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UNIVERSITY OF CAPE TOWN

DEPARTMENT OF MATHEMATICS

STRUCTURED FRAMES

BY

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CONTENTS

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INTRODUCTION

Ehresmann [14] in 1959 first articulated the view that a complete lattice with an appropriate distributivity property deserved to be studied as a generalized topological space in its own right. He called the lattice a *local* lattice. Here is the distributivity property :

$$x \wedge \vee x_\alpha = \vee(x \wedge x_\alpha) \quad .$$

A *map* of local lattices should preserve finite meets and arbitrary joins (and hence top and bottom elements).

Dowker and Papert ([11], [12], [13]) introduced the term *frame* for a local lattice and extended many results of topology to frame theory. (The above is not an exhaustive list of their papers.)

At the 1981 international conference on categorical algebra and topology at Cape Town University a suggestion was made that a study of "uniform frames" (whatever they might be!) would be an appropriate and useful start to a project concerned with examining, from a lattice theoretical point of view, the many topological structures which have gained acceptance in the topologist's arsenal of useful tools. It was felt that many of the pre-requisites for such a study had been established, and in fact one of the themes of the conference was the growing role of lattice theory in topology. ([2] , [3]). The suggestion was eagerly accepted, and this thesis is the result.

In 1983, Johnstone's paper "The point of pointless topology" ([24]) appeared. In this paper he stated that a theory of uniform *locales* was of interest. Now a locale is a frame, but a locale map goes the "opposite" way; the category of locales is the opposite of the category of frames. The project had by this time reached the stage where the category of uniform frames (chapter 2) had been established, its basic properties investigated, together with relationships with other categories. Johnstone's paper served to encourage further investigation.

Particularly attractive was the thought that a theory of quasi-uniform frames (non-symmetric uniform frames) might be established. The study of quasi-uniform spaces is known to be at least as general as the study of topological spaces (Császár [10], Pervin [33]); also well known (Salbany [36], Brümmer [6]) was the important role played by the so-called Skula modification (Skula [39]) in the relationship between quasi-uniform spaces and topological spaces, at least at a categorical or functorial level. An exciting question arose; could one hope for similar relationships between appropriate frame structures? If so, what would be the important structure underlying these relationships? The question indeed had an affirmative answer, and the structure concerned proved to be the celebrated congruence lattice or assembly of a frame; this is a structure investigated in part by Dowker and Papert [11] , more fully by Isbell [21] and latterly by Simmons [37] amongst others. This was indeed a rich find and adds a further dimension to the relationship between topology and uniform theory.

On the other hand, topological insight yielded new facts about the congruence lattice, which led to a characterization of the congruence lattice as an initial object in a certain functor category. It should be noted that a biframe point of view was particularly fruitful here.

A suggestion was made that a useful exercise would be to develop, as far as possible, the theory of the assembly from a congruence point of view. This was successfully carried out, yielding a simple and attractive construction. This particular approach, it was soon realized was also susceptible to considerable generalization, permitting the construction of an assembly-type of structure, with all its important functorial properties, for lattices more general than frames.

Two themes emerge in this thesis. The first is the use of "open" and "spectrum" functors to serve as categorical guides to the "correctness" of the categories posited. The "open" functor "forgets the points"; the spectrum of a frame is the set of completely prime filters (filters inaccessible by arbitrary joins) on the frame, together with a certain "spectral" topology. Well known are the facts that

(i) "taking the spectrum" and "forgetting the points" are adjoint on the right

(ii) a frame is representable as an open set lattice when it has a "sufficient" number of completely prime filters (called "points"!)

It was felt that a suitable theory of uniform or quasi-uniform frames should yield a similar situation in terms of functors and representability; a major part of the work in this thesis is directed to establishing an affirmative answer to this question.

The other theme is that of structures using *covers*. All the structures in this thesis are ultimately presented in terms of covers of a frame or set. (A cover, C , of a frame, L , satisfies $\bigvee C = 1$.) Isbell [22] expresses a predeliction for the use of covers in the study of uniform spaces, and the work in this thesis serves to bolster this attitude; families of covers constitute the only tool that works for frames. They even work (suitably modified) in non-symmetric situations. In fact, an understanding of the role of covers in quasi-uniform spaces/frames has yielded an unexpected bonus, namely an interesting category of non-symmetric nearness spaces which, even though considerably more general than Herrlich's category of nearness spaces ([19]), still retains much of the richness of this category. This has been developed in the last chapter of the thesis; it does not readily fall under the umbrella of "structured frames" as yet, but is a logical development of the study of quasi-uniform spaces and frames.

There is some preliminary evidence for the thesis that covers are the "right" objects to work with. The fact that alternatives to covers exist in the study of uniform and quasi-uniform spaces seems to be as a result of the existence of a "sufficient" number of "points" (a space always has enough "points"). If we consider, for example, Császár's syntopogenous structures ([10]), which, it is claimed, serve as a foundation for topology, we can construct a category of syntopogenous frames. In the absence of a certain degree of "spatiality", we find that the "uniform" syntopogenous frames need not coincide with uniform frames, and that the spectrum of a "uniform" syntopogenous frame need not be a "uniform" syntopogenous space. Further investigation of this problem is called for.

With the categories of uniform and quasi-uniform frames established, it was felt that important questions to consider would be the existence of links between "proximal" structures (similar to those of Banaschewski [1]), "quasi-proximal" structures and totally bounded uniform or quasi-uniform structures. The results achieved are as would be expected and further confirm the correctness of the notions established.

As an external application of the structures mooted, a brief chapter is devoted to their relationship to similar structures in fuzzy topology. It was felt that it would be inappropriate to pursue this subject in any detail, but it is clear that the structures developed in this thesis underpin much of the work done on the corresponding fuzzy structures. This seems to be an important fact which it is hoped will be further developed. The suggestion can be made that a clearer understanding of the relationships between topology, fuzzy topology and frame theory is becoming increasingly urgent.

To conclude, it is felt that a useful theory of uniform and related frame structures has been established, and that there is much external evidence for the correctness of these notions. By the very nature of the subject, this thesis must be regarded as a preliminary investigation of these issues; the choice of questions examined must, to a certain extent, be a matter of taste, and much work remains to be done. It is hoped that a significant groundwork has been laid.

SUMMARY

Chapter 1

This is devoted to well-known basic results concerning topology, bitopology, frames and biframes. All results here appear in the literature.

Chapter 2

The category of uniform frames is established. Uniform frames are seen to be (completely-) regular; every completely regular frame has a compatible uniform structure. It must be noted that Pultr [35] in a paper which appeared in 1984 considered frames with various uniform-type structures. He too proved that completely regular frames have a compatible uniform structure in our sense, but his method of proof is different; the proof presented here is considerably simpler and the resulting structure is shown to be functorial. Pultr does not consider categorical aspects of his constructions. Uniform "open" and "spectrum" functors are established and they are shown to be adjoint on the right; the spatial uniform frames are characterized and an alternative construction of the separated reflection of a uniform space is noted.

Chapter 3

In this chapter, a category of quasi-uniform frames is presented. Important, here, is the use of conjugate covers.

Gantner and Steinlage [16] characterized quasi-uniform spaces in terms of conjugate covers, but their formulation of these covers is somewhat cumbersome. A simpler formulation is offered and a series of technical lemmas render the use of conjugate covers (whether of spaces or frames) relatively straightforward.

A quasi-uniform frame is shown to be a completely regular biframe, and any completely regular biframe is shown to have a functorial compatible quasi-uniform structure. In fact all results of chapter 2 can be recovered from results of chapter 3, but a separation of these results seemed necessary from the point of view of clarity. Quasi-uniform "open" and "spectrum" functors are constructed (adjoint on the right) and spatial objects are discussed.

Chapter 4

Links between (quasi-) proximal frames and (quasi-) uniform frames are examined. The equivalence of the category of totally bounded (quasi-) uniform frames and the category of (quasi-) proximal frames is established in a simple manner (which specializes elegantly to the corresponding result for spaces.) For completeness, (quasi-) proximal "open" and "spectrum" functors are exhibited (but see the final note in the notes at the end of this chapter).

Chapter 5

The Pervin *covering* quasi-uniformity for a topological space is constructed. This serves to introduce the question of whether an arbitrary frame can "appear" as the "first" subframe of a canonical (functorial) quasi-uniform frame. To answer this question, the congruence lattice or assembly of a frame is introduced.

As mentioned in the introduction, a simple and attractive comprehensive treatment of this structure is presented in terms of congruences. (See Johnstone [23] for a presentation in terms of nuclei.) This treatment leads to a realization that similar structures together with important functorial properties exist for lattice structures more general than frames (σ-frames, distributive lattices).

Important in this chapter is the realization that the congruence lattice of a frame is naturally viewed as a completely regular biframe in which the "first" subframe is just (an isomorphic copy of) the given frame, a fact which has as yet passed unnoticed or unused.

The readers attention is drawn to theorem 5.17 which plays such a vital role in both the functorial nature of the congruence lattice and its characterization as an initial object in theorem 5.31. Crucial also is the observation that the congruence lattice of the 3-chain, $\underline{3}$, is simply the four element Boolean algebra, and that consequently the congruence lattice of $\underline{3}$ has, as a biframe, a unique compatible quasi-uniform structure. To differentiate between the congruence lattice as frame from its role as biframe, we have denoted the biframe structure as Sk(–), a reference to the analogy with the Skula bitopology.

Functoriality of Sk and the uniqueness of a compatible quasi-uniform structure for Sk(3) lead to the interesting result of theorem 5.35 which echoes Brümmer's observation [5] that the coarsest functorial quasi-uniformity compatible with a given topology is the Pervin quasi-uniformity (hence the notation $(Sk(L), q^*_{PL})$.) Theorem 5.31 must be seen as the frame equivalent of Salbany's result [36] that the Skula functor (from topological spaces to completely regular bitopological spaces) is the unique right inverse to the forgetful functor (going "the other way") which "forgets" the second topology.

The chapter ends with an interesting result in a similar vein for quasi-proximal frames.

It is felt that the biframe approach has yielded much important insight on this chapter.

Chapter 6

Fuzzy "open" and "spectrum" functors, adjoint on the right, are constructed. This categorical link between frames and fuzzy topologies does not seem to appear in the literature. Brief mention is made of the fact that this functorial link can be extended to all other structures considered in this thesis and their corresponding fuzzy structures. Many questions arise (fuzzy spatiality, fuzzy soberness) and it is felt that an interesting area for research has been uncovered. See the notes at the end of the chapter as well.

Chapter 7

The category of non-symmetric nearness spaces, which we will call quasi-nearness spaces, (in line with quasi-uniform spaces etc.) is established. (For a discussion of the use of "quasi" here, see the beginning of chapter 7.) Its relationship with nearness, quasi-uniform, quasi-proximal and topological spaces is investigated.

Many other structures more general than nearness spaces have been investigated (Harris [18], Morita [31]) but a notion of non-symmetric nearness space which preserves the flavour of nearness spaces seems to have escaped notice as yet. The theory developed here seems to have just such a flavour. Attention is drawn both to proposition 7.30 which is an improvement on Salbany's result mentioned above in respect of chapter 5, as well as to the interesting role of the Skula bitopology in this chapter.
A successful theory of completions or bicompletions would render the category of even more interest. The question is under consideration.

Prerequisites for reading this thesis include a basic knowledge of lattice theory, category theory and of course a familiarity with the many topological structures considered in the thesis especially uniform, and quasi-uniform spaces. Readers interested in the last chapter should also familiarize themselves at least with Herrlich's paper "A concept of nearness" [19].

A LIST OF CATEGORIES USED

BIFRM	=	(biframes, biframe maps)
BITOP	=	(bitopological spaces, bicontinuous maps)
COMP REG FRM	=	(completely regular frames, frame maps)
FRM	=	(frames, frame maps)
FUZZTOP	=	(fuzzy topological spaces, fuzzy continuous maps)
NEAR	=	(nearness spaces, nearness maps)
PROX	=	(Proximity spaces, proximity maps)
PROXFRM	=	(proximal frames, proximal maps)
Q-NEAR	=	(quasi-nearness spaces, quasi nearness maps)
QPROX	=	(quasi-proximity frames, quasi-proximity maps)
QPROXFRM	=	(quasi-proximal frames, quasi-proximal maps)
QUN	=	(quasi-uniform spaces , quasi-uniformly continuous maps)
QUNFRM	=	(quasi-uniform frames, quasi-uniform maps)
R_0	=	(R_0-topological spaces, continuous maps)
SOB	=	(sober spaces, continuous maps)
SPFRM	=	(spatial frames, frame maps)
T-NEAR	=	(topological nearness spaces, nearness maps)
TOP	=	(topological spaces, continuous maps)
TQ-NEAR	=	(topological quasi-nearness spaces, quasi-nearness maps)
UNIF	=	(uniform spaces, uniformly continuous maps)
UNIFRM	=	(uniform frames, uniform maps)
UQ-NEAR	=	(uniform quasi-nearness spaces, quasi-nearness maps)
2-R_0	=	(pairwise-R_0 bitopological spaces, bicontinuous maps)

GLOSSARY OF SOME SYMBOLS USED

$\in$	:	element of
$\cap$	:	intersection
$\cup$	:	union
$\subseteq$	:	inclusion
$\emptyset$	:	empty set
$A \times B$	:	cartesian product of A with B
$X \setminus A$	:	$\{x \in X: x \notin A\}$
$\bar{A}$	:	closure of A with respect to a topology
PX	:	power set of X
$f^{-1}(A)$	:	$\{x: f(x) \in A\}$
$\circ$	:	composition of functions
$\wedge$	:	meet (finite)
$\vee$	:	join (finite)
$\bigwedge$	:	meet (arbitrary)
$\bigvee$	:	join (arbitrary)
1	:	top element of lattice
0	:	bottom element of lattice
a'	:	complement of a $(a \wedge a' = 0, a \vee a' = 1)$
hom (A,B)	:	homomorphisms from object A to object B
$\Rightarrow$	:	implication
$\Longleftrightarrow$	:	double implication
iff	:	if and only if
$\mid$	:	negation (of a relation)
$\square$	:	end of proof or sometimes end of statement (if proof omitted).

CHAPTER 1

This chapter consists of relevant background, examples, and ends with the important adjoint situation for the categories of frames and topological spaces, and includes a discussion of the so-called fixed objects of this adjunction; similar results for bitopological spaces and biframes are briefly mentioned. All results are well known; results in the case of biframes are due to Banaschewski, Brümmer, and Hardie [3].

1.1 Definition

(i) A *frame*, L, is a complete lattice satisfying the (infinite) distributive law $a \wedge \bigvee x_\alpha = \bigvee(a \wedge x_\alpha)$ where $\alpha \in A$, an arbitrary set.

(ii) A *frame map* (homomorphism) $f: L \to M$ is a function preserving the top and bottom elements (which we will always denote by 1 and 0 respectively), finite meets, and arbitrary joins.

(iii) Frames and frame maps are the objects and arrows of the category FRM.

Remarks:

(i) A frame is clearly (finitely) distributive.

(ii) A useful way of proving that a complete lattice is indeed a frame is to show that all so-called relative pseudo-complements exist. We will need this approach on occasion, so the details are provided below.

1.2 Proposition

Let L, M be complete lattices, $f: L \to M$ and $g: M \to L$ order preserving functions. If for all $x \in L$, $y \in M$ we have :

$$f(x) \leq y \iff x \leq g(y)$$

then f preserves arbitrary joins.

Proof:

Since f is order preserving, it is clear that $\vee f(x_\alpha) \leq f(\vee x_\alpha)$ $(\alpha \in A)$. We show that $f(\vee x_\alpha) \leq \vee f(x_\alpha)$; notice that, since $f(x_\alpha) \leq f(x_\alpha)$, we must have

$$x_\alpha \leq gf(x_\alpha) .$$

Now $\vee x_\alpha \leq \vee gf(x_\alpha) \leq g(\vee f(x_\alpha))$, so

$f(\vee x_\alpha) \leq \vee f(x_\alpha)$, as desired. □

Remark:

We say f is left-adjoint to g .

1.3 Definition

Let L be a complete lattice with $a, b \in L$. The element $a \to b$ of L (if it exists) has as defining property:

$$a \wedge t \leq b \iff t \leq a \to b \quad ;$$

$a \to b$ is called the *pseudo-complement of* a *relative to* b.

1.4 Proposition

Let L be a complete lattice. If for every pair of elements a, b of L, $a \to b$ exists, then L is a frame.

Proof:

Define $f_a(t) = a \wedge t$, $g_a(t) = a \to t$; f_a, g_a are order preserving functions, and from definition 1.3

$$f_a(t) \leqslant b \iff t \leqslant g_a(b) .$$

Now by Proposition 1.2, f_a preserves arbitrary joins, which is what we wanted; $a \wedge (\bigvee x_\alpha) = \bigvee (a \wedge x_\alpha)$. The converse to this proposition is trivial. □

Remark:

We denote by a^* the element $a \to 0$, called the pseudo complement of a . In a frame $a^* = \bigvee\{t: a \wedge t = 0\}$. One should note that in general if $f: L \to M$ is a frame map, it does not follow that $f(a^*) = f(a)^*$, and so a frame map need not preserve relative pseudo complements.

1.5 Examples

(i) Every complete chain is a frame (if $a \leqslant b$, $a \to b = 1$; if $a \geqslant b$, $a \to b = b$). Of particular importance to us are the 2-chain, $\underline{2}$, and the 3-chain, $\underline{3}$.

(ii) A complete boolean algebra, B , is a frame. ($a \to b = a' \vee b$, where a' is the unique complement of a).

(iii) Let (X,T) be a topological space; T is a frame ($\wedge = \cap$, $\vee = \cup$, arbitrary meet = interior of intersection).

(iv) Frames that are not topologies exist; any complete boolean algebra without atoms is not a family of subsets of a set.

1.6 Definition

$K \subseteq L$ is a sub frame of L iff $\{0,1\} \subseteq K$ and K is closed under finite meets, arbitrary joins.

We introduce now a generalization of the notion of a frame; the motivation is bi-topological and the ideas are due to Banaschewski, Brümmer and Hardie [3].

1.7 Definition

(i) An ordered triple $L = (L_0, L_1, L_2)$ is a *biframe* iff L_1 and L_2 are sub frames of L_0 and generate L_0, so that $a \in L_0 \Rightarrow a = \bigvee(b_\alpha \wedge c_\alpha)$, where $b_\alpha \in L_1$, $c_\alpha \in L_2$, $\alpha \in A$.

(ii) A *biframe map* $f: (L_0, L_1, L_2) \to (M_0, M_1, M_2)$ is a frame map from L_0 to M_0 which maps L_1 into M_1 and L_2 into M_2.

(iii) Biframes and biframe maps are the objects and arrows of the category BIFRM.

1.8 Examples

(i) Let L be a frame; (L, L, L) is trivially a biframe.

(ii) Let (X, T_1, T_2) be a bitopological space; then $(T_1 \vee T_2, T_1, T_2)$ is a biframe.

We consider now some important subcategories of FRM; the theme is "separation".

1.9 Definition

Let L be a frame, with $a,b \in L$.

(i) We write $b \,\bar{<}\, a$ if there is a "separating" element, s , of L such that $b \wedge s = 0$, $a \vee s = 1$; b is then "rather below" a .

(ii) Let $J = \left\{\frac{m}{2^n} : m,n \in N ; \quad 0 \leqslant m \leqslant 2^n\right\}$. We write $b \,\bar{<}\bar{<}\, a$ iff there is a sequence $\{x_\alpha : \alpha \in J\}$ of "interpolating elements" satisfying

$$x_0 = b \ , \ x_1 = a \ , \ \alpha < \beta \in J \Rightarrow x_\alpha \,\bar{<}\, x_\beta \ ;$$

b is then "completely below" a .

1.10 Definition

A frame, L , is

(i) *regular* if $a = \bigvee\{b : b \,\bar{<}\, a\}$.

(ii) *completely regular* if $a = \bigvee\{b : b \,\bar{<}\bar{<}\, a\}$ for each $a \in L$.

1.11 Definition

A frame, L , is *normal* if for any pair of elements a,b of L such that $a \vee b = 1$, there are "separating" elements s,t of L such that :

$$s \wedge t = 0 \quad , \quad a \vee t = 1 \quad , \quad b \vee s = 1 \ .$$

1.12 Examples

(i) Let (X,T) be a topological space. (X,T) is regular, (completely regular, normal) iff T is regular (completely regular, normal).

(ii) Let B be a complete boolean algebra. B is normal, completely regular (hence also regular): if $a \vee b = 1$ then $a' \wedge b' = 0$ and we take as separating elements the pair a', b'. Complete regularity is easy; since $a \wedge a' = 0$, $a \vee a' = 1$, we have $a \mathrel{\overline{<}} a$ for any $a \in B$; this automatically yields $a \mathrel{\overline{<\!<}} a$.

At the risk of boring the reader, we present similar ideas for the biframe situation.

1.13 Definition

Let (L_0, L_1, L_2) be a biframe, and $a_i, b_i \in L_i$ $(i = 1 \text{ or } 2)$.

(i) We write $b_i \mathrel{\overline{<}_i} a_i$ if there is an element s_j of L_j $(j = 1 \text{ or } 2,\ j \neq i)$ such that $b_i \wedge s_j = 0$, $a_i \vee s_j = 1$.

(ii) We write $b_i \mathrel{\overline{<\!<}_i} a_i$ iff there is a sequence $\{x_{i\alpha} : \alpha \in J\} \subseteq L_i$ such that $x_{i0} = b_i$, $x_{i1} = a_i$, $\alpha < \beta \in J \Rightarrow x_{i\alpha} \mathrel{\overline{<}_i} x_{i\beta}$.

And we say (L_0, L_1, L_2) is *regular* (*completely regular*) if $a_i = \bigvee\{b_i : b_i \mathrel{\overline{<}_i} a_i\}$ $(a_i = \bigvee\{b_i : b_i \mathrel{\overline{<\!<}} a_i\})$ for each $a_i \in L_i$ $(i = 1 \text{ or } 2)$.

1.14 Definition

A biframe (L_0, L_1, L_2) is *normal* if for any pair of elements $a_1 \in L_1$, $a_2 \in L_2$ such that $a_1 \vee a_2 = 1$, there are separating elements $s_1 \in L_1$, $s_2 \in L_2$ such that:

$$s_1 \wedge s_2 = 0, \quad s_1 \vee a_2 = 1, \quad s_2 \vee a_1 = 1 .$$

Remark:

One should note that if the biframe (L_0, L_1, L_2) is regular (completely regular, normal), then L_0 is regular (completely regular, normal) as a frame. The converse is not true.

We turn now to the relationship between frames and topologies; while example 1.5(iv) shows that frames are not simply topologies, there is nevertheless an important adjoint situation for FRM and TOP, which is explained below.

1.15 Definition

(i) Let (X,T) be a topological space; set $\mathcal{Q}(X,T) = T$. If $f: (X,T) \to (Y,S)$ is continuous, define $\mathcal{Q}f: \mathcal{Q}(Y,S) \to \mathcal{Q}(X,T)$ by $\mathcal{Q}f(V) = f^{-1}(V)$, where $V \in S$, then $\mathcal{Q}$ is a contravariant functor, the "open" functor, from TOP to FRM.

(ii) Let L be a frame. The *spectrum* of L, ΣL, is the set $\hom(L,\underline{2})$ (the so-called "points" of L). For each $a \in L$, let $\Sigma_a = \{p \in \Sigma L: p(a) = 1\}$. The family $T_{\Sigma L} = \{\Sigma_a: a \in L\}$ is the *spectral topology* on ΣL. Now let $f: L \to M$ be a frame map; define $\Sigma f: \Sigma M \to \Sigma L$ by $\Sigma f(p) = p \circ f$; Σf is continuous with respect to the two spectral topologies; we thus again have a contravariant functor from FRM to TOP.

1.16 Theorem

The functors $\mathcal{Q}$: TOP $\to$ FRM and Σ: FRM $\to$ TOP are adjoint on the right.

Proof:

Let (X,T) be a topological space, L a frame and suppose $f \in \hom(L,\mathcal{Q}X)$

$g \in \hom((X,T), (\Sigma L, T_{\Sigma L}))$.

Define $\bar{f}: (X,T) \to (\Sigma L, T_{\Sigma L})$ by $\bar{f}(x)(a) = 1$ iff $x \in f(a)$

$\tilde{g}: L \to \mathcal{Q}X$ by $\tilde{g}(a) = g^{-1}(\Sigma_a)$:

we have $x \in \bar{f}^{-1}(\Sigma_a) \iff \bar{f}(x) \in \Sigma_a$

$\iff \bar{f}(x)(a) = 1$

$\iff x \in f(a)$ so $\bar{f}^{-1}(\Sigma_a) = f(a) \in \mathcal{Q}X$,

showing that $\bar{f}$ is indeed continuous; $\tilde{g}$ is almost trivially a frame map, since $\Sigma_a \cap \Sigma_b = \Sigma_{a \wedge b}$ and $\bigcup \Sigma_a = \Sigma_{\bigvee a}$.

We now see that $\tilde{\bar{f}} = f$ and $\bar{\tilde{g}} = g$:

$\tilde{\bar{f}}(a) = \bar{f}^{-1}(\Sigma_a) = f(a)$ as seen above; also

$\bar{\tilde{g}}(x)(a) = 1$ iff $x \in \tilde{g}(a)$

iff $x \in g^{-1}(\Sigma_a)$

iff $g(x) \in \Sigma_a$

iff $g(x)(a) = 1$,

so $\tilde{\bar{f}} = f$ and $\bar{\tilde{g}} = g$. Thus $\hom(L, \mathcal{Q}X) \cong \hom((X,T),(\Sigma L,T_{\Sigma L}))$.

We check also the appropriate naturality conditions; let $h \in \hom(L,M)$, $j \in \hom((X,T) , (Y,S))$; the following diagrams are helpful.

$$\begin{tikzcd}
L \arrow[d, "h"'] & \mathrm{hom}(L,\mathcal{Q}X) \arrow[r, "\overline{(-)}"] & \mathrm{hom}((X,T)\,,\,(\Sigma L,T_{\Sigma L})) \\
M & \mathrm{hom}(M,\mathcal{Q}X) \arrow[u, "(-)\circ h"] \arrow[r, "\overline{(-)}"'] & \mathrm{hom}((X,T)\,,\,(\Sigma M,T_{\Sigma M})) \arrow[u, "\Sigma h\circ(-)"']
\end{tikzcd} \qquad [1]$$

Starting with $k \in \mathrm{hom}(M,\mathcal{Q}X)$, we show that $\overline{k\circ h} = \Sigma h\circ\bar{k}$:

$\overline{k\circ h}(x)(a) = 1$ iff $x \in k\circ h(a)$, but on the other hand

$\Sigma h\circ\bar{k}(x)(a) = 1$ iff $(\bar{k}(x)\circ h)(a) = 1$

iff $\bar{k}(x)(h(a)) = 1$

iff $x \in k(h(a)) = k\circ h(a)$ as required,

so [1] commutes.

$$\begin{tikzcd}
(X,T) \arrow[d, "j"'] & \mathrm{hom}((X,T)\,,\,(\Sigma L,T_{\Sigma L})) \arrow[r, "\widetilde{(-)}"] & \mathrm{hom}(L,\mathcal{Q}X) \\
(Y,S) & \mathrm{hom}((Y,S)\,,\,(\Sigma L,T_{\Sigma L}) \arrow[u, "(-)\circ j"] \arrow[r, "\widetilde{(-)}"'] & \mathrm{hom}(L,\mathcal{Q}Y) \arrow[u, "\mathcal{Q}j\circ(-)"']
\end{tikzcd} \qquad [2]$$

Starting with $\ell \in \mathrm{hom}((Y,S)\,,\,(\Sigma L,T_{\Sigma L}))$ we show that $\widetilde{\ell\circ j} = \mathcal{Q}j\circ\tilde{\ell}$:

$\widetilde{\ell\circ j}(a) = (\ell\circ j)^{-1}(\Sigma_a) = j^{-1}(\ell^{-1}(\Sigma_a))$, but on the other hand $\mathcal{Q}j\circ\tilde{\ell}(a) = \mathcal{Q}j(\ell^{-1}(\Sigma_a)) = j^{-1}(\ell^{-1}(\Sigma_a))$ as required, so [2] commutes. □

Set $L = \mathcal{Q}X$ in the above adjunction and let $\sigma_X = \overline{id}_X$; $\sigma_X(x) = \overline{id}_{\mathcal{Q}X}(x)$, and $\overline{id}_{\mathcal{Q}X}(x)(U) = 1$ iff $x \in U$, so $\sigma_X(x)(U) = 1$ iff $x \in U$; it is perhaps more suggestive to denote $\sigma_X(x)$ by ψ_x , indicating a kind of characteristic function.

Similarly, set $X = \Sigma L$ in the above adjunction, and let

$$O_L = \widetilde{id}_{\Sigma L} \; ; \qquad O_L(a) = id^{-1}_{\Sigma L}(\Sigma_a) = \Sigma_a \; .$$

We show now that the adjoint situation described above restricts to a duality of categories when we consider sober spaces on the one hand and spatial frames on the other.

1.17 Definition

(i) A closed set, A, is *join irreducible* if B, C closed and $B \cup C = A$ implies B or C is A.

(ii) A space (X,T) is *sober* iff each join irreducible closed set is the closure of a unique singleton set.

1.18 Proposition

Let X be a topological space. The following are equivalent.

(i) X is sober.

(ii) X is T_0 and every point of $\Sigma\mathcal{Q}X$ is of the form ψ_x for some $x \in X$.

(iii) $\sigma_X : X \to \Sigma\mathcal{Q}X$ is an isomorphism.

Proof:

(i) $\Rightarrow$ (ii) : T_0 is easy. Suppose $\psi \in \Sigma\mathcal{Q}X$. Set $C = X \setminus \cup\{V : \psi(V) = 0\}$; C is a join irreducible set, so $C = \overline{\{x\}}$ for unique x ; one checks that $\psi = \psi_x$.

(ii) $\Rightarrow$ (iii): By supposition σ_X is onto; T_0 ensures 1-1; $\sigma_X(U) = \Sigma_U$, so σ_X is a topological isomorphism.

(iii) $\Rightarrow$ (i): Suppose A is closed, join irreducible. Define $\psi: \mathcal{Q}X \to \underline{2}$ by $\psi(V) = 0$ iff $V \cap A = \emptyset$; ψ is indeed a morphism, so $\psi = \psi_x$, and then it is easy to see that $A = \overline{\{x\}}$. □

1.19 Proposition

Let L be a frame; the following are equivalent.

(i) L is spatial (a topology).

(ii) The "points" of L separate L.

(iii) $O_L: L \to \mathcal{Q}\Sigma L$ is an isomorphism.

Proof:

(i) $\Rightarrow$ (ii): Suppose $L = \mathcal{Q}X$, and let $U, V \in \mathcal{Q}X$, $U \neq V$. Assume $x \in U$, $x \notin V$; $\psi_x(U) = 1$, $\psi_x(V) = 0$, so the points separate L.

(ii) $\Rightarrow$ (iii): $O_L: L \to \mathcal{Q}\Sigma L$ is easily a surjection. Suppose $a \neq b \in L$. By assumption for some $p: L \to \underline{2}$, $p(a) = 0$ and $p(b) = 1$, say. Hence $p \notin \Sigma_a$, $p \in \Sigma_b$, so O_L is injective.

(iii) $\Rightarrow$ (i): Obvious. □

1.20 Proposition

(i) On the categories of sober spaces and spatial frames $\mathcal{Q}$ and Σ induce a dual equivalence.

(ii) $\sigma_X : X \to \Sigma\mathcal{Q}X$ and $O_L : L \to \mathcal{Q}\Sigma L$ are reflections.

Proof:

(i) Clear from 1.18 and 1.19.

(ii) We prove only that σ_X is the "sobrification" reflection:

suppose $f: X \to Y$ is continuous, Y sober .

$$\begin{tikzcd} X \arrow[r, "\sigma_X"] \arrow[d, "f"'] & \Sigma\mathcal{Q}X \arrow[d, "\Sigma\mathcal{Q}f"] \\ Y \arrow[r, phantom, "\underset{\sigma_Y}{\cong}"] & \Sigma\mathcal{Q}Y \end{tikzcd}$$

One checks that $\Sigma\mathcal{Q}f \circ \sigma_X(x) = \sigma_Y(f(x))$, and the factorization of f through σ_X is clearly unique.

□

Remark:

SOB (sober spaces, continuous functions) and SPFRM (spatial frames and frame maps) are the largest subcategories of TOP and FRM respectively which remain fixed under σ and 0 ; we call these objects the *fixed objects* of the adjunction.

To close this chapter of background material, we give brief details of the "open" and "spectrum" functors between the categories BIFRM and BITOP.

Let (X, T_1, T_2) be a bitopological space; set $\mathcal{Q}(X, T_1, T_2) = (T_1 \vee T_2, T_1, T_2)$; if $f: (X, T_1, T_2) \to (Y, S_1, S_2)$ is bi-continuous, define $\mathcal{Q}f: \mathcal{Q}(Y, S_1, S_2) \to \mathcal{Q}(X, T_1, T_2)$ by $\mathcal{Q}f(V_i) = f^{-1}(V_i)$ for $V_i \in S_i$ $(i = 1 \text{ or } 2)$ and extend $\mathcal{Q}f$ to $S_1 \vee S_2$ in the obvious way; $\mathcal{Q}f$ is a biframe map.

Now let $L = (L_0, L_1, L_2)$ be a biframe; set $\Sigma L_0 = \hom(L_0, \underline{2})$; let $T_{\Sigma L_i} = \{\Sigma_a : a \in L_i\}$, $(i = 1 \text{ or } 2)$, where, as usual, $\Sigma_a = \{p \in \Sigma L_0 : p(a) = 1\}$. Then $(\Sigma L_0, T_{\Sigma L_1}, T_{\Sigma L_2})$ is a bitopological space. Let $f: (L_0, L_1, L_2) \to (M_0, M_1, M_2)$ be a biframe map and define $\Sigma f: \Sigma M_0 \to \Sigma L_0$ by $\Sigma f(p) = p \circ f$; Σf is bicontinuous from $(\Sigma M_0, T_{\Sigma M_1}, T_{\Sigma M_2})$ to $(\Sigma L_0, T_{\Sigma L_1}, T_{\Sigma L_2})$.

1.21 <u>Theorem</u>

$\mathcal{Q}$: <u>BITOP</u> $\to$ <u>BIFRM</u> and Σ: <u>BIFRM</u> $\to$ <u>BITOP</u> are adjoint on the right. The fixed objects of the adjunction are the spatial biframes on the one hand and on the other, the bitopological spaces (X, T_1, T_2) where $(X, T_1 \vee T_2)$ is sober.

<u>Proof</u>:

Banaschewski, Brümmer and Hardie [3]. □

Notes on Chapter 1

(1) The reader is referred to "Categories for the Working Mathematician" by Saunders MacLane [30] for all basic categorical notions and notation. A reference for topological notions hardly needs to be cited; Kelley [27] remains appropriate. Bitopology was first mooted by Kelly [28]. We cite Crawley and Dilworth [9] as a basic reference for lattice theory notions; a more recent reference would be Johnstone's "Stone Spaces" [23].

(2) Results in this chapter are well known. Results proved concerning relative pseudo-complemtns and adjoints are not of the utmost generality; we have merely provided the tools we will need later.

(3) Completely prime filters on a frame L (see introduction) are in 1-1 correspondence with homomorphisms from L to the 2-chain. The construction of the spectrum has been presented in terms of the latter; this is a matter of taste, to some extent, but works particularly smoothly in chapter 6.

CHAPTER 2

We first provide a brief review of uniform spaces from the covering point of view. (Tukey [40]). Let X be a set; $\mathcal{C} \subseteq PX$ is a *cover* of X if $\cup\mathcal{C} = X$; for $A \subseteq X$, $st(A,\mathcal{C}) = \cup\{C \in \mathcal{C}: A \cap C \neq \emptyset\}$ is called the *star of* A *in* $\mathcal{C}$; $st(\{x\},\mathcal{C})$ is usually written $st(x,\mathcal{C})$. If $\mathcal{C},\mathcal{D}$ are covers of X we say

(i) $\mathcal{C}$ *refines* $\mathcal{D}$ if for $C \in \mathcal{C}$ there is $D \in \mathcal{D}$ such that $C \subseteq D$; this is written $\mathcal{C} \leqslant \mathcal{D}$.

(ii) $\mathcal{C}$ *star-refines* $\mathcal{D}$ if the cover $\{st(C,\mathcal{C}): C \in \mathcal{C}\}$ refines $\mathcal{D}$; this is written $\mathcal{C} \leqslant^* \mathcal{D}$.

(iii) $\mathcal{C} \wedge \mathcal{D} = \{C \cap D: C \in \mathcal{C},\ D \in \mathcal{D}\}$ is the *meet* of $\mathcal{C}$ and $\mathcal{D}$.

A non-empty family, μ , of covers of X is a (covering) *uniformity* on X if

(i) μ is a filter with respect to $\wedge, \leqslant$ defined above, and

(ii) for each $\mathcal{C} \in \mu$, there is $\mathcal{D} \in \mu$ such that $\mathcal{D} \leqslant^* \mathcal{C}$.

The pair , (X,μ), is then called a *uniform space*.
There is a natural topology, $T(\mu)$, generated by μ as follows:

$U \in T(\mu)$ iff for each $x \in U$ there is $\mathcal{C} \in \mu$ such that $st(x,\mathcal{C}) \subseteq U$.

$T(\mu)$ and μ interact in a nice way; the $T(\mu)$-open covers of μ form a base for μ in the sense that any member of

μ is refined by a $T(\mu)$-open member of μ. Let us, for now, agree to write

$$A \lhd B \text{ if } st(A,\mathcal{C}) \subseteq B \text{ for some } \mathcal{C} \in \mu \quad (A,B \subseteq X) \ ;$$

then we have the following rather expressive result.

2.1 <u>Proposition</u>

Let (X,μ) be a uniform space, $U \in T(\mu)$. Then $U = \bigcup\{V \in T(\mu) : V \lhd U\}$.

<u>Proof</u>: Suppose $x \in U$; for some $\mathcal{C} \in \mu$, $st(x,\mathcal{C}) \subseteq U$. Now select a $T(\mu)$-open member of μ, $\mathcal{D}$, such that $\mathcal{D} \leqslant^* \mathcal{C}$, and consider $st(x,\mathcal{D})$ (which is open). We claim $st(x,\mathcal{D}) \lhd U$; to see this, let $y \in st(st(x,\mathcal{D}),\mathcal{D})$; then $y \in D_1 \in \mathcal{D}$ and $D_1 \cap st(x,\mathcal{D}) \neq \emptyset$, so $x \in D_2 \in \mathcal{D}$ and $D_1 \cap D_2 \neq \emptyset$, so $D_1 \subseteq st(D_2,\mathcal{D}) \subseteq C \in \mathcal{C}$, and $x \in C$, yielding $D_1 \subseteq U$, so $st(st(x,\mathcal{D}),\mathcal{D}) \subseteq U$, as was claimed. □

Also well known are the facts that

(i) $(X,T(\mu))$ is a completely regular topological space.

(ii) If (X,T) is completely regular, then a uniformity, μ, on X exists such that $T(\mu) = T$.

We now proceed to the main task of the chapter, which is to establish the category of uniform frames.

2.2 Definition

Let L be a frame.

(i) $\mathcal{C} \subseteq L$ is a *cover* of L if $\bigvee \mathcal{C} = 1$.

(ii) Let $\mathcal{C}, \mathcal{D}$ be covers of L; we say $\mathcal{C}$ *refines* $\mathcal{D}$, written $\mathcal{C} \leq \mathcal{D}$, if for each $c \in \mathcal{C}$, there is $d \in \mathcal{D}$ with $c \leq d$. We denote by $\mathcal{C} \wedge \mathcal{D}$ the cover $\{c \wedge d : c \in \mathcal{C},\ d \in \mathcal{D}\}$.

(iii) Let $a \in L$, $\mathcal{C}, \mathcal{D}$ covers of L; define

$st(a, \mathcal{C}) = \bigvee\{c \in \mathcal{C} : c \wedge a \neq 0\}$, and denote by $\mathcal{C}^*$ the cover $\{st(c, \mathcal{C}) : c \in \mathcal{C}\}$. We say $\mathcal{C}$ *star-refines* $\mathcal{D}$ iff $\mathcal{C}^* \leq \mathcal{D}$ (usually written $\mathcal{C} \leq^* \mathcal{D}$).

2.3 Lemma

Let L be a frame, $\mathcal{C}, \mathcal{D}$ covers of L with $\mathcal{C} \leq^* \mathcal{D}$. For $a \in L$ we have:

$$a \leq st(st(a, \mathcal{C}), \mathcal{C}) \leq st(a, \mathcal{D}) .$$

Proof:

The first inequality is just about obvious if one realizes that $a \leq st(a, \mathcal{C})$ for any cover $\mathcal{C}$. For the second, we have

$$st(st(a, \mathcal{C}), \mathcal{C}) = \bigvee\{c \in \mathcal{C} : c \wedge st(a, \mathcal{C}) \neq 0\}$$

but $c \wedge st(a, \mathcal{C}) \neq 0$ implies that for some $c_0 \in \mathcal{C}$, $c_0 \wedge c \neq 0$ and $a \wedge c_0 \neq 0$. Now $c \leq st(c_0, \mathcal{C}) \leq d$ for some $d \in \mathcal{D}$, but also $a \wedge d \neq 0$, so $c \leq st(a, \mathcal{D})$, as required.

□

2.4 Definition

1. Let L be a frame, q a non-empty family of covers of L. (L,q) is a *uniform frame* if

   (i) q is a filter with respect to $\wedge$, $\leqslant$,

   (ii) $\mathcal{D} \in q \Rightarrow$ there is $\mathcal{C} \in q$ with $\mathcal{C} \leqslant^* \mathcal{D}$,

   (iii) For each $a \in L$, $a = \bigvee\{b \in L\colon st(b,\mathcal{C}) \leqslant a$ for some $\mathcal{C} \in q\}$.

We also say; q is a compatible uniform structure or uniformity on L.

2. Let (L,q), (M,p) be uniform frames; a function $f\colon L \to M$ is a *uniform map* if

   (i) $f\colon L \to M$ is a frame map,

   (ii) $\mathcal{C} \in q \Rightarrow f[\mathcal{C}] = \{f(c)\colon c \in \mathcal{C}\} \in p$.

3. Uniform frames and maps are the objects and arrows of the category <u>UNIFRM</u>.

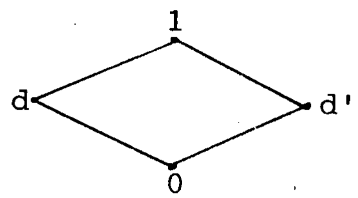
<u>Remark</u>:

Condition (iii) is motivated by Proposition 2.1. Structures satisfying (i) and (ii) only are also of some interest.

2.5 Examples

(i) Let D be the Boolean algebra (complete!) with 4 elements: $\{0, d, d', 1\}$

We represent D by the diagram on the right.



Let $C_1 = \{1\}$, $C_d = \{d,d'\}$; these are covers of D. We also have $C_1 \leqslant^* C_1$ and $C_d \leqslant^* C_d$. Also, since $d = st(d,C_d)$ and $d' = st(d',C_d)$, $\{C_1,C_d\}$ is a base for a compatible uniform structure on D . This is in fact the *only* uniform structure for D .

(ii) Let B be any complete boolean algebra. For each $b \in B$, let $C_b = \{b,b'\}$. ($C_b = C_{b'}$ of course) and let q have as filterbase the family of all finite meets of covers of the form C_b . Since $C_b \leqslant^* C_b$, and $b = st(b,C_b)$, (B,q) is a uniform frame.

(iii) Not all frames have a compatible uniform structure. The 3-chain, $\underline{3}$ has none.

(iv) Let (X,μ) be a uniform space. Let $\mathcal{Q}X = T(\mu)$ and let $\mathcal{Q}\mu$ be the $T(\mu)$-open members of μ ; $(\mathcal{Q}X,\mathcal{Q}\mu)$ is a uniform frame by Proposition 2.1 and the observations preceeding this proposition.

Remark:

The existence of non-spatial complete boolean algebras guarantees the existence of non-spatial uniform frames.

We provide some results now on the structure of the underlying frame of a uniform frame.

2.6 Proposition

Let (L,q) be a uniform frame, then L is regular.

Proof:

We need to show that $a = \bigvee\{b : b \,\overline{<}\, a\}$ for each $a \in L$; this will be the case if we can show that for $\mathcal{C} \in q$, $st(b,\mathcal{C}) \leqslant a$ implies $b \,\overline{<}\, a$. Now set $s = \bigvee\{t \in L : t \wedge b = 0\}$; $(s = b^*)$. Of course $s \wedge b = 0$, but we claim $a \vee s = 1$:

$$\begin{aligned} a \vee s \geqslant st(b,\mathcal{C}) \vee s &= \bigvee\{c \in \mathcal{C} : c \wedge b \neq 0\} \vee \bigvee\{t \in L : t \wedge b = 0\} \\ &\geqslant \bigvee\{c \in \mathcal{C} : c \wedge b \neq 0\} \vee \bigvee\{c \in \mathcal{C} : c \wedge b = 0\} \\ &= \bigvee \mathcal{C} \\ &= 1 . \end{aligned}$$

□

Remark:

We did not use 1(ii) of Definition 2.4.

2.7 Proposition

Let (L,q) be a uniform frame. If we assume the countable dependent axiom of choice, L is completely regular.

Proof:

We simply show that $st(b,\mathcal{C}) \leqslant a$ $(\mathcal{C} \in q)$ implies $b \,\overline{<}\overline{<}\, a$. Select $\mathcal{D}_1 \in q$ such that $\mathcal{D}_1 \leqslant^* \mathcal{C}$, and let $x_{\frac{1}{2}} = st(b,\mathcal{D}_1)$; we immediately have

(i) $b \,\overline{<}\, x_{\frac{1}{2}}$

(ii) $st(x_{\frac{1}{2}},\mathcal{D}_1) \leqslant st(b,\mathcal{C}) \leqslant a$ by Lemma 2.3 yielding $x_{\frac{1}{2}} \,\overline{<}\, a$.

Now select $\mathcal{D}_2 \leqslant^* \mathcal{D}_1$ and proceed in a similar manner to construct the required family $\{x_\alpha : \alpha \in J\}$ of interpolating elements. Hence $b \overline{\overline{<<}} a$, as required. □

Before we turn to the uniformizability of completely regular frames, we need a simple lemma.

2.8 Lemma

Let $f: L \to M$ be a frame map with $a \bar{<} b \in L$. Then $f(b)^* \bar{<} f(a^*)$ (and hence $f(b)^* \leqslant f(a^*)$).

Proof: We have $a \bar{<} b \Rightarrow a^* \vee b = 1$

$$\Rightarrow f(a^*) \vee f(b) = f(a^* \vee b) = 1$$

But we also have $f(b)^* \wedge f(b) = 0$, so $f(b)^* \bar{<} f(a^*)$.

□

2.9 Proposition

Let L be a completely regular frame. L has a compatible uniform structure.

Proof:

Let $a \overline{\overline{<<}} b \in L$ and let $\{x_\alpha : \alpha \in J\}$ be a family of interpolating elements, as guaranteed by Definition 1.9(ii). By a suitable relabelling of these elements we see that in fact $a \overline{\overline{<<}} x_{\frac{1}{2}} \overline{\overline{<<}} b$, say; that means that the completely below relationship interpolates.

For $a \overline{\overline{<<}} b$, define $C_a^b = \{a^*, b\}$ which is a cover of L. Select $c_1, c_2 \in L$ such that $a \overline{\overline{<<}} c_1 \overline{\overline{<<}} c_2 \overline{\overline{<<}} b$.

We can see quite easily that

$$C = C_a^{c_1} \wedge C_{c_1}^{c_2} \wedge C_{c_2}^{b} \leqslant^* C_a^b \quad \text{as follows :}$$

Listing elements of C we get $C = \{c_2^*,\ b \wedge c_1^*,\ c_2 \wedge a^*,\ c_1\}$.

Now $\mathrm{st}(c_2^*, C) \leqslant c_2^* \vee (b \wedge c_1^*) \leqslant a^*$

$\mathrm{st}(b \wedge c_1^*, C) \leqslant c_2^* \vee (b \wedge c_1^*) \vee (c_2 \wedge a^*) \leqslant a^*$

$\mathrm{st}(c_2 \wedge a^*, C) \leqslant (b \wedge c_1^*) \vee (c_2 \wedge a^*) \vee c_1 \leqslant b$

$\mathrm{st}(c_1, C) \leqslant (c_2 \wedge a^*) \vee c_1 \leqslant b$.

So $C \leqslant^* C_a^b$ as claimed. Now let $q_{\overline{<}\overline{<}}(L)$ be the family of all covers having as sub-base all covers of the form C_a^b for $a \,\overline{<}\overline{<}\, b$. We need only check that $q_{\overline{<}\overline{<}}(L)$ is compatible with L , but this is trivial since we have $b = \bigvee\{a : a \,\overline{<}\overline{<}\, b\}$ and $a \,\overline{<}\overline{<}\, b$ implies $\mathrm{st}(a, C_a^b) = b$.

□

2.10 <u>Proposition</u>

Let $f: L \to M$ be a frame map, L, M completely regular. Then $f: (L, q_{\overline{<}\overline{<}}(L)) \to (M, q_{\overline{<}\overline{<}}(M))$ is a uniform map.

<u>Proof</u>:

Suppose $a \,\overline{<}\overline{<}\, b$; we wish to show $f[C_a^b] = \{f(a^*), f(b)\}$ is a member of $q_{\overline{<}\overline{<}}(M)$. Select $c, d \in L$ such that $a \,\overline{<}\overline{<}\, c \,\overline{<}\overline{<}\, d \,\overline{<}\overline{<}\, b$. We know $f(c) \,\overline{<}\overline{<}\, f(d)$, so $\{f(c)^*, f(d)\} \in q_{\overline{<}\overline{<}}(M)$. But $f(c)^* \leqslant f(a^*)$ by Lemma 2.8

so $C_{f(c)}^{f(d)}$ refines $f[C_a^b]$ and we are finished.

□

2.11 <u>Corollary</u>

$F: L \to (L, q_{\preceq\preceq}(L))$ is a functor from <u>COMP REG FRM</u> to <u>UNIFRM</u> .

We turn now to the relationship between the categories <u>UNIF</u> and <u>UNIFRM</u> . Firstly the "open" functor, $\mathcal{Q}$: <u>UNIF</u> $\to$ <u>UNIFRM</u> assigns to each uniform space, (X,μ) , the uniform frame $(\mathcal{Q}X, \mathcal{Q}\mu)$ as mentioned in Example 2.5(iv); also if $f: (X,\mu) \to (Y,\nu)$ is uniformly continuous, then $\mathcal{Q}f: \mathcal{Q}Y \to \mathcal{Q}X$ defined by $\mathcal{Q}f(V) = f^{-1}(V)$ is a uniform map; $\mathcal{Q}$ is a contravariant functor from <u>UNIF</u> to <u>UNIFRM</u> .

2.12 <u>Definition</u>

Let (L,q) be a uniform frame. Let $\Sigma L = \mathrm{hom}(L,\underline{2})$ and for $C \in q$, let $\Sigma C = \{\Sigma_c : c \in C\}$, where $\Sigma_c = \{p \in \Sigma L : p(c) = 1\}$. Denote by Σq the family of covers of ΣL each of which is refined by some member of $\{\Sigma C : C \in q\}$. (We see below that ΣC is always a cover of ΣL.)

2.13 <u>Proposition</u>

Let (L,q) be a uniform frame; $(\Sigma L, \Sigma q)$ is a uniform space.

<u>Proof</u>:

(i) Let $C \in q$; $\bigcup\{\Sigma_c : c \in C\} = \Sigma_{\vee C} = \Sigma_1 = \Sigma L$, so Σq is indeed a non-empty family of covers.

(ii) Let $\mathcal{C},\mathcal{D} \in q$; trivially $\mathcal{C} \leqslant \mathcal{D} \Rightarrow \Sigma\mathcal{C} \leqslant \Sigma\mathcal{D}$, and $\Sigma\mathcal{D} \wedge \Sigma\mathcal{C} = \Sigma(\mathcal{C} \curlywedge \mathcal{D})$.

(iii) Suppose $\mathcal{D} \leqslant^* \mathcal{C} \in q$; then we claim $\Sigma\mathcal{D} \leqslant^* \Sigma\mathcal{C}$. To see this, suppose $d \in \mathcal{D}$ and consider

$$\begin{aligned} &st(\Sigma_d, \Sigma\mathcal{D}) \\ &= \bigcup\{\Sigma_t : t \in \mathcal{D} \text{ and } \Sigma_t \cap \Sigma_d \neq \emptyset\} \\ &= \bigcup\{\Sigma_t : t \in \mathcal{D} \text{ and } \Sigma_{t\wedge d} \neq \emptyset\} \\ &\subseteq \bigcup\{\Sigma_t : t \wedge d \neq 0\} \quad \text{since if } t \wedge d = 0 ,\ \Sigma_{t\wedge d} = \emptyset . \\ &= \Sigma_{st(d,\mathcal{D})} \leqslant \Sigma_c \quad \text{for some } c \in \mathcal{C} \text{ , as required.} \end{aligned}$$

□

We can say more than this though; we have two topologies at hand, namely the spectral topology and the topology generated by Σq .

2.14 Proposition

Let (L,q) be a uniform frame. The topologies $T(\Sigma q)$ and T_Σ on ΣL coincide.

Proof:

Suppose $U \in T(\Sigma q)$; then for each $p \in U$, there is $\mathcal{C} \in q$ such that $st(p,\Sigma\mathcal{C}) \subseteq U$. But members of $\Sigma\mathcal{C}$ are also member of the spectral topology T_Σ, so $U \in T_\Sigma$. Conversely, suppose $U \in T_\Sigma$, i.e. $U = \Sigma_a$ for some $a \in L$. Now $a = \bigvee\{b \in L: st(b,\mathcal{C}) \leqslant a \text{ for some } \mathcal{C} \in q\}$; $p \in \Sigma_a \Rightarrow p(a)=1$ so for some $b \in L$, $\mathcal{C} \in q$ with $st(b,\mathcal{C}) \leqslant a$, we must have $p(b) = 1$. We can now show that $st(p,\Sigma\mathcal{C}) \subseteq \Sigma_a$, thus showing that U is a member of $T(\Sigma q)$:

$$st(p, \Sigma C) = \cup\{\Sigma_c : c \in C \text{ and } p(c) = 1\}$$

but $p(c) = 1$ and $p(b) = 1 \Rightarrow p(c \wedge b) = 1 \Rightarrow c \wedge b \neq 0$

$$\Rightarrow c \leqslant st(b, C) \leqslant a$$

$$\Rightarrow \Sigma_c \subseteq \Sigma_a .$$

So $st(p, \Sigma C) \leqslant \Sigma_a$ as required. □

2.15 Definition

Let $f: (L,q) \to (M,p)$ be a uniform map. Define $f: \Sigma M \to \Sigma L$ by $\Sigma f(p) = p \circ f$.

2.16 Proposition

$\Sigma f: (\Sigma M, \Sigma p) \to (\Sigma L, \Sigma q)$ is uniformly continuous.

Proof:

Let $C \in q$, then $f[C] \in p$ of course. For $c \in C$,

$$\begin{aligned} p \in \Sigma f^{-1}(\Sigma_c) &\iff \Sigma f(p) \in \Sigma_c \\ &\iff p \circ f \in \Sigma_c \\ &\iff p(f(c)) = 1 \\ &\iff p \in \Sigma_{f(c)} \end{aligned}$$

so $f^{-1}(\Sigma_c) = \Sigma_{f(c)} \in \Sigma f[C] \in \Sigma p$, as required. □

2.17 Corollary

Σ is a contravariant functor from UNIFRM to UNIF.

2.18 Theorem

The two contravariant functors $\mathcal{Q}, \Sigma$ constructed above are adjoint on the right.

Proof:

Let (X,μ) be a uniform space, and let (L,q) be a uniform frame.
Also let $f \in \hom((L,q), (\mathcal{Q}X,\mathcal{Q}\mu))$
$g \in \hom((X,\mu), (\Sigma L,\Sigma q))$.

Define $\bar{f}: X \to \Sigma L$ by $\bar{f}(x)(a) = 1$ iff $x \in f(a)$; we check that $\bar{f}$ is uniformly continuous. Let $C \in q$ and consider $\bar{f}^{-1}(\Sigma_c)$ for $c \in C$: as calculated previously $\bar{f}^{-1}(\Sigma_c) = f(c)$, but $f[C] = \{f(c): c \in C\} \in \mathcal{Q}\mu$, so $\bar{f}$ is uniformly continuous.

Now define $\tilde{g}: L \to \mathcal{Q}X$ by $\tilde{g}(a) = g^{-1}(\Sigma_a)$; g is uniformly continuous, by assumption, but the spectral topology and the uniform topology $T(\Sigma q)$ coincide, so g is continuous with respect to the spectral topology, i.e. $g^{-1}(\Sigma_a) \in T(\mu) = \mathcal{Q}X$ as desired. We check that $\tilde{g}$ preserves structure: let $c \in C \in q$; then $\tilde{g}(c) = g^{-1}(\Sigma_c)$, and $\{g^{-1}(\Sigma_c): c \in C\} \in \mathcal{Q}\mu$ as required. The remaining naturality conditions are taken care of by Theorem 1.16.

□

2.19 Proposition

Let (X,μ) be a uniform space. The following are equivalent :

(i) (X,μ) is separated.

(ii) $(X,T(\mu))$ is T_0 and every point of $\Sigma\mathcal{Q}X$ is of the form ψ_x for some $x \in X$.

(iii) $\sigma_X: (X,\mu) \to (\Sigma\mathcal{Q}X,\Sigma\mathcal{Q}\mu)$ is an isomorphism.

Proof:

(i) $\Rightarrow$ (ii) A separated uniform space is T_2, hence sober and T_0.

(ii) $\Rightarrow$ (iii) σ_X is onto by assumption, 1-1 by T_0. We see also that σ_X is a uniform isomorphism as follows: Let C be a $T(\mu)$-open member of μ; for $U \in C$, $\sigma_X(U) = \Sigma_U$, so $\sigma_X[C] = \{\Sigma_U : U \in C\} \in \Sigma\Omega\bar{\mu}$.

(iii) $\Rightarrow$ (i) By Proposition 1.18, $(X, T(\mu))$ is sober, hence T_0, and so (X, μ) is separated.

2.20 Proposition

Let (L,q) be a uniform frame; the following are equivalent

(i) (L,q) is spatial.

(ii) The "points" of L separate L.

(iii) $O_L : L \to \Omega\Sigma L$ is a uniform isomorphism.

Proof:

(i) $\Rightarrow$ (ii) Trivial; L is a topology, so is separated by its "points".

(ii) $\Rightarrow$ (iii) O_L is a frame isomorphism, but also a structure morphism; For $c \in C \in q$, $O_L[C] = \{O_L(c) : c \in C\} = \{\Sigma_c : c \in C\} \in \Omega\Sigma q$ as required.

(iii) $\Rightarrow$ (i) Trivial.

□

2.21 Proposition

(i) The separated uniform spaces and the spatial uniform frames are the fixed objects of the adjunction of Theorem 2.18.

(ii) $\sigma_X : (X,\mu) \to (\Sigma\mathcal{Q}X, \Sigma\mathcal{Q}\mu)$ is the separated reflection.

Proof:

See Proposition 1.20 and the remarks after it. $\square$

Remark :

Compact regular frames (a frame, L , is compact iff $\vee x_\alpha = 1$ $(\alpha \in A) \Rightarrow \exists \alpha_1, \alpha_2, \ldots, \alpha_n \in A$ such that $\vee x_{\alpha_i} = 1$) are well known to be spatial and are in fact just the topologies of compact Hausdorff spaces. These have unique uniform structures (all covers refined by open covers), so compact regular frames also have a unique uniform frame structure. This can also be proved directly.

Notes on Chapter 2

(1) We have reserved for the notion defined in definition 2.4, 1 the title of uniform frame, guided of course by the observation that it is these frames which are related to uniform spaces by the adjoint situation of theorem 2.18. Structures satisfying 1(i) and (ii) of definition 2.4 correspond under a similar adjunction to triples (X, T, μ), where X is a set, T a topology on X and μ a uniformity on X whose topology, $T(\mu)$, is coarser than T.

(2) Alternative approaches to uniform spaces exist. To name just two, there is the approach via "entourages" or "surroundings" of Weil [41] and the approach via the syntopogenous structures of Császár [10]. Both of these approaches exhibit significant problems when viewed from the "pointless" point of view. This lends support to the view that Tukey's approach is the "best" one. (Isbell [20] claims that Tukey's approach is most convenient "nine-tenths of the time" !)

(3) The set of uniform structures compatible with a frame L forms a complete lattice. Suppose $\{q_\alpha : \alpha \in A\}$ is a set of compatible uniform structures on L; if we let $q_0 = \cup \{q_\alpha : \alpha \in A\}$ then q_0 is a sub-base for a uniform structure compatible with L. This structure would be the "coarsest" uniform structure containing each q_α.

(4) Paracompact frames (Dowker and Strauss [13]) and uniform frames behave as expected.

CHAPTER 3

In this chapter we present a theory of quasi-uniform frames, again making use of frame covers. The classical theory of quasi-uniform spaces is commonly presented in terms of so-called entourages, but without the symmetry requirement; covers are present, however, in the form of sections of entourages, although since entourages are no longer symmetric one is forced to consider two "interlocking" families of covers. A quasi-uniform space generates two (usually) distinct topologies which interact in a nice way with the "interlocking" covers; it should come as only a mild surprise that this theory of quasi-uniform frames is presented in terms of covering structures on a bi-frame.

We present first a brief sketch of the theory of conjugate covers of a set, and covering quasi-uniformities. The results are due to Gantner and Steinlage [16]. We refer the reader to Fletcher and Lindgren [15] for a general reference on quasi-uniform spaces.

3.1 Definition

Let X be a set.

(i) A *conjugate cover pair*, $\mathcal{C}$, of X is a subset of $\mathcal{P}X \times \mathcal{P}X$ satisfying $\cup\{(C_1 \cap C_2): (C_1,C_2) \in \mathcal{C}\} = X$.

(ii) A conjugate cover pair, $\mathcal{C}$, is called *strong* if $(C_1,C_2) \in \mathcal{C}$, and either $C_1 \neq \emptyset$ or $C_2 \neq \emptyset$ implies that $C_1 \cap C_2 \neq \emptyset$.

One sees that a cover pair is a "decomposition" of a cover into two covers. A strong cover pair might be termed a "non superfluous" decomposition. For brevity, we talk simply of conjugate covers.

(iii) Let $\mathcal{C}$, $\mathcal{D}$ be conjugate covers. We write $\mathcal{C} \leqslant \mathcal{D}$ if for each $(C_1,C_2) \in \mathcal{C}$, there is $(D_1,D_2) \in \mathcal{D}$ such that $\begin{cases} C_1 \subseteq D_1 \\ C_2 \subseteq D_2 \end{cases}$ and we say $\mathcal{C}$ *refines* $\mathcal{D}$.
(A "simultaneous" refinement.)

(iv) For $\mathcal{C}$, $\mathcal{D}$ conjugate covers, we set
$\mathcal{C} \wedge \mathcal{D} = \{(C_1 \cap D_1, C_2 \cap D_2) : (C_1,C_2) \in \mathcal{C}, (D_1,D_2) \in \mathcal{D}\}$;
$\mathcal{C} \wedge \mathcal{D}$ is also a conjugate cover.

(v) Let $A \subseteq X$, and let $\mathcal{C}$ be a conjugate cover.
Set $st_1(A,\mathcal{C}) = \cup\{C_{1\alpha} : (C_{1\alpha},C_{2\alpha}) \in \mathcal{C}$ and $C_{2\alpha} \cap A \neq \emptyset\}$.

$st_2(A,\mathcal{C}) = \cup\{C_{2\alpha} : (C_{1\alpha},C_{2\alpha}) \in \mathcal{C}$ and $C_{1\alpha} \cap A \neq \emptyset\}$.

As usual, we will write $st_i(x,\mathcal{C})$ for $st_i(\{x\},\mathcal{C})$ $(i = 1 \text{ or } 2)$.

(vi) For $\mathcal{C}$ a conjugate cover, set
$\mathcal{C}^* = \{(st_1(C_1,\mathcal{C}) , st_2(C_2,\mathcal{C})) : (C_1,C_2) \in \mathcal{C}\}$.
It is easy to check that

(a) $\mathcal{C}^*$ is a conjugate cover pair

(b) $\mathcal{C} \leqslant \mathcal{C}^*$

(c) $\mathcal{C} \leqslant \mathcal{D} \Rightarrow \mathcal{C}^* \leqslant \mathcal{D}^*$

To set the readers mind at ease, let us see how entourages and conjugate covers are related : let $\mathcal{C}$ be a conjugate cover; set $(x,y) \in R_{\mathcal{C}} \iff (x,y) \in C_1 \times C_2$ for some $(C_1,C_2) \in \mathcal{C}$. Conversely, given a reflexive relation $R \subseteq X \times X$, construct the conjugate cover
$\mathcal{C}_R = \{(R[x], R^{-1}[x]): x \in X\}$

$$\text{where } \begin{cases} R[x] = \{y \in X: (x,y) \in R\} \\ R^{-1}[x] = \{y \in X: (y,x) \in R\} . \end{cases}$$

If $R,S \subseteq X \times X$ are reflexive relations and $R \circ R \circ R \subseteq S$ one can easily see that $\mathcal{C}_R{}^* \leqslant \mathcal{C}_S$.

3.2 Definition

A non empty collection, μ , of conjugate covers of a set X is a *cover quasi-uniformity* on X provided that:

(i) The strong conjugate covers form a filter base with respect to $\leqslant , \wedge$ defined above.

(ii) For each $\mathcal{C} \in \mu$, there is $\mathcal{D} \in \mu$ such that $\mathcal{D}^* \leqslant \mathcal{C}$.

(X,μ) is a quasi-uniform space.

We consider now the topologies generated by a cover quasi-uniformity.

3.3 Definition

Let (X,μ) be a quasi-uniform space; we let $U \in T_i(\mu) \subseteq PX$ iff for each $x \in U$, there is $\mathcal{C} \in \mu$ such that $st_i(x,\mathcal{C}) \subseteq U$ (i = 1 or 2).
$T_i(\mu)$ is indeed a topology on X .

3.4 Proposition

Let (X,μ) be a quasi-uniform space, $x \in X$, $C \in \mu$. Then $st_i(x,C)$ is a $T_i(\mu)$-neighbourhood of x.

Proof: Omitted.

3.5 Proposition

Let (X,μ) be a quasi-uniform space. The subfamily of μ of conjugate covers $\{C: C \subseteq T_1(\mu) \times T_2(\mu)\}$ is a base for μ. (Base has the obvious meaning here.)

Proof:

For $U \subseteq X$, $int_i(U)$ denotes the interior of U with respect to $T_i(\mu)$ $(i = 1 \text{ or } 2)$. Suppose that $C, \mathcal{D} \in \mu$ such that $C^* \leqslant \mathcal{D}$. We show that

$C \leqslant \{(int_1(D_1), int_2(D_2)): (D_1,D_2) \in \mathcal{D}\}$ $\quad (= int\ \mathcal{D})$

yielding $int\ \mathcal{D} \in \mu$. This is sufficient to prove the proposition. So suppose $(C_1,C_2) \in C$; by assumption, there is $(D_1,D_2) \in \mathcal{D}$ such that $\begin{cases} st_1(C_1,C) \subseteq D_1 \\ st_2(C_2,C) \subseteq D_2 \end{cases}$

but this also shows that $\begin{cases} C_1 \subseteq int_1(D_1) \\ C_2 \subseteq int_2(D_2) \end{cases}$

since $x \in C_1$ (say) $\Rightarrow st_1(x,C) \subseteq st_1(C_1,C) \subseteq D_1$ (the other case is similar). □

The following result has the appropriate lattice-theoretic feel about it; let us agree, for now, to write

$U \lhd_i V$ iff for some conjugate cover, $\mathcal{C}$, of X, $st_i(U,\mathcal{C}) \subseteq V$, where $U,V \subseteq X$.

3.6 Proposition

Let (X,μ) be a quasi-uniform space. Then (for $i = 1$ or 2) $V \in T_i(\mu) \Rightarrow V = \cup\{U \in T_i(\mu) : U \lhd_i V\}$.

Proof:

Let $x \in V \in T_i(\mu)$; select $\mathcal{C},\mathcal{D} \in \mu$ such that

(i) $st_i(x,\mathcal{C}) \subseteq V$

(ii) $\mathcal{D} \subseteq T_1(\mu) \times T_2(\mu)$

(iii) $\mathcal{D}^* \leqslant \mathcal{C}$.

Clearly $st_i(x,\mathcal{D}) \subseteq st_i(x,\mathcal{C}) \subseteq V$, and $st_i(x,\mathcal{D})$ is, by choice of $\mathcal{D}$, a member of $T_i(\mu)$. We claim also that $x \in st_i(st_i(x,\mathcal{D}),\mathcal{D}) \subseteq V$, thus proving the result. Fix $i = 1$. Trivially $x \in st_1(st_1(x,\mathcal{D}),\mathcal{D})$. We also have : $st_1(st_1(x,\mathcal{D}),\mathcal{D}) = \cup\{D_{1\alpha}: (D_{1\alpha},D_{2\alpha}) \in \mathcal{D}$ and $D_{2\alpha} \cap st_1(x,\mathcal{D}) \neq \emptyset\}$, but $D_{2\alpha} \cap st_1(x,\mathcal{D}) \neq \emptyset \iff$ for some $(D_{1\beta},D_{2\beta}) \in \mathcal{D}$ we have

$$x \in D_{2\beta} \text{ and } D_{1\beta} \cap D_{2\alpha} \neq \emptyset .$$

This implies that $\left\{\begin{array}{l} D_{1\alpha} \subseteq st_1(D_{1\beta},\mathcal{D}) \subseteq C_{1\gamma} \\ x \in st_2(D_{2\beta},\mathcal{D}) \subseteq C_{2\gamma} \end{array}\right\}$ where $(C_{1\gamma},C_{2\gamma}) \in \mathcal{C}$.

Finally $D_{1\alpha} \subseteq st_1(x,\mathcal{C}) \subseteq V$, show that $st_1(st_1(x,\mathcal{D}),\mathcal{D}) \subseteq V$ as was required. The case $i = 2$ is similar. □

In the case that a family, μ, of conjugate cover pairs, on a set X satisfies the conditions of definition 3.2 with the exception of the "strength" condition, it is important to be able to extend such a family μ to a quasi-uniform structure. We outline a canonical way of doing so.

3.7 Definition

Let $\mathcal{C}$ be a conjugate cover of X. We define $\mathcal{C}^r = \{(C_1,C_2): (C_1,C_2) \in \mathcal{C} \text{ and } C_1 \cap C_2 \neq \emptyset\}$.
(We "reduce" $\mathcal{C}$ by removing superfluous pairs.)

3.8 Lemma

Let $\mathcal{C},\mathcal{D}$ be conjugate covers of Y. Then we have

(i) $\mathcal{C}^r$ is a strong conjugate cover of Y,

(ii) $(\mathcal{C} \wedge \mathcal{D})^r \leqslant \mathcal{C}^r \wedge \mathcal{D}^r$,

(iii) $\mathcal{C}^* \leqslant \mathcal{D} \Rightarrow (\mathcal{C}^r)^* \leqslant \mathcal{D}^r$,

(iv) Let $f: X \to Y$ be a function; then $[f^{-1}(\mathcal{C})]^r \leqslant f^{-1}(\mathcal{C}^r)$.

Proof:

(i) Obvious.

(ii) $C_1 \cap D_1 \cap C_2 \cap D_2 \neq \emptyset \Rightarrow C_1 \cap C_2 \neq \emptyset$ and $D_1 \cap D_2 \neq \emptyset$.

(iii) Let $(C_1,C_2) \in \mathcal{C}^r$;

then $\begin{cases} st_1(C_1,\mathcal{C}^r) \subseteq st_1(C_1,\mathcal{C}) \subseteq D_1 \\ st_2(C_2,\mathcal{C}^r) \subseteq st_2(C_2,\mathcal{C}) \subseteq D_2 \end{cases}$,

where $(D_1,D_2) \in \mathcal{D}$. But $C_1 \cap C_2 \neq \emptyset \Rightarrow D_1 \cap D_2 \neq \emptyset$, $(D_1,D_2) \in \mathcal{D}^r$, as required.

(iv) We have $f^{-1}(\mathcal{C}) = \{(f^{-1}(C_1), f^{-1}(C_2)): (C_1,C_2) \in \mathcal{C}\}$. Now if $f^{-1}(C_1) \cap f^{-1}(C_2) \neq \emptyset$, clearly $C_1 \cap C_2 \neq \emptyset$, so $(C_1,C_2) \in \mathcal{C}^r$, yielding $(f^{-1}(C_1),f^{-1}(C_2)) \in f^{-1}(\mathcal{C}^r)$, as required. □

3.9 Definition

Let X be a set, μ a filter of conjugate covers of X with the property that for each $C \in \mu$, there is $\mathcal{D} \in \mu$ such that $\mathcal{D}^* \leqslant C$. Set $\mu^r = \{C^r : C \in \mu\}$, and let μ^* be the family of conjugate covers of X that has μ^r as filter sub-base.

3.10 Proposition

Let (X,μ) be as in definition 3.9; then (X,μ^*) is a quasi-uniform space.

Proof:

It is clear that μ^* is a filter with respect to $\wedge, \leqslant$. Suppose $C \in \mu^*$; $C \geqslant C_1^r \wedge C_2^r \wedge \ldots \wedge C_m^r$, where $C_1, C_2, \ldots C_m \in \mu$. Now $(\bigwedge_{i=1}^{m} C_i)^r \leqslant C$, by lemma 3.8 (ii) so we have "enough" strong conjugate covers. Now select $\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_m \in \mu$ such that for $i = 1, \ldots, m$, $\mathcal{D}_i^* \leqslant C_i$. Then $((\mathcal{D}_1 \wedge \mathcal{D}_2 \wedge \ldots \wedge \mathcal{D}_m)^r)^* \leqslant C$, as is required. □

3.11 Proposition

If (X,μ) is a quasi-uniform space, then $\mu^* = \mu$.

Proof:

Let $C \in \mu^*$ as in the proof of the previous proposition. For each $i = 1, \ldots, m$ select a strong $\mathcal{D}_i \in \mu$ such that $\mathcal{D}_i \leqslant C_i$. It is easy to see that $\mathcal{D}_i \leqslant C_i^r$ and so $C \in \mu$ as required. □

3.12 Proposition

If (X, μ) , (Y, ν) are structures as in definition 3.9 and $f: X \to Y$ is a function satisfying

$$C \in \nu \Rightarrow f^{-1}[C] \in \mu$$

then $f: (X, \mu^*) \to (Y, \nu^*)$ is quasi-uniformly continuous.

Proof:

Let $C \in \nu^*$; $C \geq C_1^r \wedge C_2^r \wedge \ldots \wedge C_m^r$, where for $i = 1, \ldots, m$, $C_i \in \nu$. Now $f^{-1}[C_i]^r \leq f^{-1}[C_i^r]$ so $f^{-1}(C) \geq f^{-1}[C_1]^r \wedge f^{-1}[C_2]^r \wedge \ldots \wedge f^{-1}[C_m]^r \in \mu^*$.
So $f^{-1}(C) \in \mu^*$ as required. □

We are ready now to set up the category of quasi-uniform frames.

3.13 Definition

Let $L = (L_0, L_1, L_2)$ be a biframe.

(i) $C \subseteq L_1 \times L_2$ (the cartesian product of L_1 with L_2) is called a *conjugate cover pair of* L if

$$\bigvee\{ c_1 \wedge c_2 : (c_1, c_2) \in C\} = 1 .$$

We usually refer to C as simply a conjugate cover of L .

(ii) C (as above) will be called *strong* if $c_1 \wedge c_2 \neq 0$ whenever c_1 or $c_2 \neq 0$ (and $(c_1, c_2) \in C$) .

(iii) Let $C, \mathcal{D}$ be conjugate covers of L .
We write $C \leq \mathcal{D}$ if whenever $(c_1, c_2) \in C$, there is $(d_1, d_2) \in \mathcal{D}$ such that:

$$\begin{cases} c_1 \leq d_1 \\ c_2 \leq d_2 \end{cases} .$$

We say $\mathcal{C}$ *refines* $\mathcal{D}$.

We also set $\mathcal{C} \wedge \mathcal{D} = \{(c_1 \wedge d_1 , c_2 \wedge d_2): (c_1,c_2) \in \mathcal{C}, (d_1,d_2) \in \mathcal{D}\}$.

$\mathcal{C} \wedge \mathcal{D}$ is a conjugate cover of L .

(iv) Let $a \in L_0$, $\mathcal{C}$ a conjugate cover of L . We set

$$st_1(a,\mathcal{C}) = \bigvee\{c_1: (c_1,c_2) \in \mathcal{C} \text{ and } c_2 \wedge a \neq 0\}$$

$$st_2(a,\mathcal{C}) = \bigvee\{c_2: (c_1,c_2) \in \mathcal{C} \text{ and } c_1 \wedge a \neq 0\} .$$

Denote by $\mathcal{C}^*$ the conjugate cover

$$\{(st_1(c_1,\mathcal{C}) , st_2(c_2,\mathcal{C})) : (c_1,c_2) \in \mathcal{C}\}.$$

It is perhaps easiest to dispose of a technical lemma at this stage.

3.14 <u>Lemma</u>

Let $\mathcal{C},\mathcal{D}$ be conjugate covers of $L = (L_0,L_1,L_2)$. We have :

(i) For $a \in L_0$, $a \leqslant st_i(a,\mathcal{C})$ $(i = 1 \text{ or } 2)$.

(ii) If $\mathcal{D}^* \leqslant \mathcal{C}$, $a \in L_0$, then

$$st_i(st_i(a,\mathcal{D}),\mathcal{D}) \leqslant st_i(a,\mathcal{C}) \quad (i = 1 \text{ or } 2) .$$

<u>Proof</u>:

For definiteness, fix $i = 1$; we have

$$\begin{aligned}
a \wedge st_i(a,\mathcal{C}) &= a \wedge \bigvee\{c_1: c_2 \wedge a \neq 0 \text{ and } (c_1,c_2) \in \mathcal{C}\} \\
&\geqslant a \wedge \bigvee\{c_1 \wedge c_2: c_2 \wedge a \neq 0 \text{ and } (c_1,c_2) \in \mathcal{C}\} \\
&= \bigvee\{a \wedge c_1 \wedge c_2: c_2 \wedge a \neq 0 \text{ and } (c_1,c_2) \in \mathcal{C}\} \\
&= \bigvee\{a \wedge c_1 \wedge c_2: (c_1,c_2) \in \mathcal{C}\} \\
&= a \wedge \bigvee\{c_1 \wedge c_2: (c_1,c_2) \in \mathcal{C}\}
\end{aligned}$$

$$= a \wedge 1$$
$$= a \quad , \quad \text{proving (i)} .$$

For (ii), we have $st_1(st_1(a,\mathcal{D}),\mathcal{D}) = \bigvee\{d_{i1}:(d_{i1},d_{i2}) \in \mathcal{D}$ and $d_{i2} \wedge st_1(a,\mathcal{D}) \neq 0\}$. But $d_{i2} \wedge st_1(a,\mathcal{D}) \neq 0$ means that there is $(d_1,d_2) \in \mathcal{D}$ such that

$$\begin{cases} d_{i2} \wedge d_1 \neq 0 \\ a \wedge d_2 \neq 0 \end{cases} .$$

But then $\begin{cases} d_{i1} \leqslant st_1(d_1,\mathcal{D}) \leqslant c_1 \\ d_2 \leqslant st_2(d_2,\mathcal{D}) \leqslant c_2 \end{cases}$, where $(c_1,c_2) \in \mathcal{C}$.

But $a \wedge d_2 \neq 0$ implies $a \wedge c_2 \neq 0$, so $c_1 \leqslant st_1(a,\mathcal{C})$, yielding $d_{i1} \leqslant st_1(a,\mathcal{C})$. We thus have $st_1(st_1(a,\mathcal{D}),\mathcal{D}) \leqslant st_1(a,\mathcal{C})$ as required. The case $i = 2$ is similar. □

3.15 <u>Definition</u>

(1) Let $L = (L_0,L_1,L_2)$ be a biframe and let q be a non empty family of conjugate covers of L. (L,q) is a *quasi-uniform frame* provided that :

(i) The family of strong members of q is a filter-base for q with respect to $\wedge, \leqslant$ defined above.

(ii) $\mathcal{C} \in q \Rightarrow$ there is $\mathcal{D} \in q$ such that $\mathcal{D}^* \leqslant \mathcal{C}$.

(iii) For each $a_i \in L_i$ $(i = 1 \text{ or } 2)$

$$a_i = \bigvee\{b \in L_i : \text{for some } \mathcal{C} \in q \ \ st_i(b,\mathcal{C}) \leqslant a_i\} .$$

We also say; q is a compatible quasi-uniform structure on L.

(2) Let (L,q) , (M,p) be quasi-uniform frames. A function $f: L \to M$ is a *quasi-uniform map* iff

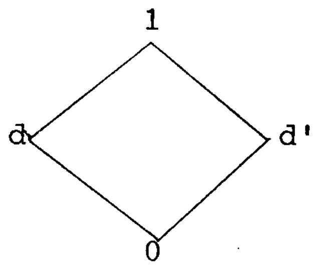
(i) $f: L \to M$ is a biframe map

(ii) $C \in q \Rightarrow f[C] \in p$ where $f[C]$ has the obvious meaning.

(3) Quasi-uniform frames and quasi-uniform maps are the objects and arrows of the category <u>QUNFRM</u> .

3.16 <u>Examples</u>

(i) Let D be the 4 element boolean algebra with Hasse diagram :



We view this as a biframe (D_0, D_1, D_2) , where $D_0 = D$, $D_1 = \{0,d,1\}$, $D_2 = \{0,d',1\}$.
$C_d = \{(d,1) , (1,d')\}$ is a strong conjugate cover.
$C_1 = \{(1,1)\}$ is a strong conjugate cover.
Let $q = \{C_1, C_d\}$. It is a simple matter to check

(i) $st_1(d,C_d) = d$

(ii) $st_2(d',C_d) = d'$

and thus $C_d^* = C_d$. Trivially $C_1^* = C_1$.
Using (i) and (ii) one sees that q is a base for a compatible quasi-uniform structure for D . In fact this is the *only* compatible quasi-uniform structure for D as an exhaustive check quickly shows.

(ii) Not all biframes have compatible quasi-uniform structure; the biframe $(\underline{2},\underline{2},\underline{2})$ has none.

(iii) Suppose (L,q) is a uniform frame. Let $q^d = \{C^d: C \in q\}$ where $C^d = \{(c,c): c \in C\}$. Then q^d is a base for a compatible quasi-uniform structure on (L,L,L).

(iv) Let (X,μ) be a (covering) quasi-uniform space. Let $\mathcal{Q}X = (T_1(\mu) \vee T_2(\mu), T_1(\mu), T_2(\mu))$. Let $\mathcal{Q}\mu = \{C \in \mu: C \subseteq T_1(\mu) \times T_2(\mu)\}$. By propositions 3.5 and 3.6, $(\mathcal{Q}X, \mathcal{Q}\mu)$ is a quasi-uniform frame.

As expected we have:

3.17 <u>Proposition</u>

Let $L = (L_0, L_1, L_2)$ be a biframe, and suppose (L,q) is a quasi-uniform frame. Then L is a regular biframe.

Proof:

By hypothesis, for each $a \in L_i$ $(i = 1 \text{ or } 2)$,

$a = \bigvee\{b \in L_i: \text{for some } C \in q, \ st_i(b,C) \leqslant a\}$.

Let us fix $i = 1$.

For a particular such $b \in L_1$, set $s = \bigvee\{x \in L_2: x \wedge b = 0\}$.

Clearly $s \in L_2$ and $b \wedge s = 0$. We also have

$$\begin{aligned} a \vee s &\geqslant st(b,C) \vee s \\ &= \bigvee\{c_1: (c_1,c_2) \in C \text{ and } c_2 \wedge b \neq 0\} \vee \\ &\quad \bigvee\{x \in L_2: x \wedge b = 0\} \end{aligned}$$

$$\geqslant \bigvee\{c_1 \wedge c_2 : (c_1,c_2) \in \mathcal{C} \text{ and } c_2 \wedge b \neq 0\} \vee$$
$$\bigvee\{c_1 \wedge c_2 : (c_1,c_2) \in \mathcal{C} \text{ and } c_2 \wedge b = 0\}$$
$$= \bigvee\{c_1 \wedge c_2 : (c_1,c_2) \in \mathcal{C}\}$$

$= 1$, so $a \vee s = 1$. This means $b \mathrel{\overline{<}_1} a$ and so $a = \bigvee\{b : b \mathrel{\overline{<}_1} a\}$. The case $i = 2$ is similar. □

Remark:

We did not use 3.15 (i),(ii) here. A characterization of regularity of biframes in terms of conjugate cover pair families is possible.

3.18 Proposition

Let (L,q) be as in proposition 3.17. If we assume the axiom of countable dependent choice, then L is a completely regular biframe.

Proof:

Suppose $a,b \in L_i$ $(i = 1 \text{ or } 2)$ and that $st_i(b,\mathcal{C}) \leqslant a$ where $\mathcal{C} \in q$. Select $\mathcal{D} \in q$ such that $\mathcal{D}^* \leqslant \mathcal{C}$. Let $x_{i_{1/2}} = st_i(b,\mathcal{D})$; we immediately have

(i) $b \mathrel{\overline{<}_i} x_{i_{1/2}}$ since $st_i(b,\mathcal{D}) \leqslant st_i(b,\mathcal{D}) = x_{i_{1/2}}$

(ii) $x_{i_{1/2}} \mathrel{\overline{<}_i} a$ since $st_i(x_{i_{1/2}},\mathcal{D}) = st_i(st_i(b,\mathcal{D}),\mathcal{D}) \leqslant st_i(b,\mathcal{C})$ by lemma 3.14 (ii).

This process may be continued (select $\mathcal{E} \in q$ such that $\mathcal{E}^* \leqslant \mathcal{D}$) and we see that in fact $b \mathrel{\overline{<\!<}_i} a$, yielding $a = \bigvee\{b \in L_i : b \mathrel{\overline{<\!<}_i} a\}$. □

We include here analogous ideas and results to those expressed in sections 3.7 to 3.12. Details are sometimes omitted.

3.19 Definition

Let $\mathcal{C}$ be a conjugate cover of the biframe $L = (L_0, L_1, L_2)$. Set $\mathcal{C}^r = \{(c_1, c_2) \in \mathcal{C}: c_1 \wedge c_2 \neq 0\}$.

3.20 Lemma

Let $\mathcal{C}, \mathcal{D}$ be conjugate covers of the biframe $L = (L_0, L_1, L_2)$.

(i) $\mathcal{C}^r$ is a strong conjugate cover pair.

(ii) $(\mathcal{C} \wedge \mathcal{D})^r \leqslant \mathcal{C}^r \wedge \mathcal{D}^r$.

(iii) $\mathcal{C}^* \leqslant \mathcal{D} \Rightarrow (\mathcal{C}^r)^* \leqslant \mathcal{D}^r$.

(iv) If $f: (L_0, L_1, L_2) \to (M_0, M_1, M_2)$ is a biframe map, then $[f(\mathcal{C})]^r \leqslant f(\mathcal{C}^r)$.

Proof:

We omit (i), (ii), (iii). For (iv) suppose $f(c_1) \wedge f(c_2) \neq 0$, where $(c_1, c_2) \in \mathcal{C}$. Then $f(c_1 \wedge c_2) \neq 0$, so $c_1 \wedge c_2 \neq 0$, so $(c_1, c_2) \in \mathcal{C}^r$, so $(f(c_1), f(c_2)) \in f(\mathcal{C}^r)$. □

3.21 Definition

Let $L = (L_0, L_1, L_2)$ be a biframe, q a filter of conjugate cover pairs of L with the property that for each $\mathcal{C} \in q$, there is $\mathcal{D} \in q$ such that $\mathcal{D}^* \leqslant \mathcal{C}$. Set

$q^r = \{C^r: C \in q\}$ and let q^* be the filter of conjugate cover pairs of L that has q^r as sub-base.

3.22 Proposition

Suppose (L,q) is as in definition 3.21 and that for each $a \in L_i$; $a = \bigvee\{b \in L_i: st_i(b,C) \leqslant a$ for some $C \in q\}$, $(i = 1$ or $2)$. Then (L,q^*) is a quasi-uniform frame.

Proof:

We have that q^* is a filter, and $q^* \supseteq q$. It is clear that we have "enough" strong conjugate cover pairs. □

3.23 Proposition

If (L,q) is a quasi-uniform frame, then $q^* = q$.

Proof:

Omitted. □

3.24 Proposition

If (L,q) , (M,p) are structures as in definition 3.21 and $f: (L_0,L_1,L_2) \to (M_0,M_1,M_2)$ is a biframe map with the property that $C \in q \Rightarrow f[C] \in p$, then $f: (L,q^*) \to (M,p^*)$ is a quasi-uniform map.

Proof:

Omitted. □

3.25 Definition

Let $L = (L_0, L_1, L_2)$ be a biframe, $a \in L_i$ $(i = 1$ or $2)$. Define $a^* = \bigvee\{t \in L_j : j = 1$ or $2,\ j \neq i,\ t \wedge a = 0\}$.

Remark:

We see below that a^* defined above plays an analogous role to the pseudo-complement in a frame.

3.26 Lemma

Let $f: L \to M$ be a biframe map, where $L = (L_0, L_1, L_2)$ and $M = (M_0, M_1, M_2)$. If $a \overline{<}_i b$ $(a, b \in L_i$ of course$)$ then $f(b)^* \overline{<}_j f(a^*)$, $(j = 1$ or $2,\ j \neq i)$.

Proof:

Since $a, b \in L_i$, it follows that $f(b)^*, f(a^*) \in L_j$.
Now $a \overline{<}_i b \Rightarrow a^* \vee b = 1$ and $a \wedge a^* = 0$.
So $f(a^*) \vee f(b) = f(1) = 1$ and $f(b) \wedge f(b)^* = 0$.
But this just means that $f(b)^* \overline{<}_j f(a^*)$ ($f(b)$ is a separating element in L_i). □

3.27 Proposition

Let $L = (L_0, L_1, L_2)$ be a completely regular biframe L has a compatible quasi-uniform structure.

Proof:

Suppose for definiteness that $a \overline{\overline{<<}}_1 b$. Set $C_a^b = \{(b,1), (1,a^*)\}$. This is a strong conjugate cover of L.

Now select $c_1, c_2 \in L_1$ such that $a \mathrel{\overline{<\!<}}_1 c_1 \mathrel{\overline{<\!<}}_1 c_2 \mathrel{\overline{<\!<}}_1 b$, and let $C = C_a^{c_1} \wedge C_{c_1}^{c_2} \wedge C_{c_2}^{b}$. Thus we have

$C = \{(c_1,1),\ (c_1,c_2^*),\ (c_1,c_1^*),\ (c_2,a^*),\ (c_2,c_2^*),\ (b,c_1^*),(1,c_2^*)\}$

By an exhaustive check we see easily that $C^* \leqslant C_a^b$. Thus if we take q to be the family of conjugate covers which has as sub-base all covers of the form C_a^b where $a \mathrel{\overline{<\!<}}_i b$ ($i = 1$ or 2), we have that q satisfies the conditions of proposition 3.22, since if $a \mathrel{\overline{<\!<}}_i b$, then $st_i(a, C_a^b) = b$. But then q^* is the required quasi-uniform structure, which we denote by $q_{\overline{<\!<}}(L)$.

□

3.28 Proposition

Let $L = (L_0, L_1, L_2)$ and $M = (M_0, M_1, M_2)$ be completely regular biframes, and suppose $f\colon L \to M$ is a biframe map. Then $f\colon (L, q_{\overline{<\!<}}(L)) \to (M, q_{\overline{<\!<}}(M))$ is a quasi-uniform map.

Proof:

Suppose $a \mathrel{\overline{<\!<}}_i b$; we show that $f(C_a^b) \in q_{\overline{<\!<}}(M)$. Select $c_1, c_2 \in L_i$ such that $a \mathrel{\overline{<\!<}}_i c_1 \mathrel{\overline{<\!<}}_i c_2 \mathrel{\overline{<\!<}}_i b$; then $f(c_1) \mathrel{\overline{<\!<}}_i f(c_2)$, and so $C_{f(c_1)}^{f(c_2)} \in q_{\overline{<\!<}}(M)$. But by lemma 3.26 $C_{f(c_1)}^{f(c_2)}$ refines $f(C_a^b)$, and we are finished.

□

3.29 Corollary

$F\colon L \to (L, q_{\overline{<\!<}}(L))$ is a functor from the category of completely regular biframes to <u>QUNFRM</u>.

We turn now to the relationship between the categories QUN and QUNFRM . As expected we construct "open" and "spectrum" functors which are adjoint on the right.

Let (X,μ) be a quasi-uniform space; set $\mathcal{Q}(X,\mu) = (\mathcal{Q}X,\mathcal{Q}\mu)$ as in example 3.16 (iv); also if $f:(X,\mu) \to (Y,\nu)$ is quasi-uniformly continuous, define $\mathcal{Q}f: \mathcal{Q}Y \to \mathcal{Q}X$ by $\mathcal{Q}f(V) = f^{-1}(V)$; $\mathcal{Q}f$ is a quasi-uniform map from $(\mathcal{Q}Y,\mathcal{Q}\nu)$ to $(\mathcal{Q}X,\mathcal{Q}\mu)$; $\mathcal{Q}$ is thus a contravariant functor from QUN to QUNFRM .

3.30 Definition

Let (L,q) be a quasi-uniform frame, where $L = (L_0,L_1,L_2)$. Let $\Sigma L = \hom(L_0,\underline{2})$ and for $C \in q$, let $\Sigma C = \{(\Sigma_{c_1},\Sigma_{c_2}): (c_1,c_2) \in C\}$. Denote by Σq the family of all conjugate cover pairs that has as base the set $\{\Sigma C: C \in q\}$.

3.31 Proposition

Let (L,q) be a quasi-uniform frame. Then $(\Sigma L,\Sigma q^*)$ is a quasi-uniform space.

Proof:

(i) Let $C \in q$;
$$\begin{aligned} \cup\{\Sigma_{c_1} \cap \Sigma_{c_2}: (c_1,c_2) \in C\} &= \cup\{\Sigma_{c_1 \wedge c_2}: (c_1,c_2) \in C\} \\ &= \Sigma_{\vee(c_1 \wedge c_2)} \quad \text{where } (c_1,c_2) \in C \\ &= \Sigma_1 = \Sigma L . \end{aligned}$$

(ii) Let $\mathcal{C},\mathcal{D} \in q$. Trivially $\mathcal{C} \leqslant \mathcal{D} \Rightarrow \Sigma\mathcal{C} \leqslant \Sigma\mathcal{D}$ and $\Sigma(\mathcal{C} \wedge \mathcal{D}) = \Sigma\mathcal{C} \wedge \Sigma\mathcal{D}$

(iii) Suppose $\mathcal{C}^* \leqslant \mathcal{D}$ $(\mathcal{C},\mathcal{D} \in q)$. Then we show $\Sigma\mathcal{D}^* \leqslant \Sigma\mathcal{C}$. To see this, suppose $(d_1,d_2) \in \mathcal{D}$.

Then $\begin{cases} st_1(d_1,\mathcal{D}) \leqslant c_1 \\ st_2(d_2,\mathcal{D}) \leqslant c_2 \end{cases}$, where $(c_1,c_2) \in \mathcal{C}$.

Consider $st_1(\Sigma_{d_1},\Sigma\mathcal{D}) = \cup\{\Sigma_{d_{i1}}: \Sigma_{d_{i2}} \cap \Sigma_{d_1} \neq \emptyset ,$

$$(d_{i1},d_{i2}) \in \mathcal{D}\}.$$

$$= \cup\{\Sigma_{d_{i1}}: \Sigma_{d_{i2} \wedge d_1} \neq \emptyset , \ (d_{i1},d_{i2}) \in \mathcal{D}\}$$

$$\subseteq \cup\{\Sigma_{d_{i1}}: d_{i2} \wedge d_1 \neq 0\}$$

$$= \Sigma_{\vee d_{i1}} \quad \text{where } d_{i2} \wedge d_1 \neq 0 , \ (d_{i1},d_{i2}) \in \mathcal{D}$$

$$= \Sigma_{st_1(d_1,\mathcal{D})} \subseteq \Sigma_{c_1} \quad \text{, as required.}$$

The case $st_2(\Sigma_{d_2},\Sigma\mathcal{D})$ is similar. □

We can, as expected say something about the topologies at hand:

3.32 <u>Proposition</u>

The topologies $T_1(\Sigma q^*)$ and $T_2(\Sigma q^*)$ coincide with the spectral topologies $T_{\Sigma L_1}$, $T_{\Sigma L_2}$ respectively.

<u>Proof</u>:

Suppose $p \in U \in T_i(\Sigma q^*)$ $(i = 1 \text{ or } 2)$; there is $\mathcal{D} \in \Sigma q^*$ such that $st_i(p,\mathcal{D}) \subseteq U$.
But $\mathcal{D} \geqslant \mathcal{C} = \Sigma\mathcal{C}_1^r \wedge \Sigma\mathcal{C}_2^r \wedge \dots \wedge \Sigma\mathcal{C}_m^r \in \Sigma q^*$ and so

$st_i(p,C) \subseteq U$. But $st_i(p,C)$ is a union of $T_{\Sigma L_i}$-open sets, so $U \in T_{\Sigma L_i}$.

Conversely, suppose $U \in T_{\Sigma L_i}$; then $U = \Sigma_a$ $(a \in L_i)$, and $a = \bigvee\{b \in L_i: st_i(b,C) \leqslant a$ for some $C \in q\}$. Select $p \in U = \Sigma_a$; then $p(a) = 1$, and for some $C \in q$, $b \in L_i$ such that $st_i(b,C) \leqslant a$, we must have $p(b) = 1$. We can now show that

$st_i(p,\Sigma C) \subseteq \Sigma_a$ (and $\Sigma C \in q^*$ of course): fix $i = 1$. We have

$$\begin{aligned} st_1(p,\Sigma C) &= \cup\{\Sigma_{c_1}: (c_1,c_2) \in C \text{ and } p \in \Sigma_{c_2}\} . \\ &= \cup\{\Sigma_{c_1}: (c_1,c_2) \in C \text{ and } p(c_2) = 1\} \end{aligned}$$

but $p(c_2) = 1$ and $p(b) = 1 \Rightarrow p(c_2 \wedge b) = 1$

$$\begin{aligned} &\Rightarrow c_2 \wedge b \neq 0 \\ &\Rightarrow c_1 \leqslant st_1(b,C) \leqslant a \end{aligned}$$

so $\Sigma_{c_1} \subseteq \Sigma_a$ as desired; the case $i = 2$ is similar.

□

3.33 Definition

Suppose $f: (L,q) \to (M,p)$ is a quasi-uniform map. Define $\Sigma f: \Sigma M \to \Sigma L$ by $\Sigma f(p) = p \circ f$.

3.34 Proposition

$\Sigma f: (\Sigma M, \Sigma p^*) \to (\Sigma L, \Sigma q^*)$ is quasi-uniformly continuous.

Proof:

Use proposition 3.12.

□

<u>Remark</u>:

We see that $\Sigma : (L,q) \to (\Sigma L, \Sigma q^*)$ is a contravariant functor from <u>QUNFRM</u> to <u>QUN</u> .

3.35 <u>Theorem</u>

The two contravariant functors $\mathcal{Q}$, Σ are adjoint on the right.

<u>Proof</u>:

Let (X,μ) be a quasi-uniform space, and let (L,q) be a quasi-uniform frame, where $L = (L_0, L_1, L_2)$.

Let $f \in \hom((L,q), (\mathcal{Q}X, \mathcal{Q}\mu))$

$g \in \hom((X,\mu), (\Sigma L, \Sigma q^*))$.

As usual, define $\overline{f}: X \to \Sigma L$ by $\overline{f}(x)(a) = 1$ iff $x \in f(a)$ $(a \in L_0)$. We check that $\overline{f}$ is quasi-uniformly continuous: let $\mathcal{C} \in q$, and consider $\overline{f}^{-1}(\Sigma \mathcal{C}) = \{(f(c_1), f(c_2)): (c_1,c_2) \in \mathcal{C}\}$. But this is a member of $\mathcal{Q}\mu$, hence a member of μ . Now, appealing to proposition 3.12, we have $\overline{f}: (X,\mu^*) \to (\Sigma L, \Sigma q^*)$ is quasi-uniformly continuous, but $\mu^* = \mu$, yielding the required result. Now define $\tilde{g}: L \to \mathcal{Q}X$ by $\tilde{g}(a) = g^{-1}(\Sigma_a)$ $(a \in L_0)$; g is continuous with repsect to the spectral topologies, so $\tilde{g}$ is a biframe map. Now let $\mathcal{C} \in q$: $\tilde{g}(\mathcal{C}) = \{(\tilde{g}(c_1), \tilde{g}(c_2)): (c_1,c_2) \in \mathcal{C}\}$

$$= \{(g^{-1}(\Sigma_{c_1}), g^{-1}(\Sigma_{c_2})): (c_1,c_2) \in \mathcal{C}\}$$

$$= g^{-1}(\Sigma \mathcal{C}) \in \mu \text{ and hence of } \mathcal{Q}\mu \ .$$

The naturality conditions are easily disposed of;

□

Remarks:

The fixed objects of the adjunction are the separated quasi-uniform spaces and the quasi-uniform frames that are spatial.

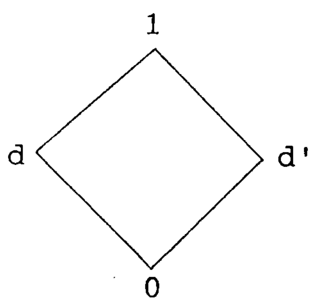
Notes on Chapter 3

(1) Quasi-uniform spaces are presented in the literature in terms of entourages or surroundings. This is undoubtably a convenient and easily applicable theory which has achieved much success. (See Fletcher and Lindgren [15].) Gantner and Steinlage [16] in a short paper provide other methods of presenting quasi-uniform spaces including the method of conjugate cover pairs. Their formulation is perhaps rather cumbersome, and this may have contributed to a certain lack of interest in pursuing or developing the "cover" approach. It seems clear now from both the frame and space point of view that this was perhaps an oversight; we cite as evidence for this statement the elegance and simplicity of the Pervin quasi-uniformity when constructed via covers, as done in Chapter 5.

(2) We note that the "strength" axiom (Definitions 3.1(ii), 3.13(ii)) does cause mild technical problems now and again. This problem is of course absent in the case of uniform spaces, since adjoining or removing the empty set to or from a given cover of a set X is of little significance; one may think of non-strong conjugate covers as those where the empty set is "decomposed" into the intersection (empty) of two non-empty subsets of X. Thus the need arises to be able to adjoin to a family of conjugate covers the canonical strong conjugate covers "underlying" this family. This explains

the content of the technical results 3.7 to 3.12 and 3.19 to 3.23. Our method of doing so presents a mild problem in the case that we are considering the empty set (which is a uniform and hence quasi-uniform space) or the frame (or biframe) in which $0 = 1$ (essentially the same thing) which is a uniform (and hence quasi-uniform) frame. But as these are already quasi-uniform spaces/frames, the procedure is unneccesary. A slightly more refined procedure can be devised which covers all cases (simply do *not* "remove" pairs of the form $(\emptyset,\emptyset)$ or $(0,0)$).

(3) We make special note of Example 3.16(i) ; the fact that D



viewed as a biframe has unique compatible quasi-uniform structure plays an important role in chapter 5. An obvious question, which we have not pursued is to ask which biframes have unique compatible quasi-uniform structures.

(4) The set of quasi-uniform structures compatible with a biframe forms a complete lattice. See remark 3 in the notes on chapter 2.

CHAPTER 4

First the relevant background for proximity spaces. We will present the notion of a proximity space in terms of "strong inclusion", although we could equally use the notion of "nearness". The former notion lends itself readily to a lattice-theoretic treatment. The reader is referred to Naimpally and Warrack [32] for a general reference.

4.1 Definition

Let X be a set, $\ll$ a relation on PX. $(X,\ll)$ is a *proximity space* if

(i) $X \ll X$, $\emptyset \ll \emptyset$

(ii) $A \ll B \Rightarrow A \subseteq B$

(iii) $A \subseteq B \ll C \subseteq D \Rightarrow A \ll D$

(iv) $A_i \ll B \Rightarrow \cup A_i \ll B$; $A \ll B_i \Rightarrow A \ll \cap B_i$ $(i = 1,2,\ldots,n)$

(v) $A \ll B$ implies that there is $C \subseteq X$ such that $A \ll C \ll B$

(vi) $A \ll B \Rightarrow X \setminus B \ll X \setminus A$.

For $A \ll B$, read "A strongly included in B".
A set $A \subseteq X$ is *open* iff for each $x \in A$, $x \ll A$ (strictly: $\{x\} \ll A$). The open sets form a topology which is completely regular. We will need the following simple lemmas.

4.2 Lemma

Let $(X,\ll)$ be a proximity space. Suppose $A \ll B$; there is an open set, U , satisfying $A \ll U \ll B$.

Proof:

Select $C \subseteq X$ such that $A \ll C \ll B$. Let $U = \{x: x \ll C\}$; U is open since for each $x \ll C$ select D such that $x \ll D \ll C$. Clearly $D \subseteq U$, so $x \ll U$, as required. (In fact, U is the interior of C.) Also $A \ll U$, since again if D is such that $A \ll D \ll C$, we have $D \subseteq U$, so $A \ll D \subseteq U$, yielding $A \ll U$. □

4.3 Lemma

Let $(X, \ll)$ be a proximity space, with $A \ll B$; then $\overline{A} \ll B$, (and $\overline{A} \subseteq B$), where $\overline{A}$ denotes the closure of A with respect to the topology mentioned above.

Proof:

We prove only that $\overline{A} \subseteq B$. If this is so, select C such that $A \ll C \ll B$. Then $\overline{A} \subseteq C \ll B$ yielding $\overline{A} \ll B$. Now suppose $x \in X \setminus B \ll X \setminus A$. But then x is in the interior of $X \setminus A$, so $x \notin \overline{A}$, as required. □

4.4 Lemma

Let $(X, \ll)$ be a proximity space, and suppose $U \subseteq X$ is open. Then $U = \bigcup\{V: V$ is open and $V \ll U\}$.

Proof:

We have that $x \in U \Rightarrow x \ll U \Rightarrow x \ll V \ll U$, where V is open by lemma 4.2 yielding $x \in V \ll U$, as required.

□

We may now proceed to the main task of setting up the category <u>PROXFRM</u> of proximal frames.

4.5 <u>Definition</u>

Let L be a frame, $\ll$ a relation on L.

(1) $(L,\ll)$ is a *proximal frame* if:

(i) $1 \ll 1$, $0 \ll 0$.

(ii) $a \ll b \Rightarrow a \leq b$.

(iii) $a \leq b \ll c \leq d \Rightarrow a \ll d$.

(iv) For $i = 1,2,\ldots,n$; $a_i \ll b \Rightarrow \vee a_i \ll b$ and $a \ll b_i \Rightarrow a \ll \wedge b_i$.

(v) $a \ll b \Rightarrow$ there is $c \in L$ such that $a \ll c \ll b$.

(vi) $a \ll b \Rightarrow b^* \ll a^*$.

(vii) $a = \vee\{b: b \ll a\}$ for each $a \in L$.

We also say $\ll$ is a compatible "strong inclusion" on L.

(2) Let $(L,\ll_1)$, $(M,\ll_2)$ be proximal frames; a function $f: L \to M$ is a *proximal map* if

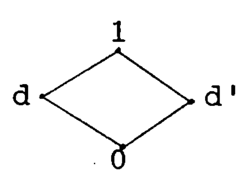
(i) $f: L \to M$ is a frame map

(ii) $a \ll_1 b \Rightarrow f(a) \ll_2 f(b)$.

(3) Proximal frames and maps are the objects and arrows of the category <u>PROXFRM</u>.

4.6 <u>Examples</u>

(i) Let D as usual be the Boolean algebra with Hasse diagram:



We let $t_1 \ll t_2$ iff $t_1 \leqslant t_2$ for any $t_1, t_2 \in D$; this is the unique strong inclusion compatible with D .

(ii) Let B be any complete Boolean algebra; for $t_1, t_2 \in B$, let $t_1 \ll t_2$ iff $t_1 \leqslant t_2$; $\ll$ is a strong inclusion compatible with B .

(iii) $\underline{3}$ has no compatible strong inclusion. ($\leqslant$ will not work since $d \not\ll d$.)

(iv) Let $(X, \ll)$ be a proximity space; denote by $\mathcal{O}X$ the open sets of $(X, \ll)$, and set $\mathcal{O}\ll \, = \, \ll$; then $(\mathcal{O}X, \mathcal{O}\ll)$ is a proximal frame; to see this use lemmas 4.2, 4.3, 4.4.

Again, non spatial complete Boolean algebras give examples of non spatial proximal frames.

The underlying frame, L , of a proximal frame $(L, \ll)$ is, as can be expected, regular (and completely regular assuming a choice principle.)

4.7 Proposition

Let $(L, \ll)$ be a proximal frame; then L is a regular frame.

Proof:

Obvious, since $a = \bigvee\{b : b \ll a\}$ and $b \ll a \Rightarrow b \mathrel{\overline{<}} a$.

□

4.8 <u>Proposition</u>

Let $(L,\preccurlyeq)$ be a proximal frame. If we assume the countable dependent axiom of choice, then L is completely regular.

<u>Proof</u>:

Also obvious; suppose $a \preccurlyeq b$; select $x_{\frac{1}{2}}$ such that $a \preccurlyeq x_{\frac{1}{2}} \preccurlyeq b$. Now select $x_{\frac{1}{4}}$, $x_{\frac{3}{4}}$ such that $a \preccurlyeq x_{\frac{1}{4}} \preccurlyeq x_{\frac{1}{2}} \preccurlyeq x_{\frac{3}{4}} \preccurlyeq b$. Proceeding in this manner and assuming the stated axiom of choice, we exhibit a family $\{x_\alpha : \alpha \in J\}$ of required interpolating elements.

□

4.9 <u>Proposition</u>

Let L be a completely regular frame; then L has a compatible strong inclusion.

<u>Proof</u>:

Set $a \preccurlyeq b$ if $a \,\overline{<}\overline{<}\, b$. We check only that $a \preccurlyeq b \Rightarrow b^* \preccurlyeq a^*$; to see this we need only show that $a \,\overline{<}\, b \Rightarrow b^* \,\overline{<}\, a^*$, but this is almost obvious: $a \,\overline{<}\, b \Rightarrow a^* \vee b = 1$, and of course $b^* \wedge b = 0$, so we immediately get that $b^* \,\overline{<}\, a^*$. □

<u>Remark</u>:

The strong inclusion above is clearly functorial. We exhibit now a pair of "open" and "spectrum" functors, adjoint on the right; firstly the "open" functor $\mathcal{O}$: <u>PROX</u> → <u>PROXFRM</u>

assigns to each proximity space $(X,\ll)$, the proximal frame $(\mathcal{Q}X,\mathcal{Q}\ll)$ as in example 4.6(iv). If $f: (X,\ll_1) \to (Y, \ll_2)$ is a proximity map, then define $\mathcal{Q}f: (\mathcal{Q}Y,\mathcal{Q}\ll_2) \to (\mathcal{Q}X,\mathcal{Q}\ll_1)$ by $\mathcal{Q}f(V) = f^{-1}(V)$; $\mathcal{Q}f$ is indeed a proximal map; the functor $\mathcal{Q}$ so defined is contravariant from <u>PROX</u> to <u>PROXFRM</u> .

4.10 <u>Definition</u>

Let $(L,\ll)$ be a proximal frame. As usual set $\Sigma L = \hom(L,\underline{2})$, and for $c \in L$, let $\Sigma_c = \{p \in \Sigma L: p(c) = 1\}$ Now, for $A,B \subseteq \Sigma L$ we set $A \ll_\Sigma B \Longleftrightarrow$ there exist $c_1,c_2 \in L$ such that $A \subseteq \Sigma_{c_1}$, $\Sigma_{c_2} \subseteq B$ and $c_1 \ll c_2$.

4.11 <u>Proposition</u>

$(\Sigma L,\ll_\Sigma)$ is a proximity space.

<u>Proof</u>:

(i) $\Sigma L = \Sigma_1 \ll_\Sigma \Sigma_1 = \Sigma L$, since $1 \ll 1$; similarly $\emptyset \ll_\Sigma \emptyset$.

(ii) $A \ll_\Sigma B \Rightarrow$ for some $c_1 \ll c_2 \in L$, $A \subseteq \Sigma_{c_1}$, $\Sigma_{c_2} \subseteq B$. But $c_1 \ll c_2 \Rightarrow c_1 \leqslant c_2 \Rightarrow \Sigma_{c_1} \subseteq \Sigma_{c_2}$, so $A \subseteq B$, as required.

(iii) $A \subseteq B \ll_\Sigma C \subseteq D$ clearly implies $A \ll_\Sigma D$.

(iv) Proof of axiom (iv) omitted; straightforward.

(v) $A \ll_\Sigma B \Rightarrow$ for some $c_1 \ll c_2 \in L$, $A \subseteq \Sigma_{c_1}$, $\Sigma_{c_2} \subseteq B$. Now select c_3 such that $c_1 \ll c_3 \ll c_2$. Then $A \ll_\Sigma \Sigma_{c_3} \ll_\Sigma B$, as required.

(vi) $A \ll_\Sigma B \Rightarrow$ for some $c_1 \ll c_2 \in L$ $A \subseteq \Sigma_{c_1}$, $\Sigma_{c_2} \subseteq B$. Now select c_3 such that $c_1 \ll c_3 \ll c_2$. Now we have $c_2^* \ll c_3^* \ll c_1^*$, so $\Sigma_{c_2^*} \ll_\Sigma \Sigma_{c_3^*} \ll_\Sigma \Sigma_{c_1^*}$.

We can now see that $\Sigma L \setminus B \subseteq \Sigma_{c_3^*}$, since $p \notin B \Rightarrow p \notin \Sigma_{c_2} \Rightarrow p(c_2) = 0$. Now if $p(c_3^*) = 0$, it follows that $p(c_2 \vee c_3^*) = 0$, but $c_2 \vee c_3^* = 1$, since $c_3 \lessapprox c_2$, a contradiction. Thus $p(c_3^*) = 1$, so $p \in \Sigma_{c_3^*}$, i.e. $\Sigma L \setminus B \subseteq \Sigma_{c_3^*}$. Also $\Sigma_{c_1^*} \subseteq \Sigma L \setminus A$, since $p(c_1^*) = 1 \Rightarrow p(c_1) = 0 \Rightarrow p \notin \Sigma_{c_1} \Rightarrow p \notin A$. This shows that $\Sigma L \setminus B \ll_\Sigma \Sigma L \setminus A$. $\square$

4.12 <u>Proposition</u>

The spectral topology on ΣL and the topology induced by $\ll_\Sigma$ coincide.

<u>Proof</u>:

Let U be open in the proximal topology; $p \in U \Rightarrow p \ll_\Sigma U$, so for some $c_1 \ll c_2 \in L$ we have $p \in \Sigma_{c_1}$ and $\Sigma_{c_2} \subseteq U$. But Σ_{c_2} is open in the spectral topology, so U is also a member of the spectral topology. Now suppose U is a member of the spectral topology; $U = \Sigma_a$ for some $a \in L$. But $a = \bigvee\{b\colon b \ll a\}$; if $p \in \Sigma_a$, then $p(a) = 1$ and for some $b \ll a$, $p(b) = 1$ also, so $p \in \Sigma_b \subseteq \Sigma_a$ and $b \ll a$, so $p \ll_\Sigma \Sigma_a$, as required.

$\square$

4.13 Definition

Let $f: (L, \leqslant_1) \to (M, \leqslant_2)$ be a proximal map. Define $\Sigma f: \Sigma M \to \Sigma L$ by $\Sigma f(p) = p \circ f$.

4.14 Proposition

$\Sigma f: (\Sigma M, \leqslant_{2\Sigma}) \to (\Sigma L, \leqslant_{1\Sigma})$ is a proximity map.

Proof:

$A \leqslant_{1\Sigma} B \Rightarrow$ for $c_1 \leqslant_1 c_2 \in L$, $A \subseteq \Sigma_{c_1}$ and $\Sigma_{c_2} \subseteq B$. So $f(c_1) \leqslant_2 f(c_2)$, and $\Sigma f^{-1}(A) \subseteq \Sigma_{f(c_1)}$, since $p \in \Sigma f^{-1}(A) \Rightarrow p \circ f \in \Sigma_{c_1}$, i.e. $p(f(c_1)) = 1$, so $p \in \Sigma_{f(c_1)}$. Also $\Sigma_{f(c_2)} \subseteq \Sigma f^{-1}(B)$, since $p(f(c_2)) = 1 \Rightarrow$ $p \circ f\ (c_2) = 1 \Rightarrow p \circ f \in \Sigma_{c_2} \subseteq B$ so $p \in \Sigma f^{-1}(B)$.

□

4.15 Corollary

Σ is a contravariant functor from PROXFRM to PROX.

4.16 Theorem

The functors $\mathcal{Q}, \Sigma$ are adjoint on the right.

Proof:

We will omit details that are similar to those in previous such theorems. Let $(X, \leqslant)$ be a proximity space, $(L, \leqslant)$ a proximal frame. Let $f \in \hom((L, \leqslant), (\mathcal{Q}X, \mathcal{Q}\leqslant))$

$g \in \hom((X, \leqslant), (\Sigma L, \leqslant_\Sigma))$.

As usual define $\overline{f}: X \to \Sigma L$ by $\overline{f}(x)(a) = 1$ iff $x \in f(a)$; $\overline{f}$ is a proximity map from $(X,\ll)$ to $(\Sigma L,\ll_\Sigma)$; to see this suppose $A \ll_\Sigma B$ and suppose $A \subseteq \Sigma_{c_1}$, $\Sigma_{c_2} \subseteq B$ and $c_1 \ll c_2 \in L$. Now $\overline{f}^{-1}(\Sigma_{c_1}) = f(c_1) \ll f(c_2) = \overline{f}^{-1}(\Sigma_{c_2})$, so $\overline{f}^{-1}(A) \ll \overline{f}^{-1}(B)$. Now define $\tilde{g}: L \to \mathcal{Q}X$ by $\tilde{g}(a) = g^{-1}(\Sigma_a)$; g is a proximity map, so g is continuous with respect to the spectral topology (by 4.12), so $g^{-1}(\Sigma_a) \in \mathcal{Q}X$; $\tilde{g}$ is easily a frame map; to see that it is a proximal map, suppose $a \ll b \in L$ then $\Sigma_a \ll_\Sigma \Sigma_b$ and $g^{-1}(\Sigma_a) \ll g^{-1}(\Sigma_b)$, so $\tilde{g}(a) \ll \tilde{g}(b)$. Naturality conditions are easily checked, as well as the fact that $\overline{\tilde{g}} = g$ and $\tilde{\overline{f}} = f$. □

4.17 Proposition

The fixed objects of the adjunction in 4.16 are the separated proximity spaces on the one hand and the spatial proximal frames on the other.

□

We examine now the relationship between uniform and proximal frames.

4.18 Proposition

Let (L,q) be a uniform frame and for $a,b \in L$ set $a \ll_q b$ iff for some $C \in q$, $st(a,C) \leq b$; then $(L,\ll_q)$ is a proximal frame.

Proof:

(i) For $\mathcal{C} \in q$, $st(1,\mathcal{C}) = 1$; $st(0,\mathcal{C}) = 0$.

(ii) For $\mathcal{C} \in q$, $st(a,\mathcal{C}) \leqslant b \Rightarrow a \mathrel{\bar{<}} b$. (See proof of proposition 2.6).

(iii) For $\mathcal{C} \in q$, $a \leqslant st(b,\mathcal{C}) \leqslant c \leqslant d \Rightarrow st(a,\mathcal{C}) \leqslant d$.

(iv) Omitted.

(v) For $\mathcal{C} \in q$; $st(a,\mathcal{C}) \leqslant b \Rightarrow$ there is $d \in L$, $\mathcal{D} \in q$ such that $st(a,\mathcal{D}) \leqslant d$, $st(d,\mathcal{D}) \leqslant b$ so $a \leqslant_q d \leqslant_q b$. (See the proof of proposition 2.7).

(vi) Suppose $st(a,\mathcal{C}) \leqslant b$ for $\mathcal{C} \in q$, then $st(b^*,\mathcal{C}) \leqslant a^*$: $c \wedge b^* \neq 0 \Rightarrow c \nleqslant b \Rightarrow c \wedge a = 0 \Rightarrow c \leqslant a^*$.

(vii) Clearly $a = \bigvee\{b: b \leqslant_q a\}$ since $a = \bigvee\{b:$ for some $\mathcal{C} \in q$, $st(b,\mathcal{C}) \leqslant a\}$.

4.19 Proposition

Let (L,q) be a uniform space; the assignation $U: (L,q) \to (L,\leqslant_q)$ is functorial.

Proof:

Suppose $f: (L,q) \to (M,p)$ is a uniform map, and that $a \leqslant_q b$, i.e. $st(a,\mathcal{C}) \leqslant b$ for some $\mathcal{C} \in q$. We then have that $f[\mathcal{C}] \in p$ and $st(f(a), f[\mathcal{C}]) \leqslant f(b)$:
$f(a) \wedge f(c) \neq 0 \Rightarrow f(a \wedge c) \neq 0 \Rightarrow a \wedge c \neq 0 \Rightarrow c \leqslant b \Rightarrow f(c) \leqslant f(b)$. □

We say that $\leqslant_q$ is the proximal relation *induced* by q .

Let us now consider the problem of endowing a proximal frame $(L,\ll)$ with a compatible uniform structure, $q_{\ll}(L)$, such that $q_{\ll}(L)$ "induces" $\ll$.

4.20 Theorem

Let $(L,\ll)$ be a proximal frame; a compatible uniform structure $q_{\ll}(L)$ exists such that $q_{\ll}(L)$ induces $\ll$.

Proof:

Suppose $a \ll b \in L$. Set $C_a^b = \{a^*, b\}$; C_a^b is a cover since $a \ll b \Rightarrow a \,\bar{<}\, b$. Now select $c_1, c_2 \in L$ such that

$$a \ll c_1 \ll c_2 \ll b .$$

It is straightforward to check that

$$C^* = (C_a^{c_1} \wedge C_{c_1}^{c_2} \wedge C_{c_2}^{b})^* \leq C_a^b$$

(we have done the check already). Thus as sub-base for $q_{\ll}(L)$ we take all covers of the form C_a^b where $a \ll b$. Since $b = \bigvee\{a : a \ll b\}$ and $st(a, C_a^b) = b$, we immediately have a compatible uniform structure. We need only show that $q_{\ll}(L)$ induces $\ll$. Suppose $a \ll b$; then $st(a, C_a^b) \leq b$ and so $a \ll_{q_{\ll}(L)} b$. Conversely, suppose for some $C \in q_{\ll}(L)$ we have $st(a, C) \leq b$. Our aim is to show that $a \ll b$. We may as well assume that $C = C_{a_1}^{b_1} \wedge C_{a_2}^{b_2} \wedge \ldots \wedge C_{a_n}^{b_n}$, where $a_i \ll b_i$ $(i = 1, \ldots, n)$. Select $c_i, d_i \in L$ such that $a_i \ll c_i \ll d_i \ll b_i$ for each $i = 1, \ldots, n$. Notice that $c_i^* \ll a_i^*$ and $d_i \ll b_i$, so we might say $C_{c_i}^{d_i}$ "strongly" refines $C_{a_i}^{b_i}$. We now observe that

$$C = C_1 \cup C_2 \quad \text{and} \quad C_1 \cap C_2 = \emptyset \quad \text{where} \quad C_1 = \{t \in C : t \wedge a = 0\}$$
$$C_2 = \{t \in C : t \wedge a \neq 0\}.$$

Suppose $t \in C$ is such that $t = \bigwedge_{i \in I} a_i^* \wedge \bigwedge_{j \in J} b_j$, $I \cup J = \{1, \ldots n\}$, $I \cap J = \emptyset$. Then define

$$t^r = \bigwedge_{i \in I} c_i^* \wedge \bigwedge_{j \in J} d_j \in C_{c_1}^{d_1} \wedge C_{c_2}^{d_2} \wedge \ldots \wedge C_{c_n}^{d_n}.$$

We have $t^r \ll t$. Now

$a \leqslant \bigwedge\{t^* : t \in C_1\} = (\bigvee_{t \in C_1} t)^* \ll (\bigvee_{t \in C_1} t^r)^* \leqslant \bigvee_{t \notin C_1} t^r$ (the t^r's form a cover) $= \bigvee_{t \in C_2} t^r \ll \bigvee_{t \in C_2} t = st(a, C) \leqslant b$, so $a \ll b$.

□

4.21 <u>Proposition</u>

If $f: (L, \ll) \to (M, \ll)$ is a proximal map then $f: (L, q_{\ll}(L)) \to (M, q_{\ll}(M))$ is a uniform map.

<u>Proof:</u>

We have $a \ll b \Rightarrow f(a) \ll f(b)$. Select $c, d \in L$ such that $a \ll c \ll d \ll b$; $C_{f(c)}^{f(d)} \in q_{\ll}(M)$ and refines $f[C_a^b] = \{f(a^*), f(b)\}$.

□

4.22 <u>Corollary</u>

$U_{\ll}: (L, \ll) \to (L, q_{\ll}(L))$ is a functor from <u>PROXFRM</u> to <u>UNIFRM</u>.

□

4.23 <u>Proposition</u>

Suppose $(L, \ll)$ is a proximal frame, q a compatible uniform structure on L that induces $\ll$. Then $q_{\ll}(L) \subseteq q$.

Proof:

Suppose $a \ll b \in L$. Then for some $\mathcal{C} \in q$, $\mathrm{st}(a,\mathcal{C}) \leq b$, since q induces $\ll$. Once again $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$ where

$\mathcal{C}_1 = \{c \in \mathcal{C}: c \wedge a = 0\}$

$\mathcal{C}_2 = \{c \in \mathcal{C}: c \wedge a \neq 0\}$.

If $c \wedge a = 0$, then $c \leq a^* \in \mathcal{C}_a^b$.

If $c \wedge a \neq 0$, then $c \leq b \in \mathcal{C}_a^b$, so $\mathcal{C}$ refines $\mathcal{C}_a^b$, so $\mathcal{C}_a^b \in q$, as required. □

4.24 Proposition

Let $(L,\ll)$ be a proximal frame. Suppose q is a totally bounded compatible uniformity which induces $\ll$; then $q \subseteq q_{\ll}(L)$.

Proof:

Let $\mathcal{C} \in q$; select $\mathcal{D} \in q$ such that $\mathcal{D}^* \leq \mathcal{C}$. Total boundedness allows us to select a finite subset, $\mathcal{E}$, of $\mathcal{D}$, which is still a cover of L. Suppose $\mathcal{E} = \{d_1, d_2, \ldots, d_n\}$. For each such d_i $(i = 1, \ldots, n)$ $\mathrm{st}(d_i,\mathcal{D}) \leq c_i \in \mathcal{C}$, so in fact $d_i \ll c_i$. Now $\mathcal{C}_{d_1}^{c_1} \wedge \mathcal{C}_{d_2}^{c_2} \wedge \ldots \wedge \mathcal{C}_{d_n}^{c_n}$ refines $\mathcal{C}$ (since $d_1^* \wedge d_2^* \wedge \ldots \wedge d_n^* = 0$!), showing that $\mathcal{C} \in q_{\ll}(L)$. □

4.25 Corollary

Let $(L,\ll)$ be a proximal frame; then $q_{\ll}(L)$ is the unique compatible totally bounded uniform structure which induces $\ll$. □

Remark:

The proofs of the results in 4.20 to 4.25 specialize rather elegantly to the "spatial" setting. Indeed a mimic of theorem 4.20 for proximity spaces and totally bounded uniform spaces seems to be simpler than the alternatives available in the literature.

We turn now to a consideration of the "non-symmetric" version of a proximal frame. The symmetry for proximal frames is encapsulated in the two axioms $a \ll b \Rightarrow b^* \ll a^*$ and $a \ll b \Rightarrow a \mathrel{\bar{<}} b$. There are two ways one can generalize to "non-symmetry".

(i) Omit these two axioms.

(ii) Replace them with appropriate "biframe" axioms.

The first option seems relatively uninteresting, especially when one realizes that in a quasi-proximity space, one is working with two topologies rather than one. The second approach seems to be rather appropriate if one hopes for links with quasi-uniform frames.

4.26 Definition

Let $L = (L_0, L_1, L_2)$ be a biframe. The ordered triple $(L, \ll_1, \ll_2)$ is a *quasi-proximal frame* if:

(i) $\ll_i$ $(i = 1 \text{ or } 2)$ is a relation on L_i .

(ii) $0 \ll_i 0$, $1 \ll_i 1$.

(iii) $a \leq b \ll_i c \leq d \quad (a,b,c,d \in L_i) \Rightarrow a \ll_i d$.

(iv) For $k = 1, 2, \ldots, n$; $a_k \ll_i b \Rightarrow \bigvee a_k \ll_i b$

$a \ll_i b_k \Rightarrow a \ll_i \bigwedge b_k$.

(v) $a \ll_i b \Rightarrow$ for some $c \in L_i$, $a \ll_i c \ll_i b$.

(vi) $a \ll_i b \Rightarrow a \,\overline{<}_i\, b$.

(vii) $a \ll_i b \Rightarrow b^* \ll_j a^*$ $(j = 1 \text{ or } 2,\ j \neq i)$.

(Recall: in a biframe $b \in L_i \Rightarrow b^* = \bigvee\{t \in L_j : t \wedge b = 0\}$.)

(viii) $a \in L_i$ $(i = 1 \text{ or } 2) \Rightarrow a = \bigvee\{b \in L_i : b \ll_i a\}$.

We will not develop the theory of such structures in detail, but provide sketches of salient results.

4.27 <u>Proposition</u>

Let $L = (L_0, L_1, L_2)$ be a biframe, $(L, \ll_1, \ll_2)$ a quasi-proximal frame; then L is completely regular.

<u>Proof</u>:

We have $a \ll_i b \Rightarrow a \,\overline{<}\overline{<}_i\, b$ $(i = 1 \text{ or } 2)$ (assuming the countable dependent choice axiom), and $b = \bigvee\{a : a \ll_i b\}$.

□

4.28 <u>Proposition</u>

Let $L = (L_0, L_1, L_2)$ be a completely regular biframe. There is a compatible quasi-proximal structure on L.

<u>Proof</u>:

Let $a \ll_i b \Leftrightarrow a \,\overline{<}\overline{<}_i\, b$ $(i = 1 \text{ or } 2)$. Then $(L, \ll_1, \ll_2)$ is a quasi-proximal frame.

□

Now let $(X, \ll)$ be a quasi-proximity space (all axioms except for (vi) of definition 4.1). A second $\ll$-type relationship exists in the form of

$$A \mathrel{\overline{\ll}} B \text{ iff } X \setminus B \ll X \setminus A .$$

Furthermore we have two topologies $T_{\ll}$ and $T_{\overline{\ll}}$; a set U is a member of $T_{\ll}(T_{\overline{\ll}})$ iff for each $x \in U$, $x \ll U$ $(x \mathrel{\overline{\ll}} U)$ One checks easily that if U is a member of $T_{\ll}$ $(T_{\overline{\ll}})$ then

$$U = \bigcup\{V \in T_{\ll}\colon V \ll U\} \qquad (U = \bigcup\{V \in T_{\overline{\ll}}\colon V \mathrel{\overline{\ll}} U\}) .$$

Finally suppose $A \ll B$; select C such that $A \ll C \ll B$, then $X \setminus B \mathrel{\overline{\ll}} X \setminus C$ and so we can find a member, D , of $T_{\overline{\ll}}$ such that $X \setminus B \mathrel{\overline{\ll}} D \mathrel{\overline{\ll}} X \setminus C$; $D \cap A = \emptyset$, since $D \cap C = \emptyset$ and $D \cup B = X$, so that takes care of (one half of) axiom (v). Other details are similar; we summarize this by stating that there is an open functor, $\mathcal{Q}$ from <u>QPROX</u> (the category of quasi-proximity spaces) to <u>QPROXFRM</u> (the category of quasi-proximal frames); the obvious definitions are left to the reader. One similarly has a "spectrum" functor in the reverse direction, these contravariant functors are adjoint on the right, and the fixed objects are as expected.

4.29 <u>Proposition</u>

Let (L,q) be a quasi-uniform frame $(L = (L_0, L_1, L_2))$. Set $a \ll_{iq} b$ $(i = 1 \text{ or } 2 \text{ ; } a,b \in L_i) \Longleftrightarrow st_i(a,C) \leqslant b$ for some $C \in q$. Then $(L, \ll_{1q}, \ll_{2q})$ is a quasi-proximal frame.

Proof:

We check only that $a \ll_{iq} b \Rightarrow b^* \ll_{jq} a^*$ $(i = 1$ or $2;$ $j = 1$ or $2;$ $i \neq j)$.

Suppose that $a, b \in L_1$ and $st_1(a, C) \leq b$, where $C \in q$. We must show that $st_2(b^*, C) \leq a^*$. Suppose $(c_1, c_2) \in C$ then $c_1 \wedge b^* \neq 0 \Rightarrow c_1 \nleq b \Rightarrow c_2 \wedge a = 0 \Rightarrow c_2 \leq a^*$, as required. The case with 1 and 2 reversed is similar. □

4.30 Proposition

Let $L = (L_0, L_1, L_2)$ be a biframe and $(L, \ll_1, \ll_2)$ a quasi-proximal frame. A compatible quasi-uniform structure $q_{\ll_1}^{\ll_2}(L)$ on L exists such that $q_{\ll_1}^{\ll_2}(L)$ induces $\ll_1, \ll_2$.

Proof:

Suppose $a \ll_1 b$ $(a, b \in L_1)$. Set $C_a^b = \{(b,1),(1,a^*)\}$. This is a strong conjugate cover of L. Select $c_1, c_2 \in L_1$ such that $a \ll_1 c_1 \ll_1 c_2 \ll_1 b$. We know that $C^* = (C_a^{c_1} \wedge C_{c_1}^{c_2} \wedge C_{c_2}^{b})^* \leq C_a^b$, so as sub-base for q, we select all covers of the form C_a^b where $a \ll_1 b$ (or $a \ll_2 b$, in which case $C_a^b = \{(a^*,1),(1,b)\}$). Finally we set $q_{\ll_1}^{\ll_2}(L) = q^*$. One must check that $st_1(a, C) \leq b \Rightarrow a \ll_i b$ $(i = 1$ or $2)$; this is a generalization of the method of proof of theorem 4.20. □

4.31 Proposition

Let $(L, \ll_1, \ll_2)$ be a quasi-proximal frame; then $q_{\ll_1}^{\ll_2}(L)$ is a functorial quasi-uniform structure and is the unique compatible totally bounded quasi-uniform structure which induces the pair $\leqslant_1, \leqslant_2$.

Proof:

Omitted. $\square$

Notes on Chapter 4

(1) For the interested reader: let $(X,\preccurlyeq)$ be a proximity space; for $A,B \subseteq X$ we set $A\,\delta\,B \iff A \not\preccurlyeq X \setminus B$, and say "A is near B" ; one can present axioms for proximity spaces in terms of properties which the pair (X,δ) must satisfy.

(2) Banaschewski [1] utilized structures very close to our proximal frames. In another paper [4] Banaschewski and Mulvey construct the Stone-Čech compactification of a frame using "completely regular" ideals; we observe that every proximal structure on a completely regular frame yields a "compactification" of the frame in a similar manner; the relationship between compactifications and compatible proximal structures needs further investigation.

(3) Császár [10] introduced the notion of a syntopogenous structure on a set X ; this is essentially a set of "orders" which satisfy the axioms of "strong inclusion" except for the interpolation axiom. This axiom is weakened in the following way: suppose $A <_\alpha B$ where $<_\alpha$ is one of the set of "orders" $(A,B \subseteq X)$; then there is $C \subseteq X$ and another "order" $<_\beta$ such that $A <_\beta C <_\beta B$. Strengthening the axioms of a syntopogenous space in various ways is shown to yield all the well known topological structures (uniform, proximity, quasi-uniform spaces etc.), thus unifying all these apparently diverse

structures. We note that the notion of a syntopogenous frame is certainly available and that "open" and "spectrum" functors (contravariant, adjoint on the right) also exist; however when one looks at the strengthening of the axioms required to yield uniform (or quasi-uniform) spaces or frames one immediately sees that the "spectrum" of a "uniform" syntopogenous frame need *not* be a uniform syntopogenous space. This is a surprising state of affairs which seems to emphasize that (in the absence of points) covers are "best". This has not been pursued any further, and awaits deeper investigation. In some sense, the spectrum functors we have constructed in this chapter, which are special cases of the more general "syntopogenous" spectrum functor mentioned above, are not "the same" as the spectrum functors constructed in chapters 2 and 3; this is revealed further in that the following diagram of functors need not commute:

$$\begin{array}{ccc}
\underline{\mathrm{PROX}} & \xleftarrow{\quad\Sigma\quad} & \underline{\mathrm{PROXFRM}} \\
{\scriptstyle \mu_{\ll}(\;)}\big\downarrow & & \big\downarrow{\scriptstyle q_{\ll}(-)} \\
\underline{\mathrm{UNIF}} & \xleftarrow{\quad\Sigma\quad} & \underline{\mathrm{UNIFRM}}
\end{array} \quad .$$

(It does, at least in the case where the proximal frame given in spatial, or even if every non-zero element of the frame is separated from 0 by some "point".)

CHAPTER 5

A well known, simple, but none-the-less surprising fact is that every topological space gives rise, quite naturally, to at least one quasi-uniformity which generates as one of its topologies the given topology. This result has been proved by many authors (Császár [10], Pervin [33]) using entourages. It will be instructive to see how the proof goes from the point of view of conjugate covers.

5.1 Definition

Let (X,T) be a topological space, $U \in T$, $(\emptyset \neq U \neq X)$. Define $C(U) = \{(U,X),(X,X\setminus U)\}$.

5.2 Proposition

(i) $C(U)$ is a strong conjugate cover pair.

(ii) $C(U)^* \leqslant C(U)$.

Proof:

(i) Obvious.

(ii) $st_1(U,\ C(U)) = U \qquad st_1(X,C(U)) = X$

$st_2(X,\ C(U)) = X \qquad st_2(X\setminus U,C(U)) = X\setminus U$. □

5.3 Proposition

Let (X,T) be a topological space, and let $\mu(T)$ be the family of conjugate covers which has as sub-base the family $\{\mathcal{C}(U) : U \in T\}$. Then $(X, \mu(T)^*)$ is a quasi-uniform space, and $T_1(\mu(T)^*) = T$.

Proof:

$(X, \mu(T)^*)$ is certainly a quasi-uniform space by proposition 5.2(ii) and proposition 3.10. It remains only to be shown that $T_1(\mu(T)^*) = T$. Select $V \in T_1(\mu(T)^*)$; for each $x \in V$, there is $\mathcal{C} \in \mu(T)^*$ such that $st_1(x,\mathcal{C}) \subseteq V$. But $\mathcal{C} \geq \mathcal{C}_1^r \wedge \mathcal{C}_2^r \wedge \ldots \wedge \mathcal{C}_n^r$ where each $\mathcal{C}_i$ $(i = 1, \ldots, n)$ is a member of $\mu(T)$. It thus suffices to show that $st_1(x,\mathcal{C}(U))$ is T-open for any $U \in T$: but this is just about obvious since $st_1(x,\mathcal{C}(U))$ is either U or X .

Conversely, select $U \in T$; for $x \in U$, $st_1(x,\mathcal{C}(U)) = U$, and since $\mathcal{C}(U) \in \mu(T)^*$, U is a member of $T_1(\mu(T)^*)$.

□

We call $\mu(T)^*$ the Pervin quasi-uniformity.
Brümmer [5] has proved (using entourages, of course) that:

(i) If $f: (X,T) \to (Y,S)$ is continuous, then f is quasi-uniformly continuous form $(X, \mu(T)^*)$ to $(Y, \mu(S)^*)$. (The Pervin quasi-uniformity is functorial.)

(ii) If F is any functor from <u>TOP</u> to <u>QUN</u> , right inverse to the forgetful functor T_1: <u>QUN</u> $\to$ <u>TOP</u> (which assigns the "first" topology), then the F quasi-uniformity is finer than the Pervin quasi-uniformity.

The second topology, $T_2(\mu(T)^*)$ is also interesting; it has as base all T-closed sets of X (since $X \setminus U$ is T-closed), and $T_1(\mu(T)^*) \vee T_2(\mu(T)^*)$ is the well known Skula topology (Skula [39]). Let F: <u>TOP</u> $\rightarrow$ <u>QUN</u> be any functor right inverse to the forgetful functor T_1: <u>QUN</u> $\rightarrow$ <u>TOP</u>. Salbany [36] has proved that the join of the two topologies generated by the F-quasi-uniformity is none other than the Skula topology. We are naturally led to ask:

(1) What is the frame analogue of the Skula topology?

(2) If it exists, can it be used to provide right inverses to the forgetful functor U : <u>QUNFRM</u> $\rightarrow$ <u>FRM</u> which assigns the "first" subframe of the underlying bi-frame.

(3) Is there a "minimal" such right inverse?

We are led now to an important structure, the congruence lattice of a frame (also called the assembly). This has been studied by many authors (Dowker [11], Isbell [21], Johnstone [23]). It is felt that a comprehensive treatment in terms of congruences is of some value. Recently (Johnstone [23], Simmons [37]) treatment of this structure has favoured use of the so-called nuclei, but working with congruences has led to advantages, especially for studying lattice structures more general than frames (σ-frames (Gilmour [17]), distributive lattices). We now present such a treatment recovering the known facts in a simple and appealing manner, which is susceptible to considerable generalization.

5.4 Definition

Let L be a frame; ρ is a *congruence* on L if ρ is an equivalence relation on L and satisfies

(I) $x_1 \rho y_1$ and $x_2 \rho y_2 \Rightarrow (x_1 \wedge x_2) \rho (y_1 \wedge y_2)$

(II) $x_\alpha \rho y_\alpha$ ($\alpha \in A$, an arbitrary set) $\Rightarrow \bigvee x_\alpha \rho \bigvee y_\alpha$.

Remark:

Each congruence on L is a sub-frame of $L \times L$.

5.5 Definition

Let L be a frame; denote by CL the set of all congruences on L .

Remark:

CL is a complete lattice; arbitrary meets exist; just take intersections. It follows that arbitrary joins exist; take meet of all upper bounds; these are usually *not* just unions (an example will be given).

5.6 Proposition

The following are examples of congruences on L . Let $a,b \in L$:

(i) $\nabla_a = \{(x,y): x \vee a = y \vee a\}$

(ii) $\Delta_a = \{(x,y): x \wedge a = y \wedge a\}$

(iii) $\rho[a,b] = \bigwedge\{\rho \in CL: a \rho b\}$

(iv) $L \times L \quad (= \nabla_1 = \Delta_0)$

(v) $\{(x,x): x \in L\} \quad (= \nabla_0 = \Delta_1)$.

Proof:

Omitted. □

Remarks:

(i) ∇_1 is the top, ∇_0 the bottom element of CL .

(ii) (iii) is just notation for the "smallest" congruence identifying a and b . The key to understanding CL is a characterization of $\rho[a,b]$, at least for $a \leqslant b$.

5.7 Lemma

Let ρ be a congruence on L , $a,b,c \in L$. Then

(i) $a \,\rho\, b \Rightarrow (a \wedge b) \,\rho\, (a \vee b)$,

(ii) $a \,\rho\, b$ and $a \leqslant c \leqslant b \Rightarrow a \,\rho\, c$ and $c \,\rho\, b$.

Proof:

Straightforward. □

Remark:

Lemma 5.7 frequently allows one, when working with a pair of elements a, b say, to assume without loss of generality that $a \leqslant b$.

5.8 Proposition

Let $a,b \in L$: (i) $\nabla_a = \rho[0,a]$

(ii) $\Delta_a = \rho[a,1]$

(iii) For $a \leqslant b$, $\rho[a,b] = \rho[0,b] \wedge \rho[a,1]$.

Proof:

(i) Since $0 \vee a = a \vee a$ we have $(0,a) \in \nabla_a$, yielding $\rho[0,a] \subseteq \nabla_a$. Conversely, suppose $(x,y) \in \nabla_a$, i.e. $x \vee a = y \vee a$; we now have

$$\begin{cases}(x \vee 0\ ,\ x \vee a) = (x, x \vee a) \in \rho[0,a] & \text{(by 5.4(II))}\\ (x \vee a\ ,\ y \vee a) \in \rho[0,a] & \\ (y \vee a\ ,\ y) \in \rho[0,a] & \text{(by 5.4(II))}\end{cases}$$

yielding (transitivity) $(x,y) \in \rho[0,a]$ as desired.

(ii) Similar to (i).

(iii) Since $a \vee b = b \vee b$ $(=b)$, we have $(a,b) \in \nabla_b$, similarly $(a,b) \in \Delta_a$, thus $\rho[a,b] \subseteq \nabla_b \wedge \Delta_a = \rho[0,b] \wedge \rho[a,1]$.

Conversely, suppose $(x,y) \in \nabla_b \wedge \Delta_a$, i.e. $x \wedge a = y \wedge a$ and $x \vee b = y \vee b$. We now have

$$\begin{cases}(x \wedge b\ ,\ x \wedge a) \in \rho[a,b] & \text{(by 5.4(I))}\\ (x \wedge a\ ,\ y \wedge a) \in \rho[a,b] & \text{(since } x \wedge a = y \wedge a)\\ (y \wedge a\ ,\ y \wedge b) \in \rho[a,b] & \text{(by 5.4(I))}\end{cases}$$

yielding $(x \wedge b, y \wedge b) \in \rho[a,b]$. Now

$$((x \wedge y) \vee (x \wedge b)\ ,\ (x \wedge y) \vee (y \wedge b)) \in \rho[a,b] \qquad \text{(by 5.4(II))},$$

so $(x \wedge (y \vee b)\ ;\ y \wedge (x \vee b)) \in \rho[a,b]$,

i.e. $(x \wedge (x \vee b)\ ,\ y \wedge (y \vee b)) \in \rho[a,b]$,

i.e. $(x,y) \in \rho[a,b]$, as desired. □

5.9 Corollary

Let $a \in L$; then $\nabla_a \wedge \Delta_a = 0$ and $\nabla_a \vee \Delta_a = 1$.

Proof:

We have $(x,y) \in \nabla_a \wedge \Delta_a$ iff $x \vee a = y \vee a$ and $x \wedge a = y \wedge a$, but since L is certainly a distributive lattice, this implies $x = y$, i.e. $\nabla_a \wedge \Delta_a = 0$. Since $(0,a) \in \nabla_a$ and $(a,1) \in \Delta_a$ it follows that $(0,1) \in \nabla_a \vee \Delta_a$. Lemma 5.7 now ensures that $(x,y) \in \nabla_a \vee \Delta_a$ for every $x,y \in L$, showing that $\nabla_a \vee \Delta_a = 1$. □

5.10 Corollary

Let $\rho \in CL$; then $\rho = \dot{\bigvee}\{\nabla_b \wedge \Delta_a : a \,\rho\, b$ and $a \leqslant b\}$.

Proof:

Since $(a,b) \in \rho$ implies $\rho[a,b] \subseteq \rho$, it follows that $\rho = \dot{\bigvee}\{\rho[a,b] : a \,\rho\, b\}$. But $a \,\rho\, b \Rightarrow (a \wedge b)\,\rho\,(a \vee b) \Rightarrow \rho[a,b] = \rho[a \wedge b, a \vee b]$, and so we need only consider $\rho[a,b]$ where $a \leqslant b$, i.e. $\rho = \dot{\bigvee}\{\rho[a,b] : a \,\rho\, b$ and $a \leqslant b\}$. Now proposition 5.8 (iii) shows that $\rho = \dot{\bigvee}\{\nabla_b \wedge \Delta_a : a \,\rho\, b$ and $a \leqslant b\}$. □

Remarks:

(i) Corollary 5.10 demonstrates the importance of congruences of the form ∇_a, Δ_a ; they are seen to generate the congruence lattice.

(ii) We do not as yet have any obvious distributivity properties. These will be established in the next few results.

(iii) As a simple example, we can now calculate the congruence lattice of the 3-chain, $\underline{3}$. Suppose the 3 distinct elements of $\underline{3}$ are $0,d,1$:

$$\nabla_d = \{(0,0),\ (d,d),\ (1,1),\ (0,d)\}$$

$$\Delta_d = \{(0,0),\ (d,d),\ (1,1),\ (d,1)\} \ .$$

Thus $C\underline{3}$ has four distinct elements, viz. $0,\nabla_d,\Delta_d,1$, and $\nabla_d{}' = \Delta_d$. This is just the Boolean algebra, D , of example 2.5(i). Notice also that $1 = \nabla_d \vee \Delta_d \neq \nabla_d \cup \Delta_d$, since $(0,1) \notin \nabla_d \cup \Delta_d$, so here is the promised example showing that join in CL is not just union.

5.11 <u>Proposition</u>

Let L be a frame, a, b, x_α $(\alpha \in A)$ elements of L .

(i) $\nabla_a \wedge \nabla_b = \nabla_{a\wedge b}$

(ii) $\bigvee_A \nabla_{x_\alpha} = \nabla_{\bigvee x_\alpha}$

(iii) $\Delta_a \vee \Delta_b = \Delta_{a\wedge b}$

(iv) $\bigwedge_A \Delta_{x_\alpha} = \Delta_{\bigvee x_\alpha}$.

<u>Proof</u>:

(i), (iii) Straightforward.

(ii) $\nabla_{x_\alpha} \leqslant \nabla_{\bigvee x_\alpha}$ so $\bigvee\nabla_{x_\alpha} \leqslant \nabla_{\bigvee x_\alpha}$. Now suppose $\rho \geqslant \nabla_{x_\alpha}$ for each α . This means that $(0,x_\alpha) \in \rho$ for each α, and so $(0,\bigvee x_\alpha) \in \rho$, so $\rho \geqslant \nabla_{\bigvee x_\alpha}$. We thus have $\bigvee\nabla_{x_\alpha} \geqslant \nabla_{\bigvee x_\alpha}$, as required.

(iv) Similar to (ii).

□

Notice that this is the first (and only) occasion on which we use the full force of 5.4(II).

We establish now some useful distributivity properties.

5.12 Proposition

Let ρ, ρ_α $(\alpha \in A)$ be elements of CL, $a \in L$ a frame. Then

(i) $\nabla_a \vee \rho = \{(x,y): (x \vee a, y \vee a) \in \rho\}$ (which we denote by ρ^a)

(ii) $\Delta_a \vee \rho = \{(x,y): (x \wedge a, y \wedge a) \in \rho\}$ (which we denote by ρ_a)

(iii) $\nabla_a \vee (\Lambda \rho_\alpha) = \Lambda(\nabla_a \vee \rho_\alpha)$

(iv) $\Delta_a \vee (\Lambda \rho_\alpha) = \Lambda(\Delta_a \vee \rho_\alpha)$.

Proof:

(i) One must check that ρ^a is indeed a congruence; this is straightforward. Clearly $\nabla_a \subseteq \rho^a$, $\rho \subseteq \rho^a$, so $\nabla_a \vee \rho \subseteq \rho^a$. Now suppose that $(x \vee a, y \vee a) \in \rho$; we have

$$\begin{cases}(x, x \vee a) \in \nabla_a \subseteq \nabla_a \vee \rho \\ (x \vee a, y \vee a) \in \rho \subseteq \nabla_a \vee \rho \\ (y \vee a, y) \in \nabla_a \subseteq \nabla_a \vee \rho\end{cases}$$

yielding $(x,y) \in \nabla_a \vee \rho$, so $\nabla_a \vee \rho = \rho^a$.

(ii) Similar to (i).

(iii) Follows easily from (i).

(iv) Follows easily from (ii).

□

It is our aim to establish that CL is indeed a frame. This we can do after noting a corollary of the above proposition.

5.13 Corollary

Let $\sigma, \rho_1, \rho_2 \in L$ and suppose $a \rho_1 b$. Then

$$\sigma \wedge \rho_1 \leqslant \rho_2 \Rightarrow \sigma \leqslant \nabla_a \vee \Delta_b \vee \rho_2 .$$

Proof:

We have
$$\begin{aligned}
\nabla_a \vee \Delta_b \vee \rho_2 &= \nabla_a \vee \Delta_b \vee \rho_2 \vee (\sigma \wedge \rho_1) \quad \text{(since } \sigma \wedge \rho_1 \leqslant \rho_2\text{)} \\
&= \rho_2 \vee [\nabla_a \vee \Delta_b \vee (\sigma \wedge \rho_1)] \\
&= \rho_2 \vee [\nabla_a \vee ((\Delta_b \vee \sigma) \wedge (\Delta_b \vee \rho_1))] \\
&= \rho_2 \vee [(\nabla_a \vee \Delta_b \vee \sigma) \wedge (\nabla_a \vee \Delta_b \vee \rho_1)]
\end{aligned} \Bigg\} \text{ Using 5.12.}$$
$$\begin{aligned}
&= \rho_2 \vee [(\nabla_a \vee \Delta_b \vee \sigma) \wedge 1] \quad \text{(since } a \rho_1 b\text{)} \\
&= \nabla_a \vee \Delta_b \vee \rho_2 \vee \sigma ,
\end{aligned}$$

so $\sigma \leqslant \nabla_a \vee \Delta_b \vee \rho_2$, as claimed. □

Remark:

The intuitive content of this result becomes more apparent if we consider the special case $\rho_2 = 0$, in which case the corollary states $\sigma \wedge \rho_1 = 0 \Rightarrow \sigma \leqslant \nabla_a \vee \Delta_b$ (where $a \rho_1 b$). This is an intuitive converse to

$a \rho_1 b \Rightarrow \nabla_b \wedge \Delta_a \leqslant \rho_1$.

We are now in a position to show that CL is a frame by exhibiting a relative pseudo-complement for any pair of congruences.

5.14 Theorem

CL is a frame.

Proof:

Let $\rho_1, \rho_2, \sigma \in CL$. Define $\rho_1 \to \rho_2 = \bigwedge\{\nabla_a \vee \Delta_b \vee \rho_2 :$ $a \leqslant b$ and $a \, \rho_1 \, b\}$. We check

(i) $(\rho_1 \to \rho_2) \wedge \rho_1 \leqslant \rho_2$;

(ii) $\sigma \wedge \rho_1 \leqslant \rho_2 \Rightarrow \sigma \leqslant \rho_1 \to \rho_2$, proving that $\rho_1 \to \rho_2$ is indeed the desired relative pseudo-complement.

(i) Suppose $a \, \rho_1 \, b$ and $a \leqslant b$ and $(a,b) \in \rho_1 \to \rho_2$.
This means that $(a,b) \in \nabla_a \vee \Delta_b \vee \rho_2$,
i.e. $(a \vee a, a \vee b) \in \Delta_b \vee \rho_2$,
i.e. $((a \vee a) \wedge b, (a \vee b) \wedge b) \in \rho_2$,
i.e. $(a,b) \in \rho_2$ as desired.

(ii) Follows from corollary 5.13. □

Remark:

It is appropriate to point out at this stage that nowhere in our proof of theorem 5.14 have we utilized the full infinite distributivity of the given frame L or the existence of arbitrary joins in L. Proposition 5.11 is not a step leading up to the proof of 5.14. The construction of CL in terms of so-called nuclei seems not to be susceptible to generalization (see notes).

We can now examine some functorial aspects of CL, which will lead us back to the questions posed at the beginning of the chapter.

5.15 Definition

(i) Define $\nabla_L: L \to CL$ by $\nabla_L(a) = \nabla_a$.

(ii) Let $\nabla L = \{\nabla_a : a \in L\}$.

(iii) Let ΔL be the subframe of CL generated by $\{\Delta_a : a \in L\}$.

5.16 Proposition

(i) ∇L is a subframe of CL.

(ii) $\nabla_L: L \to \nabla L$ is a bijection.

(iii) $\nabla_L: L \to CL$ is a bi-morphism (epi and mono).

Proof:

(i) Follows from proposition 5.11.

(ii) Suppose (without loss of generality) that $a \leqslant b$, $a \neq b$; since $(a,b) \in \nabla_b$, but $(a,b) \notin \nabla_a$, it follows that ∇_L is a bijection (∇_L is clearly "onto").

(iii) Suppose f, g are frame maps from CL to M such that $f \circ \nabla_L = g \circ \nabla_L$; now for $\rho \in CL$ we have

$$\begin{aligned} f(\rho) &= f(\bigvee\{\nabla_b \wedge \Delta_a : a \,\rho\, b \text{ and } a \leqslant b\}) \\ &= \bigvee\{f(\nabla_b) \wedge f(\Delta_a) : a \,\rho\, b \text{ and } a \leqslant b\} \\ &= \bigvee\{(f \circ \nabla_L)(b) \wedge f(\nabla_L(a)') : a \,\rho\, b \text{ and } a \leqslant b\} \\ &= \bigvee\{(g \circ \nabla_L)(b) \wedge g(\nabla_L(a)') : a \,\rho\, b \text{ and } a \leqslant b\} \\ &\qquad (\text{since } f(\nabla_L(a)') = ((f \circ \nabla_L)(a))'). \\ &= g(\rho), \text{ yielding } f = g, \text{ so } \nabla_L \text{ is an} \end{aligned}$$

epimorphism (as well as being a monomorphism, by (ii)). □

The next theorem is vital both in establishing functorial properties of CL and in other situations.

5.17 Theorem

Let $f: L \to M$ be a frame map such that each member of the image of L under f is complemented in M. There is a unique map $\overline{f}: CL \to M$ such that $\overline{f} \circ \nabla_L = f$.

$$\begin{tikzcd} L \arrow[r, "\nabla_L"] \arrow[dr, "f"'] & CL \arrow[d, dashed, "\overline{f}"] \\ & M \end{tikzcd} \qquad (f[L] \text{ complemented})$$

Proof:

Uniqueness follows from the fact that ∇_L is an epimorphism. Existence: For $\rho \in CL$, define
$\overline{f}(\rho) = \bigvee\{f(b) \wedge f(a)' : a \,\rho\, b$ and $a \leqslant b\}$. We first check that $\overline{f} \circ \nabla_L = f$; let $c \in L$:
$\overline{f} \circ \nabla_L(c) = \overline{f}(\nabla_c) = \bigvee\{f(b) \wedge f(a)' : a \vee c = b \vee c$ and $a \leqslant b\}$.
Since $0 \vee c = c \vee c$, $\overline{f} \circ \nabla_L(c) \geqslant f(c) \wedge f(0)' = f(c) \wedge 1$
$= f(c)$. On the other hand, suppose $a \vee c = b \vee c$ and $a \leqslant b$; then we have

$$\begin{aligned}(f(b) \wedge f(a)') \vee f(c) &= (f(b) \vee f(c)) \wedge (f(a)' \vee f(c)) \\ &= f(b \vee c) \wedge (f(a)' \vee f(c)) \\ &= f(a \vee c) \wedge (f(a)' \vee f(c)) \\ &= (f(a) \vee f(c)) \wedge (f(a)' \vee f(c)) \\ &= (f(a) \wedge f(a)') \vee f(c) \\ &= f(c) \text{ , so } f(b) \wedge f(a)' \leqslant f(c) \text{ .}\end{aligned}$$

So $\overline{f} \circ \nabla_L(c) = f(c)$ as required.

We must now check that $\overline{f}$ is indeed a frame map. It is straightforward to check that $\overline{f}$ preserves top and bottom elements and that $\overline{f}$ is order preserving. Thus we check that

$\overline{f}(\rho_1) \wedge \overline{f}(\rho_2) \leqslant \overline{f}(\rho_1 \wedge \rho_2)$: we have

$$\begin{aligned}\overline{f}(\rho_1) \wedge \overline{f}(\rho_2) &= \bigvee\{f(b) \wedge f(a)': a \;\rho_1\; b \text{ and } a \leqslant b\} \\ &\wedge \bigvee\{f(t) \wedge f(s)': s \;\rho_2\; t \text{ and } s \leqslant t\} \\ &= \bigvee\{f(b \wedge t) \wedge f(a \vee s)': a \;\rho_1\; b,\ s \;\rho_2\; t, \\ &\qquad\qquad a \leqslant b \text{ , } s \leqslant t\} \text{ .}\end{aligned}$$

But for such a,b,s,t notice that

$$(a \vee s,\ (b \vee s) \wedge (a \vee t)) \in \rho_1 \wedge \rho_2 \text{ ,}$$

$$a \vee s \leqslant (b \vee s) \wedge (a \vee t)$$

and that $f(b \wedge t) \wedge f(a \vee s)' \leqslant f[(b \vee s) \wedge (a \vee t)] \wedge f(a \vee s)'$,

so $\overline{f}(\rho_1) \wedge \overline{f}(\rho_2) \leqslant \overline{f}(\rho_1 \wedge \rho_2)$.

Turning to arbitrary joins, the simplest procedure seems to be to construct an order preserving right adjoint $[\ ]_f : M \to CL$ to $\overline{f}$. Proposition 1.2 does the rest.

For $m \in M$, define $[m]_f = \{(x,y): f(x) \vee m = f(y) \vee m\}$.

This is easily seen to be a congruence and $m_1 \leqslant m_2 \Rightarrow [m_1]_f \subseteq [m_2]_f$. We now show that $\overline{f}(\rho) \leqslant m \iff \rho \leqslant [m]_f$; suppose

$x \leqslant y$ and $x \rho y$; $f(y) \wedge f(x)' \leqslant m$

$\iff f(y) \leqslant m \vee f(x)$

$\iff m \vee f(y) = m \vee f(x)$. (remember $f(x) \leqslant f(y)$)

as required. □

5.18 Corollary

$C: L \to CL$ is the object part of a functor $C: \underline{FRM} \to \underline{FRM}$.

Proof:

Let $f: L \to M$ be a frame map. Then $\nabla_M \circ f: L \to CM$ is also a frame map and the image of L under $\nabla_M \circ f$ is complemented in CM , so $\overline{\nabla_M \circ f}$ is the unique map which makes the following diagram commute

$$\begin{tikzcd} L \arrow[r, "\nabla_L"] \arrow[d, "f"'] & CL \arrow[d, dashed, "\overline{\nabla_M \circ f}"] \\ M \arrow[r, "\nabla_M"] & CM \end{tikzcd}$$

Setting $Cf = \overline{\nabla_M \circ f}$ completes the proof. □

This completes the construction of the congruence lattice (or assembly) and we turn now to applications of the congruence lattice to our investigation of quasi-uniform frames. Important from our point of view is the fact that the congruence lattice can be viewed very naturally as a bi-frame, with many attractive properties. We feel that this view point sheds much new light on this structure.

5.19 Definition

Given a frame, L, let $Sk(L) = (CL, \nabla L, \Delta L)$.

Remark:

The ordered triple, $Sk(L)$, is clearly a biframe. We examine this further.

5.20 Proposition

$Sk: L \to Sk(L)$ is the object part of a functor from FRM to BIFRM.

Proof:

Suppose $f: L \to M$ is a frame map.

$$\begin{array}{ccc} L & \xrightarrow{\nabla_L} & CL \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle \overline{\nabla_M \circ f}} \\ M & \xrightarrow{\nabla_M} & CM \end{array}$$

We have already seen that $\overline{\nabla_M \circ f}$ is a frame map from CL to CM which makes the above diagram commute, but we can see that $\overline{\nabla_M \circ f}$ is actually a biframe map from $Sk(L)$ to $Sk(M)$. For $a \in L$, we have :

$$\overline{\nabla_M \circ f}\ (\nabla_a) = \overline{\nabla_M \circ f}\ (\nabla_L(a))$$
$$= \nabla_M \circ f(a) \qquad \text{(By commutativity)}$$
$$= \nabla_{f(a)} \in \nabla M\ .$$

Also $\overline{\nabla_M \circ f}\ (\Delta_a) = \overline{\nabla_M \circ f}\ (\nabla_L(a)')$

$$= \overline{\nabla_M \circ f}\ (\nabla_L(a))'$$
$$= \nabla_M \circ f(a)'$$
$$= \nabla_{f(a)}{}' \in \Delta M\ ,\ \text{as required.}$$

□

Also vital, for our purposes, are the following two results.

5.21 Proposition

Sk(L) is completely regular.

Proof:

Select $\nabla_a \in \nabla L$. Since ∇_a is complemented ($\nabla_a \wedge \Delta_a = 0$, $\nabla_a \vee \Delta_a = 1$) by an element of ΔL , we automatically have $\nabla_a \mathrel{\overline{<}\!\overline{<}_1} \nabla_a$. Similarly, select $\rho \in \Delta L$; $\rho = \bigvee_{a \in A} \Delta_a$. But we similarly have $\Delta_a \mathrel{\overline{<}\!\overline{<}_2} \Delta_a \leqslant \rho$, so

$$\rho = \bigvee\{\sigma \in \Delta L : \sigma \mathrel{\overline{<}\!\overline{<}_2} \rho\} ,$$

and we have satisfied both conditions for complete regularity.

□

5.22 Proposition

Sk(L) is normal.

Proof:

Select $\nabla_a \in \nabla L$ and $\rho \in \Delta L$ such that $\nabla_a \vee \rho = 1$; we may assume $\rho = \bigvee_{b \in B} \Delta_b$. Let $s_1 = \Delta_a$, $s_2 = \nabla_a$. Then $s_1 \wedge s_2 = 0$, $s_1 \vee \nabla_a = 1$, $s_2 \vee \rho = 1$, as required.

□

5.23 Corollary

CL is a normal, completely regular frame.

Proof:

Remark after definition 1.14. □

5.24 Proposition

Let $L = (L_0, L_1, L_2)$ be a biframe; set $U_1(L) = L_1$. Then U_1 is the object part of a functor from BIFRM to FRM.

Proof:

Suppose $f: (L_0, L_1, L_2) \to (M_0, M_1, M_2)$ is a biframe map. Define $U_1 f = f$; $U_1 f$ is then a frame map from L_1 to M_1 , since f maps L_1 into M_1 and L_1 , M_1 are closed under finite meets and arbitrary joins.

□

5.25 Definition

A pair (F, i_F) is called a *completely regular extension* if

(i) F is a functor from FRM to BIFRM ,

(ii) FL is completely regular for each frame L ,

(iii) i_F is a natural isomorphism from 1_{FRM} to U_1F .

For convenience, set $F(L) = (F_0(L), F_1(L), F_2(L))$.

5.26 Proposition

(Sk,∇) is a completely regular extension.

Proof:

Clearly $\nabla_L: L \to \nabla L$ is the required isomorphism. For naturality note that $\nabla_M \circ f(a) = \nabla_{f(a)}$ and $Skf \circ \nabla_L(a) = Skf(\nabla_a) = \nabla_{f(a)}$.

5.27 Lemma

Let (F,i_F) be a completely regular extension. Then $F(\underline{3}) \cong Sk(\underline{3})$.

Proof:

As usual $\underline{3} = \{0,d,1\}$. Since $F(\underline{3})$ is completely regular, $i_{F\underline{3}}(d) = \bigvee\{t \in F_1(\underline{3}) : t \mathrel{\overline{\overline{<\!<}}}_1 i_{F\underline{3}}(d)\}$. We must then have that $i_{F\underline{3}}(d) \mathrel{\overline{\overline{<\!<}}}_1 i_{F\underline{3}}(d)$ (the other two elements of $F_1(\underline{3})$ being ineligible), so there is $s \in F_2(\underline{3})$ such that $s \wedge i_{F\underline{3}}(d) = 0$, $s \vee i_{F\underline{3}}(d) = 1$. Thus $F_1(\underline{3})$ is complemented by elements of $F_2(\underline{3})$. Now suppose $t \in F_2(\underline{3})$; $t = \bigvee\{s \in F_2(\underline{3}) : s \mathrel{\overline{\overline{<\!<}}}_2 t\}$; for such an s , we can find $r \in F_1(\underline{3})$ such that $s \wedge r = 0$, $r \vee t = 1$.

But then $r'(\in F_2(\underline{3}))$ satisfies $s \leq r' \leq t$, so t is a join of complements of members of $F_1(\underline{3})$; this means that $t = 1$ or 0 or $i_{F\underline{3}}(d)'$, so $F(\underline{3})$ is evidently isomorphic to $Sk(\underline{3})$. □

5.28 Proposition

Let (F, i_F) be a completely regular extension. For each frame, L, $F_1(L)$ is complemented, with complements in $F_2(L)$ and every element of $F_2(L)$ is a join of complements of members of $F_1(L)$.

Proof:

For each $a \in L$, define $f_a: \underline{3} \to L$ by

$$f_a(0) = 0 , \quad f_a(d) = a , \quad f_a(1) = 1 .$$

We have $Ff_a(i_{F\underline{3}}(d)) = i_{FL}(f_a(d)) = i_{FL}(a)$ and $Ff_a(i_{F\underline{3}}(d))'$ is an element of $F_2(L)$ and is clearly the complement of $i_{FL}(a)$. We have thus exhibited a complement for each element of $F_1(L)$, since i_{FL} is an isomorphism. Now suppose $t \in F_2(L)$; $t = \bigvee\{s \in F_2(L): s \,\overline{<\!\!<}_2\, t\}$; select $r \in F_1(L)$ such that $s \wedge r = 0$, $r \vee t = 1$. Then $s \leq r' \leq t$, and t is thus a join of complements of elements of $F_1(L)$. □

5.29 Proposition

Let (F, i_F) be a completely regular extension. Then i_F induces a surjective biframe map from $Sk(L)$ to $F(L)$ for each frame L .

Proof:

We have i_{FL} a frame map from L to $F_0(L)$, but we also have the image of L under i_{FL} complemented in $F_0(L)$; i_{FL} thus factors uniquely through CL : $i_{FL} = \overline{i_{FL}} \circ \nabla_L$. Moreover, $\overline{i_{FL}}(\Delta_a) = \overline{i_{FL}}(\nabla_a)' = i_{FL}(a)' \in F_2(L)$. So $\overline{i_{FL}}$ is a biframe map. Now suppose

$$
\begin{aligned}
t \in F_2(L); \quad t &= \dot{\bigvee}\{r' : r \in F_1(L),\ r' \leqslant t\}. \\
&= \dot{\bigvee}\{\overline{i_{FL}}(\nabla_a)' : \overline{i_{FL}}(\nabla_a) \leqslant t\} \\
&= \dot{\bigvee}\{\overline{i_{FL}}(\Delta_a) : \overline{i_{FL}}(\Delta_a) \leqslant t\} \\
&= \overline{i_{FL}}\ \bigvee\{\Delta_a : \overline{i_{FL}}(\Delta_a) \leqslant t\}
\end{aligned}
$$

so t is the image under $\overline{i_{FL}}$ of an element of ΔL . Thus $\overline{i_{FL}}$ is surjective. (The extension to elements of $F_0(L)$ is easy.) □

5.30 Corollary

Let (F, i_F) be a completely regular extension. Then $\overline{i_F}$ is a natural transformation from Sk to F .

Proof:

Let $f: L \to M$ be a frame map. Consider the following diagram:

$$
\begin{tikzcd}
L \arrow[d, "f"] & SkL \arrow[r, "\overline{i_{FL}}"] \arrow[d, "Skf"] & FL \arrow[d, "Ff"] \\
M & SkM \arrow[r, "\overline{i_{FM}}"] & FM
\end{tikzcd}
$$

We have $\rho \in CL \Rightarrow \rho = \bigvee\{\nabla_b \wedge \Delta_a : a\ \rho\ b\ ,\ a \leqslant b\}$, and

$Skf(\rho) = \bigvee\{\nabla_{f(b)} \wedge \Delta_{f(a)}: a \rho b, a \leqslant b\}$.

Now $\overline{i_{FM}} \circ Skf(\rho) = \bigvee\{\overline{i_{FM}}(\nabla_{f(b)}) \wedge \overline{i_{FM}}(\Delta_{f(a)}): a \rho b, a \leqslant b\}$

$$= \bigvee\{i_{FM}(f(b)) \wedge i_{FM}(f(a))': a \rho b, a \leqslant b\}.$$

On the other hand, $Ff \circ \overline{i_{FL}}(\rho) = Ff(\bigvee\{\overline{i_{FL}}(\nabla_b) \wedge \overline{i_{FL}}(\Delta_a):$
$a \rho b, a \leqslant b\})$

$$= Ff(\bigvee\{i_{FL}(b) \wedge i_{FL}(a)': a \rho b, a \leqslant b\})$$
$$= \bigvee\{Ff \circ i_{FL}(b) \wedge Ff(i_{FL}(a)'): a \rho b, a \leqslant b\}$$
$$= \bigvee\{i_{FM}(f(b)) \wedge i_{FM}(f(a))': a \rho b, a \leqslant b\}$$

as required.

The maps are all biframe maps. □

We can see that (Sk, ∇) is a rather special member of the family of completely regular extensions. This becomes even more transparent when expressed categorically.

Let us consider the (quasi) category whose objects are completely regular extensions. An arrow, η, in this category from (F, i_F) to $(F', i_{F'})$ is a natural transformation from F to F' satisfying :

$$U_1\eta_L \circ i_{FL} = i_{F'L} \quad \text{for each frame } L .$$

5.31 <u>Theorem</u>

(Sk, ∇) is an initial object in the above category.

<u>Proof</u>:

Let (F, i_F) be a completely regular extension. We have already seen that $\overline{i_F}$ is a natural transformation from Sk

to F. But $U_1\overline{i_{FL}} \circ \nabla_L = i_{FL}$ by the very construction of $\overline{i_F}$, so $\overline{i_F}$ is an arrow from (Sk,∇) to (F,i_F).

Suppose j is another arrow; then

$U_1 j_L \circ \nabla_L = i_{FL}$ implies that

$j_L \circ \nabla_L = i_{FL}$, which by the uniqueness of $\overline{i_{FL}}$ makes $j_L = \overline{i_{FL}}$ for each frame L. □

We return now to the question of endowing $F(L)$ with a compatible quasi-uniform structure, where (F,i_F) is a completely regular extension. This is now easily done, and we treat only the "canonical" extension (Sk,∇), the others being similar. Some surprising results emerge.

5.32 Lemma

For each $a \in L$, $a \neq 0, 1$, let $C_a = \{(\nabla_a,1),(1,\Delta_a)\}$; C_a is a strong conjugate cover of $Sk(L)$, and $C^*_a \leqslant C_a$.

Proof:

Easy, since $\nabla_a \vee \Delta_a = 1$ and $\nabla_a \wedge \Delta_a = 0$. □

5.33 Definition

Let q_{PL} be the family of conjugate covers of $Sk(L)$ which has as sub-base the set $\{C_a : a \in L\}$.

5.34 Proposition

The pair $(Sk(L), q^*_{PL})$ is a quasi-uniform frame.

Proof:

Clearly q_{PL} is a filter of conjugate covers satisfying the required star-refining property. Select $\nabla_a \in \nabla L$; $st_1(\nabla_a, C_a) = \nabla_a$ so $\nabla_a = \bigvee\{\nabla_b : st_1(\nabla_b, C) \leqslant \nabla_a$ for some $C \in q_{PL}\}$. Now select $\rho = \bigvee\{\Delta_b : b \in B\} \in \Delta L$. Again, $st_2(\Delta_b, C_b) = \Delta_b$, so $\rho = \bigvee\{\Delta_b : st_2(\Delta_b, C) \leqslant \rho$ for some $C \in q_{PL}\}$. Now use proposition 3.22 and we are finished.

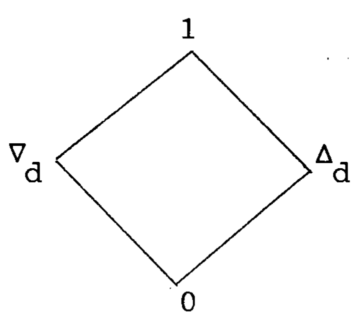
□

5.35 Theorem

Let $F: Sk(L) \to (Sk(L), q_{FL})$ be functorial from <u>BIFRM</u> to <u>QUNFRM</u> ; then $q_{FL} \supseteq q_{PL}^*$.

Proof:

For each $a \in L$, define $f_a : \underline{3} \to L$ by $f_a(0) = 0$, $f_a(d) = a$, $f_a(1) = 1$. We have already seen that $C\underline{3} = D$, the Boolean algebra represented by



D has a unique compatible quasi-uniform structure, which has as base the single strong conjugate cover

$C_d = \{(\nabla_d, 1), (1, \Delta_d)\}$.

Now let q_{FL} be any functorial quasi-uniform structure on $Sk(L)$. We have then that $Skf_a : (D, q_{F3}) \to (Sk(L), q_{FL})$ is a quasi-uniform map, so $Skf_a[C_d] \in q_{FL}$. But $Skf_a(\nabla_d) = \nabla_L \circ f_a(d) = \nabla_a$, and $Skf_a(\Delta_d)$ $Skf_a(\nabla_d') = Skf_a(\nabla_d)'$ $= \nabla_a' = \Delta_a$. That yields $\{(\nabla_a, 1), (1, \Delta_a)\} \in q_{FL}$. Hence $q_{FL} \supseteq q_{PL}$, so $q_{FL} \supseteq q^*_{PL}$, as required. □

5.36 Proposition

Let L be a frame. For $\nabla_a, \nabla_b \in \nabla L$, set $\nabla_a \lessdot_{1PL} \nabla_b$ iff $\nabla_a \subseteq \nabla_b$; for $\rho_1, \rho_2 \in \Delta L$, set $\rho_1 \lessdot_{2PL} \rho_2$ iff for some $c \in L$, $\rho_1 \leqslant \Delta_c \leqslant \rho_2$. Then $(Sk(L), \lessdot_{1PL}, \lessdot_{2PL})$ is a quasi-proximal frame.

Proof:

We check the axioms of definition 4.26. Clearly axiom (i) is satisfied.

	∇L		ΔL
(ii)	$\nabla_0 \lessdot \nabla_0$, $\nabla_1 \lessdot \nabla_1$	(ii)	same
(iii)	obvious	(iii)	obvious
(iv)	obvious	(iv)	$\rho_1 \lessdot_{2PL} \rho_2, \rho_3 \Rightarrow \begin{cases} \rho_1 \leqslant \Delta_a \leqslant \rho_2 \\ \rho_1 \leqslant \Delta_b \leqslant \rho_3 \end{cases}$ $\Rightarrow \rho_1 \leqslant \Delta_{a \vee b} \leqslant \rho_2 \wedge \rho_3$

∇L	ΔL
(v) $\nabla_a \leq \nabla_b \Rightarrow \nabla_a \leq \nabla_a \leq \nabla_b$	(v) $\rho_1 \leq \Delta_b \leq \rho_2$ $\Rightarrow \rho_1 \ll_{2PL} \Delta_b \ll_{2PL} \rho_2$
(vi) $\nabla_a \wedge \Delta_a = 0$, $\nabla_a \vee \Delta_a = 1$ so $\nabla_a \overline{<}_1 \nabla_a$.	(vi) $\rho_1 \leq \Delta_b \leq \rho_2 \Rightarrow \rho_1 \wedge \nabla_b = 0$, $\rho_2 \vee \nabla_b = 1$ so $\rho_1 \overline{<}_2 \rho_2$.
(vii) $\nabla_a \leq \nabla_b \Rightarrow \Delta_b \leq \Delta_a$, so $\nabla_b^* \ll_{2PL} \nabla_a^*$	(vii) $\rho_1 \leq \Delta_b \leq \rho_2 \Rightarrow \rho_2^* \leq \nabla_b \leq \rho_1^*$ $\Rightarrow \rho_2^* \leq \rho_1^*$.
(viii) Obvious.	(viii) $\rho = \bigvee\{\Delta_b : b \in B\}$, but then $\Delta_b \ll_{2PL} \rho$, so $\rho = \bigvee\{\sigma : \sigma \ll_{2PL} \rho\}$.

□

5.37 Proposition

Let $(SkL, \ll_1, \ll_2)$ be a quasi-proximal frame. Then

$$\begin{cases} \nabla_a \ll_1 \nabla_b \Rightarrow \nabla_a \ll_{1PL} \nabla_b \\ \rho_1 \ll_2 \rho_2 \Rightarrow \rho_1 \ll_{2PL} \rho_2 . \end{cases}$$

Proof:

(i) $\nabla_a \ll_1 \nabla_b \Rightarrow \nabla_a \leq \nabla_b$.

(ii) $\rho_1 \ll_2 \rho_2 \Rightarrow \rho_1 \overline{<}_2 \rho_2 \Rightarrow$ for some $a \in L$, $\nabla_a \wedge \rho_1 = 0$ and $\nabla_a \vee \rho_2 = 1$. So $\rho_1 \leq \Delta_a \leq \rho_2$.

□

5.38 Proposition

Let $f: L \to M$ be a frame map. Then $Skf: (Sk(L), \ll_{1PL}, \ll_{2PL}) \to (Sk(M), \ll_{1PM}, \ll_{2PM})$ is a quasi-proximal map.

Proof:

(i) $\nabla_a \leqslant \nabla_b$ $(a,b \in L) \Rightarrow Skf(\nabla_a) \leqslant Skf(\nabla_b)$.

(ii) $\rho_1 \leqslant \Delta_b \leqslant \rho_2 \Rightarrow Skf(\rho_1) \leqslant Skf(\Delta_b) \leqslant Skf(\rho_2)$

$\Rightarrow Skf(\rho_1) \leqslant \Delta_{f(b)} \leqslant Skf(\rho_2)$.

□

5.39 Proposition

$(Sk(L), \leqslant_{1PL}, \leqslant_{2PL})$ is the only functorial quasi-proximal structure on $Sk(L)$.

Proof:

For $a \in L$, define $f_a: \underline{3} \to L$ by $f_a(0) = 0$, $f_a(d) = a$, $f_a(1) = 1$. We know $Sk(\underline{3})$ has unique compatible quasi-proximal structure. Suppose $F: Sk\,L \to (Sk\,L, \leqslant_{1FL}, \leqslant_{2FL})$ is functorial from BIFRM to QPROXFRM. Then

$$\nabla_d \leqslant_{1F3} \nabla_d \Rightarrow Skf_a(\nabla_d) \leqslant_{1FL} Skf_a(\nabla_d)$$

$$\Rightarrow \nabla_a \leqslant_{1FL} \nabla_a \quad \text{for each} \quad a \in L .$$

So $\nabla_a \leqslant \nabla_b \Rightarrow \nabla_a \leqslant_{1FL} \nabla_b$, i.e. $\nabla_a \leqslant_{1PL} \nabla_b \Rightarrow \nabla_a \leqslant_{1FL} \nabla_b$.
Also $\Delta_d \leqslant_{2F3} \Delta_d \Rightarrow \Delta_a \leqslant_{2FL} \Delta_a$ in a similar manner, so $\rho_1 \leqslant_{2PL} \rho_2 \Rightarrow \rho_1 \leqslant \Delta_b \leqslant \rho_2 \Rightarrow \rho_1 \leqslant_{2FL} \rho_2$.
Now using proposition 5.37, we see $\rho_1 \leqslant_{iPL} \rho_2 \Longleftrightarrow \rho_1 \leqslant_{iFL} \rho_2$ ($i = 1$ or 2) as required.

□

5.40 Corollary

(i) $(Sk(L), q^*_{PL})$ induces $(Sk(L), \leqslant_{1PL}, \leqslant_{2PL})$.

(ii) $(Sk(L), q^*_{PL})$ is the only functorial totally bounded quasi-uniform structure on $Sk\,L$.

□

Notes on Chapter 5

(1) We note the simple construction of the Pervin quasi-uniformity via conjugate covers. One is tempted to suggest that the classical construction via entourages is really a proof using conjugate covers without making this explicit (Pervin [33]). Functorial and minimal properties of this Pervin quasi-uniformity are also easily established via conjugate covers.

(2) One cannot argue with the fact that the assembly of a frame (the congruence lattice viewed as a collection of so-called "nuclei") has been efficiently exploited and analysed, (Simmons [37,38]) and that nuclei themselves serve useful purposes in frame theory (see especially Johnstone [23] ~~[10]~~). We do feel, however, that a comprehensive treatment in terms of congruences has provided a simple and attractive alternative. The recent interest in structures more general than frames such as σ-frames (Gilmour [17]) has promoted the need to establish analogous structures to the assembly for these more general structures, together with the vital functorial properties present in the assembly. This can be achieved very naturally by replacing frame congruences by congruences appropriate to the more general structure. The resulting congruence lattice is still a *frame*, and retains its functorial and factoring properties. Theorem 5.17 remains valid with the proviso that M must still be a *frame* although f will now be a map in the more general category.

For the interested reader, a *nucleus* on a frame L is a function $j: L \to L$ which satisfies

(i) $x \leqslant j(x)$ $(x \in L)$ (expansive)

(ii) $j \circ j = j$ (idempotent)

(iii) $j(x \wedge y) = j(x) \wedge j(y)$ (meet preserving)

If j is a nucleus on L, then $\theta_j = \{(x,y): j(x) = j(y)\}$ is a congruence on L.

If θ is a congruence on L, then $j_\theta: L \to L$ defined by $j_\theta(x) = \bigvee\{t: t \;\theta\; x\}$ is a nucleus on L. The existence of arbitrary joins ensures the existence of j_θ and one easily establishes that

(i) $\theta_{j_\theta} = \theta$

(ii) $j_{\theta_j} = j$.

Moreover, pointwise ordering of nuclei corresponds to ordering by inclusion of congruences under the correspondence established above ($\theta_1 \leqslant \theta_2 \iff j_{\theta_1}(x) \leqslant j_{\theta_2}(x)$ for all $x \in L$).

The treatment of the assembly in terms of nuclei would not seem to be susceptible to generalization to the σ-frame setting, where existence of arbitrary joins is not insured. (One only has countable joins and a countable distributivity law.)

(3) We feel that viewing the congruence lattice as a biframe and in particular as (part of) a completely regular extension has thrown interesting new light on this structure. The results in 5.29 to 5.31 show that

it has a certain "free" property; it is the "free" completely regular extension. Other ways of formulating this are possible, but this seems to be adequate. One obvious question arises: are there in fact any other completely regular extensions? Another interesting problem is whether the congruence lattice or Skula biframe can be interpreted as a Kan extension. As an initial object in the stated category, we see that it is "close" to being a left Kan extension of U_1 along the identity functor from <u>FRM</u> to <u>FRM</u>.

(4) The theorem in the space setting analogous to theorem 5.35 was one of the results motivating Brümmer [6] to explore categorically the concept of initiality. The fact, then, that this theorem has a frame analogue seems to be a rather interesting result. Is there a role for initiality in frame theory?

(5) Proposition 5.36 is the frame analogue of the Pervin quasi-proximity on a topological space. (Pervin [34])

(6) In a paper by H P Künzi (to appear), topological spaces which admit a unique compatible quasi-uniformity are characterized. The corresponding question for frames seems interesting.

CHAPTER 6

There is at present considerable interest in so-called "fuzzy" topologies and "fuzzy" structures. The notion of a fuzzy topology (Chang [8]) generalizes the notion of topology, but is still an example of a frame; we provide the details below.

6.1 Definition

(i) A *fuzzy subset*, μ, of a set X is a function from X to $[0,1]$ (the unit interval).

(ii) The set of all fuzzy subsets of X, denoted by PFX, is the set of all functions from X to $[0,1]$, i.e. $PFX = [0,1]^X$.

(iii) Let X be a set; set $\begin{cases} \mu_0(x) = 0 \text{ for all } x \in X \\ \mu_1(x) = 1 \text{ for all } x \in X . \end{cases}$

Then $\mu_0, \mu_1 \in PFX$.

(iv) Let $\mu_A, \mu_B \in PFX$. Then define

$$\mu_A \wedge \mu_B(x) = \inf\{\mu_A(x), \mu_B(x)\}$$

$$\mu_A \vee \mu_B(x) = \sup\{\mu_A(x), \mu_B(x)\} .$$

(v) A *fuzzy topology*, F, *on* X is a subfamily of PFX satisfying

(1) $\mu_0, \mu_1 \in F$

(2) $\mu_A, \mu_B \in F \Rightarrow \mu_A \wedge \mu_B \in F$.

(3) $\mu_i \in F \ (i \in I) \Rightarrow \vee\mu_i \in F$.

Remarks:

(i) A fuzzy topology is in fact a frame. The distributivity property is easily verified.

(ii) If $\mathcal{F}$ is a fuzzy topology on X, then $(X,\mathcal{F})$ is called a fuzzy topological space.

6.2 Definition

Let $f\colon X \to Y$ be a function, and suppose $(X,\mathcal{F}_1)$, $(Y,\mathcal{F}_2)$ are fuzzy topological spaces. Then $f\colon (X,\mathcal{F}_1)\to(Y,\mathcal{F}_2)$ is *fuzzy continuous* if for each $\nu \in \mathcal{F}_2$, $\nu \circ f \in \mathcal{F}_1$.

In fact we have here a functor, $\mathcal{Q}$, from the category of fuzzy topological spaces and fuzzy continuous maps to the category FRM.

6.3 Definition

For each fuzzy topological space $(X,\mathcal{F})$, set $\mathcal{Q}(X,\mathcal{F}) = \mathcal{F}$. If $f\colon (X,\mathcal{F}_1) \to (Y,\mathcal{F}_2)$ is fuzzy continuous, set $\mathcal{Q}f = f^{\leftarrow}\colon \mathcal{F}_2 \to \mathcal{F}_1$ where $f^{\leftarrow}(\nu) = \nu \circ f$.

6.4 Lemma

If $f\colon (X,\mathcal{F}_1) \to (Y,\mathcal{F}_2)$ is a fuzzy continuous map, then $f^{\leftarrow}\colon \mathcal{F}_2 \to \mathcal{F}_1$ is a frame map.

Proof:

Let ν_A, ν_B, ν_i $(i \in I)$ be members of $\mathcal{F}_2$.

Then $f^{\leftarrow}(\nu_A \wedge \nu_B)(x) = (\nu_A \wedge \nu_B) \circ f(x)$

$$= \nu_A \wedge \nu_B \ (f(x))$$

$$= \inf \{\nu_A \ (f(x)), \ \nu_B \ (f(x))\}$$

$$= \inf \{f^{\leftarrow}(\nu_A)(x), \ f^{\leftarrow}(\nu_B)(x)\} ,$$

so $f^{\leftarrow}(\nu_A \wedge \nu_B) = f^{\leftarrow}(\nu_A) \wedge f^{\leftarrow}(\nu_B)$.

Also $f^{\leftarrow}(\vee \nu_i)(x) = \vee \nu_i \circ f(x)$

$$= \vee \nu_i \ (f(x))$$

$$= \vee \{\nu_i \ (f(x))\}$$

$$= \vee \{f^{\leftarrow}(\nu_i)(x)\} ,$$

so $f^{\leftarrow}(\vee \nu_i) = \vee f^{\leftarrow}(\nu_i)$. □

6.5 Corollary

$\mathcal{Q}$ is a contravariant functor from FUZZTOP (fuzzy topological spaces, fuzzy continuous maps) to FRM .

6.6 Definition

(i) Let L be a frame; set $\Sigma L = \hom(L,[0,1])$ ($[0,1]$ is a frame in its usual ordering).

(ii) For each $a \in L$, define $\Sigma_a \colon \Sigma L \to [0,1]$ by $\Sigma_a(p) = p(a)$.

(iii) Let $F_{\Sigma L} = \Sigma_a : a \in L$.

6.7 Proposition

$(\Sigma L, F_{\Sigma L})$ is a fuzzy topological space.

Proof:

We have $\Sigma_1(p) = p(1) = 1$ for all $p \in \Sigma L$; also $\Sigma_0(p) = p(0) = 0$ for all $p \in \Sigma L$, so the top and bottom elements of $PF(\Sigma L)$ are members of $F_{\Sigma L}$. Let

$$
\begin{aligned}
a,b \in L \; ; \quad \Sigma_a(p) \wedge \Sigma_b(p) &= p(a) \wedge p(b) \\
&= p(a \wedge b) \\
&= \Sigma_{a \wedge b}(p) \quad \text{for all } p \in \Sigma L .
\end{aligned}
$$

So $\Sigma_a \wedge \Sigma_b \in F_{\Sigma L}$.

$$
\begin{aligned}
\text{Let } a_i \in L \quad (i \in I) \; ; \quad \bigvee \Sigma_{a_i}(p) &= \bigvee p(a_i) \\
&= p(\bigvee a_i) \\
&= \Sigma_{\bigvee a_i}(p) \quad \text{for all } p \in \Sigma L
\end{aligned}
$$

so $\bigvee \Sigma_{a_i} \in F_{\Sigma L}$.

Thus $F_{\Sigma L}$ is a fuzzy topology. □

6.8 Proposition

Let $f: L \to M$ be a frame map. Define $\Sigma f: \Sigma M \to \Sigma L$ by $\Sigma f(p) = p \circ f$. Then $\Sigma f: (\Sigma M, F_{\Sigma M}) \to (\Sigma L, F_{\Sigma L})$ is fuzzy continuous.

Proof:

Let $a \in L$, so $\Sigma_a \in F_{\Sigma L}$. Then

$$
\begin{aligned}
&\Sigma_a \circ \Sigma f(p) \qquad (p \in \Sigma M) \\
= \; &\Sigma_a(p \circ f) \\
= \; &p \circ f(a) = \Sigma_{f(a)}(p)
\end{aligned}
$$

, but $\Sigma_{f(a)} \in F_{\Sigma M}$, and we are finished.

□

6.9 Corollary

Σ is a contravariant functor from FRM to FUZZTOP.

6.10 Theorem

Σ and $\mathcal{Q}$ are adjoint on the right.

Proof:

Let $\begin{cases} f \in \hom((X,\mathcal{F}),(\Sigma L,\mathcal{F}_{\Sigma L})) \ , \\ g \in \hom(L,\mathcal{Q}(X,\mathcal{F})) . \end{cases}$

Let $a \in L$, $x \in X$.

Define $\begin{cases} \bar{f} \in \hom(L,\mathcal{Q}(X,\mathcal{F})) \quad \text{by} \quad \bar{f}(a)(x) = f(x)(a) \ , \\ \tilde{g} \in \hom((X,\mathcal{F}),(\Sigma L,\mathcal{F}_{\Sigma L})) \quad \text{by} \quad \tilde{g}(x)(a) = g(a)(x) \ . \end{cases}$

We must of course check that $\bar{f}$, $\tilde{g}$ are as asserted. For $a \in L$, we have $\Sigma_a \in \mathcal{F}_{\Sigma L}$, so $\Sigma_a \circ f \in \mathcal{F}$. But $\Sigma_a \circ f\ (x) = \Sigma_a\ (f(x)) = f(x)(a) = \bar{f}(a)(x)$, so $\bar{f}(a) \in \mathcal{F}$. We now check that $\bar{f}$ is a frame map: for $a,b \in L$, $\bar{f}(a \wedge b)(x) = f(x)(a \wedge b) = f(x)(a) \wedge f(x)(b)$

$$= \bar{f}(a)(x) \wedge \bar{f}(b)(x) \ ,$$

so $\bar{f}(a \wedge b) = \bar{f}(a) \wedge \bar{f}(b)$; also we can prove that $\bar{f}(\vee a_i)(x) = \vee\{\bar{f}(a_i)(x)\}$, so $\bar{f}$ is a frame map. We now show that $\tilde{g}(x) \in \Sigma L$ for each $x \in X$; $\tilde{g}(x)(a \wedge b) = g(a \wedge b)(x)$

$$\begin{aligned} &= (g(a) \wedge g(b))(x) \\ &= g(a)(x) \wedge g(b)(x) \\ &= \tilde{g}(x)(a) \wedge \tilde{g}(x)(b) \ ; \end{aligned}$$

similarly $\tilde{g}(x)(\vee a_i) = \vee\{\tilde{g}(x)(a_i)\}$. Now we show that $\tilde{g}$ is fuzzy continuous; for $a \in L$ we have $\Sigma_a \in \mathcal{F}_{\Sigma L}$ and

$\Sigma_a \circ \tilde{g}(x) = \tilde{g}(x)(a) = g(a)(x)$, but $g(a) \in \mathcal{F}$, so
$\Sigma_a \circ \tilde{g} \in \mathcal{F}$ as required.

We note that $\begin{cases} \tilde{\bar{f}}(x)(a) = \bar{f}(a)(x) = f(x)(a) \\ \bar{\tilde{g}}(a)(x) = \tilde{g}(x)(a) = g(a)(x) \end{cases}$,

so $\tilde{\bar{f}} = f$ and $\bar{\tilde{g}} = g$. Naturality conditions are easily checked, and the proof is complete. □

<u>Remark</u>:

The adjoint situation for frames and topologies is very elegant when presented in terms of characteristic functions rather than open sets, as a quick re-reading of theorem 6.10 with [0,1] replaced by {0,1} shows.

The fixed objects of the adjunction established must also be of some interest, especially on the "fuzzy" side; one can establish that the fixed frames under this adjunction are just the frames which are "fuzzy-spatial" (images under Ω of some fuzzy topological space). The "sober" fuzzy topological spaces, on the other hand, are more complicated; one has $(X, \mathcal{F})$ is "sober" iff

(i) $x \neq y \in X \Rightarrow$ for some $\mu \in \mathcal{F}$, $\mu(x) \neq \mu(y)$

(ii) Each frame map, ψ , from $\mathcal{F}$ to $[0,1]$ satisfies $\psi = \psi_x$ for some $x \in X$, where $\psi_x(\mu) = \mu(x)$ for each $\mu \in \mathcal{F}$.

Further exploration of these ideas should be revealing, especially with respect to the role of fuzzy closed sets and complements.

It would be inappropriate to pursue in any detail here the relationships between the frame structures we have investigated and the fuzzy structures which abound in the literature (see notes). We feel, though, that it must be pointed out that such relationships exist and must be of considerable interest to researchers in the "fuzzy" field.

Let us take, for example, the case of uniform frames: it is easy to construct a "fuzzy spectrum" of a uniform frame, (L,q) ; this turns out to be a pair $(\Sigma L, \mu_F)$ where μ_F is a (covering) fuzzy uniformity on the set ΣL and furthermore, the fuzzy topology induced by μ_F is none other than the spectral fuzzy topology, $F_{\Sigma L}$. This is all as it should be and we are presented yet again with the realisation that it is the (fuzzy) open sets which are important (not the points, which cause trouble in fuzzy topology anyway!) Further work shows that an adjoint situation (on the right) exists as well.

Incidentally, it may be worth noting here that much of the work done on separation axioms for frames (regularity, complete regularity, normality, etc.) is of immediate interest to "fuzzy" topology.

The conclusion of this section must be that frame theory must serve as an important categorical tool in the study of fuzzy topological structures.

Notes on Chapter 6

(1) Chang [8] is the originator of the notion of a fuzzy topology. Lowen [29] presented a theory of fuzzy uniform spaces, amplified considerably by Hutton [20]. Katsaras in two papers [25,26] initiated the study of fuzzy proximity spaces. (I have not seen a result in the literature considering the relationship between totally bounded uniform and proximity fuzzy spaces, but this is assured by our results in chapter 4.)

(2) Results which assert the spatiality (topological) of a particular type of frame (inevitably?) use some choice principle. Fuzzy spatiality of a frame is a weaker condition than topological spatiality (since all topologies are fuzzy); presumably results on the fuzzy spatiality of a frame would use correspondingly weaker choice principles. This seems an interesting question to pursue.

CHAPTER 7

The need to consider a covering approach to quasi-uniform spaces arose from the desire to construct a suitable category of quasi-uniform frames. Insight gained from this endeavour has lead also to the idea of constructing a category which, while more general than the category NEAR of nearness spaces and nearness maps, still preserves some of the richness of structure available in NEAR.

The theory of nearness spaces as developed by Herrlich [19] and others has as one of its aims the unification of various types of topological structures. In particular, NEAR contains the categories of

(a) all R_0 topological spaces, continuous maps

(b) all uniform spaces, uniformly continuous functions

(c) all proximity spaces and proximity maps

amongst others. (In fact these categories are very nicely contained in NEAR; they are all either bireflective or bi-coreflective full subcategories.) Well known is the fact that TOP does not appear as a subcategory of NEAR. Other structures which are commonly regarded as useful topological structures are also lacking in the NEAR situation, primarily the category of quasi-uniform spaces (in the sense of Fletcher and Lindgren [15], a very useful reference to the subject, and *not* in the sense of Isbell [22] for whom quasi-uniform spaces are uniform spaces without the star-refinement property.)

A paper of Carlson's [7] explores the link between quasi-uniform spaces and nearness spaces (in the case that the quasi-uniform spaces are locally right symmetric) and makes the point that many results known in NEAR can also be proved in QUN (even when the property of local right symmetry is lacking). This suggests that a more general category than NEAR should be available which would encompass such "non-symmetric" categories as QUN and QPROX. We provide now a category of non-symmetric nearness spaces, to be called quasi-nearness spaces (*not* the quasi-nearness spaces of Herrlich [19]) which seems to fit the bill. It also contains TOP as a coreflective subcategory, an added bonus.

There are many (equivalent) ways of viewing the category NEAR of nearness spaces and nearness maps. We provide brief details of the approach using so-called uniform covers.

7.1 Definition

Let X be a set, μ a non-empty family of covers of X.

(i) For $A \subseteq X$, $x \in \mu\text{-int}(A)$ (usually written $\text{int}(A)$) iff $\{A, X \setminus x\} \in \mu$.

(ii) The pair (X,μ) is a *nearness space* if the following conditions are satisfied

N1: $\mathcal{C}, \mathcal{D} \in \mu \Rightarrow \mathcal{C} \wedge \mathcal{D} \in \mu$, where $\mathcal{C} \wedge \mathcal{D} = \{C \cap D : C \in \mathcal{C},\ D \in \mathcal{D}\}$.

N2: $\mathcal{C} \in \mu$, $\mathcal{C} \leqslant \mathcal{D} \Rightarrow \mathcal{D} \in \mu$.

N3: $\mathcal{C} \in \mu \Rightarrow \text{int}\, \mathcal{C} \in \mu$, where $\text{int}\, \mathcal{C} = \{\text{int}(C) : C \in \mathcal{C}\}$

(iii) A collection of covers of X, $\nu \subseteq \mu$, is a *base* for μ if (X,ν) satisfies N1, N3 and each member of μ is refined by a member of ν.

(iv) If (X,μ) and (Y,ν) are nearness spaces, then $f\colon (X,\mu) \to (Y,\nu)$ is called a *nearness map* if $f\colon X \to Y$ is a function satisfying : $\mathcal{C} \in \nu \Rightarrow f^{-1}[\mathcal{C}] \in \mu$, where $f^{-1}[\mathcal{C}] = \{f^{-1}(C): C \in \mathcal{C}\}$.

The members of μ are called *uniform covers* of X.

We provide now the details of the category Q-NEAR of quasi-nearness spaces and quasi-nearness maps.

We recall that a conjugate pair cover of X is a subset $\mathcal{C}$, of $PX \times PX$, satisfying $\bigcup\{C_1 \cap C_2 : (C_1,C_2) \in \mathcal{C}\} = X$. We say that $\mathcal{C}$ is a strong conjugate pair cover if $(C_1,C_2) \in \mathcal{C}$ and C_1 or $C_2 \neq \emptyset$ implies that $C_1 \cap C_2 \neq \emptyset$.

7.2 Definition

Let X be a set, and μ a non empty collection of conjugate covers of X.

(i) For $A \subseteq X$: (i) $x \in \mu\text{-int}_1(A)$ (usually written $\text{int}_1(A)$) iff $\{(A,X),(X,X\setminus x)\} \in \mu$.

(ii) $x \in \mu\text{-int}_2(A)$ $(\text{int}_2(A))$ iff $\{(X,A),(X\setminus x,X)\} \in \mu$.

7.3 Lemma

For A, X, μ as in definition 7.2, we have

$$\text{int}_i(A) \subseteq A \qquad (i = 1 \text{ or } 2).$$

Proof:

Assume $x \in \mathrm{int}_1(A)$; then $\{(A,X),(X,X\setminus x)\} \in \mu$, so $(A \cap X) \cup (X \cap X\setminus x) = A \cup X\setminus x = X$, yielding $x \in A$. The case $i = 2$ is similar. □

7.4 Lemma

For X, μ as in definition 7.2, and $A,B \subseteq X$, let μ satisfy :

QN1: $\mathcal{C},\mathcal{D} \in \mu \Rightarrow$ there is $\mathcal{E} \in \mu$, $\mathcal{E}$ a strong conjugate cover, satisfying $\mathcal{E} \leqslant \mathcal{C} \wedge \mathcal{D}$

QN2: $\mathcal{C} \in \mu$ and $\mathcal{C} \leqslant \mathcal{D} \Rightarrow \mathcal{D} \in \mu$.

Then $\mathrm{int}_i(A \cap B) = \mathrm{int}_i(A) \cap \mathrm{int}_i(B)$ $(i = 1$ or $2)$.

Proof:

By QN2 we have: $A \subseteq B \Rightarrow \mathrm{int}_1(A) \subseteq \mathrm{int}_1(B)$, since if $x \in \mathrm{int}_1(A)$, then $\mathcal{C}_A = \{(A,X),(X,X\setminus x)\} \in \mu$ refines $\mathcal{C}_B = \{(B,X),(X,X\setminus x)\}$, so $\mathcal{C}_B \in \mu$ and $x \in \mathrm{int}_1(B)$.
We thus prove that $\mathrm{int}_1(A) \cap \mathrm{int}_1(B) \subseteq \mathrm{int}_1(A \cap B)$
So suppose
$$\begin{cases} \mathcal{C}_A = \{(A,X),(X,X\setminus x)\} \in \mu \\ \mathcal{C}_B = \{(B,X),(X,X\setminus x)\} \in \mu . \end{cases}$$
By QN1 and QN2,
$\mathcal{C}_A \wedge \mathcal{C}_B = \{(A \cap B,X),(A,X\setminus x),(B,X\setminus x),(X,X\setminus x)\} \in \mu$ but $\mathcal{C}_A \wedge \mathcal{C}_B$ refines $\mathcal{C}_{A\cap B} = \{(A \cap B,X),(X,X\setminus x)\}$, so $\mathcal{C}_{A\cap B} \in \mu$ yielding $x \in \mathrm{int}_1(A \cap B)$. The case $i = 2$ is similar. □

7.5 Definition

Let X , μ be as in definition 7.2. For $\mathcal{C} \in \mu$, define μ-int $\mathcal{C}$ (usually just int $\mathcal{C}$) as

$$\{(\mathrm{int}_1(C_1),\ \mathrm{int}_2(C_2)) : (C_1, C_2) \in \mathcal{C}\}\ .$$

7.6 Proposition

Let X, μ be as in definition 7.2 and suppose μ satisfies QN1, QN2 and QN3: $\mathcal{C} \in \mu \Rightarrow \mathrm{int}\ \mathcal{C} \in \mu$. Then int_i $(i = 1 \text{ or } 2)$ is an interior operator.

Proof:

We need only prove that $\mathrm{int}_1(A) \subseteq \mathrm{int}_1(\mathrm{int}_1(A))$ for $A \subseteq X$. We have

$$\begin{aligned} x \in \mathrm{int}_1(A) &\iff \{(A,X),(X,X \setminus x)\} \in \mu \\ &\Rightarrow \{(\mathrm{int}_1(A),\mathrm{int}_2(X)),(\mathrm{int}_1 X,\mathrm{int}_2(X \setminus x))\} \in \mu \\ &\qquad\qquad \text{by QN3} \\ &\Rightarrow \{(\mathrm{int}_1(A),X),(X,X \setminus x)\} \in \mu \quad \text{by QN2} \end{aligned}$$

so $x \in \mathrm{int}_1(\mathrm{int}_1(A))$ as desired.
The case $i = 2$ is similar. □

7.7 Definition

(i) Let X, μ be as in definition 7.2. The pair (X, μ) is called a *quasi-nearness space* iff μ satisfies QN1, QN2 and QN3.

(ii) Let (X,μ), (Y,ν) be quasi-nearness spaces;
$f\colon (X,\mu) \to (Y,\nu)$ is a *quasi-nearness map*
if $\mathcal{C} \in \nu \Rightarrow f^{-1}[\mathcal{C}] \in \mu$, where
$f^{-1}[\mathcal{C}] = \{(f^{-1}(C_1), f^{-1}(C_2)) : (C_1, C_2) \in \mathcal{C}\}$.

(iii) Q-NEAR is the category of quasi-nearness spaces and quasi-nearness maps.

Remark:

Since int_1 and int_2 are interior operators, each quasi-nearness space, (X,μ), gives rise to a bitopological space $(X, T_1(\mu), T_2(\mu))$ where $U \in T_i(\mu)$ iff $\text{int}_i(U) = U$ $(i = 1 \text{ or } 2)$.

7.8 Definition

A bitopological space (X, T_1, T_2) is *pairwise*-R_0 if the following equivalent conditions are satisfied:

(i) $x \in \text{cl}_1\{y\} \iff y \in \text{cl}_2\{x\}$, where cl_i denotes closure with respect to T_i $(i = 1 \text{ or } 2)$.

(ii) $x \quad \text{int}_1(X \setminus \{y\}) \iff y \quad \text{int}_2(X \setminus \{x\})$ where int_i denotes interior with respect to T_i $(i = 1 \text{ or } 2)$.

7.9 Proposition

Let (X,μ) be a quasi-nearness space; then $(X, T_1(\mu), T_2(\mu))$ is pairwise-R_0.

Proof:

We have $x \in \mathrm{int}_1(X \setminus \{y\}) \iff \{(X \setminus \{y\}, X), (X, X \setminus \{x\})\} \in \mu$
$\iff y \in \mathrm{int}_2(X \setminus \{x\})$.

□

7.10 Proposition

Let (X, μ) be a quasi-nearness space. Then
$x \in \mathrm{int}_i(A) \iff \mathrm{st}_i(x, \mathcal{C}) \subseteq A$ for some $\mathcal{C} \in \mu$ $(i = 1$ or $2)$.

Proof:

Fix $i = 1$; $x \in \mathrm{int}_1(A) \iff \mathcal{C}_A = \{(A,X), (X, X \setminus x)\} \in \mu$, but then $\mathrm{st}_1(x, \mathcal{C}_A) = A$ as required. Conversely suppose $\mathrm{st}_1(x, \mathcal{C}) \subseteq A$ for some $\mathcal{C} \in \mu$; we show that $\mathcal{C} \leqslant \mathcal{C}_A$; suppose $(C_1, C_2) \in \mathcal{C}$; if $x \in C_2$, Then $\begin{cases} C_1 \subseteq A \\ C_2 \subseteq X \end{cases}$.

On the other hand, if $x \notin C_2$, then $\begin{cases} C_1 \subseteq X \\ C_2 \subseteq X \setminus x \end{cases}$.

This shows that $\mathcal{C} \leqslant \mathcal{C}_A$.

□

We now show that there is an abundance of quasi-nearness spaces.

7.11 Examples

(i) Let (X, T_1, T_2) be a pairwise R_0 bitopological space. Let μ_0 be the collection of conjugate covers of X which satisfy :

$\mathcal{C} \in \mu_0 \iff \{(\mathrm{int}_1(C_1), \mathrm{int}_2(C_2)) : (C_1, C_2) \in \mathcal{C}\}$

is a conjugate cover of X. [Warning: here int_1,

int_2 denote interior with respect to T_1, T_2 .]
One proves that μ_0 - int_i = int_i (i = 1 or 2) ; the remaining details are easy. If $x \in int_1(A)$, then $int_1 A \cup int_2 X \setminus \{x\} = X$, since if $y \notin int_1 A$, then $x \notin cl_1\{y\}$ so $y \notin cl_2\{x\}$, so $y \in int_2 X \setminus \{x\}$. Thus $\{(int_1 A, X), (X, int_2 X \setminus \{x\})\}$ is a conjugate cover of X , yielding $\{(A,X),(X,X\setminus\{x\})\} \in \mu_0$, so $x \in \mu_0$ -$int_1(A)$ as required. The converse is trivial. A moments thought shows that we have "sufficient" strong conjugate covers.

(ii) Any (covering) quasi-uniform space (X,μ) is a quasi-nearness space.

(iii) Any quasi-proximal space is a quasi-nearness space (since it "is" a totally bounded quasi-uniform space.) (Salbany [36])

(iv) Any topology T on a set X appears as the "first" topology of a quasi-nearness space, (X,μ) . This follows from example (ii) and a similar fact for quasi-uniform spaces (the Pervin quasi-uniformity) or from observing that the Skula bitopology (X,T,T^*) is pairwise R_0 and appealing to example (i).

(v) For $\mathcal{C}$ a cover of X , let $\mathcal{C}_d = \{(C,C): C \in \mathcal{C}\}$; $\mathcal{C}_d$ is clearly a strong conjugate cover of X .
For (X,μ) a nearness space let μ_d be the collection of conjugate covers of X which has as base the family $\{\mathcal{C}_d : \mathcal{C} \in \mu\}$. Then (X,μ_d) is a quasi-nearness space.

7.12 Definition

Let $\mathcal{C}$ be a conjugate pair cover of X. Define $\mathcal{C}_s = \{C_1 \cap C_2 : (C_1, C_2) \in \mathcal{C}\}$.

Remark:

$\mathcal{C}_s$ is a cover of X.

7.13 Proposition

Let (X,μ) be a quasi-nearness space. Let μ_s be the family of covers of X which has $\{\mathcal{C}_s : \mathcal{C} \in \mu\}$ as a base. Then (X,μ_s) is a nearness space.

Proof:

Clearly μ_s is a non empty family of covers of X.

N1: Let $\mathcal{C}$, $\mathcal{D} \in \mu$; then $\mathcal{C}_s \wedge \mathcal{D}_s = (\mathcal{C} \wedge \mathcal{D})_s$ since

$$\begin{aligned}\mathcal{C}_s \wedge \mathcal{D}_s &= \{(C_1 \cap C_2) \cap (D_1 \cap D_2) : (C_1, C_2) \in \mathcal{C},\\ &\qquad (D_1, D_2) \in \mathcal{D}\}\\ &= \{(C_1 \cap D_1) \cap (C_2 \cap D_2) : (C_1, C_2) \in \mathcal{C},\\ &\qquad (D_1, D_2) \in \mathcal{D}\}\\ &= (\mathcal{C} \wedge \mathcal{D})_s .\end{aligned}$$

N2: $\mathcal{C} \in \mu_s \Rightarrow$ there is $\mathcal{D} \in \mu$ such that $\mathcal{D}_s \leqslant \mu$; now $\mathcal{C} \leqslant \mathcal{E} \Rightarrow \mathcal{D}_s \leqslant \mathcal{E}$, so $\mathcal{E} \in \mu_s$.

N3: $\mathcal{C} \in \mu_s \Rightarrow$ there is $\mathcal{D} \in \mu$ such that $\mathcal{D}_s \leqslant \mathcal{C}$, but $\mathcal{D} \in \mu \Rightarrow \mu\text{-int}\,\mathcal{D} \in \mu$. We need only show that $(\mu\text{-int}\,\mathcal{D})_s \leqslant \mu_s\text{-int}\,\mathcal{C}$.

But for $(D_1, D_2) \in \mathcal{D}$ with $D_1 \cap D_2 \subseteq C \in \mathcal{C}$ we claim $\mu\text{-int}_1(D_1) \cap \mu\text{-int}_2(D_2) \subseteq \mu_s\text{-int}(C)$;

$x \in \mu\text{-int}_1(D_1) \cap \mu\text{-int}_2(D_2) \Rightarrow$

$$\begin{cases} \{(D_1, X), (X, X \setminus x)\} \in \mu \\ \{(X \setminus x, x), (X, D_2)\} \in \mu. \end{cases}$$

so $\{(D_1 \cap X \setminus x, X), (D_1, D_2), (X \setminus x, X \setminus x), (X, X \setminus x \cap D_2)\} \in \mu$.

$\Rightarrow \{D_1 \cap D_2, D_1 \cap X \setminus x, D_2 \cap X \setminus x, X \setminus x\} \in \mu_s$

$\Rightarrow \{D_1 \cap D_2, X \setminus x\} \in \mu_s$, yielding $x \in \mu_s\text{-int}(D_1 \cap D_2)$ as needed. □

7.14 Proposition

Let (X, μ) be a quasi-nearness space. Then $T_1(\mu) \vee T_2(\mu) = T(\mu_s)$, the topology generated by μ_s.

Proof:

Let $x \in U \in T_1(\mu)$; then $\{(U, X), (X, X \setminus x)\} \in \mu$ so $\{U, X \setminus x\} \in \mu_s$ yielding $x \in \mu_s\text{-int}(U)$.

So $\mu_s\text{-int}(U) = U$. Similarly $U \in T_2(\mu) \Rightarrow U \in T(\mu_s)$

So $T(\mu_s) \geqslant T_1(\mu), T_2(\mu)$. Conversely, suppose $x \in U \in T(\mu_s)$; then $\{U, X \setminus x\} \in \mu_s$. Suppose $\mathcal{C} \in \mu$ satisfies $\mathcal{C}_s \leqslant \{U, X \setminus x\}$; we must have $x \in C_1 \cap C_2$ where $(C_1, C_2) \in \mathcal{C}$: we may assume using QN3 that $C_1 \in T_1(\mu)$ and $C_2 \in T_2(\mu)$; since $x \in C_1 \cap C_2$, we must have $C_1 \cap C_2 \subseteq U$, since $C_1 \cap C_2 \nsubseteq X \setminus x$. This proves that $U \in T_1(\mu) \vee T_2(\mu)$.

□

7.15 Proposition

Let (X,μ) be a quasi-nearness space; then $S\colon (X,\mu) \to (X,\mu_s)$ is the object part of a functor from Q-NEAR to NEAR .

Proof:

Suppose $f\colon (X,\mu) \to (Y,\nu)$ is a quasi-nearness map. Let $\mathcal{C} \in \mu$; we show $f^{-1}[\mathcal{C}_s] = \{f^{-1}(C_1 \cap C_2)\colon (C_1,C_2) \in \mathcal{C}\} \in \mu_s$. But $f^{-1}[\mathcal{C}] = \{(f^{-1}(C_1), f^{-1}(C_2))\colon (C_1,C_2) \in \mathcal{C}\} \in \mu$, so $\{f^{-1}(C_1) \cap f^{-1}(C_2)\colon (C_1,C_2) \in \mathcal{C}\} \in \mu_s$ yielding $\{f^{-1}(C_1 \cap C_2)\colon (C_1,C_2) \in \mathcal{C}\} \in \mu_s$. This is sufficient to show that f is a nearness map from (X,μ_s) to (Y,ν_s) . So defining $Sf = f$, we have that S is a functor from Q-NEAR to NEAR. □

If (X,μ) is a nearness space we may regard it as a quasi-nearness space by constructing the quasi-nearness space (X,μ_d) which has as base all conjugate covers of the form $\mathcal{C}_d = \{(C,C)\colon C \in \mathcal{C}\}$. It is a trivial matter to see that $D\colon (X,\mu) \to (X,\mu_d)$ is a functor from NEAR to Q-NEAR.

7.16 Theorem

The functor D is left adjoint to the functor S .

Proof:

Let $(X,\mu) \in$ NEAR , $(Y,\nu) \in$ Q-NEAR . We show that if f is a quasi-nearness map from (X,μ_d) to (Y,ν) , then f is a nearness map from (X,μ) to (Y,ν_s) .

Select $\mathcal{C}$ a conjugate cover in ν . Then there is a cover $\mathcal{D}$ of μ such that $\mathcal{D}_d = \{(D,D): D \in \mathcal{D}\} \leqslant \{(f^{-1}(C_1), f^{-1}(C_2)): (C_1,C_2) \in \mathcal{C}\}$. (*)

We claim that $\mathcal{D} \leqslant f^{-1}[\mathcal{C}_s]$, which proves the result, since $\mu_{ds} = \mu$. But $D \in \mathcal{D} \Rightarrow \begin{cases} D \subseteq f^{-1}(C_1) \\ D \subseteq f^{-1}(C_2) \end{cases}$ for some $(C_1,C_2) \in \mathcal{C}$ by (*) , so $D \subseteq f^{-1}(C_1) \cap f^{-1}(C_2) \in f^{-1}[\mathcal{C}_s]$ as desired.

Similarly, if g is a nearness map from (X,μ) to (Y,ν_s) then it is a simple matter to see that g is a quasi-nearness map from (X,μ_d) to (Y,ν) . The naturality conditions are trivially verified. □

In fact we may isolate a subcategory of quasi-nearness spaces that is isomorphic to the category <u>NEAR</u>. A member, (X,μ) of this subcategory satisfies $\mu_{sd} = \mu$. (If (X,μ) is a nearness space, it is a trivial matter that $\mu_{ds} = \mu$.) The functors S, D provide the required isomorphism(s). <u>NEAR</u> is thus coreflective in <u>Q-NEAR</u>.

7.17 <u>Definition</u>

A quasi-nearness space, (X,μ) is called *topological* if whenever $\operatorname{int}\mathcal{C}$ is a conjugate cover of X it follows that $\mathcal{C} \in \mu$.

7.18 Proposition

(i) If (X,μ) is a topological quasi-nearness space, then (X,μ_s) is a topological nearness space.

(ii) If (X,μ) is a topological nearness space, then (X,μ_d) is a topological quasi-nearness space.

Proof:

(i) Suppose $\mathcal{C} \in \mu$ and that $\mu_s\text{-int}\, \mathcal{C}_s = \{\mu_s\text{-int}(C_1 \cap C_2) : (C_1,C_2) \in \mathcal{C}\}$ is a cover of X.

Now $\mu_s\text{-int}\,(C_1 \cap C_2) = \bigcup_{\alpha\in A} (U_{\alpha 1} \cap U_{\alpha 2})$ where $U_{\alpha 1} \in T_1(\mu)$ and $U_{\alpha 2} \in T_2(\mu)$. The set of all pairs $(U_{\alpha 1},U_{\alpha 2})$ for all pairs $(C_1,C_2) \in \mathcal{C}$ is a member, $\mathcal{D}$, of μ, since μ is topological. But $\mathcal{D}_s \leqslant \mu_s\text{-int}\, \mathcal{C}_s \leqslant \mathcal{C}_s$, so $\mathcal{C}_s \in \mu_s$ as required.

(Recall: a nearness space, (X,μ), is topological if $\mathcal{C} \in \mu$ whenever $\text{int}\, \mathcal{C}$ is a cover of X.)

(ii) This is easy to prove once one realizes that $\mu_d\text{-int}_1$ coincides with $\mu_d\text{-int}_2$.

□

7.19 Theorem

The full subcategory TQ-NEAR of Q-NEAR whose objects are the topological quasi-nearness spaces is

(i) isomorphic to the category $\underline{2\text{-}R_0}$ of pairwise-R_0 bitopological spaces,

(ii) coreflective in Q-NEAR.

<u>Proof</u>:

(i) It (X,μ) is a topological quasi-nearness space, then $(X,T_1(\mu),T_2(\mu))$ is certainly a pairwise-R_0 bitopological space, but this correspondence actually induces an isomorphism, since membership of μ is governed entirely by the two topologies.

(ii) Suppose (X,μ) is a quasi-nearness space. Define $\mu_T = \{\mathcal{C} \subseteq PX \times PX\colon \mu\text{-int } \mathcal{C}$ is a conjugate cover of $X\}$. We claim that μ_T is a quasi-nearness space.

QN1: $\mathcal{C},\mathcal{D} \in \mu_T \Rightarrow \text{int } \mathcal{C}$, $\text{int } \mathcal{D}$ are conjugate covers of X . Clearly then $\text{int } \mathcal{C} \wedge \text{int } \mathcal{D} = \text{int}(\mathcal{C} \wedge \mathcal{D})$ is a conjugate cover of X and, by eliminating superfluous pairs we may "reduce" $\text{int}(\mathcal{C} \wedge \mathcal{D})$ to a strong conjugate cover, $\mathcal{E}$. $\mathcal{E}$ is clearly a member of μ_T and obviously refines $\mathcal{C} \wedge \mathcal{D}$.

QN2: $\mathcal{C} \in \mu_T$ and $\mathcal{C} \leqslant \mathcal{D}$ implies that $\text{int } \mathcal{C}$ is a conjugate cover of X and $\text{int } \mathcal{C} \leqslant \text{int } \mathcal{D}$. But this implies $\text{int } \mathcal{D}$ is a conjugate cover of X.

QN3: Let $\mathcal{C} \in \mu_T$. Then $\text{int } \mathcal{C}$ is a conjugate cover of X and so $\text{int}(\text{int } \mathcal{C}) = \text{int } \mathcal{C}$ is a conjugate cover pair of X , yielding $\text{int } \mathcal{C} \in \mu_T$. But $\text{int } \mathcal{C} \leqslant \mu_T\text{-int } \mathcal{C}$.

(X,μ_T) is clearly topological. The identity $1_X\colon (X,\mu_T) \to (X,\mu)$ is the <u>TQ-NEAR</u> coreflection, since $\mathcal{C} \in \mu$ implies $\mathcal{C} \in \mu_T$.

□

7.20 Corollary

(i) T-NEAR is isomorphic to $\underline{R_0}$ (the category of R_0-topological spaces.)

(ii) T-NEAR is coreflective in NEAR.

Proof:

(i) If (X,μ) is such that $\mu_{sd} = \mu$, then $T_1(\mu) = T_2(\mu)$ and $(X,T_1(\mu),T_2(\mu))$ "is" an R_0 topological space.

(ii) If (X,μ) is such that $\mu_{sd} = \mu$, then $\mu_{T_{sd}} = \mu_T$, so the result follows. □

Remark:

A topological quasi-nearness space, (X,μ) , will be said to have a bitopological property, P , if the space $(X,T_1(\mu),T_2(\mu))$ has that property.

7.21 Definition

A quasi-nearness space, (X,μ) , is called *uniform* if it satisfies the following condition:

U: If $\mathcal{C} \in \mu$, there is $\mathcal{D} \in \mu$ such that $\mathcal{D}^* \leq \mathcal{C}$.

7.22 Theorem

The full subcategory UQ-NEAR of Q-NEAR whose objects are the uniform quasi-nearness spaces is isomorphic to the category QUN of quasi-uniform spaces.

Proof:

Obvious. □

7.23 Proposition

(i) If (X,μ) is a uniform quasi-nearness space, then (X,μ_s) is a uniform nearness space.

(ii) If (X,μ) is a uniform nearness space, then (X,μ_d) is a uniform quasi-nearness space.

Proof:

Obvious. □

7.24 Theorem

The category UQ-NEAR is reflective in Q-NEAR.

Proof:

Suppose (X,μ) is a quasi-nearness space. Let $\mu_U = \{C \in \mu : C \geq C_1^* \geq C_2^* \geq \dots$, where $C_1, C_2, \dots \in \mu\}$. (X,μ_U) is a uniform quasi-nearness space. The map $1_X : (X,\mu) \to (X,\mu_U)$ is the UQ-NEAR reflection. □

7.25 Definition

A quasi-nearness space (X,μ) is *contigual* if for each $C \in \mu$, there is a finite subset, $\mathcal{D}$, of C with $\mathcal{D} \in \mu$.

Remark:

This generalizes the notion of a contigual (nearness) space.

7.26 Definition

A quasi-nearness space, (X,μ) is *totally bounded* if for each $\mathcal{C} \in \mu$, there is a finite subset, $\mathcal{D}$, of $\mathcal{C}$ with $\mathcal{D}$ a conjugate cover of X.

7.27 Proposition

(1) Every contigual quasi-nearness space is totally bounded.

(2) For a topological quasi-nearness space, (X,μ), the following are equivalent.

(i) (X,μ) is contigual.

(ii) (X,μ) is totally bounded.

(iii) (X,μ) is pairwise-compact

(i.e. $(X, T_1(\mu) \vee T_2(\mu)$ is compact).

(3) For a uniform quasi-nearness space, (X,μ), the following conditions are equivalent.

(i) (X,μ) is contigual.

(ii) (X,μ) is totally bounded.

Proof:

(1) Obvious.

(2) (i) $\Rightarrow$ (ii) is clear.

(ii) $\Rightarrow$ (iii): Suppose $\mathcal{C}$ is a $T_1(\mu) \vee T_2(\mu)$ open cover of X. We may decompose $\mathcal{C}$ into a conjugate pair cover $\mathcal{D} = \{(D_{\alpha 1}, D_{\alpha 2}) : \alpha \in A\}$, where $D_{\alpha j} \in T_j(\mu)$ for all $\alpha \in A$, $j = 1$ or 2. Since (X, μ) is topological, this is a member of μ . Now select a finite subset, $\mathcal{E}$, of $\mathcal{D}$ which is a conjugate cover of X . For each pair (E_{i1}, E_{i2}) of $\mathcal{E}$ we can find a $C_i \in \mathcal{C}$ such that $E_{i1} \cap E_{i2} \subseteq C_i$. But then $\cup C_i = X$, as required.

(iii) $\Rightarrow$ (i): Let $\mathcal{C} \in \mu$. Then

$\operatorname{int} \mathcal{C} = \{(\operatorname{int}_1(C_1), \operatorname{int}_2(C_2)) : (C_1, C_2) \in \mathcal{C}\} \in \mu$. So $\cup\{\operatorname{int}_1(C_1) \cap \operatorname{int}_2(C_2) : (C_1, C_2) \in \mathcal{C}\} = X$. But each $\operatorname{int}_1(C_1) \cap \operatorname{int}_2(C_2)$ is a member of $T_1(\mu) \vee T_2(\mu)$, so we can select a *finite* subset, $\mathcal{E}$ of $\mathcal{C}$ satisfying $\cup\{(\operatorname{int}_1(E_1) \cap \operatorname{int}_2(E_2) : (E_1, E_2) \in \mathcal{E}\} = X$. But then $\mathcal{E}$ is a member of μ , since (X, μ) is topological.

(3) (i) $\Rightarrow$ (ii): Obvious.

(ii) $\Rightarrow$ (i): Suppose $\mathcal{C} \in \mu$; select $\mathcal{D} \in \mu$ a strong conjugate pair cover, such that $\mathcal{D}^* \leqslant \mathcal{C}$. Now select a finite subfamily, $\mathcal{E}$, of $\mathcal{D}$ which is a conjugate cover of X . Then

$\mathcal{D} \leqslant \{(\operatorname{st}_1(E_1, \mathcal{D}) , \operatorname{st}_2(E_2, \mathcal{D})) : (E_1, E_2) \in \mathcal{E}\} \leqslant \mathcal{C}$;

The second inequality follows from the fact that $\mathcal{D}^* \leqslant \mathcal{C}$. Now suppose $(D_1, D_2) \in \mathcal{D}$; there is a pair $(E_1, E_2) \in \mathcal{E}$ with $(D_1 \cap D_2) \cap (E_1 \cap E_2) \neq \emptyset$; hence $D_1 \subseteq \operatorname{st}_1(E_1, \mathcal{D})$ and $D_2 \subseteq \operatorname{st}_2(E_2, \mathcal{D})$ yielding the first inequality. Now for each pair $(E_1, E_2) \in \mathcal{E}$, there is a pair $(C_1, C_2) \in \mathcal{C}$ with $E_1 \subseteq C_1$, $E_2 \subseteq C_2$. The set of all such pairs is a finite subfamily of $\mathcal{C}$ which is a member of μ .

□

7.28 Definition

A quasi-nearness space is called *proximal* if it is uniform and contigual (or totally bounded).

7.29 Theorem

The category of proximal quasi-nearness spaces is isomorphic to the category of quasi-proximity spaces.

Proof:

Salbany [36] proves that the totally bounded quasi-uniform spaces are the quasi-proximity spaces. □

We feel that this establishes successfully a category which encompasses nicely all the non-symmetric categories mentioned at the beginning of the chapter. Many theorems in NEAR are seen to be theorems in Q-NEAR.

We have already seen in example 7.11 (iv) that every topology T on X appears as $T_1(\mu)$ for some quasi-nearness space (X,μ). The following result tells us that it does so in a restricted way if we impose functoriality considerations.

7.30 Proposition

The Skula functor, $Sk: (X,T) \to (X,T,T^*)$ is the unique right inverse to the functor U from $\underline{2\text{-}R_0}$ to TOP which "forgets" the second topology $(U: (X,T_1,T_2) \to (X,T_1))$.

Proof:

Let $S = \{\emptyset, \{1\}, \{0,1\}\}$ be the Sierpinski topology on $P = \{0,1\}$. If $(P,S,\bar{S})$ is a pairwise-R_0 bitopological space, then $\bar{S} = \{\emptyset, \{0\}, \{0,1\}\}$. Now suppose F is a functor, right inverse to U, such that $F(X,T) = (X,T,F(T))$ is pairwise-R_0. We show that $F(T) = T^*$ (the topology with base all T-closed subsets of X).

Suppose $x \in U \in F(T)$; since $F(X,T)$ is pairwise-R_0, there is $V \in T$ such that $x \notin V$, $V \cup U = X$. But then $x \in X \setminus V \subseteq U$, and since $X \setminus V \in T^*$, we have $U \in T^*$. Thus $F(T) \subseteq T^*$.

Conversely, suppose $x \in X$. Define $f_x\colon X \to \{0,1\}$ by $f(t) = 0 \iff t \in \mathrm{cl}_T\{x\}$; $f_x\colon (X,T) \to (P,S)$ is continuous, hence $f_x\colon (X,T,F(T)) \to (P,S,F(S)) = (P,S,\bar{S})$ is bi-continuous. Thus $f_x^{-1}\{0\} \in F(T)$, ie. $\mathrm{cl}_T\{x\} \in F(T)$. But then $F(T) \supseteq T^*$. □

7.31 Corollary

U: T-QNEAR → TOP has unique right inverse. (Here $U(X,\mu) = (X,T_1(\mu))$.)

7.32 Theorem

The functor Sk: TOP → 2-R_0 is left adjoint to the functor U: 2-R_0 → TOP.

Proof:

Let (X,T) be a topological space, (Y,T_1,T_2) a pair-wise-R_0 bitopological space. Clearly if f is bicontinuous from (X,T,T^*) to (Y,T_1,T_2), then f is continuous from (X,T) to (Y,T_1). Conversely suppose g is continuous from (X,T) to (Y,T_1). We show g is bicontinuous from (X,T,T^*) to (Y,T_1,T_2). Let $U \in T_2$; we want $f^{-1}(U) \in T^*$. Let $x \in f^{-1}(U)$; then $f(x) \in U$, and there is $V \in T_1$ such that $f(x) \notin V$ and $V \cup U = Y$. But then $f^{-1}(V) \in T$ and $x \notin f^{-1}(V)$ and $f^{-1}(V) \cup f^{-1}(U) = X$. So $cl_T\{x\} \subseteq f^{-1}(U)$, showing that $f^{-1}(U) \in T^*$. We have established a bijection from $hom((X,T,T^*),(Y,T_1,T_2))$ to $hom((X,T),(Y,T_1))$; the required naturality conditions are easily verified yielding the required adjunction. □

7.33 Corollary

TOP is coreflective in $\underline{2\text{-}R_0}$.

7.34 Corollary

TOP is coreflective in TQ-NEAR.

We consider now some general aspects of the category Q-NEAR.

7.35 Theorem

Let X be a set and let Υ be the set of all quasi-nearness structures on X ; Υ, when ordered by inclusion, is a complete lattice.

Proof:

Let $\{\mu_i : i \in I\}$ be a family of quasi-nearness structures on X. Then $\vee\mu_i$ has as base all conjugate pair covers of the form $C_{i_1} \wedge C_{i_2} \wedge \ldots \wedge C_{i_n}$ (where we have $i_j \in I$ and $C_{i_j} \in \mu_{i_j}$ for each $j = 1, 2, \ldots, n$). □

In particular $\mu_i = \{(X,X)\}$ is the smallest member of γ and $\mu_D = \{$all conjugate covers$\}$ is the largest member.

7.36 Proposition

If (Y,ν) is a quasi-nearness space and if $f: X \to Y$ is a function, then $f^{-1}(\nu) = \{f^{-1}[C]: C \in \nu\}$ is a base for the smallest structure on X such that f is a quasi-nearness map.

Proof:

Omitted. □

7.37 Theorem

Let X be a set, (X_i, μ_i) a family of quasi-nearness spaces (where $i \in I$, which may be empty, a class, or a set). Let $f_i: X \to X_i$ be a family of functions and μ the join of the set $\{\nu_i : \nu_i$ has as base $f_i^{-1}(\mu_i)\}$. Then we have:

If (Y,η) is a quasi-nearness space and $g: Y \to X$ is a function, then $g: (Y,\eta) \to (X,\mu)$ is a quasi-nearness map iff $f_i \circ g: (Y,\eta) \to (X_i,\mu_i)$ is a quasi-nearness map for each $i \in I$.

Proof:

Straightforward. □

So Q-NEAR has "initial" structures. Q-NEAR is also a properly fibred concrete category.

One of the motives for constructing the category Q-NEAR has been the recent interest in relationships between quasi-uniform spaces and nearness spaces. (Carlson, [7]). He establishes a link between locally right symmetric quasi-uniform spaces and nearness spaces. This link actually turns out to be a link between locally right symmetric quasi-nearness spaces and nearness spaces, as might be expected.

7.38 Definition

Let (X, μ) be a quasi-nearness space. Then (X, μ) is called *locally right symmetric* if for each $x \in X$, $\mathcal{C} \in \mu$ there is $\mathcal{D} \in \mu$ such that

$$st_1(st_2(x,\mathcal{D}),\mathcal{D}) \subseteq st_1(x,\mathcal{C}) .$$

7.39 Lemma

If (X, μ) is locally right symmetric, then $(X, T_1(\mu))$ is R_0 .

Proof:

If $x \in U \in T_1(\mu)$, then there is $\mathcal{C} \in \mu$ such that $st_1(x,\mathcal{C}) \subseteq U$. Now, by local symmetry, there is $\mathcal{D} \in \mu$

such that $st_1(st_2(x,\mathcal{D}),\mathcal{D}) \subseteq st_1(x,\mathcal{C})$, so $st_2(x,\mathcal{D}) \subseteq st_1(x,\mathcal{C})$, and thus $cl_{T_1(\mu)}\{x\} \subseteq st_2(x,\mathcal{D}) \subseteq st_1(x,\mathcal{C}) \subseteq U$, as required.

□

7.40 Proposition

The following are equivalent for a quasi-nearness space, (X,μ) .

(1) (X,μ) is locally right symmetric.

(2) For $A \subseteq X$, $int_1(A) = \{x:$ there exists $\mathcal{C} \in \mu$ such that $\{st_1(y,\mathcal{C}): y \in X\} \leqslant \{A, X \setminus x\}\}$ (*)

Proof:

(1) $\Rightarrow$ (2): Suppose for some $\mathcal{C} \in \mu$ that $\{st_1(y,\mathcal{C}): y \in X\} \leqslant \{A, X\setminus x\}$. We show that $\mathcal{C} \leqslant \{(A,X),(X,X\setminus x)\}$, yielding $x \in int_1(A)$. Suppose $(C_1,C_2) \in \mathcal{C}$. If $x \in C_2$, then $C_1 \subseteq st_1(x,\mathcal{C}) \subseteq A$; on the other hand, if $x \notin C_2$, then $C_1 \subseteq X$. In either case we have the required inclusions, and so $\mathcal{C} \leqslant \{(A,X),(X,X\setminus x)\}$. Conversely, suppose $x \in int_1(A)$. Then $\mathcal{C}_A = \{(A,X),(X,X\setminus x)\} \in \mu$, and by local symmetry there is $\mathcal{D} \in \mu$ such that $st_1(st_2(x,\mathcal{D}),\mathcal{D}) \subseteq st_1(x,\mathcal{C}_A) = A$. Now let $y \in X$; we show $st_1(y,\mathcal{D}) \subseteq A$ if $x \in st_1(y,\mathcal{D})$, thus showing $x \in$ (*) :

$$x \in st_1(y,\mathcal{D}) \Rightarrow \text{there is } (D_1,D_2) \in \mathcal{D} \text{ with } \begin{cases} x \in D_1 \\ y \in D_2 \end{cases} ;$$

but this shows $y \in st_2(x,\mathcal{D})$, so

$$st_1(y,\mathcal{D}) \subseteq st_1(st_2(x,\mathcal{D}),\mathcal{D}) \subseteq st_1(x,\mathcal{C}_A) = A$$

as required.

(2) $\Rightarrow$ (1): Let $x \in X$, $\mathcal{C} \in \mu$; clearly $x \in \text{int}_1(\text{st}_1(x,\mathcal{C}))$. So there is a $\mathcal{D} \in \mu$ such that

$$\{\text{st}(y,\mathcal{D}) : y \in X\} \leqslant \{X \setminus x, \ \text{st}_1(x,\mathcal{C})\} \qquad (**)$$

We now show $\text{st}_1(\text{st}_2(x,\mathcal{D}),\mathcal{D}) \subseteq \text{st}_1(x,\mathcal{C})$. Suppose $t \in \text{st}_1(\text{st}_2(x,\mathcal{D}),\mathcal{D})$; then there is $(D_1,D_2) \in \mathcal{D}$ such that

$$\begin{cases} t \in D_1 \\ \text{st}_2(x,\mathcal{D}) \cap D_2 \neq \emptyset . \end{cases}$$

So there is $(D_3,D_4) \in \mathcal{D}$, $y \in X$ such that

$$\begin{cases} y \in D_2 \\ y \in D_4 \\ x \in D_3 \end{cases}$$

hence $\begin{cases} x \in \text{st}_1(y,\mathcal{D}) \\ t \in \text{st}_1(y,\mathcal{D}) \end{cases}$. Using (**) we see $t \in \text{st}_1(x,\mathcal{C})$.

□

7.41 <u>Theorem</u>

Let (X,μ) be a quasi-nearness space. The following are equivalent :

(1) (X,μ) is locally right symmetric.

(2) $(X,\ell(\mu))$ is a nearness space and $\ell(\mu)\text{-int}\,(A) = \mu\text{-int}_1(A)$, where

$$\ell(\mu) = \{\mathcal{C} \subseteq PX : \text{for some } \mathcal{D} \in \mu \ , \ \{\text{st}_1(x,\mathcal{D}) : x \in X\} \leqslant \mathcal{C}\}.$$

<u>Proof</u>:

(1) $\Rightarrow$ (2): By proposition 7.40, we already know that $\ell(\mu)\text{-int}(A) = \mu\text{-int}_1(A)$. It remains to be shown that $(X,\ell(\mu))$ is indeed a nearness space. It is easy to see that

$\ell(\mu)$ is a non-empty family of covers of X satisfying N1 and N2. N3 is equally easy: if $\mathcal{C} \in \ell(\mu)$, and $\mathcal{D} \in \mu$ satisfies $\{st_1(x,\mathcal{D}) : x \in X\} \leqslant \mathcal{C}$, we may select μ-int $\mathcal{D}(\in \mu)$ and clearly

$\{st_1(x,\mu\text{-int } \mathcal{D}) : x \in X\} \leqslant \mathcal{C}$ as well.

But, since $\mu\text{-int}_1 A = \ell(\mu)\text{-int } A$ (for $A \subseteq X$) we have $\ell(\mu)\text{-int } \mathcal{C} \in \ell(\mu)$ as well.

(2) ⇒ (1): Follows from proposition 7.40.

□

7.42 <u>Corollary</u> (Carlson [7])

Let (X,μ) be a quasi-uniform space. The following are equivalent.

(1) (X,μ) is locally right symmetric.

(2) $(X,\ell(\mu))$ is a nearness space.

We leave the development of quasi-nearness spaces at this point. There are clearly many avenues to pursue. Of great importance is the question of completeness, bi-completeness, and existence of completions or bicompletions.

For instance a quasi-nearness space, (X,μ) would be called bi-complete iff (X, μ_s) is complete as a nearness space.

7.43 <u>Proposition</u>

Let (X,μ) be a quasi-nearness space.

(1) If (X,μ) is topological, it is bi-complete.

(2) If (X,μ) is bi-complete as a quasi-uniform space, it is bi-complete as a quasi-nearness space.

A notion of completeness for a space (X,μ) should be easy to define using subsets of $\mathcal{P}X \times \mathcal{P}X$ maximal with respect to *not* being elements of μ. Work on these questions is in progress.

Notes on Chapter 7

(1) We have been rather brief in our treatment of nearness spaces; the reader is referred to Herrlich [19] for the notions of topological, uniform and other nearness structures, although the definitions should be apparent simply by translating the appropriate definitions in Q-NEAR to NEAR.

(2) Proposition 7.27(2) is further evidence for Salbany's thesis that pairwise compactness is the appropriate bitopological notion of compactness. (Salbany [36]).

(3) Proposition 7.30 improves on a result of Salbany's [36] while theorem 7.32 establishes $\underline{2\text{-}R_0}$ (pairwise R_0 spaces, continuous functions) as an important full subcategory of BITOP , and of course has as corollary the fact that TOP is coreflective in TQ-NEAR.

(4) Results in sections 7.38 to 7.42 bear out the suggestion (at the beginning of the chapter) that links between quasi-uniform and nearness spaces are actually as a result of the situation of quasi-uniform spaces in our more general category of quasi-nearness spaces.

(5) A theory of nearness (or quasi-nearness) frames has not been established. Is this possible; there do seem to be non trivial problems.

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