UNIVERSITY OF CAPE TOWN

Department of Mathematics

Protoreflections, Relational Algebras and Topology

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CONTENTS

In accordance with the regulations governing the award of the degree of Ph.D. at the University of Cape Town, the candidate presents the following summary of the contents of the thesis to indicate in what way it constitutes a contribution to knowledge.

SUMMARY

The pattern of this thesis is broadly speaking, one of increasing specialization. We start with relations in regular categories. Then we study top categories over regular categories. Next we consider categories of relational algebras over regular categories, these being a specialized type of top category. Finally we interpret the results of the previous sections in the category of topological spaces which is itself a category of relational algebras. The concept of a protoreflection serves as a basis for the more important results of sections 5, 6, 9 and 11 on monoreflections and extremal-epi-reflections.

In sections 1 and 2 we summarize and extend the work of Grillet [1971] and Klein [1970] on relations in regular categories. In particular we show that many of the results of Klein concerning the action of a functor on relations are true without the severe conditions which he imposes on the functor. In section 3 we identify the (P) properties of the top categories of Wyler [1971a, b].

All of the results of section 4 concerning protoreflections are new although the basic definition has been used before (by Johnson [1966]) for a different purpose. Many questions concerning protoreflections remain open.

In sections 5 and 6 we obtain many pleasant results for those protoreflections which are monic and those which are extremal-epi.

In sections 7 - 10 we extend the work of Barr [1970] and Manes [1973] concerning categories of relational algebras over the category of sets to relational algebras over regular categories. In section 9 we consider generalized point separation axioms and obtain some nice results for the T_0 property.

Finally in section 11 we obtain results concerning classes of structured equivalence relations and structure functors associated with a given extremal-epireflective subcategory of the category of topological spaces. These give rise to further open problems.

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INTRODUCTION

The subject of this thesis is perhaps best explained by the titles of the papers which form its starting point. These are 'Regular Categories' [Grillet 1971], 'Relations in Categories' [Klein 1970], 'Top Categories and Categorical Topology' [Wyler 1971b and also 1971a], 'Limit Operators and Topological Coreflections' [Herrlich 1969a], 'A Universal Factorization Theorem in Topology' [Sharpe, Beattie and Marsden 1966], 'Relational Algebras' [Barr 1970] and 'Compact Hausdorff Objects' [Manes 1973].

The work in this thesis is concerned mainly with two concepts, viz. protoreflections and categories of relational algebras.

Protoreflections appear in a paper of Johnson [1966] under the name 'relative reflections'. However Johnson does not 'iterate' his relative reflections and his work, bar the definition, is entirely disjoint from ours. The term 'protoreflection' was suggested by my supervisor.

Special examples of protoreflections have been extensively studied. Extremal-epiprotoreflections are considered by Sharpe, Beattie and Marsden [1966] who did not, however form the iterates of their protoreflections. Heller and Rowe [1962] define a sheaf protoreflection and they form iterates of it in order to obtain the sheaf reflection (see also Grillet [1971]). Fakir [1970] constructs a protocoreflection and iterates it in order to obtain the 'idempotent monad' coreflection. Protoreflections appear in a slightly more implicit form in the papers of Barr [1970]

'Relational Prealgebras to Relational Algebras' and Thomas [1969] 'Topological Spaces to Regular Topological Spaces.

We have not been able to find any general treatment of protoreflections in the literature.

Relational Algebras are better known. They have been studied by Barr [1970], A. Burroni [1971], Manes [1973] and E. Burroni [1973].

Barr gives the basic result that the topological spaces are the relational algebras of the ultrafilter monad on the category of sets while Manes gives some general properties of categories of relational algebras of monads on the category of sets.

Sections 1, 2 and 3 of this thesis are mainly introductory and are concerned with relations in regular categories and top categories.

The definition of a protoreflection is given in section 4. We show how to form iterates of a protoreflection and discuss the question of when this process gives us a reflection. An interesting problem which we have not been able to settle is the equivalence or otherwise, in general of the concepts of 'weakly generating' and 'strongly generating' a reflection. Epi-protorefelections and mono-protoreflections have pleasant properties particularly in regular categories and in top categories whose base category is regular.

Section 5, inspired by a paper of Herrlich [1969a] is concerned with monoprotoreflections in top categories whose base is regular. Here we study the correspondence between monoprotoreflections and structure functors. We also obtain some results concerning the lifting of extremal-epireflective subcategories using structure functors. Section 6 inspired by Sharpe, Beattie and Marsden [1966] studies extremal-epiprotoreflections in the same setting as section 5. Here we have a correspondence between extremal-epiprotoreflections and structured equivalence relations. We obtain some results concerning the class of structured relations associated with a given extremal-epireflective subcategory. We also study the concepts of initial, hereditary and productive structured relations and show how these properties are reflected in the corresponding extremal-epiprotoreflections.

In section 9 we study the 'T_i', 'R_i' and 'S_i' protoreflections in a category of relational algebras. In section 11 the setting is the category of topological spaces and here we obtain some interesting properties of the T_i 'point separation axiom' protoreflections. The reader's attention is drawn to the result 11.4.2 where we show that the 'natural' T_2 protoreflection is in a sense the smallest protoreflection which generates the T_2 reflection. Can one construct a similar proof in the case of Urysohn spaces, Completely Hausdorff spaces etc.? See also the problem 11.6.

Relational Algebras are studied in section 7 - 10 where we extend the work of Barr and Manes from relational algebras over sets to relational algebras over regular categories. We have been able to generalise most of their results, some with rather interesting proofs.

See particularly 8.3.1 and 8.3.3. Four results remain to be generalized. See 8.4, 9.11.3, 10.5 and Manes' 'Stone-Cech compactification theorem'.

The study in section 9 of 'point separation axioms' in a category of relational algebras was inspired by some definitions of Ramaley and Wyler [1970a]. The best results here are perhaps those concerned with the T_0 property (see 9.9.2).

Finally the second part of section 11 is concerned with solutions to some problems of Thornton [1971].

0. NOTATION

Our terminology for categories is that of Mac Lane [1971] and Herrlich [1968] with few exceptions. In particular, categories are denoted by $\underline{A}$, $\underline{B}$, $\underline{C}$ etc, their objects by A, B, X, Y etc. and their morphisms by f, g, h, a etc. Monics are monomorphisms, also denoted by $\rightarrowtail$. Epis are epimorphisms and regular-epis (see §1) are denoted by $\twoheadrightarrow$. Relations are denoted by $\rightharpoonup$.

$$\begin{tikzcd} {} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{PB}"] & {} \arrow[d] \\ {} \arrow[r] & {} \end{tikzcd}$$

denotes a pullback diagram. Products are denoted by $X \times Y$, $\mathsf{X}\, X_i$ and sums by $X \sqcup Y$ and $\bigsqcup X_i$. For a family of morphisms $f_i : X_i \longrightarrow Y_i$, $\mathsf{X} f_i : \mathsf{X}\, X_i \longrightarrow \mathsf{X}\, Y_i$ is the product of the f_i. For a family $f_i : X \longrightarrow Y_i$, $\langle f_i \rangle$ is the induced morphism $X \longrightarrow \mathsf{X}\, Y_i$. Note that this notation differs from that of Grillet [1971].

$\underline{\text{Sets}}$ is the usual category of sets, $\underline{\text{Top}}$ is the category of topological spaces and $\underline{\text{Haus}}$ is the category of Hausdorff topological spaces.

I. RELATIONS IN REGULAR CATEGORIES

Relations in categories have been discussed by various authors, among them Mac Lane [1961], Klein [1970], Grillet [1971], Fay [1973] and Meisen [1973a, b]. Mac Lane is concerned with abelian categories, Klein and Meisen with relations in a category with a given bicategory structure, while Grillet as well as Meisen discuss relations in regular categories.

In this work we will follow Grillet by restricting our attention to relations in regular categories. This restriction appears to be justified for the following reasons: The axioms for a regular category are weak enough to include as examples (1) sets, (2) abelian categories, (3) varieties of universal algebras, (4) algebras of monads in regular categories, (5) functor categories of the form Funct ($\underline{X}$, $\underline{A}$) with $\underline{X}$ small and $\underline{A}$ regular, (6) every partially ordered set which has infima of finite subsets, considered as a category, (7) compact Hausdorff spaces and (8) compact Hausdorff zero dimensional spaces. The axioms are also strong enough to give relations in regular categories most of the properties required for the proof of the 'topological type' results which we present in later sections.

Our starting point is thus Grillet's paper. In this section we summarise the results of Grillet which we will use later on and also derive a number of additional properties of relations.

1.1 Definition: [Grillet 1971]

A regular-epi is a coequaliser.

A *regular category* is a finitely complete category with (regular-epi, mono) decompositions in which pullbacks preserve regular-epis.

A category is *regularly co-well-powered* if for each object X in the category there is a representative set of regular-epis with domain X.

A category is regular in our sense if and only if it is regular in the sense of Barr [1971] and is finitely complete.

For the rest of this section we work in a regular category C.

1.2 Proposition [Grillet 1971, pp. 121-138]

(a) $\underline{C}$ has a terminal object henceforth denoted by N.

(b) If $\underline{C}$ has arbitary products then it is complete.

(c) In $\underline{C}$, regular-epi is equivalent to extremal-epi.

(d) Every split-epi is regular.

(e) If ge = mf with e regular-epi and m monic, then there is a unique t such that

$$\begin{tikzcd}
\bullet \arrow[r, two heads, "e"] \arrow[d, "f"'] & \bullet \arrow[d, "g"] \arrow[dl, dashed, "t"'] \\
\bullet \arrow[r, tail, "m"'] & \bullet
\end{tikzcd}$$

commutes.

(f) A morphism is an isomorphism if and only if it is both monic and regular-epi.

(g) If f and g are regular-epi and fg is defined, then fg is regular-epi.

(h) If fg is regular-epi then f is regular-epi.

(i) Every finite product of regular-epis is regular-epi.

(j) Every morphism f has a unique*(regular-epi, mono) decomposition.

1.3 Subobjects, Direct and Inverse Images. [Grillet, 1971]

We recall that a *subobject* of an object A in $\underline{C}$ is an equivalence class of monics with codomain A.

If $m\colon \cdot \rightarrowtail A$ is monic, then Im m denotes the subobject of A which contains m. We have a partial order relation on subobjects of A given by: Im m $\leq$ Im n if and only if m = nt for some morphism t.

$$\begin{tikzcd}
\cdot \arrow[r, tail, "m"] \arrow[d, dashed, "t"'] & A \\
\cdot \arrow[ur, tail, "n"'] &
\end{tikzcd}$$

t is then monic.

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* We shall use the word "unique" when it is clear from the context that we mean "unique up to isomorphism".

$\underline{C}$ has finite intersections of subobjects and these intersections are infima relative to the order relation. If $\{\mathrm{Im}\ m_i\}$ is a family of subobjects of A, then $\bigwedge \mathrm{Im}\ m_i$ denotes their intersection $\mathrm{Im} \cap m_i$ (if it exists). If $\underline{C}$ has arbitrary intersections then suprema exist and they are denoted by $\bigvee \mathrm{Im}\ m_i$. $1_A = \mathrm{Im}\ 1_A$ is the largest subobject of A.

Each morphism $f\colon A \longrightarrow B$ has an <u>image</u> in the sense of Mitchell [1966] given by $\mathrm{Im}\ f = \mathrm{Im}\ m$ where m is the monic in the (regular-epi, mono) decomposition of f. If t is a regular-epi then $\mathrm{Im}(ft) = \mathrm{Im}\ f$. If Im m is a subobject of A then $f_*(\mathrm{Im}\ m) = \mathrm{Im}(fm)$ is a subobject of B called the <u>direct image</u> of Im(m) under f.

On the other hand if Im n is a subobject of B, we define $f^*(\mathrm{Im}\ n) = \mathrm{Im}\ m$ where

$$\begin{tikzcd}
A \arrow[r, "f"] & B \\
\bullet \arrow[u, tail, "m"] \arrow[r, "f'"] \arrow[ur, phantom, "\text{PB}"] & \bullet \arrow[u, tail, "n"']
\end{tikzcd}$$

is a pullback.

$f^*(\mathrm{Im}\ n)$ is called the <u>inverse image</u> of Im n under f.

f_* and f^* are order preserving and induce a Galois connection. [Birkhoff, 1948] between the subobjects of A and the subobjects of B. f_* preserves suprema, f^* preserves infima and $f_*(\mathrm{Im}\ m) \leq \mathrm{Im}\ n$ if and only if $\mathrm{Im}\ m \leqslant f^*(\mathrm{Im}\ n)$.

1.3.1 Proposition [Grillet, 1971]

(a) $1_* = 1$, $1^* = 1$.

(b) $(fg)_* = f_* g_*$, $(fg)^* = g^*\ f^*$.

(c) If $g = ft$ then $\mathrm{Im}\ g \leqslant \mathrm{Im}\ f$.

1.4 Relations [Grillet, 1971 pp. 143-153]

If A,B are objects of C, then a relation $\alpha : A \longrightarrow B$ is a subobject of $A \times B$, or equivalently a subobject of $A \times B$ determined by a pair

$$\begin{tikzcd}
 & A \\
\bullet \arrow[ur, "a"] \arrow[dr, "b"'] & \\
 & B
\end{tikzcd}$$

for which $\langle a, b\rangle : \bullet \rightarrowtail A \times B$ is monic.

The order on relations is that given by the order on subobjects: if $\langle a, b\rangle : R \longrightarrow A \times B$ and $\langle a', b'\rangle : R' \longrightarrow A \times B$ with $f : R \longrightarrow R'$ such that $a' f = a$ and $b' f = b$ then $\mathrm{Im}\ \langle a, b\rangle \leqslant \mathrm{Im}\ \langle a', b'\rangle$. Every morphism $f : A \longrightarrow B$ can be uniquely interpreted as a relation $A \longrightarrow B$ via the correspondence $f \longmapsto \mathrm{Im}\langle 1, f\rangle$.

If
$$\begin{tikzcd}
 & A \\
\bullet \arrow[ur, "a"] \arrow[dr, "b"'] & \\
 & B
\end{tikzcd}$$
is a pair of morphisms then the relation

$\mathrm{Im}\ \langle a, b\rangle : A \longrightarrow B$ is a morphism if and only if a is an isomorphism.

Each relation $\alpha : A \longrightarrow B$ has associated with it an <u>inverse</u> relation $\alpha^{-1} : B \longrightarrow A$ given by $\alpha^{-1} = \mathrm{Im}\langle b, a\rangle$ for any pair a, b such that $\alpha = \mathrm{Im}\langle a, b\rangle$.

<u>Composition</u> of relations is defined using pullbacks as follows:

If $\alpha : A \longrightarrow B$ and $\beta : B \longrightarrow C$ are relations with $\alpha = \mathrm{Im}\ \langle a, b\rangle$, $\beta = \mathrm{Im}\langle c, d\rangle$, then $\beta\alpha : A \longrightarrow C$ is the relation $\beta\alpha = \mathrm{Im}\langle ac', db'\rangle$ where $bc' = cb'$ is a pullback in the diagram

$$\begin{tikzcd}
 & & \bullet \arrow[dl, "c'"'] \arrow[dr, "b'"] & & \\
 & \bullet \arrow[dl, "a"'] \arrow[dr, "b"] & \text{PB} & \bullet \arrow[dl, "c"'] \arrow[dr, "d"] & \\
A & & B & & C
\end{tikzcd}$$

This definition is consistent. The composition of relations is associative, order preserving and is compatible with that of morphisms.

The formation of inverses of relations and the taking of direct and inverse images are order preserving operations. In addition forming inverses preserves both suprema and infima while taking inverse images preserve infima and direct images suprema.

I_A denotes the identity relation (= identity morphism) on an object A.

1.4.1 Proposition [Grillet, 1971]

Let $\alpha : A \longrightarrow B$ and $\beta : B \longrightarrow C$ be relations.

Then:

(a) $(\alpha^{-1})^{-1} = \alpha$, $(\beta\alpha)^{-1} = \alpha^{-1}\beta^{-1}$.

(b) $I_B\alpha = \alpha$, $\alpha I_A = \alpha$.

(c) If $\alpha = \operatorname{Im} \langle a, b\rangle$, then $\alpha = ba^{-1}$ ($\langle a, b\rangle$ need not be monic).

(d) The following are equivalent: (i) α is a morphism,

(ii) $\alpha\alpha^{-1} \leqslant I_B$ and $\alpha^{-1}\alpha \geqslant I_A$,

and then $\alpha\alpha^{-1}\alpha = \alpha$.

(e) If $f : A \longrightarrow B$ then f is monic if and only if $f^{-1}f = I_A$ and f is regular-epi if and only if $ff^{-1} = I_B$.

(f) If $f, g : A \rightrightarrows B$ and $f \leqslant g$ then $f = g$.

(g) If $\alpha = \operatorname{Im} \langle a, a\rangle : A \longrightarrow A$ and a is monic, then $I_A \leqslant \alpha$ implies $I_A = \alpha$.

(h) If

$$\begin{tikzcd}
{} \arrow[r, "h"] \arrow[d, "k"'] & {} \arrow[d, "g"] \\
{} \arrow[r, "f"'] & {}
\end{tikzcd}$$

commutes then $hk^{-1} \leqslant g^{-1}f$, and if it is a pullback then $hk^{-1} = g^{-1}f$.

1.4.2 Proposition [Klein 1970, Meisen 1973a]

Let $\alpha : A \longrightarrow B$ with $\alpha = \operatorname{Im} \langle a, b\rangle$ and $\langle a,b\rangle$ monic.

Then:

(a) $\alpha\alpha^{-1} \leqslant I_B \Longleftrightarrow$ a is monic.

$\alpha\alpha^{-1} \geqslant I_B \Longleftrightarrow$ b is regular-epi.

(b) $\alpha^{-1}\alpha \leqslant I_A \Longleftrightarrow$ b is monic.

$\alpha^{-1}\alpha \geqslant I_A \Longleftrightarrow$ a is regular-epi.

1.5 Proposition: Miscellaneous results.

(a) If $\alpha_i : B \longrightarrow C$ for each i and $f : A \longrightarrow B$ then $(\bigwedge \alpha_i)f = \bigwedge(\alpha_i f)$.

(b) If $m_i : B_i \rightarrowtail B$ is monic for each i and $m : B' \rightarrowtail B$ is their intersection, then $mm^{-1} = \bigwedge m_i m_i^{-1}$.

(c) If $m_i : B_i \rightarrowtail B$ as above and $\alpha : A \longrightarrow B$, then $\bigwedge(m_i m_i^{-1}\alpha) = (\bigwedge m_i m_i^{-1})\,\alpha$.

Proof:

(a) We certainly have $(\bigwedge \alpha_i)f \leqslant \bigwedge(\alpha_i f)$. If β is a relation with $\beta \leqslant \alpha_i f$ for each i, then $\beta f^{-1} \leqslant \alpha_i f f^{-1} \leqslant \alpha_i$.

So $\beta f^{-1} \leqslant \bigwedge \alpha_i$ and it follows that $\beta \leqslant \beta f^{-1} f \leqslant (\bigwedge \alpha_i)f$.

Thus $(\bigwedge \alpha_i)f = \bigwedge(\alpha_i f)$.

(b) Let $t_i : B' \rightarrowtail B_i$ be the canonical morphisms such that the diagrams

$$\begin{tikzcd} & B_i \arrow[dr, tail, "m_i"] & \\ B' \arrow[ur, tail, "t_i"] \arrow[rr, tail, "m"'] & & B \end{tikzcd}$$

commute.

Now $m_i m_i^{-1} = \operatorname{Im} \langle m_i, m_i \rangle$ and $mm^{-1} = \operatorname{Im} \langle m, m \rangle$.

It is easily seen that

$$\begin{tikzcd}
B_i \arrow[r, tail, "{\langle m_i, m_i\rangle}"] & B\times B \\
B' \arrow[u, tail, "t_i"] \arrow[ur, tail, "{\langle m,m\rangle}"']
\end{tikzcd}$$

(for all i) is a pullback.

Thus $mm^{-1} = \operatorname{Im}\langle m,m\rangle = \bigcap \operatorname{Im}\langle m_i, m_i\rangle = \bigwedge m_i m_i^{-1}$.

(c) Let $\alpha = \operatorname{Im}\langle a, b\rangle$ with $\langle a, b\rangle : R \rightarrowtail A\times B$ monic.

Let m and t_i be as in (b).

Now $(\bigwedge m_i m_i^{-1})\alpha = mm^{-1}\alpha$ is given by:

$$\begin{tikzcd}
 & & Q \arrow[dl, "m'"'] \arrow[dr, "b'"] & & \\
 & R \arrow[dl, "a"'] \arrow[dr, "b"] & \text{PB} & B' \arrow[dl, tail, "m"'] \arrow[dr, tail, "m"] & \\
A & & B & & B
\end{tikzcd}$$

......(i),

Using the fact that m and $\langle a, b\rangle$ are monic we see easily that $\langle am', mb'\rangle : Q \rightarrowtail A\times B$ is monic.

Thus $(\bigwedge m_i m_i^{-1})\alpha = \operatorname{Im}\langle am', mb'\rangle$.

Now for each i, $m_i m_i^{-1}\alpha$ is given by

$$\begin{tikzcd}
 & & S_i \arrow[dl, "c_i"'] \arrow[dr, "d_i"] & & \\
 & R \arrow[dl, "a"'] \arrow[dr, "b"] & \text{PB} & B_i \arrow[dl, tail, "m_i"'] \arrow[dr, tail, "m_i"] & \\
A & & B & & B
\end{tikzcd}$$

......(ii),

with $\langle ac_i, m_id_i\rangle$ monic.

Thus $m_im_i^{-1}\alpha = \mathrm{Im}\,\langle ac_i, m_id_i\rangle$.

If for each i we can find a morphism $s_i : Q \longrightarrow S_i$ such that

$$\begin{tikzcd}
S_i \arrow[r, "\langle ac_i, m_id_i\rangle"] & A \times B \\
Q \arrow[u, dashed, "s_i"] \arrow[ur, "\langle am', mb'\rangle"'] &
\end{tikzcd}$$

(for all i) ....(iii),

is a pullback, then $(\bigwedge m_im_i^{-1})\alpha = \mathrm{Im}\,\langle am', mb'\rangle$

$= \bigcap \mathrm{Im}\,\langle ac_i, m_id_i\rangle$

$= \bigwedge (m_im_i^{-1}\alpha)$ and the proof will be complete.

Now for each i, $bm' = mb' = m_it_ib'$ and so using the pullback in (ii) we have morphisms $s_i : Q \longrightarrow S_i$ such that $c_is_i = m'$ and $d_is_i = t_ib'$. So $ac_is_i = am'$ and $m_id_is_i = mb'$. Thus these s_i make (iii) commute. It remains to be shown that (iii) is a pullback.

To do this suppose that $u_i : U \longrightarrow S_i$ and $u : U \longrightarrow A \times B$ are morphisms such that $\langle ac_i, m_id_i\rangle\, u_i = u$ for each i.

$$\begin{tikzcd}
S_i \arrow[rr, "\langle ac_i, m_id_i\rangle"] & & A \times B \\
Q \arrow[u, "s_i"'] \arrow[urr, "\langle am', mb'\rangle"'] & & \\
U \arrow[u, dashed, "W"'] \arrow[uu, bend left, "u_i"] \arrow[uurr, bend right, "u"'] & &
\end{tikzcd}$$

...... (iv)

We need to fill in a w such that (iv) commutes (since $\langle am', mb' \rangle$ is monic, such a w will be unique).

Let $p_A : A \times B \longrightarrow A$ and $p_B : A \times B \longrightarrow B$ be the projections.

Now

$$\begin{tikzcd}
U \arrow[r, "u_i"] \arrow[dr, "u"'] & S_i \arrow[r, "d_i"] & B_i \arrow[dd, tail, "m_i"] \\
 & A \times B \arrow[dr, "p_B"'] & \\
 & & B
\end{tikzcd}$$

commutes for each i and so since $m : B' \rightarrowtail B$ is the intersection of the m_i.

there is a morphism $v : U \longrightarrow B'$ such that the diagrams

$$\begin{tikzcd}
U \arrow[rr, "d_i u_i"] \arrow[dr, "v"'] & & B_i \\
 & B' \arrow[ur, "t_i"'] &
\end{tikzcd}$$

commute. .... (v).

Now for all i,j $ac_i u_i = p_A u = ac_j u_j$ and

$$bc_i u_i = m_i d_i u_i = p_B u = bc_j u_j .$$

So $c_i u_i = c_j u_j = k$ (say).

Now $bk = bc_i u_i = m_i d_i u_i = m_i t_i v = mv$.

$$\begin{tikzcd}
& U \arrow[ddl, "k = c_i u_i"'] \arrow[d, "w"] \arrow[ddr, "v"] & \\
& Q \arrow[dl, "m'"'] \arrow[dr, "b'"] \arrow[dd, phantom, "\text{PB}"] & \\
R \arrow[dr, "b"'] & & B' \arrow[dl, "m"] \\
& B &
\end{tikzcd}$$

.... (vi).

Thus there is a morphism $w: U \longrightarrow Q$ which makes (vi) commute.

To complete the proof we need only show that this w makes (iv) commute.

Now $p_A \langle am', mb' \rangle w = am'w = ac_i u_i = p_A u$ and

$p_B \langle am', mb' \rangle w = mb'w = mv = m_i t_i v = m_i d_i u_i = p_B u$.

Thus $\langle am', mb' \rangle w = u$. Also for each i $(ac_i)s_i w = am'w = (ac_i)u_i$,

and $(m_i d_i)s_i w = m\,b'w = mv = m_i t_i v = (m_i d_i)u_i$.

Thus since the $\langle ac_i, m_i d_i \rangle$ are monic we have $s_i w = u_i$ for each i.

So (iv) commutes. □

Let $f:A \longrightarrow A'$ and $g:B \longrightarrow B'$ be morphisms.

If $\alpha:A \rightharpoonup B$ is a relation then $(f \times g)_* \alpha:A' \rightharpoonup B'$ is the relation obtained by regarding α as a subobject of $A \times B$. Similarly if $\alpha':A' \rightharpoonup B'$ then $(f \times g)^* \alpha':A \rightharpoonup B$. Grillet shows that for $f = g$ we have $(f \times f)_* \alpha = f\alpha f^{-1}$ and $(f \times f)^* \alpha^1 = f^{-1}\alpha' f$. We generalise this result as follows.

1.6 Proposition:

Let $f:A \longrightarrow A'$ and $g:B \longrightarrow B'$.

(a) For $\alpha:A \longrightarrow B$ we have $(f \times g)_*\alpha = g\alpha f^{-1}:A' \longrightarrow B'$.

(b) For $\alpha':A' \longrightarrow B'$ we have $(f \times g)^*\alpha' = g^{-1}\alpha' f:A \longrightarrow B$.

Proof:

(a) Let $\alpha = \operatorname{Im} \langle a,b\rangle$ with $\langle a,b\rangle : R \rightarrowtail A \times B$ monic.

Then $(f \times g)_*\alpha = \operatorname{Im} (f \times g) \langle a,b\rangle = \operatorname{Im} \langle fa,gb\rangle$. Using the description of $g\alpha f^{-1}$ via pullbacks, we see that $g\alpha f^{-1} = \operatorname{Im}\langle fa,gb\rangle$.

(b) By the Galois connection $(f \times g)^*\alpha' = \operatorname{Sup}\{\alpha:A \longrightarrow B \mid (f\times g)_*\alpha \leqslant \alpha'\}$.

Now $(f \times g)_*(g^{-1}\alpha' f) = gg^{-1}\alpha' ff^{-1} \leqslant \alpha'$ and if $(f \times g)_*\alpha \leqslant \alpha'$ then $g\alpha f^{-1} \leqslant \alpha'$, so $\alpha \leqslant g^{-1}g\alpha f^{-1}f = g^{-1}\alpha' f$.

Thus $g^{-1}\alpha' f = (f \times g)^*\alpha'$. □

1.7 Products of Relations

Let $X = \bigtimes X_i$, $Y = \bigtimes Y_i$ and suppose that for each i we have a relation $\alpha_i:X_i \longrightarrow Y_i$. We wish to define a relation $\alpha:\bigtimes X_i \longrightarrow \bigtimes Y_i$ which will be the 'product' of the relations α_i. Our definition must be compatible with the usual product of morphisms and must be a generalisation of the situation when $\underline{C}$ = sets. We make the following definition:

Let $\alpha_i = \operatorname{Im}\langle a_i,b_i\rangle$ with $\langle a_i,b_i\rangle : R_i \rightarrowtail X_i \times Y_i$ monic for each i. Let $a = \bigtimes a_i$ and $b = \bigtimes b_i$. Now define:

$\bigtimes \alpha_i = ba^{-1} = \operatorname{Im}\langle a,b\rangle$. Note that $\langle a,b\rangle$ is monic.

One sees easily that the formation of products of relations is an order preserving operation.

$$\begin{tikzcd}
X = \times X_i \arrow[d, "p_i"'] & \times R_i \arrow[l, "a = \times a_i"'] \arrow[r, "b = \times b_i"] \arrow[d, "r_i"] & \times Y_i = Y \arrow[d, "q_i"] \\
X_i & R_i \arrow[l, "a_i"] \arrow[r, "b_i"'] & Y_i
\end{tikzcd}$$

p_i, q_i, r_i are the projections.

We see easily that $\times(\alpha_i^{-1}) = (\times\alpha_i)^{-1}$.

Now suppose that $\alpha_i = \mathrm{Im}\ \langle a_i', b_i'\rangle$ with $\langle a_i', b_i'\rangle : R_i' \longrightarrow X_i \times Y_i$ not necessary monic. We want to know under what circumstances we will have $ba^{-1} = b'(a')^{-1}$ (where $a' = \times a_i'$, $b = \times b_i'$).

For each i there is a regular-epi t_i such that $a_i t_i = a_i'$ and $b_i t_i = b_i'$. So $b'(a')^{-1} = (\times b_i')(\times a_i')^{-1} = [(\times b_i)(\times t_i)][(\times a_i)(\times t_i)]^{-1}$ $= (\times b_i)(\times t_i)(\times t_i)^{-1}(\times a_i)^{-1}$.

If we are handling finite products then $\times t_i$ is a regular-epi and so $(\times t_i)(\times t_i)^{-1} = 1$. Thus in this case $b'(a')^{-1} = ba^{-1}$.

The same is true in general if $\underline{C}$ is a $\underline{\text{QI}}$ category in the following sense:

1.7.1 Definition:

$\underline{C}$ is a $\underline{\text{QI}}$ $\underline{\text{category}}$ if and only if every product of regular-epis in $\underline{C}$ is again regular-epi.

This condition is called regular C_1^* by Grillet [1971, p171]. It is satisfied if $\underline{C} = \underline{\text{sets}}$, or more generally if $\underline{C}$ is any variety •

1.7.2 Proposition

The following properties of $\underline{C}$ are equivalent:

(a) $\underline{C}$ is a QI category.

(b) Given relations α_i, $\chi\alpha_i = (\chi b_i)(\chi a_i)^{-1}$ for all pairs (a_i, b_i) such that $\alpha_i = \mathrm{Im}\,\langle a_i, b_i\rangle$ (with $\langle a_i, b_i\rangle$ not necessarily monic).

(c) Given $\alpha_i : X_i \longrightarrow Y_i$ and $\beta_i : Y_i \longrightarrow Z_i$ we have

$$(\chi\beta_i)(\chi\alpha_i) = \chi\beta_i\alpha_i .$$

Proof:

(a) $\Longrightarrow$ (b): Follows from the preceeding discussion.

(b) $\Longrightarrow$ (a): Given regular-epis $t_i : R_i \twoheadrightarrow X_i$ we have $\mathrm{Im}\,\langle t_i, t_i\rangle = 1$. So by (b), $(\chi t_i)(\chi t_i)^{-1} = 1$. Thus χt_i is regular-epi.

(c) $\Longrightarrow$ (a): For t_i as above, (c) tells us that $(\chi t_i)(\chi t_i)^{-1} = \chi(t_i t_i^{-1}) = 1$. Thus χt_i is regular-epi.

(a) $\Longrightarrow$ (c): Let $\alpha_i = \mathrm{Im}\,\langle a_i, b_i\rangle$ and $\beta_i = \mathrm{Im}\,\langle c_i, d_i\rangle$ with $\langle a_i, b_i\rangle$ and $\langle c_i, d_i\rangle$ monics.

Now $\beta_i\alpha_i$ is given by:

$$\begin{tikzcd}
 & & \bullet \arrow[dl, "c_i'"'] \arrow[dr, "b_i'"] & & \\
 & \bullet \arrow[dl, "a_i"'] \arrow[dr, "b_i"] & \text{PB} & \bullet \arrow[dl, "c_i"'] \arrow[dr, "d_i"] & \\
X_i & & Y_i & & Z_i
\end{tikzcd}$$

So $\beta_i\alpha_i = \mathrm{Im}\,\langle a_i c_i^!,\ d_i b_i^! \rangle$.

Thus $\bigtimes(\beta_i\alpha_i) = (\bigtimes d_i b_i^!)(\bigtimes a_i c_i^!)^{-1} = (\bigtimes d_i)(\bigtimes b_i^!)(\bigtimes c_i^!)^{-1}(\bigtimes a_i)^{-1}$.

By Mac Lane's interchange of limits theorem [1971 pp. 210],

$(\bigtimes b_i)(\bigtimes c_i^!) = (\bigtimes c_i)(\bigtimes b_i^!)$ is a pullback. It follows by 1.4.1(h)

that $(\bigtimes b_i^!)(\bigtimes c_i^!)^{-1} = (\bigtimes c_i)^{-1}(\bigtimes b_i)$,

So $\bigtimes(\beta_i\alpha_i) = (\bigtimes d_i)(\bigtimes c_i)^{-1}(\bigtimes b_i)(\bigtimes a_i)^{-1} = (\bigtimes \beta_i)(\bigtimes \alpha_i)$. $\square$

1.7.3 Remarks:

Our definition of the product of relations is the same as Klein's. The above result extends Klein's result 3.3 [1970].

1.7.4 Proposition: An alternative description of the product of relations.

Let $\alpha_i : X_i \longrightarrow Y_i$ for each i.

Then $\bigtimes \alpha_i = \bigwedge (p_i \times q_i)^{*}\alpha_i = \bigwedge (q_i^{-1}\alpha_i p_i)$ where

$p_i : \bigtimes X_i \longrightarrow X_i$, $q_i : \bigtimes Y_i \longrightarrow Y_i$ are the projections.

Proof:

Let $\alpha_i = \mathrm{Im}\langle a_i,b_i\rangle$ with $\langle a_i,b_i\rangle : R_i \rightarrowtail X_i \times Y_i$ monic.

By 1.6b we have $(p_i \times q_i)^*\alpha_i = (q_i^{-1}\alpha_i p_i)$.

Set $(p_i \times q_i)^*\alpha_i = \mathrm{Im}\, m_i$ as in the following pullback diagram:

$$\begin{tikzcd}
(\bigtimes X_i)\times(\bigtimes Y_i) \arrow[r, "p_i \times q_i"] \arrow[dr, phantom, "\text{PB}"] & X_i \times Y_i \\
S_i \arrow[u, tail, "m_i"] \arrow[r, "s_i"'] & R_i \arrow[u, tail, "{\langle a_i,b_i\rangle}"']
\end{tikzcd}$$

By the associativity of products the $p_i \times q_i$ are the projections $\bigtimes(X_i \times Y_i) \longrightarrow X_i \times Y_i$.

Let $a = \bigtimes a_i$ and $b = \bigtimes b_i$.

Let $r_i : \bigtimes R_i \longrightarrow R_i$ be the projections. It is easy to see that the outer square of the following diagram commutes:

$$\begin{tikzcd}
& (\bigtimes X_i)\times(\bigtimes Y_i) \arrow[r, "p_i \times q_i"] \arrow[dr, phantom, "\text{PB}"] & X_i \times Y_i \\
& S_i \arrow[u, tail, "m_i"] \arrow[r, "s_i"] & R_i \arrow[u, tail, "{\langle a_i,b_i\rangle}"'] \\
\bigtimes R_i \arrow[uur, tail, "{\langle a,b\rangle}"] \arrow[ur, dashed, "t_i"] \arrow[urr, "r_i"'] & &
\end{tikzcd}$$

.........(i).

So using the pullback we can fill in morphisms t_i so that the diagram (i) commutes.

Now by definition $\times\alpha_i = \mathrm{Im}\ \langle a,b\rangle$.

So we need only show that $\mathrm{Im}\ \langle a,b\rangle = \bigwedge \mathrm{Im}\ m_i$:

Suppose that in the diagram

$$\begin{tikzcd}
 & (\times X_i)\times(\times Y_i) \arrow[r, "p_i \times q_i"] & X_i \times Y_i \\
 & S_i \arrow[u, tail, "m_i"'] \arrow[r, "s_i"'] \arrow[ur, phantom, "\text{(PB)}"] & R_i \arrow[u, tail, "\langle a_i{,}b_i\rangle"'] \\
XR_i \arrow[uur, tail, "\langle a{,}b\rangle"] \arrow[ur, "t_i"] & & \\
Q \arrow[u, dashed, "t"] \arrow[uur, bend right, "u_i"'] \arrow[uuur, bend left=40, "u"] & &
\end{tikzcd} \qquad \ldots\ldots\text{(ii)}$$

$m_i u_i = u$ for each i. We need to find a morphism t which makes (ii) commute. (Uniqueness of t follows as $\langle a,b\rangle$ is monic).

Let $t = \langle s_i u_i\rangle$. So $r_i t = s_i u_i$.

$$\begin{aligned}
\text{Now } (p_i \times q_i)\ \langle a,b\rangle\ t &= (p_i \times q_i) m_i t_i t \\
&= \langle a_i,b_i\rangle\ s_i t_i t \\
&= \langle a_i,b_i\rangle\ r_i t \\
&= \langle a_i,b_i\rangle\ s_i u_i \\
&= (p_i \times q_i)\ m_i u_i \\
&= (p_i \times q_i) u \quad \text{for each } i .
\end{aligned}$$

Thus since the $(p_i \times q_i)$ are the projections of a product, we have $\langle a,b\rangle\ t = u$.

Now $m_i u_i = u = \langle a,b\rangle\ t = m_i t_i t$ and since m_i is monic we have

$u_i = t_i t$. Thus the diagram (ii) commutes.

This completes the proof. $\square$

1.8 Congruences [Grillet 1971 pp. 154-169]

Let $f:A \longrightarrow B$. We define $\ker f = f^{-1}f:A \longrightarrow A$ and this is called the congruence induced by f. It is clear that $\ker f = \operatorname{Im} \langle a,b\rangle$ where $\bullet \overset{a}{\underset{b}{\rightrightarrows}} A$ is the kernel pair of f.
If $f = mt$ is the (regular-epi, mono) decomposition of f, then $\ker f = \ker t$.

$\ker 1_A = 1_A$ is the least congruence on an object A. There is also a greatest congruence on A, $\ker n_A$ where $n_A:A \longrightarrow N$ is the unique morphism to the terminal object N.
In fact $\ker n_A = \operatorname{Im} 1_{A\times A}$

Every congruence $\propto$ is an equivalence relation, i.e. it is symmetric $\propto = \propto^{-1}$, reflexive $1 \leqslant \propto$ and transitive $\propto\propto \leqslant \propto$.
The converse is however not true. Following Grillet, we say that the category $\underset{\sim}{C}$ satisfies Lawvere's condition if:
(L) Every equivalence relation in $\underset{\sim}{C}$ is a congruence.

For a regular category our concept of an equivalence relation agrees with that of Kock and Wraith [1971], Barr [1971], with Lawvere's [1963] 'precongruences' and Duskin's [1969] 'equivalence pairs'. Lawvere's condition is equivalent to Duskin and Barr's conditions ' every equivalence pair is effective', and to Lawvere's condition 'every precongruence is a congruence'.

The category of sets, every elementary Topos [Kock and Wraith 1971], every category monadic over $\underline{\text{sets}}$ [Duskin 1969 5.13], every algebraic variety as well as all abelian categories satisfy Lawvere's condition. The Stone spaces do not satisfy the condition [Barr 1971].

1.8.1 Proposition. The induced morphism theorem [Grillet 1971 5.4]

If f, g are morphisms with f regular-epi and $\ker f \leqslant \ker g$ then $g = tf$ for some morphism t . t is monic if and only if $\ker f = \ker g$.

$$\begin{tikzcd}
\bullet \arrow[rr, "f"] \arrow[dr, "g"'] & & \bullet \arrow[dl, dashed, "t"] \\
& \bullet &
\end{tikzcd}$$

□

1.8.2 Proposition [Grillet 1971 5.5,6·7]

A wellpowered regular category is regularly-cowellpowered and if it has coproducts then it has intersections. □

$\underline{C}$ is complete if and only if $\underline{C}$ has products and if this is so then intersections of congruences are again congruences. In fact $\bigwedge \ker f_i = \ker \langle f_i \rangle$.

Grillet [1971 6.8] gives further conditions on $\underline{C}$ under which there is a least congruence containing a given relation.

The next result will be used frequently in this work:

1.8.3 Proposition [Grillet 1971 6.1]

Let $p_i : \times X_i \longrightarrow X_i$ be the projections of the product.

Then $\bigwedge \ker p_i = 1_{\times X_i}$. □

1.9 Difunctional Relations [Meisen 1973a]

A relation α in C is difunctional (or Von Neumann regular) if $\alpha\alpha^{-1}\alpha = \alpha$.

In abelian categories as well as in groups, rings and vector spaces all relations are difunctional. In sets there are relations which are not difunctional.

Meisen [1973a] defines a relation $\alpha = \mathrm{Im}\,\langle a,b\rangle$ to be a pullback relation if a,b is the pullback of a pair of morphisms. He proves the following result of which we present a simpler proof.

1.9.1 Proposition: [Meisen 1973a]

Let α be a relation in C.

(a) $\alpha \leqslant \alpha\alpha^{-1}\alpha$.

(b) If α is a pullback relation then α is difunctional.

Proof:

Let $\alpha = ba^{-1}$ for morphisms a and b.

(a) So $\alpha\alpha^{-1}\alpha = ba^{-1}ab^{-1}ba^{-1} \geqslant ba^{-1} = \alpha$.

(b) Since α is a pullback relation it has the form $\alpha = d^{-1}c$ for morphisms c and d. So $\alpha\alpha^{-1}\alpha = d^{-1}cc^{-1}dd^{-1}c \leqslant d^{-1}c = \alpha$. □

2. REGULAR CATEGORIES: THE ACTION OF A FUNCTOR ON RELATIONS

We now consider the following problem: Given regular categories $\underline{C}$ and $\underline{D}$ and a functor $T:\underline{C} \longrightarrow \underline{D}$, can we extend the action of T from the morphisms of $\underline{C}$ to the relations of $\underline{C}$?

Barr [1970] and Manes [1973] have obtained some results for the case $\underline{C} = \underline{D} = \underline{\text{sets}}$ without placing any restrictions on the functor, while Klein [1970] has studied this situation when T preserves finite products and monics. Klein's definition of the action of the functor on a relation uses, in an essential way, the restriction imposed on the functor. However an easy generalisation of Barr's definition enables us to give a definition which works with no restrictions on the functor.

In this section we use this definition to obtain Klein's results with minimal restrictions on the functor. In particular we are able to drop the requirement that the functor preserves finite products*. Throughout this section $\underline{C}$ and $\underline{D}$ will be regular categories and $T:\underline{C} \longrightarrow \underline{D}$ a functor.

---

* This is important because most of the functors to which we shall apply the results of this section do not preserve finite products.

2.1 Definition

Let $\alpha : X \longrightarrow Y$ be a relation in $\underline{C}$.

Define $T(\alpha) : TX \longrightarrow TY$, a relation in $\underline{D}$ by $T(\alpha) = T(b)T(a)^{-1}$ where $\alpha = \text{Im} \langle a,b \rangle$ with $\langle a,b \rangle$ monic.

2.1.1 Remarks

(a) The definition of $T(\alpha)$ is consistent:

Suppose that $\langle a',b' \rangle$ is another monic such that $\alpha = \text{Im} \langle a',b' \rangle$. Then there is an isomorphism t such that a $a = a't$ and $b = b't$. So $u = Tt$ is an isomorphism and $T(b)T(a)^{-1} = T(b')uu^{-1}(Ta')^{-1} = T(b')T(a')^{-1}$.

(b) If $f : X \longrightarrow Y$ is a morphism and we let $\alpha = \text{Im} \langle 1,f \rangle$ the $T(\alpha) = T(f)$. Thus the definition 2.1 is a proper extension of the action of T on morphisms.

(c) $T(\alpha)^{-1} = T(\alpha^{-1})$:

If $\alpha = \text{Im} \langle a,b \rangle$ with $\langle a,b \rangle$ monic then $\alpha^{-1} = \text{Im} \langle b,a \rangle$.

Thus $T(\alpha)^{-1} = [T(b)T(a)^{-1}]^{-1} = T(a)T(b)^{-1} = T(\alpha^{-1})$.

(d) T is order preserving on relations:

Let $\alpha, \beta : X \rightrightarrows Y$ with $\alpha = \text{Im} \langle a,b \rangle$, $\beta = \text{Im} \langle c,d \rangle$ and $\alpha \leqslant \beta$. There is a t such that $a = ct$ and $b = dt$.

$$\text{So } T(\alpha) = T(b)T(a)^{-1} = T(d)T(t)T(t)^{-1}T(c)^{-1}$$
$$\leqslant T(d)T(c)^{-1} = T(\beta).$$

(e) In definition 2.1, we can drop the requirement that $\langle a,b \rangle$ is monic if and only if T preserves regular-epis.

$\Longleftarrow$: Let α, a and b be as in 2.1 and $\alpha = \mathrm{Im}\,\langle a',b'\rangle$ with $\langle a',b'\rangle$ not necessarily monic. We want to show that $T(b')T(a')^{-1} = T(b)T(a)^{-1}$.

Since $\mathrm{Im}\,\langle a,b\rangle = \mathrm{Im}\,\langle a',b'\rangle$, there is a regular-epi t such that $at = a'$ and $bt = b'$. So $T(b')T(a')^{-1} = T(b)T(t)T(t)^{-1}T(a)^{-1} = T(b)T(a)^{-1}$.

$\Longrightarrow$: Suppose that the requirement can be dropped. Then if t is a regular-epi $\mathrm{Im}\,\langle t,t\rangle = \mathrm{Im}\,\langle 1,1\rangle$. So by the assumption we have made $1 = T(1) = T(t)T(t)^{-1}$.

It follows that $T(t)$ is a regular-epi. □

The following proposition gives the promised generalisation of Klein's [1970] results.

2.2 Proposition:

Let α,β be relations and f, g be morphisms in $\underline{C}$.

In each result we assume 'composability'.

(a) $T(\alpha f) \leqslant T(\alpha)T(f)$.

(b) $T(\alpha f^{-1}) \geqslant T(\alpha)T(f)^{-1}$.

(c) $T(f\alpha) \geqslant T(f)T(\alpha)$.

(d) $T(f^{-1}\alpha) \leqslant T(f)^{-1}T(\alpha)$.

(e) If f is monic then $T(\alpha f^{-1}) = T(\alpha)T(f)^{-1}$ and $T(f\alpha) = T(f)T(\alpha)$.

(f) The following are equivalent:

  (i) T preserves regular-epis,

  (ii) $T(\alpha f^{-1}) = T(\alpha)T(f)^{-1}$ for all α, f,

  (iii) $T(f\alpha) = T(f)T(\alpha)$ for all α, f,

  (iv) $T(\beta\alpha) \leqslant T(\beta)T(\alpha)$ for all α, β,

(g) If T preserves kernel pairs then $T(f^{-1}f) = T(f)^{-1}T(f)$.

(h) The following are equivalent:

(i) $T(f^{-1}g) = T(f)^{-1}T(g)$ for all f, g,

(ii) $T(\alpha f) = T(\alpha)T(f)$ for all α, f,

(iii) $T(f^{-1}\alpha) = T(f)^{-1}T(\alpha)$ for all α, f,

(iv) $T(\beta\alpha) \geqslant T(\beta)T(\alpha)$ for all α, β,

and if T preserves finite pullbacks then T has these properties.

<u>Proof</u>:

Let $\alpha = \operatorname{Im}\langle a,b\rangle$ and $\beta = \operatorname{Im}\langle c,d\rangle$ with $\langle a,b\rangle$ and $\langle c,d\rangle$ monic.

(a) αf is given by

$$\begin{tikzcd}
 & & \bullet \arrow[dl, "a'"'] \arrow[dr, "f'"] \arrow[dd, phantom, "\text{PB}"] & & \\
 & \bullet \arrow[dl, "1"'] \arrow[dr, "f"] & & \bullet \arrow[dl, "a"'] \arrow[dr, "b"] & \\
\bullet & & \bullet & & \bullet
\end{tikzcd}$$

$\alpha f = \operatorname{Im}\langle a',bf'\rangle$ and $\langle a',bf'\rangle$ is monic.

$$\begin{aligned}
\text{Thus } T(\alpha f) &= T(b)T(f')T(a')^{-1} \\
&\leqslant T(b)T(a)^{-1}T(f) \qquad (1.4.1h) \\
&= T(\alpha)T(f).
\end{aligned}$$

(b) αf^{-1} is given by:

$$\begin{tikzcd}
 & & \bullet \arrow[dl, "a"'] \arrow[dr, "1"] & & \\
 & \bullet \arrow[dl, "f"'] \arrow[dr, "1"] & & \bullet \arrow[dl, "a"'] \arrow[dr, "b"] & \\
\bullet & & \bullet & & \bullet
\end{tikzcd}$$

We factorise $\langle fa,b\rangle = \langle a',b'\rangle t$ with t regular-epi and $\langle a',b'\rangle$ monic. So $\alpha f^{-1} = \mathrm{Im}\,\langle a',b'\rangle$.

Thus
$$\begin{aligned} T(\alpha)T(f)^{-1} = T(b)T(a)^{-1}T(f)^{-1} &= T(b)T(fa)^{-1} \\ &= T(b')T(t)T(t)^{-1}T(a')^{-1} \\ &\leqslant T(b')T(a')^{-1} \\ &= T(\alpha f^{-1}). \end{aligned}$$

(c) This is obtained from (b) using the inverse operation as follows:

$$\begin{aligned} T(f\alpha) = [T(\alpha^{-1}f^{-1})]^{-1} &\geqslant [T(\alpha^{-1})T(f)^{-1}]^{-1} \\ &= T(f)T(\alpha). \end{aligned}$$

(d) $T(f^{-1}\alpha) = [T(\alpha^{-1}f)]^{-1} \leqslant [T(\alpha^{-1})T(f)]^{-1}$ (by (a))

$$= T(f)^{-1}T(\alpha).$$

(e) In the proof of (b) we see that if f is monic then so is $\langle fa,b\rangle$. Thus we can take $t = 1$ and so $T(\alpha)T(f)^{-1} = T(\alpha f^{-1})$. The second statement is obtained using inverses.

(f) Using inverses we see that (ii) and (iii) are equivalent. (ii) $\Longrightarrow$ (i) is trivial. Also by examining the proof of (b) we see that (i) $\Longrightarrow$ (ii). Using (c) we see that (iv) $\Longrightarrow$ (iii).

(iii) $\Longrightarrow$ (iv):
$$\begin{aligned} T(\beta)T(\alpha) &= T(d)T(c)^{-1}T(\alpha) \\ &\geqslant T(d)T(c^{-1}\alpha) && \text{(by (d))} \\ &= T(dc^{-1}\alpha) && \text{(by iii)} \\ &= T(\beta\alpha) \end{aligned}$$

(g) Let

$$\begin{tikzcd}
& \cdot \arrow[dl, "g"'] \arrow[dr, "h"] & \\
\cdot \arrow[dr, "f"'] & \text{PB} & \cdot \arrow[dl, "f"] \\
& \cdot &
\end{tikzcd}$$

be a pullback.

So g, h is the kernel pair of f. Also $f^{-1}f = \mathrm{Im} \langle g,h \rangle$ with $\langle g, h \rangle$ monic. Now Tg, Th is the kernel pair of Tf and so

$$\begin{tikzcd}
& \cdot \arrow[dl, "Tg"'] \arrow[dr, "Th"] & \\
\cdot \arrow[dr, "Tf"'] & \text{PB} & \cdot \arrow[dl, "Tf"] \\
& \cdot &
\end{tikzcd}$$

is a pullback.

$$\begin{aligned} \text{Hence } (Tf)^{-1}(Tf) &= (Th)(Tg)^{-1} && (1.4.1h) \\ &= T(f^{-1}f). \end{aligned}$$

(h) As in (f), (ii) and (iii) are equivalent and obviously

$(iv) \Longrightarrow (ii) \Longrightarrow (i)$.

$$\begin{aligned} (i) \Longrightarrow (iv): \quad T(\beta)T(\alpha) &= T(d)T(c)^{-1}T(b)T(a)^{-1} \\ &= T(d)T(c^{-1}b)T(a)^{-1} && \text{(by (i))} \\ &\leqslant T(dc^{-1}b)T(a)^{-1} && \text{(by (c))} \\ &\leqslant T(dc^{-1}ba^{-1}) && \text{(by (b))} \\ &= T(\beta\alpha) \ . \end{aligned}$$

Now if T preserves finite pullbacks then in the proof of (a) we see that $T(a)^{-1}T(f) = T(f')T(a')^{-1}$ and so $T(\alpha f) = T(\alpha)T(f)$. □

2.2.1 Remarks:

(a) Let $R_{\underline{C}}$ denote the category whose objects are those of $\underline{C}$ and whose morphisms are the relations of $\underline{C}$. Klein [1970 2.6] shows that the class of isomorphisms is not enlarged by passing from $\underline{C}$ to $R_{\underline{C}}$. Now 2.2 (f) and (h) show that if T preserves regular-epis and pullbacks then T extends to a functor $R_{\underline{C}} \longrightarrow R_{\underline{D}}$.

(b) Some of the results given in 1.7 about products of relations can be obtained from the results of this section by regarding the product operation as a functor on a product of categories.

(c) <u>The converse of 2.2g and 2.2h are not true</u>:
The power set functor P has the properties of 2.2(g) and (h). However it does not preserve kernel pairs (or finite pullbacks). For details see 7.10c.

(d) If however T preserves finite products then the converse of 2.2h is true [Klein 1970]. Moreover one sees easily that if T preserves the terminal object then:

(i) T preserves kernel pairs $\Longrightarrow$ T preserves finite 'self-products', i.e. $T(X \times X) \cong TX \times TX$,

and (ii) T preserves (finite) pullbacks $\Longrightarrow$ T preserves (finite) products.

3. TOP CATEGORIES: BASIC RESULTS

Top categories are defined and discussed in the papers of Wyler [1971a, b] and Ramaley and Wyler [1970 a]. In this section we repeat the basic facts about top categories which we shall require later on. We also derive some additional results.

3.1 Definition [Wyler 1971 b]

A topological theory on a category $\underline{C}$ is a contravariant functor $s^*:\underline{C} \longrightarrow \underline{ORD}$ (where $\underline{ORD}$ is the category of ordered sets and order preserving functions) such that for all objects X in $\underline{C}$, s^*X is a complete lattice and for morphisms f in $\underline{C}$, s^*f preserves infima of arbitary families including the empty family. We then define a category $\underline{C}^s$ as follows: the objects of $\underline{C}^s$ are all pairs (X,x) with X in $\underline{C}$ and $x \in s^*X$. These pairs are called spaces. The morphisms $f:(X,x) \longrightarrow (Y,y)$ in $\underline{C}^s$ (sometimes referred to as maps) are the morphisms $f:X \longrightarrow Y$ in $\underline{C}$ such that $x \leqslant (s^*f)y$. Composition of morphisms is defined as in $\underline{C}$. There is a forgetful functor $P:\underline{C}^s \longrightarrow \underline{C}$ defined by $P(X,x) = X$ and $Pf = f$. P is called the projection functor and $\underline{C}^s$ is said to be a top category over $\underline{C}$.

Let $f:X \longrightarrow Y$ in $\underline{C}$. We write $f^* = s^*f$ (when no confusion is possible) and define $s_*f = f_* : s^*X \longrightarrow s^*Y$ in $\underline{ORD}$ by:

$$f_*x = \mathrm{Inf}\{y \in s^*Y \mid x \leqslant f^*y\}, \quad \text{for all } x \in s^*X.$$

Then $x \leqslant f^*y \Longleftrightarrow f_*x \leqslant y$ and so f_*, f^* give us a Galois connection between s^*X and s^*Y. Also $f^*f_*f^* = f^*$, $f_*f^*f_* = f_*$ and f_* preserves suprema.

Examples of top categories abound. Over sets we have limit spaces, topological spaces, proximity spaces, uniform spaces, quasi-uniform spaces and uniform convergence spaces. Categories of topological algebras are top categories over the underlying categories of algebras. More examples can be found in Wyler's papers.

For the rest of this section $\underline{C}^s$ will be a top category over $\underline{C}$.

3.2 General properties of top categories [Wyler 1971a, b]

The projection functor P has a left adjoint α given by $\alpha(X) = (X, \alpha_X)$ and $\alpha(f) = f$ where α_X is the least element of $s^*(X)$. P also has a right adjoint ω given by $\omega(X) = (X, \omega_X)$ and $\omega(f) = f$, where ω_X is the largest element of s^*X. Spaces of the form (X, α_X) are said to be discrete and those of the form (X,ω_X) are indiscrete.
If $f:X \longrightarrow Y$ then $f_*(\alpha_X) = \alpha_Y$ and $f^*(\omega_Y) = \omega_X$.

For $f:(X,x) \longrightarrow (Y,y)$ in $\underline{C}^s$, f is monic (resp. epi) in $\underline{C}^s$ if and only if it is monic (resp. epi) in $\underline{C}$. f is an isomorphism in $\underline{C}^s$ if and only if it is an isomorphism in $\underline{C}$ and $x = f^*y$ (or $f_*x = y$).

$f:(X,x) \longrightarrow (Y,y)$ is extreme-monic in $\underline{C}^s$ if and only if it is extreme-monic in $\underline{C}$ and $x = f^*y$. By duality, $f:(X,x) \longrightarrow (Y,y)$

is extreme-epi in $\underline{C}^s$ if and only if it is extreme-epi in $\underline{C}$ and $y = f_*x$. If f as above is extreme-monic then we say that (X,x) is a *subspace* of (Y,y) and f is an embedding. Dually we have *quotient spaces* and *quotient maps*. A map of the form $f:(X,x) \longrightarrow (Y,f_*x)$ is said to be coinitial while $f:(X,f^*y) \longrightarrow (Y,y)$ is *initial*. Suppose that $f:(X,x) \longrightarrow (Y,f_*x)$ is also initial and $fg = 1$ for some $g:Y \longrightarrow X$. Then $g:(Y,f_*x) \longrightarrow (X,x)$. The proof is as follows: $g^*x = g^*f^*f_*x = f_*x$.

If C is a cogenerator of $\underline{C}$ then (C, ω_C) is a cogenerator of $\underline{C}^s$. The projection functor creates limits and colimits and so $\underline{C}^s$ is complete (resp. co-complete) if and only if $\underline{C}$ is complete (resp. co-complete). A similar statement is true for 'wellpowered' and 'co-wellpowered'. If N is the terminal object of $\underline{C}$ then (N, ω_N) is the terminal object of $\underline{C}^s$. Further if N has a unique s-structure and $\underline{C}$ is connected (i.e. it has non-empty morphism sets) then $\underline{C}^s$ is also connected.

Let $\underline{A}$ be a full subcategory of $\underline{C}^s$ and let
$r^*A = \{x \varepsilon s^*A \mid (A,x) \text{ is an object of } \underline{A}\}$ (for all A in $\underline{C}$).
$\underline{A}$ is said to be a *top subcategory* of $\underline{C}^s$ if the following two conditions are satisfied:

(i) r^*A is closed under infima in s^*A for all A in $\underline{C}$.

(ii) f^* maps r^*B into r^*A for all $f:A \longrightarrow B$ in $\underline{C}$.

$\underline{A}$ is then a reflective subcategory of $\underline{C}^s$ with reflector of the form $(X,x) \xmapsto[\iota_x]{} (X,Lx)$ where $Lx = \operatorname{Inf}\{x' \varepsilon s^*X \mid x \leqslant x' \text{ and } (X,x') \varepsilon \underline{A}\}$.
For $f:X \longrightarrow Y$ define $r^*f = s^*f$.

Then $\underline{C}^r$ is a top category over $\underline{C}$ which is isomorphic to $\underline{A}$ and $r_*f = L(s_*f)$. L then preserves suprema and $L(s_*f) = (r_*f)L$.

All the indiscrete spaces of $\underline{C}^s$ belong to $\underline{A}$ and are the indiscrete spaces of $\underline{C}^r$. This may not be true of the discrete spaces.

Every top category has a dual top category $(\underline{C}^{op})^{s^{op}}$ over $\underline{C}^{op}$ defined in the obvious way. Thus the general theory of top categories is self-dual.

3.3 Construction of limits and colimits in top categories

The constructions given below can be found in Wyler's paper [1971a].

(a) Products:

Let $\{(X_i,x_i)\}$ be a family of spaces from $\underline{C}^s$.
If $\times X_i$ exists in $\underline{C}$ then $\times(X_i,x_i) = (\times X_i, \bigwedge p_i^* x_i)$,
where $p_i : \times X_i \longrightarrow X_i$ are the projections of the product in $\underline{C}$.

(b) Sums:

Let the (X_i,x_i) be as in (a). If the sum $\bigsqcup X_i$ exists in $\underline{C}$ then $\bigsqcup(X_i,x_i) = (\bigsqcup X_i, \bigvee(k_i)_* x_i)$ where $k_i : X_i \longrightarrow \bigsqcup X_i$ are the injections of the sum in $\underline{C}$.

(c) Equalizers

Let $f, g : (X,x) \rightrightarrows (Y,y)$ have equalizer $k : K \longrightarrow X$ in $\underline{C}$.
Then $k : (K, k^* x) \longrightarrow (X,x)$ is the equalizer in $\underline{C}^s$.

(d) Coequalizers

Let f, g be as in (c). If $c:Y \longrightarrow C$ is the coequalizer of f, g in $\underline{C}$ then $c:(Y,y) \longrightarrow (C,c_*y)$ is the coequalizer in $\underline{C}^s$.

3.4 Proposition:

'Products of indiscrete spaces are indiscrete'

'Sums of discrete spaces are discrete'

Let $\{X_i\}$ be a family of objects of $\underline{C}$.

(a) If $\times X_i$ exists then $\times(X_i, \omega_{X_i}) = (\times X_i, \omega_{\times X_i})$.

(b) If $\sqcup X_i$ exists then $\sqcup(X_i, \alpha_{X_i}) = (\sqcup X_i, \alpha_{\sqcup X_i})$.

Proof:

$p_i^*\omega_{X_i} = \omega_{\times X_i}$ and $(k_i)_*\alpha_{X_i} = \alpha_{\sqcup X_i}$ where the pi are the projections of the product and the k_i are the injections of the sum. □

3.5 Some properties which a top category may or may not have:

The properties listed below are not true in every top category. They will be used later as assumptions required in order to obtain various results.

(finite)(PI): Every (finite) product of discrete spaces in $\underline{C}^s$ is discrete. The dual is (PI)*: Every sum of indiscrete spaces is indiscrete.

Since $\underline{TOP}$ satisfies (finite) (PI) but not (finite) (PI)* we see that the top category $\underline{TOP}^{OP}$ satisfies (finite) (PI)* but not (finite) (P1).

(P2): If $f:X \rightarrowtail Y$ is monic then $f^*\alpha_Y = \alpha_X$.

As the 'dual' we take (P2)*:

If $f:X \twoheadrightarrow Y$ is regular-epi then $f_*\omega_X = \omega_Y$.

(P3) If $f:X \rightarrowtail Y$ is monic then $f^*f_* = 1$.

As the 'dual' we take (P3)*:

If $f:X \twoheadrightarrow Y$ is regular-epi then $f_*f^* = 1$.

3.5.1 Remarks:

(a) These properties are related as follows:

(P3) $\Longrightarrow$ (P2) and (P3)* $\Longrightarrow$ (P2)*:

Proof: (P3) $\Longrightarrow$ (P2): Let $f:X \rightarrowtail Y$ be monic. Since $f_*\alpha_X = \alpha_Y$ and $f^*f_* = 1$, we have

$\alpha_X = f^*f_*\alpha_X = f^*\alpha_Y$ and so (P2) holds.

The 'dual' proof is similar. ☐

(b) If $f:X \rightarrowtail Y$ is a split-monic (resp. a split-epi) then it satisfies (P3) (resp. (P3)*) [Ramaley and Wyler 1970a].

Proof: For (P3), $gf = 1$ with $g:Y \longrightarrow X$ implies that $g_*f_* = 1$.

Thus f_* is monic in ORD and the result follows from $f_*f^*f_* = f_*$. Again the 'dual' is similar. ☐

(c) Every top category over sets has the properties (P3)* and (P2)*, and furthermore has (P3) and (P2) for all monics with non-empty domain. If the empty set has a unique structure then this is true for every monic. In particular TOP has all four properties. The proof is by (b).

(d) Top subcategories:

Let $\underline{A}$ be a top subcategory of $\underline{C}^s$.

We have the following results, the proofs of which are straightforward:

(i) If $\underline{C}^s$ has the property (PI)* (resp. (P2)*, (P3)*), then $\underline{A}$ has the property (PI)* (resp. (P2)*, (P3)*).

(ii) If $\underline{C}^s$ has the property (PI)(resp.(P2)) and $\underline{A}$ contains all the discrete spaces of $\underline{C}^s$, then $\underline{A}$ also has the property (PI) (resp.(P2))

3.6 Proposition:

Let $\underline{A}$ be a full subcategory of $\underline{C}^s$.

The following are equivalent:

(a) $\underline{A}$ is a reflective subcategory of $\underline{C}^s$ and the reflection maps are $\underline{C}$-isomorphisms.

(b) $\underline{A}$ is a reflective subcategory of $\underline{C}^s$ which contains all the indiscrete spaces.

(c) $\underline{A}$ is a top subcategory of $\underline{C}^s$.

Proof: (a) $\Longrightarrow$ (b): We can assume that the reflector has the form $1_X:(X,x) \longrightarrow (X,Lx)$. So for $X\varepsilon\underline{C}$, $\omega_X \leqslant L\omega_X$. Thus $\omega_X = L\omega_X$ and so $(X,\omega_X)\varepsilon\underline{A}$.

(b) $\Longrightarrow$ (a): For $(X,x)\varepsilon\underline{C}^s$, let $\eta_X:(X,x) \longrightarrow L(X,x)$ be the reflector. Since $(X,\omega_X)\varepsilon\underline{A}$ there is a f such that

$$\begin{tikzcd}
(x,x) \arrow[r, "\eta_X"] \arrow[dr, "1_X"'] & L(X,x) \arrow[d, "f"] \\
& (X,\omega_X)
\end{tikzcd}$$

commutes.

So η_X is split monic for all $(X,x)\varepsilon\underline{C}^s$. Now by [Herrlich 1968 8.2.4] each η_X is also epi and hence is a $\underline{C}$-isomorphism.

(a) $\Longrightarrow$ (c): Let L be as in (a) $\Longrightarrow$ (b). We need to verify the conditions (i) and (ii) in section 3.2. For (ii), let $f:X \longrightarrow Y$ with $(Y,y)\varepsilon\underline{A}$. Then since $(Y,y)\varepsilon\underline{A}$, the diagram

$$\begin{tikzcd}
(X,f^*y) \arrow[r, "1_X"] \arrow[dr, "f"'] & (X,Lf^*y) \arrow[d, "f"] \\
& (Y,y)
\end{tikzcd}$$

commutes.

So $Lf^*y \leqslant f^*y \leqslant Lf^*y$. Thus we have equality and so $(X,f^*y)\varepsilon\underline{A}$. The proof for the condition (i) is similar,

(c) $\Longrightarrow$ (a): trivial. □

<u>3.7 Proposition</u>: Suppose that $\underline{C}$ is complete, well and co-well-powered and $\underline{A}$ is a subcategory of $\underline{C}^s$.

Then: $\underline{A}$ is a monoreflective subcategory of $\underline{C}^s$ if and only if $\underline{A}$ is closed under products, subspaces and it contains all the indiscrete spaces.

<u>Proof</u>: By [Herrlich 1968 10.2.1] and the result 3.6 above. □

<u>3.8 Proposition</u>:

Every top category $\underline{C}^s$ over a regular category $\underline{C}$ has (regular-epi, mono) decompositions.

<u>Proof</u>: Let $f:(X,x) \longrightarrow (Y,y)$ in $\underline{C}^s$ have as its decomposition in $\underline{C}$, $X \underset{e}{\twoheadrightarrow} Z \underset{m}{\rightarrowtail} Y$. Let e be the coequalizer of g, $h:W \longrightarrow X$ in $\underline{C}$. Let $w = g^*x \wedge h^*x$. Then $e:(X,x) \longrightarrow (Z,e_*x)$ is the coequalizer of g, $h:(W,w) \longrightarrow (X,x)$ in $\underline{C}^s$. Thus $(X,x) \overset{e}{\twoheadrightarrow} (Z,e_*x) \overset{m}{\rightarrowtail} (Y,y)$ is a (regular-epi, mono) decomposition of f in $\underline{C}^s$. □

<u>3.8.1 Remark</u>:

It follows that in a top category over a regular category the concepts of regular-epi and extremal-epi are equivalent [Grillet 1971 1.4]. Also the codomain of an extremal-epi (= regular-epi) has the quotient structure. In particular $f:(X,x) \longrightarrow (Y,y)$ in

$\underline{C}^s$ is extremal-epi in $\underline{C}^s$ if and only if f is regular-epi in $\underline{C}$ and $y = f_* x$.

3.9 Proposition:

Let $\underline{C}$ be a complete category which is regularly-co-wellpowered. Then a subcategory $\underline{A}$ of $\underline{C}^s$ is extremal-epireflective in $\underline{C}^s$ if and only if it is closed under products and subobjects.

Proof: $\underline{C}^s$ is complete and extremal-epi-co-wellpowered

We now apply Kennison's result 1.2 [1968]. ☐

3.10 Example: The top category $\underline{C}^{r\ell}$

Let $\underline{C}$ be a wellpowered regular category with intersections. For each X in $\underline{C}$ define $r\ell^*(X)$ to be the complete lattice of reflexive relations on X. For $f:X \longrightarrow Y$ define

$f^* = r\ell^*(f) = (f \times f)^*$. It follows by 1.6 that for $\beta:Y \longrightarrow Y$ we have $f^*\beta = f^{-1}\beta f$, and for $\alpha:X \longrightarrow X$ we have $f_*\alpha = f\alpha f^{-1}$.

$\underline{C}^{r\ell}$ is then a top category over $\underline{C}$ and $\underline{C}^{r\ell}$ has the properties (P1), (P2), (P3), (P2)* and (P3)* (but not (P1)* in the case $\underline{C}$ = sets).

This example is important for the work of § 6.

$\underline{C}^{r\ell}$ is also obtained as the category of relational prealgebras of the identity monad on $\underline{C}$. (see 7.10a).

3.11 Connected spaces

In this subsection we assume that the base category $\underline{C}$ is regular and that the terminal object N has a unique s-structure.

3.11.1 Definition:

A space (X,x) is <u>connected</u> if and only if every map in $\underline{C}^s$ with domain (X,x) and codomain a discrete space factors through (N,α_N) (or equivalently through N).

$$\begin{tikzcd}
(X,x) \arrow[rr, "f"] \arrow[dr, "g"'] & & (Y, \alpha_Y) \\
& (N, \alpha_N) \arrow[ur, "t"'] &
\end{tikzcd}$$

3.11.2 Remarks:

(a) A space (S,s) is connected if and only if given $f:(X,x) \twoheadrightarrow (Y, \alpha_Y)$ with f regular-epi, it follows that $Y \cong N$.

(b) If all the indiscrete spaces are connected then N is the only object up to iso with a unique s-structure.

(c) If $\underline{C}^s$ has the properties (P2) and (P2)* and N is the only object up to iso with a unique structure, then all the indiscrete spaces are connected.

<u>Proof</u>: Let $f:(X, \omega_X) \longrightarrow (Y,\alpha_Y)$ have the decomposition

$X \overset{e}{\twoheadrightarrow} Z \overset{m}{\rightarrowtail} Y$. Since (P2) holds we have

$m^*\alpha_Y = \alpha_Z$ and by (P2)* we have $e_*\omega_X = \omega_Z$.

Thus $\alpha_Z = \omega_Z$ and it follows that $Z \cong N$. □

3.11.3 Proposition: 'A continuous image of a connected space is connected'

If (X,x) is connected and $f:(X,x) \twoheadrightarrow (Y,y)$ be regular-epi, then (Y,y) is connected.

Proof:

Let $g:(Y,y) \longrightarrow (Z, \alpha_Z)$. In the diagram below we can fill in h, k with k monic so that the diagram commutes.

$$\begin{tikzcd}
(X,x) \arrow[r, two heads, "f"] \arrow[dr, "h"'] & (Y,y) \arrow[r, "g"] \arrow[d, dashed, "t"] & (Z, \alpha_Z) \\
& (N, \alpha_N) \arrow[ur, tail, "k"'] &
\end{tikzcd}$$

Now using 1.2(e) we can fill in $t:Y \longrightarrow N$ so that $kt = g$. □

3.11.4 Proposition

Let $\{(A_i, m_i^* x)\}$ be a family of connected subspaces of (X,x). Suppose further that $\bigcap A_i$ and $\bigcup A_i$ exist and there is a

$u:N \longrightarrow \bigcap A_i$.

Then $(\bigcup A_i, m^* x)$ is connected, where $m:\bigcup A_i \rightarrowtail X$ is the natural monic associated with the union.

Proof:

Let

$$\begin{tikzcd}
\bigcap A_i \arrow[r, tail, "u_i"] & A_i \arrow[r, tail, "n_i"] \arrow[dr, tail, "m_i"'] & \bigcup A_i \arrow[d, tail, "m"] \\
& & X
\end{tikzcd}$$

be the diagram of monics associated with $\bigcap A_i$ and $\bigcup A_i$.

So $n_i u_i = n_j u_j$ for all i, j.

Let $g: (\bigcup A_i, m^*x) \longrightarrow (Y, \alpha_Y)$. We need to show that g factors through N. Now for each i, $gn_i : (A_i, m_i^* x) \longrightarrow (Y, \alpha_Y)$ and so there are morphisms h_i, t_i such that

$$\begin{tikzcd}
(A_i, m_i^* x) \arrow[rr, "gn_i"] \arrow[dr, "h_i"'] & & (Y, \alpha_Y) \\
& (N, \alpha_N) \arrow[ur, "t_i"'] &
\end{tikzcd}$$

commutes.

So $t_i = t_i h_i u_i u = gn_i u_i u = gn_j u_j u = t_j$ for all i, j.

Write $t = t_i$ and $s = ng$ where $n: Y \longrightarrow N$. We need to show that $ts = g$. Now $(ts)n_i = t_i n g n_i = t_i n t_i h_i = t_i h_i = gn_i$ for all i. It follows [Mitchell 1965 1.9.1] that $ts = g$. □

4. PROTOREFLECTIONS AND PROTOCOREFLECTIONS

Many reflections and coreflections are obtained by ordinal iteration of a functorial process. We give a number of examples of such processes in 4.5 below. In this section we note that all these are instances of what we shall call a protoreflection (or its dual). Under certain circumstances ordinal iteration of a protoreflection gives us a reflection.

Corresponding to each result in this section there is an obvious dual result which we usually do not state explicitly.

4.1 Definition:

A *protoreflection* in a category $\underline{C}$ is a triad $(F, \eta, \underline{A})$, where $F:\underline{C} \longrightarrow \underline{C}$ is a functor, $\eta: I \longrightarrow F$ is a natural transformation and $\underline{A}$ is a subcategory of $\underline{C}$ such that for each X in $\underline{C}$ and $f:X \longrightarrow A$ with $A \varepsilon \underline{A}$, there is a unique $f':Fx \longrightarrow A$ such that $f'\eta_X = f$.

$$\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[dr, "f"'] & FX \arrow[d, dashed, "f'"] \\
 & A
\end{tikzcd}$$

The dual concept is that of a *protocoreflection*.

If $(F_1, \eta_1, \underline{A})$ and $(F_2, \eta_2, \underline{A})$ are protoreflections in $\underline{C}$ then we say that:

(i) $(F_1, \eta_1, \underline{A}) \leqslant (F_2, \eta_2, \underline{A})$ if and only if there is a natural transformation ρ such that

$$\begin{tikzcd} I \arrow[r, "\eta_1"] \arrow[dr, "\eta_2"'] & F_1 \arrow[d, "\rho"] \\ & F_2 \end{tikzcd}$$

commutes.

(ii) $(F_1, \eta_1, \underline{A}) \sim (F_2, \eta_2, \underline{A})$ if and only if there is a natural isomorphism ρ such that the previous diagram commutes.

The relation $\leqslant$ is reflexive and symmetric.

$(F, \eta, \underline{A})$ is said to be a *mono*-protoreflection (*resp*. *epi*,-*regular-epi*-etc) if all the morphisms η_X are mono, (epi, regular-epi etc.)

A protoreflection is said to be *strong* if it is a reflection.

4.1.1 Remarks:

Protoreflections appear in [Johnson 1966] as 'relative reflections'. Johnson uses them to study commutativity of reflectors and coreflectors. His work is entirely disjoint from ours.

The obvious thing to do with a protoreflection is to iterate it.

4.2 Iteration of a protoreflection.

Let $(F, \eta, \underline{A})$ be a protoreflection in $\underline{C}$. Write $F^0 = I_{\underline{C}}$,

$\eta^0 = 1$, $F^1 = F$, $\eta^1 = \eta$. Define $F^2 = FF$, $\eta^2 = \eta_F\cdot\eta$ and inductively for each integer n, $F^{n+1} = F.F^n$, $\eta^{n+1} = \eta_{F^n}\cdot\eta^n$

$\{(F^n, \eta^n, \underline{A})\}$ $n = 0,1,2 \ldots$ is then an ascending sequence of protoreflections in $\underline{C}$.

Let ω denote the first infinite ordinal. Suppose that for X in $\underline{C}$, the diagram

$$\begin{tikzcd}
X = F^0X \arrow[r, "\eta_X"'] & FX \arrow[r, "\eta_{FX}"'] & F^2X \arrow[r] & \cdots\cdots\cdots\cdot
\end{tikzcd}$$

has colimit (LX, ℓ^n_X) with $\ell^n_X : F^n_X \longrightarrow LX$.

For each $f:X \longrightarrow Y$, the colimit construction gives us a morphism $Lf:LX \longrightarrow LY$ in such a way that L becomes a functor and $\ell^n:F^n \longrightarrow L$ a natural transformation. Define $F^\omega = FL$ and $\eta^\omega = \eta L.\ell^0$. Then $(F^\omega, \eta^\omega, \underline{A})$ is a protoreflection in $\underline{C}$.

More generally, if the appropiate colimits exist, then for each ordinal α, we have a protoreflection $(F^\alpha, \eta^\alpha, \underline{A})$ and for each pair of ordinals α, β with $\alpha \leqslant \beta$ we have a natural transformation $\eta^{\alpha,\beta}:F^\alpha \longrightarrow F^\beta$ with the following five properties:

(a) $F^0 = 1$, $F^1 = F$, $\eta^0 = 1$, $\eta^1 = \eta$.

(b) $F^{\alpha+1} = FF^\alpha$, $\eta^{\alpha+1} = \eta_{F^\alpha}.\eta^\alpha$, $\eta^{0,\alpha} = \eta^\alpha$, $\eta^{\alpha,\alpha} = 1$,
$\eta^{\alpha,\alpha+1} = \eta_{F^\alpha}$ for all ordinals α.

(c) For $\alpha \leqslant \beta \leqslant \gamma$, $\eta^{\beta,\gamma}\eta^{\alpha,\beta} = \eta^{\alpha,\gamma}$.

(d) Suppose that α is a limit ordinal. For each X in $\underline{C}$, let $\{(LX, \ell^\beta_X)\}_{\beta<\alpha}$ be the colimit of the diagram

$$\begin{tikzcd} X \arrow[r, "\eta^{\beta}_{X}"] \arrow[dr, "\eta^{\beta'}_{X}"'] & F^{\beta}X \arrow[d, "\eta^{\beta,\beta'}_{X}"] \\ & F^{\beta'}X \end{tikzcd}$$

(For all $\beta \leqslant \beta' < \alpha$

$\ell^{\beta}_{X}\colon\ F^{\beta}X \longrightarrow LX$

Then $F^{\alpha}X = F(LX)$, $\eta^{\alpha}_{X} = \eta_{LX}\cdot\ell^{0}_{X}$ and $\eta^{\beta,\alpha}_{X} = \eta_{LX}\cdot\ell^{\beta}_{X}$.

(e) If $\alpha \leqslant \beta$, then $(F^{\alpha}, \eta^{\alpha}, \underline{A}) \leqslant (F^{\beta}, \eta^{\beta}, \underline{A})$.

The details of the constructions are straightforward and are omitted.

Dually, the existence of certain limits in $\underline{C}$ allows us to obtain iterates of protocoreflection.

4.2.1. Remarks: Preservation of properties under iteration.

(a) All iterates of an epi-protoreflection are epi.

(b) All finite iterates of a monic-protoreflection are monic. For the infinite case the question is open. Further results are given in 4.6.

4.3 Definition: Generation of reflections

A protoreflection $(F, \eta, \underline{A})$ is said to weakly generate a reflection if for each X in $\underline{C}$ there is an ordinal α such that $F^{\alpha}(X) \in \underline{A}$. If this is so, we denote by α_X, the least ordinal with this property. α_X is called the order of X relative to $(F, \eta, \underline{A})$. It follows that

$\{ X \xrightarrow{\eta^{\alpha_X}_{X}} F^{\alpha_X}(X)\}$ is a reflection $\underline{C} \longrightarrow \underline{A}$. Further, we say that $(F, \eta, \underline{A})$ strongly generates a reflection if for each $X \varepsilon \underline{C}$ we can find an α such that $F^{\beta}(X) \varepsilon \underline{A}$ whenever $\beta \geqslant \alpha$.

For epi-protoreflections these two concepts are equivalent.

4.3.1 Proposition:

If an epi-protoreflection weakly generates a reflection then it strongly generates a reflection .

Proof:

Let $(F, \eta, \underline{A})$ be epi. For each $X \varepsilon \underline{C}$ and $\alpha \geqslant \alpha_X$ there is a morphism t such that $t\eta_X^{\alpha} = \eta_X^{\alpha_X}$.

$$\begin{tikzcd}
X \arrow[r, two heads, "\eta_X^{\alpha_X}"] \arrow[dr, "\eta_X^{\alpha}"'] & F^{\alpha_X}(X) \arrow[d, "\eta_X^{\alpha_X,\alpha}"] \\
& F^{\alpha}(X) \arrow[u, bend right, "t"']
\end{tikzcd}$$

So $t\eta_X^{\alpha_X,\alpha}\,\eta_X^{\alpha_X} = t\eta_X^{\alpha} = \eta_X^{\alpha_X}$. Thus $t\eta_X^{\alpha_X,\alpha} = 1$ and since $\eta_X^{\alpha_X,\alpha}$ is epi, it is an isomorphism. Thus $F^{\alpha}(X) \varepsilon \underline{A}$. □

4.3.2 An open problem.

Proposition 4.3.1 tells us that for epi-protoreflections, the properties of weak generation and strong generation are equivalent. We do not know whether this is true in general.

We have the following partial results:

(a) If $F^{\alpha}(X)$ and $F^{\beta}(X)$ both lie in $\underline{A}$ for some ordinals $\alpha \leqslant \beta$ then $\eta_X^{\alpha,\beta} : F^{\alpha}(X) \longrightarrow F^{\beta}(X)$ is an isomorphism.

(b) If $(F, \eta, \underline{A})$ strongly generates a reflection, then

$\alpha \leqslant \alpha_X \Longrightarrow F^{\alpha}(X) \varepsilon \underline{A}$.

(c) If $(F, \eta, \underline{A})$ weakly generates a reflection then for each X and α, there is an ordinal $\beta \geqslant \alpha$ such that $F^{\beta}(X) \varepsilon \underline{A}$.

Proofs:

(a) There is a morphism t such that $t\eta_X^{\beta} = \eta_X^{\alpha}$.

$$\begin{tikzcd}
 & F^{\alpha}(X) \arrow[d, shift right, "\eta_X^{\alpha,\beta}"'] \\
X \arrow[ur, "\eta_X^{\alpha}"] \arrow[dr, "\eta_X^{\beta}"'] & \\
 & F^{\beta}(X) \arrow[uu, shift right, "t"']
\end{tikzcd}$$

$(t\eta_X^{\alpha,\beta})\,\eta_X^{\alpha} = t\eta_X^{\beta} = \eta_X^{\alpha} = 1.\eta_X^{\alpha}$ and by uniqueness it follows that $t\eta_X^{\alpha,\beta} = 1$.

Also $(\eta_X^{\alpha,\beta}t)\,\eta_X^{\beta} = \eta_X^{\alpha,\beta}\, t\eta_X^{\alpha,\beta}\, \eta_X^{\alpha} = \eta_X^{\alpha,\beta}\eta_X^{\alpha} = \eta_X^{\beta}$ and again by uniqueness we have $\eta_X^{\alpha,\beta}t = 1$.

(b) For $X \varepsilon \underline{C}$, let α be such that $\beta \geqslant \alpha \Longrightarrow F^{\beta}(X) \varepsilon \underline{A}$. By (a) $F^{\alpha_X}(X) \cong F^{\alpha}(X)$. So $F^{\alpha_X+1}(X) \cong F^{\alpha+1}(X)$. Thus $F^{\alpha_X+1}(X) \varepsilon \underline{A}$ and so by (a), $\eta_X^{\alpha_X,\alpha_X+1}$ is an isomorphism. The proof now follows by induction.

(c) Trivial. □

4.4 Definition:

Let $(F, \eta, \underline{A})$ be a protoreflection in $\underline{C}$.

Denote by $\underline{C}(F, \eta, \underline{A})$ the full subcategory of $\underline{C}$ of objects X such $\eta_X : X \longrightarrow FX$ is an isomorphism.

4.4.1 Proposition:

If $(F, \eta, \underline{A})$ is an epi-protoreflection then:

(a) $\underline{A} \subseteq \underline{C}(F, \eta, \underline{A})$.

(b) $(F, \eta, \underline{C}(F, \eta, \underline{A}))$ is an epi-protoreflection.

(c) For each α for which the iterate exists, we have

$$\underline{C}(F^{\alpha}, \eta^{\alpha}, \underline{A}) = \underline{C}(F, \eta, \underline{A}).$$

Proof:

(a) For $X \varepsilon \underline{A}$, there is a morphism t such that $t\eta_X = 1_X$. But η_X is epi, so it is an isomorphism.

(b) Trivial.

(c) Let $X \varepsilon \underline{C}(F^{\alpha}, \eta^{\alpha}, \underline{A})$ Now $\eta_X^{1,\alpha}\eta_X = \eta_X^{\alpha}$ and since η_X^{α} is an isomorphism and η_X is epi, η_X is an isomorphism. Conversely let $X \varepsilon \underline{C}(F, \eta, \underline{A})$. Then η_X and $F(\eta_X)$ are isomorphisms. Since $F(\eta_X)\eta_X = \eta_{FX}\eta_X$, η_{FX} is an isomorphism. Now using induction we see that η_X^{α} is an isomorphism. □

4.4.2 Proposition:

If $(F, \eta, \underline{A})$ weakly generates a reflection then $\underline{C}(F, \eta, \underline{A}) \subseteq \underline{A}$ and if it strongly generates a reflection then we have $\underline{C}(F, \eta, \underline{A}) = \underline{A}$.

Proof: If η_X is an isomorphism, then as in the proof of 4.4.1(c) we see that $\eta_X^{\alpha_X}$ is an isomorphism. But $F^{\alpha_X}(X) \varepsilon \underline{A}$ and so $X \varepsilon \underline{A}$.

If we have 'strongly generates', then $X \varepsilon \underline{A}$ implies that $\alpha_X = 0$ and by 4.3.2(a), (b) we see that η_X is an isomorphism. □

4.5 Examples:

(1) The identity protoreflection.

If $\underline{A}$ is a subcategory of a category $\underline{C}$ then $(1, 1, \underline{A})$ is both a protoreflection and a protocoreflection.

(2) Reflections and coreflections.

If $\underline{A}$ is a reflective subcategory of $\underline{C}$ and $R:\underline{C} \longrightarrow \underline{A}$ is the reflection with unit η, then $(R, \eta, \underline{A})$ is a protoreflection.

(3) The idempotent monad associated with a monad.

Fakir [1970] give a process of iteration over the ordinals whereby one modifies an arbitary monad on a category $\underline{C}$ to obtain an associated idempotent monad. This process is a protocoreflection in the category of monads on $\underline{C}$ and it generates a coreflection if $\underline{C}$ is wellpowered and has limits.

Let $\pi = (T, \eta, \mu)$ be a monad on sets such that $T\emptyset = \emptyset$ and for some set X, TX has at least 2 points. Then by the results of Frei and MacDonald [1971 p.3] we see that the first modification of π gives us the identity monad. Frei and MacDonald give an example of a monad on groups for which the first modification is not the identity.

(4) The sheaf associated with a presheaf.

Heller and Rowe's [1962] iterative construction of the sheaf associated with a given presheaf on a topological space X (with values in an exact category) is a protoreflection in the category of presheaves on X. Grillet [1971] shows that with certain conditions on $\underline{C}$, this construction terminates after at most 2 steps.

(5) Topological modification of a closure space. [Cech 1966]

Let C be the category of closure spaces as defined by Cech. Let $F: \underline{C} \longrightarrow \underline{C}$ be the functor defined by $F(X,\tau) = (X, \tau^2)$ where τ is a closure operator on a set X. Then $(X, \tau) \xrightarrow{1_X} (X, \tau^2)$ is a natural transformation $1 \longrightarrow F$ and (F, 1, Top) is a protoreflection which generates the topological modification.

Kent [1968] shows that for each ordinal α, there is a closure space (S_α, τ_α) whose order with respect to the protoreflection is exactly α.

(6) Structure functors on top categories:

In §5 we show that every structure functor on a top category has associated with it a protoreflection (as well as a protocoreflection) which is not in general a reflection. This example is quite general and includes the regularity properties R_0, R_1 and regular in Top, coreflections and monoreflections in Top, the topological modification as described above, the generation of the relational algebra associated with a relational prealgebra and the transitive relation associated with a reflexive relation (see §7).

(7) Structured equivalence relations, point separation axioms.

In §6 we show that every structured equivalence relation on a top category has associated with it a protoreflection which generates the reflection into the associated extremal-epi-reflective subcategory. As examples we have the T_0, T_1, T_2 and Urysohn protoreflections in Top. In §11 we show that the T_1 and T_2 protoreflections are not strong.

4.6 Protoreflections and regular categories.

In this subsection we consider a regular category $\underline{C}$.

4.6.1 Proposition:

Let $(F, \eta, \underline{A})$ be a regular-epi-protoreflection in the regular category $\underline{C}$. Then each iterate of $(F, \eta, \underline{A})$ which exists is again regular-epi.

Proof:

We proceed by induction and prove that all $\eta_X^{\alpha,\beta}$ are regular-epi.

For non-limit ordinals the induction step is trivial (use 1.2g).

So let γ be a limit ordinal and assume that $\eta_X^{\alpha,\beta}$ is regular-epi for all $0 \leqslant \alpha \leqslant \beta \leqslant \gamma$. We now use the notation of 4.2d.

By [Grillet 1971 6.6] $\bigvee_{0\leqslant\alpha<\gamma} \mathrm{Im}\ \ell_X^{\alpha} = 1$ ....(i)

Now for each $\alpha<\gamma$, $\ell_X^0 = \ell_X^{\alpha}\eta_X^{\alpha}$ . ...(ii)

By assumption η_X^{α} is regular-epi. It follows that $\mathrm{Im}\ \ell_X^0 = \mathrm{Im}\ \ell_X^{\alpha}$ for each α. So by (i) it follows that $\mathrm{Im}\ \ell_X^0 = 1$ and hence ℓ_X^0 is regular-epi. Now by (ii) and 1.2h we see that ℓ_X^{α} is regular-epi for each α. The rest of the proof is trivial. □

Extremal-epi-protoreflections in top categories over 'nice' regular categories always generate reflections and have other pleasant properties.

4.6.2 Proposition:

Let $(F, \eta, \underline{A})$ be an extremal-epi-protoreflection in a top category $\underline{C}^S$ where $\underline{C}$ is regular.

Then:

(a) Every iterate of $(F, \eta, \underline{A})$ which exists is extremal-epi.

Now suppose that every relation in $\underline{C}$ is contained in a least congruence and that $\underline{C}$ has unions (see 1.8):

(b) Every iterate of $(F, \eta, \underline{A})$ exists.

(c) If further $\underline{C}$ is regularly-co-wellpowered and $\underline{A} = \underline{C}^s(F, \eta, \underline{A})$, then $(F, \eta, \underline{A})$ strongly generates a reflection.

Proof:

(a) Use 4.6.1 and the properties of extremal-epis and colimits in top categories.

(b) For $(X,x)\in\underline{C}^s$, write $F(X,x) = (FX,Fx)$ and $\eta_X = \eta_{(X,x)}$.

We need only show that the appropiate colimits exist in $\underline{C}$.

So let α be a limit ordinal and for $X\in\underline{C}$ let $\phi_X^0 : X \twoheadrightarrow X'$ be regular-epi such that $\ker \phi_X^0$ is the least congruence on X containing $\bigvee_{\beta<\alpha} \ker \eta_X^{0,\beta}$. Now by 1.8.1 there is for each $\beta<\alpha$ a morphism ϕ_X^β such that

$$\begin{tikzcd}
X \arrow[rr, two heads, "\eta_X^{0,\beta}"] \arrow[dr, two heads, "\phi_X^0"'] & & F^\beta X \arrow[dl, "\phi_X^\beta"] \\
& X' &
\end{tikzcd}$$

commutes.

We see easily that $\{(X', \phi_X^\beta)\}_{\beta<\alpha}$ is the colimit of the diagram

$$\begin{tikzcd}
X \arrow[r, two heads, "\eta_X^{0,\beta}"] \arrow[dr, "\eta_X^{0,\beta'}"'] & F^\beta X \arrow[d, "\eta_X^{\beta,\beta'}"] \\
& F^{\beta'} X
\end{tikzcd}$$

$, \beta\leqslant\beta'<\alpha.$

(c) By (a) and (b) all iterates exist and all $\eta_X^{\alpha,\beta}$ are extreme-epis. Since $\underline{C}^S$ is regularly-co-wellpowered there are for each $(X,x)\varepsilon\underline{C}^S$, ordinals $\alpha<\beta$ such that $\eta_X^{\alpha,\beta}$ is an isomorphism in $\underline{C}$. It follows that $\eta_X^{\alpha,\alpha+1} = \eta_{F^\alpha X}$ is an isomorphism in $\underline{C}^S$ and so $F^\alpha X\varepsilon\underline{A}$. □

4.6.3 Proposition

Let $\underline{C}$ be regular and cocomplete and satisfy Grillet's conditions C_3', C_3'', C_3''' [1971 pp. 170].

Let $(F, \eta, \underline{A})$ be a mono-protoreflection in $\underline{C}^S$. Then:

(a) Every iterate of $(F, \eta, \underline{A})$ is a mono-protoreflection.

(b) If $(F, \eta, \underline{A})$ weakly generates a reflection then $(F, \eta, \underline{A})$ is also an epi-epiprotoreflection.

Proof:

(a) is proved by induction using Grillet's result 2.5 [1971 pp. 178].

(b) The proof is similar to that of the corresponding result for monoreflections. □

5. TOP CATEGORIES: STRUCTURE FUNCTORS, MONOREFLECTIONS AND COREFLECTIONS

In this section we generalise some results of Herrlich [1969a] and Kannan [1970a] concerning monoreflections and coreflections in Top to apply to general top categories. We show that these reflections are in general obtained by iteration of a protoreflection. We also obtain some results concerning the lifting of extremal-epireflective subcategories thus partially generalising a result of Brümmer [1971] on the lifting of epireflections.

Throughout this section $\underline{C}^s$ and $\underline{C}^t$ will be top categories over $\underline{C}$ (which need not be regular). P will denote the projection functor of both $\underline{C}^s$ and $\underline{C}^t$.

5.1 Definition : [Wyler 1971a]

A functor $L:\underline{C}^s \longrightarrow \underline{C}^t$ such that $PL = P$ is called a structure functor.

In Wyler's terminology L 'lifts' the identity on $\underline{C}$. For (X,x) in $\underline{C}^s$, we write $L(X,x) = (X,Lx)$. For each object X of $\underline{C}$ the mapping $s^*X \longrightarrow t^*X$ induced by L is called the structure morphism induced by L.

It follows that if $f:(X,x) \longrightarrow (Y,y)$ in $\underline{C}^s$, then

$$Lx \leqslant f^*Ly.$$

For the rest of this section L, L' will be structure functors $\underline{C}^s \longrightarrow \underline{C}^t$.

5.1.1 Remarks:

There are two trivial structure functors $\underline{C}^s \longrightarrow \underline{C}^t$. These are α and ω where $\alpha(X,x) = (X, \alpha_X)$ and $\omega(X,x) = (X, \omega_X)$.
If $\underline{C}^s = \underline{C}^t$ then the identity functor is another one.

The following elementary properties of structure functors will be used frequently later on.

5.1.2 Lemma:

(a) If $f:X \longrightarrow Y$ and $(Y,y)\varepsilon\underline{C}^s$ then $Lf^*y \leqslant f^*Ly$.

(b) If $f:X \longrightarrow Y$ and $(X,x)\varepsilon\underline{C}^s$ then $f_*Lx \leqslant Lf_*x$.

(c) If (X_i,x_i) is a family of spaces from $\underline{C}^s$ and $\mathbf{X} X_i$ exists in $\underline{C}$, then $L(\bigwedge p_i^* x_i) \leqslant \bigwedge p_i^* Lx_i$.

(d) If (X_i,x_i) is a family of spaces from $\underline{C}^s$ and $\mathbf{U} X_i$ exists in $\underline{C}$, then $L(\bigvee (k_i)_* x_i) \geqslant \bigvee (k_i)_* Lx_i$.

Proof:

Trivial, see 3.3. □

5.2 Definition: Special properties of structure functors.

A structure functor L is said to preserve subspaces (resp. quotients) if we have equality in 5.1.2(a) (resp. (b)) for f extreme-monic (resp. extreme-epi) in $\underline{C}$.

L preserves products (resp. sums) if we have equality in (c) (resp. (d)).

5.3 Definition: Subcategories associated with a structure functor.

L and L' have associated with them two full replete subcategories of $\underline{C}^s$ defined as follows:

(a) $\underline{C}^s(L \leqslant L')$ has as spaces all $(X,x)\varepsilon\underline{C}^s$ for which $Lx \leqslant L'x$.

(b) $\underline{C}^s(L = L')$ has as spaces all $(X,x)\varepsilon\underline{C}^s$ for which $Lx = L'x$.

In the next few results we examine some properties of the subcategories defined above.

5.3.1 Proposition:

(a) If L' preserves subspaces (resp. products) then $\underline{C}^s(L \leqslant L')$ is closed under subspaces (resp. products).

(b) If L preserves quotients (resp. sums) then $\underline{C}^s(L \leqslant L')$ is closed under quotients (resp. sums).

Proof:

(a) Let $f: X \rightarrowtail Y$ be extreme-monic, $(Y,y)\varepsilon\underline{C}^s$ and $Ly \leqslant L'y$. Then $L(f^*y) \leqslant f^*Ly \leqslant f^*L'y = L'f^*y$. Thus $(X, f^*y)\varepsilon\underline{C}^s(L \leqslant L')$.

For products, let (X_i, x_i) be a family of spaces from $\underline{C}^s$.
Then $L(\bigwedge p_i^*x_i) \leqslant \bigwedge p_i^*Lx_i \leqslant \bigwedge p_i^*L'x_i = L'(\bigwedge p_i^*x_i)$.
It follows that $\mathsf{X}(X_i,x_i)\varepsilon\underline{C}^s(L \leqslant L')$.
The proof of (b) is similar to that of (a). □

5.3.2 Corollary:

If L and L' both preserve subspaces (resp. products, quotients, sums) then $\underline{C}^s(L = L')$ is closed under formation of subspaces (resp.

products, quotients, sums). □

The next result generalises the main result of Herrlich's paper [1969a] concerning coreflections in *Top*.

5.3.3 Proposition:

Let $L:\underline{C}^S \longrightarrow \underline{C}^S$ be a structure functor.

(a) $\underline{C}^S(L \leqslant 1)$ is closed under products, subspaces and contains all indiscrete spaces. It is a top subcategory of $\underline{C}^S$ and the reflection is strongly generated by the protoreflection $(F, 1, \underline{C}^S(L \leqslant 1))$ where F is the functor:

$$(X,x) \longmapsto (X, x \vee Lx), \quad f \longmapsto f.$$

(b) $\underline{C}^S(1 \leqslant L)$ is closed under sums and quotients and contains all discrete spaces. It is a cotop subcategory of $\underline{C}^S$ and the coreflection is strongly generated by the protocoreflection $(F', 1, \underline{C}^S(1 \leqslant L))$ where F' is the functor:

$$(X,x) \longmapsto (X, x \wedge Lx), \quad f \longmapsto f.$$

Proof:

(a) The identity functor preserves subspaces and products and it follows (by 5.3.1(a)) that $\underline{C}^S(L \leqslant 1)$ is closed under these operations. $\underline{C}^S(L \leqslant 1)$ obviously contains the indiscrete spaces. If $f:(X,x) \longrightarrow (Y,y)$ then $x \leqslant f^*y$ and $Lx \leqslant f^*Ly$. So $x \vee Lx \leqslant f^*y \vee f^*Ly \leqslant f^*(y \vee Ly)$. So F is a structurefunctor and $1:1 \longrightarrow F$ is a natural transformation. If $(Y,y) \varepsilon \underline{C}^S(L \leqslant 1)$ then $F(Y,y) = (Y,y)$. Thus $(F, 1, \underline{C}^S(L \leqslant 1))$ is a protoreflection.

Write $Fx = x \vee Lx$. Then $F^n(X,x) = (X, F^n x)$ for each integer n and for α a limit ordinal the colimit is given by $(X, \bigvee_{\beta<\alpha} F^\beta x)$. Thus $F^\alpha(X,x) = (X, F(\bigvee_{\beta<\alpha} F^\beta x))$. Thus all the iterates of the protoreflection exist and $\{F^\alpha x\}_\alpha$ is an increasing chain of structures on X. It follows that there is an ordinal α such that $LF^\alpha x \leqslant F^\alpha x$.

(b) This is dual to (a). □

5.3.4 Remarks:

Herrlich [1969a] states in effect that for $\underline{C}$ = Top, the protocoreflection associated with a structure functor is always strong (i.e. it is already the coreflection). This is not true. Herrlich has also made this observation in a letter to the author. We give an example: For $(X,x) \varepsilon$ Top, let Lx be the topology on X with subbase x together with the complements of members of x. Then $\underline{C}^s(1 \leqslant L)$ is the full subcategory of spaces in which every open set is closed. If we let u be the upper topology on $\mathbb{R}$ (the reals), then Lu is a T_1 topology but is not discrete and hence $(\mathbb{R}, Lu) \not\in \underline{C}^s(L \leqslant 1)$. Thus L is not the coreflection.

5.4 Examples:

We list below a number of examples of structure functors for which the associated protoreflection is not strong:

(1) The generation of the relational algebra associated with a relational prealgebra (see 7.3).

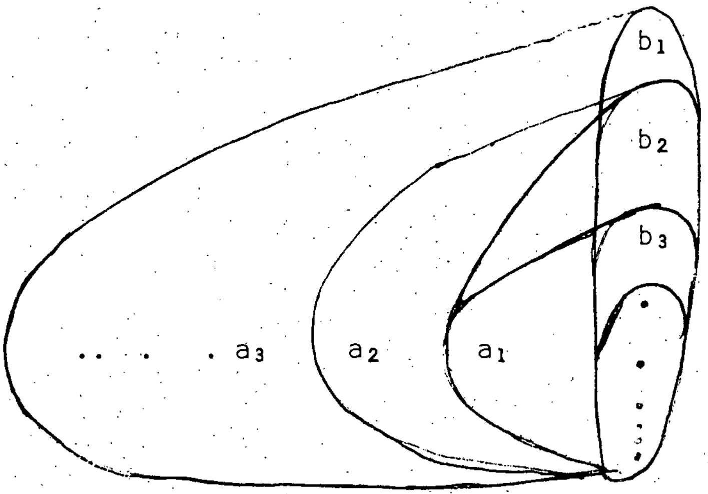
(2) The topological modification of a closure space (see 4.5(5)).

(3) The regular modification of a topological space [Thomas 1968].

(4) R_0-topological spaces [see Murdeshwar and Naimpally 1966].

Let $L:\underline{Top} \longrightarrow \underline{Top}$ be the structure functor which selects from each topology x (on a set X) the x-open sets U which have the property: $a \in U \Longrightarrow c\ell_x a \subset U$.
It is easy to check that these open sets do form a topology Lx on X and $x \leqslant Lx$. Then $\underline{C}^s(L \leqslant 1) = R_0$-spaces.
Thus the protoreflection associated with L strongly generates the R_0 reflection (5.3.3). It is not strong however. We give an example to show this:

Let $X = A \cup B$ where $A = \{a_n\}$ and $B = \{b_n\}$ are countably infinite sets. Let x be the topology on X with open sets $\emptyset$, X, B, B_m, $A^n \cup B_m$ for all m,n where $A^n = \{a_1, a_2, \ldots\ldots\ldots, a_n\}$ and $B_m = \{b_m, b_{m+1}, b_{m+2} \ldots\}$.



Then $Lx = \{\emptyset, X, B\}$ and so (X,Lx) is not yet a R_0 space. □

5.4.1 Problem:

For $\underline{M}$ a monoreflective subcategory of $\underline{C}^s$, denote by $\underline{C}^s(M)$ the class of structure functors $L:\underline{C}^s \longrightarrow \underline{C}^s$ such that $I \leqslant L$ and $\underline{C}^s(L = I) = \underline{M}$.

Study the class $\underline{C}(\underline{M})$. In particular consider the following questions:

(a) The functor of the monoreflection is always the largest member of $\underline{C}(\underline{M})$. When does $\underline{C}(\underline{M})$ have a least member?

(b) When does $\underline{C}(\underline{M})$ have exactly one member?

(c) Are any of the structure functors described in 5.4 the least members of the corresponding class $\underline{C}(\underline{M})$?

Note that in §6 and §11 we obtain some results concerning the corresponding problems for structured relations.

Consider also the 'dual' problems for coreflective subcategories.

5.5 Monoreflective and Coreflective hulls.

Let $\underline{A}$ be a full subcategory of $\underline{C}^s$. We define structure functors L, L' on $\underline{C}^s$ as follows:

$$L(X,x) = \bigwedge\{f_\alpha^* \ a_\alpha \mid f_\alpha :(X,x) \longrightarrow (A_\alpha, a_\alpha), \text{ for all } f_\alpha \ (A_\alpha, a_\alpha) \text{ in } \underline{A}\}$$

$$L'(X,x) = \bigvee\{(f_\alpha)_* a_\alpha \mid f_\alpha :(A_\alpha, a_\alpha) \longrightarrow (X,x), \text{for all } f_\alpha, \ (A_\alpha, a_\alpha) \text{ in } \underline{A})$$

$\underline{C}^s(L \leqslant I) = \underline{C}^s(L = I)$ has spaces all (X,x) such that $x = \bigwedge f^*_\alpha a_\alpha$ in the above notation. A similar statement is true for L'.

The protoreflection and protocoreflections induced by L and L' are both strong. Further $\underline{C}^s(L = 1)$ is the smallest top subcategory of $\underline{C}^s$ which contains $\underline{A}$, and if $\underline{C}$ is balanced then it is the monoreflective hull of $\underline{A}$. A corresponding result is true for the dual case.

5.6 Definition: Lifting subcategories via structure functors

Let $L:\underline{C}^s \longrightarrow \underline{C}^t$ be a structure functor and $\underline{A}$ be a full subcategory of $\underline{C}^t$. Define $\underline{C}^s(L,\underline{A})$ to be the full subcategory of $\underline{C}^s$ consisting of those spaces whose image under L lies in $\underline{A}$

The subcategory $\underline{C}^s(L,\underline{A})$ is a generalisation of Kannan's subcategory $(\underline{B},\underline{A})$ where $\underline{B}$ is a coreflective subcategory of Top and L is the $\underline{B}$-coreflection [1970a]. In §6 we will apply some of the results given below to the study of structured equivalence relations.

5.6.1 Proposition:

If $\underline{A}$ is closed under subobjects and (finite) products, then $\underline{C}^s(L,\underline{A})$ is also closed under these operations.

Proof:

Let $f: (X,x) \rightarrowtail (Y,y)$ be monic with $(Y,y)\varepsilon\underline{C}^s(L,\underline{A})$. Since $f:(X,Lx) \rightarrowtail (Y,Ly)$ we have $(X,Lx)\varepsilon\underline{A}$. Thus $(X,x)\varepsilon\underline{C}^s(L,\underline{A})$.

For a product $\mathsf{X}(X_i,x_i)$, we have $1: L(\mathsf{X}(X_i,x_i)) \rightarrowtail \mathsf{X}L(X_i,x_i)$. So if each $(X_i,x_i)\varepsilon\underline{C}^s(L,\underline{A})$, we have the product also in $\underline{C}^s(L,\underline{A})$. $\square$

5.6.2 Corollary:

(a) If $\underline{C}^t$ has the property (finite) (P1) of 3.5, then $\underline{C}^s(L = \alpha)$ is closed under (finite) products.

(b) If $\underline{C}^t$ has the property (P2) then $\underline{C}^s(L = \alpha)$ is closed under subobjects.

Proof: $\underline{C}^s(L = \alpha) = \underline{C}^s(L,$ discrete spaces$)$. Now use 5.6.1. □

5.6.3 Remarks:

(a) Let $\underline{C}^s = \underline{C}^t$.

If $L \leqslant 1$ then $\underline{C}^s(L = \alpha)$ contains all discrete spaces.

If $1 \leqslant L$ then $(X,x) \varepsilon \underline{C}^s(L = \alpha)$ implies $x = \alpha_X$.

(b) If the injections of the sum are monic then $\bigsqcup(X_i,x_i) \varepsilon \underline{C}^s(L = \alpha)$ implies that each $(X_i,x_i) \varepsilon \underline{C}^s(L = \alpha)$.

(c) If $\underline{C}$ is connected and has a terminal object N with a unique s-structure then $\times(X_i,x_i) \varepsilon \underline{C}^s(L = \alpha)$, implies that each $(X_i, x_i) \varepsilon \underline{C}^s(L = \alpha)$.

5.7 Lifting of extremal-epireflective subcategories.

Brümmer [1971 1.9] gives the following result: Let $L: \underline{C}' \longrightarrow \underline{B}$ be a functor with $\underline{C}'$ complete, wellpowered and co-wellpowered and let $\underline{C}'$ be an epireflective subcategory of $\underline{B}$. If L preserves limits or products and extreme-monics, then the subcategory $L^{-1}(\underline{B}')$ of $\underline{C}'$ of objects whose image under L lies in $\underline{B}'$, is an epireflective subcategory of $\underline{C}'$.

We show in this subsection that Brümmer's conditions on L can be relaxed somewhat if we insist that $\underline{B}'$ be an extremal-epireflective subcategory. Later on in §6 we will obtain a method of lifting the extremal-epireflector directly so that we can avoid the use of completeness conditions on the base category.

Suppose that $\underline{C}$ is complete, extremal-epi-co-wellpowered and $L:\underline{C}^s \longrightarrow \underline{C}^t$ is a structure functor:

5.7.1 Proposition:

If $\underline{A}$ is an extremal-epireflective subcategory of $\underline{C}^t$, then $\underline{C}^s(L,\underline{A})$ is an extremal-epireflective subcategory of $\underline{C}^s$.

Proof: 3.9,5.6.1. □

5.7.2 Problem:

Which top categories $\underline{C}^s$ have a 'universal' extremal-epireflective subcategory, in the sense that all other extremal-epireflective subcategories are obtained from it by lifting via a structure functor? In particular are T_2-spaces universal in Top? See also 11.9.

The result 5.7.1 handles the lifting of the point separation properties T_0, T_1, T_2 from Top to the categories of Uniform spaces, Proximity spaces etc. However it does not tell us how these properties lift from Top to categories of topological algebras. For this we need to work in a more general situation as follows: Let $\underline{C}^s$ and $\underline{B}^t$ be top categories,, $L:\underline{C}^s \longrightarrow \underline{B}^t$ a functor which lifts the functor

$L':\underline{C} \longrightarrow \underline{B}$ [Wyler 1971a].

5.7.3 Proposition:

For each full subcategory of $\underline{A}$ of $\underline{B}^t$ let $\underline{C}^s(L,L',\underline{A})$ denote the subcategory of $\underline{C}$ of spaces whose image under L lies in $\underline{A}$.

Let $\underline{C}$, $\underline{B}$ be complete and extremal-epi-co-wellpowered. Suppose further that L' preserves monics and that for each product in $\underline{C}$, the induced morphism $\langle L'p_i\rangle$: $L'(\times X_i) \longrightarrow \times L'(X_i)$ is monic.

Then if $\underline{A}$ is an extremal-epireflective subcategory of $\underline{B}^t$, $\underline{C}^s(L,L',\underline{A})$ is an extremal-epireflective subcategory of $\underline{C}^s$.

Proof: Similar to 5.7.1. □

6. TOP CATEGORIES: STRUCTURED EQUIVALENCE RELATIONS, EXTREMAL-EPIREFLECTIONS.

Throughout this section $\underline{C}$ will be a wellpowered regular category with intersections which satisfies Lawvere's condition 'every equivalence relation is a congruence (see 1.8)'. It follows by 1.8.2 that $\underline{C}$ will be regularly-co-wellpowered and so by a result of Grillet [1971, 6.8], $\underline{C}$ has the property: for every relation $\alpha : X \longrightarrow X$ there is a least congruence on X which contains α. We recall the description of $\underline{C}^{r\ell}$ (3.10) as the category with structures the reflexive relations on the objects of $\underline{C}$.

We will define 'structured relations' and 'structured equivalence relations' on a top category $\underline{C}^s$ as a generalisation of the 'topological equivalence relations' of Sharpe, Beattie and Marsden [1966]. We show that there is a one to one correspondence between the structured equivalence relations and the extremal-epiprotoreflections. We examine in some detail the relationship between the properties of a structured equivalence relation and those of the corresponding protoreflection. We also examine the lifting of an extremal-epiprotoreflection using a structure functor, as promised in §5.

The assumption that $\underline{C}$ be wellpowered can be weakened to regularly-co-wellpowered and all the results of this section will still be true with essentially the identical proofs. The reason for the stronger assumption is to ensure that $\underline{C}^{r\ell}$ is a top category (the structures on each object must form a set). However with the weaker assumption there is an obvious way by which one can reformulate the basic definition, 6.1.

6.1 Definition: Structured Relations. [cf. Sharpe, Beattie and Marsden 1966].

A *structured relation* E on a top category $\underline{C}^s$ is a structure functor $E:\underline{C}^s \longrightarrow \underline{C}^{r\ell}$.
As usual we write $E(X,x) = (X,Ex)$. If for all $(X,x)\varepsilon\underline{C}^s$, Ex is an equivalence relation, then we say that E is a *structured equivalence relation* on $\underline{C}^s$.

Note that for $f:(X,x) \longrightarrow (Y,y)$ in $\underline{C}^s$ we have $f_*(Ex) \leqslant Ey$ and equivalently $Ex \leqslant f^*(Ey)$, i.e. $f(Ex)f^{-1} \leqslant Ey$ and $Ex \leqslant f^{-1}(Ey)f$.

The category $\underline{C}^s(E = 1)$ is then the full subcategory of $\underline{C}^s$ consisting of those spaces (X,x) for which $Ex = 1_X$.

We say that the structured relations E and E' are compatible if $\underline{C}^s(E = 1) = \underline{C}^s(E' = 1)$.

6.2 Definition: Proper structured relations.

A structured relation E on $\underline{C}^s$ is said to be *proper* if $E \neq 1$ and $E \neq *$ where 1 and $*$ are the structured relations defined by:
$1(X,x) = (X, 1_X)$ and $*(X,x) = (X, \mathrm{Im}\, 1_{X \times X})$.

6.2.1 Remarks

(a) For each X in $\underline{C}$ the following are equivalent:

(i) For each Y in $\underline{C}$ there is at most one morphism $Y \longrightarrow X$.

(ii) The diagonal $\Delta : X \longrightarrow X \times X$ is an isomorphism.

(iii) The two projections $X \times X \longrightarrow X$ are equal.

(iv) The two projections $X \times X \longrightarrow X$ are isomorphisms.

If these conditions are satisfied then X is called a <u>partial terminal</u> object (see also Barr 1971, III 2.4).

(b) $\underline{C}^s(1 = 1) = \underline{C}^s$.

$\underline{C}^s(* = 1)$ is the full subcategory of spaces (X,x) for which X satisfies the conditions of (a).

(c) Let C be a cogenerator of $\underline{C}$. If $(C, \omega_C) \varepsilon \underline{C}^s(E = 1)$ then $E = 1$.

<u>Proofs:</u> (a) and (b) are trivial.

(c) For $(X,x)\varepsilon\underline{C}^s$ let $Ex = \mathrm{Im}\langle a, b\rangle$ with $\langle a, b\rangle : R \rightarrowtail X \times X$ monic. For $f : X \longrightarrow C$, $f : (X,x) \longrightarrow (C, \omega_C)$ and so $f_*(Ex) = f(ba^{-1})f^{-1} \leqslant 1_C$. Thus we have $(fb)(fa)^{-1} \leqslant 1$. It follows that $fb \leqslant fa$ and so $fb = fa$. Since C is a cogenerator it follows that $a = b$. Now by 1.4.1(g) we have $Ex = 1_X$. □

<u>6.3 Proposition:</u> [cf. Sharpe, Beattie and Marsden 1966]

Let E be a structured relation on $\underline{C}^s$.

(a) $\underline{C}^s(E = 1)$ is closed under products and subobjects. Further it contains the space (N, n) for all structures n on N.

(b) If $\underline{C}$ is complete then $\underline{C}^s(E = 1)$ is an extremal-epireflective subcategory of $\underline{C}^s$.

(c) Suppose that $\underline{C}$ is connected and that the terminal object N has a unique s-structure. Then if a product lies in $\underline{C}^s(E = 1)$ so does each factor space.

Proof:

(a) $\underline{C}^{r\ell}$ has the properties (P1) and (P2). We now apply 5.6.2.

Now since N is terminal, $N \times N \cong N$ and so $1_N = \omega_N$. Thus $E(n) = 1_N$.

(b) Use 3.9.

(c) 5.6.3(c). $\square$

6.3.1 Remark:

In 6.5.1 we will prove that $\underline{C}^s(E = 1)$ is an extremal-epi-reflective subcategory of $\underline{C}^s$ without the assumption that $\underline{C}$ be complete.

6.3.2 The case C = sets.

Let E be a structured relation on a top category $\underline{\text{sets}}^s$ in which the singleton has a unique structure and suppose that $E \neq *$.

Then $\underline{\text{sets}}^s(E = 1)$ contains all the discrete spaces and is closed under the formation of sums.

Proof:

Since $E \neq *$, we see that $\underline{\text{sets}}^s(E = 1)$ contains a space with at least two points. Thus taking products we see that it contains a space of any given cardinality and hence all discrete spaces.

Now let $(X_i, x_i) \varepsilon\ \underline{\text{sets}}^s(E = 1)$ for all $i \varepsilon I$. Write $(X,x) = \coprod_I (X_i, x_i)$. Suppose that $(a,b) \varepsilon\ Ex$. We need to show that $a = b$.

Let $f: \coprod (X_i, x_i) \longrightarrow (I, \alpha_I)$ be the map defined by: $f(X_i) = \{i\}$ for each i. Then $(fa, fb) \varepsilon\ E_{\alpha_I} = 1_I$,

and so $a, b \varepsilon X_j$ for some $j \varepsilon I$.

Now let $g: \coprod_I (X_i, x_i) \longrightarrow (X_j, x_j)$ by $g_{X_j} = 1_{X_j}$ and

$g(X_i) = \{a\}$ for $i \neq j$.

Then $(a,b) = (ga, gb) \varepsilon\ Ex_j = 1_{X_j}$ and so $a = b$. $\square$

6.4 The Structured equivalence relation associated with a structured relation.

Every structured relation E on $\underline{C}^s$ has associated with it a least (see 6.6) structured equivalence relation E^q such that for all $(X,x)\varepsilon\underline{C}^s$ we have $Ex \leqslant E^q x$. $E^q x$ is defined to be the least equivalence relation on X which is larger than Ex. It is easy to check that E^q is indeed a structured equivalence relation and that E and E^q are compatible.

6.5 The correspondence between structured equivalence relations and extremal-epiprotoreflections

Let E be a structured equivalence relation on $\underline{C}^s$.
Since $\underline{C}$ satisfies Lawvere's condition, there is for each space (X,x), a regular-epi in $\underline{C}$ which we denote by ρ_X^E, such that

$$Ex = \ker \rho_X^E .$$

For each space (X,x) let $E_X^r X$ be the codomain of ρ_X^E and <u>define</u> $E^r(X,x) = (E_X^r X, (\rho_X^E)_* x)$. For $f:(X,x) \longrightarrow (Y,y)$ the induced morphism theorem (1.8.1) gives us a map $E^r f: E^r(X,x) \longrightarrow E^r(Y,y)$.

The construction above gives us an extremal-epiprotoreflection $(E^r, \rho^E, \underline{C}^s(E = 1))$ in $\underline{C}^s$. We say that the structured equivalence relation is <u>strong</u> if the corresponding protoreflection is strong (i.e. it is the reflection).

It is easy to see that this construction in fact gives us a one to one correspondence between the structured equivalence relations on $\underline{C}^s$ and the (isomorphism classes of) extremal-epiprotoreflections on $\underline{C}^s$.

For each ordinal α, we define E^{α}, the α'th power of the structured equivalence relation by : $E^{\alpha}X = \ker(\rho^{E})^{\alpha}_{X}$ where $(\rho^{E})^{\alpha}$ is the natural transformation of the α'th iteration of the protoreflection associated with E (see §4).

6.5.1 Proposition:

Let E be a structured relation on $\underline{C}^{s}$.
Then $\underline{C}^{s}(E = I)$ is an extremal-epireflective subcategory of $\underline{C}^{s}$ and the reflection is strongly generated by the protoreflection corresponding to E^{q}.

Proof:
Use 4.6.2. $\square$

6.5.2 Remarks:

The result above will be applied in §9 and §11 to give 'internal' descriptions of the reflections associated with the T_0, T_1 and T_2 point separation axioms in Top as well as in other categories of relational algebras.

6.6 The 'complete lattice' of structured relations.

The collection of all structured relations on $\underline{C}^{s}$ admits a partial order induced by the ordering of relations. We give below some basic properties of this order.

(a) The order:

Let E and E' be structured relations on $\underline{C}^s$.

We define: $E \leqslant E'$ if and only if $Ex \leqslant E'x$ for all $(X,x) \varepsilon \underline{C}^s$. The order $\leqslant$ is reflexive, anti-symmetric and transitive.

(b) Infima and suprema:

These are formed 'pointwisely'. I is the least structured relation and $*$ is the greatest structured relation.

(c) Compatibility classes:

Let $\underline{A}$ be an extremal-epireflective subcategory of $\underline{C}^s$. The collection of all structured relations E on $\underline{C}^s$ such that $\underline{C}^s(E = I) = \underline{A}$ is called the compatibility class of structured relations associated with $\underline{A}$.

6.6.1 Proposition: Properties of the order.

Let E_1 and E_2 be structured relations on $\underline{C}^s$.

(a) $E_1 \leqslant E_2 \Longrightarrow \underline{C}^s(E_2 = I) \subseteq \underline{C}^s(E_1 = I)$.

(b) Each compatibility class is convex, is closed under suprema and has a largest element which is the strong structured equivalence relation associated with the extremal epireflection which corresponds (see 6.5) to the compatibility class.

(c) $E_1 \leqslant E_2 \Longrightarrow E_1^q \leqslant E_2^q$.

(d) The correspondence between structured equivalence relations and protoreflections is order preserving.

Proof:

Trivial. □

6.6.2 Open problems:

The results given above suggest that one should study the relationship between the properties of an extremal-epireflective subcategory and the properties of the corresponding class of structured relations.

More specifically we have the following open questions:

(a) The compatibility class of structured relations associated with the T_1 topological spaces does not have a least member (see II.3.2a). However the compatibility class of structured equivalence relations associated with the T_1-spaces does have a least member and the same result is true for the T_2-topological spaces (II.3.1, II.4.2). Is this true in general?

(b) Are compatibility classes always well ordered?

(c) When do compatibility classes have exactly one member? In Top the subcategory of T_0-spaces has this property (see II.2.1)

(d) When does a compatibility class consist simply of powers of a single structured equivalence relation? (see II.3.2c).

In the next three subsections we study initial, hereditary and productive structured relations and show how these properties transfer to the associated protoreflections.

6.7 Initial Structured equivalence relations.

A structured equivalence relation E is initial if the induced protoreflection maps are initial, i.e. for all spaces (X,x), $x = (\rho_X^E)^*(\rho_X^E)_*x$.

6.7.1 Proposition:

Suppose that every regular-epi in $\underline{C}$ is split-epi.

Then every initial structured equivalence relation on $\underline{C}^s$ is strong.

Proof:

We use the notation of 6.5. For $(X,x)\varepsilon\underline{C}^s$ write $g = \rho^E_x$ and let $f:E^r X \longrightarrow X$ be such that $gf = 1$. Since E is initial, $f:(E^r(x), g_* x) \longrightarrow (X,x)$ (see 3.2). Thus

$$\begin{aligned} E(g_* x) &\leqslant f^*(Ex) \\ &\leqslant f^{-1}g^{-1}gf \\ &= (gf)^{-1}(gf) = 1 \end{aligned}$$

and so $E^r(X,x) = (E^r_x X,\ g_* x)\varepsilon\underline{C}^s\ (E = 1)$. □

6.7.2. Remarks:

The converse of 6.7.1 is not true. In fact in §11 we will prove that T_0 is the only initial structured equivalence relation on $\underline{\mathrm{Top}}$ (11.2.3).

6.7.3 Proposition:

Suppose that $\underline{C}^s$ has the property (P3)* of 3.3.

Let E be an initial structured equivalence relation on $\underline{C}^s$.

Then the functor $E^r:\underline{C}^s \longrightarrow \underline{C}^s$ induced by E preserves initial maps.

Proof:

Let $f:X \longrightarrow Y$ and $(Y,y)\varepsilon\underline{C}^s$. Write $x = f^*y$, $\rho_x = \rho^E_x$ and $\rho_y = \rho^E_y$.

$$\begin{aligned} \text{Then } (\rho_x)_* x = (\rho_x)_* f^* y &= (\rho_x)_* f^*(\rho_y)^*(\rho_y)_* y \quad \text{(E is initial)} \\ &= (\rho_x)_*(\rho_x)^*(E^r f)^*(\rho_y)_* y \\ &= (E^r f)^*(\rho_y)_* y \qquad \text{(by (P3)*)}. \end{aligned}$$

□

6.8 Hereditary structured relations.

A structured relation E is hereditary if for all $f:X \longrightarrow Y$ and $(Y,y)\varepsilon \underline{C}^s$ we have $E(f^*y) = f^*(Ey)$.

(Notice that we do not restrict to f monic.)

6.8.1 Proposition:

Let E be a structured equivalence relation on $\underline{C}^s$.

The following are equivalent:

(a) E is hereditary.

(b) For $f:X \longrightarrow Y$ and $(Y,y)\varepsilon\underline{C}^s$ the morphism $E^r f:E^r(X,x) \longrightarrow E^r(Y,y)$ is monic.

Proof:

Write $x = f^*y$, $\rho_x = \rho_x^E$ and $\rho_y = \rho_y^E$.

(a)$\Longrightarrow$(b):

$$\begin{tikzcd}
(X,x) \arrow[r, two heads, "\rho_x"] \arrow[d, "f"] & E^r(X,x) \arrow[d, "E^r f"] \\
(Y,y) \arrow[r, two heads, "\rho_y"'] & E^r(Y,y)
\end{tikzcd}$$

$E(f^*y) = f^*(Ey)$ gives us $\ker(\rho_x) = \ker(\rho_y f)$ and so by 1.8.1 we see that $E^r f$ is monic.

(b)$\Longrightarrow$(a): Now $f^*(Ey) = f^{-1}\rho_y^{-1}\rho_y f = \rho_x^{-1}(E^r f)^{-1}(E^r f)\rho_x$

$= \rho_x^{-1}\rho_x = Ex.$ □

6.8.2 Corollary:

Let $\underline{C}$ be balanced and $\underline{C}^s$ have the property (P3)*.

Then the protoreflection functor induced by a structured equivalence relation which is initial and hereditary preserves subspaces in $\underline{C}^s$.

Proof: 6.7.3 and 6.8.1. $\square$

6.8.3 Remarks:

In §11 we will prove that T_0 is the only proper structured equivalence relation on Top which is hereditary (11.2.3). The structured relation T_1 is also hereditary but is not an equivalence relation.

6.8.4 The case $\underline{C}$ = sets.

We say that a structured relation E on a top category $\underline{\text{sets}}^s$ is weakly hereditary if for all spaces (X,x) and (a,b)ε Ex there is a finite subset $\{c_1, c_2, \ldots c_n\}$ of X such that $a = c_1$, $b = c_n$ and $(c_i, c_{i+1}) \varepsilon E(j_i^* x)$ for $1 \leqslant i \leqslant n-1$ where $j_i: \{c_i, c_{i+1}\} \rightarrowtail X$ are the inclusions.

We have the following result which is a contribution towards the solution of the problem stated in 6.6.2(a).

Proposition:

Let E and E' be structured relations on $\underline{\text{sets}}^s$.

(a) E hereditary $\Longrightarrow$ E weakly hereditary.

(b) E hereditary $\Longrightarrow$ E^q is weakly hereditary.

(c) Let E and E' be compatible.

If either (i) E is hereditary and E' is symmetric,

or (ii) E is weakly hereditary and E' is an equivalence,

then $E \leqslant E'$.

If further E is strong then E = E'.

Proof:

(a) and (b) are trivial.

(c) (i) Let $(X,x) \varepsilon$ sets^s and $(a,b) \varepsilon$ Ex with $a \neq b$.

Let $X' = \{a,b\}$ and $j : X' \rightarrowtail X$ be the inclusion.

Since E is hereditary we have $(a,b) \varepsilon E(j^*x)$.

So $(X', j^*x) \not\varepsilon$ Top$(E = 1)$ and since E and E' are compatible $(X', j^*x) \not\varepsilon$ Top$(E' = 1)$. So $E'(j^*x) \neq 1$ and since E' is symmetric we have $E'(j^*x) = X' \times X'$.

So $(a,b) \varepsilon E'(j^*x)$ and it follows that $(a,b) = (ja,jb) \varepsilon E'x$.

So $E \leqslant E'$.

The proof in the case of (ii) is similar.

Now if E is strong then by 6.6.1b we have $E' \leqslant E$ and so $E = E'$. □

In Top, the structured equivalence relation T_1^q is weakly hereditary but not hereditary. The structured equivalence relation T_2^q on Top is the least structured equivalence relation associated with the T_2-spaces but T_2^q is not weakly hereditary (consider $\mathbb{N}$ with the cofinite topology). So we do not have a converse for (c). For details see §11.

6.9 Productive Structured Relations.

A structured relation E on $\underline{C}^s$ is (finitely) productive if as a functor $\underline{C}^s \longrightarrow \underline{C}^{r\ell}$ it preserves (finite) products.

Let (X_i, x_i) be a family of spaces and $(X,x) = \bigtimes(X_i, x_i)$ with $P_i : \bigtimes X_i \longrightarrow X_i$ the projections.

Then if E is productive, we have

$$\begin{aligned} Ex &= \bigwedge p_i^*(Ex_i) \\ &= \bigwedge p_i^{-1}(Ex_i)p_i \qquad \text{(see 3.10)} \\ &= \bigtimes Ex_i \qquad \text{(by 1.7.4).} \end{aligned}$$

6.9.1 Remarks:

(a) The structured relation T_1 on Top is productive (9.8.3) but the associated equivalence relation T_1^q is not productive (11.3.2d).

(b) The case C = sets.

If sets^s is a top category in which the singleton has a unique structure then every structured equivalence relation on sets^s is finitely productive (for proof see [Sharpe, Beattie and Marsden 1966, prop: 4]).

Let E be a structured equivalence relation.

For $(X,x) = \bigtimes(X_i, x_i)$, let $p_i : \bigtimes X_i \longrightarrow X_i$ and $q_i : \bigtimes E^r X_i \longrightarrow E^r X_i$ be the projections.

Let $\phi = \langle E^r p_i \rangle : E^r(\bigtimes X_i) \longrightarrow \bigtimes E^r X_i$.

Then

$$\begin{tikzcd}
\bigtimes X_i \arrow[r, "\rho_X"'] \arrow[d, "p_i"'] & E^r(\bigtimes X_i) \arrow[d, "E^r p_i"] \arrow[dr, "\phi"] & \\
X_i \arrow[r, "\rho_{X_i}"'] & E^r X_i & \bigtimes E^r X_i \arrow[l, "q_i"]
\end{tikzcd}$$

commutes for each i.

Now $q_i(\phi\rho_X) = (E^r p_i)\rho_X = \rho_{X_i} p_i = q_i(\bigtimes \rho_{X_i})$ for each i, and so $\phi\rho_X = \bigtimes \rho_{X_i}$. Thus if the index set for i is finite or C is a QI-category (1.7.1) then $\phi\rho_X$ is regular-epi and so ϕ is regular-epi.

6.9.2 Proposition:

Suppose that $\underline{C}^S$ has the property (P3)* and that $\underline{C}$ is a $\underline{QI}$-category. Let E be a structured equivalence relation on $\underline{C}^S$ which is productive and initial.

Then the associated protoreflection functor E^r preserves products in $\underline{C}^S$.

Proof:

We use the notation of the preceeding discussion.

Firstly we show that ϕ is an isomorphism in $\underline{C}$. By the above discussion it is regular-epi and so we need only to show that it is monic. This is true since:

$$
\begin{aligned}
\phi\phi^{-1} &= \rho_X\rho_X^{-1}\phi^{-1}\phi\rho_X\rho_X^{-1} \\
&= \rho_X\rho_X^{-1}\phi^{-1}(\bigwedge q_i^{-1}q_i)\phi\rho_X\rho_X^{-1} \qquad (1.8.3) \\
&= \rho_X\bigwedge_i(\rho_X^{-1}\phi^{-1}q_i^{-1}q_i\phi\rho_X)\rho_X^{-1} \\
&= \rho_X(\bigwedge_i(p_i^{-1}\rho_{X_i}^{-1}\rho_{X_i}p_i))\rho_X^{-1} \\
&= \rho_X(\bigwedge_i p_i^*(Ex_i))\rho_X^{-1} = \rho_X(Ex)\rho_X^{-1} \\
&= \rho_X\rho_X^{-1}\rho_X\rho_X^{-1} \quad = 1.
\end{aligned}
$$

To complete the proof we need to show that ϕ is initial.

This follows by:

$$
\begin{aligned}
\phi^*(\bigwedge_i(q_i{}^*(\rho_{X_i})_* x_i)) &= (\rho_X)_*(\rho_X)^*\phi^*(\bigwedge q_i^*(\rho_{X_i})_*x_i) \\
&= (\rho_X)_*\bigwedge_i((q_i\phi\rho_X)^*(\rho_{X_i})_*x_i) \\
&= (\rho_X)_*\bigwedge_i(p_i^*\rho_{X_i}^*(\rho_{X_i})_* x_i) \\
&= (\rho_X)_*(\bigwedge_i p_i^* x_i) \quad \text{(since E is initial)}
\end{aligned}
$$

see 6.7. □

6.9.3 Remarks:

The structured equivalence relation T_0 in Top (and more generally in certain categories of relational algebras) satisfies all the conditions specified in the preceeding result (see 11.2.1).

6.10 Extremal-epireflective hulls

Let $\underline{A}$ be a full subcategory of $\underline{C}^s$.

For each $(X,x)\varepsilon\underline{C}^s$, let $Ex = \bigwedge\{f_i^{-1}f_i \mid f_i:(X,x) \longrightarrow (A_i,a_i),$
$\text{all } f_i,\ (A_i,a_i)\varepsilon\underline{A}\}$

It is easily seen that E is a structured equivalence relation on $\underline{C}^s$.

With the above notation we have:

6.10.1 Proposition:

The structured equivalence relation E is strong and $\underline{C}^s(E = 1)$ is the extremal-epireflective hull of $\underline{A}$.

Proof:

By 1.8.1 each morphism in the definition of Ex factors through $E^r(X,x)$. Thus for each f_i there is a $g_i:E^r(X,x) \longrightarrow (A_i,a_i)$ such that $g_i\rho_x = f_i$. So

$$\begin{aligned}\bigwedge g_i^{-1}g_i &= \rho_x\rho_x^{-1}(\bigwedge g_i^{-1}g_i)\rho_x\rho_x^{-1}\\ &= \rho_x(\bigwedge (g_i\rho_x)^{-1}(g_i\rho_x))\rho_x^{-1}\\ &= \rho_x(\bigwedge f_i^{-1}f_i)\rho_x^{-1}\\ &= \rho_x(\rho_x^{-1}\rho_x)\rho_x^{-1} = 1.\end{aligned}$$

It follows that $E(\rho_x x) = 1$ and so $E^r(X,x)\varepsilon\underline{C}^s(E = 1)$.

So we have proved that E is strong.

Now let $\underline{B}$ be an extremal-epireflective subcategory of $\underline{C}^s$ which contains $\underline{A}$ and let $\rho': I \twoheadrightarrow R$ be the $\underline{B}$-reflector. For $(X,x)\varepsilon\underline{C}^s(E = I)$ we need to show that ρ'_x is an isomorphism. Now there are maps $f_i:(X,x) \longrightarrow (A_i,a_i)$ with $(A_i,a_i)\varepsilon\underline{A}$ and $\bigwedge f_i^{-1}f_i = I$. For each i there is a map $g_i:R(X,x) \longrightarrow (A_i,a_i)$ such that $g_i\rho'_x = f_i$.

So $(\rho'_x)^{-1}\rho'_x \leqslant \rho'^{-1}_x \bigwedge (g_i^{-1}g_i)\rho'_x = \bigwedge f_i^{-1}f_i = I$. Thus ρ'_x is monic and since it is regular-epi, it is an isomorphism. □

6.10.2 Remarks:

It is well known that if $\underline{C}$ is complete then the extremal-epireflective hull of $\underline{A}$ consists precisely of all subobjects of products of spaces from $\underline{A}$.

6.11 Monoreflective hulls of extremal-epireflective subcategories.

Let E be a structured relation on $\underline{C}^s$. We define a structure functor $L:\underline{C}^s \longrightarrow \underline{C}^s$ by:

$$L(X,x) = (X, (\rho_x)^*(\rho_x)_*x).$$

6.11.1 Proposition:

If E^q is strong then $\underline{C}^s(L = I)$ is the smallest top subcategory of $\underline{C}^s$ which contains $\underline{C}^s(E = I)$ and L is strong.

Proof:

Let $\underline{M}$ be a top subcategory of $\underline{C}^s$ which contains $\underline{C}^s(E = I)$. For $(X,x)\varepsilon\underline{C}^s(L = I)$ we have $x = (\rho_x)^*(\rho_x)_*x$. Also $E^r(X,x) = (E^rx, (\rho_x)_*x)\varepsilon\ \underline{M}$ and so by the condition (ii) for top subcategories (3.2) we have $(X,x)\varepsilon\underline{M}$.

To show that L is strong one uses the fact that ρ_X and ρ_{LX} differ by an isomorphism. □

6.12 Lifting of extremal-epiprotoreflections.

We now continue the discussion which was started in 5.7.

Let $L:\underline{C}^s \longrightarrow \underline{C}^t$ be a structure functor. We can use L to lift structured relations from $\underline{C}^t$ to $\underline{C}^s$ as follows:

For E a structured relation on $\underline{C}^t$ the composition EL is a structured relation on $\underline{C}^s$ and

$$\underline{C}^s(EL = 1) = \underline{C}^s(L, \underline{C}^t(E = 1))$$

6.12.1 Proposition:

(a) The protoreflection induced by the lifted structured relation strongly generates an extremal-epiprotoreflection and $\underline{C}^s(L,\underline{C}^t(E = 1))$ is an extremal-epireflective subcategory of $\underline{C}^s$.

(b) Suppose that $Lf_* = f_*L$ for all regular epi's f. Then if E is strong so is EL.

Proof:

(a) 6.5.1.

(b) For $(X,x)\varepsilon\underline{C}^s$ we have

$$\begin{aligned} L(EL)^r(X,x) &= L(E^r_{Lx}, (\rho^E_{Lx})_* x) \\ &= (E^r_{Lx}, (\rho^E_{Lx})_* Lx) \\ &= E^r(X,Lx)\varepsilon\underline{C}^t(E = 1). \end{aligned}$$

□

6.12.2 Remarks:

The conditions on L in (b) is satisfied if in Wyler's terminology

L is 'cotaut'. Every top subcategory has a cotop reflector which is cotaut [Wyler 1971b, 2.7, 2.9.2].

7. RELATIONAL ALGEBRAS : GENERALITIES

Barr [1970] introduced the concept of relational algebras of a monad (= triple = co-standard construction) on the category of sets. Barr's main result is that Top is the category of relational algebras of the ultrafilter monad.

A. Burroni [1971] has further generalizations of this concept and obtains relational algebras for a monad in a general category. There is little overlap between his work and ours. We will define relational algebras of a monad on a regular category using the obvious generalization of Barr's definition.

Manes [1973] has further results about relational algebras over Sets. Many of our results in §7 - 10 are straight generalizations of the results of Manes from Sets to regular categories.

In this section we give generalizations of the results in §2 of Barr's paper [1970] concerning the reflection from relational prealgebras to relational algebras. We show that the relational algebras (and the prealgebras) form a top category. We give conditions on the functor of the monad which ensure that the 'P' properties hold. We discuss the formation of quotient structures and give a generalization of Michaels' theorem on products of biquotient maps. We also give some examples.

Our terminology for monads follows Mac Lane [1971]. $\underline{C}$ will always be a wellpowered regular category with intersections and which satisfies Lawvere's condition. $\pi = (T, \eta, \mu)$ will be a monad on $\underline{C}$.

<u>7.1 Definition</u>: [cf. Barr 1970, A. Burroni 1971].

We define categories $\underline{C}^{P(\pi)}$ and $\underline{C}^{R(\pi)}$ as follows:

(a) $\underline{C}^{P(\pi)}$, the category of <u>relational π-prealgebra's</u> has as objects all pairs (X,x) with $X \in \underline{C}$ and $x : TX \longrightarrow X$ a relation such that $1_X \leqslant x\eta_X$.

$$\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[dr, "1_X"'] \arrow[dr, phantom, "\leqq", shift left=4ex] & TX \arrow[d, "x"] \\
 & X
\end{tikzcd}$$

and as morphisms $f:(X,x) \longrightarrow (Y,y)$ the $\underline{C}$-morphisms $f:X \longrightarrow Y$ such that $fx \leqslant yTf$.

$$\begin{tikzcd}
TX \arrow[r, "Tf"] \arrow[d, "x"'] \arrow[dr, phantom, "\leqslant"] & TY \arrow[d, "y"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

(b) $\underline{C}^{R(\pi)}$, the category of <u>relational π-algebras</u> is the full subcategory of $\underline{C}^{P(\pi)}$ consisting of all (X,x) for which $x\,Tx \leqslant x\,\mu_X$.

$$\begin{tikzcd}
T^2X \arrow[r, "\mu_X"] \arrow[d, "Tx"'] \arrow[dr, phantom, "\leqslant"] & TX \arrow[d, "x"] \\
TX \arrow[r, "x"'] & X
\end{tikzcd}$$

7.1.1 Remarks:

(a) Our definition of $\underline{C}^{R(\pi)}$ is a straight generalization of Barr's definition. However our relational prealgebras are more special than Barr's. His prealgebras do not even form a top category and so from our viewpoint are too general to be interesting.

(b) A. Burroni's [1971] regular π-graphs are equivalent to Barr's prealgebras, his regular pointed π-graphs are our relational π-prealgebras while his preorders are our relational algebras.

(c) The category $\underline{C}^{\pi}$ of π-algebras is a full subcategory of $\underline{C}^{R(\pi)}$.

(d) Let $(X,x)\varepsilon\underline{C}^{R(\pi)}$. Then $(X,x)\varepsilon\underline{C}^{\pi} \Longleftrightarrow x$ is a morphism.

(e) $1_X \leqslant x\,\eta_X \Longleftrightarrow \eta_X^{-1} \leqslant x$.

7.2 Proposition: [cf. Manes 1973 pp 27, A. Burroni 1971]

$\underline{C}^{P(\pi)}$ is a top category over $\underline{C}$.

Proof:

For each $X\varepsilon\underline{C}$, we define $s^*X = \{x:TX \longrightarrow X \mid 1_X \leqslant x\,\eta_X\}$, and for $f:X \longrightarrow Y$, $f^*=s^*f:s^*Y \longrightarrow s^*X$ by

$$(s^*f)y = f^{-1}y\ Tf = (Tf\times f)^*y \quad \text{(see 1.6b)}$$

We give s^*X the usual ordering on relations.

One checks easily that s^*f is order-preserving, it preserves infima and $f:(X,x) \longrightarrow (Y,y) \Longleftrightarrow x \leqslant (s^*f)y$

It remains for us to show that each s^*X is a complete lattice:

The largest element of s^*X is $\omega_X = \mathrm{Im}\ 1_{TX \times X}$.

For $x_i \in s^*X$, we have $1 \leqslant \bigwedge(x_i\eta_X) = (\bigwedge x_i)\eta_X$ (1.5a)

and so s^*X is closed under infima.

Since $1 \leqslant \eta_X^{-1}\eta_X$ and $1 \leqslant x\eta_X \Longleftrightarrow \eta_X^{-1} \leqslant x$ we see that η_X^{-1} is the least member of s^*X. □

<u>7.3 Proposition</u>: [cf. Barr 1970 2.3]. The reflection $\underline{C}^{P(\pi)} \longrightarrow \underline{C}^{R(\pi)}$. $\underline{C}^{R(\pi)}$ is a top subcategory of $\underline{C}^{P(\pi)}$ and the reflection is strongly generated by the protoreflection $(L, 1, \underline{C}^{R(\pi)})$, where $L(X,x) = (X, x\ Tx\ \mu_X^{-1})$.

<u>Proof</u>:

(i) L is a structure functor and $1 \leqslant L$.

For $(X,x) \in \underline{C}^{P(\pi)}$ we have $1 \leqslant T(x\ \eta_X)$ and so

$$\begin{aligned} x \leqslant xT(x\eta_X) &\leqslant x\ Tx\ T\eta_X \qquad (2.2a)\\ &\leqslant x\ Tx\ \mu_X^{-1}\ \mu_X\ T\eta_X\\ &= x\ Tx\ \mu_X^{-1}. \end{aligned}$$

Thus $L(X,x) \in \underline{C}^{P(\pi)}$ and $1 \leqslant L$.

For $f: (X,x) \longrightarrow (Y,y)$ in $\underline{C}^{P(\pi)}$,

$$\begin{aligned} f\,Lx = f(x\,Tx\,\mu_X^{-1}) &\leq y\,Tf\,Tx\,\mu_X^{-1} \\ &\leq y\,T(fx)\,\mu_X^{-1} && (2.2c) \\ &\leq y\,T(y\,Tf)\,\mu_X^{-1} \\ &\leq y\,Ty\,T^2f\,\mu_X^{-1} && (2.2a) \\ &\leq y\,Ty\,\mu_Y^{-1}\,Tf && (1.4.1h) \\ &= Ly\,Tf, \text{ and so } L \text{ is a} \end{aligned}$$

structure functor.

(ii) $\underline{C}^{P(\pi)}(L \leq 1) = \underline{C}^{R(\pi)}$.

$x\,Tx\,\mu_X^{-1} \leq x \Longleftrightarrow x\,Tx \leq x\,\mu_X$.

(iii) We now apply 5.3.3(a) to obtain the result. $\square$

7.3.1 Remarks:

(a) Since $\underline{C}^{R(\pi)}$ is a top subcategory, it contains all the indiscrete spaces of $\underline{C}^{P(\pi)}$.

$\underline{C}^{R(\pi)}$ also contains all the discrete spaces of $\underline{C}^{P(\pi)}$ since $\eta_X^{-1}\,T(\eta_X^{-1}) \leq \eta_X^{-1}\,T(\eta_X^{-1})\,\mu_X^{-1}\,\mu_X$

$$= \eta_X^{-1}\,(\mu_X T\eta_X)^{-1}\,\mu_X = \eta_X^{-1}\,\mu_X.$$

(b) In 7.10a we give an example which shows that in general an infinite number of iterations are required in order to generate the reflection. In fact in that example there is for each ordinal α, a space in $\underline{C}^{P(\pi)}$ whose order relative to the protoreflection is exactly α.

Finally we obtain the generalization of the last result in §2 of Barr's paper.

<u>7.4 Proposition:</u> [Barr 1970 2.4, A. Burroni 1971 1.3.10]

The inclusion functor $\underline{C}^{\pi} \longrightarrow \underline{C}^{R(\pi)}$ has a left adjoint.

<u>Proof:</u>

Since $\underline{C}$ is regularly co-wellpowered we can use Barr's proof. $\square$

<u>7.5 Proposition:</u> [cf. Barr 1970 pp. 45] <u>Formation of products.</u>

Let $(X,x) = \bigtimes(X_i, x_i)$ with $(X_i, x_i) \varepsilon \underline{C}^{P(\pi)}$ and

$p_i : \bigtimes X_i \longrightarrow X_i$ the projections.

Then $x = (\bigtimes x_i) <Tp_i>$.

<u>Proof:</u>

We have

$$\begin{tikzcd}
T(\bigtimes X_i) \arrow[r, "<Tp_i>"] \arrow[dr, "Tp_i"'] & \bigtimes T(X_i) \arrow[r, "\bigtimes x_i"] \arrow[d, "q_i"] & \bigtimes X_i \arrow[d, "p_i"] \\
& TX_i \arrow[r, "x_i"'] & X_i
\end{tikzcd}$$

(q_i are the projections).

Write $\phi = <Tp_i>$.

We have $(\bigtimes x_i)\phi = (\bigwedge p_i^{-1} x_i q_i)\phi$ (1.7.4)

$= \bigwedge(p_i^{-1} x_i q_i \phi)$ (1.5a)

$= \bigwedge(p_i^{-1}\ x_i\ Tp_i)$

$= \bigwedge p_i^{*} x_i = x.$ $\square$

7.6 Formation of Coinitial structures in $C^{P(\pi)}$.

For $f:X \longrightarrow Y$ and $(Y,y) \varepsilon \underline{C}^{P(\pi)}$, we have defined $f^*y = f^{-1}y\, Tf$ (see 7.2). By the Galois connection result we have: for $(X,x)\varepsilon\, \underline{C}^{P(\pi)}$, $f_*x = \text{Inf}\,\{\, y \mid x \leqslant f^*y \text{ and } (Y,y)\varepsilon\, \underline{C}^{P(\pi)}\}$

We now give an explicit description of f_*:

7.6.1 Proposition:

Let $f:X \longrightarrow Y$ in $\underline{C}$ and $(X,x)\varepsilon\, \underline{C}^{P(\pi)}$.

Then (a) $f_*x = (f\,x\,(Tf)^{-1}) \vee \eta_Y^{-1}$.

(b) If f is regular-epi then $f_*x = f\,x\,(Tf)^{-1}$.

Proof:

(a) Write $y' = (f\,x\,(Tf)^{-1}) \vee \eta_Y^{-1}$.

Then $1 \leqslant \eta_Y^{-1}\,\eta_Y \leqslant y'\eta_Y$ and so $(Y,y')\varepsilon\, \underline{C}^{P(\pi)}$.

If on the other hand $(Y,y)\varepsilon\, \underline{C}^{P(\pi)}$ and $x \leqslant f^*y$ then

$$\begin{aligned}\eta_Y^{-1} \leqslant y \text{ (by 7.1.1e) and } fx\,(Tf)^{-1} &\leqslant f\,f^*y\,(Tf)^{-1}\\ &= ff^{-1}y\,Tf\,(Tf)^{-1}\\ &\leqslant y\end{aligned}$$

and so $y' \leqslant y$.

(b) $$\begin{aligned}\eta_Y^{-1} = f\,f^{-1}\eta_Y^{-1} &= f\eta_X^{-1}\,(Tf)^{-1}\\ &\leqslant f\,x\,(Tf)^{-1},\end{aligned}$$

and the result now follows by (a). □

7.6.2 Remarks:

If we define $\underline{C}^{P'(\pi)}$ as the category of prealgebras in the sense of Barr [1970] (where the requirement $1 \leqslant x\eta_X$) is dropped) then in $\underline{C}^{P'(\pi)}$, coinitial structures are formed as in (b) for all f. Further, the reflection $\underline{C}^{P'(\pi)} \longrightarrow \underline{C}^{P(\pi)}$ is given by $x \longrightarrow x \vee \eta_X^{-1}$.

In this way we recover (a) from (b).

7.7 Products of quotients in $\underline{C}^{P(\pi)}$ are quotients.

In this section we show that with certain conditions on $\underline{C}$ and T, every product of quotient maps in $\underline{C}^{P(\pi)}$ is again a quotient map. First we formulate the condition on T:

A functor $T:\underline{C} \longrightarrow \underline{C}$ is said to have the property Q2 if for each family $\{f_i : X_i \twoheadrightarrow Y_i\}$ of regular-epis we have $\langle Tp_i \rangle \; T(\times f_i)^{-1} = (\times Tf_i)^{-1} \; \langle Tq_i \rangle$ where $p_i : \times X_i \longrightarrow X_i$ and $q_i : \times Y_i \longrightarrow Y_i$ are the projections.

$$\begin{tikzcd}
T(\times X_i) \arrow[r, "T(\times f_i)"'] \arrow[d, "\langle Tp_i \rangle"'] & T(\times Y_i) \arrow[d, "\langle Tq_i \rangle"] \\
\times TX_i \arrow[r, "\times Tf_i"'] & \times TY_i
\end{tikzcd}$$

Note that since the diagram commutes we always have

$$\langle Tp_i \rangle \; T(\times f_i)^{-1} \leq (\times Tf_i)^{-1} \; \langle Tq_i \rangle \quad \text{and so}$$

we need only consider the reverse inequality.

7.7.1 Proposition:

Suppose that $\underline{C}$ is a QI category (see 1.7.1) and that T has the property Q2.

Then every product of quotient in $\underline{C}^{P(\pi)}$ is again a quotient map.

Proof:

Let $\{ f_i : (X_i, x_i) \twoheadrightarrow (Y_i, y_i)\}$ be a family of quotient maps in $\underline{C}^{P(\pi)}$. So each f_i is regular-epi and $y_i = f\, x_i\, (Tf)^{-1}$ (7.6.1). Write $\phi_X = \langle Tp_i \rangle$ and $\phi_Y = \langle Tq_i \rangle$ where $p_i : \mathbf{X}\, X_i \longrightarrow X_i$ and $q_i : \mathbf{X} Y_i \longrightarrow Y_i$ are the projections.

By Q1, $\mathbf{X} f_i$ is regular-epi and by 7.5

$$\mathbf{X}(X_i, x_i) = (\mathbf{X} X_i,\ (\mathbf{X} x_i)\phi_X) \text{ and}$$
$$\mathbf{X}(Y_i, y_i) = (\mathbf{X} Y_i,\ (\mathbf{X} y_i)\phi_Y).$$

Also $(\mathbf{X} f_i)(\mathbf{X} x_i\ \phi_X)\ T(\mathbf{X} f_i)^{-1} = \mathbf{X}(f_i\ x_i)(\mathbf{X} Tf_i)^{-1}\ \phi_Y$ (1.7.2c and Q2)

$= \mathbf{X}(f_i\ x_i\ Tf_i^{-1})\ \phi_Y$,

and so by 7.6.1b we see that $\mathbf{X} f_i$ is a quotient map. □

7.7.2 Remarks: Biquotient maps, Michael's Theorem.

Michael [1968b] defines biquotient maps in Top and proves that every product of biquotient maps in Top is again a biquotient map. By a result of Kent [1969 Theorem 5] the biquotient maps in Top are precisely those maps in Top which are quotient in the category of Choquet spaces [Wyler 1973a, 3.2.2].

Now the Choquet spaces are the relational prealgebras of the ultrafilter monad and the topological spaces are the relational algebras of this monad. Also the ultrafilter functor has the property Q2 (7.10(b)(vi) and so we obtain Michael's result as a special case of 7.7.1.

7.8 The 'P' properties

We now consider the question of whether the various 'P' properties (3.5) are true of a category of relational algebras. The results given below are somewhat sharper than those given in 3.5.1a, b, c. Note that 3.5.1d shows how these properties transfer to top subcategories.

7.8.1 Proposition:

(a) If T preserves monics then $\underline{C}^{P(\pi)}$ has the property (P2).

(b) If T preserves regular-epis then $\underline{C}^{P(\pi)}$ has the properties (P2)* and (P3)*.

Proof:

(a) For $f: X \rightarrowtail Y$ a monic we have

$$\begin{aligned} f^*(\eta_Y^{-1}) &= f^{-1}\,\eta_Y^{-1}\,Tf \\ &= \eta_X^{-1}\,(Tf)^{-1}\,Tf \\ &= \eta_X^{-1}. \end{aligned}$$

(b) We need only prove (P3)* (3.5a).

If f is regular-epi then

$$\begin{aligned} f_* f^* y &= f\,f^{-1} y\,Tf\,(Tf)^{-1} \\ &= y. \end{aligned}$$

□

7.8.2 Remarks:

(a) If $\underline{C}$ is connected and $\underline{C}^{P(\pi)}$ has the property (P2)*, then T preserves regular-epis.

Proof:

Let $f: X \twoheadrightarrow Y$ be regular-epi.

Then

$$\begin{aligned} f\,p_X p_{TX}^{-1}\,Tf^{-1} &= f\omega_X\,Tf^{-1} \\ &= \omega_Y && \text{(by (P2)*)} \\ &= p_Y\,p_{TY}^{-1} \end{aligned}$$

where the p's are the projections.

So there is a regular-epi ϕ such that

$$\begin{tikzcd}
 & TX \times X \arrow[dl, "(Tf\ p_{TX})"'] \arrow[dr, "f\ p_X"] \arrow[dd, two heads, "\phi"] & \\
TY & & Y \\
 & TY \times Y \arrow[ul, "p_{TY}"] \arrow[ur, "p_Y"'] &
\end{tikzcd}$$

commutes.

Since $\underline{C}$ is connected, p_{TY} is regular-epi.

Thus Tf p_{TX} is regular-epi and so Tf is regular-epi.

(b) If for every (finite product $\{p_i : \times X_i \longrightarrow X_i\}$ the morphism $\langle Tp_i \rangle$ is monic, then $\underline{C}^{P(\pi)}$ is (finitely) (PI).

Proof:

$$(\times \eta_{X_i}^{-1}) \langle Tp_i \rangle = (\langle Tp_i \rangle \eta_X)^{-1} \langle Tp_i \rangle = \eta_X^{-1}. \qquad \square$$

The condition on T above is not satisfied by the ultrafilter functor (see 7.10(b)(v)) and so this result for (PI) is not very satisfactory.

Many of our results concerning top categories were proved using the assumption that the terminal object has a unique structure.

7.9 Proposition:

T preserves the terminal object of $\underline{C}$ if and only if the terminal object has a unique $\underline{C}^{P(\pi)}$-structure.

Proof:

As usual N denotes the terminal object.

If $TN \cong N$ then $TN \times N \cong N$ and so there is exactly one reflexive relation $TN \rightharpoonup N$. Conversely if N has a unique structure then there is an isomorphism $\phi : N \longrightarrow TN \times N$ such that

$$\begin{tikzcd}
 & TN \times N \arrow[dl, "p_{TN}"'] \arrow[dr, "p_N"] & \\
TN & & N \\
 & N \arrow[uu, two heads, "\phi"] \arrow[ul, "\eta_N"] \arrow[ur, "1"'] &
\end{tikzcd}$$

commutes.

Since N is terminal, p_{TN} is regular-epi and so η_N is regular-epi. Since N is terminal η_N is monic and so it is an isomorphism. □

7.10 Examples of categories of relational algebras.

(a) Reflexive and transitive relations [Barr 1970].

Let π be the identity monad on $\underline{C}$. Then $\underline{C}^{P(\pi)}$ is the category $\underline{C}^{r\ell}$ of reflexive relations (see 3.10) and $\underline{C}^{R(\pi)}$ the reflexive and transitive relations. $\underline{C}^{P(\pi)}$ has all the 'P' properties. The protoreflection of 7.3 is just $(X,x) \longmapsto (X,xx)$.

To fulfill the promise made in 7.3.1b, let $\underline{C}$ = sets and for each ordinal α, let X_α be the set of ordinals

$\leqslant \alpha + 2$. Let $x_\alpha = \Delta_{X_\alpha} \cup \{(\beta, \beta+1), (\beta+1, \beta) \mid 1 \leqslant \beta \leqslant \alpha+1\}$

Then the order of (X_α, x_α) relative to the protoreflection is exactly α.

(b) Topological spaces.

Let $\underline{C}$ = sets and $\pi = (\beta, \eta, \mu)$ be the ultrafilter monad.

(i) The details of the definition of π are as follows [Manes 1969 pp 109]: For each set X, βX is the set of ultrafilters on X. For $f : X \longrightarrow Y$, $\beta f : \beta X \longrightarrow \beta Y$ by: $\beta f(\mathcal{U})$ is the (unique) ultrafilter on Y generated by $\{f(U) \mid U \in \mathcal{U}\}$.

The unit $\eta : 1 \longrightarrow \beta$ is given by:

$$\text{for } x \in X,\ \eta_X x = \dot{x} = \{A \subset X \mid x \in A\},$$

while the multiplication $\mu : \beta\beta \longrightarrow \beta$ is given by:

$$\text{for } A \subset X, \text{ define } \dot{A} = \{\mathcal{U} \in \beta X \mid A \in \mathcal{U}\} \text{ and}$$

$$\text{for } \mathcal{H} \in \beta\beta,\ \mu_X = \{A \subset X \mid \dot{A} \in \mathcal{H}\}.$$

(μ is the well-known 'contraction' of filters).

(ii) Manes [1969] showed that the algebras of this monad are the compact Hausdorff spaces and Barr [1970] showed that the relational algebras are just all topological spaces. It is trivial to see that the relational prealgebras are the Choquet spaces of Wyler [1973a 3.2.2].

(iii) The functor β has the following property:

For $f: X \longrightarrow Y$ we have $\beta(f^{-1}f) = (\beta f)^{-1}\beta f$. (see 2.2g)

Proof: By 2.2a we always have $\beta(f^{-1}f) \leqslant (\beta f)^{-1}\beta f$.

For the converse, let $K = \{(a, b) \varepsilon X \times X \mid fa = fb\}$.

Let $p_1, p_2 : K \rightrightarrows X$ be the projections.

Then $f^{-1}f = p_2 p_1{}^{-1}$ and $\beta(f^{-1}f) = (\beta p_2)(\beta p_1)^{-1}$.

Let $(\mathcal{U}, \mathcal{V}) \varepsilon (\beta f)^{-1}\beta f$, i.e. $\mathcal{U}, \mathcal{V}$ are ultrafilters on X such that $\beta f(\mathcal{U}) = \beta f(\mathcal{V})$.

We need to find an ultrafilter $\mathcal{W}$ on K such that $\beta p_1(\mathcal{W}) = \mathcal{U}$ and $\beta p_2(\mathcal{W}) = \mathcal{V}$. For $U \varepsilon \mathcal{U}$ and $V \varepsilon \mathcal{V}$, it follows from $\beta f(\mathcal{U}) = \beta f(\mathcal{V})$ that there is a $V' \varepsilon \mathcal{V}$ such that $fV' \subset fU$. Thus $f(V \cap V') \subset f(V') \subset f(U)$ from which it follows that $(U \times V) \cap K \neq \emptyset$. So $\{(U \times V) \cap K \mid U \varepsilon \mathcal{U}, V \varepsilon \mathcal{V}\}$ is a filter base on K. Let $\mathcal{W}$ be an ultrafilter on K containing this filter base. Then $U \varepsilon \mathcal{U}$ implies $(U \times X) \cap K \varepsilon \mathcal{W}$, so that $U \varepsilon \beta p_1(\mathcal{W})$. Thus $\mathcal{U} = \beta p_1(\mathcal{W})$, and similarly we have $\mathcal{V} = \beta p_2(\mathcal{W})$. □

(iv) The functor β does not preserve kernel pairs.

Proof:

We know [Gillman and Jerison p97, 6N.2] that $\beta(\mathbb{N} \times \mathbb{N}) \neq \beta\mathbb{N} \times \beta\mathbb{N}$.

It follows by 2.2.1(d)(i), that β does not preserve kernel pairs. □

(v) The functor β does not have the property of 7.8.2(b).

Proof: $\langle\beta p_1, \beta p_2\rangle : \beta(\mathbb{N}\times\mathbb{N}) \longrightarrow \beta(\mathbb{N})\times\beta(\mathbb{N})$ is easily seen to be epi (see proof of (iii)) and so since it is not an isomorphism [ibid] it cannot be monic. □

(vi) The functor β has the property Q2 (see 7.7).

We use the notation of 7.7.

$$\begin{tikzcd}
\overset{\mathcal{W}}{\beta(\mathbf{X} X_i)} \arrow[rr, "\beta(\mathbf{X} f_i)"] \arrow[d, "\langle\beta p_i\rangle"'] & & \overset{\mathcal{V}}{\beta(\mathbf{X} Y_i)} \arrow[d, "\langle\beta q_i\rangle"] \\
\underset{\mathcal{U}_i}{\mathbf{X}\,\beta X_i} \arrow[rr, "\mathbf{X}\,\beta f_i"'] & & \mathbf{X}\,\beta Y_i
\end{tikzcd}$$

For each i, let $\mathcal{U}_i$ be an ultrafilter on X_i and let $\mathcal{V}$ be an ultrafilter on $\mathbf{X} Y_i$ such that $\beta f_i(\mathcal{U}_i) = \beta q_i(\mathcal{V})$. We need to find an ultrafilter $\mathcal{W}$ on $\mathbf{X} X_i$ such that for each i, $\beta p_i(\mathcal{W}) = \mathcal{U}_i$ and also $\beta(\mathbf{X} f_i)(\mathcal{W}) = \mathcal{V}$.

Let $\mathcal{S} = \{\mathbf{X} U_i \mid U_i \varepsilon \mathcal{U}_i$ and for all but finitely many i we have $U_i = X_i\}$.

We now show that $\mathcal{S} \cup \{(\mathbf{X} f_i)^{-1} V \mid V \varepsilon \mathcal{V}\}$ is a filter subbase on $\mathbf{X} X_i$. To do this we need only show that for each $\mathbf{X} U_i \varepsilon \mathcal{S}$ and $V \varepsilon \mathcal{V}$ we have $\mathbf{X} U_i \cap (\mathbf{X} f_i)^{-1} V \neq \emptyset$.

Let I' be the finite subset of the index set I such that $i \notin I' \Longrightarrow U_i = X_i$ and let $V \varepsilon \mathcal{V}$. For $i \varepsilon I'$, $f_i(U_i) \varepsilon q_i(\mathcal{V})$, so that there are $V_i \varepsilon \mathcal{V}$ such that $q_i(V_i) \subset f_i(U_i)$.

Let $V' = \bigcap_{I'} V_i \varepsilon \mathcal{V}$. So $V' \subset \bigcap_{I'} q_i^{-1} f_i(U_i) = \mathbf{X}_{I} f_i(U_i)$

(since f_i is epi, $i \notin I' \Longrightarrow f_i(U_i) = Y_i$).

Now $V \cap V' \neq \emptyset$, so $V \cap \underset{I}{\times} f_i(U_i) \neq \emptyset$ and it follows that

$(\times f_i)^{-1}(V) \cap \times U_i \neq \emptyset$.

Let $\mathcal{W}$ be any ultrafilter on $\times X_i$ which contains

$\mathcal{S} \cup \{(\times f_i)^{-1}V \mid V \varepsilon \mathcal{V}\}$.

Now $\times f_i$ is epi and so for each $V \varepsilon \mathcal{V}$ we have

$V = (\times f_i)(\times f_i)^{-1}V \; \varepsilon \beta(\times f_i)(\mathcal{W})$. Thus $\mathcal{V} \subset \beta(\times f_i)(\mathcal{W})$

and since $\mathcal{V}$ is maximal we have $\mathcal{V} = \beta(\times f_i)(\mathcal{W})$.

For each i and $U_i \; \varepsilon \mathcal{U}_i$, $p_i^{-1}U_i \; \varepsilon \mathcal{S} \subset \mathcal{W}$.

Thus $U_i = p_i(p_i^{-1}U_i) \; \varepsilon \; \beta p_i(\mathcal{W})$. It follows as before that

$\mathcal{U}_i = \beta p_i(\mathcal{W})$. So $\mathcal{W}$ has the required properties. $\square$

(c) Generalized sup-semilattices. [Manes 1973]

Let C = sets and $\pi = (P, \eta, \mu)$ be the power set monad on sets.

(i) The details of the definition of π are as follows:

P:sets $\longrightarrow$ sets is the power set functor.

$\eta : I \longrightarrow P$ is given by : for $x \varepsilon X$, $\eta_X x = \{x\}$.

$\mu : PP \longrightarrow P$ is given by $\mu_X \mathcal{A} = \cup \mathcal{A}$.

(ii) The relational prealgebras are the pairs (X,sup) where sup is a relation $PX \longrightarrow X$ such that x is a supremum of $\{x\}$,

i.e. $(\{x\}, x) \varepsilon$ sup.

The morphisms are the supremum preserving functions,

i.e. $f:(X,\text{sup}) \longrightarrow (Y,\text{sup})$ if $a \varepsilon \text{ sup } A \Longrightarrow fa \varepsilon \text{ sup } fA$.

The relational algebras are the prealgebras (X,sup) which have the property:

for all $a, a_i \varepsilon X$ and $A_i \subset X$, if $a_i \varepsilon \text{ sup } A_i$ and $a \; \varepsilon \text{ sup } \{a_i\}$

then $a \varepsilon \text{sup} \bigcup A_i$.

Manes calls these the generalized sup-semilattices.

The category of algebras ofthis monad is the category with the complete lattices as objects and with morphisms the supremum preserving functions.

(iii) The functor P does not preserve products or kernel pairs.

(iv) The functor P has the property:

$$P(f^{-1}g) = Pf^{-1}Pg \text{ for all } f,\ g \text{ (see 2.2(h)).}$$

(v) The functor P has the property Q2. (see 7.7).

We use the notation of 7.7.

Let $A_i \subset X_i$ and $B \subset \mathbf{X} Y_i$ satisfy : $f_i(A_i) = q_i(B)$ for each i.

Let $W = (\mathbf{X} A_i) \cap (\mathbf{X} f_i^{-1}(B))$. Then $p_i W = A_i$

and $(\mathbf{X} f_i)W = B$. □

(d) Generalized semi-groups.[Manes 1973]

Let $\pi = (T, \eta, \mu)$ be the monad on sets defined as follows:

$$TX = \coprod_{n \in \mathbb{N}} X^n, \quad \eta_X x = (x) \text{ and}$$

$$\mu_X((x_1^1 \ldots\ldots x_{m(1)}^1) \ldots\ldots\ldots (x_1^n \ldots\ldots\ldots x_{m(n)}^n)) =$$
$$= x_1^1 \ldots\ldots\ldots x_{m(1)}^1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots x_{m(n)}^n .$$

So TX is the set of all 'strings' in X. η is the injection $X^1 \longrightarrow \coprod X^n$ and μ is the operation of 'concatenation' of strings.

(i) The relational algebras of this monad are the pairs (X,comp) where comp: $\coprod X^n \rightharpoonup X$ is a relation which satisfies: $(x,x) \in$ comp for all $x \in X$ and if $(x_1^i \ldots\ldots x_{m(i)}^i, y_i) \in$ comp for $1 \leqslant i \leqslant n$ and $(y_1 \ldots\ldots y_n, y) \in$ comp then $(x_1^1 \ldots\ldots x_{m(1)}^1\ x_1^2 \ldots\ldots x_{m(n)}^n, y) \in$ comp.

These are called generalized semi-groups.

(ii) The algebras of this monad are of course just the semigroups.

(iii) The functor T preserves kernel pairs but it does not preserve the terminal object.

(iv) The functor T has the property Q2 (see 7.7).

Generalizing this example to the case where we were given a set Ω of relation symbols each with an assigned 'arity', we can construct a monad on sets and the relational algebras are then the generalized Ω-algebras. For details see [Manes 1973 pp 35] and [Neumann 1962 pp50].

(e) Generalized R-modules:

Let R be a ring with identity. Let $\pi = (T, \eta, \mu)$ be the monad on sets defined as follows (see Mac Lane 1971 pp. 142 ex. 2): TX is the set of finitely non-zero functions $X \longrightarrow R$. For $r \varepsilon R$ and $x \varepsilon X$ let $<r,x>$ denote the function $X \longrightarrow R$ which has the value r at x and is zero elsewhere. Using the addition R we can add members of TX and so $\sum <r_i,x_i>$ is the function with value r_i at x_i (for each i) and value zero elsewhere. For $t:X \longrightarrow Y$ and $f \varepsilon TX$, define $Tt(f)(y) = \sum_{\substack{x \varepsilon X \\ tx=y}} f(x)$. So $Tt(\sum <r_i,x_i>) = \sum <r_i,tx_i>$.

We see easily that T is a functor.

Define $\eta: I \longrightarrow T$ by $\eta_X x = <1,x>$.

and $\mu: T^2 \longrightarrow T$ by $(\mu_X k)(x) = \sum_{f \varepsilon TX} k(f)f(x)$, for $k \varepsilon T^2X$, $x \varepsilon X$.

The algebras of this monad are the left R-modules.

Let $\xi: TX \longrightarrow X$ be a relation. Write $rx = \xi(<r,x>)$ and $\sum r_i x_i = \xi(\sum <r_i,x_i>)$. So rx and $\sum r_i x_i$ are subsets of X.

(i) The relational algebras:

The pair (X, ξ) is a relational π-algebra if and only if

(1) $x \in 1.x$ for all $x \in X$ and

(2) for all $a, a_i, x_{ij} \in X$, $r_i, s_{ij} \in R$,

$$a_i \in \sum_j s_{ij}x_{ij} \text{ and } a \in \sum r_i a_i \implies a \in \sum_{i,j} r_i s_{ij} x_{ij}.$$

These are called generalized R-modules.

Taking $R = \mathbb{Z}$ we obtain the abelian group monad and the relational algebras are then called generalized abelian-groups.

(ii) The functor T does not preserve the terminal object.

(iii) The functor T has the property:

$$T(g^{-1}f) = Tg^{-1}Tf \text{ for all functions } f, g.$$

Proof: Let $f, g : X \rightrightarrows Y$ and $K = \{(x,x') \mid fx = gx'\}$.

Let $k = \sum_1^n \langle r_i, x_i\rangle$ and $k' = \sum_1^{n'} \langle r'_i, x'_i\rangle$ and suppose that $Tf(k) = Tg(k')$. So $\sum_1^n \langle r_i, fx_i\rangle = \sum_1^{n'} \langle r'_i\; gx'_i\rangle$.

$$\begin{tikzcd}
& TK \arrow[dl, "Tp_1"'] \arrow[dr, "Tp_2"] & \\
TX \arrow[dr, "Tf"'] & & TX \arrow[dl, "Tg"] \\
& TY &
\end{tikzcd}$$

We need to find $h \in TK$ such that $Tp_1(h) = k$ and

$Tp_2(h) = k'$.

We can assume that we have $fx_i = gx'_i = y$ for all x_i, x'_i.

So we have $\sum r_i = \sum r'_i$.

We can also assume that $n = n'$ (define the extra r_i or r_i' to be zero).

Let $h = \langle r_1,(x_1,x_1')\rangle + \langle r_1' - r_1, (x_2,x_1')\rangle +$

$$\langle r_2 - r_1' + r_1,(x_2,x_2')\rangle + \ldots\ldots\ldots\ldots +$$

$$\langle r_n - r'_{n-1} + r_{n-1} - \ldots\ldots r_2 - r_1' + r_1,\ (x_n,x_n')\rangle$$

Then h has the required properties. □

7.10.1 Problem:

(a) Give an example of a monad whose functor T does not satisfy:

$$T(f^{-1}f) = Tf^{-1}Tf \text{ for all regular-epis } f.$$

(b) Find 'nice' sufficient conditions for a functor to have this property.

7.11 Morphisms of Monads

Let $\pi = (T, \eta, \mu)$ and $\pi' = (T', \eta', \mu')$ be monads on $\underline{C}$. A morphism from π to π' [Frei 1969] is a natural transformation $\phi : T \longrightarrow T'$ such that

$$\begin{tikzcd}
I \arrow[r, "\eta"] \arrow[dr, "\eta'"'] & T \arrow[d, "\phi"] \\
 & T'
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
 & TT' \arrow[dr, "\phi T'"] & \\
T^2 \arrow[ur, "T\phi"] \arrow[dr, "\phi T"'] \arrow[ddd, "\mu"'] & & (T')^2 \arrow[ddd, "\mu'"] \\
 & T'T \arrow[ur, "T'\phi"'] & \\
 & & \\
T \arrow[rr, "\phi"'] & & T'
\end{tikzcd}$$

commute.

ϕ then induces a functor $\bar{\phi} : \underline{C}^{\pi'} \longrightarrow \underline{C}^{\pi}$ by:

$$\bar{\phi}\ (X,x) = (X, x\ \phi_X) \text{ and } \bar{\phi}(f) = f.$$

For relational algebras we have the following result:

7.11.1 Proposition:

The correspondence $(X,x) \longmapsto (X,x\ \phi_X)$ and $f \longmapsto f$ gives us top functors $\phi_P : \underline{C}^{P(\pi')} \longrightarrow \underline{C}^{P(\pi)}$ and

$\phi_R : \underline{C}^{R(\pi')} \longrightarrow \underline{C}^{R(\pi)}$ with left adjoints

ψ_P and ψ_R given by: $\psi_P(X,x) = (X,x\ \phi_X^{-1})$ and

$\psi_R(X,x) = L(X,x\ \phi_X^{-1})$ where

$L : \underline{C}^{P(\pi')} \longrightarrow \underline{C}^{R(\pi')}$ is the reflector.

ψ_P and ψ_R have the obvious actions on morphisms.

Proof:

(i) For $(X,x) \in \underline{C}^{P(\pi')}$, $(x\phi_X)\ \eta_X = x\ \eta'_X \geqslant 1_X$.

So $(X,x\ \phi_X) \in \underline{C}^{P(\pi)}$.

(ii) For $(X,x) \in \underline{C}^{R(\pi')}$, let $x = \mathrm{Im} \langle x_a, x_b \rangle$ with $\langle x_a, x_b \rangle$ monic.

$$\begin{tikzcd} & R \arrow[dl, "x_a"'] \arrow[dr, "x_b"] & \\ T'X & & X \end{tikzcd}$$

$$\begin{tikzcd} TR \arrow[rr, dash, "\phi_R"] \arrow[d, "Tx_a"] & & T'R \arrow[d, "T'x_a"] \\ TT'X \arrow[rr, dash, "\phi T'X"'] & & T'T'X \end{tikzcd}$$

$$\begin{aligned} \text{Then } (x\,\phi_X)T(x\,\phi_X) &\leqslant x\,\phi_X Tx\,T\,\phi_X \qquad (2.2a)\\ &= x\,\phi_X\,(Tx_b)(Tx_a)^{-1}\,T\,\phi_X\\ &= x\,T'x_b\,\phi_R(Tx_a)^{-1}\,T\,\phi_X\\ &\leqslant x(T'x_b)(T'x_a)^{-1}\,\phi_{T'X}\,T\,\phi_X\\ &= x\,T'x\,\phi_{T'X}\,T\,\phi_X\\ &\leqslant x\,\mu'_X\,\phi_{T'X}\,T\,\phi_X = (x\,\phi_X)\mu_X. \end{aligned}$$

So $(X, x\,\phi_X) \in \underline{C}^{R(\pi)}$.

By 1.5(a), $(\bigwedge x_i)\phi_X = \bigwedge(x_i\,\phi_X)$ and $\omega'_X\,\phi_X = \omega_X$. Thus the structure maps of ϕ_P and ϕ_R preserve infima.

Also for $f:X \longrightarrow Y$ and $y:T'Y \longrightarrow Y$ we have

$(f^*y)\phi_X = f^{-1}yT'f\,\phi_X = f^{-1}\,(y\,\phi_Y)\,Tf = f^*(y\,\phi_Y)$.

Thus ϕ_P and ϕ_R are top functors [Wyler 1971b 2.8].

By [ibid 2.8.1] the functors ψ are the left adjoints. $\square$

8. RELATIONAL ALGEBRAS: CLOSURE

In this section we work in a category $\underline{C}^{P(\pi)}$ of relational prealgebras where $\underline{C}$ and $\pi = (T, \eta, \mu)$ are as in §7.

This section is devoted to showing that almost all of the results of Manes [1973] about closed subspaces, closed and strongly closed maps in the case $\underline{C}$ = sets are valid for monads on regular categories. However in our situation the proofs are much more involved than in Manes' case where because of the availability of points, the proofs are very straightforward.

8.1 Definition [cf. Manes 1973, §3 and pp. 30]

$f(X,x) \longrightarrow (Y,y)$ in $\underline{C}^{P(\pi)}$ is strongly closed if $f\,x = y\,Tf$.

A subobject $m:A \rightarrowtail X$ of X is closed in (X,x) if $m:(A, m^*x) \rightarrowtail (X,x)$ is strongly closed.

$f:(X,x) \longrightarrow (Y,y)$ is a closed map if the image under f of closed subobjects of (X,x) are closed in (Y,y) i.e. for $m:A \rightarrowtail X$ monic with $m:(A,m^*x) \rightarrowtail (X,x)$ strongly closed and $A \xrightarrow{t} A' \xrightarrow{n} Y$ the (regular-epi, mono) decomposition of fm, the map $n:(A',n^*y) \rightarrowtail (Y,y)$ is strongly closed.

8.1.1 Remarks:

The functors induced by morphisms of monads preserve strongly closed maps (see 7.11).

8.2 Strongly Closed maps.

We now give some general properties of strongly closed maps.

8.2.1 Proposition: [cf. Manes 1973]

(a) If f and g are strongly closed, so is gf.

(b) Identity maps are strongly closed.

(c) Strongly closed monics are initial in $\underline{C}^{P(\pi)}$.

(d) Each regular-epi which is initial in $\underline{C}^{P(\pi)}$ is strongly closed.

(e) Finite products of strongly closed maps are strongly closed and if $\underline{C}$ has the property $\underline{QI}$ then the same is true for finite products.

Proof:

(a), (b) trivial.

(c) $f\,x = y\,Tf$ and f monic $\Longrightarrow$ $x = f^{-1}fx = f^{-1}y\ Tf = f^{*}y$.

(d) f regular-epi and $x = f^{*}y$ $\Longrightarrow$ $fx = f\ f^{*}y = f\ f^{-1}\ y\ Tf = yTf$.

(e) Let $f_i : (X_i, x_i) \longrightarrow (Y_i, y_i)$ be strongly closed for each i.

Let $(X,x) = \mathbf{X}(X_i, x_i)$ and $(Y,y) = \mathbf{X}(Y_i, y_i)$.

$$\begin{tikzcd}
T(\mathbf{X}X_i) \arrow[rrr, "T(\mathbf{X}f_i)"] \arrow[dr, "<Tp_i>"] \arrow[dd, "x"'] & & & T(\mathbf{X}Y_i) \arrow[dl, "<Tq_i>"'] \arrow[dd, "y"] \\
& \mathbf{X}TX_i \arrow[r, "\mathbf{X}Tf_i"] \arrow[dl, "\mathbf{X}x_i"] & \mathbf{X}TY_i \arrow[dr, "\mathbf{X}y_i"'] & \\
\mathbf{X}X_i \arrow[rrr, "\mathbf{X}f_i"'] & & & \mathbf{X}Y_i
\end{tikzcd}$$

($p_i : \mathbf{X}X_i \longrightarrow X_i$ and $q_i : \mathbf{X}Y_i \longrightarrow Y_i$ are the projections)

By 7.5 and 1.7.2 the diagram commutes and so $\mathbf{X}f_i$ is strongly closed. □

With conditions on T we obtain further results.

8.2.2 Proposition: [cf. Manes 1973 pp. 9]

Suppose that T preserves regular-epis.

(a) If gf is strongly closed and f is regular-epi then g is strongly closed.

(b) A strongly closed regular-epi is a quotient (in $\underline{C}^{P(\pi)}$).

(c) Each strongly closed map is a closed map.

Proof:

(a) Let $f:(X,x) \longrightarrow (Y,y)$ and $g:(Y,y) \longrightarrow (Z,z)$ with gf strongly closed. Then

$$zTgTf = z\,Tgf = gfx \leq g\,y\,Tf \leq z\,Tg\,Tf$$

So $g\,y\,Tf = z\,Tg\,Tf$ and since Tf is regular-epi we have $gy = z\,Tg$.

(b) Let $f:(X,x) \longrightarrow (Y,y)$ be strongly closed with f regular-epi. Then $f\,x\,(Tf)^{-1} = y\,Tf\,(Tf)^{-1} = y$.

(c) Let $f:(X,x) \longrightarrow (Y,y)$ be strongly closed and $m: A \rightarrowtail X$ be a closed subobject of (X,x). Let $A \overset{}{\underset{t}{\twoheadrightarrow}} A' \underset{n}{\rightarrowtail} Y$ be the (regular-epi,mono) decomposition of $f\,m$. By 8.2.1(a) the map $fm = nt: (A,m^*x) \longrightarrow (Y,y)$ is strongly closed and so by (a), $n:(A',n^*y) \rightarrowtail (Y,y)$ is strongly closed. So $n : A' \rightarrowtail Y$ is a closed subobject of (Y,y). □

8.3 Closed subobjects.

We now show that the closed subobjects of a space in $\underline{C}^{P(\pi)}$ satisfy Manes' conditions $C\ell 1$, $C\ell 2$ and $C\ell 3$ [1973 pp. 8].

8.3.1 Proposition: [cf. Manes 1973 pp. 8 and 7.8]

(a) Every intersection of closed subobjects is closed.

(b) Inverse images of closed subobjects are closed.

Proof:

(a) Let $\{m_i : A_i \rightarrowtail X\}$ be a family of closed subobjects of (X,x) and $m: \bigcap A_i \rightarrowtail X$ be their intersection. Let $g_i : \bigcap A_i \rightarrowtail A_i$ be the canonical monics associated with the intersection.

$$\begin{tikzcd}
A_i \arrow[rr, tail, "m_i"] & & X \\
\bigcap A_i \arrow[u, tail, "g_i"] \arrow[urr, "m"'] & &
\end{tikzcd}$$

$$\begin{aligned}
\text{Then } m\,m^{*}x = m\,m^{-1}x\;Tm &= (\textstyle\bigwedge m_i m_i^{-1})x\;Tm && (1.5b)\\
&= \textstyle\bigwedge (m_i m_i^{-1}x\;Tm) && (1.5c)\\
&= \textstyle\bigwedge (m_i m_i^{-1}\;x\;Tm_i\;Tg_i)\\
&= \textstyle\bigwedge (m_i m_i^{*}(x)\;Tg_i)\\
&= \textstyle\bigwedge (x\;Tm_i\;Tg_i) && (m_i \text{ strongly closed})\\
&= \textstyle\bigwedge (x\;Tm) \qquad = x\;Tm,
\end{aligned}$$

and so $m:(\bigcap A_i, m^{*}x) \rightarrowtail (X,x)$ is strongly closed.

(b) Let $f:(X,x) \longrightarrow (Y,y)$ and $j:A \rightarrowtail Y$ be a closed subobject of (Y,y). Let $k:B \rightarrowtail X$ be a monic which is the inverse image under f of j. So the diagram

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \\
B \arrow[u, tail, "k"] \arrow[r, "g"'] \arrow[ur, phantom, "\text{PB}"] & A \arrow[u, tail, "j"']
\end{tikzcd}$$

is a pullback.

We want to show that $k: (B,k^*x) \rightarrowtail (X,x)$ is strongly closed.

Let $x = \text{Im} <x_a,x_b>$ and $y = \text{Im} <y_a,y_b>$ with $<x_a,x_b>$ and $<y_a,y_b>$ monic.

The composition $f\ x$ is represented by

$$\begin{tikzcd}
 & R \arrow[dl, "x_a"'] \arrow[r, "x_b"] & X \arrow[d, "f"] \\
TX & & Y
\end{tikzcd}$$

and $y\ Tf$ by

$$\begin{tikzcd}
 & & R'' \arrow[dl, "y_1"'] \arrow[dr, "y_2"] \arrow[dd, phantom, "\text{PB}"] & & \\
 & TX \arrow[dl, "1"'] \arrow[dr, "Tf"] & & R' \arrow[dl, "y_a"'] \arrow[dr, "y_b"] & \\
TX & & TY & & Y
\end{tikzcd}$$

It is easy to see that $< y_1, y_b y_2>$ is monic.

Since $f\, x \leqslant y\, Tf$, there is a morphism c such

$$\begin{tikzcd}
& R \arrow[dl, "x_a"'] \arrow[r, "x_b"] \arrow[dd, "c"] & X \arrow[dr, "f"] & \\
TX & & & Y \\
& R'' \arrow[ul, "y_1"] \arrow[r, "y_2"'] & R' \arrow[ur, "y_b"'] &
\end{tikzcd}$$

commutes.

We need to prove that $k\, k^{*}x \geqslant x\, Tk$, i.e. $k\, k^{-1}x\, Tk \geqslant x\, Tk$.

Now $k\, k^{-1}x\, Tk$ is given by

$$\begin{tikzcd}
& & & & B_3 \arrow[dl, "b_5"'] \arrow[dr, "b_6"] & & & \\
& & & B_1 \arrow[dl, "b_1"'] \arrow[dr, "b_2"] & \text{PB} & B_2 \arrow[dl, "b_3"'] \arrow[dr, "b_4"] & & \\
& & TB \arrow[dl, "1"'] \arrow[dr, "Tk"] & \text{PB} & R \arrow[dl, "x_a"'] \arrow[dr, "x_b"] & \text{PB} & B \arrow[dl, "k"'] \arrow[dr, "k"] & \\
& TB & & TX & & X & & X
\end{tikzcd}$$

We need to find a morphism d_B such that

$$\begin{tikzcd}
& & B_3 \arrow[dl, "b_5"'] \arrow[dr, "b_6"] & & & \\
& B_1 \arrow[ddl, "b_1"'] & & B_2 \arrow[dr, "b_4"] & & \\
& & & & B \arrow[dr, "k"] & \\
TB & & & & & X \\
& & & & R \arrow[ur, "x_b"'] & \\
& & B_1 \arrow[uull, "b_1"] \arrow[urr, "b_2"'] \arrow[uuuuu, dashed, "d_B"] & & &
\end{tikzcd}$$

.... (i).

commutes (see 1.3.1c)

Now $j\ j^*y$ is represented by

$$\begin{tikzcd}
 & & & A_3 \arrow[dl, "a_5"'] \arrow[dr, "a_6"] \arrow[dd, phantom, "PB"] & & & \\
 & & A_1 \arrow[dl, "a_1"'] \arrow[dr, "a_2"] \arrow[dd, phantom, "PB"] & & A_2 \arrow[dl, "a_3"'] \arrow[dr, "a_4"] \arrow[dd, phantom, "PB"] & & \\
 & TA \arrow[dl, "1"'] \arrow[dr, "Tj"] & & R' \arrow[dl, "y_a"'] \arrow[dr, "y_b"] & & A \arrow[dl, "j"'] \arrow[dr, "j"] & \\
TA & & TY & & Y & & Y
\end{tikzcd}$$

with $\langle a_1a_5\ ,\ ja_4\ a_6\rangle$ monic. Since $j(j^*y) = y\ Tj$ there is a morphism d_A such that

$$\begin{tikzcd}
 & & A_3 \arrow[dl, "a_5"'] \arrow[dr, "a_6"] & & & \\
 & A_1 \arrow[ddl, "a_1"'] & & A_2 \arrow[dr, "a_4"] & & \\
 & & & & A \arrow[dr, "j"] & \\
TA & & & & & Y \\
 & & & R' \arrow[urr, "y_b"'] & & \\
 & & A_1 \arrow[uull, "a_1"] \arrow[uuuuu, "d_A"] \arrow[ur, "a_2"'] & & &
\end{tikzcd}$$

commutes.

In each of the following diagrams the outer paths are equal and the morphism d_i is induced by the pullback:

$$\begin{tikzcd}
 & R \arrow[r, "c"] & R'' \arrow[rr, "y_2"] & & R' \arrow[dr, "y_a"] & \\
B_1 \arrow[ur, "b_2"] \arrow[dr, "b_1"'] \arrow[rrr, dashed, "d_1"] & & & A_1 \arrow[ur, "a_2"] \arrow[dr, "a_1"'] & \text{PB} & TY \\
 & TB \arrow[rrr, "Tg"] & & & TA \arrow[ur, "Tj"'] &
\end{tikzcd}$$

$(y_a y_2\, c\, b_2 = Tf\; y_1 c\, b_2 = Tf\; x_a b_2 = Tf\; Tk\; b_1 = Tj\; Tg\; b_1.)$

$$\begin{tikzcd}
 & A_1 \arrow[r, "d_A"] & A_3 \arrow[r, "a_6"] & A_2 \arrow[r, "a_4"] & A \arrow[dr, "j"] & \\
B_1 \arrow[ur, "d_1"] \arrow[dr, "b_2"'] \arrow[rrr, dashed, "d_2"] & & & B \arrow[ur, "g"] \arrow[dr, "k"'] & \text{PB} & Y \\
 & R \arrow[rrr, "x_b"] & & & X \arrow[ur, "f"'] &
\end{tikzcd}$$

$(j\; a_4 a_6 d_A\; d_1 = y_b a_2 d_1 = y_b y_2 c\; b_2 = f\; x_b b_2.)$

$$\begin{tikzcd}
 & & B \arrow[rrd, "k"] & & \\
B_1 \arrow[rru, "d_2"] \arrow[rr, dashed, "d_3"] \arrow[rrd, "b_2"'] & & B_2 \arrow[u, "b_4"] \arrow[d, "b_3"] & \text{PB} & X \\
 & & R \arrow[rru, "x_b"'] & &
\end{tikzcd}$$

$(k\; d_2 = x_b b_2.)$

$$\begin{tikzcd}
 & & B_2 \arrow[dr, "b_3"] & & \\
B_1 \arrow[urr, "d_3"] \arrow[r, dashed, "d_4"] \arrow[drr, "1"'] & B_3 \arrow[ur, "b_6"] \arrow[dr, "b_5"] & \text{PB} & R & (b_3d_3 - = b_2) \\
 & & B_1 \arrow[ur, "b_2"'] & &
\end{tikzcd}$$

So let $d_B = d_4$ in (i). Then $k\,b_4 b_6 d_B = k\,b_4\,d_3 = kd_2 = x_b\,b_2$ and $b_1 b_5 d_B = b_1$. Thus d_B has the properties required of it. □

8.3.2 Definition: Closure.

Let $j:A \rightarrowtail X$ be a subobject of (X,x).

By 8.3.1a we can define the closure of (A,j) in (X,x) to be the smallest closed subobject of (X,x) which is larger than (A,j). (A,j) is dense in (X,x) if its closure in (X,x) is (X,x).

Relational algebras can be characterised as those prealgebras for which the structure relation is closed. We have the following result:

8.3.3 Proposition: [cf. Manes 1973, 7.7]

Let $(X,x) \in \underline{C}^{P(\pi)}$.

The following are equivalent:

(a) $(X,x) \in \underline{C}^{R(\pi)}$.

(b) x is a closed subobject of $(TX, \mu_X) \times (X,x)$.

Furthermore if these conditions are satisfied then x is the closure of η_X^{-1} in $(TX, \mu_X) \times (X,x)$.

Proof:

Let $x = \mathrm{Im} \langle x_a, x_b \rangle$ with $\langle x_a, x_b \rangle$ monic. Write μ for μ_X, $j = \langle x_a, x_b \rangle$ and $\phi = \langle Tp_{TX}, TP_X \rangle$. By 7.5

$$(TX, \mu_X) \times (X,x) = (TX \times X, (\mu \times x)\, \phi).$$

$$\begin{tikzcd}
TR \arrow[r, "Tj"] \arrow[dd, "j^{-1}(\mu \times x)\, \phi\, Tj"'] & T(TX \times X) \arrow[d, "\phi"] \\
 & T^2X \times TX \arrow[d, "\mu \times x"] \\
R \arrow[r, tail] & TX \times X
\end{tikzcd}
\qquad
\begin{tikzcd}
 & R \arrow[dl, "x_a"'] \arrow[dr, "x_b"] & \\
TX & & X
\end{tikzcd}$$

Now $\phi\, Tj = \langle Tx_a, Tx_b \rangle$.

$x\, Tx$ is given by:

$$\begin{tikzcd}
 & & X_1 \arrow[dl, "x_1"'] \arrow[dr, "x_2"] \arrow[dd, phantom, "PB"] & & \\
 & TR \arrow[dl, "Tx_a"'] \arrow[dr, "Tx_b"] & & R \arrow[dl, "x_a"'] \arrow[dr, "x_b"] & \\
T^2X & & TX & & X
\end{tikzcd}$$

$x\,\mu_X$ is given by:

$$\begin{tikzcd}
 & & X_2 \arrow[dl, "x_3"'] \arrow[dr, "x_4"] \arrow[dd, phantom, "\text{PB}"] & & \\
 & T^2X \arrow[dl, "1"'] \arrow[dr, "\mu_X"] & & R \arrow[dl, "x_a"'] \arrow[dr, "x_b"] & \\
T^2X & & TX & & X
\end{tikzcd}$$

Now $< x_3, x_b x_4 >$ is monic.

The statement (a) is equivalent to $x\,Tx \leqslant x\,\mu_X$ which is true if and only if there is a morphism c such that

$$\begin{tikzcd}
 & & X_1 \arrow[dl, "x_1"'] \arrow[dr, "x_2"] \arrow[dddd, dashed, "c"] & & \\
 & TR \arrow[dl, "Tx_a"'] & & R \arrow[dr, "x_b"] & \\
T^2X & & & & X \\
 & & & R \arrow[ur, "x_b"'] & \\
 & & X_2 \arrow[uull, "x_3"] \arrow[ur, "x_4"'] & &
\end{tikzcd}$$

commutes.

.......(i).

Now $j\, j^{-1}\, (\mu \times x)\, \phi T j$ is given by:

$$\begin{tikzcd}
 & & & Y_3 \arrow[dl, "y_5"'] \arrow[dr, "y_6"] & & & \\
 & & Y_1 \arrow[dl, "y_1"'] \arrow[dr, "y_2"] & \text{PB} & Y_2 \arrow[dl, "y_3"'] \arrow[dr, "y_4"] & & \\
 & TR \arrow[dl, "1"'] \arrow[dr, "{<Tx_a, Tx_b>}"] & \text{PB} & T^2X \times R \arrow[dl, "1 \times x_a"'] \arrow[dr, "\mu \times x_b"] & \text{PB} & R \arrow[dl, "j"'] \arrow[dr, "j"] & \\
TR & & T^2X \times TX & & TX \times X & & TX \times X
\end{tikzcd}$$

The statement (b) is equivalent to $j\, j^{-1}(\mu \times x)\, \phi\, Tj \geqslant (\mu \times x)\, \phi\, Tj$ which is true (note that $\langle y_1 y_5, j y_4 y_6 \rangle$ is monic) if and only if there is a morphism d such that

$$\begin{tikzcd}
 & & & Y_3 \arrow[dl, "y_5"'] \arrow[dr, "y_6"] & & & \\
 & & Y_1 \arrow[ddll, "y_1"'] & & Y_2 \arrow[dr, "y_4"] & & \\
 & & & & & R \arrow[dr, "j"] & \\
TR & & & & & & TX \times X \\
 & & & & & T^2X \times R \arrow[ur, "\mu \times x_b"'] & \\
 & & & Y_1 \arrow[uulll, "y_1"] \arrow[urr, "y_2"'] \arrow[uuuuu, dashed, "d"] & & &
\end{tikzcd}$$

commutes.

......(ii)

$\underline{(a) \Longrightarrow (b):}$ Suppose that $(X,x) \varepsilon \underline{C}^{R(\pi)}$. So there is c which makes (i) commute. We need to find a morphism d which makes (ii) commute. In each of the diagrams below the outer paths are equal and the morphism d_i is induced by the pullback.

$$\begin{tikzcd}
 & T^2X \times R \arrow[rr, "p_R"] & & R \arrow[dr, "x_a"] & \\
Y_1 \arrow[ur, "y_2"] \arrow[rr, dashed, "d_1"] \arrow[drrr, "y_1"'] & & X_1 \arrow[ur, "x_2"] \arrow[dr, "x_1"] & \text{PB} & TX \\
 & & & TR \arrow[ur, "Tx_b"'] &
\end{tikzcd}$$

$(Tx_b\ y_1 = p_{TX}(1 \times x_a)y_2 = x_a p_R y_2)$

$$\begin{tikzcd}
 & X_1 \arrow[r, "c"] & X_2 \arrow[r, "x_4"] & R \arrow[dr, "j"] & \\
Y_1 \arrow[ur, "d_1"] \arrow[rr, dashed, "d_2"] \arrow[drrr, "y_2"'] & & Y_2 \arrow[ur, "y_4"] \arrow[dr, "y_3"] & \text{PB} & TX \times X \\
 & & & T^2X \times R \arrow[ur, "\mu \times x_b"'] &
\end{tikzcd}$$

$$(p_{TX}\ j\ x_4\ c\ d_1 = x_a x_4 c d_1 = \mu\ x_3 c\ d_1 = \mu\ Tx_a\ x_1 d_1 = \mu\ Tx_a\ y_1$$
$$= \mu\ p_{T^2X}(1 \times x_a)y_2 = \mu\ p_{T^2X} y_2 = p_{TX}(\mu \times x_b)y_2,$$

and $p_X j\ x_4 c\ d_1 = x_b x_4\ c\ d_1 = x_b x_2 d_1 = x_b p_R y_2 = p_X(\mu \times x_b)y_2$)

$$\begin{tikzcd}
 & & Y_1 \arrow[dr, "y_2"] & \\
Y_1 \arrow[urr, "1"] \arrow[r, dashed, "d_3"] \arrow[drr, "d_2"'] & Y_3 \arrow[ur, "y_5"] \arrow[dr, "y_6"'] & \text{PB} & T^2X \times R \\
 & & Y_2 \arrow[ur, "y_3"'] &
\end{tikzcd}$$

$(y_3d_2 = y_2)$

Let $d = d_3$ in (ii). Then $y_1y_5d = y_1$ and $j\ y_4y_6d = (\mu \times x_b)y_2y_5d = (\mu \times x_b)y_2$. Thus d has the required properties.

(b) $\Longrightarrow$ (a): We assume the existence of d and try to find c.

The procedure is as above.

$$\begin{tikzcd}
 & & T^2X \times R \arrow[dr, "1 \times x_a"] & \\
X_1 \arrow[urr, "<Tx_a x_1{,}x_2>"] \arrow[r, dashed, "c_1"'] \arrow[drr, "x_1"'] & Y_1 \arrow[ur, "y_2"] \arrow[dr, "y_1"] & \text{PB} & T^2X \times TX \\
 & & TR \arrow[ur, "<Tx_a{,}Tx_b>"'] &
\end{tikzcd}$$

($P_{T^2X}\langle Tx_a,Tx_b\rangle x_1 = Tx_a x_1 = p_{T^2X}\ (1 \times x_a)\langle Tx_a\, x_1, x_2\rangle$

and $p_{TX}\langle Tx_a,Tx_b\rangle x_1 = Tx_b\, x_1 = x_a\, x_2 = p_{TX}\ (1 \times x_a)\langle Tx_a\, x_1, x_2\rangle$)

$$\begin{tikzcd}
& TR \arrow[rrr, "Tx_a"] & & & T^2X \arrow[dr, "\mu"] & \\
X_1 \arrow[ur, "x_1"] \arrow[rrr, dashed, "c_2"] \arrow[dr, "c_1"'] & & & X_2 \arrow[ur, "x_3"] \arrow[dr, "x_4"'] \arrow[rr, phantom, "PB"] & & TX \\
& Y_1 \arrow[r, "d"'] & Y_3 \arrow[r, "y_6"'] & Y_2 \arrow[r, "y_4"'] & R \arrow[ur, "x_a"'] &
\end{tikzcd}$$

$$(\mu\, Tx_a\, x_1 = \mu\, p_{T^2X}\, y_2 c_1 = p_{TX}\,(\mu \times x_b) y_2 c_1 = p_{TX}\, j\, y_4 y_6 d\, c_1$$

$$= x_a y_4 y_6 d\, c_1)$$

So we take $c = c_2$ in (i). Then $x_3 c = Tx_a x_1$ and

$x_b x_4 c = x_b\, y_4 y_6 d\, c_1 = p_X(\mu \times x_b) y_2 c_1 = x_b p_R y_2 c_1 = x_b x_2$ and

so c has the required properties.

This completes the proof that (a) $\Longleftrightarrow$ (b).

To prove the last statement let $y : TX \longrightarrow X$ with $\eta_X^{-1} \leqslant y$ and y closed in $(TX, \mu_X) \times (X,x)$.

We want to show that $x \leqslant y$.

Let $y = \mathrm{Im}\, \langle y_a, y_b \rangle$ with $\langle y_a, y_b \rangle$ monic.

Write $k = \langle y_a, y_b \rangle$.

$$\begin{tikzcd}
& Q \arrow[dl, "y_a"'] \arrow[dr, "y_b"] & \\
TX & & X
\end{tikzcd}$$

Since $\eta_X^{-1} \leqslant y$ there is a t such that

$$\begin{tikzcd}
 & X \arrow[dl, "\eta_X"'] \arrow[dd, "t"] \arrow[dr, "1"] & \\
TX & & X \\
 & Q \arrow[ul, "y_a"] \arrow[ur, "y_b"'] &
\end{tikzcd}$$

commutes

Since y is closed the following diagram commutes.

$$\begin{tikzcd}
TQ \arrow[r, "Tk"] \arrow[dd, "k^{-1}(\mu \times x)\,\phi\, Tk"'] & T(TX \times X) \arrow[d, "\phi"] \\
 & T^2X \times TX \arrow[d, "\mu \times x"] \\
Q \arrow[r, "k"] & TX \times X
\end{tikzcd}$$

Now $kk^{-1}(\mu \times x)\ \phi\ Tk$ is given by:

$$\begin{tikzcd}
 & & & Q_3 \arrow[dl, "q_5"'] \arrow[dr, "q_6"] \arrow[dd, phantom, "PB"] & & & \\
 & & Q_1 \arrow[dl, "q_1"'] \arrow[dr, "q_2"] \arrow[dd, phantom, "PB"] & & Q_2 \arrow[dl, "q_3"'] \arrow[dr, "q_4"] \arrow[dd, phantom, "PB"] & & \\
 & TQ \arrow[dl, "1"'] \arrow[dr, "{<Ty_a, Ty_b>}"] & & T^2X \times R \arrow[dl, "1 \times x_a"'] \arrow[dr, "\mu \times x_b"] & & Q \arrow[dl, "k"'] \arrow[dr, "k"] & \\
TQ & & T^2X \times TX & & TX \times X & & TX \times X
\end{tikzcd}$$

with $<q_1q_5, kq_4q_6>$ monic.

So there is a morphism v such that

$$\begin{tikzcd}
 & & Q_3 \arrow[dl, "q_5"'] \arrow[dr, "q_6"] & & & \\
 & Q_1 \arrow[dl, "q_1"'] & & Q_2 \arrow[dr, "q_4"] & & \\
TQ & & & & Q \arrow[dr, "k"] & \\
 & & & & & TX \times X \\
 & & & T^2X \times R \arrow[urr, "\mu \times x_b"'] & & \\
 & & Q_1 \arrow[uuuuu, "v"] \arrow[uull, "q_1"] \arrow[ur, "q_2"'] & & &
\end{tikzcd}$$

commutes.

Consider.

$$\begin{tikzcd}
TX \arrow[dd, "Tt"'] & R \arrow[l, "x_a"'] \arrow[d, dashed, "w"] \arrow[ddr, "{<Ty_a Tt\, x_a, 1>}"] & \\
 & Q_1 \arrow[dl, "q_1"'] \arrow[dr, "q_2"] & \\
TQ \arrow[dr, "{<Ty_a, Ty_b>}"'] & \arrow[phantom, "\text{(PB')}"] & T^2X \times R \arrow[dl, "1 \times x_a"] \\
 & T^2X \times TX &
\end{tikzcd}$$

We have

$$p_{T^2X} <Ty_a, Ty_b> Tt\, x_a = Ty_a\, Tt\, x_a = p_{T^2X} (1 \times x_a) < Ty_a Tt\, x_a, 1>,$$

and $p_{TX} <Ty_a, Ty_b> Tt\, x_a = Ty_b Tt\, x_a = x_a = p_{TX} (1 \times x_a) <Ty_a\, Tt\, x_a, 1>$.

So the outer paths of the diagram are equal. Thus there is a morphism w which makes the diagram commute.

Consider the diagram

$$\begin{tikzcd}
 & R \arrow[ddl, "x_a"'] \arrow[d, "w"] \arrow[ddr, "x_b"] & \\
 & Q_1 \arrow[d, "v"] & \\
TX & Q_3 \arrow[d, "q_6"] & X \\
 & Q_2 \arrow[d, "q_4"] & \\
 & Q \arrow[uul, "y_a"] \arrow[uur, "y_b"'] &
\end{tikzcd}$$

We have $y_b q_4 q_6 v\, w = p_X(\mu \times x_b) q_2 w = x_b p_R \langle Ty_a Tt\, x_a, 1\rangle = x_b$

and $y_a q_4 q_6 v\, w = p_{TX}(\mu \times x_b) q_2 w = \mu\, p_{T^2X}\, q_2 w = \mu\, Ty_a Tt\, x_a$

$= \mu\, T\, \eta_X x_a = 1\, x_a = x_a$. So the diagram

commutes and this tells us that $x \leqslant y$.

□

The following characterization of strongly closed maps in a category of relational algebras is given by Manes [1973] for the case $\underline{C}$ = sets. We have not been able to generalize this result.

8.4 Proposition: [ibid pp. 9 and 7.9]

Suppose that $\underline{C}$ = sets.

Let (X,x) and (Y,y) be relational algebras and $f:(X,x) \longrightarrow (Y,y)$ be a map. Then f is strongly closed if and only if for all relational algebras (Z,z) the map $f \times 1:(X,x) \times (Z,z) \longrightarrow (Y,y) \times (Z,z)$ is closed.

8.5 Examples:

(a) Reflexive and transitive relations.

Let x be a reflexive relation on a set X. A subset $A \subset X$ is closed in (X,x) if and only if A is a right ray, i.e. $a \in A$, $b \in X$ and $(a,b) \in x \Longrightarrow b \in A$. The closure of a subset is the right ray which it generates.

A map $f:(X,x) \longrightarrow (Y,y)$ is strongly closed if and only if for all $a \in X$ and $b \in Y$, $(f(a), b) \in y \Longrightarrow$ there exists $a' \in X$ such that $(a,a') \in x$ and $f(a') = b$.

(b) Topological Spaces.

The concept of a closed subset coincides with the usual topological concept.

Manes [1973 pp. 23] gives the following characterizations of strongly closed maps in Top :

$$f:(X,x) \longrightarrow (y,y) \text{ is strongly closed}$$

$$\Longleftrightarrow \quad 1 \times f:(\beta X, \mu_X) \times (X,x) \longrightarrow (\beta X, \mu_X) \times (Y,y) \text{ is a closed map,}$$

$\Longleftrightarrow$ fx is a closed subset of the space $(\beta X, \mu_X) \times (Y,y)$.

(c) Generalized sup-semilattices.

A subset A of a generalized sup-semilattice $(X,\sup)$ is closed if and only if all suprema of subsets of A lie in A.

The closure of a subset A of $(X,\sup)$ is the set of all suprema of all subsets of A.

(d) Generalized semi-groups.

A subset A of a generalized semi-group (X,comp) is closed if and only if for every string $x_1 \ldots\ldots x_n$ in A,

$(x_1 \ldots\ldots x_n, x) \in \mathrm{comp} \Longrightarrow x \in A$.

In particular a subset of a semi-group is closed if and only if it is a sub-semi-group.

(e) Generalized R-modules.

A subset A of a generalized R-module (X, ζ) is closed if and only if for every function $\sum \langle r_i, x_i \rangle$ with all $x_i \in A$ we have $\sum r_i x_i \subset A$.

In particular a subset of an R-module is closed if and only if it is a sub-module.

8.6 Proposition:

Let $\underline{C}$ = sets and suppose that the functor T has the property:

for all subsets $A_1, A_2 \subset X$, the induced function $TA_1 \sqcup TA_2 \longrightarrow T(A_1 \cup A_2)$ is epi.

Then every finite union of closed subsets of a relational π-prealgebra is again closed.

Proof: Follows easily from definitions. $\square$

8.6.1 Remarks:

The identity functor and the ultrafilter functor satisfy the condition required for the use of 8.6. None of the other functors given in the examples in 7.10 have this property and the corresponding categories of relational prealgebras do not satisfy the conclusion of 8.6.

8.6.2 Problem:

There is a top functor $T^* : \underline{C}^{P(\pi)} \longrightarrow \underline{\mathrm{Top}}$ which is given by associating with each relational prealgebra (X,x), the topology on X with the closed subsets of (X,x) as base for closed sets.
By 8.3.1, T* is indeed a functor.

The problem is: Study the relationship between properties of π and of T*. In particular, find conditions on π which are sufficient for T* restricted to $\underline{C}^{R(\pi)}$ to be an isomorphism of categories.
(See John W. Gray's problem $\underline{[\mathrm{T}]}$D in [Illinois Conference 1973]).

9. RELATIONAL ALGEBRAS: SEPARATION AXIOMS

Ramaley and Wyler [1970a] extend the separation axioms T_0, T_1, T_2, R_0 and R_1 from Top to categories of limit spaces. Manes [1973] defines Hausdorff relational algebras in terms of closure conditions on the diagonal and shows that this is equivalent to the structure relation being 'monic'. He also proves that products and subobjects of Hausdorff relational algebras are again Hausdorff.

Our setting is again as in §7: $\pi = (T, \eta, \mu)$ is a monad on a regular category $\underline{C}$. First of all we define T_i, R_i and S_i separation axioms in a category of relational algebras and produce a protoreflection corresponding to each property. We show that in a certain subcategory of $\underline{C}^{R(\pi)}$, the T_0-protoreflection is strong and it preserves products and subspaces. This generalises the known properties of the T_0-reflection in Top. For T_2, we generalise the results of Manes.

The basic definition for this section is the following:

9.1 Definition:

Let $(X,x) \in \underline{C}^{P(\pi)}$.

We say that (X,x) is: T_0 if $(x\,\eta_X) \wedge (x\,\eta_X)^{-1} \leqslant 1_X$,

T_1 if $x\,\eta_X \leqslant 1_X$,

T_2 if $x\,x^{-1} \leqslant 1_X$,

R_0^- if $(x\,\eta_X)^{-1} \leqslant (x\,\eta_X)$,

R_0 if $(x\,\eta_X)\,(x\,\eta_X)^{-1} \leqslant (x\,\eta_X)$,

R_1 if $x\,x^{-1} \leqslant x\,\eta_X$,

S_0^{--} if $(x\,\eta_X)\,(x\,\eta_X) \leqslant (x\,\eta_X)$,

S_0^{-} if $(x\,\eta_X)\,x \leqslant x$,

S_0 if $(x\,\eta_X)^{-1}\,x \leqslant x$,

S_1 if $x\,x^{-1}\,x \leqslant x$.

Using $1 \leqslant \eta_X$ and $\eta_X \leqslant x^{-1}$, we see easily that equivalent definitions are obtained if we replace the inequalities above by = .

9.1.1 Remarks:

(a) The axiom S_0^{-} corresponds to Wyler's 'quasi-uniformizable' [1973a] 3.2.6. S_0^{--} and R_0^{-} are as far as we know new, while the other axioms are taken from Ramaley and Wyler [1970a 3.2].

(b) Our definition of T_2 is equivalent to Manes' 'Hausdorff' [1973] and A. Burroni's 'séparée' [1971].

(c) The functors induced by morphisms of monads (see 7.11) preserve all the 'T', 'R' and 'S' properties.

The following simple lemma will be useful in the sequel.

9.2 Lemma: [Ramaley and Wyler 1970a, 3.3]

For $(X,x) \in \underline{C}^{P(\pi)}$ the following are equivalent:

(a) (X,x) is R_0;

(b) $x\,\eta_X$ is an equivalence relation;

(c) $(x\,\eta_X)^{-1}(x\,\eta_X) \leqslant x\,\eta_X$.

Proof:

(a) $\Longrightarrow$ (b): $1 \leqslant x\,\eta_X$ give us $(x\,\eta_X)^{-1} \leqslant (x\,\eta_X)(x\,\eta_X)^{-1} \leqslant (x\,\eta_X)$.

Taking inverses we get $(x\,\eta_X) = (x\,\eta_X)^{-1}$ and the result follows.

The rest of the proof is straightforward. □

9.3 Proposition: [cf. ibid 3.4]

The following implications are true:

$$\begin{tikzcd}
T_2 \arrow[r, Rightarrow] \arrow[d, Rightarrow] & T_1 \arrow[r, Rightarrow] \arrow[d, Rightarrow] & T_0 & \\
S_1 \arrow[r, Rightarrow] \arrow[d, Rightarrow] & S_0 \arrow[r, Rightarrow] \arrow[d, Rightarrow] & S_0^- \arrow[r, Rightarrow] & S_0^{--} \\
R_1 \arrow[r, Rightarrow] & R_0 \arrow[r, Rightarrow] \arrow[rru, Rightarrow] & R_0^- &
\end{tikzcd}$$

R^-_0 and $T_0 \Longleftrightarrow T_1$

R_0^- and $S_0^{--} \Longleftrightarrow R_0$

R_1 and $T_0 \Longleftrightarrow T_2$

R_1 and $S_0^- \Longleftrightarrow S_1$

R_0^- and $S_0^- \Longleftrightarrow S_0$

Proof:

Let $(X,x) \in \underline{C}^{P(\pi)}$. So $1 \leqslant x\,\eta_X$, $\eta_X^{-1} \leqslant x$ and $\eta_X \leqslant x^{-1}$.

$T_2 \Longrightarrow T_1$: $x\,\eta_X \leqslant x\,x^{-1} \leqslant 1$.

$S_1 \Longrightarrow S_0$: $(x\,\eta_X)^{-1}x = \eta_X^{-1}\,x^{-1}\,x \leqslant x\,x^{-1}x \leqslant x$.

$R_1 \Longrightarrow R_0$: $(x\,\eta_X)(x\,\eta_X)^{-1} = x\,\eta_X\,\eta_X^{-1}x^{-1} \leqslant x\,x^{-1} \leqslant x\,\eta_X$.

$S_1 \Longrightarrow R_1$: $x\,x^{-1} \leqslant x\,x^{-1}\,\eta_X^{-1}\,\eta_X \leqslant x\,x^{-1}x\,\eta_X \leqslant x\,\eta_X$.

$S_0 \Longrightarrow R_0$: $S_0 \Longrightarrow 9.2(c) \Longrightarrow R_0$.

The remaining proofs are straightforward. □

We now obtain a result concerning the closure of the 'T', 'S' and 'R' properties under strongly closed quotients.

9.4 Proposition:

Suppose that $f:(X,x) \twoheadrightarrow (Y,f_*x)$ is strongly closed and f is regular-epi.

(a) If (X,x) is T_2 (resp. $S_{\bar{0}}$, R_1) then (Y, f_*x) has the same property.

(b) If T preserves regular-epis and $(X,x) \in \underline{C}^{R(\pi)}$ then $(Y,f_*x) \in \underline{C}^{R(\pi)}$.

Proof: By 7.6.1 we have $f_*x = f\ x\ Tf^{-1}$ and so since f is strongly closed, $f\ x = f\ x\ Tf^{-1}Tf$.

(a) If (X,x) is T_2 then
$$(f_*x)(f_*x)^{-1} = f\ x\ Tf^{-1}\ Tf\ x^{-1}\ f^{-1} = f\ x\ x^{-1}\ f^{-1} \leqslant f\ f^{-1} = 1$$
and so (Y,f_*x) is T_2.

If (X,x) is $S_{\bar{0}}$ then
$$\begin{aligned} f_*x\ \eta_Y\ f_*x &= f\ x\ Tf^{-1}\ \eta_Y\ f\ x\ Tf^{-1} \\ &= f\ x\ Tf^{-1}Tf\ \eta_X\ x\ Tf^{-1} \\ &= f\ x\ \eta_X\ x\ Tf^{-1} \\ &\leqslant f\ x\ Tf^{-1} = f_*x \end{aligned}$$
and so (Y,f_*x) is $S_{\bar{0}}$.

If (X,x) is R_1 then
$$\begin{aligned} (f_*x)(f_*x)^{-1} &= f\ x\ Tf^{-1}\ Tf\ x^{-1}\ f^{-1} \\ &= f\ x\ x^{-1}\ f^{-1} \leqslant f\ x\ \eta_X\ f^{-1} \\ &= f\ x\ Tf^{-1}Tf\ \eta_X f^{-1} \\ &= f\ x\ Tf^{-1}\ \eta_Y f\ f^{-1} = f_*x\ \eta_Y \end{aligned}$$
and so (Y,f_*x) is R_1.

(b) If $(X,x) \varepsilon \underline{C}^{R(\pi)}$ then $(f_*x)T(f_*x) \leqslant f\ x(Tf)^{-1}\ Tf\ Tx(T^2f)^{-1}$ (2.2(f))

$$= f\ x\ Tx(T^2f)^{-1} \leqslant f\ x\ \mu_X(T^2f)^{-1}$$

$$\leqslant f\ x\ (Tf)^{-1}\ \mu_Y$$

$= f_*x\ \mu_Y$ and so $(Y,f_*x) \varepsilon \underline{C}^{R(\pi)}$. □

9.5 Examples:

We now examine the meaning of the 'T', 'R' and 'S' properties in the categories of relational (pre)algebras described in 7.10.

(a) Reflexive and transitive relations.

Here $(X,x) \varepsilon \underline{C}^{P(\pi)}$ is $T_0 \Longleftrightarrow$ x is antisymmetric,

is T_1 and $T_2 \Longleftrightarrow x = 1_X$,

is S_0^{--} and $S_0^{-} \Longleftrightarrow$ x is transitive,

$\Longleftrightarrow (X,x) \varepsilon \underline{C}^{R(\pi)}$,

is $R_0^{-} \Longleftrightarrow$ x is symmetric,

is R_0, S_0, S_1 and $R_1 \Longleftrightarrow$ x is an equivalence relation.

(b) Topological Spaces.

In Top = $\underline{\text{sets}}^{R(\beta)}$, T_0, T_1 and T_2 are the usual point separation axioms. All topological spaces are S_0^{-} and S_0^{--} and so by 9.3, $R_1 \Longleftrightarrow S_1$, $R_0^{-} \Longleftrightarrow R_0 \Longleftrightarrow S_0$. R_0 and R_1 are the regularity conditions of Davis [Murdeshwar and Naimpally 1966, 3.5, 3.11].

(c) Generalized sup-semilattices.

A generalized sup-semilattice is T_2 if and only if every subset has at most one supremum, and is T_1 if and only if the only supremum of a singleton {a} is the point a.

(d) Generalized semi-groups.

The T_2 generalized semi-groups are the usual partial semi-groups. More generally, the T_2 generalized Ω-algebras are the usual partial Ω-algebras [see Manes 1973 pp. 35].

(e) Generalized R-modules.

A T_2 generalized R-module is just a 'partial R-module', i.e. a semi-abelian-group with identity which is a partial group and has a partially defined R-action which satisfies the usual R-module laws on its domain of definition.

9.6 Remarks:

Suppose that $\underline{C}$ = groups, rings, vector spaces or any abelian category. Then every relation in $\underline{C}$ is difunctional (see 1.9). Thus all relational-prealgebras of a monad on $\underline{C}$ are S_1 and so in this situation the T_0, T_1 and T_2 axioms are equivalent.

9.7 The 'S' and 'R' properties.

Each 'S' and 'R' property has associated with it a structure functor $L:\underline{C}^{P(\pi)} \longrightarrow \underline{C}^{P(\pi)}$ with $1 \leqslant L$ and which has the corresponding 'S' and 'R' subcategory as the associated top subcategory $\underline{C}^{P(\pi)}$ $(L = 1)$ (see §5). These structure functors are defined as follows:

$$R_0^- : \quad x \longmapsto x\, v((x\, \eta_X)^{-1}\, \eta_X^{-1}),$$

$$R_0 : \quad x \longmapsto x\, v((x\, \eta_X)(x\, \eta_X)^{-1}\, \eta_X^{-1}),$$

$$R_1 : \quad x \longmapsto x\, v(x\, x^{-1}\, \eta_X^{-1}),$$

$$S_0^{--} : \quad x \longmapsto x\, v((x\, \eta_X)(x\, \eta_X)\eta_X^{-1}),$$

$$S_0^- : \quad x \longmapsto x\, \eta_X\, x$$

$$S_0 : \quad x \longmapsto (x\, \eta_X)^{-1} x,$$

$$S_1 : \quad x \longmapsto x\, x^{-1} x,$$

with the obvious actions on morphisms.

We need to check that the associated top subcategories are the correct ones (this is easy), and that in each case for $f:(X,x) \longrightarrow (Y,y)$ we have $f\, Lx \leqslant Ly\, Tf$. In fact we need only show this for L the second term in each 'v' (see 5.3.3).

We have $f\, \eta_X^{-1} \leqslant \eta_Y^{-1} Tf$.

For $R_{\bar{0}}$, we have $f(x\, \eta_X)^{-1}\, \eta_X^{-1} \leqslant \eta_Y^{-1} Tf\, x^{-1} f^{-1} f\, \eta_X^{-1}$

$$\leqslant \eta_Y^{-1} Tf\; Tf^{-1} y^{-1} \eta_Y^{-1}\, Tf$$

$$\leqslant \eta_Y^{-1}\, y^{-1} \eta_Y^{-1}\, Tf = (y\, \eta_Y)^{-1}\, \eta_Y^{-1}\, Tf.$$

For R_0, we have $f(x\, \eta_X)(x\, \eta_X)^{-1}\, \eta_X^{-1}$

$$\leqslant y\, Tf\, \eta_X\, \eta_X^{-1}\, x^{-1}\, f^{-1} f\, \eta_X^{-1} \leqslant y\, \eta_Y f\, \eta_X^{-1}\, (y\, Tf)^{-1}\, \eta_Y^{-1}\, Tf$$

$$\leqslant y\, \eta_Y\, \eta_Y^{-1}\, Tf\; Tf^{-1}\, y^{-1}\, \eta_Y^{-1}\, Tf = (y\, \eta_Y)(y\, \eta_Y)^{-1}\, \eta_Y^{-1}\, Tf.$$

The proofs for the other properties are similar.

Moreover the discrete spaces satisfy all the 'R' and 'S' properties. □

Now we have the following result:

9.7.1 Proposition:

The 'R' and 'S' properties define top subcategories of $\underline{C}^{P(\pi)}$ which contain all the discrete spaces and the structure functors defined above give us protoreflections which strongly generate the reflections. □

Proof: 5.3.3

9.7.2 Remarks:

By 7.3.1(b), 7.10(a) and 9.5(a) we see that none of the protoreflections given above can be strong except possibly $R_{\bar{0}}$.

9.8 The point separation axioms T_0, T_1 and T_2.

Each property T_i has associated with it a structured relation T_i (see §6) which has the corresponding subcategory of T_i-spaces as the associated extremal-epireflective subcategory. These structured relations are defined as follows:

For $(X,x) \varepsilon \underline{C}^{P(\pi)}$, $\quad T_0 x = (x\,\eta_X) \wedge (x\,\eta_X)^{-1}$,

$$T_1 x = (x\,\eta_X),$$

$$T_2 x = x\,x^{-1}.$$

We need to check that these are indeed structured relations.

So let $f:(X,x) \longrightarrow (Y,y)$. Then $f\,x \leqslant y\,Tf$ and $\eta_X\, f^{-1} \leqslant (Tf)^{-1}\,\eta_Y$.

We need to show in each case that $f(T_i x)f^{-1} \leqslant T_i y$ (see 6.1).

For T_1 we have $f(x\,\eta_X)f^{-1} \leqslant y\,Tf\,(Tf)^{-1}\,\eta_Y \leqslant y\,\eta_Y$.

For T_0, we now have $f(T_0 x)f^{-1} \leqslant y\,\eta_Y$ and since the left hand term is symmetric we also have $f(T_0 x)f^{-1} \leqslant (y\,\eta_Y)^{-1}$.

Thus $f(T_0 x)f^{-1} \leqslant T_0 y$.

For T_2, we have $fx\,x^{-1}\,f^{-1} \leqslant y\,Tf\,(y\,Tf)^{-1} \leqslant y\,y^{-1}$.

We now apply 6.3 and 6.5.1 to obtain the following result:

9.8.1 Proposition:

For $i = 0$, 1 and 2 the subcategory of $\underline{C}^{P(\pi)}$ consisting of the T_i-spaces is closed under products and subobjects. It is an extremal-epireflective subcategory. The structured equivalence relation generated by the structured relation T_i induces a protoreflection which strongly generates the reflection.

$\square$

9.8.2 Miscellaneous Results

(a) The structured relations T_i are order related as follows:

$$T_0 \leqslant T_1 \leqslant T_2.$$

(b) $\underline{C}^{P(\pi)}(T_0 = T_1) = R_0^-$ - spaces, $\underline{C}^{P(\pi)}(T_0 = T_2) = R_1$-spaces.

(c) If for $f:X \longrightarrow Y$ and $(Y,y) \varepsilon \underline{C}^{P(\pi)}$ the space (X,f^*y) is T_0, then f is monic. Cf. [Brümmer 1971, 1.3.8-9]

Proof: $f^{-1}f \leqslant f^{-1}T_0y\ f = (f^{-1}y\ \eta_Y f) \wedge (f^{-1}y\ \eta_Y f)^{-1}$

$= T_0(f^*y) = 1_X$ and so f is monic.

(d) (X, η_X^{-1}) is $T_0 \Longleftrightarrow$ it is $T_1 \Longleftrightarrow$ it is $T_2 \Longleftrightarrow \eta_X$ is monic.

Frei and MacDonald [1971 lemma 3] show that for $\underline{C}$ = sets, all the η_X are monic $\Longleftrightarrow$ there is a set X for which TX has cardinality $\geqslant 2$.

(e) The indiscrete space (X, ω_X) is $T_0 \Longleftrightarrow$ it is T_1

$\Longleftrightarrow$ it is T_2

$\Longleftrightarrow$ X is a partial terminal object. (see 6.2.1).

(f) Let $(X,x) \varepsilon \underline{C}^{P(\pi)}$ and $x = \mathrm{Im}\langle x_a, x_b\rangle$ with $\langle x_a, x_b\rangle$ monic.

Then (X,x) is $T_2 \Longleftrightarrow x_a$ is monic.

Proof: 1.4.2. □

(g) We say that $X \varepsilon \underline{C}$ is π-finite if η_X is regular-epi.

(i) A π-finite T_1 space is discrete.

(ii) A π-finite subobject of a T_1 space is closed.

Proof: (i) $x = x\ \eta_X\ \eta_X^{-1} = \eta_X^{-1}$.

(ii) Let $f:X \rightarrowtail Y$ be monic with X π-finite and (Y,y) a T_1-space. Then $f\ f^*y\ \eta_X = f\ f^{-1}y\ Tf\ \eta_X$

$= f\ f^{-1}y\ \eta_Y f = f\ f^{-1}f = f$

$= y\ \eta_Y f = y\ Tf\ \eta_X$ and

since η_X is regular-epi we have $f\ f^*y = yTf$. □

9.8.3 Proposition:

(a) The structured relations T_0 and T_1 are hereditary and productive.

(b) If (X,x) is S_0^{--} then T_0x is an equivalence relation. T_1x is an equivalence relation if and only if (X,x) is R_0.

Proof:

(a) T_1 is hereditary: Let $f:X \longrightarrow Y$ and $(Y,y) \varepsilon \underline{C}^{P(\pi)}$.

Then $T_1(f^*y) = f^{-1}yTf\,\eta_X = f^{-1}y\,\eta_Y f = f^*(T_1y)$.

T_1 is productive: Let $(X,x) = \mathbf{X}(X_i,x_i)$. So $x = \bigwedge p_i^*x_i$ and $T_1(p_i^*x_i) = p_i^*(T_1x_i)$. Thus $T_1x = (\bigwedge p_i^*x_i)\,\eta_X$

$= \bigwedge(p_i^*x_i\;\eta_X) = \bigwedge p_i^*(T_1x_i)$.

The proofs for T_0 follow using the identity $T_0y = (T_1y)\wedge(T_1y)^{-1}$.

The first part of (b) is straightforward and the rest follows by 9.2. $\square$

9.9 The T_0-protoreflection.

The last result suggests that the T_0-protoreflection is likely to have nice properties when we restrict ourselves to the subcategory of S_0^{--}-spaces and that the T_1-protoreflection may not be so nice unless we work in the subcategory of R_0 spaces where T_0 and T_1 are equal. In this subsection we prove that the T_0-protoreflection is indeed 'nice' if we restrict further to the S_0^{-} relational algebras.

9.9.1 Lemma:

Suppose that T preserves regular-epis and that for all regular-epis f we have $T(f^{-1}f) = Tf^{-1}\,Tf$.

Then for (X,x) a S_0^{-} relational algebra the following statements are true: (a) the T_0-protoreflection map $\rho_X : (X,x) \twoheadrightarrow (T_0X, (\rho_X)_*x)$ is initial,

(b) $(T_0X, (\rho_X)_*x)$ is an S_0^--relational algebra,

and (c) $(T_0X, (\rho_X)_*x)$ is a T_0-space.

Proof:

T_0x is an equivalence relation on X (9.8.3) and so

$T_0x = (x\,\eta_X)\wedge(x\,\eta_X)^{-1} = \rho_X^{-1}\,\rho_X$. Also $(\rho_X)_*x = \rho_X x(T\,\rho_X)^{-1}$

(a) $x \leqslant \rho_X^*\,(\rho_X)_*x = \rho_X^{-1}\,\rho_X\,x\,(T\,\rho_X)^{-1}T\,\rho_X$ (7.6.1b).

$$= \rho_X^{-1}\,\rho_X\,x\,T(\rho_X^{-1}\,\rho_X)$$

$$\leqslant (x\,\eta_X)\,x\,T(x\,\eta_X)$$

$$\leqslant x\,Tx\,T\,\eta_X \qquad (S_0^- \text{ and } 2.2(a))$$

$$\leqslant x\,\mu_X\,T\,\eta_X = x, \text{ and so } x = (\rho_X)^*(\rho_X)_*x$$

So ρ_X is initial.

Using (a) and 8.2.1(d) we see that ρ_X is strongly closed and using 9.4 we obtain (b).

(c) Since ρ_X is strongly closed we have

$$\rho_X x = \rho_X x(T\,\rho_X)^{-1}(T\,\rho_X).$$

Write $Y = T_0X$.

$$\text{So } (\rho_X)_*x\,\eta_Y = \rho_X\,x(T\,\rho_X)^{-1}\,\eta_Y = \rho_X\,x(T\,\rho_X)^{-1}\,\eta_Y\,\rho_X\,\rho_X^{-1}$$

$$= \rho_X\,x(T\,\rho_X)^{-1}(T\,\rho_X)\,\eta_X\,\rho_X^{-1}$$

$$= \rho_X(x\,\eta_X)\,\rho_X^{-1}.$$

It follows that $T_0\,(\,(\rho_X)_*x) = (\rho_X(x\,\eta_X)\,\rho_X^{-1})\wedge(\rho_X\,(x\,\eta_X)^{-1}\,\rho_X^{-1})$.

Now $\rho_X(\,(x\,\eta_X)\wedge(x\,\eta_X)^{-1})\,\rho_X^{-1} \leqslant (\rho_X(x\,\eta_X)\,\rho_X^{-1})\wedge(\rho_X(x\,\eta_X)^{-1}\,\rho_X^{-1})$ ... (*).

If $y \leqslant (\rho_X (x\,\eta_X)\,\rho_X^{-1}) \wedge (\rho_X (x\,\eta_X)^{-1}\,\rho_X^{-1})$ then

$\rho_X^{-1} y\,\rho_X \leqslant (\rho_X^{-1}\,\rho_X)(x\,\eta_X)(\rho_X^{-1}\,\rho_X)$ and

$\rho_X^{-1} y\,\rho_X \leqslant (\rho_X^{-1}\,\rho_X)(x\,\eta_X)^{-1}(\rho_X^{-1}\,\rho_X)$.

So $\rho_X^{-1} y\,\rho_X \leqslant (x\,\eta_X)(x\,\eta_X)(x\,\eta_X) \leqslant (x\,\eta_X)$ $\quad (S_0^{--})$

and similarly $\rho_X^{-1} y\,\rho_X \leqslant (x\,\eta_X)^{-1}$.

Thus $\rho_X^{-1} y\,\rho_X \leqslant (x\,\eta_X) \wedge (x\,\eta_X)^{-1}$ and so since ρ_X is regular-epi we have $y \leqslant$ left hand side of the inequality (*).

It follows that we have equality in (*) and so

$$T_0(\,(\rho_X)_* x) = \rho_X(\,(x\,\eta_X) \wedge (x\,\eta_X)^{-1})\,\rho_X^{-1}$$

$$= \rho_X\,\rho_X^{-1}\,\rho_X\,\rho_X^{-1} = 1_Y. \qquad \square$$

9.9.2 Proposition:

Suppose that T preserves regular-epis and that for all regular-epis f we have $T(f^{-1}f) = Tf^{-1}\,Tf$.

Let $\underline{A}$ be the top category of S_0^{-}-relational algebras.

Then the T_0-protoreflection in the category $\underline{A}$ is strong and the T_0-reflection in $\underline{A}$ has the following properties:

(a) the reflection maps are initial,

(b) If $\underline{C}$ is a $\underline{QI}$ category then the reflection preserves products and

(c) If $\underline{C}$ is balanced then the reflection preserves subspaces.

Proof: Use 9.9.1, the fact that the reflection maps are coinitial in $\underline{C}^{P(\pi)}$ and so satisfy the condition (P3)*, 9.8.3, 6.9.2 and 6.8.2 (see also proof of 6.7.3). $\square$

9.9.3 Remarks:

(a) In each of the examples of monads given in 7.10, the functor satisfies the conditions for 9.9.2. In particular the ultrafilter functor satisfies the conditions for 9.9.2 (see 7.10(b)) and so this result gives as an application the well known properties of the T_0-reflection in Top.

(b) The T_0-protoreflection in $C^{P(\pi)}$ is not strong.

Let C = sets and π be the identity monad (see 7.10a), A space (X,x) is T_0 $\Longleftrightarrow$ x is antisymmetric.

For the example, let $X = \{a,b,c\}$ and

$$x = \Delta_X \cup \{(a,b),(b,a),(a,c),(c,b)\}.$$

Then $T_0 x = x \wedge x^{-1} = \Delta_X \cup \{(a,b),(b,a)\}$.

So the T_0-protoreflection of (X,x) is an indiscrete space with 2 points and so is not T_0.

(c) In §11 we give an example which shows that the T_1 and T_2 protoreflections in Top are not strong.

(d) Suppose that C = sets and T has the property that $T(f^{-1}f) = Tf^{-1}\,Tf$ for all regular-epis f.

Then the structured equivalence relation T_0 is the only symmetric structuredrelation on the category of $\bar{S}_0$-relational algebras which has the T_0-spaces as the associated subcategory (use 6.8.4). For the identity monad or the ultrafilter monad we can drop the word 'symmetric' (see 11.2.1).

9.10 Proposition:

Let $\underline{C}$ = $\underline{\text{sets}}$, $\underline{A}$ be the category of S_0^- relational algebras and suppose that $T(f^{-1}f) = Tf^{-1}\,Tf$ for all regular-epis f.

Let E be a structured relation on $\underline{A}$ with $T_0 \leqslant E$.

Then (a) $\underline{A}(E = T_0)$ is the monoreflective hull in $\underline{A}$ of $\underline{A}(E = 1)$.

and (b) the subcategory of R_0^- (resp. R_1) spaces is the monoreflective hull in $\underline{A}$ of the subcategory of T_1 (resp. T_2) spaces.

Proof:

(a) Obviously $\underline{A}(E = 1) \subset \underline{A}(E = T_0)$. By 9.8.2(e) we have $T_0 \neq 1$ and so we have $E(\omega_X) = T_0(\omega_X) = \text{Im}\ 1_{X\times X}$ for all $X\varepsilon\underline{C}$. By 9.8.3(a) and 5.3.1 we see that $\underline{A}(E = T_0)$ is closed under products and subspaces and so it is a monoreflective subcategory of $\underline{A}$ (3.7). Let $(X,x)\varepsilon\ \underline{A}(E = T_0)$. Then by 9.9.2, $\rho_X : (X,x) \twoheadrightarrow T_0^r(X,x) = E^r(X,x)$ is initial in $\underline{A}$ and so by 6.7.1 we see that $E^r(X,x)\varepsilon\ \underline{A}(E = 1)$. Thus $\langle\rho_X, 1_X\rangle : (X,x) \rightarrowtail E^r(X,x) \times (X, \omega_X)$ is an extreme-monic whose codomain lies in every monoreflective subcategory of $\underline{A}$ which contains $\underline{A}(E = 1)$.

(b) follows now by 9.8.2(b). □

9.11 T_2 - spaces.

A topological space is Hausdorff if and only if the diagonal is closed. The next theorem shows that this is true also for relational algebras over a regular category.

First we need a lemma:

9.11.1 Lemma:

Let $a:R \longrightarrow Y$ in $\underline{C}$ have kernel pair $K \overset{c_1}{\underset{c_2}{\rightrightarrows}} R$ and let

$k = ac_1 = ac_2$.

Then the diagram

$$\begin{tikzcd}
 & K \arrow[dl, "k"'] \arrow[dr, tail, "{<c_1,c_2>}"] & \\
Y \arrow[dr, "\Delta_Y"'] & & R \times R \arrow[dl, "a \times a"] \\
 & Y \times Y &
\end{tikzcd} \qquad \ldots.\ (i)$$

is a pullback.

Proof:

It is clear that the diagram (i) commutes. Let $r_i : R \times R \longrightarrow R$ and $q_i : Y \times Y \longrightarrow Y$ be the projections. Now suppose that we have morphisms $g: \cdot \longrightarrow Y$ and $h: \cdot \longrightarrow R \times R$ which satisfy $\Delta_Y g = (a \times a)h$. We need to find a morphism $f: \cdot \longrightarrow K$ such that $kf = g$ and $<c_1,c_2>\ f = h$.

Now $a\,r_i h = q_i (a \times a)\, h = q_i\, \Delta_Y\, g = g$ for $i = 1,2$.

So there is a morphism f which makes

$$\begin{tikzcd}
 & \cdot \arrow[ddl, "r_1 h"'] \arrow[d, "f"] \arrow[ddr, "r_2 h"] & \\
 & K \arrow[dl, "c_1"] \arrow[dr, "c_2"'] & \\
R \arrow[dr, "a"'] & \mathrm{PB} & R \arrow[dl, "a"] \\
 & Y &
\end{tikzcd}$$

commute.

Then $kf = ac_1f = ar_1h = g$ and $r_i\langle c_1,c_2\rangle f = c_if = r_ih$ for $i = 1,2$ and so f has the required properties. □

9.11.2 Proposition: [cf. Manes 1973, 4.5]

For $(X,x) \varepsilon \underline{C}^{P(\pi)}$ the following are equivalent:

(a) (X,x) is T_2.

(b) $\Delta_X : X \rightarrowtail X \times X$ is closed in $(X,x) \times (X,x)$.

(c) For all (Y,y) and $f, g : (Y,y) \rightrightarrows (X,x)$, the kernel of f, g is closed in (X,x).

(d) For all $f : (Y,y) \longrightarrow (X,x)$, the subobject $\langle 1,f\rangle : Y \rightarrowtail Y \times X$ is closed in $(Y,y) \times (X,x)$.

Proof:

Let $x = \mathrm{Im}\langle x_a,x_b\rangle$ with $\langle x_a,x_b\rangle$ monic. Let $p_i : X \times X \longrightarrow X$ be the projections and $\phi = \langle Tp_1,Tp_2\rangle$.

By 7.5 we have $(X,x) \times (X,x) = (X \times X,\ p_1^*x \wedge p_2^*x) = (X \times X,\ (x \times x)\phi)$.

also $\Delta_X^*(p_1^*x \wedge p_2^*x) = x$.

So $\Delta_X : X \rightarrowtail X \times X$ is closed in $(X,x) \times (X,x)$ if and only if

$$\begin{tikzcd}
TX \arrow[r, "T(\Delta_X)"] \arrow[dd, "x"'] & T(X \times X) \arrow[d, "\phi"] \\
 & TX \times TX \arrow[d, "x \times x"] \\
X \arrow[r, tail, "\Delta_X"'] & X \times X
\end{tikzcd}$$

commutes. ...... (i)

We see easily that $\phi\, T\Delta_X = \Delta_{TX}$ and $\Delta_X x_b = (x_b \times x_b)\, \Delta_R$.

Consider the diagram

$$\begin{tikzcd}
TX \arrow[rr, "T\Delta_X"] \arrow[drr, "\Delta_{TX}"] \arrow[ddd, "x"'] & & T(X \times X) \arrow[d, "\phi"] \\
& & TX \times TX \\
& R \arrow[uul, "x_a"] \arrow[r, "\Delta_R"'] \arrow[dl, "x_b"] & R \times R \arrow[u, "x_a \times x_a"'] \arrow[d, "x_b \times x_b"] \\
X \arrow[rr, "\Delta_X"'] & & X \times X
\end{tikzcd}$$

We see that (i) commutes if and only if

$$(x_b \times x_b)\, \Delta_R\, x_a^{-1} = (x_b \times x_b)(x_a \times x_a)^{-1}\, \Delta_{TX} \qquad \text{.....(ii).}$$

(a) $\Longrightarrow$ (b): Let (X,x) be a T_2 space. Then by 9.8.2(f) we know that x_a is monic and so the kernel pair of x_a is $R \overset{1}{\underset{1}{\rightrightarrows}} R$. Now by 9.11.1 and 1.4.1h we have $\Delta_R\, x_a^{-1} = (x_a \times x_a)^{-1}\, \Delta_{TX}$ and so (ii) is true.

(b) $\Longrightarrow$ (a): We suppose now that $\Delta_X : X \longrightarrow X \times X$ is closed, i.e. that (ii) is true. Let $K \overset{c_1}{\underset{c_2}{\rightrightarrows}} R$ be the kernel pair of x_a and let $k = x_a c_1 = x_a c_2$.

Now by 9.11.1 and 1.4.1h we have $(x_a \times x_a)^{-1}\, \Delta_{TX} = \langle c_1, c_2 \rangle\, k^{-1}$.

So by (ii) we have

$$\begin{aligned} \mathrm{Im}\langle x_a, \Delta_X x_b\rangle &= \Delta_X x_b x_a^{-1} = (x_b \times x_b)\, \Delta_R\, x_a^{-1} \\ &= (x_b \times x_b)(x_a \times x_a)^{-1}\, \Delta_{TX} = (x_b \times x_b)\langle c_1, c_2\rangle\, k^{-1} \\ &= \langle x_b c_1, x_b c_2\rangle\, k^{-1} = \mathrm{Im}\,\langle k, \langle x_b c_1, x_b c_2\rangle\,\rangle. \end{aligned}$$

Now $\langle x_a, \Delta_X x_b\rangle$ is monic and so there is a morphism f such that

$$\begin{tikzcd} & R \arrow[ld, "x_a"'] \arrow[rd, "\Delta_X x_b"] & \\ TX & & X \times X \\ & K \arrow[lu, "k = x_a c_i"] \arrow[ru, "\langle x_b c_1, x_b c_2\rangle"'] \arrow[uu, "f"] & \end{tikzcd}$$

commutes.

So $x_b c_1 = x_b f = x_b c_2$. Also $x_a c_1 = x_a c_2$ and so since $\langle x_a, x_b\rangle$ is monic, we have $c_1 = c_2$ which tells us that x_a is monic. Now by 9.8.2(f) we see that (X,x) is T_2.

(b) $\Longrightarrow$ (c): $\langle f,g\rangle^{-1}(\Delta_X)$ is the equalizer of f,g and so by 8.3.1b, if Δ_X is closed then so is the equalizer.

(c) $\Longrightarrow$ (d): $\langle 1, f\rangle$ is the equalizer of $f\,P_Y$, P_X.

(d) $\Longrightarrow$ (b): $\Delta_X = \langle 1_X, 1_X\rangle$. ☐

9.11.3 Problem:

It is well known that for (X,x) a topological space, the relation $x\, x^{-1}$ is the closure of the diagonal Δ_X in $(X,x) \times (X,x)$.

This is also true for relational algebras in the case $\underline{C}$ = sets.

The proof is as follows: By 8.3.3 the relation $x : (TX, \mu_X) \longrightarrow (X,x)$ is closed. The map $(TX, \mu_X) \times (X,x) \longrightarrow (X,x) \times (TX, \mu_X)$

which 'interchange coordinates' is an isomorphism and so the inverse relation x^{-1} is also closed. Now since (TX, μ_X) is compact, 10.5(iii) tells us that $x\, x^{-1}$ is closed. It remains to be shown that $\Delta_X \subset xx^{-1} \subset \mathrm{cl}\,\Delta_X$. The first part follows from $1 \leqslant x\, \eta_X$.

For the other inclusion let $(a,b) \in xx^{-1}$ amd $c \in TX$ be such that $(c,a) \in x$ and $(c,b) \in x$. Let $c' = T(\langle 1_X, 1_X \rangle)\, c$.

$$\begin{tikzcd}
c' & & & \\
T(\Delta_X) \arrow[rr] & & T(X \times X) \arrow[d, "\langle TP_1{,}TP_2 \rangle"] & \\
 & & TX \times TX \arrow[d, "x \times x"] & (c\,,\,c) \arrow[d] \\
\Delta_X \arrow[rr] & & X \times X & (a\,,\,b)
\end{tikzcd}$$

From the diagram we see that $(a,b) \in \mathrm{cl}\ \Delta_X$. □

<u>The problem is:</u> is this result true for $\underline{C}$ a regular category?

<u>For prealgebras the result is not true:</u> Let π be the identity monad on <u>sets</u>. Let $X = \{1,2,3\}$ and $x = \Delta_X \cup \{(1,2), (2,3)\}$. Then (X,x) is a prealgebra but not a relational algebra. The closure of Δ_X in $(X,x) \times (X,x)$ is $X \times X$ (see 8.5a) but $(1,3) \notin x\, x^{-1}$.

<u>9.12 Proposition:</u> <u>'The closure of a connected space is connected'</u>
Suppose that for all $X \in \underline{C}$ the morphism η_X is monic and that T preserves the terminal object N.

Let $j:A \rightarrowtail X$ be a dense subobject of (X,x) and suppose that (A,j^*x) is connected.

Then (X,x) is connected.

Proof:

By 7.9 we have $\alpha_N = \omega_N$. Let $f:(X,x) \longrightarrow (Y, \alpha_Y)$ Since (A, j^*x) is connected there are morphisms t and h such that $th = fj$.

$$\begin{tikzcd}
(A,j^*x) \arrow[r, tail, "j"] \arrow[dr, "h"'] & (X,x) \arrow[r, "f"] \arrow[d, "t'"] & (Y, \alpha_Y) \\
 & (N, \alpha_N) \arrow[ur, "t"'] &
\end{tikzcd}$$

Let $t':X \longrightarrow N$ be the unique morphism. Since $\alpha_N = \omega_N$, $t':(X,x) \longrightarrow (N, \alpha_N)$. Since $t\, t'\, j = th = f\, j$ we have $\mathrm{Im}\, j \leqslant$ kernel $(t\, t', f)$. Also by 9.8.2d and 9.11.2c kernel $(t\, t', f)$ is closed in (X,x) and so since $\mathrm{Im}\, j$ is dense in (X,x) we have $t\, t' = f$. $\square$

10 RELATIONAL ALGEBRAS: COMPACTNESS

Compactness for relational algebras over sets is discussed by Manes [1973]. The results given in this subsection are, as in §8, mainly the generalization to regular categories of Manes' results. We have been able to do this for all of his work on compactness except for his 'Stone-Čech' compactification theorem [1973, 8.4] and his characterization of compactness as stated here in 10.5.

10.1 Definition: [cf. Manes 1973, 7.10]

We say that a space $(X,x) \in \underline{C}^{P(\pi)}$ is compact if $x^{-1}x \geqslant 1_{TX}$.

10.1.1 Remarks:

(a) Let $x = \text{Im}\,\langle x_a, x_b\rangle$ with $\langle x_a, x_b\rangle$ monic. By 1.4.2 we see that (X,x) is compact if and only if x_a is regular-epi. So for the ultrafilter monad (7.10b) 'compact' is equivalent to 'every ultrafilter converges' which agrees with the usual idea of compactness in Top.

(b) (X,x) is a compact T_2 space if and only if (X,x) is a π-algebra. Proof: 1.4.1d.

(c) The indiscrete structure is the only compact T_2 structure on the terminal object. If $\underline{C}$ is connected then all the indiscrete spaces are compact.

(d) The compact discrete spaces are precisely those whose underlying object is π-finite (see 9.8.2g). Also, all π-finite spaces are compact.

(e) The functors induced by morphisms of monads preserve compactness.

We now give the generalizations of some of the basic results of Manes [1973] about compactness in topology.

10.2 Proposition:

(a) 'Tychonoff theorem'

Every finite product of compact spaces is compact and if C has the property QI then the same is true for infinite products.

(b) Suppose that T preserves regular-epis. Then a map in $\underline{C}^{P(\pi)}$ which is regular-epi in C and has a compact domain has a compact codomain.

(c) Every closed subspace of a compact space is compact.

(d) Every map from a compact space to a T_2 space is strongly closed.

Proof:

(a) Let $(X,x) = \mathsf{X}(X_i,x_i)$ with (X_i,x_i) compact for each i.

By 7.5 we have $x = (\mathsf{X}\, x_i)\phi$ with $\phi = \langle Tp_i \rangle$.

So $x^{-1}x = \phi^{-1}(\mathsf{X}\, x_i)^{-1}(\mathsf{X}\, x_i)\phi = \phi^{-1}(\; x_i^{-1}x_i)\phi$ (1.7.2)

$\geqslant \phi^{-1}\phi \geqslant 1.$

(b) Let $f:(X,x) \twoheadrightarrow (Y,y)$ be regular-epi in C with (X,x) compact. Then $y^{-1}y \geqslant (Tf)(Tf)^{-1}\, y^{-1}f\, f^{-1}y\, (Tf)(Tf)^{-1}$

$= (Tf)(f^*y)^{-1}(f^*y)\,(Tf)^{-1}$

$\geqslant (Tf)(x^{-1}x)(Tf)^{-1} \geqslant (Tf)(Tf)^{-1} = 1.$

(c) Let $m:(A,m^*x) \rightarrowtail (X,x)$ be strongly closed with (X,x) compact. Then

$$(m^*x)^{-1}(m^*x) = (Tm)^{-1}x^{-1}m\,m^*x = (Tm)^{-1}x^{-1}x\,Tm$$
$$\geqslant (Tm)^{-1}(Tm) \geqslant 1.$$

(d) Let $f:(X,x) \longrightarrow (Y,y)$ with (X,x) compact and (Y,y) T_2. Let $x = \mathrm{Im}\,\langle x_a,x_b\rangle$ and $y = \mathrm{Im}\,\langle y_a,y_b\rangle$ with $\langle x_a,x_b\rangle$ and $\langle y_a,y_b\rangle$ both monic. Now

$$f\,x = \mathrm{Im}\,\langle x_a,fx_b\rangle$$
$$= \mathrm{Im}\,\langle a,b\rangle$$

where $\langle x_a,fx_b\rangle = \langle a,b\rangle\,t$ is the (regular-epi, mono) decomposition of $\langle x_a,fx_b\rangle$.

Also $y\,Tf$ is given by

$$\begin{tikzcd}
 & & \cdot \arrow[dl, tail, "a'"'] \arrow[dr, "b'"] & & \\
 & TX \arrow[dl, "1"'] \arrow[dr, "Tf"] & \text{PB} & \cdot \arrow[dl, tail, "y_a"'] \arrow[dr, "y_b"] & \\
TX & & TY & & Y
\end{tikzcd}$$

Since (Y,y) is a T_2-space the morphism y_a is monic (9.8.2f) and it follows that a' is monic. Thus $\langle a',y_b b'\rangle$ is monic. Now $f\,x \leqslant y\,Tf$ implies that there is a monic u such that the following diagram commutes.

$$\begin{tikzcd}
 & \cdot \arrow[dl, "x_a"'] \arrow[d, "t"] \arrow[dr, "fx_b"] & \\
TX & \cdot \arrow[l, "a"'] \arrow[r, "b"] \arrow[d, "u"] & Y \\
 & \cdot \arrow[ul, tail, "a'"] \arrow[ur, "y_b b'"'] &
\end{tikzcd}$$

Since (X,x) is compact the morphism x_a is regular-epi(10.1.1a.). It follows that a and a' are both monic and regular-epi and so they are isomorphisms. Thus u is an isomorphism and so $fx = yTf$. □

10.3 Proposition: [cf. Manes 1973, 4.2]

A space is compact if and only if the unique map from the space to the terminal object with the indiscrete structure is strongly closed.

Proof:

Let n be the indiscrete structure on the terminal object N.

By 10.1.1c $n:TN \longrightarrow N$ is a morphism.

Let $f:(X,x) \longrightarrow (N,n)$. By 9.8.2e, (N,n) is a T_2 space.

If (X,x) is compact then 10.2d tells us that f is strongly closed.

Conversely assume that f is strongly closed.

Let $x = \mathrm{Im}\,\langle x_a,x_b\rangle$ with $\langle x_a,x_b\rangle$ monic.

Then $fx = \mathrm{Im}\,\langle x_a,fx_b\rangle$ and since $fx = n\,Tf$ there is a regular-epi $\mathbf{t}$ which makes the following diagram commute.

$$\begin{tikzcd}
 & \bullet \arrow[dl, "x_a"'] \arrow[dr, "fx_b"] \arrow[dd, two heads, "\mathbf{t}"] & \\
TX & & N \\
 & TX \arrow[ul, "1"] \arrow[ur, "n\,Tf"'] &
\end{tikzcd}$$

So x_a is regular-epi and hence (X,x) is compact (10.1.1a). □

10.4 Examples:

(a) Every relational prealgebra of the identity monad is compact.

(b) A generalized sup-semilattice is compact if and only if every subset has at least one supremum.

The following characterizations of compactness in a category of relational algebras are given by Manes [1973] for the case $\underline{C}$ = sets. We have not been able to generalize these results. We say that a relation is closed if it is closed as a subobject of the product space.

10.5 Proposition: [ibid 4.2 and 7.10]

Let (X,x) be a relational algebra.

The following are equivalent:

(i) (X,x) is compact,

(ii) for all relational algebras (Y,y), the projection $(Y,y)\times(X,x) \longrightarrow (Y,y)$ is closed,

(iii) for all closed relations $\alpha:(Y,y) \longrightarrow (X,x)$, $\beta:(X,x) \longrightarrow (Z,z)$ the composite $\beta\alpha:(Y,y) \longrightarrow (Z,z)$ is closed. □

11. TOPOLOGICAL SPACES

In this section we will study in some detail the 'lattice' of structured relations on Top. We obtain some strong results concerning the structured relations associated with the subcategories of T_0, T_1 and T_2-spaces.

We also use the concept of structured relations to make a contribution to the solution of problem of Wyler [1971c] on function space topologies and to solve a number of problems of Thornton [1971] concerning equalizers, closure operators and the double construction.

11.1 Examples: Structured relations on Top.

We list below a number of structured relations on Top and their associated subcategories which we will study in this section. Some of these are also described by Sharpe, Beattie and Marsden [1966]. The descriptions of T_0, T_1 and T_2 are the 'open set' versions of those given in 9.8.

Let $(X,x) \in$ Top.

(a) T_0-spaces: $(a,b) \in T_0x \Longleftrightarrow$ (for all U open in X, $a \in U \Longleftrightarrow b \in U$).

(b) T_1-spaces: $(a,b) \in T_1x \Longleftrightarrow$ (for all U open in X, $b \in U \Longrightarrow a \in U$)

(c) <u>T_2-spaces:</u>

$(a,b) \in T_2 X \iff$ (for all U,V open in X, $a \in U$ and $b \in V$

$\implies U \cap V \neq \emptyset$).

(d) <u>$\bar{T}_n$-spaces:</u> [Viglino 1971, Porter and Votaw 1973]

For each integer n, a structured relation $\bar{T}_n$ is defined by

$(a,b) \in \bar{T}_n X \iff$ (for each sequence $O_1, O_2, \ldots\ldots, O_n$

of n open sets such that

$a \in O_1 \subset \bar{O}_1 \subset O_2 . \;\; .. \; \bar{O}_{n-1} \subset O_n$

we have $b \in \bar{O}_n$).

Then (i) $\bar{T}_1 = T_2$,

(ii) <u>$\bar{T}_2$-spaces</u> = <u>Urysohn spaces</u>,

(iii) $\bar{T}_1 \leqslant \bar{T}_2 \leqslant \bar{T}_3 \; \ldots$,

Viglino shows that the categories of <u>$\bar{T}_n$-spaces</u> are all distinct.

(e) <u>Completely Hausdorff spaces:</u>

$(a,b) \in T_{[0,1]} X \iff$ (for all $f:(X,x) \longrightarrow [0,1]$ with usual topology,

we have $f(a) = f(b)$).

(f) <u>T_*-spaces:</u> (quasi-components are singletons)

$(a,b) \in T_* X \iff$ (for all $f:(X,x) \longrightarrow (\odot\,\odot)$, (the two point discrete space)

we have $f(a) = f(b)$).

(g) <u>Semi-Hausdorff spaces:</u> (sequences have unique limits)

$(a,b) \in T_{SH} X \iff$ (There is a sequence $\{x_n\}$ in X such

that $x_n \longrightarrow a$ and $x_n \longrightarrow b$).

11.2 T_0-spaces.

For the T_0 structured relation the work done in §9 gives us the following nice results.

11.2.1 Proposition:

The structured relation T_0 is a structured equivalence relation which is hereditary, productive, initial and strong and the T_0-reflection preserves products and subspaces. Also T_0 is the only structured relation associated with the T_0-spaces.

Proof:

All topological spaces are $S_{\bar{0}}$ and the ultrafilter functor satisfies the conditions required for the use of 9.9.2. Now apply 9.8.3 and 9.9.2. For the last assertion, let E be a structured relation which is compatible with T_0. By 6.6.1b we have $E \leqslant T_0$. If E were symmetric then 6.8.4(ci) tells us that $T_0 \leqslant E$ and so $T_0 = E$. However the proof of 6.8.4(ci) only uses the symmetry of E on a certain 2-point space which in this case is just the indiscrete space on which every structured relation is symmetric. So the same result is true even if E is not assumed to be symmetric. □

11.2.2 Proposition:

Let E be a proper structured relation on Top.

Then $T_0 \leqslant E \leqslant T_*$

Proof:

By 6.3.2 the two-point discrete space (D,d) lies in Top (E = I).

Let $(X,x) \varepsilon$ Top. For $(a,b) \notin T_*x$ there is a map $f:(X,x) \longrightarrow (D,d)$ such that $f(a) = 0$ and $f(b) = 1$. It follows that $(a,b) \notin Ex$. So $E \leqslant T_*$.

By 6.2.1c, the 2-point indiscrete space $(D, d_i) \notin$ Top $(E = 1)$.

The proof that $T_0 \leqslant E$ is now similar to the proof of the last assertion in II.2.1. $\square$

II.2.3 Proposition: 'Characterization of T_0'

Let E be a proper structured equivalence relation on Top.

The following are equivalent:

(a) $E = T_0$,

(b) E is hereditary,

(c) E is initial,

(d) For all $(X,x) \varepsilon$ Top, each open set is E-saturated,

i.e. $U \subset X$ open, $(a,b) \varepsilon Ex$ and $a \varepsilon U \Longrightarrow b \varepsilon U$.

Proof:

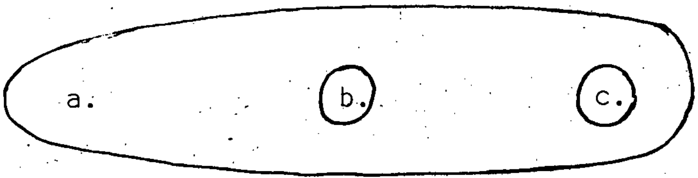
(a) $\Longrightarrow$ (b) and (a) $\Longrightarrow$ (c): II.2.1.

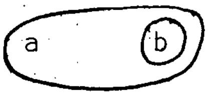
(b) $\Longrightarrow$ (a): By II.2.2 we need only show that $E \leqslant T_0$.

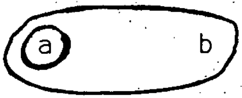
Let $(X,x) \varepsilon$ Top and $(a,b) \varepsilon Ex$ with $a \neq b$.

Since E is hereditary we have $(a,b) \varepsilon E(j^*x)$ where $j:\{a,b\} \rightarrowtail X$ is the inclusion map.

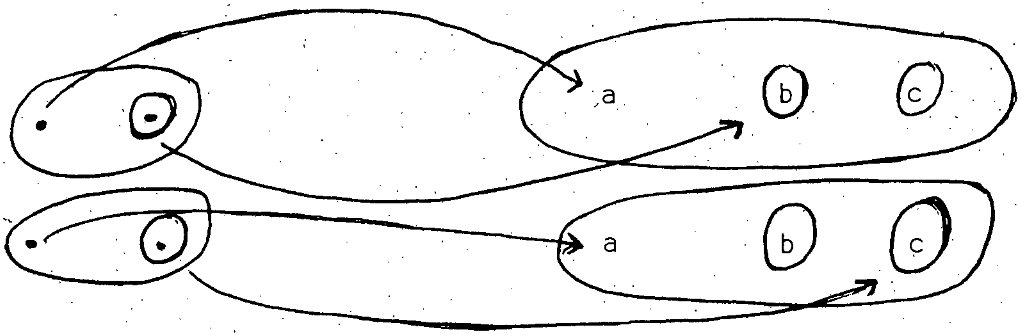
Since E is proper, j^*x cannot be the discrete topology.

Let (B,b) be the space 

If j^*x is one of the T_0-topologies 

or 

then using the maps



we see that (b,c) ε Eb and so since E is hereditary we have (b,c) ε Ed (where d is the discrete topology on the 2-point space) which is not true. So j^*x must be the indiscrete topology.

Thus (a,b) ε T_0 (j^*x) and it follows that (a,b) ε T_0x.

(c) $\Longleftrightarrow$ (d): Trivial.

(d) $\Longrightarrow$ (a): By (d) the space (D,d_1) =  ε *Top* (E = 1) and so *T_0-spaces* $\subset$ *Top*(E = 1). By 11.2.2 we have $T_0 \leqslant$ E and so *Top*(E = 1) $\subset$ *T_0-spaces*. So we have equality here and by 11.2.1 we have E = T_0. □

11.2.4 Remarks:

For the equivalence (a) $\Longleftrightarrow$ (d) we can drop the requirement that E be an equivalence. However for (a) $\Longleftrightarrow$ (b), E must be an equivalence relation. For a counter example we note that T_1 is an hereditary structured relation (9.8.3).

11.3 T_1-spaces.

The structured relation T_1 is not as 'nice' as T_0. This seems to be due to the fact that it is not a structured equivalence relation. We will use T_1 to produce counter-examples to various possible conjectures concerning structured relations.

11.3.1 Proposition:

The structured relation T_1 is hereditary and productive. $T_1 \vee T_1^{-1}$ is the least symmetric structured relation associated with the T_1-spaces and T_1^{q} (the structured equivalence relation generated by T_1) is the least structured equivalence relation associated with the T_1-spaces.

Proof:

9.8.3, 6.8.4b,c. $\square$

11.3.2 Counter-examples (see 6.6.2):

(a) The class of structured relations associated with the T_1-spaces has no least member.

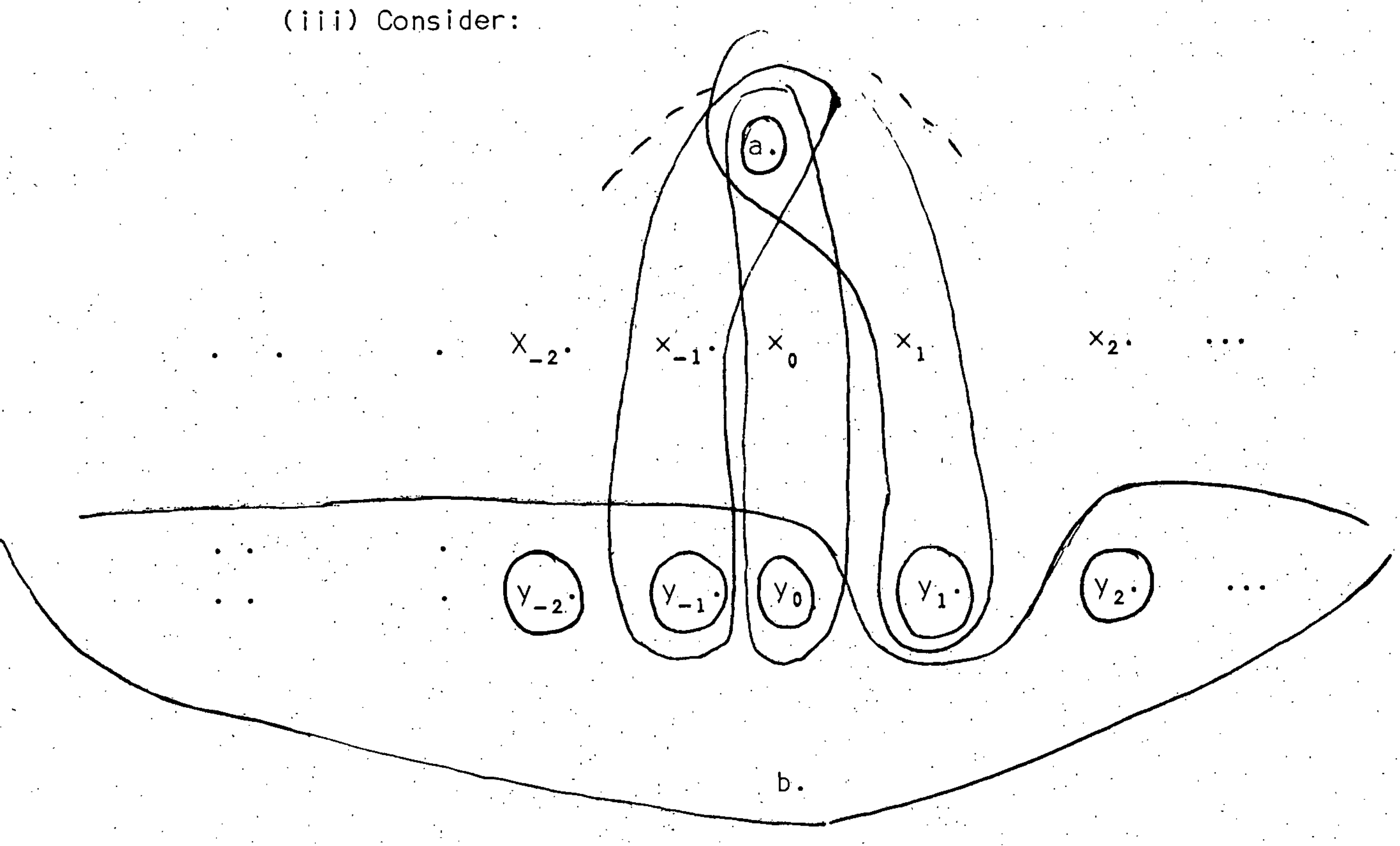
Proof: T_1 and T_1^{-1} are distinct and not comparable under the order relation. If $E \leqslant T_1$ and $E \leqslant T_1^{-1}$ then $E \leqslant T_0$ and so E is not associated with the T_1-spaces.

(b) <u>The structured equivalence relation T_1^q is not strong.</u>

(i) The following example shows that if E is a structured equivalence relation with $T_1^q \leqslant E \leqslant T_2^q$ then E is not strong.

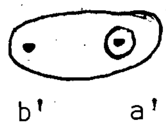
(ii) Sharpe, Beattie and Marsden give an example for T_2 [1966] but their example is false. We give another example.

(iii) Consider:



Let $X = \{a, b, x_0, x_{\pm 1}, x_{\pm 2}, \ldots, y_0, y_{\pm 1}, \ldots\}$

and let the topology x have subbase the sets $\{a\}$, $\{y_i\}$, $\{a, x_i, y_i\}$, $\{b, y_0, y_{\pm 1}, y_{\pm 2}, \ldots\} - \{y_i\}$, all $i \in \mathbb{Z}$.

The induced protoreflections T_1^r and T_2^r give us, after the first iteration the space  which is neither T_1 nor T_2.

(c) Not every structured equivalence relation associated with the T_1-spaces is a power of T_1^q.

Let $E = (T_1^q)^2 \wedge T_2^q$. Then $T_1^q \leqslant E \leqslant (T_1^q)^2$ and so Top$(E = 1)$ = T_1-spaces.

Let $X = \mathbb{Z} \cup \{a,b\}$ with the topology x on X having as subbase the sets $\mathbb{Z} \cup \{a\}$, $\{b\} \cup \mathbb{Z} - F$ for all finite $F \subset \mathbb{Z}$.

Then $T_2^q(x) = X \times X$ and $T_1^q(x) = \Delta_X \cup \{(a,n),(n,a) \mid n \in \mathbb{Z}\}$.

The first iteration of the T_1 protoreflection gives the space

and so $(T_1^q)^2 = X \times X$. Thus $Ex = X \times X$ and $Ex \neq T_1^q x$.

Now for the space (X,x) described in (b) we have $Ex \neq (T_1^q)^2$.

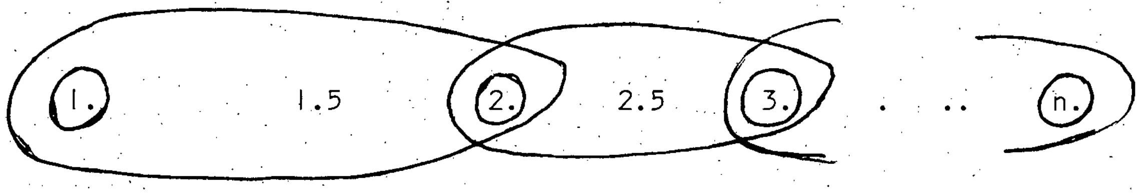
So E is not a power of T_1^q.

(d) The structured equivalence relation T_1^q is not productive although T_1 is productive.

The example given below also shows that T_2^q is not productive.

For $n \in \mathbb{N}$, let $X_n = \{1, 1.5, 2, 2.5, \ldots, n\}$

with base for the topology x_n as in the diagram below:



Let $(X,x) = \underset{\mathbb{N}}{\times}(X_n,x_n)$. Let $a = (1, 1, 1, 1, 1, 1 \ldots)$

and $b = (1, 2, 3, 4, 5, 6, \ldots)$.

So $a_n = 1$ and $b_n = n$.

Using transitivity we see that for each n we have $(a_n, b_n) \in T_1^q x_n$.

We show by contradiction that $(a,b) \notin T_1^q x$.

If $(a,b) \in T_1^q x$ then there is a finite sequence $a = c_0, c_1 \; .. \; c_m = b$ in X such that $(c_i, c_{i+1}) \in T_1x \vee T_1^{-1}x$ for $1 \leqslant i \leqslant m-1$. ....(*)

For $0 \leqslant i \leqslant m$, let d_i be the supremum of the coordinates of c_i (or '∞' if it does not exist).

So $d_0 = 1$ and $d_m = \infty$.

If for $1 \leqslant i \leqslant m$ we have $d_i \leqslant d_{i-1} + 1$ then $d_m \leqslant m + 1$.

So there must be an i with $1 \leqslant i \leqslant m$, $d_i > d_{i-1} + 1$ and d_{i-1} finite. It follows that for some $n \in \mathbb{N}$,

$(c_i)_n > d_{i-1} + 1 \geqslant (c_{i-1})_n + 1$ (where $(c_i)_r$ denotes the r'th coordinate of c_i).

So $(c_i)_n \geqslant (c_{i-1})_n + 2$ and thus $(c_i)_n$ and $(c_{i-1})_n$ have disjoint open x_n-neighbourhoods.

So $((c_i)_n, (c_{i-1})_n) \notin T_1x_n$ and hence $(c_i, c_{i-1}) \notin T_1x \vee T_1^{-1}x$.

This contradicts (*). □

The following result supplements II.2.2.

II.3.3 Proposition:

Let E be a proper symmetric structured relation on Top.

If $E \neq T_0$ then $T_1 \leqslant E$.

Proof:

By II.2.2 we have $T_0 \leqslant E$ and so $\underline{\text{Top}}\ (E = I) \subset T_0\text{-}\underline{\text{spaces}}$.

Let (D,d_1) be the space  and (D,d_i) be the 2-point indiscrete space.

So $E(d_i) = D \times D$.

If $E(d_1) = I$ then $T_0\text{-}\underline{\text{spaces}} \subset \underline{\text{Top}}\ (E = I)$. Thus we have equality here and by II.2.1 we get $E = T_0$. This is not so.

Hence since E is symmetric we must have $E(d_1) = D \times D$.

Now for $(X,x) \in \underline{\text{Top}}$, $(a,b) \in T_1x$ and $a \neq b$ implies that the subspace $\{a,b\}$ of (X,x) is either of the form (D, d_1) or (D,d_i).

So in either case we see, by using the inclusion map $\{a,b\} \rightarrowtail (X,x)$ that $(a,b) \in Ex$. Thus we have $T_1 \leqslant E$. □

II.4 T_2-spaces.

We have already shown that T_2^q is neither strong nor productive. It is also not weakly hereditary (consider $\mathbb{N}$ with the cofinite topology) but nevertheless T_2^q is the least structured equivalence relation associated with the T_2-spaces. To prove this we first need the following result.

II.4.1. Lemma:

Let $(X,x) \in \underline{\text{Top}}$.

If $T_2^q x - \Delta_X$ is finite then $T_2^r(X,x)$ is a T_2-space.

Proof:

Let $a, b \in X$ and $(a,b) \notin T_2^q x$. We need only to find a pair of T_2^q-saturated (see II.2.3d) disjoint open neighbourhoods of a and b.

Let $Y = \{ y \in X \mid \text{there is a } z \in X - y \text{ with } (y,z) \in T_2^q \} \cup \{a,b\}$.

By the hypothesis, Y is finite.

Let $A = \{ y \in X \mid (a,y) \in T_2^q x \}$ and

$B = \{ y \in X \mid (b,y) \in T_2^q x \}$.

Then $A \cap B = \emptyset$ and A, B are both finite subsets of Y.

Consider any point $c \in A$. For each $y \in Y - A$ there is a pair of disjoint open neighbourhoods of c and y, say U_c^y and V_c^y.

Let $U_c = \bigcap_{y \in Y-A} U_c^y$ and $V_c = \bigcup_{y \in Y-A} V_c^y$

Then U_c, V_c are disjoint, open and $c \in U_c$, $B \subset Y - A \subset V_c$, $U_c \cap (Y-A) = \emptyset$.

Let $U = \bigcup_{c \in A} U_c$ and $V = \bigcap_{c \in A} V_c$.

Then $A \subset U$, $B \subset V$, $U \cap V = \emptyset$, $U \cap (Y-A) = \emptyset$ and U, V are both open.

In a similar manner we obtain open sets U' and V' such that

$A \subset U'$, $B \subset V'$, $U' \cap V' = \emptyset$ and $V' \cap (Y-B) = \emptyset$.

Let $U'' = U \cap U'$ and $V'' = V \cap V'$. Then U'' and V'' are the required disjoint, T_2^q-saturated open neighbourhoods of a, b. □

II.4.2 Proposition:

T_2^q is the least structured equivalence relation associated with the T_2-spaces.

Proof:

Suppose that E is a structured equivalence relation associated with the T_2-spaces. Let $(X,x) \in$ Top and $(a_1, a_2) \in \overset{q}{T_2}x$. We need to show that $(a_1, a_2) \in Ex$.

We can assume that (i) $a_1 \neq a_2$,

(ii) a_1, a_2 do not have disjoint open neighbourhoods,

(iii) The subspace $\{a_1, a_2\}$ of (X,x) is discrete.

For $i = 1, 2$, let $A_i = \{a \in X-\{a_1, a_2\} \mid$ for all U open in (X,x), $a_i \in U \Longrightarrow a \in U\}$.

Case I: $A_1 \cap A_2 \neq \emptyset$.

Let $a \in A_1 \cap A_2$. It follows that for $i = 1,2$ the subspace $\{a_i, a\}$ is not discrete, that $(a_i, a) \in E(\text{subspace } \{a_i,a\})$ and so $(a_i,a) \in Ex$. Thus $(a_1, a_2) \in Ex$.

Case II: $A_1 \cap A_2 = \emptyset$.

Let x' be the topology on X with subbase x together with the sets $\{a\}$, $X-\{a\}$ for all $a \in X-\{a_1, a_2\}$.

So $1:(X,x') \longrightarrow (X,x)$ is a map.

We now show that $\overset{q}{T_2}x' - \Delta_X = \{(a_1,a_2),(a_2,a_1)\}$ .... (*).

The inclusion $\subset$ is trivial.

If $(a_1,a_2) \notin \overset{q}{T_2}x'$ then there are disjoint x'-open neighbourhoods U_1, U_2 of a_1, a_2. Now by the definition of x', we can find x-open neighbourhoods V_1, V_2 of a_1, a_2 with $V_1 \cap V_2$ finite.

For $i = 1, 2$, let $F_i = (V_1 \cap V_2) \cap (X-A_i)$.

For each $a \varepsilon F_i$ we have $a \not\varepsilon A_i$ and so there is an x-open set U_i^a with $a_i \varepsilon U_i^a$, $a \not\varepsilon U_i^a$.

Let $W_i = (\bigcap_{a \varepsilon F_i} U_i^a) \cap V_i$ for $i = 1, 2$.

We have $W_1 \cap W_2 = \emptyset$, W_i is x-open and $a_i \varepsilon W_i$.

This contradicts (ii).

So we must have $(a_1,a_2) \varepsilon T_2^q x'$.

This completes the proof of (*).

Now by II.4.1, $T_2^r(X,x')$ is a T_2-space. So we have a map t which makes the following diagram commute.

$$\begin{tikzcd}
(X,x') \arrow[rr, two heads] \arrow[dr, two heads] & & E^r(X,x') \arrow[dl, "t"] \\
& T_2^r(X,x') &
\end{tikzcd}$$

It follows that $Ex' \leqslant T_2^q x'$.

Since (X,x') is not a T_2-space, $Ex' \neq 1$ and so we must have $Ex' = T_2^q x'$. Thus $(a_1,a_2) \varepsilon Ex'$ and it follows that $(a_1,a_2) \varepsilon Ex$. $\Box$

II.5 The lattice of structured relations.

The following Hasse diagram illustrates some of the order relationships between the various structured equivalence relations on Top which we have studied here.

The notation in the diagram is as follows:

(i) For E a structured relation E^r is the associated strong structured relation.

(ii) The order $\leqslant$ increases as we descend.

(iii) $\begin{matrix} E \\ + \\ F \end{matrix}$ means that no structured (equivalence) relation lies properly between E and F.

$$\begin{tikzcd}
 & & {} \arrow[d, "+" marking] & & & & \\
{} \arrow[dr, dashed, "?"] & & T_0 = T_0^r \arrow[dl] \arrow[dr] & & {} \arrow[dl, dashed, "?"'] & & \\
 & T_1^{-1} \arrow[dl, dashed] \arrow[dr] & & T_1 \arrow[dl] \arrow[dr, dashed] & & & \\
{} & & T_1 \vee T_1^{-1} \arrow[dl, dashed] \arrow[d] \arrow[rrr] & & {} & T_1^q \arrow[r] \arrow[d] & T_1^r \arrow[d] \\
 & {} & T_{SH} \arrow[d] \arrow[rrr] & & & T_{SH}^q \arrow[r] \arrow[d] & T_{SH}^r \arrow[d] \\
 & & T_2 \arrow[d] \arrow[rrr] & & & T_2^q \arrow[r] \arrow[d] & T_2^r \arrow[d] \\
{} \arrow[drr, dashed] & {} \arrow[r, dashed, "?"] & \bar{T}_n \arrow[d, dash] \arrow[rrr] & & & \bar{T}_n^q \arrow[r] & \bar{T}_n^r \\
 & & T_* = T_*^r \arrow[d, "+" marking] & & & & \\
 & & * & & & &
\end{tikzcd}$$

11.6 Open problems concerning structured relations (see 6.6.2)

(a) For $\underline{A}$ an extremal-epireflective subcategory of Top let $C_q(\underline{A})$ (resp. $C_r(\underline{A})$) denote the family of structured equivalence relations (resp. structured relations) associated with $\underline{A}$.

(i) Does $C_q(\underline{A})$ always have a least member? Equivalently, is $C_q(\underline{A})$ closed under infima?

(ii) Is $C_q(\underline{A})$ totally ordered by $\leq$?

(iii) Answer (i) for $\underline{A}$ = Urysohn spaces, completely Hausdorff spaces, T_*-spaces.

(iv) Is T_2 the least member of $C_r(T_2\text{-spaces})$?

(v) Is it true that : $C_r(\underline{A})$ (or $C_q(\underline{A})$ perhaps) has a unique member if and only if $\underline{A}$ is the extremal-epireflective (or perhaps just epireflective) hull of a single space?

(vi) Is the subcategory of T_0-spaces the only extremal-epireflective subcategory of Top which is the epireflective hull of a single space?

(b) Is 11.4.1 true for all structured equivalence relations on Top?

(c) Is every strong structured equivalence relation on Top productive?

11.7 On a problem of Wyler.

The setting is as follows:

Let $\underline{A}$ be a full subcategory of $\underline{\text{Top}}$ which contains at least one non-empty space, A_0 say.

Let $C_{\underline{A}}(X,Y)$ denote the set of maps $X \longrightarrow Y$ with the topology which has as subbase for open sets all sets of the form

$$W(u,V) = \{f:X \longrightarrow Y \mid u(A) \subset f^{-1}V\},$$

for all $A \varepsilon \underline{A}$, $u:A \longrightarrow X$ and V open in Y.

Let $\underline{B}$ be a full, replete epireflective subcategory of $\underline{\text{Top}}$.

Wyler's problem [1971c, 3.7.1] is: Give conditions on $\underline{B}$ which will ensure that $X, Y \varepsilon \underline{B} \Longrightarrow C_{\underline{A}}(X,Y) \varepsilon \underline{B}$.

Wyler shows that if $\underline{B} = T_0$, T_1, T_2 or T_3-spaces, then this is true.

We extend these results as follows:

11.7.1 Proposition:

Let $\underline{B}$ be an extremal-epireflective subcategory of $\underline{\text{Top}}$.

Then $Y \varepsilon \underline{B} \Longrightarrow C_{\underline{A}}(X,Y) \varepsilon \underline{B}$.

Proof:

Let E be the strong structured equivalence relation associated with $\underline{B}$. Let X^D denote the set X with the discrete topology and $e_{XY}: C_{\underline{A}}(X,Y) \times X^D \longrightarrow Y$ denote the evaluation function.

For $V \subset Y$ open, we have $e_{XY}^{-1}(V) = \bigcup_{x \varepsilon X} W(u_x, V) \times \{x\}$

where $u_x: A_0 \longrightarrow X$ is the constant map with value x.

Thus e_{XY} is continuous.

$$\begin{aligned} \text{Now}\quad (f,g) \;\varepsilon\; E(C_{\underline{A}}(X,Y)) &\Longrightarrow \text{for all } x\varepsilon X,\ ((f,x),(g,x)) \;\varepsilon\; E(C_{\underline{A}}(X,Y)\times X^{D}) \\ &\Longrightarrow \text{for all } x\varepsilon X,\ (fx,gx) \;\varepsilon\; Ey, \\ &\Longrightarrow \text{for all } x\varepsilon X,\ fx = gx \\ &\Longrightarrow \qquad f = g. \end{aligned}$$

Thus $E(C_{\underline{A}}(X,Y)) = I$ and so $C_{\underline{A}}(X,Y) \;\varepsilon\; \underline{B}$. □

II.8 On some problems of Thornton.

In the next part of this section we give solutions to some problems posed by M.C. Thornton [1971].

First we have a definition.

II.8.1 Definition:

Let $L:\underline{\mathrm{Top}} \longrightarrow \underline{\mathrm{Top}}$ be a structure functor. We define full subcategories of *Top* as follows:

$\underline{\mathrm{Top}}(\Delta\ L) = \{(X,x) \;\varepsilon\; \underline{\mathrm{Top}} \mid \Delta_X$ is closed in $(X\times X,\ L(x\times x)\}$.

$\underline{\mathrm{Top}}(\ker\ L) = \{(X,x) \;\varepsilon\; \underline{\mathrm{Top}} \mid$ for all f, g, $(Z,z) \;\varepsilon\; \underline{\mathrm{Top}}$ such that $f,g\colon (Z,z) \rightrightarrows (X,x)$, the set kernel$(f,g)$ is closed in (Z,Lz) $\}$.

$\underline{\mathrm{Top}}(c\ell\Delta,\ L) = \{(X,x) \;\varepsilon\; \underline{\mathrm{Top}} \mid (c\ell_{x\times x}\ \Delta_X)-\Delta_X$ is closed in $(X\times X,\ L(x\times x)\ \}$.

We also have a structured relation E^L on *Top* given by

$$E^L(X,x) = c\ell_{L(x\times x)}\Delta_X.$$

For $(X,x) \;\varepsilon\; \underline{\mathrm{Top}}$ and $A \subset X$, we say that A is *L-closed* in (X,x) if A is closed in the topology Lx.

11.8.2 Remarks:

(a) One verifies easily that: Top$(\Delta\ L)$ = Top(ker L) = Top$(E^L = 1)$ and it follows that these are extremal-epireflective subcategories of Top.

(b) Top$(c\ell\Delta,\ L)$ is closed under formation of open subspaces.

Proof: Let $f:(X,x) \rightarrowtail (Y,y)$ be an embedding in Top with $(Y,y) \in$ Top$(c\ell\Delta,\ L)$. For $(a,b) \in c\ell_{L(x \times x)}(c\ell_{x\times x}\Delta_X - \Delta_X)$ we have

$$(fa,fb) \in c\ell_{L(y\times y)}(f\times f)(c\ell_{x\times x}\Delta_X - \Delta_X)$$

and so

$$(fa,fb) \in c\ell_{L(y\times y)}(c\ell_{y\times y}\Delta_Y - \Delta_Y) = c\ell_{y\times y}\Delta_Y - \Delta_Y.$$

So $a \neq b$ and since f is an open embedding we have $(a,b) \in c\ell_{x\times x}\Delta_X$. So $c\ell_{x\times x}\Delta_X - \Delta_X$ is L-closed in $(X\times X,\ x\times x)$. □

Thornton [1971 problems 2 and 3] asks:

Which subcategories of Top are obtained by imposing closure restrictions on the diagonal and which subcategories are characterized by necessary conditions on the equalizers of maps into them?

We offer partial solutions to these problems.

11.8.3 Proposition:

Let A be a full subcategory of Top.

The following are equivalent:

(a) A is an extremal-epireflective subcategory of Top.

(b) There is a structure functor L such that Top$(\Delta\ L)$ = A.

(c) There is a structure functor L such that Top(ker L) = A.

Proof:

(a) $\Longrightarrow$ (b): For $(X,x) \in$ Top, let Lx be the topology on X with subbase for closed sets, the family of sets of the form kernel(f,g) for all $f,g: (X,x) \rightrightarrows (Z,z)$ with $(Z,z) \in$ A.

One checks easily that L is indeed a structure functor.

If $(X,x) \in$ A, then $(X,x) \in$ Top(Δ L) since Δ_X = kernel(p_1, p_2).

Conversely suppose that $(X,x) \in$ Top(Δ L).

Let $a, b \in X$ with $a \neq b$. Since Δ_X is L-closed there are maps $f, g:(X \times X, x \times x) \rightrightarrows (Z,z)$ with $(Z,z) \in$ A, $(a,b) \notin$ kernel(f,g) and $(a,a) \in$ kernel(f,g).

Define a map $F_{a,b}:(X,x) \longrightarrow (Z,z) \times (Z,z)$ by:

$$F_{a,b}(c) = (f(a,c), g(a,c)) \text{ for all } c \in X.$$

Now $F_{a,b}(a) \neq F_{a,b}(b)$, for if not then

$f(a,b) = f(a,a) = g(a,a) = g(a,b)$ which is not true.

Also $(Z,z) \times (Z,z) \in$ A.

Thus the family of maps of the form $F_{a,b}$ for $a, b \in X$, $a \neq b$, induces a monic on (X,x) with codomain in A. Now since A is closed under formation of subobjects it follows that $(X,x) \in$ A.

(b) $\Longleftrightarrow$ (c) $\Longrightarrow$(a) follows by II.8.2a. □

II.8.4 Remarks:

The structure functor L constructed in II.8.3 has the following properties:

(a) The associated structured equivalence relation E^L is strong and Top(E^L = I) = A.

(b) If L' is any other structure functor on Top such that $\mathrm{Top}(\Delta\, L') = \underline{A}$, then $L' \leqslant L$.

Proof:

(a) If $(a,b) \notin c\ell_{L(x\times x)}\, \Delta_X$ then as in the proof of 11.8.3 there is a map $F:(X,x) \longrightarrow (A,a)$ with $F(a) \neq F(b)$ and $(A,a) \in \underline{A}$. Conversely if we have such a map F, then $f(c,d) = F(c)$ and $g(c,d) = F(d)$ defines a pair of maps $f, g:(X\times X, x\times x) \rightrightarrows (A,a) \in \underline{A}$ such that $(a,b) \notin \mathrm{kernel}(f,g)$ and $\Delta_X \subset \mathrm{kernel}(f,g)$. It follows that $(a,b) \notin c\ell_{L(x\times x)}\Delta_X$. Thus E^L is the structured equivalence relation associated with $\underline{A}$ which is defined in 6.10 and by 6.10.1 it is strong.

(b) Let $(X,x) \in$ Top and $f, g:(X,x) \rightrightarrows (A,a) \in \underline{A}$.

We need to show that $\mathrm{kernel}(f,g)$ is L'-closed in X. This follows immediately from $\mathrm{kernel}(f,g) = \langle f,g\rangle^{-1}\, \Delta_A$ and the fact that Δ_A is L'-closed in $(A,a)\times(A,a)$. □

Thornton's problem 4 has the following rather trivial solution. The double construction is explained in Thornton's paper.

11.8.5 Proposition:

Let $\underline{A}$ be a subcategory of Top which is closed under finite products and subobjects.

Then for $(X,x) \in \underline{A}$ and $B \subset X$, the following are equivalent:

(a) $(X,x)_B$, the double of X along B belong to $\underline{A}$.

(b) B is the kernel of a pair of maps with codomain in $\underline{A}$.

Proof:

(a) $\Longrightarrow$(b): $B = \text{kernel}(\ell, r)$ where ℓ and r are the left and right-hand-side maps (see Thornton 1971 p3).

(b) $\Longrightarrow$(a): Let $k:(X,x)_B \longrightarrow (X,x)$ be the folding map [ibid p5].

Let $B = \text{kernel}(f,g)$, $f,g:(X,x) \rightrightarrows (A,a) \in \underline{A}$.

Then f, g induce a map $h: (X,x)_B \longrightarrow (A,a)$

and the map $\langle h,k\rangle:(X,x)_B \rightarrowtail (A,a) \times (X,x)$ is monic.

So $(X,x)_B \in \underline{A}$. □

II.8.6 Definition: Locally Hausdorff spaces.

A space $(X,x) \in \underline{\text{Top}}$ is said to be locally Hausdorff if every point of X has an x-neighbourhood which is itself a Hausdorff space.

We can now give the solution to the last part of problem 2 in Thornton's paper [1971].

II.8.7 Proposition:

$\underline{\text{Top}}(c\ell\Delta, I)$ = Locally Hausdorff spaces.

Proof:

Suppose that $(X,x) \in \underline{\text{Top}}(c\ell\Delta, 1)$. So $c\ell_{x \times x}\, \Delta_X - \Delta_X$ is closed in $(X \times X, x \times x)$. For each $x \in X$, $(x,x) \notin c\ell\, \Delta_X - \Delta_X$

and so there is an open set $U \subset X$ such that $x \in U$ and $U \times U \cap (c\ell\Delta_X - \Delta_X) = \emptyset$. We show that U is a Hausdorff space.

Let a, b $\in$ U with $a \neq b$. So $(a,b) \in U \times U$ and $(a,b) \notin \Delta_X$. Thus $(a,b) \notin c\ell\, \Delta_X$ and so there are open neighbourhoods V_a, V_b of a, b with $(V_a \times V_b) \cap \Delta_X = \emptyset$. So $V_a \cap V_b = \emptyset$ and $U \cap V_a$, $U \cap V_b$ are disjoint open neighbourhoods of a, b in U.

Conversely let (X,x) be a locally Hausdorff space.

Let $(a,b) \in X$ with $(a,b) \notin c\ell\, \Delta_X - \Delta_X$.

We have two cases:

Case I: $a = b$. Let U be an open Hausdorff neighbourhood of a. Then $U \times U \cap (c\ell\Delta_X - \Delta_X) = \emptyset$.

Case II: $a \neq b$. Since $(a, b) \notin c\ell\Delta_X$, a and b have disjoint open neighbourhoods V_a and V_b say. Now $(V_a \times V_b) \cap c\ell\, \Delta_X = \emptyset$ and so $(V_a \times V_b) \cap (c\ell\, \Delta_X - \Delta_X) = \emptyset$.

So in both cases we have shown that $(a,b) \notin c\ell(c\ell\, \Delta_X - \Delta_X)$. Thus $c\ell\, \Delta_X - \Delta_X$ is closed in $(X,x) \times (X,x)$. □

II.8.8 Remarks:

(a) Hausdorff $\Longrightarrow$ locally Hausdorff $\Longrightarrow T_1$.

The converses are not true. We give examples. Let U be a proper open subset of a Hausdorff space which is not closed. Then $(X,x)_U$, the double of X along U is locally Hausdorff but not Hausdorff. For the second implication, consider $\mathbb{N}$ with the cofinite topology.

(b) The subcategory of locally Hausdorff spaces is closed under finite products, arbitary sums and subobjects. If an infinite product is locally Hausdorff then all but finitely many of the factors are Hausdorff.

11.9 Problem: Reflective hulls in Top.

(1) Let A be an extremal-epireflective subcategory of Top. Suppose that the functor L constructed in 11.8.3 has the following two properties:

(a) $\underline{A} \subset \underline{Top}(L, \underline{Haus})$,

(b) the family of equaliser subsets kernel(f,g) of (X,x) (see 11.8.3 (a) $\Longrightarrow$ (b)) are closed under finite unions when $(X,x) \in \underline{A}$ and so form a topology.

Then:

(c) $\underline{A} = \underline{Top}(L, \underline{Haus})$,

(d) epi in A means L-dense,

(e) A is cowellpowered.

Proof:

(c) Let $(X,x) \in \underline{Top}(L, \underline{Haus})$. So (X, Lx) is a Hausdorff space and thus Δ_X is closed in $Lx \times Lx$. Also $L(x \times x) \leq Lx \times Lx$ and so Δ_X is closed in $L(x \times x)$. It follows that $(X,x) \in \underline{Top}(\Delta L) = \underline{A}$.

(d) Use 11.8.5.

(e) Let $(X,x) \varepsilon \underline{A}$ and Y be L-dense in (X,x). So (X,Lx) is a Hausdorff space and Y is dense in (X,Lx).

Now by the usual argument for Hausdorff spaces (see Herrlich 1968, 15.3.1(4)) we see that cardinal $(X) \leqslant 2^{2^{\text{cardinal}(Y)}}$.

(II) The following facts are easily verified:

(a) If $\underline{A}$ is a monoreflective subcategory of a cowellpowered category $\underline{B}$ then $\underline{A}$ is also cowellpowered.

(b) Every epireflective subcategory of Top is a monoreflective subcategory of its extremal-epireflective hull.

(III) Thus if the assumptions on L given in (I) are true for every extremal-epireflective subcategory $\underline{A}$ of Top, then every epireflective subcategory of Top is cowellpowered. Further, by theorems 7 and 8 in Baron [1969] we could then conclude that;

(a) every intersection of reflective subcategories of Top is a reflective subcategory and

(b) reflective hulls exist in Top.

This would give affirmative solutions to problems 2 and 3 in Herrlich's paper [1969b] in the case of Top.

We do not know of an epireflective subcategory of Top which is not cowellpowered. We have checked that the properties (a) and (b) of (I) do hold when $\underline{A}$ is the category of T_0, T_1, T_2, T_* or completely-Hausdorff spaces.

Appendix A: List of definitions and formulae.

1.1 : N, the terminal object.

1.7.1 : The property $\underline{Q1}$.

1.8 : (L): Lawvere's condition.

3.5 : The 'P' properties (P1), (P2), (P3), (P1)*, (P2)*, (P3)*.

3.10 : The top category $\underline{C}^{r\ell}$ of reflexive relations on objects of $\underline{C}$.

4.4 : $\underline{C}(F, \eta, \underline{A})$.

5.1.1 : The structure functors α and ω.

5.3 : $\underline{C}^s(L \leqslant L')$, $\underline{C}^s(L = L')$ for L and L' structure functors.

5.6 : $\underline{C}^s(L, \underline{A})$ for L a structure functor.

6.1 : $\underline{C}^s(E = 1)$ for E a structured relation.

6.4 : E^q, the structured equivalence relation associated with a structured relation.

6.5 : The functor E^r and the natural transformation ρ^E associated with a structured relation E.

E^α, the α'th power of the structured relation E.

7.1 : $\underline{C}^{P(\pi)}$ and $\underline{C}^{R(\pi)}$.

7.7 : The property $\underline{Q2}$.

9.1 : The 'T', 'R' and 'S' properties.

9.8 : The structured relations T_0, T_1 and T_2.

9.8.2(g): π-finite objects.

11.8.1 : $\underline{\text{Top}}(\Delta\, L)$, $\underline{\text{Top}}(\ker L)$, $\underline{\text{Top}}(c\ell\Delta, L)$ and E^L.

Appendix B: Summary of open problems.

4.3.2 : Are the concepts of 'strongly generating a reflection' and 'weakly generating a reflection' equivalent?

5.4.1 : Study the class of structure functors associated with a given monoreflective subcategory.

5.7.2 : Characterise those top categories which have a 'universal' extremal-epireflective subcategory.

6.6.2 : Various questions concerning the class of structured relations associated with an extremal-epireflective subcategory of a top category.

7.10.1 : Find 'nice' sufficient conditions which ensure that a functor T has the property:

$$T(f^{-1}f) = Tf^{-1}Tf \text{ for all regular-epis } f.$$

8.4 : Is the result 8.4 true for $\underline{C}$ a regular category?

8.6.2 : Study the functor $T^*:\underline{C}^{P(\pi)} \longrightarrow \underline{\text{Top}}$.

9.11.3 : Is xx^{-1} the closure of the diagonal for relational prealgebras of monads on a regular category?

§10 : Prove Manes' compactification theorem [1973 Theorem 8.4] with sets replaced by a regular category.

10.5 : Is the result 10.5 true for $\underline{C}$ a regular category?

11.6 : Problems concerning the structured relations on Top. (see also 6.6.2).

11.9 : Is every extremal-epireflective subcategory of Top co-wellpowered? Do reflective hulls exist in Top? Is Haus a 'universal' extremal-epireflective subcategory of Top?

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