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UNIFORM ALGEBRAS AND MEASURES

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CONTENTS

INTRODUCTION

The purpose of this work is to demonstrate the vital role played by measures in the study of Uniform Algebras. Most authors in the subject of Uniform Algebras are, essentially, Functional Analysts and this emphasis is apparent in their treatment of the subject. Usually the basic Measure Theoretic assumptions are mentioned at the outset and thereafter underplayed or taken for granted so that one is left with very little impression of the continuous role played by, for instance, the Borel conditions or the regularity condition, in the development of the subject. We here endeavour to present this aspect.

In the first chapter we introduce Uniform Algebras and some basic examples are dealt with. We proceed in the second chapter to provide the basic measure theoretic tools needed and to show the existence of representing measures and non-trivial annihilating measures for uniform algebras. The crucial theorems used here are the Hahn-Banach Theorem and, particularly, the Riesz Representation Theorem. Starting from the case of real, positive Borel measures and positive, definite linear functionals we extend directly to the case of complex Borel measures and bounded linear functionals. Here, especially, the importance of regularity is apparent and a substantial part is also played by Lusin's Theorem. Various results relating to representing measures and annihilating measures are proved. These are used extensively later on. Perhaps the most important of these is the Riesz-Banach result (2.11(c)(ii)) whose application is critical in the proving of many major approximation theorems. (e.g. of Bishop and Glicksberg.)

In chapter three we define peak sets and peak points and p-sets and p-points. We characterize p-sets in terms of annihilating measures and, in the case where the underlying space is metric, we characterize peak points in terms of their representing measures. It is precisely these characterizations which help us to exploit these concepts so usefully.

The main result of chapter four is Bishop's Theorem on Rational Approximation and, later, its extension to all T-invariant Uniform Algebras. Here we introduce the powerful idea of the Cauchy transform of an annihilating measure. It's role is seen to be almost ubiquitous in Rational and, indeed, all T-invariant approximation. Fubini's Theorem is needed in the development of this idea.

In chapter five we find that Gleason Parts, particularly, are characterized by representing measures. The Radon-Nikodym theorem is useful here. In considering Gleason parts of T-invariant uniform algebras we combine the measure theoretic aspects of peak points, Cauchy transforms and Gleason parts, to obtain a restatement of Bishop's theorem and a partition of the underlying space which is related to the representing measures of the various Gleason parts.

Although this is by no means an exhaustive work, we feel that it would be "incomplete" without mention of the generalized F. and M. Riesz Theorem. This is proved in chapter six and, as it's consequence, the Decomposition Theorem for Orthogonal measures. These results are then related to Gleason parts and used to extend a previous result about Gleason parts of T-invariant algebras.

In chapter seven we deal with Glicksberg's Theorem on closed Restrictions - another example of a major approximation theorem, the crux of whose proof is the Riesz-Banach result and in which the role of regularity is clearly illustrated.

A fairly comprehensive bibliography is provided. This is preceeded by several pages of notes which give a detailed reference to source material and some historical information. Any reference given should not be considered as the origin of the relevant results, unless explicitly stated as such. The bibliography could probably have been streamlined, particularly with regard to the textbooks. There are two reasons for this diversity; firstly because the given reference often contained the particular form of the result required and secondly because some of the texts played an important part in our study of background material for this subject and related topics.

We envisage that this work will be particularly useful to a student of Honours level whose measure theoretic background is similar to that of Bartle's "The Elements of Integration" [7] or the first seven chapters of Halmos' "Measure Theory". [8] .

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CHAPTER I

Uniform Algebras

1.1 We shall use the following notation throughout :

X is a compact, Hausdorff, topological space.

$C(X)$ is the set of all continuous, complex-valued functions on X .

$C_R(X)$ is the set of all continuous, real-valued functions on X .

1.2 We regard $C(X)$ as an algebra, being closed under pointwise addition, multiplication and scalar multiplication.

It is easily established that the relation :

$$\|f\| = \sup_{x \in X} |f(x)|$$

defines a norm on $C(X)$, called the uniform norm. We regard $C(X)$ as having the topology generated by this norm. Hence, as is well known, $C(X)$, and, indeed, any closed subalgebra of $C(X)$, becomes a Banach Algebra.

Note that, in this context, the terms "closed" and "uniformly closed" are synonymous.

1.3 Definition A is a Uniform algebra on X iff :

(i) A is a uniformly closed subalgebra of $C(X)$

(ii) A separates points of X ; i.e. $\forall x , y \in X$, $x \neq y$, $\exists f \in A : f(x) \neq f(y)$

(iii) A contains the constants, or, equivalently, $1 \in A$.

Note that Uniform Algebras coincide with Function Algebras, as defined by Browder [1].

1.4 Note also that the given topology on X is equal to the weak topology induced by A .

Proof Call these topologies $\mathcal{S}$ and $\mathcal{W}$ respectively. Clearly $\mathcal{S} \supseteq \mathcal{W}$. Thus the identity map, $i : (X,\mathcal{S}) \longrightarrow (X,\mathcal{W})$ is continuous. By condition (ii) in 1.3, $(X,\mathcal{W})$ is Hausdorff and, since $(X,\mathcal{S})$ is compact, i is a homeomorphism as required. (See [2], 0.12)

1.5 The following three examples of Uniform algebras will interest us :

Let K be a compact subset of $\mathbb{C}$:

$P(K)$ - the set of functions in $C(K)$ which can be approximated uniformly on K by polynomials.

$R(K)$ - the set of functions in $C(K)$ which can be approximated uniformly on K by rational functions with poles off K .

$A(K)$ - the set of functions in $C(K)$ which are analytic (or holomorphic) on the interior of K .

The relation : $P(K) \subseteq R(K) \subseteq A(K) \subseteq C(K)$ is trivial. Less trivial are the conditions under which strict inclusion or equality hold. Some results relating to these will be developed later.

Hereafter, K, unqualified, will denote a compact subset of $\mathbb{C}$.

1.6 We now investigate this relationship for some examples of K. In order to do this we need the following three well-known results, which we state without proof.

(a) The Stone-Weierstrass Theorem.

If A is a subalgebra of $C(X)$ with the properties :

(i) $1 \in A$

(ii) A separates points of X

(iii) A is closed under complex conjugation.

Then A is uniformly dense in $C(X)$. (See [3], 17. Thm. 1)

(b) The Maximum Modulus Theorem.

If f is analytic throughout a bounded domain D and continuous on the closure $\bar{D}$ and if M denotes the maximum value of $|f(z)|$ in $\bar{D}$, then, unless f is a constant function,

$$|f(z)| < M \text{ for every point } z \text{ in } D.$$

(See [4], section 54)

This means that any such f could only assume its maximum modulus on the topological boundary of D.

(c) If $f(z)$ is holomorphic throughout a domain D ,it is uniquely determined by its values on an arc in D .

(See [5], 7.68)

(d) We define : $\Delta = \{s \in \mathbb{C} : |s| \leqslant 1\}$, the closed unit disk

$\Gamma = \{s \in \mathbb{C} : |s| = 1\}$, the unit circle.

We refer to $P(\Delta)$ as the disk algebra on the disk

and to $P(\Gamma)$ as the disk algebra on the circle.

We maintain this notation as these examples will be used again.

(e) Taylor Series Expansion

Let $f(z)$ be holomorphic at all points within a circle C_0 with centre at z_0 . Then, at each point z inside C_0 :

$$f(z) = f(z_0) + \sum_{n=1}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$$

(See [5], 7.47)

During the course of the proof hereof it becomes clear that f is, in fact, uniformly approximated in C_0 by polynomials of the form :

$$f(z_0) + \sum_{n=1}^{m} \frac{f^{(n)}(z_0)}{n!} (z-z_0)^n$$

(f) If f is a continuous mapping from a compact metric space X into a metric space Y , then f is uniformly continuous on X. (See [5], 3.56)

(g) We now consider the space Δ , as defined above in (d). Firstly we show that $P(\Delta) = A(\Delta)$:

Take $f \in A(\Delta)$ and, for r st. $0 < r < 1$, define $f_r(z) = f(rz)$. Clearly each f_r is holomorphic in a neighbourhood of Δ and hence, by (e) , can be approximated uniformly on Δ by polynomials. i.e. $f_r \in P(\Delta)$. Now, by (f) , we know that f is uniformly continuous on Δ . Thus :

$$\forall \varepsilon > 0, \exists\, \delta > 0 \text{ st.} \forall z_1, z_2 \in \Delta , |z_1 - z_2| < \delta \Rightarrow |f(z_1) - f(z_2)| < \varepsilon$$

Now for any such ε and corresponding δ , choose r st. $1 - r < \delta$.

Then $|z - zr| = |1 - r||z| < \delta$, since $|z| \leqslant 1$ on Δ .

So $\forall \varepsilon > 0 , \exists r_1$ st. $\forall z \in \Delta , |f(z) - f_r(z)| < \varepsilon , \forall r$ st. $r_1 \leqslant r < 1$

So we see that $f_r \longrightarrow f$ uniformly on Δ as $r \longrightarrow 1^-$.

Thus $f \in P(\Delta)$ as required.

Secondly we show that $A(\Delta) \neq C(\Delta)$:

Consider $f(z) = |z|^2$. f is continuous at any point $z_0 \in \Delta$.

Let $|z - z_0| < \delta$ then :

$$\begin{aligned} |f(z) - f(z_0)| &= \big||z|^2 - |z_0|^2\big| = |z\bar{z} - z_0\bar{z}_0| \\ &= |\bar{z}(z - z_0) + z_0(\bar{z} - \bar{z}_0)| \\ &\leqslant |\bar{z}||z - z_0| + |z_0||\overline{z - z_0}| \\ &< 2\delta \end{aligned}$$

since $|z| \leqslant 1$ on Δ . Continuity follows easily.

However, f is holomorphic nowhere, except at zero :

We have : $\dfrac{f(z) - f(z_0)}{z - z_0} = \bar{z} + z_0 \dfrac{\overline{z - z_0}}{z - z_0}$

(set $z - z_0 = re^{i\theta}$) $= \bar{z} + z_0 e^{-2i\theta}$

Thus $\lim\limits_{z \to z_0} \dfrac{f(z) - f(z_0)}{z - z_0}$ depends on θ except when $z_0 = 0$.

So $f \in C(\Delta)$ but $f \notin A(\Delta)$.

Now for Δ we have established that :

$P(\Delta) = R(\Delta) = A(\Delta) \subsetneqq C(\Delta)$.

(h) Suppose that f_n is holomorphic in a domain D , $\forall_n$ and that the sequence $\{f_n(z)\}$ converges uniformly to a function $f(z)$ in D . Then $f(z)$ is holomorphic in D . Also $f_n'(z)$ converges uniformly to $f'(z)$ in D .

(See [6], Chap. 5, Thm.1)

(i) Now we consider the space Γ , defined in (d) .

Firstly we show that $R(\Gamma) = C(\Gamma)$:

Certainly $R(\Gamma)$ contains the constants and separates points of Γ . Let $f \in R(\Gamma)$. Then there are rational functions, f_n , with poles off Γ such that $f_n \longrightarrow f$ uniformly on Γ . It is trivial to check that $\overline{f}_n \longrightarrow \overline{f}$ uniformly on Γ . So it remains only to show that $\overline{f_n} \in R(\Gamma)$ for each n . In fact, we can show that $\overline{f_n}$ is a rational function with poles off Γ .

Let $f_n(z) = \dfrac{p(z)}{q(z)}$, where p, q are polynomials on Γ and q has zeroes off Γ .

Now $\overline{f_n(z)} = \dfrac{\overline{p(z)}}{\overline{q(z)}} = \dfrac{p_1(\overline{z})}{q_1(\overline{z})} = \dfrac{p_1(\frac{1}{z})}{q_1(\frac{1}{z})}$, since $\overline{z} = \dfrac{1}{z}$ on Γ .

Here $q_1(z) = \overline{q(\overline{z})} \neq 0$ on Γ , since $z \in \Gamma \iff \overline{z} \in \Gamma$. So $q_1(z)$ has zeroes off Γ . Say that $q_1(z)$ has zeroes at $z_1, z_2, \ldots z_n$, where $z_i \notin \Gamma$, $1 \leqslant i \leqslant n$. We see that the only possible poles of $\overline{f}_n$ would be $z = \dfrac{1}{z_i}$, $1 \leqslant i \leqslant n$. None of these points would be in Γ , so $\overline{f_n} \in R(\Gamma) \Rightarrow \overline{f} \in R(\Gamma)$. So $R(\Gamma)$ is closed under complex conjugation and satisfies the conditions of (a) . Hence $R(\Gamma) = C(\Gamma)$.

To investigate the relation between $P(\Gamma)$ and $R(\Gamma)$ we first establish a relation between $P(\Delta)$ and $P(\Gamma)$. Now (h) tells us that $f \in P(K) \Rightarrow f$ is holomorphic in a neighbourhood of K , for any compact subset, K , of $\mathbb{C}$. The maximum modulus principle, (b) , tells us that the restriction mapping from $P(\Delta)$ into $P(\Gamma)$ is an isometry. Furthermore, this isometry is one-to-one : Take f_1 , $f_2 \in P(\Delta)$ and let $f_1|_\Gamma = f_2|_\Gamma$. Now, by a previous comment, f_1 , f_2 are holomorphic on a neighbourhood of Δ , hence, by (c) , $f_1 = f_2$ on Δ .

Finally, we show that the restriction mapping is onto :
Let $g \in P(\Gamma)$; then there is a sequence of polynomials, f_n , which converge uniformly on Γ to g . Now (b) tells us that the f_n converge uniformly on Δ to some $f \in P(\Delta)$. Certainly $f|_\Gamma = g$. So we know that $P(\Gamma)$ is isometrically isomorphic to $P(\Delta)$.

We denote by $B(z_0,\varepsilon)$ the open disk, centre z_0 , radius ε . Clearly $\frac{1}{z} \in R(\Gamma)$ and , for any positive ε , $\frac{1}{z}$ is holomorphic in the domain, $D_1 = B(0,1+\varepsilon) \sim \{0\}$. Assume that $\frac{1}{z} \in P(\Gamma)$. Then there is a unique $f \in P(\Delta)$ such that $f|_\Gamma = \frac{1}{z}$ and for some $\varepsilon < 0$, f is holomorphic on $B(0,1+\varepsilon)$. Now, by (c), f coincides with $\frac{1}{z}$ on D_1 . This contradicts the continuity of f at zero. Thus $\frac{1}{z} \notin P(\Gamma)$, and we have established that :

$$P(\Gamma) \subsetneq R(\Gamma) = A(\Gamma) = C(\Gamma) \quad .$$

CHAPTER II

Representing Measures

2.1 Much of the material in the first few sections of this chapter is devoted to a comprehensive definition of terms. This is mainly for the sake of clarity, as many authors use these terms with different definitions, which are usually, but not always equivalent, depending often upon the limits of his field of work; e.g. "just positive measures" or "complex measures" etc.

We have dealt rather briefly with some of the measure theoretic results as we wish only to establish them for use in the treatment of the Riesz Representation Theorem. Where proofs are omitted and references given, the statement of the result can easily be seen to be valid in the contexts of both the source and this essay. Where we have used terms such as "clearly", "easily" or "trivially", they are used, we hope, meaningfully.

2.2 Definitions

(a) Signed Measure An extended, real-valued, countably additive set function, μ, on the σ-ring $\mathbb{S}$ of measurable sets of a measurable space $(Y,\mathbb{S})$, such that $\mu(\emptyset) = 0$ and μ assumes, at most, one of the values $+\infty$ or $-\infty$

Note The Jordan decomposition Theorem tells us that a signed measure, μ, may be expressed as $\mu = \mu^+ - \mu^-$, where μ^+ and μ^- are positive measures.

(See [7], 8.5)

(b) Complex Measure A set function $\mu = \mu_1 + i\mu_2$ where μ_1 and μ_2 are signed measures on some measurable space, $(Y,\mathbb{S})$.

(c) Borel Sets Let Y be a locally compact, Hausdorff space. Let $\mathbb{C}$ be the set of all compact subsets of Y; $\mathbb{S}$ the σ-ring generated by $\mathbb{C}$; and $\mathbb{U}$ the class of all open sets belonging to $\mathbb{S}$. We say that $\mathbb{S}$ is the family of Borel sets of Y.

(d) Borel Measure A positive measure, μ, defined on the class, $\mathbb{S}$, of Borel sets of Y such that $\mu(C) < \infty$, $\forall C \in \mathbb{C}$.

(e) Regular Measure Let μ be a positive Borel measure and $E \in \mathbb{S}$:-

$\mu(E) = \inf\{\mu(U) : E \subset U \in \mathbb{U}\} \Rightarrow E$ is outer regular w.r.t. μ.

$\mu(E) = \sup\{\mu(C) : E \supset C \in \mathbb{C}\} \Rightarrow E$ is inner regular w.r.t. μ.

E is regular if it is both inner and outer regular.

μ is regular if every $E \in \mathbb{S}$ is regular.

Note that if μ is a positive Borel measure, then $E_1, E_2 \in \mathbb{S} \wedge E_1 \subset E_2 \Rightarrow \mu(E_1) \leqslant \mu(E_2)$.

(f) Signed, regular, Borel Measure A signed measure whose positive and negative parts (see note in (a)) are regular Borel measures.

(g) Complex, regular, Borel Measure A complex measure whose real and imaginary parts are signed, regular, Borel measures.

2.3 In our context, where $Y = X$, a compact space, we have $X \in \mathbb{C}$. Thus $\mathbb{S}$ is a σ-algebra and $|\mu(X)| < \infty$ for any Borel measure, . Also, since every closed set in X is compact, we know that every closed set, hence also every open set, is in $\mathbb{S}$.

Hereafter, "measure", unqualified, shall mean "finite, complex, regular, Borel measure" .

We define the following :

(a) Probability Measure Positive measure of total mass 1 ; i.e. $\mu(X) = 1$.

(b) $|\mu|$; Positive total variation measure

Let $E \in \mathbb{S}$ and let π_E denote any finite, pairwise-disjoint collection $E_1, \ldots, E_n$ such that $E_i \in \mathbb{S} \wedge E_i \subset E$; $1 \leqslant i \leqslant n$.

$$\text{Then : } |\mu|(E) = \sup_{\pi_E} \left\{ \sum_{i=1}^{n} \left|\mu(E_i)\right| \right\}$$

(the supremum taken over all possible collections, π_E , with the required property) .

Observe that:

$$|\mu(E)| = \sup_{\pi_E} \left| \sum_{i=1}^{n} \mu(E_i) \right| \leq \sup_{\pi_E} \sum |\mu(E_i)| = |\mu|(E) .$$

(c) Here we should check that $|\mu|$ is, in fact, a positive measure.

Proof The non-negativity and the property that $|\mu|(\emptyset) = 0$ follow trivially from the definition. It remains to show countable additivity.

Let $\{A_n | n \in \mathbb{N}\}$ be any countable collection of mutually disjoint measurable sets. Let $\left\{E_k\right\}_{k=1}^{k=m}$ be a $\pi_{\cup A_n}$ as defined in (b).

Then, for every $n \in \mathbb{N}$, $\left\{E_k \cap A_n\right\}_{k=1}^{m}$ is a π_{A_n}, and

$$\sum_{k=1}^{m} | \mu(E_k) | = \sum_{k=1}^{m} \left| \sum_{n\in\mathbb{N}} \mu(E_k \cap A_n) \right| \text{ by countable additivity of } \mu$$

$$\leq \sum_{k=1}^{m} \sum_{n\in\mathbb{N}} | \mu(E_k \cap A_n) |$$

$$= \sum_{n\in\mathbb{N}} \sum_{k=1}^{m} | \mu(E_k \cap A_n) | \leq \sum_{n\in\mathbb{N}} |\mu|(A_n) .$$

Taking the supremum over all $\pi_{\cup A_n}$ we obtain :

$$|\mu|(\cup A_n) \leq \sum_{n\in\mathbb{N}} |\mu|(A_n) .$$

For the reverse inequality, choose real numbers, r_n, such that $r_n < |\mu|(A_n)$.

Then $\forall_n \in \mathbb{N}$, $\exists\ \pi_{A_n} = \left\{E_{nk}\right\}_{k=1}^{m_n}$ such that

$$r_n < \sum_{k=1}^{m_n} | \mu(E_{nk}) | .$$

Since $\bigcup_{n\in\mathbb{N}} \pi_{UA_n}$ is a π_{UA_n}, we have

$$\sum_{n\in\mathbb{N}} r_n < \sum_{n\in\mathbb{N}} \sum_{k=1}^{m_n} \left| \mu(E_{nk}) \right| \leq |\mu|(UA_n) .$$

Taking the supremum over all possible choices of $\{r_n\}$ on the left hand side, we obtain

$$\sum_{n\in\mathbb{N}} |\mu|(A_n) \leq |\mu|(UA_n) \quad \text{as required.}$$

(d) If μ is a signed measure, then $|\mu| = \mu^+ + \mu^-$.

<u>Proof</u> Firstly note the relation between a Hahn decomposition of X for μ and the Jordan decomposition of μ . (see [7], 8.2 - 8.5) .

Let A, B be a Hahn decompostion of X for μ . Then :

For $E \in \mathbb{S}$; $|\mu|(E) = \sup_{\pi_E}\left\{\sum_1^n\left(\left|\mu(E_k)\right|\right)\right\}$

$$= \sup_{\pi_E}\left\{\sum_1^n\left(\left|\mu(E_k \cap A) + \mu(E_k \cap B)\right|\right)\right\}$$

$$\leq \sup_{\pi_E}\left\{\sum_1^n\left[\mu^+(E_k) + \mu^-(E_k)\right]\right\}$$

since μ^+ and μ^- are positive measures

$$\leq \mu^+(E) + \mu^-(E) .$$

We see that equality holds if we consider $\pi_E = \{A \cap E , B \cap E\}$.

(e) $\|\mu\| = |\mu|(X)$: It is easily seen that this defines a norm. We check the triangle inequality :

$$\|\mu+\nu\| = |\mu+\nu|(X) = \sup_{\pi_X}\left(\sum_1^n\left|(\mu+\nu)(E_i)\right|\right)$$

$$= \sup_{\pi X}\left(\sum_1^n |\mu(E_i) + \nu(E_i)|\right)$$

$$\leq \sup_{\pi X}\left\{\sum_1^n\left(|\mu(E_i)| + |\nu(E_i)|\right)\right\}$$

$$\leq |\mu|(X) + |\nu|(X) = \|\mu\| + \|\nu\| .$$

(f) We show that : μ is a regular Borel measure $\Longleftrightarrow$ $|\mu|$ is a regular Borel measure.

<u>Proof</u> If μ is positive, this is trivial.

Necessity : If μ is a signed measure, then μ^+ and μ^- are regular Borel measures. Then, by (d), $|\mu| = \mu^+ + \mu^-$ is regular Borel.

If μ is complex, say $\mu = \mu_1 + i\mu_2$ where μ_1 and μ_2 are regular, signed Borel measures. From the definition of $|\mu|$ it is clear that $|\mu| \leq |\mu_1| + |\mu_2|$. Hence $|\mu|$ is Borel. Assume that $|\mu|$ is not regular. So there exists some Borel set E which is not outer regular (a similar argument may be applied to inner regularity, but is not really necessary, in view of [8], §52 F). Thus $\exists\ \varepsilon > 0$ such that $|\mu|(U \sim E) > \varepsilon$ for all U such that $E \subset U \in \mathbb{U}$. By the previous inequality, we have : $(|\mu_1| + |\mu_2|)(U \sim E) > \varepsilon$ for all such U . This contradicts the regularity of $|\mu_1| + |\mu_2|$.

Sufficiency : This last argument gives the sufficiency, using the obvious inequalities $\mu^+ \leq |\mu|$ for signed measures and $|\mu_1| \leq |\mu|$ for complex measures.

(g) δ_x ; the unit point mass at x : The measure defined by : -

$$\delta_x(E) = \begin{cases} 0 \ ; \ x \notin E \\ 1 \ ; \ x \in E \end{cases} .$$

(h) supp μ ; the support of μ : The complement of the union of all open sets U , such that $|\mu|(U) = 0$.

2.4 Until this stage we have used definitions found in Halmos [8] and Taylor [9] . However, we wish to refer to the treatment in Rudin [10] of the Riesz Representation Theorem. It is easily seen that in our case, i.e. X compact, the definitions of Borel set and Borel measure in Halmos and Rudin are equivalent (see [8] , §51,3.). We shall use the definition of measurability of a function given by Rudin :

(a) If X is a measurable space and Y is a topological space, and f is a mapping from X into Y , then f is said to be measurable provided that $f^{-1}(V)$ is a measurable set in X for every open set V in Y .

An immediate consequence of this definition is that every $f \in C(X)$ is Borel measurable. In view of [10] , Thm 1.12(a) , it is clear that, for $f \in C(X)$, this definition of measurability is equivalent to that of Halmos (see [8] , §18) . (Here we must bear in mind that $f = u+iv$ is measurable if and only if u and v are real measurable functions (see [10], 1.9)) .

We are now in a position to introduce the concept of the integral. The following definitions hold for a positive measure, μ :

(b) We define the integral, w.r.t. μ , of a simple, measurable function, φ , as follows :

$$\int \varphi d\mu = \sum_1^n \alpha_i \mu(E_i)$$

where $\varphi = \sum_1^n \alpha_i \chi_{E_i}$; the α_i being finite and distinct, the E_i disjoint. We say that such a φ is integrable.

(c) We say that a property is "almost everywhere (μ)" true if it is true on the whole space with the exception only of a set of μ-measure zero. (abbreviate : a.e.(μ)) .

(d) We say that a sequence, $\{f_n\}$, of a.e.(μ) finite-valued, measurable functions converges in measure (μ) to the measurable function f if

$$\forall \varepsilon > 0 \; ; \; \lim_{n\to\infty} \mu\Big(\{x : |f_n(x) - f(x)| \geqslant \varepsilon\}\Big) = 0$$

We briefly present a different aspect of this latter definition. Here we may consider "convergence in measure" in a topological, rather than measure theoretical, sense.

Firstly we associate with every positive regular Borel measure, μ , a topology, $\mathcal{J}_\mu$, on the class, $\mathcal{M}$, of measurable functions. We define a neighbourhood system as follows:

Let $f \in \mathcal{M}$. $U \subset \mathcal{M}$ is a neighbourhood of f if it contains a set of the form :

$$N_f(\delta) = \Big\{g \in \mathcal{M} \mid \mu\,\{x : f(x)-g(x) \neq 0\} < \delta\Big\} .$$

Let $\mathcal{N}_f$ be the family of neighbourhoods of f . It is easily shown that :

(i) $U \in \mathcal{N}_f \Rightarrow f \in U$

(ii) $U , V \in \mathcal{N}_f \Rightarrow U \cap V \in \mathcal{N}_f$

(iii) $U \in \mathcal{N}_f$ and $V \supset U \Rightarrow V \in \mathcal{N}_f$

(iv) $U \in \mathcal{N}_f \Rightarrow \exists\, W \subset U$ such that $W \in \mathcal{N}_f$ and $\forall_g \in W,\ W \in \mathcal{N}_g$.

[To see this, let $W = N_f(\delta)$ (by definition) . Take $g \in N_f(\delta)$, then $\mu\{x : f(x) - g(x) \neq 0\} = \delta_1 < \delta$. Then we have : $N_f(\delta) \supset N_g(\delta-\delta_1) \in \mathcal{N}_g$ as required .]

Now we have :

> The sequence, $\{f_n\}$, of a.e.(μ) finite-valued, measurable functions converges in measure (μ) to the measurable function f if and only if $\{f_n\}$ converges to f in terms of the topology $\mathcal{T}_\mu$.

<u>Proof</u> $\{f_n\}$ converges in measure (μ) to f

$\Longleftrightarrow \forall \varepsilon > 0, \lim_{n\to\infty} \mu(\{x : | f_n(x) - f(x) | \geqslant \varepsilon\}) = 0$

$\Longleftrightarrow \forall \varepsilon > 0, \forall \delta > 0, \exists\, N \in \mathbb{N}$ st. $n > N \Rightarrow \mu(\{x : |f_n(x)-f(x)| \geqslant \varepsilon\}) < \delta$

$\Longleftrightarrow \forall \delta > 0, \exists\, N \in \mathbb{N}$ st. $n > N \Rightarrow \mu(\{x : |f_n(x)-f(x)| \neq 0\}) < \delta$

$\Longleftrightarrow \forall \delta > 0, \exists\, N \in \mathbb{N}$ st. $n > N \Rightarrow f_n \in N_f(\delta)$

$\Longleftrightarrow \{f_n\}$ converges in terms of the topology, $\mathcal{T}_\mu$, to f .

(e) A sequence $\{f_n\}$ of integrable functions is mean fundamental (μ) if $\int | f_n - f_m | \, d\mu \longrightarrow 0$ as $m,n \longrightarrow \infty$.

(f) An a.e.(μ) finite-valued, measurable function, f , is integrable with respect to μ if there exists a mean fundamental (μ) sequence $\{f_n\}$ of integrable simple functions which converge in measure (μ) to f .

The integral of f w.r.t. μ is defined by

$$\int f d\mu = \lim_{n\to\infty} \int f_n d\mu \quad .$$

(g) It is easily seen that these definitions lead to the following characterization :

A complex-valued function, $f = f_1 + if_2$ is integrable (μ) if and only if its real and imaginary parts, f_1 and f_2 , are integrable (μ) .

Also, in general; $\int f d\mu = \int f_1 d\mu + i \int f_2 d\mu$.

(h) Another easy result is that if f is integrable w.r.t.μ and $\{f_n\}$ is the corresponding sequence fulfilling the conditions of (f) , then so does the sequence $\{|f_n|\}$ satisfy those conditions and $\{|f_n|\} \xrightarrow[\infty]{n} |f|$ which is then integrable w.r.t. μ . (see [10], 1.9(b)) .

(i) We state the following result without proof :
If f is measurable, g is integrable w.r.t.μ and $|f| \leq |g|$ a.e. (μ) , then f is integrable w.r.t.μ .
(see [8], §27.A)

Although this result is proved only for real-valued functions in Halmos, the characterization in (g) allows us to extend it easily to complex-valued functions.

An immediate consequence of this result is that every $f \in C(X)$ is integrable w.r.t. any positive Borel measure μ .

To see this we note that X is compact, hence every $f \in C(X)$ is bounded, and that certainly every constant function is integrable w.r.t. μ .

2.5 (a) Now we wish to extend these notions to signed measures, μ : Bearing in mind the following two facts, from 2.3(d) that $|\mu| = \mu^+ + \mu^-$ and that, in general, $\int f d(\mu+\nu) = \int f d\mu + \int f d\nu$, (This can be seen by looking firstly at characteristic functions, then extending to simple and integrable functions.), it can easily be seen that if f is integrable w.r.t. $|\mu|$ then it is also integrable w.r.t. μ^+ and μ^- and we can define :

$$\int f d\mu = \int f d\mu^+ - \int f d\mu^- .$$

(b) Finally we look at complex measures, $\mu = \mu_1 + i\mu_2$: It is easily seen that $|\mu_1| \leqslant |\mu|$ and that $|\mu_2| \leqslant |\mu|$. Hence it follows easily that if $f = f_1 + if_2$ is integrable wrt. $|\mu|$, then it is integrable wrt. $|\mu_1|$ and $|\mu_2|$, and we define :

$$\int f d\mu = \int f d\mu_1 + i \int f d\mu_2$$

$$= \int f_1 d\mu_1 - \int f_2 d\mu_2 + i \int f_2 d\mu_1 + i \int f_1 \, d\mu_2 .$$

(c) Now we wish to establish the following useful property :

$$\left| \int f d\mu \right| \leqslant \int |f| \, d \, |\mu| .$$

<u>Proof</u> Corresponding to f we have a sequence $\{f_n\}$ satisfying the conditions of 2.4(f) wrt. $|\mu|$. Each f_n is a simple function ;

$$\text{Let} \quad |f_n| = \left| \sum_{k=1}^{p_n} A_{k_n} \chi_{E_{k_n}} \right|$$

$$= \sum_{k=1}^{p_n} \left| A_{k_n} \right| \chi_{E_{k_n}}$$

since the E_{k_n} are disjoint .

$$\text{Now :} \int |f| \, d\,|\mu| = \lim_{n\to\infty} \int |f_n| \, d\,|\mu|$$

$$= \lim_{n\to\infty} \left(\sum_{k=1}^{p_n} |A_{k_n}| \; |\mu|(E_{k_n}) \right)$$

$$\geqslant \lim_{n\to\infty} \left(\sum_{k=1}^{p_n} |A_{k_n}| \; |\mu(E_{k_n})| \right)$$

$$\geqslant \lim_{n\to\infty} \left(\left| \sum_{k=1}^{p_n} A_{k_n} \mu(E_{k_n}) \right| \right)$$

$$\geqslant \lim_{n\to\infty} \left| \int f_n \, d\mu \right| = \left| \int f d\mu \right| \quad \text{as required}$$

This last step follows by breaking the integral into parts, as in (a) and (b) .

(d) We use (c) to establish a result which we shall use later :

$$|\mu|(X) = \sup_{|x|\leqslant 1} \left| \int_X x \, d\mu \right| ,$$ where the supremum is taken over all $|\mu|$-integrable functions, x, such that $|x(t)| \leqslant 1$, $\forall\ t \in X$.

<u>Proof</u> We have $|\mu|(X) = \sup_{\pi_X}\left(\sum_1^n |\mu(E_k)| \right) = \sup_{\pi_X}\left(\sum_1^n \left| \int_X \chi_{E_k} d\mu \right| \right)$

$$= \sup_{\pi_X} \left\{ \sum_1^n (A_k \int_X \chi_{E_k} d\mu) \right\}$$

for suitable A_k , we have $|A_k| = 1, \forall k$.

$$= \sup_{\pi_X} \left\{ \int_X \sum_1^n (A_k \chi_{E_k}) d\mu \right\}$$

$$= \sup_{\pi_X} \left\{ \left| \int_X \sum_1^n (A_k \chi_{E_k}) d\mu \right| \right\}$$

$$\leqslant \sup_{|x|\leqslant 1} \left| \int_X x \, d\mu \right| ,$$ where x is $|\mu|$-integrable.

On the other hand ; $\sup_{|x|\leqslant 1} \left| \int_X x \, d\mu \right| \leqslant \sup_{|x|\leqslant 1} \int_X |x| d|\mu|$ by (c)

$$\leq \int_X d|\mu| \text{ , since } |\mu| \text{ is a positive measure}$$

$$\leq |\mu|(X) \text{ as required.}$$

In fact, the same argument will give a more general result by substituting E for X, where E is measurable. (In fact, one would need a more general form of (c) as well. This, too, is easily shown by a similar argument.)

2.6

(a) We say that a linear functional, φ, on $C(X)$ is a positive linear functional if it has the property : $f \geq 0 \Rightarrow \varphi(f) \geq 0$.

(b) Now, by virtue of what appears in Rudin [10], 2.14 - 2.17 , we may state the Riesz Representation Theorem in the following form :

Let φ be a positive linear functional on $C(X)$, then there exists a unique, positive, regular Borel measure, μ , on X , which represents φ in the sense that :

$$\varphi(f) = \int_X f d\mu \quad ; \ \forall f \in C(X) .$$

with the property that $\|\varphi\| = \|\mu\|$.

(c) This latter property can be proved easily as follows :

$$\begin{aligned}\|\varphi\| &= \sup_{\|f\|\leq 1}\left\{|\varphi(f)|\right\} = \sup_{\|f\|\leq 1}\left(\left|\int_X f d\mu\right|\right)\\ &\leq \sup_{\|f\|\leq 1}\left(\int_X |f| d|\mu|\right) \quad \text{(by 2.5(c))}\\ &\leq \int_X d|\mu| = \|\mu\| \quad (\text{here } |\mu| = \mu)\end{aligned}$$

but for $f \equiv 1$, we have equality.

In fact, this tells us that any positive linear functional on $C(X)$ is bounded, hence continuous. This result holds, of course, because X is compact and because of property 2.2(d).

(d) Looking at a positive linear functional, φ, more closely, we see that $\varphi(f)$ is necessarily real-valued for all $f \in C_R(X)$. So we may conceivably generalize the statement of (b) in two directions : -

(i) To those linear functionals, φ, such that $\varphi(f) \in \mathbb{R}$, $\forall f \in C_R(X)$, but that property (a) does not necessarily hold ; and

(ii) To those linear functionals, φ, such that $\varphi(f) = \varphi_1(f) + i\varphi_2(f)$, $\forall f \in C(X)$, where φ_1, φ_2 belong to the class described in (i).

The question is : Can a statement similar to that of (b) be made for these wider classes of linear functionals ? The answer is : Yes, provided that we consider only those linear functionals that are continuous. (i.e. bounded in the sense that $\|\varphi\| = \sup_{\|f\| \leq 1} |\varphi(f)|$ is finite.) This is shown in the following sections.

<u>2.7</u> Firstly we look at those linear functionals mentioned in 2.6(d)(i).

(a) <u>Lemma</u> Let φ be a bounded linear functional on $C(X)$ such that $\varphi : C_R(X) \longrightarrow \mathbb{R}$. Then there exist two positive linear functionals, φ^+ and φ^-, such that $\varphi(f) = \varphi^+(f) - \varphi^-(f)$, $\forall f \in C(X)$.

Proof If $f \in C_R(X)$ and $f \geqslant 0$ define

$$\varphi^+(f) = \sup \{\varphi(g) : g \in C_R(X) \wedge 0 \leqslant g \leqslant f\}$$

(hence the necessity for the boundedness condition) .

It is clear that $\varphi^+(cf) = c\varphi^+(f)$, $\forall\, c \geqslant 0$.

If $0 \leqslant g_i \leqslant f_i$, then $\varphi(g_1) + \varphi(g_2) = \varphi(g_1 + g_2) \leqslant \varphi^+(f_1 + f_2)$.

Taking the suprema over all such g_i we have :

$$\varphi^+(f_1) + \varphi^+(f_2) \leqslant \varphi^+(f_1 + f_2) .$$

Conversely ; if $0 \leqslant h \leqslant f_1 + f_2$, let $g_1 = \sup(h-f_2 , 0)$ and let $g_2 = \inf(h,f_2)$. Then $g_1 + g_2 = h$ and $0 \leqslant g_i \leqslant f_i$.

Thus $\varphi(h) = \varphi(g_1) + \varphi(g_2) \leqslant \varphi^+(f_1) + \varphi^+(f_2)$.

Taking the supremum over all such $h \in C_R(X)$, we have :

$$\varphi^+(f_1 + f_2) \leqslant \varphi^+(f_1) + \varphi^+(f_2) .$$

Now let f be an arbitrary element of $C(X)$; say $f = f_1 + if_2$:

Then set : $\varphi^+(f) = \varphi^+(f_1) + i\varphi^+(f_2)$

$$= \varphi^+(f_1^+) - \varphi^+(f_1^-) + i\varphi^+(f_2^+) - i\varphi^+(f_2^-) ,$$

f_i^+ and f_i^- being the positive and negative parts, respectively, of f_i .

It is now trivial to show that φ^+ is, in fact, a positive linear functional . We now define : $\varphi^-(f) = \varphi^+(f) - \varphi(f)$, $\forall\, f \in C(X)$. Clearly φ^- is a linear functional and, if $f \geqslant 0$, then $\varphi^-(f) \geqslant 0$ since $\varphi^+(f) \geqslant \varphi(f)$ by definition. So φ^- is a positive linear functional and the lemma is established.

(cf. [7], 8.13)

We shall need the following topological results which we state without proof :

(b) Definition A set E is σ-compact if there exists a sequence, $\{C_n\}$, of compact sets such that $E = \bigcup_{n=1}^{\infty} C_n$.

(c) If $C \subset X$ is compact and $\mathcal{G}_\delta$, then $\exists f \in C_R(X)$ such that $C = \{x : f(x) = 0\}$. (see [8] , §50. c)

(d) If C is compact, U is open, and $C \subset U$, then there exist sets, C_0 , compact and $\mathcal{G}_\delta$, and U_0 open and σ-compact, such that

$$C \subset U_0 \subset C_0 \subset U .$$

(see [8], §50, D)

(e) Any σ-compact set is a Borel set. (trivial)

(f) We also need Lebesgue's Bounded Convergence Theorem :

If $\{f_n\}$ is a sequence of integrable, real-valued functions which converge to f a.e.(μ) and g is an integrable function such that $|f_n(x)| \leq |g(x)|$ a.e. (μ) , $\forall_n \in \mathbb{N}$; then f is integrable and $\int f d\mu = \lim_{n\to\infty} \int f_n d\mu$. (μ a positive measure) .

(see [8], §26. D.)

(Strictly speaking, we have used this result before ; e.g. to arrive at 2.4(i)) .

We wish, however, to use a more general form of this result : Firstly we wish to extend it to include complex-valued functions $\{f_n\}$; By considering real and imaginary parts of f_n and f we easily see that the same statement would hold, replacing "real" by "complex" .

Secondly, we wish to consider complex-valued measures ; $\mu = \mu_1 + i\mu_2$: Here we can express $\mu = \mu_1^+ - \mu_1^- + i\mu_2^+ - i\mu_2^-$ and, bearing in mind that each such component is less than $|\mu|$, it is easily seen that the following statement is true :

If $\{f_n\}$ is a sequence of $|\mu|$-integrable, complex-valued functions which converge to f a.e. $(|\mu|)$ and g is a $|\mu|$-integrable function such that $|f_n(x)| \leqslant |g(x)|$ a.e. $(|\mu|)$, $\forall n \in \mathbb{N}$; then f_n is $|\mu|$-integrable and $\int f d\mu = \lim_{n\to\infty} \int f_n d\mu$.

(g) Now, by virtue of 2.6(b) and with φ a bounded linear functional as in 2.6(d)(i), we have :

$$\varphi(f) = \varphi^+(f) - \varphi^-(f) \quad \text{(by (a))}$$

$$= \int f d\mu_1 - \int f d\mu_2 = \int f d\mu .$$

where $\mu = \mu_1 - \mu_2$ (μ_i a positive measure.)

So φ is represented by a signed, regular, Borel measure μ.

But is μ unique ? : Let λ be a signed, regular, Borel measure which represents φ. Then $\mu-\lambda$ is a signed, regular, Borel measure representing the zero functional ; i.e. $\int f d(\mu-\lambda) = 0$, $\forall f \in C(X)$.

Let $\mu-\lambda = \nu_1 - \nu_2$ where the ν_i are positive measures.

Now let C_0 be compact and $\mathcal{G}_\delta$ and let f be the function associated with C_0 as in (c). Clearly we may choose $0 \leqslant f \leqslant 1$. (Simply take $(f \vee 0) \wedge 1$). Form the sequence of functions $f_n = nf \wedge 1$. Clearly $\{f_n\}$ converges pointwise to $\chi_{X\sim C_0}$.

Now set $g_n = 1 - f_n$, then $\{g_n\} \longrightarrow \chi_{C_0}$, and, by (f), we have $0 = \lim_{n\to\infty} \int g_n d(\nu_1-\nu_2) = \int \chi_{C_0} d(\nu_1-\nu_2) = \nu_1(C_0) - \nu_2(C_0)$

$$\Rightarrow \nu_1(C_0) = \nu_2(C_0) \text{ for all compact and } \mathcal{G}_\delta \text{ sets, } C_0 .$$

Let C be any compact set. We have :

$\nu_1(C) = \inf\{\nu_1(U) : C \subset U \in \mathbb{U}\} = \otimes$, by the outer regularity of ν_1.

But, by (d) , there exists, for every such U , a compact and $\mathcal{G}_\delta$ set, C_0 , such that $C \subset C_0 \subset U$.
Similarly, for any compact and $\mathcal{G}_\delta$ set , $C_0 \supset C$, there exists by (d) and (e) an open Borel set $U \supset C_0$. Thus, by the monotonicity of ν_1 :

$$\begin{aligned} \otimes &= \inf\{\nu_1(C_0) : C \subset C_0 \text{ , compact and } \mathcal{G}_\delta\} \\ &= \inf\{\nu_2(C_0) : C \subset C_0 \text{ , compact and } \mathcal{G}_\delta\} \quad \text{by the above} \\ &= \nu_2(C) \quad \text{by the same argument .} \end{aligned}$$

Thus the ν_i coincide on compact sets

Now let E be an arbitrary Borel set. Then :

$$\begin{aligned} \nu_1(E) &= \sup\{\nu_1(C) : E \supset C \in \mathbb{C}\} \text{ by the inner regularity of } \nu_1 \\ &= \sup\{\nu_2(C) : E \supset C \in \mathbb{C}\} \quad \text{by the above} \\ &= \nu_2(E) \quad \text{by the inner regularity of } \nu_2 \text{ .} \end{aligned}$$

Thus $\nu_1 = \nu_2 \Rightarrow \mu = \lambda$. So we have uniqueness.

(h) In fact, we have shown more than just the uniqueness required. This argument also shows that for a positive linear functional φ , the positive measure representing φ is unique , not only as a positive measure, but as a signed measure as well.

(i) Secondly we look at the class of bounded linear functionals described in 2.6(d)(ii) . Let $\varphi = \varphi_1 + i\varphi_2$. We see that the boundedness condition tells us that φ_1 , φ_2 are bounded linear functionals from the class 2.6(d)(i) . So, by (g) we have :

$$\varphi(f) = \varphi_1(f) + i\varphi_2(f)$$
$$= \int f d\mu_1 + i \int f d\mu_2 = \int f d(\mu_1 + i\mu_2) \quad .$$

So $\mu = \mu_1 + i\mu_2$ represents φ, where μ_1 and μ_2 are unique signed measures. Since the decomposition of functions into real and imaginary parts is unique, we may say that μ is unique.

2.8

(a) Thirdly ; We must show that $\|\varphi\| = \|\mu\|$. We deal with both cases at once :

We shall need the following result, which we state, without proof, in simplified form for X compact :

<u>Lusin's Theorem</u> Suppose that f is a complex, measurable function on X and $\varepsilon > 0$. Then $\exists\, g \in C(X)$ such that

$$\mu\Big(\{x : f(x) \neq g(x)\}\Big) < \varepsilon \quad (\mu \text{ is a positive measure}) .$$

Furthermore, g may be chosen so that $\|g\| \leqslant \sup_{x \in X} \Big| f(x) \Big|$.

<u>Corollary</u> : Assume that the hypotheses of Lusin's theorem are met and that $|f| \leqslant 1$. Then there is a sequence $\{g_n\}$ such that $g_n \in C(X) \wedge |g_n| \leqslant 1$, $\forall n \in \mathbb{N}$ and :

$$f(x) = \lim_{n\to\infty} g_n(x) \text{ a.e. } (\mu) \quad (\mu \text{ a positive measure}) .$$

(see [10], 2.23)

Since this holds for all positive, regular, Borel measures, we may, in view of 2.3(b) and 2.3(f), say that this result holds, in a sense, for all complex-valued, regular Borel measures. (More specifically : In Lusin's theorem we may write $|\mu(\{x : f(x) \neq g(x)\})| < \varepsilon$ and in the Corollary we may write :

$$f(x) = \lim_{n\to\infty} g_n(x) \text{ a.e. } (|\mu|) \quad . \quad)$$

In view of the Lebesgue bounded convergence theorem (2.7(f)) we may say that : $\int f d\mu = \lim_{n\to\infty} \int g_n d\mu$; the f and g_n as in the above corollary .

Then, since any f which is $|\mu|$-integrable is measurable, we use this and 2.5(d) to get :

$$\|\mu\| = |\mu|(X) = \sup_{|f|\leq 1} \left| \int_X f d\mu \right| \quad , f \text{ is } |\mu|\text{-integrable}$$

$$= \sup_{|f|\leq 1} \left| \lim_{n\to\infty} \int_X g_n d\mu \right|$$

$$= \sup_{\|g\|\leq 1} \left| \int_X g d\mu \right| \quad , \quad g \in C(X)$$

$$= \sup_{\|g\|\leq 1} |\varphi(g)| = \|\varphi\| \quad \text{as required .}$$

(b) Since this last result is found independently of whether $|\mu|$ or φ are bounded, it gives us some extra information. Namely that, since X is compact, hence $|\mu|$ finite, it is clear that the boundedness condition on φ is necessary, and has not just been introduced by limitations in our method of proof.

(c) Thus we may state the Riesz Representation Thm. as follows :

Regarding the set of continuous linear functionals on $C(X)$ and the set of regular, Borel measures on X as symmetric Banach spaces (easily checked), we may say that they are isometrically *-isomorphic, the relationship being given by $\varphi(f) = \int f d\mu \ , \forall f \in C(X) \iff \varphi \leftrightarrow \mu$.

(d) One immediate and useful result of this is that if a regular Borel measure, μ , is such that $\int f d\mu = 0 \ , \forall f \in C_R(X)$, then $\mu \equiv 0$.

2.9 Clearly, if μ is a probability measure, the associated linear functional, φ, on $C(X)$ has the property : $\varphi(1) = \|\varphi\| = 1$.

Conversely ; if φ is a continuous linear functional on $C(X)$ such that $\varphi(1) = \|\varphi\| = 1$, then the associated measure is a probability measure.

Proof In view of 2.6(b) it will be sufficient to show that φ is a positive linear functional. Since, $\forall f \in C_R(X) \wedge f \geq 0$ we have $0 \leq \frac{f}{\|f\|} \leq 1$, it will be sufficient to show that $0 \leq f \leq 1 \Rightarrow \varphi(f) \geq 0$. Now let $\varphi(f) = a + ib$; a,b real . For each $t \in \mathbb{R}$, set $f_t = f + ibt$.

Then : $\|f_t\|^2 = \|f + ibt\|^2 = \left(\sup_{x\in X} | f(x) + ibt |\right)^2$

$$= \left(\sup_{x\in X} \sqrt{f(x)^2 + b^2t^2}\right)^2$$

$$= \left(\sqrt{\sup_{x\in X} f(x)^2 + b^2t^2}\right)^2$$

$$\leq \left(\sqrt{1 + b^2t^2}\right)^2 = 1 + b^2t^2$$

Thus $|\varphi(f_t)|^2 \leq \|\varphi\|^2\|f_t\|^2 \leq 1 + b^2t^2$

Hence : $a^2 + b^2(1+t)^2 = |a + ib + ibt|^2 = |\varphi(f_t)|^2 \leq 1 + b^2t^2$

Thus $\forall t \in \mathbb{R}$; $2tb^2 \leq 1 - a^2 - b^2 \Rightarrow b = 0$.

Now : $\|1 - f\| \leq 1 \Rightarrow |\varphi(1 - f)| \leq 1 \Rightarrow |1 - a| \leq 1 \Rightarrow a \geq 0$ as required .

2.10 Let R be a linear subspace of $C(X)$ and R^* the space of continuous linear functionals on R .

Definitions

(a) μ is a complex representing measure for $\varphi \in R^*$ if and only if

$$\int f d\mu = \varphi(f) \quad ; \forall f \in R .$$

(b) μ is a representing measure for φ if and only if

$$\int f d\mu = \varphi(f) \ , \forall f \in R \quad \text{and} \quad \|\varphi\| = \|\mu\| \ .$$

Clearly a complex representing measure is a representing measure if and only if $\|\varphi\| = \|\mu\|$.

2.11

(a) We state this famous theorem without proof :

Hahn-Banach Theorem

If E is a normed linear space and R any linear subspace of E , then any bounded linear functional, ℓ , on R may be extended to a linear functional L on E such that $\|\ell\|_R = \|L\|_E$.

(see [10] , 5.16 - 5.18)

(b) We state, also without proof, a result which is a consequence of the Hahn-Banach Theorem :

If E is a normed linear space and R any linear subspace of E , then $x_0 \in E$ is in the closure $\overline{R}$ of R if and only if there is no bounded linear functional L on E such that $L(x) = 0$, $\forall x \in R$ but $L(x_0) \neq 0$. (see [10], 5.19)

(c) Now let R be a linear subspace of $C(X)$. Then the Hahn-Banach and Riesz Representation theorems give us the following two results :

(i) Each $\varphi \in R^*$ has a representing measure.

Since φ can be extended with preservation of norm to $C(X)$, by the Hahn-Banach Theorem, and the resulting extension has a representing measure, by the Riesz Representation Theorem, which will clearly be a representing measure for φ as well.

<u>Definition</u> $R^{\perp} = \{\mu : \int g d\mu = 0 ; \forall g \in R\}$, the set of annihilating measures for R . For $\mu \in R^{\perp}$ we also write $\mu \perp R$.

(ii) $\overline{R} = \{f : f \in C(X) \wedge \int f d\mu = 0 , \forall \mu \in R^{\perp}\}$.

Bearing in mind that any $\mu \in R^{\perp}$ represents a linear functional which is zero on R , we see that (b) gives us this result immediately.

(d) Let $1 \in R$ and $\varphi \in R^*$ such that $\|\varphi\| = \varphi(1) = 1$. Clearly any Hahn-Banach extension of φ to $C(X)$ will also have this property. So, by the arguments in 2.9 , any representing measure for φ is a probability measure .

<u>2.12</u> Again let $1 \in R \subset C(X)$. With any $x \in X$ we may associate the evaluation functional , τ_x , defined by : $\tau_x(f) = f(x)$, $\forall f \in R$. Certainly $\tau_x \in R^*$ and $\tau_x(1) = 1$.

Also $\|\tau_x\| = \sup\limits_{R\sim\{0\}} \dfrac{|\tau_x(f)|}{\|f\|} = \sup\limits_{R\sim\{0\}} \dfrac{|f(x)|}{\|f\|} = 1$.

So any representing measure for τ_x is a probability measure . We say that μ is a representing measure for x if and only if μ is a representing measure for τ_x .

i.e. μ represents $x \iff \mu$ represents τ_x .

<u>2.13</u>
(a) Let f be $|\mu|$-integrable. Then the relation $g \to \int gfd\mu$ defines a linear functional φ . φ is bounded since :

$$\|\varphi\| = \sup_{\|g\|\leq 1} \left| \int gfd\mu \right|$$

$$\leq \int |f| \, d|\mu| \quad \text{by } 2.5(c)$$

$$< \infty \quad \text{since any integral is necessarily finite .}$$

Thus there is a regular, Borel measure, representing φ, which we denote by $f\mu$.

Consider the set function defined by $(f\mu)'(E) = \int_E fd\mu$ for any Borel set E . It is easily shown that $(f\mu)'$ is a measure . (See [7], 5.2; easily extended to arbitrary measures and complex functions.) Now we show that $(f\mu)'$ is finite, hence a Borel measure :

$$|(f\mu)'(E)| = \left| \int_E fd\mu \right| \leq \int_E |f| \, d|\mu| \text{ , by note after 2.5(d)}$$

$$\leq \int_X |f| \, d|\mu| < \infty \quad \text{by the above .}$$

Now we claim that $(f\mu)'$ represents φ . We have :

$$\int gd(f\mu)' = \lim_{n\to\infty} \int g_n d(f\mu)' \text{ ; where the } g_n \text{ are as the } f_n \text{ in } 2.4(f)$$

$$= \lim_{n\to\infty} \left\{ \sum_{k=1}^{P_n} A_{k_n} (f\mu)'(E_{k_n}) \right\}$$

$$= \lim_{n\to\infty} \left\{ \sum_{k=1}^{P_n} A_{k_n} \int_{E_{k_n}} fd\mu \right\}$$

$$= \lim_{n\to\infty} \int g_n fd\mu$$

$$= \int fgd\mu \text{ , since we may choose the } g_n \text{ st. } |g_n| \leq |g| \text{ and then by } 2.7(f) \text{ .}$$

It still remains to show that $(f\mu)'$ is regular.

Firstly note that
$$|(f\mu)'|(E) = \sup_{\pi_E} \left[\sum_1^n |(f\mu)'(E_n)| \right]$$

$$= \sup_{\pi_E} \left[\sum_1^n \left| \int_{E_n} fd\mu \right| \right]$$

$$\leq \sup_{\pi_E} \left[\sum_1^n \int_{E_n} |f| \, d|\mu| \right] \text{ by } 2.5(c)$$

$$\leq \int_E |f| \, d|\mu|$$

$$\leq (|f| \, |\mu|)'(E) \text{ , } \forall \text{ Borel sets, } E$$

Now we show that $|(f\mu)'|$ is regular. Assume that it is not. In view of [8] §52 F., we may assume that there exists a Borel set, E, which is not outer regular with respect to $|(f\mu)'|$. Thus $\exists\ \varepsilon > 0$ such that $|(f\mu)'|(U \sim E) > \varepsilon$, $\forall\, U$ such that $E \subset U \in \mathbb{U}$.

So we have $(|f||\mu|)'(U \sim E) > \varepsilon$, $\forall\ U$ st. $E \subset U \in \mathbb{U}$ (see above)

$$\Rightarrow \int_{U\sim E} |f| d|\mu| > \varepsilon, \forall U \text{ st. } E \subset U \in \mathbb{U} .$$

Now let $\{f_n\}$ be the sequence corresponding to f as in 2.4(f). It is easily seen that there exists $n \in \mathbb{N}$ such that

$$\int_{U\sim E} |f_n| \, d|\mu| > \varepsilon/2 \quad \forall\ U \text{ st. } E \subset U \in \mathbb{U} .$$

So we have $\sup_{x\in X} |f_n(x)| \ |\mu|(U\sim E) > \varepsilon/2 \quad \forall\ U$ st. $E \subset U \in \mathbb{U}$.

This contradicts the regularity of $|\mu|$ (by 2.3(f)).
Thus $|(f\mu)'|$ is regular. Again by 2.3(f), $(f\mu)'$ is regular.

So, by the uniqueness of 2.8(c) we have $f\mu = (f\mu)'$.

(b) Clearly, R is an algebra $\Rightarrow (f \in R \wedge \mu \perp R \Rightarrow f\mu \perp R)$

Conversely; $[R$ closed $\wedge\ (f \in R \wedge \mu \perp R \Rightarrow f\mu \perp R)] \Rightarrow R$ is an algebra. We show closure under multiplication: Take $f, g \in R$ and $\mu \in R^{\perp}$.
Then $\int gf d\mu = \int g d(f\mu) = 0 \Rightarrow gf \in R$ by 2.11(c)(ii)

2.14
(a) <u>Definition</u> Let R be a Banach Algebra with identity. Then $\varphi \in R^*$ is a multiplicative linear functional if $\varphi \not\equiv 0$ and $\varphi(fg) = \varphi(f).\varphi(g)$; $\forall f, g \in R$. Set of all such φ denoted by Spec R.

Clearly ; $\varphi \in \text{Spec } R \Rightarrow \varphi(1) = 1$ (since $\varphi(1.1) = \varphi(1).\varphi(1)$) .
It can also be shown that $\varphi \in \text{Spec } R \Rightarrow \|\varphi\| = 1$.
(see [1], L.1.2.9). Thus $\varphi \in \text{Spec } R \Rightarrow$ any representing measure for φ is a probability measure .

(b) It is clear that $\varphi \in \text{Spec } R$ is a homomorphism of R onto the field of complex numbers.
In the case where $R \subset C(X)$ we see, from [11] , §11,1, I and II that Spec R coincides with the maximal ideal space of R .

(c) From now on we shall mostly consider $R = A$, a uniform algebra on X and $\varphi \in M_A$, the maximal ideal space of A . All of the above remarks, referring to R , carry naturally over to A , and, in some cases, in view of the closure of A , may be immediately extended.

(d) Let μ be a representing measure for $\varphi \in M_A$.

$$\text{Then : } \left(f \in A \wedge \int f d\mu = 0\right) \Rightarrow f\mu \perp A .$$

(since $\int gfd\mu = \varphi(gf) = \varphi(g)\,\varphi(f) = \int gd\mu \,.\, \int fd\mu = 0$) .

2.15

(a) Let $\varphi \in M_A$. Denote by M_φ the set of representing measures for φ . By virtue of 12.11(d) , these are all probability measures.

(b) <u>Definition</u> Let E be a real or complex normed linear space. Consider the conjugate space E^* . For each $L \in E^*$, each finite subset $\{x_1 , \ldots , x_n\}$ of E and each $\varepsilon > 0$, define :

$$U(L , x_1,\ldots x_n;\varepsilon) = \{M \in E^*: |M(x_k) - L(x_k)| < \varepsilon \; ; \; 1 \leqslant k \leqslant n\}$$

These sets form an open basis for a topology for E^* , called the weak-* topology for E^* . It is the weakest topology on E^* for which each $x' \in E^{**}$ is continuous on E^* . The weak and weak-* topologies on E^* clearly coincide if E is reflexive .

(c) We state the following theorem of Alaoglu without proof :

Let E be a real or complex normed linear space. Then the closed unit ball , $S^* = \{L \in E^* : \|L\| \leq 1\}$ in E^* is compact in the weak-* topology of E^* .

(see [12] , B24, B25)

(d) M_φ is weak-* compact and convex .

<u>Proof</u> By 2.8(c) we have a 1-1 correspondence between $(C(X))^*$, the set of continuous linear functionals on $C(X)$ and the set $M(X)$ of all regular, Borel measures on X . Since each member of M_φ is a probability measure, we may regard M_φ as being a subset of S^* , the closed unit ball in $(C(X))^*$. By Alaoglu's Theorem, S^* is weak-* compact in $(C(X))^*$, so it remains to show that M_φ is a weak-* closed subset of S^* .

Let L be a limit point of M_φ , then for any $\varepsilon > 0$ and any finite set $\{f_i\}$ in $C(X)$, $U(L, f_1 , \ldots f_n; \varepsilon)$ (as defined in (b)) will contain members of M_φ . Now take a fixed $f \in C(X)$ and an arbitrary $\varepsilon > 0$. Then any neighbourhood $U(L, f, \varepsilon)$ will contain a member of M_φ , say M . So we have $|L(f) - M(f)| < \varepsilon$. Firstly, we may take $f \in A$ and , noting that $M(f) = \varphi(f)$, we can say that

$L(f) = \varphi(f)$ for $f \in A$.

Secondly, for an arbitrary $f \in C(X)$ we have $\big| |L(f)| - |M(f)| \big| < \varepsilon$. But $|M(f)| \leqslant \|f\|$ for any $M \in M_\varphi$. Thus $|L(f)| \leqslant \|f\| \Rightarrow \|L\| \leqslant 1$, but we have $L(1) = \varphi(1) = 1 \Rightarrow \|L\| = 1$ as required. Thus $L \in M_\varphi$ as required. The convexity of M_φ is trivially confirmed.

2.16 (a) It is clear from definitions 2.10 that M_φ coincides with the set of norm-preserving extensions of φ from A to C(X) and that the set of complex representing measures for φ coincides with the set of continuous extensions of φ from A to C(X) .

(b) Let $Re(A) = \{Re(f) : f \in A\}$. φ can be defined on Re(A) as follows :

$$\varphi(Re(f)) = Re(\varphi(f)) \quad ; \forall f \in A .$$

Then $\varphi(u) = \int u d\mu$ for all $u \in Re(A)$ and all $\mu \in M_\varphi$. (This is easily checked by looking at real and imaginary parts of functions in A). Thus φ is bounded, hence continuous, on Re(A) and $\varphi(u) \geqslant 0$ if $u \in Re(A)$ and $u \geqslant 0$ (since μ is positive) . From this, it follows easily that φ is monotone on Re A .

(c) Now let L be a monotone extension of φ from Re(A) to $C_R(X)$. Then : $L(v) \leqslant L(\|v\|) = \|v\|$, $\forall v \in C_R(X)$. Thus L is bounded, hence continuous. It is easily seen that L is norm-preserving (in fact $\|L\| = 1$) . By the Hahn-Banach Theorem, we may now extend L , with preservation of norm, to C(X) . It is easily checked that this extension of L is, in fact, a Hahn-Banach extension of φ from A to $C(X)$. (again, look at real and imaginary parts of functions in A.) Thus there is some $\mu \in M_\varphi$ which represents this extension of L .

Conversely, it is clear that any $\mu \in M_\varphi$ will represent a monotone extension of φ from $Re(A)$ to $C_R(X)$.

Thus M_φ coincides with the set of monotone extensions of φ from $Re(A)$ to $C_R(X)$.

(d) We state, without proof, a well-known result :

Zorn's Lemma If each chain in a partially ordered set has an upper bound, then there is a maximal element of the set.

(see [13], 1, 25.)

(e) Now let $v \in C_R(X)$ and $v \notin Re(A)$. Let B be the subspace of $C_R(X)$ spanned by v and $Re(A)$. We can extend φ monotonically to B by letting $\varphi(v) = c$, where c is any number such that :

$\sup\{\varphi(u) : u \in Re(A) \text{ and } u \leqslant v\} \leqslant c \leqslant \inf\{\varphi(u) : u \in Re(A) \wedge u \geqslant v\}$.

We partially order the set of monotone extensions of φ by :

$f_1 \prec f_2$ if f_2 is a monotone extension of f_1 .

Any chain $\{f_i\}$ in this set will have a maximum f , defined on $\bigcup_i \mathcal{D}(f_i)$, ($\mathcal{D}(f_i)$ = domain of f_i) , by $f(x) = f_i(x)$ if $x \in \mathcal{D}(f_i)$. So, by Zorn's Lemma, there is a maximal element in this set, which, by the first four lines of this section, extends φ to $C_R(X)$. Thus φ can be extended monotonically to $C_R(X)$ such that $\varphi(v) = c$. By virtue of our conclusions in (c) , for any v and c as above, there exists $\mu \in M_\varphi$ st. $\int v\,d\mu = c$.

(f) Let A be a uniform algebra on X and let $\varphi \in M_A$.

If $v \in C_R(X)$, then :

$$\sup\{\varphi(u) : u \in \mathrm{Re}(A) \text{ and } u \leqslant v\} = \inf\{\int v\,d\mu : \mu \in M_\varphi\}$$

$$\inf\{\varphi(u) : u \in \mathrm{Re}(A) \text{ and } u \geqslant v\} = \sup\{\int v\,d\mu : \mu \in M_\varphi\} .$$

Proof For the first equality : By (e) we can choose c equal to the left hand side and have some $\mu \in M_\varphi$ such that : $\varphi(u) = c = \int v\,d\mu$. (The equality is, of course, trivial if $v \in \mathrm{Re}(A)$). On the other hand, for any $u \in \mathrm{Re}(A)$, $v \in C_R(X)$ and $\mu \in M_\varphi$ we have $\varphi(u) = \int u\,d\mu \leqslant \int v\,d\mu$. This establishes the equality. The second is similarly proved.

2.17

(a) Definition Let μ and ν be measures. We say that ν is absolutely continuous with respect to μ if $|\mu|(E) = 0 \Rightarrow \nu(E) = 0$, for all Borel sets E. We denote this by : $\nu << \mu$.

(b) If f is $|\mu|$-integrable and ν is the representing measure for the linear functional defined by : $\varphi(g) = \int gf\,d\mu \;; \forall g \in C(X)$, then $\nu << \mu$.

Proof We know from 2.13(a) that ν has the property :

$$\nu(E) = \int_E f\,d\mu , \text{ for all Borel sets } E .$$

Now let $\{f_n\}$ be the sequence of $|\mu|$-integrable, simple functions of definition 2.4(f), such that

$fd|\mu| = \lim_{n\to\infty} \int f_n d|\mu|$. We note that the sequence $\{f_n\}$ also has the property $\int f\,d\mu = \lim_{n\to\infty} \int f_n d\mu$, and that, by similar arguments $\int_E f\,d\mu = \int \chi_E f\,d\mu = \lim_{n\to\infty} \int \chi_E f_n d\mu = \lim_{n\to\infty} \int_E f_n d\mu$.

(see note at end of 2.5(c)) . We also note that any simple function , ψ , can be written : $\psi = \alpha\psi'$, where α is the maximum of the moduli of the finite values which make up ψ, and ψ' is a simple function such that $|\psi'| \leq 1$.

Following from the note at the end of 2.5(d) , we can say that :

$$|\mu|(E) = \sup_{|x|\leq 1} \left|\int x d\mu\right| \ ; \ \forall \text{ Borel sets } E \ ; \ x \text{ is}$$

$|\mu|$-integrable.

So we have :

$|\mu|(E) = 0 \Rightarrow \int_E x d\mu = 0$; where x is $|\mu|$-integrable such that $|x| \leq 1$

$\Rightarrow \int_E f_n d\mu = 0$; f_n any $|\mu|$-integrable simple function.

$\Rightarrow \int_E f d\mu = \int_X \chi_E f d\mu = \lim_{n\to\infty} \int_X \chi_E f_n d\mu = \lim_{n\to\infty} \int_E f_n d\mu = 0$.

$\Rightarrow \nu(E) = \int_E f d\mu = 0$ as required.

(c) We define : $L^2(|\mu|) = \{f : f$ is measurable and $|f|^2$ is $|\mu|$-integrable$\}$.

Clearly f is $|\mu|$-integrable $\Rightarrow f \in L^2(|\mu|)$ and also :

$f \in L^2(|\mu|) \Rightarrow \bar{f} \in L^2(|\mu|)$; $\bar{f}$ being the complex conjugate of f .

We claim that $f , g \in L^2(|\mu|) \Rightarrow fg$ is $|\mu|$-integrable ;

This follows from the easily proven inequality :

$$ab \leq \frac{a^2}{2} + \frac{b^2}{2} \ , \ \forall \ a, b \geq 0 \ .$$

So we have $|fg| \leq \frac{|f|^2}{2} + \frac{|g|^2}{2}$. Hence, by 2.4(i) , fg is integrable with respect to $|\mu|$.

It is well-known that $L^2(|\mu|)$ is, in fact, an inner product space (see [12] , 12.8) . The inner product is given by : $\langle f,g\rangle = \int f\bar{g}d|\mu|$. By the claim above, this is well-defined, and the inner-product properties are easily checked .

(d) We state, without proof, a well-known result for inner product spaces :

<u>The Projection Theorem</u> If M is a closed subspace in the Hilbert space H and x is an arbitrary vector in H, then M contains a unique vector x' such that $x-x' \perp M$ i.e. $\langle x-x', y\rangle = 0$, $\forall\, y \in M$.

(see [11] §5, 2 II)

(e) Let A be a uniform algebra and let $\varphi \in M_A$. Let μ be a complex representing measure for φ. Then there is a representing measure for φ which is absolutely continuous with respect to μ.

<u>Proof</u> Let A_φ denote the kernel of φ. Let H and H_φ be the closures of A and A_φ respectively in $L^2(|\mu|)$. Here the closure is in terms of the metric topology on $L^2(|\mu|)$ generated by the norm given by $\|f\| = \left(\int |f|^2 d|\mu|\right)^{\frac{1}{2}}$.

Now if $f \in A_\varphi$, then $\int |1-f|^2 d|\mu| \geqslant \int (1-f)^2 d\mu$ (by 2.5(c))

$= 1 \Rightarrow 1 \notin H_\varphi \Rightarrow H_\varphi \neq H$.

We also claim that $A_\varphi H \subseteq H_\varphi$: Let $f \in A_\varphi$ and $g \in H$. Then there is a sequence $\{g_n\}$ in A which converges in the $L^2(|\mu|)$ norm to g. It is easily seen that the sequence $\{fg_n\}$ converges in the $L^2(|\mu|)$ norm to fg. But, since A_φ is an ideal in A, $fg_n \in A_\varphi$, $\forall\, n \in \mathbb{N}$. Thus $fg \in H_\varphi$ as claimed.

Now, by the projection theorem we may choose $F \in H$ and $F \notin H_\varphi$ such that $F \perp H_\varphi$. Multiplying by a constant if necessary, we may say that $\int |F|^2 d|\mu| = 1$. Thus $|F|^2|\mu|$ is a probability measure, which, by (b), is absolutely continuous with respect to $|\mu|$. Clearly, then, we also have $|F|^2|\mu| \ll \mu$.

It remains to show that $|F|^2|\mu|$ represents φ . Let $f \in A_\varphi$, then $fF \in H_\varphi \Rightarrow fF \perp F \Rightarrow \int fF\bar{F}d|\mu| = 0 \Rightarrow \int f|F|^2 d|\mu| = 0$. Now take any $h \in A$, then :

$$\begin{aligned}\int h|F|^2 d|\mu| &= \int[\varphi(h) + h - \varphi(h)]|F|^2 d|\mu| \\ &= \varphi(h) + \int[h - \varphi(h)]|F|^2 d|\mu| \\ &= \varphi(h) \text{ , since } h - \varphi(h) \in A_\varphi .\end{aligned}$$

Thus $|F|^2\ |\mu|$ is the required representing measure.

CHAPTER III

Peak Sets

3.1

(a) Definition A subset K of X is said to be a peak set for A, (a uniform algebra on X), if there exists $f \in A$ such that :

$$x \in K \Rightarrow f(x) = 1 \wedge x \notin K \Rightarrow |f(x)| < 1 .$$

Any such f is said to peak on K.

(b) We call K a p-set (or generalized peak set) if it is the intersection of a collection of peak sets.

$x \in X$ is called a peak point if the set $\{x\}$ is a peak set.

$x \in X$ is called a p-point if the set $\{x\}$ is a p-set.

(c) Note that a peak set K is a $\mathcal{G}_\delta$ set, since it is the countable intersection of a family of open sets, U_n, where :

$$U_n = \left\{x : |f(x)| > 1 - \frac{1}{n}\right\}$$

Also ; peak sets and p-sets are compact, being closed subsets of a compact space.

(d) A countable intersection of peak sets for A is a peak set for A. Clearly, if $f_n \in A$ peaks on E_n, then $f = \sum_{n=1}^{\infty} f_n/_{2^n}$ peaks on $\bigcap_1^\infty E_n$. Furthermore, since A is uniformly closed, $f \in A$ as required.

3.2

(a) Let E be a p-set and W a $\mathcal{G}_\delta$-set containing E. Then there is a peak set F such that : $E \subseteq F \subseteq W$.

<u>Proof</u> Let $W = \bigcap_{n=1}^{\infty} W_n$; W_n open $\forall n \in \mathbb{N}$. Clearly W_n', the complement of W_n, is closed, hence compact. Also $E \subseteq W_n$. Certainly the complements of the peak sets, whose intersection gives E , form an open covering for W_n' . By the compactness of W_n' , we may choose from this covering, a finite subcovering. Let $F_1, \ldots, F_m$ be the peak sets corresponding to this subcovering, then we have : $E \subseteq F_1 \cap F_2 \ldots \cap F_m \subseteq W_n$. Let $E_m = \bigcap_1^m F_i$, clearly a peak set. This argument holds $\forall n \in \mathbb{N}$. Thus $F = \bigcap_{m=1}^{\infty} E_m$ is the desired peak set (3.1(d)) .

(b) In particular, we note that ; If E is $\mathcal{G}_\delta$, then E is a peak set. This, together with the previous note (3.1(c)), gives

A p-set is a peak set if and only if it is $\mathcal{G}_\delta$.

(c) The concepts of peak set and p-set coincide when X is metrizable, since every closed set in a metric space is $\mathcal{G}_\delta$: E closed in X , then $x \in E \Leftrightarrow d(x,E) = 0$, d the metric on X . We construct a sequence of open sets, $D_n = \left\{x : d(x,E) < \frac{1}{n}\right\}$ $= \bigcup_{x \in E} B(x,\frac{1}{n})$, $n \in \mathbb{N}$ whose intersection is E.

<u>3.3</u>
(a) If B_0 is a closed subspace of a Banach space B , then $B/_{B_0}$ is a Banach space with the quotient norm given by :

$$\|x + B_0\| = \inf_{y \in B_0} \|x + y\| \ ; \quad x \in B .$$

B^* , the dual space of B , is the set of continuous linear functionals on B .

(see [11] §9, 4(iv))

(b) B_0^* is isometrically isomorphic to $B^*/_{B_0^\perp}$; $B_0^\perp$ being the set of functionals in B^* which annihilate B_0 .

$B^*/_{B_0^\perp}$ consists of equivalence classes of functions which coincide on B_0 . Since B_0 is closed, we may use the Hahn-Banach Theorem (see 2.11(a)) to extend any $f \in B_0^*$, with preservation of norm to $\bar{f} \in B^*$.

We define a relation $\alpha : B_0^* \longrightarrow B^*/_{B_0^\perp}$ by $\alpha(f) = \bar{f} + B_0^\perp$.

Since any two extensions of f will belong to the same equivalence class in $B^*/_{B_0^\perp}$, α is a function.

If f_1 , $f_2 \in B_0^* \wedge f_1 \neq f_2$ then $f_1 + B_0^\perp \neq \bar{f}_2 + B_0^\perp$, so α is one-one. Finally, it is clear that

$\|\bar{f} + B_0^\perp\| = \inf_{g \in B_0^\perp} \|\bar{f} + g\| = \|\bar{f}\|$, since g annihilates B_0 and also $0 \in B_0^\perp$. Since $\|f\| = \|\bar{f}\|$ (Hahn-Banach Thm.) , α is an isometry. (α is clearly onto, since $g \in B^* \Rightarrow g|_{B_0} \in B_0^*$.)

(c) If T is a bounded linear operator between Banach spaces E & F ($T:E \rightarrow F$) and $T(f) = g$, then we define the adjoint operator $T^* : F^* \longrightarrow E^*$ by $(T^*(\psi))(f) = \psi(g)$; $\forall \psi \in F^*$.

We note the following :

(i) $\|T\| = \|T^*\|$ (see [5], 5.44)

(ii) T norm-decreasing $\Rightarrow \|T\| \leqslant 1 \Rightarrow \|T^*\| \leqslant 1 \Rightarrow T^*$ is norm-decreasing.

(iii) If T is one-one and onto, then T^{-1} exists and, by the Open Mapping Theorem (see [5],5.41), is bounded .

(iv) If T is one-one and onto, so is T^* . Furthermore $T^{*-1} = T^{-1*}$.

The first claim is easily seen from the definition and by noting that if $\varphi \in E^*$ then $\varphi T^{-1} \in F^*$. For the second, let $T^*\psi = \varphi$, then we have :

$$(T^{-1}{}^*\varphi)(g) = \varphi(T^{-1}(g)) = \varphi(f) = \psi(g) = (T^{*-1}\varphi)(g) .$$

Since this is true for all pairs f, g and φ, ψ we have $T^{-1*} = T^{*-1}$ as required.

(d) Let B be a closed subspace of $C(X)$. Suppose E is a closed subset of X such that $\mu \in B^{\perp} \Rightarrow \mu_E \in B^{\perp}$ (μ_E is the restriction of μ to E). Then $B_{|E}$ is closed in $C(E)$. In fact, $B_{|E}$ is isometric to $B/_{I_E}$, where I_E is the subspace of functions in B which vanish on E.

<u>Proof</u> We have that I_E is the kernel of the restriction operator $B \rightarrow B_{|E}$. It is easily seen that this operator can be factored into two linear operators S and T as follows : $B \xrightarrow{S} B/_{I_E} \xrightarrow{T} B_{|E}$. Where S takes $f \in B$ to the equivalence class, of functions which coincide on E, containing f. T then takes this equivalence class to the coincident function, i.e. $f_{|E}$. It is clear that both S and T are surjective and that T is one-one. It is also clear, in view of (a), that both S and T are norm-decreasing. Hence $\|S\| \leqslant 1$ and $\|T\| \leqslant 1$. It remains to show that T is an isometry, i.e. also norm-preserving. We look at the adjoint operators of T and S :

$$\left(B_{|E}\right)^* \xrightarrow{T^*} \left(B/_{I_E}\right)^* \xrightarrow{S^*} B^* .$$

Now by (c)(i) we know that : $\|T^*\| = \|T\| \leqslant 1$ and $\|S^*\| = \|S\| \leqslant 1$. Thus S^* and T^* are both norm-decreasing, hence $S^* \circ T^*$ is norm-decreasing.

Now we show that $S^* \circ T^*$ is, in fact, norm-preserving :

Let $M(X)$ be the set of finite, regular, Borel measures on X, then by 2.8(c) and 3.3(b) we may express B^* as $C(X)^*/B^\perp = M(X)/B^\perp$ and $(B|_E)^*$ as $M(E)/(B|_E)^\perp$.

Now we can express $S^* \circ T^*$ explicitly in the form :

$$(S^* \circ T^*)\left(\mu + (B|_E)^\perp\right) = \mu + B^\perp \quad ; \quad \mu \in M(E) .$$

(Note we need only consider on the R.H.S. measures restricted to E since a functional in B^* takes on $f \in B$ the same value as the corresponding functional in $(B|_E)^*$ takes on $f|_E$.)

Now let $\nu \in M(X)$ and $\mu \in M(E)$, then

$\|\mu + \nu\| = \|\mu + \nu_E\| + \|\nu_{X\sim E}\|$.

To see this : R.H.S. $= |\mu + \nu_E|(E) + |\nu_{X\sim E}|(X\sim E)$

$$= \sup_{\pi_E}\left\{\sum_1^{n_1} |(\mu+\nu_E)(E_n)|\right\} + \sup_{\pi_{X\sim E}} \left\{\sum_1^{m_1} |\nu_{X\sim E}(F_m)|\right\}$$

$$= \sup_{\pi_X} \sum_1^{n_2} |(\mu + \nu_E + \nu_{X\sim E})(E_n)|$$

since E is compact, hence Borel measurable,

hence $\pi_X \supset \pi_E \cup \pi_{X\sim E}$

$= |\mu + \nu|(X) = \|\mu + \nu\| =$ L.H.S.

(also using the triangle inequality for norms.)

Now let $\mu \in M(E)$ and $\nu \in B^\perp$; By hypothesis :

$\nu_E \in B^\perp \Rightarrow \nu_E \in (B|_E)^\perp$

Thus : $\|\mu + (B|_E)^\perp\| \leq \|\mu + \nu_E\|$ by definition

$\leq \|\mu + \nu\| - \|\nu_{X\sim E}\|$, by the previous paragraph

$\leq \|\mu + \nu\|$.

Since this is true $\forall\ \nu \in B^\perp$ we have : $\|\mu+(B|_E)^\perp\| \leq \|\mu+B^\perp\|$.

The reverse inequality holds since $S^* \circ T^*$ is norm-decreasing.

Thus $S^* \circ T^*$ is norm-preserving, as claimed.

Now we know that T^* is norm-preserving and, by (c)(iv) is an isometry. By (c)(iii) and (iv) we know that T^{-1} and T^{*-1}

exist. Clearly T^{-1} is norm-increasing and T^{*-1} is an isometry. But $T^{*-1} = T^{-1*}$. Thus $\|T^{-1}\| = \|T^{-1*}\| = 1$. Since T^{-1} is norm increasing, it follows that T^{-1} is an isometry. Thus T is an isometry as required.

(e) Let B and E be as in (d). Let p be any positive continuous function on X and let $f \in B|_E$ be such that $|f(y)| \leqslant p(y)$, $\forall\, y \in E$. Then, given $\varepsilon > 0$, there exists $g \in B$ such that $g|_E = f$ and $|g(x)| \leqslant p(x) + \varepsilon$, $\forall\, x \in X$.

<u>Proof</u> Define $h/_p(x) = \dfrac{h(x)}{p(x)}$, $\forall\, x \in X$. Now let $B/_p$ be the subspace of functions of the form $h/_p$, $h \in B$. It is easily seen that $B/_p$ is closed. Clearly, also, $\nu \in (B/_p)^\perp$ if and only if $\nu/_p \in B/_p$ (cf.2.13).

Thus we have :

$$\nu \in (B/_p)^\perp \Rightarrow \nu/_p \in B^\perp \Rightarrow \nu/p|_E \in B^\perp (\text{hypothesis}) \Rightarrow \nu_E/_p \in B^\perp \Rightarrow \nu_E \in (B/_p)^\perp .$$

Thus $B/_p$ and E satisfy the requirements of (d) and we may say that $B/_p|_E$ is isometric to $B/_p \big/ I_E/_p$. So, $\forall\, \varepsilon > 0$, we may find a function $g \in B$ such that $g|_E = f$ and

$$\|g/_p\|_X \leqslant \|f/_p\|_E + \varepsilon/\|p\|_X \quad \ldots\ldots\ldots\ldots \quad (1)$$

(To see this, just look at the isometry, T, in (d)). Now for any $x \in X$, multiply through by $p(x)$ to get :

$$|g(x)| \leqslant \|g/_p\|_X\, p(x) \leqslant \|f/_p\|_E\, p(x) + \varepsilon \frac{p(x)}{\|p\|_X} \leqslant p(x) + \varepsilon .$$

Since this will hold for all $x \in X$, we have the required result.

(f) <u>Note</u> Since $B|_E$ is isometric to $B/_{I_E}$ (isometry, T, in (d)), we have : Given $\varepsilon > 0$ and $f \in B|_E$, there exists $g \in B$ such that $g|_E = f$ and $\|g\|_X \leqslant \|f\|_E + \varepsilon$, i.e. the results of (e) hold in the case $p \equiv 1$. In (e) we have, in effect, by arriving at the inequality (1), reduced the general case to the case $p \equiv 1$.

(g) Let B be a closed subspace of $C(X)$ and let E be a closed subset of X such that : $\mu \in B^{\perp} \Rightarrow \mu_E \in B^{\perp}$. Let $f \in B|_E$ and p a positive continuous function on X such that $|f(y)| \leq p(y)$, $\forall\, y \in E$. Then there exists $g \in B$ such that $g|_E = f$ and $|g(x)| \leq p(x)$, $\forall\, x \in X$.

Proof Proceeding as in (e) (cf. Note (f)), we see that we need only prove this result in the case $p \equiv 1$. Thus assume that $p \equiv 1$ and that $\|f\|_E \leq 1$. By (e), there exists $g_1 \in B$ such that $g_1|_E = f$ and $\|g_1\|_X \leq {}^5/_4$.

Claim : Given any open set U containing E we can find a continuous positive function $h(x)$ such that $0 \leq h(x) \leq 1$; $h(x) = 1$, $x \in E$; $h(x) = \frac{1}{2}, x \in X \sim U$. Note that X is compact Hausdorff, hence normal. So we have a Urysohn function $u(x)$ such that $0 \leq u(x) \leq 1$ and $u(x) = 1$, $x \in E$; $u(x) = 0$ $x \in X \sim U$. Then, clearly $h = u/2 + \frac{1}{2}$ will do. This establishes the claim. (see [11] §2, 8(I)).

Suppose $g_1 , \ldots , g_{n-1}$ are chosen so that $g_j|_E = f$, $1 \leq j \leq n - 1$.

Let $U_n = \{x \in X : |g_j(x)| \leq 1 + (\frac{1}{2})^{n+1};\ 1 \leq j \leq n-1\}$.

Note that $\forall_n \in \mathbb{N}$, U_n is an open set containing E and that $U_1 = X$. By the above claim and (e) , there exists $g_n \in B$ such that :

$g_n|_E = f$ and $\|g_n\|_X \leq 1 + (\frac{1}{2})^{n+1}$ and $|g_n(x)| \leq \frac{1}{2}$ for $x \notin U_n$. (just take h as a dominating function.)

Now let $g = \sum_{n=1}^{\infty} g_n/2^n$; Then $g \in B$, since B is closed under the uniform norm topology. Also $g|_E = f$.

Now take any $x \in X$. We know that $x \in U_n$, some $n \in \mathbb{N}$. (since $U_1 = X$.) Say $x \in U_n$ and $x \notin U_{n+1}$:

Then $|g_j(x)| \leqslant 1 + (\frac{1}{2})^{n+1}$ for $1 \leqslant j \leqslant n$,

while $|g_j(x)| \leqslant \frac{1}{2}$ for $j > n$.

$$\text{Thus } |g(x)| \leqslant (1 + (\tfrac{1}{2})^{n+1}) \sum_{j=1}^{n} \frac{1}{2^j} + \tfrac{1}{2} \sum_{j=n+1}^{\infty} \frac{1}{2^j}$$

$$\leqslant 1 - (\tfrac{1}{2})^{2n+1}$$

$$< 1$$

Say $x \in U_n$, $\forall n \in \mathbb{N}$, then $|g_n(x)| \leqslant 1$, $\forall n \in \mathbb{N}$, thus $|g(x)| \leqslant 1$. So we have $|g(x)| \leqslant 1$, $\forall x \in X$ as required.

(h) The results so far established in 3.3 have depended only on the linear structure of $C(X)$. We now relate these results to peak sets to get a characterization for p-sets:

<u>Glicksberg Peak Set Theorem</u>

Let A be a uniform algebra on X and E a closed subset of X. Then E is a p-set if and only if $\mu \in A^{\perp} \Rightarrow \mu_E \in A^{\perp}$.

<u>Proof</u> In order to show the sufficiency of the condition : $\mu \in A^{\perp} \Rightarrow \mu_E \in A^{\perp}$, we shall extend the function $1 \in A|_E$ to A in such a way that the resulting function has a peak set which contains E but is a proper subset of X. In fact, we shall construct a family of extensions whose corresponding peak sets have intersection equal to E. Thus E will be known to be a p-set. By the claim in (g), we can, for any $y \notin E$ and $y \in X$, find a positive continuous function, $p_y(x)$, such that $0 \leqslant p_y(x) \leqslant 1$, $\forall x \in X$; $p_y(x) = 1$, $\forall x \in E$ and $p_y(y) = \frac{1}{2}$. By (g), it is clear that the family of extensions corresponding to all such dominating functions will do.

Conversely : Let F be a peak set of A and let $f \in A$ peak on F. Now if $\mu \in A^{\perp}$ then $f^n\mu \in A^{\perp}$, $\forall\, n \in \mathbb{N}$ (see 2.13(b)). It is clear that $f^n \rightarrow \chi_F$ pointwise a.e. $(|\mu|)$ and that each f^n is dominated by f. In the same way, for any $g \in A$, we have : $f^n g \rightarrow \chi_F g$ pointwise a.e.$(|\mu|)$ and each $f^n g$ is dominated by g. So the requirements of Lebesgue's Bounded convergence theorem are satisfied (see 2.7(f)) and we may say :

$\int g d\mu_F = \int \chi_F g d\mu = \lim_{n\to\infty} \int f^n g d\mu = 0$, $\forall\, g \in A$. Thus $\mu_F \in A^{\perp}$.

Now suppose that E is a p-set. Let $\mu \in A^{\perp}$, then μ is regular, thus, by 2.3(f), $|\mu|$ is regular. By the outer regularity of $|\mu|$, we may find, for every $n \in \mathbb{N}$, an open set, U_n, such that $E \subset U_n$ and $|\mu|(U_n \sim E) < (\frac{1}{2})^n$. Clearly the set, $U = \bigcap_{n=1}^{\infty} U_n$ is $\mathcal{G}_\delta$ and contains E. Furthermore $|\mu|(U \sim E) = 0$. Now, by 3.2(a), we may find a peak set, F, such that $E \subseteq F \subseteq U$. So we have : $\mu_E = \mu_F \in A^{\perp}$ as required.

(i) If E and F are p-sets, then $E \cup F$ is a p-set. If E_j, $j \in \mathbb{N}$ are p-sets and $E = \bigcup_{n=1}^{\infty} E_n$ is closed, then E is a p-set.

<u>Proof</u> We note that $E \cap F$ is a p-set. Let $\mu \in A^{\perp}$, then, by (h), μ_E, μ_F and $\mu_{E\cap F}$ are in $A^{\perp}$. But clearly: $\mu_{E\cup F} = \mu_E + \mu_F - \mu_{E\cap F}$. Thus $\mu_{E\cup F} \in A^{\perp}$ and $E \cup F$ is a p-set (again by (h)).

For the second part, let $\mu \in A^{\perp}$. We note that $\mu_{E_1 \cup E_2 \cup \ldots \cup E_n} \in A^{\perp}$ for all $n \in \mathbb{N}$. Also $\chi_{E_1 \cup E_2 \cup \ldots \cup E_n} \rightarrow \chi_E$ pointwise a.e. $(|\mu|)$. So we use the Lebesgue dominated convergence theorem, as in (h), to get:

$\int g d\mu_E = \int \chi_E g d\mu = \lim_{n\to\infty} \int g \chi_{E_1 \cup E_2 \ldots \cup E_n} d\mu = 0 \ , \ \forall \ g \in A$.

Thus $\mu_E \in A^\perp$ as required.

(j) If E is a p-set of A and $F \subseteq E$ is a p-set of $A|_E$, then F is a p-set of A .

Proof We have : $\mu \in A^\perp \Rightarrow \mu_E \in A^\perp \Rightarrow \mu_E \in (A|_E)^\perp$
$\Rightarrow \mu_F \in (A|_E)^\perp \Rightarrow \mu_F \in A^\perp$, since $F \subseteq E$.

(k) We see, from the above, that p-sets behave like closed sets and hence generate the closed sets of a topology on X , which, since each p-set is closed in terms of the given topology, must be weaker than the given topology on X . By virtue of 1.4, it is clear that this topology is Hausdorff if and only if it is equal to the given topology. Furthermore, we claim :

The topology generated by the p-sets of A is equal to the given topology on X if and only if $A = C(X)$.

Proof For the necessity, we show that any closed subset of X is a p-set of $C(X)$. Let E be a closed set in X . By the claim in (g) , we can, for any $y \notin E \wedge y \in X$ find a positive continuous function, $P_y(x)$, such that $0 \leqslant P_y(x) \leqslant 1$, $\forall \ x \in X$ and $P_y(x) = 1$, $\forall \ x \in E$ and $P_y(y) = \frac{1}{2}$. Clearly E will be the intersection of the peak sets corresponding to all such functions. (i.e. $\forall \ y \notin E$) . [More simply, we could note that $\mu \perp C(X) \Rightarrow \mu \equiv 0$ (see 2.8(d)) .

Now, applying Glicksberg's Peak Set theorem, we see that every closed subset of X is a p-set .] For the sufficiency, Glicksberg's Peak set theorem tells us that :
$\mu \in A^{\perp} \Rightarrow \mu_E \in A^{\perp}$ for every closed set , E , in X . Thus every closed set in X , and, in particular, X itself, has μ-measure zero. Thus every open set in X has μ-measure zero. If μ is a complex measure and $\mu(U) = 0$ for some Borel set U , it is clear that the real and imaginary parts of μ also take value zero on U . So we need only consider the case of a signed measure, $\mu = \mu^+ - \mu^-$. By the outer regularity of μ^+ and μ^- we have : $\mu^+(U) = \mu^-(U)$ for any open set, $U \Rightarrow \mu^+(E) = \mu^-(E)$ for any Borel set E . Thus $\mu(E) = 0$ for any Borel set E . So the only measure which annihilates A is the zero measure. Thus by 2.11(c)(ii) , $A = C(X)$ as required.

3.4 We now consider the more specialized setting where X is a metrizable space.

(a) Let $x \in X$ and c, M constants such that $0 < c < 1$; $M \geq 1$ and with the property that : For all neighbourhoods, U , of x , there exists $f_u \in A$ such that $f_u(x) = 1$ and $\|f_u\|_X \leq M$ and $|f_u(y)| \leq c$, $\forall\, y \in X \sim U$. If such constants, c and M , exist, then x is a peak point of A .

Proof Choose $0 < s < 1$ sufficiently close to 1 so that: $M - 1 + s(c-M) < 0$. Choose a sequence $\{\varepsilon_n\}_{n=0}^{\infty}$ decreasing to zero, such that :

$$\varepsilon_{n-1}(1 - s^n) + s^n[M - 1 + s(c - M)] < 0 \ , \ \forall\, n \geq 1 \ .$$

Let $\{F_n\}_{n=0}^{\infty}$ be a sequence of closed subsets of X such that : $\bigcup_{n=0}^{\infty} F_n = X \sim \{x\}$. Such a sequence exists since :

X metric $\Rightarrow \{x\}$ is $\mathcal{G}_\delta$. (see 3.2(c)).

Choose a sequence of functions $\left\{h_n\right\}_{n=0}^{\infty}$ by induction, as follows : Take $h_0 \equiv 1$. Suppose that $h_0, \ldots, h_n$ have been chosen;

$$\text{Let} \quad W_n = \left\{y : \max_{1\leqslant j\leqslant n} |h_j(y)| \geqslant 1 + \varepsilon_n\right\} .$$

Clearly $W_0 = \emptyset$ and the W_n form an increasing sequence of closed sets. Now take $h_{n+1} \in A$ such that $h_{n+1}(x) = 1$ and $\|h_{n+1}\|_X \leqslant M$ and $\|h_{n+1}\|_{W_n \cup F_n} \leqslant c$. This we can do by hypothesis, since $X \sim W_n \cup F_n$ is an open set containing x , hence, a neighbourhood of x .

$$\text{Define :} \quad h = (1 - s) \sum_{j=0}^{\infty} s^j h_j$$

Then $h \in A$, since A is uniformly closed, and, clearly, $h(x) = 1$. Suppose $y \neq x$ does not belong to $\bigcup_{n=0}^{\infty} W_n$. Then $|h_j(y)| \leqslant 1$, $0 \leqslant j < \infty$. Also, $y \in F_j$, some j , thus $|h_{j+1}(y)| \leqslant c < 1$ for some j . Thus $|h(y)| < 1$. Now suppose that $y \neq x$ and $y \in \bigcup_{n=0}^{\infty} W_n$. So there is an index, m , such that $y \in W_m$ and $y \notin W_{m-1}$. Then the following inequalities hold:

$$|h_j(y)| \leqslant 1 + \varepsilon_{m-1} \; ; \quad 1 \leqslant j \leqslant m-1 \quad (\text{since } y \notin W_{m-1})$$

$$|h_m(y)| \leqslant M \quad ; \quad \text{by definition}$$

$$|h_j(y)| \leqslant c \quad ; \quad j > m \quad ; \quad \text{since } y \in W_m \cup F_m .$$

$$\text{Thus} \quad |h(y)| \leqslant (1 - s)\left\{(1+\varepsilon_{m-1}) \sum_{j=0}^{m-1} s^j + Ms^m + c \sum_{j=m+1}^{\infty} s^j\right\}$$

$$\leqslant 1 + \varepsilon_{m-1}(1 - s^m) + s^m[M - 1 + s(c - M)]$$

$$< 1$$

Thus $|h(y)| < 1$, $\forall\, y \neq x$. So we can say that h peaks at x .

(b) The set of peak points of A is a $\mathcal{G}_\delta$ set .

<u>Proof</u> Let d be a metric for X . Let U_n , $n \geq 1$, be the set of all $x \in X$ for which there exists a continuous function, $f_{xn} \in A$, such that $\|f_{xn}\| = 1$, $|f_{xn}(x)| > \frac{1}{2}$ and $|f_{xn}(y)| < \frac{1}{4}$ ⊛ when $d(x,y) \geq {}^1/n$.

We want to show that U_n is open, $\forall\, n \in \mathbb{N}$: Consider $x \in U_n$. Then the closed set, $D = \{y : d(x,y) \geq {}^1/n\}$ is a subset of the open set, $O = \{y : |f_{xn}(y)| < \frac{1}{4}\}$; So, centered at any point y such that $d(x,y) = {}^1/n$, we can construct an open ball contained in O . Now we form an open covering for D , which is compact, by taking all such open balls and the set $\{y : d(x,y) > {}^1/n\}$. Looking at the finite subcover, we choose the radius of the smallest open ball in it. Clearly the open ball, centered at x , with this radius, will satisfy ⊛ . Now the intersection of this open ball with the open set $\{x : |f_{xn}(x)| > \frac{1}{2}\}$ will give a neighbourhood of x contained in U_n , as required. (In fact, we have a neighbourhood of points, for each of which the function f_{xn} will do.)

Now to show that the set of peak points coincides with $\bigcap_{n=1}^{\infty} U_n$: Let x be a peak point. Let $f \in A$ be a function which peaks at x . Let U be any neighbourhood of x of the form $B(x, {}^1/n)$, $n \in \mathbb{N}$. Then $X \sim U$ is closed, hence compact. Now, since f attains the value $\|f\|_{X\sim U}$ on $X \sim U$ and $|f(x)| \neq 1$, $x \in X \sim U$, we have $\|f\|_{X\sim U} = c < 1$. It is easily seen that the function f , taken to an appropriate power, will serve to establish the membership of x in U_n . Since this argument holds for all $n \in \mathbb{N}$, we have : $x \in \bigcap_{n=1}^{\infty} U_n$ as required .

Conversely, say $x \in U_n$, $\forall\, n \in \mathbb{N}$. Consider any neighbourhood, U, of x. Then there exists $n \in \mathbb{N}$ such that the open ball $B(x, 1/n)$ is a subset of U. Now, referring to (a), take $M = 2$, $c = \frac{1}{2}$, then, with f_{xn} as described above, (since $x \in U_n$), it is clear that the function $f = \dfrac{f_{xn}}{f_{xn}(x)}$ will satisfy the conditions of (a). Thus x is a peak point as required.

(c) $x \in X$ is a peak point for A if and only if the point mass at x is the only representing measure for x. This occurs if and only if $\mu(\{x\}) = 1$ for every complex representing measure for x.

<u>Proof</u> We wish to prove :

(i) x is a peak point of A.

$\Longleftrightarrow$ (ii) δ_x is the only representing measure for x.

$\Longleftrightarrow$ (iii) $\mu(\{x\}) = 1$ for every complex repn. meas. for x.

We do this as follows : (i) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) .

(i) $\Rightarrow$ (iii) : Suppose that $f \in A$ peaks on x. Then f^n tends pointwise a.e. and boundedly ($|f^n| \leq 1$) to $\chi_{\{x\}}$ as $n \to \infty$. If μ is a complex representing measure for x, then : $1 = (f(x))^n = \int f^n d\mu \longrightarrow \mu(\{x\})$ (Lebesgue dominated convergence). Thus $\mu(\{x\}) = 1$ as required.

(iii) $\Rightarrow$ (ii): Trivial.

(ii) $\Rightarrow$ (i) : Suppose that δ_x is the only representing measure for x. Let U be any neighbourhood of x. Choose $v \in C_R(X)$ such that $v \leq 0$, $v(x) = 0$ and $v(y) < -2$ for $y \notin U$.

(This we can do by the claim in 3.3(g)).

Now, by 2.16(f), we have :

$$\sup\{\tau_x(u) : u \in Re(A) \text{ and } u \leqslant \nu\} = \inf\{\int \nu d\mu : \mu \in M_{\tau_x}\} .$$

$$\Rightarrow \sup\{u(x) : u \in Re(A) \text{ and } u \leqslant \nu\} = 0 .$$

Thus we can choose $g \in A$ such that $Re(g) \leqslant 0$; $Re(g(x)) > -1$ and $Re(g(y)) < -2$ for $y \notin U$. Then $f = e^{g-g(x)} \in A$, (since A is uniformly closed), and $f(x) = 1$; $|f(y)| \leqslant \frac{1}{e}$ for $y \notin U$ and $\|f\|_X = \sup_X |e^{g-g(x)}| \leqslant |e^{-g(x)}| \leqslant e$. Thus the conditions of (a) are satisfied with $c = \frac{1}{e}$ and $M = e$. So x is a peak point as required.

(d) $x \in X$ is a peak point for A if and only if $\mu(\{x\}) > 0$ for every complex representing measure for x .

Proof If x is a peak point of A , then, by (c) , we have, for all complex representing measures, μ , of x : $\mu(\{x\}) = 1 > 0$ as required. Conversely, say that x is not a peak point of A . Then, by (c) , there exists a representing measure, ν , for τ_x such that $\nu \neq \delta_x$. Since ν must be a probability measure we can say that $\nu(\{x\}) = c < 1$. Set $\mu = (1-c)^{-1}(\nu - c\delta_x)$. It is easily seen that μ is a representing measure for τ_x and that $\mu(\{x\}) = 0$.

CHAPTER IV

Bishop's Peak Point Theorem and T-invariance

4.1 We firstly produce the set of results which lead to Bishop's Peak point Theorem.

(a) Initially, some definitions and notation :

$C^{(K)}(U)$ - set of all complex-valued functions on U having continuous partial derivatives up to order K .

$C_c^{(K)}(U) = \{ f \in C^{(K)}(U) : f$ has compact support on U .$\}$

$$\frac{\partial f}{\partial z} = \tfrac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) \quad ; \quad \frac{\partial f}{\partial \bar{z}} = \tfrac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right).$$

Denote $\frac{\partial f}{\partial z}$ by f_z and $\frac{\partial f}{\partial \bar{z}}$ by $f_{\bar{z}}$.

Note that : f is holomorphic $\Longleftrightarrow f \in C^{(1)}(U) \wedge \frac{\partial f}{\partial \bar{z}} = 0$ in U .
(Look at the Cauchy-Riemann condition; see [5] , 7.5) .

Definition : Let X be a compact set in $\mathbb{C}$ and μ a measure on X . For all $w \in \mathbb{C}$ we set : $\tilde{\mu}(w) = \int_X \frac{d|\mu|}{|w-z|}$

Definition : Let X be a compact set in $\mathbb{C}$ and μ a measure on X . For each $w \in \mathbb{C}$ such that $\tilde{\mu}(w) < \infty$ define;

$$\hat{\mu}(w) = \int_X \frac{d\mu}{(z-w)}$$

We denote Lebesgue two-dimensional measure in $\mathbb{C}$ by m .

i.e. $\int dx\, dy \equiv \int dm$

(b) We state without proof a well-known result which we need :

<u>Green's Theorem</u> If two functions $P(x,y)$ and $Q(x,y)$, together with their first-order partial derivatives are continuous in a region R of $\mathbb{C}$ bounded by a closed curve C (where R includes C) ,

then : $\int_C (Pdx + Qdy) = \int_R \int \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dm$

(c) Let $\mathcal{G}$ be a bounded plane domain with smooth positively oriented boundary γ . Let $f \in C^{(1)}(U)$, where U is a neighbourhood of $\overline{\mathcal{G}}$.

Then for all $w \in \mathcal{G}$;

$$f(w) = \frac{1}{2\pi i} \int_\gamma \frac{f dz}{z-w} - \frac{1}{\pi} \int_{\mathcal{G}} \frac{f_{\bar{z}}}{z-w} dm \quad .$$

<u>Proof</u> Choose $\varepsilon > 0$ small enough so that $\overline{B(w,\varepsilon)} \subset \mathcal{G}$ ($B(w,\varepsilon)$ - the open ball, centred at w , radius ε) .

Let $\mathcal{G}_\varepsilon = \mathcal{G} \sim \overline{B(w,\varepsilon)}$. Then $\mathcal{G}_\varepsilon$ is a smoothly bounded domain with boundary $\gamma - \gamma_\varepsilon$, where γ_ε denotes the circle $\{ s : |s - w| = \varepsilon \}$ with positive orientation . Let $g = \frac{f}{z-w}$.

Then $g \in C^{(1)}(U_1)$, where U_1 is a neighbourhood of $\mathcal{G}_\varepsilon$.

$$\text{Also } g_{\bar{z}} = \tfrac{1}{2}\left(\frac{\partial g}{\partial x} + i\frac{\partial g}{\partial y}\right) = \tfrac{1}{2}\left(\frac{(z-w)\frac{\partial f}{\partial x} - f\left[\frac{\partial z}{\partial x}\right]}{(z-w)^2} + i\frac{(z-w)\frac{\partial f}{\partial y} - f\left[\frac{\partial z}{\partial y}\right]}{(z-w)^2}\right)$$

$$= \frac{f_{\bar{z}}}{z-w} - \frac{f}{(z-w)^2}\, \tfrac{1}{2}\left(\frac{\partial z}{\partial x} + i\frac{\partial z}{\partial y}\right) = \frac{f_{\bar{z}}}{z-w}$$

Now we have: $\int_{\gamma-\gamma_\varepsilon} g\,dz = \int_{\gamma-\gamma_\varepsilon} (g_1+ig_2)(dx + idy)$

$$= \int_{\gamma-\gamma_\varepsilon} \left[(g_1 + ig_2)dx + (ig_1 - g_2)dy\right]$$

$$= \iint_{\mathcal{G}_\varepsilon} \left[\partial/_{\partial x}(ig_1 - g_2) - \partial/_{\partial y}(g_1+ig_2)\right] dx\,dy \quad \text{(by (b))}$$

$$= \int_{\mathcal{G}_\varepsilon} \left[i\frac{\partial g}{\partial x} - \frac{\partial g}{\partial y}\right] dm$$

$$= 2i\int_{\mathcal{G}_\varepsilon} g_{\bar z}\, dm$$

$$= 2i\int_{\mathcal{G}_\varepsilon} \frac{f_{\bar z}}{z-w}\, dm \qquad \text{(by the above)}$$

So we have : $\displaystyle\int_\gamma \frac{fdz}{z-w} - \int_{\gamma_\varepsilon} \frac{fdz}{z-w} = \int_{\gamma-\gamma_\varepsilon} gdz$

$$= 2i\int_{\mathcal{G}_\varepsilon} \frac{f_{\bar z}}{z-w}\, dm$$

Taking the limit as $\varepsilon \to 0$ in this identity, we note that :

(i) $\displaystyle\int_{\gamma_\varepsilon} \frac{fdz}{z-w} = i\int_0^{2\pi} f(w + \varepsilon e^{i\theta})\, d\theta$ (set $z-w = \varepsilon e^{i\theta}$)

$$\xrightarrow[0]{\varepsilon} 2\pi i\, f(w)$$

(ii) $\displaystyle\int_{\mathcal{G}_\varepsilon} \frac{f_{\bar z}}{z-w}\, dm \xrightarrow[0]{\varepsilon} \int_{\mathcal{G}} \frac{f_{\bar z}}{z-w}\, dm$

The required result now follows immediately.

We note that, in the case where f is holomorphic in a neighbourhood of $\mathcal{G}$, i.e. $f_{\bar z} \equiv 0$, this formula reduces to

$f(w) = \dfrac{1}{2\pi i}\displaystyle\int_\gamma \frac{fdz}{z-w}$ (Cauchy integral formula) .

(d) Let $f \in C_c^{(1)}(\mathbb{C})$. Then for all $w \in \mathbb{C}$, $f(W) = \frac{1}{\pi}\int_{\mathbb{C}} \frac{f_{\bar{z}}}{w-z}\, dm$.

Proof We simply choose γ to be a circle large enough to properly include both w and supp f. (This is possible since supp f is compact, hence bounded.) Thus $\frac{f}{z-w}$ is zero on γ and the first integral of result (c) falls away, as required.

(e) We state, without proof, this well-known result :

Fubini's Theorem If $(X, \mathbb{S}, \mu)$ and $(Y, \mathbb{T}, \nu)$ are σ-finite measure spaces and h is an integrable function such that $h : X \times Y \longrightarrow \mathbb{R}/\mathbb{C}$, then almost every section of h is integrable. Also, if functions f and g are defined by $f(x) = \int h(x,y)\, d\nu(y)$ and $g(y) = \int h(x,y)\, d\mu(x)$, then f and g are integrable and ;

$$\int h \, d(\mu \times \nu) = \int f \, d\mu = \int g \, d\nu .$$

(see [8] §36.c)

(f) Let X be a compact plane set, μ a regular Borel measure on X and $f \in C_c^{(1)}(\mathbb{C})$. Then

$$\int_X f d\mu = \int_{\mathbb{C}} f_{\bar{z}}\, \hat{\mu}\, dm .$$

Proof $\int_X f(w)d\mu(w) = \int_X \left(\frac{1}{\pi} \int_{\mathbb{C}} \frac{f_{\bar{z}}(z)}{w-z} dm(z)\right) d\mu(w)$ (by (d))

$$= \frac{1}{\pi} \int_{\mathbb{C}} \left(\int_X \frac{f_{\bar{z}}(z)}{w-z} d\mu(w) \right) dm(z) \text{ by Fubini's Thm.}$$

$$= \frac{1}{\pi} \int_{\mathbb{C}} f_{\bar{z}}(z) \left(\int_X \frac{d\mu(w)}{w-z} \right) dm(z)$$

$$= \frac{1}{\pi} \int_{\mathbb{C}} f_{\bar{z}}(z)\, \hat{\mu}(z)\, dm(z) \quad \text{as required.}$$

Note that we may use Fubini since μ and m are σ-finite and

$\frac{f_{\bar{z}}}{w-z}(w,z) : X \times \mathbb{C} \longrightarrow \mathbb{C}$ is integrable.

(g) Let μ be a regular Borel measure on $\mathbb{C}$, with compact support. If U is an open set and $\hat{\mu} = 0$ a.e. (m) in U, then $|\mu|(U) = 0$ (i.e. $\operatorname{supp} \mu \subset \operatorname{supp} \hat{\mu}m$). In particular : $\mu = 0$ iff $\hat{\mu} = 0$ a.e.(m) .

Proof Let $f \in C_c^{(1)}(U)$, then $\int f d\mu = 0$ by (f) and since $\hat{\mu} = 0$ a.e. (m) on U. Let K be a compact subset of U. Then, since K is $\mathcal{G}_\delta$, and by the argument in the claim of 3.3(g), we can find a sequence $f_n \in C_c^{(1)}(U)$, $\forall_n \in \mathbb{N}$, decreasing to χ_K. Thus $\mu(K) = 0$. Now, writing μ in terms of real, positive components and arguing as in 2.7(g), we see that $|\mu|(U) = 0$. The second part of the result follows trivially.

(h) Let X be a compact plane set and let μ be a measure on X. Then $\mu \perp R(X)$ iff $\hat{\mu}$ vanishes off X.

Proof Let $\mu \perp R(X)$. If $w \notin X$ then $\frac{1}{z-w} \in R(X)$. Thus $\hat{\mu}(w) = 0$. Conversely, suppose that $\hat{\mu} = 0$ off X and that f is holomorphic in a neighbourhood U of X. Now, by virtue of the note at the end of (c), we may choose a boundary, γ, of U such that $\gamma \cap X = \emptyset$ and for each $w \in X$ we have :

$$f(w) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-w} dz .$$

Now, integrating with respect to μ :

$$\int_X f(w) d\mu(w) = \frac{1}{2\pi i} \int_X \left(\int_\gamma \frac{f(z)}{z-w} dz \right) d\mu(w)$$

$$= \frac{1}{2\pi i} \int_\gamma f(z) \left(\int_X \frac{d\mu(w)}{z-w} \right) dz \qquad \text{(Fubini)}$$

$$= \frac{-1}{2\pi i} \int_\gamma f(z)\, \hat{\mu}(z)\, dz$$

$$= 0 \quad \text{since} \quad \hat{\mu} = 0 \quad \text{on} \quad \gamma .$$

Thus μ annihilates the restriction to X of any function holomorphic in a neighbourhood of X ; In particular, any rational function whose poles lie off X . Thus $\mu \perp R(X)$ as required.

(i) If $f \in C(K)$ and f extends to be analytic in a neighbourhood of K , then $f \in R(K)$.

Proof By Cauchy's Integral formula we have :

$$f(w) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z-w}\, dz ,$$ for Γ a suitable contour surrounding K. (see (c))

Say that Γ is defined by $\varphi(t)$, $0 \leqslant t \leqslant 1$, then we have :

$$f(w) = \frac{1}{2\pi i} \int_0^1 \frac{f(\varphi(t))}{\varphi(t) - w}\, \varphi'(t)\, dt$$

Now we define a sequence of Riemann sums which approximate this integral :

$$g_n(w) = \frac{1}{2\pi i n} \sum_{m=0}^{n-1} \frac{f\left\{\varphi\left(\frac{m}{n}\right)\right\}}{\varphi\left(\frac{m}{n}\right) - w}\, \varphi'\left(\frac{m}{n}\right)$$

Clearly g_n is a rational function of w , with poles off K and the g_n approximate f uniformly on K .

(j) If $f \in R(K)$ and $x \in K$, then $\dfrac{f(z) - f(x)}{z - x} \in R(K)$.

Proof Clearly $\dfrac{f(z) - f(x)}{z - x}$ can be extended to be analytic in a neighbourhood of K with the possible exception of only the point x . In view of (i) , we need only show that

$\frac{f(z) - f(x)}{z - x}$ is analytic at the point x . Since $f(z) - f(x)$

is analytic at x , we may, for a suitably small δ, find a Taylor series expansion (see 1.6(e)) as follows :

$$f(z) - f(x) = f'(x)(z-x) + \frac{f''(x)}{2}(z-x)^2 + \frac{f^{(3)}(x)}{3!}(z-x)^3 + \ldots, \forall |z-x| < \delta .$$

$$\Rightarrow \frac{f(z)-f(x)}{z - x} = f'(x) + \frac{f''(x)}{2}(z-x) + \frac{f^{(3)}(x)}{3!}(z-x)^2 + \ldots, \quad \forall |z-x| < \delta .$$

It is easily seen that this series is, in fact, convergent in the given disk. (see [15] , 3.39) . Thus $\frac{f(z) - f(x)}{z - x}$ is analytic at x , as required.

(k) Let X be a compact plane set and $\mu \perp R(X)$. If $x \in X$, $\tilde{\mu}(x) < \infty$ and $\hat{\mu}(x) \neq 0$, then $\nu = \frac{1}{\hat{\mu}(x)} \frac{\mu}{z-x}$ is a complex measure which represents x .

Proof By 2.5(c) we know that $|\hat{\mu}(x)| < \tilde{\mu}(x) < \infty$. So ν is well defined . We have $\int_X d\nu = \frac{1}{\hat{\mu}(x)} \int_X \frac{d\mu}{z-x} = 1$.

We note that ν is, in fact, a regular, Borel measure (see 2.13(a)). If $f \in R(X)$, then $\frac{f - f(x)}{z - x} \in R(X)$ (by (j)) .

Thus $\int_X \frac{f - f(x)}{z - x} d\mu = 0$

So, for each $f \in R(X)$, we have :

$$f(x) = \int f(x)\, d\nu(z) = \int [f - (f - f(x))]\, d\nu$$

$$= \int f d\nu - \frac{1}{\hat{\mu}(x)} \int \frac{f - f(x)}{z - x} d\mu = \int f d\nu .$$

Thus $f(x) = \int f d\nu$, $\forall f \in R(X)$ as required .

(1) Let X be a compact plane set and $x \in X$. Then :

x is a peak point for $R(X) \iff (\mu \perp R(X) \wedge \tilde{\mu}(x) < \infty \Rightarrow \hat{\mu}(x) = 0)$.

Proof Let $\mu \perp R(X)$ and $\tilde{\mu}(x) < \infty$. Suppose that x is a peak point of $R(X)$. Say that $\hat{\mu}(x) \neq 0$.

Firstly, since x is a peak point of $R(X)$ and $\mu \perp R(X)$, we can say by 3.3(h) that $\mu|_{\{x\}} \perp R(X)$. Hence $\mu(\{x\}) = 0$.

We know that there exists $g \in R(X)$ such that $g(x) = 1$, $|g| < 1$ on $X \sim \{x\}$. Now, for every $n \in \mathbb{N}$, we have :

$$\int_X \frac{g^n(z)}{z - x} \frac{d\mu(z)}{\hat{\mu}(x)} = \int_X g^n(z)\, d\nu(z) , \quad (\nu \text{ as in } (k))$$

$$= (g(x))^n = 1$$

since ν is a complex representing measure for x.

But $g^n \longrightarrow \chi_{\{x\}}$ a.e. (μ) pointwise on X and, since $\mu(\{x\}) = 0$ we can say $g^n \longrightarrow 0$ a.e. (μ) pointwise on X.

Now since $\frac{1}{|z-x|}$ is μ-integrable (by hypothesis $\tilde{\mu}(x) < \infty$) and and since $\left|\frac{g^n(z)}{z-x}\right| \leqslant \frac{1}{|z-x|}$, $\forall_n \in \mathbb{N}$; we may use the Lebesgue dominated convergence theorem to get :

$$\int_X \frac{g^n(z)}{z-x} \frac{d\mu(z)}{\hat{\mu}(x)} \longrightarrow 0 .$$

This contradicts the previous statement ; So x is not a peak point of $R(X)$ when $\hat{\mu}(x) \neq 0$.

Conversely, say that x is not a peak point of $R(X)$. Then by 3.4(d), there exists a representing measure, λ, for x with $\lambda(\{x\}) = 0$. Then it is easily seen that $\mu = (z - x)\lambda$ annihilates $R(X)$. Furthermore, $\hat{\mu}(x) = \tilde{\mu}(x) = 1$.

This completes the proof .

(m) Bishop's Peak point criterion for Rational approximation : $R(X) = C(X)$ if and only if almost all points (m) of X are peak points of $R(X)$.

<u>Proof</u> Say almost all points (m) of X are peak points for $R(X)$. Then, by (l) , $\hat{\mu} = 0$ a.e.(m) on X for every $\mu \perp R(X)$. Then, by (h) , $\hat{\mu} = 0$ a.e. (m) on $\mathbb{C}$. Thus, by (g) , $\mu = 0$ is the only annihilating measure of $R(X)$. So, by 2.11(c)(ii) ; $R(X) = C(X)$. Conversely, if $R(X) = C(X)$, the peak points of $R(X)$ coincide with the peak points of $C(X)$. Now, since X is metric, hence normal, any $x \in X$ can be considered as the intersection of peak sets of all Urysohn functions U_y such that $U_y(x) = 1 \wedge U_y(y) = 0$, $\forall y \neq x$. Hence x is a p-set , and since X is metric, a peak set. Thus X is the set of peak points of $R(X)$.

<u>4.2</u> We introduce a concept which will help us to generalize Bishop's Theorem on Rational approximation. We still, however, restrict our attention to compact subsets, K , of the complex plane.

(a) <u>Definition</u> T_g - operators :

Let $g \in C_c^{(1)}(\mathbb{C})$ and f be a bounded Borel function (i.e. Borel measurable) . We define an operator on the space of complex-valued bounded Borel functions as follows :

$$(T_g f)(w) = g(w) f(w) + \frac{1}{\pi} \int_{\mathbb{C}} \frac{f(z)}{z-w} \frac{\partial g}{\partial \bar{z}} \, dm(z) \quad ; \quad w \in \mathbb{C} .$$

(b) Definition T - invariance :-

A subalgebra A of $C(K)$ is T-invariant if, whenever $g \in C_c^{(1)}$ and f is a bounded Borel function such that $f_{|K} \in A$, then $(T_g f)_{|K} \in A$.

4.3 We will show that $A(K)$ and $R(K)$ are, in fact, examples of T-invariant algebras. To do this, we must look at some results and definitions relating to T_g-operators

(a) Definition Local Integrability:-

Let μ be a regular Borel measure on $\mathbb{C}$. We shall say that f is locally μ-integrable if it is a Borel function such that $\int_K |f| \, d|\mu| < \infty$ for every compact subset K of $\mathbb{C}$.

By considering the volume of revolution generated by the function, $y = {}^1/x$ in the xy-plane, it is easily seen that the function ${}^1/z$ is, in fact, locally integrable with respect to Lebesgue measure on the complex plane. This, and more, follows from the next result.

(b) $\int_K \dfrac{dm}{|z - w|} \leqslant 2\sqrt{\pi m(K)}$, $\forall\, w \in \mathbb{C}$

Proof Let $R = \sqrt{m(K)/\pi} \Rightarrow m(K) = \pi R^2$. Let $D = B(w,R)$.

Then $\int_D \dfrac{dm}{|z-w|} = \int_0^{2\pi} \int_1^R \dfrac{1}{r}\, r dr\, d\theta$; changing to polar coordinates

$= 2\pi R = 2\sqrt{\pi m(K)}$

We have : $K = (K \cap D) \cup (K \sim D)$ and $D = (K \cap D) \cup (D \sim K)$.

Since $m(K) = m(D)$ we have $m(K \sim D) = m(D \sim K)$.

Since $\dfrac{1}{|z-w|} \geqslant \dfrac{1}{R}$ on $D \sim K$ and $\dfrac{1}{|z-w|} \leqslant \dfrac{1}{R}$ on $K \sim D$

we have $\int_{K\sim D} \frac{dm}{|z-w|} \leqslant \int_{D\sim K} \frac{dm}{|z-w|}$

So
$$\int_K \frac{dm}{|z-w|} = \int_{K\cap D} \frac{dm}{|z-w|} + \int_{K\sim D} \frac{dm}{|z-w|}$$
$$\leqslant \int_{K\cap D} \frac{dm}{|z-w|} + \int_{D\sim K} \frac{dm}{|z-w|}$$
$$\leqslant \int_D \frac{dm}{|z-w|} = 2\sqrt{\pi m(K)}$$
as required .

Note If our only restriction on K is that it be m-measurable, rather than compact, the above proof still holds, subject to the additional observation that if $m(K) = \infty$, the result is trivial.

(c) We state the following result without proof :

If K is a compact subset of $\mathbb{C}$ and V is an open set containing K , then $\exists$ $g \in C_c^{(1)}(\mathbb{C})$ such that

(i) $g(x) = 1$, $\forall\, x \in K$

(ii) $\text{supp}\, g \subset K$

(iii) $0 \leqslant g(z) \leqslant 1$, $\forall\, z \in \mathbb{C}$.

(see e.g. [14] , 3.10)

(d) If $f \in C(K)$ and f can be extended to be analytic in a neighbourhood, V , of K , then f can be extended to $\tilde{f} \in C_c^{(1)}(\mathbb{C})$.

Proof Produce a function, g, as in (c) . Then $\tilde{f} = f_1 g$ will do, where $f_1 = f^*$, the analytic extension of f , on V and $f_1 = 0$ on $\mathbb{C} \sim V$.

(e) Let g be a bounded Borel function on K , a compact subset of $\mathbb{C}$.

Let $f(w) = \int_K \frac{g}{z-w}\, dm(z)$, $\forall\, w \in \mathbb{C}$.

Then f is continuous on $\mathbb{C}$ and analytic on $\mathbb{C} \sim K$.

<u>Proof</u> If $m(K) = 0$ we have $f \equiv 0$ trivially . Otherwise, we may assume, without loss of generality, that $m(K) = \|g\| = 1$. Let $\varepsilon > 0$ and choose R so that $K \subset B(0,{}^{R}/_{2})$. Now, proceeding as in (d) , obtain $h \in C_c^{(1)}(\mathbb{C})$ such that $h(z) = {}^{1}/_{z}$ on $B(0,R) \sim B(0,\delta)$ and $h = 0$ on $B(0,{}^{\delta}/_{2})$. Clearly, by choosing δ small enough, we may, in view of (b) , obtain h so that $\int_{B(0,R)} |h - {}^{1}/z|\, dm < \varepsilon$.

Choose $\zeta > 0$ such that $|h(s) - h(t)| < \zeta$, whenever $|s - t| < \zeta$ (1.6(f)) .

Now, for $|t| < {}^{R}/_{2}$, we have :

$$\begin{aligned} &\left| f(t) - \int_K g(s)\, h(s-t)\, dm(s) \right| \\ &\leqslant \int_K |g(s)| \left| \frac{1}{s-t} - h(s-t) \right| dm(s) \qquad (2.5\ (c)) \\ &\leqslant \|g\| \int_{B(0,R)} \left| \frac{1}{z} - h \right| dm < \varepsilon \quad (s \in K \text{ and } |t| < {}^{R}/_{2}) . \end{aligned}$$

Thus, for s , $t \in B(0,{}^{R}/_{2})$ and $|s - t| < \zeta$, we have :

$$\begin{aligned} |f(s) - f(t)| &= \Big| f(s) - \int_K g(u)\, h(u-s)\, dm(u) + \int_K g(u)\, h(u-s)\, dm(u) \\ &\quad - \int_K g(u)\, h(u-t)\, dm(u) + \int_K g(u) h(u-t) dm(u) - f(t) \Big| \\ &< \varepsilon + \varepsilon + \int_K |g(u)|\, |h(u-s) - h(u-t)|\, dm(u) \\ &< 2\varepsilon + \|g\|\, m(K) \sup\{ |h(s)-h(t)| : |s-t| < \zeta \} \\ &< 3\varepsilon . \end{aligned}$$

Thus f is (uniformly) continuous on $B(0,{}^{R}/_{2})$ and hence on all of $\mathbb{C}$, since ${}^{R}/_{2}$ can be chosen large enough to include any given point. Now consider $w \notin K$:

$$\text{we have} \quad \frac{f(w+h) - f(w)}{h} = \int_K \frac{dm(z)}{(z-w-h)(z-w)}$$

Provided that we can show that this latter integral varies continuously with respect to h, we have $f'(w) = \int_K \frac{dm(z)}{(z-w)^2}$ which exists for $w \notin K$.

Consider $\varphi(h) = \int_K \frac{\left(\frac{1}{z-w}\right)dm(z)}{(z - w - h)}$. Since $w \notin K$, $\frac{1}{z-w}$ is a bounded Borel function on K. Hence, by the first part of the proof, bearing in mind that the translation by addition of a compact set is again a compact set, we see that $\varphi(h)$ is continuous on $\mathbb{C}$, as required.

(f) We define a function on $\mathbb{C}$ as follows :

$$G(w) = \frac{1}{\pi} \quad \frac{f(z) - f(w)}{z - w} h(z)\, dm(z)$$

where f is a bounded Borel function and h is a bounded Borel function with compact support. Since $(f(z) - f(w))\, h(z)$ is a bounded Borel function with compact support, we see, by (e), that G is continuous on $\mathbb{C}$.

(g) We now investigate the analyticity of G :

Considering the Newton Quotient

$$\frac{G(w+t) - G(w)}{t} = \frac{1}{t\pi}\int_{\mathbb{C}} \left[\frac{f(z) - f(w+t)}{z - w - t} - \frac{f(z) - f(w)}{z - w}\right] h(z)\, dm(z)$$

$$= \frac{1}{t\pi}\int_{\mathbb{C}} \frac{f(w)(z-w) - f(w+t)(z-w) + t(f(z)-f(w))}{(z-w)(z-w-t)} h(z)\, dm(z)$$

$$= \frac{1}{\pi}\int_{\mathbb{C}} \frac{f(z) - f(w)}{(z-w)(z-w-t)} h(z)\, dm(z) - \frac{1}{\pi}\,\frac{f(w+t)-f(w)}{t}\int_{\mathbb{C}} \frac{h(z)\, dm(z)}{z-w-t}$$

By virtue of (b) and (e) the second integral exists and is continuous with respect to t. If f is analytic at w, then $\frac{f(z) - f(w)}{z-w}$ is continuous in a neighbourhood of w and is thus a bounded Borel function. Now, by (b) and (e) again, we may say that the first integral exists and varies continuously with t . So we see that G is analytic whenever f is analytic.

(h) Set $h = \frac{\partial g}{\partial \bar{z}}$, where $g \in C_c^{(1)}(\mathbb{C})$. Clearly h is a bounded Borel function with compact support. Substituting this into the expression for G , we get :

$$G(w) = f(w)\, g(w) + \frac{1}{\pi}\int_{\mathbb{C}} \frac{f(z)}{z-w}\, {}^{\partial g}/_{\partial \bar{z}}\, dm(z)\ .$$

<u>Proof</u> We need to show that $g(w) = \frac{-1}{\pi}\int_{\mathbb{C}} {}^{\partial g}/_{\partial \bar{z}}(z-w)^{-1}\, dm(z)$. This is shown in 4.1(d) .

(i) So, under the conditions of (h) , $G(w) = (T_g f)(w)$. Thus we may say that $T_g f$ is continuous and is analytic at every point where f is analytic.

Considering the uniform algebra $A(K)$, we see that if we have an appropriate f such that $f_{|K} \in A(K)$, then $T_g f$ is analytic on the interior of K and continuous on K . i.e. $T_g f_{|K} \in A(K)$. Thus we have the T-invariance of $A(K)$.

(j) T-invariance of $R(K)$: Take $f \in R(K)$ and $\varepsilon > 0$ st. $V_\varepsilon = \{x : d(K,x) < \varepsilon\}$ admits an analytic extension, $\tilde{f}$, of f . Take a bounded Borel f_1 st. $f_{1|K} = f$. Let $f_2 = \tilde{f}$ on V_ε and $f_2 = f_1$ on $\mathbb{C} \sim V_\varepsilon$. Then f_2 is bounded and Borel and by (g) and (h), $T_g f_2$ is analytic in V_ε . So by 4.1(i), $T_g f_{2|K} \in R(K)$. We may choose ε st. $\sup\left|(f_1 - f_2)\frac{\partial g}{\partial \bar{z}}\right| \leq M < \infty$. Thus : $\|T_g f_1 - T_g f_2\|_K = \left\|\frac{1}{\pi}\int \frac{(f_1-f_2)(z)}{z-w}\, \frac{\partial g}{\partial \bar{z}}\, dm\right\|_K \leq \frac{M}{\pi}\left\|\int_{V_\varepsilon \sim K} \frac{dm}{z-w}\right\|_K \xrightarrow[0]{\varepsilon} 0$. As $\varepsilon \to 0$, we just restrict the original f_2 to the smaller sets V_ε . Thus M is constant and we use (b) .

<u>4.4</u> We now state a more general version of Bishop's Theorem on Rational approximation :

If A is a T-invariant uniform algebra on K and if almost all

(m) points of K are peak points of A , then $A = C(K)$. The converse also holds.

Before proving the theorem, we need several lesser results.

(a) If A is a T-invariant uniform algebra on K , then the functions $z_{|K}$ and $\left.\frac{1}{z-w}\right|_K$, $w \notin K$, are in A .

Proof Consider $\tilde{K} = \{x : d(x,K) \leqslant \varepsilon\}$ and $V = \{x : d(x,K) < 2\varepsilon\}$ Then, by 4.3(d) , $\exists\, g \in C_c^{(1)}(\mathbb{C})$ such that $g(z) = z$, $z \in \tilde{K}$ and $\operatorname{supp} g \subset V$. Let $f = \chi_K$. Then f, g certainly satisfy the conditions for T-invariance (cf. 4.2(b)) . So by hypothesis, $T_g f_{|K} \in A$.

But $$(T_g f)(w) = f(w)\, g(w) + \frac{1}{\pi}\int_{\mathbb{C}} \frac{f(z)}{z-w}\ \frac{\partial g}{\partial \bar{z}}\, dm(z)$$

$$= w\chi_K$$

since $f = 0$ off K and $\frac{\partial g}{\partial \bar{z}} = 0$ on K , g being analytic on K . Thus $P(K) \subset A$.

Similarly, for $\zeta \notin K$, set $4\varepsilon = d(\zeta,K)$ and take $\tilde{K}$ and V as above. Now, by 4.3(d), $\exists\, g \in C_c^{(1)}(\mathbb{C})$ such that $\operatorname{supp} g \subset V$ and $g(z) = \frac{1}{z-\zeta}$ on $\tilde{K}$. Let $f = \chi_K$, then, as before :

$$(T_g f)(w) = f(w)\, g(w) + \frac{1}{\pi}\int_{\mathbb{C}} \frac{f(z)}{z-w}\,\frac{\partial g}{\partial \bar{z}}\, dm(z)$$

$$= \frac{1}{w-\zeta}\ \chi_K \in A\ .$$

Thus we can say that $R(K) \subset A$ and, in view of 4.3(j) , we see that $R(K)$ is the smallest possible T-invariant uniform algebra on K .

(b) One can find $\varphi \in C_c^{(1)}(\mathbb{C})$ such that $\|\varphi\| = 1$, $\varphi = 0$ on $B(0,\delta)$, $\varphi = 1$ on $B(0,R) \sim B(0,2\delta)$ for some $R > 2\delta$ and some given $\delta > 0$. Also $\left\| \frac{\partial \varphi}{\partial \bar{z}} \right\|_{B(0,R)} \leqslant 2/\delta$.

Proof All conditions but the last could be satisfied by the function whose existence is given by 4.3(c). However, we proceed constructively by considering the joining of two suitable parabola's in the xy-plane in such a way that the derivative of the new function remains continuous. Then, letting $z = x + iy$, we arrive at :

$$f(z) = \begin{cases} 0 & ; \quad 0 \leqslant \sqrt{x^2+y^2} < \delta \\[2ex] \dfrac{2(x^2+y^2)}{\delta^2} - \dfrac{4\sqrt{x^2+y^2}}{\delta} + 2 & ; \quad \delta \leqslant \sqrt{x^2+y^2} \leqslant \dfrac{3\delta}{2} \\[2ex] \dfrac{-2(x^2+y^2)}{\delta^2} + \dfrac{8\sqrt{x^2+y^2}}{\delta} - 7 & ; \quad 3\delta/2 < \sqrt{x^2+y^2} < 2\delta \\[2ex] 1 & ; \quad \sqrt{x^2+y^2} \geqslant 2\delta . \end{cases}$$

Now by 4.3(c) we have $f_1 \in C_c^{(1)}(\mathbb{C})$ such that $f_1 = 1$ on $B(0,R)$ and $f_1 = 0$ off $B(0, R+\delta)$. Now set $\varphi = f f_1$. It is easily checked that φ has all of the required properties.

(c) Let A be a T-invariant uniform algebra on K . Let $p \in K$ and $\varepsilon > 0$ and $f \in A$. Then $\exists\, g \in A$ which is analytic at p and such that $\|f-g\| < \varepsilon$.

Proof We may assume without loss of generality that $f(p) = 0$ and that $p \doteq 0$. Choose $\delta > 0$ so that $|f(s)| < \varepsilon/9$ whenever $|s| < 2\delta$. Take φ as in (b) with R so large that $K \subset B(0, R)$. Let $\tilde{f} = f\chi_K$.

Let $\tilde{g}(w) = (T_\varphi \tilde{f})(w) = \frac{1}{\pi}\int_{\mathbb{C}} \frac{\tilde{f}(z) - \tilde{f}(w)}{z - w} \frac{\partial\varphi}{\partial\bar{z}}\, dm(z)$ (4.3(i))

$$= \frac{1}{\pi}\int_{B(0,2\delta) - B(0,\delta)} \frac{\tilde{f}(z) - \tilde{f}(w)}{z - w} \frac{\partial\varphi}{\partial\bar{z}}\, dm(z).$$

Let $g = \tilde{g}|_K$. By hypothesis, $g \in A$ and by 4.3(e) g is analytic at 0 . For any $w \in \mathbb{C}$

$$|\tilde{f}(w) - \tilde{g}(w)| = \left|\tilde{f}(w)(1-\varphi(w)) + \frac{1}{\pi}\int_{\mathbb{C}} \frac{\tilde{f}(z)}{z-w}\frac{\partial\varphi}{\partial\bar{z}}\, dm(z)\right|$$

$$\leq \|\tilde{f}\|_{B(0,2\delta)} + \|\tilde{f}\|_{B(0,2\delta)} \left\|\frac{\partial\varphi}{\partial\bar{z}}\right\|_{B(0,2\delta)} \frac{1}{\pi}\int_{B(0,2\delta)} |z-w|^{-1}\, dm(z) .$$

$$\leq \|\tilde{f}\|_{B(0,2\delta)}(1 + \frac{2}{\delta} \cdot 4\delta) \qquad (4.3(b))$$

$$< \varepsilon .$$

Thus $\|f - g\| < \varepsilon$ as required.

(d) Let A, f, p and ε be as in (c) . Then $\exists\, g \in A$ such that $\|f - f(p) - (z - p)g\| < \varepsilon$

<u>Proof</u> By (c) we have $\tilde{g} \in A$ st. $\|f - \tilde{g}\| < \varepsilon/2$ and $\tilde{g}$ is analytic at p . Then $\|f - f(p) - (\tilde{g} - \tilde{g}(p))\| < \varepsilon$ Now set

$g = \dfrac{\tilde{g} - \tilde{g}(p)}{z - p}$. Since $\tilde{g}$ is analytic at p , g is continuous at p and, in fact, on all of K . Clearly it remains only to show that $g \in A$.

Again, without loss of generality, we assume that $\tilde{g}(p) = 0$ and that $p = 0$. Choose φ as in (c) , set $h(z) = \frac{1}{z}\varphi(z)$.

Then $h \in C_c^{(1)}(\mathbb{C})$. Let $\hat{g}(z) = (\tilde{g}(z) - z\, g(0))\chi_K$.

Choose δ such that $|z| < 2\delta \Rightarrow |g(z) - g(0)| < \varepsilon/9$.

By (a) we know that $\tilde{g}(z) - z\, g(0) \in A$. Thus, by hypothesis, $T_h \hat{g} \in A$.

Now consider for $w \in K$:

$$|g(w)-g(0)-(T_h\hat{g})(w)| = |g(w)-g(0) - \hat{g}(w)h(w) - \frac{1}{\pi}\int_{\mathbb{C}} \frac{\hat{g}(z)}{z-w} \frac{\partial h}{\partial \bar{z}} \, dm(z)|$$

$$= \left|\left(g(w)-g(0)\right)\left(1-\varphi(w)\right)-\frac{1}{\pi}\int_{\mathbb{C}} \frac{\tilde{g}(z)-zg(0)}{z-w} \cdot \frac{1}{z}\varphi_{\bar{z}} \, dm(z)\right|$$

$$\leqslant \|g-g_0\|_{B(0,2\delta)} + \|g-g(0)\|_{B(0,2\delta)} \|\varphi_{\bar{z}}\|_{B(0,2\delta)} \frac{1}{\pi}\int_{B(0,2\delta)} \frac{dm(z)}{|z-w|}$$

$$< \|g-g_0\|_{B(0,2\delta)} \left[1 + \frac{2}{\delta} \cdot 4\delta\right] \qquad (4.3(b))$$

$$< \varepsilon \ .$$

Thus $g(z) - g(0)$ is uniformly approximated on K by members of A. So $g \in A$.

(e) Let A be a T-invariant uniform algebra on K, $p \in K$ and $\mu \in A^{\perp}$ such that $\tilde{\mu}(p) < \infty$. Then, if $f \in A$, we have

$$\int_K \frac{f(z) - f(p)}{z - p} \, d\mu(z) = 0$$

<u>Proof</u> Let $\varepsilon > 0$. By (d), we choose $g \in A$ such that $\|f - f(p) - (z-p)g\| < \varepsilon/\tilde{\mu}(p)$.

Then we have :

$$\left| \int_K \frac{f - f(p)}{z - p} \, d\mu \right| = \left| \int_K \frac{f - f(p)}{z - p} \, d\mu - \int_K g \, d\mu \right| \quad \text{since } \mu \perp g$$

$$= \left| \int_K \frac{f - f(p) - (z - p)g}{z - p} \, d\mu \right|$$

$$\leqslant \int_K \left| \frac{f - f(p) - (z - p)g}{z - p} \right| d|\mu| \quad (2.5(c))$$

$$< \varepsilon/_{\tilde{\mu}(p)} \int_K \frac{1}{|z-p|} \, d|\mu|$$

$$< \varepsilon \ .$$

Since ε is arbitrarily small, we have the required result.

(f) Proof of the Theorem :

The proof of the general version of the theorem is analogous to that given in section 4.1. We substitute A for $R(K)$ throughout, noting that :

(i) In 4.1(h) we need only the forward implication. This follows by (a).

(ii) Section 4.1(k) is proved in (e).

(iii) In section 4.1(l) we need only the forward implication (although, in fact, the equivalence is easily shown.)

4.5 We make some further comments about T-invariance.

(a) Let A be a T-invariant uniform algebra on K. Let $f = \chi_K$ and $g \in C_c^{(1)}(\mathbb{C})$.

Then

$$(T_g f)(w) = g(w)\chi_K + \frac{1}{\pi}\int_{\mathbb{C}} \frac{f(z)}{z-w}\frac{\partial g}{\partial \bar{z}}\, dm(z)$$

$$= g(w)\chi_K + \frac{1}{\pi}\int_K \frac{1}{z-w}\frac{\partial g}{\partial \bar{z}}\, dm(z)$$

So we see that if $\tilde{g} \in C(K)$ can be extended to $g \in C_c^{(1)}(\mathbb{C})$ such that g is analytic almost everywhere (m) in K, then $\tilde{g} \in A$. This tells us immediately that $R(K) \subset A$, a result which we used before.

(b) This also gives immediately a result known as the Hartogs-Rosenthal Theorem :

If $m(K) = 0$, then $A = C(K)$. In particular, $R(K) = C(K)$.

Of course, this is also just a corollary to Bishop's theorem, 4.4(f).

(c) In view of (a) and 4.3(c), it is easily seen that A contains the characteristic function of any open-closed subset of K.

(d) Let G be an open subset of K. Define $A(K,G)$ as being the set of all $f \in C(K)$ such that f is analytic on G. It is easily checked that $A(K,G)$ is a uniform algebra and that it is T-invariant (4.3(i)). By the Maximum Modulus Principle (1.6(b)) it is clear that if x is a peak point of $A(K,G)$, then $x \notin G$. Since G clearly has positive planar measure if it is non-empty, we have by Bishop's theorem (the converse, in fact, will do) :

$$G \neq \emptyset \Rightarrow A(K,G) \neq C(K) .$$

In particular, $\text{int}(K) \neq \emptyset \Rightarrow A(K) \neq C(K)$ and, by a similar argument, $\text{int}(K) \neq \emptyset \Rightarrow R(K) \neq C(K)$.

(e) We know that the intersection of uniform algebras on a given space is not necessarily a uniform algebra, since the separation property may not be preserved. However, for T-invariant algebras it is easily seen that :

The intersection of T-invariant uniform algebras on a set K is again a T-invariant uniform algebra on K.
The intersection of all T-invariant uniform algebras on K is $R(K)$.

(f) This enables us to make the following definition :

Let B be a subset of $C(K)$. Then the T-invariant hull of B, $TH(B)$, is the smallest T-invariant algebra containing B. i.e. $TH(B)$ is the intersection of all T-invariant uniform algebras containing B .

Further, we may denote by $T(B)$ the set

$$\{ T_g f : f \in B \wedge g \in C_c^{(1)}(\mathbb{C}) \}$$

In general $T(B) \subseteq TH(B)$. It is also easily checked that :

$$T(B) \subseteq T(\overline{B}) \subseteq \overline{T(B)} \subseteq TH(B) = TH(\overline{B}) .$$

If $B \subset R(K)$ then $TH(B) = R(K)$. Let c be a constant, then $T\{c\} = R(K)$. It may be interesting to investigate the conditions under which $T(B) = TH(B)$.

(g) We may restate Bishop's theorem (4.4(f)) as follows :

Let E be such that $m(K \sim E) = 0$. Let B be such that $\forall x \in E$, $\exists f \in B$ such that f peaks at x . Then $TH(B) = C(K)$.

CHAPTER V

Gleason Parts

5.1 Many concepts relating to uniform algebras find their origin in algebras on the disk and on the circle. So it is with Gleason Parts. (Introduced by A.M. Gleason in 1956, see [16]). We briefly examine the algebra $P(\Delta)$:

(a) Schwartz's Lemma

If $f \in P(\Delta)$ and $f(0) = 0$, then $|f(s)| \leqslant |s| \, \|f\|$, $\forall\, s \in \Delta$.

Proof Firstly we show that if $f \in P(\Delta)$ and $f(0) = 0$ then $\frac{f}{z} \in P(\Delta)$.

By Taylor series expansion (see 1.6(e) we have :

$$f(z) = f(0) + f'(0)z + f''(0)\frac{z^2}{2!} + f^{(3)}(0)\frac{z^3}{3!} + \ldots, \forall |z| < 1 .$$

$$\Rightarrow \frac{f(z)}{z} = f'(0) + f''(0)\frac{z}{2!} + f^{(3)}(0)\frac{z^2}{3!} + \ldots \qquad ,\forall |z| < 1 .$$

It is easily seen that this series is, in fact, convergent (see [15] 3.39). Thus $\frac{f}{z}$ is analytic on the interior of Δ and clearly $\frac{f}{z}$ is continuous on Γ. Thus $\frac{f}{z} \in A(\Delta) = P(\Delta)$ (1.6(h)). Now we apply the maximum modulus principle (see 1.6(b)), to get: $\left|\frac{f(z)}{z}\right| \leqslant \left\|\frac{f}{z}\right\|_\Gamma$, $\forall\, z \in \Delta$.

$\Rightarrow |f(z)| \leqslant |z| \, \|f\|_\Gamma = |z| \, \|f\|$, by the maximum modulus principle again.

(b) The mapping $\varphi_a(z) = \dfrac{a - z}{1 - \bar{a}z}$ is a linear fractional transformation. (see [10], 14.3)

It has the following properties : (i) $\varphi_a(a) = 0$.

(ii) $\varphi_a : \Gamma \longrightarrow \Gamma$, if $|a| \neq 1$.

Take $z = e^{i\theta}$, then $|\varphi_a(z)| = \left|\dfrac{a - e^{i\theta}}{1 - \bar{a}e^{i\theta}}\right|$

$$= |-e^{i\theta}| \frac{|1 - ae^{-i\theta}|}{|1 - \bar{a}\, e^{i\theta}|} = 1.$$

(iii) φ_a maps a circle to straight line or circle.

(iv) φ_a preserves boundaries.

Thus, for $|a| < 1$, φ_a is a homeomorphism on Δ which maps Γ to Γ , Δ° to Δ° and a to 0 .

Note that the inverse function φ_a^{-1} is given by

$$\varphi_a^{-1}(w) = \frac{w - a}{\bar{a}w - 1} .$$

(c) Take $|a| < 1$ and $f \in P(\Delta)$ such that $\| f \| \leq 1$.

We define the function $g = \varphi_{f(a)} \circ f \circ \varphi_a^{-1}$. Noting that $\varphi_{f(a)}$ and φ_a^{-1} are analytic in a neighbourhood of Δ (since $|f(a)| < 1$ by Schwartz's Lemma) and remembering that $P(\Delta) = A(\Delta)$, it is easily seen that $g \in P(\Delta)$. Clearly, also, $\| g \| \leq 1$ and $g(0) = 0$. Applying Schwartz's Lemma :

$$|g(z)| \leq |z| \, \| g \| \leq |z| .$$

Now let $\varphi_a^{-1}(z) = t$, then $z = \dfrac{t - a}{\bar{a}t - 1}$

Thus $|(\varphi_{f(a)} \circ f)(t)| \leq \left|\dfrac{t - a}{\bar{a}t - 1}\right|$

$$\Rightarrow \left|\frac{f(a) - f(t)}{1 - \overline{f(a)}f(t)}\right| \leq \left|\frac{a - t}{1 - \bar{a}t}\right|$$

$$\Rightarrow |f(a) - f(t)| \leq \left|\frac{a - t}{1 - \bar{a}t}\right| \; |1 - \overline{f(a)}\, f(t)|$$

(d) Now if $|a| < 1$ and $|t| < 1$, the first factor on the right-hand side of this inequality is strictly less than 1 . (by the properties of φ_a) . Since $\| f \| \leq 1$, the second factor is clearly less than or equal to two. So we have :

$\sup \{ | f(a) - f(t) | : f \in P(\Delta), \| f \| \leq 1 \} < 2$ if $|a| < 1$ and $|t| < 1$.

On the other hand, if $|a| = 1$ and $|t| \leq 1$ we can define

$f_r(z) = \bar{a} \dfrac{z - ra}{1 - r\bar{a}z}$, $\forall z \in \Delta$ and $0 < r < 1$. Then, clearly, $f_r \in A(\Delta)$, hence $f_r \in P(\Delta)$ and also $f_r(a) = 1$, $\|f_r\| = 1$. But $f_r(t) = \dfrac{\bar{a}t - r}{1 - r\bar{a}t} \longrightarrow -1$ as $r \longrightarrow 1$. So we have:

$$\sup\{|f(a) - f(t)| : f \in P(\Delta) , \|f\| \leqslant 1\} = 2 \text{ if } |a| = 1 , |t| \leqslant 1 .$$

Now, identifying points in Δ with the corresponding evaluation functionals, we have established the relation:

$$\|s - t\| < 2 \text{ iff } [\text{both } s \text{ and } t \text{ are in } \Delta^\circ \text{ or } s = t] .$$

Now let $\varphi \in \text{Spec } P(\Delta)$ (the maximal ideal space of $P(\Delta)$ see 2.14(b)) and let $g(z) = z$, $g \in P(\Delta)$. Suppose $\varphi(g) = z_0$, then $|\varphi(g)| = |z_0| \leqslant \|\varphi\| \|g\| = 1$ since $\|\varphi\| = 1 \wedge \|g\| = 1$.
So for any polynomial $p(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n$ we have : $\varphi(p) = a_0 + a_1 z_0 + a_2 z_0^2 + \ldots + a_n z_0^n = p(z_0)$.
So $\forall f \in P(\Delta)$, $\varphi(f) = f(z_0)$ i.e. φ is the evaluation functional at $z_0 \in \Delta$. So $\text{Spec } P(\Delta) = \Delta$, i.e. every $\varphi \in \text{Spec } P(\Delta)$ is an evaluation functional at some point in Δ .
So we see that the relation $\|s - t\| < 2$ divides Spec $P(\Delta)$ into a family of equivalence classes.

(e) In fact, the maximal ideal space of any function algebra, Spec A (or M_A), can be split into parts, also called Gleason parts, in this way:

Definition Let A be a function algebra. The Gleason parts of A are the equivalence classes defined by the relation : $\varphi \sim \psi$ iff $\|\varphi - \psi\| < 2$.

It remains, of course, to show that $\sim$ is, in fact, an equivalence relation. We shall do this shortly.

5.2 Characterization of Parts

Before exhibiting a characterization of Gleason parts, we need an elementary result from Banach algebra theory.

(a) Let R be a Banach algebra with unit. If $f \in R$, $s \in \mathbb{C}$ and $\|f\| < |s|$, then $s - f$ has an inverse in R, given by the norm-convergent series $\sum_{n=0}^{\infty} \frac{f^n}{s^{n+1}}$.

<u>Proof</u> Let $g = f/s$, so $\|g\| < 1$. The above result is equivalent to saying that $1 - g$ has an inverse, given by $\sum_{n=0}^{\infty} g^n$ $\left(\text{since then } (1-g)^{-1} = \left(\frac{s-f}{s}\right)^{-1} = s\sum_{n=0}^{\infty} \frac{f^n}{s^{n+1}}\right)$.

Let $G_n = \sum_{k=0}^{n} g^k$. So for $m < n$ we have :

$\|G_n - G_m\| = \|\sum_{k=m+1}^{n} g^k\| \leqslant \sum_{m+1}^{n} \|g\|^k \leqslant \frac{\|g\|^{m+1}}{1 - \|g\|}$, by properties of the norm.

Thus since $\|g\| < 1$, G_n is a Cauchy sequence and has a limit, G, in R.

Since $(1 - g)G_n = G_n(1-g) = 1-g^{n+1}$ and $g^{n+1} \to 0$ as $n \to \infty$, we have : $(1 - g)G = G(1 - g) = 1$, as required.

(b) If A is a function algebra and φ, ψ are in M_A, then the following are equivalent : (i) φ, ψ are in the same part of M_A.

(ii) Let $A_\psi = \{f \in A : \psi(f) = 0\}$. Then $\sup\{|\varphi(f)| : \|f\| \leqslant 1 \text{ and } f \in A_\psi\} < 1$ i.e. the norm of the restriction of φ to A_ψ is less than one.

(iii) If $\{f_n\}$ is a sequence in A such that $\|f_n\| < 1$, $\forall n \in \mathbb{N}$ and $|\varphi(f_n)| \to 1$, then $|\psi(f_n)| \to 1$.

(iv) $\exists\, c > 0$ such that Harnack's inequality is valid, i.e. :

$$\frac{1}{c} < \frac{u(\varphi)}{u(\psi)} < c, \quad \forall\, u \in \mathrm{Re}(A) \text{ st. } u > 0 .$$

Proof The proof will proceed as follows : (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i) .

(i) $\Rightarrow$ (ii) : Let $\|\varphi - \psi\| = 2c < 2$. Let $f \in A$ st. $\|f\| \leqslant 1$ and $\psi(f) = 0$. We can assume that $\varphi(f) > 0$ (multiply f by a rotating factor.) . Put $g = \dfrac{c - f}{1 - cf}$. By (a) , we know that $g \in A$. By (5.1)(b) , since c is real, we see that $\|g\| \leqslant 1$. By hypothesis $|\varphi(g) - \psi(g)| \leqslant 2c$

$$\Rightarrow \left| \frac{c - \varphi(f)}{1 - c\varphi(f)} - c \right| \leqslant 2c$$

$$\Rightarrow (1-c^2)\varphi(f) \leqslant 2c(1-c\varphi(f)) \quad \text{by (5.1)(b) again.}$$

$$\Rightarrow \varphi(f) \leqslant \frac{2c}{1+c^2} < 1 \quad \text{as required.}$$

(ii) $\Rightarrow$ (iii) : Suppose that (iii) does not hold, i.e. there is a sequence $f_n \in A$ st. $\|f_n\| < 1$, $\forall_n$ and $\varphi(f_n) \longrightarrow 1$ (having multiplied f_n by a rotating factor) , but $|\psi(f_n)| \leqslant c < 1$, $\forall_n$.

Now we must show that (ii) does not hold:

Let $q_n(z) = \dfrac{z - \psi(f_n)}{1 - \overline{\psi(f_n)}z}$. Set $g_n = q_n \circ f_n$. By (a) and (5.1 (b)) $g_n \in A$ and $\|g_n\| < 1$. We have : $\psi(g_n) = 0$ and $|\varphi(g_n)| \longrightarrow 1$ (using 5.1(b) again and continuity.) So the norm of φ on A_ψ is one , as required.

(iii) $\Rightarrow$ (iv): We show: $\sim$ (iv) $\Rightarrow$ $\sim$(iii) . Suppose we have a sequence $u_n \in \mathrm{Re}(A)$ such that $u_n > 0$ and $\dfrac{\varphi(u_n)}{\psi(u_n)} \geqslant n$, $\forall_n \in \mathbb{N}$ (we may assume this without loss of generality). Now let $g_n \in A$ be such that $\mathrm{Re}(g_n) = u_n$, $\forall_n \in \mathbb{N}$. Then set

$f_n = \frac{g_n}{\psi(u_n)\sqrt{n}}$. Then it is easily seen that $Re(f_n) > 0$ and $\psi(Re(f_n)) \longrightarrow 0$ and $\varphi(Re(f_n)) \longrightarrow \infty$. Now set $h_n = e^{-f_n}$, then $h_n \in A$; $\|h_n\| < 1$ and $|\psi(h_n)| \longrightarrow 1$ while $\varphi(h_n) \longrightarrow 0$, contradicting (iii) .

(iv) $\Rightarrow$ (i) : We show : $\sim$(i) $\Rightarrow$ $\sim$(iv) : Assume $\|\varphi - \psi\| = 2$. So we may choose a sequence $g_n \in A$ such that $\|g_n\| \leq 1$ and $|\varphi(g_n) - \psi(g_n)| \longrightarrow 2$. If we set $f_n = g_n\alpha_n(1 + \frac{1}{n})^{-1}$, α_n a suitable rotating factor, we get : $f_n \in A$, $\|f_n\| < 1$, $|\varphi(f_n) - \psi(f_n)| \longrightarrow 2$ and $\varphi(f_n) \longrightarrow 1$ so that $\psi(f_n) \longrightarrow -1$. Now set $u_n = Re(1 - f_n)$, then $u_n > 0$, $\varphi(u_n) \longrightarrow 0$ and $\psi(u_n) \longrightarrow 2$. So (iv) is contradicted and the chain of equivalences proved.

(c) As an immediate corollary to this, we see that the relation defining Gleason parts is, in fact, an equivalence relation. Symmetry and reflexivity are obvious. For transitivity, take $\varphi \sim \psi$ and $\psi \sim \theta$. Then by (b)(iv) we have : $\exists\ c_1 > 0$ such that $\frac{1}{c_1} < \frac{\varphi(u)}{\psi(u)} < c_1$ and $\exists\ c_2 > 0$ such that $\frac{1}{c_2} < \frac{\psi(u)}{\theta(u)} < c_2$,$\forall\ u \in Re(A)$ st. $u > 0$. Then clearly :

$$\frac{1}{c_1c_2} < \frac{\varphi(u)}{\theta(u)} < c_1c_2 , \quad \forall\ u \in Re(A) \text{ st. } u > 0 \text{ and } c_1c_2 > 0 .$$

So $\varphi \sim \theta$.

(d) <u>Definition</u> We say that two measures μ and ν are mutually singular if there exist two disjoint sets A and B such that $A \cup B = X$ and for every measurable set E , $A \cap E$ and $B \cap E$ are measurable and $|\mu|\ (A \cap E) = |\nu|\ (B \cap E) = 0$. We often say "μ is singular with respect to ν" or "ν is singular to μ" .

Let A be a function algebra on X and $\varphi, \psi \in M_A$ such that $\|\varphi - \psi\| = 2$. If μ and ν are representing measures for φ and ψ respectively, then $\mu - \nu$ represents $\varphi - \psi$. By virtue of 2.5(d), we see that $\|\mu - \nu\| \geqslant 2$. Let $\lambda^+ - \lambda^-$ be the Hahn-decomposition of $\mu - \nu$. Then we have $X = A \cup B$ such that $\|\mu - \nu\| = \lambda^+(A) + \lambda^-(B)$. Bearing in mind that μ , ν are probability measures, we see that $\lambda^+(A) = 1 = \mu(A)$ and $\lambda^-(B) = 1 = \nu(B)$. So clearly μ , ν are mutually singular . Thus we can state:

If φ and ψ have representing measures which are not mutually singular, then they lie in the same Gleason part.

(e) We state a well known measure theoretic result :

<u>The Radon-Nikodym Theorem</u> If $(X, \mathbb{S}, \mu)$ is a totally σ-finite measure space (μ a positive measure) and if a σ-finite signed measure ν on $\mathbb{S}$ is absolutely continuous with respect to μ, then there exists a finite-valued measurable function, f , on X such that $\nu(E) = \int_E f d\mu$ for every measurable set E. f is unique in the sense that if also $\nu(E) = \int_E g d\mu$, $\forall E \in \mathbb{S}$, then $f = g$ a.e.(μ).

(see [8] §31.B)

(f) Let θ , φ belong to the same part of M_A , the maximal ideal space of a function algebra A . Let $b(\theta , \varphi)$ be the infimum of all positive numbers, c , for which Harnack's inequality holds (cf. (b)(iv)). It is easily seen that $b(\theta, \varphi)$ has the following properties :

(i) $\frac{1}{b(\theta,\varphi)} \leqslant \frac{\theta(u)}{\varphi(u)} \leqslant b(\theta,\varphi)$, $\forall\ u \in Re(A)$, $u > 0$.

(ii) $b(\theta,\varphi) \geqslant 1$.

(iii) $b(\theta,\varphi) = 1$ iff $\theta = \varphi$.

(iv) $b(\theta,\varphi) = b(\varphi,\theta)$

(v) $b(\theta,\varphi)\ b(\varphi,\psi) \leqslant b(\theta,\psi)$ if ψ also belongs to the same part as θ .

Clearly $\log b(\theta,\varphi)$ is a metric on each part of M_A .

(g) We now state and prove a result which includes the converse of (d) :

Let A be a function algebra on X and let θ , φ be in the same part of M_A . Then there exist constants a and b , $0 < a \leqslant b < \infty$ and a Borel function , h , on X such that $a \leqslant h \leqslant b$ and representing measures μ , ν for θ and φ respectively such that $\mu = h\,\nu$.

<u>Proof</u> Write b for $b(\theta,\varphi)$. By (f)(i) we have :
$b\varphi(u) - \theta(u) \geqslant 0$ for all $u \in Re(A)$ st. $u > 0$. Since the L.H.S. of the inequality represents, in fact, a linear functional on A , we have, by 2.6(b) , a positive measure, α , on X , such that : $b\varphi(u) - \theta(u) = \int u\,d\alpha$, $u \in Re(A)$.
Similarly we have a probability measure, β , on X , such that $b\theta(U) - \varphi(u) = \int u\,d\beta$, $u \in Re(A)$.
Solving these equations simultaneously for $\theta(u)$ and $\varphi(u)$ we get measures, $\mu = \frac{(b\beta + \alpha)}{b^2 - 1}$ and $\nu = \frac{b\alpha + \beta}{b^2 - 1}$, which represent θ and φ respectively. Clearly these are positive, hence, representing measures. Clearly also they are mutually absolutely continuous.

So, by (e) , there is a Borel function, h , on X st. $\mu(E) = \int_E h\,d\nu$, for every Borel subset, E , of X . By the arguments of 2.13(a) , $\mu = h\nu$. Now, let E be the Borel set on which $h > b$. Then :

$$\mu(E) = \int_E h\,d\nu$$

$$\Rightarrow \mu(E) > \int_E b\,d\nu \quad \text{if} \quad \nu(E) \neq 0$$

$$\Rightarrow (b\beta + \alpha)(E) > b(b\alpha + \beta)(E)$$

$\Rightarrow \alpha(E) > b^2\alpha(E)$, which by (f)(ii) is absurd unless $b = 1$ or $\alpha(E) = 0$. By (f)(iii) we may discard the possibility that $b = 1$. But, if $\alpha(E) = 0$, the second last inequality is contradictory. So we must assume that $\nu(E) = 0$.

Now, bearing in mind that h is unique only in the sense that it is a member of an equivalence class of a.e. (ν or μ) equal functions, we may say that $h \leqslant b$. A similar argument gives $h \geqslant \frac{1}{b}$. So the conditions of the theorem are satisfied.

(h) <u>Corollary</u> If θ, φ lie in the same part of M_A and η is a representing measure for θ , then there is a representing measure λ for φ , such that η is absolutely continuous with respect to λ .

<u>Proof</u> Let $\lambda = \eta/b + \nu - \mu/b$ (b, ν, μ as in (g)).

(i) Suppose θ and φ are not in the same part of M_A . Then there are disjoint Borel sets, E_1 and E_2 , such that every representing measure for θ is supported on E_1 and every representing measure for φ is supported on E_2 . In particular, representing measures for θ are singular to representing measures for φ .

Proof Proceeding as in (b) (proof of (ii) $\Rightarrow$ (iii)) we have a sequence $\{g_n\} \subset A$ such that $\|g_n\| < 1$, $\theta(g_n) \xrightarrow[\infty]{n} 1$ and $\varphi(g_n) = 0$, $\forall_n \in \mathbb{N}$.

Now let $q_n(z) = \dfrac{\frac{n-1}{n} - z}{\frac{n-1}{n}z - 1}$. Set $f_n = q_n \circ g_n$. By (a) and 5.1(b).

we have : $f_n \in A$, $\|f_n\| < 1$, $\theta(f_n) \xrightarrow[\infty]{n} 1$ and

$\varphi(f_n) = \frac{1-n}{n}$. So $\varphi(f_n) \xrightarrow[\infty]{n} -1$. Clearly we may choose the f_n so that $|1 - \theta(f_n)| < \frac{1}{n^4}$ and $|1 + \varphi(f_n)| < \frac{1}{n^4}$.

Now let E_1 be the set of x in X such that $f_n(x) \longrightarrow 1$, and E_2 the set of x in X such that $f_n(x) \longrightarrow -1$. Then E_1 and E_2 are disjoint Borel sets.

[Let $V_{pn} = \{x \in X : 1 - f_n(x)| \leqslant \frac{1}{p}\}$. Since f_n is continuous, this is a closed set, hence Borel. Now set $V_p = \bigcup_{n\in\mathbb{N}} V_{pn}$. Then $E_1 \subset V_p$, a Borel set. Then clearly $E_1 = \bigcap_{p\in\mathbb{N}} V_p$. Hence E_1 is Borel as required.]

Now let μ be a representing measure for θ.

$$
\begin{aligned}
\text{Then} \quad \int |f_n - 1|^2 d\mu &= \int [(\operatorname{Re} f_n - 1)^2 + (\operatorname{Im} f_n)^2]\, d\mu \\
&= \int |f_n|^2 d\mu + 1 - 2 \operatorname{Re} \int f_n d\mu \\
&\leqslant 2 - 2 \operatorname{Re}\theta(f_n) \\
&\leqslant 2/n^4 .
\end{aligned}
$$

Thus $\int \sum_{n=1}^{\infty} |f_n - 1|^2 d\mu < \infty$.

Now we claim that $f_n \longrightarrow 1$ a.e.(μ). Assume that this is not true. Then there exists a Borel set, E, such that $\mu(E) > 0$ and $\forall x \in E \ \exists\, \varepsilon_x > 0$ such that $\forall_N \in \mathbb{N}$, $\exists\, n > N$ st. $|1-f_n(x)| \geqslant \varepsilon_x$.

Thus $\sum_{n=1}^{\infty} |1 - f_n(x)| = \infty$, $\forall\, x \in E$. In fact, we may say that

$\sum_{n=1}^{\infty} |1 - f_n(x)|^2 = \infty, \forall x \in E$. Note that $\sum_{n=1}^{\infty} |1 - f_n|$ is measurable (see [7] , 2.9) . It follows that $\int \sum_{n=1}^{\infty} |1 - f_n|^2 d\mu = \infty$. As this contradicts the above result we see that $f_n \longrightarrow 1$ a.e.(μ) . Thus μ is supported on E_1 Similarly every representing measure for φ is supported on E_2 , as required.

5.3 We now leave the general case and again restrict our attention to uniform algebras on plane sets. In particular, we look at T-invariant uniform algebras. A shall denote such an algebra throughout this section.

(a) The following result is essentially due to Arens.

Let A be a T-invariant uniform algebra on K . Then $M_A = K$.
(see 2.14)

Proof Consider $\varphi \in M_A$. By 4.4(a) we have $z \in A$ and $\frac{1}{z-w} \in A$, if $w \notin K$. Let $\varphi(z) = z_0$. Assume that $z_0 \notin K$.

$$\text{Then} \quad 1 = \varphi(1) = \varphi\left(\frac{z - z_0}{z - z_0}\right) = \varphi(z-z_0).\varphi\left(\frac{1}{z-z_0}\right) = 0.$$

Contradiction. So we have $z_0 \in K$. Take $f \in A$. By 4.4(c), we have a sequence $\{f_n\} \subset A$ which converges uniformly to f and is such that f_n is analytic at z_0 . By the proof of 4.4(d) we know that $\frac{f_n(z) - f_n(z_0)}{z - z_0} \in A$, $\forall n \in \mathbb{N}$.

$$\text{So we have} \quad \varphi(f) - f(z_0) = \lim_{n\to\infty} (\varphi(f_n) - f_n(z_0))$$

$$= \lim_{n\to\infty} \varphi(f_n - f_n(z_0))$$

$$= \lim_{n\to\infty} \varphi\left(\frac{f_n(z) - f_n(z_0)}{z - z_0}\right)\varphi\,(z - z_0)$$

$$= 0 \, .$$

So φ evaluates every $f \in A$ at $z_0 \in K$. i.e. each $\varphi \in M_A$ is of the form τ_x, $x \in K$.

(b) Peak points of A are one-point parts of A.

Proof Let x be a peak point of A. By 3.4(c), δ_x, the unit mass at x, is the only representing measure for x (i.e. for τ_x). Consider $\varphi \in M_A$, $\varphi \neq \tau_x$. We wish to show that φ is not in the same Gleason part as τ_x.
Let μ represent φ. Then, by (a), we have
$\int f d\mu = f(z_0)$, $\forall\ f \in A$, some $z_0 \in K$. Assume that
$\mu(\{x\}) = c > 0$ and choose $f \in A$ which peaks at x. Now
$f^n \in A$ and f^n tends pointwise boundedly to the characteristic function of $\{x\}$. Then :

$$0 = \lim_{n\to\infty} \varphi(f_n) = \lim_{n\to\infty} \int f_n \, d\mu$$

$$= \int \lim_{n\to\infty} f_n d\mu \quad (2.6(f))$$

$$= c > 0$$

By this contradiction we see that $\mu\{x\} = 0$. Thus μ is mutually singular with δ_x. Now, by the conclusion of 5.2(d), φ is not in the same part as τ_x.

(c) We must digress briefly to establish a property of $\tilde{\mu}(w)$, namely :

$\tilde{\mu}$ is integrable with respect to Lebesgue planar measure, m, over any bounded set. In particular, $\tilde{\mu} < \infty$ a.e.(m) on any bounded set.

<u>Proof</u> Say that μ is a measure on K and note that any bounded set is contained in $D = B(0, R)$ for R large enough. Now, using Fubini's theorem (4.1(e)) whose conditions are clearly satisfied, and 4.3(b) we have :

$$\int_D \int_K \frac{d|\mu|(z)}{|w - z|} \, dm(w) = \int_K \int_D \frac{dm(w)}{|w-z|} \, d|\mu|(z)$$

$$\leqslant 2\pi R \, |\mu|(K) .$$

(d) Let A be a uniform, T-invariant, algebra on K . If $p \in K$ is not a peak point of A , then the Gleason part containing p has positive planar measure.

<u>Proof</u> Let ν be a representing measure for p which has no mass at p (by 3.4(d)) . Clearly $\mu = (z-p)\nu$ is a regular Borel measure (2.13(a)) which annihilates A . Let $q \in K$. Provided that $\tilde{\mu}(q) < \infty$ and $\hat{\mu}(q) \neq 0$ we know that $\lambda_q = \frac{1}{\hat{\mu}(q)} \cdot \frac{\mu(z)}{z - q}$ is a complex representing measure for q . (This is mentioned in 4.4(f)(ii)). Now, since ν has no mass at p , we know that $\mu \neq 0$. Thus, in view of 4.1(g), we know that $\hat{\mu}(q) \neq 0$ for $q \in Q$ a set such that $m(Q) > 0$. Furthermore, by (c), $\tilde{\mu}(q) < \infty$ a.e.(m) in Q . So we may assume that both the conditions, $\tilde{\mu}(q) < \infty$ and $\hat{\mu}(q) \neq 0$, are satisfied on Q . Thus every $q \in Q$ has a complex representing measure, λ_q . Now, by 2.17(e), there exists, for every $q \in Q$, a representing measure, α_q , such that $\alpha_q << \lambda_q$. It is easily seen that $\alpha_q << \nu$. (see 2.17(b)). Now, by 5.2(d), q lies in the same part as p . Since this is true for every $q \in Q$, we have the result.

(e) Now we have:

p is a peak point of A iff it is a Gleason part of A, or equivalently, the one-point parts of A are precisely the peak points of A.

<u>Proof</u> Immediately from (b) and (d).

(f) In view of (e), we may make the following restatement of Bishop's Theorem (4.4).

Every point of K is a Gleason part of A iff $A = C(K)$.

(g) The number of non-trivial parts of A (i.e. not one-point parts) is, at most, countable.

<u>Proof</u> Follows from (d).

(h) Let P be a Gleason part of A and ν a representing measure for $p \in P$. Then ν is supported on $\overline{P}$.

<u>Proof</u> By (e), we may assume that P is not a one-point part and, by 3.4(d), that $\nu(\{p\}) = 0$. Let $\mu = (z - p)\nu$. Arguing as in (d) we see that if $\hat{\mu} \neq 0$ on a set of positive m-measure off P then there would be a point, $q \notin P$, such that $\hat{\mu}(q) \neq 0$ and $\hat{\mu}(q) < \infty$. But, as demonstrated before, such a point must be in P. So we know that $\hat{\mu} = 0$ a.e.(m) off P. Now, by 3.5(g), $|\mu|(\mathbb{C} \sim \overline{P}) = 0$. Thus μ, and hence ν, are supported on $\overline{P}$.

(i) Let P_i, $i \in \mathbb{N}$ be the nontrivial Gleason parts of a T-invariant uniform algebra, A, on K. (cf.(g)). Then there exist Borel sets E_i, $i \in \mathbb{N}$ such that

(i) The E_i are pairwise disjoint

(ii) $P_i \subset E_i \subset \overline{P}_i$

(iii) Every representing measure for any point in P_i is supported on E_i .

Proof Consider the parts P_1 and P_j , $j \neq 1$. Let $x \in P_1$ and $y \in P_j$. By 5.2(i) , we have disjoint Borel sets B_{1j} and D_{1j} such that any representing measure for x is supported on B_{1j} and any representing measure for y is supported on D_{1j} . In particular δ_x is supported on B_{1j} . Hence $x \in B_{1j}$. Consider $z \in P_1$, $z \neq x$. By 5.2(h) , there is a representing measure λ for x such that $\delta_z \ll \lambda$. Since λ is supported on B_{1j} , so is δ_z . Hence $z \in B_{1j}$. Thus $P_1 \subset B_{1j}$. Similarly $P_j \subset D_{1j}$. Now set $E_1 = \bigcap_{j>1} B_{1j}$. Clearly E_1 is Borel, supports any representing measure of any point of P_1 , in particular $P_1 \subset E_1$, and $E_1 \cap D_{1j} = \emptyset$, $j \neq 1$. Now for P_2 we have corresponding sets B_{2j} and D_{2j} ; $j \neq 1, 2$. We set $E_2 = B_{12} \cap \bigcap_{j>2} B_{2j}$. Clearly, if we continue in this way we obtain a sequence E_i satisfying properties (i) and (iii) . In view of (h) , we may replace E_i by $E_i \cap \overline{P}_i$. Thus property (ii) is also satisfied.

CHAPTER VI

The Generalised F. and M. Riesz Theorem

6.1 In this chapter we find that the classical F. and M. Riesz Theorem can be generalised to apply to all uniform algebras.

(a) Firstly we must formulate more general concepts of singularity and absolute continuity of measures. We do this as follows:

Definition The measure μ is singular with respect to a set M of probability measures if μ is carried by some Borel set F such that $\lambda(F) = 0$, $\forall \lambda \in M$. By "carried by" we mean that if E is a Borel set such that $E \cap F = \emptyset$, then $|\mu|(E) = 0$. We say that μ is M-singular. Such an F will be called an M-null set.

Definition If μ vanishes on all M-null sets of a set M of probability measures (i.e. $|\mu|(F) = 0$) , we say that μ is M-absolutely continuous. We denote this by $\mu << M$.

(b) Definition Let μ , λ be two measures. We say that $\mu_s + \mu_a$ is the Lebesgue decomposition of μ with respect to λ if $\mu = \mu_s + \mu_a$ and μ_s is singular with respect to λ and $\mu_a << \lambda$. This decomposition exists and is, in fact, unique. (see [8] §32.c)

(c) Extending this idea, we say that for μ , a measure, and M , a set of probability measures, there exists a unique Lebesgue decomposition of μ with respect to M . i.e.

$\mu = \mu_F + \mu_{F'}$, where μ_F is M-singular and $\mu_{F'} << M$. (Here, by μ_F and $\mu_{F'}$, we mean the restriction of μ to F and F' respectively.)

Proof Let $C = \sup\{|\mu|(E)$, E an M-null set$\}$. Choose a sequence $\{E_n\}$ of M-null sets such that $|\mu|(E_n) \xrightarrow[\infty]{n} C$. Let $F = \bigcup_1^\infty E_n$. By the regularity of $|\mu|$ (2.3(f)) and of $\lambda \in M$ we have $|\mu|(F) = C$ and F is M-null. Then, if E is M-null, so is $E \cup F$ and we have

$$\|\mu_{E\cup F}\| = \|\mu_F\| + \|\mu_{E\cap F'}\| \leq \|\mu_F\|$$

Thus $|\mu_{F'}|(E) = 0$ and $\mu_{F'} << M$.

For the uniqueness, let $\mu = \mu_s + \mu_a$ be a Lebesgue decomposition of μ with respect to M such that $\mu_a << M$. Now let E be any M-null set. We have

$0 = (\mu_F - \mu_s + \mu_{F'} - \mu_a)(E) = (\mu_F - \mu_s)(E) = |\mu_F - \mu_s|(E)$.

(The last step follows since any Borel subset of E is also M-null). So μ_F and μ_s coincide on any M-null set. Since, by definition, they are both carried on M-null sets, we have $\mu_F = \mu_s$ as required.

6.2 We need one result before proving the Generalised F. and M. Riesz Theorem.

(a) Let A be a uniform algebra on X . Let $\varphi \in M_A$ and let M_φ be the set of representing measures of φ . Let F be an M-null set which is F_σ . Then there exists a sequence $\{a_n\} \subset A$ such that $\|a_n\| \leq 1$, $\forall n \in \mathbb{N}$ and $a_n \longrightarrow 0$ on F and $a_n \longrightarrow 1$ a.e. (λ) , $\forall \lambda \in M_\varphi$.

<u>Proof</u> Let $F = \bigcup_1^\infty K_n$ be an increasing sequence of closed sets in X. Firstly, let n be fixed. Then, since K_n is compact and λ is outer regular, we obtain, using 2.7(c) and 2.7(d), a sequence $\{g_k\} \subset C_R(X)$ such that $g_k \longrightarrow -n\chi_{K_n}$ point-wise from below and a.e.(λ), $\forall \lambda \in M_\varphi$. Furthermore $\| g_k \| \leqslant n$, $\forall k \in \mathbb{N}$. Thus $\int g_k \, d\lambda \xrightarrow[\infty]{k} 0$ from below $\forall \lambda \in M_\varphi$, by Lebesgue dominated convergence theorem (2.7(f)).

In the weak-* topology on M_φ, $f_k : \lambda \longrightarrow \int g_k \, d\lambda$ is continuous (see 2.15(b)) and M_φ is compact (2.15(d)). By 1.6(f), f_k is uniformly continuous on M_φ. So we have:

$$\forall \varepsilon > 0, \ |f_k(\lambda_i) - f_k(\lambda_j)| < \varepsilon ; \text{ for } \lambda_i \text{ close enough to } \lambda_j$$

$$\Rightarrow |f_k(\lambda_i)| < \varepsilon + |f_k(\lambda_j)| ; \text{ for } \lambda_i \text{ close enough to } \lambda_j$$

$$\Rightarrow |f_{k_j}(\lambda_i)| < 2\varepsilon ; \text{ for } k_j \text{ large enough and } \lambda_i \text{ close enough to } \lambda_j .$$

Now the sets $\{\lambda_i : \| \lambda_i - \lambda_j \| < \delta\}$, $\lambda_j \in M_\varphi$ form an open cover for M_φ. Since M_φ is compact, we can choose a finite subcover, $j = 1, 2, \ldots, n$. Then if we set $k = \max\{k_1, \ldots, k_n\}$ we have :

$$\forall \varepsilon > 0, \exists k \in \mathbb{N} \text{ such that } \| f_k(\lambda) \| < 2\varepsilon , \ \forall \lambda \in M_\varphi .$$

So we have $\int g_k d\lambda > -\tfrac{1}{2}n^{-4}$, $\forall \lambda \in M_\varphi$, for some $k \in \mathbb{N}$. Now by the first part of 2.16(f) we have $h_n \in A$ such that

$$\text{Re } h_n \leqslant -n\chi_{K_n} \quad \text{and} \quad \text{Re } \varphi(h_n) > -n^{-4} .$$

(We introduce the notation : $\text{sgn } z = \frac{z}{|z|}$ and $\overline{\text{sgn}}(z) = \frac{\bar{z}}{|z|}$; $z \in \mathbb{C}$.) Now set $a_n = e^{h_n}\, \overline{\text{sgn}}\,(e^{\varphi(h_n)})$. Then $a_n \in A$. (since A contains the constants). $\| a_n \| \leqslant 1$, since $\text{Re } h_n \leqslant 0$ and $|a_n| \leqslant e^{-n}$ on K_n. Thus $a_n \longrightarrow 0$ on F and

$$1 \geqslant \varphi(a_n) = |e^{\varphi(h_n)}| = \exp[\text{Re}\,\varphi(h_n)] \geqslant e^{-n^{-4}} \geqslant 1 - n^{-4}$$

(using Taylor's expansion 1.6(e))

Thus $\int|a_n - 1|^2 d\lambda \leq \int[(\mathrm{Re}\, a_n - 1)^2 + (\mathrm{Im}\, a_n)^2]\, d\lambda$

$$\leq 1 + 1 - 2\,\mathrm{Re}\,\varphi(a_n)$$

$$\leq 2n^{-4}\ , \ \forall\lambda \in M_\varphi\ .$$

Arguing as in the proof of 5.2(i) , we see that $a_n \longrightarrow 1$ a.e.(λ) ,$\forall\, \lambda \in M_\varphi$.

(b) The Generalized F. and M. Riesz Theorem :

Let A be a uniform algebra on X . Let $\varphi \in M_A$ and $\mu \in A^\perp$. If $\mu = \mu_F + \mu_{F'}$ is the Lebesgue decomposition of μ with respect to M_φ , then $\mu_F\ ,\mu_{F'} \in A^\perp$.

<u>Proof</u> We know that F is an M_φ-null set. As we saw in 6.1(c), F can be taken as an F_σ set . So, applying (a), we get $\{a_n\} \subset A$ such that $\|a_n\| \leq 1$, $\forall n \in \mathbb{N}$ and $a_n \longrightarrow 0$ on F and $a_n \longrightarrow 1$ a.e.(λ) , $\forall\lambda \in M_\varphi$. Let E be the set where $a_n \not\rightarrow 1$. Then $\lambda(E) = 0$, $\forall\lambda \in M_\varphi$. So E is M_φ - null . Thus $|\mu_{F'}|(E) = 0$ since $\mu_{F'} << M_\varphi$. Hence $a_n \longrightarrow 1$ a.e.$(|\mu_{F'}|)$. Now let $g \in A$.

We have $\int g\, d\mu_{F'} = \lim_{n\to\infty} \int g\, a_n d\mu$ (2.7(f))

$$= 0\ .$$

Thus $\mu_{F'} \in A^\perp$ and hence $\mu_F \in A^\perp$ as required.

Note that the separation property of A and the closure of A were not used. So the theorem may, in fact, be stated for algebras which contain the constants.

(c) It is clear from the formulation of the Lebesgue decomposition with respect to a set, M, of probability measures (6.1(c)) that the components, μ_F and $\mu_{F'}$, are mutually singular .

(d) Use of the Generalized F. and M. Riesz Theorem enables us to formulate the following decomposition theorem for orthogonal measures.

Let A be a uniform algebra on X and let $\mu \in A^{\perp}$. Then there are linear functionals $\varphi_n \in M_A$ and measures μ_s and μ_n, $n \in \mathbb{N}$, all in $A^{\perp}$ such that μ_s is M_{φ}-singular, $\forall \varphi \in M_A$ and $\mu_n << M_{\varphi_n}$, $\forall n \in \mathbb{N}$. The μ_n are pairwise mutually singular and

$$\mu = \mu_s + \sum_1^{\infty} \mu_n \text{ , the series being norm-convergent .}$$

<u>Proof</u> If $\varphi \in M_A$, let μ_φ be the M_φ-absolutely continuous component of the Lebesgue decomposition of μ with respect to M_φ.

Now let $C_1 = \sup\{\|\mu_\varphi\|;\varphi \in M_A\}$.

If $C_1 = 0$, let $\mu = \mu_s$. Clearly μ_s is M_φ-singular, any $\varphi \in M_A$. If $C_1 \neq 0$, choose $\varphi_1 \in M_A$ such that $\mu_1 = \mu_{\varphi_1}$ satisfies $\|\mu_1\| \geqslant C_1/2$. Say $\mu = \mu_{s1} + \mu_1$.

We now repeat this process.

Let $\mu^1 = \mu - \mu_1$. Set $C_2 = \sup\{\|\mu^1_\varphi ; \varphi \in M_A\}$

If $C_2 = 0$, let $\mu = \mu_s + \mu_1$. Clearly μ_s is M_φ-singular, $\forall \varphi \in M_A$ and $\mu_1 << M_{\varphi_1}$.

If $C_2 \neq 0$, choose $\varphi_2 \in M_A$ such that $\mu^1 = \mu_{s2} + \mu_2$, the Lebesgue decomposition of μ^1 with respect to M_{φ_2}, satisfies $\mu_2 << M_{\varphi_2}$ and $\|\mu_2\| \geqslant C_2/2$.

Now let $\mu^2 = \mu^1 - \mu_2 = \mu - (\mu_1 + \mu_2)$. etc.

We carry on repeating this process as indicated. If it terminates after a finite number of steps, we clearly get a finite decomposition satisfying the statement of the theorem. If not, we obtain a sequence, $\{\mu_n\}$, $n \in \mathbb{N}$, with the following properties.

μ_n is singular to $\mu - \sum_{j=1}^{n} \mu_j$. To see this we have $\mu - \sum_{j=1}^{n} \mu_j = \mu_{sn}$, using the terminology established above. By (c), μ_n is singular to μ_{sn} . In fact, μ_n is carried on a set disjoint from the set which carries μ_{sn} . But any μ_k , $k > n$ will be carried on a set which is a subset of the set carrying μ_{sn} . Thus μ_k is singular to μ_n , for $k > n$. Since this is true $\forall n \in \mathbb{N}$, we see that the μ_n are pairwise mutually singular.

It follows that $\sum_{n=1}^{\infty} \mu_n$ converges in norm and that $\left\| \sum_{n=1}^{\infty} \mu_n \right\| = \sum_{n=1}^{\infty} \| \mu_n \|$.

Now let $\mu_s = \mu - \sum_{n=1}^{\infty} \mu_n$. By the generalized F. and M. Riesz theorem $\mu_n \perp A$, $\forall n \in \mathbb{N}$. So $\mu_s \perp A$.
For any fixed n and any $\varphi \in M_A$ we have : The M_φ-absolutely continuous part of μ^{n-1} has norm less than or equal to $C_n \leq 2\|\mu_n\|$, $\forall \varphi \in M_A$. Now, since μ_s is the restriction of μ^{n-1} to some subset of supp μ^{n-1}, we know that the M_φ-absolutely continuous part of μ_s does not exceed $2\|\mu_n\|$, $\forall \varphi \in M$. But we know that $\|\mu_n\| \xrightarrow[\infty]{n} 0$. Thus μ_s is M_φ-singular , $\forall \varphi \in M_A$ as required.

6.3 We now relate the generalized F. and M. Riesz theorem to some of our earlier results on Gleason parts and T-invariance.

(a) Let A be a uniform algebra on X and let φ , ψ lie in the same Gleason part of M_A . Then the Lebesgue decompositions of a measure, μ , with respect to M_φ and M_ψ coincide.

Proof Clearly it will suffice to show:
E is M_φ-null $\Leftrightarrow$ E is M_ψ-null. Say that E is M_φ-null. Let $\lambda \in M_\psi$. By 5.2(h), $\exists\, \lambda' \in M_\varphi$ such that $\lambda << \lambda'$. Thus $\lambda'(E) = 0 \Rightarrow \lambda(E) = 0$. So E is M_ψ-null. The converse is similarly proved.

(b) Let A be a uniform algebra on X. Let φ and ψ lie in different Gleason parts. For a measure μ, let μ_a be M_φ-absolutely continuous. Then μ_a is M_ψ-singular.

Proof By 5.2(i) we have disjoint Borel sets B_1 and B_2 such that M_φ is carried on B_1 and M_ψ is carried on B_2. Thus μ_a is carried on B_1. Clearly then, μ_a is M_ψ-singular.

(c) It is easily seen from (a) that each φ_n of 6.2(d) lies in a different Gleason part. Now, using (b) as well, we may say:

Let A be a uniform algebra on X and $\mu \perp A$. Then there is a sequence, $F_0, F_1, F_2, \ldots$ of pairwise disjoint Borel sets and a sequence $P_1, P_2, P_3, \ldots$ of Gleason parts for which :

(i) $\mu = \sum_{i=0}^{\infty} \mu_{F_i}$; $\mu_{F_i} \in A^\perp$

(ii) μ_{F_0} is M_φ-singular, $\forall \varphi \in M_A$.

(iii) For $i \geq 1$, μ_{F_i} is M_φ-singular for $\varphi \notin P_i$ and $\mu_{F_i} << M_\varphi$ for $\varphi \in P_i$.

(d) In view of these latter comments, we may restate the decomposition theorem for orthogonal measures as follows :

Let A be a uniform algebra on X. Let $\{\varphi_\alpha\}$ be a subset of M_A containing exactly one linear functional from each part of M_A. Let $\mu \perp A$ and let μ_α be the absolutely continuous component of μ with respect to M_{φ_α}. Then $\mu_\alpha \in A^\perp$ and the μ_α are mutually singular.

Furthermore : (i) $\mu = \mu_s + \sum_\alpha \mu_\alpha$

(ii) $\|\mu\| = \|\mu_s\| + \sum_\alpha \|\mu_\alpha\|$

(iii) $\mu_s \perp A$ and μ_s is M_φ-singular, $\forall \varphi \in M_A$.

(e) We digress briefly to prove a theorem of Wilken :

Let A be a T-invariant uniform algebra on K. If $\mu \in A^\perp$ is M_φ-singular for every $\varphi \in M_A$, then $\mu \equiv 0$.

Proof Assume that $\mu \neq 0$ and $\mu \perp A$. Choose $z_0 \in K$ such that $\tilde{\mu}(z_0) < \infty$ and $\hat{\mu}(z_0) \neq 0$ (such a z_0 exists by 3.5(g) and 5.3(c)). Then we know that $\nu = \dfrac{\mu}{\hat{\mu}(z_0)(z-z_0)}$ is a complex representing measure for z_0 (see 4.4(f)(ii)). Now by 2.17(e), we have a representing measure, λ, for z_0 such that $\lambda << \nu$. Hence $\lambda << \mu$. If λ is also singular with respect to μ, then $\lambda \equiv 0$, which is a contradiction. So the theorem is proved.

(f) We use this and the decomposition theorem in order to extend result 5.3(i).

Let P_i, $i \in \mathbb{N}$, be the non-trivial Gleason parts of a T-invariant uniform algebra A, on K. Then there exist Borel sets E_i , $i \in \mathbb{N}$ such that :

(i) The E_i are pairwise disjoint

(ii) $P_i \subset E_i \subset \overline{P_i}$, $\forall i \in \mathbb{N}$

(iii) Every representing measure for every point in P_i is supported on E_i .

(iv) If $\mu \perp A$, then $\mu_{E_i} \in A^{\perp}$ and $\mu = \sum_{i=1}^{\infty} \mu_{E_i}$.

Proof The first three properties are proved in 5.3(i) . For part (iv) we apply (d) to A , noting that if $\varphi_\alpha = \tau_x$, x a peak point of A , then $\mu_\alpha = 0$. This follows from the fact (3.4(c)) that $M_{\tau_x} = \{\delta_x\}$ and that $\mu\{x\} = 0$. (3.3(h) . By (e) , $\mu_s \equiv 0$.

So, as in (c) we have Borel sets $F_1, F_2, F_3, \ldots$ such that $\mu_{F_i} << M_{\varphi_i}, \forall \varphi_i \in P_i$ and $\mu = \sum_{i=1}^{\infty} \mu_{F_i}$. Now, since $K \sim E_i$ is M_{φ_i}-null , $\forall \varphi_i \in P_i$, we have $F_i \subset E_i$. Since the E_i are pairwise disjoint, we have $\mu(E_i) = \mu_{F_i}(E_i)$, $\forall i \in \mathbb{N}$. Thus $\mu_{E_i} = \mu_{F_i}$ as required.

CHAPTER VII

Glicksberg's theorem on closed restrictions

7.1 In this section we shall prove a result first shown by Glicksberg in [17] . The proof given here is based on the work of Katznelson in [18] on Idempotents in Quotient Algebras. Firstly, some notation and definitions:

(a) In what follows, A is a function algebra on X, a compact, T_2 set. Let F be closed in X ; Then $kF = \{f \in A : f_{|F} = 0\}$ is a closed 2-sided ideal in A . Then $A/_{kF}$ is a Banach algebra with respect to the norm :

$$\|[f]\| = \|f + kF\| = \inf\{\|f + g\| ; g \in kF\}.$$

$[f]$ being the equivalence class containing f . These well-known results may be found, for example, in [11] .

It is easily seen that $\forall x \in F$, the evaluation map : $\varphi_x([f]) = f(x)$ is a multiplicative homomorphism of $A/_{kF}$ into $\mathbb{C}$. i.e. $\varphi_x \in \text{Spec } A/_{kF}$.

Thus $\|\varphi_x\| = 1$ (see 2.14(a)). So we have :

$$\|f\|_F = \|f_{|F}\| \leq \|[f]\| = \|f + kF\| .$$

(b) We state, without proof, the Open-Mapping Theorem :

If X and Y are Banach Spaces and T is a bounded linear transformation which maps X onto all of Y , then T is an open mapping. In particular, if T is also a one-one map, then T^{-1} is continuous.

(see e.g. [19];2,20)

(c) Consider the map $T : A_{/kF} \to A|_F$ given by $T([f + kF]) = f|_F$. This is clearly one-one and onto and, by the inequality in (a), it is continuous, hence bounded. Since T satisfies the conditions of the Open-Mapping theorem, we see that T^{-1} is continuous. So there exists a constant $C_F > 0$ such that $\| f + kF \| \leq C_F \| f \|_F$.

(d) An idempotent in $A_{/kF}$ is an equivalence class of functions in A which take, on F, only the values 0 or 1; i.e. $[f^2] = [f]^2 = [f]$. We can now define a certain concept of boundedness :

A is bounded on $V \subset X$ if $\exists\, C_V > 0$ such that : If F is a closed subset of V and if $[f]$ is an idempotent in $A_{/kF}$, then $\| [f] \| \leq C_V$. A is said to be bounded at a point, $x \in X$, if it is bounded on some neighbourhood of that point.

(e) Let A be such that $A_{|F}$ is closed in $C(F)$ for every closed subset F of X. Then, if F and K are disjoint, closed subsets of X, $\exists\, f \in A$ such that

$$f_{|F} = 1 \quad \text{and} \quad f_{|K} = 0 .$$

<u>Proof</u> Take $x \in F$ and $y \in K$. By the separation property of A, we have $g \in A$ such that $g(x) = 0$ and $g(y) = 1$. . Choose neighbourhoods V_x and W_{xy} of x and y respectively such that $|g(V_x)| \leq \frac{1}{4}$ and $|g(W_{xy}) - 1| \leq \frac{1}{4}$. Since X is normal (see [11] §2,8) we may easily choose V_x such that $V_x \cap K = \emptyset$.

Now we require the existence of a sequence of polynomials $\{p_n(z)\}$ which converge uniformly on $\{z : |z| \leq \frac{1}{4}\} \cup \{z : |1-z| \leq \frac{1}{4}\}$ to a function which is 1 on $\{z : |z| \leq \frac{1}{4}\}$ and 0 on $\{z : |1-z| \leq \frac{1}{4}\}$. This sequence can be constructed as follows : Consider the functions $q_n(z) = \exp\{-e^{nz - n/2}\}$, $n \in \mathbb{N}$. It is easily seen that the sequence, $\{q_n(z)\}$ converges uniformly on $\{z : |z| \leq \frac{1}{4}\} \cup \{z : |1-z| \leq \frac{1}{4}\}$ to the required function. Since each $q_n(z)$ is holomorphic throughout the plane, we have a Taylor series expansion for each one. Now let $p_n(z)$ be the partial sum to n terms of the expansion for $q_n(z)$. The sequence $\{p_n(z)\}$ will do. (see 1.6(f)). Now $\{p_n(g)\}$ will be a sequence in A which converges uniformly on $V_x \cup W_{xy}$ to the function which is 1 on V_x and 0 on W_{xy}. But since, by hypothesis, $A|_{V_x \cup W_{xy}}$ is closed in $C(V_x \cup W_{xy})$, this function belongs to $A|_{V_x \cup W_{xy}}$. Hence $\exists\ e \in A$ st. $e(V_x) = 1$ and $e(W_{xy}) = 0$. Since F is compact, it may be covered by finitely many of the sets V_x, $x \in F$. Call these $V_{x_1}, \ldots, V_{x_n}$. Let $e_1, \ldots, e_n$ be the corresponding elements of A. Then $e^1 = e_1 + e_2 - e_1 e_2$ is 1 on $V_{x_1} \cup V_{x_2}$ and 0 on $W_{x_1 y} \cap W_{x_2 y}$. Proceeding in this way we obtain $f_y \in A$ such that f_y is 1 on F and 0 on $\bigcap_{i=1}^{n} W_{x_i y} = U_y$. But U_y is a neighbourhood of y and, since K is compact, we can find $U_{y_1}, \ldots, U_{y_m}$ which cover K. Let $f_{y_1}, \ldots, f_{y_m} \in A$ be the corresponding functions, achieved as above. Then $f = f_{y_1} f_{y_2} \ldots f_{y_m} \in A$ is 1 on F and 0 on $\bigcup_{j=1}^{m} U_{y_j} \supset K$, as required.

(f) Let A be as in (e). Then, if A is bounded on the open subsets, V_1, V_2, of X, it is bounded on every closed set contained in $V_1 \cup V_2$.

<u>Proof</u> Take a closed $F \subset V_1 \cup V_2$. Then $F \sim V_2$ is closed and so, by normality, $\exists\, W_1$ open, such that $F \sim V_2 \subset W_1$ and $\overline{W}_1 \subset V_1$. Similarly, $F \sim W_1$ is closed and there exists an open set W_2 such that $F \sim W_1 \subset W_2 \subset \overline{W}_2 \subset V_2$. Clearly if $F \subset V_i$; $i = 1$ or 2, then the result follows trivially. So we may assume that $F \cap W_i \neq \emptyset$, $i = 1,2$.

Now the closed sets $F \sim W_2$ and $F \sim W_1$ are disjoint and so, by (e): $\exists\, g \in A$ such that $g|_{F \sim W_2} \equiv 1$ and $g|_{F \sim W_1} \equiv 0$.

Now take a closed set $F_0 \subset F$ and let $f + kF_0$ be an idempotent in $A/_{kF_0}$. Then $f + k(F_0 \cap \overline{W}_j)$ is an idempotent in $A/_{k(F_0 \cap \overline{W}_j)}$, $j = 1,2$ (Since $F_0 \cap \overline{W}_j \subset F_0$, $j = 1,2$).

So there exists $f_j \in f + k(F_0 \cap \overline{W}_j)$ such that $\| f_j \| \leqslant C_{V_j} + 0.1$, say. $i = 1,2$. C_{V_j} being the constant referred to in (d).

On F_0 we have : $f = gf_1 + (1-g)f_2$, since $f_j = f$ on $F_0 \cap \overline{W}_j$ and $F_0 \subset F \subset W_1 \cup W_2$.

So $f + kF_0 = gf_1 + (1-g)f_2 + kF_0$ and

$$\| f + kF_0 \| = \| gf_1 + (1-g)f_2 + kF_0 \| \leqslant \| gf_1 + (1-g)f_2 \|$$
$$\leqslant \| g \| (C_{V_1} + 0.1) + (1 + \| g \|)(C_{V_2} + 0.1) .$$

Thus, this constant, independent of F_0, is an upper bound for any idempotent on F_0. So F is bounded as required.

(g) Let A be as in (e). If $F \subset X$ is closed and A is bounded at every point of F, then there exists an open set,

$V \supset F$, on which A is bounded.

Proof For every $x \in F$ we have an open set V_x , on which A is bounded, such that $x \in V_x$. By the compactness of F , we have a finite number of bounded open sets $V_1, \ldots, V_n$ such that $F \subset \bigcup_1^n V_i$.

Claim: Any closed set E , contained in $\bigcup_1^n V_i$ is bounded. To see this we proceed by induction.
The claim is true for $n = 2$ (by (f)).
Assume that it is true for $n = m$. Consider
$E \subset \bigcup_1^{m+1} V_i$.
If $E \cap V_{m+1} = \emptyset$, then E is bounded, by hypothesis.
If $E \cap V_{m+1} \neq \emptyset$, then $E \sim V_{m+1}$ is a closed set contained in $\bigcup_1^m V_i$.
By the normality of X , there exists an open set
U such that : $E \sim V_{m+1} \subset U \subset \overline{U} \subset \bigcup_1^m V_i$.
By hypothesis, $\overline{U}$ is bounded. Hence U is bounded.
So we have : $E \subset U \cup V_{m+1}$, which is the case $n = 2$. Thus E is bounded as required.

Now, by normality, we have an open set, V , such that:
$F \subset V \subset \overline{V} \subset \bigcup_1^n V_i$. By the claim above, $\overline{V}$ is bounded, hence V is bounded, as required.

(h) Let A be as in (e) . There exist only finitely many points at which A is not bounded.

Proof Suppose the contrary. Let $\{x_i \; ; \; i \in I\}$ be an infinite set of points at which A is not bounded. We can select a sequence $\{x_n , n \in \mathbb{N}\}$ of distinct points and a corresponding sequence $\{V_n\}$ of mutually disjoint neighbourhoods such that $x_n \in V_n$. To see this we proceed by induction :

Assume that we have m points, $x_1 , \ldots , x_m$ with corresponding open neighbourhoods $V_1 , \ldots , V_m$ such that $\overline{V}_i \cap \overline{V}_j = \emptyset$, $i \neq j$ and an infinite number of the x_i lie outside of $\bigcup_1^m \overline{V}_i$. Choose one of these. If it has a neighbourhood such that an infinite number of the x_i lie outside of that neighbourhood, call this point x_{m+1} . If no such neighbourhood exists, choose any other of the $x_i (i \neq 1, \ldots , m)$ and call this x_{m+1} . By the Hausdorff property of X , this will have a suitable open neighbourhood, say U_1 . By the normality of X , there exists an open set, U_2 , such that $x_{m+1} \in U_2$ and $U_2 \cap \bigcup_1^m \overline{V} = \emptyset$. By normality again, $\exists$ an open set V_{m+1} such that $x \in V_{m+1} \subset \overline{V}_{m+1} \subset U_1 \cap U_2$. We now have m + 1 points, with corresponding neighbourhoods, which satisfy the conditions of the above assumption. The case, m = 1 , is easily established, using elements of this argument.

Now, since some neighbourhood of x_n is not bounded, we may assume that V_n is not bounded for all $n \in \mathbb{N}$ (if not, the intersection of V_n with the unbounded neighbourhood will satisfy all of the conditions.) So $\forall n \in \mathbb{N}$, $\exists$ a closed set $F_n \subset V_n$ and an idempotent $f_n + kF_n$ in $A/_{kF_n}$ such that $\| f_n + kF_n \| > n$. Now let $F = \left(\overline{\bigcup_{\mathbb{N}} F_n}\right)$. Then $F = F_n \cup (F \sim F_n)$ is a decomposition of F into disjoint closed sets. To see this we note the following :

$F = F_n \cup \left(\overline{\bigcup_{m\neq n} F_m}\right)$ and $F_n \cap \left(\overline{\bigcup_{m\neq n} F_m}\right) = \emptyset$ since $F_n \subset V_n$ which is open and such that $V_n \cap \left(\bigcup_{m\neq n} F_m\right) = \emptyset$. Furthermore $\left(\overline{\bigcup_{m\neq n} F_m}\right) = (\overline{F \sim F_n})$ and, since this does not intersect with F_n, we have $F \sim F_n = (\overline{F \sim F_n})$ as required. Now by (e), $\exists\ g \in A$ such that $g|_{F_n} = 1$ and $g|_{F\sim F_n} = 0$.

Then $gf_n + kF$ is an idempotent in $A/_{kF}$ and we have :

$\|gf_n + kF\| \geq \|gf_n + kF_n\| = \|f_n + kF_n\| > n$. ($gf_n = f_n$ on F_n)

But, by (c) : $\|gf_n + kF\| \leq C_F\|gf_n\|_F = C_F\|f_n\|_{F_n} = C_F$ or $0, \forall n \in \mathbb{N}$. This gives the required contradiction.

(i) (Glicksberg): Let A be such that $A|_F$ is closed in $C(F)$ for every closed subset, F, of X. Then $A = C(X)$.

<u>Proof</u> Let $T = \{x_1, \ldots, x_m\}$ be the finite set of points on which A is not bounded (by (h)). Let F be closed such that $F \subset X \sim T$.

(1) Firstly we show that for such an F, $A|_F = C(F)$.

By (g) we have an open set V such that $F \subset V$ and $\overline{V} \cap T = \emptyset$ and A is bounded on V. So there exists a constant, $C_V > 0$, such that, if K is a closed subset of V, then every idempotent in $A/_{kK}$ has norm less than or equal to C_V.

Let μ be a regular Borel measure on F which annihilates $A|_F$. Choose any closed set $K \subset F$ and $\varepsilon > 0$. Now we have a closed set, $K_0 \subset F \sim K$ such that $|\mu|(F \sim (K \cup K_0)) < \varepsilon/2C_V$. (by regularity; see 2.1(f)).

By (e), we have : $f \in A$ such that $f|_K = 1$ and $f|_{K_0} = 0$.

Then $f + k(K \cup K_0)$ is an idempotent in $A/_{k(K\cup K_0)}$;

Hence $\|f + k(K \cup K_0)\| \leq C_V$. So there exists $f^* \in A$ such that $f^* \in f + k(K \cup K_0)$ and $\|f^*\| \leq 2C_V$. Also, clearly $f^*|_K = 1$ and $f^*|_{K_0} = 0$. Then

$$0 = \int_F f^* d\mu = \int_K f^* d\mu + \int_{F\sim(K\cup K_0)} f^* d\mu$$

$$\Rightarrow |\mu(K)| = |\int_{F\sim(K\cup K_0)} f^* d\mu| \leq 2C_V \cdot \varepsilon/2C_V = \varepsilon .$$

Since ε is arbitrary, we have $|\mu(K)| = 0$ for all closed sets $K \subset F$. Now, by regularity and the definition of $|\mu|$, we have $|\mu|(F) = 0 \Rightarrow \mu = 0$. So, by 2.11(c) and since $A|_F$ is closed, we have $A|_F = C(F)$.

(2) Secondly we show that every measure, $\mu \in A^\perp$, is supported on T . Take F and V as in (1). Then F and $X \sim V$ are two disjoint closed sets. So, by (e), $\exists f \in A$ such that $f|_F = 1$ and $f|_{X\sim V} = 0$. Then the measure $f\mu$ annihilates A , (2.13(b)) , and is supported on $\bar{V}$. Since $\bar{V} \subset X \sim T$, we have, by (1) : $A|_{\bar{V}} = C(\bar{V})$. So $f\mu \perp C(\bar{V}) \Rightarrow f\mu = 0$ (2.11(c)).
However $f|_F = 1 \Rightarrow f\mu(F) = \mu(F) = 0$. By regularity again, we have $|\mu|(X \sim T) = 0$. Thus μ is supported on T , as claimed.

(3) Finally we show : $\mu \in A^\perp \Rightarrow \mu \equiv 0$. Thus by 2.11(c) , $A = C(X)$.
By (2): $\mu \in A^\perp$ and $f \in A \Rightarrow 0 = \int_X f d\mu = \int_T f d\mu = \sum_1^m f(x_i)\,\mu(x_i)$.

By the separation property, we have, for $i = 2, \ldots, m$; $f_i \in A$ such that :

$f_i(x) = \begin{cases} 1, x = x_1 \\ 0, x = x_i \end{cases}$. Then $\tilde{f}_1 = f_2 f_3 \ldots f_m \in A$ and

$$\tilde{f}_1(x) = \begin{cases} 1, x = x_1 \\ 0, x = x_2, \ldots, x_m \end{cases}$$

So : $0 = | \int_T \tilde{f}_1 \, d\mu | = | \int_{\{x_1\}} d\mu | = |\mu|\{x_1\}$.

Similarly $|\mu|\{x_i\} = 0$, $i = 1, 2, \ldots, m. \Rightarrow |\mu|(T) = 0 \Rightarrow \mu \equiv 0$ as required.

BIBLIOGRAPHICAL NOTES

CHAPTER I

Our definition of a uniform algebra is taken from Gamelin [21]. This coincides with the definition of a uniform algebra in Stout [29] and the definition of a function algebra in Browder [1]. Stout, in fact, defines a function algebra in a slightly wider sense, which does not however coincide with the definition of Browder, in the case of a compact underlying space, since the condition of uniform closure is missing. The predecessor of these notions is probably the "sup-norm algebra" as defined by Hoffman in [31]. To be found in [1] are section 1.4, the notation involving Γ and Δ and the first part of 1.6(g). The second part is taken from Professor Kotzé's notes [5]. The work of 1.6(i) is mainly our own, except the isometric isomorphism, $P(\Gamma) \cong P(\Delta)$, which is mentioned by Browder.

CHAPTER II

The initial Measure theoretic definitions are taken from Halmos [8] and Taylor [9]. The proof of 2.3(c) is substantially due to Professor Kotzé. The proofs of 2.3(d), (e) and (f), although probably well-known, are our own. The compatibility of various definitions and results taken from different sources was carefully investigated and comments with regard to this occur in several places in the text, particularly in section 2.4. The definition of integrability is that of Halmos, while its extension to complex measures follows the outline of Taylor. The proofs of 2.5(c) and (d) are our own, as are the comments in 2.6(d). In 2.7(a), we generalize a result which appears in Bartle [7]. The proof of the uniqueness in 2.7(g) is our own, as is the method of using Lusin's theorem in 2.8(a) to obtain the norm property. The results of sections 2.9 to 2.14 inclusive are outlined in Browder [1], with the exception of the proofs of 2.13, which are our own. Sections 2.15 and 2.16 elaborate on the outline given by Gamelin [21], which excludes our proof of 2.15(d). The proof of 2.17(b) is our own, while that of 2.17(e) appears in [21].

CHAPTER III

The definitions of peak sets and generalized peak sets (and the notation "p-sets") are those used by Gamelin [21]. The approach to the main result, 3.3(h), which was first proved by Glicksberg [26], is, with some modification, basically that presented by Gamelin in [21]. The remark (3.3(k)) about topologies of p-sets, appear in Browder [1] . The proof given is our own. The results about peak points in 3.4 stem mainly from work by Bishop [28] . In Browder [1], result 3.4(d) actually precedes 3.4(c)

CHAPTER IV

The notation and approach to Bishop's Theorem are basically those of Browder [1], with the exception of 4.1(h) which appears in Gamelin [21]. The actual theorem, 4.1(m), is originally due to Bishop [28], who was the first to make extensive use of the Cauchy transform in the area of Rational approximation. The formulation of T-invariance is presented by Gamelin in [25]. The investigation of T-operators and the T-invariance of R(K) and A(K) (a fact stated without proof by Gamelin in [25]) contains elements of arguments from both Gamelin [21] and Browder [1]. Bishop's theorem for T-invariant algebras is stated in [25], where an outline of the proof appears. By means of results 4.4(a), (b) and (e), which we have proved independently, and 4.4(c) and (d), where we have generalized results appearing in [1], we are able to present a full proof. The comments in 4.5, with the exception of 4.5(b), where we generalize a result of Hartogs and Rosenthal [30], are mainly ours.

CHAPTER V

As stated in the text, this idea originates with Gleason [16]. The approach in 5.1 is basically that outlined by Browder [1] and the definition of Gleason parts is the original one, which arises naturally, rather than given by Gamelin [21]. The well-known result, 5.2(a), may be found in [1] and the main characterization theorem, 5.2(b), contains arguments from both [1] and [21] . The remainder of section 5.2 follows the outline of Gamelin [21], which arises mostly from work by Bishop [33]. We generalize a result of Arens [32] in 5.3(a) The proof of 5.3(b) is our own, whereas 5.3(c) may be found in [1]. Results of 5.3(d), (g) and (h) are generalizations of those found in Wilken [34] and [35]. The formulation of Bishop's theorem in 5.3(f) is suggested in Stout [29]. The proof of 5.3(i) generalizes the argument outlined in Gamelin [21]. This and most of the other results in this section are stated without proof in Gamelin [25].

CHAPTER VI

The definitions we give for M-singularity and M-absolute continuity are those of Glickberg [20] rather than those of Gamelin ([21], II, 7) although, as Gamelin points out, the two approaches coincide in our context. The proof of 6.1(c) contains elements of arguments from both Glicksberg and Gamelin. A form of 6.2(a) was first proved by Forelli [22] in the context of Dirichlet algebras, a subject with which we do not deal. The statement we use is due to Glicksberg [20] and the proof, although modified somewhat by us, is essentially his. The classical F. and M. Riesz theorem appears in [23]. The statement and proof of 6.2(b) is essentially that of Glicksberg [20]. This followed on work by Ahern [27]. It can be shown (see [21] II, 7.10 or [1], 4.2.4) that the classical result follows from the abstract result. The Decomposition theorem (6.2(d)) appears in [21] and our proof follows the outline of the proof given there. Results 6.3(a), (b) are mentioned in both [20] and [21]. Result 6.3(c) is stated in [20] and 6.3(d) appears in [21] . In 6.3(e) we have generalized a result of Wilken [24], to apply to T-invariant uniform algebras. The statement of 6.3(f) is given by Gamelin [25] without proof. The proof we give follows naturally from the results previously developed.

CHAPTER VII

As mentioned in the text, the main result is due to Glicksberg [17]. The proof given here appears in Stout [29] and is based on the work of Katznelson [18]. Only the generation of a sequence of polynomials as described in 7.1(e) is our own. The lucidity of this section is due to the exposition thereof by Professor Kotzé in the M.Sc. seminar on uniform algebras, held during 1976 at the University of Cape Town.

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