore

$$\begin{aligned}
 T^{-1} \left(\|T\|^{-1} B_E \right) &= \|T\|^{-1} T^{-1}(B_E) \\
 &\subset \|T\|^{-1} \|T^{-1}\| B_E \\
 (1.5.1) \qquad &= \alpha B_E \quad \text{for some } 0 < \alpha < \infty, \text{ since } 0 < \|T\|, \|T^{-1}\| < \infty
 \end{aligned}$$

It follows that

$$\begin{aligned}
 \|T\|^{-1} B_E &= T \left(T^{-1} \left(\|T\|^{-1} B_E \right) \right) \quad \text{since } TT^{-1} = I \\
 &\subset T(\alpha B_E) \quad \text{by (1.5.1)} \\
 &\subset \overline{\alpha T(B_E)}
 \end{aligned}$$

$\|T\|^{-1} B_E$ is closed and has been shown to be a subset of a compact set $\overline{\alpha T(B_E)}$ and is therefore compact itself. It follows that B_E is compact and thus E is finite dimensional. \square

We will use the theorem above in conjunction with the inverse mapping theorem, which we state here for ease of reference.

THEOREM 1.5.13 (Inverse mapping theorem). [3, p.91] *Let E and F be Banach spaces and $T : E \rightarrow F$ a bounded linear transformation. If T is bijective then T^{-1} is bounded.*

Spaces of measurable functions

2.1. Topological vector spaces of measurable functions

Recall that $L^0(\mu) = L^0(\Omega, \Sigma, \mu)$ will be used to denote the set of all equivalence classes of real or complex-valued measurable functions (finite almost everywhere) on Ω , where any two functions which are equal μ -a.e. have been identified, and that for simplicity we will use f to denote the equivalence class containing f . In addition to studying the continuity and compactness of composition operators on Banach function spaces in later sections we will also investigate transformations $\tau : \Omega \rightarrow \Omega$ which induce composition operators on $L^0(\mu)$ and on two important subspaces of $L^0(\mu)$ in Section 7.2. To this end we consider the topological and order structures of these spaces. A useful consequence of these results will be that positive operators on such spaces are automatically continuous.

2.1.1. $L^0(\Omega, \Sigma, \mu)$. We define topological and order structures on $L^0(\mu)$. We start by defining a family \mathcal{N} of subsets of $L^0(\mu)$ and showing that there exists a linear topology on $L^0(\mu)$ such that \mathcal{N} forms a basis for the neighbourhoods of the origin. It will then be shown that this topology is complete, metrizable and described by convergence of sequences in measure on sets of finite measure. Finally, in the particular example where $L^0(\mu)$ consists of equivalence classes of real-valued, almost everywhere finite, measurable functions on Ω , it will be demonstrated that a partial order can be defined on $L^0(\mu)$ such that $L^0(\mu)$, equipped with this partial order and the topology of local convergence in measure, is a complete metrizable locally solid Riesz space.

Recall that $\Sigma_f := \{A \in \Sigma : \mu(A) < \infty\}$. For $A \in \Sigma_f$ and $\epsilon, \delta > 0$, let

$$\begin{aligned} N(A, \epsilon, \delta) &:= \{f \in L^0 : \mu(\{x \in A : |f(x)| \geq \epsilon\}) < \delta\} \quad \text{and} \\ \mathcal{N} &:= \{N(A, \epsilon, \delta) : A \in \Sigma_f, \epsilon, \delta > 0\} \end{aligned}$$

We list some of the properties of these subsets of $L^0(\mu)$.

PROPOSITION 2.1.1. *Let \mathcal{N} be as defined above, then*

- (1) $A_1, A_2 \in \Sigma$ with $A_1 \subset A_2 \Rightarrow N(A_2, \epsilon, \delta) \subset N(A_1, \epsilon, \delta)$ for all $\epsilon, \delta > 0$
- (2) $0 < \epsilon_1 \leq \epsilon_2 \Rightarrow N(A, \epsilon_1, \delta) \subset N(A, \epsilon_2, \delta)$ for all $A \in \Sigma_f, \delta > 0$
- (3) $0 < \delta_1 \leq \delta_2 \Rightarrow N(A, \epsilon, \delta_1) \subset N(A, \epsilon, \delta_2)$ for all $A \in \Sigma_f, \epsilon > 0$
- (4) Let $\alpha > 0$ and $N \in \mathcal{N}$, then $f \in \alpha N \Rightarrow f \in \lambda N$ for every $\lambda \in \mathbb{F}$ such that $|\lambda| \geq \alpha$

PROOF. 1) Let $A_1, A_2 \in \Sigma_f$ with $A_1 \subset A_2$, $\epsilon, \delta > 0$ and $f \in N(A_2, \epsilon, \delta)$, then

$$\begin{aligned} \{x \in A_1 : |f(x)| \geq \epsilon\} &\subset \{x \in A_2 : |f(x)| \geq \epsilon\} \\ \Rightarrow \mu(\{x \in A_1 : |f(x)| \geq \epsilon\}) &\leq \mu(\{x \in A_2 : |f(x)| \geq \epsilon\}) \\ &< \delta \quad \text{since } f \in N(A_2, \epsilon, \delta) \end{aligned}$$

and so $f \in N(A_1, \epsilon, \delta)$

2) Let $0 < \epsilon_1 \leq \epsilon_2$, $A \in \Sigma_f$ and $f \in N(A, \epsilon_1, \delta)$, then

$$\begin{aligned} \{x \in A : |f(x)| \geq \epsilon_2\} &\subset \{x \in A : |f(x)| \geq \epsilon_1\} \\ \Rightarrow \mu(\{x \in A : |f(x)| \geq \epsilon_2\}) &\leq \mu(\{x \in A : |f(x)| \geq \epsilon_1\}) \\ &< \delta \quad \text{since } f \in N(A, \epsilon_1, \delta) \end{aligned}$$

and so $f \in N(A, \epsilon_2, \delta)$

3) Let $0 < \delta_1 \leq \delta_2$, $A \in \Sigma_f$ and $f \in N(A, \epsilon, \delta_1)$, then

$$\mu(\{x \in A : |f(x)| \geq \epsilon\}) < \delta_1 \leq \delta_2$$

and so $f \in N(A, \epsilon, \delta_2)$

4) Let $\alpha > 0$, $\lambda \in \mathbb{F}$ with $|\lambda| \geq \alpha$, $N = N(A, \epsilon, \delta) \in \mathcal{N}$ and $f \in \alpha N$, then

$$\begin{aligned} \left| \frac{f}{\lambda}(x) \right| &\leq \left| \frac{f}{\alpha}(x) \right| \quad \forall x \in A \\ \Rightarrow \{x \in A : \left| \frac{f}{\lambda}(x) \right| \geq \epsilon\} &\subset \{x \in A : \left| \frac{f}{\alpha}(x) \right| \geq \epsilon\} \\ \Rightarrow \mu\left(\{x \in A : \left| \frac{f}{\lambda}(x) \right| \geq \epsilon\}\right) &\leq \mu\left(\{x \in A : \left| \frac{f}{\alpha}(x) \right| \geq \epsilon\}\right) \\ &< \delta \quad \text{since } \frac{f}{\alpha} \in N \end{aligned}$$

and so $\frac{f}{\lambda} \in N$, i.e. $f \in \lambda N$ □

PROPOSITION 2.1.2. *There is a vector space topology \mathcal{T} on $L^0(\mu)$ such that \mathcal{N} is a neighbourhood basis for the origin.*

PROOF. We show that \mathcal{N} is a filter basis on Ω satisfying the three conditions of Theorem 1.4.1. To show that \mathcal{N} is a filter basis, let $N_1 = N(A_1, \epsilon_1, \delta_1)$, $N_2 = N(A_2, \epsilon_2, \delta_2) \in \mathcal{N}$. Set $A = A_1 \cup A_2$, $\epsilon = \min\{\epsilon_1, \epsilon_2\}$ and $\delta = \min\{\delta_1, \delta_2\}$. Then $N = N(A, \epsilon, \delta) \in \mathcal{N}$, since $\epsilon, \delta > 0$ and

$$\mu(A) = \mu(A_1 \cup A_2) \leq \mu(A_1) + \mu(A_2) < \infty.$$

Furthermore $N \subset N_1 \cap N_2$ since for $i = 1, 2$

$$\begin{aligned} N(A, \epsilon, \delta) &\subset N(A_i, \epsilon, \delta) && \text{by Proposition 2.1.1 (1)} \\ &\subset N(A_i, \epsilon_i, \delta) && \text{by Proposition 2.1.1 (2)} \\ &\subset N(A_i, \epsilon_i, \delta_i) && \text{by Proposition 2.1.1 (3)} \end{aligned}$$

\mathcal{N} is therefore a filter basis.

1) Every $N \in \mathcal{N}$ is balanced: Let $N = N(A, \epsilon, \delta) \in \mathcal{N}$, $f \in N$ and $\lambda \in \mathbb{F}$ such that $|\lambda| \leq 1$. If $\lambda = 0$, then $\lambda f = 0 \in N$. If $0 < |\lambda| \leq 1$, then $f \in \frac{1}{\lambda}N$ by Proposition 2.1.1(4) and so $\lambda f \in N$.

2) For every $N \in \mathcal{N}$ there exists an $M \in \mathcal{N}$ such that $M + M \subset N$: Let $N = N(A, \epsilon, \delta) \in \mathcal{N}$. Then choosing $M = N(A, \frac{\epsilon}{2}, \frac{\delta}{2})$, we obtain $M \in \mathcal{N}$ and for any $f, g \in M$

$$\begin{aligned}
|(f+g)(x)| \geq \epsilon &\Rightarrow |f(x)| \geq \frac{\epsilon}{2} \text{ or } |g(x)| \geq \frac{\epsilon}{2} \\
\Rightarrow \{x \in A : |(f+g)(x)| \geq \epsilon\} &\subset \{x \in A : |f(x)| \geq \frac{\epsilon}{2}\} \cup \{x \in A : |g(x)| \geq \frac{\epsilon}{2}\} \\
\Rightarrow \mu(\{x \in A : |(f+g)(x)| \geq \epsilon\}) &\leq \mu\left(\{x \in A : |f(x)| \geq \frac{\epsilon}{2}\} \cup \{x \in A : |g(x)| \geq \frac{\epsilon}{2}\}\right) \\
&\leq \mu\left(\{x \in A : |f(x)| \geq \frac{\epsilon}{2}\}\right) + \mu\left(\{x \in A : |g(x)| \geq \frac{\epsilon}{2}\}\right) \\
&< \frac{\delta}{2} + \frac{\delta}{2} \quad \text{since } f, g \in M
\end{aligned}$$

and so $f+g \in N$. Since $f, g \in M$ were arbitrary we obtain $M+M \subset N$.

3) Every $N = N(A, \epsilon, \delta) \in \mathcal{N}$ is absorbing: Let $f \in L^0(\mu)$ and let $A_n := \{x \in A : |f(x)| \geq n\}$. Then $\mu(A_n) < \infty$ for each n , since $A_n \subset A$ and $\mu(A) < \infty$. Furthermore $(A_n)_{n=1}^\infty$ is a decreasing sequence and thus by Theorem 1.1.3

$$\begin{aligned}
\lim_{n \rightarrow \infty} \mu(A_n) &= \mu\left(\bigcap_{n=1}^\infty A_n\right) \\
&= \mu(\{x \in A : |f(x)| = \infty\}) \\
&= 0 \quad \text{since } f \in L^0(\mu)
\end{aligned}$$

We can therefore choose an $n_\delta \in \mathbb{N}^+$ such that $\mu(A_{n_\delta}) < \delta$. This implies that

$$\mu\left(\left\{x \in A : \frac{\epsilon}{n_\delta} |f(x)| \geq \epsilon\right\}\right) < \delta$$

and hence $f \in \frac{n_\delta}{\epsilon} N$. Thus by Proposition 2.1.1 (4) $f \in \lambda N$ for all $\lambda \in \mathbb{F}$ with $|\lambda| \geq \frac{n_\delta}{\epsilon}$, i.e. N is absorbing.

It therefore follows by Theorem 1.4.1 that there exists a unique vector space topology \mathcal{T} on L^0 such that \mathcal{N} is a basis for the neighbourhoods of the origin. \square

We want to show that this topology is complete, metrizable and described by convergence in measure on sets of finite measure. To facilitate this we define two other families of subsets of $L^0(\mu)$ and show that these form neighbourhood bases at the origin for the topology \mathcal{T} . Since we have assumed throughout that (Ω, Σ, μ) is a σ -finite measure space we can write $\Omega = \bigcup_{n=1}^\infty \Omega_n$ where $\mu(\Omega_n) < \infty$ and $\Omega_n \subset \Omega_{n+1}$ for every $n \in \mathbb{N}^+$. For $A \in \Sigma_f$ and $\epsilon > 0$, let $N(A, \epsilon) := N(A, \epsilon, \epsilon)$. Put

$$\begin{aligned}
\mathcal{N}_1 &:= \{N(A, \epsilon) : A \in \Sigma_f, \epsilon > 0\} \\
\mathcal{N}_2 &:= \left\{N\left(\Omega_m, \frac{1}{n}\right) : m, n \in \mathbb{N}^+\right\}
\end{aligned}$$

PROPOSITION 2.1.3. *Let \mathcal{T} be the topology on $L^0(\mu)$ defined by \mathcal{N} . \mathcal{N}_1 and \mathcal{N}_2 are neighbourhood bases of the origin for \mathcal{T} .*

PROOF. It is clear that $\mathcal{N}_2 \subset \mathcal{N}_1 \subset \mathcal{N}$. Let $N = N(A, \epsilon, \delta) \in \mathcal{N}$. We show that there is an $M \in \mathcal{N}_2$ such that $M \subset N$. Since $(\Omega_n)_{n=1}^\infty$ is an increasing sequence such that $\bigcup_{n=1}^\infty \Omega_n = \Omega$, we have that $B_n := A \cap \Omega_n^c$ defines a decreasing sequence and $\mu(B_n) < \infty$ for each n , since $B_n \subset A$ for each n and $\mu(A) < \infty$. By Theorem 1.1.3

we have that

$$\begin{aligned}
\lim_{n \rightarrow \infty} \mu(B_n) &= \mu\left(\bigcap_{n=1}^{\infty} B_n\right) \\
&= \mu\left(A \cap \left(\bigcap_{n=1}^{\infty} \Omega_n^c\right)\right) \\
&= \mu\left(A \cap \left(\bigcup_{n=1}^{\infty} \Omega_n\right)^c\right) \\
&= \mu(A \cap \emptyset) \quad \text{since } \bigcup_{n=1}^{\infty} \Omega_n = \Omega \\
&= 0.
\end{aligned}$$

It follows that there exists a $m_\delta \in \mathbb{N}^+$ such that $\mu(A \cap \Omega_{m_\delta}) < \frac{\delta}{2}$. Let $n \in \mathbb{N}^+$ such that $\frac{1}{n} \leq \min\{\epsilon, \frac{\delta}{2}\}$, and let $f \in N(\Omega_{m_\delta}, \frac{1}{n})$. Then

$$\begin{aligned}
\{x \in A : |f(x)| \geq \epsilon\} &\subset \{x \in A : |f(x)| \geq \frac{1}{n}\} \\
&= \{x \in A \cap \Omega_{m_\delta} : |f(x)| \geq \frac{1}{n}\} \cup \{x \in A \cap \Omega_{m_\delta}^c : |f(x)| \geq \frac{1}{n}\} \\
\Rightarrow \mu(\{x \in A : |f(x)| \geq \epsilon\}) &\leq \mu\left(\{x \in A \cap \Omega_{m_\delta} : |f(x)| \geq \frac{1}{n}\} \cup \{x \in A \cap \Omega_{m_\delta}^c : |f(x)| \geq \frac{1}{n}\}\right) \\
&= \mu\left(\{x \in A \cap \Omega_{m_\delta} : |f(x)| \geq \frac{1}{n}\}\right) + \mu\left(\{x \in A \cap \Omega_{m_\delta}^c : |f(x)| \geq \frac{1}{n}\}\right) \\
&\leq \mu\left(\{x \in \Omega_{m_\delta} : |f(x)| \geq \frac{1}{n}\}\right) + \mu(A \cap \Omega_{m_\delta}^c) \\
&\leq \frac{1}{n} + \frac{\delta}{2} \quad \text{since } f \in N(\Omega_{m_\delta}, \frac{1}{n}) \\
&\leq \frac{\delta}{2} + \frac{\delta}{2}.
\end{aligned}$$

It follows that $f \in N(A, \epsilon, \delta)$ and therefore that $N(\Omega_{m_\delta}, \frac{1}{n}) \subset N(A, \epsilon, \delta)$. \square

It is shown in [1, p.28] that a Hausdorff topological vector space (E, \mathcal{T}) has a metrizable topology if and only if \mathcal{T} has a countable basis for the neighbourhoods of the origin.

PROPOSITION 2.1.4. *Let \mathcal{T} be the topology defined above. $L^0(\mu)$ equipped with this topology is metrizable.*

PROOF. The proposition above shows that \mathcal{T} has a countable base \mathcal{N}_2 for the neighbourhoods of the origin. Since \mathcal{T} is a vector space topology on $L^0(\mu)$, it follows by our previous discussion that it suffices to show that this topology is Hausdorff. Let $f \in L^0$ such that $f \neq 0$. It is therefore possible (Proposition 1.2.1) to find an $\epsilon > 0$ and an $A \in \Sigma$ such that $\mu(A) > 0$ and $|f(x)| \geq \epsilon$ for all $x \in A$. Since $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$ is σ -finite, we can find an $n \in \mathbb{N}^+$ with $0 < \mu(A \cap \Omega_n) = \delta < \infty$. Let $\tilde{\epsilon} = \min\{\epsilon, \delta\}$, then

$$\begin{aligned}
\{x \in A \cap \Omega_n : |f(x)| \geq \tilde{\epsilon}\} &\supset \{x \in A \cap \Omega_n : |f(x)| \geq \epsilon\} \\
\Rightarrow \mu(\{x \in A \cap \Omega_n : |f(x)| \geq \tilde{\epsilon}\}) &\geq \mu(\{x \in A \cap \Omega_n : |f(x)| \geq \epsilon\}) \\
&= \mu(A \cap \Omega_n) \quad \text{since } |f(x)| \geq \epsilon \forall x \in A \\
&= \delta \\
&\geq \tilde{\epsilon}
\end{aligned}$$

and so $f \notin N(A \cap \Omega_n, \tilde{\epsilon})$. It is shown in [13, p.83] that a topological vector space E is Hausdorff if for every $x \in E$ such that $x \neq 0$ there exists a neighbourhood of 0 which does not contain x . It follows that $L^0(\mu)$ is Hausdorff. \square

Next we show how this topology can be described in measure theoretic terms.

PROPOSITION 2.1.5. *A sequence $(f_n)_{n=1}^\infty$ in $(L^0(\mu), \mathcal{T})$ converges to $f \in L^0(\mu)$ if and only if $f_n \rightarrow f$ in measure on sets of finite measure.*

PROOF. \mathcal{T} is a linear topology, so $f_n \xrightarrow{\mathcal{T}} f$ if and only if $(f_n - f) \xrightarrow{\mathcal{T}} 0$. It can be shown that similarly $f_n \rightarrow f$ in measure on a particular set if and only if $(f_n - f) \rightarrow 0$ in measure on that set. Without loss of generality we can therefore consider a sequence of functions tending to the zero function. We have that $f_n \rightarrow 0$ in measure on sets of finite measure if and only if for every $A \in \Sigma_f$ and for every $\epsilon, \delta > 0$ there exists an $n_{A, \epsilon, \delta} \in \mathbb{N}^+$ such that $n \geq n_{A, \epsilon, \delta}$ implies $\mu(\{x \in A : |f_n(x)| \geq \epsilon\}) < \delta$. This happens if and only if, for all $A \in \Sigma_f$ and for every $\epsilon, \delta > 0$, there exists an $n_{A, \epsilon, \delta} \in \mathbb{N}^+$ such that $n \geq n_{A, \epsilon, \delta}$ implies $f_n \in N(A, \epsilon, \delta)$. This last statement is equivalent to $f_n \rightarrow 0$ in $(L^0(\mu), \mathcal{T})$. \square

REMARK 2.1.6. Since this topology is metrizable and hence fully described by sequences, this topology is called the topology of local convergence in measure.

PROPOSITION 2.1.7. *$(L^0(\mu), \mathcal{T})$ is complete.*

PROOF. Let $(f_n)_{n=1}^\infty$ be a Cauchy sequence in $(L^0(\mu), \mathcal{T})$. A similar argument to the one employed above can be used to show that $(f_n)_{n=1}^\infty$ is Cauchy in measure on every finite subset of Ω . Since Ω is σ -finite, we can write $\Omega = \bigcup_{n=1}^\infty \Omega_n$, where $\mu(\Omega_n) < \infty$ for each n and $\Omega_n \cap \Omega_m = \emptyset$ if $n \neq m$. Let $m \in \mathbb{N}^+$. Since $(f_n \upharpoonright \Omega_m)_{n=1}^\infty$ is Cauchy in measure on Ω_m , there exists by Theorem 1.2.5 a measurable function f^m on Ω_m such that $f_n \upharpoonright \Omega_m \rightarrow f^m$ in measure as $n \rightarrow \infty$. Since this can be done for each $m \in \mathbb{N}^+$ and the Ω_m 's are disjoint one can define a function f on Ω by letting $f \upharpoonright \Omega_m = f^m$. f defined in this way is a measurable function since if G is an open subset of \mathbb{F} , then $(f^m)^{-1}(G)$ is a measurable subset of Ω_m and hence of Ω for each m . $f^{-1}(G) = \bigcup_{m=1}^\infty (f^m)^{-1}(G)$ is therefore measurable and hence f is a measurable function. We show that $f_n \rightarrow f$ in measure on sets of finite measure and hence that $f_n \rightarrow f$ in $(L^0(\mu), \mathcal{T})$. Let $A \in \Sigma$ such that $\mu(A) = \alpha < \infty$ and let $\epsilon > 0$. Let $\alpha_m = \mu(A \cap \Omega_m)$. Then

$$\begin{aligned} \mu(A) &= \mu\left(\bigcup_{m=1}^\infty A \cap \Omega_m\right) \\ &= \sum_{m=1}^\infty \mu(A \cap \Omega_m) \quad \text{since } (A \cap \Omega_m) \cap (A \cap \Omega_n) = \emptyset \text{ if } m \neq n \\ &= \sum_{m=1}^\infty \alpha_m \end{aligned}$$

So there exists an $m_\epsilon \in \mathbb{N}^+$ such that $m \geq m_\epsilon$ implies

$$\sum_{m=m_\epsilon+1}^\infty \alpha_m = \alpha - \sum_{m=1}^{m_\epsilon} \alpha_m < \frac{\epsilon}{2}.$$

Since $f_n \rightarrow f$ in measure on each Ω_m , we can choose for each m an $n_{\epsilon, m} \in \mathbb{N}^+$ such that $n \geq n_{\epsilon, m}$ implies

$$\begin{aligned} \mu\left(\left\{x \in \Omega_m : |f_n(x) - f(x)| \geq \frac{\epsilon}{2m_\epsilon}\right\}\right) &< \frac{\epsilon}{2m_\epsilon} \\ \Rightarrow \mu\left(\left\{x \in \Omega_m : |f_n(x) - f(x)| \geq \epsilon\right\}\right) &< \frac{\epsilon}{2m_\epsilon} \end{aligned}$$

Let $n_\epsilon = \max\{n_{\epsilon,m} : 1 \leq m \leq m_\epsilon\}$, then for $n \geq n_\epsilon$

$$\begin{aligned} & \mu(\{x \in A : |f_n(x) - f(x)| \geq \epsilon\}) \\ &= \mu\left(\left(\bigcup_{m=1}^{m_\epsilon} \{x \in A \cap \Omega_m : |f_n(x) - f(x)| \geq \epsilon\}\right) \cup \left(\bigcup_{m=m_\epsilon+1}^{\infty} \{x \in A \cap \Omega_m : |f_n(x) - f(x)| \geq \epsilon\}\right)\right) \\ &= \sum_{m=1}^{m_\epsilon} \mu(\{x \in A \cap \Omega_m : |f_n(x) - f(x)| \geq \epsilon\}) + \sum_{m=m_\epsilon+1}^{\infty} \mu(\{x \in A \cap \Omega_m : |f_n(x) - f(x)| \geq \epsilon\}) \\ &< m_\epsilon \left(\frac{\epsilon}{2m_\epsilon}\right) + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

and so $f_n \rightarrow f$ in measure on A . By the Proposition above this implies that $f_n \rightarrow f$ in $(L^0(\mu), \mathcal{T})$. \square

For the remainder of this subsection we will assume all scalars are real and that $L^0(\mu)$ consists of equivalence classes of real-valued, almost everywhere finite, measurable functions. We show that we can define an order structure on $L^0(\mu)$ such that it becomes a Riesz space.

PROPOSITION 2.1.8. *Define \leq on $L^0(\mu)$ by $f \leq g \Leftrightarrow f(x) \leq g(x)$ μ -a.e. Then $(L^0(\mu), +, \cdot, \leq)$ is a Riesz space.*

PROOF. It is easily checked that \leq is a partial order on $L^0(\mu)$. Furthermore, for any $f, g, h \in L^0(\mu)$ and $\alpha \geq 0$ it is easily checked that

- (1) $f \leq g \Rightarrow f + h \leq g + h$
- (2) $0 \leq f \Rightarrow 0 \leq \alpha f$

So $(L^0(\mu), +, \cdot, \leq)$ is a partially ordered linear space. It is easily checked that if we let

$$\begin{aligned} h(x) &:= \max\{f(x), g(x)\} \quad \text{and} \\ j(x) &:= \min\{f(x), g(x)\}, \end{aligned}$$

then $h, j \in L^0(\mu)$ and $f \vee g = h$ and $f \wedge g = j$, i.e. $h \geq f, g$ and if $k \in L^0(\mu)$ such that $k \geq f, g$, then $k \geq h$; similarly $j \leq f, g$ and if $k \in L^0(\mu)$ such that $k \leq f, g$, then $k \leq j$. So $(L^0(\mu), \leq)$ is a lattice and hence $L^0(\mu)$ is a Riesz space. \square

Since the proofs in the previous section on the topological structure of $L^0(\mu)$ did not depend on the field of scalars or whether the functions were real or complex valued, the families \mathcal{N} , \mathcal{N}_1 and \mathcal{N}_2 define a complete metrizable linear topology \mathcal{T} on $L^0(\mu)$. Next we investigate the interaction between the topological and order structures on $L^0(\mu)$.

PROPOSITION 2.1.9. *$(L^0(\mu), \mathcal{T})$ is a complete metrizable locally solid Riesz space.*

PROOF. It has been shown that $(L^0(\mu), \mathcal{T})$ is a Riesz space equipped with a complete metrizable vector space topology. It remains to prove that \mathcal{T} has a basis for the neighbourhoods of the origin consisting of solid sets. Let $N = N(A, \epsilon) \in \mathcal{N}$, $f \in N$ and $g \in L^0(\mu)$ with $|g| \leq |f|$, then $|g(x)| \leq |f(x)|$ μ -a.e. and so

$$\begin{aligned} \{x \in A : |g(x)| \geq \epsilon\} &\subset \{x \in A : |f(x)| \geq \epsilon\} \cup \{x \in A : |g(x)| > |f(x)|\} \\ \Rightarrow \mu(\{x \in A : |g(x)| \geq \epsilon\}) &\leq \mu(\{x \in A : |f(x)| \geq \epsilon\}) + \mu(\{x \in A : |g(x)| > |f(x)|\}) \\ &\leq \mu(\{x \in A : |f(x)| \geq \epsilon\}) + \mu(\{x \in A : |g(x)| > |f(x)|\}) \\ &< \epsilon + 0 \end{aligned}$$

So $g \in N$ and hence N is a solid set. \mathcal{N} therefore consists of solid sets and thus \mathcal{T} is locally solid. \square

2.1.1.2. $L^{00}(\Omega, \Sigma, \mu)$. Recall that

$$\begin{aligned} L^{00}(\Omega, \Sigma, \mu) &= \{f \in L^0(\Omega, \Sigma, \mu) : \exists A \in \Sigma_f, M > 0 : |f(x)| \leq M \ \forall x \in A^c\} \\ &= \{f \in L^0(\Omega, \Sigma, \mu) : f^*(t) < \infty \ \forall t > 0\} \quad \text{by Remark 1.3.6} \end{aligned}$$

It will be shown that $L^{00}(\mu)$ equipped with the topology of convergence in measure is a complete metrizable topological vector space. Furthermore, in the case where $L^0(\mu)$ consists of equivalence classes of real-valued, almost everywhere finite, measurable functions, it will be shown that $L^{00}(\mu)$ is a complete metrizable locally solid Riesz space.

PROPOSITION 2.1.10. $L^{00}(\mu)$ is a vector subspace of $L^0(\mu)$.

PROOF. Let $f, g \in L^{00}(\mu)$ and $\alpha \in \mathbb{F}$. Then for any $t > 0$

$$\begin{aligned} (\alpha f)^*(t) &= |\alpha| f^*(t) \quad \text{by Proposition 1.3.5(2)} \\ &< \infty \quad \text{since } f \in L^{00}(\mu) \end{aligned}$$

Furthermore,

$$\begin{aligned} (f+g)^*(t) &= (f+g)^*\left(\frac{t}{2} + \frac{t}{2}\right) \\ &\leq f^*\left(\frac{t}{2}\right) + g^*\left(\frac{t}{2}\right) \quad \text{by Proposition 1.3.5(3)} \\ &< \infty \quad \text{since } f, g \in L^{00}(\mu) \end{aligned}$$

So αf and $f+g \in L^{00}(\mu)$. $L^{00}(\mu)$ is therefore a vector subspace of $L^0(\mu)$. \square

Next we define a linear topology on $L^{00}(\mu)$. Let $N(\epsilon, \delta) := \{f \in L^{00}(\mu) : \mu(\{x \in \Omega : |f(x)| > \epsilon\}) \leq \delta\}$ and let $\mathcal{M} := \{N(\epsilon, \delta) : \epsilon, \delta > 0\}$.

REMARK 2.1.11. Using the definition of a distribution function and Proposition 1.3.5(14), we have that $N(\epsilon, \delta) = \{f \in L^{00}(\mu) : \mu_f(\epsilon) \leq \delta\} = \{f \in L^{00}(\mu) : f^*(\delta) \leq \epsilon\}$.

PROPOSITION 2.1.12. \mathcal{M} is a basis for the neighbourhoods of the origin for a vector space topology \mathcal{U} on $L^{00}(\mu)$. Furthermore, $L^{00}(\mu)$ equipped with this topology is a complete metrizable topological vector space.

PROOF. It can be shown in a similar way to the $L^0(\mu)$ case that \mathcal{M} is a basis for the neighbourhoods of the origin for a vector space topology on $L^{00}(\mu)$. The one important difference is that in the $L^0(\mu)$ case we do not have to restrict ourselves to measurable sets of finite measure when defining the sets $N(\epsilon, \delta)$. The reason for this can be seen in proving that every $N(\epsilon, \delta) \in \mathcal{M}$ is absorbing: Let $f \in L^{00}(\mu)$. In this case we have the additional information that this implies that $f^*(t) < \infty$ for every $t > 0$. In particular $f^*(\delta) = \beta < \infty$. Let $\lambda \in \mathbb{F}$ such that $|\lambda| \geq \frac{\beta}{\epsilon}$. Then

$$\begin{aligned} \left(\frac{1}{\lambda}f\right)^*(\delta) &= \left|\frac{1}{\lambda}\right| f^*(\delta) \quad \text{by Proposition 1.3.5(2)} \\ &\leq \frac{\epsilon}{\beta} \beta \\ &= \epsilon \end{aligned}$$

Therefore $\frac{1}{\lambda}f \in N(\epsilon, \delta)$ by Remark 2.1.11, i.e. $f \in \lambda N(\epsilon, \delta)$.

A similar process to the one employed in the $L^0(\mu)$ case can be used to show that this topology is metrizable and complete. \square

REMARK 2.1.13. Since this topology \mathcal{U} is metrizable and a sequence converges in this topology if and only if it converges in measure, this topology is called the topology of convergence in measure.

To investigate the order structure of $L^{00}(\mu)$ we consider the order structure it inherits from $L^0(\mu)$. For the remainder of this subsection we will assume that all scalars are real and that $L^0(\mu)$ consists of equivalence classes of real-valued, almost everywhere finite, measurable functions. Let E be a Riesz space and $A \subset E$ a vector subspace. If A is also a sublattice of E , then A is called a *Riesz subspace*. If A is a solid set and also a vector subspace of a Riesz space E , then A is an order ideal and hence a Riesz subspace [8, p.12].

PROPOSITION 2.1.14. $L^{00}(\mu)$ is a Riesz subspace of $L^0(\mu)$.

PROOF. It has been shown (Proposition 2.1.10) that $L^{00}(\mu)$ is a vector subspace of $L^0(\mu)$. To show that $L^{00}(\mu)$ is a Riesz subspace it therefore suffices by the discussion above to show that $L^{00}(\mu)$ is solid in $L^0(\mu)$. Let $f \in L^0(\mu)$ and $g \in L^{00}(\mu)$ such that $|f| \leq |g|$. Since $g \in L^{00}(\mu)$ there exists an $M > 0$ and an $A \in \Sigma$ with $\mu(A) < \infty$ such that $|g(x)| \leq M$ for all $x \in A^c$. Let

$$B = \{x \in \Omega : |g(x)| < |f(x)|\}.$$

Then $|f(x)| \leq M$ for all $x \in (A^c \cap B^c) = (A \cup B)^c$ and $\mu(B) = 0$ since $|f(x)| \leq |g(x)|$ μ -a.e. Hence $g \in L^{00}(\mu)$ and so $L^{00}(\mu)$ is a solid set. \square

PROPOSITION 2.1.15. $L^{00}(\mu)$ equipped with the topology of convergence in measure is a complete metrizable locally solid Riesz space.

PROOF. Similar to the proof of Proposition 2.1.9 \square

2.1.3. $L^{01}(\Omega, \Sigma, \mu)$. Let

$$L^{01}(\Omega, \Sigma, \mu) := \{f \in L^{00}(\Omega, \Sigma, \mu) : f^*(t) \rightarrow 0 \text{ as } t \rightarrow \infty\}$$

It will be shown that $L^{01}(\mu)$ equipped with the topology of convergence in measure is a complete metrizable topological vector space and in the case where $L^0(\mu)$ consists of equivalence classes of real-valued, almost everywhere finite, measurable functions, then $L^{01}(\mu)$ is a complete metrizable locally solid Riesz space.

PROPOSITION 2.1.16. $L^{01}(\Omega, \Sigma, \mu)$ is a closed subspace of $L^{00}(\Omega, \Sigma, \mu)$ equipped with the topology of convergence in measure.

PROOF. Let $f, g \in L^{01}$ and $\alpha \in \mathbb{F}$. Then

$$\begin{aligned} (\alpha f)^*(t) &= |\alpha| f^*(t) && \text{by Proposition 1.3.5(2)} \\ &\rightarrow 0 && \text{since } f^*(t) \rightarrow 0 \text{ as } t \rightarrow \infty \end{aligned}$$

Furthermore,

$$\begin{aligned} (f+g)^*(t) &= (f+g)^*\left(\frac{t}{2} + \frac{t}{2}\right) \\ &\leq f^*\left(\frac{t}{2}\right) + g^*\left(\frac{t}{2}\right) && \text{by Proposition 1.3.5(3)} \end{aligned}$$

Since this holds for all $t > 0$ and $f^*(\frac{t}{2})$ and $g^*(\frac{t}{2})$ tend to zero as t tends to infinity, we have that $(f+g)^*(t) \rightarrow 0$ as $t \rightarrow \infty$. So $L^{01}(\mu)$ is a vector subspace of $L^{00}(\mu)$.

Let $(f_n)_{n=1}^\infty \subset L^{01}(\mu)$ such that $f_n \rightarrow f$ in measure and $f \in L^{00}(\mu)$. This implies, using the continuity of addition, that $f - f_n \rightarrow 0$ in measure and hence for any $t > 0$, $(f - f_n)^*(t) \rightarrow 0$ as $n \rightarrow \infty$ by Proposition 1.3.8. We can write $f = f - f_n + f_n$ and therefore for a fixed $t > 0$

$$\begin{aligned} f^*(t) &= (f - f_n + f_n)^*(t) \\ (2.1.1) \quad &\leq (f - f_n)^*\left(\frac{t}{2}\right) + f_n^*\left(\frac{t}{2}\right) \end{aligned}$$

Let $\epsilon > 0$. Since $(f - f_n)^*(1) \rightarrow 0$ as $n \rightarrow \infty$, there exists an $n_\epsilon \in \mathbb{N}^+$ such that $n \geq n_\epsilon$ implies that $(f - f_n)^*(1) < \epsilon/2$. Furthermore since $f_{n_\epsilon} \in L^{01}(\mu)$, there exists a $t_0 \geq 2$ such that $t \geq t_0$ implies that $f_{n_\epsilon}^*(\frac{t}{2}) < \epsilon/2$. It follows using (2.1.1) that $f^*(t) < \epsilon$ for $t \geq t_0$ and hence that $f^*(t) \rightarrow 0$ as $t \rightarrow \infty$.

Therefore $f \in L^{01}(\mu)$ and hence $L^{01}(\mu)$ is a closed subspace of $L^{00}(\mu)$ equipped with the topology of convergence in measure. \square

COROLLARY 2.1.17. *$L^{01}(\mu)$ equipped with the topology of convergence in measure is a complete metrizable topological vector space.*

In the case where $L^0(\mu)$ consists of equivalence classes of real-valued, almost everywhere finite, measurable functions we have the following result.

PROPOSITION 2.1.18. *$L^{01}(\mu)$ equipped with the topology of convergence in measure is a complete metrizable locally solid Riesz space.*

2.2. Banach spaces of measurable functions

The Lebesgue spaces are well-known examples of Banach spaces whose elements are equivalence classes of measurable functions. The Lebesgue function norms are lattice norms and possess the Fatou property. Furthermore Lebesgue spaces always contain all simple functions and there is a relationship between the integral of a measurable function over a set of finite measure and the L^p -norm of that function. We study Orlicz spaces, Lorentz spaces and Orlicz-Lorentz spaces, which all form generalizations of the Lebesgue spaces. Some salient properties required for the sequel will be documented and it will be shown that each of these spaces possess the properties listed for the Lebesgue spaces. This will serve as motivation for our definition of Banach function spaces, which will provide a common generalization for all of these spaces.

2.2.1. Lorentz spaces. We define four types of Lorentz spaces, namely $L^{p,q}(\mu)$, $L^{(p,q)}(\mu)$, M_φ and Λ_η . The $L^{p,q}$ spaces generalize the L^p spaces and will be used extensively for the study of composition operators. The $L^{(p,q)}$ spaces will be important when considering duality properties of Lorentz spaces. The remaining two Lorentz spaces are included predominantly for completeness, but are also significant when considering the embeddings of rearrangement invariant Banach function spaces.

DEFINITION 2.2.1. [2, p.216] Let $f \in L^0(\mu)$ and define

$$\|f\|_{p,q} := \begin{cases} \left(\int_0^\infty (t^{1/p} f^*(t))^q \frac{dt}{t} \right)^{1/q} & \text{if } 0 < p < \infty, 0 < q < \infty \\ \sup_{0 < t < \infty} t^{1/p} f^*(t) & \text{if } 0 < p < \infty, q = \infty \\ f^*(0) & \text{if } p = q = \infty \end{cases}$$

The Lorentz space $L^{p,q}(\mu)$ is defined as the set of all $f \in L^0(\mu)$ such that $\|f\|_{p,q} < \infty$.

The functional $\|\cdot\|_{p,q}$ is not always a norm, but we have that if $1 \leq q \leq p < \infty$ (or $p = q = \infty$), then $L^{p,q}(\mu)$ is a normed space ([2, p.218]).

We show that the $L^{p,q}$ spaces generalize the L^p spaces.

EXAMPLE 2.2.2. The Lorentz space $L^{p,p}(\mu)$ coincides with the Lebesgue space $L^p(\mu)$ with equality of norms for $(1 \leq p \leq \infty)$.

PROOF. For $1 \leq p < \infty$, we have that

$$\begin{aligned} \|f\|_{p,p}^p &= \int_0^\infty (t^{1/p} f^*(t))^p \frac{dt}{t} \\ &= \int_0^\infty (f^*(t))^p dt \\ &= \|f\|_p^p \quad \text{by Proposition 1.3.7} \end{aligned}$$

Similarly $\|f\|_{\infty,\infty} = f^*(0) = \|f\|_\infty$ by Proposition 1.3.7. \square

The following result will be useful when considering the boundedness of composition operators on Lorentz spaces.

PROPOSITION 2.2.3. *Let $1 \leq q \leq p < \infty$, $M > 0$, $f \in L^{p,q}(\mu)$ and $g \in L^0(\mu)$. If*

$$g^*(t) \leq f^*(t/M) \quad \forall t \geq 0$$

then $\|g\|_{p,q} \leq M^{1/p} \|f\|_{p,q}$.

PROOF. We have that

$$\begin{aligned} \|g\|_{p,q} &= \left(\int_0^\infty (t^{1/p} g^*(t))^q \frac{dt}{t} \right)^{1/q} \\ &\leq \left(\int_0^\infty (t^{1/p} f^*(t/M))^q \frac{dt}{t} \right)^{1/q} \quad \text{since } t^{1/p} \geq 0 \\ &= \left(\int_0^\infty ((sM)^{1/p} f^*(s))^q \frac{M ds}{Ms} \right)^{1/q} \quad \text{letting } s = t/M \\ &= M^{1/p} \left(\int_0^\infty (s^{1/p} f^*(s))^q \frac{ds}{s} \right)^{1/q} \\ (2.2.1) \quad &= M^{1/p} \|f\|_{p,q} \end{aligned}$$

\square

We are able to obtain normed spaces for all p and q such that $1 < p < \infty$ and $1 \leq q \leq \infty$ if we replace the decreasing rearrangements in the definition of the $L^{p,q}$ spaces with maximal functions.

DEFINITION 2.2.4. [2, p.219] Let $f \in L^0(\mu)$ and define

$$\|f\|_{(p,q)} := \begin{cases} \left(\int_0^\infty (t^{1/p} f^{**}(t))^q \frac{dt}{t} \right)^{1/q} & \text{if } 1 \leq p \leq \infty, 0 < q < \infty \\ \sup_{0 < t < \infty} t^{1/p} f^{**}(t) & \text{if } 1 \leq p < \infty, q = \infty \end{cases}$$

The Lorentz space $L^{(p,q)}(\mu)$ is defined as the set of all $f \in L^0(\mu)$ such that $\|f\|_{(p,q)} < \infty$.

It can be shown ([2, p.219]) that

$$\|f\|_{p,q} \leq \|f\|_{(p,q)} \leq \frac{p}{p-1} \|f\|_{p,q} \quad \forall f \in L^0(\mu)$$

and hence that $L^{p,q}(\mu) = L^{(p,q)}(\mu)$ for all p and q such that $1 < p < \infty$ and $1 \leq q \leq \infty$.

We define two other types of Lorentz spaces. Let φ be a nonnegative function defined on $[0, \infty)$. If φ is such that $\varphi(t) = 0$ if and only if $t = 0$, φ is increasing and $\frac{\varphi(t)}{t}$ is decreasing, then φ is called a *quasiconcave* function. Let φ be a quasiconcave function and let η be a nonnegative, increasing concave function. Consider

the following two functionals on $L^0(\mathbb{R}^+, \mathcal{M}, m)$

$$\begin{aligned}\|f\|_{M_\varphi} &:= \sup_{0 < t < \infty} \{f^{**}(t)\varphi(t)\} \quad \text{and} \\ \|f\|_{\Lambda_\eta} &:= \int_0^\infty f^*(s)\eta'(s) ds + f^*(0)\eta(0+),\end{aligned}$$

where the latter is a Lebesgue-Riemann Stieltjes integral. The Lorentz spaces M_φ and Λ_η are the spaces of all functions f in $L^0(\mathbb{R}^+, \mathcal{M}, m)$ for which the respective functionals above are finite. M_φ is sometimes also called a *Marcinkiewicz space*.

The following result shows that the different types of Lorentz spaces defined above all share some important properties.

THEOREM 2.2.5. [2, p.219][2, p.72] *Let $X = L^{p,q}(\mu)$, ($1 \leq q \leq p < \infty$) or $X = L^{(p,q)}(\mu)$, ($1 < p < \infty$, $1 \leq q \leq \infty$) or $X = M_\varphi$, where φ is a quasiconcave function, or $X = \Lambda_\eta$, where η is a nonnegative, increasing concave function. Then X is a normed space and furthermore*

- (1) *If $0 \leq f \leq g$ μ -a.e. then $\|f\|_X \leq \|g\|_X$ and $\|\|f\|\|_X = \|f\|_X$ for any $f \in L^0(\mu)$ ($f \in L^0(m)$ for $X = M_\varphi, \Lambda_\eta$), i.e. $\|\cdot\|_X$ is a lattice norm on X*
- (2) *If $0 \leq f_n \uparrow f$, μ -a.e. then $\|f_n\|_X \uparrow \|f\|_X$, i.e. $\|\cdot\|_X$ possesses the Fatou property*
- (3) *If $A \in \Sigma_f$, then $\|\chi_A\|_X < \infty$*
- (4) *If $A \in \Sigma_f$, then there exists a constant C_A depending only on A , such that*

$$\int_A f d\mu \leq C_A \|f\|_X \quad \forall f \in X$$

2.2.2. Orlicz spaces. The Lebesgue space $L^p(\mu)$ ($1 \leq p < \infty$) is the set of all $f \in L^0(\mu)$ such that

$$\int_\Omega \phi(|f(x)|) d\mu < \infty$$

where $\phi(t) = t^p$. It is natural to consider replacing ϕ with a more general function. This leads one to consider the restrictions that have to be imposed on such a function in order to generate a Banach space of equivalence classes of measurable functions. We note that if $\phi(t) = t^p$, then ϕ is a convex function, $\phi(0) = 0$ and $\lim_{t \rightarrow \infty} \phi(t) = \infty$. We define Orlicz functions based on these attributes and describe some of their properties. In particular, consequences of a growth condition, namely the Δ_2 -condition will be investigated. Thereafter Orlicz spaces will be defined and it will be shown that the Lebesgue spaces are examples of Orlicz spaces.

DEFINITION 2.2.6. [5, p.190] $\phi : [0, \infty) \rightarrow [0, \infty]$ is called an Orlicz (Young) function if ϕ is convex, $\phi(0) = 0$ and $\lim_{t \rightarrow \infty} \phi(t) = \infty$. We assume furthermore that ϕ is neither identically zero nor infinite on $(0, \infty)$ and that ϕ is left continuous at $b_\phi := \sup\{t > 0 : \phi(t) < \infty\}$. We also define the constant, $a_\phi := \inf\{t > 0 : \phi(t) > 0\}$.

The following theorem yields an equivalent definition of an Orlicz function.

THEOREM 2.2.7. [2, p.265] $\phi : [0, \infty) \rightarrow [0, \infty]$ is an Orlicz function if and only if there exists a $\varphi : [0, \infty) \rightarrow [0, \infty]$ which is increasing, left continuous, not identically infinite nor identically zero such that $\varphi(0) = 0$ and

$$(2.2.2) \quad \phi(t) = \int_0^t \varphi(s) ds$$

We use this to show the following result.

PROPOSITION 2.2.8. [2, p.276] *If ϕ is an Orlicz function, then ϕ is increasing on $[0, \infty)$ and strictly increasing on (a_ϕ, b_ϕ) .*

PROOF. Let φ be as in Theorem 2.2.7. Using (2.2.2) and the fact that $\varphi \geq 0$, it is clear that ϕ is increasing. Let $a_\phi < t_1 < t_2 < b_\phi$. Assume that $\varphi(t_1) = 0$, then

$$\begin{aligned}\phi(t_1) &= \int_0^{t_1} \varphi(s) ds \\ &= 0 \quad \text{since } \varphi \text{ is increasing} \\ \Rightarrow a_\phi &\geq t_1\end{aligned}$$

This is a contradiction and therefore $\varphi(t_1) > 0$. Therefore

$$\begin{aligned}\phi(t_2) &= \int_0^{t_2} \varphi(s) ds \\ &= \int_0^{t_1} \varphi(s) ds + \int_{t_1}^{t_2} \varphi(s) ds \\ &\geq \phi(t_1) + (t_2 - t_1)\varphi(t_1) \quad \text{since } \varphi \text{ is increasing} \\ &> \phi(t_1) \quad \text{since } \varphi(t_1) > 0\end{aligned}$$

□

REMARK 2.2.9. It follows that if ϕ is any Orlicz function, then ϕ is invertible on (a_ϕ, b_ϕ) .

We define the right continuous inverse of ϕ as follows

$$\phi^{-1}(t) := \inf\{s > 0 : \phi(s) > t\}.$$

REMARK 2.2.10. It is easily checked that ϕ^{-1} is increasing and that

$$\inf\{s > 0 : \phi(s) > t\} = \sup\{s > 0 : \phi(s) \leq t\}$$

and so

$$\phi^{-1}(t) = \sup\{s > 0 : \phi(s) \leq t\}.$$

LEMMA 2.2.11. *Let $0 \leq t < \infty$, then*

$$\phi(\phi^{-1}(t)) \leq t \leq \phi^{-1}(\phi(t))$$

PROOF. Let $B = \{s > 0 : \phi(s) > \phi(t)\}$ and let $s \in B$. Then, since ϕ is increasing, it is easily shown by contradiction that $s > t$. Since t is therefore a lower bound for B , it follows that

$$\begin{aligned}\phi^{-1}(\phi(t)) &= \inf B \\ &\geq t\end{aligned}$$

Let $C = \{s > 0 : \phi(s) \leq \phi(t)\}$, then clearly $t \in C$ and so

$$\begin{aligned}t &\leq \sup C \\ &= \phi^{-1}(\phi(t)) \quad \text{by Remark 2.2.10}\end{aligned}$$

□

COROLLARY 2.2.12. (cf. [24, p.4]) *Let ϕ be an Orlicz function and let $f \in L^0(\mu)$ be such that $\phi(|f|)$ is finite almost everywhere. Then*

$$\phi(f^*(t)) = (\phi(|f|))^*(t)$$

PROOF. Let $\lambda \geq 0$ and let $A_\lambda = \{x \in \Omega : \phi(|f(x)|) > \lambda\}$, $B_\lambda = \{x \in \Omega : |f(x)| > \phi^{-1}(\lambda)\}$. If $x \notin A_\lambda$, then

$$\begin{aligned} \phi(|f(x)|) &\leq \lambda \\ \Rightarrow \phi^{-1}(\phi(|f(x)|)) &\leq \phi^{-1}(\lambda) && \text{since } \phi^{-1} \text{ is increasing} \\ \Rightarrow |f(x)| &\leq \phi^{-1}(\lambda) && \text{by Lemma 2.2.11} \\ \Rightarrow x &\notin B_\lambda \end{aligned}$$

It follows that $B_\lambda \subset A_\lambda$. Let $x \notin B_\lambda$, then

$$\begin{aligned} |f(x)| &\leq \phi^{-1}(\lambda) \\ \Rightarrow \phi(|f(x)|) &\leq \phi(\phi^{-1}(\lambda)) && \text{since } \phi \text{ is increasing} \\ &\leq \lambda && \text{by Lemma 2.2.11} \end{aligned}$$

It follows that $A_\lambda \subset B_\lambda$ and hence that $A_\lambda = B_\lambda$. We therefore have that $\mu_{\phi(|f|)}(\lambda) = \mu_f(\phi^{-1}(\lambda))$. It can similarly be shown that $m_{\phi(f^*)}(\lambda) = m_{f^*}(\phi^{-1}(\lambda))$. Therefore

$$\begin{aligned} \mu_{\phi(|f|)}(\lambda) &= \mu_f(\phi^{-1}(\lambda)) \\ &= m_{f^*}(\phi^{-1}(\lambda)) && \text{since } f \text{ and } f^* \text{ are equimeasurable by Proposition 1.3.5(7)} \\ &= m_{\phi(f^*)}(\lambda) \end{aligned}$$

□

Next, we consider an important growth parameter on an Orlicz function ϕ . If there exists a $t_0 > 0$ and a $C > 0$ such that

$$(2.2.3) \quad \phi(2t) \leq C\phi(t) < \infty \quad \forall t \text{ such that } t_0 \leq t < \infty$$

then ϕ is said to satisfy the Δ_2 -condition for large t . If $t_0 = 0$, then ϕ is said to satisfy the Δ_2 -condition globally. We will sometimes write $\phi \in \Delta_2$ for large t (respectively globally) if this is the case. It is easily checked that if $\phi(x) = |x|^p$, then ϕ is an Orlicz function satisfying the Δ_2 -condition globally. We give two important consequences of an Orlicz function satisfying this condition.

PROPOSITION 2.2.13. *Let ϕ be an Orlicz function satisfying the Δ_2 -condition globally, then*

- (1) $\forall k \geq 1, \exists M_k > 0$ such that $\phi(kt) \leq M_k \phi(t)$ for all $t \geq 0$
- (2) ϕ is invertible

PROOF. 1): Let $k \geq 1$ and let $C > 0$ be such that

$$(2.2.4) \quad \phi(2t) \leq C\phi(t) < \infty \quad \forall t \geq 0$$

There exists an $n \in \mathbb{N}^+$ such that $k \leq 2^n$. It follows that

$$\begin{aligned} \phi(kt) &\leq \phi(2^n t) && \text{since } \phi \text{ is increasing} \\ &\leq C^n \phi(t) && \text{by repeated application of (2.2.4)} \end{aligned}$$

Let $M_k = C^n$.

2): Since ϕ is an Orlicz function and hence not identically zero, we have that $a_\phi < \infty$. We first show that $a_\phi = 0$. Assume that $0 < a_\phi$. Since ϕ is increasing we have that $\phi\left(\frac{3}{4}a_\phi\right) = 0$, but

$$\begin{aligned} \phi\left(2\left(\frac{3}{4}a_\phi\right)\right) &> 0 \\ &= C\phi\left(\frac{3}{4}a_\phi\right) && \forall C > 0 \end{aligned}$$

This is a contradiction since ϕ satisfies the Δ_2 -condition globally. It follows that $a_\phi = 0$. If ϕ satisfies the Δ_2 -condition for large t , then $b_\phi = \infty$, since $\phi(t) < \infty$ for all $t \geq t_0 = 0$ by (2.2.3). It follows by Proposition 2.2.8 that ϕ is strictly increasing on $(0, \infty)$ and hence invertible. \square

Next we investigate the spaces generated by Orlicz functions. If ϕ is an Orlicz function, it seems natural to consider the modular I_ϕ , defined by

$$I_\phi(f) := \int_{\Omega} \phi(|f(x)|) d\mu,$$

and the space generated by the collection of all measurable functions f such that $I_\phi(f) < \infty$. This collection of functions is denoted $P(\phi)$ and is called the *Orlicz class* corresponding to ϕ .

Orlicz classes would be nice generalizations of $L^p(\mu)$ spaces, but unfortunately an Orlicz class $P(\phi)$ is a linear subspace of $L^0(\mu)$ if and only if ϕ satisfies the Δ_2 -condition globally ([2, p.267]). Even though $P(\phi)$ is not linear in general, the next result shows that one can generate a linear space based on $P(\phi)$ using the Minkowski functional.

PROPOSITION 2.2.14. [17, p.70] *Let ϕ be an Orlicz function. The Orlicz class $P(\phi)$ is a convex set containing the origin.*

Let $A = \{f \in L^0(\mu) : I_\phi(f) \leq 1\}$. Then A is a non-empty convex, balanced subset which is absorbing at each of its points. A therefore has a Minkowski functional ρ associated with it such that $A = \{f \in L^0(\mu) : \rho(f) < 1\}$ and

$$\begin{aligned} \rho(f) &= \inf\{\lambda : \lambda \geq 0 \text{ and } f \in \lambda A\} \\ &= \inf\{k^{-1} > 0 : f \in k^{-1} A\} \\ &= \inf\{k^{-1} : kf \in A\} \\ &= \inf\{k^{-1} : I_\phi(kf) \leq 1\} \end{aligned}$$

DEFINITION 2.2.15. Let ϕ be an Orlicz function. The collection of all $f \in L^0(\mu)$ such that $I_\phi(\lambda f) < \infty$ for some $\lambda > 0$ is called an *Orlicz space* and is denoted by $L^\phi(\mu)$. Restricted to $L^\phi(\mu)$, the functional $\|\cdot\|_\phi : L^0(\mu)^+ \rightarrow [0, \infty)$ defined by

$$\|f\|_\phi = \inf\{k^{-1} : I_\phi(kf) \leq 1\}$$

is a norm. $\|\cdot\|_\phi$ as defined above is called the *Luxemburg norm*.

REMARK 2.2.16. [2, p.268] It can be shown that if $0 < \|f\|_\phi < \infty$, then

$$\|f\|_\phi = \min\{\lambda > 0 : I_\phi(f/\lambda) \leq 1\}$$

When studying the compactness of composition operators we will also be interested in weighted Orlicz sequence spaces, which are defined as follows.

DEFINITION 2.2.17. Let $\mathbf{w} = (w_n)_{n=0}^\infty$ be a sequence of positive real numbers and let μ_c be the counting measure defined on the powerset of all natural numbers. We define the weighted Orlicz sequence space $L_w^\phi(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu_c)$ as follows:

$$L_w^\phi(\mu_c) := \{f \in L^0(\mu) : I_\phi^w(f/\lambda) < \infty \text{ for some } \lambda > 0\},$$

where

$$I_\phi^w(f) := \sum_{n=0}^{\infty} \phi(|f(n)|) w_n$$

REMARK 2.2.18. Letting

$$\|f\|_\phi^w := \inf\{\lambda > 0 : I_\phi^w(f/\lambda) \leq 1\},$$

$(L_w^\phi, \|\cdot\|_\phi^w)$ is a Banach space and L_w^ϕ can be identified with $L^\phi(\mu_w) = L^\phi(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu_w)$ with equality of modulars and hence norms, where μ_w is defined by

$$\mu_w(\{n\}) = w_n$$

We demonstrate that the Orlicz spaces generalize the L^p spaces.

EXAMPLE 2.2.19. [2, p.266] The following are examples of Orlicz spaces:

- (1) If $1 \leq p < \infty$ and $\phi(t) := t^p$, then ϕ is an Orlicz function satisfying the Δ_2 -condition globally and $L^\phi(\mu) = L^p(\mu)$ with equality of norms.
- (2) If $\phi(t) := \begin{cases} 0 & 0 \leq t \leq 1 \\ \infty & t > 1 \end{cases}$ then ϕ is an Orlicz function and $L^\phi(\mu) = L^\infty(\mu)$ with equality of norms.

PROOF. 1): It is easily verified that ϕ is an Orlicz function satisfying the Δ_2 -condition globally. Furthermore

$$\begin{aligned} I_\phi(f/\lambda) &= \int_\Omega \left(\frac{|f|}{\lambda} \right)^p d\mu \\ &= \frac{\|f\|_p^p}{\lambda^p} \end{aligned}$$

It follows that $\|f\|_p = \|f\|_\phi$.

2): It is easily checked that ϕ is an Orlicz function. Let $f \in L^\infty(\mu)$ and $\|f\|_\infty = k$. Then

$$\begin{aligned} \frac{1}{k}|f(x)| &\leq 1 \quad \mu\text{-a.e.} \\ \Rightarrow \phi\left(\frac{1}{k}|f(x)|\right) &= 0 \quad \mu\text{-a.e.} \\ \Rightarrow \int_\Omega \phi\left(\frac{1}{k}|f(x)|\right) d\mu &= 0 \end{aligned}$$

Therefore $f \in L^\phi(\mu)$ and $\|f\|_\phi \leq k$.

Let $f \in L^\phi(\mu)$ and let $A = \{\lambda > 0 : I_\phi(f/\lambda) \leq 1\}$. Since $f \in L^\phi(\mu)$ we have that $A \neq \emptyset$. Let $\lambda \in A$, then

$$\begin{aligned} \int_\Omega \phi\left(\frac{1}{\lambda}|f(x)|\right) d\mu &\leq 1 \\ \Rightarrow \phi\left(\frac{1}{\lambda}|f(x)|\right) &< \infty \quad \mu\text{-a.e.} \\ \Rightarrow \frac{1}{\lambda}|f(x)| &\leq 1 \quad \mu\text{-a.e.} \end{aligned}$$

Therefore $f \in L^\infty(\mu)$ and $\|f\|_\infty \leq \lambda$. Since this holds for all $\lambda \in A$ and $\|f\|_\phi = \inf A$, we have that $\|f\|_\infty \leq \|f\|_\phi$. It follows that $f \in L^\infty(\mu)$ if and only if $f \in L^\phi(\mu)$ and $\|f\|_\infty = \|f\|_\phi$. \square

THEOREM 2.2.20. [2, p.269] *The Luxemburg norm has the following properties:*

- (1) If $0 \leq f \leq g$ μ -a.e. then $\|f\|_\phi \leq \|g\|_\phi$ and $\| |f| \|_\phi = \|f\|_\phi$ for any $f \in L^0(\mu)$, i.e. $\|\cdot\|_\phi$ is a lattice norm on $L^{\phi,w}(\mu)$
- (2) If $0 \leq f_n \uparrow f$, μ -a.e. then $\|f_n\|_\phi \uparrow \|f\|_\phi$
- (3) If $A \in \Sigma_f$, then $\|\chi_A\|_\phi < \infty$
- (4) If $A \in \Sigma_f$, then there exists a constant C_A depending only on A , such that

$$\int_A f d\mu \leq C_A \|f\|_\phi \quad \forall f \in L^\phi(\mu)$$

We finish this subsection by defining complementary Orlicz functions and stating an important result linking them.

DEFINITION 2.2.21. [2, p.271] Let ϕ be an Orlicz function and let φ be as in Theorem 2.2.7. Let

$$\begin{aligned}\psi(u) &:= \inf\{t : \varphi(t) \geq u\} \quad u \in [0, \infty) \text{ and} \\ \Psi(t) &:= \int_0^t \psi(u) du \quad t \in [0, \infty),\end{aligned}$$

then Ψ is called the *complementary Orlicz function of ϕ* .

REMARK 2.2.22. It is easily checked that ψ is increasing, left continuous, not identically infinite nor identically zero such that $\psi(0) = 0$ and therefore Ψ is an Orlicz function by Theorem 2.2.7.

The complementary Orlicz function will become important when considering duality properties of Orlicz spaces.

THEOREM 2.2.23 (Young's inequality). [2, p.271] *Let ϕ and Ψ be complementary Orlicz functions. Then*

$$st \leq \phi(s) + \Psi(t) \quad (0 \leq s, t < \infty)$$

2.2.3. Orlicz-Lorentz spaces. As the name suggests, Orlicz-Lorentz spaces are a common generalization of Orlicz spaces and Lorentz spaces. The process of defining an Orlicz-Lorentz space is similar to the one employed for Orlicz spaces, except that decreasing rearrangements are used and a weight function is introduced. After defining Orlicz-Lorentz spaces, we demonstrate that Lorentz $L^{p,q}$ spaces and Orlicz spaces are examples of Orlicz-Lorentz spaces and we explore some of the important relationships between the modular and the norm. Finally, some important properties of the Orlicz-Lorentz norm will be proved.

Throughout this section let (Ω, Σ, μ) be a σ -finite measure space and let $I = [0, \mu(\Omega)]$ if $\mu(\Omega) < \infty$ and $I = [0, \infty)$ if $\mu(\Omega) = \infty$. A map $w : I \rightarrow (0, \infty)$ is called a weight function if w is decreasing, locally integrable with respect to the Lebesgue measure and $\int_I w(s) ds = \infty$ if $\mu(\Omega) = \infty$. For $t \geq 0$ we define

$$\Gamma(t) = \int_0^t w(s) ds$$

If there exists a $k > 1$ such that

$$\Gamma(2t) \geq k\Gamma(t) \quad \forall t > 0 \text{ (or for all } t \in (0, \frac{\mu(\Omega)}{2}) \text{ if } \mu(\Omega) < \infty)$$

then w is called a *regular weight*.

REMARK 2.2.24. [14, p.1646] It can be shown that a weight function w is regular if and only if there exists a $k > 0$ such that

$$tw(t) \leq \Gamma(t) \leq ktw(t) \quad \forall t \in I$$

To define Orlicz-Lorentz spaces, let $\phi : [0, \infty) \rightarrow [0, \infty]$ be an Orlicz function, and let $w : I \rightarrow (0, \infty)$ be a weight function. Consider the following modular on $L^0(\mu)$:

$$I_{\phi,w}(f) := \int_I \phi(f^*(t))w(t)dt$$

The Orlicz-Lorentz space $L^{\phi,w}(\mu)$ is defined as the set of all $f \in L^0(\mu)$ such that $I_{\phi,w}(\lambda f) < \infty$ for some $\lambda > 0$. It can be shown ([22, p.80]) that the Orlicz-Lorentz space $L^{\phi,w}(\mu)$ is a Banach space when equipped with the norm

$$\|f\|_{\phi,w} := \inf\{\lambda > 0 : I_{\phi,w}(f/\lambda) \leq 1\}$$

REMARK 2.2.25. If $f^*(t) = \infty$ for any $t > 0$, then it is easily checked that $I_{\phi,w}(f/\lambda) = \infty$ for all $\lambda > 0$ and hence $f \notin L^{\phi,w}(\mu)$. It follows that $L^{\phi,w}(\mu) \subset L^{00}(\mu)$.

LEMMA 2.2.26. *If $0 < \|f\|_{\phi,w} < \infty$, then*

$$\|f\|_{\phi,w} = \min\{\lambda > 0 : I_{\phi,w}(f/\lambda) \leq 1\}$$

PROOF. Let $0 < \|f\|_{\phi,w} = \alpha < \infty$. For each $n \in \mathbb{N}^+$ put $f_n := \frac{f}{\alpha+1/n}$, then

$$(2.2.5) \quad I_{\phi,w}(f_n) = I_{\phi,w}(f/(\alpha+1/n)) \leq 1$$

Furthermore,

$$\begin{aligned} & f_n(t) \uparrow \alpha^{-1}f(t) \quad \forall t \\ \Rightarrow & (f_n)^* \uparrow (\alpha^{-1}f)^* \quad \text{by Proposition 1.3.5(5)} \\ \Rightarrow & \phi((f_n)^*(t)) \uparrow \phi(\alpha^{-1}f^*(t)) \quad \forall t \\ \Rightarrow & \phi((f_n)^*(t))w(t) \uparrow \phi\left(\frac{f^*}{\alpha}(t)\right)w(t) \quad \forall t \text{ since } w \geq 0 \\ \Rightarrow & \int_0^{\mu(\Omega)} \phi((f_n)^*(t))w(t) dt \uparrow \int_0^{\mu(\Omega)} \phi\left(\frac{f^*}{\alpha}(t)\right)w(t) dt \quad \text{by Theorem 1.2.6.} \end{aligned}$$

The third line in the argument above follows since ϕ is increasing and left continuous and $\left(\frac{f}{\alpha}\right)^* = \frac{f^*}{\alpha}$ by Proposition 1.3.5(2). Furthermore, $\int_0^{\mu(\Omega)} \phi((f_n)^*(t))w(t) dt \leq 1$ for all $n \in \mathbb{N}^+$ by (2.2.5) and so

$$I_{\phi,w}\left(\|f\|_{\phi,w}^{-1}f\right) = \int_0^{\mu(\Omega)} \phi\left(\frac{f^*}{\alpha}(t)\right)w(t) dt \leq 1.$$

□

We show that the Orlicz-Lorentz spaces generalize the Orlicz spaces and the Lorentz spaces. For the Orlicz spaces, we initially show the result for Orlicz spaces over a resonant measure space. Once the properties of the Orlicz-Lorentz norm have been verified, it will be demonstrated that this claim can be extended to more general measure spaces.

EXAMPLE 2.2.27. (cf. [22, p.80]) Let ϕ be an Orlicz function. If (Ω, Σ, μ) is a resonant measure space and $w \equiv 1$, then $L^{\phi,w}(\mu) = L^{\phi}(\mu)$ with equality of norms.

PROOF. Since $w \equiv 1$, we can write $w = (\chi_{\Omega})^*$. We first show that if $\phi(|f|)$ is finite almost everywhere, then $I_{\phi,w}(f) = I_{\phi}(f)$:

$$\begin{aligned} \int_0^{\mu(\Omega)} \phi(f^*(t))w(t) dt &= \int_0^{\infty} (\phi(|f|))^*(t) (\chi_{\Omega})^*(t) dt \quad \text{using Corollary 2.2.12} \\ &= \sup \left\{ \int_{\Omega} |\phi(|f|(x))g(x)| d\mu : g \sim \chi_{\Omega} \right\} \\ &= \int_{\Omega} \phi(|f|(x)) d\mu, \end{aligned}$$

where the second line follows using the defining property of a resonant measure space and the last line follows, since $g \sim \chi_{\Omega}$ implies that $|g(x)| = 1$ μ -a.e. $x \in \Omega$. Therefore $I_{\phi,w}(f) = I_{\phi}(f)$.

If $g \in L^{\phi}(\mu)$, then either $g = 0$ or $I_{\phi}(g/\|g\|_{\phi}) \leq 1$ (see Remark 2.2.16). If $g = 0$, then $\|g\|_{\phi} = 0 = \|g\|_{\phi,w}$. If $I_{\phi}(g/\|g\|_{\phi}) \leq 1$, then $\phi(|g|/\|g\|_{\phi})$ is finite almost everywhere and hence $I_{\phi,w}(g/\|g\|_{\phi}) = I_{\phi}(g/\|g\|_{\phi}) \leq 1$. This implies that $g \in L^{\phi,w}(\mu)$ and $\|g\|_{\phi,w} \leq \|g\|_{\phi}$. If $g \in L^{\phi,w}(\mu)$, then either $g = 0$ or $I_{\phi,w}(g/\|g\|_{\phi,w}) \leq 1$ (see Lemma 2.2.26). If $g = 0$, then $\|g\|_{\phi} = 0 = \|g\|_{\phi,w}$. If $I_{\phi,w}(g/\|g\|_{\phi,w}) \leq 1$, then $\phi\left(\left(g/\|g\|_{\phi,w}\right)^*(t)\right)$ is finite for all $t > 0$. This implies that ϕ is finite on $[0, \left(g/\|g\|_{\phi,w}\right)^*(0)) = [0, \frac{\|g\|_{\infty}}{\|g\|_{\phi,w}})$ and hence, for any $0 < \epsilon < 1$,

$\phi\left(\frac{\epsilon g}{\|g\|_{\phi,w}}\right)$ is finite almost everywhere. By the first part of the proof of this corollary, we therefore have that $I_{\phi}\left(\frac{\epsilon g}{\|g\|_{\phi,w}}\right) = I_{\phi,w}\left(\frac{\epsilon g}{\|g\|_{\phi,w}}\right) \leq 1$, for every $0 < \epsilon < 1$. It follows that $\|g\|_{\phi} \leq \|g\|_{\phi,w}/\epsilon$ for every $0 < \epsilon < 1$, whence $\|g\|_{\phi} \leq \|g\|_{\phi,w}$. We therefore obtain that $L^{\phi,w}(\mu) = L^{\phi}(\mu)$ with equality of norms. \square

EXAMPLE 2.2.28. [15, p.30] Let $1 \leq q \leq p < \infty$, $\phi(t) = t^q$ and $w(t) = t^{\frac{q}{p}-1}$, then $L^{\phi,w}(\mu) = L^{p,q}(\mu)$ with equality of norms.

PROOF. It is easily checked that ϕ is an Orlicz function and w is a weight function since $q \leq p$. Furthermore, for any $\lambda > 0$, $(\lambda^{-1}f)^* = \lambda^{-1}f^*$ and so

$$\begin{aligned} \|f\|_{\phi,w} &= \inf \{ \lambda > 0 : I_{\phi,w}(f/\lambda) \leq 1 \} \\ &= \inf \left\{ \lambda > 0 : \int_0^{\infty} \phi(\lambda^{-1}f^*(t)) w(t) dt \leq 1 \right\} \\ &= \inf \left\{ \lambda > 0 : \int_0^{\infty} \lambda^{-q} (f^*(t))^q t^{\frac{q}{p}-1} dt \leq 1 \right\} \\ &= \inf \left\{ \lambda > 0 : \int_0^{\infty} (t^{1/p} f^*(t))^q \frac{dt}{t} \leq \lambda^q \right\} \\ &= \|f\|_{p,q} \quad \text{since } \int_0^{\infty} (t^{1/p} f^*(t))^q \frac{dt}{t} = \|f\|_{p,q}^q \end{aligned}$$

\square

REMARK 2.2.29. It is easily shown that ϕ as defined in the example above satisfies the Δ_2 -condition globally and if $w(t) = t^{\frac{q}{p}-1}$, then $\Gamma(t) = \int_0^t s^{\frac{q}{p}-1} ds = \frac{p}{q} t^{q/p}$. It follows that $\Gamma(2t) = 2^{q/p} \Gamma(t)$ and hence that w is a regular weight. Furthermore, w is not bounded away from zero.

EXAMPLE 2.2.30. Let $\phi = \begin{cases} 0 & 0 \leq t \leq 1 \\ \infty & t > 1 \end{cases}$ and $w \equiv 1$, then $L^{\phi,w}(\mu) = L^{\infty}(\mu)$ with equality of norms.

PROOF. It is easily checked that ϕ is an Orlicz function. By Example 2.2.38, $L^{\phi,w}(\mu) = L^{\phi}(\mu)$ with equality of norms. Moreover, by Example 2.2.19, $L^{\phi}(\mu) = L^{\infty}(\mu)$ with equality of norms. \square

The next series of results details the relationship between the modular and the norm.

LEMMA 2.2.31. [2, p.268] *If ϕ is an Orlicz function and w is a weight function, then*

$$f = 0 \text{ } \mu\text{-a.e.} \iff I_{\phi,w}(\lambda f) \leq 1 \quad \forall \lambda > 0$$

PROOF. Let $\lambda > 0$. If $f = 0$ μ -a.e. then $f^* = 0$ by Example 1.3.4. Therefore $\phi(\lambda f^*(t)) = \phi(0) = 0$ for all $t \geq 0$ and hence $I_{\phi,w}(\lambda f) = 0$.

The converse will be shown using the contrapositive. Let $f \neq 0$. By Proposition 1.2.1 this implies the existence of an $\epsilon > 0$ and $A \in \Sigma$ such that $\mu(A) = \alpha > 0$ and $|f(x)| \geq \epsilon$ for all x in A . Therefore for $0 \leq \lambda < \epsilon$

$$\begin{aligned} A &\subset \{x \in \Omega : |f(x)| > \lambda\} \\ \Rightarrow \alpha = \mu(A) &\leq \mu_f(\lambda) \\ \Rightarrow \lambda &\notin \{t > 0 : \mu_f(t) \leq \frac{\alpha}{2}\} \\ \Rightarrow f^*\left(\frac{\alpha}{2}\right) &\geq \epsilon \\ (2.2.6) \quad \Rightarrow f^*(t) &\geq \epsilon \quad 0 \leq t \leq \frac{\alpha}{2} \quad \text{since } f^* \text{ is decreasing} \end{aligned}$$

Since ϕ is an Orlicz function there exists a $u \in (0, \infty)$ such that

$$(2.2.7) \quad \phi(u) > \frac{2}{w(\alpha/2)}$$

Letting $\lambda = u/\epsilon$ we have that

$$\begin{aligned}
I_{\phi,w}(\lambda f) &= \int_0^\infty \phi((\lambda f)^*(t)) w(t) dt \\
&\geq \int_0^{\alpha/2} \phi(\lambda(f^*(t))) w(t) dt \\
&\geq \int_0^{\alpha/2} \phi(\lambda\epsilon) w(t) dt \quad \text{using (2.2.6), } \phi \text{ increasing and } w \geq 0 \\
&\geq \phi(u) w(\alpha/2) \quad \text{since } w \text{ is decreasing} \\
&> 2 \quad \text{by (2.2.7)}
\end{aligned}$$

□

LEMMA 2.2.32. [2, p.268] *If ϕ is an Orlicz function and w is a weight function, then*

$$(2.2.8) \quad \|f\|_{\phi,w} \leq 1 \Rightarrow I_{\phi,w}(f) \leq \|f\|_{\phi,w}$$

and

$$(2.2.9) \quad \|f\|_{\phi,w} > 1 \Rightarrow \|f\|_{\phi,w} \leq I_{\phi,w}(f)$$

PROOF. Let $\|f\|_{\phi,w} \leq 1$. If $\|f\|_{\phi,w} = 0$, then $f = 0$ μ -a.e. and so $I_{\phi,w}(f) = 0$ as in the proof of the lemma above (using $\lambda = 1$). Therefore $I_{\phi,w}(f) \leq \|f\|_{\phi,w}$. If $0 < \|f\|_{\phi,w} \leq 1$ then by Lemma 2.2.26

$$(2.2.10) \quad I_{\phi,w}(\alpha f) \leq 1,$$

where $\alpha = (\|f\|_{\phi,w})^{-1}$. Furthermore, if $s > 0$

$$\begin{aligned}
\frac{\phi(s)}{s} &\leq \frac{\phi(\alpha s)}{\alpha s} \quad \text{since } \alpha \geq 1 \text{ and} \\
&\quad \frac{\phi(t)}{t} \text{ is an increasing function of } t \\
(2.2.11) \quad &\Rightarrow \alpha \phi(s) \leq \phi(\alpha s) \quad \forall s \geq 0
\end{aligned}$$

So

$$\begin{aligned}
1 &\geq I_{\phi,w}(\alpha f) \\
&= \int_0^\infty \phi((\alpha f)^*(t)) w(t) dt \\
&= \int_0^\infty \phi(\alpha f^*(t)) w(t) dt \\
&\geq \int_0^\infty \alpha \phi(f^*(t)) w(t) dt \quad \text{using (2.2.11) and } w \geq 0 \\
&= \alpha I_{\phi,w}(f) \\
\Rightarrow I_{\phi,w}(f) &\leq \alpha^{-1} = \|f\|_{\phi,w}
\end{aligned}$$

Conversely, if $\|f\|_{\phi,w} = \alpha > 1$, then there exists an $N \in \mathbb{N}^+$ such that $N^{-1} < \alpha - 1$. Let $n \geq N$, then $1 < \alpha - \frac{1}{n} < \|f\|_{\phi,w}$. If we assume that $I_{\phi,w}\left(\frac{f}{\alpha - 1/n}\right) \leq 1$ then we obtain

$$\|f\|_{\phi,w} = \inf\{\lambda > 0 : I_{\phi,w}(f/\lambda) \leq 1\} \leq \alpha - \frac{1}{n},$$

which is a contradiction and so

$$(2.2.12) \quad I_{\phi,w}\left(\frac{f}{\alpha - 1/n}\right) > 1$$

We have that $\frac{\phi(t)}{t}$ is an increasing function of t and so

$$(2.2.13) \quad \begin{aligned} \frac{\phi((\alpha - 1/n)^{-1}s)}{(\alpha - 1/n)^{-1}s} &\leq \frac{\phi(s)}{s} \quad \forall s > 0 \\ &\text{since } (\alpha - 1/n)^{-1} \leq 1 \\ \Rightarrow (\alpha - 1/n)\phi\left(\frac{s}{\alpha - 1/n}\right) &\leq \phi(s) \quad \forall s > 0 \end{aligned}$$

and so

$$\begin{aligned} I_{\phi,w}(f) &= \int_0^\infty \phi(f^*(t)) w(t) dt \\ &\geq \int_0^\infty (\alpha - 1/n)\phi\left(\frac{1}{\alpha - 1/n}f^*(t)\right) w(t) dt \quad \text{using (2.2.13) and } w \geq 0 \\ &= (\alpha - 1/n) \int_0^\infty \phi\left(\left(\frac{f}{\alpha - 1/n}\right)^*(t)\right) w(t) dt \\ &= (\alpha - 1/n)I_{\phi,w}\left(\frac{f}{\alpha - 1/n}\right) \\ &> \alpha - 1/n \quad \text{by (2.2.12)} \end{aligned}$$

Since this holds for every $n \geq N$, we have that $I_{\phi,w}(f) \geq \alpha = \|f\|_{\phi,w}$. □

COROLLARY 2.2.33. $I_{\phi,w}(f) \leq 1 \Leftrightarrow \|f\|_{\phi,w} \leq 1$

PROOF. We have by Lemma 2.2.32 (2.2.8) that

$$\|f\|_{\phi,w} \leq 1 \Rightarrow I_{\phi,w}(f) \leq \|f\|_{\phi,w} \leq 1$$

Let $I_{\phi,w}(f) \leq 1$. If we assume that $\|f\|_{\phi,w} > 1$, then

$$\begin{aligned} I_{\phi,w}(f) &\geq \|f\|_{\phi,w} \quad \text{by Lemma 2.2.32 (2.2.9)} \\ &> 1 \end{aligned}$$

which is a contradiction and so $\|f\|_{\phi,w} \leq 1$. □

LEMMA 2.2.34. *Let ϕ be an Orlicz function, w a weight function, $k > 0$ and $f \in L^0(\mu)$. If $I_{\phi,w}(f) \geq k$, then $\|f\|_{\phi,w} \geq \min\{1, k\}$. If $I_{\phi,w}(f) \leq k$, then $\|f\|_{\phi,w} \leq \max\{1, k\}$.*

PROOF. If $I_{\phi,w}(f) > 1$, then $\|f\|_{\phi,w} > 1$ by Corollary 2.2.33. If $k \leq I_{\phi,w}(f) \leq 1$, then $\|f\|_{\phi,w} \leq 1$ by the same corollary and so $\|f\|_{\phi,w} \geq I_{\phi,w}(f) \geq k$ by Lemma 2.2.32. Therefore $\|f\|_{\phi,w} \geq \min\{1, k\}$. It can similarly be shown that if $I_{\phi,w}(f) \leq k$, then $\|f\|_{\phi,w} \leq \max\{1, k\}$. □

PROPOSITION 2.2.35. *Let ϕ be an Orlicz function satisfying the Δ_2 -condition globally and let w be a weight function. Then for every $\epsilon > 0$ there exists a $\delta > 0$ such that*

$$I_{\phi,w}(f) < \delta \Rightarrow \|f\|_{\phi,w} < \epsilon$$

PROOF. If $\phi \in \Delta_2$ globally then for every $k \geq 1$ there exists an $M_k > 0$ such that

$$(2.2.14) \quad \phi(ku) \leq M_k \phi(u) \quad \forall u \geq 0$$

by Proposition 2.2.13. Let $\epsilon > 0$. We can assume that $0 < \epsilon < 1$. Let $M_{2/\epsilon}$ be such that (2.2.14) holds for $k = 2/\epsilon$ and let $\delta = \frac{1}{M_{2/\epsilon}}$. Then $I_{\phi,w}(f) < \delta$ implies

$$\begin{aligned} I_{\phi,w}\left(\frac{f}{\epsilon/2}\right) &= \int_0^{\mu(\Omega)} \phi\left(\frac{2}{\epsilon}f^*(t)\right) w(t) dt \\ &\leq \int_0^{\mu(\Omega)} M_{2/\epsilon} \phi(f^*(t)) w(t) dt \quad \text{since } w \geq 0 \\ &= M_{2/\epsilon} I_{\phi,w}(f) \\ &< 1 \end{aligned}$$

and so

$$\begin{aligned} \|f\|_{\phi,w} &= \inf\{\lambda > 0 : I_{\phi,w}(f/\lambda) \leq 1\} \\ &\leq \epsilon/2 \\ &< \epsilon \end{aligned}$$

□

PROPOSITION 2.2.36. *Let w be a weight function and ϕ_1, ϕ_2 Orlicz functions. Then*

$$\begin{aligned} \phi_1 \leq \phi_2 &\Rightarrow I_{\phi_1,w}(f) \leq I_{\phi_2,w}(f) \quad \forall f \in L^0(\mu) \\ &\Rightarrow \|f\|_{\phi_1,w} \leq \|f\|_{\phi_2,w} \quad \forall f \in L^0(\mu) \end{aligned}$$

PROOF. Let $f \in L^0(\mu)$. If $\phi_1 \leq \phi_2$, then

$$\begin{aligned} I_{\phi_1,w}(f) &= \int_0^{\mu(\Omega)} \phi_1(f^*(t)) w(t) dt \\ &\leq \int_0^{\mu(\Omega)} \phi_2(f^*(t)) w(t) dt \quad \text{since } \phi_1 \leq \phi_2 \text{ and } w \geq 0 \\ &= I_{\phi_2,w}(f) \end{aligned}$$

We show that the second implication holds. Let $I_{\phi_1,w}(f) \leq I_{\phi_2,w}(f)$ for all $f \in L^0(\mu)$. If $\|f\|_{\phi_2,w} = 0$, then $f = 0$ and so $\|f\|_{\phi_1,w} = 0 \leq \|f\|_{\phi_2,w}$. If $\|f\|_{\phi_2,w} = \infty$, then $\|f\|_{\phi_1,w} \leq \|f\|_{\phi_2,w}$ trivially. Let $0 < \|f\|_{\phi_2,w} < \infty$, then

$$\begin{aligned} I_{\phi_1,w}\left(f/\|f\|_{\phi_2,w}\right) &\leq I_{\phi_2,w}\left(f/\|f\|_{\phi_2,w}\right) \\ &\leq 1 \quad \text{by Lemma 2.2.26} \end{aligned}$$

and therefore $\|f\|_{\phi_1,w} \leq \|f\|_{\phi_2,w}$. □

The following result details the properties of the Orlicz-Lorentz norm.

THEOREM 2.2.37. *Let ϕ be an Orlicz function and w a weight function, then the following hold:*

- (1) *If $0 \leq f \leq g$ μ -a.e. then $\|f\|_{\phi,w} \leq \|g\|_{\phi,w}$ and $\|f\|_{\phi,w} = \|f\|_{\phi,w}$ for any $f \in L^0(\mu)$, i.e. $\|\cdot\|_{\phi,w}$ is a lattice norm on $L^{\phi,w}(\mu)$*
- (2) *If $0 \leq f_n \uparrow f$, μ -a.e. then $\|f_n\|_{\phi,w} \uparrow \|f\|_{\phi,w}$*
- (3) *If $A \in \Sigma_f$, then $\|\chi_A\|_{\phi,w} < \infty$*
- (4) *If $A \in \Sigma_f$, then there exists a constant C_A depending only on A , such that*

$$\int_A f d\mu \leq C_A \|f\|_{\phi,w} \quad \forall f \in L^{\phi,w}(\mu)$$

PROOF. 1): Let $f, g \in L^0(\mu)$ such that $0 \leq f \leq g$ μ -a.e.

If $\|g\|_{\phi,w} = 0$, then $g = 0$ μ -a.e. This implies that $f = 0$ μ -a.e. and hence $\|f\|_{\phi,w} = 0 = \|g\|_{\phi,w}$.

If $0 < \|g\|_{\phi,w} < \infty$, then

$$(2.2.15) \quad \|g\|_{\phi,w} = \min\{\lambda > 0 : I_{\phi,w}(g/\lambda) \leq 1\} = \alpha \quad \text{by Lemma 2.2.26}$$

We therefore have that

$$\begin{aligned} \frac{f}{\alpha}(x) &\leq \frac{g}{\alpha}(x) \quad \mu\text{-a.e.} \\ \Rightarrow \left(\frac{f}{\alpha}\right)^*(t) &\leq \left(\frac{g}{\alpha}\right)^*(t) \quad \forall t > 0 \text{ by Proposition 1.3.5(1)} \\ \Rightarrow \phi\left(\left(\frac{f}{\alpha}\right)^*(t)\right) w(t) &\leq \phi\left(\left(\frac{g}{\alpha}\right)^*(t)\right) w(t) \quad \forall t > 0 \text{ since } \phi \text{ is increasing and } w \geq 0 \\ \Rightarrow I_{\phi,w}(f/\alpha) &= \int_0^{\mu(\Omega)} \phi\left(\left(\frac{f}{\alpha}\right)^*(t)\right) w(t) dt \leq \int_0^{\mu(\Omega)} \phi\left(\left(\frac{g}{\alpha}\right)^*(t)\right) w(t) dt = I_{\phi,w}(g/\alpha) \\ &\leq 1 \quad \text{by (2.2.15)} \end{aligned}$$

It follows that $\|f\|_{\phi,w} := \inf\{\lambda > 0 : I_{\phi,w}(f/\lambda) \leq 1\} \leq \alpha = \|g\|_{\phi,w}$

If $\|g\|_{\phi,w} = \infty$, then $\|f\|_{\phi,w} \leq \|g\|_{\phi,w}$.

Since $|f|^* = f^*$, it follows that $\|f\|_{\phi,w} = \|f\|_{\phi,w}$. Therefore $\|\cdot\|_{\phi,w}$ is a lattice norm.

2): Let $0 \leq f_n \uparrow f$, $\alpha_n := \|f_n\|_{\phi,w}$ and $\alpha := \sup \alpha_n$. Then since $\|\cdot\|_{\phi,w}$ is a lattice norm we have that $0 \leq \alpha_n \leq \|f\|_{\phi,w}$ for each n . It follows that $\alpha \leq \|f\|_{\phi,w}$. If $\alpha = 0$ or $\alpha = \infty$, then we trivially have that $\alpha_n \uparrow \|f\|_{\phi,w}$. Let $0 < \alpha_n \leq \alpha < \infty$, then

$$\begin{aligned} 0 \leq \frac{f_n}{\alpha} &\leq \frac{f_n}{\alpha_n} \quad \forall n \in \mathbb{N}^+ \\ \Rightarrow I_{\phi,w}\left(\frac{f_n}{\alpha}\right) &\leq I_{\phi,w}\left(\frac{f_n}{\alpha_n}\right) \quad \forall n \in \mathbb{N}^+ \\ (2.2.16) \quad &\leq 1 \quad \text{for each } n \text{ by Lemma 2.2.26} \end{aligned}$$

Furthermore

$$\begin{aligned} f_n/\alpha &\uparrow f/\alpha \quad \mu\text{-a.e.} \\ \Rightarrow (f_n/\alpha)^*(t) &\uparrow (f/\alpha)^*(t) \quad \forall t > 0 \quad \text{by Proposition 1.3.5(5)} \\ \Rightarrow \phi((f_n/\alpha)^*(t)) w(t) &\uparrow \phi((f/\alpha)^*(t)) w(t) \\ &\text{since } \phi \text{ is increasing, continuous from the left and } w \geq 0 \\ \Rightarrow I_{\phi,w}(f_n/\alpha) &= \int_0^{\mu(\Omega)} \phi((f_n/\alpha)^*(t)) w(t) dt \\ &\uparrow \int_0^{\mu(\Omega)} \phi((f/\alpha)^*(t)) w(t) dt \\ &\text{by the Monotone Convergence theorem 1.2.6} \\ &= I_{\phi,w}(f/\alpha) \end{aligned}$$

Using (2.2.16) this implies that $I_{\phi,w}(f/\alpha) \leq 1$ and hence that $\|f\|_{\phi,w} \leq \alpha$ by the definition of $\|\cdot\|_{\phi,w}$. Therefore $\|f\|_{\phi,w} = \alpha$ and thus $\|f_n\|_{\phi,w} \uparrow \|f\|_{\phi,w}$.

3): Let $A \in \Sigma_f$ and let $\lambda > 0$. Then

$$\begin{aligned}
 I_{\phi,w}(\lambda\chi_A) &= \int_0^{\mu(\Omega)} \phi((\lambda\chi_A)^*(t)) w(t) dt \\
 &= \int_0^{\mu(\Omega)} \phi(\lambda\chi_{[0,\mu(A)]}(t)) w(t) dt \quad \text{by Proposition 1.3.5(2) and Example 1.3.4} \\
 (2.2.17) \quad &= \phi(\lambda) \int_0^{\mu(A)} w(t) dt
 \end{aligned}$$

Since w is locally integrable and $\mu(A) < \infty$, it follows that $\int_0^{\mu(A)} w(t) dt < \infty$. Furthermore, since ϕ is an Orlicz function it is not identically infinite on $(0, \infty)$. Therefore there exists a $\lambda > 0$ such that $\phi(\lambda) < \infty$ and hence $\phi(\lambda) \int_0^{\mu(A)} w(t) dt < \infty$. Thus $\chi_A \in L^{\phi,w}(\mu)$ and so $\|\chi_A\| < \infty$.

4): Let Ψ be the complementary Orlicz function to ϕ and let $0 < \mu(A) < \infty$. Then $0 < \|\chi_A\|_{\Psi,w} < \infty$, since $\|\cdot\|_{\Psi,w}$ is a norm and it has been established that every Orlicz-Lorentz space contains all characteristic functions of measurable sets of finite measure. If $0 < \|f\|_{\phi,w} < \infty$, then

$$\begin{aligned}
 \frac{f^*(t)}{\|f\|_{\phi,w}} \cdot \frac{\chi_A^*(t)}{\|\chi_A\|_{\Psi,w}} \cdot w(t) &= \left(\|f\|_{\phi,w}^{-1} f\right)^*(t) \left(\|\chi_A\|_{\Psi,w}^{-1} \chi_A\right)^*(t) \cdot w(t) \quad \text{by Proposition 1.3.5(2)} \\
 (2.2.18) \quad &\leq \phi\left(\left(\|f\|_{\phi,w}^{-1} f\right)^*(t)\right) w(t) + \Psi\left(\left(\|\chi_A\|_{\Psi,w}^{-1} \chi_A\right)^*(t)\right) w(t) \quad \text{by Theorem 2.2.23}
 \end{aligned}$$

Furthermore

$$\begin{aligned}
 \frac{w(\mu(A))}{\|f\|_{\phi,w} \|\chi_A\|_{\Psi,w}} \int_0^{\mu(A)} f^*(t) dt &\leq \frac{1}{\|f\|_{\phi,w} \|\chi_A\|_{\Psi,w}} \int_0^{\mu(A)} f^*(t) w(t) dt \\
 &\quad \text{since } w \text{ is decreasing} \\
 &= \frac{1}{\|f\|_{\phi,w} \|\chi_A\|_{\Psi,w}} \int_0^{\infty} f^*(t) \chi_A^*(t) w(t) dt \\
 &\quad \text{by Example 1.3.4} \\
 &\leq \int_0^{\infty} \phi\left(\left(\|f\|_{\phi,w}^{-1} f\right)^*(t)\right) w(t) dt + \\
 &\quad \int_0^{\infty} \Psi\left(\left(\|\chi_A\|_{\Psi,w}^{-1} \chi_A\right)^*(t)\right) w(t) dt \quad \text{by (2.2.18)} \\
 (2.2.19) \quad &\leq 1 + 1 \quad \text{by Lemma 2.2.26}
 \end{aligned}$$

It follows that

$$\begin{aligned}
 \int_A |f| d\mu &\leq \int_0^{\mu(A)} f^*(t) dt \quad \text{by (1.3.3)} \\
 &\leq \frac{2\|f\|_{\phi,w} \|\chi_A\|_{\Psi,w}}{w(\mu(A))} \quad \text{by (2.2.19)} \\
 &= C_A \|f\|_{\phi,w}
 \end{aligned}$$

where $C_A = \frac{2\|\chi_A\|_{\Psi,w}}{w(\mu(A))}$. If $\mu(A) = 0$ or $\|f\|_{\phi,w} = 0$, then

$$\begin{aligned}
 \int_A |f| d\mu &= 0 \\
 &\leq C_A \|f\|_{\phi,w}
 \end{aligned}$$

□

EXAMPLE 2.2.38. Let (Ω, Σ, μ) be any σ -finite measure space. If ϕ is an Orlicz function and $w \equiv 1$, then $L^{\phi,w}(\mu) = L^{\phi}(\mu)$ with equality of norms.

PROOF. If f is a positive simple function, then f can be written in the form $f = \sum_{i=1}^n \alpha_i \chi_{A_i}$, where the A_i 's are pairwise disjoint. Then by Example 1.3.4, $f^* = \sum_{i=1}^n \alpha_i \chi_{[k_{i-1}, k_i]}$, where $k_0 = 0$ and $k_j = \sum_{i=1}^j \mu(A_i)$. It follows that

$$\begin{aligned}
 \int_0^{\mu(\Omega)} \phi(f^*(t))w(t) dt &= \int_0^{\mu(\Omega)} \phi\left(\sum_{i=1}^n \alpha_i \chi_{[k_{i-1}, k_i]}\right) dt \\
 &= \sum_{i=1}^n \int_{k_{i-1}}^{k_i} \phi(\alpha_i) dt \\
 &= \sum_{i=1}^n \phi(\alpha_i)(k_i - k_{i-1}) \\
 &= \sum_{i=1}^n \phi(\alpha_i)\mu(A_i) \quad \text{since } k_i - k_{i-1} = \mu(A_i) \\
 &= \sum_{i=1}^n \int_{A_i} \phi(\alpha_i) d\mu \\
 &= \int_{\Omega} \phi\left(\sum_{i=1}^n \alpha_i \chi_{A_i}\right) d\mu \quad \text{since } A_k \cap A_m = \emptyset \text{ if } k \neq m
 \end{aligned}$$

Therefore $I_{\phi, w}(f) = I_{\phi}(f)$, and hence $\|f\|_{\phi, w} = \|f\|_{\phi}$ for every positive simple function f . Let $g \in L^{00}(\mu)$. There exists a sequence $(f_n)_{n=1}^{\infty}$ of positive simple functions such that $g_n \uparrow |g|$ pointwise μ -a.e. It follows that $\|g_n\|_{\phi, w} \uparrow \| |g| \|_{\phi, w} = \|g\|_{\phi, w}$ by Theorem 2.2.37. Similarly $\|g_n\|_{\phi} \uparrow \| |g| \|_{\phi} = \|g\|_{\phi}$ by Theorem 2.2.20. Furthermore $\|g_n\|_{\phi, w} = \|g_n\|_{\phi}$ for every n and so $\|g\|_{\phi, w} = \|g\|_{\phi}$. \square

Banach function spaces

In Theorem 2.2.37 we saw that the Orlicz-Lorentz norm, and hence Lorentz, Orlicz and Lebesgue norms, all satisfy a particular set of properties. Motivated by these properties we define a Banach function norm, which will enable us to generalize the aforementioned spaces and provide a framework for studying composition operators on Banach spaces of (equivalence classes) of real or complex-valued measurable functions.

This first section consists of a formal definition of Banach function spaces and some elementary properties derived from this definition. Furthermore, associate spaces are introduced and some examples given. In the second section we identify the conditions under which the dual space of a Banach function space X can be identified with the associate space of X .

3.1. Banach function spaces

Authors differ somewhat on how to define a Banach function space. We discuss this briefly and provide some motivation for our particular choice of definition. We then list some properties of Banach function spaces and demonstrate some examples. Furthermore, the associate space of a Banach function space will be discussed and some explicit descriptions given of the associate spaces of Orlicz spaces, Lorentz spaces and Lebesgue spaces. Lastly, it will be shown that our definition of an associate space yields a generalization of Hölder's inequality and that the second associate space of a Banach function space coincides with the original space.

DEFINITION 3.1.1. [26, p.138] Let $\rho : L^0(\mu)^+ \rightarrow [0, \infty]$ be a mapping. If ρ satisfies properties 1) to 4), given below, for all f, g , in $L^0_+(\mu)$ and for all $\alpha \in \mathbb{F}$ then ρ is called a *function seminorm*.

- (1) $\rho(0) = 0$
- (2) $\rho(|\alpha|f) = |\alpha|\rho(f)$
- (3) $\rho(f + g) \leq \rho(f) + \rho(g)$
- (4) $g \leq f$ μ -a.e. implies that $\rho(g) \leq \rho(f)$

If in addition $\rho(f) = 0$ implies that $f = 0$ μ -a.e. then ρ is called a *function norm*.

Let ρ be a function norm on $L^0_+(\mu)$. The Köthe function space $X(\rho)$ is defined to be the set of all $f \in L^0(\mu)$ such that $\|f\|_X := \rho(|f|) < \infty$. It is clear from the definition of a function norm that a Köthe function space is a normed space. We list three important properties which a function norm may possess.

DEFINITION 3.1.2. [26, p.141,148] Let $\rho : L^0_+(\mu) \rightarrow [0, \infty]$ be a function norm and let $(f_n)_{n=1}^\infty \cup \{f\} \subset L^0_+(\mu)$.

- (1) If $f_n \uparrow f$ pointwise μ -a.e. implies that $\rho(f_n) \uparrow \rho(f)$ then ρ is said to have the *Fatou property*.
- (2) If $f_n \uparrow f$ pointwise μ -a.e. and $\lim_{n \rightarrow \infty} \rho(f_n) < \infty$ implies that $\rho(f) < \infty$, then ρ is said to have the *weak Fatou property*.
- (3) If $\sum_{n=1}^\infty \rho(f_n) < \infty$ implies that $\rho\left(\sum_{n=1}^\infty f_n\right) < \infty$, then ρ is said to have the *Riesz-Fischer property*.

It is clear that the Fatou property in the above definition implies the weak Fatou property. Luxemburg and Zaanen ([26, p.150,143]) showed that the weak Fatou property implies the Riesz-Fischer property and that a Köthe function space $X(\rho)$ is complete if and only if ρ possesses the Riesz-Fischer property. Furthermore,

examples are given to demonstrate that the Riesz-Fischer property does not necessarily imply the weak Fatou property and that the weak Fatou property does not necessarily imply the Fatou property.

Some authors ([26], [27]) define a Banach function space to be a K i et e function space $X(\rho)$ which is complete with respect to the norm induced by ρ . Motivated by Theorem 2.2.37, which shows that the Orlicz-Lorentz norms (and hence the Lorentz, Orlicz and Lebesgue norms) have the Fatou property, and a desire to use results obtained in [2], we use the following definition for a Banach function norm.

DEFINITION 3.1.3. [2, p.2] Let ρ be a function norm on $L_+^0(\mu)$. If in addition ρ satisfies the following two properties, then ρ is called a *Banach function norm*.

- (1) $f_n \uparrow f$ pointwise μ -a.e. implies that $\rho(f_n) \uparrow \rho(f)$
- (2) $\mu(A) < \infty$ implies that $\rho(\chi_A) < \infty$ and there exists a $0 < C_A < \infty$ such that

$$\int_A f d\mu \leq C_A \rho(f) \quad \forall f \in L_+^0(\mu)$$

Let ρ be a Banach function norm on $L_+^0(\mu)$. The collection of all $f \in L^0(\mu)$ such that $\rho(|f|) < \infty$ is called a *Banach function space* and is denoted by $X = X(\rho)$. For each $f \in X$, define

$$\|f\|_X = \rho(|f|)$$

We list some properties of a Banach function space.

PROPOSITION 3.1.4. [2, p.4] *Let $X(\rho)$ be a Banach function space, then the following hold:*

- (1) $(X, \|\cdot\|_X)$ is a Banach space
- (2) $0 \leq f \leq g$ μ -a.e. implies that $\|f\|_X \leq \|g\|_X$
- (3) X contains the simple functions with supports of finite measure
- (4) $A \in \Sigma_f$ implies there exists a $0 < C_A < \infty$ such that

$$\int_A |f| d\mu \leq C_A \|f\|_X \quad \forall f \in L^0(\mu)$$

- (5) (the Fatou property) Let $\{f_n\}_{n=1}^\infty \subset X$ such that $0 \leq f_n \uparrow f$ μ -a.e.. Then $f \in X$ and $\|f_n\|_X \uparrow \|f\|_X$ or $f \notin X$ and $\|f_n\|_X \uparrow \infty$
- (6) If $A, B \in \Sigma$ and $A \cap B = \emptyset$, then

$$\|\alpha\chi_A - \beta\chi_B\|_X = \|\alpha\chi_A + \beta\chi_B\|_X \quad \forall \alpha, \beta \in \mathbb{F}$$

and if $\tau : \Omega \rightarrow \Omega$ is a measurable transformation, then

$$\|\chi_{\tau^{-1}(A)} - \chi_{\tau^{-1}(B)}\|_X = \|\chi_{\tau^{-1}(A)} + \chi_{\tau^{-1}(B)}\|_X$$

PROOF. For the most part, these properties follow immediately from the definition of a Banach function norm. We make two comments however.

1) It is clear that $X(\rho)$ is a K i et e function space and therefore a normed space. We have defined a Banach function norm in such a way that it possesses the Fatou property. By our earlier discussion this implies the Riesz-Fischer property, which in turn is equivalent to the completeness of the K i et e function space.

6) If $A \cap B = \emptyset$ then it is easily shown that $|(\alpha\chi_A - \beta\chi_B)(x)| = |(\alpha\chi_A + \beta\chi_B)(x)|$ for all $x \in \Omega$ and hence $\|\alpha\chi_A - \beta\chi_B\|_X = \|\alpha\chi_A + \beta\chi_B\|_X$. The second part of the lemma follows since

$$A \cap B = \emptyset \Rightarrow \tau^{-1}(A) \cap \tau^{-1}(B) = \emptyset$$

The other properties follow directly from the definition of a Banach function norm. \square

It can be shown ([8, p.12]) that if A is an order ideal (i.e. a solid set which is also a vector subspace of a Riesz space E), then A is a Riesz subspace. We use this to show that X is a Banach lattice.

PROPOSITION 3.1.5. *If $L^0(\mu)$ consists of equivalence classes of real-valued, almost everywhere finite, measurable functions, and $X \subset L^0(\mu)$ is a Banach function space, then $(X, +, \cdot, \leq, \|\cdot\|_X)$ is a Banach lattice.*

PROOF. This follows since if $g \in X$ and $f \in L^0(\mu)$ such that $|f| \leq |g|$, then $|f|(x) \leq |g|(x)$ μ -a.e. and hence

$$\begin{aligned} \|f\|_X &= \||f|\|_X \\ &\leq \||g|\|_X \\ &= \|g\|_X < \infty \quad \text{since } g \in X, \end{aligned}$$

i.e. X is a solid subset of the Riesz space $L^0(\mu)$ (Proposition 2.1.8). X is therefore a Riesz subspace of $L^0(\mu)$. Since $|f(x)| \leq |g(x)|$ μ -a.e. implies that $\|f\|_X \leq \|g\|_X$, $\|\cdot\|_X$ is a lattice norm. Furthermore, $(X, \|\cdot\|_X)$ is complete and hence a Banach lattice. \square

We will need the following result to apply interpolation theory to Banach function spaces.

THEOREM 3.1.6. [2, p.3] *If X is a Banach function space, then X is continuously embedded in $L^0(\mu)$ equipped with the topology of local convergence in measure, i.e. if $f_n \rightarrow f$ in X then $f_n \rightarrow f$ in measure on sets of finite measure.*

PROOF. Let $f_n \rightarrow f$ in X and let $\epsilon > 0$. Let $A \in \Sigma_f$ and define $A_{\epsilon,n} := \{x \in A : |f(x) - f_n(x)| > \epsilon\}$. It follows that

$$\begin{aligned} \mu(A_{\epsilon,n}) \cdot \epsilon &\leq \int_{A_{\epsilon,n}} |f(x) - f_n(x)| d\mu \\ \Rightarrow \mu(A_{\epsilon,n}) &\leq \frac{1}{\epsilon} \int_A |f(x) - f_n(x)| d\mu \quad \text{since } A_{\epsilon,n} \subset A \\ &\leq \frac{1}{\epsilon} C_A \|f - f_n\|_X \quad \text{by Proposition 3.1.4(4)} \\ &\rightarrow 0 \quad \text{as } n \rightarrow \infty \end{aligned}$$

\square

We show that the function spaces described in Section 2.2 are Banach function spaces.

EXAMPLE 3.1.7. Let (Ω, Σ, μ) be a σ -finite measure space, ϕ an Orlicz function and w a weight function. The Orlicz-Lorentz space $L^{\phi,w}(\mu)$ is a Banach function space by Theorem 2.2.37. The Lorentz spaces, Orlicz spaces and Lebesgue spaces are therefore also Banach function spaces, since these are examples of Orlicz-Lorentz spaces (Examples 2.2.28, 2.2.38 and 2.2.2).

EXAMPLE 3.1.8. By Theorem 2.2.5 the Lorentz space $M_\varphi(\mathbb{R}^+, \mathcal{M}, m)$ is a Banach function space if φ is a quasiconcave function.

EXAMPLE 3.1.9. The Lorentz space $L^{(p,q)}(\mu)$ ($1 < p < \infty$, $1 \leq q \leq \infty$) is a Banach function space by Theorem 2.2.5.

Based on Hölder's inequality for L^p spaces, it can be shown that if $\frac{1}{p} + \frac{1}{p'} = 1$, then

$$\|g\|_{L^{p'}} = \sup \left\{ \int_\Omega |fg| d\mu : f \in B_{L^p} \right\}$$

Let ρ be a Banach function norm on $L_+^0(\mu)$. Motivated by the discussion above, we define a functional ρ' on $L_+^0(\mu)$ by

$$\rho'(g) := \sup \left\{ \int_\Omega fg d\mu : f \in L^0(\mu)^+, \rho(f) \leq 1 \right\}.$$

This functional ρ' is called the *associate norm* of ρ . It can be shown ([2, p.8]) that if ρ is a Banach function norm, then its associate norm ρ' is also a Banach function norm. The Banach function space generated by the associate norm ρ' is called the *associate space* of X and is denoted X' . Furthermore

$$(3.1.1) \quad \|g\|_{X'} = \sup \left\{ \int_{\Omega} |fg| d\mu : f \in B_X \right\} \quad \forall g \in X'$$

The following theorems show that we can give more explicit descriptions of the associate spaces of Orlicz spaces, Lorentz spaces and Lebesgue spaces.

THEOREM 3.1.10. [2, p.275] *Let (Ω, Σ, μ) be a resonant measure space and let ϕ be an Orlicz function. Then $L^{\phi}(\mu)$ equipped with the Luxemburg norm has for its associate space $L^{\psi}(\mu)$ equipped with the Orlicz norm, where ψ is the complementary Orlicz function of ϕ .*

THEOREM 3.1.11. [2, p.220] *Let (Ω, Σ, μ) be a resonant measure space and let $1 < p \leq q < \infty$. Then the associate space of $L^{p,q}(\mu)$ is, up to equivalence of norms, the Lorentz space $L^{p',q'}(\mu)$, where $1/p + 1/p' = 1 = 1/q + 1/q'$.*

COROLLARY 3.1.12. *Let (Ω, Σ, μ) be a resonant measure space and let $1 < p < \infty$. Then the associate space of $L^p(\mu)$ is, up to equivalence of norms, the Lebesgue space $L^{p'}(\mu)$, where $1/p + 1/p' = 1$.*

Holder's inequality for L^p spaces has the following generalization to Banach function spaces:

THEOREM 3.1.13 (Hölder's inequality). [41, p.418] *Let X be a Banach function space and X' its associate space. If $f \in X$ and $g \in X'$, then fg is integrable and*

$$\int_{\Omega} |fg| d\mu \leq \|f\|_X \|g\|_{X'}$$

The following theorem due to Lorentz and Luxemburg asserts that every Banach function space coincides with its second associate space.

THEOREM 3.1.14. [2, p.10] *Let X be a Banach function space, then $X = (X')'$ and*

$$\|f\|_X = \|f\|_{(X')'}$$

for every $f \in L^0(\mu)$.

If X and Y are Banach function spaces and X is continuously embedded in Y , then the associate space of Y will be continuously embedded in the associate space of X as shown by the following proposition.

PROPOSITION 3.1.15. [2, p.13] *If X and Y are Banach function spaces and*

$$\|f\|_Y \leq k \|f\|_X \quad \forall f \in X$$

for some $k > 0$, then

$$\|f\|_{X'} \leq k \|f\|_{Y'} \quad \forall f \in Y'$$

3.2. Duals of Banach function spaces

Let X be a Banach function space and let $g \in X'$, then

$$L_g : f \mapsto \int_{\Omega} fg d\mu$$

defines a bounded linear functional on X . We are interested in the relationship between the dual space X^* and the associate space X' of a Banach function space. In general, identifying $g \in X'$ with L_g , it can be shown ([2, p.13]) that X' is isometrically isomorphic to a closed subspace of X^* . To determine the conditions under which

the associate space can be identified with the dual space we introduce two important subspaces of X , namely X_a , the set of all functions in X with absolutely continuous norm, and X_b , the closure in X of the set of simple functions. We give some properties of these subspaces before detailing their significance in describing the dual of X and in determining if X is reflexive. We start by defining what it means for a function to have absolutely continuous norm.

Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of μ -measurable sets. We shall write $A_n \rightarrow \emptyset$ if the characteristic functions $\chi_{A_n} \rightarrow 0$ μ -a.e. If in addition the sequence $\{A_n\}_{n=1}^{\infty}$ is decreasing, then we shall write $A_n \downarrow \emptyset$. Let X be a Banach function space and let $f \in X$. If $\|f\chi_{A_n}\|_X \rightarrow 0$ for every sequence $\{A_n\}_{n=1}^{\infty}$ satisfying $A_n \rightarrow \emptyset$, then f is said to have *absolutely continuous norm* in X . If $X_a = X$, then X is said to have absolutely continuous norm. It can be shown ([2, p.14]) that $f \in X$ has absolutely continuous norm if and only if $\|f\chi_{A_n}\|_X \rightarrow 0$ for every sequence $\{A_n\}_{n=1}^{\infty}$ satisfying $A_n \downarrow \emptyset$. The following is another characterization of functions with absolutely continuous norm and provides a generalization of the Lebesgue Dominated Convergence Theorem.

PROPOSITION 3.2.1. [2, p.16] *A function g in a Banach function space X has absolutely continuous norm if and only if whenever $(f_n)_{n=1}^{\infty} \cup \{f\} \subset L^0(\mu)$ are such that $|f_n| \leq |g|$ for all n and $f_n \rightarrow f$ μ -a.e. then $f_n \rightarrow f$ in X .*

We give some examples of spaces with absolutely continuous norm.

PROPOSITION 3.2.2. [29, p.77, 85] *If ϕ is an Orlicz function satisfying the Δ_2 -condition globally, then $L^\phi(\mu)$ has absolutely continuous norm.*

PROPOSITION 3.2.3. (cf. [2, p.222]) *The Lorentz space $L^{p,q}(\mu)$ ($1 \leq q \leq p < \infty$ or $p = q = \infty$) has absolutely continuous norm.*

COROLLARY 3.2.4. *$L^p(\mu)$ ($1 \leq p < \infty$) has absolutely continuous norm.*

Next we characterize X_b and explore some of the relationships between X_a and X_b .

PROPOSITION 3.2.5. [2, p.17] *X_b is the closure in X of the set of bounded functions with supports of finite measure.*

THEOREM 3.2.6. (cf. [2, p.18,19]) *Let X be a Banach function space. Then*

- (1) X_a and X_b are closed order ideals of X
- (2) $X_a \subset X_b \subset X$
- (3) If X has absolutely continuous norm, then $X = X_b$
- (4) $X_a = X_b$ if and only if

$$\chi_A \in X_a \quad \forall A \in \Sigma_f$$

The main result linking dual spaces and associate spaces is as follows.

THEOREM 3.2.7. [2, p.20] *Let Y be a closed order ideal of X such that Y contains the simple functions. Then Y^* is isometrically isomorphic to X' if and only if $Y \subset X_a$. In this case,*

$$Y = X_a = X_b$$

We will, however, be more interested in the following corollary.

COROLLARY 3.2.8. (cf. [2, p.23]) *If X_a contains the simple functions, then $(X_a)^*$ is isometrically isomorphic to X' . Furthermore, X^* is canonically isometrically isomorphic to X' if and only if X has absolutely continuous norm and therefore X is reflexive if and only if X and X' both have absolutely continuous norm.*

The Orlicz spaces (for Orlicz functions satisfying the Δ_2 -condition globally), Lorentz spaces and Lebesgue spaces have absolutely continuous norm (Propositions 3.2.2 and 3.2.3 and Corollary 3.2.4). Furthermore, explicit descriptions of the associate spaces of these Banach function spaces were given in Theorems 3.1.10 and 3.1.11 and Corollary 3.1.12. Using the previous corollary, we can therefore identify the dual spaces of the aforementioned spaces.

COROLLARY 3.2.9. *Let (Ω, Σ, μ) be a resonant measure space and suppose $1 \leq q \leq p < \infty$, (or $p = q = \infty$). Then the dual space of $L^{p,q}(\mu)$ can be identified with the space $L^{p',q'}(\mu)$ (with equivalent norms), where $1/p + 1/p' = 1 = 1/q + 1/q'$.*

COROLLARY 3.2.10. *Let (Ω, Σ, μ) be a resonant measure space and let ϕ be an Orlicz function satisfying the Δ_2 -condition globally. Then the dual space of $L^\phi(\mu)$ equipped with the Luxemburg norm can be identified with $L^\varphi(\mu)$ equipped with the Orlicz norm, where φ is the complementary Orlicz function of ϕ .*

COROLLARY 3.2.11. *Let (Ω, Σ, μ) be a resonant measure space and let $1 < p < \infty$. Then the dual space of $L^p(\mu)$ can be identified with $L^{p'}(\mu)$, where $1/p + 1/p' = 1$.*

We present one result regarding the separability of a Banach function space.

THEOREM 3.2.12. [2, p.27] *Let X be a Banach function space over a measure space (Ω, Σ, μ) and let Y be a closed order ideal of X containing the simple functions. Then Y is separable if and only if Y has absolutely continuous norm and μ is a separable measure.*

Rearrangement invariant Banach function spaces

An important class of Banach function spaces are those which have the property that equimeasurable functions have the same norm. These are called rearrangement invariant Banach function spaces. The Lebesgue, Lorentz, Orlicz and Orlicz-Lorentz spaces are examples of such spaces. In Section 4.1 we show that if X is a rearrangement invariant Banach function space over a resonant measure space, then the norm of X has a description in terms of decreasing rearrangements and that the norm of such a space can be represented in terms of the norm of a rearrangement invariant space over $(\mathbb{R}^+, \mathcal{M}, m)$. In Sections 4.2 and 4.3 we discuss two of the useful features of rearrangement invariant spaces, namely that we are able to define the notion of a fundamental function for such spaces, and that we are able to provide conditions under which such spaces have preduals.

4.1. Rearrangement invariant spaces

Recall that two functions f and g are called equimeasurable (written $f \sim g$) if they yield the same distribution functions, i.e. if $\mu_f = \mu_g$. If X is a Banach function space and $\mu_f = \mu_g$ implies that $\|f\|_X = \|g\|_X$, then X is called rearrangement invariant. We show that in the particular case where X is a rearrangement invariant space over a resonant measure space, the associate space is also rearrangement invariant. Significantly, if this is the case, we are able to give descriptions for the norms of X and X' . This leads to a generalization of Hölder's inequality and some useful properties of rearrangement invariant Banach function spaces over resonant measure spaces. Finally we state and prove the Luxemburg representation theorem which will enable us to transfer results obtained for rearrangement invariant Banach function spaces over $(\mathbb{R}^+, \mathcal{M}, m)$ to rearrangement invariant spaces over more general resonant measure spaces.

We start by showing that Orlicz-Lorentz spaces (and hence Lorentz, Orlicz and Lebesgue spaces) are rearrangement invariant.

EXAMPLE 4.1.1. Let (Ω, Σ, μ) be a σ -finite measure space, ϕ an Orlicz function and w a weight function. The Orlicz-Lorentz space $L^{\phi, w}(\mu)$ is a rearrangement invariant Banach function space.

PROOF. $L^{\phi, w}(\mu)$ is a Banach function space (Example 3.1.7). If f and g are equimeasurable functions, then $f^* = g^*$ and so by Proposition 1.3.5(2), $(\lambda^{-1}f)^* = \lambda^{-1}f^* = \lambda^{-1}g^* = (\lambda^{-1}g)^*$ for every $\lambda > 0$. Hence

$$\begin{aligned} \|f\|_{L^{\phi, w}} &= \inf \left\{ \lambda > 0 : \int_I \phi \left(\left(\frac{f}{\lambda} \right)^* (t) \right) w(t) dt \leq 1 \right\} \\ &= \inf \left\{ \lambda > 0 : \int_I \phi \left(\left(\frac{g}{\lambda} \right)^* (t) \right) w(t) dt \leq 1 \right\} \\ &= \|g\|_{L^{\phi, w}} \end{aligned}$$

$L^{\phi, w}(\mu)$ is therefore rearrangement invariant. \square

EXAMPLE 4.1.2. [2, p.69] Let φ be a quasiconcave function on \mathbb{R}^+ . The Lorentz space M_φ is a Banach function space by Example 3.1.8. Since the norm of the Lorentz space $M_\varphi(\mathbb{R}^+, \mathcal{M}, m)$ is defined in terms of maximal functions and the equimeasurability of two functions implies the equality of their decreasing rearrangements, and hence of their maximal functions, it follows that M_φ is rearrangement invariant.

It has previously been mentioned that there is often an important interplay between a Banach function space and the underlying measure space. Of particular significance is the following which shows that for a rearrangement invariant Banach function space over a resonant measure space we have an explicit description of the norm.

PROPOSITION 4.1.3. [2, p.59] *If X is a rearrangement invariant Banach function space over a resonant measure space (Ω, Σ, μ) , then X' , the associate space, is also rearrangement invariant. Furthermore, for all $f, g \in L^0(\mu)$*

$$(4.1.1) \quad \|f\|_X = \sup \left\{ \int_0^\infty f^*(s)g^*(s)ds : \|g\|_{X'} \leq 1 \right\}$$

and

$$(4.1.2) \quad \|g\|_{X'} = \sup \left\{ \int_0^\infty f^*(s)g^*(s)ds : \|f\|_X \leq 1 \right\}$$

PROOF. We show that (4.1.2) holds. Let $g \in X'$ and $h \in B_X$. Then

$$(4.1.3) \quad \begin{aligned} \int_0^\infty h^*(s)g^*(s)ds &\geq \int_\Omega |hg| d\mu \\ &\text{by the Hardy-Littlewood inequality (1.3.2).} \\ \Rightarrow \sup \left\{ \int_0^\infty h^*(s)g^*(s)ds : \|h\|_X \leq 1 \right\} &\geq \sup \left\{ \int_\Omega |hg| d\mu : \|h\|_X \leq 1 \right\} \end{aligned}$$

Furthermore, if $h \in B_X$ and f is equimeasurable with h , then $\|f\|_X = \|h\|_X \leq 1$ since X is rearrangement invariant. It follows that

$$\sup \left\{ \int_\Omega |fg| d\mu : f \sim h \right\} \leq \sup \left\{ \int_\Omega |fg| d\mu : \|f\|_X \leq 1 \right\}$$

Therefore

$$\begin{aligned} \int_0^\infty h^*(s)g^*(s)ds &= \sup \left\{ \int_\Omega |fg| d\mu : f \sim h \right\} \quad \text{since } (\Omega, \Sigma, \mu) \text{ is resonant} \\ &\leq \sup \left\{ \int_\Omega |fg| d\mu : \|f\|_X \leq 1 \right\} \end{aligned}$$

Since this holds for every $h \in B_X$, we have that

$$(4.1.4) \quad \sup \left\{ \int_0^\infty h^*(s)g^*(s)ds : \|h\|_X \leq 1 \right\} \leq \sup \left\{ \int_\Omega |fg| d\mu : \|f\|_X \leq 1 \right\}$$

Combining (4.1.3) and (4.1.4) and recalling that $\sup \left\{ \int_\Omega |fg| d\mu : \|f\|_X \leq 1 \right\} = \|g\|_{X'}$, we obtain (4.1.2). Since the decreasing rearrangements of equimeasurable functions are equal it follows that X' is rearrangement invariant.

Furthermore,

$$\begin{aligned} \|f\|_X &= \|f\|_{(X')'} \quad \text{by Theorem 3.1.14} \\ &= \sup \left\{ \int_0^\infty f^*(s)g^*(s)ds : \|g\|_{X'} \leq 1 \right\} \quad \text{by (4.1.2)} \end{aligned}$$

□

REMARK 4.1.4. We note that if X is a rearrangement invariant Banach function space, then $X \subset L^{00}(\mu)$, i.e. $f^*(t) < \infty$ for all $t > 0$ and for all $f \in X$. We give a brief proof by contradiction. Let $f \in X$. Assume that there exists a $t_0 > 0$ such that $f^*(t_0) = \infty$. Then $f^*(t) = \infty$ for all $t \leq t_0$, since f^* is decreasing. Let $A \in \Sigma$ such that $0 < \mu(A) < \infty$, and let $\alpha = \min\{t_0, \mu(A)\}$. Then $\|\chi_A\|_{X'}^{-1} \chi_A \in B_{X'}$ and $\alpha > 0$. Furthermore

$(\|\chi_A\|_{X'}^{-1}\chi_A)^* = \|\chi_A\|_{X'\chi_{[0,\mu(A)]}}^{-1}$ and so

$$\begin{aligned} \|f\|_X &\geq \int_0^\infty (\|\chi_A\|_{X'}^{-1}\chi_A)^*(s)f^*(s) ds && \text{by (4.1.1)} \\ &= \int_0^{\mu(A)} f^*(s) ds \\ &\geq \int_0^\alpha f^*(s) ds && \text{since } f^* \geq 0 \\ &\geq \alpha f^*(\alpha) && \text{since } f^* \text{ is decreasing} \\ &= \infty \end{aligned}$$

Therefore $f \notin X$, which is a contradiction.

We have the following generalization of Hölder's inequality (Theorem 3.1.13) if the underlying measure space is resonant.

COROLLARY 4.1.5 (Hölder's inequality). [2, p.60] *Let X be a rearrangement invariant Banach function space over a resonant measure space. If $f, g \in L^0(\mu)$, then*

$$(4.1.5) \quad \begin{aligned} \int_\Omega |fg| d\mu &\leq \int_0^\infty f^*(s)g^*(s) ds \\ &\leq \|f\|_X \|g\|_{X'} \end{aligned}$$

PROOF. By the Hardy-Littlewood inequality (1.3.2), we have that $\int_\Omega |fg| d\mu \leq \int_0^\infty f^*(s)g^*(s) ds$. If $f^*(t) = \infty$ for some $t > 0$, then $f \notin X$ by Remark 4.1.4, i.e. $\|f\|_X = \infty$ and so (4.1.5) holds. Let $f \in L^{01}(\mu)$. If $\|g\|_{X'} = 0$ (i.e. $g = 0$) or $\|g\|_{X'} = \infty$, then the second inequality follows trivially. If $0 < \|g\|_{X'} < \infty$, then $(\|g\|_{X'}^{-1}g)^* = \|g\|_{X'}^{-1}g^*$ by Proposition 1.3.5(2) and so

$$\begin{aligned} \int_0^\infty f^*(s)g^*(s) ds &= \|g\|_{X'} \int_0^\infty f^*(s) (\|g\|_{X'}^{-1}g)^*(s) ds \\ &\leq \|g\|_{X'} \|f\|_X && \text{by (4.1.1)} \end{aligned}$$

□

Let $f, g \in L^0(\mu)$. Recall that we will write $f \prec g$ if $f^{**} \leq g^{**}$. We list some useful properties of rearrangement invariant Banach function spaces.

PROPOSITION 4.1.6. (cf. [2, p.61]) *Let X be a rearrangement invariant Banach function space over a resonant measure space (Ω, Σ, μ) .*

- (1) *If $A_1, A_2 \in \Sigma$ with $\mu(A_1) \leq \mu(A_2)$, then $\|\chi_{A_1}\|_X \leq \|\chi_{A_2}\|_X$*
- (2) *If $f, g \in L^0(\mu)$ and $f \prec g$, then $\|f\|_X \leq \|g\|_X$*
- (3) *Let $(A_n)_{n \in \mathbb{N}}$ be a collection of pairwise disjoint subsets of Ω , each with finite positive measure and let $A = \Omega \setminus \bigcup_{n \in \mathbb{N}} A_n$. If we define*

$$Tf = f\chi_A + \sum_{n \in \mathbb{N}} \left(\frac{1}{\mu(A_n)} \int_{A_n} f d\mu \right) \chi_{A_n} \quad f \in L^0(\mu)$$

then

$$\|Tf\|_X \leq \|f\|_X \quad \forall f \in X$$

PROOF. 1) Let $\mu(A_1) = t_1$ and $\mu(A_2) = t_2$, then

$$\begin{aligned}
\|\chi_{A_2}\|_X &= \sup \left\{ \int_0^\infty (\chi_{A_2})^*(s)g^*(s)ds : \|g\|_{X'} \leq 1 \right\} && \text{by Proposition 4.1.3} \\
&= \sup \left\{ \int_0^{t_2} g^*(s)ds : \|g\|_{X'} \leq 1 \right\} && \text{using } \chi_{A_2}^* = \chi_{[0, \mu(A_2)]} \\
&\geq \sup \left\{ \int_0^{t_1} g^*(s)ds : \|g\|_{X'} \leq 1 \right\} && \text{since } g^* \geq 0 \\
&= \sup \left\{ \int_0^\infty (\chi_{A_1})^*(s)g^*(s)ds : \|g\|_{X'} \leq 1 \right\} \\
&= \|\chi_{A_1}\|_X
\end{aligned}$$

2) $f \prec g$ implies that

$$\begin{aligned}
\int_0^\infty f^*(s)h^*(s)ds &\leq \int_0^\infty g^*(s)h^*(s)ds \quad \forall h \in B_{X'} \text{ by Corollary 1.3.13} \\
\Rightarrow \|f\|_X &\leq \|g\|_X \quad \text{by (4.1.1)}
\end{aligned}$$

3) If $f \in X$, then f is locally integrable by Proposition 3.1.4(4) and so $Tf \prec f$ by Proposition 1.3.15. It follows by 2) that $\|Tf\|_X \leq \|f\|_X$. \square

It can be shown ([2, p.62]) that if ρ is a rearrangement invariant function norm on $L_+^0(\mathbb{R}^+, \mathcal{M}, m)$ and (Ω, Σ, μ) is an arbitrary σ -finite measure space, then defining a functional $\underline{\rho}$ on $L_+^0(\mu)$ by

$$\underline{\rho}(f) := \rho(f^*)$$

yields a rearrangement invariant function norm on $L_+^0(\mu)$. The theorem below shows that the converse of the above is also true, i.e. that the rearrangement invariant spaces over resonant measure spaces are completely determined by the rearrangement invariant spaces over $(\mathbb{R}^+, \mathcal{M}, m)$.

THEOREM 4.1.7 (Luxemburg representation Theorem). [2, p.62] *Let (Ω, Σ, μ) be a resonant measure space and let X be a rearrangement invariant function space over (Ω, Σ, μ) . Then there exists a rearrangement invariant function norm ρ on $L_+^0(\mathbb{R}^+, \mathcal{M}, m)$ which can be considered a representation of $\|\cdot\|_X$ i.e.*

$$\|f\|_X = \rho(f^*) \quad \forall f \in L^0(\mu)$$

Furthermore, if σ is any function norm on $(\mathbb{R}^+, \mathcal{M}, m)$ representing $\|\cdot\|_X$, then the associate norm $\|\cdot\|_{X'}$ is represented by σ' , i.e.

$$\|f\|_{X'} = \sigma'(f^*) \quad \forall f \in L^0(\mu)$$

PROOF. Define ρ on $L_+^0(\mathbb{R}^+, \mathcal{M}, m)$ by

$$\rho(f) = \sup \left\{ \int_0^\infty f^*(s)g^*(s)ds : \|g\|_{X'} \leq 1 \right\}$$

It follows by (4.1.1) that $\|f\|_X = \rho(f^*)$ for all $f \in L^0(\mu)$ and ρ is clearly rearrangement invariant. We show that ρ is a Banach function norm on $L_+^0(\mathbb{R}^+, \mathcal{M}, m)$. It is clear that if $f = 0$ m -a.e. then $\rho(f) = 0$. Let $\rho(f) = 0$, then $\int_0^\infty f^*(s)g^*(s)ds = 0$ for all $g \in B_{X'}$. There exists an $A \in \Sigma$ such that $0 < \mu(A) < \infty$. Then $0 < \|\chi_A\|_{X'} < \infty$ by Proposition 3.1.4(3) and $\left(\|\chi_A\|_{X'}^{-1}\chi_A\right)^* = \|\chi_A\|_{X'}^{-1}\chi_{[0, \mu(A)]}$ by Proposition 1.3.5(2) and Example 1.3.4. It follows that

$$\begin{aligned}
0 &= \int_0^\infty f^*(s) \left(\|\chi_A\|_{X'}^{-1}\chi_{[0, \mu(A)]} \right)^*(s) ds \\
&= \frac{1}{\|\chi_A\|_{X'}} \int_0^{\mu(A)} f^*(s) ds
\end{aligned}$$

and therefore $f^*(s) = 0$ for all $s < \mu(A)$ since $f^* \geq 0$. Thus $f^* = 0$, since f^* is decreasing, and hence $f = 0$ m -a.e. It is easily checked that $\rho(\alpha f) = |\alpha| \rho(f)$ for all $\alpha \in \mathbb{F}$. We show that the triangle inequality holds. Let $f, g \in L^0(\mathbb{R}^+, \mathcal{M}, m)$, then

$$\begin{aligned} (f+g)^{**} &\leq f^{**} + g^{**} && \text{by Proposition 1.3.11(6)} \\ &= (f^* + g^*)^{**} && \text{by Propostion 1.3.11(9) since } f^* \text{ and } g^* \text{ are positive and decreasing} \\ \Rightarrow (f+g) &< (f^* + g^*) \\ \Rightarrow \int_0^\infty (f+g)^*(s)h^*(s) ds &\leq \int_0^\infty (f^*(s) + g^*(s))h^*(s) ds && \forall h \in X' \text{ by Corollary 1.3.13} \\ &= \int_0^\infty f^*(s)h^*(s) ds + \int_0^\infty g^*(s)h^*(s) ds \end{aligned}$$

It can easily be shown that this implies that

$$\begin{aligned} \sup \left\{ \int_0^\infty (f+g)^*(s)h^*(s) ds : \|h\|_{X'} \leq 1 \right\} &\leq \sup \left\{ \int_0^\infty f^*(s)h^*(s) ds : \|h\|_{X'} \leq 1 \right\} + \\ &\quad \sup \left\{ \int_0^\infty g^*(s)h^*(s) ds : \|h\|_{X'} \leq 1 \right\} \end{aligned}$$

and hence that $\rho(f+g) \leq \rho(f) + \rho(g)$. Since $f \leq g$ m -a.e. implies that $f^* \leq g^*$ by Proposition 1.3.5(1), it follows that $f \leq g$ m -a.e. implies that $\rho(f) \leq \rho(g)$. To show the Fatou property, let $\epsilon > 0$, $h \in X'$ and $(f_n)_{n=1}^\infty \cup \{f\} \subset L^0_+(\mathbb{R}^+, \mathcal{M}, m)$ such that $0 \leq f_n \uparrow f$ m -a.e. Then

$$\begin{aligned} f_n^* &\uparrow f^* && \text{by Proposition 1.3.5(5)} \\ \Rightarrow f_n^* h^* &\uparrow f^* h^* && \text{since } h^* \geq 0 \\ (4.1.6) \Rightarrow \int_0^\infty f_n^*(s)h^*(s) ds &\uparrow \int_0^\infty f^*(s)h^*(s) ds && \text{by the Monotone Convergence theorem (1.2.6)} \end{aligned}$$

By definition of $\rho(f)$ there exists a sequence $(h_m)_{m=1}^\infty \subset B_{X'}$ such that $\int_0^\infty f^*(s)h_m^*(s) ds \rightarrow \rho(f)$. Let $m_\epsilon \in \mathbb{N}^+$ such that $m \geq m_\epsilon$ implies

$$(4.1.7) \quad \rho(f) - \int_0^\infty f^*(s)h_{m_\epsilon}^*(s) ds < \epsilon/2$$

By (4.1.6) there exists an $n_\epsilon \in \mathbb{N}^+$ such that $n \geq n_\epsilon$ implies that

$$(4.1.8) \quad \int_0^\infty f_n^*(s)h_{m_\epsilon}^*(s) ds - \int_0^\infty f_n^*(s)h_m^*(s) ds < \epsilon/2$$

Combining (4.1.7) and (4.1.8), we obtain

$$\rho(f) - \int_0^\infty f_n^*(s)h_m^*(s) ds < \epsilon$$

Furthermore $\rho(f) \geq \rho(f_n)$ and, by definition of ρ , we have that $\rho(f_n) \geq \int_0^\infty f_n^*(s)h_m^*(s) ds$. Hence $\rho(f) - \rho(f_n) < \epsilon$ for $n \geq n_\epsilon$.

Let $A \in \mathcal{M}$ such that $m(A) = s < \infty$. We show that $\rho(\chi_A) < \infty$ and there exists a $0 < C_A < \infty$ such that $\int_A f dm \leq C_A \rho(f)$ for all $f \in L^0_+(\mathbb{R}^+, \mathcal{M}, m)$: There exists a $B \in \Sigma$ such that $0 < \mu(B) = t < \infty$. It follows by

Proposition 3.1.4(3) that $\|\chi_B\|_X < \infty$. Let $n \in \mathbb{N}^+$ such that $nt \geq s$. Therefore,

$$\begin{aligned}
\rho(\chi_A) &= \rho(\chi_{[0,s]}) && \text{since } \rho \text{ is rearrangement invariant} \\
&\leq \rho(\chi_{[0,nt]}) && \text{since } \rho \text{ is a lattice norm} \\
&= \rho(\chi_{[0,t]} + \chi_{[t,2t]} + \dots + \chi_{[(n-1)t, nt]}) \\
&\leq \rho(\chi_{[0,t]}) + \rho(\chi_{[t,2t]}) + \dots + \rho(\chi_{[(n-1)t, nt]}) && \text{using the triangle inequality} \\
&= n\rho(\chi_{[0,t]}) && \text{since } \rho \text{ is rearrangement invariant} \\
&< \infty
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\int_A f \, dm &= \int_{\mathbb{R}^+} \chi_A f \, dm \\
&\leq \int_0^s f^*(s) \, ds && \text{by (1.3.3)} \\
&\leq \int_0^{nt} f^*(s) \, ds && \text{since } f^* \geq 0 \\
&\leq n \int_0^t f^*(s) \, ds && \text{since } f^* \text{ is decreasing} \\
&= n \|\chi_B\|_{X'} \int_0^\infty \left(\frac{\chi_B}{\|\chi_B\|_{X'}} \right)^* (s) f^*(s) \, ds && \text{since } \left(\frac{\chi_B}{\|\chi_B\|_{X'}} \right)^* = \frac{\chi_{[0,t]}}{\|\chi_B\|_{X'}} \\
&\leq n \|\chi_B\|_{X'} \rho(f) && \text{by definition of } \rho
\end{aligned}$$

Put $C_A = n \|\chi_B\|_{X'}$, then $0 < C_A < \infty$,

$$\int_A f \, dm \leq C_A \rho(f) \quad \forall f \in L_+^0(\mathbb{R}^+, \mathcal{M}, m)$$

and C_A is only dependent on A . The second part of the theorem has been included for completeness, but will not be used and therefore the proof thereof is omitted. \square

4.2. The fundamental function

Certain boundedness and compactness properties of composition operators can be expressed in terms of the norms of characteristic functions of sets of finite measure. This motivates our definition of the fundamental function. We describe and prove some of the salient properties of fundamental functions and derive expressions for the fundamental functions of the Orlicz-Lorentz spaces, Orlicz spaces, Lorentz spaces and Lebesgue spaces. Throughout this section we will assume that X is a rearrangement invariant Banach function space over a σ -finite measure space (Ω, Σ, μ) . For the definition of the fundamental function we will not restrict ourselves to resonant measure spaces. It will be shown that the fundamental function has certain useful properties if the underlying space is resonant.

DEFINITION 4.2.1. [18, p.101] The fundamental function $\Phi_X : [0, \infty) \cap \text{ran } \mu \rightarrow [0, \infty)$ of a rearrangement invariant Banach function space X is defined as follows:

$$\Phi_X(t) := \|\chi_A\|_X \quad \text{where } A \in \Sigma \text{ such that } \mu(A) = t$$

REMARK 4.2.2. Φ_X is well-defined since if $A, B \in \Sigma$ with $\mu(A) = t = \mu(B)$, then χ_A and χ_B are equimeasurable and so $\|\chi_A\|_X = \|\chi_B\|_X$ by the rearrangement invariance of X .

REMARK 4.2.3. If (Ω, Σ, μ) is nonatomic, then $\Phi_X(t)$ is defined for all $t \in [0, \mu(\Omega))$, since $\text{ran } \mu = [0, \mu(\Omega))$ for nonatomic spaces. If (Ω, Σ, μ) is purely atomic with all atoms having equal measure $\alpha > 0$, then $\text{ran } \mu = \alpha\mathbb{N}$.

PROPOSITION 4.2.4. [2, p.67] Φ_X has the following properties:

- (1) $\Phi_X(t) = 0 \Leftrightarrow t = 0$
- (2) Φ_X is increasing if (Ω, Σ, μ) is resonant

PROOF. i) By the definition of a Banach function norm

$$\begin{aligned} \|\chi_A\|_X = 0 &\Leftrightarrow \chi_A = 0 \text{ } \mu\text{-a.e.} \\ &\Leftrightarrow \mu(A) = 0 \end{aligned}$$

ii) Let $t_1, t_2 \in \text{ran}\mu$ such that $0 < t_1 \leq t_2 < \infty$, and let $A_1, A_2 \in \Sigma$ such that $\mu(A_1) = t_1$ and $\mu(A_2) = t_2$, then

$$\begin{aligned} \Phi_X(t_2) &= \|\chi_{A_2}\|_X \\ &= \sup \left\{ \int_0^\infty (\chi_{A_2})^*(s)g^*(s)ds : \|g\|_{X'} \leq 1 \right\} \quad \text{by Proposition 4.1.3} \\ &= \sup \left\{ \int_0^{t_2} g^*(s)ds : \|g\|_{X'} \leq 1 \right\} \quad \text{since } (\chi_{A_2})^* = \chi_{[0, t_2]} \\ &\geq \sup \left\{ \int_0^{t_1} g^*(s)ds : \|g\|_{X'} \leq 1 \right\} \quad \text{since } g^* \geq 0 \text{ and } t_1 \leq t_2 \\ &= \sup \left\{ \int_0^\infty (\chi_{A_1})^*(s)g^*(s)ds : \|g\|_{X'} \leq 1 \right\} \\ &= \Phi_X(t_1) \end{aligned}$$

□

Before listing some of the other properties of fundamental functions, we state and prove the following result, which will be used to prove subsequent properties, but will also find application in other settings.

THEOREM 4.2.5. [2, p.66] Let X be a rearrangement invariant Banach function space over a resonant measure space (Ω, Σ, μ) . Then

$$\Phi_X(t)\Phi_{X'}(t) = t \quad \forall t \in \text{ran}\mu \cap [0, \infty)$$

PROOF. We note first that if X is a rearrangement invariant Banach function space then so is X' by Proposition 4.1.3 and so $\Phi_{X'}$ is well-defined. If $t = 0$, then the result follows by Corollary 4.2.4. Let $0 < t < \infty$ and let $A \in \Sigma$ such that $\mu(A) = t$. Then

$$\begin{aligned} \Phi_X(t) &= \|\chi_A\|_X \\ &= \|\chi_A\|_{(X')'} \quad \text{by Theorem 3.1.14} \\ (4.2.1) \quad &= \sup \left\{ \int_A |g| d\mu : \|g\|_{X'} \leq 1 \right\} \quad \text{by definition of the associate norm} \end{aligned}$$

Let $0 \leq g \in X'$ such that $\|g\|_{X'} \leq 1$ and define

$$Tg := g\chi_{A^c} + \left(\frac{1}{t} \int_A |g| d\mu \right) \chi_A$$

Then $\left\| \left(\frac{1}{t} \int_A |g| d\mu \right) \chi_A \right\|_{X'} = \left(\frac{1}{t} \int_A |g| d\mu \right) \Phi_{X'}(t)$. Furthermore

$$\left(\frac{1}{t} \int_A |g| d\mu \right) \chi_A \leq Tg$$

since $g \geq 0$, and so

$$\begin{aligned}
 \left(\frac{1}{t} \int_A |g| d\mu\right) \Phi_{X'}(t) &\leq \|Tg\|_{X'}, && \text{since } X' \text{ is a Banach lattice} \\
 &\leq \|g\|_{X'}, && \text{by Proposition 4.1.6(3)} \\
 (4.2.2) \qquad &\leq 1
 \end{aligned}$$

So

$$\begin{aligned}
 \frac{\Phi_{X'}(t)}{t} \Phi_X(t) &= \frac{\Phi_{X'}(t)}{t} \sup \left\{ \int_A |g| d\mu : \|g\|_{X'} \leq 1 \right\} && \text{by (4.2.1)} \\
 &= \sup \left\{ \left(\frac{1}{t} \int_A |g| d\mu\right) \Phi_{X'}(t) : \|g\|_{X'} \leq 1 \right\} \\
 &\leq 1 && \text{by (4.2.2)}
 \end{aligned}$$

In the last line we have also used the fact that g and $|g|$ have the same norm and therefore

$$\sup \left\{ \left(\frac{1}{t} \int_A |g| d\mu\right) \Phi_{X'}(t) : \|g\|_{X'} \leq 1 \right\} = \sup \left\{ \left(\frac{1}{t} \int_A |g| d\mu\right) \Phi_{X'}(t) : 0 \leq g \text{ and } \|g\|_{X'} \leq 1 \right\}$$

We therefore have that $\Phi_X(t)\Phi_{X'}(t) \leq t$. For the reverse inequality we note that

$$\begin{aligned}
 t &= \mu(A) \\
 &= \int_{\Omega} |\chi_A \chi_A| d\mu \\
 &\leq \|\chi_A\|_X \|\chi_A\|_{X'}, && \text{by Hölder's inequality (Corollary 4.1.5)} \\
 &= \Phi_X(t)\Phi_{X'}(t)
 \end{aligned}$$

Note that application of Proposition 4.1.6 requires the resonance of the underlying measure space. \square

Some further properties of the fundamental function are shown by the following result.

COROLLARY 4.2.6. [2, p.67] *Let X be a rearrangement invariant Banach function space over a resonant measure space. Then*

- (1) $\frac{\Phi_X(t)}{t}$ is decreasing for all $0 < t < \infty$
- (2) Φ_X is continuous, except perhaps at the origin.

PROOF. 1) Let $t_1, t_2 \in \text{ran } \mu$ such that $0 < t_1 \leq t_2 < \infty$, and let $A_1, A_2 \in \Sigma$ such that $\mu(A_1) = t_1$ and $\mu(A_2) = t_2$. Then by Theorem 4.2.5

$$\begin{aligned}
 \frac{\Phi_X(t_2)}{t_2} &= \frac{1}{\Phi_{X'}(t_2)} \\
 &\leq \frac{1}{\Phi_{X'}(t_1)} && \text{since } \Phi_{X'} \text{ is increasing by Proposition 4.2.4} \\
 &= \frac{\Phi_X(t_1)}{t_1}
 \end{aligned}$$

2) (Ω, Σ, μ) is resonant, so by Theorem 1.3.9 (Ω, Σ, μ) is either nonatomic or purely atomic with all atoms having the same measure. In the latter case Φ_X is continuous since it is defined on a discrete subset of $[0, \infty)$. Let (Ω, Σ, μ) be nonatomic. Assume there exists some $t_0 > 0$ such that Φ_X is not continuous at t_0 . Since Φ_X is increasing, the discontinuity at t_0 must be a jump discontinuity. There are two possibilities, namely

- a) $\exists \alpha > 0$ such that $\Phi_X(t_0 + \epsilon) - \Phi_X(t_0) \geq \alpha$ for all $\epsilon > 0$ or
- b) $\exists \alpha > 0$ such that $\Phi_X(t_0) - \Phi_X(t_0 - \epsilon) \geq \alpha$ for all $\epsilon > 0$.

Assume a) holds. Let $\epsilon = \frac{1}{2} \left(\frac{t_0(\Phi_X(t_0) + \alpha)}{\Phi_X(t_0)} - t_0 \right)$. Then $\epsilon > 0$ and

$$\begin{aligned} \frac{\Phi_X(t_0 + \epsilon)}{t_0 + \epsilon} &\geq \frac{\Phi_X(t_0) + \alpha}{\epsilon + t_0} && \text{since a) holds} \\ &= \frac{(2\epsilon + t_0)\Phi_X(t_0)/t_0}{\epsilon + t_0} && \text{by definition of } \epsilon \\ &> \frac{\Phi_X(t_0)}{t_0} \end{aligned}$$

But this contradicts the fact that $\frac{\Phi_X(t)}{t}$ is decreasing. A contradiction can similarly be obtained if b) holds. We therefore have that Φ_X is continuous at every $t > 0$. \square

We mention briefly as a matter of interest that rearrangement invariant Banach function spaces can be roughly classified according to their fundamental functions. It is clear from Proposition 4.2.4 and Corollary 4.2.6 that every fundamental function is a quasiconcave function. It can be shown ([2, Proposition 2.5.11]) that if X is a rearrangement invariant Banach function space over a resonant measure space, then X can be equivalently renormed with a rearrangement invariant function norm in such a way that the resulting fundamental function is concave. Let X be a function space which has been renormed as described. We define the Lorentz spaces $\Lambda(X)$ and $M(X)$ to be the spaces Λ_{Φ_X} and M_{Φ_X} as described in Subsection (2.2.1). Then ([2, Theorem 2.5.13]) $\Lambda(X)$ and $M(X)$ are rearrangement invariant Banach function spaces, each with fundamental function Φ_X and

$$\Lambda(X) \hookrightarrow X \hookrightarrow M(X)$$

where each embedding has norm 1.

We turn now to explicit descriptions of the fundamental functions of particular rearrangement invariant Banach function spaces.

EXAMPLE 4.2.7. Let (Ω, Σ, μ) be a general σ -finite measure space. Listed below are expressions for the fundamental functions of specific rearrangement invariant Banach function spaces over (Ω, Σ, μ) . Let ϕ be an Orlicz function and let w be a weight function. Recall that $\Gamma(t) := \int_0^t w(s) ds$ and $\phi^{-1}(t) := \inf\{s > 0 : \phi(s) > t\}$. Let $t \in \text{ran } \mu \cap [0, \infty)$, then

$$\begin{aligned} (1) \quad \Phi_{L^{\phi, w}}(t) &= \begin{cases} 0 & t = 0 \\ \frac{1}{\phi^{-1}(1/\Gamma(t))} & t > 0 \end{cases} \\ (2) \quad \Phi_{L^{\phi}}(t) &= \begin{cases} 0 & t = 0 \\ \frac{1}{\phi^{-1}(1/t)} & t > 0 \end{cases} \\ (3) \quad \Phi_{L^{p, q}}(t) &= \left(\frac{p}{q}\right)^{1/q} t^{1/p} \text{ for } 1 \leq q \leq p < \infty \\ (4) \quad \Phi_{L^p}(t) &= t^{1/p} \text{ for } 1 \leq p < \infty \\ (5) \quad \Phi_{L^\infty}(t) &= \begin{cases} 0 & \text{if } t = 0 \\ 1 & \text{if } t > 0 \end{cases} \end{aligned}$$

PROOF. Let X be one of the spaces above. All of the above spaces are rearrangement invariant Banach function spaces so by Corollary 4.2.6, $\Phi_X(t) = 0 \Leftrightarrow t = 0$. Let $t \in (0, \infty) \cap \text{ran } \mu$ and let $A \in \Sigma$ such that $\mu(A) = t$. It follows that $0 < \|\chi_A\| < \infty$ since all Banach function spaces contain characteristic functions of sets of finite measure.

1) Let $\lambda > 0$, then

$$\begin{aligned}
 I_{\phi,w}(\lambda\chi_A) &= \int_0^{\mu(\Omega)} \phi((\lambda\chi_A)^*(s)) w(s) ds \\
 &= \int_0^{\mu(A)} \phi(\lambda)w(s) ds \quad \text{since } (\lambda\chi_A)^* = |\lambda|(\chi_A)^* = \lambda\chi_{[0,\mu(A)]} \\
 (4.2.3) \quad &= \phi(\lambda)\Gamma(t)
 \end{aligned}$$

So

$$\begin{aligned}
 \|\chi_A\|_{L^{\phi,w}} &= \min \{ \lambda > 0 : I_{\phi,w}(\chi_A/\lambda) \leq 1 \} \quad \text{by Lemma 2.2.26} \\
 &= \max \{ \lambda^{-1} > 0 : I_{\phi,w}(\lambda\chi_A) \leq 1 \} \\
 &= \max \left\{ \lambda^{-1} > 0 : \phi(\lambda) \leq \frac{1}{\Gamma(t)} \right\} \quad \text{using (4.2.3)} \\
 &= \left(\sup \left\{ \lambda > 0 : \phi(\lambda) \leq \frac{1}{\Gamma(t)} \right\} \right)^{-1} \\
 &= \left(\phi^{-1} \left(\frac{1}{\Gamma(t)} \right) \right)^{-1} \quad \text{by Remark 2.2.10}
 \end{aligned}$$

2) We note that $L^{\phi}(\mu) = L^{\phi,w}(\mu)$ with equality of norms if $w \equiv 1$ (Example 2.2.38). Therefore

$$\begin{aligned}
 \Phi_{L^{\phi}}(t) &= \Phi_{L^{\phi,w}}(t) \\
 &= \frac{1}{\phi^{-1}(1/\Gamma(t))} \quad \text{by 1)} \\
 &= \frac{1}{\phi^{-1}(1/t)} \quad \text{since } \Gamma(t) = t \text{ if } w \equiv 1
 \end{aligned}$$

3) Recall (Example 2.2.28) that $L^{p,q}(\mu) = L^{\phi,w}(\mu)$ with equality of norms if $\phi(s) = s^q$ and $w(s) = s^{q/p-1}$. If this is the case then $\Gamma(t) = \int_0^t s^{q/p-1} ds = \frac{p}{q}t^{q/p}$ and $\phi^{-1}(s) = s^{1/q}$. Therefore

$$\begin{aligned}
 \Phi_{L^{p,q}}(t) &= \Phi_{L^{\phi,w}}(t) \\
 &= \frac{1}{\left(1/\left(\frac{p}{q}t^{q/p}\right)\right)^{1/q}} \quad \text{by 1)} \\
 &= \left(\frac{p}{q}\right)^{1/q} t^{1/p}
 \end{aligned}$$

4) $L^p(\mu) = L^{p,p}(\mu)$ with equality of norms (Example 2.2.2), therefore

$$\begin{aligned}
 \Phi_{L^p}(t) &= \Phi_{L^{p,p}}(t) \\
 &= \left(\frac{p}{p}\right)^{1/p} t^{1/p} \quad \text{by 3)} \\
 &= t^{1/p}
 \end{aligned}$$

5) L^∞ : $\|\chi_A\|_{L^\infty} = 1$ □

4.3. Preduals of rearrangement invariant Banach function spaces

We wish to show that under certain conditions a rearrangement invariant Banach function space will have a predual. This will enable us, using Alaoglu's Theorem, to conclude that the unit ball in such a space is weak-star compact. This will be useful when considering the compactness of composition operators on certain Orlicz spaces.

Recall that a σ -finite measure space is resonant if it is either nonatomic or purely atomic with all atoms having equal measure (Theorem 1.3.9). We give two preliminary theorems which state similar results, but in the two different contexts characterizing resonant measure spaces. For a purely atomic measure space we have the following result.

THEOREM 4.3.1. [2, p.67] *Let (Ω, Σ, μ) be an atomic σ -finite measure space with all atoms having the same measure. If X is any rearrangement invariant measure space over (Ω, Σ, μ) , then*

- (1) $X_a = X_b$
- (2) $(X_b)^*$ is isometrically isomorphic to X'
- (3) X_b is separable

The result for nonatomic measure spaces is as follows.

THEOREM 4.3.2. [2, p.67] *Let X be a rearrangement invariant space over a nonatomic σ -finite measure space. Then the following are equivalent*

- (1) $\lim_{t \rightarrow 0^+} \Phi_X(t) = 0$
- (2) $X_a = X_b$
- (3) $(X_b)^*$ is isometrically isomorphic to X'

The following result is given in [16] for rearrangement invariant spaces over $(\mathbb{R}^+, \mathcal{M}, m)$. We generalise it to rearrangement invariant spaces over resonant measure spaces.

PROPOSITION 4.3.3. [16, p.353] *Let X be a rearrangement invariant Banach function space over a σ -finite measure space (Ω, Σ, μ) . If (Ω, Σ, μ) is nonatomic and $\lim_{t \rightarrow 0^+} \frac{t}{\Phi_X(t)} = 0$ or if (Ω, Σ, μ) is atomic with all atoms having the same measure, then*

$$((X')_a)^* \cong ((X')_a)' = X$$

with equality of norms. Furthermore, $(X')_a$ contains every bounded function with support on a set of finite measure.

PROOF. By Corollary 3.2.8 $((X')_a)^*$ is isometrically isomorphic to $((X')_a)'$. We show that $((X')_a)^*$ is isometrically isomorphic to $(X)'$. The result will then follow since $(X)' = X'' = X$ and $\|\cdot\|_{X''} = \|\cdot\|_X$ by Theorem 3.1.14. If (Ω, Σ, μ) is atomic, then $(X')_a = (X')_b$ by Theorem 4.3.1 and so $((X')_a)^* = ((X')_b)^*$ and $((X')_b)^*$ is isometrically isomorphic to $(X)'$ by the same theorem. For the nonatomic case we use Theorem 4.2.5 which states that

$$(4.3.1) \quad \Phi_X(t)\Phi_{X'}(t) = t \quad \forall t \geq 0$$

and so

$$\begin{aligned} \lim_{t \rightarrow 0^+} \Phi_{X'}(t) &= \lim_{t \rightarrow 0^+} \frac{t}{\Phi_X(t)} \\ &= 0 \quad \text{by assumption} \end{aligned}$$

By Theorem 4.3.2 this implies that $(X')_a = (X')_b$ and so $((X')_a)^* = ((X')_b)^*$ and $((X')_b)^*$ is isometrically isomorphic to $(X)'$ by the same theorem. $(X')_a = (X')_b$ and so $(X')_a$ contains every bounded function with support on a set of finite measure by Proposition 3.2.5. \square

Interpolation theory

The interpolation theory developed in this chapter will provide a technique for evaluating the boundedness of composition operators on rearrangement invariant Banach function spaces over resonant measure spaces. We start in Section 5.1 by defining the sum and intersection of two Banach spaces and describing some of the properties of the sum and intersection spaces generated by the L^1 and L^∞ spaces. In Section 5.2 we show that every rearrangement invariant Banach function space over a resonant measure space (Ω, Σ, μ) is an exact interpolation space between $L^1(\mu)$ and $L^\infty(\mu)$.

5.1. The sum and intersection of two Banach spaces

Let X_0 and X_1 be Banach spaces. To define the sum and intersection of these two spaces (in particular to be able to speak meaningfully about the sum of elements from X_0 and X_1), the pair must satisfy a suitable condition. More specifically, (X_0, X_1) is called a compatible couple if there is a Hausdorff topological vector space (E, \mathcal{T}) such that X_0 and X_1 are continuously embedded in (E, \mathcal{T}) , i.e. X_0 and X_1 are linear subspaces of E and the topologies induced by \mathcal{T} on X_0 and X_1 are weaker than the norm topologies. For any compatible couple (X_0, X_1) , the sum and intersection of these two spaces will be defined. As an example of such spaces we consider the sum and intersection spaces generated by the L^1 and L^∞ spaces. We give expressions for the norms of $(L^1 + L^\infty)(\mu)$ and $(L^1 \cap L^\infty)(\mu)$ and show that these two spaces are mutually associate. Furthermore, an Orlicz space which can be written as an intersection space will also be investigated. Apart from exhibiting another example of an intersection space, this Orlicz space will also be useful when considering the compactness of composition operators.

DEFINITION 5.1.1. [18, p.9] Let (X_0, X_1) be a compatible couple. The space $X_0 \cap X_1$ consists of all elements in the intersection of X_0 and X_1 and the space $X_0 + X_1$ consists of all elements of the form $x = x_0 + x_1$ where $x_0 \in X_0$ and $x_1 \in X_1$.

Let (Ω, Σ, μ) be a σ -finite measure space. By Theorem 3.1.6 $L^1(\mu)$ and $L^\infty(\mu)$ are continuously embedded in the topological Hausdorff space, $L^0(\mu)$, equipped with the topology of local convergence in measure. We therefore have the following example.

EXAMPLE 5.1.2. [2, p.74] Let (Ω, Σ, μ) be a σ -finite measure space.

- (1) The space of all functions f in $L^0(\mu)$ which are representable in the form $f = g + h$ where $g \in L^1(\mu)$ and $h \in L^\infty(\mu)$ will be denoted by $(L^1 + L^\infty)(\mu) = (L^1 + L^\infty)(\Omega, \Sigma, \mu)$. For each such f , let

$$\|f\|_{L^1 + L^\infty} := \inf \{ \|g\|_1 + \|h\|_\infty : f = g + h, g \in L^1(\mu), h \in L^\infty(\mu) \}$$

- (2) The space of all functions in the intersection of $L^1(\mu)$ and $L^\infty(\mu)$ will be denoted by $(L^1 \cap L^\infty)(\mu)$ and we define

$$\|f\|_{L^1 \cap L^\infty} := \max \{ \|f\|_{L^1}, \|f\|_{L^\infty} \}$$

for each $f \in (L^1 \cap L^\infty)(\mu)$

It can be shown in general ([18, p.9]) that $X_0 \cap X_1$ equipped with $\|x\|_{X_0 \cap X_1} := \inf \{ \|x\|_{X_0}, \|x\|_{X_1} \}$ and $X_0 + X_1$ equipped with $\|x\|_{X_0 + X_1} := \inf \{ \|x_0\|_{X_0} + \|x_1\|_{X_1} : x = x_0 + x_1 \in X_0 + X_1 \}$ are Banach spaces.

However, for the sum and intersection spaces defined in Example 5.1.2, it will be shown that these can be identified with certain Lorentz spaces and are therefore rearrangement invariant Banach function spaces by Example 4.1.2.

THEOREM 5.1.3. [2, p.74, 76] *Let (Ω, Σ, μ) be a σ -finite resonant measure space. The spaces $(L^1 + L^\infty)(\mu)$ and $(L^1 \cap L^\infty)(\mu)$ are rearrangement invariant Banach function spaces and for any $f \in L^0(\mu)$, the following holds:*

- (1) $\alpha_t := \inf \{ \|g\|_{L^1} + t\|h\|_{L^\infty} : f = g + h \} = \int_0^t f^*(s) ds$ for all $t > 0$.
- (2) $\|f\|_{L^1 + L^\infty} = \int_0^1 f^*(s) ds = f^{**}(1)$
- (3) $\|f\|_{L^1 \cap L^\infty} = \sup_{0 < t < \infty} \frac{1}{\min(1, t)} \int_0^t f^*(s) ds = \sup_{0 < t < \infty} \{ f^{**}(t) \times \max(1, t) \}$

PROOF. (1): Let $f \in L^0(\mu)$ and $t > 0$. We show first that

$$(5.1.1) \quad \int_0^t f^*(s) ds \leq \alpha_t$$

If $f \notin (L^1 + L^\infty)(\mu)$, then f cannot be represented as $g + h$ for $g \in L^1(\mu)$, $h \in L^\infty(\mu)$ and therefore $\alpha_t = \inf \emptyset = \infty$ and so (5.1.1) holds. Let $f \in (L^1 + L^\infty)(\mu)$. Let $g \in L^1(\mu)$ and $h \in L^\infty(\mu)$ such that $f = g + h$, then

$$\begin{aligned} \int_0^t f^*(s) ds &= t f^{**}(t) \quad \text{by definition of } f^{**} \\ &= t(g + h)^{**}(t) \\ &\leq t g^{**}(t) + t h^{**}(t) \quad \text{by the subadditivity of the operator } f \mapsto f^{**} \\ &= \int_0^t g^*(s) ds + t \left(\frac{1}{t} \int_0^t h^*(s) ds \right) \\ &\leq \int_0^\infty g^*(s) ds + t h^*(0) \quad \text{since } g^* \geq 0 \text{ and } h^* \text{ is decreasing} \\ &= \|g\|_{L^1} + t \|h\|_\infty \quad \text{by Proposition 1.3.7} \end{aligned}$$

Since this holds for any $g \in L^1(\mu)$, $h \in L^\infty(\mu)$ such that $f = g + h$, we obtain (5.1.1) by taking the infimum over all such representations of f . Next we show that

$$(5.1.2) \quad \alpha_t \leq \int_0^t f^*(s) ds$$

If $\int_0^t f^*(s) ds = \infty$, then (5.1.2) follows immediately. Let $\int_0^t f^*(s) ds < \infty$. We note first that this implies that $f^*(t) < \infty$ since if not we would have that $f^*(s) = \infty$ for $0 \leq s \leq t$ since f^* decreasing and hence $\int_0^t f^*(s) ds = \infty$, a contradiction. Let $E = \{x \in \Omega : |f(x)| > f^*(t)\}$ and $t_0 = \mu(E)$, then

$$\begin{aligned} t_0 &= \mu_f(f^*(t)) \\ &\leq t \quad \text{by Proposition 1.3.5(9)} \end{aligned}$$

Let $g(x) := \max\{|f(x)| - f^*(t), 0\} \cdot \text{sgn}(f(x))$. Then $0 < |g(x)| \leq |f(x)|$ for all $x \in E$ and $g(x) = 0$ for all $x \in E^c$. We therefore have that

$$\begin{aligned} \int_{\Omega} |g(x)| d\mu &= \int_E |g(x)| d\mu \\ &\leq \int_E |f(x)| d\mu \\ &\leq \int_0^{t_0} f^*(s) ds \quad \text{by (1.3.3)} \\ &\leq \int_0^t f^*(s) ds \quad \text{since } f^* \geq 0 \text{ and } t_0 \leq t \\ &< \infty \end{aligned}$$

so $g \in L^1(\mu)$. Let $h(x) := \min\{|f(x)|, f^*(t)\} \cdot \text{sgn}(f(x))$, then $|h(x)| \leq f^*(t)$ for all $x \in \Omega$ and so $h \in L^\infty(\mu)$ and

$$(5.1.3) \quad \|h\|_{L^\infty} \leq f^*(t).$$

Furthermore, $f = g + h$, as can easily be shown by considering the appropriate case distinctions. Now,

$$\begin{aligned} \|g\|_{L^1} + t\|h\|_{L^\infty} &\leq \int_E (|f(x)| - f^*(t)) d\mu + t f^*(t) \quad \text{since } g(x) = 0 \forall x \in E^c \text{ and (5.1.3) holds} \\ &= \int_E |f(x)| d\mu - t_0 f^*(t) + t f^*(t) \quad \text{since } \mu(E) = t_0 \\ &\leq \int_0^{t_0} f^*(s) ds + (t - t_0) f^*(t) \quad \text{by (1.3.3)} \\ &\leq \int_0^{t_0} f^*(s) ds + \int_{t_0}^t f^*(s) ds \quad \text{since } f^* \text{ is decreasing} \\ &= \int_0^t f^*(s) ds \end{aligned}$$

and so $\alpha_t = \inf\{\|g\|_{L^1} + t\|h\|_{L^\infty} : f = g + h\} \leq \int_0^t f^*(s) ds$. Combining (5.1.1) and (5.1.2) and noting that $t > 0$ was arbitrary, we obtain

$$(5.1.4) \quad \inf\{\|g\|_{L^1} + t\|h\|_{L^\infty} : f = g + h\} = \int_0^t f^*(s) ds \quad \forall t > 0$$

$$(5.1.5) \quad = t f^{**}(t) \quad \forall t > 0$$

2): This follows immediately from the definition of $\|\cdot\|_{L^1+L^\infty}$ and letting $t = 1$ in (5.1.4).

3): Let $f \in (L^1 \cap L^\infty)(\mu)$, then

$$\begin{aligned} \|f\|_{L^1 \cap L^\infty} &:= \max\{\|f\|_{L^1}, \|f\|_{L^\infty}\} \\ &= \max\left\{\int_0^\infty f^*(s) ds, f^*(0)\right\} \quad \text{by Proposition 1.3.7} \\ (5.1.6) \quad &= \max\left\{\sup_{1 \leq t < \infty} \int_0^t f^*(s) ds, \sup_{0 < t < 1} \frac{1}{t} \int_0^t f^*(s) ds\right\} \\ &\quad \text{using } f^* \geq 0, \text{ Proposition 1.3.11(8) and } f^{**} \text{ decreasing} \\ &= \sup_{0 < t < \infty} \frac{1}{\min\{1, t\}} \int_0^t f^*(s) ds \\ &= \sup_{0 < t < \infty} \{f^{**}(t) \times \max\{1, t\}\} \quad \text{from (5.1.6)} \end{aligned}$$

To show that $(L^1 + L^\infty)(\mu)$ and $(L^1 \cap L^\infty)(\mu)$ are rearrangement invariant Banach function spaces let

$$\begin{aligned} \varphi(t) &:= \min\{1, t\} & \text{and} \\ \psi(t) &:= \begin{cases} 0 & t = 0 \\ \max\{1, t\} & t > 0 \end{cases} \end{aligned}$$

It is easily checked that φ and ψ are quasiconcave functions. Furthermore, if $t \leq 1$, then

$$\begin{aligned} f^{**}(t)\varphi(t) &= \int_0^t f^*(s) ds \\ &\leq \int_0^1 f^*(s) ds & \text{since } f^* \geq 0 \\ &= f^{**}(1) \end{aligned}$$

and if $t > 1$, then $f^{**}(t)\varphi(t) = f^{**}(t) \leq f^{**}(1)$ since f^{**} is decreasing. It follows, using (2), that $\|f\|_{L^1+L^\infty} = \sup_{0 < t < \infty} \{f^{**}(t)\varphi(t)\}$. We also trivially have that $\|f\|_{L^1 \cap L^\infty} = \sup_{0 < t < \infty} \{f^{**}(t)\psi(t)\}$. $(L^1+L^\infty)(\mu)$ and $(L^1 \cap L^\infty)(\mu)$ therefore correspond to the Lorentz spaces M_φ and M_ψ respectively and are therefore rearrangement invariant Banach function spaces by Example 4.1.2. \square

REMARK 5.1.4. Since $(L^1 + L^\infty)(\mu)$ and $(L^1 \cap L^\infty)(\mu)$ are rearrangement invariant Banach function spaces, we have that these spaces are subsets of $L^{00}(\mu)$ by Remark 4.1.4.

As an application of the description of the norms given by the theorem above we give expressions for the fundamental functions of these spaces.

PROPOSITION 5.1.5. [2, p.77] *The fundamental functions of $(L^1 + L^\infty)(\mu)$ and $(L^1 \cap L^\infty)(\mu)$ are given by*

$$\begin{aligned} \Phi_{L^1+L^\infty}(t) &= \min\{1, t\} & \text{and} \\ \Phi_{L^1 \cap L^\infty}(t) &= \begin{cases} 0 & t = 0 \\ \max\{1, t\} & t > 0 \end{cases} \end{aligned}$$

PROOF. We note first that by Theorem 5.1.3 $(L^1 + L^\infty)(\mu)$ and $(L^1 \cap L^\infty)(\mu)$ are rearrangement invariant Banach function spaces and therefore their respective fundamental functions are defined for all $t \in [0, \infty) \cap \text{ran } \mu$. Furthermore

$$\begin{aligned} \Phi_{L^1+L^\infty}(t) &= \|\chi_A\|_{L^1+L^\infty} & \text{for some } A \in \Sigma \text{ such that } \mu(A) = t \\ &= \int_0^1 (\chi_A)^*(s) ds & \text{by Theorem 5.1.3(2)} \\ &= \int_0^1 \chi_{[0,t]}(s) ds \\ &= \min\{1, t\} \end{aligned}$$

and

$$\begin{aligned} \Phi_{L^1 \cap L^\infty}(t) &= \|\chi_A\| & \text{for some } A \in \Sigma \text{ such that } \mu(A) = t \\ &= \max\{\|\chi_A\|_{L^1}, \|\chi_A\|_{L^\infty}\} \\ &= \begin{cases} 0 & t = 0 \\ \max\{1, t\} & t > 0 \end{cases} \end{aligned}$$

\square

THEOREM 5.1.6. [2, p.76] *Let (Ω, Σ, μ) be a resonant measure space. $(L^1 + L^\infty)(\mu)$ and $(L^1 \cap L^\infty)(\mu)$ are mutually associate rearrangement invariant Banach function spaces.*

PROOF. We first prove the result for $(\Omega, \Sigma, \mu) = (\mathbb{R}^+, \mathcal{M}, m)$. Let $f, g \in L^0(m)$, then

$$(5.1.7) \quad \begin{aligned} \int_0^\infty f^*(s)g^*(s) ds &\leq \left(\int_0^1 f^*(s) ds \right) \cdot \left(\sup_{0 < t < \infty} \frac{1}{\min(1, t)} \int_0^t g^*(s) ds \right) \quad \text{by Lemma 1.3.14} \\ &= \|f\|_{L^1+L^\infty} \|g\|_{L^1 \cap L^\infty} \quad \text{by Theorem 5.1.3(2 and 3)} \end{aligned}$$

Therefore

$$\begin{aligned} \|g\|_{(L^1+L^\infty)'} &= \sup \left\{ \int_0^\infty f^*(s)g^*(s) ds : \|f\|_{L^1+L^\infty} \leq 1 \right\} \quad \text{by Proposition 4.1.3} \\ &\leq \|g\|_{L^1 \cap L^\infty} \quad \text{by (5.1.7)} \end{aligned}$$

To show that $\|g\|_{(L^1+L^\infty)'} \geq \|g\|_{L^1 \cap L^\infty}$ consider the following:

$$\begin{aligned} \int_0^t g^*(s) ds &\leq \| \chi_{[0,t]} \|_{L^1+L^\infty} \|g\|_{(L^1+L^\infty)'} \quad \forall t > 0 \text{ by Hölder's inequality (Corollary 4.1.5)} \\ &= \min\{1, t\} \|g\|_{(L^1+L^\infty)'} \quad \forall t > 0 \text{ by Proposition 5.1.5} \\ \Rightarrow \frac{\int_0^t g^*(s) ds}{\min\{1, t\}} &\leq \|g\|_{(L^1+L^\infty)'} \quad \forall t > 0 \\ \Rightarrow \|g\|_{L^1 \cap L^\infty} &\leq \|g\|_{(L^1+L^\infty)'} \quad \text{by Theorem 5.1.3(3)} \end{aligned}$$

Therefore $\|g\|_{(L^1+L^\infty)'} = \|g\|_{L^1 \cap L^\infty}$ for all $g \in L^0(m)$ and hence for all $g \in L^0(m)$ since $\|g\|_{(L^1+L^\infty)'} = \infty = \|g\|_{L^1 \cap L^\infty}$ if $g \in L^0(m) \setminus L^0(m)$ by Remark 5.1.4. It follows that $(L^1 + L^\infty)'(m) = (L^1 \cap L^\infty)(m)$. Thus, by Theorem 3.1.14, $(L^1 + L^\infty)(m) = (L^1 + L^\infty)''(m) = (L^1 \cap L^\infty)'(m)$. If (Ω, Σ, μ) is an arbitrary resonant measure space, then the result follows by the Luxemburg representation theorem 4.1.7. \square

Finally, we consider an example of an Orlicz space which can be identified with an intersection space.

PROPOSITION 5.1.7. [29, p.130] *Let ϕ be an Orlicz function which is continuous on the closed interval $[0, b_\phi]$ with $\phi(b_\phi) > 0$ and $\varphi(b_\phi) = \beta < \infty$, where φ is the left derivative of ϕ . Let*

$$\phi_1(u) := \begin{cases} \phi(u) & \text{if } 0 \leq u \leq b_\phi \\ \beta u + \phi(b_\phi) - \beta b_\phi & \text{if } u > b_\phi \end{cases}$$

and

$$\phi_2(u) := \begin{cases} 0 & \text{if } 0 \leq u \leq b_\phi \\ \infty & \text{if } u > b_\phi \end{cases}$$

Then $L^\phi(\mu) = (L^{\phi_1} \cap L^{\phi_2})(\mu)$ and $\|f\|_\phi = \max \{ \|f\|_{\phi_1}, \|f\|_{\phi_2} \}$ for every $f \in L^\phi(\mu)$.

PROOF. It is easily checked that ϕ_1 and ϕ_2 are Orlicz functions and $\phi(u) = \max \{ \phi_1(u), \phi_2(u) \}$ for all $u \geq 0$. Since ϕ is continuous on the closed interval $[0, b_\phi]$, we have that $\phi(b_\phi) < \infty$. Furthermore $\beta < \infty$ and so it is easily checked that ϕ_1 is finite-valued. Since $\phi_i \leq \phi$ for $i = 1, 2$, it follows by Proposition 2.2.36 that $\|f\|_{\phi_i} \leq \|f\|_\phi$ for all $f \in L^0(\mu)$ for $i = 1, 2$ and so

$$(5.1.8) \quad \|f\|_\phi \geq \max \{ \|f\|_{\phi_1}, \|f\|_{\phi_2} \}$$

To show the reverse inequality, let $\alpha > 0$ and $f \in L^0(\mu)$. If $|f(x)/\alpha| \leq b_\phi$ for μ -a.e. $x \in \Omega$, then

$$\begin{aligned}
 I_\phi(f/\alpha) &= \int_{\Omega} \phi(|f(x)/\alpha|) d\mu \\
 &= \int_{\Omega} \phi_1(|f(x)/\alpha|) d\mu \\
 (5.1.9) \qquad &= I_{\phi_1}(f/\alpha) \quad \text{and } I_{\phi_2}(f/\alpha) = 0
 \end{aligned}$$

If there exists an $A \in \Sigma$ with $\mu(A) > 0$ such that $|f(x)/\alpha| > b_\phi$ for all $x \in A$, then

$$\begin{aligned}
 I_\phi(f/\alpha) &= \infty \\
 (5.1.10) \qquad &= I_{\phi_2}(f/\alpha) \quad \text{and } I_{\phi_1}(f/\alpha) \leq \infty
 \end{aligned}$$

Combining (5.1.9) and (5.1.10), we have that $I_\phi(f/\alpha) = \max\{I_{\phi_1}(f/\alpha), I_{\phi_2}(f/\alpha)\}$ for all $\alpha > 0$. We show that

$$(5.1.11) \qquad \|f\|_\phi \leq \max\{\|f\|_{\phi_1}, \|f\|_{\phi_2}\}$$

Clearly (5.1.11) holds if $\max\{\|f\|_{\phi_1}, \|f\|_{\phi_2}\} = \infty$. If $\max\{\|f\|_{\phi_1}, \|f\|_{\phi_2}\} = 0$, then $f = 0$ μ -a.e. and so $\|f\|_\phi = 0$, i.e. (5.1.11) holds. If $0 < \max\{\|f\|_{\phi_1}, \|f\|_{\phi_2}\} = \lambda < \infty$, then $I_{\phi_i}(f/\lambda) \leq 1$ if $\|f\|_{\phi_i} = \lambda$ (using Lemma 2.2.26) and $I_{\phi_i}(f/\lambda) \leq 1$ if $\|f\|_{\phi_i} < \lambda$ using the definition of $\|f\|_{\phi_i}$. It follows that

$$I_\phi(f/\lambda) = \max\{I_{\phi_1}(f/\lambda), I_{\phi_2}(f/\lambda)\} \leq 1$$

and hence that $\|f\|_\phi \leq \lambda$, i.e. (5.1.11) holds. Combining (5.1.11) and (5.1.8), we have that

$$\|f\|_\phi = \max\{\|f\|_{\phi_1}, \|f\|_{\phi_2}\}$$

Since this holds for all $f \in L^0(\mu)$, it follows that $L^\phi(\mu) = (L^{\phi_1} \cap L^{\phi_2})(\mu)$. □

5.2. Interpolation spaces

Let (X_0, X_1) be a compatible couple. A Banach space X is called an intermediate space between X_0 and X_1 if $X_0 \cap X_1$ is continuously embedded in X and X is continuously embedded in $X_0 + X_1$, i.e.

$$X_0 \cap X_1 \hookrightarrow X \hookrightarrow X_0 + X_1$$

It will be shown that if (Ω, Σ, μ) is a resonant measure space, then rearrangement invariant Banach function spaces over (Ω, Σ, μ) are intermediate spaces between $L^1(\mu)$ and $L^\infty(\mu)$. Admissible operators will then be defined and the admissible operators relative to the couple $(L^1(\mu), L^\infty(\mu))$ characterized. Furthermore, since interpolation spaces will be defined in such a way that the restriction of an admissible operator to an interpolation space X is a bounded operator on X , the value of the characterization of the admissible operators relative to the couple $(L^1(\mu), L^\infty(\mu))$ will be demonstrated by showing that every rearrangement invariant Banach function space is an interpolation space between $L^1(\mu)$ and $L^\infty(\mu)$.

THEOREM 5.2.1. [2, p.77] *If X is a rearrangement invariant Banach function space over a resonant measure space (Ω, Σ, μ) , then X is an intermediate space between $L^1(\mu)$ and $L^\infty(\mu)$.*

PROOF. We first prove the Theorem for rearrangement invariant Banach function spaces over $(\mathbb{R}^+, \mathcal{M}, m)$. Let $f \in X$, then

$$\begin{aligned}
 \|f\|_{L^1+L^\infty} &= \int_0^1 f^*(s) ds \quad \text{by Theorem 5.1.3(2)} \\
 &= \int_0^\infty \chi_{[0,1)} f^*(s) ds \\
 &\leq \Phi_{X'}(1) \|f\|_X \quad \text{using Hölder's inequality (Corollary 4.1.5) and } \chi_{[0,1)} \in X'
 \end{aligned}$$

It follows that $X \hookrightarrow (L^1 + L^\infty)(m)$. It can similarly be shown that if $f \in X'$, then

$$\|f\|_{L^1 + L^\infty} \leq \Phi_X(1) \|f\|_{X'}$$

and therefore $X' \hookrightarrow (L^1 + L^\infty)(m)$. It follows by Proposition 3.1.15 that

$$\begin{aligned} (L^1 + L^\infty)'(m) &\hookrightarrow (X')' \\ \Leftrightarrow (L^1 \cap L^\infty)(m) &\hookrightarrow X \quad \text{using Theorems 3.1.14 and 5.1.6} \end{aligned}$$

If (Ω, Σ, μ) is an arbitrary resonant measure space, then the result follows by the Luxemburg representation theorem 4.1.7. \square

Let (X_0, X_1) and (Y_0, Y_1) be two compatible couples and let T be a linear map defined on $X_0 + X_1$ and taking values in $Y_0 + Y_1$. T is called an *admissible operator* if T maps X_i into Y_i for $i = 0, 1$ and $T \in \mathcal{B}(X_i, Y_i)$ for $i = 0, 1$. Let $\mathcal{A} = \mathcal{A}((X_0, X_1); (Y_0, Y_1))$ denote the set of all admissible operators and define the norm of an admissible operator as follows

$$\|T\|_{\mathcal{A}} := \max_{i=0,1} \|T\|_{\mathcal{B}(X_i, Y_i)}$$

It can be shown ([2, p.100]) that if T is an admissible operator, then $T \in \mathcal{B}(X_0 + X_1, Y_0 + Y_1)$ and furthermore $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ is a Banach space which is continuously embedded in $\mathcal{B}(X_0 + X_1, Y_0 + Y_1)$. We characterize the admissible operators for the couple $(L^1(\mu), L^\infty(\mu))$.

PROPOSITION 5.2.2. [2, p.106] *Let T be a linear operator defined on $(L^1 + L^\infty)(\mu)$ and taking values in $L^0(\mu)$. Then T is an admissible operator for $(L^1(\mu), L^\infty(\mu))$ if and only if there exists a $k > 0$ such that*

$$(5.2.1) \quad Tf \prec kf \quad \forall f \in (L^1 + L^\infty)(\mu)$$

Furthermore,

$$\|T\|_{\mathcal{A}} = \inf \{k > 0 : Tf \prec kf \forall f \in (L^1 + L^\infty)(\mu)\}$$

PROOF. (Necessity): Let T be admissible and let $f = g + h \in L^1(\mu) + L^\infty(\mu)$ be a representation of f . Then

$$\begin{aligned} Tf &= T(g + h) \\ &= Tg + Th \quad \text{since } T \text{ is linear} \\ &\in L^1(\mu) + L^\infty(\mu) \quad \text{since } T \text{ is admissible} \end{aligned}$$

Let $t > 0$. Then

$$\begin{aligned} \int_0^t (Tf)^*(s) ds &\leq \|Tg\|_{L^1} + t\|Th\|_{L^\infty} \quad \text{by Theorem 5.1.3(1)} \\ &\leq \|T\|_{\mathcal{B}(L^1(\mu))} \|g\|_{L^1} + t\|T\|_{\mathcal{B}(L^\infty(\mu))} \|h\|_{L^\infty} \quad \text{since } T \text{ is admissible} \\ (5.2.2) \quad &\leq \|T\|_{\mathcal{A}} (\|g\|_{L^1} + t\|h\|_{L^\infty}) \quad \text{by definition of } \|\cdot\|_{\mathcal{A}} \end{aligned}$$

Since (5.2.2) holds for all representations $f = g + h \in L^1(\mu) + L^\infty(\mu)$, we have

$$\begin{aligned} \int_0^t (Tf)^*(s) ds &\leq \|T\|_{\mathcal{A}} \inf \{\|g\|_{L^1} + t\|h\|_{L^\infty} : f = g + h \in L^1(\mu) + L^\infty(\mu)\} \\ &= \|T\|_{\mathcal{A}} \int_0^t f^*(s) ds \quad \text{by Theorem 5.1.3(1),} \end{aligned}$$

Since the above inequality holds for all $t > 0$, we have that $(Tf)^{**} \leq \|T\|_{\mathcal{A}} f^{**}$, i.e. $Tf \prec \|T\|_{\mathcal{A}} f$ and hence

$$(5.2.3) \quad \|T\|_{\mathcal{A}} \geq \inf \{k > 0 : Tf \prec kf \forall f \in (L^1 + L^\infty)(\mu)\}$$

(Sufficiency): Let $k > 0$ such that (5.2.1) holds and let $f \in (L^1 + L^\infty)(\mu)$. Then

$$\begin{aligned} \|Tf\|_{L^1+L^\infty} &= (Tf)^{**}(1) \quad \text{by Theorem 5.1.3(2)} \\ &\leq kf^{**}(1) \quad \text{since } Tf \prec kf \\ &= k\|f\|_{L^1+L^\infty} \end{aligned}$$

So $Tf \in (L^1 + L^\infty)(\mu)$. We show that $T \upharpoonright L^1(\mu) \in \mathcal{B}(L^1(\mu))$ and $T \upharpoonright L^\infty(\mu) \in \mathcal{B}(L^\infty(\mu))$. Let $g \in L^1(\mu)$, then

$$\begin{aligned} \|Tg\|_{L^1} &= \lim_{t \rightarrow \infty} \int_0^t (Tg)^*(s) ds \quad \text{by Proposition 1.3.7} \\ &\leq \lim_{t \rightarrow \infty} k \int_0^t g^*(s) ds \quad \text{if this limit exists (using } (Tg)^{**}(t) \leq kg^{**}(t) \forall t > 0) \\ &= k\|g\|_{L^1} \\ &< \infty \quad \text{since } g \in L^1(\mu) \end{aligned}$$

So $Tg \in L^1(\mu)$ and $\|T\|_{\mathcal{B}(L^1(\mu))} \leq k$.

Let $h \in L^\infty(\mu)$, then $h^*(0) = \|h\|_{L^\infty} < \infty$ by Proposition 1.3.7. Therefore $h^*(0) = \lim_{t \rightarrow 0^+} h^{**}(t)$ by Proposition 1.3.11(8). Since $(Th)^{**}(t) \leq kh^{**}(t)$ for all $t > 0$, it follows that $\lim_{t \rightarrow 0^+} (Th)^{**}(t)$ exists and

$$\begin{aligned} \|Th\|_{L^\infty} = (Th)^*(0) &= \lim_{t \rightarrow 0^+} (Th)^{**}(t) \\ &\leq \lim_{t \rightarrow 0^+} kh^{**}(t) \\ &= k\|h\|_{L^\infty} \end{aligned}$$

Therefore $Th \in L^\infty(\mu)$ and $\|T\|_{\mathcal{B}(L^\infty(\mu))} \leq k$. It follows that

$$\begin{aligned} \|T\|_{\mathcal{A}} &:= \max \left(\|T\|_{\mathcal{B}(L^1(\mu))}, \|T\|_{\mathcal{B}(L^\infty(\mu))} \right) \\ &\leq k \quad \forall k > 0 \text{ such that (5.2.1) holds} \\ (5.2.4) \quad \Rightarrow \|T\|_{\mathcal{A}} &\leq \inf \{k > 0 : Tf \prec kf \forall f \in (L^1 + L^\infty)(\mu)\} \end{aligned}$$

Combining (5.2.3) and (5.2.4) we obtain equality in (5.2.4). \square

Let X be an intermediate space between X_0 and X_1 and let Y be an intermediate space between Y_0 and Y_1 . (X, Y) is called an *interpolation pair* if every admissible operator maps X into Y . The following lemma will be used to demonstrate that if (X, Y) is an interpolation pair and T is an admissible operator, then $T \upharpoonright X$ is automatically a bounded operator from X into Y .

LEMMA 5.2.3. [2, p.102] *Let A and B be Banach spaces continuously embedded into Banach spaces C and D respectively. If $T \in \mathcal{B}(C, D)$ and T maps A into B , then $T \in \mathcal{B}(A, B)$.*

PROOF. The closed graph Theorem [31, p.114] will be used to prove the boundedness of T . Let $(a_n)_{n=1}^\infty \subset A$ be such that $a = \lim_{n \rightarrow \infty} a_n$ and $b = \lim_{n \rightarrow \infty} Ta_n$ exist in A and B respectively. Now $a_n \rightarrow a$ in A implies that $a_n \rightarrow a$ in C since A is continuously embedded in C . Similarly $Ta_n \rightarrow b$ in D . Since T is a bounded operator from C into D , we have that $Ta_n \rightarrow Ta$ in D and so $Ta = b$ since D is Hausdorff. T is therefore continuous or equivalently bounded from A into B by the closed graph theorem. \square

PROPOSITION 5.2.4. [2, p.103] *Let (X, Y) be an interpolation pair. If $T \in \mathcal{A}$, then $T \upharpoonright X \in \mathcal{B}(X, Y)$.*

PROOF. By the definitions of an interpolation pair and intermediate spaces, X and Y are continuously embedded in $X_0 + X_1$ and $Y_0 + Y_1$ respectively and T maps X into Y . By the lemma above, T is bounded since $X_0 + X_1$ and $Y_0 + Y_1$ are Banach spaces. \square

It is shown in [18, p.20] that if (X, Y) is an interpolation pair relative to the compatible couples (X_0, X_1) and (Y_0, Y_1) , then there exists a $c > 0$ such that

$$(5.2.5) \quad \|T \upharpoonright X\|_{\mathcal{B}(X,Y)} \leq c \|T\|_{\mathcal{A}}$$

If $c \leq 1$ in (5.2.5) then (X, Y) is called an *exact interpolation pair*. We are particularly interested in the case where $X_i = Y_i$ for $i = 0, 1$ and $X = Y$.

DEFINITION 5.2.5. [2, p.105] If X is an intermediate space for the compatible couple (X_0, X_1) and every admissible operator $T \in \mathcal{A}((X_0, X_1); (X_0, X_1))$ maps X into itself, then X is called an *interpolation space* between X_0 and X_1 .

If X is an interpolation space relative to the compatible couple (X_0, X_1) , then (X, X) is an interpolation pair relative to the compatible couples (X_0, X_1) and (X_0, X_1) . We therefore have the following application of Proposition 5.2.4

COROLLARY 5.2.6. *Let X be an interpolation space relative to the compatible couple (X_0, X_1) , then every admissible operator is bounded on X .*

If $\|T \upharpoonright X\|_{\mathcal{B}(X)} \leq \|T\|_{\mathcal{A}}$ for all $T \in \mathcal{A}$, then X is called an *exact interpolation space*. The following yields a family of exact interpolation spaces.

THEOREM 5.2.7. [2, p.106] *Let X be a rearrangement invariant Banach function space over a resonant measure space. Then X is an exact interpolation space between $L^1(\mu)$ and $L^\infty(\mu)$.*

PROOF. By Theorem 5.2.1 X is an intermediate space relative to the couple $(L^1(\mu), L^\infty(\mu))$. Let T be an admissible operator for the couple $(L^1(\mu), L^\infty(\mu))$ and let $f \in X$. Then $f \in (L^1 + L^\infty)(\mu)$ since $X \hookrightarrow (L^1 + L^\infty)(\mu)$. Furthermore, by Proposition 5.2.2, there exists a $k > 0$ such that

$$(5.2.6) \quad Tg \prec kg \quad \forall g \in (L^1 + L^\infty)(\mu)$$

and so by Proposition 4.1.6(2),

$$(5.2.7) \quad \begin{aligned} \|Tf\|_X &\leq \|kf\|_X \\ &= k\|f\|_X \end{aligned}$$

Therefore $T(X) \subset X$ and hence X is an interpolation space relative to the couple $(L^1(\mu), L^\infty(\mu))$. Since (5.2.7) holds for every $k > 0$ such that (5.2.6) holds, we have that

$$\begin{aligned} \|T\|_{\mathcal{B}(X)} &\leq \inf \{k > 0 : Tg \prec kg \forall g \in (L^1 + L^\infty)(\mu)\} \\ &= \|T\|_{\mathcal{A}} \quad \text{by Proposition 5.2.2} \end{aligned}$$

and hence X is an exact interpolation space. □

Multiplication operators

Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be σ -finite measure spaces. In section 7.4 we wish to investigate the conditions on the symbol function τ such that $C_\tau(L^p(\mu_1)) \subset L^q(\mu_2)$, where $1 \leq p, q \leq \infty$. In this chapter the connection between composition operators and multiplication operators will be detailed and the multiplication operators between $L^p(\mu)$ and $L^q(\mu)$ will be investigated. Furthermore we study the compactness of multiplication operators on L^p spaces.

6.1. Multiplication operators between $L^p(\mu)$ and $L^q(\mu)$

Let $\pi \in L^0(\mu)$ and $1 \leq p, q \leq \infty$. π induces a linear transformation M_π from $L^p(\mu)$ into $L^q(\mu)$ defined by

$$(M_\pi f)(x) := \pi(x) \cdot f(x) \quad f \in L^p(\mu), x \in \Omega$$

If M_π takes $L^p(\mu)$ into $L^q(\mu)$, then M_π is called a *multiplication operator* from $L^p(\mu)$ into $L^q(\mu)$.

The following describes the connection between composition operators and multiplication operators.

LEMMA 6.1.1. [39, p.330] *Let $1 \leq p \leq \infty$, $1 \leq q < \infty$ and $\tau : \Omega_2 \rightarrow \Omega_1$ a nonsingular measurable transformation, then*

$$\|C_\tau f\|_{L^q(\mu_2)} = \|M_{f_\tau^{1/q}} f\|_{L^q(\mu_1)} \quad \forall f \in L^p(\mu_1)$$

where f_τ is the Radon-Nikodym derivative of $\mu_2 \circ \tau^{-1}$ with respect to μ_1 (see Corollary 1.2.16).

PROOF. Let $f \in L^p(\mu_1)$, then

$$\begin{aligned} \|C_\tau f\|_{L^q(\mu_2)}^q &= \int_{\Omega_2} |(C_\tau f)(x)|^q d\mu_2 \\ &= \int_{\Omega_2} g(\tau(x)) d\mu_2 \quad \text{where } g(x) := |f(x)|^q \\ &= \int_{\Omega_1} g(y) f_\tau(y) d\mu_1 \quad \text{by Theorem 1.2.18} \\ &= \int_{\Omega_1} (|f(y)| f_\tau(y)^{1/q})^q d\mu_1 \\ &= \|M_{f_\tau^{1/q}} f\|_{L^q(\mu_1)}^q \end{aligned}$$

□

COROLLARY 6.1.2. *Let $1 \leq p \leq \infty$, $1 \leq q < \infty$ and $\tau : \Omega_2 \rightarrow \Omega_1$ a nonsingular measurable transformation. Then τ induces a bounded composition operator from $L^p(\mu_1)$ into $L^q(\mu_2)$ if and only if $f_\tau^{1/q}$ induces a bounded multiplication operator from $L^p(\mu_1)$ into $L^q(\mu_1)$. Furthermore,*

$$\|C_\tau\| = \|M_{f_\tau^{1/q}}\|$$

We therefore investigate the conditions τ must satisfy to induce a bounded multiplication operator between $L^p(\mu)$ and $L^q(\mu)$, where $\mu = \mu_1 = \mu_2$. The symbol functions $\pi \in L^0(\mu)$ which induce bounded multiplication operators from $L^p(\mu)$ into $L^q(\mu)$ will be characterized. The three cases $p > q$, $p = q$ and $p < q$ will be treated separately. We start by showing that every multiplication operator from $L^p(\mu)$ into $L^q(\mu)$ is necessarily bounded.

PROPOSITION 6.1.3. [39, p.323] *Let $1 \leq q, p \leq \infty$. Every multiplication operator M_π from $L^p(\mu)$ into $L^q(\mu)$ is a bounded linear operator from $L^p(\mu)$ into $L^q(\mu)$.*

PROOF. Let $f_n \rightarrow f$ in $L^p(\mu)$ and assume that $M_\pi f_n \rightarrow g$ in $L^q(\mu)$. By Theorem 1.2.8 there exists a subsequence $(f_{n_k})_{k=1}^\infty$ such that $f_{n_k} \rightarrow f$ pointwise μ -a.e. Since $M_\pi f_{n_k} \rightarrow g$ in $L^q(\mu)$, we similarly have a subsequence $(f_{n_{k_m}})_{m=1}^\infty$ such that $M_\pi f_{n_{k_m}} \rightarrow g$ pointwise μ -a.e. It follows that for almost every $x \in \Omega$

$$\begin{aligned} (M_\pi f)(x) &= \pi(x) \cdot \lim_{m \rightarrow \infty} f_{n_{k_m}}(x) \\ &= \lim_{m \rightarrow \infty} (M_\pi f_{n_{k_m}})(x) \\ &= g(x) \end{aligned}$$

Therefore $M_\pi f = g$ and hence M_π is continuous by the Closed Graph Theorem ([31, p.114]). \square

When considering multiplication operators from $L^p(\mu)$ into $L^q(\mu)$ we will treat the cases $p < q$ and $p > q$ separately. We start with $q < p$.

THEOREM 6.1.4. [39, p.323] *Let $1 \leq q < p < \infty$ and $\pi \in L^0(\mu)$. π induces a bounded multiplication operator from $L^p(\mu)$ into $L^q(\mu)$ if and only if $\pi \in L^r(\mu)$, where $\frac{1}{p} + \frac{1}{r} = \frac{1}{q}$. Furthermore*

$$\|M_\pi\| = \|\pi\|_r$$

PROOF. (Sufficiency): Since $\frac{1}{p} + \frac{1}{r} = \frac{1}{q}$, we have that $\left(\frac{p}{q}\right)^{-1} + \left(\frac{r}{q}\right)^{-1} = 1$. Therefore $(L^{r/q}(\mu))' = L^{p/q}(\mu)$ by Corollary 3.1.12. Let $\pi \in L^r(\mu)$ and $f \in L^p(\mu)$, then

$$\begin{aligned} \|M_\pi f\|_q &= \left(\int_\Omega |\pi(x)|^q |f(x)|^q d\mu \right)^{1/q} \\ &\leq \left(\left(\int_\Omega (|\pi(x)|^q)^{r/q} d\mu \right)^{q/r} \left(\int_\Omega (|f(x)|^q)^{p/q} d\mu \right)^{q/p} \right)^{1/q} \\ &= \|\pi\|_r \|f\|_p \end{aligned}$$

where the second line follows using Hölder's inequality (Theorem 3.1.13) and the fact that $\pi^q \in L^{r/q}(\mu)$ and $f^q \in L^{p/q}(\mu)$. We conclude that M_π is a bounded multiplication operator from $L^p(\mu)$ into $L^q(\mu)$ and

$$(6.1.1) \quad \|M_\pi\| \leq \|\pi\|_r$$

(Necessity): Let π induce a bounded multiplication operator from $L^p(\mu)$ into $L^q(\mu)$. Let r' be such that $\frac{1}{r'} + \frac{1}{p} = 1$. Define a map ϕ on $L^{r'}(\mu)$ by

$$\phi(f) := \int_\Omega f \pi d\mu \quad f \in L^{r'}(\mu)$$

We show that ϕ is a bounded linear functional on $L^{r'}(\mu)$: Let $f \in L^{r'}(\mu)$ and let q' be such that $\frac{1}{q'} + \frac{1}{q} = 1$. It is easily checked that $\frac{1}{p} + \frac{1}{q'} = \frac{1}{r'}$. Let

$$\begin{aligned} f_1(x) &:= |f(x)|^{r'/p} \quad \text{and} \\ f_2(x) &:= \begin{cases} f(x)/f_1(x) & \text{if } f_1(x) \neq 0 \\ 0 & \text{if } f_1(x) = 0 \end{cases} \end{aligned}$$

Then $f_1 \in L^p(\mu)$ and $f_2 \in L^{q'}(\mu)$ since

$$\begin{aligned} \int_{\Omega} |f_1(x)|^p d\mu &= \int_{\Omega} |f(x)|^{r'} d\mu < \infty \quad \text{and} \\ \int_{\Omega} |f_2(x)|^{q'} d\mu &= \int_{\Omega} |f(x)|^{(1-r'/p)q'} d\mu \\ &= \int_{\Omega} |f(x)|^{r'} d\mu < \infty \end{aligned}$$

Furthermore

$$(6.1.2) \quad \|f_1\|_p^p = \|f\|_{r'}^{r'} = \|f_2\|_{q'}^{q'}$$

and $f = f_1 f_2$, so

$$\begin{aligned} |\phi(f)| &= \left| \int_{\Omega} \pi \cdot f d\mu \right| \\ &= \left| \int_{\Omega} \pi \cdot f_1 \cdot f_2 d\mu \right| \\ &\leq \int_{\Omega} |M_{\pi} f_1| |f_2| d\mu \\ &\leq \|M_{\pi} f_1\|_q \|f_2\|_{q'} \quad \text{using Hölder's inequality and } f_1 \in L^p(\mu) \Rightarrow M_{\pi} f_1 \in L^q(\mu) \\ &\leq \|M_{\pi}\| \|f_1\|_p \|f_2\|_{q'} \quad \text{since } M_{\pi} \text{ is bounded} \\ &= \|M_{\pi}\| \|f\|_{r'}^{r'/p} \|f\|_{r'}^{r'/q'} \quad \text{by (6.1.2)} \\ (6.1.3) \quad &= \|M_{\pi}\| \|f\|_{r'} \quad \text{since } \frac{r'}{p} + \frac{r'}{q'} = 1 \end{aligned}$$

This implies that $\phi(f) \in \mathbb{F}$ for all $f \in L^{r'}(\mu)$. ϕ is linear by the linearity of the integral and is therefore a bounded linear functional on $L^{r'}(\mu)$ by (6.1.3). Furthermore

$$(6.1.4) \quad \|\phi\| \leq \|M_{\pi}\|$$

By Theorem 1.2.9 there exists a $g \in L^r(\mu)$ such that $\phi = \phi_g$, where

$$\phi_g(f) := \int_{\Omega} f g d\mu \quad \forall f \in L^{r'}(\mu)$$

It follows that

$$\int_{\Omega} f g d\mu = \int_{\Omega} f \pi d\mu \quad \forall f \in L^{r'}(\mu)$$

In particular, for every $A \in \Sigma_f$, $\chi_A \in L^{r'}(\mu)$ and so

$$\int_A g d\mu = \int_A \pi d\mu \quad \forall A \in \Sigma_f$$

It follows by Proposition 1.2.2 that $g = \pi$ μ -a.e. and hence $\pi \in L^r(\mu)$. Furthermore

$$\begin{aligned} \|\pi\|_r &= \|g\|_r \\ &= \|\phi_g\| \\ &= \|\phi\| \quad \text{since } \phi_g = \phi \\ (6.1.5) \quad &\leq \|M_{\pi}\| \quad \text{by (6.1.4)} \end{aligned}$$

Combining (6.1.1) and (6.1.5), we obtain $\|\pi\|_r = \|M_{\pi}\|$. □

Next we consider the case $p = q$.

THEOREM 6.1.5. (cf. [10, p.211-213]) *Let $\pi \in L^0(\mu)$ and $1 \leq p \leq \infty$. π induces a bounded multiplication operator on $L^p(\mu)$ if and only if $\pi \in L^\infty(\mu)$. Furthermore, if M_π is a bounded multiplication operator on $L^p(\mu)$, then $\|M_\pi\| = \|\pi\|_\infty$.*

PROOF. Let $\pi \in L^\infty(\mu)$ and $f \in L^p(\mu)$. If $1 \leq p < \infty$, then

$$\begin{aligned} \|M_\pi f\|_p^p &= \int_\Omega |\pi(x) \cdot f(x)|^p d\mu \\ &\leq \|\pi\|_\infty^p \int_\Omega |f(x)|^p d\mu \\ &= \|\pi\|_\infty^p \|f\|_p^p \end{aligned}$$

If $p = \infty$, then it is easily checked that we obtain $\|M_\pi f\|_\infty \leq \|\pi\|_\infty \|f\|_\infty$. Therefore M_π is a bounded multiplication operator and

$$(6.1.6) \quad \|M_\pi\| \leq \|\pi\|_\infty$$

Conversely, let M_π be a bounded multiplication operator on $L^p(\mu)$ and assume that $\pi \notin L^\infty(\mu)$. Let $\epsilon > 0$, then there exists an $A \in \Sigma_f$ with $\mu(A) > 0$ such that $|\pi(x)| > \|M_\pi\| + \epsilon$ for all $x \in A$. Therefore $\chi_A \in L^p(\mu)$. If $1 \leq p < \infty$, then

$$\begin{aligned} \|M_\pi \chi_A\|_p^p &= \int_\Omega |\pi(x) \cdot \chi_A(x)|^p d\mu \\ &= \int_A |\pi(x)|^p d\mu \\ &\geq (\|M_\pi\| + \epsilon)^p \mu(A) \\ &= (\|M_\pi\| + \epsilon)^p \|\chi_A\|_p^p \end{aligned}$$

If $p = \infty$, then $\|M_\pi \chi_A\|_\infty \geq (\|M_\pi\| + \epsilon) \|\chi_A\|_\infty$. In both cases we obtain a contradiction and therefore $\pi \in L^\infty(\mu)$ and

$$(6.1.7) \quad \|M_\pi\| \geq \|\pi\|_\infty$$

Combining (6.1.6) and (6.1.7), we obtain $\|M_\pi\| = \|\pi\|_\infty$. \square

Finally, we consider the case $p < q$. Recall that $\Omega = \Omega_c \cup \Omega_a$ where Ω_c is nonatomic and $\Omega_a = \bigcup_{n=1}^\infty A_n$ is purely atomic and each A_n is an atom of strictly positive measure.

THEOREM 6.1.6. [39, p.324] *Let $1 \leq p < q < \infty$ and $\pi \in L^0(\mu)$. Then π induces a bounded multiplication operator from $L^p(\mu)$ into $L^q(\mu)$ if and only if π satisfies the following conditions:*

- (1) $\pi(x) = 0$ for μ -a.e. $x \in \Omega_c$
- (2) $\sup_{n \in \mathbb{N}^+} \frac{|\pi(A_n)|^s}{\mu(A_n)} = \beta < \infty$ where $\frac{1}{q} + \frac{1}{s} = \frac{1}{p}$

If this is the case then

$$\|M_\pi\| = \sup_{n \in \mathbb{N}^+} \frac{|\pi(A_n)|}{\mu(A_n)^{1/s}}$$

PROOF. (Sufficiency): Let $f \in L^p(\mu)$ such that $\|f\|_p \leq 1$. We note first that for each $n \in \mathbb{N}^+$ this implies

$$\begin{aligned} |f(A_n)|^p \mu(A_n) &= \int_{A_n} |f|^p d\mu \\ &\leq \int_\Omega |f|^p d\mu \\ &= \|f\|_p^p \leq 1 \end{aligned} \tag{6.1.8}$$

Therefore

$$\begin{aligned}
\|M_\pi f\|_q^q &= \int_\Omega |\pi \cdot f|^q d\mu \\
&= \int_{\Omega_c} |\pi \cdot f|^q d\mu + \int_{\Omega_a} |\pi \cdot f|^q d\mu \\
&= 0 + \sum_{n=1}^{\infty} |\pi(A_n) \cdot f(A_n)|^q \mu(A_n) \quad \text{using Condition 1} \\
&= \sum_{n=1}^{\infty} \left(\frac{|\pi(A_n)|^s}{\mu(A_n)} \right)^{q/s} |f(A_n)|^q \mu(A_n)^{1+q/s} \\
&\leq \beta^{q/s} \sum_{n=1}^{\infty} (|f(A_n)|^p \mu(A_n))^{q/p} \\
&\quad \text{using Condition 2 and } \frac{1}{q} + \frac{1}{s} = \frac{1}{p} \\
&\leq \beta^{q/s} \sum_{n=1}^{\infty} (|f(A_n)|^p \mu(A_n)) \\
&\quad \text{since } |f(A_n)|^p \mu(A_n) \leq 1 \text{ by (6.1.8) and } \frac{q}{p} > 1 \\
&= \beta^{q/s} \int_{\Omega_a} |f|^p d\mu \\
&\leq \beta^{q/s} \int_\Omega |f|^p d\mu \\
(6.1.9) \quad &\leq \beta^{q/s} \quad \text{since } \int_\Omega |f|^p d\mu = \|f\|_p^p \leq 1
\end{aligned}$$

More generally, if $f \in L^p(\mu)$, then $\| |f| \|_p^{-1} f \|_p = 1$ and so

$$\begin{aligned}
\|M_\pi f\|_q^q &= \| |f| \|_p^q \|M_\pi \| |f|^{-1} f \|_p^{-1} f \|_p^q \quad \text{since } M_\pi \text{ is linear} \\
&\leq \| |f| \|_p^q \beta^{q/s} \quad \text{by (6.1.9)}
\end{aligned}$$

M_π is therefore a bounded multiplication operator and

$$(6.1.10) \quad \|M_\pi\| \leq \beta^{1/s}$$

(Necessity): Let π induce a bounded multiplication operator from $L^p(\mu)$ into $L^q(\mu)$ and let $B = \{x \in \Omega_c : \pi(x) \neq 0\}$. Assume that $\mu(B) > 0$. Then by Proposition 1.2.1 there exists an $\epsilon > 0$ and an $A \in \Sigma \cap \Omega_c$ with $\mu(A) > 0$ such that $|\pi(x)| \geq \epsilon$ for all $x \in A$. Since A is nonatomic we have by Lemma 1.2.3 an $f \in L^p(\mu)$ such that $\int_A |f|^q d\mu = \infty$. Then

$$\begin{aligned}
\infty &= \left(\epsilon^q \int_A |f|^q d\mu \right)^{1/q} \\
&\leq \left(\int_A |\pi(x) \cdot f(x)|^q d\mu \right)^{1/q} \\
&\leq \left(\int_\Omega |(M_\pi f)(x)|^q d\mu \right)^{1/q} \\
&= \|M_\pi f\|_q \\
&< \infty \quad \text{since } M_\pi(L^p(\mu)) \subset L^q(\mu)
\end{aligned}$$

This is a contradiction and so $\mu(B) = 0$, i.e. $\pi(x) = 0$ for μ -a.e. $x \in \Omega_c$. To show that $\sup_{n \in \mathbb{N}^+} \frac{|\pi(A_n)|^s}{\mu(A_n)} < \infty$, let

$$f_n := \frac{\chi_{A_n}}{\mu(A_n)^{1/p}}$$

Then $f_n \in L^p(\mu)$ and $\|f_n\|_p = 1$ for each $n \in \mathbb{N}^+$. So

$$\begin{aligned} \frac{|\pi(A_n)|^s}{\mu(A_n)} &= \left(\frac{|\pi(A_n)|^q}{\mu(A_n)^{q/p} \mu(A_n)} \right)^{s/q} && \text{since } 1 - \frac{q}{p} = -\frac{q}{s} \\ &= \left(\int_{A_n} |\pi f_n|^q d\mu \right)^{s/q} \\ &= \|M_\pi f_n\|_q^s \\ &\leq \|M_\pi\|^s \|f_n\|_p^s \\ &= \|M_\pi\|^s \quad \text{since } \|f_n\|_p = 1 \end{aligned}$$

Since this holds for all $n \in \mathbb{N}^+$, we have

$$\begin{aligned} (6.1.11) \quad \beta &= \sup_{n \in \mathbb{N}^+} \frac{|\pi(A_n)|^s}{\mu(A_n)} \leq \|M_\pi\|^s < \infty \\ \Rightarrow \beta^{1/s} &= \sup_{n \in \mathbb{N}^+} \frac{|\pi(A_n)|}{\mu(A_n)^{1/s}} \leq \|M_\pi\| \end{aligned}$$

□

COROLLARY 6.1.7. *Let $1 \leq p < q < \infty$, $\pi \in L^0(\mu)$ and Ω nonatomic. Then the zero operator is the only multiplication operator from $L^p(\mu)$ into $L^q(\mu)$.*

PROOF. Ω is nonatomic, so $\Omega = \Omega_c$. Condition 2 of the theorem above is satisfied since

$$\begin{aligned} \sup_{n \in \mathbb{N}^+} \frac{|\pi(A_n)|^s}{\mu(A_n)} &= \sup \emptyset \\ &= 0 < \infty \end{aligned}$$

It follows by the Theorem above that π induces a bounded multiplication operator from $L^p(\mu)$ into $L^q(\mu)$ if and only if $\pi(x) = 0$ for μ -a.e. $x \in \Omega_c = \Omega$. □

6.2. Compactness of multiplication operators

Let $\phi : \Omega \rightarrow \mathbb{F}$ be a measurable function and define

$$\begin{aligned} \Omega_\epsilon^\phi &:= \{x \in \Omega : |\phi(x)| > \epsilon\} \quad \text{and} \\ Z_\epsilon^\phi &:= \left\{ f \in L^p(\mu) : f(x) = 0 \text{ for } \mu\text{-a.e. } x \in (\Omega_\epsilon^\phi)^c \right\}. \end{aligned}$$

If $\tau : \Omega \rightarrow \Omega$ is a nonsingular measurable transformation, then we show that the compactness of the multiplication operator $M_{f_\tau^{1/p}}$ on an L^p space can be characterized in terms of the compactness of the composition operator C_τ or in terms of the dimension of $Z_\epsilon^{f_\tau^{1/p}}$. Furthermore we apply this result to the case where the symbol function τ maps the entire underlying measure space into the atomic part of the measure space to obtain a sufficient condition for $M_{f_\tau^{1/p}}$ to be compact on $L^p(\mu)$. We start by showing that Z_ϵ^ϕ is a Banach space.

PROPOSITION 6.2.1. *Let $\phi : \Omega \rightarrow \mathbb{F}$ be a measurable function and $\epsilon > 0$. Z_ϵ^ϕ is a Banach space.*

PROOF. It is easily checked that Z_ϵ^ϕ is a subspace of $L^p(\mu)$. We show that Z_ϵ^ϕ is closed. Let $(f_n)_{n=1}^\infty \subset Z_\epsilon^\phi$ such that $f_n \rightarrow f \in L^p(\mu)$. If we assume that $f \notin Z_\epsilon^\phi$, i.e. there exists a $B \in \Sigma \cap (\Omega_\epsilon^\phi)^c$ with $\mu(B) > 0$, such

that $f(x) \neq 0$ on B , then by Proposition 1.2.1 there exists a $\delta > 0$ and an $A \in \Sigma \cap (\Omega_\epsilon^\phi)^c$ with $\mu(A) > 0$ such that $|f(x)| \geq \delta$ for all $x \in A$. It follows that for all $n \in \mathbb{N}^+$ we have

$$\begin{aligned} \|f - f_n\|_{L^p}^p &= \int_{\Omega} |f(x) - f_n(x)|^p d\mu \\ &\geq \int_A |f(x) - f_n(x)|^p d\mu \\ &= \int_A |f(x)|^p d\mu \quad \text{since } f_n = 0 \text{ on } (\Omega_\epsilon^\phi)^c \supset A \\ &\geq \delta^p \mu(A) \end{aligned}$$

This is a contradiction and hence $f \in Z_\epsilon^\phi$. It follows that Z_ϵ^ϕ is closed and therefore complete since it is a closed subset of a Banach space. \square

The main characterization is given by the following result.

LEMMA 6.2.2. [38, p.506] *Let (Ω, Σ, μ) be a σ -finite measure space, $\tau : \Omega \rightarrow \Omega$ a nonsingular measurable transformation and $\phi = f_\tau^{1/p}$. The following are equivalent:*

- (1) C_τ is a compact operator on $L^p(\mu)$
- (2) M_ϕ is a compact operator on $L^p(\mu)$
- (3) For any $\epsilon > 0$, Z_ϵ^ϕ is finite dimensional

PROOF. (1) \Leftrightarrow (2): Since C_τ and M_ϕ are linear and $\|C_\tau f\|_{L^p} = \|M_\phi f\|_{L^p}$ for all $f \in L^p(\mu)$ by Lemma 6.1.1, $(C_\tau f_n)_{n=1}^\infty$ is Cauchy if and only if $(M_\phi f_n)_{n=1}^\infty$ is a Cauchy sequence in $L^p(\mu)$, where $(f_n)_{n=1}^\infty \subset B_{L^p}$. Therefore C_τ is compact if and only if M_ϕ is compact.

(2) \Rightarrow (3): Since M_ϕ is compact and Z_ϵ^ϕ is invariant under M_ϕ , $M_\phi^\epsilon := M_\phi \upharpoonright Z_\epsilon^\phi$ is compact by Remark 1.5.7. We show that M_ϕ^ϵ is invertible: Let $f, g \in Z_\epsilon^\phi$, such that $M_\phi^\epsilon f = M_\phi^\epsilon g$, then

$$\begin{aligned} \phi(x).f(x) &= \phi(x).g(x) \quad \mu\text{-a.e. on } \Omega_\epsilon^\phi \\ \Rightarrow f(x) &= g(x) \quad \mu\text{-a.e. on } \Omega_\epsilon^\phi, \text{ since } \phi(x) > \epsilon \forall x \in \Omega_\epsilon^\phi \\ \Rightarrow f(x) &= g(x) \quad \mu\text{-a.e. on } \Omega, \text{ since } f = g = 0 \text{ on } (\Omega_\epsilon^\phi)^c \\ \Rightarrow f &= g \end{aligned}$$

and so M_ϕ^ϵ is injective. To show that M_ϕ^ϵ is surjective, let $f \in Z_\epsilon^\phi$ and define

$$g(x) = \begin{cases} \frac{f(x)}{\phi(x)} & x \in \Omega_\epsilon^\phi \\ 0 & x \notin \Omega_\epsilon^\phi \end{cases}$$

Then $M_\phi^\epsilon g = f$ since $f = 0$ on $(\Omega_\epsilon^\phi)^c$. Furthermore it is easily checked that $g \in L^p(\mu)$ using $\phi(x) > \epsilon$ for all $x \in \Omega_\epsilon^\phi$. It follows that $g \in Z_\epsilon^\phi$ and therefore that M_ϕ^ϵ is surjective. Since Z_ϵ^ϕ is a Banach space (Proposition 6.2.1), it follows by the inverse mapping theorem (1.5.13) that $M_\phi^\epsilon : Z_\epsilon^\phi \rightarrow Z_\epsilon^\phi$ is invertible. Z_ϵ^ϕ is therefore finite dimensional by Proposition 1.5.12 since M_ϕ^ϵ is compact and invertible.

(3) \Rightarrow (2): If (3) holds for each $\epsilon > 0$, then $Z_{1/n}^\phi$ is finite dimensional for each $n \in \mathbb{N}^+$. Let

$$\phi_n(x) := \begin{cases} \phi(x) & x \in \Omega_{1/n}^\phi \\ 0 & x \notin \Omega_{1/n}^\phi \end{cases}$$

Then M_{ϕ_n} is a finite rank operator for each n since $\text{ran } M_{\phi_n} \subset Z_{1/n}^\phi$. Furthermore we show that $\|M_\phi - M_{\phi_n}\| \rightarrow 0$: Let $f \in L^p(\mu)$, then

$$\begin{aligned} \|(M_\phi - M_{\phi_n})f\|_{L^p(\mu)}^p &= \int_{\Omega} |\phi(x) \cdot f(x) - \phi_n(x) \cdot f(x)|^p d\mu \\ &= \int_{(\Omega_{1/n}^\phi)^c} |\phi(x) \cdot f(x)|^p d\mu \\ &\leq \frac{1}{n^p} \int_{(\Omega_{1/n}^\phi)^c} |f(x)|^p d\mu \quad \text{since } \phi(x) \leq \frac{1}{n^p} \forall x \in (\Omega_{1/n}^\phi)^c \\ &\leq \frac{1}{n^p} \int_{\Omega} |f(x)|^p d\mu \\ &= \frac{1}{n^p} \|f\|_{L^p(\mu)}^p \end{aligned}$$

Since this holds for any $f \in L^p(\mu)$, it follows that $\|M_\phi - M_{\phi_n}\| \leq \frac{1}{n}$. M_ϕ is therefore the limit of a sequence of finite rank operators and hence compact by Proposition 1.5.11. \square

COROLLARY 6.2.3. . *If $\tau : \Omega \rightarrow \Omega$ is a nonsingular measurable transformation such that $\tau^{-1}(\Omega_a) = \Omega$ and $f_\tau^{1/p}(A_n) \rightarrow 0$, then $M_{f_\tau^{1/p}}$ is compact on $L^p(\mu)$.*

PROOF. Let $\phi = f_\tau^{1/p}$ and let $\epsilon > 0$. There exists an $N_\epsilon \in \mathbb{N}^+$ such that

$$(6.2.1) \quad \begin{aligned} n \geq N_\epsilon &\Rightarrow |\phi(A_n)| < \epsilon \\ &\Rightarrow A_n \subset (\Omega_\epsilon^\phi)^c \quad \text{for } n \geq N_\epsilon \end{aligned}$$

Furthermore $\tau^{-1}(\Omega_a) = \Omega$ implies $\mu \circ \tau^{-1}(\Omega_c) = 0$ and hence

$$0 = \mu \circ \tau^{-1}(\Omega_c) = \int_{\Omega_c} f_\tau d\mu \quad \text{by Corollary 1.2.16}$$

Therefore $\phi = f_\tau^{1/p} = 0$ μ -a.e. on Ω_c . It follows from the above and (6.2.1) that

$$\Omega_\epsilon^\phi \subset \bigcup_{n=1}^{N_\epsilon-1} A_n$$

Therefore for any $f \in Z_\epsilon^\phi$, $f(x) = 0$ for all $x \notin \bigcup_{n=1}^{N_\epsilon-1} A_n$. It follows that $\{\chi_{A_n} : 1 \leq n \leq N_\epsilon - 1\}$ is a linearly independent set spanning Z_ϵ^ϕ , i.e. Z_ϵ^ϕ is finite dimensional. Since $\epsilon > 0$ was arbitrary, we have by Lemma 6.2.2 that M_ϕ is compact. \square

Composition operators on spaces of measurable functions

Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces, $\tau : \Omega_2 \rightarrow \Omega_1$ a map and $F_i(\Omega_i)$ ($i = 1, 2$) vector spaces of real or complex-valued functions on Ω_i ($i = 1, 2$). Recall that we can define a composition transformation C_τ on $F_1(\Omega_1)$ by

$$(C_\tau f)(x) := f(\tau(x)) \quad x \in \Omega_2, f \in F_1(\Omega_1)$$

In the setting of spaces of measurable functions, the first natural question to ask is “what conditions must the symbol function τ satisfy to ensure that $C_\tau f$ is a Σ_2 -measurable function for every Σ_1 -measurable function f ?” If we then identify functions which are equal μ_1 -a.e. and define the composition transformation C_τ on equivalence classes of measurable functions (i.e. elements of $L^0(\mu_1)$) via

$$C_\tau[f] := [C_\tau f],$$

then it is necessary to determine the condition(s) on the symbol function τ such that C_τ is well-defined as a mapping of equivalence classes. Note that for simplicity we will often abuse notation and use f to denote the equivalence class generated by f , and as has been done above, we will not distinguish between the composition transformation acting on measurable functions and the composition transformation acting on equivalence classes of measurable functions.

Recall further that if $C_\tau(F_1(\Omega_1)) \subset F_2(\Omega_2)$, then C_τ is called composition operator from $F_1(\Omega_1)$ into $F_2(\Omega_2)$. We are interested in the conditions τ must satisfy such that

- (1) $C_\tau(F_1(\Omega_1)) \subset F_2(\Omega_2)$ and
- (2) C_τ is continuous from $F_1(\Omega_1)$ into $F_2(\Omega_2)$, where $F_j(\Omega_j)$ ($j = 1, 2$) are equipped with suitable topologies (in the case where $F_j(\Omega_j)$ ($j = 1, 2$) are Banach function spaces the topologies will be the ones induced by the Banach function norms).

In Sections 7.2 and 7.3 we consider these questions for topological vector spaces and rearrangement invariant Banach function spaces respectively under the simplifying assumption that $F_1(\Omega_1) = F_2(\Omega_2)$. In Section 7.4 these questions will be considered for composition operators between $L^p(\mu_1)$ and $L^q(\mu_2)$, i.e. in a setting where the simplifying assumption is not made. Finally, as examples of composition operators, we characterize those rational functions on the real line which induce bounded composition operators.

7.1. Composition operators

We start by addressing the preliminary questions stated earlier and then give a few simple examples and properties of composition operators.

PROPOSITION 7.1.1. *Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces and let $\tau : \Omega_2 \rightarrow \Omega_1$ be a map. $C_\tau f$ is a Σ_2 -measurable function for every Σ_1 -measurable function f if and only if τ is a measurable transformation.*

PROOF. (Sufficiency): Let $G \subset \mathbb{F}$, then

$$\begin{aligned} (C_\tau f)^{-1}(G) &= (f \circ \tau)^{-1}(G) \\ &= \tau^{-1}(f^{-1}(G)) \end{aligned}$$

If G is open, then $f^{-1}(G) \in \Sigma_1$, since f is a Σ_1 -measurable function, and so $\tau^{-1}(f^{-1}(G)) \in \Sigma_2$, since τ is a measurable transformation. $C_\tau f$ is therefore a Σ_2 -measurable function since $G \subset \mathbb{F}$ was an arbitrary open set.

(Necessity): Let $A \in \Sigma_1$, then it is easily checked that χ_A is a Σ_1 -measurable function. $\chi_{\tau^{-1}(A)} = C_\tau \chi_A$ is therefore a Σ_2 -measurable function by assumption. It follows that $\tau^{-1}(A) = (C_\tau \chi_A)^{-1}((0, 2)) \in \Sigma_2$ and hence that τ is a measurable transformation by the arbitrariness of $A \in \Sigma_1$. \square

PROPOSITION 7.1.2. *Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces and $\tau : \Omega_2 \rightarrow \Omega_1$ a measurable transformation. C_τ is well-defined as a mapping of equivalence classes if and only if τ is nonsingular.*

PROOF. (Sufficiency): Let $[f] = [g]$, $A = \{x \in \Omega_1 : f(x) \neq g(x)\}$ and $B = \{x \in \Omega_2 : (C_\tau f)(x) \neq (C_\tau g)(x)\}$. Then

$$\begin{aligned} B &= \{x \in \Omega_2 : f(\tau(x)) \neq g(\tau(x))\} && \text{by definition of } C_\tau \\ &= \{x \in \Omega_2 : \tau(x) \in A\} \\ &= \tau^{-1}(A) \end{aligned}$$

and so $\mu_2(B) = \mu_2 \circ \tau^{-1}(A) = 0$ since $\mu_1(A) = 0$ and τ nonsingular. We therefore have that $[C_\tau f] = [C_\tau g]$, i.e. $C_\tau[f] = C_\tau[g]$.

(Necessity): We show the contrapositive. Let $A \in \Sigma_1$ such that $\mu_1(A) = 0$, but $\mu_2 \circ \tau^{-1}(A) \neq 0$. Then $[0] = [\chi_A]$ since $0 = \chi_A$ μ_1 -a.e. However,

$$\begin{aligned} [C_\tau \chi_A] &= [\chi_{\tau^{-1}(A)}] \\ &\neq [0] \\ &= [C_\tau 0] \end{aligned}$$

i.e. $C_\tau[\chi_A] \neq C_\tau[0]$ and so C_τ is not well-defined. \square

COROLLARY 7.1.3. *Let (Ω, Σ, μ) be a measure space and $\tau : \Omega \rightarrow \Omega$ a map. C_τ is a composition operator on $L^0(\mu)$ if and only if τ is a nonsingular measurable transformation.*

We provide some simple examples of composition operators.

EXAMPLE 7.1.4. Recall that \mathcal{M} is the σ -algebra of Lebesgue measurable subsets of the real line and m is the Lebesgue measure defined on \mathcal{M} . Fix $y \in \mathbb{R}$ and let $\tau_y : \mathbb{R} \rightarrow \mathbb{R}$ be defined by $\tau_y(x) = y + x$. Then $\tau_y^{-1}(A) = A - y \in \mathcal{M}$ for all $A \in \mathcal{M}$ and so τ_y is a nonsingular measurable transformation, since the Lebesgue measure is translation invariant. τ_y therefore induces a composition operator on $L^0(\mathbb{R}, \mathcal{M}, m)$, called a *translation operator*.

EXAMPLE 7.1.5. Let $\mathbf{w} := (w_i)_{i \in \mathbb{Z}}$ be a sequence of strictly positive real numbers and recall that μ_w is defined by $\mu_w(\{i\}) = w_i$ for $i \in \mathbb{Z}$. Consider the maps τ_b and τ_s defined on the integers by $\tau_b(i) = i - 1$ and $\tau_s(i) = -i$. $\tau_b^{-1}(A) = A + 1 \in \mathcal{P}(\mathbb{Z})$ and $\tau_s^{-1}(A) = -A \in \mathcal{P}(\mathbb{Z})$ for any $A \in \mathcal{P}(\mathbb{Z})$, i.e. τ_b and τ_s are measurable. Furthermore $\mu_w(A) = 0$ if and only if $A = \emptyset$. It therefore follows trivially that τ_b and τ_s are nonsingular. The composition operators on $L^0(\mathbb{Z}, \mathcal{P}(\mathbb{Z}), \mu_w)$ induced by τ_b and τ_s are the bilateral shift operator and symmetry operator respectively.

It is easily checked that composition operators have the properties listed in the following proposition.

PROPOSITION 7.1.6. *Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be σ -finite measure spaces and let $\tau : \Omega_2 \rightarrow \Omega_1$ be a nonsingular measurable transformation. Let $f, g \in L^0(\mu_1)$ and $(f_n)_{n=1}^\infty \subset L^0(\mu_1)$. Then*

- (1) $C_\tau(\alpha f + \beta g) = \alpha C_\tau f + \beta C_\tau g$ for all $\alpha, \beta \in \mathbb{F}$
- (2) $C_\tau f \in L^0_{\mathbb{R}}(\mu_2)$ if $f \in L^0_{\mathbb{R}}(\mu_1)$

- (3) $C_\tau f \geq 0$ if $f \geq 0$
(4) $C_\tau f_n \rightarrow C_\tau f$ pointwise μ_2 -a.e. if $f_n \rightarrow f$ pointwise μ_1 -a.e.

7.2. Composition operators on topological vector spaces

Let (Ω, Σ, μ) be a σ -finite measure space. Some results concerning composition operators on $L^0(\mu)$ have been obtained in [23]. We characterize the maps $\tau : \Omega \rightarrow \Omega$ which induce composition operators on $L^0(\mu)$ and two of its important subspaces, namely $L^{00}(\mu)$ and $L^{01}(\mu)$. In section 2.1 it was shown how these spaces can be equipped with complete metrizable linear topologies. Furthermore, it was shown that in the case where these spaces consist of equivalence classes of real-valued measurable functions, these spaces are complete metrizable locally solid Riesz spaces. This implies that whenever τ induces a composition operator on any of these spaces, then the induced composition operator is continuous with respect to the topology on that space.

7.2.1. Composition operators on $L^0(\Omega, \Sigma, \mu)$. By Corollary 7.1.3, the symbol function τ induces a composition operator on $L^0(\mu)$ if and only if τ is a nonsingular measurable transformation. It remains to be shown that the topology of local convergence in measure defined on $L^0(\mu)$ satisfies the conditions of Proposition 1.5.5 and hence that continuity of the composition operator C_τ on $L^0(\mu)$ follows automatically if τ induces a composition operator on $L^0(\mu)$.

PROPOSITION 7.2.1. *Let (Ω, Σ, μ) be a σ -finite measure space and $\tau : \Omega \rightarrow \Omega$ a map. C_τ is a continuous composition operator on $L^0(\Omega, \Sigma, \mu)$ equipped with the topology of local convergence in measure \mathcal{T} if and only if τ is a nonsingular measurable transformation.*

PROOF. (Sufficiency): By Corollary 7.1.3 C_τ maps $L^0(\mu)$ into itself and is therefore a composition operator. Furthermore, $L^0(\mu)$ is a metrizable topological vector space when equipped with the topology \mathcal{T} of local convergence in measure (Propositions 2.1.2, 2.1.4 and 2.1.5). We show that the conjugation map is continuous with respect to \mathcal{T} : Let $(f_n)_{n=1}^\infty \subset L^0(\mu)$ such that $f_n \rightarrow f$ in $(L^0(\mu), \mathcal{T})$, let $A \in \Sigma_f$ and let $\epsilon, \delta > 0$. Since $f_n \rightarrow f$ in measure on A we can find an $n_{\epsilon, \delta} \in \mathbb{N}^+$ such that $n \geq n_{\epsilon, \delta}$ implies

$$\mu(\{x \in A : |f_n(x) - f(x)| \geq \delta\}) < \epsilon,$$

but since $|\overline{f_n(x)} - \overline{f(x)}| = |\overline{f_n(x) - f(x)}| = |f_n(x) - f(x)|$ for all $x \in A$, we have that $n \geq n_{\epsilon, \delta}$ implies

$$\mu(\{x \in A : |\overline{f_n(x)} - \overline{f(x)}| \geq \delta\}) < \epsilon$$

and hence $\overline{f_n} \rightarrow \overline{f}$ in measure on A . Since $A \in \Sigma_f$ was arbitrary we have that $\overline{f_n} \rightarrow \overline{f}$. $L_{\mathbb{R}}^0(\mu)$, viewed as a vector space over \mathbb{R} , is a complete metrizable locally solid Riesz space when equipped with the topology of local convergence in measure restricted to $L_{\mathbb{R}}^0(\mu)$ (Proposition 2.1.9). It follows by Proposition 1.5.5 that C_τ is continuous on $L^0(\mu)$ with respect to \mathcal{T} .

(Necessity): If C_τ is a bounded composition operator, then τ is a nonsingular measurable transformation by Corollary 7.1.3. \square

If C_τ is a composition operator on $L^0(\mu)$ and $f \in L^0(\mu)$, then $C_\tau f \in L^0(\mu)$ and so we can calculate the distribution function of $C_\tau f$.

PROPOSITION 7.2.2. *Let (Ω, Σ, μ) be a σ -finite measure space, $\tau : \Omega \rightarrow \Omega$ a nonsingular measurable transformation and $f \in L^0(\mu)$. Then $\mu_{C_\tau f}(\lambda) = \mu \circ \tau^{-1}(A_{f, \lambda})$ where $A_{f, \lambda} = \{x \in \Omega : |f(x)| > \lambda\}$.*

PROOF. Since τ is a nonsingular measurable transformation, $C_\tau f \in L^0(\mu)$ by Corollary 7.1.3. Let $\lambda \geq 0$, then

$$\begin{aligned} \mu_{C_\tau f}(\lambda) &= \mu(\{x \in \Omega : |(C_\tau f)(x)| > \lambda\}) \\ &= \mu(\{x \in \Omega : |f(\tau(x))| > \lambda\}) \\ &= \mu(\tau^{-1}(\{x \in \Omega : |f(x)| > \lambda\})) \\ &= \mu \circ \tau^{-1}(A_{f,\lambda}) \end{aligned}$$

□

When considering the compactness of composition operators we will often work with functions of the form $C_\tau \chi_A - C_\tau \chi_B$ and their decreasing rearrangements.

PROPOSITION 7.2.3. *If (Ω, Σ, μ) is a σ -finite measure space, $\tau : \Omega \rightarrow \Omega$ a measurable nonsingular transformation, $\alpha, \beta > 0$ and $A, B \in \Sigma$ such that $A \cap B = \emptyset$, then*

$$(C_\tau \alpha \chi_A - C_\tau \beta \chi_B)^* = \beta \chi_{[0, \mu \circ \tau^{-1}(B))} + \alpha \chi_{[\mu \circ \tau^{-1}(B), \mu \circ \tau^{-1}(A \cup B))}$$

where we have assumed that $\alpha \leq \beta$.

PROOF. We note first that since $A \cap B = \emptyset$, we have that $\tau^{-1}(A) \cap \tau^{-1}(B) = \emptyset$ and hence

$$\alpha \chi_{\tau^{-1}(A)} \cdot \beta \chi_{\tau^{-1}(B)} \equiv 0$$

Therefore

$$\begin{aligned} |C_\tau \alpha \chi_A - C_\tau \beta \chi_B| &= |\alpha \chi_{\tau^{-1}(A)} - \beta \chi_{\tau^{-1}(B)}| \\ &= \alpha \chi_{\tau^{-1}(A)} + \beta \chi_{\tau^{-1}(B)} \end{aligned}$$

An argument similar to the one used for simple functions can therefore be used to obtain the desired result. □

7.2.2. Composition operators on $L^{00}(\Omega, \Sigma, \mu)$. Recall that $L^{00}(\mu)$ is the set of all $f \in L^0(\mu)$ such that f is bounded except possibly on a set of finite measure. In light of this description of $L^{00}(\mu)$ it is perhaps not surprising that if τ is such that the pre-image under τ of a set of finite measure has finite measure, then τ induces a composition operator on $L^{00}(\mu)$. It will be shown that this is the case, but an example will be given to demonstrate that this is not a necessary condition for τ to induce a composition operator on $L^{00}(\mu)$. The maps which induce continuous composition operators on $L^{00}(\mu)$, equipped with the topology of convergence in measure, will then be characterized. Equipped with this topology, $L^{00}(\mu)$ is a complete metrizable topological vector space and satisfies the conditions of Proposition 1.5.5 and therefore a composition operator on $L^{00}(\mu)$ will automatically be continuous. Recall (Remark 1.3.6) that $L^{00}(\mu)$ can also be described as follows:

$$L^{00}(\Omega, \Sigma, \mu) := \{f \in L^0(\Omega, \Sigma, \mu) : f^*(t) < \infty \forall t > 0\}$$

PROPOSITION 7.2.4. *Let (Ω, Σ, μ) be a σ -finite measure space and $\tau : \Omega \rightarrow \Omega$ a nonsingular measurable transformation. τ defines a continuous composition operator on $L^{00}(\Omega, \Sigma, \mu)$ equipped with the topology of convergence in measure if the following condition holds*

$$(7.2.1) \quad A \in \Sigma, \mu(A) < \infty \Rightarrow \mu \circ \tau^{-1}(A) < \infty$$

PROOF. τ nonsingular and measurable implies that C_τ is well-defined by Proposition 7.1.2. We show that $C_\tau(L^{00}(\mu)) \subset L^{00}(\mu)$. Let $f \in L^{00}(\mu)$; there exists an $M > 0$ and an $A \in \Sigma$ with $\mu(A) < \infty$ such that

$|f(x)| \leq M$ for all $x \in A^c$. Then

$$\begin{aligned} |(C_\tau f)(x)| &= |f(\tau(x))| \\ &\leq M \quad \forall x \in \tau^{-1}(A^c), \end{aligned}$$

but $\tau^{-1}(A^c) = (\tau^{-1}(A))^c$ and $\mu \circ \tau^{-1}(A) < \infty$ by (7.2.1), so $C_\tau f$ is bounded except possibly on $\tau^{-1}(A)$, which has finite measure. Hence $C_\tau f \in L^0(\mu)$. The continuity of C_τ follows as in the $L^0(\mu)$ case. \square

REMARK 7.2.5. In [9, p.442-444], condition (7.2.1) is called local absolute continuity of $\mu \circ \tau^{-1}$ with respect to μ . This condition is explored in detail in the non-commutative context in this paper.

REMARK 7.2.6. Condition (7.2.1) is not necessary for C_τ to be a composition operator on $L^0(\mu)$ as the following example shows: Let $(\Omega, \Sigma, \mu) = (\mathbb{N}, P(\mathbb{N}), \mu_c)$ and $\tau(n) = 1$ for all $n \in \mathbb{N}$. Then τ is a nonsingular measurable transformation and furthermore C_τ is a composition operator on $L^0(\mathbb{N}, P(\mathbb{N}), \mu_c)$ since if $f \in L^0(\mathbb{N}, P(\mathbb{N}), \mu_c)$, then

$$\begin{aligned} |(C_\tau f)(n)| &= |f(\tau(n))| \\ &= |f(1)| \quad \forall n \in \mathbb{N} \end{aligned}$$

and so $C_\tau f \in L^\infty(\mathbb{N}, P(\mathbb{N}), \mu_c) \subset L^0(\mathbb{N}, P(\mathbb{N}), \mu_c)$. Condition (7.2.1) does not hold however since $\mu_c(\{1\}) = 1$, but $\mu \circ \tau^{-1}(\{1\}) = \mu(\mathbb{N}) = \infty$.

PROPOSITION 7.2.7. *Let (Ω, Σ, μ) be a σ -finite measure space and $\tau : \Omega \rightarrow \Omega$ a nonsingular measurable transformation. τ defines a continuous composition operator on $L^0(\Omega, \Sigma, \mu)$ equipped with the topology of convergence in measure if and only if*

$$(7.2.2) \quad \forall \epsilon > 0 \exists \delta_\epsilon > 0 \text{ such that } A \in \Sigma, \mu(A) < \delta \Rightarrow \mu \circ \tau^{-1}(A) < \epsilon$$

PROOF. (Sufficiency): τ is a nonsingular measurable transformation and hence C_τ is well-defined. Let $f \in L^0(\mu)$ and let $t > 0$. By (7.2.2) there exists a $\delta_t > 0$ such that

$$(7.2.3) \quad A \in \Sigma, \mu(A) < \delta_t \Rightarrow \mu \circ \tau^{-1}(A) < t$$

And so by Remark 1.3.1

$$\begin{aligned} \inf \left\{ \lambda > 0 : \mu(A_{f,\lambda}) \leq \frac{\delta_t}{2} \right\} &= \inf \left\{ \lambda > 0 : \mu_f(\lambda) \leq \frac{\delta_t}{2} \right\} \\ &= f^* \left(\frac{\delta_t}{2} \right) \\ &< \infty \quad \text{since } f \in L^0(\mu) \\ \Rightarrow \left\{ \lambda > 0 : \mu(A_{f,\lambda}) \leq \frac{\delta_t}{2} \right\} &\neq \emptyset \end{aligned}$$

Let $\alpha \in \left\{ \lambda > 0 : \mu(A_{f,\lambda}) \leq \frac{\delta_t}{2} \right\}$, then $\mu(A_{f,\alpha}) \leq \frac{\delta_t}{2} < \delta_t$ and so $\mu \circ \tau^{-1}(A_{f,\alpha}) < t$ by (7.2.3). Therefore

$$\begin{aligned} (C_\tau f)^*(t) &= \inf \{ \lambda > 0 : \mu_{C_\tau f}(\lambda) \leq t \} \\ &= \inf \{ \lambda > 0 : \mu \circ \tau^{-1}(A_{f,\lambda}) \leq t \} \quad \text{by Proposition 7.2.2} \\ &\leq \alpha \quad \text{since } \mu \circ \tau^{-1}(A_{f,\alpha}) < t \\ &< \infty \end{aligned}$$

Since $t > 0$ was arbitrary, we have $(C_\tau f)^*(t) < \infty$ for all $t > 0$ and hence $C_\tau f \in L^0(\mu)$. Furthermore, $f \in L^0(\mu)$ was arbitrary and so $C_\tau(L^0(\mu)) \subset L^0(\mu)$. The continuity of C_τ follows as in the $L^0(\mu)$ case.

(Necessity): We show the contrapositive. If condition (7.2.2) does not hold then there exists an $\epsilon > 0$ such that

$$\forall \delta > 0, \exists A \in \Sigma \text{ with } \mu(A) < \delta, \text{ but } \mu \circ \tau^{-1}(A) \geq \epsilon$$

In particular

$$(7.2.4) \quad \forall n \in \mathbb{N}^+, \exists A_n \in \Sigma \text{ with } \mu(A_n) < \frac{1}{2^n}, \text{ but } \mu \circ \tau^{-1}(A_n) \geq \epsilon$$

Let $f_k := \sum_{n=1}^k n \chi_{A_n}$. We show that this yields a Cauchy sequence in $L^{00}(\mu)$ and hence that the limit $f = \lim_{k \rightarrow \infty} f_k$ exists by the completeness of $L^{00}(\mu)$. Let $\tilde{\delta}, \tilde{\epsilon} > 0$ and let $\tilde{n} \in \mathbb{N}^+$ such that $\frac{1}{2^{\tilde{n}}} < \tilde{\epsilon}$. Without loss of generality we can assume that $\tilde{\delta}, \tilde{\epsilon} < 1$. Then for $n, m \in \mathbb{N}^+$ and $n > m$

$$\begin{aligned} (f_n - f_m)(x) &= \sum_{k=m+1}^n k \chi_{A_k} \\ \Rightarrow \{x \in \Omega : |(f_n - f_m)(x)| \geq \tilde{\delta}\} &\subset \bigcup_{k=m+1}^n A_k \\ \Rightarrow \mu\left(\{x \in \Omega : |(f_n - f_m)(x)| \geq \tilde{\delta}\}\right) &\leq \mu\left(\bigcup_{k=m+1}^n A_k\right) \\ &\leq \sum_{k=m+1}^n \mu(A_k) \\ &\leq \sum_{k=m+1}^n \frac{1}{2^k} \\ &< \tilde{\epsilon} \quad \text{for } n, m \geq \tilde{n} + 1 \end{aligned}$$

Therefore $(f_k)_{k=1}^{\infty}$ is Cauchy in $L^{00}(\mu)$ equipped with the topology of convergence in measure. Let $A = \bigcup_{n=1}^{\infty} A_n$. Then $f(x) = 0$ for all $x \in A^c$ and $\mu(A) \leq \sum_{n=1}^{\infty} \mu(A_n) \leq \sum_{n=1}^{\infty} \frac{1}{2^n} = 1$ so f is bounded except possibly on a set of finite measure, i.e. $f \in L^{00}(\mu)$. We show that $C_{\tau}f \notin L^{00}(\mu)$: Let $B_k = \{x \in \Omega : |(C_{\tau}f)(x)| > k\}$, then $B_{k+1} \subset B_k$ for all k and

$$(7.2.5) \quad \begin{aligned} \tau^{-1}(A_{k+1}) &\subset B_k \quad \forall k \\ \Rightarrow \epsilon &\leq \mu \circ \tau^{-1}(A_{k+1}) \leq \mu(B_k) \quad \forall k \end{aligned}$$

If $\mu(B_k) < \infty$ for some $k \in \mathbb{N}^+$, then

$$\begin{aligned} \mu(\{x \in \Omega : |(C_{\tau}f)(x)| = \infty\}) &= \mu\left(\bigcap_{k=1}^{\infty} B_k\right) \\ &= \lim_{k \rightarrow \infty} \mu(B_k) \quad \text{by Theorem 1.1.3} \\ &\geq \epsilon \quad \text{by (7.2.5)} \end{aligned}$$

and so $C_{\tau}f \notin L^0(\mu) \supset L^{00}(\mu)$. If $\mu(B_k) = \infty$ for all $k \in \mathbb{N}^+$, then $\mu_{C_{\tau}f}(k) = \mu(B_k) > 1$ for all $k \in \mathbb{N}^+$ and so

$$\begin{aligned} k &\notin \{\lambda > 0 : \mu_{C_{\tau}f}(\lambda) \leq 1\} \quad \forall k \\ \Rightarrow (C_{\tau}f)^*(1) &:= \inf\{\lambda > 0 : \mu_{C_{\tau}f}(\lambda) \leq 1\} \geq k \quad \forall k \in \mathbb{N}^+ \\ \Rightarrow (C_{\tau}f)^*(1) &= \infty \end{aligned}$$

and so $C_{\tau}f \notin L^{00}(\mu)$. Hence C_{τ} is not a composition operator on $L^{00}(\mu)$. \square

We note that we could have removed from the statement of the proposition above the requirement that τ be nonsingular, taking into account the following result.

PROPOSITION 7.2.8. *If for every $\epsilon > 0$ there exists a $\delta_\epsilon > 0$ such that $A \in \Sigma$ and $\mu(A) < \delta_\epsilon$ implies that $\mu \circ \tau^{-1}(A) < \epsilon$, then τ is a nonsingular transformation.*

PROOF. Let $A \in \Sigma$ such that $\mu(A) = 0$ and let $\epsilon > 0$. By assumption we can find a $\delta_\epsilon > 0$ such that $B \in \Sigma$, $\mu(B) < \delta_\epsilon$ implies $\mu \circ \tau^{-1}(B) < \epsilon$. And so $\mu(A) = 0 < \delta_\epsilon$ implies that $\mu \circ \tau^{-1}(A) < \epsilon$. Since $\epsilon > 0$ was arbitrary, we have that $\mu \circ \tau^{-1}(A) < \epsilon$ for all $\epsilon > 0$ and so $\mu \circ \tau^{-1}(A) = 0$. τ is therefore nonsingular. \square

7.2.3. Composition operators on $L^{01}(\Omega, \Sigma, \mu)$. We define $L^{01}(\mu)$ as follows:

$$L^{01}(\Omega, \Sigma, \mu) := \{f \in L^{00}(\Omega, \Sigma, \mu) : f^*(t) \rightarrow 0 \text{ as } t \rightarrow \infty\}$$

As was the case for $L^0(\mu)$, it can be shown that a composition operator on $L^{01}(\mu)$ is automatically continuous on $L^{01}(\mu)$ equipped with the topology of convergence in measure. The maps which induce composition operators on $L^{01}(\mu)$ are characterized in the following proposition.

PROPOSITION 7.2.9. *Let (Ω, Σ, μ) be a σ -finite measure space and $\tau : \Omega \rightarrow \Omega$ a nonsingular measurable transformation. τ defines a continuous composition operator on $L^{01}(\Omega, \Sigma, \mu)$ equipped with the topology of convergence in measure if and only if*

$$(7.2.6) \quad A \in \Sigma, \mu(A) < \infty \Rightarrow \mu \circ \tau^{-1}(A) < \infty$$

PROOF. (Sufficiency): Let $f \in L^{01}(\mu)$. By Remark 1.3.1 and Proposition 1.3.5(13) this implies that $\mu(A_{f,\lambda}) = \mu_f(\lambda) < \infty$ for all $\lambda > 0$. By Proposition 7.2.2 and condition (7.2.6) this implies that $\mu_{C_\tau f}(\lambda) = \mu \circ \tau^{-1}(A_{f,\lambda}) < \infty$ for all $\lambda > 0$ and hence $(C_\tau f)^*(t) \rightarrow 0$ as $t \rightarrow \infty$ by Proposition 1.3.5(13). C_τ is therefore a composition operator on $L^{01}(\mu)$. The continuity of C_τ follows as in the $L^0(\mu)$ case.

(Necessity): We show the contrapositive. Let condition (7.2.6) not hold, i.e. there exists some $E \in \Sigma_f$ such that $\mu \circ \tau^{-1}(E) = \infty$. Let $f = \chi_E$, then

$$\mu_f(\lambda) = \begin{cases} \mu(E) & \lambda < 1 \\ 0 & \lambda \geq 1 \end{cases}$$

so $f \in L^{01}(\mu)$ by Proposition 1.3.5(13). However, $\mu_{C_\tau f}(\frac{1}{2}) = \mu \circ \tau^{-1}(E) = \infty$. So $C_\tau f \notin L^{01}(\mu)$ by Proposition 1.3.5(13) and hence C_τ is not a composition operator on $L^{01}(\mu)$. \square

7.3. Composition operators on rearrangement invariant Banach function spaces

Throughout this section we will assume that X is a rearrangement invariant Banach function space over a σ -finite measure space (Ω, Σ, μ) and that τ is a measurable transformation from Ω into itself. It was demonstrated (Example 4.1.1) that Orlicz-Lorentz spaces, and hence Lorentz, Orlicz and Lebesgue spaces, are examples of rearrangement invariant Banach function spaces. The maps which induce bounded composition operators on these examples of rearrangement invariant Banach function spaces and on rearrangement invariant Banach function spaces in general will be characterized. Furthermore, estimates for the norm of the composition operator will be provided. The motivation for focusing on rearrangement invariant spaces is that in this setting we are able to use the notion of a fundamental function.

Historically, the L^p -spaces were among the first spaces of measurable functions used as settings for studying composition operators. Consider the following example. Let $\tau : \Omega \rightarrow \Omega$ be a nonsingular measurable transformation such that C_τ is a bounded composition operator on $L^1(\mu)$. It follows that if $A \in \Sigma_f$, then $\chi_A \in L^1(\mu)$ and

$$\begin{aligned} \|C_\tau \chi_A\|_{L^1} &\leq \|C_\tau\| \|\chi_A\|_{L^1} \\ \Leftrightarrow \Phi_{L^1}(\mu \circ \tau^{-1}(A)) &\leq \|C_\tau\| \Phi_{L^1}(\mu(A)) \quad \text{since } C_\tau \chi_A = \chi_{\tau^{-1}(A)} \\ \Leftrightarrow \mu \circ \tau^{-1}(A) &\leq \|C_\tau\| \mu(A) \quad \text{by Example 4.2.7} \end{aligned}$$

Furthermore since τ is a nonsingular measurable transformation, the Radon-Nikodym derivative f_τ exists by Corollary 1.2.16. If f_τ is essentially bounded, then

$$\begin{aligned}
\|C_\tau f\|_{L^1} &= \int_{\Omega} |(C_\tau f)(x)| d\mu(x) \\
&= \int_{\Omega} |f(\tau(x))| d\mu(x) \\
&= \int_{\Omega} |f(y)| \cdot f_\tau(y) d\mu(y) \quad \text{by Theorem 1.2.18 and Remark 1.2.19} \\
&\leq \int_{\Omega} |f(y)| \cdot \|f_\tau\|_{\infty} d\mu(y) \\
&= \|f_\tau\|_{\infty} \|f\|_{L^1}
\end{aligned}$$

This motivates considering the conditions

$$(7.3.1) \quad \exists M > 0 : \mu \circ \tau^{-1}(A) \leq M\mu(A) \quad \forall A \in \Sigma_f$$

$$(7.3.2) \quad \exists k > 0 : \Phi_X(\mu \circ \tau^{-1}(A)) \leq k\Phi_X(\mu(A)) \quad \forall A \in \Sigma_f$$

$$(7.3.3) \quad \tau \text{ is nonsingular and } f_\tau \in L^\infty(\mu)$$

when trying to generalize results obtained for L^p -spaces. It is easily shown that if C_τ is a bounded composition operator on X , then (7.3.2) holds. Two important aspects of this section will be to

- (1) show that (7.3.1) is sufficient to ensure that τ induces a bounded composition operator on any rearrangement invariant Banach function space over a resonant measure space or on an Orlicz-Lorentz space over any σ -finite measure space, and to
- (2) establish the conditions under which (7.3.1) is also a necessary condition.

It will therefore be useful to know the relationship between these conditions (to simplify matters, we will refer to the first condition as “the condition on the measure” and the second condition as “the condition on the fundamental function”). We therefore start this section by investigating the relationship between these conditions and some of the conditions encountered previously.

7.3.1. Conditions on the symbol function. We start by showing that “the condition on the fundamental function” and “the condition on the measure” are sufficient to ensure that the symbol function τ is nonsingular (and hence to ensure that the composition transformation is well-defined as a mapping of equivalence classes). The relationship between “the condition on the measure” and “the condition on the fundamental function” is then considered. Lastly we present two lemmas which demonstrate the relationships between “the condition on the measure” and a condition on decreasing rearrangements and a condition on maximal functions respectively.

PROPOSITION 7.3.1. *If there exists a $k > 0$ such that*

$$\Phi_X(\mu \circ \tau^{-1}(A)) \leq k\Phi_X(\mu(A)) \quad \forall A \in \Sigma_f$$

then τ is nonsingular. The converse is not necessarily true, however.

PROOF. Let $A \in \Sigma$ such that $\mu(A) = 0$, then

$$\begin{aligned}
\|\chi_{\tau^{-1}(A)}\|_X &= \Phi_X(\mu \circ \tau^{-1}(A)) \\
&\leq k\Phi_X(\mu(A)) \\
&= k\|\chi_A\| \\
&= 0 \quad \text{since } \chi_A = 0 \mu\text{-a.e.}
\end{aligned}$$

Hence $\chi_{\tau^{-1}(A)} = 0$ μ -a.e. and so $\mu \circ \tau^{-1}(A) = 0$

To show that the converse is not necessarily true consider the following example. Let $(\Omega, \Sigma, \mu) = (\mathbb{N}, P(\mathbb{N}), \mu_c)$, where μ_c is the counting measure on \mathbb{N} and let $X = l_1(\mathbb{N}, P(\mathbb{N}), \mu_c)$, then X is a Banach function space. Define $\tau : \Omega \rightarrow \Omega$ by $\tau(n) = 1$ for all $n \in \mathbb{N}$. Then τ is trivially measurable since every subset of \mathbb{N} is measurable. Furthermore if $A \subset \mathbb{N}$ such that $\mu(A) = 0$, then $A = \emptyset$ and so

$$\begin{aligned} \mu \circ \tau^{-1}(A) &= \mu \circ \tau^{-1}(\emptyset) \\ &= \mu(\emptyset) \\ &= 0 \end{aligned}$$

and so τ is nonsingular. We show that for every $k > 0$ there exists an $A \subset \mathbb{N}$ such that $\Phi_{l_1}(\mu \circ \tau^{-1}(A)) > k\Phi_{l_1}(\mu(A))$:

$$\begin{aligned} \Phi_{l_1}(\mu \circ \tau^{-1}(\{1\})) &= \|\chi_{\tau^{-1}(\{1\})}\|_{l_1} \\ &= \|\chi_{\mathbb{N}}\|_{l_1} \\ &= \infty \\ &> k\|\chi_{\{1\}}\|_{l_1} \quad \forall k > 0 \text{ since } \|\chi_{\{1\}}\|_{l_1} = 1 \\ &= k\Phi_{l_1}(\mu(\{1\})) \end{aligned}$$

□

Recall that $\Sigma_{f,0}$ is the set of all measurable sets with finite, nonzero measure.

PROPOSITION 7.3.2. *Let $\tau : \Omega \rightarrow \Omega$ be a measurable transformation such that there exists a $k > 0$ such that*

$$\mu \circ \tau^{-1}(A) \leq k\mu(A) \quad \forall A \in \Sigma_f$$

Let $\mathcal{A} = \{k > 0 : \mu \circ \tau^{-1}(A) \leq k\mu(A) \forall A \in \Sigma_{f,0}\}$ and $\mathcal{B} = \left\{ \frac{\mu \circ \tau^{-1}(A)}{\mu(A)} : A \in \Sigma_{f,0} \right\}$, then $\inf \mathcal{A} = \sup \mathcal{B}$.

PROOF. Let $k \in \mathcal{A}$, then

$$\begin{aligned} k &\geq \frac{\mu \circ \tau^{-1}(A)}{\mu(A)} \quad \forall A \in \Sigma_{f,0} \\ \Rightarrow k &\geq \sup \mathcal{B} \end{aligned}$$

Since this holds for all k in \mathcal{A} , we have that $\inf \mathcal{A} \geq \sup \mathcal{B}$. Furthermore,

$$\begin{aligned} \frac{\mu \circ \tau^{-1}(A)}{\mu(A)} &\leq \sup \mathcal{B} \quad \forall A \in \Sigma_{f,0} \\ \Rightarrow \mu \circ \tau^{-1}(A) &\leq (\sup \mathcal{B})\mu(A) \quad \forall A \in \Sigma_{f,0} \\ \Rightarrow \sup \mathcal{B} &\in \mathcal{A} \\ \Rightarrow \sup \mathcal{B} &\geq \inf \mathcal{A} \end{aligned}$$

□

PROPOSITION 7.3.3. *There exists a $k > 0$ such that*

$$(7.3.4) \quad \mu \circ \tau^{-1}(A) \leq k\mu(A) \quad \forall A \in \Sigma_f$$

if and only if τ is nonsingular and $f_\tau \in L^\infty(\mu)$. Furthermore if either of these conditions hold then

$$\|f_\tau\|_\infty = \sup \left\{ \frac{\mu \circ \tau^{-1}(A)}{\mu(A)} : A \in \Sigma_{f,0} \right\}$$

PROOF. (Necessity): Let \mathcal{A} and \mathcal{B} be defined as in Proposition 7.3.2. Let $k > 0$ be such that condition (7.3.4) holds, i.e. $k \in \mathcal{A}$. It is clear that $\mu \circ \tau^{-1}(A) = 0$ whenever $\mu(A) = 0$ and so τ is nonsingular. Let f_τ be

the Radon-Nikodym derivative of $\mu \circ \tau^{-1}$ with respect to μ . We claim that $f_\tau \leq k$ μ -a.e. Assume that this is not the case. Then we can find (using a slight modification of Proposition 1.2.1) an $\epsilon > 0$ and an $A \in \Sigma$ with $\mu(A) > 0$ such that $f_\tau(x) \geq k + \epsilon$ for all $x \in A$ and so

$$\begin{aligned} \mu \circ \tau^{-1}(A) &= \int_A f_\tau d\mu \\ &\geq (k + \epsilon)\mu(A) \\ &> k\mu(A) \end{aligned}$$

This contradicts condition (7.3.4) and so $f_\tau \in L^\infty(\mu)$ and $\|f_\tau\|_\infty \leq k$. Since this holds for any $k \in \mathcal{A}$, we have $\|f_\tau\|_\infty \leq \inf \mathcal{A}$

(Sufficiency): Since the Radon-Nikodym derivative is bounded we have the following:

$$\begin{aligned} \mu \circ \tau^{-1}(A) &= \int_A f_\tau d\mu \quad \forall A \in \Sigma_f \\ &\leq \int_A \|f_\tau\|_\infty d\mu \quad \forall A \in \Sigma_f \\ &= \|f_\tau\|_\infty \mu(A) \end{aligned}$$

and so $\|f_\tau\|_\infty \in \mathcal{A}$. Therefore $\|f_\tau\|_\infty \geq \inf \mathcal{A}$ and hence

$$\begin{aligned} \|f_\tau\|_\infty &= \inf \mathcal{A} \\ &= \sup \mathcal{B} \quad \text{by Proposition 7.3.2} \\ &= \sup \left\{ \frac{\mu \circ \tau^{-1}(A)}{\mu(A)} : A \in \Sigma_f, 0 < \mu(A) \right\} \end{aligned}$$

□

We show that if the underlying measure space is resonant, then “the condition on the measure” implies “the condition on the fundamental function”.

PROPOSITION 7.3.4. *Let X be a rearrangement invariant Banach function space over a resonant measure space (Ω, Σ, μ) . If there exists a $k > 0$ such that*

$$(7.3.5) \quad \mu \circ \tau^{-1}(A) \leq k\mu(A) \quad \forall A \in \Sigma_f$$

then there exists an $M > 0$ such that

$$(7.3.6) \quad \Phi_X(\mu \circ \tau^{-1}(A)) \leq M\Phi_X(\mu(A)) \quad \forall A \in \Sigma_f$$

The converse, however, is not true in general.

PROOF. Let $k > 0$ such that (7.3.5) holds. Without loss of generality we can assume that $k \geq 1$. Let $g \in X'$ and $A \in \Sigma_f$, then

$$\begin{aligned}
\int_0^\infty (C_\tau \chi_A)^*(t) g^*(t) dt &= \int_0^\infty (\chi_{\tau^{-1}(A)})^*(t) g^*(t) dt \\
&= \int_0^\infty \chi_{[0, \mu \circ \tau^{-1}(A)]}(t) g^*(t) dt \quad \text{by Example 1.3.4} \\
&= \int_0^{\mu \circ \tau^{-1}(A)} g^*(t) dt \\
&\leq \int_0^{k\mu(A)} g^*(t) dt \quad \text{since } g^* \geq 0 \\
&= \int_0^{\mu(A)} g^*(sk) k ds \quad \text{using } s = \frac{t}{k} \\
&\leq k \int_0^{\mu(A)} g^*(s) ds \quad \text{since } g^* \text{ is decreasing and } s \leq sk \forall s \\
&= k \int_0^\infty (\chi_A)^*(s) g^*(s) ds
\end{aligned}$$

Taking the supremum over all $g \in X'$ such that $\|g\|_{X'} \leq 1$, we obtain

$$\|C_\tau \chi_A\|_X \leq k \|\chi_A\|_X$$

or equivalently

$$\Phi_X(\mu \circ \tau^{-1}(A)) \leq k \Phi_X(\mu(A))$$

where the resonance of (Ω, Σ, μ) was used to apply Proposition 4.1.3 to obtain the norms in terms of the decreasing rearrangements.

To show that the converse does not necessarily hold, consider the following example. Let $(\mathbb{N}, P(\mathbb{N}), \mu_c)$ and define $\tau : \mathbb{N} \rightarrow \mathbb{N}$ by $\tau(n) = 1$ for all $n \in \mathbb{N}$. Let $X = l_\infty(\mathbb{N}, P(\mathbb{N}), \mu_c)$. We note that if $A = \emptyset$, then

$$\begin{aligned}
\|\chi_{\tau^{-1}(A)}\|_{l_\infty} &= \|\chi_\emptyset\|_{l_\infty} \\
&= \|\chi_A\|_{l_\infty}
\end{aligned}$$

If $A \neq \emptyset$, then

$$\begin{aligned}
\|\chi_{\tau^{-1}(A)}\|_{l_\infty} &\leq 1 \\
&= \|\chi_A\|_{l_\infty}
\end{aligned}$$

So (7.3.6) holds with $k = 1$. Condition (7.3.5) does not hold however since

$$\begin{aligned}
\mu \circ \tau^{-1}(\{1\}) &= \mu(\mathbb{N}) \\
&= \infty \\
&> k\mu(\{1\}) \quad \forall k > 0 \text{ since } \mu(\{1\}) = 1
\end{aligned}$$

□

The following two examples illustrate that there are particular spaces where “the condition on the fundamental function” does imply “the condition on the measure”.

EXAMPLE 7.3.5. Let $1 \leq q \leq p < \infty$ and let $\tau : \Omega \rightarrow \Omega$ be a measurable transformation. If there exists a $k > 0$ such that

$$(7.3.7) \quad \Phi_{L^{p,q}}(\mu \circ \tau^{-1}(A)) \leq k \Phi_{L^{p,q}}(\mu(A)) \quad \forall A \in \Sigma_f$$

then

$$\mu \circ \tau^{-1}(A) \leq k^p \mu(A) \quad \forall A \in \Sigma_f$$

PROOF. Let $A \in \Sigma_f$ and let $k > 0$ such that (7.3.7) holds. Then

$$(7.3.8) \quad \left(\frac{p}{q}\right)^{1/q} (\mu \circ \tau^{-1}(A))^{1/p} \leq k \left(\frac{p}{q}\right)^{1/q} (\mu(A))^{1/p} \quad \text{by Example 4.2.7}$$

$$(7.3.9) \quad \Rightarrow \mu \circ \tau^{-1}(A) \leq k^p \mu(A) \quad \forall A \in \Sigma_{f,0}$$

τ is nonsingular by Proposition 7.3.1 and therefore (7.3.8) holds even if $\mu(A) = 0$. \square

EXAMPLE 7.3.6. Let ϕ be an Orlicz function satisfying the Δ_2 -condition globally and let $\tau : \Omega \rightarrow \Omega$ be a measurable transformation. If there exists a $k > 0$ such that

$$(7.3.10) \quad \Phi_{L^\phi}(\mu \circ \tau^{-1}(A)) \leq k \Phi_{L^\phi}(\mu(A)) \quad \forall A \in \Sigma_f$$

then there exists an $M > 0$ such that

$$(7.3.11) \quad \mu \circ \tau^{-1}(A) \leq M \mu(A) \quad \forall A \in \Sigma_f$$

PROOF. Let $k > 0$ such that (7.3.10) holds. Without loss of generality we can assume that $k \geq 1$. By Proposition 2.2.13 ϕ is invertible and there exists an $M_k > 0$ such that

$$(7.3.12) \quad \phi(kt) \leq M_k \phi(t) \quad \forall t \geq 0$$

Let $A \in \Sigma_{f,0}$. Condition (7.3.10) implies that

$$\begin{aligned} \phi^{-1}(1/\mu(A)) &\leq k \phi^{-1}(1/\mu \circ \tau^{-1}(A)) && \text{by Example 4.2.7} \\ \Rightarrow \phi(\phi^{-1}(1/\mu(A))) &\leq \phi(k \phi^{-1}(1/\mu \circ \tau^{-1}(A))) && \text{since } \phi \text{ is increasing} \\ \Rightarrow 1/\mu(A) &\leq M_k \phi(\phi^{-1}(1/\mu \circ \tau^{-1}(A))) && \text{since } \phi \text{ is invertible and (7.3.12) holds} \\ &= M_k / \mu \circ \tau^{-1}(A) && \text{since } \phi \text{ is invertible} \\ \Rightarrow \mu \circ \tau^{-1}(A) &\leq M_k \mu(A) \end{aligned}$$

If $\mu(A) = 0$, then

$$\begin{aligned} \mu \circ \tau^{-1}(A) &= 0 && \text{since } \tau \text{ is nonsingular by Proposition 7.3.1} \\ &\leq M_k \mu(A) \end{aligned}$$

Therefore (7.3.11) holds with $M = M_k$. \square

Motivated by these two examples and the significance of the relationship between the two conditions under discussion we give the following definition.

DEFINITION 7.3.7. Let X be a Banach function space over a σ -finite measure space (Ω, Σ, μ) . X will be called a Banach function space of type 1 if, whenever $\tau : \Omega \rightarrow \Omega$ is a measurable transformation such that “the condition on the fundamental function” holds, then “the condition on the measure” holds.

EXAMPLE 7.3.8. It follows by Examples 7.3.5 and 7.3.6 that the $L^{p,q}$ -spaces, Orlicz spaces (for $\phi \in \Delta_2$ globally) and therefore L^p -spaces are examples of Banach function spaces of type 1.

PROPOSITION 7.3.9. Let X be a rearrangement invariant Banach function space over a nonatomic measure space (Ω, Σ, μ) . If for every $k > 1$ there exists a $C_k > 0$ such that

$$(7.3.13) \quad k \Phi_X(t) \leq \Phi_X(C_k t) \quad \forall t > 0$$

and there exists an $M > 0$ such that

$$(7.3.14) \quad \Phi_X(\mu \circ \tau^{-1}(A)) \leq M\Phi_X(\mu(A)) \quad \forall A \in \Sigma_f$$

then there exists a $K > 0$ such that

$$(7.3.15) \quad \mu \circ \tau^{-1}(A) \leq K\mu(A) \quad \forall A \in \Sigma_f$$

PROOF. Let $M > 0$ be such that (7.3.14) holds. Let C_{M+1} be such that

$$(7.3.16) \quad (M+1)\Phi_X(t) \leq \Phi_X(C_{M+1}t) \quad \forall t > 0$$

Therefore if $A \in \Sigma_{f,0}$, then $\Phi_X(\mu(A)) > 0$ by Proposition 4.2.4 and so

$$\begin{aligned} \Phi_X(\mu \circ \tau^{-1}(A)) &< (M+1)\Phi_X(\mu(A)) && \text{by (7.3.14)} \\ &\leq \Phi_X(C_{M+1}\mu(A)) && \text{by (7.3.16)} \\ \Rightarrow \mu \circ \tau^{-1}(A) &\leq C_{M+1}\mu(A) && \text{since } \Phi_X \text{ is increasing} \end{aligned}$$

If $\mu(A) = 0$, then $\Phi_X(\mu(A)) = 0$ by Proposition 4.2.4 and so

$$\begin{aligned} \mu \circ \tau^{-1}(A) &= 0 && \text{since } \tau \text{ is nonsingular by Proposition 7.3.1} \\ &\leq C_{M+1}\mu(A), \end{aligned}$$

i.e. (7.3.15) holds with $K = C_{M+1}$. Note that we needed the underlying measure space to be nonatomic to ensure that there exists a $B \in \Sigma$ with $\mu(B) = C_{M+1}\mu(A)$, i.e. to ensure that $\Phi_X(C_{M+1}\mu(A))$ is defined. \square

REMARK 7.3.10. The above shows that if X is a Banach function space over a nonatomic measure space such that

$$\forall k > 1 \exists M_k > 0 \text{ such that } k\Phi_X(t) \leq \Phi_X(M_k t) \quad \forall t > 0$$

then X is a Banach function space of type 1 by Propositions 7.3.9 and 7.3.4 since a nonatomic measure space is resonant by Theorem 1.3.9.

Next we show the relationship between the decreasing rearrangement of a function $f \in L^{00}(\mu)$ and the decreasing rearrangement of the image of f under a composition transformation if τ is such that “the condition on the measure” holds.

LEMMA 7.3.11. *If there exists an $M > 0$ such that*

$$\mu \circ \tau^{-1}(A) \leq M\mu(A) \quad \forall A \in \Sigma_f$$

then for every $f \in L^{00}(\mu)$

$$(C_\tau f)^*(t) \leq f^*(t/M) \quad \forall t \in [0, \infty)$$

PROOF. Let $f \in L^{00}(\mu)$, then

$$\begin{aligned} \mu_{C_\tau f}(\lambda) &= \mu(\{x \in \Omega : |(C_\tau f)(x)| > \lambda\}) \\ &= \mu(\{x \in \Omega : |f(\tau(x))| > \lambda\}) \\ &= \mu(\cup \{\tau^{-1}(\{y\}) : y \in \Omega : |f(y)| > \lambda\}) \\ &= \mu(\tau^{-1}(\cup \{y \in \Omega : |f(y)| > \lambda\})) \\ &= \mu \circ \tau^{-1}(\{y \in \Omega : |f(y)| > \lambda\}) \\ &\leq M\mu(\{y \in \Omega : |f(y)| > \lambda\}) \\ &= M\mu_f(\lambda) \end{aligned}$$

Therefore $(C_\tau f)^*(t) \leq f^*(t/M)$ for all $t \geq 0$ by Proposition 1.3.5(11). \square

The following is a similar result to the one above, but in terms of maximal functions.

LEMMA 7.3.12. *If there exists an $M > 0$ such that*

$$(7.3.17) \quad \mu \circ \tau^{-1}(A) \leq M\mu(A) \quad \forall A \in \Sigma_f$$

then

$$(C_\tau f)^{**}(t) \leq f^{**}(t/M) \quad \forall t > 0, f \in L^{00}(\mu)$$

PROOF. We have by the lemma above that $(C_\tau f)^*(t) \leq f^*(t/M)$ for all $t \in [0, \infty)$ and so for $t \in (0, \infty)$

$$\begin{aligned} (C_\tau f)^{**}(t) &= \frac{1}{t} \int_0^t (C_\tau f)^*(s) ds \\ &\leq \frac{1}{t} \int_0^t f^*(s/M) ds \\ &= \frac{1}{t} \int_0^{t/M} f^*(u) M du \\ &= \frac{1}{t/M} \int_0^{t/M} f^*(u) du \\ &= f^{**}(t/M) \end{aligned}$$

\square

REMARK 7.3.13. Let $M \geq 1$. Since $f^* \geq 0$ and $t/M \leq t$, we have that

$$\frac{1}{t/M} \int_0^{t/M} f^*(u) du \leq M \frac{1}{t} \int_0^t f^*(u) du = M f^{**}(t)$$

and hence condition (7.3.17) for $M \geq 1$ implies $C_\tau f \prec k f$.

7.3.2. Composition operators on rearrangement invariant spaces. The strategy employed in this section is to start by proving a general result for composition operators on arbitrary rearrangement invariant Banach function spaces and then to determine which of the results for composition operators on specific examples of rearrangement invariant Banach function spaces can be derived from the general result. It will often be the case that a result in a more general setting will yield (when applied to a special case) a somewhat weaker result than can be obtained using a direct proof in the context of the special case. In these situations elements of the original proofs will be incorporated to obtain the desired results. Furthermore, translation operators, the bilateral shift operator and the symmetry operator will be used to illustrate some of the main results. Based on Proposition 1.5.5, we start by showing that if τ induces a composition operator on an appropriate function space, then the induced composition operator is automatically continuous.

COROLLARY 7.3.14. *Let (Ω, Σ, μ) be a measure space and $\tau : \Omega \rightarrow \Omega$ a map. If X is a Banach function space and C_τ maps X into itself, then C_τ is bounded on X .*

PROOF. $X_{\mathbb{R}}$ can be viewed as a Banach function space over \mathbb{R} and therefore $(X_{\mathbb{R}}, +, \cdot, \leq, \|\cdot\|_{X_{\mathbb{R}}})$ is a Banach lattice by Proposition 3.1.5. By Remark 1.5.6 this implies that $X_{\mathbb{R}}$ is a complete metrizable locally solid Riesz space. Furthermore, the conjugate map is continuous with respect to the norm topology since $|\bar{f}| = |f|$ and $\|\bar{f}\|_X = \|f\|_X$. The result thus follows by Proposition 1.5.5. \square

In the case where the underlying measure space is resonant we are able to use Proposition 4.1.3, which allows us to express the norms of $C_\tau f$ and f in terms of their decreasing rearrangements, or Theorem 5.2.1, which allows us to use interpolation theory, to obtain the following general result.

THEOREM 7.3.15. (cf. [21, p.2112]) Let X be a rearrangement invariant Banach function space over a resonant measure space (Ω, Σ, μ) and let $\tau : \Omega \rightarrow \Omega$ be a measurable transformation. If there exists an $M > 0$ such that

$$(7.3.18) \quad \mu \circ \tau^{-1}(A) \leq M\mu(A) \quad \forall A \in \Sigma_f$$

then C_τ is a bounded composition operator on X . Conversely, if C_τ is a bounded composition operator on X , then there exists a $k > 0$ such that

$$(7.3.19) \quad \Phi_X(\mu \circ \tau^{-1}(A)) \leq k\Phi_X(\mu(A)) \quad \forall A \in \Sigma_f$$

Furthermore,

$$\sup_{A \in \Sigma_{f,0}} \frac{\Phi_X(\mu \circ \tau^{-1}(A))}{\Phi_X(\mu(A))} \leq \|C_\tau\| \leq \max \left\{ 1, \sup_{A \in \Sigma_{f,0}} \frac{\mu \circ \tau^{-1}(A)}{\mu(A)} \right\}$$

PROOF. Let $f \in X$. Without loss of generality it may be assumed that $M \geq 1$. By Lemma 7.3.11

$$(7.3.20) \quad (C_\tau f)^*(t) \leq f^*(t/M) \quad \forall t \in [0, \infty)$$

Let $g \in X'$. Then using the above and the fact that $g^* \geq 0$, we obtain

$$\begin{aligned} \int_0^\infty (C_\tau f)^*(t)g^*(t)dt &\leq \int_0^\infty f^*(t/M)g^*(t)dt \\ &= \int_0^\infty f^*(s)g^*(Ms)M ds \quad \text{using } s = t/M \\ &\leq M \int_0^\infty f^*(s)g^*(s)ds, \end{aligned}$$

since g^* decreasing and $M \geq 1$ implies that $s \leq Ms$ for all $s > 0$. Then, using the above and Proposition 4.1.3,

$$\begin{aligned} \|C_\tau f\| &= \sup \left\{ \int_0^\infty (C_\tau f)^*(t)g^*(t)dt : \|g\|_{X'} \leq 1 \right\} \\ &\leq \sup \left\{ M \int_0^\infty f^*(s)g^*(s)ds : \|g\|_{X'} \leq 1 \right\} \\ (7.3.21) \quad &= M\|f\|_X \end{aligned}$$

Since this holds for every $f \in X$, we have that C_τ is a bounded composition operator on X . Furthermore,

$$\begin{aligned} \|C_\tau\| &= \inf \{ k > 0 : \|C_\tau f\| \leq k\|f\| \forall f \in X \} \\ &\leq \inf \{ M \geq 1 : \mu \circ \tau^{-1}(A) \leq M\mu(A) \forall A \in \Sigma_{f,0} \} \quad \text{since (7.3.21) holds } \forall M \geq 1 \text{ such that (7.3.18) holds} \\ &= \max \left\{ 1, \sup_{A \in \Sigma_{f,0}} \frac{\mu \circ \tau^{-1}(A)}{\mu(A)} \right\} \quad \text{by Proposition 7.3.2} \end{aligned}$$

For the converse, let C_τ be a bounded composition operator on X and let $k > 0$ such that

$$(7.3.22) \quad \|C_\tau f\|_X \leq k\|f\|_X \quad \forall f \in X$$

Let $A \in \Sigma_f$. Then $\chi_A \in X$, since X contains all characteristic functions of sets with finite measure, and so

$$\begin{aligned} \Phi_X(\mu \circ \tau^{-1}(A)) &= \|\chi_{\tau^{-1}(A)}\|_X \\ &= \|C_\tau \chi_A\|_X \\ &\leq k\|\chi_A\|_X \quad \text{by (7.3.22)} \\ &= k\Phi_X(\mu(A)) \end{aligned}$$

In particular

$$\begin{aligned}
\|C_\tau f\|_X &\leq \|C_\tau\| \|f\|_X \\
\Rightarrow \Phi_X(\mu \circ \tau^{-1}(A)) &\leq \|C_\tau\| \Phi_X(\mu(A)) \quad \forall A \in \Sigma_f \\
\Rightarrow \|C_\tau\| &\geq \frac{\Phi_X(\mu \circ \tau^{-1}(A))}{\Phi_X(\mu(A))} \quad \forall A \in \Sigma_{f,0} \\
\Rightarrow \|C_\tau\| &\geq \sup_{A \in \Sigma_{f,0}} \frac{\Phi_X(\mu \circ \tau^{-1}(A))}{\Phi_X(\mu(A))}
\end{aligned}$$

The following is an alternative proof of the sufficiency of condition (7.3.18) for the boundedness of the composition operator using interpolation theory:

By Proposition 7.3.3, condition (7.3.18) implies that τ is nonsingular and hence $C_\tau(L^0(\mu)) \subset L^0(\mu)$ by Corollary 7.1.3. C_τ is therefore defined on $(L^1 + L^\infty)(\mu) \subset L^0(\mu)$ and takes values in $L^0(\mu)$. Furthermore, by remark 7.3.13, condition (7.3.18) implies that $C_\tau f \prec Mf$ for all $f \in (L^1 + L^\infty)(\mu)$. C_τ is therefore an admissible operator for (L^1, L^∞) by Proposition 5.2.2. By Theorem 5.2.7 X is an exact interpolation space relative to (L^1, L^∞) and so C_τ is bounded by Corollary 5.2.6. \square

REMARK 7.3.16. To prove the sufficiency of condition (7.3.18) for the boundedness of the composition operator we required the resonance of the underlying measure space in order to apply Proposition 4.1.3. The proof of the necessity of condition (7.3.19) however did not require the underlying measure space to be resonant.

REMARK 7.3.17. Recall that X' is the associate space of X . The original result ([21, Theorem 2.4]) states that “ C_τ is a bounded composition operator on X and X' if and only if (7.3.18) holds”. If X is a rearrangement invariant Banach function space over a resonant measure space (Ω, Σ, μ) , then so is X' by Proposition 4.1.3. Therefore, by the theorem above, C_τ is a composition operator on X' if (7.3.18) holds. If C_τ is a composition operator on X and X' , then by the theorem above there exist $k, k' > 0$ such that

$$\begin{aligned}
\Phi_X(\mu \circ \tau^{-1}(A)) &\leq k \Phi_X(\mu(A)) \quad \forall A \in \Sigma_f \text{ and} \\
\Phi_{X'}(\mu \circ \tau^{-1}(A)) &\leq k' \Phi_{X'}(\mu(A)) \quad \forall A \in \Sigma_f
\end{aligned}$$

Therefore

$$\begin{aligned}
\Phi_X(\mu \circ \tau^{-1}(A)) \Phi_{X'}(\mu \circ \tau^{-1}(A)) &\leq k k' \Phi_X(\mu(A)) \Phi_{X'}(\mu(A)) \quad \forall A \in \Sigma_f \\
\Rightarrow \mu \circ \tau^{-1}(A) &\leq k k' \mu(A) \quad \forall A \in \Sigma_f \text{ by Theorem 4.2.5}
\end{aligned}$$

The original result has been modified to obtain a result which is independent of the associate space.

We have defined Banach function spaces of type 1 to be those rearrangement invariant spaces such that condition (7.3.1) holds if τ is a measurable transformation such that condition (7.3.2) holds. We therefore have the following corollary.

COROLLARY 7.3.18. *Let X be a rearrangement invariant Banach function space of type 1 over a resonant measure space (Ω, Σ, μ) and let $\tau : \Omega \rightarrow \Omega$ be a measurable transformation. C_τ is a bounded composition operator on X if and only if there exists an $M > 0$ such that*

$$(7.3.23) \quad \mu \circ \tau^{-1}(A) \leq M \mu(A) \quad \forall A \in \Sigma_f$$

Next we consider the boundedness of the composition operator C_τ on specific examples of rearrangement invariant Banach function spaces over a general σ -finite measure space.

THEOREM 7.3.19. (cf. [22, p.81]) *Let (Ω, Σ, μ) be a σ -finite measure space, $\tau : \Omega \rightarrow \Omega$ a measurable transformation, w a weight function and ϕ an Orlicz function. Define $\Gamma(t) := \int_0^t w(s) ds$. If there exists an*

$M > 0$ such that

$$(7.3.24) \quad \mu \circ \tau^{-1}(A) \leq M\mu(A) \quad \forall A \in \Sigma_f$$

then C_τ is a bounded composition operator on the Orlicz-Lorentz space $L^{\phi,w}(\mu)$. Conversely if C_τ is a bounded composition operator on $L^{\phi,w}(\mu)$ then there exists a $k > 0$ such that

$$(7.3.25) \quad \frac{1}{\phi^{-1}(1/\Gamma(\mu \circ \tau^{-1}(A)))} \leq \frac{k}{\phi^{-1}(1/\Gamma(\mu(A)))} \quad \forall A \in \Sigma_{f,0}$$

In particular, if C_τ is a bounded composition operator on $L^{\phi,w}(\mu)$ and ϕ satisfies the Δ_2 -condition globally, then there exists a $k > 0$ such that

$$(7.3.26) \quad \Gamma(\mu \circ \tau^{-1}(A)) \leq k\Gamma(\mu(A)) \quad \forall A \in \Sigma_{f,0}$$

If C_τ is a bounded composition operator on $L^{\phi,w}(\mu)$, then

$$\sup_{A \in \Sigma_{f,0}} \frac{\phi^{-1}(1/\Gamma(\mu(A)))}{\phi^{-1}(1/\Gamma(\mu \circ \tau^{-1}(A)))} \leq \|C_\tau\| \leq \max \left\{ 1, \sup_{A \in \Sigma_{f,0}} \frac{\mu \circ \tau^{-1}(A)}{\mu(A)} \right\}$$

PROOF. (Sufficiency): Let (7.3.24) hold. If the underlying measure space is resonant, the result follows by Theorem 7.3.15, since $L^{\phi,w}(\mu)$ is a rearrangement invariant Banach function space (Example 4.1.1). Let (Ω, Σ, μ) be a general σ -finite measure space. Without loss of generality it can be assumed that $M \geq 1$ in (7.3.24) and so by Lemma 7.3.11

$$(7.3.27) \quad (C_\tau f)^*(t) \leq f^*(t/M) \quad \forall t > 0$$

We therefore have that if $\|f\|_{\phi,w} > 0$, then

$$\begin{aligned} I_{\phi,w}(C_\tau f / (M\|f\|_{\phi,w})) &= \int_0^{\mu(\Omega)} \phi \left(\frac{(C_\tau f)^*(t)}{M\|f\|_{\phi,w}} \right) w(t) dt \\ (7.3.28) \quad &\leq \int_0^{\mu(\Omega)} \phi \left(\frac{f^*(t/M)}{M\|f\|_{\phi,w}} \right) w(t) dt \quad \text{using (7.3.27) and } \phi \text{ increasing} \\ &= \int_0^{\frac{1}{M}\mu(\Omega)} \phi \left(\frac{1}{M} \frac{f^*(s)}{\|f\|_{\phi,w}} \right) w(sM)M ds \quad \text{using } s = t/M \\ &\leq \int_0^{\mu(\Omega)} \frac{1}{M} \phi \left(\frac{f^*(s)}{\|f\|_{\phi,w}} \right) w(sM)M ds \quad \text{using Proposition 1.2.26, } w \geq 0 \text{ and } M \geq 1 \\ (7.3.29) \quad &\leq \int_0^{\mu(\Omega)} \phi \left(\frac{f^*(s)}{\|f\|_{\phi,w}} \right) w(s) ds \quad \text{since } w \text{ is decreasing and } M \geq 1 \\ &\leq 1 \quad \text{by Lemma 2.2.26} \end{aligned}$$

and so $\|C_\tau f\|_{\phi,w} = \inf\{\lambda > 0 : I_{\phi,w}(C_\tau f/\lambda) \leq 1\} \leq M\|f\|_{\phi,w}$. If $\|f\|_{\phi,w} = 0$, then $f = 0$ and so $C_\tau f = 0$ by the linearity of C_τ , whence $\|C_\tau f\|_{\phi,w} = 0 \leq M\|f\|_{\phi,w}$. We therefore have that

$$\begin{aligned} \|C_\tau f\|_{\phi,w} &\leq M\|f\|_{\phi,w} \quad \forall f \in L^{\phi,w}(\mu) \\ \Rightarrow \|C_\tau\| &\leq M \quad \forall M \geq 1 \text{ such that condition (7.3.24) holds} \\ \Rightarrow \|C_\tau\| &\leq \inf \{k \geq 1 : \mu \circ \tau^{-1}(A) \leq k\mu(A) \forall A \in \Sigma_{f,0}\} \\ &= \max \{1, \inf \{k > 0 : \mu \circ \tau^{-1}(A) \leq k\mu(A) \forall A \in \Sigma_{f,0}\}\} \\ &= \max \left\{ 1, \sup_{A \in \Sigma_{f,0}} \frac{\mu \circ \tau^{-1}(A)}{\mu(A)} \right\} \quad \text{by Proposition 7.3.2} \end{aligned}$$

We note that our proof required $M \geq 1$ in order to use the convexity of ϕ (see (1.2.7) in Proposition 1.2.26) and apply w decreasing (see (7.3.29)).

(Necessity): Recall (Example 4.2.7) that $\Phi_{L^{\phi,w}}(t) = \frac{1}{\phi^{-1}(1/\Gamma(t))}$. If C_τ is bounded on $L^{\phi,w}(\mu)$, then (7.3.25) holds by Theorem 7.3.15 and Remark 7.3.16. Furthermore,

$$\|C_\tau\| \geq \sup_{A \in \Sigma_{f,0}} \frac{\phi^{-1}(1/\Gamma(\mu(A)))}{\phi^{-1}(1/\Gamma(\mu \circ \tau^{-1}(A)))}$$

If (7.3.25) holds for some $k > 0$, then

$$(7.3.30) \quad \phi^{-1}(1/\Gamma(\mu(A))) \leq k\phi^{-1}(1/\Gamma(\mu \circ \tau^{-1}(A))) \quad \forall A \in \Sigma_{f,0}$$

If in addition ϕ satisfies the Δ_2 -condition for all $u > 0$, then by Proposition 2.2.13 ϕ is invertible. So applying ϕ to both sides of equation (7.3.30) and using the monotonicity of ϕ we obtain

$$\begin{aligned} 1/\Gamma(\mu(A)) &\leq \phi(k\phi^{-1}(1/\Gamma(\mu \circ \tau^{-1}(A)))) \\ &\leq M_k\phi(\phi^{-1}(1/\Gamma(\mu \circ \tau^{-1}(A)))) \quad \text{by Proposition 2.2.13} \\ &= M_k/\Gamma(\mu \circ \tau^{-1}(A)) \quad \text{since } \phi \text{ is invertible} \end{aligned}$$

and so

$$\Gamma(\mu \circ \tau^{-1}(A)) \leq M_k\Gamma(\mu(A)) \quad \forall A \in \Sigma_{f,0}$$

□

The following is an application of the theorems above and yields a family of bounded composition operators.

THEOREM 7.3.20. [37, p.19] *Let Ω be a locally compact abelian topological group and let μ be a Haar measure on (Ω, Σ) where Σ is the family of Borel subsets of Ω . Let $y \in \Omega$ and define $\tau_y : \Omega \rightarrow \Omega$ by*

$$\tau_y(x) = yx$$

for $x \in \Omega$. Then C_{τ_y} is a bounded composition operator on X if X is an Orlicz-Lorentz space or X is a rearrangement invariant Banach function space and (Ω, Σ, μ) is resonant. Furthermore $\|C_\tau\| = 1$.

PROOF. We note first that by Theorem 1.1.2 there exists a Haar measure μ on Ω . Furthermore if $G \subset \Omega$, then

$$(7.3.31) \quad \begin{aligned} \tau_y^{-1}(G) &= \{x \in \Omega : yx \in G\} \\ &= \{x \in \Omega : x \in y^{-1}G\} \quad y^{-1} \text{ exists since } \Omega \text{ is a group} \\ &= y^{-1}G \end{aligned}$$

Since Ω is a topological group, $y^{-1}G$ is open if G is open and so

$$\tau_y^{-1}(G) \in \Sigma \quad \forall G \text{ open,}$$

since Σ contains the open sets. τ_y is therefore measurable. We also have that if $G \in \Sigma$, then

$$(7.3.32) \quad \begin{aligned} \mu \circ \tau_y^{-1}(G) &= \mu(y^{-1}G) \quad \text{by (7.3.31)} \\ &= \mu(G) \quad \text{since } \mu \text{ is a Haar measure} \end{aligned}$$

C_τ is therefore a bounded composition operator on X by Theorem 7.3.19 if X is an Orlicz-Lorentz space or by Theorem 7.3.15 if X is a rearrangement invariant Banach function space and (Ω, Σ, μ) is resonant. Furthermore,

$$\begin{aligned} \sup_{A \in \Sigma_{f,0}} \frac{\Phi_X(\mu \circ \tau_y^{-1}(A))}{\Phi_X(\mu(A))} &\leq \|C_\tau\| \leq \max \left\{ 1, \sup_{A \in \Sigma_{f,0}} \frac{\mu \circ \tau_y^{-1}(A)}{\mu(A)} \right\} \\ &\Rightarrow 1 \leq \|C_\tau\| \leq 1 \quad \text{by (7.3.32)} \end{aligned}$$

□

COROLLARY 7.3.21. *Let X be a rearrangement invariant Banach function space over $(\mathbb{R}, \mathcal{B}, m)$. Then the translation operators on X are bounded composition operators of norm 1.*

PROOF. If \mathbb{R} is equipped with its usual topology and the operation of addition, then $(\mathbb{R}, +)$ is a locally compact abelian topological group and $(\mathbb{R}, \mathcal{B}, m)$ is resonant, so the mapping $\tau_y : \mathbb{R} \rightarrow \mathbb{R}$ defined by $\tau_y(x) = x + y$ induces a bounded composition operator by Theorem 7.3.20. □

Since Orlicz spaces and Lorentz spaces are special cases of Orlicz-Lorentz spaces we have the following two corollaries to Theorem 7.3.19.

COROLLARY 7.3.22. (cf. [5, p.193]) *Let $\tau : \Omega \rightarrow \Omega$ be a measurable transformation and ϕ an Orlicz function. If there exists an $M > 0$ such that*

$$(7.3.33) \quad \mu \circ \tau^{-1}(A) \leq M\mu(A) \quad \forall A \in \Sigma_f$$

then C_τ is a bounded composition operator on the Orlicz space $L^\phi(\mu)$. Conversely if C_τ is a bounded composition operator on $L^\phi(\mu)$ then there exists a $k > 0$ such that

$$(7.3.34) \quad \frac{1}{\phi^{-1}(1/\mu \circ \tau^{-1}(A))} \leq \frac{k}{\phi^{-1}(1/\mu(A))} \quad \forall A \in \Sigma_{f,0}$$

In particular, if ϕ satisfies the Δ_2 -condition globally, then C_τ is a bounded composition operator on $L^\phi(\mu)$ if and only if (7.3.33) holds for some $M > 0$. In general, if C_τ is a composition operator on $L^\phi(\mu)$, then

$$\sup_{A \in \Sigma_{f,0}} \frac{\phi^{-1}(1/\mu(A))}{\phi^{-1}(1/\mu \circ \tau^{-1}(A))} \leq \|C_\tau\| \leq \max \left\{ 1, \sup_{A \in \Sigma_{f,0}} \frac{\mu \circ \tau^{-1}(A)}{\mu(A)} \right\}$$

PROOF. If $w \equiv 1$, then $L^{\phi,w}(\mu) = L^\phi(\mu)$ with equality of norms (Example 2.2.38) and so the result follows by Theorem 7.3.19 and the fact that if $w \equiv 1$, then $\Gamma(t) = \int_0^t w(s) ds = t$. Furthermore if $\phi \in \Delta_2$ globally and C_τ is bounded on $L^\phi(\mu)$, then

$$\begin{aligned} \Gamma(\mu \circ \tau^{-1}(A)) &\leq M_k \Gamma(\mu(A)) \quad \forall A \in \Sigma_{f,0} \quad \text{by (7.3.26)} \\ \Rightarrow \mu \circ \tau^{-1}(A) &\leq M_k \mu(A) \quad \forall A \in \Sigma_{f,0} \quad \text{since } \Gamma(t) = t \end{aligned}$$

Alternatively, we could also use the fact that if $\phi \in \Delta_2$ globally, then “the condition on the fundamental function” implies “the condition on the measure” (see Example 7.3.6). □

COROLLARY 7.3.23. (cf. [21, p.2111]) *Let (Ω, Σ, μ) be a σ -finite measure space and $\tau : \Omega \rightarrow \Omega$ a measurable transformation. C_τ is a bounded composition operator on $L^{p,q}(\mu)$ ($1 \leq q \leq p < \infty$) if and only if there exists an $M > 0$ such that*

$$(7.3.35) \quad \mu \circ \tau^{-1}(A) \leq M\mu(A) \quad \forall A \in \Sigma_f$$

and

$$\|C_\tau\| = \left(\sup_{A \in \Sigma_{f,0}} \frac{\mu \circ \tau^{-1}(A)}{\mu(A)} \right)^{1/p} = \|f_\tau\|_\infty^{1/p}$$

PROOF. Let $\phi(t) := t^q$ and $w(t) = t^{\frac{q}{p}-1}$, then $L^{\phi,w}(\mu)$ is an Orlicz-Lorentz space and $L^{\phi,w}(\mu) = L^{p,q}(\mu)$ with equality of norms by Example 2.2.28. (Sufficiency): If condition (7.3.35) is satisfied then C_τ is a bounded composition operator on $L^{\phi,w}(\mu)$ by Theorem 7.3.19. We obtain a better estimate of the norm though by considering the following direct proof: Let $f \in L^{p,q}(\mu)$. Then $(C_\tau f)^*(t) \leq f^*(t/M)$ for all $t \geq 0$ by Lemma 7.3.11. It follows by Proposition 2.2.3 that $\|C_\tau f\|_{p,q} \leq M^{1/p} \|f\|_{p,q}$.

Similar to the Orlicz-Lorentz case it can be shown that

$$\begin{aligned} \|C_\tau\|^p &\leq M \quad \forall M > 0 \text{ such that (7.3.35) holds} \\ \Rightarrow \|C_\tau\|^p &\leq \sup_{A \in \Sigma_{f,0}} \frac{\mu \circ \tau^{-1}(A)}{\mu(A)} \\ &= \|f_\tau\|_\infty \quad \text{by Proposition 7.3.3} \end{aligned}$$

(Necessity): By Theorem 7.3.15 and Remark 7.3.16 we can find some $k > 0$ such that

$$(7.3.36) \quad \Phi_{L^{p,q}}(\mu \circ \tau^{-1}(A)) \leq k \Phi_{L^{p,q}}(\mu(A)) \quad \forall A \in \Sigma_f$$

By Example 7.3.5, this implies that

$$\mu \circ \tau^{-1}(A) \leq k^p \mu(A) \quad \forall A \in \Sigma_f$$

Since (7.3.36) holds for $k = \|C_\tau\|$, we have $\|C_\tau\|^p \geq \sup_{A \in \Sigma_{f,0}} \frac{\mu \circ \tau^{-1}(A)}{\mu(A)} = \|f_\tau\|_\infty$. \square

Since every Lebesgue space is a special case of a Lorentz space, the previous Corollary yields

COROLLARY 7.3.24. (cf. [37, p.19]) *Let (Ω, Σ, μ) be a σ -finite measure space and $\tau : \Omega \rightarrow \Omega$ a measurable transformation. C_τ is a bounded composition operator on $L^p(\mu)$ ($1 \leq p < \infty$) if and only if there exists an $M > 0$ such that*

$$(7.3.37) \quad \mu \circ \tau^{-1}(A) \leq M \mu(A) \quad \forall A \in \Sigma_f$$

Furthermore,

$$\|C_\tau\| = \sup_{A \in \Sigma_{f,0}} \left(\frac{\mu \circ \tau^{-1}(A)}{\mu(A)} \right)^{1/p} = \|f_\tau\|^{1/p}$$

PROOF. Let $q = p$, then $L^{p,p}(\mu) = L^p(\mu)$ by Example 2.2.2 and so the result follows by Corollary 7.3.23. \square

We give two examples to illustrate the results on Lorentz spaces and Orlicz spaces. These are generalizations of results appearing in [37] for Lebesgue spaces. Recall that the bilateral shift operator and symmetry operator are examples of composition operators (Example 7.1.5).

EXAMPLE 7.3.25. (cf. [37, p.20]) Let $\mathbf{w} := (w_i)_{i \in \mathbb{Z}}$ be a sequence of strictly positive real numbers and let X be an Orlicz space (with $\phi \in \Delta_2$ globally) or a Lorentz $L^{p,q}$ ($1 \leq q \leq p < \infty$) space over $(\mathbb{Z}, \mathcal{P}(\mathbb{Z}), \mu_w)$. The bilateral shift operator C_{τ_b} is a bounded composition operator on X if and only if

$$(7.3.38) \quad \exists b > 0 \text{ such that } w_{i+1} \leq b w_i \quad \forall i \in \mathbb{Z}$$

PROOF. If C_{τ_b} is bounded on X , then there exists, by Corollary 7.3.22 or Corollary 7.3.23 an $M > 0$ such that

$$\mu_w(\tau_b^{-1}(A)) \leq M \mu_w(A) \quad \forall A \in \mathcal{P}(\mathbb{Z})_{f,0}$$

In particular, if $i \in \mathbb{Z}$ and $A = \{i\}$, then

$$\begin{aligned} \mu_w(\tau_b^{-1}(\{i\})) &\leq M \mu_w(\{i\}) \\ \Rightarrow \mu_w(\{i+1\}) &\leq M \mu_w(\{i\}) \\ \Rightarrow w_{i+1} &\leq M w_i \quad \forall i \in \mathbb{Z} \text{ since } i \text{ was arbitrary} \end{aligned}$$

Conversely if (7.3.38) holds and $A \in \mathcal{P}(\mathbb{Z})$, then

$$\begin{aligned}
\mu_w \circ \tau_b^{-1}(A) &= \sum_{i \in \tau_b^{-1}(A)} w_i \\
&= \sum_{i \in (A+1)} w_i \\
&= \sum_{i \in A} w_{i+1} \\
&\leq \sum_{i \in A} b w_i \quad \text{since (7.3.38) holds} \\
&= b \sum_{i \in A} w_i \\
&= b \mu_w(A)
\end{aligned}$$

Since $A \in \mathcal{P}(\mathbb{Z})$ was arbitrary, we have C_{τ_b} bounded on X by Corollary 7.3.22 or 7.3.23. \square

COROLLARY 7.3.26. *The bilateral shift operator is bounded on $X(\mathbb{Z}, \mathcal{P}(\mathbb{Z}), \mu_c)$ where μ_c is the counting measure.*

Next we characterize the weighted sequence spaces on which the symmetry operators are bounded composition operators. Let $\alpha := (\alpha_i)_{i \in \mathbb{Z}}$ where $\alpha_i := \frac{w_{-i}}{w_i}$.

EXAMPLE 7.3.27. (cf. [37, p.20]) Let X be as in Example 7.3.25. The symmetry operator C_{τ_s} is bounded on X if and only if $\alpha \in L^\infty(\mu_w)$.

PROOF. If C_{τ_s} is bounded on X , then by Corollary 7.3.18 there exists an $M > 0$ such that

$$\mu_w(\tau_s^{-1}(A)) \leq M \mu_w(A) \quad \forall A \in \mathcal{P}(\mathbb{Z})_{f,0}$$

In particular

$$\begin{aligned}
\mu_w(\tau_s^{-1}(\{i\})) &\leq M \mu(\{i\}) \quad \forall i \in \mathbb{Z} \\
\Rightarrow \mu_w(\{-i\}) &\leq M \mu(\{i\}) \quad \forall i \in \mathbb{Z} \\
\Rightarrow w_{-i} &\leq M w_i \quad \forall i \in \mathbb{Z} \\
\Rightarrow \frac{w_{-i}}{w_i} &\leq M \quad \forall i \in \mathbb{Z} \\
\Rightarrow \alpha &\in L^\infty(\mu_w)
\end{aligned}$$

Conversely, if $\alpha \in L^\infty(\mu_w)$ and $A \subset \mathbb{Z}$, then

$$\begin{aligned}
\mu_w(\tau_s^{-1}(A)) &= \sum_{i \in \tau_s^{-1}(A)} w_i \\
&= \sum_{i \in -A} w_i \\
&= \sum_{i \in A} w_{-i} \\
&\leq \sum_{i \in A} \|\alpha\|_\infty w_i \quad \text{since } \alpha \in L^\infty(\mu_w) \\
&= \|\alpha\|_\infty \sum_{i \in A} w_i \\
&= \|\alpha\|_\infty \mu_w(A)
\end{aligned}$$

Since this holds for all $A \in \mathcal{P}(\mathbb{Z})$, C_{τ_s} is bounded on X by Corollary 7.3.18. \square

COROLLARY 7.3.28. *The symmetry operator is bounded on $X(\mathbb{Z}, \mathcal{P}(\mathbb{Z}), \mu_c)$.*

Finally, we characterize the bounded composition operators on $L^\infty(\mu)$. Even though condition (7.3.37) is sufficient to induce a bounded composition operator on $L^\infty(\mu)$, using Corollary 7.3.22 and $L^\phi(\mu) = L^\infty(\mu)$, with equality of norms (Example 2.2.19), if

$$\phi(t) = \begin{cases} 0 & 0 \leq t \leq 1 \\ \infty & t > 1 \end{cases},$$

a stronger result can be obtained using a direct proof for the $L^\infty(\mu)$ case.

THEOREM 7.3.29. (cf. [37, p.18]) *Let (Ω, Σ, μ) be a σ -finite measure space and $\tau : \Omega \rightarrow \Omega$ a measurable transformation. C_τ is a bounded composition operator on $L^\infty(\mu)$ if and only if τ is nonsingular. Furthermore, $\|C_\tau\| = 1$ if C_τ is a bounded composition operator on $L^\infty(\mu)$.*

PROOF. (Sufficiency): Let $f \in L^\infty(\mu)$. Then there exists an $A \in \Sigma$ such that $\mu(A) = 0$ and $|f(x)| \leq \|f\|_\infty < \infty$ for all $x \in A^c$. Therefore

$$\begin{aligned} |(C_\tau f)(x)| &= |f(\tau(x))| \leq \|f\|_\infty \quad \forall x \in \tau^{-1}(A^c) = (\tau^{-1}(A))^c \\ \Rightarrow \|C_\tau f\|_\infty &\leq \|f\|_\infty \quad \text{since } \tau \text{ nonsingular implies that } \mu \circ \tau^{-1}(A) = 0, \end{aligned}$$

and so $\|C_\tau\| \leq 1$ since $f \in L^\infty(\mu)$ was arbitrary. Let $f(x) = 1$ for all $x \in \Omega$, then $f \in L^\infty(\mu)$ and $|(C_\tau f)(x)| = 1$ for all $x \in \Omega$. So $\|C_\tau f\|_\infty = 1 = \|f\|_\infty$ and hence $\|C_\tau\| \geq 1$.

(Necessity): If τ is not nonsingular then C_τ is not well-defined by Proposition 7.1.2 and hence not a bounded composition operator. \square

7.4. Composition operators between $L^p(\mu_1)$ and $L^q(\mu_2)$

The focus thus far has been on composition operators which map a function space into itself. In the present section we turn our attention to composition operators acting between two distinct Lebesgue spaces over potentially different measure spaces. More specifically, let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be σ -finite measure spaces and let $\tau : \Omega_2 \rightarrow \Omega_1$ be a nonsingular measurable transformation. We investigate the conditions on the symbol function f_τ such that $C_\tau(L^p(\mu_1)) \subset L^q(\mu_2)$, where $1 \leq p, q \leq \infty$.

Recall Corollary 1.2.16. There exists a locally integrable nonnegative function $f_\tau \in L^0(\mu_1)$ such that

$$\mu_2 \circ \tau^{-1}(A) = \int_A f_\tau d\mu_1 \quad \forall A \in (\Sigma_1)_f$$

It was shown in Section 6.1 that τ induces a bounded composition operator from $L^p(\mu_1)$ into $L^q(\mu_2)$ if and only if f_τ induces a bounded multiplication operator from $L^p(\mu_1)$ into $L^q(\mu_1)$. It follows that the necessary and sufficient conditions for a function $\pi \in L^0(\mu)$ to induce a bounded multiplication operator M_π from $L^p(\mu_1)$ into $L^q(\mu_1)$, obtained in Section 6.1, will be used extensively. We treat the two cases $q < p$ and $p < q$ separately and provide conditions in terms of f_τ , the relationship between $\mu_2 \circ \tau^{-1}$ and μ_1 , and the quantities $Q_\tau(A)$ and I_τ to be defined below. Lastly, the difficulties of trying to generalize the methods employed in this section to more general Banach function spaces will be explored briefly.

For any $A \in \Sigma_1$ define $Q_\tau(A)$ as follows:

$$Q_\tau(A) := \inf\{b \geq 0 : \mu_2 \circ \tau^{-1}(B) \leq b\mu_1(B) : B \subset A, B \in \Sigma_1\}$$

LEMMA 7.4.1. [39, p.331] *For any $A \in \Sigma_1$, $Q_\tau(A) = \operatorname{ess\,sup}_{x \in A} f_\tau(x)$.*

PROOF. Let $B \in \Sigma_1$ such that $B \subset A$, then

$$\begin{aligned} \mu_2 \circ \tau^{-1}(B) &= \int_B f_\tau d\mu_1 \\ &\leq \left(\operatorname{ess\,sup}_{x \in B} f_\tau(x) \right) \mu_1(B) \\ &\leq \left(\operatorname{ess\,sup}_{x \in A} f_\tau(x) \right) \mu_1(B) \end{aligned}$$

Since this holds for any $B \in \Sigma_1$ such that $B \subset A$, we have $Q_\tau(A) \leq \operatorname{ess\,sup}_{x \in A} f_\tau(x)$. If $Q_\tau(A) = \infty$, then clearly $\operatorname{ess\,sup}_{x \in A} f_\tau(x) \leq Q_\tau(A)$. Let $Q_\tau(A) < \infty$ and assume that $Q_\tau(A) < \operatorname{ess\,sup}_{x \in A} f_\tau(x)$. This implies that $\mu_1(\{x \in A : f_\tau(x) > Q_\tau(A)\}) > 0$ and therefore by Proposition 1.2.1 there exists an $\epsilon > 0$ such that $\mu_1(B) > 0$, where $B = \{x \in \Omega_1 : f_\tau(x) \geq Q_\tau(A) + \epsilon\} \cap A$. Since B is the intersection of two measurable sets we have that $B \in \Sigma_1$ and clearly $B \subset A$. Let $n \in \mathbb{N}^+$ such that $\frac{1}{n} < \epsilon$, then

$$\begin{aligned} \left(Q_\tau(A) + \frac{1}{n} \right) \mu_1(B) &\geq \mu_2 \circ \tau^{-1}(B) \quad \text{by definition of } Q_\tau(A) \\ &= \int_B f_\tau d\mu_1 \\ &\geq (Q_\tau(A) + \epsilon) \mu_1(B) \end{aligned}$$

This is a contradiction since $\frac{1}{n} < \epsilon$ and therefore $Q_\tau(A) \geq \operatorname{ess\,sup}_{x \in A} f_\tau(x)$. □

Let $1 \leq q < p < \infty$ and let r be such that $\frac{1}{p} + \frac{1}{r} = \frac{1}{q}$. Define

$$I_\tau := \inf \left\{ \sum_{j=1}^{\infty} Q_\tau(A_j)^{r/q} \mu_1(A_j) : (A_j)_{j=1}^{\infty} \text{ is a partition of } \Omega_1 \right\}$$

LEMMA 7.4.2. [39, p.332] $\int_{\Omega_1} f_\tau^{r/q} d\mu = I_\tau$

PROOF. Let $(F_j)_{j=1}^{\infty}$ be a partition of Ω_1 . Then

$$\begin{aligned} \int_{\Omega_1} f_\tau^{r/q} d\mu &= \sum_{j=1}^{\infty} \int_{F_j} f_\tau^{r/q} d\mu \\ &\leq \sum_{j=1}^{\infty} \left(\operatorname{ess\,sup}_{x \in F_j} f_\tau(x) \right)^{r/q} \mu(F_j) \\ &= \sum_{j=1}^{\infty} (Q_\tau(F_j))^{r/q} \mu(F_j) \quad \text{by Lemma 7.4.1} \end{aligned}$$

Since this holds for every partition of Ω_1 , we obtain

$$\int_{\Omega_1} f_\tau^{r/q} d\mu \leq I_\tau$$

For the reverse inequality, let $\alpha > 1$ and for $m \in \mathbb{Z}$ let $G_m := \{x \in \Omega_1 : \alpha^{m-1} \leq f_\tau^{r/q}(x) < \alpha^m\}$. Let $F_1 = \{x \in \Omega_1 : f_\tau^{r/q}(x) = 0\}$ and for $n \geq 2$, define

$$F_n = \begin{cases} G_{n/2} & n \text{ even} \\ G_{(-n+3)/2} & n \text{ odd} \end{cases}$$

Then $(F_n)_{n=1}^\infty$ is a partition of Ω_1 since f_τ is nonnegative. Furthermore,

$$\begin{aligned}
I_\tau &\leq \sum_{n=1}^{\infty} (Q_\tau(F_n))^{r/q} \mu(F_n) && \text{by definition of } I_\tau \\
&= \sum_{n=1}^{\infty} \left(\operatorname{ess\,sup}_{x \in F_n} f_\tau(x) \right)^{r/q} \mu(F_n) && \text{by Lemma 7.4.1} \\
&= \sum_{m=-\infty}^{\infty} \left(\operatorname{ess\,sup}_{x \in G_m} f_\tau(x) \right)^{r/q} \mu(G_m) \\
&\leq \sum_{m=-\infty}^{\infty} \alpha^m \mu(G_m) && \text{by definition of } G_m \\
&= \alpha \left(\sum_{m=-\infty}^{\infty} \alpha^{m-1} \mu(G_m) \right) \\
&\leq \alpha \left(\sum_{m=-\infty}^{\infty} \int_{G_m} f_\tau^{r/q} d\mu \right) && \text{since } \alpha^{m-1} \leq f_\tau^{r/q}(x) \forall x \in G_m \\
&= \alpha \left(\sum_{n=1}^{\infty} \int_{F_n} f_\tau^{r/q} d\mu \right) \\
&= \alpha \int_{\Omega_1} f_\tau^{r/q} d\mu
\end{aligned}$$

Since this holds for all $\alpha > 1$, we have that $I_\tau \leq \int_{\Omega_1} f_\tau^{r/q} d\mu$. \square

We are now in a position to characterize those transformations $\tau : \Omega_2 \rightarrow \Omega_1$ which induce bounded composition operators from $L^p(\mu_1)$ into $L^q(\mu_2)$. We start with the case $q < p$.

THEOREM 7.4.3. [39, p.331] *Let $1 \leq q < p < \infty$ and let r be such that $\frac{1}{p} + \frac{1}{r} = \frac{1}{q}$. If $\tau : \Omega_2 \rightarrow \Omega_1$ is a nonsingular measurable transformation, then the following are equivalent:*

- (1) τ induces a bounded composition operator from $L^p(\mu_1)$ into $L^q(\mu_2)$,
- (2) $f_\tau \in L^{r/q}(\mu_1)$ and
- (3) there exists a partition $(A_j)_{j=1}^\infty$ of Ω_1 such that

$$\sum_{j=1}^{\infty} Q_\tau(A_j) \mu_1(A_j) < \infty$$

Moreover, if any of these assertions hold, then

$$\|C_\tau\| = \|f_\tau^{1/q}\|_{L^r(\mu_1)} = I_\tau^{1/r}$$

PROOF. 1) \Leftrightarrow 2): By Corollary 6.1.2, 1) holds if and only if $f_\tau^{1/q}$ induces a bounded multiplication operator from $L^p(\mu_1)$ into $L^q(\mu_1)$ and

$$(7.4.1) \quad \|C_\tau\| = \|M_{f_\tau^{1/q}}\|$$

By Theorem 6.1.4, $f_\tau^{1/q}$ induces a bounded multiplication operator from $L^p(\mu_1)$ into $L^q(\mu_1)$ if and only if $f_\tau^{1/q} \in L^r(\mu)$ and in this case

$$(7.4.2) \quad \|M_{f_\tau^{1/q}}\| = \|f_\tau^{1/q}\|_{L^r(\mu_1)}$$

But

$$(7.4.3) \quad \begin{aligned} \|f_\tau^{1/q}\|_{L^r(\mu_1)} &= \left(\int_{\Omega_1} f_\tau^{r/q} d\mu_1 \right)^{1/r} \\ &= \|f_\tau\|_{L^{r/q}(\mu_1)}^{1/q} \end{aligned}$$

So $f_\tau^{1/q} \in L^r(\mu) \Leftrightarrow f_\tau \in L^{r/q}(\mu_1)$.

2) \Leftrightarrow 3) :

$$\begin{aligned} f_\tau \in L^{r/q}(\mu_1) &\Leftrightarrow \int_{\Omega_1} f_\tau^{r/q} d\mu_1 < \infty \\ &\Leftrightarrow I_\tau < \infty \quad \text{by Lemma 7.4.2} \\ &\Leftrightarrow 3) \text{ holds} \quad \text{by definition of } I_\tau \end{aligned}$$

It follows that if any of these hold then C_τ is a bounded composition operator from $L^p(\mu_1)$ into $L^q(\mu_2)$ and

$$\begin{aligned} \|C_\tau\| &= \|f_\tau^{1/q}\|_{L^r(\mu_1)} \quad \text{by (7.4.1) and (7.4.2)} \\ &= I_\tau^{1/r} \quad \text{by (7.4.3) and Lemma 7.4.2} \end{aligned}$$

□

If $p < q$, we have the following result.

THEOREM 7.4.4. [39, p.332] *Let $1 \leq p < q < \infty$ and let $\tau : \Omega_2 \rightarrow \Omega_1$ be a nonsingular measurable transformation. The following are equivalent:*

- (1) τ induces a bounded composition operator from $L^p(\mu_1)$ into $L^q(\mu_2)$,
- (2) $f_\tau(x) = 0$ for μ_1 -a.e. $x \in (\Omega_1)_c$ and $\sup_{n \in \mathbb{N}^+} \frac{f_\tau(A_n)^p}{\mu_1(A_n)^{q-p}} < \infty$, where $(\Omega_1)_c$ is the nonatomic part of Ω_1 and $(A_n)_{n=1}^\infty$ are the atoms of Ω_1
- (3) $\mu_2 \circ \tau^{-1}((\Omega_1)_c) = 0$ and there exists a $b > 0$ such that

$$(\mu_2 \circ \tau^{-1}(A_n))^p \leq b(\mu_1(A_n))^q \quad \forall n \in \mathbb{N}^+$$

Moreover, if any of these assertions hold then

$$\begin{aligned} \|C_\tau\|^{pq} &= \sup_{n \in \mathbb{N}^+} \frac{f_\tau(A_n)^p}{\mu_1(A_n)^{q-p}} \\ &= \inf\{b \geq 0 : (\mu_2 \circ \tau^{-1}(A_n))^p \leq b(\mu_1(A_n))^q \quad \forall n \in \mathbb{N}^+\} \end{aligned}$$

PROOF. 1) \Leftrightarrow 2) : As before, 1) holds if and only if $f_\tau^{1/q}$ induces a bounded multiplication operator from $L^p(\mu_1)$ into $L^q(\mu_1)$ and

$$(7.4.4) \quad \|C_\tau\| = \|M_{f_\tau^{1/q}}\|$$

By Theorem 6.1.6 $f_\tau^{1/q}$ induces a bounded multiplication operator from $L^p(\mu_1)$ into $L^q(\mu_1)$ if and only if

$$(7.4.5) \quad f_\tau^{1/q}(x) = 0 \quad \mu_1\text{-a.e. } x \in (\Omega_1)_c \quad \text{and}$$

$$(7.4.6) \quad \sup_{n \in \mathbb{N}^+} \frac{f_\tau(A_n)^{s/q}}{\mu_1(A_n)} < \infty \quad \text{where } \frac{1}{q} + \frac{1}{s} = \frac{1}{p}$$

But

$$\begin{aligned} (7.4.6) \text{ holds} &\Leftrightarrow \left(\sup_{n \in \mathbb{N}^+} \frac{f_\tau(A_n)^{s/q}}{\mu_1(A_n)} \right)^{\frac{pq}{s}} < \infty \quad \text{since } \frac{pq}{s} > 0 \\ &\Leftrightarrow \sup_{n \in \mathbb{N}^+} \frac{f_\tau(A_n)^p}{\mu_1(A_n)^{q-p}} < \infty \quad \text{since } \frac{pq}{s} = q - p \end{aligned}$$

Furthermore,

$$\begin{aligned} \|C_\tau\|^{pq} &= \|M_{f_\tau^{1/q}}\|^{pq} \quad \text{by Corollary 6.1.2} \\ &= \left(\sup_{n \in \mathbb{N}^+} \frac{f_\tau^{1/q}(A_n)}{\mu_1(A_n)^{1/s}} \right)^{pq} \quad \text{by Theorem 6.1.6} \\ &= \sup_{n \in \mathbb{N}^+} \frac{f_\tau(A_n)^p}{\mu_1(A_n)^{q-p}} \end{aligned}$$

2) \Leftrightarrow 3): $f_\tau(x) = 0$ μ -a.e. $x \in (\Omega_1)_c$ if and only if

$$\mu_2 \circ \tau^{-1}((\Omega_1)_c) = \int_{(\Omega_1)_c} f_\tau d\mu_1 = 0 \quad \text{since } f_\tau \geq 0$$

Furthermore, since

$$\begin{aligned} \mu_2 \circ \tau^{-1}(A_n) &= \int_{A_n} f_\tau d\mu_1 \\ &= f_\tau(A_n)\mu_1(A_n), \end{aligned}$$

we have for each $n \in \mathbb{N}^+$

$$\begin{aligned} \frac{f_\tau(A_n)^p}{\mu_1(A_n)^{q-p}} &= \frac{(f_\tau(A_n)\mu_1(A_n))^p}{\mu_1(A_n)^q} \\ &= \frac{(\mu_2 \circ \tau^{-1}(A_n))^p}{\mu_1(A_n)^q} := \alpha_n \end{aligned}$$

Let $C = \{b \geq 0 : (\mu_2 \circ \tau^{-1}(A_n))^p \leq b(\mu_1(A_n))^q \forall n \in \mathbb{N}^+\}$. If $\beta := \inf\{b \in C\} < \infty$, then $C \neq \emptyset$ and so for any $b \in C$

$$\begin{aligned} b &\geq \alpha_n \quad \forall n \in \mathbb{N}^+ \\ \Rightarrow b &\geq \sup \alpha_n := \alpha \\ \Rightarrow \beta &\geq \alpha \end{aligned}$$

Furthermore, if $\alpha < \infty$, then $\alpha \in C$ and so $\beta \leq \alpha$. It follows that

$$\sup_{n \in \mathbb{N}^+} \frac{f_\tau(A_n)^p}{\mu_1(A_n)^{q-p}} = \inf\{b \geq 0 : (\mu_2 \circ \tau^{-1}(A_n))^p \leq b(\mu_1(A_n))^q \forall n \in \mathbb{N}^+\}$$

Therefore condition 2) is equivalent to condition 3) and

$$\begin{aligned} \|C_\tau\|^{pq} &= \sup_{n \in \mathbb{N}^+} \frac{f_\tau(A_n)^p}{\mu_1(A_n)^{q-p}} \\ &= \inf\{b \geq 0 : (\mu_2 \circ \tau^{-1}(A_n))^p \leq b(\mu_1(A_n))^q \forall n \in \mathbb{N}^+\} \end{aligned}$$

□

A natural question to ask is whether this method can be generalized to other Banach function spaces. More precisely, can this method be used to determine the necessary and sufficient conditions on a map $\tau : \Omega_2 \rightarrow \Omega_1$ such that C_τ is a bounded composition operator from $X(\Omega_1, \Sigma_1, \mu_1)$ into $Y(\Omega_2, \Sigma_2, \mu_2)$ where X and Y are Banach function spaces over $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ respectively? The strategy employed for the Lebesgue spaces depended fundamentally on the following:

- τ induces a bounded composition operator from $L^p(\mu_1)$ into $L^q(\mu_2)$ if and only if f_τ induces a bounded multiplication operator from $L^p(\mu_1)$ into $L^q(\mu_1)$, and
- the conditions such that a function $\pi \in L^0(\mu_1)$ induces a bounded multiplication operator from $L^p(\mu_1)$ into $L^q(\mu_1)$ can be determined.

The first result depended on the fact that

$$\|C_\tau f\|_{L^q(\mu_2)} = \|M_{f^{\tau^{1/q}}}\|_{L^q(\mu_1)} \quad \forall f \in L^p(\mu_1)$$

which was shown by direct calculation and application of Theorem 1.2.18. Since the multiplicativity of the function $|\cdot|$ was used and Orlicz functions and decreasing rearrangements, for example, do not possess this property, it is not clear how this might be generalized to other Banach function spaces. The second difficulty is that even if a more general form of 1) could be obtained, it is not clear how one could determine the conditions on a function $\pi \in L^0(\mu_1)$ such that π induces a bounded multiplication operator from a Banach function space X into a different Banach function space Y if X and Y are not Lebesgue spaces.

7.5. Composition operators induced by rational functions on $(\mathbb{R}, \mathcal{B}, m)$

Recall that $(\mathbb{R}, \mathcal{B}, m)$ denotes the set of real numbers equipped with the Lebesgue measure on the Borel subsets of \mathbb{R} . Throughout this section X will denote a Banach function space of type 1 over $(\mathbb{R}, \mathcal{B}, m)$. As examples of composition operators, we will characterize those rational functions $\tau : \mathbb{R} \rightarrow \mathbb{R}$ which induce bounded composition operators on Banach function spaces of type 1 over $(\mathbb{R}, \mathcal{B}, m)$. This is a generalization of results presented in [35] for L^p -spaces, since all L^p -spaces are Banach function spaces of type 1 (Example 7.3.8). The following assumptions will be made:

- (1) $\tau = \phi/\psi$ where ϕ and ψ are polynomials with real coefficients.
- (2) ϕ and ψ have no common real zeros
- (3) The coefficients of the leading terms of ϕ and ψ have magnitude equal to 1
- (4) $\deg \psi > 1$

It will be shown that a rational function induces a bounded composition operator on a Banach function space of type 1 over $(\mathbb{R}, \mathcal{B}, m)$ if and only if the degree of the polynomial ϕ is strictly greater than the degree of the polynomial ψ and τ' is nonzero whenever it exists. The process to be followed is first to characterize those monotone functions and polynomials with real coefficients which induce bounded composition operators. These results, the properties of rational functions listed in Subsection 1.2.2 and the Radon-Nikodym derivative induced by a rational function (Lemma 1.2.24) will then be used to deduce the main result. At various junctures the specific results for L^p -spaces will be mentioned and for these special cases of the more general theory, estimates of the norm of C_τ will be given in terms of $1/\tau'$.

A rational function is continuous in a neighbourhood of every point where its derivative exists. We first show that a necessary condition for a rational function to induce a bounded composition operator is that the derivative of the rational function must be nonzero wherever it exists.

THEOREM 7.5.1. [35, p.330] *Let $\tau : \mathbb{R} \rightarrow \mathbb{R}$ be a measurable function which is continuous in a neighbourhood of every point where its derivative exists. Then C_τ is a bounded composition operator on X only if τ' , the derivative of τ , is nonzero whenever it exists.*

PROOF. Since X is a Banach function space of type 1 and $(\mathbb{R}, \mathcal{B}, m)$ is resonant, the boundedness of C_τ on X implies (see Corollary 7.3.18) that there exists some $M > 0$ such that

$$(7.5.1) \quad m(\tau^{-1}(A)) \leq Mm(A) \quad \forall A \in \mathcal{B}_f$$

Let a_0 be a point of \mathbb{R} where τ' exists. By assumption it is possible to find a neighbourhood N of a_0 such that τ is continuous on N . Since N is a neighbourhood of a_0 we can find an $\epsilon > 0$ such that $B(a_0, \epsilon) \subset N$ and so $A_1 := [a_0, a_0 + \frac{\epsilon}{2}] \subset N$. τ is continuous on N and hence on A_1 . This implies that $\tilde{\tau}$, defined by $\tilde{\tau}(x) := |\tau(x) - \tau(a_0)|$, is continuous on A_1 and hence attains its maximum relative to A_1 at a point $a_1 \in A_1$. Clearly $|\tau(a_1) - \tau(a_0)|$ is also the maximum with respect to the interval $[a_0, a_1]$. Since $\tilde{\tau}(a_0) = 0 \leq \tilde{\tau}(x)$ for all

$x \in A_1$, it follows that a_1 can be chosen such that $a_0 < a_1$. Let $A_2 = [a_0, a_1 - \frac{a_1 - a_0}{2}]$. By the same arguments as used above we can find an $a_2 \in A_2$ such that $a_0 < a_2 < a_1$ and $\tilde{\tau}$ has a maximum with respect to the interval $[a_0, a_2]$ at a_2 . Let $A_3 = [a_0, a_2 - \frac{a_2 - a_0}{2}]$. Continuing in this fashion, we obtain a sequence $(a_n)_{n=1}^\infty$ such that

- (1) $a_n \downarrow a_0$
- (2) $\tilde{\tau}$ has a maximum with respect to the interval $[a_0, a_n]$ at a_n and so $|\tau(x) - \tau(a_0)| \leq |\tau(a_n) - \tau(a_0)|$ for all $x \in [a_0, a_n]$

Let

$$\begin{aligned}\Delta a_n &:= a_n - a_0, \\ \Delta \tau_n &:= \tau(a_n) - \tau(a_0) \quad \text{and} \\ E_n &:= [\tau(a_0) - |\Delta \tau_n|, \tau(a_0) + |\Delta \tau_n|].\end{aligned}$$

Since $\tilde{\tau}$ has a maximum with respect to the interval $[a_0, a_n]$ at $x = a_n \forall n \in \mathbb{N}^+$, we have

$$\begin{aligned}(7.5.2) \quad & \tau([a_0, a_n]) \subset E_n \quad \forall n \in \mathbb{N}^+ \\ & \Rightarrow [a_0, a_n] \subset \tau^{-1}(E_n) \quad \forall n \in \mathbb{N}^+ \\ & \Rightarrow \Delta a_n = m([a_0, a_n]) \leq m(\tau^{-1}(E_n)) \quad \forall n \in \mathbb{N}^+ \\ & \Rightarrow |\Delta \tau_n| / \Delta a_n = \frac{1}{2} m(E_n) / \Delta a_n \\ & \geq \frac{1}{2} m(E_n) / m(\tau^{-1}(E_n)) \quad \text{by (7.5.2)} \\ & \geq \frac{1}{2M} \quad \text{by (7.5.1)}\end{aligned}$$

and so

$$\begin{aligned}\left| \frac{\tau(a_n) - \tau(a_0)}{a_n - a_0} \right| &= \left| \frac{\Delta \tau_n}{\Delta a_n} \right| \geq \frac{1}{2M} \quad \forall n \in \mathbb{N}^+ \\ &\Rightarrow |\tau'(a_0)| \geq \frac{1}{2M} > 0\end{aligned}$$

□

Next we characterize those monotone functions which induce bounded composition operators.

THEOREM 7.5.2. [35, p.330] *Let $\tau : \mathbb{R} \rightarrow \mathbb{R}$ be a monotone function. Then C_τ is a bounded composition operator on X if and only if $1/\tau'$ is essentially bounded.*

PROOF. (Necessity): Let C_τ be bounded on X . Then since X is a Banach function space of type 1, there exists, by Corollary 7.3.18, an $M > 0$ such that

$$(7.5.3) \quad m(\tau^{-1}(A)) \leq Mm(A) \quad \forall A \in \mathcal{B}_f$$

We first consider the case where τ is increasing. Let $x, y \in \mathbb{R}$ with $x < y$. τ increasing implies that $\tau(x) \leq \tau(y)$ for all s in $[x, y]$ and so

$$\begin{aligned}
 [x, y] &\subset \tau^{-1}([\tau(x), \tau(y)]) \\
 \Rightarrow m([x, y]) &\leq m(\tau^{-1}([\tau(x), \tau(y)])) \\
 (7.5.4) \quad &\leq Mm([\tau(x), \tau(y)]) \quad \text{by (7.5.3)} \\
 &\Rightarrow y - x \leq M(\tau(y) - \tau(x)) \\
 &\Rightarrow \frac{\tau(y) - \tau(x)}{y - x} \geq \frac{1}{M} \\
 &\Rightarrow \tau'(x) \geq \frac{1}{M} \quad \text{if it exists} \\
 (7.5.5) \quad &\Rightarrow \left| \frac{1}{\tau'(x)} \right| \leq M \quad \text{whenever } \tau'(x) \text{ exists}
 \end{aligned}$$

By Corollary 1.2.12 τ' exists m -a.e. and so $\|1/\tau'\|_\infty \leq M$ by (7.5.5). It can similarly be shown that if τ is decreasing then $\|1/\tau'\|_\infty \leq M$.

Conversely, suppose that $1/\tau'$ is essentially bounded and consider the case where τ is increasing, i.e. $\tau' \geq 0$. Let $A = [c, d] \subset \mathbb{R}$, $\beta = \inf\{x \in \mathbb{R} : \tau(x) \geq c\}$ and $\gamma = \sup\{x \in \mathbb{R} : \tau(x) < d\}$.

We show that $\beta \leq \gamma$ and $\tau^{-1}(A) \subset [\beta, \gamma]$: Assume $\beta > \gamma$ and let $z \in \mathbb{R}$ such that $\beta > z > \gamma$, then by the definition of β and γ and using the fact that τ is increasing, we have $c \geq \tau(z) \geq d$, which is a contradiction since $c < d$. To show that $\tau^{-1}(A) \subset [\beta, \gamma]$, we note

$$\begin{aligned}
 z \in \tau^{-1}(A) &\Rightarrow c \leq \tau(z) < d \\
 &\Rightarrow z \in \{x \in \mathbb{R} : f(x) \geq c\} \cap \{x \in \mathbb{R} : f(x) < d\} \\
 &\Rightarrow \beta \leq z \leq \gamma
 \end{aligned}$$

By Lemma 1.2.11 there exists a left continuous function $f : [\beta, \gamma] \rightarrow \mathbb{R}$ such that $\{x \in [\beta, \gamma] : f(x) \neq \tau(x)\}$ is at most countable, f' exists m -a.e. and

$$f(x) - f(\beta) = \int_\beta^x f'(t) dt + s(x) \quad x \in [\beta, \gamma]$$

where s is increasing and $s'(x) = 0$ m -a.e. $x \in [\beta, \gamma]$. $1/\tau'$ essentially bounded implies that there exists a $k > 0$ such that

$$\begin{aligned}
 |1/\tau'(x)| &< k \quad m\text{-a.e. } x \in [\beta, \gamma] \\
 \Rightarrow 1/\tau'(x) &< k \quad m\text{-a.e. } x \in [\beta, \gamma], \text{ since } \tau \text{ increasing implies } \tau' \text{ positive} \\
 (7.5.6) \quad \Rightarrow f'(x) &> \frac{1}{k} \quad m\text{-a.e. } x \in [\beta, \gamma]
 \end{aligned}$$

since $f'(x) = \tau'(x)$ m -a.e. $x \in [\beta, \gamma]$ by Lemma 1.2.11. Let $x, y \in [\beta, \gamma]$ such that $x < y$ and let

$$\begin{aligned}
 f_a(x) &:= \int_\beta^x f'(t) dt \\
 f_s(x) &:= s(x) + f(c)
 \end{aligned}$$

Then $f = f_a + f_s$ where f_a is absolutely continuous by Theorem 1.2.10 and $f' = f'_a$ m -a.e. since $f'_s = 0$ m -a.e. So

$$\begin{aligned}
 f(y) - f(x) &= (f_a + f_s)(y) - (f_a + f_s)(x) \\
 &= f_a(y) - f_a(x) + (s(y) - s(x)) \\
 &\geq f_a(y) - f_a(x) \quad \text{since } s \text{ is increasing} \\
 &= \int_x^y f'_a(t) dt \\
 &= \int_x^y f'(t) dt \\
 &> (y - x)/k \quad \text{by (7.5.6)} \\
 (7.5.7) \quad &\Rightarrow y - x < k(f(y) - f(x))
 \end{aligned}$$

We show that $m(\tau^{-1}(A)) \leq km(A)$: Noting that $\tau^{-1}(A) \subset [\beta, \gamma]$ and $f(x) = \tau(x)$ m -a.e. $x \in [\beta, \gamma]$, we have

$$(7.5.8) \quad m(f^{-1}(A)) = m(\tau^{-1}(A))$$

$$(7.5.9) \quad \leq m([\beta, \gamma]) \quad \text{by ii) and the monotonicity of } m$$

It follows that $m(f^{-1}(A)) = 0$ if $\beta = \gamma$. If $\beta \neq \gamma$, then $\forall \epsilon > 0$ such that $2\epsilon < \gamma - \beta$ we have $\beta < \beta + \epsilon < \gamma - \epsilon < \gamma$ and therefore

$$(7.5.10) \quad c \leq f(\beta + \epsilon) \leq f(\gamma - \epsilon) < d$$

by the definition of β and γ and the monotonicity of f . It follows that $\beta + \epsilon, \gamma - \epsilon \in f^{-1}(A)$ and so

$$\begin{aligned}
 \gamma - \beta - 2\epsilon &= (\gamma - \epsilon) - (\beta + \epsilon) \\
 &\leq k(f(\gamma - \epsilon) - f(\beta + \epsilon)) \quad \text{by (7.5.7)} \\
 &\leq k(d - c) \quad \text{by (7.5.10)} \\
 (7.5.11) \quad &= km(A)
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 (7.5.12) \quad m(\tau^{-1}(A)) &= m(f^{-1}(A)) \\
 &\leq m([\beta, \gamma]) \quad \text{as before} \\
 &= \gamma - \beta
 \end{aligned}$$

$$(7.5.13) \quad \leq km(A) \quad \text{by (7.5.11) and letting } \epsilon \rightarrow 0$$

For $n \in \mathbb{N}^+$ define

$$\begin{aligned}
 B_{2n} &= [n - 1, n) \\
 B_{2n-1} &= [-n, -n + 1),
 \end{aligned}$$

and let $\mathcal{F}_n := \{\bigcup_{i=1}^N [a_i, b_i) : a_i, b_i \in B_n, N \in \mathbb{N}^+\}$. We show that \mathcal{F}_n is a ring and (7.5.13) holds for all $E \in \mathcal{M}(\mathcal{F}_n)$, the monotone class generated by \mathcal{F}_n for each n .

i) \mathcal{F}_n is a ring: Since the union of two non-disjoint intervals of the form $[a, b)$ yields another interval of the same form it follows that $E \cup F \in \mathcal{F}_n$ for all $E, F \in \mathcal{F}_n$. Similarly $E \cap F, B_n \setminus E \in \mathcal{F}_n$ for all $E, F \in \mathcal{F}_n$. The algebraic ring properties of $(\mathcal{F}_n, \cup, \cap)$ follow since \mathcal{F}_n is a subfamily of $\mathcal{P}(\mathbb{R})$.

ii) (7.5.13) holds for all $E \in \mathcal{F}_n$: E can be written in the form $\bigcup_{i=1}^N [a_i, b_i]$ where the intervals are disjoint. Therefore,

$$\begin{aligned}
m\left(\tau^{-1}\left(\bigcup_{i=1}^N [a_i, b_i]\right)\right) &= m\left(\bigcup_{i=1}^N \tau^{-1}([a_i, b_i])\right) \\
&= \sum_{i=1}^N m(\tau^{-1}([a_i, b_i])) \quad \text{since the intervals are disjoint} \\
&\leq \sum_{i=1}^N km([a_i, b_i]) \quad \text{by (7.5.13)} \\
&= km\left(\sum_{i=1}^N [a_i, b_i]\right) \\
&= km\left(\bigcup_{i=1}^N [a_i, b_i]\right) \quad \text{since the intervals are disjoint}
\end{aligned}$$

Let \mathcal{G}_n be the family of all subsets of B_n for which (7.5.13) holds.

iii) \mathcal{G}_n is a monotone class: Let $(E_N)_{N=1}^\infty$ be an increasing sequence of sets in \mathcal{G}_n . By Theorem 1.1.3

$$\begin{aligned}
m(E_N) &\rightarrow m\left(\bigcup_{N=1}^\infty E_N\right) \quad \text{and} \\
m(\tau^{-1}(E_N)) &\rightarrow m\left(\tau^{-1}\left(\bigcup_{N=1}^\infty E_N\right)\right)
\end{aligned}$$

Since these limits exist and $m(\tau^{-1}(E_N)) \leq km(E_N)$ for all $N \in \mathbb{N}^+$, we have that

$$m\left(\tau^{-1}\left(\bigcup_{N=1}^\infty E_N\right)\right) = \lim_{N \rightarrow \infty} m(\tau^{-1}(E_N)) \leq \lim_{N \rightarrow \infty} km(E_N) = km\left(\bigcup_{N=1}^\infty E_N\right)$$

and so $\bigcup_{N=1}^\infty E_N \in \mathcal{G}_n$. Similarly, if $(E_N)_{N=1}^\infty$ is a decreasing sequence of sets in \mathcal{G}_n , then, using the fact that $E_1 \subset B_n$ and hence that $m(E_1) \leq 1 < \infty$, we obtain $\bigcap_{N=1}^\infty E_N \in \mathcal{G}_n$. We have therefore shown that \mathcal{G}_n is a monotone class containing \mathcal{F}_n (and thus $\mathcal{M}(\mathcal{F}_n)$, the monotone class generated by \mathcal{F}_n), i.e. (7.5.13) holds for all $E \in \mathcal{M}(\mathcal{F}_n)$.

By Theorem 1.1.13 $\mathcal{M}(\mathcal{F}_n) = \mathcal{S}(\mathcal{F}_n)$, the σ -ring generated by \mathcal{F}_n . Let

$$\mathcal{F}'_n := \{[a_i, b_i] : a_i, b_i \in B_n\},$$

then clearly $\mathcal{F}'_n \subset \mathcal{F}_n$ and so $\mathcal{S}(\mathcal{F}'_n) \subset \mathcal{S}(\mathcal{F}_n)$. By Remark 1.1.14 $\mathcal{S}(\mathcal{F}'_n)$ is the family of all Borel subsets of B_n and therefore (7.5.13) holds for all Borel subsets of B_n . Let C be a Borel subset of the real line. Let $n \in \mathbb{N}^+$, then $C \cap B_n$ is a Borel subset of B_n and so (7.5.13) holds for $C \cap B_n$. Therefore

$$\begin{aligned}
m(\tau^{-1}(C)) &= m\left(\tau^{-1}\left(\bigcup_{n=1}^\infty C \cap B_n\right)\right) \quad \text{since } \bigcup_{n=1}^\infty B_n = \mathbb{R} \\
&= m\left(\bigcup_{n=1}^\infty \tau^{-1}(C \cap B_n)\right) \\
&= \sum_{n=1}^\infty m(\tau^{-1}(C \cap B_n)) \quad \text{since } (C \cap B_n) \cap (C \cap B_p) = \emptyset \text{ if } n \neq p \\
&\leq \sum_{n=1}^\infty km(C \cap B_n) \quad \text{since (7.5.13) holds for each } n \\
&= km\left(\bigcup_{n=1}^\infty C \cap B_n\right) \quad \text{since } (C \cap B_n) \cap (C \cap B_p) = \emptyset \text{ if } n \neq p \\
&= km(C)
\end{aligned}$$

C_τ is therefore a bounded composition operator on X by Theorem 7.3.15. \square

REMARK 7.5.3. If the conditions of the above theorem are satisfied for the case $X = L^p(m)$, then $\|C_\tau\|^p = \|1/\tau'\|_\infty$. This follows since we have shown that $\|1/\tau'\|_\infty \leq M$ for all $M > 0$ such that (7.5.3) holds. In particular, (7.5.3) holds for $M = \|C_\tau\|^p$ and so $\|1/\tau'\|_\infty \leq \|C_\tau\|^p$. It has also been shown that if $1/\tau' \leq k$ m -a.e., then $m(\tau^{-1}(A)) \leq km(A)$ for all $A \in \mathcal{B}$ and so

$$\begin{aligned} \|1/\tau'\|_\infty &\leq \sup_{A \in \mathcal{B}_{f,0}} \left(\frac{\mu \circ \tau^{-1}(A)}{\mu(A)} \right) \\ &= \|C_\tau\|^p \quad \text{by Corollary (7.3.24)} \end{aligned}$$

The polynomials on \mathbb{R} which induce bounded composition operators on Banach function spaces of type 1 can be characterized as follows:

THEOREM 7.5.4. [35, p.331] *Let $\tau : \mathbb{R} \rightarrow \mathbb{R}$ be a polynomial with real coefficients. Then C_τ is a bounded composition operator on X if and only if $\tau'(x) \neq 0$ for all $x \in \mathbb{R}$.*

PROOF. (Sufficiency): Let τ be a polynomial such that $\tau'(x) \neq 0$ for all $x \in \mathbb{R}$. Since τ is a polynomial, so is τ' and hence τ' is continuous. Combining these two facts we have $\tau' > 0$ or $\tau' < 0$, i.e. τ is monotone. Next it will be shown that $1/\tau'$ is bounded. Since τ' is a polynomial we have that either

$$\begin{aligned} 1/\tau'(x) &\rightarrow 0 \text{ as } |x| \rightarrow \infty \quad \text{or} \\ 1/\tau'(x) &= k \quad \text{if } \tau' \text{ is constant} \end{aligned}$$

We can therefore find an $M > 0$ and a $t > 0$ such that

$$|x| \geq t \Rightarrow |1/\tau'(x)| \leq M$$

Furthermore, $\tau'(x) \neq 0$ for all $x \in \mathbb{R}$ and the continuity of τ' implies that $1/\tau'$ is continuous and so bounded on $[-t, t]$. $1/\tau'$ is therefore bounded for all $x \in \mathbb{R}$. By Theorem 7.5.2 C_τ is a bounded composition operator on X .

(Necessity): τ' exists for all $x \in \mathbb{R}$ since τ is a polynomial. Furthermore τ is continuous for all $x \in \mathbb{R}$ and so $\tau'(x) \neq 0$ for all $x \in \mathbb{R}$ by Theorem 7.5.1. \square

REMARK 7.5.5. If the conditions of the above theorem are satisfied for the case $X = L^p(m)$, then $\|C_\tau\|^p = \|1/\tau'\|_\infty$.

Finally we characterize those rational functions which induce bounded composition operators on X . We start by elucidating two necessary conditions.

THEOREM 7.5.6. [35, p.332] *If C_τ is a composition operator induced by a rational function $\tau = \phi/\psi$, then $\deg \phi > \deg \psi$.*

PROOF. The contrapositive will be shown. Let $\deg \phi \leq \deg \psi$. Then

$$\begin{aligned} |\tau(x)| &\rightarrow 1 \text{ as } |x| \rightarrow \infty \quad \text{if } \deg \phi = \deg \psi \\ |\tau(x)| &\rightarrow 0 \text{ as } |x| \rightarrow \infty \quad \text{if } \deg \phi < \deg \psi \end{aligned}$$

In either case, we can find an $M > 0$ and a $t > 0$ such that $|\tau(x)| \leq M$ for all $x \geq t$. Let $A = [t, \infty)$. Then

$$\begin{aligned} \tau(x) &\in [-M, M] \quad \forall x \in A \\ \Rightarrow A &\subset \tau^{-1}([-M, M]) \\ \Rightarrow \infty = m(A) &\leq m(\tau^{-1}([-M, M])) \quad \text{since } m \text{ is monotone} \\ \Rightarrow m(\tau^{-1}([-M, M])) &> 2bM \quad \forall b > 0 \\ &= bm([-M, M]) \end{aligned}$$

Since X is a Banach function space of type 1 this implies that the following condition does not hold

$$\exists k > 0 \text{ such that } \Phi_X(m(\tau^{-1}(A))) \leq k\Phi_X(m(A)) \quad \forall A \in \mathcal{B}_f$$

and so C_τ is not a bounded composition operator on X by Theorem 7.3.15. \square

THEOREM 7.5.7. [35, p.332] *Let τ be a rational function. If τ induces a bounded composition operator on X , then τ has at least one real zero.*

PROOF. The contrapositive will be shown. Let τ be a rational function without any real zeros. By Theorem 7.5.6 if $\deg \phi \leq \deg \psi$ then C_τ is not bounded. Let $\deg \phi > \deg \psi$. Then

$$(7.5.14) \quad |\tau(x)| \rightarrow \infty \text{ as } |x| \rightarrow \infty$$

We consider the cases τ continuous and τ not continuous separately.

τ continuous: $\tau(x) \neq 0$ for all $x \in \mathbb{R}$ implies that $\tau > 0$ or $\tau < 0$. If τ is strictly positive for all $x \in \mathbb{R}$, then there exists a $t_0 \in [-1, 1]$ such that

$$\tau(x) \leq \tau(t_0) = M \quad \forall x \in [-1, 1]$$

since τ is continuous. By (7.5.14) there exist $t_1 < -1$ and $t_2 > 1$ such that

$$\tau(t_i) > M + 2 \quad \text{for } i = 1, 2.$$

By the Intermediate Value Theorem we can find $x_1 \in (t_1, t_0)$, $x_2 \in (t_0, t_2)$ such that $\tau(x_i) = M + 1$ for $i = 1, 2$. By Rolle's Theorem there exists an $x_0 \in [x_1, x_2]$ such that $\tau'(x_0) = 0$ and so by Theorem 7.5.1 C_τ is not bounded. It can similarly be shown that if τ is negative for all $x \in \mathbb{R}$, then C_τ is not bounded.

τ not continuous: This implies that τ has at least one real pole. Let t_0 be the largest real pole of τ , then τ is continuous on (t_0, ∞) and

$$\begin{aligned} |\tau(x)| &\rightarrow \infty \quad \text{as } x \rightarrow \infty, \\ |\tau(x)| &\rightarrow \infty \quad \text{as } x \rightarrow t_0^+ \end{aligned}$$

A similar argument to the one used above for the interval $(-\infty, \infty)$ can be used for the interval (t_0, ∞) to obtain an x_0 such that $\tau'(x_0) = 0$ and hence show that C_τ is not bounded. \square

COROLLARY 7.5.8. [35, p.332] *Let τ be a continuous rational function. If the composition operator induced by τ is bounded on X , then τ has one and only one zero.*

PROOF. Theorem 7.5.7 showed that if C_τ is bounded on X , then τ has at least one real zero. Assume that τ has more than one real zero and let x_1 and x_2 be two such zeros. Then $\tau(x_1) = \tau(x_2)$ and so by Rolle's Theorem there exists an x_0 such that $\tau'(x_0) = 0$. C_τ is therefore not continuous by Theorem 7.5.1. \square

We state and prove the main result of this section, namely the characterization of those rational functions which induce bounded composition operators on Banach function spaces of type 1 over $(\mathbb{R}, \mathcal{B}, m)$.

THEOREM 7.5.9. *A rational function $\tau = \phi/\psi$ induces a bounded composition operator on X if and only if $\deg \phi > \deg \psi$ and $\tau'(x) \neq 0$ for every x where it exists.*

PROOF. (necessity): Let C_τ be a bounded composition operator on X . Then $\deg \phi > \deg \psi$, by Theorem 7.5.6. Since τ is continuous in a neighbourhood of every point where its derivative exists, $\tau'(x) \neq 0$ whenever it exists, by Theorem 7.5.1.

(sufficiency): We consider two cases.

i) If τ is continuous then $\psi(x) \neq 0$ for all $x \in \mathbb{R}$ and so

$$\tau'(x) = \frac{\phi'(x)\psi(x) - \phi(x)\psi'(x)}{(\psi(x))^2}$$

is continuous at every $x \in \mathbb{R}$. Since $\tau'(x) \neq 0$ for all $x \in \mathbb{R}$, we have that $\tau' > 0$ or $\tau' < 0$ since τ' is continuous. Furthermore $\deg \phi > \deg \psi$ and so by Lemma 1.2.21, $|\tau'(x)| \rightarrow \infty$ as $|x| \rightarrow \infty$ or $|\tau'(x)| \rightarrow 1$ as $|x| \rightarrow \infty$. Therefore if $\tau' > 0$ there exists an $M > 0$ such that

$$|x| > M \Rightarrow \tau'(x) > \frac{1}{2}$$

τ' is continuous on $[-M, M]$, so there exists an $x_0 \in [-M, M]$ such that

$$\tau'(x) \geq \tau'(x_0) = \alpha > 0 \quad \forall x \in [-M, M]$$

We therefore have that $|\tau'(x)| \geq \min\{\frac{1}{2}, \alpha\}$ for all $x \in \mathbb{R}$ and so $\frac{1}{\tau'}$ is bounded. Furthermore, $\tau' > 0$ implies that τ is monotone and so C_τ is a bounded composition operator on X by Theorem 7.5.2. It can similarly be shown that if $\tau' < 0$, then C_τ is a bounded composition operator on X .

ii) If τ is discontinuous, then by Lemma 1.2.22 $\#(P) = \#(Z)$ (where $\#(Z)$ denotes the number of zeros and $\#(P)$ the number of poles of τ) and the poles and zeros occur alternately. Let x_1, \dots, x_n be the distinct real zeros of ψ arranged in increasing order. Define τ_k as in Lemma 1.2.24. Then by Lemma 1.2.24

$$(7.5.15) \quad f_\tau(x) = \sum_{k=1}^{n+1} \frac{1}{\tau'_k(\tau_k^{-1}(x))} \quad m\text{-a.e. } x \in \mathbb{R}$$

Let $2 \leq k \leq n$. By Lemma 1.2.20 $|\tau'(x)| \rightarrow \infty$ as $x \rightarrow x_k$. τ' is continuous on (x_{k-1}, x_k) and $\tau'(x) \neq 0$ for all $x \in (x_{k-1}, x_k)$ implies that either $\tau'_k > 0$ on (x_{k-1}, x_k) or $\tau'_k < 0$ on (x_{k-1}, x_k) . If $\tau'_k > 0$, then by Lemma 1.2.20 there exists an $\epsilon_k > 0$ such that $2\epsilon_k < x_k - x_{k-1}$ and

$$\tau'_k(x) \geq \frac{1}{2} \quad \forall x \in (x_{k-1}, x_{k-1} + \epsilon_k) \cup (x_k - \epsilon_k, x_k)$$

τ'_k is continuous on $A_k := [x_{k-1} + \epsilon_k, x_k - \epsilon_k]$ and so there exists a $t_k \in A_k$ such that

$$\tau'_k(x) \geq \tau'_k(t_k) = \alpha_k > 0 \quad \forall x \in A_k$$

$\frac{1}{\tau'_k}$ is therefore bounded on (x_{k-1}, x_k) . The case $\tau'_k < 0$ is similar. For $k = 1$ or $k = n + 1$, we use Lemma 1.2.21 as before to obtain an $M > \max\{|x_1|, |x_n|\}$ such that

$$|x| > M \Rightarrow |\tau'_k(x)| \geq \frac{1}{2}$$

Since for $k = 1, n + 1$ we also have $|\tau'_k(x)| \rightarrow \infty$ as $x \rightarrow x_k$, we can use a similar argument to that used for $2 \leq k \leq n$ to show that $\frac{1}{\tau'_k}$ is also bounded for $k = 1, n + 1$. Let $x_0 = -\infty$ and $x_{n+1} = \infty$, then

$$\beta := \max_{1 \leq k \leq n+1} \left\{ \max \left\{ \frac{1}{|\tau'_k(x)|} : x \in (x_{k-1}, x_k) \right\} \right\} < \infty$$

and $|f_\tau(x)| \leq \beta$ m -a.e. by (7.5.15). We therefore have that $\|f_\tau\|_\infty \leq \beta$ and

$$m \circ \tau^{-1}(A) \leq \beta m(A) \quad \forall A \in \mathcal{B}_f$$

by Proposition 7.3.3. C_τ is therefore a bounded composition operator on X by Theorem 7.3.15. \square

Compactness of composition operators

The necessary and sufficient conditions for a map $\tau : \Omega \rightarrow \Omega$ to induce a compact composition operator on various Banach function spaces will be investigated. Of particular significance is the interplay between the map τ and the underlying measure space. We show first that if X is a Banach function space over a nonatomic measure space then no composition operator is compact and subsequently focus on the necessary conditions for a composition operator to be compact on Banach function spaces over more general measure spaces. In particular it will be shown that for certain Orlicz-Lorentz and Orlicz spaces and all Lorentz and Lebesgue spaces, a necessary condition is that τ must map the underlying measure space (modulo a set of measure zero) into the purely atomic part of the measure space. Some examples are subsequently given of compact composition operators on Banach function spaces for which the underlying measure spaces are not purely atomic. The remainder of the section will focus on Banach function spaces over purely atomic measure spaces.

Recall (Proposition 1.1.1) that if (Ω, Σ, μ) is a σ -finite measure space then Ω can be uniquely decomposed as $\Omega = \Omega_c \cup \Omega_a$, where Ω_c is nonatomic and Ω_a is purely atomic. Throughout this chapter, whenever (Ω, Σ, μ) is a purely atomic measure space, we will use $(A_n)_{n=1}^\infty$ to denote the countable set of atoms of strictly positive measure whose union is Ω . Furthermore, whenever this is the case, we will define for each $n \in \mathbb{N}^+$ the quantities a_n , b_n and c_n , where

$$\begin{aligned} a_n &:= \mu(A_n), \\ b_n &:= \frac{\mu \circ \tau^{-1}(A_n)}{\mu(A_n)} \quad \text{and} \\ c_n &:= \frac{\Phi_X(\mu \circ \tau^{-1}(A_n))}{\Phi_X(\mu(A_n))} \end{aligned}$$

In general it will be shown that if C_τ is a compact composition operator on a Banach function space X over a purely atomic measure space (Ω, Σ, μ) , then $\lim_{n \rightarrow \infty} c_n = 0$. For certain Orlicz-Lorentz and Orlicz spaces and all Lorentz and Lebesgue spaces this can be improved to having $\lim_{n \rightarrow \infty} b_n = 0$ as a necessary condition. Lastly, the conditions under which $\lim_{n \rightarrow \infty} b_n = 0$ is sufficient to ensure that τ induces a compact composition operator will be described.

THEOREM 8.1.1. (cf. [21, p.2116]) *Let X be a rearrangement invariant Banach function space over a σ -finite measure space (Ω, Σ, μ) and let $\tau : \Omega \rightarrow \Omega$ be a nonsingular measurable transformation. If (Ω, Σ, μ) is nonatomic, then C_τ is not compact.*

PROOF. We note that $\tau^{-1}(\Omega) = \Omega$ and hence

$$0 < \mu(\Omega) = \mu \circ \tau^{-1}(\Omega) = \int_{\Omega} f_\tau d\mu \quad \text{by Corollary 1.2.16}$$

This implies that there exists an $\epsilon > 0$ and an $A \in \Sigma$ such that $\mu(A) = \delta > 0$ and $f_\tau(x) \geq \epsilon$ for all $x \in A$. We can choose ϵ such that $0 < \epsilon < 1$. Since Ω is nonatomic, there exists a pairwise disjoint sequence $(E_n)_{n=1}^\infty \subset \Sigma \cap A$ such that $\mu(E_n) = \frac{\delta}{2^n}$ for each n . For each $n \in \mathbb{N}^+$ set

$$f_n := \frac{\chi_{E_n}}{\|\chi_{E_n}\|_X}$$

Then $\|f_n\| = 1$ for all n . It will be shown that $(C_\tau(f_n))_{n=1}^\infty$ does not contain a Cauchy subsequence. We first note that since (Ω, Σ, μ) is nonatomic it is resonant by Theorem 1.3.9 and so by Corollary 4.2.6 the fundamental function Φ_X is increasing and $\Phi_X(t)/t$ decreasing. Furthermore $\Phi_X(t)$ is defined for all $t \in [0, \mu(\Omega))$ by Remark 4.2.3. Since $0 < \epsilon < 1$, we therefore have that

$$(8.1.1) \quad \frac{\Phi_X(\epsilon\mu(E_n))}{\epsilon\mu(E_n)} \geq \frac{\Phi_X(\mu(E_n))}{\mu(E_n)} \quad \forall n \in \mathbb{N}^+$$

Let $m, n \in \mathbb{N}^+$, $m \neq n$, then

$$(8.1.2) \quad \frac{\chi_{\tau^{-1}(E_n)}}{\|\chi_{E_n}\|_X} \leq \frac{\chi_{\tau^{-1}(E_n)}}{\|\chi_{E_n}\|_X} + \frac{\chi_{\tau^{-1}(E_m)}}{\|\chi_{E_m}\|_X}$$

and

$$(8.1.3) \quad \begin{aligned} \|C_\tau f_n - C_\tau f_m\|_X &= \left\| \frac{\chi_{\tau^{-1}(E_n)}}{\|\chi_{E_n}\|_X} - \frac{\chi_{\tau^{-1}(E_m)}}{\|\chi_{E_m}\|_X} \right\|_X \\ &= \left\| \frac{\chi_{\tau^{-1}(E_n)}}{\|\chi_{E_n}\|_X} + \frac{\chi_{\tau^{-1}(E_m)}}{\|\chi_{E_m}\|_X} \right\|_X \quad \text{by Proposition 3.1.4(6)} \\ &\geq \left\| \frac{\chi_{\tau^{-1}(E_n)}}{\|\chi_{E_n}\|_X} \right\|_X \quad \text{since } \|\cdot\|_X \text{ is a lattice norm and (8.1.2) holds} \end{aligned}$$

$$(8.1.4) \quad \begin{aligned} &= \frac{\Phi_X(\mu \circ \tau^{-1}(E_n))}{\Phi_X(\mu(E_n))} \\ &= \frac{\Phi_X\left(\int_{E_n} f_\tau d\mu\right)}{\Phi_X(\mu(E_n))} \quad \text{by Corollary 1.2.16} \\ (8.1.5) \quad &\geq \frac{\Phi_X(\epsilon\mu(E_n))}{\Phi_X(\mu(E_n))} \quad \text{since } f_\tau(x) \geq \epsilon \forall x \in A \supset E_n \text{ and } \Phi_X \text{ is increasing} \\ &= \frac{\Phi_X(\epsilon\mu(E_n))/\epsilon\mu(E_n)}{\Phi_X(\mu(E_n))/\epsilon\mu(E_n)} \\ &\geq \frac{\Phi_X(\mu(E_n))/\mu(E_n)}{\Phi_X(\mu(E_n))/\epsilon\mu(E_n)} \quad \text{by (8.1.1)} \\ &= \epsilon, \end{aligned}$$

It follows that $(C_\tau f_n)$ does not have a Cauchy subsequence and hence that C_τ is not compact by Proposition 1.5.8. \square

REMARK 8.1.2. This is a slight modification of the result appearing in the literature. The result in [21] required a Banach function space which has a concave fundamental function.

Let $A, B \in \Sigma$. Recall that $[A] = [B]$ if and only if $\mu(A \Delta B) = 0$ (see Remark 1.1.8) and that a weight function w is regular if and only if there exists a $k > 0$ such that

$$tw(t) \leq \Gamma(t) \leq ktw(t) \quad \forall t \in I \quad (\text{Remark 2.2.24})$$

where $I = [0, \mu(\Omega)]$ if $\mu(\Omega) < \infty$ and $I = [0, \infty)$ if $\mu(\Omega) = \infty$. In the case of Orlicz-Lorentz spaces we have the following result.

THEOREM 8.1.3. (cf. [4, p.1779], [22, p.83]) *Let ϕ be an Orlicz function satisfying the Δ_2 -condition globally. Let $\tau : \Omega \rightarrow \Omega$ be a nonsingular measurable transformation and let w be a weight function which is either regular or bounded away from zero. If C_τ is a compact operator from $L^{\phi, w}(\mu)$ into itself, then for every $\epsilon > 0$, there exist a finite number of atoms $A_1^\epsilon, A_2^\epsilon, \dots, A_{n_\epsilon}^\epsilon$ of strictly positive finite measure and a set $B_\epsilon \subset \Omega_c$ with $\mu(B_\epsilon) = 0$ such that*

$$(8.1.6) \quad N_\epsilon := \{x \in \Omega : f_\tau(x) > \epsilon\} = \left(\bigcup_{k=1}^{n_\epsilon} A_k^\epsilon \right) \cup B_\epsilon$$

and therefore $[\Omega] = [\tau^{-1}(\Omega_a)]$ and $\lim_{n \rightarrow \infty} b_n = 0$.

PROOF. We show the contrapositive. If (8.1.6) does not hold for some $\epsilon > 0$, then either there exists a $B \subset \Omega_c$ with $\mu(B) > 0$ such that $B \subset N_\epsilon$ or there exists a sequence $(A_n^\epsilon)_{n=1}^\infty$ of atoms of strictly positive finite measure such that $\bigcup_{n=1}^\infty A_n^\epsilon \subset N_\epsilon$, or both. It is possible in all cases to obtain a sequence $(B_n)_{n=1}^\infty$ of pairwise disjoint measurable sets of strictly positive finite measure such that

$$(8.1.7) \quad \begin{aligned} \mu \circ \tau^{-1}(B_n) &= \int_{B_n} f_\tau d\mu \quad \text{by Corollary 1.2.16} \\ &\geq \epsilon \mu(B_n) \quad \forall n \in \mathbb{N}^+ \end{aligned}$$

For each n in \mathbb{N}^+ , define

$$(8.1.8) \quad \begin{aligned} \alpha_n &:= 1/\Phi_{L\phi, w}(\mu(B_n)) \\ &= \phi^{-1}(1/\Gamma(\mu(B_n))) \quad \text{by Example 4.2.7} \end{aligned}$$

and $f_n := \alpha_n \chi_{B_n}$. Then it is easily checked that $\|f_n\|_{\phi, w} = 1$ for each n . We therefore have that $(f_n)_{n=1}^\infty \subset B_{L\phi, w}$. Let $m, n \in \mathbb{N}^+$ such that $n \neq m$ and assume that $\alpha_m \leq \alpha_n$. Then

$$\begin{aligned} I_{\phi, w}(C_\tau f_n - C_\tau f_m) &= \int_0^{\mu(\Omega)} \phi((C_\tau f_n - C_\tau f_m)^*(t)) w(t) dt \\ &= \int_0^{\mu(\Omega)} \phi\left(\alpha_n \chi_{[0, \mu \circ \tau^{-1}(B_n))}(t) + \alpha_m \chi_{[\mu \circ \tau^{-1}(B_n), \mu \circ \tau^{-1}(B_n \cup B_m))}(t)\right) w(t) dt \\ &\quad \text{by Proposition 7.2.3} \\ &\geq \int_0^{\mu \circ \tau^{-1}(B_n)} \phi(\alpha_n) w(t) dt \quad \text{since } \phi \text{ is increasing and } w \geq 0 \\ &\geq \phi(\alpha_n) \int_0^{\epsilon \mu(B_n)} w(t) dt \quad \text{by (8.1.7) and using } w \geq 0 \\ &\geq \phi(\phi^{-1}(1/\Gamma(\mu(B_n)))) [\epsilon \mu(B_n) w(\epsilon \mu(B_n))] \\ &\quad \text{using (8.1.8) for } \alpha_n \text{ and } w \text{ decreasing for the integral} \\ &= \frac{\epsilon \mu(B_n) w(\epsilon \mu(B_n))}{\Gamma(\mu(B_n))} \quad \text{since } \phi \text{ is invertible by Proposition 2.2.13.} \end{aligned}$$

If w is bounded away from zero, there exists an $\alpha > 0$ such that $w(t) \geq \alpha$ for all $t > 0$. It follows that

$$\begin{aligned} \frac{\epsilon \mu(B_n) w(\epsilon \mu(B_n))}{\Gamma(\mu(B_n))} &\geq \frac{\epsilon \mu(B_n) w(\epsilon \mu(B_n))}{\mu(B_n) w(0)} \quad \text{since } w \text{ is decreasing} \\ &\geq \frac{\epsilon \alpha}{w(0)} \quad \text{since } w \text{ is bounded below by } \alpha \end{aligned}$$

If w is a regular weight function then by Remark 2.2.24, there exists a $k > 0$ such that

$$(8.1.9) \quad \Gamma(t) \leq ktw(t) \quad \forall t \in I$$

We therefore have that

$$\begin{aligned} \frac{\epsilon \mu(B_n) w(\epsilon \mu(B_n))}{\Gamma(\mu(B_n))} &\geq \frac{\epsilon \mu(B_n) w(\epsilon \mu(B_n))}{k \mu(B_n) w(\mu(B_n))} \quad \text{using (8.1.9)} \\ &\geq \frac{\epsilon w(\mu(B_n))}{k w(\mu(B_n))} \quad \text{since } w \text{ is decreasing} \\ &= \frac{\epsilon}{k} \end{aligned}$$

It follows that if w is either regular or bounded away from zero then there exists a $\beta > 0$ such that

$$I_{\phi, w}(C_\tau f_n - C_\tau f_m) \geq \beta$$

Therefore $\|C_\tau f_n - C_\tau f_m\| \geq \min\{1, \beta\}$ by Lemma 2.2.34 and hence $(C_\tau(f_n))_{n=1}^\infty$ does not have a Cauchy subsequence. Thus C_τ is not compact by Proposition 1.5.8. We therefore have that for any $\epsilon > 0$, N_ϵ consists of the union of finitely many atoms and a nonatomic set of measure zero. In particular for every $n \in \mathbb{N}^+$ there exist a finite number of atoms and a $B_n \subset \Omega_c$ with $\mu(B_n) = 0$ such that

$$N_{1/n} = \left(\bigcup_{n=1}^{k_n} A_n \right) \cup B_n$$

Since $\{x \in \Omega_c : f_\tau(x) > 0\} = \bigcup_{n=1}^\infty B_n$, it follows that

$$\begin{aligned} \mu(\{x \in \Omega_c : f_\tau(x) > 0\}) &\leq \sum_{n=1}^\infty \mu(B_n) \\ (8.1.10) \qquad \qquad \qquad &= 0 \quad \text{since } \mu(B_n) = 0 \quad \forall n \end{aligned}$$

We therefore have that

$$\begin{aligned} \mu \circ \tau^{-1}(\Omega_c) &= \int_{\Omega_c} f_\tau d\mu \quad \text{by Corollary 1.2.16} \\ &= 0 \end{aligned}$$

It follows that

$$\begin{aligned} [\Omega] &= [\tau^{-1}(\Omega)] \\ &= [\tau^{-1}(\Omega_c)] \cup [\tau^{-1}(\Omega_a)] \\ &= [\tau^{-1}(\Omega_a)] \quad \text{since } \mu \circ \tau^{-1}(\Omega_c) = 0 \end{aligned}$$

Furthermore, since $\mu \circ \tau^{-1}(A_n) = \int_{A_n} f_\tau d\mu = f_\tau(A_n)\mu(A_n)$ for any atom, we have that

$$\begin{aligned} b_n &= \frac{\mu \circ \tau^{-1}(A_n)}{\mu(A_n)} \\ (8.1.11) \qquad \qquad \qquad &= f_\tau(A_n) \end{aligned}$$

Let $\epsilon > 0$. Since (8.1.6) holds, there exists an $n_\epsilon \in \mathbb{N}^+$ such that $n \geq n_\epsilon$ implies $A_n \notin N_{\epsilon/2}$ and therefore $f_\tau(A_n) \leq \epsilon/2 < \epsilon$ for $n \geq n_\epsilon$. Using (8.1.11) it follows that $\lim_{n \rightarrow \infty} b_n = 0$. \square

REMARK 8.1.4. We have imposed restrictions on the Orlicz function and the weight function which do not form part of the original result ([4, Theorem 2.1]). Furthermore, it is claimed in [4] that the converse also holds. As part of the proof of this assertion it is claimed that

$$\mu_g(\lambda) \leq \epsilon \mu_f(\lambda) \quad \forall \lambda > 0$$

implies that

$$g^*(t) \leq \epsilon f^*(t) \quad \forall t \geq 0$$

This need not be the case (consider $g = \chi_{[0, \epsilon]}$ and $f = \chi_{[0, 1]}$). In general all that can be said is that $g^*(t) \leq f^*(t/\epsilon)$ for all $t > 0$.

REMARK 8.1.5. This is a significantly weaker result than [22, Theorem 3.1]. It is claimed there that the compactness of a composition operator on any Orlicz-Lorentz space implies that the underlying measure space is purely atomic. The argument employed in [22] is to show that the sequence $(f_n)_{n=1}^\infty$, as defined in the proof above, converges weakly to the zero function and then to show that $(C_\tau f_n)_{n=1}^\infty$ does not converge to zero strongly. This would contradict the compactness of C_τ (see Proposition 1.5.9). The claim that $f_n \rightarrow 0$ weakly may however not be true. Consider the case $\phi(t) = t$ and $w \equiv 1$, i.e. $L^{\phi, w}(\mu) = L^1(\mu)$ with equality of norms.

Then $(L^1(\mu))^* = L^\infty(\mu)$ and $\Phi_{L^1}(t) = t$. So if f_n is defined as in [22, Theorem 3.1] then $f_n = (\mu(B_n))^{-1} \chi_{B_n}$. Furthermore, letting $B = \bigcup_{n=1}^{\infty} B_n$ we obtain $\chi_B \in L^\infty(\mu)$, but

$$\begin{aligned} \int_{\Omega} \chi_B f_n d\mu &= (\mu(B_n))^{-1} \mu(B_n) \quad \text{since } f_n = (\mu(B_n))^{-1} \chi_{B_n} \\ &= 1 \quad \forall n \in \mathbb{N}^+ \end{aligned}$$

and so $f_n \rightharpoonup 0$ weakly. Furthermore, Theorems 8.1.8 and 8.1.12 exhibit compact composition operators on rearrangement invariant spaces over measure spaces which are not purely atomic. The proof I have employed above is a modification of the proof given in [5, Theorem 3.1] for Orlicz spaces.

If, however, $X = L^{\phi,w}(\mu)$ and X' both have absolutely continuous norm then one can show that $f_n \rightarrow 0$ weakly as follows: X has absolutely continuous norm and therefore X^* is isometrically isomorphic to X' , by Corollary 3.2.8. Therefore if $\phi \in X^*$ then $\phi = \phi_g$ for some $g \in X'$ where

$$\phi_g(f) := \int_{\Omega} fg d\mu \quad \forall f \in X$$

We note that $f_n g = \|\chi_{B_n}\|_X^{-1} \chi_{B_n} g = \|\chi_{B_n}\|_X^{-1} \chi_{B_n} g \chi_{B_n}$ since $\chi_{B_n} \cdot \chi_{B_n} = \chi_{B_n}$ and furthermore $g \chi_{B_n} \in X'$ since $g \in X'$ and $|g \chi_{B_n}| \leq |g|$. Therefore

$$\begin{aligned} |\phi_g(f_n)| &= \left| \int_{\Omega} f_n g \chi_{B_n} d\mu \right| \\ &\leq \int_{\Omega} |f_n g \chi_{B_n}| d\mu \\ &\leq \|f_n\|_X \|g \chi_{B_n}\|_{X'} \quad \text{by Hölder's inequality (Theorem 3.1.13)} \\ &\rightarrow 0 \quad \text{since } \|f_n\|_X = 1, X' \text{ has absolutely continuous norm and } B_n \downarrow \emptyset \end{aligned}$$

Continuing with the proof as in [22, Theorem 3.1], one obtains

$$\|C_{\tau} f_n\|_{\phi,w} \geq \frac{\phi^{-1}(1/\Gamma(\mu(B_n)))}{\phi^{-1}(1/\Gamma(\epsilon\mu(B_n)))}$$

I am unable to show that this last quantity is bounded away from zero, unless w satisfies the same extra conditions as in Theorem 8.1.3.

Since $L^{\phi,w}(\mu) = L^{\phi}(\mu)$ with equality of norms if $w \equiv 1$ (Example 2.2.38), the same result holds for Orlicz spaces where the Orlicz function satisfies the Δ_2 -condition globally. Furthermore, $L^{p,q}(\mu) = L^{\phi,w}(\mu)$ with equality of norms if $\phi(t) = t^q$ and $w(t) = t^{\frac{q}{p}-1}$ (Example 2.2.28). It follows by Remark 2.2.29 that Theorem 8.1.3 also holds for Lorentz spaces.

REMARK 8.1.6. In [21, Theorem 3.1] it is claimed that the compactness of the composition operator on a Lorentz space implies that the underlying measure space is purely atomic. It uses the argument that $\mu(E) = 0$ (where $E = \{x \in \Omega_c : f_{\tau}(x) \neq 0\}$) implies

$$(8.1.12) \quad \mu \circ \tau^{-1}(\Omega_c) = 0 = \int_{\Omega_c} f_{\tau} d\mu$$

$$(8.1.13) \quad \geq \epsilon^p \mu(\Omega_c)$$

The equalities follow since $\mu(E) = 0$ implies $f_{\tau} = 0$ μ -a.e. on Ω_c and from (8.1.12) it can be deduced that $[\Omega] = [\tau^{-1}(\Omega_a)]$, as in the proof of Theorem 8.1.15. However, the inequality (8.1.13) seems to be flawed, since $E_{\epsilon} := \{x \in \Omega_c : f_{\tau}(x) \geq \epsilon^p\} \subset E$ (and hence $\mu(E_{\epsilon}) = 0$) implies that $f_{\tau} < \epsilon^p$ μ -a.e. on Ω_c and therefore

$$\int_{\Omega_c} f_{\tau} d\mu \leq \epsilon^p \mu(\Omega_c)$$

It is not clear from [21] how the reverse inequality might be obtained.

REMARK 8.1.7. Since $L^{p,p}(\mu) = L^p(\mu)$ ($1 \leq p < \infty$) with equality of norms (Example 2.2.2), the above result holds for Lebesgue spaces.

We give two sets of examples to show that the compactness of a composition operator does not necessarily imply that the underlying measure space is purely atomic.

THEOREM 8.1.8. (cf. [36, p.314]) *Let (Ω, Σ, μ) be a finite measure space with the property that the atomic part Ω_a consists of finitely many atoms of finite measure. Let X be a Banach function space over (Ω, Σ, μ) and let $\tau : \Omega \rightarrow \Omega$ be a nonsingular measurable transformation. If $[\Omega] = [\tau^{-1}(\Omega_a)]$, then C_τ is a finite rank operator and hence compact.*

PROOF. Let A_1, \dots, A_n be the atoms of Ω and let $f \in X$. $[\Omega] = [\tau^{-1}(\Omega_a)]$, so $\Omega = \tau^{-1}(\Omega_a) \cup A$, where $\mu(A) = 0$ and $A \subset \tau^{-1}(\Omega_c)$. So if $x \in \Omega$, then either $\tau(x) \in \Omega_a$ or $x \in A$. If $\tau(x) \in \Omega_a$, then

$$\begin{aligned} (C_\tau f)(x) &= f(\tau(x)) \\ &= f(A_k) \quad \text{since } \tau(x) \in A_k \text{ for some } k \end{aligned}$$

If $x \in A$, then $\tau(x) = \tilde{x}$ for some $\tilde{x} \in \Omega_c$. We therefore have that $C_\tau f(x) = \sum_{k=1}^n f(A_k) \chi_{\tau^{-1}(A_k)}(x) + f(\tilde{x}) \chi_A$ and hence $[C_\tau f] = [\sum_{k=1}^n f(A_k) \chi_{\tau^{-1}(A_k)}]$, since $\mu(A) = 0$. It follows that $\mathcal{B} = \{\chi_{\tau^{-1}(A_k)} : 1 \leq k \leq n\}$ is a basis for $\text{ran}(C_\tau)$ and hence that C_τ is a finite rank operator. We have used the fact that (Ω, Σ, μ) is a finite measure space to ensure that $\mu(\tau^{-1}(A_k)) < \infty$ for each k and hence that $\mathcal{B} \subset X$. Since $C_\tau f$ is a linear combination of elements from \mathcal{B} for each $f \in X$, this also ensures that $C_\tau(X) \subset X$, i.e. that C_τ is a composition operator. \square

REMARK 8.1.9. The original theorem ([36, Theorem 2.10]) is a characterization of compact operators on L^2 -spaces over this particular class of measure spaces. Their claim regarding the necessary condition should be adjusted however to include the possibility $\Omega = \tau^{-1}(\Omega_a) \cup A$ where $\mu(A) = 0$ and $A \subset \Omega_c$. The theorem as stated above is therefore a generalization of the sufficient condition to arbitrary Banach function spaces. A generalization of the necessary condition to a particular class of Orlicz-Lorentz spaces follows from Theorem 8.1.3.

We give an example to demonstrate the theorem above.

EXAMPLE 8.1.10. (cf. [36, p.314]) Let $n \in \mathbb{N}^+$ and $\Omega = [-n, 0] \cup \{1, 2, \dots, n\}$. Let μ be the Lebesgue measure on $[-n, 0]$ and the counting measure on $\{1, 2, \dots, n\}$. Then Ω is a finite measure space with finitely many atoms of finite measure. Let X be a rearrangement invariant Banach function space over (Ω, Σ, μ) and

$$\tau(x) := \begin{cases} 1 & \text{if } x \in [-n, 0] \cup \{n\} \\ x+1 & \text{if } x \in \{1, 2, \dots, n-1\} \end{cases}$$

τ is measurable: Let A be a measurable subset of Ω . Then $A = B \cup C$, where B is a measurable subset of $[-n, 0]$ and $C \subset \{1, 2, \dots, n\}$. It follows that

$$\begin{aligned} \tau^{-1}(A) &= \tau^{-1}(B) \cup \tau^{-1}(C) \\ &= \emptyset \cup \tau^{-1}(C) \\ &= \begin{cases} [-n, 0] \cup ((C \setminus \{1\}) - 1) \cup \{n\} & \text{if } 1 \in C \\ C - 1 & \text{if } 1 \notin C \end{cases} \end{aligned}$$

which yields a measurable set in either case. τ is nonsingular: Let $A \subset \Omega$ be such that $\mu(A) = 0$, then $A = \emptyset$ or $A \subset [-n, 0]$. In both cases we have that $\tau^{-1}(A) = \emptyset$ and hence $\mu \circ \tau^{-1}(A) = 0$. Furthermore, $\tau^{-1}(\{1, 2, \dots, n\}) = \Omega$ and so C_τ is a finite rank operator on X by Theorem 8.1.8.

REMARK 8.1.11. The original example ([36, Example 2.5]) is given for $\Omega = [0, 1] \cup \{2, 3\}$.

The next theorem due to Singh ([36, Theorem 2.9]) demonstrates another class of σ -finite measure spaces which are not purely atomic, but over which L^p -spaces ($1 \leq p < \infty$) may admit compact composition operators.

THEOREM 8.1.12. (cf. [36, p.313]) *Let (Ω, Σ, μ) be a σ -finite measure space with atoms $(A_n)_{n=1}^\infty$ of finite non-zero measure. If $\sup \{\mu(A_n) : n \in \mathbb{N}^+\} = \infty$, then $L^p(\mu)$ ($1 \leq p < \infty$) admits a compact composition operator.*

PROOF. Let $\Omega = \Omega_c \cup \Omega_a$ be the usual decomposition of Ω . Since Ω is σ -finite and Ω_c is nonatomic, we can write $\Omega_c = \bigcup_{n=1}^\infty B_n$, where $B_n \neq \emptyset$, $\mu(B_n) \leq 1$ for each n and $B_n \cap B_m = \emptyset$ if $n \neq m$. We define $\tau : \Omega \rightarrow \Omega$ inductively. Since $\sup \{\mu(A_n) : n \in \mathbb{N}^+\} = \infty$, we can find a $k_1 \in \mathbb{N}^+$ such that

$$\mu(A_{k_1}) > 1(\mu(A_1) + 1)$$

Let $\tau(A_1) = A_{k_1}$. Similarly, we can find a $k_2 > k_1$ such that

$$\mu(A_{k_2}) > 2(\mu(A_2) + 1)$$

Let $\tau(A_2) = A_{k_2}$. Continuing in this fashion we obtain a strictly increasing sequence of k_n 's such that

$$(8.1.14) \quad \mu(\tau(A_n)) = \mu(A_{k_n}) > n(\mu(A_n) + 1)$$

For each n , choose $y_n \in B_n$. To define τ on the nonatomic part, let

$$\tau(x) = \begin{cases} A_{k_n} & x \in B_n \setminus \{y_n\} \\ y_n & x = y_n \end{cases}$$

Then $\tau^{-1}(\Omega_c) = \bigcup_{n=1}^\infty \{y_n\}$ and therefore $\mu \circ \tau^{-1}(\Omega_c) = \sum_{n=1}^\infty \mu(\{y_n\}) = 0$. By Proposition 1.2.17, $f_\tau = 0$ μ -a.e. on Ω_c and $f_\tau(A_m) = \frac{\mu \circ \tau^{-1}(A_m)}{\mu(A_m)}$ for every atom. Therefore, if $\tau^{-1}(A_m) = \emptyset$, then $f_\tau(A_m) = 0$. If $\tau^{-1}(A_m) \neq \emptyset$, i.e. $m = k_n$ for some n , then $\tau^{-1}(A_m) = A_n \cup (B_n \setminus \{y_n\})$ and so

$$\begin{aligned} \mu \circ \tau^{-1}(A_m) &= \mu(A_n \cup (B_n \setminus \{y_n\})) \\ &= \mu(A_n) + \mu(B_n \setminus \{y_n\}) \quad \text{since } A_n \cap B_n = \emptyset \\ &\leq \mu(A_n) + 1 \\ &< \frac{1}{n} \mu(A_{k_n}) \\ &= \frac{1}{n} \mu(A_m) \end{aligned}$$

Therefore $f_\tau(A_m) < \frac{1}{n}$, where $m = k_n$. Let $\epsilon > 0$. There exists an $N_\epsilon \in \mathbb{N}^+$ such that $\frac{1}{N_\epsilon} < \epsilon$. Let $M_\epsilon = k_{N_\epsilon}$, then $m > M_\epsilon$ implies that $f_\tau(A_m) = 0$ if $m \neq k_n$ for any n and $f_\tau(A_m) < \frac{1}{N_\epsilon} < \epsilon$ if $m = k_n$ for some n , since $(k_n)_{n=1}^\infty$ strictly increasing and $m = k_n > k_{N_\epsilon}$ implies that $n > N_\epsilon$. Therefore $f_\tau(A_m) \rightarrow 0$ and so $f_\tau^{1/p}(A_m) \rightarrow 0$. It follows that $M_{f_\tau^{1/p}}$ is compact by Corollary 6.2.3 and hence C_τ is compact by Lemma 6.2.2. \square

REMARK 8.1.13. This is a slight generalization of [36, Theorem 2.9], which is given for L^2 -spaces. Furthermore, the theorem exhibits another compact composition operator C_τ with the property that $\tau^{-1}(\Omega_c) \neq \emptyset$ and hence illustrates that the necessary condition “[Ω] = [$\tau^{-1}(\Omega_a)$]” in Theorem 8.1.3 cannot be replaced with “[Ω] = [$\tau^{-1}(\Omega_a)$]”.

We give an example to illustrate the theorem above.

EXAMPLE 8.1.14. (cf. [36, p.314]) Let $\Omega = (-\infty, 0] \cup \mathbb{N}^+$ and let μ be the Lebesgue measure on $(-\infty, 0]$, and

$$\mu(\{n\}) = \begin{cases} n^2 & \text{if } n \text{ is an even natural number} \\ 1 & \text{if } n \text{ is an odd natural number} \end{cases}$$

For $A \subset \mathbb{N}^+$, we define $\mu(A) := \sum_{k \in A} \mu(\{k\})$. If $A \subset \Omega$, then we can write $A = B \cup C$ where $B \subset (-\infty, 0]$ and $C \subset \mathbb{N}^+$. A will be called measurable if B is a Lebesgue measurable subset of $(-\infty, 0]$ and we will define $\mu(A) := \mu(B) + \mu(C)$. It is easily checked that μ is a positive measure.

Let $n \in \mathbb{N}^+$ and define

$$\tau(x) = \begin{cases} 2n & -n < x \leq -n + 1 \\ n^2 & \text{if } x = n \text{ and } n \text{ even} \\ (n+1)^2 & \text{if } x = n \text{ and } n \text{ odd} \end{cases}$$

It is clear that $\tau^{-1}(\mathbb{N}^+) = \Omega$. τ is measurable: Let A be a measurable subset of Ω , then

$$\tau^{-1}(A) = \tau^{-1}(C)$$

for some $C \subset \mathbb{N}^+$. Since $\tau^{-1}(C)$ is a countable union of subsets of the form $(-n, -n + 1]$ and elements of \mathbb{N}^+ , it is measurable as a countable union of measurable sets. τ is nonsingular: let A be a measurable subset such that $\mu(A) = 0$, then $A = \emptyset$ or $A \subset (-\infty, 0]$. In both cases, $\tau^{-1}(A) = \emptyset$ and so $\mu \circ \tau^{-1}(A) = 0$.

Furthermore $\tau^{-1}(\{n\}) = \emptyset$, if n is an odd natural number; $\tau^{-1}(\{n\}) = (\frac{-n}{2}, \frac{-n+2}{2}] \cup \{\sqrt{n}, \sqrt{n} - 1\}$, if n is an even natural number and a square of an even number; and $\tau^{-1}(\{n\}) = (\frac{-n}{2}, \frac{-n+2}{2}]$ if n is an even number but not a square of an even number. It follows that $f_\tau(\{n\}) = \frac{\mu \circ \tau^{-1}(\{n\})}{\mu(\{n\})} \leq \frac{1+n+1}{n^2} \rightarrow 0$ and therefore it can be shown as in Theorem 8.1.12 that C_τ is compact on $L^p(\mu)$.

For the remainder of the section the focus will be Banach function spaces over purely atomic σ -finite measure spaces. The following is a generalization of [22, Theorem 3.1], which is originally given for Orlicz-Lorentz spaces.

THEOREM 8.1.15. (cf. [22, p.83]) Let X be a rearrangement invariant Banach function space over a purely atomic σ -finite measure space (Ω, Σ, μ) and let $\tau : \Omega \rightarrow \Omega$ be a measurable nonsingular transformation. If C_τ is a compact composition operator on X , then $\lim_{n \rightarrow \infty} c_n = 0$.

PROOF. We show the contrapositive. Let $c_n \not\rightarrow 0$. Then there exists an $\epsilon > 0$ such that for each $k \in \mathbb{N}^+$, there exists an $n_k \geq k$ with the property $c_{n_k} \geq \epsilon$. For each $k \in \mathbb{N}^+$ set

$$f_k = \frac{\chi_{A_{n_k}}}{\Phi_X(\mu \circ \tau^{-1}(A_{n_k}))}$$

Then

$$\begin{aligned} \|f_k\| &= \frac{\Phi_X(\mu(A_{n_k}))}{\Phi_X(\mu \circ \tau^{-1}(A_{n_k}))} \\ &= \frac{1}{c_{n_k}} \\ &\leq \frac{1}{\epsilon} \quad \forall k \in \mathbb{N}^+ \end{aligned}$$

So $f_k \in \frac{1}{\epsilon} B_X$ for each $k \in \mathbb{N}^+$. Let $k, m \in \mathbb{N}^+$ with $k \neq m$. Then

$$\begin{aligned}
\|C_\tau f_k - C_\tau f_m\| &= \left\| \frac{\chi_{\tau^{-1}(A_{n_k})}}{\Phi_X(\mu \circ \tau^{-1}(A_{n_k}))} - \frac{\chi_{\tau^{-1}(A_{n_m})}}{\Phi_X(\mu \circ \tau^{-1}(A_{n_m}))} \right\| \\
&= \left\| \frac{\chi_{\tau^{-1}(A_{n_k})}}{\Phi_X(\mu \circ \tau^{-1}(A_{n_k}))} + \frac{\chi_{\tau^{-1}(A_{n_m})}}{\Phi_X(\mu \circ \tau^{-1}(A_{n_m}))} \right\| \quad \text{by Proposition 3.1.4(6)} \\
&\geq \left\| \frac{\chi_{\tau^{-1}(A_{n_k})}}{\Phi_X(\mu \circ \tau^{-1}(A_{n_k}))} \right\| \\
&= \frac{\Phi_X(\mu \circ \tau^{-1}(A_{n_k}))}{\Phi_X(\mu \circ \tau^{-1}(A_{n_k}))} \\
&= 1
\end{aligned}$$

Therefore $(C_\tau f_k)_{k=1}^\infty$ does not have a Cauchy subsequence and hence C_τ is not compact by Proposition 1.5.8. \square

If (Ω, Σ, μ) is a purely atomic σ -finite measure space which is resonant, i.e. all atoms have equal measure (Theorem 1.3.9), then we have the following.

THEOREM 8.1.16. *Let X be a rearrangement invariant Banach function space over a σ -finite purely atomic measure space (Ω, Σ, μ) with atoms $(A_n)_{n=1}^\infty$ of equal measure. Let $\tau : \Omega \rightarrow \Omega$ be a nonsingular measurable transformation. If C_τ is a compact operator from X into itself, then $\mu \circ \tau^{-1}(A_n) = 0$ except for finitely many $n \in \mathbb{N}^+$.*

PROOF. We show the contrapositive. Assume that there are infinitely many $n \in \mathbb{N}^+$ such that $\mu \circ \tau^{-1}(A_n) > 0$, i.e. there exists a subsequence $(A_{n_k})_{k=1}^\infty$ of $(A_n)_{n=1}^\infty$ such that $\mu \circ \tau^{-1}(A_{n_k}) > 0$ for each $k \in \mathbb{N}^+$. All the atoms have equal measure; let α be the measure of an atom. Since $\mu \circ \tau^{-1}(A_{n_k}) > 0$, we have that $\tau^{-1}(A_{n_k}) \neq \emptyset$. This implies that $\tau^{-1}(A_{n_k})$ can be written as a union of atoms and hence $\mu \circ \tau^{-1}(A_{n_k}) \geq \alpha$ for each $k \in \mathbb{N}^+$. It follows that

$$b_{n_k} := \frac{\mu \circ \tau^{-1}(A_{n_k})}{\mu(A_{n_k})} \geq 1 \quad \forall k \in \mathbb{N}^+$$

For each $k \in \mathbb{N}^+$ set

$$f_k := \frac{\chi_{A_{n_k}}}{\|\chi_{A_{n_k}}\|}$$

Then $\|f_k\| = 1$ for each k and

$$C_\tau f_k = \frac{\chi_{\tau^{-1}(A_{n_k})}}{\|\chi_{A_{n_k}}\|}$$

Let $k, m \in \mathbb{N}^+$, with $k \neq m$, then

$$\begin{aligned}
\|C_\tau f_k - C_\tau f_m\|_X &\geq \frac{\Phi_X(\mu \circ \tau^{-1}(A_{n_k}))}{\Phi_X(\mu(A_{n_k}))} \quad \text{as in Theorem 8.1.1 (8.1.4)} \\
&= \frac{\Phi_X(b_{n_k} \mu(A_{n_k}))}{\Phi_X(\mu(A_{n_k}))} \quad \text{since } b_{n_k} \mu(A_{n_k}) = \mu \circ \tau^{-1}(A_{n_k}) \text{ and } \tau \text{ measurable implies} \\
&\quad \tau^{-1}(A_{n_k}) \in \Sigma, \text{ i.e. } b_{n_k} \mu(A_{n_k}) \in \text{ran } \mu \\
&\geq \frac{\Phi_X(\mu(A_{n_k}))}{\Phi_X(\mu(A_{n_k}))} \quad \text{since } b_{n_k} \geq 1 \text{ and } \Phi_X \text{ is increasing} \\
&= 1
\end{aligned}$$

$(C_\tau f_k)_{k=1}^\infty$ therefore does not have a Cauchy subsequence and thus C_τ is not compact by Proposition 1.5.8. \square

REMARK 8.1.17. Since (Ω, Σ, μ) is purely atomic and all atoms have the same measure, we have that the condition “ $\mu \circ \tau^{-1}(A_n) = 0$, except for finitely many $n \in \mathbb{N}^+$ ” is equivalent to $\lim_{n \rightarrow \infty} b_n \rightarrow 0$.

We show that for specific Banach function spaces over more general purely atomic σ -finite measure spaces, the condition $\lim_{n \rightarrow \infty} b_n \rightarrow 0$ is necessary for the composition operator to be compact, provided the underlying

measure space is purely atomic. The desired result for Orlicz-Lorentz spaces, where the Orlicz function satisfies the Δ_2 -condition globally and the weight function is either regular or bounded away from zero, follows from Theorem 8.1.3.

We show that a similar result can be obtained for Orlicz spaces by relaxing the condition on the Orlicz function somewhat and placing a restriction on the underlying measure space.

THEOREM 8.1.18. [5, p. 204] *Let (Ω, Σ, μ) be a purely atomic σ -finite measure space with atoms of measure $\mu(A_n) = a_n > 0$ with the additional property that $\liminf \mu(A_n) = \alpha > 0$. Let ϕ be an Orlicz function with $a_\phi = 0$, $b_\phi < \infty$ and $\varphi(b_\phi) = \beta < \infty$, where φ is the left derivative of ϕ . If $\tau : \Omega \rightarrow \Omega$ is a nonsingular measurable transformation such that C_τ is a compact operator from $L^\phi(\mu)$ into itself, then $\lim_{n \rightarrow \infty} b_n = 0$.*

PROOF. Let

$$\phi_1(u) := \begin{cases} \phi(u) & \text{if } 0 \leq u \leq b_\phi \\ \beta u + \phi(b_\phi) - \beta b_\phi & \text{if } u > b_\phi \end{cases}$$

and

$$\phi_2(u) := \begin{cases} 0 & \text{if } 0 \leq u \leq b_\phi \\ \infty & \text{if } u > b_\phi \end{cases}$$

Then by Proposition 5.1.7 $L^\phi(\mu) = (L^{\phi_1} \cap L^{\phi_2})(\mu)$ and $\|f\|_\phi = \max\{\|f\|_{\phi_1}, \|f\|_{\phi_2}\}$ for all f in $L^\phi(\mu)$. Furthermore $a_{\phi_1} = a_\phi = 0$ and $b_{\phi_1} = \infty$ since ϕ_1 is finite valued. Therefore ϕ_1 is invertible.

Let C_τ be a compact operator and assume that $b_n \rightarrow 0$. This implies that there exists an $\epsilon > 0$ and a subsequence $(b_{n_k})_{k=1}^\infty$ such that $b_{n_k} \geq \epsilon$ for every $k \in \mathbb{N}^+$. Furthermore, $\liminf_{n \rightarrow \infty} \mu(A_n) = \alpha$ implies that $\liminf_{k \rightarrow \infty} \mu(A_{n_k}) \geq \alpha$ and so there exists a subsequence $(A_{n_{k_m}})_{m=1}^\infty \subset (A_{n_k})_{k=1}^\infty$ such that $\mu(A_{n_{k_m}}) \geq \alpha/2$ for all $m \in \mathbb{N}^+$. We can therefore assume, without loss of generality, that $b_n \geq \epsilon$ and $a_n = \mu(A_n) \geq \alpha/2$ for all $n \in \mathbb{N}^+$. For each $n \in \mathbb{N}^+$ define

$$f_n := \min\{b_\phi, \phi_1^{-1}(1/a_n)\} \chi_{A_n}$$

Then

$$\begin{aligned} I_\phi(f_n) &= \int_\Omega \phi(|f_n(x)|) d\mu \\ &= \int_\Omega \phi_1(f_n(x)) d\mu \quad \text{since } 0 \leq f_n(x) \leq b_\phi \quad \forall x \in \Omega \\ &= \phi_1(\min\{b_\phi, \phi_1^{-1}(1/a_n)\}) \mu(A_n) \\ &\leq \phi_1(\phi_1^{-1}(1/a_n)) a_n \\ &= (1/a_n) a_n \quad \text{since } \phi_1 \text{ is invertible} \\ &= 1 \end{aligned}$$

Therefore $\|f_n\|_\phi \leq 1$ for all $n \in \mathbb{N}^+$, by Corollary 2.2.33 (and using the fact that every Orlicz space is an Orlicz-Lorentz space). Let $\beta_k := \min\{b_\phi, \phi_1^{-1}(1/a_k)\}$ and let $m, n \in \mathbb{N}^+$ such that $m \neq n$, then

$$\begin{aligned}
I_\phi(C_\tau f_m - C_\tau f_n) &= \int_\Omega \phi \left(|\beta_m \chi_{\tau^{-1}(A_m)} - \beta_n \chi_{\tau^{-1}(A_n)}| \right) d\mu \\
&= \int_{\tau^{-1}(A_m)} \phi(\beta_m) d\mu + \int_{\tau^{-1}(A_n)} \phi(\beta_n) d\mu \quad \text{since } \tau^{-1}(A_m) \cap \tau^{-1}(A_n) = \emptyset \\
&= \phi_1(\beta_m) \mu \circ \tau^{-1}(A_m) + \phi_1(\beta_n) \mu \circ \tau^{-1}(A_n) \quad \text{since } \beta_k \leq b_\phi \forall k \in \mathbb{N}^+ \\
&= \min \{ \phi_1(b_\phi), \phi_1(\phi_1^{-1}(1/a_m)) \} b_m \mu(A_m) + \min \{ \phi_1(b_\phi), \phi_1(\phi_1^{-1}(1/a_n)) \} b_n \mu(A_n) \\
&\quad \text{since } \phi_1 \text{ is increasing} \\
&= \min \{ \phi_1(b_\phi), 1/a_m \} b_m a_m + \min \{ \phi_1(b_\phi), 1/a_n \} b_n a_n \quad \text{since } \phi_1 \text{ is invertible} \\
&\geq \min \{ a_m \phi_1(b_\phi), 1 \} \epsilon + \min \{ a_n \phi_1(b_\phi), 1 \} \epsilon \quad \text{since } b_k \geq \epsilon \forall k \\
&\geq 2\epsilon \min \left\{ \frac{\alpha}{2} \phi_1(b_\phi), 1 \right\} \quad \text{since } a_k \geq \alpha/2 \forall k
\end{aligned}$$

Therefore $(C_\tau f_n)_{n=1}^\infty \subset C_\tau(B_{L^\phi})$ does not have a Cauchy subsequence and hence C_τ is not compact, which is a contradiction. \square

The last few results in this section show that for Orlicz spaces, where the Orlicz function satisfies the Δ_2 -condition globally, and Lorentz spaces (and hence Lebesgue spaces), the condition $\lim_{n \rightarrow \infty} b_n = 0$ is both necessary and sufficient for a composition operator to be compact, provided the underlying measure space is purely atomic.

THEOREM 8.1.19. (cf. [19, p.203]) *Let ϕ be an Orlicz function satisfying the Δ_2 -condition globally. Let (Ω, Σ, μ) be a purely atomic σ -finite measure space and let $\tau : \Omega \rightarrow \Omega$ be a nonsingular measurable transformation. The composition operator C_τ is a compact operator from $X = L^\phi(\mu)$ into itself if and only if $\lim_{n \rightarrow \infty} b_n = 0$.*

PROOF. (Sufficiency): We show that $C_\tau(B_X)$ is compact by showing that every sequence in $C_\tau(B_X)$ has a Cauchy subsequence. Let $\epsilon > 0$, $(g_n)_{n=1}^\infty \subset C_\tau(B_X)$ and for each $n \in \mathbb{N}^+$, let $f_n \in B_X$ such that $C_\tau f_n = g_n$. By Proposition 2.2.35 there exists a $\delta > 0$ such that

$$(8.1.15) \quad I_\phi(f) < \delta \Rightarrow \|f\|_\phi < \frac{\epsilon}{2}$$

We wish to show that X has a predual. In order to apply Proposition 4.3.3 we require a Banach function space over a resonant measure space. By Remark 2.2.18, $X = L_w^\phi(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu_c)$, the weighted Orlicz sequence space over the natural numbers equipped with counting measure, where $w = (a_n)_{n=1}^\infty$. Furthermore, by the same Remark we have that $I_\phi^w = I_\phi$ and hence $\|\cdot\|_\phi^w = \|\cdot\|_\phi$. By Proposition 4.3.3, we therefore have that $(X')_a$ is a predual of X . By Theorem 3.2.6, $(X')_a$ is a closed subspace of X' and is therefore a Banach space. Furthermore μ_c is separable by Remark 1.1.12 and so $(X')_a$ is separable by Theorem 3.2.12. It follows, by Theorem 1.4.5, that $B_{((X')_a)^*} = B_X$ is $\sigma(X, (X')_a)$ -metrizable. Furthermore, by Alaoglu's Theorem 1.4.4, B_X is $\sigma(X, (X')_a)$ -compact. Since B_X is $\sigma(X, (X')_a)$ -metrizable this is equivalent to B_X being $\sigma(X, (X')_a)$ -sequentially compact. We can therefore find a subsequence $(f_{n_k})_{k=1}^\infty$ of $(f_n)_{n=1}^\infty$ such that $f_{n_k} \rightarrow f \in B_X$ in the $\sigma(X, (X')_a)$ topology for some $f \in X$. If it can be shown that for all $(h_n)_{n=1}^\infty \subset 2B_X$ such that $h_n \rightarrow 0$ in the $\sigma(X, (X')_a)$ topology we have $C_\tau h_n \rightarrow 0$ in the norm topology, then it will follow that

$$\begin{aligned}
C_\tau f_{n_k} - C_\tau f &= C_\tau(f_{n_k} - f) \\
&\rightarrow 0 \quad \text{in the norm topology, since } f_{n_k}, f \in B_X \Rightarrow f_{n_k} - f \in 2B_X
\end{aligned}$$

and therefore that $(g_n)_{n=1}^\infty$ has a Cauchy subsequence. Let $(h_n)_{n=1}^\infty \subset 2B_X$ be such that $h_n \rightarrow 0$ in the $\sigma(X, (X')_a)$ topology. Since $b_m \rightarrow 0$ there exists an $m_\delta \in \mathbb{N}^+$ such that

$$(8.1.16) \quad m \geq m_\delta \Rightarrow b_m < \frac{\delta}{2}$$

We show that for all $k \in \mathbb{N}^+$, $h_n(A_k) \rightarrow 0$ as $n \rightarrow \infty$: For any $v \in (X')_a$, φ_v defined by

$$\varphi_v(u) := \sum_{n=1}^{\infty} v(A_n)u(A_n)\mu(A_n) \quad u \in X$$

is a linear functional on X . For each $k \in \mathbb{N}^+$, define $e_k : \mathbb{N}^+ \rightarrow \mathbb{F}$ by

$$e_k(A_n) = \begin{cases} 1 & n = k \\ 0 & n \neq k \end{cases}$$

Then, since e_k is a bounded function with support in a set of finite measure, $e_k \in (X')_a$ by Proposition 4.3.3. We therefore have that

$$\begin{aligned} h_n(A_k) &= \varphi_{e_k}(h_n) \frac{1}{\mu(A_k)} \\ &\rightarrow 0 \text{ as } n \rightarrow \infty \end{aligned}$$

since $h_n \rightarrow 0$ in the $\sigma(X, (X')_a)$ topology. So for all $k \in \mathbb{N}^+$ such that $1 \leq k < m_\delta$ there exists an $n_{\delta,k} \in \mathbb{N}^+$ such that

$$(8.1.17) \quad n \geq n_{\delta,k} \Rightarrow \left| \frac{h_n(A_k)}{2} \right| < \phi^{-1} \left(\frac{\delta}{2m_\delta \cdot b_k a_k} \right)$$

where we have used the fact that ϕ vanishes only at zero to ensure that $\phi^{-1} \left(\frac{\delta}{2m_\delta \cdot a_k} \right) > 0$. Let $n_\delta = \max_{1 \leq k < m_\delta} \{n_{\delta,k}\}$ and $\mathcal{A} = \{\tau(A_m) : m \in \mathbb{N}^+\}$. For $n \geq n_\delta$ we have

$$\begin{aligned} I_\phi \left(C_\tau \frac{h_n}{2} \right) &= \sum_{p=1}^{\infty} \phi \left(\left| (C_\tau \frac{h_n}{2})(A_p) \right| \right) \mu(A_p) \\ &= \sum_{p=1}^{\infty} \phi \left(\left| \frac{h_n}{2}(\tau(A_p)) \right| \right) \mu(A_p) \\ &\leq \sum_{A_m \in \mathcal{A}} \phi \left(\left| \frac{h_n}{2}(A_m) \right| \right) \mu \circ \tau^{-1}(A_m) \quad \text{by letting } A_m = \tau(A_p) \\ &= \sum_{m=1}^{\infty} \phi \left(\left| \frac{h_n}{2}(A_m) \right| \right) \mu \circ \tau^{-1}(A_m) \quad \text{since } A_m \notin \mathcal{A} \Rightarrow \mu \circ \tau^{-1}(A_m) = 0 \\ &= \sum_{m=1}^{m_\delta-1} \phi \left(\left| \frac{h_n}{2}(A_m) \right| \right) b_m \mu(A_m) + \sum_{m=m_\delta}^{\infty} \phi \left(\left| \frac{h_n}{2}(A_m) \right| \right) b_m \mu(A_m) \\ &< \sum_{m=1}^{m_\delta-1} \phi \left(\phi^{-1} \left(\frac{\delta}{2m_\delta b_m a_m} \right) \right) b_m \mu(A_m) + \frac{\delta}{2} \sum_{m=m_\delta}^{\infty} \phi \left(\left| \frac{h_n}{2}(A_m) \right| \right) \mu(A_m) \quad \text{using (8.1.17) and (8.1.16)} \\ &= \sum_{m=1}^{m_\delta-1} \frac{\delta}{2m_\delta b_m a_m} \cdot b_m a_m + \frac{\delta}{2} \sum_{m=m_\delta}^{\infty} \phi \left(\left| \frac{h_n}{2}(A_m) \right| \right) \mu(A_m) \\ &\quad \text{since } \phi \in \Delta_2 \Rightarrow \phi \text{ invertible (Proposition 2.2.13)} \\ &\leq \frac{\delta}{2} + \frac{\delta}{2} I_\phi \left(\frac{h_n}{2} \right) \\ &\leq \delta \end{aligned}$$

since $\left\| \frac{h_n}{2} \right\| \leq 1$ implies $I_\phi \left(\frac{h_n}{2} \right) \leq 1$ by Corollary 2.2.33. So by (8.1.15) we have

$$\|C_\tau h_n\|_\phi = 2 \left\| C_\tau \frac{h_n}{2} \right\|_\phi < \epsilon \quad \forall n \geq n_\delta$$

and hence $C_\tau h_n \rightarrow 0$.

(Necessity): Since $L^{\phi,w}(\mu) = L^{\phi}(\mu)$ with equality of norms if $w \equiv 1$ (Proposition 2.2.38) and furthermore ϕ is invertible if ϕ satisfies the Δ_2 -condition globally (Proposition 2.2.13), the result follows by Theorem 8.1.3. \square

We give an example to illustrate the theorem above.

EXAMPLE 8.1.20. [37, p.36] Let $w > 1$ and let $\mu(A_n) = w^n$ for all $n \in \mathbb{N}^+$. Furthermore, let ϕ be an Orlicz function vanishing only at zero and satisfying the Δ_2 -condition globally. If $\tau(n) := 2n$ for all $n \in \mathbb{N}^+$, then C_τ is a compact operator on $L^\phi(\mu)$.

PROOF. τ is clearly measurable and nonsingular. Furthermore

$$b_n = \frac{\mu \circ \tau^{-1}(A_n)}{\mu(A_n)} = \begin{cases} 0 & \text{if } n \text{ is odd} \\ w^{-n/2} & \text{if } n \text{ is even} \end{cases} \\ \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

and so C_τ is compact by Theorem 8.1.19. \square

COROLLARY 8.1.21. (cf. [37, p.35]) Let (Ω, Σ, μ) be a purely atomic σ -finite measure space and let $\tau : \Omega \rightarrow \Omega$ be a nonsingular measurable transformation. The composition operator C_τ is a compact operator from $X = L^p(\mu)$ ($1 \leq p < \infty$) into itself if and only if $\lim_{n \rightarrow \infty} b_n = 0$.

PROOF. $\phi(t) = t^p$ is an Orlicz function satisfying the Δ_2 -condition globally and in this case $L^\phi(\mu) = L^p(\mu)$ with equality of norms by Proposition 2.2.19. The result therefore follows by the previous theorem. \square

The result for Lorentz spaces is as follows:

THEOREM 8.1.22. (cf. [21, p.2114]) Let (Ω, Σ, μ) be a purely atomic σ -finite measure space and let $\tau : \Omega \rightarrow \Omega$ be a nonsingular measurable transformation. Then C_τ is a compact composition operator on the Lorentz space $L^{p,q}(\mu)$ ($1 \leq q \leq p < \infty$) if and only if $\lim_{n \rightarrow \infty} b_n = 0$.

PROOF. (Necessity): Note that $L^{p,q}(\mu) = L^{\phi,w}(\mu)$, with equality of norms, if $\phi(t) = t^q$ and $w(t) = t^{\frac{q}{p}-1}$ (Example 2.2.28). The necessity therefore follows by Theorem 8.1.3 and Remark 2.2.29.

(Sufficiency): It will be shown that C_τ is the limit of a sequence of finite rank operators and hence compact by Proposition 1.5.11. Let $f \in L^{p,q}(\mu)$. Since f is measurable, f is constant on each atom and so using the fact that $\Omega = \bigcup_{n=1}^{\infty} A_n$, we have that

$$f = \sum_{n=1}^{\infty} f(A_n) \chi_{A_n}$$

For each $k \in \mathbb{N}^+$, let $C_\tau^{(k)}$ be defined by

$$C_\tau^{(k)} f := \sum_{n=1}^k f(A_n) \chi_{\tau^{-1}(A_n)}$$

Since $\{\chi_{\tau^{-1}(A_n)} : n \leq k\}$ forms a basis for $\text{ran } C_\tau^{(k)}$, $C_\tau^{(k)}$ is a finite rank operator for each $k \in \mathbb{N}^+$. It will be shown that $C_\tau^{(k)} \rightarrow C_\tau$ in $\mathcal{B}(L^{p,q}(\mu))$. We note first that

$$(8.1.18) \quad \begin{aligned} \mu_f(\lambda) &= \mu(\{x \in \Omega : |f(x)| > \lambda\}) \\ &= \mu(\cup \{A_n : |f(A_n)| > \lambda\}) \\ &= \sum_{\substack{A_n \\ |f(A_n)| > \lambda}} \mu(A_n) \quad \text{since the } A_n \text{'s are disjoint} \end{aligned}$$

If we let $A^{(k)} := \left\{ x \in \Omega : \left| \sum_{n=k+1}^{\infty} f(A_n) \chi_{\tau^{-1}(A_n)}(x) \right| > \lambda \right\}$, then $y \in A^{(k)}$ implies that $y \in \tau^{-1}(A_n)$ for some $n > k$ and $|f(A_n)| > \lambda$. Therefore

$$y \in A^{(k)} \Rightarrow y \in \cup \{ \tau^{-1}(A_n) : n > k \text{ and } |f(A_n)| > \lambda \}$$

and so

$$\begin{aligned} \mu_{(C_\tau - C_\tau^{(k)})(f)}(\lambda) &= \mu(A^{(k)}) \\ &\leq \mu(\cup \{ \tau^{-1}(A_n) : n > k \text{ and } |f(A_n)| > \lambda \}) \\ &\leq \sum_{n > k, |f(A_n)| > \lambda} \mu \circ \tau^{-1}(A_n) \\ &= \sum_{n > k, |f(A_n)| > \lambda} b_n \mu(A_n) \\ &\leq \sup \{ b_n : n > k \} \sum_{n > k, |f(A_n)| > \lambda} \mu(A_n) \\ (8.1.19) \quad &\leq \sup \{ b_n : n > k \} \mu_f(\lambda) \quad \text{by (8.1.18)} \end{aligned}$$

By Proposition 1.3.5(11) this implies that $\left((C_\tau - C_\tau^{(k)})(f) \right)^*(t) \leq f^*(t/\beta_k)$, where $\beta_k := \sup \{ b_n : n > k \}$. Therefore

$$\| (C_\tau - C_\tau^{(k)})(f) \|_{pq} \leq \beta_k^{1/p} \|f\|_{pq} \quad \text{by Proposition 2.2.3}$$

Since this holds for all $f \in L^{p,q}(\mu)$, we have

$$\begin{aligned} \| (C_\tau - C_\tau^{(k)}) \| &\leq (\sup \{ b_n : n > k \})^{1/p} \\ &\rightarrow 0 \text{ as } k \rightarrow \infty \quad \text{since } b_n \rightarrow 0 \end{aligned}$$

and so $C_\tau^{(k)} \rightarrow C_\tau$. □

REMARK 8.1.23. The claim is made in [22, Theorem 3.2] that the condition “ $\lim_{n \rightarrow \infty} b_n = 0$ ” is sufficient to ensure that the composition operator is compact on Orlicz-Lorentz spaces. The argument is as follows. A sequence $(C_\tau^{(k)})_{k=1}^\infty$ is defined as in the theorem above. Furthermore it is shown as above that if $f \in L^0(\mu)$, then

$$\mu_{(C_\tau - C_\tau^{(k)})(f)}(\lambda) \leq \beta_k \mu_f(\lambda)$$

For each $k \in \mathbb{N}^+$, an $n_k \in \mathbb{N}^+$ is then chosen to ensure that $\beta_k n_k > 1$. It is then demonstrated that if $0 < \|f\|_{\phi,w} < \infty$,

$$(8.1.20) \quad I_{\phi,w} \left(\frac{(C_\tau - C_\tau^{(k)})(f)}{\beta_k n_k \|f\|_{\phi,w}} \right) \leq 1$$

and hence that

$$\| (C_\tau - C_\tau^{(k)})(f) \|_{\phi,w} \leq \beta_k n_k \|f\|_{\phi,w}$$

The claim is then made that $\beta_k n_k \|f\|_{\phi,w} \rightarrow 0$, since $\beta_k \rightarrow 0$. This claim is false, however, since for each $k \in \mathbb{N}^+$, n_k was specifically chosen to ensure that $\beta_k n_k > 1$, so $(n_k)_{k=1}^\infty$ is not a bounded sequence. Unfortunately, since w is a decreasing function, one requires $\beta_k n_k > 1$ to show that (8.1.20) holds.

Characterizations of composition operators

In this section two conditions will be investigated which can be used to determine if a linear operator on a Banach function space is a composition operator. In general we have identified a function f with its equivalence class $[f]$. For the majority of this section we wish to be more explicit and will therefore take care to distinguish between a function and its equivalence class. Let X be a rearrangement invariant Banach function space over a σ -finite measure space (Ω, Σ, μ) and let $\mathcal{K} = \{[\chi_A] : A \in \Sigma\}$, $\mathcal{K}' = \mathcal{K} \cap X$ and $[f], [g], [fg] \in X$. If $\tau : \Omega \rightarrow \Omega$ is a measurable transformation such that C_τ is a composition operator on X , then it will be demonstrated that $C_\tau(\mathcal{K}') \subset \mathcal{K}'$ (i.e. \mathcal{K}' is invariant under C_τ) and $C_\tau([f] \cdot [g]) = C_\tau([f]) \cdot C_\tau([g])$ (i.e. C_τ is multiplicative) for any composition operator C_τ on X . It will be shown that under particular restrictions on the Banach function space X and the underlying measure, a linear operator under which \mathcal{K}' is invariant or which is multiplicative is in fact a (generalized) composition operator. We start by defining a generalized composition operator.

DEFINITION 9.1.1. [37, p.21] Let X be a Banach function space over a measure space (Ω, Σ, μ) . Let $\Omega' \in \Sigma$, $\tau : \Omega' \rightarrow \Omega$ a measurable transformation and $[f] \in X$. Define the generalized composition transformation C_τ by

$$\begin{aligned} C_\tau[f] &:= [\tilde{C}_\tau f] \quad \text{where} \\ (\tilde{C}_\tau f)(x) &:= \begin{cases} f(\tau(x)) & x \in \Omega' \\ 0 & x \in \Omega \setminus \Omega' \end{cases} \end{aligned}$$

If $C_\tau(X) \subset X$ then C_τ is called the generalized composition operator induced by τ and Ω' .

REMARK 9.1.2. Slight modifications to the proofs of Propositions 7.1.2 and 7.1.1 show that a generalized composition transformation is well-defined as a mapping of equivalence classes and maps (equivalence classes of) measurable functions onto (equivalence classes of) measurable functions if and only if τ is a non-singular measurable transformation. Furthermore, it is easily checked that we still have $C_\tau[\chi_A] = [\chi_{\tau^{-1}(A)}]$.

The first result to be proved characterizes composition operators in terms of the invariance of \mathcal{K}' under the operator.

THEOREM 9.1.3. [21, 37, p. 2112, p.23] *If T is a generalized composition operator on a Banach function space X , then \mathcal{K}' is invariant under T . Conversely, if X is a Banach function space with an absolutely continuous norm over a σ -finite absolute Borel space (Ω, Σ, μ) and T is a bounded linear operator on X , such that \mathcal{K}' is invariant under T , then T is a generalized composition operator on X .*

PROOF. Let T be a generalized composition operator on X , i.e. $T = C_\tau$ for some measurable transformation $\tau : \Omega' \rightarrow \Omega$, where $\Omega' \in \Sigma$. Let $[\chi_A] \in \mathcal{K}'$. Then

$$\begin{aligned} T[\chi_A] &= C_\tau[\chi_A] \\ &= [\chi_{\tau^{-1}(A)}] \quad \text{by Remark 9.1.2,} \end{aligned}$$

where the first equality shows that $T[\chi_A] \in X$, since C_τ is a composition operator and therefore maps X into itself, and the second equality shows that $T[\chi_A]$ is a characteristic function of a measurable set, since τ is measurable. Therefore $T[\chi_A] \in \mathcal{K}'$ and $T(\mathcal{K}') \subset \mathcal{K}'$.

Recall (Proposition 1.1.7) that Σ/Σ_0 is a Boolean algebra. For the converse, we will first define Ω' and Σ' before constructing a σ -homomorphism from Σ/Σ_0 to Σ'/Σ'_0 . Sikorski's Theorem (1.1.10) will then be applied to yield the desired measurable function $\tau : \Omega' \rightarrow \Omega$ such that $T = C_\tau$. Let $A \in \Sigma_f$. Then $\|[\chi_A]\|_X < \infty$ by definition of the Banach function norm and so $[\chi_A] \in \mathcal{K}'$. By our assumption, $T[\chi_A] = [\chi_B]$ for some $B \in \Sigma$ such that $\|[\chi_B]\| < \infty$. We note that if $T[\chi_A] = [\chi_B] = [\chi_C]$, then $[B] = [C]$ by Proposition 1.1.9. This means we can use this association to define a function $\phi_0 : \Sigma_f \rightarrow \Sigma/\Sigma_0$ by letting

$$\phi_0(A) := [B], \text{ where } T[\chi_A] = [\chi_B].$$

We will use $[\chi_{\phi_0(A)}]$ to denote $[\chi_B]$. Then $T[\chi_A] = [\chi_{\phi_0(A)}]$. We show that ϕ_0 is a σ -homomorphism. Let $A_1, A_2 \in \Sigma_f$.

i) ϕ_0 is monotone: Let $A_1 \subset A_2$. Then

$$\begin{aligned} [\chi_{\phi_0(A_2)}] &= T[\chi_{A_2}] \\ &= T([\chi_{A_1}] + [\chi_{A_2 \setminus A_1}]) \\ &= T[\chi_{A_1}] + T[\chi_{A_2 \setminus A_1}] \quad \text{by the linearity of } T \\ &= [\chi_{\phi_0(A_1)}] + [\chi_{\phi_0(A_2 \setminus A_1)}] \\ &\geq [\chi_{\phi_0(A_1)}] \quad \text{since } [\chi_{\phi_0(A_2 \setminus A_1)}] \geq 0 \\ \Rightarrow \phi_0(A_2) &\supset \phi_0(A_1) \quad \text{by Proposition 1.1.9} \end{aligned}$$

ii) ϕ_0 preserves disjoint unions: Let $A_1 \cap A_2 = \emptyset$. Then $\chi_{A_1 \cup A_2} = \chi_{A_1} + \chi_{A_2}$ and so

$$\begin{aligned} [\chi_{\phi_0(A_1 \cup A_2)}] &= T[\chi_{A_1 \cup A_2}] \\ &= T([\chi_{A_1}] + [\chi_{A_2}]) \\ &= T[\chi_{A_1}] + T[\chi_{A_2}] \\ &= [\chi_{\phi_0(A_1)}] + [\chi_{\phi_0(A_2)}] \end{aligned}$$

and therefore $\phi_0(A_1) \cap \phi_0(A_2) = [\emptyset]$, since $[\chi_{\phi_0(A_1 \cup A_2)}] \leq 1$. This yields

$$[\chi_{\phi_0(A_1) \cup \phi_0(A_2)}] = [\chi_{\phi_0(A_1)}] + [\chi_{\phi_0(A_2)}] = [\chi_{\phi_0(A_1 \cup A_2)}]$$

and therefore $\phi_0(A_1) \cup \phi_0(A_2) = \phi_0(A_1 \cup A_2)$ by Proposition 1.1.9.

iii) ϕ_0 preserves finite unions: We have from i) that

$$\begin{aligned} \phi_0(A_1) &\subset \phi_0(A_1 \cup A_2) \quad \text{and} \\ \phi_0(A_2) &\subset \phi_0(A_1 \cup A_2) \end{aligned}$$

So

$$\phi_0(A_1) \cup \phi_0(A_2) \subset \phi_0(A_1 \cup A_2)$$

Furthermore,

$$\begin{aligned} \phi_0(A_1 \cup A_2) &= \phi_0((A_1 \setminus A_2) \cup (A_2 \setminus A_1) \cup (A_1 \cap A_2)) \\ &= \phi_0(A_1 \setminus A_2) \cup \phi_0(A_2 \setminus A_1) \cup \phi_0(A_1 \cap A_2) \quad \text{by ii)} \\ &\subset \phi_0(A_1) \cup \phi_0(A_2) \cup \phi_0(A_1) \quad \text{by i)} \\ &= \phi_0(A_1) \cup \phi_0(A_2) \end{aligned}$$

iv) ϕ_0 preserves intersections:

$$\begin{aligned}
[\chi_{\phi_0(A_1 \cap A_2)}] &= T[\chi_{A_1 \cap A_2}] \\
&= T([\chi_{A_1}] + [\chi_{A_2}] - [\chi_{A_1 \cup A_2}]) \\
&= T[\chi_{A_1}] + T[\chi_{A_2}] - T[\chi_{A_1 \cup A_2}] \\
&= [\chi_{\phi_0(A_1)}] + [\chi_{\phi_0(A_2)}] - [\chi_{\phi_0(A_1 \cup A_2)}] \\
&= [\chi_{\phi_0(A_1)}] + [\chi_{\phi_0(A_2)}] - [\chi_{\phi_0(A_1) \cup \phi_0(A_2)}] \quad \text{by iii)} \\
&= [\chi_{\phi_0(A_1) \cap \phi_0(A_2)}]
\end{aligned}$$

Therefore $\phi_0(A_1 \cap A_2) = \phi_0(A_1) \cap \phi_0(A_2)$, by Proposition 1.1.9.

v) ϕ_0 preserves differences:

$$\begin{aligned}
[\chi_{\phi_0(A_1 \setminus A_2)}] &= T[\chi_{A_1 \setminus A_2}] \\
&= T([\chi_{A_1}] - [\chi_{A_1 \cap A_2}]) \\
&= T[\chi_{A_1}] - T[\chi_{A_1 \cap A_2}] \\
&= [\chi_{\phi_0(A_1)}] - [\chi_{\phi_0(A_1 \cap A_2)}] \\
&= [\chi_{\phi_0(A_1)}] - [\chi_{\phi_0(A_1) \cap \phi_0(A_2)}] \quad \text{by iv)} \\
&= [\chi_{\phi_0(A_1) \setminus \phi_0(A_2)}]
\end{aligned}$$

and so $\phi_0(A_1 \setminus A_2) = \phi_0(A_1) \setminus \phi_0(A_2)$ by Proposition 1.1.9.

vi) ϕ_0 preserves countable disjoint unions: Let $(A_n)_{n=1}^{\infty} \subset \Sigma_f$ be pairwise disjoint such that $A = \bigcup_{n=1}^{\infty} A_n \in \Sigma_f$. Then $[\chi_A] \in X$. We note that since X is a Banach function space with absolutely continuous norm, $f_n \uparrow f$ pointwise μ -a.e. and $f \in X$ implies that $f_n \uparrow f$ in X (Proposition 3.2.1). Let $f_n := \sum_{k=1}^n \chi_{A_k}$, then $f_n \uparrow \chi_A$ pointwise μ -a.e. (since $n \neq m$ implies $A_n \cap A_m = \emptyset$) and so $f_n \uparrow \chi_A$ in X by the previous argument. Therefore

$$\begin{aligned}
T[\chi_A] &= T\left(\lim_{n \rightarrow \infty} [f_n]\right) \\
&= \lim_{n \rightarrow \infty} (T[f_n]) \quad \text{since } T \text{ is continuous} \\
&= \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n T[\chi_{A_k}]\right) \quad \text{since } T \text{ is linear} \\
&= \sum_{k=1}^{\infty} [\chi_{\phi_0(A_k)}] \\
&= [\chi_B] \quad \text{where } [B] := \bigcup_{k=1}^{\infty} \phi_0(A_k)
\end{aligned}$$

The last equality follows since

$$\begin{aligned}
\phi_0(A_n) \cap \phi_0(A_m) &= \phi_0(A_n \cap A_m) \quad \text{by iv)} \\
&= \phi_0(\emptyset) \quad \text{if } n \neq m \\
&= [\emptyset] \quad \text{since } T[\chi_{\emptyset}] = T[0] = [0] = [\chi_{\emptyset}]
\end{aligned}$$

Hence $\phi_0(A) = [B] = \bigcup_{k=1}^{\infty} \phi_0(A_k)$ by Proposition 1.1.9. Since (Ω, Σ, μ) is σ -finite, we can find a sequence $(\Omega_n)_{n=1}^{\infty}$ of pairwise disjoint measurable subsets of finite measure such that $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$. Let $[\Omega'_n] := \phi_0(\Omega_n)$ and $[\Omega'] := \bigcup_{n=1}^{\infty} [\Omega'_n]$. Then $[\Omega'_n] \in \Sigma/\Sigma_0$ for each n by definition of ϕ_0 and hence $[\Omega'] \in \Sigma/\Sigma_0$. Let Σ' represent the family of measurable subsets of Ω' (i.e. $\Sigma' = \{A \cap \Omega' : A \in \Sigma\}$) and $\Sigma'_0 := \{B \in \Sigma' : \mu(B) = 0\}$. It is easily

checked that Σ' is a σ -field and Σ'_0 is a σ -ideal. Define $\phi_1 : \Sigma \rightarrow \Sigma/\Sigma_0$ by

$$\phi_1(A) := \bigcup_{n=1}^{\infty} \phi_0(A \cap \Omega_n).$$

Then $\phi_1 : \Sigma \rightarrow \Sigma'/\Sigma'_0$ since

$$\begin{aligned} \phi_0(A \cap \Omega_n) &\subset \phi_0(\Omega_n) = [\Omega'_n] \quad \text{for each } n \text{ by i)} \\ \Rightarrow \bigcup_{n=1}^{\infty} \phi_0(A \cap \Omega_n) &\subset \bigcup_{n=1}^{\infty} [\Omega'_n] = [\Omega'] \\ \Rightarrow \phi_1(A) &= \left(\bigcup_{n=1}^{\infty} \phi_0(A \cap \Omega_n) \right) \cap [\Omega'] \\ &= \bigcup_{n=1}^{\infty} \left((\phi_0(A \cap \Omega_n)) \cap [\Omega'] \right) \\ &\in \Sigma'/\Sigma'_0 \end{aligned}$$

and ϕ_1 extends ϕ_0 since if $A \in \Sigma_f$ then $A = \bigcup_{n=1}^{\infty} A \cap \Omega_n$

$$\begin{aligned} \phi_1(A) &= \bigcup_{n=1}^{\infty} \phi_0(A \cap \Omega_n) \\ &= \phi_0 \left(\bigcup_{n=1}^{\infty} A \cap \Omega_n \right) \quad \text{by (vi)} \\ (9.1.1) \quad &= \phi_0(A) \end{aligned}$$

ϕ_1 is a σ -homomorphism:

a) ϕ_1 preserves countable unions: Let $(A_n)_{n=1}^{\infty} \subset \Sigma$ and let $A = \bigcup_{n=1}^{\infty} A_n$, then

$$\begin{aligned} \phi_1(A) &= \bigcup_{m=1}^{\infty} \phi_0 \left(\left(\bigcup_{n=1}^{\infty} A_n \right) \cap \Omega_m \right) \\ &= \bigcup_{m=1}^{\infty} \phi_0 \left(\bigcup_{n=1}^{\infty} (A_n \cap \Omega_m) \right) \\ &= \bigcup_{m=1}^{\infty} \left(\bigcup_{n=1}^{\infty} \phi_0(A_n \cap \Omega_m) \right) \quad \text{by (vi)} \\ &= \bigcup_{n=1}^{\infty} \left(\bigcup_{m=1}^{\infty} \phi_0(A_n \cap \Omega_m) \right) \\ &= \bigcup_{n=1}^{\infty} \phi_1(A_n) \end{aligned}$$

b) ϕ_1 preserves intersections: Let $A_1, A_2 \in \Sigma$, then

$$\begin{aligned} \phi_1(A_1 \cap A_2) &= \bigcup_{n=1}^{\infty} \phi_0 \left((A_1 \cap A_2) \cap \Omega_n \right) \\ &= \bigcup_{n=1}^{\infty} \phi_0 \left((A_1 \cap \Omega_n) \cap (A_2 \cap \Omega_n) \right) \\ &= \bigcup_{n=1}^{\infty} \left(\phi_0(A_1 \cap \Omega_n) \cap \phi_0(A_2 \cap \Omega_n) \right) \\ &= \left(\bigcup_{n=1}^{\infty} \phi_0(A_1 \cap \Omega_n) \right) \cap \left(\bigcup_{n=1}^{\infty} \phi_0(A_2 \cap \Omega_n) \right) \\ &= \phi_1(A_1) \cap \phi_1(A_2) \end{aligned}$$

c) ϕ_1 preserves complements: This follows since $A^c = \Omega \setminus A$, ϕ_0 preserves difference and $\phi_1(\Omega) = [\Omega']$.

ϕ_1 induces a map $\phi : \Sigma/\Sigma_0 \rightarrow \Sigma'/\Sigma'_0$ defined by

$$(9.1.2) \quad \phi([A]) := \phi_1(A).$$

ϕ is well-defined since if $[A] = [B]$, then

$$\begin{aligned} [A \cap \Omega_m] &= [B \cap \Omega_m] \quad \forall m \\ \Rightarrow T[\chi_{A \cap \Omega_m}] &= T[\chi_{B \cap \Omega_m}] \quad \forall m \text{ since } [\chi_{A \cap \Omega_m}] = [\chi_{B \cap \Omega_m}] \\ \Rightarrow \phi_0(A \cap \Omega_m) &= \phi_0(B \cap \Omega_m) \quad \forall m \\ \Rightarrow \bigcup_{m=1}^{\infty} \phi_0(A \cap \Omega_m) &= \bigcup_{m=1}^{\infty} \phi_0(B \cap \Omega_m), \end{aligned}$$

and therefore $\phi_1(A) = \phi_1(B)$. Furthermore, it can be checked (using the properties of ϕ_1 and similar strategies to those used already) that ϕ is a σ -homomorphism. By Sikorski's Theorem 1.1.10 there exists a map $\tau : \Omega' \rightarrow \Omega$ such that

$$(9.1.3) \quad \phi([A]) = [\tau^{-1}(A)] \quad \forall [A] \in \Sigma/\Sigma_0$$

τ is measurable since $\phi : \Sigma/\Sigma_0 \rightarrow \Sigma'/\Sigma'_0$ and so $\tau^{-1}(A) \in \Sigma'$ for every $A \in \Sigma$. Furthermore τ is nonsingular:

$$\begin{aligned} \mu(A) = 0 &\Rightarrow [A] = [\emptyset] \\ \Rightarrow \phi([A]) &= \phi_1(\emptyset) = [\emptyset] \\ \Rightarrow \mu(\tau^{-1}(A)) &= 0 \quad \text{since } [\tau^{-1}(A)] = \phi([A]) \end{aligned}$$

Let C_τ be the generalized composition transformation induced by τ . We show that $T = C_\tau$: If $A \in \Sigma_f$, then $\|\chi_A\| < \infty$ and

$$\begin{aligned} T[\chi_A] &= [\chi_{\phi_0(A)}] \quad \text{by definition of } \phi_0 \\ &= [\chi_{\phi_1(A)}] \quad \text{by (9.1.1)} \\ &= [\chi_{\tau^{-1}(A)}] \quad \text{by (9.1.2) and (9.1.3)} \\ &= [C_\tau \chi_A] \quad \text{by Remark 9.1.2} \\ &= C_\tau[\chi_A] \end{aligned}$$

Therefore T and C_τ agree on the set of all characteristic functions of measurable sets of finite measure. By linearity, T and C_τ agree on the set of all simple functions. Let $f \in X$, $f = f_1 + if_2$, where $f_1, f_2 \in X_{\mathbb{R}}$ and $f_j = f_j^+ - f_j^-$, where $f_j^+, f_j^- \geq 0$ for $j = 1, 2$. By Theorem 1.2.4 there exists a sequence $(f_n)_{n=1}^{\infty}$ of positive simple functions such that $f_n \uparrow f_1^+$ pointwise μ -a.e. This implies that

$$(9.1.4) \quad 0 \leq C_\tau f_n \uparrow C_\tau f_1^+ \quad \text{pointwise } \mu\text{-a.e.}$$

$$(9.1.5) \quad \Rightarrow \|Tf_n\| = \|C_\tau f_n\| \uparrow \|C_\tau f_1^+\| \quad \text{since } T \text{ and } C_\tau \text{ agree on simple functions} \\ \text{and using Proposition 3.1.4(5)}$$

However, since X has absolutely continuous norm, $f_n \uparrow f_1^+$ in X by Proposition 3.2.1. It follows that $Tf_n \rightarrow Tf_{1+}$ in X since T is continuous and hence

$$(9.1.6) \quad \begin{aligned} \|Tf_n\| &\rightarrow \|Tf_1^+\| \quad \text{since } \|\|g\| - \|h\|\| \leq \|g - h\| \quad \forall g, h \in X \\ \Rightarrow \|Tf_1^+\| &= \|C_\tau f_1^+\| \quad \text{using (9.1.5), (9.1.6) and the uniqueness of limits in } \mathbb{R} \end{aligned}$$

Since $T(X) \subset X$, we therefore have that $C_\tau f_1^+ \in X$. Application of Proposition 3.2.1 to (9.1.4) yields $C_\tau f_n \rightarrow C_\tau f_1^+$ in X . But since $Tf_n \rightarrow Tf_1^+$ in X and $C_\tau f_n = Tf_n$ for each n , it follows by the uniqueness of limits in X that $C_\tau f_1^+ = Tf_1^+$. It can similarly be shown that $C_\tau f_2^+ = Tf_2^+$ and $C_\tau f_j^- = Tf_j^-$ for $j = 1, 2$. Using the linearity of T and C_τ , we therefore have that $Tf = C_\tau f$. \square

The next result shows that composition operators can also be characterised in terms of a multiplicativity property. For this result we will not distinguish between equivalence classes and their representatives.

COROLLARY 9.1.4. [21, p.2111] *Let T be a composition operator on a Banach function space X . Then $T(f.g) = T(f)T(g)$ whenever $f, g, fg \in X$. Conversely, if X is a Banach function space with an absolutely continuous norm over a σ -finite absolute Borel space (Ω, Σ, μ) and T is a bounded linear operator on X , such that $T(f.g) = T(f)T(g)$ whenever $f, g, fg \in X$, then T is a generalized composition operator.*

PROOF. Let $f, g, fg \in X$ and let $T = C_\tau$ for some measurable transformation $\tau : \Omega \rightarrow \Omega$, then

$$\begin{aligned} (T(f.g))(x) &= (C_\tau(f.g))(x) \\ &= (f.g)(\tau(x)) \\ &= f(\tau(x)).g(\tau(x)) \\ &= ((C_\tau f)(x)).((C_\tau g)(x)) \\ &= ((C_\tau f).(C_\tau g))(x) \\ &= ((Tf).(Tg))(x) \end{aligned}$$

Since this holds for all $x \in \Omega$, we have that $T(f.g) = T(f)T(g)$.

For the converse, let $\chi_A \in \mathcal{K}'$. Then $T\chi_A \in X$ since T is an operator on X . Furthermore,

$$\begin{aligned} T\chi_A &= T(\chi_A \cdot \chi_A) \\ &= (T\chi_A).(T\chi_A) \quad \text{by assumption} \\ &= (T\chi_A)^2 \\ \Rightarrow ((T\chi_A)(x)) \cdot (1 - (T\chi_A)(x)) &= 0 \quad \forall x \in \Omega \\ \Rightarrow (T\chi_A)(x) &= 0 \text{ or } 1 \quad \forall x \in \Omega \end{aligned}$$

This means that $T\chi_A$ is a characteristic function and since $T\chi_A \in X$, we have $T\chi_A \in \mathcal{K}'$. It follows that $T(\mathcal{K}') \subset \mathcal{K}'$ and hence T is a generalized composition operator by Theorem 9.1.3. \square

Isometries and composition operators

As an application of composition operators the link between composition operators and isometries will be explored. We have the following theorem by Banach and Stone which represents isometries on spaces of continuous functions as the products of unimodular functions and composition operators.

THEOREM 10.1.1. [7, p.25] *Let Ω_1 and Ω_2 be compact Hausdorff spaces. If T is an isometric isomorphism of $C_{\mathbb{F}}(\Omega_1)$ onto $C_{\mathbb{F}}(\Omega_2)$, then there is a homeomorphism τ from Ω_2 onto Ω_1 and a continuous unimodular function $h : \Omega_2 \rightarrow \mathbb{F}$ such that for each $f \in C_{\mathbb{F}}(\Omega_1)$,*

$$(Tf)(x) = h(x).f(\tau(x)) \quad x \in \Omega_2$$

We will investigate the conditions under which a similar result holds for Lebesgue spaces. First we detail some notation to be used and present requisite information regarding regular set isomorphisms and the linear transformations induced by them.

Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces. Recall (Proposition 1.1.7) that if Σ is a σ -algebra, then Σ/Σ_0 is a Boolean algebra, where Σ_0 is the σ -ideal of sets with zero measure. We will use $[\Sigma_i]$ to represent $\Sigma_i/(\Sigma_i)_0$. Recall further that $[A] = [B]$ if and only if $\mu_i(A \Delta B) = 0$. We will call a sequence $([A_n])_{n=1}^{\infty} \subset [\Sigma_1]$ pairwise disjoint if $[A_n] \cap [A_m] = [\emptyset]$ whenever $n \neq m$. If $[A] \in [\Sigma_i]$ and $B, C \in [A]$, then $\mu_1(B \Delta C) = 0$ and so $\mu_1(C) = \mu_1(B)$. We will abuse notation somewhat and write

$$\mu_i([A]) := \mu_i(B) \quad \text{for some } B \in [A]$$

We note that if $(f_n)_{n=1}^{\infty} \cup \{f\} \subset L^0(\mu_i)$ and $f_n \rightarrow f$ pointwise μ_i -a.e., $g \in [f]$ and $g_n \in [f_n]$ for each n , then $g_n \rightarrow g$ pointwise μ_i -a.e. This follows since $f_n(x) \rightarrow f(x)$ for all $x \in \Omega_i \setminus A$, where $\mu_i(A) = 0$, and $f_n(x) = g_n(x)$ for all $x \in \Omega_i \setminus B_n$, where $\mu_i(B_n) = 0$, and $f(x) = g(x)$ for all $x \in \Omega_i \setminus B$, where $\mu_i(B) = 0$. So for any $x \in \Omega_i \setminus \left(\left(\bigcup_{n=1}^{\infty} B_n \right) \cup A \cup B \right)$ we have that $g_n(x) = f_n(x)$ for all n and $f_n \rightarrow f$. Furthermore since a countable union of sets with measure zero has measure zero, it follows that $g_n \rightarrow g$ pointwise μ_i -a.e. We will therefore write " $[f_n] \rightarrow [f]$ pointwise" if $f_n \rightarrow f$ pointwise μ_i -a.e. If $[f], [g] \in L^0(\mu)$, then we can define $[f][g] := [fg]$ since if $f(x) = \tilde{f}(x)$ for all $x \in A^c$ and $g(x) = \tilde{g}(x)$ for all $x \in B^c$, then $f(x).g(x) = \tilde{f}(x).\tilde{g}(x)$ for all $x \in (A \cup B)^c$. We define the essential support of an equivalence class of functions $[f] \in L^0(\mu_i)$ as follows:

$$[\text{supp } [f]] := \{x \in \Omega_i : g(x) \neq 0 \text{ for some } g \in [f]\}$$

It is easily checked that this notion is independent of the choice of $g \in [f]$.

DEFINITION 10.1.2. [25, p.461] A map $\eta : [\Sigma_1] \rightarrow [\Sigma_2]$ is called a regular set isomorphism if

- (1) $\eta \left(\left[\bigcup_{n=1}^{\infty} A_n \right] \right) = \bigcup_{n=1}^{\infty} \eta([A_n])$ for any pairwise disjoint sequence $([A_n])_{n=1}^{\infty} \subset [\Sigma_1]$
- (2) $\eta([\Omega_1 \setminus A]) = \eta([\Omega_1]) \setminus \eta([A])$ for all $[A] \in [\Sigma_1]$ and
- (3) $\eta([A]) = [\emptyset]$ if and only if $[A] = [\emptyset]$.

The following properties follow from the definition of a regular set isomorphism.

PROPOSITION 10.1.3. [7, p.52] *Let $[A], [B] \in [\Sigma_1]$ and $\eta : [\Sigma_1] \rightarrow [\Sigma_2]$ be a regular set isomorphism. Then*

- (1) *If $[A] \subset [B]$, then $\eta([A]) \subset \eta([B])$*

- (2) $\eta\left(\bigcup_{n=1}^{\infty} A_n\right) = \bigcup_{n=1}^{\infty} \eta([A_n])$ for any sequence $([A_n])_{n=1}^{\infty} \subset [\Sigma_1]$
(3) $\eta\left(\bigcap_{n=1}^{\infty} A_n\right) = \bigcap_{n=1}^{\infty} \eta([A_n])$ for any sequence $([A_n])_{n=1}^{\infty} \subset [\Sigma_1]$
(4) $\eta([A]) \cap \eta([B]) = [\emptyset]$ if and only if $[A \cap B] = [\emptyset]$

REMARK 10.1.4. η is injective (as a mapping of equivalence classes), i.e. if $[A], [B] \in [\Sigma_1]$ such that $\eta([A]) = \eta([B])$, then $[A] = [B]$

η is injective: Let $\eta([A]) = \eta([B])$, then

$$\begin{aligned} \eta([\Omega_1 \setminus A]) \cap \eta([B]) &= (\eta([\Omega_1]) \setminus \eta([A])) \cap \eta([B]) \quad \text{since } \eta \text{ is a regular set isomorphism} \\ &= [\emptyset] \quad \text{since } \eta([A]) = \eta([B]) \\ \Rightarrow [\Omega_1 \setminus A] \cap [B] &= [\emptyset] \quad \text{by Proposition 10.1.3(4)} \\ \Rightarrow [B] &\subset [A] \end{aligned}$$

It can similarly be shown that $[A] \subset [B]$ and hence $[A] = [B]$.

REMARK 10.1.5. Let $A \in \Sigma_1$, then $[A] \in [\Sigma_1]$ and $\eta([A]) \in \eta([\Sigma_1])$. Define

$$\Sigma_2^\eta := \{B \in \Sigma_2 : B \in \eta([A]) \text{ for some } A \in \Sigma_1\}$$

It can be shown that Σ_2^η is a σ -algebra. If $B \in \Sigma_2^\eta$ and $A \in \Sigma_1$ is such that $B \in \eta([A])$, then put

$$\mu_1 \circ \eta^{-1}(B) := \mu_1(A)$$

$\mu_1 \circ \eta^{-1}$ is well-defined since if $A' \in \Sigma_1$ such that $\eta([A]) = \eta([A'])$, then $[A] = [A']$ since η is injective. It follows that $\mu_1(A) = \mu_1(A')$. It can be shown that $\mu_1 \circ \eta^{-1}$ is a positive measure on Σ_2^η . We show that $\mu_1 \circ \eta^{-1}$ is absolutely continuous with respect to $\mu_2 \upharpoonright \Sigma_2^\eta$. Let $B \in \Sigma_2^\eta$ such that $\mu_2(B) = 0$ and let $A \in \Sigma_1$ such that $B \in \eta([A])$. $\mu_2(B) = 0$ implies that $[\emptyset] = [B] = \eta([A])$ and so $[A] = [\emptyset]$ by definition of a regular set isomorphism. It follows that $0 = \mu_1(A) := \mu_1 \circ \eta^{-1}(B)$.

Let $\eta : [\Sigma_1] \rightarrow [\Sigma_2]$ be a regular set isomorphism. We define a linear transformation T_η on $L^1(\mu)$ using η . We start by defining T_η on characteristic functions. If $A \in \Sigma_1$, then let $[\chi_{\eta([A])}] := [\chi_E]$, where $E \in \eta([A])$, and define T_η for $[\chi_A] \in L^0(\mu_1)$ by

$$T_\eta[\chi_A] = [\chi_{\eta([A])}]$$

We extend T_η to positive simple functions by letting

$$T_\eta[f] := \sum_{i=1}^n \alpha_i \chi_{\eta([A_i])},$$

if $[f] = \left[\sum_{i=1}^n \alpha_i \chi_{A_i}\right]$, with $\alpha_i \geq 0$, $A_i \in \Sigma_1$ and $A_i \cap A_j = \emptyset$ if $i \neq j$. For $[f] \in L^0_+(\mu_1)$, define

$$T_\eta[f] := \sup \{T_\eta[g] : g \text{ simple, } 0 \leq [g] \leq [f]\}$$

To extend T_η to $L^0_{\mathbb{R}}(\mu_1)$, note that if $[f] \in L^0_{\mathbb{R}}(\mu_1)$, then $[f] = [f^+] - [f^-]$, so put

$$T_\eta[f] := T_\eta[f^+] - T_\eta[f^-]$$

Finally, since $[f] = [\operatorname{Re}f] + i[\operatorname{Im}f]$ if $[f] \in L^0(\mu)$, T_η can similarly be extended to all of $L^0(\mu_1)$. The details are omitted, but it can be checked that T_η defined in this way is well-defined and linear.

PROPOSITION 10.1.6. (cf. [7, p.52]) Let $\eta : [\Sigma_1] \rightarrow [\Sigma_2]$ be a regular set isomorphism and let $A \in \Sigma_1$. If T_η is defined as above then $T_\eta : L^0(\Omega_1, \Sigma_1, \mu_1) \rightarrow L^0(\eta([\Omega_1]), \Sigma_2 \cap \eta([\Omega_1]), \mu_2 \upharpoonright \eta([\Omega_1]))$ is a linear transformation having the following properties: for any $[f], [g] \in L^0(\Omega_1, \Sigma_1, \mu_1)$,

- (1) $T_\eta[f_n] \rightarrow T_\eta[f]$ pointwise μ_2 -a.e. if $([f_n])_{n=1}^{\infty} \subset L^0(\Omega_1, \Sigma_1, \mu_1)$ is such that $[f_n] \rightarrow [f]$ pointwise

- (2) $T_\eta([f \cdot g]) = (T_\eta[f]) \cdot (T_\eta[g])$
- (3) $T_\eta(|[f]|) = |T_\eta[f]|$
- (4) If $[f] \leq [g]$ then $T_\eta[f] \leq T_\eta[g]$
- (5) $T_\eta[f] \cdot [\chi_A] = [0]$ if $[A] = \eta(\Omega_1) \setminus \eta(\text{supp}[f])$

REMARK 10.1.7. T_η is a positive operator, since T_η is linear and $T_\eta[f] \leq T_\eta[g]$ if $[f] \leq [g]$.

If η is a regular set isomorphism, $f \in L^0(\mu_2)$ and $A \in \Sigma$, then $\mu_2(E \Delta F) = 0$ for all $E, F \in \eta(A)$ and so

$$\int_E f d\mu_2 = \int_F f d\mu_2$$

We will therefore define

$$\int_{\eta(A)} f d\mu_2 := \int_E f d\mu_2 \quad \text{for some } E \in \eta(A)$$

If $g, h \in [f] \in L^0(\mu_i)$, then

$$\int_\Omega g d\mu = \int_\Omega h d\mu$$

and so we will use the notation $\int_\Omega [f] d\mu := \int_\Omega g d\mu$ for some $g \in [f]$.

We need one further result before proving the main result of this chapter.

LEMMA 10.1.8. [25, p.461] If $[f]$ and $[g]$ belong to $L^p(\mu)$, then

$$\begin{aligned} \|[f] + [g]\|^p + \|[f] - [g]\|^p &\leq 2\|[f]\|^p + 2\|[g]\|^p & 0 < p \leq 2 \\ \|[f] + [g]\|^p + \|[f] - [g]\|^p &\geq 2\|[f]\|^p + 2\|[g]\|^p & p \geq 2 \end{aligned}$$

In either case, if $p \neq 2$, then equality occurs if and only if $[f] \cdot [g] = 0$.

REMARK 10.1.9. If $p = 2$ in the above Corollary, then $L^p(\mu)$ is a Hilbert space and so by the parallelogram law ([3, p.8])

$$\|[f] + [g]\|^2 + \|[f] - [g]\|^2 = 2\|[f]\|^2 + 2\|[g]\|^2$$

for any $[f], [g] \in L^2(\mu)$, not just for $[f]$ and $[g]$ such that $[f] \cdot [g] = 0$.

THEOREM 10.1.10. [25, p.461] [7, p.53] Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be σ -finite measure spaces and let $U : L^p(\Omega_1, \Sigma_1, \mu_1) \rightarrow L^p(\Omega_2, \Sigma_2, \mu_2)$ ($1 \leq p < \infty$, $p \neq 2$) be a linear isometry. Then there exists a regular set isomorphism $\eta : [\Sigma_1] \rightarrow [\Sigma_2]$ and a function $h : \Omega_2 \rightarrow \mathbb{F}$ such that for all $[f] \in L^p(\Omega_1, \Sigma_1, \mu_1)$

$$(10.1.1) \quad U[f] = [h] \cdot (T_\eta[f])$$

where T_η is the transformation induced by η . Furthermore,

$$(10.1.2) \quad \int_{\eta([A])} |[h]|^p d\mu_2 = \int_{\eta([A])} \frac{d(\mu_1 \circ \eta^{-1})}{d\mu_2} d\mu_2 = \mu_1(A) \quad \forall A \in \Sigma_1.$$

Conversely, if η is a regular set isomorphism and $h : \Omega_2 \rightarrow \mathbb{F}$ is such that (10.1.2) holds, then the operator U defined by (10.1.1) is a linear isometry.

PROOF. We first prove the sufficiency for finite measure spaces. Let $\mu_1(\Omega_1) < \infty$. Define a set mapping $\eta : [\Sigma_1] \rightarrow [\Sigma_2]$ by

$$\begin{aligned} \eta([A]) &:= [\text{supp}(U[\chi_A])] \\ &= [\{x \in \Omega_2 : g(x) \neq 0\}] \quad \text{where } g \in U[\chi_A] \end{aligned}$$

We show that η is a regular set isomorphism. η preserves finite disjoint union: Let $[A], [B] \in [\Sigma_1]$ such that $[A] \cap [B] = [\emptyset]$. Then

$$\begin{aligned}
\|U[\chi_A] + U[\chi_B]\|^p + \|U[\chi_A] - U[\chi_B]\|^p &= \|U([\chi_A + \chi_B])\|^p + \|U([\chi_A - \chi_B])\|^p \\
&\text{since } U \text{ is linear} \\
&= \|[\chi_A] + [\chi_B]\|^p + \|[\chi_A] - [\chi_B]\|^p \\
&\text{since } U \text{ is an isometry} \\
&= 2\|[\chi_A]\|^p + 2\|[\chi_B]\|^p \\
&\text{by Lemma 10.1.8, since } [\chi_A] \cdot [\chi_B] = [0] \\
&= 2\|U[\chi_A]\|^p + 2\|U[\chi_B]\|^p \\
&\text{since } U \text{ is an isometry}
\end{aligned}$$

and so

$$(10.1.3) \quad (U[\chi_A]) \cdot (U[\chi_B]) = [0] \quad \text{by Lemma 10.1.8}$$

Therefore

$$(10.1.4) \quad \begin{aligned} \eta([A]) \cap \eta([B]) &= [\text{supp}(U[\chi_A])] \cap [\text{supp}(U[\chi_B])] \\ &= [\emptyset] \end{aligned}$$

From this it follows that

$$(10.1.5) \quad \begin{aligned} \eta([A \cup B]) &= [\text{supp}(U[\chi_{A \cup B}])] \\ &= [\text{supp}(U[\chi_A] + U[\chi_B])] \quad \text{using } [A] \cap [B] = [\emptyset] \text{ and } U \text{ linear} \\ &= [\text{supp}(U[\chi_A])] \cup [\text{supp}(U[\chi_B])] \quad \text{by (10.1.4)} \\ &= \eta([A]) \cup \eta([B]) \end{aligned}$$

η preserves countable disjoint union: Let $([A_n])_{n=1}^\infty \subset [\Sigma_1]$ be a pairwise disjoint sequence. Let $[f_n] := \sum_{k=1}^n [\chi_{[A_k]}] = [\chi_{[\bigcup_{k=1}^n A_k]}]$, where the latter equality follows since the $[A_k]$'s are pairwise disjoint. Let $[f] = [\chi_{[\bigcup_{k=1}^\infty A_k]}]$, then $[f_n] \uparrow [f]$ pointwise. We show that $[f_n] \uparrow [f]$ in $L^p(\mu_1)$. Since $\mu_1(\Omega_1) < \infty$, we have that $[\chi_{\Omega_1}] \in L^p(\mu_1)$. Furthermore $[f] \leq [\chi_{\Omega_1}]$ and so $[f_n] \uparrow [f]$ in $L^p(\mu_1)$, using the Fatou and Banach lattice properties of the L^p norm. Therefore

$$\begin{aligned}
U[f] &= U(\lim_{n \rightarrow \infty} [f_n]) \\
&= \lim_{n \rightarrow \infty} U[f_n] \quad \text{since } U \text{ is continuous} \\
&= \lim_{n \rightarrow \infty} \sum_{k=1}^n U[\chi_{A_k}]
\end{aligned}$$

By (10.1.4) $[\text{supp}(U[\chi_{A_k}])] \cap [\text{supp}(U[\chi_{A_m}])] = \emptyset$ if $k \neq m$ and so

$$\begin{aligned}
\eta\left([\bigcup_{n=1}^\infty A_n]\right) &= [\text{supp}(U[f])] \\
&= \bigcup_{n=1}^\infty [\text{supp}(U[\chi_{A_n}])] \\
&= \bigcup_{n=1}^\infty \eta([A_n])
\end{aligned}$$

η preserves complements: Let $[A] \in [\Sigma_1]$

$$\begin{aligned}\eta([\Omega_1]) &= \eta([A] \cup [\Omega_1 \setminus A]) \\ &= \eta([A]) \cup \eta([\Omega_1 \setminus A]) \quad \text{by (10.1.5)}\end{aligned}$$

Since $\eta([A])$ and $\eta([A^c])$ are disjoint by (10.1.4), we have that

$$\eta([\Omega_1] \setminus [A]) = \eta([\Omega_1]) \setminus \eta([A])$$

$\eta([A]) = [\emptyset]$ if and only if $[A] = [\emptyset]$:

$$\begin{aligned}[A] = [\emptyset] &\Leftrightarrow [\chi_A] = 0 \\ &\Leftrightarrow \|\chi_A\|^p = 0 \\ &\Leftrightarrow \|U[\chi_A]\|^p = 0 \quad \text{since } U \text{ is an isometry} \\ &\Leftrightarrow U[\chi_A] = 0 \\ &\Leftrightarrow [\text{supp}(U[\chi_A])] = [\emptyset] \\ &\Leftrightarrow \eta([A]) = [\emptyset]\end{aligned}$$

So η is a regular set isomorphism. Let T_η be the transformation induced by η .

Since $\mu_1(\Omega_1) < \infty$, $[\chi_{\Omega_1}] \in L^p(\Omega_1, \Sigma_1, \mu_1)$ and so $U[\chi_{\Omega_1}] \in L^p(\Omega_2, \Sigma_2, \mu_2)$. Let $[h] := U[\chi_{\Omega_1}]$ and let $A \in \Sigma_1$. Then

$$\begin{aligned}[h] &= U[\chi_{\Omega_1}] \\ &= U([\chi_A] + [\chi_{A^c}]) \\ &= U[\chi_A] + U[\chi_{A^c}]\end{aligned}$$

We also have that $U[\chi_A]$ and $U[\chi_{A^c}]$ have disjoint support by (10.1.4) and so if $\tilde{h} \in [h]$, $g \in U[\chi_A]$ and $B \in [\text{supp}(U[\chi_A])] = \eta([A])$, then

$$\tilde{h}(x) = g(x) \quad \text{for } \mu_2\text{-a.e. } x \in B$$

Therefore

$$\begin{aligned}U[\chi_A] &= [h] \cdot [\chi_{\eta(A)}] \\ (10.1.6) \quad &= [h] \cdot T_\eta[\chi_A]\end{aligned}$$

It follows that (10.1.1) holds for characteristic functions. By the linearity of U , (10.1.1) also holds for simple functions. Therefore (10.1.1) holds for all $[f] \in L^p(\Omega_1, \Sigma_1, \mu_1)$ since simple functions are dense in $L^p(\Omega_1, \Sigma_1, \mu_1)$ and U is continuous.

Next we extend the result to σ -finite measure spaces. Let $\Omega_1 = \bigcup_{n=1}^{\infty} \Omega'_n$, where $\mu_1(\Omega'_n) < \infty$ for each n and $\Omega'_n \cap \Omega'_m = \emptyset$ if $n \neq m$ and let $\Sigma'_n := \{A \cap \Omega'_n : A \in \Sigma\}$. For each n , U induces an isometry from

$L^p_n(\mu_1) := L^p(\Omega'_n, \Sigma'_n, \mu_1 \upharpoonright \Omega'_n)$ into $L^p(\mu_2)$ in the following way: Let $[f] \in L^p_n(\mu_1)$ and let $\tilde{f}(t) = \begin{cases} f(t) & t \in \Omega'_n \\ 0 & t \notin \Omega'_n \end{cases}$.

Then $[\tilde{f}] \in L^p(\mu_1)$. Define $U_n : L^p_n(\mu_1) \rightarrow L^p(\mu_2)$ by

$$(10.1.7) \quad U_n[f] := U[\tilde{f}]$$

Then U_n is an isometry, since

$$\begin{aligned}
\|U_n[f]\|_{L^p(\mu_2)}^p &= \|U[\tilde{f}]\|_{L^p(\mu_2)}^p \\
&= \|\tilde{f}\|_{L^p(\mu_1)}^p \quad \text{since } U \text{ is an isometry} \\
&= \int_{\Omega_1} |\tilde{f}(x)|^p d\mu_1 \\
&= \int_{\Omega'_n} |f(x)|^p d\mu_1 \quad \text{since } \tilde{f}(x) = 0 \text{ for } \mu_1\text{-a.e. } x \in (\Omega'_n)^c \\
&= \|f\|_{L^p_n(\mu_1)}^p
\end{aligned}$$

Furthermore, it is easily checked that U_n is linear. From what has been shown in the case where $\mu(\Omega_1) < \infty$, we therefore have for each $n \in \mathbb{N}^+$, a regular set isomorphism $\eta_n : [\Sigma'_n] \rightarrow [\Sigma_2]$ and a function h_n defined on Ω_2 such that

$$(10.1.8) \quad U_n[f] = [h_n]T_{\eta_n}[f] \quad [f] \in L^p_n(\mu_1)$$

Define $\eta : [\Sigma_1] \rightarrow [\Sigma_2]$ by

$$(10.1.9) \quad \eta([A]) := \bigcup_{n=1}^{\infty} \eta_n([A \cap \Omega'_n])$$

It is easily checked that η is a regular set isomorphism. Let

$$[h] := \sum_{n=1}^{\infty} [h_n]$$

We show that $[h]$ is well-defined and for any $n \in \mathbb{N}^+$ and that $[h_n] = [h][\chi_{\eta(\Omega'_n)}]$: We note that

$$\begin{aligned}
[h_n] &:= U_n[\chi_{\Omega'_n}] \\
&= [h_n] \cdot T_{\eta_n}[\chi_{\Omega'_n}] \quad \text{by (10.1.8)} \\
&= [h_n] \cdot [\chi_{\eta_n(\Omega'_n)}] \quad \text{by Proposition 10.1.6(5)} \\
(10.1.10) \quad &= [h_n] \cdot [\chi_{\eta(\Omega'_n)}] \quad \text{by (10.1.9)}
\end{aligned}$$

Furthermore, $\Omega'_n \cap \Omega'_m = \emptyset$ if $n \neq m$ implies that $\eta(\Omega'_n) \cap \eta(\Omega'_m) = [\emptyset]$ if $n \neq m$ by Proposition 10.1.3(4). Therefore $[h]$ is well-defined and

$$\begin{aligned}
[h] \cdot [\chi_{\eta(\Omega'_n)}] &:= \left(\sum_{n=1}^{\infty} [h_n] \right) \cdot [\chi_{\eta(\Omega'_n)}] \\
(10.1.11) \quad &= [h_n] \quad \text{by (10.1.10)}
\end{aligned}$$

Let $[f] \in L^p(\mu_1)$. We show that

$$(10.1.12) \quad U[f] = [h] \cdot T_{\eta}[f]$$

Let

$$\tilde{f}_n(x) := \begin{cases} f(x) & x \in \Omega'_n \\ 0 & x \notin \Omega'_n \end{cases}$$

and $f_n := f \upharpoonright \Omega'_n$. Since the Ω'_n 's are pairwise disjoint, we have that $f(x) = \left(\sum_{n=1}^{\infty} \tilde{f}_n \right)(x)$. Furthermore $L^p(\mu_1)$

has absolutely continuous norm by Corollary 3.2.4. So if we let $g_k = \sum_{n=1}^k \tilde{f}_n$, then $|g_k| \leq |f|$ for all k and $g_k \rightarrow f$ μ -a.e. and hence $[g_k] \rightarrow [f]$ in L^p by Proposition 3.2.1. We show first that

$$(10.1.13) \quad (T_{\eta}[f]) \cdot [\chi_{\eta(\Omega'_m)}] = (T_{\eta_m}[f_m]) \cdot [\chi_{\eta(\Omega'_m)}] \quad \forall m$$

Let $[A] \in [\Sigma_1]$, then $\chi_{[A]} \upharpoonright \Omega'_m = \chi_{[A \cap \Omega'_m]}$. Thus

$$\begin{aligned}
(T_\eta[\chi_{[A]}) \cdot [\chi_{\eta(\Omega'_m)}] &= [\chi_{\eta([A])}] \cdot [\chi_{\eta(\Omega'_m)}] \\
&= [\chi_{\eta([A]) \cap \eta(\Omega'_m)}] \\
&= [\chi_{\eta([A \cap \Omega'_m])}] \quad \text{by Proposition 10.1.3(3)} \\
&= [\chi_{\eta_m([A] \cap \Omega'_m)}] \quad \text{by (10.1.9)} \\
&= T_{\eta_m}[\chi_{[A \cap \Omega'_m]}] \\
\Rightarrow (T_\eta[\chi_{[A]}) \cdot [\chi_{\eta(\Omega'_m)}] &= (T_{\eta_m}[\chi_{[A \cap \Omega'_m]}]) \cdot [\chi_{\eta(\Omega'_m)}]
\end{aligned}$$

This implies that (10.1.13) holds for $[f] = [\chi_{[A]}]$. We can therefore extend this result to all $[f] \in L^p(\mu_1)$ as before. Next we show that (10.1.12) holds:

$$\begin{aligned}
(10.1.14) \quad U[f] &= U\left(\lim_{k \rightarrow \infty} \sum_{n=1}^k [\tilde{f}_n]\right) \\
&= \lim_{k \rightarrow \infty} \sum_{n=1}^k U[\tilde{f}_n] \quad \text{since } U \text{ is continuous and linear} \\
&= \lim_{k \rightarrow \infty} \sum_{n=1}^k U_n[f_n] \quad \text{by (10.1.7),}
\end{aligned}$$

where the limits above are L^p limits. By Theorem 1.2.8 the sequence defined by $[j_k] = \sum_{n=1}^k U_n[f_n]$ has a subsequence $([j_{k_p}])_{p=1}^\infty$ which converges pointwise to $\lim_{k \rightarrow \infty} \sum_{n=1}^k U_n[f_n] = U[f]$ (where the equality follows by (10.1.14)). Without loss of generality we can therefore assume that $[j_k] \rightarrow U[f]$ pointwise. It follows that the following pointwise limits are equal

$$(10.1.15) \quad \left(\lim_{k \rightarrow \infty} \sum_{n=1}^k U_n[f_n]\right) \cdot [\chi_{\eta(\Omega'_m)}] = \lim_{k \rightarrow \infty} \sum_{n=1}^k U_n[f_n] \cdot [\chi_{\eta(\Omega'_m)}]$$

and therefore using (10.1.14) and (10.1.15), we have

$$\begin{aligned}
(U[f]) \cdot [\chi_{\eta(\Omega'_m)}] &= \lim_{k \rightarrow \infty} \left(\sum_{n=1}^k [h_n] \cdot (T_{\eta_n}[f_n]) \cdot [\chi_{\eta(\Omega'_m)}]\right) \quad \text{by (10.1.8)} \\
&= [h_m] \cdot (T_{\eta_m}[f_m]) \cdot [\chi_{\eta(\Omega'_m)}] \quad \text{using (10.1.10) and } \eta(\Omega'_m)\text{'s pairwise disjoint} \\
&= [h] \cdot (T_\eta[f]) \cdot [\chi_{\eta(\Omega'_m)}] \quad \text{by (10.1.13)}
\end{aligned}$$

Since the $\eta(\Omega'_m)$'s are pairwise disjoint we have that $U[f] = [h] \cdot (T_\eta[f])$. Furthermore, by Remark 10.1.5 $\mu_1 \circ \eta^{-1}$ defined on Σ_2^η is absolutely continuous with respect to $\mu_2 \upharpoonright \Sigma_2^\eta$. The Radon-Nikodym derivative $\frac{d(\mu_1 \circ \eta^{-1})}{d\mu_2}$ therefore exists. Let $A \in \Sigma_1$ and let $B \in \eta([A])$. Then

$$\begin{aligned}
\int_{\eta([A])} |[h]|^p d\mu_2 &= \int_{\Omega_2} |[h][\chi_{\eta([A])}]|^p d\mu_2 \\
&= \|U[\chi_A]\|^p \quad \text{by (10.1.1)} \\
&= \|\chi_A\|^p \quad \text{since } U \text{ is an isometry} \\
&= \mu_1(A) \\
&= \mu_1 \circ \eta^{-1}(B) \quad \text{by definition of } \mu_1 \circ \eta^{-1} \\
&= \int_B \frac{d(\mu_1 \circ \eta^{-1})}{d\mu_2} d\mu_2 \quad \text{by Theorem 1.2.15} \\
&= \int_{\eta(A)} \frac{d(\mu_1 \circ \eta^{-1})}{d\mu_2} d\mu_2
\end{aligned}$$

For the converse, let η be a regular set isomorphism and let h be such that (10.1.2) holds. Define U by (10.1.1) and let f be a simple function. Then f can be written in the form $\sum_{i=1}^n \alpha_i \chi_{A_i}$ where the A_i 's are pairwise disjoint. It is easily checked that

$$(10.1.16) \quad \left| \sum_{i=1}^n \alpha_i \chi_{A_i} \right|^p = \sum_{i=1}^n |\alpha_i|^p \chi_{A_i}$$

Furthermore, the same equality holds if the A_i 's are replaced by $\eta([A_i])$'s since η preserves disjoint unions. It follows that

$$\begin{aligned}
\|U[f]\|^p &= \int_{\Omega_2} \left| [h] \cdot T_\eta \left[\sum_{i=1}^n \alpha_i \chi_{A_i} \right] \right|^p d\mu_2 \\
&= \int_{\Omega_2} |[h]|^p \left| \left(\sum_{i=1}^n \alpha_i [\chi_{\eta(A_i)}] \right) \right|^p d\mu_2 \quad \text{by the linearity of } T_\eta \\
&= \int_{\Omega_2} |[h]|^p \left(\sum_{i=1}^n |\alpha_i|^p \chi_{\eta(A_i)} \right) d\mu_2 \\
&= \sum_{i=1}^n |\alpha_i|^p \int_{\eta([A_i])} |[h]|^p d\mu_2 \\
&= \sum_{i=1}^n |\alpha_i|^p \int_{\eta([A_i])} \frac{d(\mu_1 \circ \eta^{-1})}{d\mu_2} d\mu_2 \quad \text{by (10.1.2)} \\
&= \sum_{i=1}^n |\alpha_i|^p \mu_1([A_i]) \quad \text{as before} \\
&= \int_{\Omega} \left(\sum_{i=1}^n |\alpha_i|^p \chi_{A_i} \right) d\mu_1 \\
&= \int_{\Omega} \left| \sum_{i=1}^n \alpha_i \chi_{A_i} \right|^p d\mu_1 \quad \text{by the R-N theorem (1.2.15)} \\
&= \|f\|^p
\end{aligned}$$

Let $[f] \in L^p(\Omega_1, \Sigma_1, \mu_1)$. There exists a sequence $(f_n)_{n=1}^\infty$ of positive simple functions such that $f_n \uparrow |f|$ pointwise μ_1 -a.e. By Proposition 3.1.4(5) $\|[f_n]\|_p \uparrow \|[f]\|_p$, since $L^p(\mu_1)$ is a Banach function space. Then by Proposition

10.1.6(1),

$$\begin{aligned}
& T_\eta[f_n] \uparrow T_\eta[|f|] \quad \text{pointwise} \\
\Rightarrow & |[h].T_\eta[f_n]|^p \uparrow |[h].T_\eta[|f|]|^p \quad \text{pointwise} \\
\Rightarrow \|U[f_n]\|^p = & \int_{\Omega_2} |[h].T_\eta[f_n]|^p d\mu_2 \uparrow \int_{\Omega_2} |[h].T_\eta[|f|]|^p d\mu_2 \\
& \text{by the Monotone Convergence theorem} \\
& = \|U[f]\|^p
\end{aligned}$$

But $\|U[f_n]\| = \|[f_n]\|$ for each n , since U preserves the norm of positive simple functions. We therefore have that $\|U[f_n]\| = \|[f_n]\| \uparrow \|[f]\|$ and $\|U[f_n]\| \uparrow \|U[f]\|$ and so $\|U[f]\| = \|[f]\|$, since limits are unique in any Hausdorff space. \square

REMARK 10.1.11. Since $T_\eta[f_n] \rightarrow T[f]$ pointwise if $[f_n] \rightarrow [f]$ pointwise (Proposition 10.1.6(1)), a similar method to the one employed in the proof of Proposition 6.1.3 can be used to show that T_η is bounded.

Conditions under which an isometry can be represented using a composition operator are given by the result below.

COROLLARY 10.1.12. *Let (Ω, Σ, μ) be a σ -finite absolute Borel space and let $U : L^p(\Omega, \Sigma, \mu) \rightarrow L^p(\Omega, \Sigma, \mu)$ ($1 \leq p < \infty$, $p \neq 2$) be a linear isometry. Then there exists a generalized composition operator C_τ and a function $h : \Omega \rightarrow \mathbb{F}$ such that for all $[f] \in L^p(\Omega, \Sigma, \mu)$*

$$(10.1.17) \quad [Uf] = [h].C_\tau[f]$$

PROOF. By the theorem above there exists a regular set isomorphism $\eta : [\Sigma] \rightarrow [\Sigma]$ such that

$$[Uf] = [h]T_\eta[f],$$

where T_η is the transformation induced by η . By Corollary 3.2.4, $L^p(\mu)$ is a Banach function space with absolutely continuous norm. Furthermore, $T_\eta([f].[g]) = T_\eta[f].T_\eta[g]$ whenever $[f], [g], [f.g] \in L^p(\mu)$, by Proposition 10.1.6, and T_η is bounded by Remark 10.1.11. T_η is therefore a generalized composition operator by Corollary 9.1.4. \square

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