

UNIVERSITY OF CAPE TOWN

DEPARTMENT OF MATHEMATICS

A categorial study of initiality  
in uniform topology

by

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A thesis prepared under the  
supervision of Dr. K.A. Hardie  
for the degree of Doctor of  
Philosophy in Mathematics.

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1971

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. . . en bevestig die werk  
van ons hande oor ons,  
ja die werk van ons hande,  
bevestig dit!

Opgedra  
aan die nagedagtenis  
van my pa  
JOHAN NIKOLAAS BRÜMMER  
1899 - 1967

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Together with this thesis the candidate presents copies of his two papers:

Initial quasi-uniformities 1969a .

Note on a compactification due to  
Nielsen and Sloyer [(to appear)]a .

In accordance with the regulations governing the award of the degree of Ph.D. at the University of Cape Town, the candidate presents the following summary of the contents of the thesis to indicate in what way it constitutes a contribution of knowledge.

### S U M M A R Y

This thesis consists of two chapters, of which the first presents a categorical study of the concept of initiality (also known as projective generation) and the second gives applications in the theory of uniform and quasi-uniform spaces.

The first three sections of chapter 1 expound basic aspects of initiality, such as its relation to categorical limits and to embeddings, the latter being defined with respect to a faithful functor to a bicategory. The notion of a separated object with respect to such a functor is defined. The main contribution of the chapter lies in the fourth section, where, given two amnestic functors  $L : \underline{A} \rightarrow \underline{B}$  and  $M : \underline{B} \rightarrow \underline{C}$ , the functors  $F : \underline{B} \rightarrow \underline{A}$  with  $MLF = M$  are studied. These functors were characterized by Hušek [1967a]. Our investigations branch away from Hušek's by proceeding to further specialization of the type of functor  $F$ . We characterize the functors  $F$  for which the initiality construction depends only on a class of objects of  $\underline{A}$ . Among these  $F$  we study the right inverses of  $F$  in

detail. The results are then applied in the sixth section to find conditions for the existence of an adjoint right inverse of  $L$ , in the seventh section to explore the natural order structure of the class of all right inverses of  $L$ , and in the eighth section to study monoreflectors in  $\underline{B}$ . The chapter ends with a brief investigation of the lifting of epireflectors from  $\underline{B}$  to  $\underline{A}$  against a functor  $L : \underline{A} \rightarrow \underline{B}$ .

Chapter 2 starts with a minimal self-contained introduction to the categories of the quasi-uniform spaces ( $\underline{Qun}$ ), the bitopological spaces ( $\underline{2\ Top}$ ), and the quasi-uniformizable bitopological spaces ( $\underline{Bcrg}$ ); their classical analogues are  $\underline{Un}$ ,  $\underline{Top}$ ,  $\underline{Crg}$ . We show that the notion of separated object, abstractly introduced in chapter 1, corresponds in each of these categories to the epireflective property obtained by lifting the  $T_0$ -epireflector from  $\underline{Top}$ . We settle a number of questions concerning right inverses of forgetful functors among the mentioned categories and among their subcategories of separated objects. We also characterize the epimorphisms, extreme monomorphisms and equalizers in these categories. Thus equipped, we study total boundedness and completeness in  $\underline{Qun}$ , and compactness in  $\underline{2\ Top}$ , by regarding these properties as lifted from  $\underline{Un}$  and  $\underline{Top}$ , respectively; we thereby obtain some new results and perspectives on quasi-uniform spaces.

ACKNOWLEDGEMENTS

I thank my supervisor, Dr. K.A. Hardie, for very much that he has done for me. He took an interest in my work, and instilled faith into it, long before he became my supervisor. He was always accessible, ready with a host of stimulating ideas. His example has sharpened my ideal of research and teaching as an integral human activity that takes place in a community. He taught me never to be satisfied that a result is definite enough or that an approach is universal enough, but he also urged me to write up and to get on. He read and criticized various drafts of chapter 1 of this thesis; the work would have been incomparably better had I been able to follow many of the suggestions with which he so often harrassed me. I therefore assume full responsibility for all respects in which approach and results fall short of Dr. Hardie's standards.

I gratefully acknowledge the influence of my earlier teachers, especially Mr. J.P. Theart in Beaufort West, Prof. Dr. S.R.F. Göldner in Stellenbosch, and Prof. Dr. J de Groot as well as the late Prof. Dr. E.W. Beth in Amsterdam.

I thank Dr. H. Schlagbauer for the benefit of our joint teaching in 1965 and 1966, and for his help with many things, such as the proof-reading and reproduction of this thesis.

I thank Dr. S. de O. Salbany for a great deal of very stimulating exchange of ideas in our largely overlapping fields of research, especially since 1969; and for help with the proof-reading.

I thank Prof. K.O. Househam, head of our Department of Mathematics, for his aid and support in many matters. I thank him and all other members of the Department for the pleasure of working together in such a congenial milieu.

I thank the South African Council for Scientific and Industrial Research for generous bursaries in 1963, 1964 and 1971; the University of Cape Town for a sabbatical year (1.2.1971 to 31.1.1971) and for research and travel grants; and the University of Würzburg in Bavaria, through Prof. Dr. W. Velte of the Institut für Angewandte Mathematik, for an appointment during the latter half of my sabbatical and for facilities which aided the completion of this thesis.

My indebtedness to those closest to me has to be expressed elsewhere.

CHAPTER O. INTRODUCTION AND PRELIMINARIES.O.1 INTRODUCTION

The naive concept of initiality is that of a space having the coarsest structure that makes certain functions continuous (or uniformly continuous, etc.) into certain given spaces. Initiality constructions so permeate general topology that they resort more to the folklore than to specific items of literature; but one may mention that the concept was formalized by Bourbaki [1957], Sonner [1965] and Taylor [1965], and that certain degrees of freedom in the definition have become apparent ([Čech 1966], [Hušek 1966, 1967a, 1967b], [Wyler 1971b]). Relatively few other papers had initiality as a major concern; such were e.g. [Hušek 1964a, 1964b, 1965], [Antoine 1966, 1966b] and [Roberts 1968]. Considerations of initiality are more or less implicit in the theory of concrete or structured categories ([Isbell 1957], [Semadeni 1963], [Ehresmann 1964, 1968], [Pultr 1968], [Calenko 1969], [Wyler 1971a]).

This dissertation is not an essay on concrete or structured categories, but has something in common with that subject. The motivation is that not single categories, but rather categories together with certain faithful functors present themselves for study in categorial topology [Wyler 1971a]. Thus our typical object of study is a faithful functor  $K : \underline{A} \rightarrow \underline{C}$ , with  $\underline{C}$  fixed. Given another such entity

$M : \underline{B} \rightarrow \underline{C}$ , the natural task is to consider the functors  $L : \underline{A} \rightarrow \underline{B}$  with  $K = ML$ , and the functors  $F : \underline{B} \rightarrow \underline{A}$  with  $M = KF$ . (These functors  $L$  and  $F$  are readily characterized in terms of initiality; this was done by Hušek [1967a].)

In practice at least one such functor, say  $L : \underline{A} \rightarrow \underline{B}$ , is given and distinguished. One may call  $L$  a forgetful functor (a term which I do not define, preferring to keep the notion naive and open), but one should realize that other forgetful functors  $\underline{A} \rightarrow \underline{B}$  as well as  $\underline{B} \rightarrow \underline{A}$  may be equally distinguished. However, given  $L : \underline{A} \rightarrow \underline{B}$  with  $K = ML$ , one question is about the  $F : \underline{B} \rightarrow \underline{A}$  with  $M = KF$ .

Thus our point of departure is simply that we are given two faithful functors  $L : \underline{A} \rightarrow \underline{B}$  and  $M : \underline{B} \rightarrow \underline{C}$  and we have to study the  $F : \underline{B} \rightarrow \underline{A}$  with  $MLF = M$ . Our main contribution lies in characterizing a number of increasingly special conditions on these functors  $F$ . It is hoped that these characterizations will be of service in categorical topology beyond the confines of the (bi-)topological and (quasi-)uniform spaces to which the given applications are restricted. The approach does uncover a number of new facts and perspectives on quasi-uniform spaces, and it may be claimed that the very successful investigations of Salbany [1970, 1971b] in this field were partly motivated by just such an approach.

This work is not intended to be or to develop into a monolithic theory, and I do not strive after maximal generality of assumptions. I keep to one simple definition of initiality (the Bourbaki - Söner - Taylor form) and, as indicated above, I derive special results. This may make the work readable, but risks that some of the results may by some be written off as mere consequences of other (high-powered) theories such as those of Ehresmann [1964, 1968] or Gray [1965].

A survey of the contents now follows.

Chapter 1 deals with initiality. In its first two sections, 1.1 and 1.2, basic and largely known aspects of initiality, such as its relation to limits, are expounded.

In section 1.3 we consider a faithful functor  $M : \underline{B} \rightarrow \underline{C}$ , with the simplest assumptions about given "injections" and "surjections" in  $\underline{C}$  such that the resulting M-embeddings and M-quotients in  $\underline{B}$  suffice for our later purposes; eventually the assumption is that  $\underline{C}$  is an Isbell bicategory. The innovation in this section is the definition of M-separated objects in  $\underline{B}$ ; these are shown to be an abstraction of  $T_0$ -spaces. The curious "dual" of  $T_0$  in  $\underline{Top}$  is also characterized.

In section 1.4 we consider two amnestic functors  $L : \underline{A} \rightarrow \underline{B}$  and  $M : \underline{B} \rightarrow \underline{C}$ . (The restriction to amnestic rather than

faithful functors is a practical but removable simplification.) We study the functors  $F : \underline{B} \rightarrow \underline{A}$  with  $MLF = M$ . Under the assumption henceforth that  $L$  preserves initiality for  $M$ , we isolate first those  $F$  which are right  $M$ -subinverses of  $L$ . Then we consider the  $F$  for which the initiality construction depends just on a class  $\underline{U}$  of objects of  $\underline{A}$ ; they receive the notation  $F = (L, M : (\underline{U}))$ , and are shown to be the right  $M$ -subinverses of  $L$  which satisfy  $FLF = F$ . Then a basic necessary and sufficient condition for  $(L, M : (\underline{U}))$  to be a right inverse of  $L$  is given, and further adapted to forms involving the theory of embeddings from section 1.3.

Section 1.5 discusses the special form  $(L, \mathbf{1}_{\underline{B}} : (\underline{U}))$  and its relations to  $(L, M : (\underline{U}))$ .

We then have a little calculus of formal properties of the functor  $(L, M : (\underline{U}))$ , which we apply in section 1.6 to the finding of sufficient conditions for the existence of an adjoint right inverse of  $L$ . A postscript also gives a necessary condition :  $L$  should preserve initiality for every  $M$  (the simple proof is omitted).

In section 1.7 the "little calculus" enables us to study the natural order structure of the class of all right inverses of  $L$ , thereby extending the results of the paper [Brümmer 1969a].

In section 1.8 we observe that, by a result stated above, the functors  $(\mathbf{1}_{\underline{B}}, M : (\underline{V}))$  are just the functors  $F : \underline{B} \rightarrow \underline{B}$  with  $FF = F \leq_M \mathbf{1}_{\underline{B}}$ . Under conditions of wide validity these are, up to natural equivalence, just the monoreflectors in  $\underline{B}$ .

Section 1.9 gives ad hoc conditions on a category  $\underline{A}$  and a functor  $L : \underline{A} \rightarrow \underline{B}$  in order that each epireflector in  $\underline{B}$  can be lifted against  $L$  to an epireflector in  $\underline{A}$ . These conditions are applied several times in chapter 2, but they seem still susceptible to a good deal of sharpening.

Chapter 2 consists of applications.

Section 2.1 gives a minimal self-contained introduction to the categories of the quasi-uniform spaces ( $\underline{Qun}$ ), the bitopological spaces ( $\underline{2 Top}$ ), and the quasi-uniformizable bitopological spaces ( $\underline{Bcrg}$ ). The classical analogues are  $\underline{Un}$ ,  $\underline{Top}$ ,  $\underline{Crg}$ .

Section 2.2 shows that the separated objects (in the sense of section 1.3) in the categories  $\underline{Qun}$ ,  $\underline{Un}$ ,  $\underline{Bcrg}$ ,  $\underline{Crg}$ ,  $\underline{2 Top}$  are obtained by lifting the property  $T_0$  from  $\underline{Top}$  against the forgetful functors.

Section 2.3 dwells on right inverses of various forgetful functors among these categories and among their subcategories of the separated objects; a problem of Salbany [1971b] is solved.

Section 2.4 discusses the total boundedness monoreflector in  $\underline{Qun}$ ; this is an instance of lifting from  $\underline{Un}$ . Some new results for  $\underline{Qun}$  are obtained.

Section 2.5 concerns the categories of separated objects of  $\underline{Qun}$ ,  $\underline{Un}$ ,  $\underline{Bcrg}$ ,  $\underline{Crg}$ . We find their (partly known) epimorphisms, equalizers and extreme monomorphisms. Then all is ready for applying the lifting theorems of section 1.9.

Section 2.6 studies completeness in  $\underline{Qun}$  and compactness in  $\underline{2 Top}$  as properties lifted from  $\underline{Un}$  and  $\underline{Top}$ . Thus some of the results of [Salbany 1970] are retrieved by "external" methods. It is observed that a very important property of the completeness epireflector in  $\underline{Qun}$  is not obtained from the present results on the lifting of epireflections. This points to the desirability of studying the preservation of special properties of an epireflector under the lifting. The section concludes with problems, one on the lifting of realcompactness.

## O.2. NOMENCLATURE AND NOTATIONS.

What in the sequel is called "initiality" is termed "projective generation" in the Prague school ([Čech 1966], [Hušek 1967a]). However, I needed a basic term allowing certain combinations and inflections, and I could not be sure how "projective generation" is inflected in Prague. Therefore it was safer to adapt my terminology from Bourbaki's [1957] "structures initiales". This has the further justification that Bourbaki's definition is a precategory form of the single concept that I employ, whereas Hušek uses a whole scale of concepts. Confusion with initial objects need not occur. Definitions follow in chapter 1.

The terminology on categories is that of Freyd [1964] and Herrlich [1968], with few exceptions: In a category  $\underline{C}$ ,  $\underline{C}(X, Y)$  denotes the set of all morphisms  $X \rightarrow Y$ . Functor always means covariant functor. If  $\underline{B}$  is a reflective subcategory of  $\underline{A}$ , we speak of the reflection functor  $\underline{A} \rightarrow \underline{B}$ , but of the reflector  $\underline{A} \rightarrow \underline{A}$  (or reflector in  $\underline{A}$ ) which is obtained from the reflection functor by composition with the inclusion functor. We have the general topologist's bad habit of writing small letters for reflection functors and reflectors; these should not be confused with the corresponding natural transformations.

The categories

Qun, Un, Bcrg, Crg, 2 Top, Top

which are mentioned in examples throughout chapter 1, are defined in section 2.1. So also are the spaces

$I$ ,  $I_q$ ,  $I_s$ ,  $I_b$ ,  $I_u$ ,  $D_u$ .

The categories

Qprox and Prox

of the non-separated quasi-proximity and proximity spaces, respectively, are not defined; for Prox see [Isbell 1964] and for Qprox see [Salbany 1970, chapter 4].

Discrete topologies and uniformities are well known; we use accrete (instead of "indiscrete") for the opposite extreme.

Complete regularity does not imply  $T_0$ . Tihonov spaces are completely regular  $T_0$ -spaces.

O.3 ERRATA AND ADDENDA

The references to [Antoine 1962] in 1.1.13(2) and after 1.2.2 should read [Antoine 1966]. Proofs of 1.2.2 and 1.2.6 for the case  $\underline{C} = \underline{Ens}$  were given in [Antoine 1966b].

In 1.3.2 the reference to [Isbell 1963] should read [Isbell 1964b].

The theory of sets and classes that we use throughout the thesis is the Bernays - von Neumann - Gödel system as developed in the appendix of [Kelley 1955]. We make only one exception, which is indicated in section 1.7.

## CHAPTER 1 : INITIALITY

### 1.1 BASIC CONCEPTS.

This section contains basic definitions and a few simple, largely known basic results. The definitions agree in content with standard usage, especially with that of the Prague school, e.g. [Čech 1966]. The nomenclature, however, is adapted from that of Bourbaki [1957] for reasons stated in section 0.2. (Further references to the literature are to be found at the end of the section.)

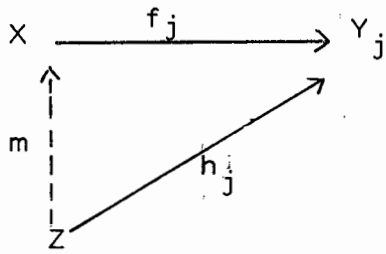
Throughout this section  $\underline{B}$  and  $\underline{C}$  will be categories with a faithful functor  $M : \underline{B} \rightarrow \underline{C}$ .

1.1.1. Definition. Let  $X \in \text{obj } \underline{B}$ . A class  $\{f_j : j \in J\}$  of  $\underline{B}$ -morphisms  $f_j : X \rightarrow Y_j$  (each with  $X$  as domain) is an M-initiality (at  $X$ ) if, whenever  $\underline{B}$ -morphisms  $h_j : Z \rightarrow Y_j$  (with the same arbitrary  $Z$  as domain) are given together with  $k : MZ \rightarrow MX$  such that  $Mf_j \cdot k = Mh_j$  ( $j \in J$ ), there exists  $m : Z \rightarrow X$  with  $Mm = k$ . An M-initiality  $\{f\}$  consisting of a single morphism is called simple and we may then abuse the language, saying that  $f$  is a simple M-initiality.

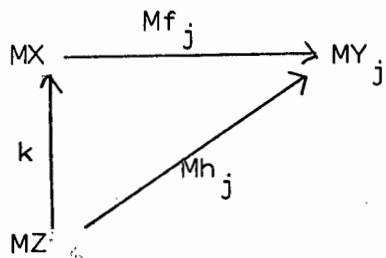
We observe that the faithfulness of  $M$  implies uniqueness of  $m$ , and implies that  $f_j \cdot m = h_j$ . Each class of  $\underline{B}$ -morphisms with common domain is a  $1_{\underline{B}}$ -initiality.

The dual concepts are M-cointinality (at  $X$ ) and simple M-cointinality.

We shall always set out the diagrams for the situation of definition 1.1.1 as in figure 1:



(upper diagram)



(lower diagram)

figure 1

Indeed, we like to think of  $\underline{B}$  as the "upper",  $\underline{C}$  as the "lower" category.

1.1.2. Proposition. If  $\{f_j : j \in J\}$  is an M-initiality and if  $\{f'_j : j \in J\}$  is an equivalent family in the sense that there are isomorphisms  $t, t_j$  such that  $f'_j t = t_j f_j$ , then  $\{f'_j : j \in J\}$  is an M-initiality.

□

1.1.3. Definition. Given objects  $Y_j$  in  $\underline{B}$  and  $\underline{C}$ -morphisms  $g_j : C \rightarrow MY_j$  ( $j \in J$ ). We shall say that the  $\underline{B}$ -object  $X$  is M-initial (over C) for the  $g_j$  to the  $Y_j$  if there exist  $f_j : X \rightarrow Y_j$  such that  $Mf_j = g_j$  and  $\{f_j : j \in J\}$  is an M-initiality at  $X$ . Occasionally it will be economic to say:  $X$  is a solution to the M-initiality problem  $(C, g_j, Y_j : j \in J)$ . If the index class  $J$  is non-void, so that  $C$  is determined by the  $g_j$ , we take the liberty of abbreviating this last notation

to  $(g_j, Y_j : j \in J)$ . (Dual : X is M-coinitial for the  $g_j$  from the  $Y_j$ .)

1.1.4. Remarks. (a) It is very important to note that in the phrase "X is M-initial for the  $g_j$  to the  $Y_j$ ", X and the  $Y_j$  are B-objects whereas the  $g_j$  are C-morphisms. Thus the words "to the  $Y_j$ " should not mislead one to think of  $Y_j$  as codomain of  $g_j$ ; to guard against this, the reader may replace our "to" mentally by the lengthier "with respect to".

(b) It would be in line with the usage on limits to call  $(X, (f_j : j \in J))$  a solution to the M-initiality problem. However, in this context the  $f_j$  are determined by the other data, and our convention seems on the whole more convenient for our purposes.

(c) In the phrase "X is M-initial over C for the  $g_j$  to the  $Y_j$ " we may and do omit the object C when the index class J is non-void. However, when J is void, we shorten the phrase to "X is M-initial over C", which means : For every  $Z \in \text{obj } \underline{B}$  and every  $k : MZ \rightarrow C$ , there exists  $m : Z \rightarrow X$  with  $Mm = k$ . We then also say : X is a solution to the M-initiality problem (C). Thus when we speak of a void M-initiality problem, we do require that its source C (or MX) be prescribed. By grace of this restriction, the following proposition remains valid for void M-initiality problems.

1.1.5. Proposition. If X and X' are solutions to the same M-initiality problem, there exists a unique isomorphism

$h : X' \rightarrow X$  such that  $Mh = 1$ .

□

1.1.6. Definition. The faithful functor  $M : \underline{B} \rightarrow \underline{C}$  is amnesitic if, whenever  $h$  is an isomorphism in  $\underline{B}$  with  $Mh = 1$ , then  $h = 1$ .

We observe that amnesiticity is a self-dual notion, closed under composition and natural equivalence, and true of any identity functor.

If  $M$  is amnesitic, then by 1.1.5. each  $M$ -initiality problem has at most one solution, and then the following notation is permissible:

1.1.7. Definition. The notation

$$X = \text{in}_M (C, g_j, Y_j : j \in J)$$

means that  $X$  is  $M$ -initial over  $C$  for the  $g_j$  to the  $Y_j$ . If  $J$  is non-void, we may briefly write  $X = \text{in}_M (g_j, Y_j : j \in J)$ .

No confusion will result from using this notation when  $M$  is not amnesitic.

1.1.8. Definition. The relation  $\leq_M$  among objects of  $\underline{B}$  is given by:  $X \leq_M Y$  (read :  $X$  is  $M$ -coarser than  $Y$ ) if and only if there exists a morphism  $f : Y \rightarrow X$  with  $Mf = 1$ . (Dual :  $X \supseteq_M Y$  —  $X$  is  $M$ -finer than  $Y$ ).

Habit and the set-theoretic inclusion between two topologies are to blame for the above notation. The Prague school [Čech, Hušek] has the ordering sign the opposite way.

1.1.9. Proposition. The faithful functor  $M : \underline{B} \rightarrow \underline{C}$  is amnesitic if and only if  $X \leq_M Y$  and  $Y \leq_M X$  always implies  $X = Y$ .

□

1.1.10. Associativity Theorem [Čech 1966, p.876]

Let  $X = \text{in}_M (C, g_j, Y_j : j \in J)$  and, for each  $j \in J$ , let  $Y_j = \text{in}_M (D_j, h_{jk}, Z_k : k \in K_j)$ . Then  $X = \text{in}_M (C, h_{jk}g_j, Z_k : j \in J, k \in K_j)$ .

Proof. This is straightforward from figure 2 when all index classes are non-void.

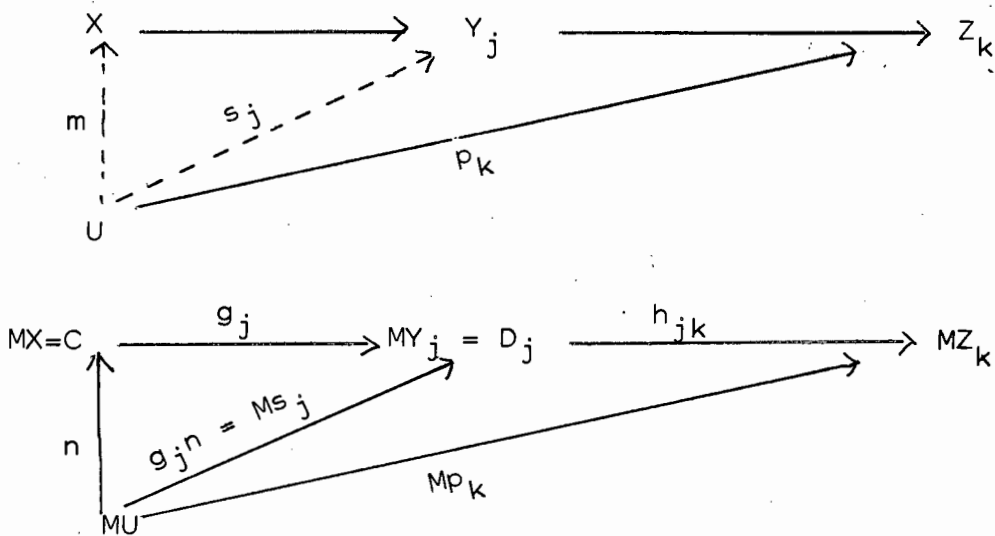


figure 2

The proof is trivial when  $J$  is void. If  $J \neq \emptyset$  and each  $K_j = \emptyset$ , it is easy to see that  $X = \text{in}_M (C)$ . Finally, with

$J \neq \emptyset$ , let  $J = J' \cup J''$  where  $K_j \neq \emptyset$  for  $j \in J'$ ,  $K_j = \emptyset$  for  $j \in J''$ ; then one readily sees that  $X = \text{in}_M(C, h_{jk}g_j, Z_k : j \in J', k \in K_j)$ .

□

1.1.11. Proposition : Augmenting an initiality

Let  $\{f_k : k \in K\}$  be a family of  $\underline{B}$ -morphisms with domain  $X$ . If  $J \subset K$  and  $\{f_j : j \in J\}$  is an  $M$ -initiality at  $X$ , then  $\{f_k : k \in K\}$  is an  $M$ -initiality at  $X$ .

□

It is worth observing why the following apparently similar proposition has a markedly different outcome. In the one proposition, morphisms in the upper category are given; in the other, they are given in the lower category.

1.1.12. Proposition : Decreasing an initiality

Let  $X$  be  $M$ -initial (over  $C$ ) for the  $g_k$  to the  $Y_k$  ( $k \in K$ ). Let  $J \subset K$ , and let  $X'$  be  $M$ -initial (over  $C$ ) for the  $g_k$  to the  $Y_k$  ( $k \in J$ ). Then  $X' \leq_M X$ .

□

1.1.13. Remarks (1) The definition of  $M$ -initiality can be generalized as follows. Let  $\underline{C}'$  be a subcategory of  $\underline{C}$ . The morphisms  $f_j : X \rightarrow Y_j$  form an  $(M, \underline{C}')$ -initiality at  $X$  if, whenever  $h_j : Z \rightarrow Y_j$  and  $k \in \underline{C}'(MZ, MX)$  are given with  $Mf_j \cdot k = Mh_j$ , there exists  $m : Z \rightarrow X$  with  $Mm = k$ . This is the concept used by Hušek [1966, 1967a, 1967b] under the name

$\langle M, \underline{C}' \rangle$  -projective generation. When  $\underline{C}' = \underline{C}$ , this is M-initiality; the other extreme case is when  $\underline{C}'$  consists of just the identity morphisms of  $\underline{C}$  (write  $\underline{C}' = \text{obj } \underline{C}$ ). In [Čech, 1966],  $(M, \text{obj } \underline{C})$ -initiality is termed M-projective generation, while M-initiality is termed strong M-projective generation. Certainly,  $(M, \text{obj } \underline{C})$ -initiality is a concept very ready to hand in Top or Un, if M forgets to Ens. Fortunately, in these examples  $(M, \text{obj } \underline{C})$ -initiality and M-initiality coincide, but there are examples [Hušek 1967b, p. 128] where they do not coincide.

M-initiality has given us the associativity theorem 1.1.10, and it bears straightforward relations to limits (section 1.2) and embeddings (section 1.3). These benefits we cannot claim, at least not so easily, from  $(M, \underline{C}')$ - or  $(M, \text{obj } \underline{C})$ -initiality. Considering also that M-initiality gives an interesting theory of great clarity and of sufficient generality to cover the applications we have in mind, we shall henceforth discuss only M-initiality.

(2) Our concept of M-initiality agrees, modulo terminology, with those of [Antoine 1962], [Bourbaki 1957], [Brümmer 1969a, 1969c], [Roberts 1968], [Sonner 1965], and [Taylor 1965]. The name amnesic comes from [Čech 1966], and the concept is used by [Antoine 1962]. An explicit treatment of void initiality problems does not seem to occur in the literature. We shall find void problems forced on us in sections 1.2 and 1.4.

## 1.2 INITIALITY AND LIMITS.

The self-dual concept of M-initiality completeness is introduced and related to limits and completeness.

Throughout this section  $M : \underline{B} \rightarrow \underline{C}$  will be a faithful functor.

1.2.1. Definition The category  $\underline{B}$  is M-initiality complete if every M-initiality problem has a solution (in  $\underline{B}$ , of course). (Void M-initiality problems in the sense of remark 1.1.4(c) are admitted.)

This concept is self-dual:

1.2.2. Theorem The category  $\underline{B}$  is M-initiality complete if and only if every M-cointinality problem has a solution.

First Proof [Roberts 1968, proposition 4.6].



The first proof employs universes, which is not necessary here:

Second Proof Let  $\underline{B}$  be M-initiality complete. Let objects  $Y_j$  of  $\underline{B}$  and  $\underline{C}$ -morphisms  $g_j : MY_j \rightarrow C$  ( $j \in J$ ), with  $J \neq \emptyset$ , be given. Thus a non-void M-cointinality problem is posed (figure 1). For each object  $Z$  of  $\underline{B}$  we define a class  $\underline{A}(Z)$  of

C-morphisms as follows:

$k \in \underline{A}(Z)$  iff  $k : C \rightarrow MZ$  and there exists a function  $h^k : J \rightarrow \underline{B}$  such that, for each  $j \in J$ ,  $h^k(j) : Y_j \rightarrow Z$  and  $kg_j = Mh^k(j)$ .

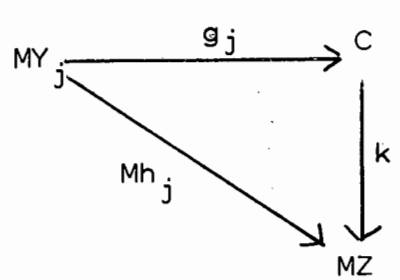
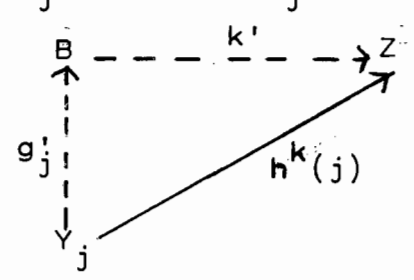
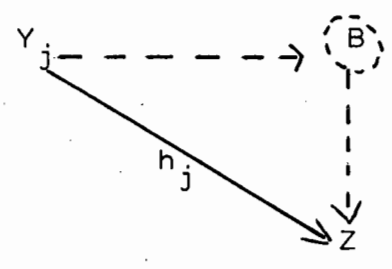


figure 1

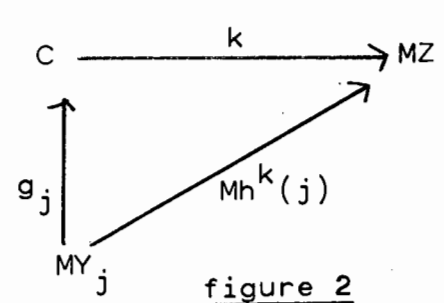


figure 2

Let  $B = \text{in}_M(C, k, Z : Z \in \text{obj } \underline{B}, k \in \underline{A}(Z))$ .

B exists by hypothesis, even though  $\underline{A}(Z)$  may be void. For each  $k \in \underline{A}(Z)$  there is then  $k' : B \rightarrow Z$  with  $Mk' = k$ . Also, thinking of  $j$  as fixed and  $k$  ranging through  $\underline{A}(Z)$ ,  $Z$  ranging through  $\text{obj } \underline{B}$ , we see (figure 2) that there is  $g'_j : Y_j \rightarrow B$  with  $Mg'_j = g_j$ . Now, given the commutative lower diagram in figure 1, we complete its upper diagram with  $g'_j$  and  $k'$ , and the M-cointiality problem is solved. If we start with a void M-cointiality problem over C, the above procedure applies in appropriately simplified form. Finally, the converse implication follows by duality. □

The above theorem was also stated, without proof, for the special case  $\underline{C} = \underline{\text{Ens}}$ , by Antoine [1962 proposition 1].

Our term "M-initiality complete" is close to the term used by Antoine. Roberts [1968] expresses the same concept by calling M an initial functor.

1.2.3. Example. Each of the categories Top, Crg, 2Top, Un, Qun, Prox, Qprox is M-initiality complete, where M is in each case the forgetful functor to Ens. (The first five are obvious. For the others see [Hušek 1967b, p.128] or [Alfsen and Fenstad 1959, p.359] and [Salbany 1970, p.84].) Further, any category B is  $1_{\underline{B}}$ -initiality complete.

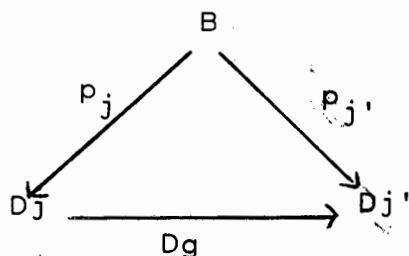
□

1.2.4. Theorem. Let J be a small category. Consider a diagram  $D : \underline{J} \rightarrow \underline{B}$  and let  $(C, q_j)$  be limit of MD. For any object B of B we have:

$B = \text{in}_M (C, q_j, D_j)$  if and only if  $MB = C$  and there exist  $p_j : B \rightarrow D_j$  with  $Mp_j = q_j$  such that  $(B, p_j)$  is limit of D.

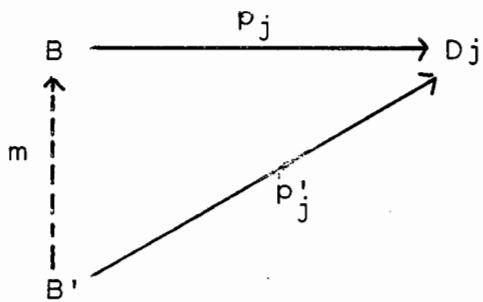
Proof It is understood that the  $j$  range through obj J.

Assume that  $B = \text{in}_M (C, q_j, D_j)$ . Thus there are  $p_j : B \rightarrow D_j$  with  $Mp_j = q_j$ , and hence  $MB = C$ . For any  $g : j \rightarrow j'$  the diagram



commutes because its image under the faithful M commutes. Thus  $(B, p_j)$  is a lower bound of D. Let  $(B', p'_j)$  be any lower

bound of D. Then  $(MB', Mp'_j)$  is lower bound of MD, and thus



there is unique  $k : MB' \rightarrow C$  making the lower diagram in figure 3 commute. By the initiality of B it follows that there is  $m : B' \rightarrow B$  with  $Mm = k$ , and then  $p_j m = p'_j$ . If also  $p_j m' = p'_j$  then the uniqueness of  $k$  in the lower diagram would imply  $Mm' = k$ , whence by the faithfulness of  $M$ ,  $m' = m$ .

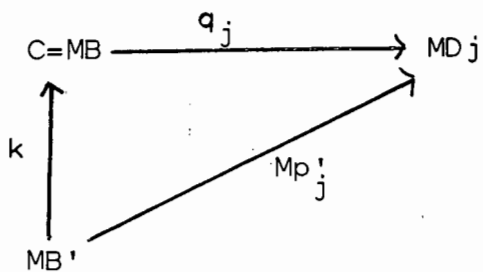
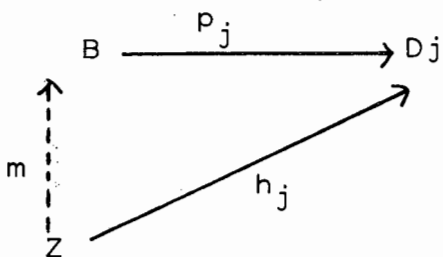


figure 3

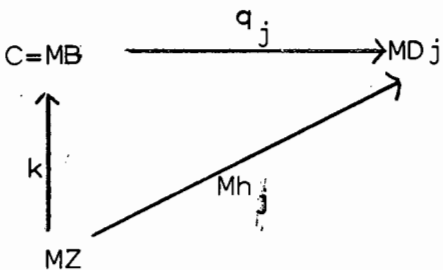
Hence  $(B, p_j)$  is limit of D.

Conversely, let  $(B, p_j)$  be limit of D, with  $Mp_j = q_j$  and

$MB = C$ . To prove that  $B = \text{in}_M (C, q_j, Dj)$ , we have to consider



$h_j : Z \rightarrow Dj$  and  $k : MZ \rightarrow C$  such that the lower diagram in figure 4 commutes. We claim that  $(Z, h_j)$  is lower bound of D.



Given  $g : j \rightarrow j'$  then, as  $(C, q_j)$

is limit of MD, we have

$$MDg \cdot q_j = q_{j'}$$

Hence

$$\begin{aligned} M(Dg \cdot h_j) &= MDg \cdot Mh_j = MDg \cdot q_j \cdot k = \\ &= q_{j'} \cdot k = Mh_{j'}, \text{ and thus, as } M \text{ is faith-} \\ &\text{ful, } Dg \cdot h_j = h_{j'}. \text{ Thus } (Z, h_j) \end{aligned}$$

figure 4

is lower bound of D, and there is unique  $m : Z \rightarrow B$  with  $p_j m = h_j$ .

Hence  $q_j Mm = Mh_j$ . But  $(MZ, Mh_j)$  is lower bound of MD, so  $k$  is

unique in  $q_j k = Mh_j$ . Thus  $Mm = k$  and  $B = \text{in}_M (C, q_j, D_j)$ .

In case  $\underline{J}$  is void,  $C$  is a terminal object of  $\underline{C}$ . It is easy to see that  $B = \text{in}_M (C)$  if and only if  $MB = C$  and  $B$  is a terminal object of  $\underline{B}$ . This is the sense to which the theorem reduces when  $\underline{J}$  is void.

□

1.2.5. Corollary. Let  $\underline{B}$  be  $M$ -initiality complete. Let  $\underline{J}$  be a small category. If  $\underline{C}$  has  $\underline{J}$ -limits, then  $\underline{B}$  has  $\underline{J}$ -limits and  $M$  preserves them. If  $\underline{C}$  has  $\underline{J}$ -colimits, then  $\underline{B}$  has  $\underline{J}$ -colimits and  $M$  preserves them.

Proof. Given a diagram  $D : \underline{J} \rightarrow \underline{B}$ , and let  $\underline{C}$  have  $\underline{J}$ -limits. Then  $MD$  has limit  $(C, q_j)$ . Letting  $B = \text{in}_M (C, q_j, D_j)$ , we have by theorem 1.2.4 a limit  $(B, p_j)$  of  $D$ , with  $Mp_j = q_j$ . To see that  $M$  preserves  $\underline{J}$ -limits, consider any given limit  $(Z, s_j)$  of  $D$ . There is an isomorphism  $t : Z \rightarrow B$  with  $p_j t = s_j$ . Then  $Mt$  is an isomorphism with  $Mp_j \cdot Mt = q_j Mt = Ms_j$ , whence  $(MZ, Ms_j)$  is limit of  $MD$ .

The dual statement follows from theorem 1.2.2.

□

1.2.6. Corollary Let  $\underline{B}$  be  $M$ -initiality complete. If  $\underline{C}$  is complete, then  $\underline{B}$  is complete and  $M$  preserves limits. If  $\underline{C}$  is cocomplete, then  $\underline{B}$  is cocomplete and  $M$  preserves colimits.

□

Theorem 1.2.4 and its two corollaries may well be generally known: Corollary 1.2.6 was stated without proof for the case  $\underline{C} = \underline{Ens}$  by Antoine [1966, proposition 2]. The special case of theorem 1.2.4 for products was stated in [Čech 1966, p.876], and essentially the form given here was announced in [Hušek 1966, theorem 3].

The following result, however, seems to be new. It has an illuminating application in the next section (1.3.13).

1.2.7. Proposition Each equalizer in  $\underline{B}$  is a simple  $M$ -initiality if one of the following two conditions is satisfied:

- (1)  $M$  sends equalizers to monomorphisms
- (2)  $\underline{B}$  is  $M$ -initiality complete and  $\underline{C}$  has equalizers.

Proof Let  $(B, d)$  be equalizer of  $f, g : X \rightarrow Y$  in  $\underline{B}$ . Given  $h : Z \rightarrow X$  and  $k : MZ \rightarrow MB$  with  $Md.k = Mh$ , we have to find  $m : Z \rightarrow B$  with  $Mm = k$ . Now  $M(fh) = Mf.Md.k = Mg.Md.k = M(gh)$ , whence  $fh = gh$  and there is unique  $m : Z \rightarrow B$  with  $dm = h$ . Then  $Md.Mm = Mh = Md.k$ . Under either of the given conditions,  $Md$  is mono (use 1.2.5 in case (2)) and thus  $Mm = k$ .

□

1.2.8. Remark. Initiality completeness can be a very strong requirement. Let  $L$  be the forgetful functor  $\underline{Un} \rightarrow \underline{Crg}$ . It will be shown in 1.5.3 that  $\underline{Un}$  is not  $L$ -initiality complete. Yet  $\underline{Un}$  and  $\underline{Crg}$  are both complete and cocomplete (this follows

by applying corollary 1.2.6 to the forgetful functors to Ens, see 1.2.3) and  $L$  preserves limits. A great deal of our work in section 1.4 and later owes its applicability to the fact that though a category  $\underline{A}$  may not be  $L$ -initiality complete ( $L : \underline{A} \rightarrow \underline{B}$ ), there may be a functor  $M : \underline{B} \rightarrow \underline{C}$  such that  $\underline{A}$  is  $ML$ -initiality complete.

Clearly desired is a characterization of initiality completeness in terms of limits. There is a discrepancy between the convention of taking limits over small categories and our insistence on solving large initiality problems. We leave the matter here.

### 1.3 EMBEDDING, SEPARATION AND FACTORIZATION.

Throughout this section  $M : \underline{B} \rightarrow \underline{C}$  will be a faithful functor.

It is safe (because vague) enough to claim that "true" subobjects should correspond to the intuitive notion of embedding. Assuming some settlement of the "true" subobject problem in the lower category  $\underline{C}$ , one can use the faithful functor  $M$  to define "injections" and "embeddings" (to name but two types) in the upper category  $\underline{B}$ . This is a widespread idea in the theory of concrete or structured categories, cf. [Isbell 1957, 1958], [Semadeni 1963], [Sonner 1965], [Taylor 1965, p. 780], [Cech 1966, p.877], [Hušek 1966, definition 2], [Calenko 1969, p.502].

We shall start with rather rudimentary assumptions about "injections" and "surjections" in  $\underline{C}$ , and exhibit some basic properties of the consequent "injections", "embeddings" and their "duals" in  $\underline{B}$ .

Next we give an apparently new definition of M-separated objects in the category  $\underline{B}$ , and show that these become the  $T_0$ -spaces when  $M$  is the forgetful functor  $\underline{Top} \rightarrow \underline{Ens}$ ; we also isolate the property which can be regarded as dual to  $T_0$  in  $\underline{Top}$ .

Finally we assume that  $\underline{C}$  is an Isbell bicategory and obtain a standard factorization theorem for morphisms in  $\underline{B}$ ; we also find a reason why equalizers in  $\underline{B}$  are often embeddings.

We do not claim any canonicity for the assumption of bicategory structure on  $\underline{C}$ ; finer distinctions can be drawn for various purposes [Isbell 1957, 1958], or the initiality mechanism can be modified [Wyler 1971, b]; but what we do, suffices for our later applications. We do claim that our exposition shows that the concept of initiality deals with these matters in a simple and clear way.

1.3.1. Motivation. It is known [Taylor 1965] that the topological "embedding lemma" (as Kelley [1955] calls it) can be phrased in terms of initiality. I have found the following a very teachable analysis of this lemma.

Let  $E : \underline{Top} \rightarrow \underline{Ens}$  be the forgetful functor. Given mappings  $f_j : X \rightarrow Y_j$  ( $j \in J$ ) in  $\underline{Top}$ .

- (i) Let  $f : X \rightarrow \prod Y_j$  be the unique mapping given by  $p_j f = f_j$ , with the  $p_j$  the projections.
- (ii)  $f$  is an  $E$ -initiality if and only if  $\{f_j : j \in J\}$  is an  $E$ -initiality.
- (iii)  $f$  is an embedding if and only if  $f$  is an  $E$ -initiality and  $Ef$  an injection.
- (iv)  $f$  is an injection if and only if the  $f_j$  distinguish points.

- (v) If  $f$  is an E-initiality, then  $f$  sends sets open in  $X$  onto sets open in  $fX$  (as subspace of  $\prod Y_j$ ). The converse fails.
- (vii) The  $f_j$  distinguish points and closed sets if and only if the preimages of open sets under the  $f_j$  form a base for open sets of  $X$ . This condition is sufficient for  $f$  to be an E-initiality, but not necessary (only a subbase is necessary). [Brümmer 1968 a].

□

We are only concerned here with generalizing (i) (ii) (iii), which is almost merely a matter of definition. It is easy to generalize (iv) and seems feasible to generalize (v) (vi) along the lines of [Semadeni 1963, §5].

1.3.2. Conventions. We assume throughout this section that in  $\underline{C}$  there is given a class  $\underline{S}(\underline{C})$  of morphisms which we shall call surjections, as well as a class  $\underline{I}(\underline{C})$  of morphisms which we shall call injections, such that:

- (1)  $\underline{S}(\underline{C})$  consists of epimorphisms,  $\underline{I}(\underline{C})$  consists of monomorphisms, and both classes are closed under composition;
- (2)  $\underline{S}(\underline{C}) \cap \underline{I}(\underline{C})$  is the class of all isomorphisms in  $\underline{C}$ ;
- (3) in case  $\underline{C}$  is equivalent to Ens,  $\underline{S}(\underline{C})$  shall consist of all epimorphisms and  $\underline{I}(\underline{C})$  of all monomorphisms of  $\underline{C}$ .

We call  $\underline{C}$  (more precisely,  $\underline{C}$  equipped with  $\underline{S}(\underline{C})$  and  $\underline{I}(\underline{C})$ ) an Isbell bicategory ([Isbell 1957, 1958, 1963],

[Sonner 1965], [Calenko 1969] ) if, in addition, the following condition holds:

- (4) Each  $\underline{C}$ -morphism factors, uniquely up to isomorphism, into a member of  $\underline{S}(\underline{C})$  followed by a member of  $\underline{I}(\underline{C})$ .

1.3.3. Remark. For an Isbell bicategory the condition (3) is redundant. Indeed, any epimorphism  $e$  factors as  $e = js$  with  $s \in \underline{S}(\underline{C})$ ,  $j \in \underline{I}(\underline{C})$ ;  $j$  is then mono and epi, and supposing  $\underline{C}$  equivalent to  $\underline{Ens}$ , (2) implies that  $j \in \underline{S}(\underline{C})$ , and hence by (1),  $e \in \underline{S}(\underline{C})$ ; dually any monomorphism will be in  $\underline{I}(\underline{C})$ .

□

1.3.4. Definition. A morphism  $f$  in  $\underline{B}$  is an

M-injection iff  $Mf$  is an injection,

M-surjection iff  $Mf$  is a surjection,

M-embedding iff  $f$  is an M-initiality and M-injection,

M-quotient iff  $f$  is an M-coinitiality and M-surjection.

1.3.5. Proposition Each of the four classes:

M-injections, M-surjections, M-embeddings, M-quotients, is closed under composition and under equivalence (i.e. composition with isomorphisms). Each M-injection is a monomorphism, and each M-surjection is an epimorphism in  $\underline{B}$ .

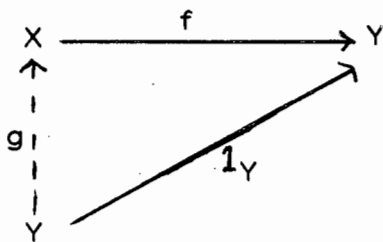
Proof The class of M-initialities is closed under equivalence by 1.1.2, and under composition by the associativity theorem 1.1.10.

□

1.3.6. Proposition The following conditions on a morphism  $f$  in  $\underline{B}$  are equivalent:

- (a)  $f$  is an isomorphism
- (b)  $f$  is an  $M$ -surjective  $M$ -embedding
- (c)  $f$  is an  $M$ -injective  $M$ -quotient.

Proof. By 1.3.2(2), (a) implies (b) and (c).



Let (b) hold, then  $Mf$  is iso. Initiality gives  $g : Y \rightarrow X$  with  $Mg = (Mf)^{-1}$  and then  $fg = 1_Y$ .

$$\begin{aligned} M(gf) &= Mg.Mf = (Mf)^{-1}.Mf \\ &= 1_{MX} \end{aligned}$$

Hence  $gf = 1_X$ . Thus (a) follows.

Dually, (c) implies (a).

□

We now reach the promised generalization of 1.3.1 (i)(ii)(iii):

1.3.7. Proposition Given  $\underline{B}$ -morphisms  $f_j : X \rightarrow Y_j$  ( $j \in J$ ), let the product  $(\prod Y_j, p_j)$  exist, and let  $M$  preserve this product.

- (i) Let  $f : X \rightarrow \prod Y_j$  be the unique morphism with  $p_j f = f_j$  ( $j \in J$ ).
- (ii)  $f$  is an  $M$ -initiality if and only if  $\{f_j : j \in J\}$  is an  $M$ -initiality.
- (iii)  $f$  is an  $M$ -embedding if and only if  $f$  is an  $M$ -initiality and  $Mf$  an injection.

Proof (i) and (iii) are matters of definition and (ii) is essentially the special case of theorem 1.2.4 for products.



1.3.8. Definition An object  $X$  of  $\underline{B}$  is  $M$ -separated if every simple  $M$ -initiality at  $X$  is an  $M$ -embedding.  $X$  is  $M$ -coseparated if every simple  $M$ -cointinality at  $X$  is an  $M$ -quotient.

This definition and the following result seem to be new. The approach differs from the way in which [Sharpe, Beattie, Marsden 1966] generalized certain lower separation axioms.

1.3.9. Theorem Let  $E : \underline{Top} \rightarrow \underline{Ens}$  be the forgetful functor.

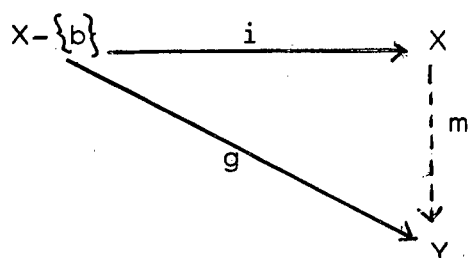
- (i) The  $E$ -separated objects in  $\underline{Top}$  are precisely the  $T_0$ -spaces.
- (ii) The  $E$ -coseparated objects in  $\underline{Top}$  are precisely the spaces without closed isolated points. (A one-point space is regarded as having no isolated point.)

Proof (i) The  $T_0$ -reflection  $sX$  of a topological space  $X$  can be obtained by taking the quotient space with respect to the relation:  $x \equiv y$  iff  $x$  and  $y$  have the same neighbourhoods.

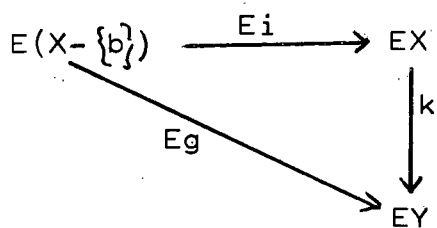
It is then clear that the quotient mapping  $s_X : X \rightarrow sX$  is an  $E$ -initiality. Therefore, if  $X$  is  $E$ -separated,  $s_X$  is an  $E$ -embedding onto and thus a homeomorphism, so that  $X$  is a  $T_0$ -space. Conversely, let  $f$  be a simple  $E$ -initiality at the  $T_0$ -space  $X$ . The open sets of  $X$  are preimages of open

sets, and therefore  $f$  distinguishes points.

(ii) Consider a non-surjective  $E$ -cointinality  $f : Z \rightarrow X$ . We have some  $b \in X - fZ$  and claim  $b$  is a closed isolated point. If we refine the topology of  $X$  by adding  $\{b\}$  and  $X - \{b\}$  as new subbasic open sets, the relative topology on  $fZ$  is unaltered, so that  $f$  remains continuous to the new space. As  $X$  has the finest topology making  $f$  continuous,  $\{b\}$  and  $X - \{b\}$  are open sets of  $X$ , i.e.  $b$  is a closed isolated point. Conversely, given  $X$  with at least one closed isolated point  $b$ , then by agreement  $X - \{b\} \neq \emptyset$ .



The inclusion mapping  $i : X - \{b\} \rightarrow X$  is non-surjective and an  $E$ -cointinality. Indeed, given  $k$  and  $g$  such that  $k \cdot Ei = Eg$ , then there is  $m : X \rightarrow Y$  with  $Em = k$  because  $k$  is continuous at the isolated point  $b$  and its restriction to the open set  $X - \{b\}$  coincides with  $g$ .



□

1.3.10 Remarks. (1) We shall see in section 3.

that, in all the categories we discuss there, the (forgetful functor to Ens)-separated objects are precisely the spaces with  $T_0$ -topology or its natural substitute.

(2) The conjecture that under mild conditions the  $M$ -separated objects will form a reflective subcategory of  $\underline{B}$ , is quickly subdued by the example  $E^{\text{op}} : \underline{\text{Top}}^{\text{op}} \rightarrow \underline{\text{Ens}}^{\text{op}}$ . The  $E^{\text{op}}$ -separated objects do not form a reflective subcategory of  $\underline{\text{Top}}^{\text{op}}$  as it is clear from 1.3.9 (ii) that they do not

form a coreflective subcategory of Top.

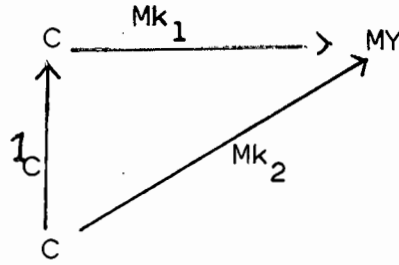
(3) It is natural to ask whether other separation properties in Top, e.g.  $T_1$ ,  $T_2$ ,  $T_3$ ,  $T_{3a}$ , can be represented as K-separation for suitable functors K. This question leads to the following observation. Consider any category D between Top and Ens in the sense that the forgetful functor  $E : \text{Top} \rightarrow \text{Ens}$  decomposes into two amnestic functors  $K : \text{Top} \rightarrow \underline{D}$ ,  $L : \underline{D} \rightarrow \text{Ens}$ . (For example, any coreflective or monoreflective subcategory of Top is between Top and Ens.)

Then each K-separated object of Top is E-separated; i.e. K-separation implies  $T_0$ . (Proof: Each E-initiality is a K-initiality. The K-injections are defined in the obvious way :  $f$  is a K-injection iff  $Kf$  is an L-injection. Thus the K-injections are precisely the E-injections, i.e. the one-to-one mappings. It is immaterial that we do not here apply all the conventions 1.3.2 to D). I have to leave this question with the following conjecture: For any epireflective subcategory S of Top between the subcategory of the  $T_0$ -spaces and that of the Tihonov spaces, there is a decomposition of the forgetful functor  $E : \text{Top} \rightarrow \text{Ens}$  into two amnestic functors, K followed by L, such that the objects of S are precisely the K-separated objects of Top.

The following factorization result generalizes a well-known situation in topology. Isbell [1964, p. 15] gives it for uniform spaces.



triangle



commutes, there is  $n : B_2 \rightarrow B_1$  with  $Mn = 1_C$ , and then  $k_1 n = k_2$ . Dually, we have  $n' : B_2 \rightarrow B_1$  (sic!) with  $Mn' = 1_C$  and  $n' m_2 = m_1$ . As  $n, n' : B_2 \rightarrow B_1$  with  $Mn = Mn'$ , we have  $n = n'$ . Finally the uniqueness of  $n$  in the commutative diagram (3) immediately follows from the fact that  $k_1$  is mono and  $m_2$  is epi.

□

1.3.12. Corollary. Let  $\underline{B}$  be  $M$ -initiality complete and let  $\underline{C}$  be an Isbell bicategory. Then each  $f : X \rightarrow Y$  in  $\underline{B}$  has factorizations of the forms (1) and (2) above, each unique up to isomorphism. Further, given factorizations (1) and (2) of  $f$ , there exists a unique  $n : B_2 \rightarrow B_1$  making the diagram (3) commute, and then  $Mn$  is an isomorphism.

□

Finally we use a result from the previous section to shed light on the fact that an equalizer is very often an embedding.

1.3.13. Proposition. Either of the following two conditions is sufficient for each equalizer in  $\underline{B}$  to be an  $M$ -embedding :

- (a)  $M$  sends equalizers to injections
- (b)  $\underline{B}$  is  $M$ -initiality complete and  $\underline{C}$  is an Isbell bicategory having equalizers.

Proof Apply corollary 1.2.5. to the definition of M-embedding. In case (b), observe that each equalizer in an Isbell bicategory is an injection [Calenko 1969, p. 498].

□

1.3.14. Remark Certain results in this section did not require the factorization condition 1.3.2(4). It is therefore worth observing that, if we only require the conditions 1.3.2 (1)(2)(3) to be satisfied, it is possible to define "species"  $\underline{I}$  and  $\underline{S}$  of morphisms for all categories  $\underline{C}$ ; e.g. let

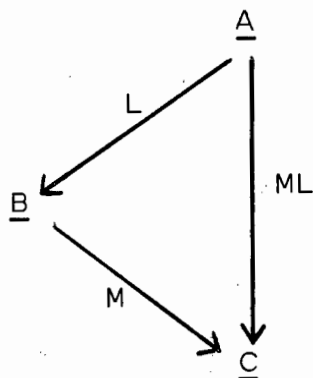
$$\underline{I}(\underline{C}) = \{\text{all } \underline{C}\text{-monomorphisms which are iso or not epi}\}$$

$$\underline{S}(\underline{C}) = \{\text{all } \underline{C}\text{-epimorphisms which are iso or not mono}\}.$$

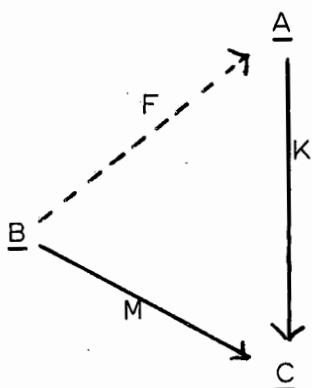
In this example,  $\underline{I}$  and  $\underline{S}$  are dual to each other and contain, respectively, the species of the extreme monomorphisms and of the extreme epimorphisms. It may be interesting to seek still wider species  $\underline{I}$  and  $\underline{S}$ , dual to each other, satisfying 1.3.2.(1)(2)(3) for all  $\underline{C}$ .

#### 1.4 RIGHT INVERSES OF AMNESTIC FUNCTORS.

The main purpose of this section is to give several characterizations of the right inverses of an amnestic functor  $L : \underline{A} \rightarrow \underline{B}$ , in terms of initiality. For reasons touched upon in remark 1.2.8 and to become clear in the next section, it is advantageous to consider also an amnestic functor  $M : \underline{B} \rightarrow \underline{C}$  and work with  $ML$ -initiality (and later also with  $M$ -initiality). Thus we may start with the situation



Clearly, if  $F : \underline{B} \rightarrow \underline{A}$  is a right inverse of  $L$  (i.e.  $LF = \mathbf{1}_{\underline{B}}$ ), then  $MLF = M$ . We shall become interested in the class of functors  $F$  for which  $MLF = M$ . Obviously, for this class  $M$  and  $ML$  are relevant, but  $L$  itself is irrelevant. Thus we shall gain generality, with no extra effort, by studying at first the functors  $F : \underline{B} \rightarrow \underline{A}$  for which the triangle



commutes. There is a basic construction which characterizes such  $F$ , as follows. Given any subclass  $\underline{V}$  of  $\text{obj } \underline{B}$  and any function  $w : \underline{V} \rightarrow \text{obj } \underline{A}$  such that the expression

$$FX = \text{in}_K (Mf, wB : B \in \underline{V}, f \in \underline{B}(X, B))$$

exist for each object  $X$  of  $\underline{B}$ ; then  $F$ , thus defined, can be uniquely extended to a functor  $\underline{B} \rightarrow \underline{A}$  satisfying  $KF = M$ . Conversely, any  $F : \underline{B} \rightarrow \underline{A}$  for which  $KF = M$  can be obtained in this way (proposition 1.4.4).

Having done that, we have to assume  $K = ML$  in order to proceed to our objectives. First we give a notation to the above construction:

$$F = (L, M : w).$$

Thus the functors of this form are precisely those that satisfy  $MLF = M$ . Next we characterize those  $w$  for which  $F$  is a right  $M$ -subinverse of  $L$  - (that is,  $LF \leq_M 1_{\underline{B}}$ , which means  $LFx \leq_M x$  for each  $x$  in  $\text{obj } \underline{B}$ ). It is also necessary from here on to impose a certain preservation condition on  $L$ . The following stage of narrowing down the class of functors introduces a more unexpected procedure : we consider a type of function  $w$  which is determined in a special way by a subclass  $\underline{U}$  of  $\text{obj } \underline{A}$ , and we set down the notation  $w = (\underline{U})$  (definition 1.4.17). It then turns out that  $F$  is of the form  $(L, M : (\underline{U}))$  if and only if  $FLF = F$  and  $LF \leq_M 1_{\underline{B}}$ . Then we look for a condition on  $\underline{U}$  in order that  $(L, M : (\underline{U}))$  be a right inverse of  $L$ . The necessary and sufficient condition is that the objects  $LU$ , with  $U \in \underline{U}$ , be sufficiently plentiful so that each object of  $\underline{B}$  is  $M$ -initial for its morphisms into these objects. We name the

condition :  $LU$  is  $M$ -sufficient in  $B$ . Thus the right inverses of  $L$  are characterized. Finally, we obtain certain interesting modifications of the characterization theorem under additional assumptions; in particular we express the  $M$ -sufficiency of  $LU$  in terms of the notion of  $M$ -embedding from section 1.3, thereby making intensive use of the theory of that section.

Lest the above described excursion be suspected of being an exercise for its own sake, we promise here that each of the four successive layers of generality of the functor  $(L, M : w)$  will find its applications in later sections. It will be helpful to think of  $A$ ,  $B$ ,  $C$  as  $Un$ ,  $Crg$ ,  $Ens$  respectively, with  $L$  and  $M$  the forgetful functors.

The restriction to amnestic (rather than faithful) functors is for convenience and clarity. The restriction can be dropped at the cost of almost everywhere replacing equalities by natural equivalences, but no essential gain in information would result. It is no great concern here that one encounters in the literature many "forgetful" functors which fail to be amnestic : for example the various "operations" of [Császár 1963]. Indeed in this example one can make all the "operations" amnestic by identifying equally coarse objects. If a man seems driven by reasons other than convenience to distinguish between equally coarse objects, one can usually suggest that he should really be dealing with a category "above" the one he professes to study (and that if he takes this step, his forgetful functors will become amnestic).

1.4.1. Definition Given functors  $F, G : \underline{D} \rightarrow \underline{B}$  and  $M : \underline{B} \rightarrow \underline{C}$  we write  $F \leq_M G$  (and say  $F$  is M-coarser than  $G$ ,  $G$  is M-finer than  $F$ ) if there is a natural transformation  $t : G \rightarrow F$  such that  $Mt$  is the identity natural transformation  $MF = MG$ .

We use the same symbol  $\leq_M$  for an ordering of functors here, and of objects in definition 1.1.8; no confusion need arise, since the two orderings are related as follows.

1.4.2. Proposition In the situation of definition 1.4.1, let  $M$  be faithful. Then  $F \leq_M G$  if and only if  $FX \leq_M GX$  for each object  $X$  of  $\underline{D}$ .

□

1.4.3. Proposition Given functors  $F, G : \underline{D} \rightarrow \underline{B}$  and  $M : \underline{B} \rightarrow \underline{C}$ . Let  $F \leq_M G$  and  $G \leq_M F$ . Then  $F$  and  $G$  are naturally equivalent. If further  $M$  is amnestic, then  $F = G$ .

□

1.4.4. Proposition Given two amnestic functors  $K : \underline{A} \rightarrow \underline{C}$ ,  $M : \underline{B} \rightarrow \underline{C}$ .

(i) Given a class  $\underline{V} \subset \text{obj } \underline{B}$  and a function  $w : \underline{V} \rightarrow \text{obj } \underline{A}$  such that for each object  $X$  of  $\underline{B}$ , the  $K$ -initiality problem

$$(Mf, wB : B \in \underline{V}, f \in \underline{B}(X, B))$$

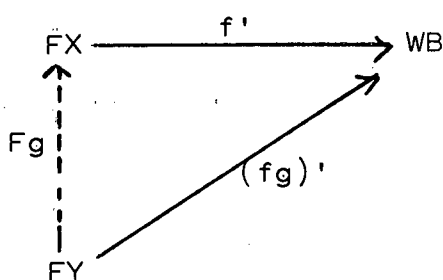
has a solution  $FX$  (necessarily unique). Then the correspondence  $X \rightsquigarrow FX$  can be uniquely extended to a functor  $F : \underline{B} \rightarrow \underline{A}$  satisfying  $KF = M$ .

(ii) Each functor  $F : \underline{B} \rightarrow \underline{A}$  with  $KF = M$  can be obtained as in (i) above.

Remark It is possible for the  $K$ -initiality problem in (i) to be void for certain  $X$  (of course not when  $X \in \underline{V}$ ). When the problem is void, it is necessary to mention its source (see remark 1.1.4(c)). Here the notation tells us that the source,  $\text{dom } Mf$ , is  $MX$  since  $f \in \underline{B}(X, B)$ . Even if there is no  $f$ , we agree that  $MX$  is the source.

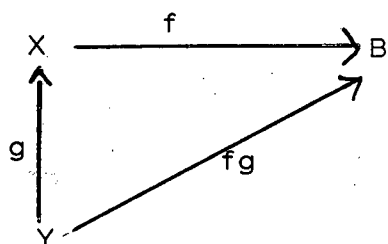
Proof of 1.4.4. (i) We have

$$FX = \text{in}_K (Mf, w_B : B \in \underline{V}, f \in \underline{B}(X, B)).$$



Hence  $KFX = MX$  (see above remark).

For each  $f : X \rightarrow B$  with  $B \in \underline{V}$ , there exists unique  $f' : FX \rightarrow w_B$  with  $Kf' = Mf$ . Hence also  $Kw_B = MB$  (even if the problem is void).



Further, given  $g : Y \rightarrow X$ , there is unique  $(fg)' : FY \rightarrow w_B$  with  $K(fg)' = M(fg)$ . As the lower

diagram in figure 1 commutes, there is unique  $Fg : FY \rightarrow FX$  with

$K(Fg) = Mg$ . We show that  $F$  thus extended to morphisms is a functor.

Consider also  $h : Z \rightarrow Y$ . Then  $K(F(gh)) = M(gh) = Mg.Mh =$

$= K(Fg).K(Fh) = K(Fg.Fh)$  whence by

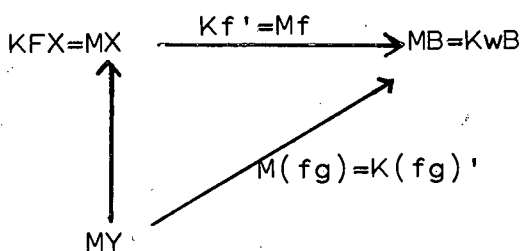
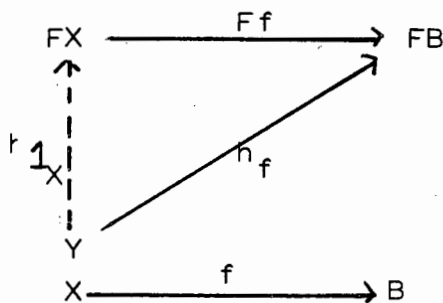


figure 1

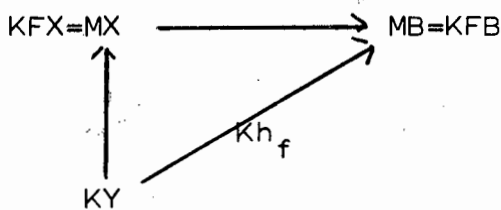
the faithfulness of  $K$ ,  $F(gh) = Fg.Fh$ . Likewise  $K(F1_X) = M1_X = 1_{MX} = 1_{KFX} = K(1_{FX})$ , so  $F1_X = 1_{FX}$ . Thus  $F : \underline{B} \rightarrow \underline{A}$  is a functor with  $KF = M$ . If  $F' : \underline{B} \rightarrow \underline{A}$  is also a functor with  $KF' = M$  and  $F'X = FX$  for each object  $X$  of  $\underline{B}$ , then it follows at once from the faithfulness of  $K$  that  $F' = F$ .

(ii) Given  $F : \underline{B} \rightarrow \underline{A}$  with  $KF = M$ . We take  $\underline{V} = \text{obj } \underline{B}$  and



$w = F|_{\underline{V}}$ . For each  $f$  in  $\underline{B}(X, B)$  the morphism  $Ff : FX \rightarrow FB$  is such that  $K(Ff) = Mf$ .

If morphisms  $h_f : Y \rightarrow FB$  and  $k : KY \rightarrow MX$  are given such that the lower triangle in figure 2 commutes for each  $f$ , then as  $1_X$



is one of the considered  $f$ , we have  $K(h_{1_X}) = k$ . Thus

figure 2

$FX = \text{in}_K (Mf, wB : B \in \underline{V}, f \in \underline{B}(X, B))$ . (We remark that our choice of  $\underline{V}$  made this initiality problem non-void.)



In (ii) above we see the relevance of "large" initiality problems : it might be impossible to have  $\underline{V}$  a set; this matter is considered further in section 1.7.

1.4.5. Corollary Given two amnestic functors  $K : \underline{A} \rightarrow \underline{C}$

and  $M : \underline{B} \rightarrow \underline{C}$ . If  $\underline{A}$  is  $K$ -initiality complete, there exists at least one functor  $F : \underline{B} \rightarrow \underline{C}$  with  $KF = M$ .

Proof Any choice of  $w$  in 1.4.4(i) will do; in particular we may choose  $\underline{V}$  void.

□

Hušek [1967a, p.557; 1967b, p.129] made intensive use of the construction of proposition 1.4.4, but in a more complicated form due to his more general concept of initiality (see remark 1.1.13(1)). Our use of the construction branches off from Hušek's investigations at this point, because we are aiming at describing the right inverses of an amnestic functor  $L : \underline{A} \rightarrow \underline{B}$ . From now on the functor  $K$  takes the more special form  $ML$ .

THROUGHOUT THE REMAINDER OF THIS SECTION WE CONSIDER  
TWO GIVEN AMNESTIC FUNCTORS

$$L : \underline{A} \rightarrow \underline{B} \quad \text{AND} \quad M : \underline{B} \rightarrow \underline{C}.$$

1.4.6. Definition Given a subclass  $\underline{V}$  of  $\text{obj } \underline{B}$  and a function  $w : \underline{V} \rightarrow \text{obj } \underline{A}$ . The unique functor (if it exists)  $F : \underline{B} \rightarrow \underline{A}$  determined by the two conditions

$$(1) \quad MLF = M$$

(2) for each object  $X$  of  $\underline{B}$ ,

$$FX = \text{in}_{ML} (Mf, w_B : B \in \underline{V}, f \in \underline{B}(X, B))$$

will be written  $F = (L, M : w)$ .

1.4.7. Theorem (i) The functors  $F : \underline{B} \rightarrow \underline{A}$  with  $MLF = M$  are precisely the functors  $F = (L, M : w)$ .

(ii) If  $\underline{A}$  is ML-initiality complete, then  $(L, M : w)$  exists for each function  $w : \underline{V} \rightarrow \text{obj } \underline{A}$  with  $\underline{V} \subset \text{obj } \underline{B}$ .

Proof 1.4.4. □

1.4.8. Remark Our choice of the notation  $(L, M : w)$  invites criticism at this point where, as remarked at the beginning of the section,  $M$  and  $ML$  are relevant but  $L$  is irrelevant. However, from 1.4.17 onward we concentrate on special  $w$  to which  $L$  is relevant, and then our notation is very efficient. Further, there is a formal trick by which the notation can still be used in the more general situation

$$\underline{A} \xrightarrow{K} \underline{C} \xleftarrow{M} \underline{B}$$

of proposition 1.4.4, namely: write  $(M^{-1}K, M : w)$  for the functor of 1.4.4. and agree that, formally,  $M^{-1}K = L$  when  $K = ML$ .

1.4.9. Proposition. If the functor  $F : \underline{B} \rightarrow \underline{A}$  satisfies  $MLF = M$ , then  $F = (L, M : F|_{\text{obj } \underline{B}})$ .

Proof This was obtained in the proof of 1.4.4(ii). □

1.4.10. Proposition If the functor  $F = (L, M : w)$  exists, then  $wB \leq_{ML} FB$  for each  $B$  in  $\text{dom } w$ .

Proof Take  $X = B$ ,  $f = 1_B$  in figure 1.

□

1.4.11. Remark One might be misled by 1.4.9 to expect 1.4.10 to be an equality. An easy counter example is given by the forgetful functors  $T : \underline{Qun} \rightarrow \underline{Top}$ ,  $E : \underline{Top} \rightarrow \underline{Ens}$  and the Pervin functor  $P$  of chapter 2, given by  $P = (T; E : w)$  with  $\text{dom } w = \{I_U\}$  and  $wI_U = I_q$ , while  $PI_U \neq I_q$  (see 2.3.7).

1.4.12 Definition The functor  $F : \underline{B} \rightarrow \underline{A}$  is a right M-subinverse of  $L : \underline{A} \rightarrow \underline{B}$  if  $LF \leq_M 1_B$ .

Clearly such  $F$  has to satisfy  $MLF = M$ .

1.4.13 Remark There has been recent interest in continuous uniformities for a topological space, i.e. uniformities whose topologies are coarser than the given topology [Alò and Shapiro 1970]. If  $L : \underline{Un} \rightarrow \underline{Top}$  and  $M : \underline{Top} \rightarrow \underline{Ens}$  are the forgetful functors, then the right M-subinverses of  $L$  give continuous uniformities to topological spaces. Hence we see also that the functors  $F$  with  $MLF = M$  need not be right M-subinverses of  $L$  (consider  $F : \underline{Top} \rightarrow \underline{Un}$  giving each space the discrete uniformity).

1.4.14. Definition  $L$  preserves initiality for  $M$  if,

whenever

$$Y = \text{in}_{ML} (C, g_j, Y_j : j \in J)$$

then

$$LY = \text{in}_M (C, g_j, LY_j : j \in J).$$

1.4.15. Example If  $L$  and  $M$  are the forgetful functors

$\underline{\text{Un}} \rightarrow \underline{\text{Crg}}, \underline{\text{Crg}} \rightarrow \underline{\text{Ens}}$ , then clearly  $L$  preserves initiality

for  $M$ . However, if  $L$  and  $M$  are the forgetful functors

$\underline{\text{Un}} \rightarrow \underline{\text{Prox}}, \underline{\text{Prox}} \rightarrow \underline{\text{Crg}}$ , then  $L$  does not preserve initiality

for  $M$  ([Alfsen and Fenstad 1959, p.359], [Dowker 1961],

[Császár 1963, p.96]).

1.4.16 Theorem (i) If  $F : \underline{B} \rightarrow \underline{A}$  is a right  $M$ -subinverse

of  $L$ , there exists a function  $w$  such that  $F = (L, M : w)$  and,

for each  $B$  in  $\text{dom } w$ ,  $LwB \leq_M B$ .

(ii) Let  $L$  preserve initiality for  $M$ . If  $F = (L, M : w)$

and  $LwB \leq_M B$  for each  $B$  in  $\text{dom } w$ , then  $F$  is a right

$M$ -subinverse of  $L$ .

Proof (i) We may by 1.4.9 take  $w = F|_{\text{obj } \underline{B}}$ . By 1.4.10,

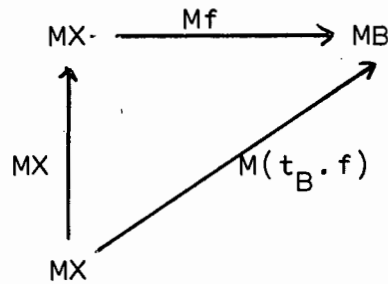
$wB \leq_{ML} FB$ , whence  $LwB \leq_M LFB \leq_M B$  (definition 1.4.12).

(ii) Given  $w : \underline{V} \rightarrow \text{obj } \underline{A}$  with  $F = (L, M : w)$  and, for

each  $B$  in  $\underline{V}$ ,  $LwB \leq_M B$ . Thus for  $B \in \underline{V}$  we have

$t_B : B \rightarrow LwB$  with  $Mt_B = 1_{MB}$ . For each  $f : X \rightarrow B$ , with

$B \in \underline{V}$ , we have  $t_B \cdot f : X \rightarrow LwB$  such that the triangle



commutes; also by the condition on  $L$  we have

$$LFX = \text{in}_M (Mf, Lw_B : B \in \underline{V}, f \in \underline{B}(X, B)).$$

Thus there is  $m_X : X \rightarrow LFX$  such that  $Mm_X = 1_{MX}$ ,

i.e.  $LF \leq_M 1_{\underline{B}}$ .

□

We now introduce an important special form of the functions  $w$  hitherto considered.

1.4.17 Definition Let  $\underline{U}$  be a subclass of  $\text{obj } \underline{A}$ . Then the function  $(\underline{U}) : \underline{L}\underline{U} \rightarrow \text{obj } \underline{A}$  is given by

$$(\underline{U})B = \text{in}_{ML} (Mf, A : A \in \underline{U}, f \in \underline{B}(B, LA))$$

provided this expression exists for each  $B$  in  $\underline{L}\underline{U}$ ; otherwise the function  $(\underline{U})$  is not defined.

We now analyse the construction of the functor  $(L, M : w)$  when  $w = (\underline{U})$ .

1.4.18. Proposition Let  $\underline{U}$  be a subclass of  $\text{obj } \underline{A}$ .

(i) If the expression

$$FX = \text{in}_{ML} (Mf, A : A \in \underline{U}, f \in \underline{B}(X, LA))$$

exists for each  $X$  in  $\text{obj } \underline{B}$ , then the correspondence  $X \rightsquigarrow FX$  can be uniquely extended to a functor  $F : \underline{B} \rightarrow \underline{A}$  with  $MLF = M$ ; and indeed  $F = (L, M : (\underline{U}))$ .

(ii) If  $F = (L, M : (\underline{U}))$  exists, then  $MLF = M$  and for each object  $X$  of  $\underline{B}$ ,

$$FX = \text{in}_{ML} (Mf, A : A \in \underline{U}, f \in \underline{B}(X, LA)).$$

Proof (i) Applying the procedure of the proof of proposition 1.4.4(i) to figure 3, we see that there is a unique functor  $F$  with  $MLF = M$  such that, for each  $X \in \text{obj } \underline{B}$ ,

$$FX = \text{in}_{ML} (Mf, A : A \in \underline{U}, f \in \underline{B}(X, LA)).$$

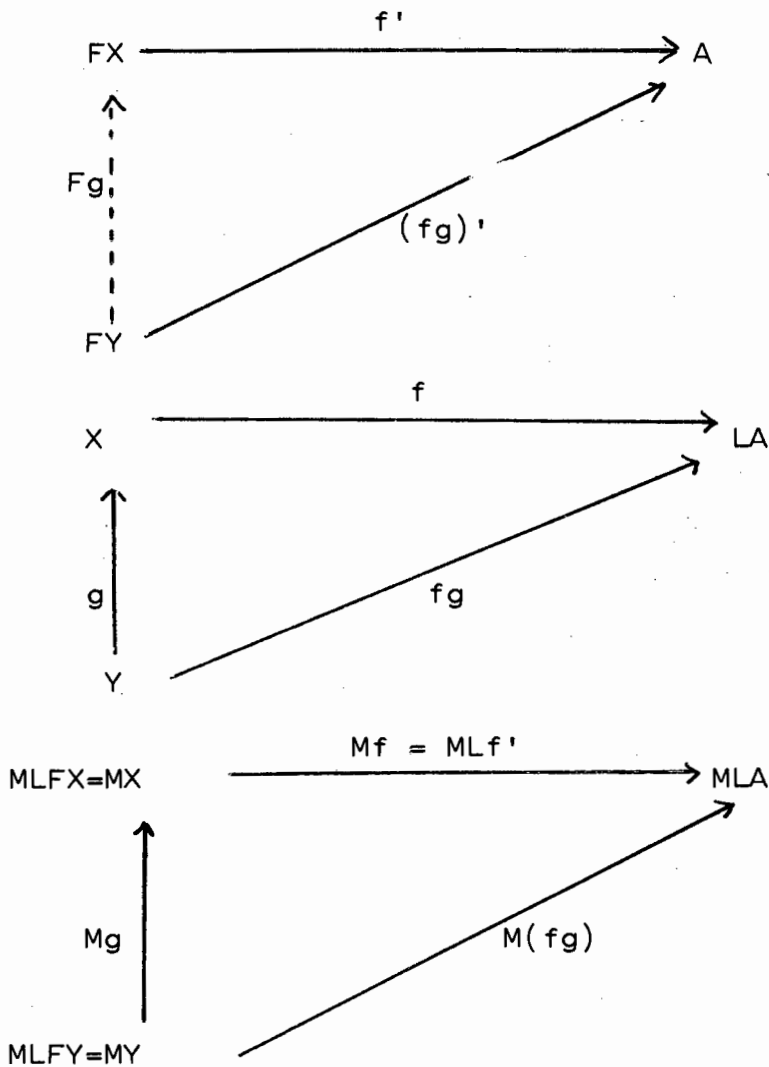


figure 3

Now the function  $(\underline{U})$  exists; indeed from definition 1.4.17 and the given expression for  $FX$ , we see that  $(\underline{U}) = F|_{\underline{LU}}$ . We have to show that  $F = (L, M : (\underline{U}))$ ; by 1.4.4(i) it suffices to show that

$$FX = \text{in}_{ML} (Mf, (\underline{U})B : B \in \underline{LU}, f \in \underline{B}(X, B))$$

that is,

$$FX = \text{in}_{ML} (Mf, FLA : A \in \underline{U}, f \in \underline{B}(X, LA)).$$

Putting  $X = LA$  (with  $A \in \underline{U}$ ) and  $f = 1_{LA}$  in figure 3, we see that there is  $t_A : FLA \rightarrow A$  with  $MLt_A = Mf = 1_{MLA}$ ; in other words  $A \leq_{ML} FLA$  for each  $A \in \underline{U}$ .

For each

$$f : X \rightarrow LA \text{ we have}$$

$Ff : FX \rightarrow FLA$  with  $ML(Ff) = Mf$ . Further, given  $h_f : Z \rightarrow FLA$  and  $k : MLZ \rightarrow MX$  such that the lower triangle in figure 4 commutes for each  $f$ , we seek

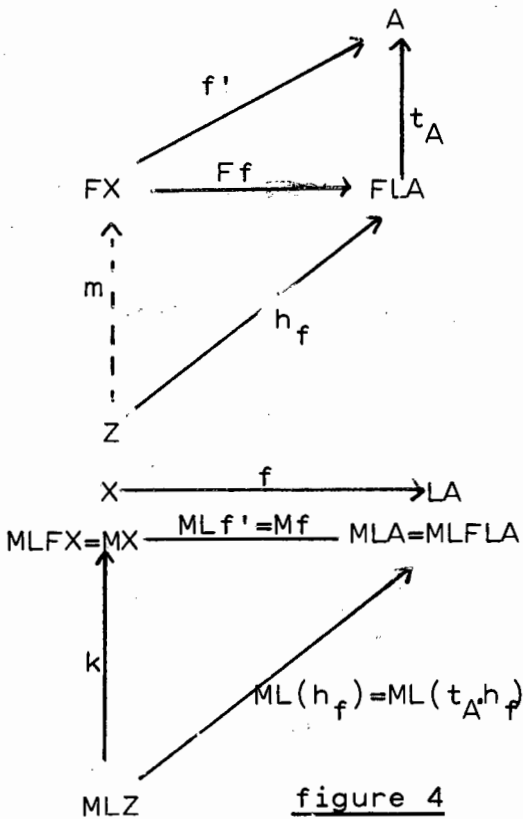


figure 4

$m : Z \rightarrow FX$  with  $MLm = k$ . We observe that  $t_A \cdot h_f : Z \rightarrow A$  is such that  $ML(t_A \cdot h_f)$  makes the lower triangle in figure 4 commute; then from  $FX = \text{in}_{ML} (Mf, A : A \in \underline{U}, f \in \underline{B}(X, LA))$  follows the existence of  $m : Z \rightarrow FX$  with  $MLm = k$ .

(ii) Let  $F = (L, M : (\underline{U}))$ . Thus the function  $(\underline{U})$  has to exist and by definition 1.4.17,

$$(\underline{U})B = \text{in}_{\text{ML}} (\text{Mg}, U : U \in \underline{U}, g \in \underline{B}(B, \text{LU}))$$

for each  $B \in \underline{\text{LU}}$ . Also by definition 1.4.6 we have

$\text{MLF} = \text{M}$  and, for each  $X \in \text{obj } \underline{B}$ ,

$$\text{FX} = \text{in}_{\text{ML}} (\text{Mf}, (\underline{U})B : B \in \underline{\text{LU}}, f \in \underline{B}(X, B)).$$

From these two equations the associativity theorem 1.1.10 gives

$$\text{FX} = \text{in}_{\text{ML}} (\text{Mg.Mf}, U : U \in \underline{U}, B \in \underline{\text{LU}}, X \xrightarrow{f} B \xrightarrow{g} \text{LU}).$$

We wish to augment this initiality (proposition 1.1.11) to

$$\text{yield } \text{FX} = \text{in}_{\text{ML}} (\text{Mh}, U : U \in \underline{U}, X \xrightarrow{h} \text{LU})$$

which is the desired result. But to apply proposition

1.1.11 we need to know that for each  $h : X \rightarrow \text{LU}$  there is

$h' : \text{FX} \rightarrow U$  with  $\text{ML}h' = \text{Mh}$ . Now from the above equation

for  $(\underline{U})B$  we readily see that there is  $r_U : (\underline{U})\text{LU} \rightarrow U$  with

$\text{ML}r_U = \mathbf{1}_{\text{MLU}}$ , and from 1.4.10 we have the existence of

$s_U : \text{FLU} \rightarrow (\underline{U})\text{LU}$  with  $\text{ML}s_U = \mathbf{1}_{\text{MLU}}$ . Taking  $h' = r_U \circ s_U \circ \text{Fh} :$

$\text{FX} \rightarrow U$ , we have  $\text{ML}h' = \mathbf{1.1.MLFh} = \text{Mh}$ , as required.

□

An approximate summary of the above fundamental proposition is:

$$F = (L, M : (\underline{U})) \text{ iff}$$

$\text{MLF} = \text{M}$  and, for each  $X \in \text{obj } \underline{B}$ ,

$$\text{FX} = \text{in}_{\text{ML}} (\text{Mf}, A : A \in \underline{U}, f \in \underline{B}(X, \text{LA})).$$

It is essential to observe the contrast between this result and definition 1.4.6 which approximately says:

$F = (L, M : w)$  iff

$MLF = M$  and, for each  $X \in \text{obj } \underline{B}$ ,

$FX = \text{in}_{ML} (Mf, wB : B \in \text{dom } w, f \in \underline{B}(X, B))$ .

1.4.19 Proposition Let  $\underline{U}$  be a subclass of  $\text{obj } \underline{A}$ . If the functor  $F = (L, M : (\underline{U}))$  exists, then

- (i) for each  $A \in \underline{U}$ ,  $A \leq_{ML} FLA$
- (ii)  $(\underline{U}) = F|_{L\underline{U}}$ .

Proof By 1.4.18(ii), the assumptions of 1.4.18(i) hold. In the proof of 1.4.18(i) we explicitly obtained the two results here asserted.

□

We have been developing a hierarchy of functors of increasing specialization : First come the  $F$  with  $MLF = M$ , which are precisely the  $(L, M : w)$ . Next come the right  $M$ -subinverses of  $L$ , which are precisely the  $(L, M : w)$  with a certain condition on  $w$  (theorem 1.4.16), assuming that  $L$  preserves initiality for  $M$ . We have now reached the third level, that of the  $(L, M : (\underline{U}))$ , which we characterize as follows.

1.4.20 Theorem Let  $F : \underline{B} \rightarrow \underline{A}$  be a functor. Consider the following two conditions:

- (a)  $LF \leq_M 1_{\underline{B}}$  and  $FLF = F$

(b) there is a subclass  $\underline{U}$  of  $\text{obj } \underline{A}$  such that  
 $F = (L, M : (\underline{U}))$ .

Then (a) implies (b). If  $L$  preserves initiality for  $M$ , then (b) implies (a).

Proof Given (a), we then have  $MLF = M$ . Letting  
 $\underline{U} = \{FB : B \in \text{obj } \underline{B}\}$ , we wish to prove  $F = (L, M : (\underline{U}))$ ,  
 that is (1.4.18) :

$$FX = \text{in}_{ML} (Mf, FB : B \in \text{obj } \underline{B}, f \in \underline{B}(X, LFB)).$$

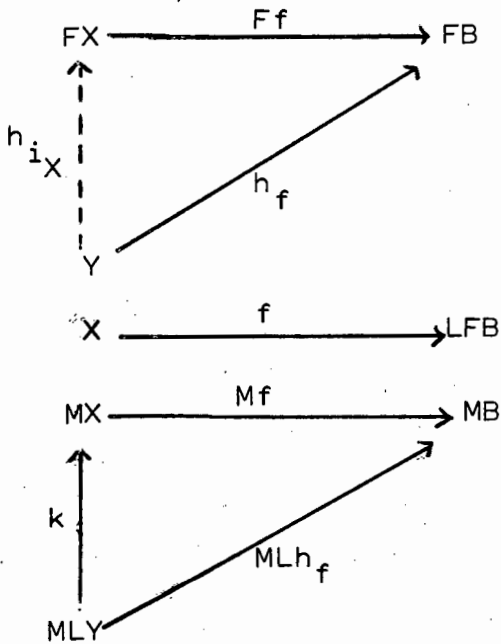


figure 5

Given  $f : X \rightarrow LFB$  we have

$Ff : FX \rightarrow FLFB = FB$ , and

$ML(Ff) = Mf$ . Further, consider

$h_f : Y \rightarrow FB$  and  $k : MLY \rightarrow MX$

such that the lower triangle in

figure 5 commutes for all  $f$ . As

$LF \leq_M \mathbf{1}_{\underline{B}}$  we have  $i_X : X \rightarrow LFX$

with  $Mi_X = \mathbf{1}_{MX}$ ; with the commuta-

tivity of the lower triangle for

$f = i_X$  this gives  $k = MLh_{i_X}$ , which

proves the required initiality.

Conversely, given (b), i.e.  $F = (L, M : (\underline{U}))$ , whence  
 $FX = \text{in}_{ML} (Mf, A : A \in \underline{U}, f \in \underline{B}(X, LA))$ . Assuming that  $L$   
 preserves initiality for  $M$ , we have

$LFX = \text{in}_M (Mf, LA : A \in \underline{U}, f \in \underline{B}(X, LA))$ . Thus for  
 each  $f : X \rightarrow LA$  ( $A \in \underline{U}$ ) we have  $f' : LFX \rightarrow LA$  with  $Mf' = Mf$ .

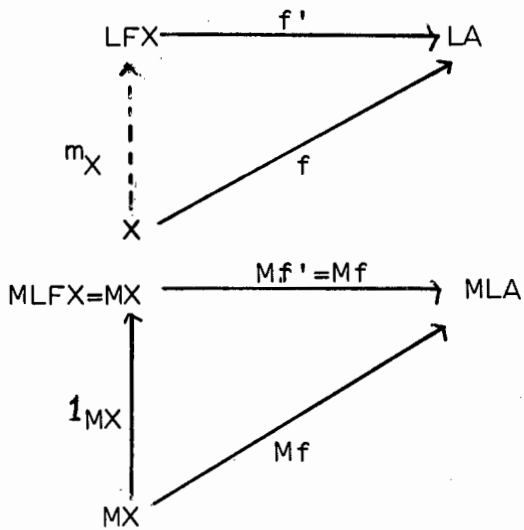


figure 6

Consider again that  $FX = \text{in}_{ML}(Mf, U : U \in \underline{U}, f \in \underline{B}(X, LU))$ .

Hence for  $f : X \rightarrow LU$  with  $U \in \underline{U}$  we have  $f' : FX \rightarrow U$  with

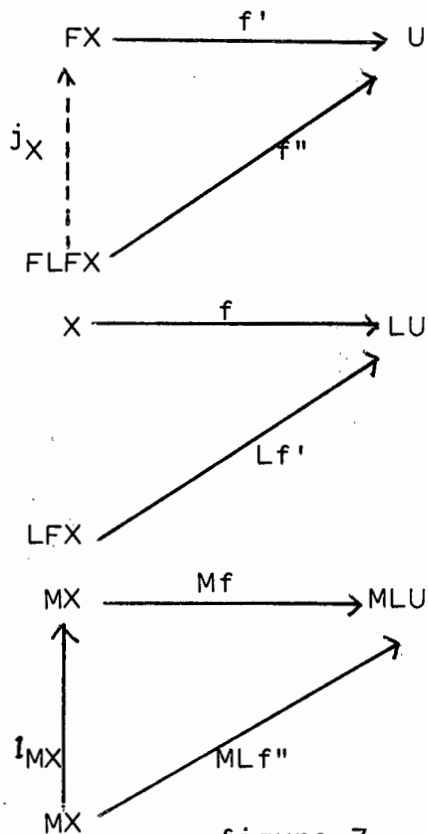


figure 7

As the lower triangle in figure 6 commutes, we have  $m_X : X \rightarrow LFX$  with  $Mm_X = 1_{MX}$ .

Thus  $LFX \leq_M X$ , and hence from  $MLF = M$  follows  $LF \leq_M 1_B$ . Hence also  $FLF \leq_{ML} F$ , and it remains to prove  $F \leq_{ML} FLF$ , i.e. to find

$j_X : FLFX \rightarrow FX$  with  $MLj_X = 1_{MX}$ .

$MLf' = Mf$ . Likewise, as

$Lf' : LFX \rightarrow LU$  we have

$f'' : FLFX \rightarrow U$  with  $MLf'' = M(Lf') = Mf$ . Thus the lower triangle in

figure 7 commutes and there exists

$j_X : FLFX \rightarrow FX$  with  $MLj_X = 1_{MX}$ .



We remark that the third level in the hierarchy, which we have just characterized, is properly distinct from the second level. Indeed we shall give an example (1.8.4) in which L

preserves initiality for  $M$  and there is a right  $M$ -subinverse  $F$  of  $L$  which does not satisfy  $FLF = F$ , i.e. is not of the form  $(L, M : (\underline{U}))$ .

We now prepare for the fourth and last level in the hierarchy, that of the right inverses of  $L$ .

Let us for a moment consider the special situation that  $\underline{A} = \underline{B}$  and  $L = \mathbf{1}_{\underline{B}}$ . In this situation we can deal with the functor  $(\mathbf{1}_{\underline{B}}, M : (\underline{V}))$ , where  $\underline{V}$  is a subclass of  $\text{obj } \underline{B}$ . This functor will be further considered in section 1.8, but here we need to observe just one fact : by 1.4.18,

$$(\mathbf{1}_{\underline{B}}, M : (\underline{V})) = \mathbf{1}_{\underline{B}} \text{ if and only if, for each object } X \text{ of } \underline{B}, X = \text{in}_M(Mf, B : B \in \underline{V}; f \in \underline{B}(X, B)).$$

This will certainly be the case when  $\underline{V} = \text{obj } \underline{B}$ . When  $M$  is the forgetful functor  $\underline{\text{Top}} \rightarrow \underline{\text{Ens}}$ , then already  $\underline{V} = \{D_U\}$  suffices, where  $D_U$  is the space of two points and three open sets. We shall show in 1.4.28 that the condition  $(\mathbf{1}_{\underline{B}}, M : (\underline{V})) = \mathbf{1}_{\underline{B}}$  closely resembles that of  $\underline{V}$  being a "cogenerating class" for  $\underline{B}$ .

1.4.21. Definition A subclass  $\underline{V}$  of  $\text{obj } \underline{B}$  is  $M$ -sufficient (in  $\underline{B}$ ) if  $(\mathbf{1}_{\underline{B}}, M : (\underline{V})) = \mathbf{1}_{\underline{B}}$ . An object  $B$  of  $\underline{B}$  is  $M$ -sufficient if the class  $\{B\}$  is  $M$ -sufficient.

We recall that we are dealing with two amnestic functors  $L : \underline{A} \rightarrow \underline{B}$  and  $M : \underline{B} \rightarrow \underline{C}$ .

1.4.22 Theorem Let  $\underline{U} \subset \text{obj } \underline{A}$ . If the functor  $(L, M : (\underline{U}))$  is a right inverse of  $L$ , then the class  $\underline{L}\underline{U}$  is  $M$ -sufficient in  $\underline{B}$ .

Proof Let  $F = (L, M : (\underline{U}))$  and  $LF = \mathbf{1}_{\underline{B}}$ . To prove, for  $X \in \text{obj } \underline{B}$ , that

$$X = \text{in}_M(Mf, LA : A \in \underline{U}, f \in \underline{B}(X, LA)).$$

For  $f : X \rightarrow LA$  with  $A \in \underline{U}$ ,

consider  $g_f : Y \rightarrow LA$  and

$k : MY \rightarrow MX$  such that the lower diagram in figure 8 commutes.

By 1.4.19(i) we have  $j_A : FLA \rightarrow A$  with  $MLj_A = \mathbf{1}_{MLA}$ . Let  $f' = j_A \cdot Ff$  then  $MLf' = Mf$  and  $ML(j_A \cdot Fg_f) = Mg_f$ .

Hence, considering that

$$FX = \text{in}_{ML}(Mf, A : A \in \underline{U}, f \in \underline{B}(X, LA))$$

(1.4.18), there exists  $m : FY \rightarrow FX$

such that  $MLm = k$ . Now since

$LF = \mathbf{1}_{\underline{B}}$ , we have  $Lm : Y \rightarrow X$ , and

$M(Lm) = k$ , as required.

□

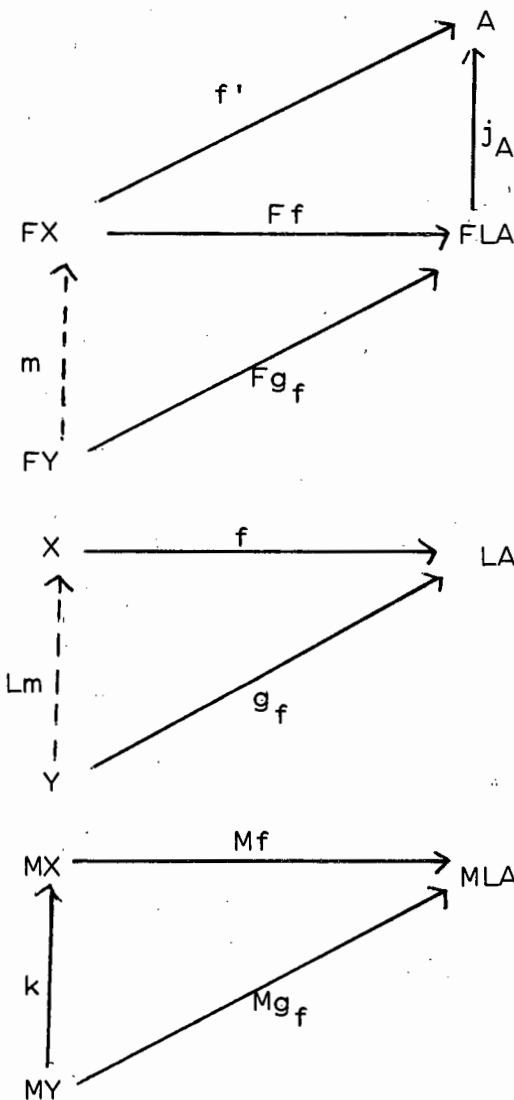


figure 8

1.4.23 Theorem (i) Let  $F : \underline{B} \rightarrow \underline{A}$  be a right inverse of  $L$ . Then there is a subclass  $\underline{U}$  of  $\text{obj } \underline{A}$  such that  $F = (L, M : (\underline{U}))$  and  $L\underline{U}$  is  $M$ -sufficient in  $\underline{B}$ .

(ii) Let  $L$  preserve initiality for  $M$ . If  $\underline{U}$  is a subclass of  $\text{obj } \underline{A}$  such that  $L\underline{U}$  is  $M$ -sufficient in  $\underline{B}$ , then the functor  $(L, M : (\underline{U}))$ , if it exists, is a right inverse of  $L$ .

Proof (i) Immediate from 1.4.20 and 1.4.22.

(ii) Let  $F = (L, M : (\underline{U}))$ ,

i.e.  $FX = \text{in}_{ML}(Mf, A : A \in \underline{U}, f \in \underline{B}(X, LA))$ .

As  $L$  preserves initiality for  $M$ , we have

$LFX = \text{in}_M(Mf, LA : A \in \underline{U}, f \in \underline{B}(X, LA))$ ,

i.e.  $LF = (\underline{1}_{\underline{B}}, M : (L\underline{U}))$ . As  $L\underline{U}$  is  $M$ -sufficient in  $\underline{B}$ ,

$LF = \underline{1}_{\underline{B}}$ .

□

1.4.24 Corollary Let  $L$  preserve initiality for  $M$ , and let  $F = (L, M : (\underline{U}))$  with  $\underline{U} \subset \text{obj } \underline{A}$ . Then  $F$  is a right inverse of  $L$  if and only if  $L\underline{U}$  is  $M$ -sufficient in  $\underline{B}$ .

Proof 1.4.22 and 1.4.23.

□

1.4.25 Theorem Let  $L$  preserve initiality for  $M$ , let  $\underline{V}$  be an  $M$ -sufficient subclass of  $\text{obj } \underline{B}$ , and let the functor  $F : \underline{B} \rightarrow \underline{A}$  be a right  $M$ -subinverse of  $L$ . Then  $F$  is a right inverse of  $L$  if and only if  $LFV = V$  for each  $V \in \underline{V}$ .

Proof If  $LF = \underline{1}_B$ , then trivially  $LFV = V$ . Conversely, let  $LF \leq_M \underline{1}_B$  with  $LFV = V$  for each  $V \in \underline{V}$ . By 1.4.16 there is a function  $w$  with  $F = (L, M : w)$  and, for each  $B \in \text{dom } w$ ,  $LwB \leq_M B$ . By 1.4.6. we have, for  $X \in \text{obj } \underline{B}$ ,

$$FX = \text{in}_{ML} (Mf, wB : B \in \text{dom } w, f \in \underline{B}(X, B))$$

so that by the condition on  $L$ ,

$$LFX = \text{in}_M (Mf, LwB : B \in \text{dom } w, f \in \underline{B}(X, B)).$$

For  $V \in \underline{V}$  this equation gives

$$V = LFV = \text{in}_M (Mh, LwB : B \in \text{dom } w, h \in \underline{B}(V, B)).$$

As  $\underline{V}$  is  $M$ -sufficient,

$$X = \text{in}_M (Mg, V : V \in \underline{V}, g \in \underline{B}(X, V)).$$

By the associativity theorem 1.1.10, the equations for  $X$  and  $V$  together imply

$$X = \text{in}_M (Mh.Mg, LwB : V \in \underline{V}, B \in \text{dom } w, \\ X \xrightarrow{g} V \xrightarrow{h} B).$$

Since  $LwB \leq_M B$ , we can augment (1.1.11) the class of morphisms  $X \xrightarrow{g} V \xrightarrow{h} B$  to the class of all  $f : X \rightarrow B$ ; in other words

$$X = \text{in}_M (Mf, LwB : B \in \text{dom } w, f \in \underline{B}(X, B)).$$

Thus  $X = LFX$ . As  $MLF = M$ , it follows that  $LF = \underline{1}_B$ .

□

1.4.26 Remark The above theorem fails if the assumption that  $F$  is a right  $M$ -subinverse of  $L$  is weakened to  $MLF = M$ . This follows from [Brümmer 1969a, example 3.2.]. On the other hand, we can give an extreme example [ibidem, p. 407] of an  $F$  with  $LF \leq_M \underline{1}_B$  and  $FLF = F$ , such that  $F$  is not a right inverse

of  $L$ . Indeed, let  $L$  and  $M$  be the forgetful functors  $\underline{Qun} \rightarrow \underline{Top}$ ,  $\underline{Top} \rightarrow \underline{Ens}$ . Let  $F = L, M : (\underline{U})$  where  $\underline{U}$  is void or, just as well, consists of a single quasi-uniform space of one point. By 1.4.20 and 1.4.22,  $F$  has the stated properties. It is further easy to see that  $F$  is the ML-coarsest of all the functors  $G : \underline{Top} \rightarrow \underline{Qun}$  for which  $MLG = M$ .

□

The following result strengthens theorem 1.4.20.

1.4.27 Theorem Let the functor  $F : \underline{B} \rightarrow \underline{A}$  be a right  $M$ -subinverse of  $L$  such that  $FLF = F$ . Then there is a largest subclass  $\hat{U}$  of  $\text{obj } \underline{A}$  satisfying  $F = (L, M : (\hat{U}))$ , namely

$$\hat{U} = \{A \in \text{obj } \underline{A} : A \leq_{ML} FLA\}.$$

Proof

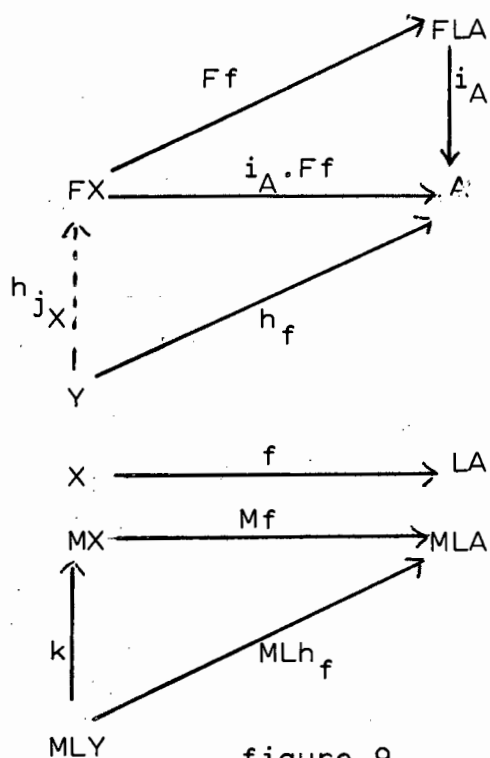


figure 9

Taking  $\hat{U}$  as stated, we first show that  $F = (L, M : (\hat{U}))$ . For each  $A \in \hat{U}$  we have  $i_A : FLA \rightarrow A$  with  $MLi_A = 1_{MLA}$ . Then  $ML(i_A.Ff) = Mf$ . Consider  $h_f : Y \rightarrow A$  and  $k : MLY \rightarrow MX$  such that the lower triangle in figure 9 commutes for each  $f : X \rightarrow LA$  ( $A \in \hat{U}$ ). As  $LF \leq_M 1_{\underline{B}}$ , we have  $j_X : X \rightarrow LFX$  with  $Mj_X = 1_{MX}$ . As  $FLF = F$ , we have  $FX = FL(FX)$  so that  $FX \in \hat{U}$ ; hence  $j_X : X \rightarrow L(FX)$  is one of

the considered  $f$ . We have  $h_{j_X} : Y \rightarrow FX$ , and (with  $f = h_{j_X}$ ) the lower triangle gives  $k = MLh_{j_X}$ . Thus  $F = (L, M : (\hat{U}))$ . Now consider any  $\underline{U} \subset \text{obj } \underline{A}$  such that  $F = (L, M : (\underline{U}))$ . By 1.4.19(i) immediately  $\underline{U} \subset \hat{\underline{U}}$ . □

The following proposition elaborates our earlier remark that  $M$ -sufficient classes resemble "cogenerating classes".

1.4.28 Proposition Given an amnestic functor  $M : \underline{B} \rightarrow \underline{C}$ , assume that

- (1)  $\underline{B}$  is  $M$ -initiality complete
- (2)  $\underline{B}$  is co-wellpowered
- (3)  $\underline{C}$  has products
- (4)  $\underline{C}$  is an Isbell bicategory.

Let  $X$  be an  $M$ -separated object of  $\underline{B}$ , and let  $\underline{V} \subset \text{obj } \underline{B}$ . Then the following two statements are equivalent:

- (a)  $X = \text{in}_M (Mg, V : V \in \underline{V}, g \in \underline{B}(X, V))$
- (b) There is an  $M$ -embedding of  $X$  into a product of a set of members of  $\underline{V}$ .

Proof By 1.2.5,  $\underline{B}$  has products and  $M$  preserves them. Let (b) hold. Thus we have an index set  $J$  and  $V_j \in \underline{V}$  ( $j \in J$ ) and an  $M$ -embedding

$$f : X \rightarrow \prod V_j,$$

the product having projections  $p_j$ . By 1.2.4 the  $p_j$  form an

M-initiality, and hence by the associativity theorem 1.1.10 the  $p_j f$  form an M-initiality at X. We can clearly augment this M-initiality by 1.1.11 to obtain (a).

Conversely, let (a) hold. For each  $f : X \rightarrow V$  with  $V \in \underline{V}$  we have a distinguished factorization

$$X \xrightarrow[\text{M-surjection}]{f'} X_f \xrightarrow[\text{M-embedding}]{f''} V$$

namely that of 1.3.11(1); the proof there provides a means of distinguishing this factorization among equivalent ones. By assumption (2), since the  $f'$  are epimorphisms, there are, up to isomorphism, only a set of them, which we denote  $\{f'_j : j \in J\}$ . Thus we have

$$X \xrightarrow{f_j} V_j = X \xrightarrow{f'_j} X_{f'_j} \xrightarrow{f''_j} V_j.$$

We claim that  $\{f_j : j \in J\}$  is an M-initiality. Given

$$\begin{array}{ccc} X & \xrightarrow{f_j} & V_j \\ & \nearrow h_j & \\ Y & & \end{array}$$

$$\begin{array}{ccc} MX & \xrightarrow{Mf_j} & MV_j \\ k \uparrow & \nearrow Mh_j & \\ MY & & \end{array}$$

figure 10

$h_j : Y \rightarrow V_j$  and  $k : MY \rightarrow MX$  such that the lower triangle in figure 10 commutes, we seek  $m : Y \rightarrow X$  such that  $Mm = k$ .

Consider any  $f : X \rightarrow V$ ,  $V \in \underline{V}$ .

Then  $f'$  is equivalent to some

$f'_j$ , i.e. there is an isomorphism  $d_j$  such that  $f' = d_j f'_j$ , and we set  $d_j = 1$  in case  $f = f'_j$ . Since

$Mh_j = Mf_j'' \cdot (Mf_j' \cdot k)$  and since the  $f_j''$  as constructed in the

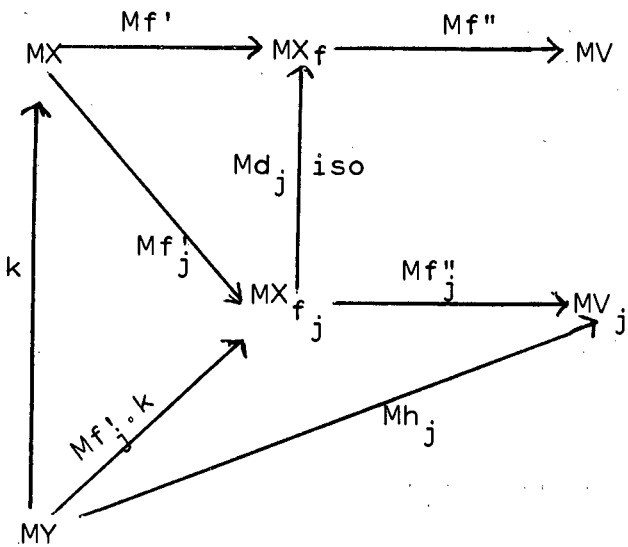
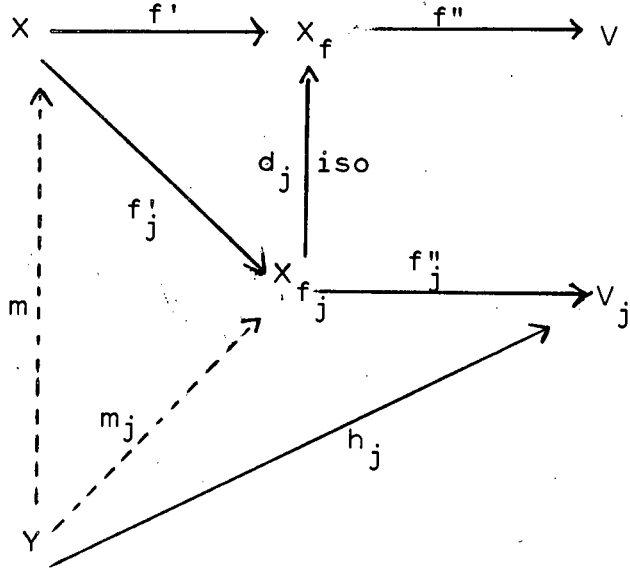


figure 11

proof of 1.3.11(1) is an  $M$ -initiality, there is a morphism  $m_j : Y \rightarrow X_{f_j}$  with  $Mm_j = Mf'_j \cdot k$ . Now let  $h_f = f'' \cdot d_j \cdot m_j$ . Thus  $h_f : Y \rightarrow V$  is such that  $h_f = h_j$  when  $f = f_j$ . Hence  $Mh_f = Mf'' \cdot Md_j \cdot Mf'_j \cdot k = Mf'' \cdot Mf' \cdot k = Mf \cdot k$

and therefore, by condition (a), there is  $m : Y \rightarrow X$  such that  $Mm = k$ . Thus  $\{f_j : j \in J\}$  is an  $M$ -initiality at  $X$ . By proposition 1.3.7 there is then a simple  $M$ -initiality  $e : X \rightarrow \prod V_j$ , and as  $X$  is  $M$ -separated,  $e$  is an  $M$ -embedding.



**1.4.29 Theorem** Let the following six assumptions hold.

- (1)  $\underline{A}$  is  $ML$ -initiality complete
- (2)  $L$  preserves initiality for  $M$

- (3)  $\underline{B}$  is co-wellpowered
- (4)  $\underline{C}$  has products
- (5)  $\underline{C}$  is an Isbell bicategory
- (6) There is an M-sufficient M-separated object  $B_0$  in  $\underline{B}$ .

Then for any subclass  $\underline{U}$  of  $\text{obj } \underline{A}$  the functor  $F = (L, M : (\underline{U}))$  exists and the following five statements are equivalent:

- (a)  $F$  is a right inverse of  $L$
- (b)  $\underline{LU}$  is M-sufficient in  $\underline{B}$ .
- (c)  $LFB_0 = B_0$
- (d)  $B_0 = \text{in}_M(Mf, LU : U \in \underline{U}, f \in \underline{B}(B_0, LU))$
- (e) There is an M-embedding of  $B_0$  into a product of a set of members of  $\underline{LU}$ .

Proof By 1.4.24, (a) and (b) are equivalent. Since  $F$  is a right M-subinverse of  $L$ , the equivalence of (a) and (c) follows immediately from 1.4.25. By assumption (2),

$$LFB_0 = \text{in}_M(Mf, LU : U \in \underline{U}, f \in \underline{B}(B_0, LU)),$$

so that (c) and (d) are equivalent. From assumptions (1) and (2) clearly follows that  $\underline{B}$  is M-initiality complete; thus 1.4.28 is applicable and (d) and (e) are equivalent.

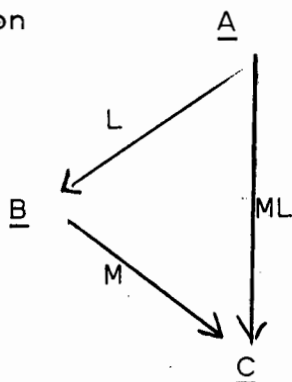
□

Remark The assumptions (3), (4) and (5) were only used to prove the equivalence of (d) and (e).

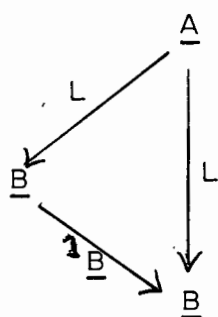
The theory of the four-level hierarchy in this section originated in a result [Brümmer 1969a, theorem 3.1] which described the right inverses of the forgetful functor  $\underline{Qun} \rightarrow \underline{Top}$ . That theorem was obtained in ignorance of the fact that the construction of the lowest level of the hierarchy overlaps with one given by Hušek (see our remark after 1.4.5). A preliminary attempt at developing the present theory was made in [Brümmer 1969c].

1.5. THE SPECIAL RIGHT INVERSE THEORY

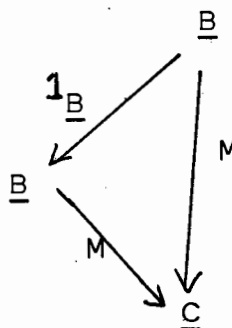
The situation



in which right inverses of  $L$  and their generalizations were studied in the previous section, admits the following two specializations:



and



The first will be studied in this section, the second in section 1.8. We show here that the theory of the previous section applies very readily to yield a description of right inverses of  $L$  in terms of  $L$ -initiality. We also give certain relations between  $L$ -initiality and  $ML$ -initiality from which it becomes clear why the more complicated theory of the previous section was necessary.

THROUGHOUT THE PRESENT SECTION  $L : \underline{A} \rightarrow \underline{B}$  IS  
AN AMNESTIC FUNCTOR.

We observe two trivialities:

- (1)  $L$  preserves initiality for  $\mathbf{1}_{\underline{B}}$ .
- (2) For functors  $F : \underline{B} \rightarrow \underline{A}$ , the condition  $MLF = M$  of the previous section here reduces to  $LF = \mathbf{1}_{\underline{B}}$ , so that the entire hierarchy of section 1.4 here collapses to the class of right inverses of  $L$ .

Thus theorem 1.4.7. gives:

- 1.5.1. Proposition (i) The right inverses of  $L$  are precisely the functors  $(L, \mathbf{1}_{\underline{B}} : w)$  with  $\text{dom } w \subset \text{obj } \underline{B}$ ,  $\text{codom } w \subset \text{obj } \underline{A}$ .
- (ii) If  $\underline{A}$  is  $L$ -initiality complete, then  $(L, \mathbf{1}_{\underline{B}} : w)$  exists for each function  $w$  with  $\text{dom } w \subset \text{obj } \underline{B}$ ,  $\text{codom } w \subset \text{obj } \underline{A}$ .

□

Likewise theorem 1.4.20 yields:

- 1.5.2. Theorem The right inverses of  $L$  are precisely the functors  $(L, \mathbf{1}_{\underline{B}} : (\underline{U}))$  with  $\underline{U} \subset \text{obj } \underline{A}$ .

□

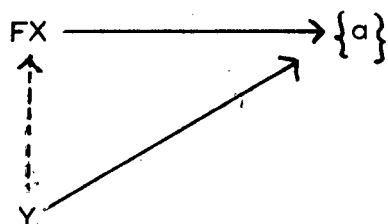
This simple description of the right inverses of  $L$  raises the question of whether our more complicated descriptions in section 1.4 were necessary at all. The following example and remark answer this question.

- 1.5.3. Example Let  $L : \underline{Un} \rightarrow \underline{Crg}$  and  $M : \underline{Crg} \rightarrow \underline{Ens}$  be the usual forgetful functors. Then  $\underline{Un}$  is  $ML$ -initiality complete

but not L-initiality complete.

Proof. We said in 1.2.3 that it is obvious that  $\underline{Un}$  is ML-initiality complete. (Remark This is obvious exactly because ML-initiality is so easy to handle in  $\underline{Un}$  : given uniform spaces  $A_j$  and functions  $g_j : C \rightarrow ML A_j$ , one takes the preimages under the  $g_j$  of the entourages in the  $A_j$ , and observes that one thereby has on  $C$  a subbase for the uniform space  $in_{ML}(C, g_j, A_j)$ . On the other hand, I know of no such simple way of dealing with L-initiality for  $\underline{Un}$ ).

To continue with the proof, suppose that  $\underline{Un}$  is L-initiality complete. Let  $\underline{U}$  be the class consisting of just one uniform space of one single point  $a$ . Then the functor  $F = (L, 1_B : (\underline{U}))$  exists and is a right inverse of  $L$  (1.5.2). For any object  $X$  of  $\underline{Crg}$ ,  $FX$  is then the L-coarsest, and hence the ML-coarsest, among all the uniform spaces  $Y$  for which  $LY = X$ . This is



clear from figure 1.) But it is well known [Gillman and Jerison 1960, problem 15K] that a Tihonov space  $X$  admits an ML-coarsest uniformity if and only if  $X$  is locally compact.

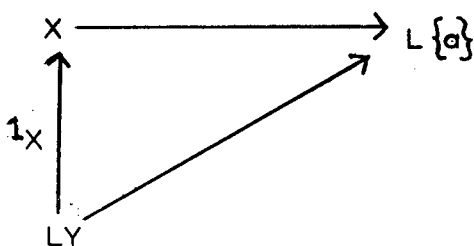


figure 1


□

The following result is a positive and typical application of theorem 1.5.2.

1.5.4. Example The forgetful functor  $E : \underline{\text{Top}} \rightarrow \underline{\text{Ens}}$  has exactly two right inverses : one giving the accrete, another giving the discrete topology to each set.

Proof By 1.5.2 each right inverse  $F$  of  $E$  is of the form  $F = (E, \mathbf{1}_{\underline{\text{Ens}}} : (\underline{U}))$  with  $\underline{U} \subset \text{obj } \underline{\text{Top}}$ . By 1.4.27 we may take  $\underline{U}$  maximal, namely

$$\underline{U} = \{A \in \text{obj } \underline{\text{Top}} : A \leq_E \text{FEA}\}.$$

Let  $D_U$  denote the topological space . If  $D_U \in \underline{U}$ , then  $FX$  is always discrete — indeed each  $f : X \rightarrow LD_U$  becomes a continuous mapping  $FX \rightarrow D_U$  and hence every subset of  $FX$  is open in  $FX$ . On the other hand, suppose  $D_U \notin \underline{U}$ . From the above form of  $\underline{U}$ , it follows that no space finer than  $D_U$  can be in  $\underline{U}$ . Writing  $D$  for a set of two points, we see that  $FD$  is an accrete space. For any set  $X$ , any function  $f : D \rightarrow X$  becomes a continuous mapping  $Ff : FD \rightarrow FX$ . As  $FD$  is accrete, so is  $FX$ .

□

Finally we give two useful general relations between the functors  $(L, M : (\underline{U}))$  and  $(L, \mathbf{1}_{\underline{B}} : (\underline{U}))$ .

1.5.5. Proposition. Let  $\underline{U} \subset \text{obj } \underline{A}$ . If the functor  $F = (L, M : (\underline{U}))$  is a right inverse of  $L$ , then  $F = (L, \mathbf{1}_{\underline{B}} : (\underline{U}))$ .

Proof. By 1.4.18, it suffices to prove that

$$FX = \text{in}_L(f, A : A \in \underline{U}, f \in \underline{B}(X, LA)).$$

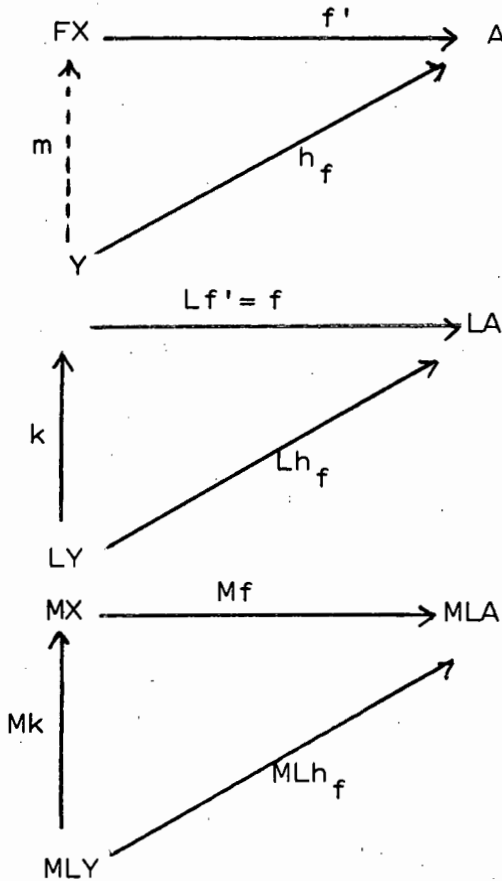


figure 2

We have

$$FX = \text{in}_{ML}(Mf, A : A \in \underline{U}, f \in \underline{B}(X, LA)).$$

Given  $f : X \rightarrow LA$ , there is

$f' : FX \rightarrow A$  with  $MLf' = Mf$ . As

$LFX = X$ , we have  $Lf' : X \rightarrow LA$  and

then by the faithfulness of  $M$ ,

$Lf' = f$ . Consider  $h_f : Y \rightarrow A$

and  $k : LY \rightarrow X$  such that the middle

triangle in figure 2 commutes. Then

the lower triangle commutes and there

is  $m : Y \rightarrow FX$  with  $MLm = k$ . As

both  $Lm, k : LY \rightarrow X$  and  $M$  is

faithful, we have  $Lm = k$ .

□

In 1.5.5. we cannot assume less than that  $F$  is a right inverse of  $L$ ; this is immediate from 1.5.2.

1.5.6. Proposition Let  $\underline{U} \subset \text{obj } \underline{A}$ . If the functor  $F = (L, 1_{\underline{B}} : (\underline{U}))$  exists, then there is a class  $\underline{U}'$  with  $\underline{U} \subset \underline{U}' \subset \text{obj } \underline{A}$  such that  $F = (L, M : (\underline{U}'))$ .

Proof By 1.5.2  $F$  is a right inverse of  $L$ . Hence by 1.4.23 there is  $\underline{U}_0$  with  $F = (L, M : (\underline{U}_0))$ . For  $f : X \rightarrow LA$  with

$A \in \underline{U}$  there is by hypothesis  $f' : FX \rightarrow A$  with  $Lf' = f$ , and hence  $MLf' = Mf$ . Thus by 1.1.11 we can augment the initiality.

$$FX = \text{in}_{ML}(Mf, A : A \in \underline{U}_0, f \in \underline{B}(X, LA))$$

$$\text{to } FX = \text{in}_{ML}(Mf, A : A \in \underline{U}', f \in \underline{B}(X, LA))$$

where  $\underline{U}' = \underline{U}_0 \cup \underline{U}$ .



## 1.6. RIGHT INVERSES AND ADJOINTS

In this section we give sufficient conditions for an amnestic functor  $L : \underline{A} \rightarrow \underline{B}$  to have an adjoint right inverse, that is, a right inverse which is also a (left) adjoint in the sense of [Kan 1958], [Freyd 1964], [Herrlich 1968], (and termed a "coadjoint" by Mitchell [1965]). An adjoint right inverse was called a lari by Gray [1965].

The basic relations between adjoint right inverses and finest right inverses are clarified with the aid of the special theory of the previous section.

THROUGHOUT THIS SECTION,  $L : \underline{A} \rightarrow \underline{B}$  AND  $M : \underline{B} \rightarrow \underline{C}$  ARE GIVEN AMNESTIC FUNCTORS.

1.6.1. Proposition If  $F$  and  $G$  are right inverses of  $L$ , then  $F \leq_L G$  if and only if  $F \leq_{ML} G$ .

□

Hence in dealing with right inverses of  $L$ , we may use the terms finer, finest, coarser, coarsest without the prefixes  $L$ - or  $ML$ -.

We shall occasionally impose the condition on the category  $\underline{B}$  that there is only one natural transformation of the identity functor  $1_{\underline{B}}$  to itself. This is no great restriction, as it is

fulfilled when  $\underline{B}$  has a generator (in the sense of [Mitchell 1965]) with just one endomorphism; such is the case when  $\underline{B}$  is a concrete category containing a one-point object. (I owe this remark to Dr. K.A. Hardie.)

1.6.2. Proposition Let  $\underline{B}$  be such that there is only one natural transformation of  $1_{\underline{B}}$  to itself. Then:

- (i)  $L : \underline{A} \rightarrow \underline{B}$  has at most one adjoint right inverse.
- (ii) If  $R$  is an adjoint right inverse of  $L$  with the adjunctions  $r : 1_{\underline{B}} \rightarrow LR$  and  $s : RL \rightarrow 1_{\underline{A}}$ , then  $r = (1_{\underline{B}})_{\underline{B}} \in \underline{B}$  and  $Ls = (1_{\underline{A}})_{\underline{A}} \in \underline{A}$ .
- (iii) If  $R$  is an adjoint right inverse of  $L$  then  $1_{\underline{A}} \leq_L RL$  and hence  $R$  is the finest right inverse of  $L$ .

Proof (i) If  $R$  and  $R'$  are adjoint right inverses of  $L$ , there is a natural equivalence  $t : R' \rightarrow R$ . Then  $Lt : 1_{\underline{B}} \rightarrow 1_{\underline{B}}$ , so that  $Lt = (1_{\underline{B}})$ , and thus  $R \leq_L R'$ . Considering  $t^{-1}$  gives  $R' \leq_L R$ . As  $L$  is amnestic,  $R' = R$ .

(ii)  $r = (1_{\underline{B}})$  since  $LR = 1_{\underline{B}}$ . For each  $A \in \text{obj } \underline{A}$  we have

$$Ls_A \cdot r_{LA} = 1_{LA}$$

(see e.g. [Herrlich 1968, Satz 6.4.3]) and as  $r_{LA} = 1_{LA}$ ,

$$Ls_A = 1_{LA}.$$

(iii) Immediate from (ii).

□

The following example shows that it is possible for the functor  $L$  to have a finest right inverse but no adjoint. However, I have not investigated whether it is possible for  $L$  (in 1.6.2) to have a finest right inverse and an adjoint, but no adjoint right inverse.

1.6.3. Example Let  $L$  be the forgetful functor  $\underline{Un} \rightarrow \underline{Prox}$ . The condition in 1.6.2 is satisfied. There is a well-known right inverse  $R$  of  $L$ , assigning to each proximity space the coarsest compatible uniform space, which is also the unique compatible precompact uniform space [Alfsen and Fenstad 1959]. Hence if  $A$  is a non-precompact uniform space, then  $A \not\ll_L RLA$ . Thus  $1_A \not\ll_L RL$ , so that by 1.6.2(iii)  $R$  is not adjoint to  $L$ . However there is an unpublished proof by S. Salbany (and an implicit claim in [Hušek 1967a, example 5]) that  $R$  is the only (and thus the finest) right inverse of  $L$ . Further,  $L$  has no adjoint. Indeed,  $R$  induces an equivalence between  $\underline{Prox}$  and  $\underline{Pc}$  (the full subcategory of precompact spaces in  $\underline{Un}$ ) carrying  $L$  to the precompact reflection  $\underline{Un} \rightarrow \underline{Pc}$  and  $R$  to the inclusion  $\underline{Pc} \rightarrow \underline{Un}$ ; thus  $L$  is adjoint to  $R$ . Hence, if  $L$  had an adjoint  $K$ , then  $LK$  would be adjoint to  $LR$  (which is  $1_{\underline{Prox}}$ ) and thus  $LK$  would be naturally equivalent to  $1_{\underline{Prox}}$ . By considering the underlying sets one then readily sees that  $K$  would be naturally equivalent to  $R$ , in contradiction to the fact that  $R$  is not adjoint to  $L$ .

□

1.6.4. Proposition If  $R$  is a right inverse of  $L$  such that  $1_A \leq_L RL$ , then  $R$  is adjoint to  $L$ .

Proof. The adjunctions  $(1_B) : 1_B \rightarrow LR$  and  $s : RL \rightarrow 1_A$  (given by  $1_A \leq_L RL$ ) satisfy the required conditions (e.g. [Herrlich 1968, Satz 6.4.3]) for  $R$  to be adjoint to  $L$ .

□

1.6.5. Proposition Let  $R$  be a right inverse of  $L$ . Then  $1_A \leq_L RL$  if and only if  $R = (L, 1_B : (\text{obj } \underline{A}))$ .

Proof Let  $1_A \leq_L RL$ . By 1.4.27 there is a largest subclass  $\underline{U}$  of  $\text{obj } \underline{A}$  with  $R = (L, 1_B : (\underline{U}))$ , namely

$$\begin{aligned} \underline{U} &= \{A \in \text{obj } \underline{A} : A \leq_L RLA\} \\ &= \text{obj } \underline{A}. \end{aligned}$$

Conversely, from  $R = (L, 1_B : (\text{obj } \underline{A}))$  follows  $RLA$

$$RLA = \text{in}_L (g, U : U \in \text{obj } \underline{A}, g \in \underline{B} (LA, LU)).$$

Taking  $U = A$  and  $g = 1_{LA}$  we have  $f : RLA \rightarrow A$  with  $Lf = g$ , i.e.  $A \leq_L RLA$ .

□

1.6.6. Proposition Let there be only one natural transformation of  $1_B$  to itself.

- (i) If  $L$  has an adjoint right inverse  $R$ , then  $R = (L, 1_B : (\text{obj } \underline{A}))$ .
- (ii) If the functor  $(L, 1_B : (\text{obj } \underline{A}))$  exists, then it is the adjoint right inverse of  $L$ .

Proof (i) 1.6.2. and 1.6.5.

(ii) 1.5.2, 1.6.5 and 1.6.4.

□

1.6.7. Proposition If  $\underline{A}$  is L-initiality complete, then L has an adjoint right inverse.

Proof 1.5.2, 1.6.5, 1.6.4.

□

1.6.8. Example Let  $L : \underline{Un} \rightarrow \underline{Prox}$  be the forgetful functor. Then  $\underline{Un}$  is not L-initiality complete. (This follows from 1.6.3 and 1.6.7.)

□

We now extend proposition 1.6.7. by bringing in the second amnestic functor M (the motivation again being example 1.5.3 and the discussion in section 1.5).

1.6.9 Proposition Let  $\underline{A}$  be ML-initiality complete, let L preserve initiality for M and let  $L(\text{obj } \underline{A})$  be M-sufficient in  $\underline{B}$ . Then  $(L, M : (\text{obj } \underline{A}))$  is the adjoint right inverse of L.

Proof By 1.4.23,  $F = (L, M : (\text{obj } \underline{A}))$  is a right inverse of L. By 1.4.19, for each  $A \in \text{obj } \underline{A}$ ,  $A \leq_{ML} FLA$ . Thus there is  $j_A : FLA \rightarrow A$  with  $MLj_A = 1_{MLA}$ . As F is right inverse to

Postscript. There is a partial converse to proposition 1.6.9: If there is a right inverse R of L such that  $1_{\underline{A}} \leq_L RL$ , then L preserves initiality for M.

$L$ , we have  $Lj_A : LA \rightarrow LA$  and it follows that  $Lj_A = 1_{LA}$ .

Thus  $1_A \leq_L FL$ , and by 1.6.4  $F$  is adjoint to  $L$ .

□

1.6.10 Remarks (1) The condition that  $L(\text{obj } \underline{A})$  be  $M$ -sufficient in  $\underline{B}$  is often trivially satisfied, namely when  $L(\text{obj } \underline{A}) = \text{obj } \underline{B}$ .

(2) Let  $L$  be the forgetful functor  $\underline{Un} \rightarrow \underline{Crg}$ . The existence of an adjoint right inverse of  $L$  follows immediately from 1.6.9 (using the forgetful functor  $M : \underline{Crg} \rightarrow \underline{Ens}$  and the  $ML$ -initiality completeness of  $\underline{Un}$ , proved in 1.5.3). We observe that the existence of this adjoint right inverse is thus purely an elementary consequence of initiality considerations, though some texts (e.g. [Kelley 1955]) construct this functor by means of pseudometrics.

(3) We do not discuss the analogy between proposition 1.6.9 and Freyd's Adjoint Functor Theorem [Freyd 1964]. We do observe that each right inverse of a functor  $L$  can be regarded as an adjoint right inverse of a suitable restriction of  $L$  :

1.6.11 Proposition Let  $R$  be a right inverse of  $L$ . Let  $\underline{A}'$  be the full subcategory of  $\underline{A}$  whose class of objects is  $\{U \in \text{obj } \underline{A} : U \leq_L RLU\}$ .  $L$  and  $R$  induce functors  $L' : \underline{A}' \rightarrow \underline{B}$  and  $R' : \underline{B} \rightarrow \underline{A}'$ . Then  $R'$  is an adjoint right inverse to  $L'$ .

Proof Clearly  $L'R' = 1_{\underline{B}}$  and  $1_{\underline{A}} \cong L'R'L'$ . Thus by 1.6.4.

$R'$  is adjoint to  $L'$ .

□

### 1.7. STRUCTURE OF THE CLASS OF RIGHT INVERSES.

We again consider an amnestic functor  $L : \underline{A} \rightarrow \underline{B}$ , and show that, under certain conditions involving a second amnestic functor  $M : \underline{B} \rightarrow \underline{C}$ , the class of right inverses of  $L$  with the ordering "coarser than" is a complete "lattice". We say ' "lattice" ' because the class may be an unset, i.e. not a set. This occurs when  $L$  is the forgetful functor  $\underline{Un} \rightarrow \underline{Crg}$  and in some analogous cases. For these cases we raise two appealing problems concerning the occurrence of right inverses of the form  $(L, \mathbf{1}_{\underline{B}} : (\underline{U}))$  with  $\underline{U}$  having only one member.

1.7.1. Definition  $\underline{ri}(L)$  is the class of all right inverses of  $L$ , partially ordered by the relation  $\leq_L$ .

1.7.2. Remarks (1) In dealing with classes of functors  $\underline{B} \rightarrow \underline{A}$  we use the obvious symbols  $\sup_{ML}$ ,  $\inf_{ML}$  (resp.  $\sup_L$ ,  $\inf_L$ ) determined by the ordering  $\leq_{ML}$  (resp.  $\leq_L$ ). However by 1.6.1,  $\leq_{ML}$  and  $\leq_L$  coincide on  $\underline{ri}(L)$ , so that here we may simply write  $\sup$ ,  $\inf$ ,  $\leq$ .

(2) Since it is in general impossible for a functor to be a member of a class in a Bernays - von Neumann - Gödel theory of sets and classes, a device such as a scale of universes would be needed for handling  $\underline{ri}(L)$  and its subclasses. This objection only properly applies to part (ii) of our theorem

1.7.6; we treat that part naively as it will not be applied subsequently. All other occurrences of the symbol  $\underline{ri}(L)$  can be removed by the device : replace " $F \in \underline{ri}(L)$ " by " $F$  is a right inverse of  $L$ ".

1.7.3. Proposition Let  $F, G \in \underline{ri}(L)$  with  $F = (L, M : (\underline{U}))$  and  $G = (L, M : (\underline{V}))$ .

(i) If  $\underline{U} \subset \underline{V}$  then  $F \leq G$ .

(ii) If  $F \leq G$  and  $\underline{V}$  is maximal for  $G$ , then  $\underline{U} \subset \underline{V}$ .

Proof (i) is obvious.

(ii) If  $U \in \underline{U}$  then by 1.4.19

$$U \leq_{ML} FLU \leq_{ML} GLU$$

so that by 1.4.27,  $U \in \underline{V}$ .

□

1.7.4. Proposition Let  $\underline{A}$  be ML-initiality complete and let  $L$  preserve initiality for  $M$ .

Let  $F_j = (L, M : (\underline{U}_j)) \in \underline{ri}(L)$  ( $j \in J$ )

and let  $\underline{U} = \bigcup \{ \underline{U}_j : j \in J \}$ .

Then  $(L, M : (\underline{U})) = \sup \{ F_j : j \in J \} \in \underline{ri}(L)$ .

Proof Let  $F = (L, M : (\underline{U}))$ . Since by 1.4.22 each  $L\underline{U}_j$  is  $M$ -sufficient, so is  $L\underline{U}$ , and hence by 1.4.23  $F \in \underline{ri}(L)$ . By 1.7.3,  $F$  is an upper bound of the  $F_j$ . If  $G$  is any upper bound of the  $F_j$  in  $\underline{ri}(L)$ , then by 1.4.27 we have maximal  $\underline{V}$  with  $G = (L, M : (\underline{V}))$ , and 1.7.3 gives  $F \leq G$ .

□

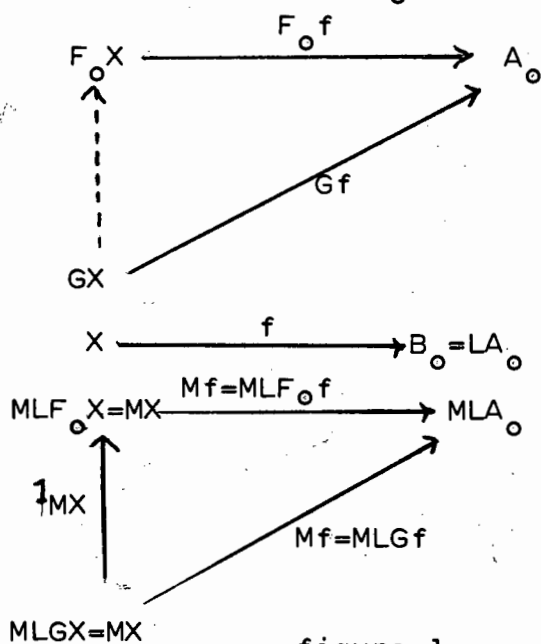
The following definition is motivated by three important examples : the space  $[0, 1]$  in  $\underline{Crg}$  which admits a unique uniformity, the space  $D_U$  in  $\underline{Top}$  which admits a unique quasi-uniformity [Brümmer 1969a] and the bitopological space  $I_b$  which admits a unique quasi-uniformity [Salbany 1970].

1.7.5. Definition An object  $B_0$  of  $\underline{B}$  is an L,M-pivot if  $B_0$  is M-sufficient in  $\underline{B}$  and there is a unique object  $A_0$  of  $\underline{A}$  such that  $LA_0 = B_0$ .

1.7.6. Theorem Let  $\underline{A}$  be ML-initiality complete, let  $L$  preserve initiality for  $M$ , and let  $\underline{B}$  have an L,M-pivot  $B_0$ . Then

- (i)  $L$  has a coarsest right inverse, namely  $(L, M : (\{L^{-1}(B_0)\}))$ .
- (ii) Each non-void subclass of  $\underline{ri}(L)$  has a supremum and an infimum in  $\underline{ri}(L)$ .

Proof (i) Let  $A_0 = L^{-1}(B_0)$  and let  $F_0 = (L, M : (\{A_0\}))$ , then



by 1.4.23  $F_0$  is a right inverse of  $L$ . Consider any right inverse  $G$  of  $L$ . For  $f : X \rightarrow LA_0$  we have  $Gf : GX \rightarrow A_0$  since, by the pivot property of  $B_0$ ,  $GLA_0 = GB_0 = A_0$ . It is now clear from figure 1 that  $F_0X \leq_{ML} GX$ , so that  $F_0 \leq G$ .

figure 1

(ii) Suprema are given by 1.7.4. Further, for each non-void subclass  $\underline{S}$  of  $\underline{ri}(L)$ , the class of lower bounds of  $\underline{S}$  is non-void (having  $F_0$  in it) and so has a supremum, which is then  $\inf \underline{S}$ .

□

The above theorem is an immediate generalization of [Brümmer 1969a, propositions 1.5 and 1.7].

1.7.7. Proposition Let  $J$  be an index set, let  $\underline{U} = \{U_j : j \in J\}$  be a subfamily of  $\text{obj } \underline{A}$ , let the product  $A_0 = \prod_{j \in J} U_j$  exist in  $\underline{A}$ , and let  $ML$  preserve this product.

(i) If  $(L, M : (\underline{U}))$  is a right inverse of  $L$ , then

$$(L, M : (\underline{U})) = (L, M : (\{A_0\})).$$

(ii) Suppose that  $L$  (as well as  $ML$ ) preserves the product  $\prod_{j \in J} U_j$ . Then, assuming only that  $(L, M : (\underline{U}))$  exists, we have

$$(L, M : (\underline{U})) = (L, M : (\{A_0\})).$$

Proof. Let  $F = (L, M : (\underline{U}))$ . To prove  $F = (L, M : (\{A_0\}))$ .

Given  $f : X \rightarrow LA_0$ , let  $f_j = Lp_j \cdot f$ , where  $p_j : A_0 \rightarrow U_j$  are the projections. There are  $f'_j : FX \rightarrow U_j$  with  $MLf'_j = Mf_j$ , and hence there is  $f' : FX \rightarrow A_0$  with  $p_j f' = f'_j$ . Then

$$\begin{aligned} MLP_j \cdot MLf' &= MLf'_j = \\ &= Mf_j = \\ &= MLP_j \cdot Mf, \end{aligned}$$

and as  $ML$  preserves the product, the  $MLp_j$  are projections, whence

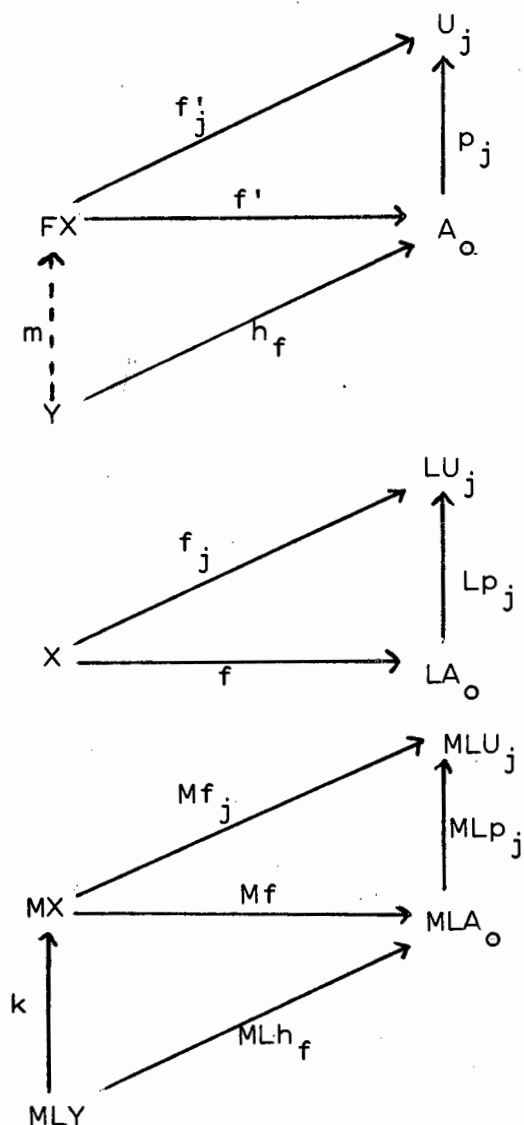


figure 2

$$MLf' = Mf.$$

Now consider  $h_f : Y \rightarrow A_0$  such that  $MLh_f = Mf.k$  for each  $f : X \rightarrow LA_0$ . Then

$Mf_j.k = ML(p_j.h_f)$ . If we assume that  $L$  preserves the product  $A_0$ ,

then  $LA_0$  is a product with projections  $Lp_j$  and then the  $f_j$  range through all possible

morphisms  $g : X \rightarrow LU_j$ . Since

$FX = \text{in}_{ML}(Mg_j, U_j : j \in J, g_j \in \underline{B}(X, LU_j))$

and we have the  $p_j.h_f : Y \rightarrow U_j$

such that the lower triangle

$(Mf_j.k = ML(p_j.h_f))$  commutes for

each  $g_j : X \rightarrow LU_j$  and each

$j \in J$ , it follows that there

exists  $m : Y \rightarrow FX$  with  $MLm = k$ .

Thus (ii) is proved. To prove (i)

we drop the assumption that  $L$  preserves the product, and we assume that  $F = (L, M : (\underline{U}))$  is right inverse to  $L$ . We need only prove that each  $g_j : X \rightarrow LU_j$  is of the form  $Lp_j.f$ , with  $f : X \rightarrow LA_0$ . Consider  $g_j : X \rightarrow LU_j$  given for each  $j \in J$ . There exist  $g'_j : FX \rightarrow LU_j$  with  $MLg'_j = Mg_j$ . Hence there exists  $g' : FX \rightarrow A_0$  with  $p_j.g' = g'_j$ . Let  $f = Lg'$ . Since  $LFX = X$ , we have  $f : X \rightarrow LA_0$  and both  $Lg'_j, g_j : X \rightarrow LU_j$ . Thus by the faithfulness of  $M$ ,  $Lg'_j = g_j$ . Now  $g_j = Lg'_j = L(p_j.g') = Lp_j.f$ , with  $f : X \rightarrow LA_0$ , as required.

□

1.7.8. Definition. A right inverse  $F$  of  $L$  is simple if there is an object  $A_0$  in  $\underline{A}$  such that  $F = (L, 1_{\underline{B}} : (\{A_0\}))$ .

We point out that in this definition we expressly avoid the second functor  $M$  in order to make the definition intrinsic. It is relevant that functors of the form  $(L, 1_{\underline{B}} : (\underline{U}))$  are always right inverses of  $L$  (1.5.2).

1.7.9. Proposition. If  $A_0$  is an object of  $\underline{A}$  such that  $F = (L, M : (\{A_0\}))$  is a right inverse of  $L$ , then  $F$  is simple.

Proof. By 1.5.5,  $F = (L, 1_{\underline{B}} : (\{A_0\}))$ .

□

1.7.10 Proposition. Let  $\underline{A}$  have products and let  $ML$  preserve products. If there is a set  $\underline{U}$  of objects of  $\underline{A}$  such that  $F = (L, M : (\underline{U}))$  is a right inverse of  $L$ , then  $F$  is simple.

Proof. 1.7.7(i) and 1.7.9.

□

It is possible for  $L$  to have no right inverse, or exactly one (example 1.6.3), or exactly two (example 1.5.4). We now discuss a case where there is an unset of right inverses. For the rest of this section we consider the two forgetful functors

$$T : \underline{Un} \rightarrow \underline{Crg}, \quad E : \underline{Crg} \rightarrow \underline{Ens}.$$

1.7.11 Theorem. For each infinite cardinal number

$m \geq 2^{\aleph_0}$ , let

$$\underline{U}_m = \{A : A \in \text{obj } \underline{Un} \text{ and } \text{card } \text{ETA} \leq m\}$$

and  $F_m = (T, E : (\underline{U}_m))$ .

Then:

- (i) Each  $F_m$  is a simple right inverse of  $T$ .
- (ii)  $m < n$  implies  $F_m < F_n$  (i.e.  $F_m \neq F_n$  and  $F_m \leq F_n$ ).  
Thus  $T$  has an uncountable set of right inverses.
- (iii) For each simple right inverse  $G$  of  $T$ , there exists  $m$  such that  $G \leq F_m$ .

Proof (i) The condition  $m \geq 2^{\aleph_0}$  ensures that the uniform space on  $[0, 1]$  is in  $\underline{U}_m$ , and as  $[0, 1]$  is  $E$ -sufficient in  $\underline{Crg}$ , it follows from 1.4.23 that  $F_m$  is a right inverse of  $L$ .

That  $F_m$  is simple, is seen by taking one representative from each uniform isomorphism class in  $\underline{U}_m$  and applying 1.7.10.

(ii) follows as in [Brümmer 1969a, proposition 1.9].

(iii) Let  $G = (L, \mathbb{1}_B : (\{A_\alpha\}))$ . By 1.5.5,

$$F_m = (L, \mathbb{1}_B : (\underline{U}_m)),$$

and we take  $m$  large enough that  $A_\alpha \in \underline{U}_m$ . Then by 1.7.3

$$G \leq F_m.$$

□

1.7.12 Corollary. The adjoint right inverse of the forgetful functor  $T : \underline{Un} \rightarrow \underline{Crg}$  is not simple.

Proof By 1.6.10(2) the adjoint right inverse exists; it is the finest right inverse of  $T$  and hence by 1.7.11(iii) and (ii) cannot be simple.

□

We observe that, for a simple right inverse  $G$  of  $T$ , the smallest cardinal  $m$  such that  $G \leq F_m$  gives a rough estimate of the "height" of  $G$  in the "lattice"  $\underline{ri}(T)$ . This leads to the following question.

1.7.13 Problem (1) Is the adjoint right inverse the only non-simple right inverse of the forgetful functor

$T : \underline{Un} \rightarrow \underline{Crg}$ ?

(2) Suppose  $F, G \in \underline{ri}(T)$  and  $F < G$ . If  $G$  is simple, is then  $F$  simple?

1.7.14 Remarks (1) All results of this section are very readily applicable, mutatis mutandis, when  $T$  and  $E$  are the following pairs of forgetful functors:

$\underline{Un} \rightarrow \underline{Crg} \rightarrow \underline{Ens}$

$\underline{Qun} \rightarrow \underline{Top} \rightarrow \underline{Ens}$

$\underline{Qun} \rightarrow \underline{Bcrg} \rightarrow \underline{Ens}$

and also when these are restricted to separated spaces. For the sake of brevity we have not given a generalization of theorem 1.7.11 that would contain all these particular cases. Such a generalization can be made if one introduces the notion of ML-discrete objects in  $\underline{A}$  and follows the idea of the proof of the quasi-uniform analogue of 1.7.11(ii) in the paper [Brümmer 1969a].

(2) Hušek [1967a, example 2] gives an upset of full embeddings of the category of semi-uniformizable spaces into the category of semi-uniform spaces. His construction involves discrete

objects but is further different from our construction for 1.7.11(ii). Our construction gives right inverses, and they of course are full embeddings.

(3) Problem 1.7.13 was raised in the talk [Brümmer 1969c].

1.8. MONOREFLECTIONS

The results of section 1.4 for the situation

$$\underline{A} \xrightarrow{L} \underline{B} \xrightarrow{M} \underline{C}$$

are here applied to the special situation

$$\underline{B} \xrightarrow{1_{\underline{B}}} \underline{B} \xrightarrow{M} \underline{C}.$$

Thus it is immediate that the functors  $(1_{\underline{B}}, M : (\underline{U}))$  with  $\underline{U} \subset \text{obj } \underline{B}$  are precisely the functors  $F : \underline{B} \rightarrow \underline{B}$  with  $FF = F \underset{M}{\cong} 1_{\underline{B}}$ . Further we find, under simple and useful conditions, that these functors are, up to natural equivalence, precisely the monoreflectors in  $\underline{B}$ . Finally, these results are used to obtain a general result that covers the behaviour of the totally bounded monoreflectors in  $\underline{Un}$  and  $\underline{Qun}$ .

We adhere to the convention that reflective as well as coreflective subcategories are replete (i.e. closed under isomorphism) and full. A full, replete subcategory  $\underline{B}'$  of  $\underline{B}$  is reflective iff the inclusion functor  $\underline{B}' \rightarrow \underline{B}$  has an adjoint. In agreement with [Herrlich 1968], [Kennison 1965], [Freyd 1964] (but not with [Mitchell 1965]) we call such an adjoint a reflection functor, and note that it is a functor  $\underline{B} \rightarrow \underline{B}'$ . It will be convenient to agree that a reflector  $\underline{B} \rightarrow \underline{B}$ , or reflector in  $\underline{B}$ , is the composition of the inclusion functor  $\underline{B}' \rightarrow \underline{B}$  with a reflection functor  $\underline{B} \rightarrow \underline{B}'$ .

THROUGHOUT THIS SECTION  $L : \underline{A} \rightarrow \underline{B}$

AND  $M : \underline{B} \rightarrow \underline{C}$  ARE GIVEN AMNESTIC FUNCTORS.

We start with the situation

$$\underline{B} \xrightarrow{1_{\underline{B}}} \underline{B} \xrightarrow{M} \underline{C}$$

It is clear that

$1_{\underline{B}}$  preserves initiality for  $M$ .

There is little to say about the right inverse of  $1_{\underline{B}}$ , so we consider only the first three levels of the hierarchy of functors of section 1.4.

From theorem 1.4.7. immediately follows:

1.8.1. Proposition Consider functions  $w : \underline{V} \rightarrow \text{obj } \underline{B}$  with  $\underline{V} \subset \text{obj } \underline{B}$ .

- (i) The functors  $F : \underline{B} \rightarrow \underline{B}$  with  $MF = M$  are precisely the functors  $F = (1_{\underline{B}}, M : w)$ .
- (ii) If  $\underline{B}$  is  $M$ -initiality complete, then  $(1_{\underline{B}}, M : w)$  exists for each function  $w$ .

Likewise theorem 1.4.16 gives:

1.8.2 Proposition The functors  $F : \underline{B} \rightarrow \underline{B}$  with  $F \leq_M 1_{\underline{B}}$  are precisely the functors  $F = (1_{\underline{B}}, M : w)$  such that, for each  $B \in \text{dom } w$ ,  $wB \leq_M B$ .

□

We observe that propositions 1.8.1. and 1.8.2 treat two distinct classes of functors. Indeed,  $F \leq_M 1_{\underline{B}}$  implies  $MF = M$ , but the converse fails, witness the forgetful functor

$M : \underline{\text{Top}} \rightarrow \underline{\text{Ens}}$  and the discrete coreflector  $F : \underline{\text{Top}} \rightarrow \underline{\text{Top}}$  (we may arrange that  $MF = M$ ). In fact, up to natural equivalence every coreflector  $F : \underline{\text{Top}} \rightarrow \underline{\text{Top}}$  satisfies  $MF = M$  [Herrlich 1968, p. 105], so that it would be interesting to seek conditions on  $w$  for  $(1_{\underline{B}}, M ; w)$  to be a coreflector in  $\underline{B}$ ; but we do not investigate that question (nor its interesting dual).

Again immediately, theorem 1.4.20 yields:

1.8.3. Theorem The functors  $F : \underline{B} \rightarrow \underline{B}$  with  $FF = F \leq_M 1_{\underline{B}}$  are precisely the functors  $(1_{\underline{B}}, M : (\underline{U}))$  with  $\underline{U} \subset \text{obj } \underline{B}$ .

□

To distinguish between the classes of 1.8.2 and 1.8.3 we have to show that  $F \leq_M 1_{\underline{B}}$  does not imply  $FF = F$ . I am indebted to Mr. I. Mordant for a conversation which led to my finding the following example which also fulfils the promise made after the proof of 1.4.20.

1.8.4. Example Let  $\underline{B}$  be the category of metric spaces and non-expansive mappings, i.e. mappings  $f : (X, d) \rightarrow (Y, e)$  such that  $e(f(x), f(y)) \leq d(x, y)$  for all  $x, y \in X$ . Let  $\underline{C}$  be the category of metrizable topological spaces and continuous mappings. Let  $M : \underline{B} \rightarrow \underline{C}$  be the forgetful functor which gives the metric topology to each metric space. One easily proves  $M$  amnestic. There is a functor  $F : \underline{B} \rightarrow \underline{B}$  with  $F \leq_M 1_{\underline{B}}$  determined by

$F(X, d) = (X, \frac{1}{2}d)$  for each metric space  $(X, d)$ . Clearly  $F$  is not idempotent.

□

1.8.5. Theorem Given a functor  $F : \underline{B} \rightarrow \underline{B}$ ,

- (i) If  $FF = F \leq_M \mathbf{1}_{\underline{B}}$  then  $F$  is a monoreflector in  $\underline{B}$ .
- (ii) Suppose  $M$  sends bimorphisms to isomorphisms and  $\underline{B}$  is  $M$ -initiality complete. If  $F$  is a monoreflector, then  $F$  is naturally equivalent to a functor  $G : \underline{B} \rightarrow \underline{B}$  with  $GG = G \leq_M \mathbf{1}_{\underline{B}}$ .

Proof (1) We have  $i_X : X \rightarrow FX$  with  $Mi_X = \mathbf{1}_{MX}$ . Let  $\underline{B}_0$  be the full subcategory of  $\underline{B}$  of objects which are fixed under  $F$ . From the amensticity of  $M$  one readily sees that  $\underline{B}_0$  is replete. Also  $FX$  is in  $\underline{B}_0$ . Given  $f : X \rightarrow Y$  with  $Y$  in  $\underline{B}_0$ , then since  $Mi_X = \mathbf{1}_{MX}$ ,  $Ff$  is the unique morphism  $FX \rightarrow Y$  satisfying  $Ff \cdot i_X = f$ . As moreover  $i_X$  is mono,  $(i_X, FX)$  is a  $\underline{B}_0$ -monoreflection.

(ii) Given a monoreflector  $F$ , let the reflection morphisms be  $i_X : X \rightarrow FX$ . Then  $i_X$  is a bimorphism [Herrlich 1968, p.79] and by hypothesis  $Mi_X$  is iso. Also by hypothesis, for each  $X$  there exists an object

$$GX = \text{in}_M(Mi_X, FX).$$

Hence we have unique  $t_X : GX \rightarrow FX$  with  $Mt_X = Mi_X$ , and the form of proof of 1.3.6 shows that  $t_X$  is an isomorphism. For  $f : Y' \rightarrow X$  let  $Gf = t_X^{-1} \cdot Ff \cdot t_Y$  then it is immediate that  $G$

is a functor and  $t = (t_X) : G \rightarrow F$  a natural equivalence. As

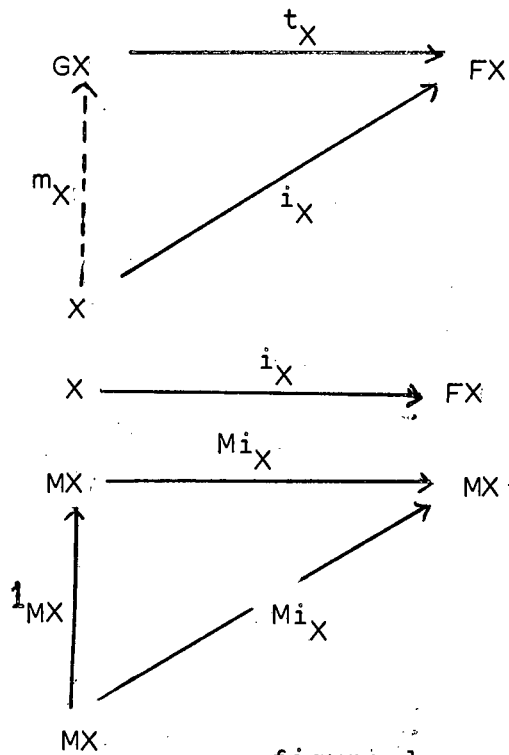


figure 1

the lower diagram in figure 1 commutes, there is  $m_X : X \rightarrow GX$  with  $Mm_X = 1_{MX}$ . Thus  $G \leq_M 1_{\underline{B}}$ .

Finally to see that  $GG = G$ .

From  $G \leq_M 1_{\underline{B}}$  follows  $GG \leq_M G$

and it remains to show that

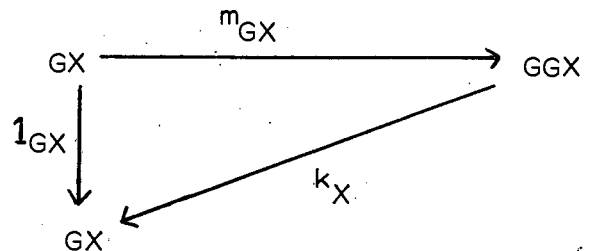
$G \leq_M GG$ . Clearly  $G$  is a

reflector with reflection

morphisms  $m_X : X \rightarrow GX$ . Thus

there exists  $k_X$  making the

diagram



commute, and then

$$1_{MX} = 1_{MGX} = M1_{GX} = M(k_X \cdot m_{GX}) = Mk_X \cdot 1_{MGX} = Mk_X,$$

whence  $G \leq_M GG$ . (Alternatively, to see that  $GG = G$  we may consider the triple associated with the reflector  $G$ .)



The above theorem is remarkable in that an intrinsic property of  $F : \underline{B} \rightarrow \underline{B}$  — that of being a monoreflector — is characterized in terms of an extrinsic datum, that of a functor  $M : \underline{B} \rightarrow \underline{C}$ . This is another instance of the usefulness of the "second amnestic functor."

follows:

$$\underline{A} \xrightarrow{1_{\underline{A}}} \underline{A} \xrightarrow{ML} \underline{C}.$$

Thus we replace  $M$  in 1.8.6 by  $ML$ , and only in order to apply 1.8.6(ii) do we need  $\underline{A}$  to be  $ML$ -initiality complete and  $ML$  to send bimerphisms to isomorphisms. Now a monoreflector  $(1_{\underline{A}}, ML : (\underline{U}))$  with  $\underline{U} \subset \text{obj } \underline{A}$  relates in some way to the functor  $(L, M : (\underline{U}))$ , and it is tempting to look for correspondences independent of the particular  $\underline{U}$ , but we do not propose to do that.

We recall that in 1.7.6(i) we found, under certain assumptions, that  $L$  had a coarsest right inverse  $(L, M : \{A_0\})$ , provided  $LA_0$  was an  $L, M$ -pivot. We also introduce for the first time the obvious infimum symbols  $\wedge_{ML}$  and  $\wedge_L$  determined by the orderings  $\leq_{ML}$  and  $\leq_L$  respectively.

1.8.8. Proposition Let the following five conditions hold.

- (1)  $B_0 = LA_0$  is an  $L, M$ -pivot.
- (2)  $F_0 = (L, M : (\{A_0\}))$  is the coarsest right inverse of  $L$ , and  $A'_0$  is also an object of  $\underline{A}$  such that  $F_0 = (L, M : (\{A'_0\}))$ .
- (3) The monoreflector  $s = (1_{\underline{A}}, ML : (\{A'_0\}))$  exists.
- (4)  $Ls = L$ .
- (5) For all objects  $X, Y$  of  $\underline{A}$ ,  $Y \leq_{ML} X = sX$  implies  $Y = sY$ .

Then

- (a)  $sF = F_0$  for each right inverse  $F$  of  $L$ .
- (b)  $s = 1_{\underline{A}} \wedge_{ML} F_0 L = 1_{\underline{A}} \wedge_L F_0 L$ .

Proof (a) Let  $F$  be right inverse to  $L$ . By (4),  $sF$  is right inverse to  $L$ , hence  $F_0 \leq_{ML} sF$ . Consider that

$$FX = \text{in}_{ML} (Mf, A'_0 : f \in \underline{B}(X, LA'_0))$$

$$\text{and } sFX = \text{in}_{ML} (MLg, A'_0 : g \in \underline{A}(FX, A'_0)).$$

Since each  $Lg$  is an  $f$ , we have  $sFX \leq_{ML} F_0X$  by 1.1.12. Thus  $sF = F_0$ .

(b) Consider that

$$F_0LY = \text{in}_{ML} (Mf, A'_0 : f \in \underline{B}(LY, LA'_0))$$

$$\text{and } sY = \text{in}_{ML} (MLg, A'_0 : g \in \underline{A}(Y, A'_0)).$$

Since each  $Lg$  is an  $f$ , we have  $s \leq_{ML} F_0L$ . Of course

$s \leq_{ML} \underline{1}_A$ . Given any ML-lower bound  $t$  of  $\underline{1}_A$  and  $F_0L$ , we have

to show  $t \leq_{ML} s$ . Now  $tY \leq_{ML} F_0LY = sF_0LY$  by (i), hence

$stY = tY$  by (5). As  $t \leq_{ML} \underline{1}_A$ , we have  $j_Y : Y \rightarrow tY$  with

$MLj_Y = \underline{1}$ ; also the reflection morphism  $i_Y : Y \rightarrow sY$  is such that  $MLi_Y = \underline{1}$ . The reflection property of  $(i_Y, sY)$  yields

$k_Y : sY \rightarrow tY$  with  $k_Y i_Y = j_Y$ . Hence  $MLk_Y = \underline{1}$ , i.e.  $t \leq_{ML} s$ ,

so that  $s = \underline{1}_A \wedge_{ML} F_0L$ . Using (4) we readily get

$$s = \underline{1}_A \wedge_L F_0L.$$

□

1.8.9. Remarks In proposition 1.8.8 we proved (a) without using (5).

If  $A'_0 = A_0$ , we can prove (a) without using (4) or (5).

(Proof: Since  $LA_0$  is an L,M-pivot, the morphisms  $f : X \rightarrow LA_0$  are precisely the  $Lg$  with  $g : FX \rightarrow A_0$ , and thus the initiality problems for  $F_0X$  and  $sFX$  are the same.)

Finally, if  $A'_0 = A_0$ , we have  $s = \mathbb{1}_{\underline{A}} \wedge_{ML} F_0 L$  without using (4).

The quasi-uniform case of (b) was proved in [Brümmer 1970].

## 1.9 LIFTING EPIREFLECTIONS

Let  $L : \underline{A} \rightarrow \underline{B}$  be a functor. We shall give sufficient conditions on  $L$  and  $\underline{A}$  to ensure that each epireflector  $r$  in  $\underline{B}$  is lifted against  $L$  to an epireflector  $r^L$  in  $\underline{A}$ . Further we show that if  $L$  preserves epimorphisms and has a right inverse  $R$ , then  $r \simeq Lr^LR$ .

1.9.1. Definition Given a reflector  $r$  in a category  $\underline{B}$ .

We write  $\underline{B}(r)$  for the full, replete subcategory of  $\underline{B}$  onto which  $r$  reflects  $\underline{B}$  (i.e.  $B \in \text{obj } \underline{B}(r)$  iff  $rB \simeq B$ ). Given a functor  $L : \underline{A} \rightarrow \underline{B}$ , then  $L^{-1}(\underline{B}(r))$  denotes the full, replete subcategory of  $\underline{A}$  such that

$A \in \text{obj } L^{-1}(\underline{B}(r))$  if and only if  $LA \in \text{obj } \underline{B}(r)$ .

We recall that replete means closed under isomorphism.

The repleteness of  $\underline{B}(r)$  is a matter of convention, that of  $L^{-1}(\underline{B}(r))$  is then implied. We shall informally speak of  $L^{-1}(\underline{B}(r))$  as the subcategory of  $\underline{A}$  lifted (against  $L$ ) from  $\underline{B}(r)$ .

If we think of  $\text{obj } \underline{B}(r)$  as a property, then we speak of  $\text{obj } L^{-1}(\underline{B}(r))$  as the lifted property.

1.9.2. Theorem Let the category  $\underline{A}$  be complete, well- and co-well-powered. Given a functor  $L : \underline{A} \rightarrow \underline{B}$  and an epireflector  $r$  in  $\underline{B}$ . Either of the following conditions is sufficient for  $L^{-1}(\underline{B}(r))$  to be an epireflective subcategory of  $\underline{A}$ .

- (1)  $L$  preserves products and extreme monomorphisms.
- (2)  $L$  preserves limits.

Proof. Let (1) hold. We show that  $L^{-1}(\underline{B}(r))$  is closed for the taking of products in  $\underline{A}$ . Let  $A$  be the  $\underline{A}$ -product of objects  $A_j$  from  $L^{-1}(\underline{B}(r))$ . Then  $LA$  is  $\underline{B}$ -product of the  $LA_j$ , and these are in  $\underline{B}(r)$ . As  $\underline{B}(r)$  is reflective in  $\underline{B}$  it follows (without any assumptions on  $\underline{B}$ ) that  $LA$  is in  $\underline{B}(r)$  [Herrlich 1968, Satz 9.1.2]. Thus  $A$  is in  $L^{-1}(\underline{B}(r))$ . Similarly, using [Herrlich 1968, Satz 10.1.2] we see that  $L^{-1}(\underline{B}(r))$  is closed for the taking of extreme subobjects, i.e. whenever  $f : X \rightarrow Y$  is an  $\underline{A}$ -extreme monomorphism with  $Y$  in  $L^{-1}(\underline{B}(r))$ , then  $X$  is in  $L^{-1}(\underline{B}(r))$ . Thus  $L^{-1}(\underline{B}(r))$  is closed for the taking of products and extreme subobjects in  $\underline{A}$ , and by the assumptions on  $\underline{A}$  it follows from [Herrlich 1968, Satz 10.2.1] that  $L^{-1}(\underline{B}(r))$  is epireflective in  $\underline{A}$ .

Let (2) hold, i.e. let  $L$  preserve limits. We show that  $L^{-1}(\underline{B}(r))$  is strongly closed for the taking of limits in  $\underline{A}$ ; then epireflectivity of  $L^{-1}(\underline{B}(r))$  follows from [Herrlich 1968, Satz 10.2.1]. Let  $D : \underline{J} \rightarrow \underline{A}$  be a diagram partially in  $L^{-1}(\underline{B}(r))$  [Herrlich 1968, definition 9.3.1], and let  $(A, a_j)$  be  $\underline{A}$ -limit of  $D$ . Then  $(LA, La_j)$  is  $\underline{B}$ -limit of the diagram  $LD : \underline{J} \rightarrow \underline{B}$  which is partially in  $\underline{B}(r)$ . By [Herrlich 1968, Satz 9.3.2]  $LA$  is in  $\underline{B}(r)$ , i.e.  $A$  is in  $L^{-1}(\underline{B}(r))$ . Thus  $L^{-1}(\underline{B}(r))$  is strongly closed for the taking of limits in  $\underline{A}$ .

□

The above theorem seems simpler than the corresponding result of Hušek [1967c, theorem 2] which deals with reflections.

1.9.3. Definition Given a functor  $L : \underline{A} \rightarrow \underline{B}$  and an epireflector  $r$  in  $\underline{B}$ . If the subcategory  $L^{-1}(\underline{B}(r))$  is epireflective in  $\underline{A}$ , we shall denote by  $r^L$  the epireflector in  $\underline{A}$ , determined uniquely up to natural equivalences, which is such that  $L^{-1}(\underline{B}(r)) = \underline{A}(r^L)$ . We shall then say that  $r^L$  exists, or that  $r$  can be lifted against  $L$ .

1.9.4. Theorem Given a functor  $L : \underline{A} \rightarrow \underline{B}$  and an epireflector  $r$  in  $\underline{B}$  such that the epireflector  $r^L$  in  $\underline{A}$  exists. If  $L$  preserves epimorphisms and has a right inverse  $R : \underline{B} \rightarrow \underline{A}$ , then we have a natural equivalence.

$$r \simeq Lr^LR.$$

Proof Consider  $f : X \rightarrow Y$  in  $\underline{B}$  with  $Y$  in  $\text{obj } \underline{B}(r)$ . Then  $Rf : RX \rightarrow RY$  with  $RY$  in  $\text{obj } \underline{A}(r^L)$  (since  $LR Y = Y$ ). We have

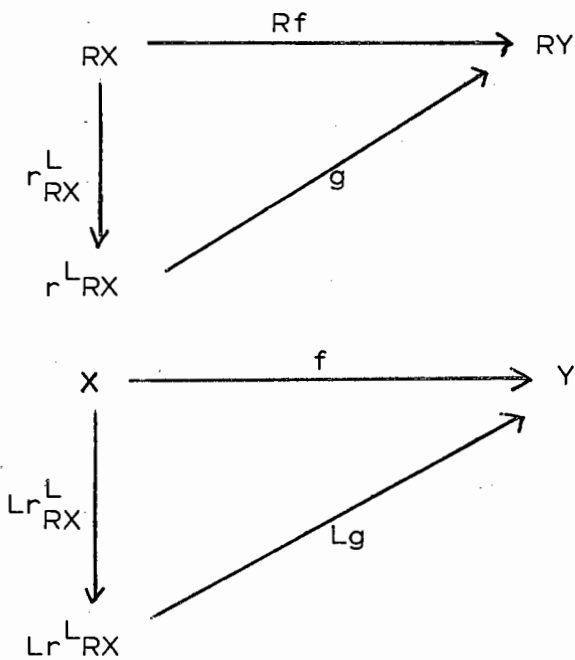


figure 1

the reflection morphism  $r^L_{RX} : RX \rightarrow r^L RX$  and a unique morphism  $g : r^L RX \rightarrow RY$  making the upper triangle in figure 1 commute. As  $L$  sends  $r^L_{RX}$  to an epimorphism,  $Lg$  is unique in making the lower triangle commute. By definition of  $r^L$  we have  $Lr^L RX \in \text{obj } \underline{B}(r)$ . Hence  $(Lr^L_{RX}, Lr^L RX)$  is a  $\underline{B}(r)$ -reflection of  $X$  and therefore equivalent to  $(r_X, rX)$ . Thus  $r \simeq Lr^LR$ .



The above theorem was inspired by the corresponding result of Salbany [1970, p.37] for the lifting of the Stone-Čech compactification to the category of pairwise Tihonov bitopological spaces. This application, among others, will be given in Chapter 2.

I have not investigated sufficient conditions for the natural equivalence

$$rL \cong Lr^L$$

to hold. We observe that this relation implies  $r \cong Lr^L R$  for any right inverse  $R$  of  $L$ .

CHAPTER 2. APPLICATIONS IN THE THEORY  
OF UNIFORM AND QUASI-UNIFORM SPACES.

2.1. THE BASIC CATEGORIES AND FORGETFUL FUNCTORS.

In this section the categories of the (quasi-) uniform spaces, the (bi-) topological spaces and the (pairwise) completely regular spaces are defined for the convenience of the reader. Basic properties of the associated forgetful functors are proved.

All definitions given coincide with or are equivalent to those in [Murdeswar and Naimpally 1966], [Lane 1967] and [Salbany 1970].

2.1.1. Definitions. A quasi-uniform space  $X$  is a set  $|X|$  together with a filter  $\text{ent } X$  of subsets of  $|X| \times |X|$ , called entourages in  $X$ , such that :

- (i) each entourage contains the diagonal  

$$\{(x, x) : x \in X\}$$
- (ii) for each entourage  $H$  there exists an entourage  $K$  such that  $K \circ K \subset H$ .

In the category  $\underline{\text{Qun}}$  of the quasi-uniform spaces, the morphisms  $f : X \rightarrow Y$  are the functions  $f : |X| \rightarrow |Y|$  such that, for each  $K$  in  $\text{ent } Y$ ,  $(f \times f)^{-1}K$  is in  $\text{ent } X$ .

The conjugation functor  $(-1) : \underline{\text{Qun}} \rightarrow \underline{\text{Qun}}$  is given by  $X \rightsquigarrow X^{(-1)}$ , functions being preserved, where  $X^{(-1)}$  has as entourages the sets  $H^{-1}$  with  $H$  in  $\text{ent } X$ .

The category Un of the uniform spaces is the full subcategory of Qun which stays fixed under the conjugation functor.

Top is the category of the topological spaces and continuous mappings; it has the obvious forgetful functor

$$E : \underline{\text{Top}} \rightarrow \underline{\text{Ens.}}$$

The forgetful functor

$$T_1 : \underline{\text{Qun}} \rightarrow \underline{\text{Top}}$$

is given thus: for  $X \in \text{obj } \underline{\text{Qun}}$  and each point  $x \in |X|$ , the neighbourhoods of  $x$  in  $T_1 X$  are the sets  $H[x]$  with  $H$  in  $\text{ent } X$ ; and  $T_1$  preserves functions.

The forgetful functor

$$T : \underline{\text{Un}} \rightarrow \underline{\text{Top}}$$

is the restriction of  $T_1$  to Un, written

$$T = T_1|_{\underline{\text{Un}}}.$$

A bitopological space  $X$  is an ordered pair  $(X_1, X_2)$  of topological spaces with the same underlying set  $EX_1 = EX_2$ ; the functions  $f : EX_1 \rightarrow EY_1$  which are continuous mappings  $X_k \rightarrow Y_k$  (for  $k = 1, 2$ ) are the morphisms for the category 2 Top of the bitopological spaces. The forgetful functor

$$\bar{E} : \underline{2 \text{ Top}} \rightarrow \underline{\text{Ens}}$$

assigns the underlying sets and functions.

The important forgetful functor

$$\bar{T} : \underline{\text{Qun}} \rightarrow \underline{2 \text{ Top}}$$

is given by  $X \rightsquigarrow (T_1 X, T_1(X^{(-1)}))$ , functions being preserved.

There is a forgetful functor

$$s : \underline{\text{Qun}} \rightarrow \underline{\text{Un}}$$

which can be defined as that coreflection functor of Qun onto Un which preserves underlying sets. Clearly

$$sX = X \vee X^{(-1)}$$

(we should write  $\vee_{\overline{E}\overline{T}}$  but we omit mention of a forgetful functor to Ens.)

The forgetful functor

$$S : \underline{2 \text{ Top}} \rightarrow \underline{\text{Top}}$$

is given by  $SX = X_1 \vee X_2$ , functions being preserved.

The following proposition recapitulates all the forgetful functors defined above.

2.1.2. Proposition

The diagram of forgetful functors in figure 1 is commutative.

Commutativity is lost if the functor

$T_1 : \underline{\text{Qun}} \rightarrow \underline{\text{Top}}$  is also entered.

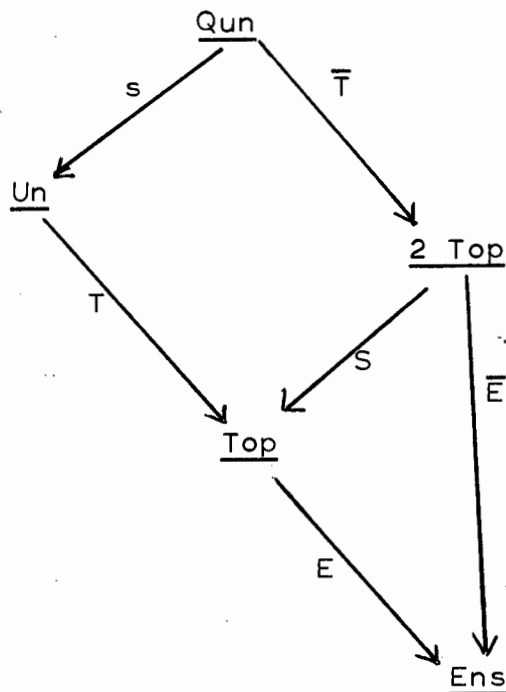


figure 1



We need symbols for a number of special spaces.

2.1.3. Definition. The real line and its subsets  $[0, 1]$  and  $\{0, 1\}$ , with their usual topology, are denoted by  $\mathbb{R}, I, D$  respectively.

These sets, equipped with entourages for which a filter base consists of the sets

$$\{(x, y) : |x - y| < r\} \quad \text{with } r > 0,$$

become uniform spaces, which we denote by

$$\mathbb{R}_s, I_s, D_s \quad \text{respectively.}$$

(Thus these spaces have the standard or usual uniformity.)

We may equip the same sets with the so-called upper quasi-uniformity, basic entourages being the sets

$$\{(x, y) : y < x + r\}, \quad r > 0.$$

We denote the resulting quasi-uniform spaces by

$$\mathbb{R}_q, I_q, D_q \quad \text{respectively.}$$

Finally we name the bitopological spaces

$$\mathbb{R}_b = \bar{T}\mathbb{R}_q, \quad I_b = \bar{T}I_q, \quad D_b = \bar{T}D_q$$

and the topological spaces (having the so-called upper topology)

$$\mathbb{R}_u = T_1\mathbb{R}_q, \quad I_u = T_1I_q, \quad D_u = T_1D_q.$$

2.1.4. Proposition. For each quasi-uniform space  $Y$ ,

$$T_1Y = \text{in}_E (ET_1f, I_u : f \in \underline{\text{Qun}}(Y, I_q)).$$

Proof The right hand side clearly exists and is  $E$ -coarser than  $T_1Y$ ; we have to show it also  $E$ -finer than  $T_1Y$ . Let  $Q$

be the set of quasi-pseudometrics which determines  $Y$  ([Brümmer 1969a, proposition 2.1.]). Consider any open set  $U$  in  $T_1 Y$  and any point  $b$  in  $U$ . There exist  $d_1, \dots, d_n$  in  $Q$ , bounded above by 1, with

$$\left( \bigcap_{k=1}^n d_k^{-1} [0, 1) \right) [b] \subset U.$$

Let  $d = d_1 \vee \dots \vee d_n$ . Since for each  $r > 0$ ,

$$d^{-1} [0, r) = \bigcap_{k=1}^n d_k^{-1} [0, r)$$

is an entourage in  $Y$ , we have  $d \in Q$ . There is a morphism  $g : Y \rightarrow I_q$  given by

$$g(y) = d(b, y)$$

(verification only involves the triangle inequality). The proof is completed by observing that

$$b \in g^{-1} [0, 1) \subset U.$$

□

The above proof is from [Brümmer 1970]. Essentially the same result was obtained, with a longer proof, by Fletcher [1970, theorem 6]. The idea is from [Bourbaki 1948, p.9].

2.1.5. Theorem For each quasi-uniform space  $Y$ ,

$$\bar{T}Y = \text{in}_E (\bar{E}\bar{T}f, I_b : f \in \underline{Qun} (Y, I_q)).$$

Proof. Noting that  $I_U = T_1 I_q$ , we see that the proof of 2.1.4 can readily be adapted to yield

$$T_1 X = \text{in}_E (ET_1 f, T_1(I_q^{(-1)})) : f \in \underline{Qun} (X, I_q^{(-1)})$$

which upon replacing  $X$  by  $Y^{(-1)}$  becomes

$$T_1(Y^{(-1)}) = \text{in}_E (ET_1 f, T_1(I_q^{(-1)})) : f \in \underline{Qun} (Y, I_q).$$

This together with 2.1.4 immediately gives the stated equality.

□

We now retrieve a result of Lane [1967] corresponding to a classical theorem of Weil [1937].

2.1.6. Proposition Let  $X$  be a bitopological space. Then

$$X = \text{in}_{\bar{E}} (\bar{E}g, I_b : g \in \underline{2\text{Top}}(X, I_b))$$

if and only if there exists a quasi-uniform space  $Y$  with  $\bar{T}Y = X$ .

Proof. Let  $\bar{T}Y = X$ . As  $\bar{T}$  preserves initiality for  $E$ ,

2.1.5 implies

$$\begin{aligned} X &= \text{in}_{\bar{E}} (\bar{E}\bar{T}f, I_b : f \in \underline{\text{Qun}}(Y, I_q)) \\ &= \text{in}_{\bar{E}} (\bar{E}g, I_b : g \in \underline{2\text{Top}}(X, I_b)) \end{aligned}$$

(using 1.1.11). Conversely, taking the quasi-uniform space

$$Y = \text{in}_{\bar{E}} (\bar{E}g, I_q : g \in \underline{2\text{Top}}(X, I_b))$$

we have  $\bar{T}Y = X$  provided  $X$  satisfies the given condition.

□

2.1.7. Definition  $\underline{\text{Bcrg}}$  is the full subcategory of  $\underline{2\text{Top}}$  whose objects  $X$  satisfy

$$X = \text{in}_{\bar{E}} (\bar{E}g, I_b : g \in \underline{2\text{Top}}(X, I_b)).$$

These spaces are called pairwise completely regular [Lane 1967], [Salbany 1970].

2.1.8. Proposition  $\underline{\text{Bcrg}}$  is a monoreflective subcategory of  $\underline{2\text{Top}}$ .

Proof. The monoreflector is  $(\underline{1}_{\underline{2\text{Top}}}, \bar{E} : (\{I_b\}))$ , (see 1.8.6)

□

2.1.9. Proposition A bitopological space  $X$  is in Bcrg if and only if there exists a quasi-uniform space  $Y$  with  $\overline{TY} = X$ .

(One then says :  $X$  is quasi-uniformizable,  $X$  admits  $Y$ ,  $Y$  is compatible with  $X$ .)

Proof. 2.1.6, 2.1.7. □

Applying the functors  $s$  and  $S$  to 2.1.5 - 2.1.9 readily yields their classical analogues, which we summarize as follows.

2.1.10. Proposition , (i) For each uniform space  $Y$ ,

$$TY = \text{in}_{ET} (ETf, I : f \in \underline{\text{Un}} (Y, I_s)).$$

(ii) The category Crg of the completely regular spaces is monoreflective in Top, with monoreflector  $(\underline{1}_{\text{Top}}, E : (\{I\}))$ .

(iii) The objects of Crg are the uniformizable topological spaces. □

2.1.11. CONVENTION In terms such as :  $M$ -coarser,  $M$ -initiality,  $M$ -embedding,  $M$ -injection,  $M$ -separated object,  $M$ -sufficient class,  $\text{sup}_M$  etc. we omit mention of  $M$  if  $M$  is a forgetful functor to Ens.

2.1.12. Theorem The categories

Qun, Un, Bcrg, Crg, 2 Top, Top  
are initiality complete.

Proof. See 1.5.3 for Un. Qun is similar. Trivial for Top and 2 Top. For Crg: an initiality problem in Crg can be regarded as an initiality problem in Top, where it then has a solution  $X$ . The Crg-monoreflection of  $X$  is then the solution in Crg; this works because the monoreflector preserves the underlying set.

Likewise for Bcrg.

□

The functor  $S : \underline{2\ Top} \rightarrow \underline{Top}$  sends Bcrg-objects to Crg-objects. (Indeed if  $X = \bar{T}Y$ , then  $SX = S\bar{T}Y = TsY$ , and  $sY \in \text{obj } \underline{Un}$ .) This together with 2.1.9 and 2.1.10(iii) warrants us to define:

### 2.1.13 Definition

$S' : \underline{Bcrg} \rightarrow \underline{Crg}$  is the functor induced by  $S : \underline{2\ Top} \rightarrow \underline{Top}$ .

$\bar{T}' : \underline{Qun} \rightarrow \underline{Bcrg}$  is the functor induced by  $\bar{T} : \underline{Qun} \rightarrow \underline{2\ Top}$ .

$T' : \underline{Un} \rightarrow \underline{Crg}$  is the functor induced by  $T : \underline{Un} \rightarrow \underline{Top}$ .

$$\bar{E}' = \bar{E} \lfloor \underline{Bcrg} \quad E' = E \lfloor \underline{Crg}.$$

2.1.14 Proposition. Throughout the diagram of figure 2 it is true that  $L$  preserves initiality for  $M$ . Here  $L$  and  $M$  can be taken from the displayed functors or their compositions, provided that the target of  $L$  is the source of  $M$ , and the target of  $M$  is Ens. Further, all functors in figure 2 are amnestic.

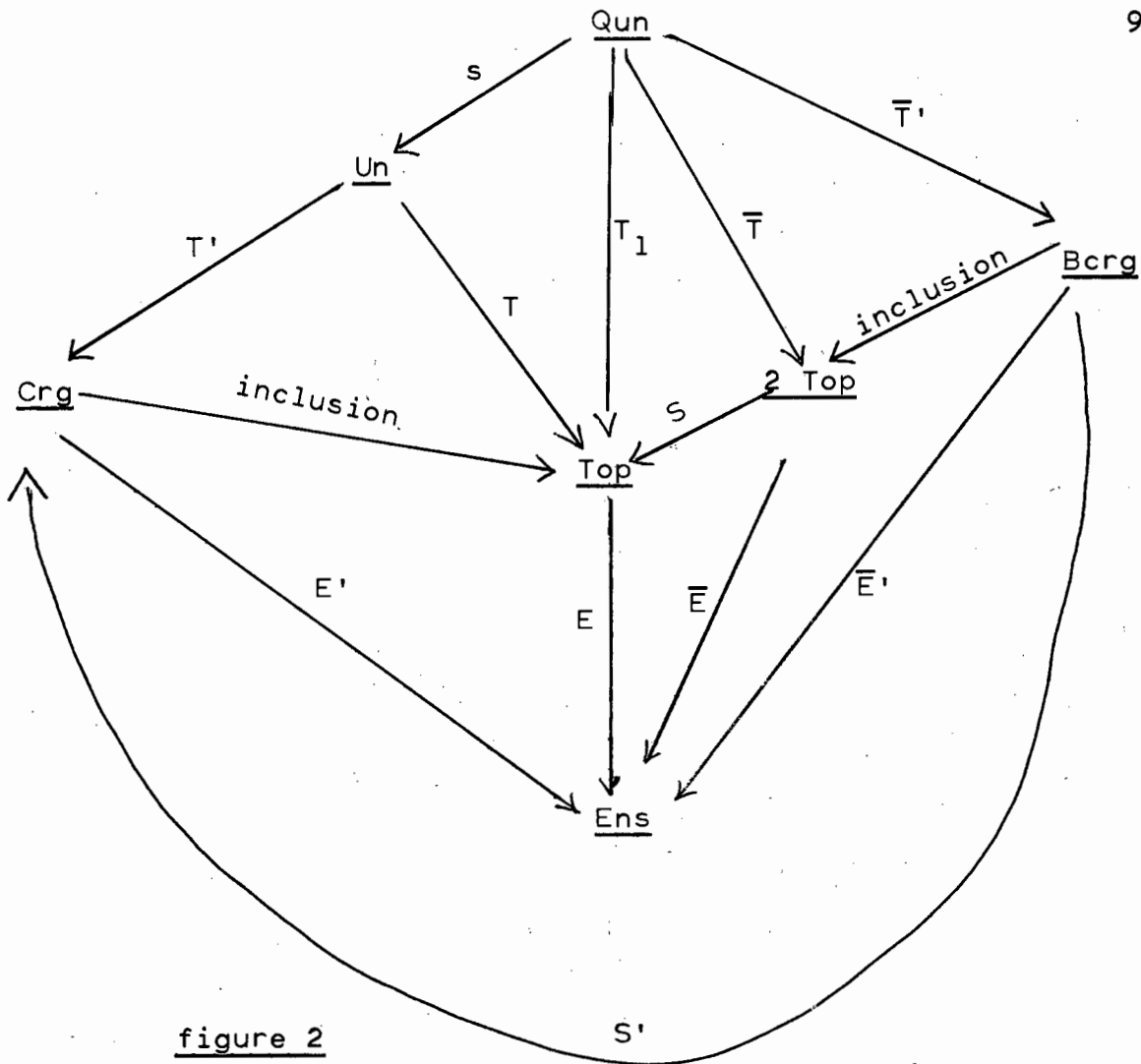


figure 2

Proof To show that the inclusion functor Crg  $\rightarrow$  Top preserves initiality, one considers its adjoint, the mono-reflection functor. Likewise for Bcrg  $\rightarrow$  2 Top. The proofs for the other functors are straightforward. One observes that preservation of initiality respects composition of functors.

□

2.1.15 Proposition The categories

Qun, Un, Bcrg, Crg, 2 Top, Top

are complete and co-complete, well- and co-well-powered. In

each of them, the epimorphisms coincide with the surjections, and the monomorphisms as well as the extreme monomorphisms and the equalizers coincide with the injective mappings.

Proof. The (co-) completeness follows from 2.1.12 and 1.2.6. The epimorphisms can be checked by means of accrete two-point spaces, the monomorphisms by means of discrete two-point spaces. The rest is straightforward.

□

2.1.16 Corollary. All the functors in figure 2 preserve limits (hence products), epimorphisms and extreme monomorphisms.

Proof Preservation of products and equalizers is here equivalent to preservation of limits [Mitchell 1965, p. 55].

□

2.1.17 Remarks (1) The functors in figure 2 thus satisfy the conditions for the lifting of epireflectors in theorem 1.9.2. They also satisfy the conditions of theorem 1.9.4 in so far as they have right inverses. We shall see that  $T$ ,  $\bar{T}$  and the inclusion functors  $\underline{Crg} \rightarrow \underline{Top}$ ,  $\underline{Bcrg} \rightarrow \underline{2 Top}$  have no right inverses.

(2) The subcategory  $\underline{Bcrg}$  of  $\underline{2 Top}$  is not the epireflective subcategory lifted from  $\underline{Crg}$  against  $S$ , for  $SX$  may be completely regular without  $X$  being pairwise completely regular. (Take  $X_1$  discrete with more than one point, and  $X_2$  accrete [Salbany 1970 p.5].)

(3) Instead of regarding Crg as "above" Top (figure 2) one might wish to regard the monoreflection functor  $\text{Top} \rightarrow \text{Crg}$  as a forgetful functor. This, however, would be less convenient in the next section. Moreover, this functor does not preserve initiality for  $E'$ . (Proof : it does not preserve the initiality Top  $(I, D_u)$ .)

A similar remark applies to Bcrg and 2-Top.

## 2.2. THE SEPARATED CATEGORIES

It is customary to call a uniform space separated iff it has  $T_0$ -topology. A Tihonov space is a completely regular  $T_0$ -space. (In both cases, one then has  $T_2$ -topology.) Accordingly, we lift the  $T_0$ -separation property from Top to the categories Qun, Un, Bcrgr, Crg and 2 Top, thus obtaining epireflective subcategories by virtue of the lifting theorem 1.9.2. Moreover, we show that the objects of these subcategories are the separated objects (in the sense of definition 1.3.8) of the given categories.

**2.2.1. Definition.** Quns, Uns, Btih, Tih and 2 T<sub>0</sub>

are the full, epireflective subcategories of

Qun, Un, Bcrgr, Crg and 2 Top

respectively, obtained by lifting from Top the  $T_0$ -epireflector against the forgetful functors in the commutative diagram of figure 1:

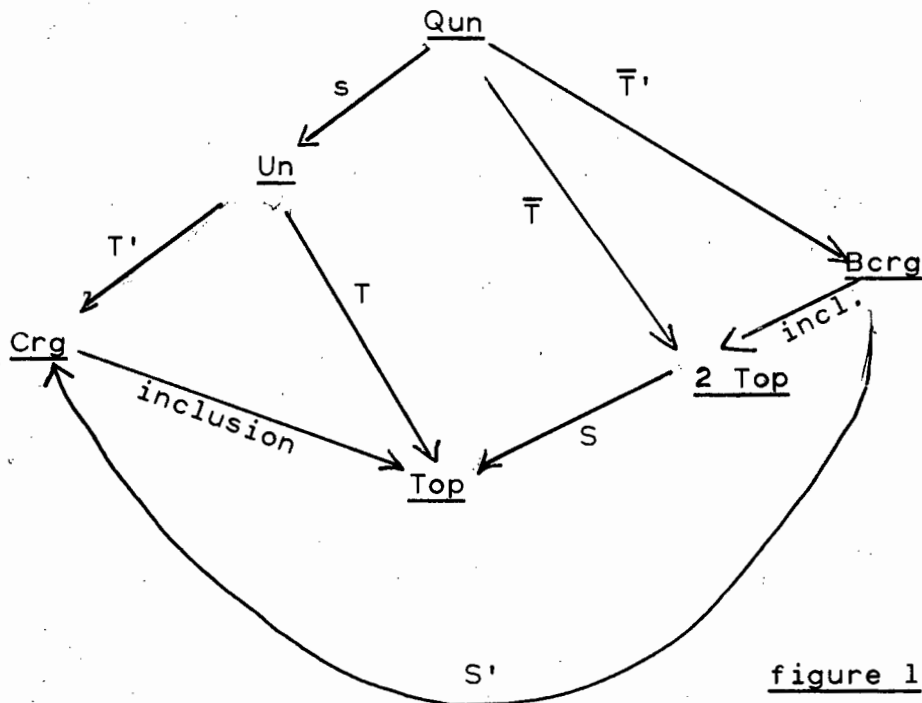


figure 1

The claim of epireflectivity in the above definition is immediately verified by remark 2.1.17(1).

2.2.2. Theorem The objects of the categories

Quns, Uns, Btih, Iih, 2T<sub>0</sub>, I<sub>0</sub>

are the separated objects (in the sense of definition 1.3.8 and convention 2.1.11) of the categories

Qun, Un, Bcrg, Crg, 2 Top, Top

respectively.

Proof (1) For the case of I<sub>0</sub> in Top the proof was given in 1.3.9(i).

The same pattern of proof can be used for the other cases, but the necessity half can each time be short-cut as follows.

(2a) Let  $X \in \text{obj } \underline{2T_0}$ , i.e.  $SX$  is a  $T_0$ -space. To show that  $X$  is an  $\bar{E}$ -separated object of 2 Top, i.e. that each simple  $\bar{E}$ -initiality  $f$  at  $X$  is injective. Now, as  $S$  preserves initiality,  $Sf$  is an  $E$ -initiality and thus, by (1), injective, i.e.  $f$  is injective.

(2b) The converse of (2a) has to be proved as in 1.3.9(i).

One has to show that, for the epireflection  $r : \underline{2 Top} \rightarrow \underline{2T_0}$ , each reflection mapping  $r_X : X \rightarrow rX$  is an  $\bar{E}$ -initiality. This is easy if one constructs  $r_X$  as the quotient w.r.t. the equivalence relation

$x \equiv y$  iff  $x$  and  $y$  have the same  $SX$ -neighbourhoods.

(3) The proof for Uns in Qun is like (2a) and (2b) above. The verification that the reflection mapping is an initiality may

be found in [Salbany 1970, p.67].

The other proofs are exactly similar.

□

We shall henceforth informally speak of

Qun, Un, Bcrg, Crg, 2 Top, Top

as "the non-separated categories" and of

Quns, Uns, Btih, Tih, 2T<sub>0</sub>, I<sub>0</sub>

as "the separated categories".

2.2.3. Remark. We have seen (2.1.12) that the non-separated categories are initiality complete. None of the separated categories is initiality complete.

Proof. In each case, the initiality problem posed by the constant mapping of a two-point set to a one-point space has no solution.

□

2.2.4. Question. Are the objects of a separated category the separated objects of that category (i.e. with respect to the restricted forgetful functor)? That the answer seems to be negative, is no obstacle to the further development.

### 2.3. RIGHT INVERSES OF THE FORGETFUL FUNCTORS

In the previous section we introduced certain forgetful functors among the "non-separated" categories Qun, Un, Bcrg, Crg, 2 Top, Top, and from these to Ens. In this section we point out that we can fully characterize the right inverses of all these functors. These right inverses induce right inverses of the restricted forgetful functors among the "separated categories" Quns, Uns, Btih, Tih, 2T<sub>0</sub>, I<sub>0</sub> — an advantage due to the fact that the separation was achieved by lifting I<sub>0</sub> against the forgetful functors. There are two exceptions to this latter statement : the forgetful functor  $T_1 : \underline{Qun} \rightarrow \underline{Top}$ , and a functor  $C_1 : \underline{Bcrg} \rightarrow \underline{Top}$  to be introduced. We devote a number of remarks to these two rather curious functors, and solve a problem raised by Salbany.

2.3.1. We consider the forgetful functors in figure 2 of 2.1.14. The categories are initiality complete (2.1.12) and the functors preserve initiality (2.1.14). Thus sufficient conditions reign for applying all our general characterizations of right inverses (in fact of all four types of functors discussed in section 1.4). In particular, in each situation

$$\underline{A} \xrightarrow{L} \underline{B} \xrightarrow{M} \underline{Ens}$$

in figure 2 of 2.1.14, the right inverses of L are the functors  $(L, M : (\underline{U}))$  with  $L \underline{U} M$ -sufficient in B.

We see immediately that  $\bar{T}$ , T and the inclusion functors  $\underline{Bcrg} \rightarrow \underline{2 Top}$ ,  $\underline{Crg} \rightarrow \underline{Top}$  have no right inverses (as they are not onto). The functors  $\bar{E}$ , E,  $\bar{E}'$ , E' have rather trivial

right inverses (cf. 1.5.4). To show that  $\bar{T}'$ ,  $T'$  and  $T_1$  have right inverses, we prove:

- 2.3.2. Lemma (i)  $I_b$  is a  $\bar{T}'$ ,  $\bar{E}'$ -pivot in Bcrg  
 (ii)  $I$  is a  $T'$ ,  $E^L$ -pivot in Crg  
 (iii)  $D_u$  is a  $T_1$ ,  $E$ -pivot in Top.

Proof (i) By 2.1.7 each space in Bcrg is  $\bar{E}$ -initial, and therefore  $\bar{E}'$ -initial, for its mappings to  $I_b$ . That  $I_b$  admits a unique quasi-uniformity was proved by Salbany [1970 p.76].

(ii) [Brümmer 1968b; 1969a].

(iii) Well known.

□

- 2.3.3. Definition
- $$\bar{F}_0 = (\bar{T}', \bar{E}' : (\{I_q\}))$$
- $$F_0 = (T', E' : (\{I_s\}))$$
- $$P = (T_1, E : (\{D_q\})).$$

2.3.4. Proposition  $\bar{F}_0$ ,  $F_0$ ,  $P$  are the coarsest right inverses of the forgetful functors

$$\bar{T}' : \underline{Qun} \rightarrow \underline{Bcrg}$$

$$T' : \underline{Un} \rightarrow \underline{Crg}$$

$$T_1 : \underline{Qun} \rightarrow \underline{Top}$$

respectively.

Proof. 2.3.2, 1.7.6(i)

□

We mention a simple result which sometimes facilitates calculations.

2.3.5. Proposition (i) For any subclass  $\underline{U}$  of obj  $\underline{Qun}$   
 $(\bar{T}', \bar{E}' : (\underline{U})) = (\bar{T}, \bar{E} : (\underline{U})) \Big|_{\underline{Bcrg}}$ .

(ii) For any subclass  $\underline{V}$  of obj  $\underline{Un}$ ,

$$(T', E' : (\underline{V})) = (T, E : (\underline{V})) \Big|_{\underline{Crg}}.$$

Proof Direct from definitions.

□

2.3.6. Remarks (1) The functors  $S : \underline{2 Top} \rightarrow \underline{Top}$ ,  
 $S' : \underline{Bcrg} \rightarrow \underline{Crg}$ ,  $s : \underline{Qun} \rightarrow \underline{Un}$  are "onto" and thus by  
 1.6.9 and 1.6.10(1) have adjoint right inverses. (In  
 fact all the "onto" functors in figure 2 of 2.1.14 have  
 adjoint right inverses.)

(2) Herewith we have completely settled the question of  
 existence of right inverses in figure 2 of 2.1.14.

(3) The adjoint right inverse of  $S$ , as well as that of  $S'$ ,  
 is given by  $X \rightsquigarrow (X, X)$ .

(4) The inclusion functor  $\underline{Un} \rightarrow \underline{Qun}$  is the adjoint right  
 inverse of the forgetful functor  $s : \underline{Qun} \rightarrow \underline{Un}$ .

(5) Proposition 2.3.4, for the cases of  $P$  and  $F_0$ , was given  
 in [Brümmer 1969a], where  $P$  was called the Pervin functor since  
 it was shown there that  $P$  gives each topological space the  
Pervin quasi-uniformity, due to Pervin [1962]. Of course  
 the functoriality of  $F_0$  had long been known [Isbell 1964].

Usually, as in [Gillman and Jerison 1960, chapter 15],  $F_0$  was given by

$$F_0 X = \text{in}_{\text{ET}}(E_f, R_S : f \in \underline{\text{Top}}(X, \mathbb{R}) \text{ and } f \text{ bounded})$$

- a construction which obviously coincides with ours. Our use of  $I_S$  instead of  $R_S$  makes it very easy to see that  $F_0$  is the coarsest right inverse of  $T'$ , and to prove the well-known connection with the Stone-Čech compactification (section 2.6). The functor  $\bar{F}_0$  was used by Salbany [1970; 1971b].

We may replace the two-point space  $D_q$  in the definition of the Pervin functor  $P$  by the space  $I_q$ :

2.3.7. Proposition  $P = (T_1, E : (\{I_q\}))$ .

Proof. We first show that  $I_q \leq PI_U$ .

A defining basic entourage of  $I_q$  is of the form

$$B_r = \{(x, y) \in I \times I : y < x + r\}$$

with  $r > 0$ . Given  $r > 0$ , we choose points  $x_1 < x_2 < \dots < x_n$  in  $(0, 1)$  such that  $\max(x_{k+1} - x_k) < r/2$ . Then the entourages

$$([0, x_k) \times [0, x_k]) \cup ([x_k, 1] \times [0, 1])$$

belong to the Pervin quasi-uniformity on  $I_U$ , and their intersection is contained in  $B_r$ . Thus  $I_q \leq PI_U$ . (In fact, none of these entourages can contain  $B_r$ , whence  $I_q \neq PI_U$ .)

Since  $P = (T_1, E : (\{D_q\}))$

and  $D_q$  is a subspace of  $I_q$ , we have  $P \leq (T_1, E : (\{I_q\}))$ .

It remains to show that above each  $f \in \underline{\text{Top}}(X, I_U)$  lies a morphism  $PX \rightarrow I_q$ . But this is immediate as  $Pf : PX \rightarrow PI_U$

and  $I_q \leq PI_U$ .

□

The above result was obtained by Hunsaker and Lindgren [1970] via quasi-proximities. I found the above proof in late 1969. A different proof was later obtained by Salbany [1970, p.75]. Császár [1960, p. 171] showed that each topological space is quasi-uniformizable; his construction consisted in taking the initial quasi-uniformity for the bounded continuous mappings into  $\mathbb{R}$ ; this is tantamount to taking the continuous mappings to  $I_q$ ; thus by 2.3.7, Pervin's quasi-uniformity equals that of Császár.

2.3.8. Definition. The functor  $C_1 : \text{Bcrg} \rightarrow \text{Top}$  is given by  $(X_1, X_2) \rightsquigarrow X_1$ , functions being preserved.

2.3.9. Salbany [1970 pp. 37-40] made interesting use of the functor  $C_1$ ; among other things, he proved that  $C_1$  has a unique right inverse. Let

$$Q : \text{Top} \rightarrow \text{Bcrg}$$

denote the unique right inverse of  $C_1$ . Since  $C_1 \bar{T}' = T_1$ , it follows that if  $F$  is a right inverse of  $T_1$ , then  $\bar{T}'F$  is a right inverse of  $C_1$ . Thus:

$\bar{T}'F = Q$  for each right inverse  $F$  of  $T_1$ . We now give the following affirmative answer to a question posed by Salbany [1971b].

2.3.10. Proposition For each right inverse  $F$  of the forgetful functor  $T_1 : \text{Qun} \rightarrow \text{Top}$ , there exists a right inverse  $\bar{F}$  of the forgetful functor  $\bar{T}' : \text{Qun} \rightarrow \text{Bcrg}$  such that

$$\bar{F}Q = F$$

where  $Q$  is the unique right inverse of the forgetful functor  $C_1 : \underline{\text{Bcrg}} \rightarrow \underline{\text{Top}}$ .

Proof. By 1.4.27 we have a maximal class  $\underline{U}$  of quasi-uniform spaces for which

$$F = (T_1, E' : (\underline{U})),$$

namely  $\underline{U} = \{A \in \text{obj } \underline{\text{Qun}} : A \leq FT_1A\}$ . Using 2.3.4,

we have  $I_q \leq PI_U \leq FI_U = FT_1I_q$ , hence  $I_q \in \underline{U}$ . Let

$\bar{F} = (\bar{T}', \bar{E}' : (\underline{U}))$ . The class  $\bar{T}'U$  contains  $\bar{T}'I_q = I_b$  and is therefore sufficient in  $\underline{\text{Bcr}}$ ; hence  $\bar{F}$  is right inverse to  $\bar{T}'$ .

To show  $\bar{F}Q = F$ , we consider  $X \in \text{obj } \underline{\text{Top}}$  and

$$\bar{F}QX = \text{in}_{\bar{E}', \bar{T}'} (\bar{E}'f, A : A \in \underline{U}, f \in \underline{\text{Bcrg}} (QX, \bar{T}'A)).$$

$$FX = \text{in}_{ET_1} (Eg, A : A \in \underline{U}, g \in \underline{\text{Top}} (X, T_1A)).$$

Since  $ET_1 = \bar{E}'\bar{T}'$ , it will follow that  $FX = \bar{F}QX$  if we can show that, for  $A \in \underline{U}$ ,

$$\{\bar{E}'f : f \in \underline{\text{Bcrg}} (QX, \bar{T}'A)\} = \{Eg : g \in \underline{\text{Top}} (X, T_1A)\}.$$

As  $A \leq FT_1A$  and  $\bar{T}'F = Q$ , we have  $\bar{T}'A \leq QT_1A$ . The required step now readily follows.

□

2.3.11. We now consider the separated categories:

$\underline{\text{Quns}}$ ,  $\underline{\text{Uns}}$ ,  $\underline{\text{Btih}}$ ,  $\underline{\text{Tih}}$ ,  $\underline{2T_0}$ ,  $\underline{I_0}$ , and we exclude the functors  $T_1$  and  $C_1$  from our considerations. The separated categories were obtained by lifting the property  $T_0$  from  $\underline{\text{Top}}$  against the forgetful functors in figure 1 of section 2.2 (see definition 2.2.1 and note that  $T_1$  and  $C_1$  could not be admitted there). It is immediately

clear that the given forgetful functors among the non-separated categories induce forgetful functors among the separated categories. Moreover, the right inverses of the forgetful functors between the non-separated categories then induce right inverses of the induced forgetful functors between the separated categories.

On the other hand, I do not know whether every right inverse of, say, the forgetful functor  $\underline{Uns} \rightarrow \underline{Tih}$  is induced by a right inverse of  $T' : \underline{Un} \rightarrow \underline{Crg}$ . (The reader is warned against a too easy affirmative answer.) At present, this ignorance does no harm, for experience shows that one carries out initiality constructions in the non-separated categories - because there one has initiality completeness - and then restricts the resulting constructs to the separated categories if one wishes. (Two possible reasons for such a wish are : to get embeddings, or to get compactness epireflective.)

## 2.4. TOTAL BOUNDEDNESS.

Total boundedness, as it is usually defined in  $\underline{Qun}$ , is a property lifted from the familiar total boundedness in  $\underline{Un}$  against the forgetful functor  $s : \underline{Qun} \rightarrow \underline{Un}$ . Salbany [1970, p.82] constructed the total boundedness monoreflector in  $\underline{Qun}$  in terms of initiality, basing his proof on his construction of the quasi-uniform completion. We give a different, direct proof of the same construction. We use a result from our section 1.8. to deduce other formulae concerning this monoreflector in  $\underline{Qun}$ . We also indicate how the (largely known) analogues in  $\underline{Un}$  can be retrieved from these results.

2.4.1 Definition ([Császár 1963], [Murdeswar and Naimpally 1966], [Fletcher 1970].)

A (quasi-)uniform space  $X$  is totally bounded iff for each entourage  $H$  in  $X$  there exists a finite covering  $\{A_k\}_k$  of  $X$  such that  $\bigcup_k A_k \times A_k \subset H$ .

2.4.2. Proposition [Murdeswar and Naimpally 1966, p. 50].)

The quasi-uniform space  $X$  is totally bounded if and only if the uniform space  $sX$  is totally bounded.

□

Thus we are dealing with a lifting against the functor  $s : \underline{Qun} \rightarrow \underline{Un}$ .

2.4.3. Proposition The monoreflector  $(1_{\underline{Qun}}, ET_1 : (\{I_q\}))$  is the total boundedness reflector in  $\underline{Qun}$ .

Proof This follows from 1.8.6 if we show that a space  $Y$  is totally bounded if and only if

$$Y = \text{in}_{ET_1} (ET_1 f, I_q : f \in \underline{Qun}(Y, I_q)).$$

One direction is trivial; conversely, assume  $Y$  totally bounded. It suffices to show that any entourage  $H$  of  $Y$  is an entourage of the initial space. Given  $H \in \text{ent } Y$ , form a sequence of entourages  $H_n$  of  $Y$  with  $H_0 = H$  and  $H_{n+1} \circ H_{n+1} \circ H_{n+1} \subset H_n$  ( $n \geq 0$ ) and find a quasi-pseudometric  $d$  with  $H_{n+1} \subset d^{-1} [0, 2^{-n}) \subset H_n$  for  $n \geq 0$  [Kelley 1955, p.185]. There is a finite covering  $\{A_k\}_k$  of  $Y$  with  $\bigcup_k A_k \times A_k \subset H_2$ . There are morphisms  $f_k : Y \rightarrow I_q$  given by  $f_k(y) = 1 \wedge d(A_k, y)$  - the verification involves only the triangle inequality. The proof is completed by showing that

$$\bigcap_k (f_k \times f_k)^{-1} \{(x, y) \in I \times I : y < x + 1/4\} \subset H$$

□

The above proof is from [Brümmer 1970].

Since the functor  $s$  preserves initiality, we retrieve from the above proposition the following well known result.

2.4.4. Proposition The monoreflector  $(1_{\underline{Un}}, ET : (\{I_s\}))$  is the total boundedness reflector in  $\underline{Un}$ .

□

2.4.5. Definition We shall write

$$p^S = (1_{\underline{Qun}}, ET_1 : (\{I_q\}))$$

$$p = (1_{\underline{Un}}, ET : (\{I_s\}))$$

for the total boundedness reflectors in Qun and Un respectively.

2.4.6. Proposition Let  $J$  denote the inclusion functor

Un  $\rightarrow$  Qun. Then  $p = sp^S J$ .

Proof 1.9.4 and 2.1.17(1)

□

This result has the curious consequence that for a uniform space  $X$  one has

$$pX = \text{in}_{ET}(EG_1 f, I_q : f \in \underline{Qun}(JX, I_q)).$$

□

2.4.7. Proposition  $\bar{T}' p^S = \bar{T}'$  and  $T_1 p^S = T_1$ .

Proof From 2.4.5 by 2.1.5 and 2.1.4.

□

We recall that the forgetful functors

$\bar{T}' : \underline{Qun} \rightarrow \underline{Bcrg}$ ,  $T' : \underline{Un} \rightarrow \underline{Crg}$ ,  $T_1 : \underline{Qun} \rightarrow \underline{Top}$   
have coarsest right inverses denoted by  $\bar{F}_o$ ,  $F_o$ ,  $P$  respectively.

2.4.8. Proposition  $p^S = 1_{\underline{Qun}} \wedge \bar{F}_o \bar{T}' = 1_{\underline{Qun}} \wedge P T_1$ .

Proof 2.4.7 and 1.8.8.

□

2.4.9. Proposition (i) For each right inverse  $F$  of  $\bar{T}'$  we have  $p^S F = \bar{F}_0$ .

(ii) For each right inverse  $G$  of  $T_1$ ,  $p^S G = P$ .

Proof. 2.4.7 and 1.8.8.

□

2.4.10. Remark The uniform analogues of the last three propositions can be proved similarly. The uniform analogue of 2.4.7 is well known, and so (but for the functional garb) is that of 2.4.8 [Gillman and Jerison 1960, p. 235].

The following theorem, whose proof we omit, extends our proposition 2.3.7. Part (i) is from [Brümmer 1970].

2.4.11. Theorem (i) Let  $A$  be a quasi-uniform space.

Then  $P = (T_1, E : (\{A\}))$  if and only if  $A$  is totally bounded and  $T_1 A$  contains a homeomorph of  $D_U$ .

(ii) Let  $B$  be a separated uniform space. Then

$F_0 = (T', E' : (\{B\}))$  if and only if  $B$  is totally bounded and  $T'B$  contains a homeomorph of  $I$ .

□

2.5. SPECIAL MORPHISMS IN THE  
SEPARATED CATEGORIES.

In this section we characterize the epimorphisms, equalizers and extreme monomorphisms in the "separated" categories Quns, Uns, Btih, Tih and deduce that these categories are complete, well- and co-well-powered. It then follows that sufficient conditions obtain for applying the two theorems of section 1.9, on the lifting of epi-reflections, to the forgetful functors

$$\begin{array}{ll} s : \underline{Quns} \rightarrow \underline{Uns} & S : \underline{Btih} \rightarrow \underline{Tih} \\ \bar{T} : \underline{Quns} \rightarrow \underline{Btih} & T : \underline{Uns} \rightarrow \underline{Tih} \end{array}$$

2.5.1. Convention We defined the functors  $s$  and  $S'$  in 2.1.1. and 2.1.13. between non-separated categories; by 2.3.11. they induce functors between the corresponding separated categories; these induced functors we shall henceforth for convenience denote by

$$s : \underline{Quns} \rightarrow \underline{Uns} \quad S : \underline{Btih} \rightarrow \underline{Tih}$$

Similarly the functors induced by  $\bar{T}$  and  $T$  will simply be denoted by

$$\bar{T} : \underline{Quns} \rightarrow \underline{Btih} \quad T : \underline{Uns} \rightarrow \underline{Tih}.$$

2.5.2. Lemma If  $Y$  is a separated quasi-uniform space, then the Qun-coproduct of two copies of  $Y$ , identified along a set closed in  $TsY$ , is again a separated quasi-uniform space.

Proof By 2.1.12 and 1.2.2 Qun is cointiality complete.

As Ens has coproducts, the dual of theorem 1.2.4 shows that we can construct the coproduct  $Y \amalg Y$  in Qun (say with canonical injections  $i_1, i_2 : Y \rightarrow Y \amalg Y$ ) as the solution of a cointiality problem. Given a set  $C$  closed in  $TsY$ , let  $R$  be the equivalence relation in  $Y \amalg Y$  defined as follows:

$$x R y \text{ iff } x = y \text{ or } i_1^{-1}(x) = i_2^{-1}(y) \in C$$

$$\text{or } i_1^{-1}(y) = i_2^{-1}(x) \in C$$

Denote by  $Z = (Y \amalg Y)/R$  the space in Qun which is cointial for the canonical identification function. We have to show  $Z$  separated, i.e. to show the uniform space  $sZ$  separated.

The functor  $s$  preserves cointiality for the forgetful functor to Ens. Proof:

Let  $f_j : B_j \rightarrow A$  be a cointiality in Qun. A subbase for ent  $A$  consists of those subsets  $H$  of  $|A| \times |A|$  such that  $(f_j \times f_j)^{-1}H \in \text{ent } B_j$  for all  $j$ . The  $H \cap H^{-1}$  form a subbase for ent  $sA$ . Now

$$(f_j \times f_j)^{-1}(H \cap H^{-1}) = ((f_j \times f_j)^{-1}H) \cap ((f_j \times f_j)^{-1}H)^{-1}$$

and hence the  $sf_j : sB_j \rightarrow sA$  form a cointiality in Un.

$$\begin{aligned} \text{Consequently } sZ &= (s(Y \amalg Y))/R \\ &= (sY \amalg sY)/R. \end{aligned}$$

Our problem is thus reduced to that of showing that if the uniform space  $X = sY$  is separated, so is  $(X \amalg X)/R$ . This can be seen by presenting  $X$  in terms of its family of uniform covers [Isbell 1964, chapter 1] rather than entourages. For each uniform cover  $\underline{U}$  of  $X$ , we form a cover  $\hat{\underline{U}}$  of  $(X \amalg X)/R$  by taking two copies of  $\underline{U}$  and identifying their members along

the closed set  $C$ . It is readily seen that the covers  $\hat{U}$  form a base for the uniform covers of  $(X \sqcup X)/R$  and hence that this space has  $T_0$ -separation.

□

2.5.3. Theorem A morphism  $f : X \rightarrow Y$  is Quns is epic in Quns if and only if the image set  $fX$  is dense in  $TsY$ .

Proof Let  $fX$  be dense in  $TsY$ . Suppose  $X \xrightarrow{f} Y \begin{array}{c} \xrightarrow{g} \\ \xrightarrow{h} \end{array} Z$  in Quns commutes. In the diagram

$$TsX \xrightarrow{Tsf} TsY \begin{array}{c} \xrightarrow{Tsg} \\ \xrightarrow{Tsh} \end{array} TsZ \text{ in } \underline{Tih},$$

$Tsf$  has dense image, hence  $Tsg = Tsh$ , hence  $g = h$ . Thus  $f$  is epic in Quns.

The converse can be proved by an immediate adaptation of a standard procedure, [Burgess 1965] or [Herrlich 1968, Satz 15.2.2], using lemma 2.5.2.

□

2.5.4. Theorem. Let  $f : X \rightarrow Y$  be a Quns-morphism. The following three conditions are equivalent:

- (1)  $f$  is an equalizer in Quns
- (2)  $f$  is an extreme monomorphism in Quns
- (3)  $f$  is equivalent to an inclusion mapping  $A \rightarrow Y$ , where  $A$  as subset of  $Y$  is closed in  $TsY$ .

Proof. Using lemma 2.5.2, the proof of [Herrlich 1968, Satz 15.2.3] can be readily adapted.

□

2.5.5. Remarks (1) As corollaries we obtain the known characterizations of epimorphisms, equalizers and extreme monomorphisms in Uns if we replace Quns by Uns and  $TsY$  by  $TY$  in the statement of theorems 2.5.3 and 2.5.4.

(2) Let  $L : \underline{A} \rightarrow \underline{B}$  be a faithful functor with a right inverse  $R : \underline{B} \rightarrow \underline{A}$ . It is clear that if  $f$  is an epimorphism in B, then  $Rf$  is an epimorphism in A.

(3) Salbany [1970, p.40] used remark (2) to derive the known characterization of epi in  $\underline{I}_0$  from his characterization of epi in Btih; his functor  $L : \underline{Btih} \rightarrow \underline{I}_0$  was given by  $(X_1, X_2) \rightsquigarrow X_1$  (cf. 2.3.9). We shall now use the same idea to retrieve Salbany's characterization of epi in Btih from our characterization of epi in Quns.

2.5.6. Theorem [Salbany 1970, p.32]

A morphism  $f : X \rightarrow Y$  in Btih is epic if and only if the image set  $fX$  is dense in  $SY$ .

Proof Let  $f : X \rightarrow Y$  be epic in Btih. Choose any right inverse  $F$  of  $\bar{T} : \underline{Quns} \rightarrow \underline{Btih}$ . By 2.5.5(2),  $Ff : FX \rightarrow FY$  is epic in Quns. By 2.5.3 the image set of  $Ff$  (i.e. of  $f$ ) is dense in  $TsFY = S\bar{T}FY = SY$ .

Conversely, let  $fX$  be dense in  $SY$ . Suppose  $X \xrightarrow{f} Y \begin{matrix} \xrightarrow{g} \\ \xrightarrow{h} \end{matrix} Z$  commutes. As  $Sf$  has dense image in  $SY$ , we have  $Sg = Sh$ , i.e.  $g = h$ . Thus  $f$  is epic in Btih.

□

2.5.7. Theorem. For a morphism  $f : X \rightarrow Y$  in Btih the following three conditions are equivalent:

- (1)  $f$  is an equalizer in Btih
- (2)  $f$  is an extreme monomorphism in Btih
- (3)  $f$  is equivalent to an inclusion mapping  $A \rightarrow Y$ , where  $A$  as subset of  $Y$  is closed in  $SY$ .

Proof. If  $Y \in \text{obj } \text{Btih}$  and  $A$  is closed in  $SY$ , then the Bcrg-coproduct of two copies of  $Y$ , identified along  $A$ , is in Btih. This can be proved like lemma 2.5.2 since  $S'$  preserves coinitiality. For the standard rest of the proof we again refer to [Herrlich 1968, Satz 15.2.3].

□

2.5.8. Remark As corollaries one has the known characterizations of the epimorphisms, equalizers and extreme monomorphisms in Tih if one replaces Btih by Tih and  $ST$  by  $Y$  in the statement of 2.5.6 and 2.5.7.

□

2.5.9. Proposition The categories Quns, Uns, Btih, Tih are complete, well- and co-well-powered.

Proof. Completeness follows from the existence of products (for Btih and Tih : reflectivity in 2 Top and Top, respectively) and equalizers. Co-well-poweredness follows from our knowledge of the epimorphisms. The monomorphisms are the injections

(as there are discrete two-point objects) and hence well-poweredness follows.

□

2.5.10. Proposition The forgetful functors

$$s : \underline{Quns} \rightarrow \underline{Uns}, \quad S : \underline{Btih} \rightarrow \underline{Tih},$$

$$\bar{T} : \underline{Quns} \rightarrow \underline{Btih}, \quad T : \underline{Uns} \rightarrow \underline{Tih}$$

preserve products and equalizers (hence limits), as well as epimorphisms and extreme monomorphisms.

Proof Preservation of products is clear [Salbany 1970]. For the morphism types, the statement is immediate from the characterizations of these types in this section.

□

2.5.11. Remark By propositions 2.5.10 and 2.5.11 the functors  $s, S, \bar{T}, T$  satisfy the conditions of theorems 1.9.2 and 1.9.4 for the lifting of epireflections. As right inverses of  $s$  and  $S$ , respectively, one can take the following two functors.

2.5.12. Definition We shall write

$$J : \underline{Un} \rightarrow \underline{Qun}$$

for the inclusion functor, and

$$D : \underline{Tih} \rightarrow \underline{Btih}$$

for the functor given by  $DX = (X, X)$ , functions being preserved.

## 2.6. COMPACTNESS AND COMPLETENESS.

Salbany [1970] defined a bitopological space  $X$  to be pairwise compact iff the topological space  $SX = X_1 \vee X_2$  is compact, and a quasi-uniform space  $Y$  to be complete iff the uniform coreflection  $sY$  is complete. These definitions gave rise to reflections and thereby to a theory systematically analogous to the basic classical theory of compactifications and completions.

Salbany's pairwise compactness and quasi-uniform completeness are instances of properties lifted against forgetful functors. Their (epi)reflectivity immediately follows from the results of our previous section. However, (epi)reflectivity does not exhaust what can be said about Salbany's results in this context, and therefore we shall point out which of the results are straight consequences of the lifting, and which do not seem to be so. We further obtain the bitopological analogue of the classical relation between the Stone- $\check{C}$ ech compactification and the coarsest right inverse of the forgetful functor  $\underline{Un} \rightarrow \underline{Crg}$ . We end by raising a few problems, one of which concerns the lifting of realcompactness to bitopological spaces.

2.6.1. Henceforth we shall mainly deal with the "separated" categories  $\underline{Quns}$ ,  $\underline{Uns}$ ,  $\underline{Btih}$ ,  $\underline{Tih}$ . However, the term embedding is always to be taken relative to the ambient "non-separated" category  $\underline{Qun}$ ,  $\underline{Un}$ ,  $\underline{Bcrg}$  (or equivalently :  $\underline{2 Top}$ ),  $\underline{Crg}$  (or equivalently :  $\underline{Top}$ ), respectively. This is

to ensure that our concept of embedding from section 1.3 coincides with the ordinary notion of embedding. A similar remark applies to the notion of subspace, which means that the inclusion mapping is an initiality.

2.6.2. Definition A bitopological space  $X = (X_1, X_2)$  is compact iff the topological space  $SX = X_1 \vee X_2$  is compact (i.e. iff every open cover of  $SX$  admits a finite subcover). A quasi-uniform space  $Y$  is complete iff its uniform coreflection  $sY$  is complete.

(We drop the word "pairwise" as it is only appropriate to properties like pairwise Hausdorff, pairwise regularity, pairwise complete regularity and pairwise normality [Salbany 1970] which are not obtained by lifting against  $S$ .)

2.6.3. The following results [Salbany 1970, pp. 79, 18, 61, 19, 61, 19, 61] are immediate consequences of the corresponding results in Top and Un.

- (i) A quasi-uniform space  $Y$  is complete and totally bounded if and only if the bitopological space  $\bar{Y}$  is compact.
- (ii) Compactness is productive in 2 Top.
- (iii) Completeness is productive in Qun.
- (iv) If the subspace  $A$  of the compact bitopological space  $X$  is closed in  $SX$ , then  $A$  is compact.

- (v) If the subspace  $B$  of the complete quasi-uniform space  $Y$  is closed in  $TsY$ , then  $B$  is complete.
- (vi) If  $A$  is a compact subspace of the Btih-space  $X$ , then  $A$  is a closed subset of  $SX$ .
- (vii) If  $B$  is a complete subspace of the Quns-space  $Y$ , then  $B$  is a closed subset of  $TsY$ .

Proof For (i) use 2.4.2. The rest is trivial. (Salbany [1970, p. 19] has a stronger version of (vi), with "pairwise Hausdorff" instead of "pairwise Tihonov".)

□

Likewise the following proposition is a straight application of the lifting theorems of section 1.9.

2.6.4. Proposition (i) Let  $\beta$  denote the compactness epireflector in Tih. The compact spaces in Btih form an epireflective subcategory with epireflector  $\beta^S$ , and

$$\beta = s\beta^S_D.$$

(ii) Let  $c$  denote the completeness epireflector in Uns. The complete spaces in Quns form an epireflective subcategory with epireflector  $c^S$ , and  $c = sc^S_J$ .

Proof. Immediate from definition 2.6.2 and remark 2.5.11.

□

The result (i) above is due to Salbany [1970, p.37]. Apart from constructing the epireflection  $c^S$  [1970, p.66] he also showed that each quasi-uniform space  $X$  has a completion, i.e. an embedding  $f : X \rightarrow Y$  into a complete space  $Y$  such that  $fX$  is dense in  $TsY$  [1970, p.63]; but did not explicitly mention  $c = sc^S J$ . The completion was anticipated by Császár [1963], whose "double completeness", for the case of quasi-uniform spaces, coincides with what is here termed completeness.

The following analogue of a well-known theorem about uniform spaces does not seem to be an immediate consequence of the lifting against the functor  $s$ .

2.6.5. Proposition. Let  $Y$  be a complete separated quasi-uniform space. Let  $f : X \rightarrow Y$  be an embedding such that the image set  $fX$  is dense in  $TsY$ . Then the pair  $(f, Y)$  is equivalent to the completeness reflection  $(c_X^S, c^S X)$ .

Proof. In constructing the completeness reflection, Salbany [1970, p.64] showed that  $c_X^S$  is an embedding. Therefore  $(c_X^S, c^S X)$  is a "double completion" in the sense of [Császár 1963]. Császár [1963; 1967, p.53] showed that any two separated double completions of a syntopogenous structure are equivalent; this result immediately specializes to quasi-uniform spaces.

□

(A direct proof, without recourse to syntopogenous structures, is desirable.)

2.6.6. Proposition There is a natural equivalence  $sc^S \simeq cs$ .

Proof For  $X$  in  $\text{obj } \underline{\text{Quns}}$ , consider  $c_X^S : X \rightarrow c^S X$ . As this is an embedding and  $\underline{\text{Quns}}$ -epimorphism,  $sc_X^S : sX \rightarrow sc^S X$  is a dense embedding into the complete separated uniform space  $sc^S X$ . Now  $c_{sX} : sX \rightarrow csX$  is another separated uniform completion of  $sX$ ; two separated uniform completions are equivalent; the equivalence is easily shown to be natural.

□

On the other hand:

2.6.7. Proposition  $S\beta^S$  and  $\beta S$  are not naturally equivalent.

Proof Let the interval  $[0, 1)$ , as bitopological subspace of  $I_b$ , be denoted by  $X$ . Then  $sX$  is  $[0, 1)$  with the usual topology. Clearly  $\beta sX$  is not  $I$ , as there are mappings  $sX \rightarrow I$  which have no extensions  $I \rightarrow I$ . However,  $\beta^S X = I_b$ , as a mapping  $X \rightarrow I_b$  is monotone increasing and therefore has a unique extension  $I_b \rightarrow I_b$ . Hence  $S\beta^S X = I$ .

□

I owe the above proof to Dr. S. Salbany.

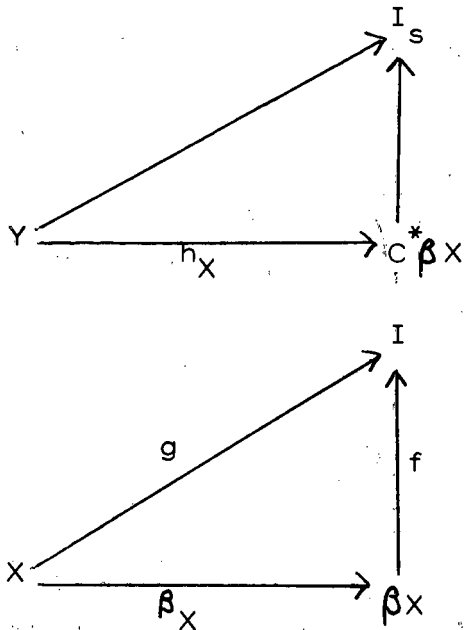
Let  $C^* : \underline{Tih} \rightarrow \underline{Uns}$  denote the functor induced (2.3.11) by the coarsest right inverse  $F_0$  of the forgetful functor  $T' : \underline{Un} \rightarrow \underline{Crg}$ . Similarly let  $\bar{C}^* : \underline{Btih} \rightarrow \underline{Quns}$  be the functor induced by  $\bar{F}_0$ .

2.6.8. Proposition

(i)  $C^*\beta \simeq cC^*$

(ii)  $\bar{C}^*\beta^S \simeq c^S\bar{C}^*$

Proof (i) Consider a Tihonov space  $X$ . As  $\underline{Un}$  is initiality



complete, there exists a uniform space

$$Y = \text{in}_{E, T'} (\beta_X, C^*\beta X)$$

and a mapping

$$h_X : Y \rightarrow C^*\beta X$$

such that  $T'h_X = \beta_X$ .

Now we have  $C^*\beta X = F_0\beta X = \text{in}_{E, T'} (E'f, I_s : f \in \underline{Crg}(\beta X, I))$ .

By the associativity theorem 1.1.10, it follows that  $Y$  is initial for the mappings  $f \cdot \beta_X$  to  $I_s$ , where  $f$  ranges through  $\underline{Crg}(\beta X, I)$ . But by the reflection property of  $\beta X$ , the mappings  $f \cdot \beta_X$  are precisely all the  $g : X \rightarrow I$ .

Thus  $Y = \text{in}_{E, T'} (E'g, I_s : g \in \underline{Crg}(X, I)) = F_0X = C^*X$ .

Hence  $h_X = C^*\beta_X$ ; as this mapping is an initiality and also injective (since  $\beta_X$  is), it is an embedding. By remark 2.5.5(2), as  $\beta_X$  is epic in  $\underline{Tih}$ ,  $C^*\beta_X$  is epic in  $\underline{Uns}$ .

Further  $C^*\beta X$  is complete since  $TC^*\beta X = \beta X$  is compact. Thus  $(C^*\beta_X, C^*\beta X)$  is a completion of  $Y = C^*X$ . This completion is equivalent to  $(c_C^*X, cC^*X)$ . Again the equivalence is easily shown to be natural.

(ii) The proof of (i) can be used verbatim to prove (ii), with only changes of symbols, e.g.  $I_a$  and  $I_b$  take the places of  $I_s$  and  $I$ .

□

The result (i) above is well known [Isbell 1964]; the proof given above is essentially that sketched in [Gillman and Jerison 1960, pp. 225-226].

2.6.9. Problems (1) It is well known [Herrlich 1968] that the realcompact spaces form the smallest epireflective subcategory of Tih which contains the space  $\mathbb{R}$ . Let  $v$  denote the realcompactness epireflector in Tih. Let

$$C : \underline{\text{Tih}} \rightarrow \underline{\text{Un}}$$

be the right inverse of  $T : \underline{\text{Un}} \rightarrow \underline{\text{Tih}}$  induced by  $(T', E' : (\{\mathbb{R}_s\}))$ . It is known [Gillman and Jerison 1960, p. 225] that

$$Cv \simeq cC.$$

(The form of proof of 2.6.8 does not quite suffice for this; one needs a different argument for the completeness of  $CvX$ .)

The lifted epireflector  $v^S$  in Btih exists, and it would be interesting to have a formula of the form

$$\bar{C}v^S \simeq c^S\bar{C}$$

with  $\bar{C}$  right inverse to the forgetful functor  $\bar{T} : \underline{\text{Quns}} \rightarrow \underline{\text{Btih}}$ .

Indeed, from such a formula one would immediately deduce the following bitopological analogue of the celebrated Shirota theorem [Gillman and Jerison 1960, p. 229]:

Let  $X$  be a Btih-space having no subset of measurable cardinality which is closed and discrete in  $SX$ . Then  $X$  admits a complete quasi-uniformity if and only if

$$v^S X \simeq X.$$

Remains the question of suitably defining  $\bar{C}$ . Inducing  $\bar{C}$  by  $(\bar{T}', \bar{E}' : (\{R_q\}))$  would be suitable if one could show that the epireflective subcategory of Btih given by  $v^S$  is the smallest epireflective subcategory containing the space  $\mathbb{R}_b$ . A more promising definition uses  $(\bar{T}', \bar{E}' : (\underline{U}))$  where  $\underline{U}$  is the class of all those complete quasi-uniform spaces  $Y$  such that  $S\bar{T}Y = R$ .

(2) Each right inverse  $F$  of  $T' : \underline{Un} \rightarrow \underline{Crg}$  can be written in the form  $F = (T', E' : (\underline{U}))$  with  $\underline{U}$  a class of separated uniform spaces. Hence it seems interesting to consider the  $\mathcal{E}$ -compact spaces in the sense of [Herrlich 1967] with  $\mathcal{E} = T\underline{U}$ , and to seek a relation between  $F$  and the  $\mathcal{E}$ -compactness reflector.

(3) The Pervin functor  $P : \underline{Top} \rightarrow \underline{Qun}$  is faintly analogous to the functor  $C^*$ . The functor  $P$  has a far from fully elucidated relation to the Wallman compactification, cf. [Nielsen and Sloyer 1970] and [Brümmer (to appear) a].

(4) We have not considered the functors

$$X \rightsquigarrow X \wedge X^{(-1)} : \underline{Qun} \rightarrow \underline{Un}$$

and

$$X \rightsquigarrow X_1 \wedge X_2 : \underline{2 Top} \rightarrow \underline{Top}.$$

Are these suited to the lifting of coreflections but not  
of reflections?

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Note on a Compactification due to  
Nielsen and Sloyer

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We show that a compactification in terms of ideals of lower semi-continuous functions recently given by Nielsen and Sloyer is equivalent to the Wallman compactification.

We follow the notation of [1]:  $X$  is a  $T_1$ -space,  $L^+(X)$  is the semi-ring of all non-negative lower semi-continuous functions on  $X$ , and  $M(X)$  is the space of all maximal ideals in  $L^+(X)$  with the Stone topology in which a subbase (soon to be seen a base) for the closed sets consists of the sets  $\tilde{f} = \{I \in M(X) \mid f \in I\}$  with  $f \in L^+(X)$ . There is a mapping  $e : X \rightarrow M(X)$ , whose continuity we need not yet check, given by  $e(x) = I_x = \{f \in L^+(X) \mid f(x) = 0\}$ .

$Cl(X)$  is the collection of all closed sets in  $X$ . Let  $wX$ , the set of all ultrafilters in  $Cl(X)$ , have the topology in which a base (sic!) for the closed sets consists of the sets  $\hat{A} = \{F \in wX \mid A \in F\}$  with  $A \in Cl(X)$ . There is a dense embedding  $i : X \rightarrow wX$  given by  $i(x) = \{A \in Cl(X) \mid x \in A\}$ . The pair  $(i, wX)$  is well known as the Wallman compactification of  $X$ .

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\*Aided by a University of Cape Town Staff Research Grant and by a bursary from the South African CSIR.

Nielsen and Sloyer proved [1 theorem 2] that there is a bijection  $b$  of the set  $M(X)$  onto the set  $wX$ , given by  $b(I) = Z[I] = \{f^{-1}(0) \mid f \in I\}$ . Since  $b[\tilde{f}] = \hat{A}$  where  $A = f^{-1}(0)$ ,  $b$  induces a bijection between the given subbase (which is thus a base) for  $M(X)$  and the given base for  $wX$ . Hence  $b$  is a homeomorphism. Clearly  $b \circ e = i$ , so that  $e$  is also a dense embedding. Thus we have at once:

Theorem. The pair  $(e, M(X))$  is a compactification of the  $T_1$ -space  $X$ , equivalent to the Wallman compactification of  $X$ .

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MATHEMATICS

INITIAL QUASI-UNIFORMITIES

BY

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(Communicated by Prof. H. FREUDENTHAL at the meeting of April 26, 1969)

*Introduction*

This note discusses functors which are right inverses of the forgetful functor  $T: Qun \rightarrow Top$ , where  $Qun$  and  $Top$  are the categories of quasi-uniform and of topological spaces respectively.

In Section 1 we show that the class  $\mathcal{K}$  of all such functors, though not a set, has complete lattice structure with respect to the partial ordering "coarser than". The coarsest functor in  $\mathcal{K}$  endows each topological space with its PERVIN quasi-uniformity [1]. The finest member of  $\mathcal{K}$ , the adjoint of  $T$ , is characterized in a number of ways, one of them requiring section 2 on quasi-pseudo-metrics.

Section 3 justifies our title: it shows that the right inverses of  $T$  are exactly those functors from  $Top$  to  $Qun$  which are obtained from initial structures in a certain way.

Section 4, as an application, gives an unusual characterization of the  $T_D$ -spaces and answers a question of THRON.

The results of sections 1, 2 and 3 have precise (and mostly known) analogues for uniform spaces. The analogy hinges partly on the fact that  $D$ , the space of two points and three open sets, admits a unique structure in  $Qun$ , as does  $[0, 1]$  in  $Unif$ .

It may be said here that the Pervin quasi-uniformity, which has now been shown to be functorial, induces the PERVIN quasi-proximity [2] via a forgetful functor (but see [3]). We intend to pursue this in a note on initial quasi-proximities.

*Notations*

$Qun$  is the category of quasi-uniform spaces [1, 4, 5, 6] with the mappings which in [4, 5, 6] are called *quasi-uniformly continuous* as morphisms. The uniform spaces with uniformly continuous mappings form a full subcategory  $Unif$  of  $Qun$ .  $Top$  is the category of topological spaces and continuous mappings,  $Ens$  that of sets and functions. We have the forget-

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<sup>1)</sup> The author is indebted to Dr. K. A. Hardie for many elucidating and stimulating conversations. This work was partly supported by University of Cape Town Staff Research Grant 46-079.

ful functors  $T: Qun \rightarrow Top$ ,  $\|\dots\|: Qun \rightarrow Ens$ ,  $\|\dots\|: Top \rightarrow Ens$ . The topological space having two points 0 and 1 and three open sets  $\emptyset$ ,  $\{0\}$ ,  $\{0, 1\}$  is denoted by  $D$ .

If we have a topological space  $X$  and quasi-uniform spaces  $Y$  and  $Z_\alpha$  with  $\|Y\| = |X|$ , and if  $Y$  is initial [7] in  $Qun$  with respect to the  $Z_\alpha$  for the continuous mappings  $X \rightarrow T(Z_\alpha)$ , then the following abuse of language will cut down notation:  $Y$  is initial for the continuous mappings from  $X$  to the  $Z_\alpha$ .

### 1. Right inverses of $T$

The following trivial result is well known.

1.1 Proposition. *Each topological space  $X$  is initial for the set of all continuous mappings from  $X$  to  $D$ .*

We recall from [1] that the Pervin quasi-uniformity on a topological space  $X$  has a subbase consisting of the entourages

$$S(A) = (A \times A) \cup ((|X| - A) \times |X|)$$

with  $A$  ranging through all the open sets of  $X$ . One readily sees that the Pervin is the only quasi-uniformity compatible with the topology of  $D$ .

1.2 Theorem. *On any topological space  $X$  the Pervin quasi-uniformity is initial for the set of all continuous mappings  $X \rightarrow D$  with respect to the unique quasi-uniformity of  $D$ .*

Proof. The quasi-uniformity on  $D$  is generated by one member,

$$S(\{0\}) = \{(0, 0), (0, 1), (1, 1)\}.$$

Hence the initial quasi-uniformity on  $X$  is generated by the entourages

$$(f \times f)^{-1}S(\{0\}) = S(f^{-1}\{0\}),$$

$f$  ranging through the continuous mappings  $X \rightarrow D$ . As the  $f^{-1}\{0\}$  are exactly the open sets of  $X$ , the theorem is proved.

Observing that for quasi-uniformities (as for uniformities) the topology of an initial structure is the same as the initial topology, we retrieve from 1.1 and 1.2 the theorem of PERVIN [1]:

1.3 Corollary. *On any topological space the Pervin quasi-uniformity is compatible with the topology.*

From 1.2 and 1.3 immediately follows:

1.4 Proposition. *There is a unique functor  $P: Top \rightarrow Qun$  (herein after to be called the Pervin functor) which satisfies  $TP = I$  and equips each object of  $Top$  with its Pervin quasi-uniformity.*

Let  $\mathcal{X}$  denote the class of all right inverses of  $T$ , that is, functors  $F: Top \rightarrow Qun$  for which  $TF = I$ . These functors satisfy  $\|\dots\|F = |\dots|$ , that

is, they “preserve sets and functions”. Hence we can define a partial ordering of  $\mathcal{K}$  as follows:

$F \leq G$  ( $F$  coarser than  $G$ ,  $G$  finer than  $F$ ) iff for every topological space  $X$ ,  $F(X)$  has a coarser structure than  $G(X)$ .

1.5 Proposition. *The Pervin functor  $P$  is the coarsest member of the class  $\mathcal{K}$  of all right inverses of  $T$ .*

Proof. Consider any  $F \in \mathcal{K}$  and  $X \in \text{Obj } Top$ . For any morphism  $f: X \rightarrow D$  we have  $F(f): F(X) \rightarrow F(D) = P(D)$ . Since  $\|F(f)\| = \|f\|$ , it follows from 1.2 that the structure of  $P(X)$  is coarser than that of  $F(X)$ .

1.6 Remark.  $P(X)$  need not have the coarsest quasi-uniformity compatible with the topology of  $X$ . For example, let  $X$  be an infinite discrete space. The entourages  $S(\{x\})$  with  $x \in |X|$  generate a compatible quasi-uniformity strictly coarser than the Pervin.

1.7 Proposition. *Every non-void subclass of  $\mathcal{K}$  has a supremum and an infimum in  $\mathcal{K}$  with respect to  $\leq$ .*

Proof. Consider any non-void subclass  $\mathcal{C}$  of  $\mathcal{K}$ . We construct its supremum  $G$ . For any object  $X$  of  $Top$ , the class  $\{F(X): F \in \mathcal{C}\}$  is a set (of cardinal  $\leq \exp \exp \text{card } |X \times X|$ ) and non-void. The union of the quasi-uniformities of these  $F(X)$  generates the quasi-uniformity of  $G(X)$ . Letting  $G$  preserve functions, we have to show that  $G$  is a functor. Consider  $f \in Top(X, Y)$ . An entourage  $M$  from the chosen subbase for  $G(Y)$  is an entourage in  $F(Y)$  for some  $F \in \mathcal{C}$ , hence  $(f \times f)^{-1}M$  will be an entourage in  $F(X)$ ; this proves that  $G(f) \in Qun(G(X), G(Y))$ . Thus  $G$  is a functor. Clearly  $G \in \mathcal{K}$  and  $G = \sup \mathcal{C}$ . To construct the infimum of  $\mathcal{C}$  one takes the supremum of the (by 1.5 non-void) class of all lower bounds of  $\mathcal{C}$ .

In particular  $\mathcal{K}$  has a finest member  $T^*$ , so that the following is almost obvious:

1.8 Proposition. *The finest member  $T^*$  of  $\mathcal{K}$  is the adjoint of the forgetful functor  $T$ . For any topological space  $X$ ,  $T^*(X)$  can be constructed in each of the following ways:*

- (i)  $T^*(X)$  has the finest quasi-uniformity compatible with the topology of  $X$ .
- (ii)  $T^*(X)$  is initial for the class of all continuous mappings of  $X$  into arbitrary quasi-uniform spaces.
- (iii)  $T^*(X)$  is initial for the set of all those identical functions which are continuous from  $X$  to quasi-pseudometric spaces on the point set of  $X$ .

Only (iii) still needs proof, which we defer to section 2.

1.9 Proposition. *The class  $\mathcal{K}$  is not a set. In fact we can assign to each infinite cardinal  $m$  a functor  $F_m \in \mathcal{K}$  in such a way that  $m < n$  implies  $F_m < F_n$ .*

Proof. For any object  $X$  in  $Top$ , let  $F_m(X)$  be the initial quasi-uniform space for the class of all continuous mappings from  $X$  into quasi-uniform spaces of cardinality  $\leq m$ . Letting  $F_m$  preserve functions, we readily see that  $F_m$  is a functor. As  $TF_m(X)$  is initial in  $Top$  for the relevant mappings and  $P(D)$  is one of the spaces of cardinality  $\leq m$ , we have  $TF_m(X) = X$ . Thus  $F_m \in \mathcal{K}$ . Now let  $m < n$ . Clearly  $F_m \leq F_n$ . To show  $F_m \neq F_n$ , consider a discrete topological space  $X$  of cardinality  $n$ . By 1.8 (ii) or (iii) we have  $F_n(X) = T^*(X)$ , so  $F_n(X)$  has the discrete uniformity. However, if  $F_m(X)$  also had the discrete uniformity, there would be a discrete uniform space  $Y$  of cardinality  $m$  and finitely many functions  $f_1, \dots, f_k: |X| \rightarrow \|Y\|$  with

$$\bigcap_{1 \leq j \leq k} \bigcup_{y \in \|Y\|} f_j^{-1}\{y\} \times f_j^{-1}\{y\} = \{(x, x): x \in |X|\}$$

which is impossible with infinite  $m < n$ .

## 2. Quasi-pseudo-metrics

Wherever possible, it is desirable that results about *Unif* be presented in such a way that they correspond to results about *Qun*; and the correspondence should extend to the proofs or constructions. The following proposition is crucial in obtaining the equivalence between uniformities and certain families ("gages" [8]) of pseudo-metrics:

(\*) For any uniform space  $Y$ , the uniformity of  $Y$  is induced by the set of those pseudo-metrics which are uniformly continuous on  $Y \times Y$ .

However, (\*) does not carry over to *Qun* [6 p. 254]. The fact that the following proposition (†) [8 theorem 6.11] also has no simple counterpart in *Qun*, seems to have obstructed the extension of the above-mentioned equivalence to quasi-uniformities.

(†) Let  $Y$  be a uniform space and  $d$  a pseudo-metric for  $\|Y\|$ . Then  $d$  is uniformly continuous on  $Y \times Y$  if and only if the identical function from  $Y$  to  $(\|Y\|, d)$  is uniformly continuous.

The remedy is simple: one observes that (†) is not needed for the equivalence, and one reformulates (\*) as follows (together with its quasi-uniform version):

**2.1 Proposition.** Each (quasi-)uniform space  $Y$  is initial for the set of those identical functions which are (quasi-)uniformly continuous from  $Y$  to (quasi-)pseudo-metric spaces on the point set of  $Y$ .

Save for terminology the uniform half of this proposition occurs in [9 theorem 24 A. 9]. Since the Metrization Lemma can be made independent of symmetry [8 p. 185], a direct proof as in [10 theorem 20.13] suffices to establish both halves of proposition 2.1 (see also [5 p. 228]).

Proof of 1.8 (iii): This is immediate from 2.1 and the fact that  $T^*$  is the finest functor in  $\mathcal{K}$ . The same argument establishes the uniform analogue [9 theorem 24 B. 12] of 1.8 (iii).

### 3. Characterization of right inverses of $T$

Let  $\mathcal{L}$  be the class of those functors  $F: Top \rightarrow Qun$  which satisfy  $\|\dots\|F=\|\dots\|$ . Clearly  $\mathcal{K} \subset \mathcal{L}$ .

**3.1 Theorem.** *Let  $F \in \mathcal{L}$ . Then  $F \in \mathcal{K}$  if and only if the following two conditions are satisfied:*

- ( $\alpha$ )  $TF(D') = D'$
- ( $\beta$ ) *There exists a class  $\mathcal{B}$  of quasi-uniform spaces such that for each topological space  $X$ ,  $F(X)$  is initial in  $Qun$  for the continuous mappings of  $X$  into members of  $\mathcal{B}$ .*

**Proof.** If  $F \in \mathcal{K}$  then ( $\alpha$ ) holds trivially and we choose  $\mathcal{B} = \{F(A) : A \in \text{Obj } Top\}$ . To see that the initial quasi-uniformity is coarser than that of  $F(X)$ , consider a morphism  $f \in Top(X, T(B))$  where  $B \in \mathcal{B}$ . For some  $A$ ,  $B = F(A)$ , hence  $FT(B) = FTF(A) = F(A) = B$ . Thus  $F(f) \in Qun(F(X), B)$ . To see that the initial structure is finer than that of  $F(X)$ , one observes that  $1_X \in Top(X, X) = Top(X, TF(X))$  so that  $1_X$  is one of the generating mappings.

Conversely, let  $F \in \mathcal{L}$  satisfy ( $\alpha$ ) and ( $\beta$ ).  $TF(X)$  is initial for the continuous mappings of  $X$  into spaces  $T(B)$  with  $B \in \mathcal{B}$ . As this holds in particular for  $X = D'$ , one sees that there exists  $B_0 \in \mathcal{B}$  such that  $D'$  is initial for mappings  $D' \rightarrow T(B_0)$ . Hence  $TF(X) = X$ .

Clearly the uniform analogue of 3.1, in which  $D'$  is replaced by the closed unit interval, is valid; in the proof one replaces  $B_0$  by a set of members of  $\mathcal{B}$ . Categorical generalizations of 3.1 lie outside the concrete scope of this note. The reader will observe that the proofs in section 1 could (with some loss of simplicity) be presented as applications of 3.1.

Extending the ordering of  $\mathcal{K}$  by coarseness to  $\mathcal{L}$ , one sees from the proof of 1.5 that  $P$  is the coarsest functor satisfying ( $\alpha$ ) in  $\mathcal{L}$ . Further,  $\mathcal{L}$  has a coarsest member — which satisfies ( $\beta$ ) but not ( $\alpha$ ).

**3.2 Example of  $F \in \mathcal{L}$  satisfying ( $\alpha$ ) but not ( $\beta$ ).** Let  $F = PS$  where  $S: Top \rightarrow Top$  is the sequential coreflector (the open sets of  $S(X)$  are the sequentially open sets of  $X$ , and  $S$  preserves functions). The functoriality of  $S$  is proved in [11 theorem 2.2]. Clearly  $TF(D') = D'$ , but  $TF(X) \neq X$  whenever  $X$  is not a sequential space.

**3.3 Example of  $F \in \mathcal{L}$  satisfying neither ( $\alpha$ ) nor ( $\beta$ ).** Let  $F$  be the finest member of  $\mathcal{L}$ . Consider an indiscrete space  $X$  to see that  $F$  does not satisfy ( $\beta$ ).

### 4. Application to $T_D$ -spaces

The  $T_D$ -spaces were introduced by AULL and THRON [12]; see also [13]. We characterize these spaces with the aid of a functor  $E$  from  $Top$  to the subcategory of uniformizable spaces as follows.

For any object  $X$  in  $Top$ , form the uniform space whose structure is the supremum of the structures of  $P(X)$  and its conjugate. It follows from 1.4 and [4 theorem 1.26] that this assignment is functorial. Then apply the forgetful functor  $T$  to obtain  $E(X)$ . Clearly  $E(X)$  has a uniformizable topology finer than that of  $X$ , and one expects that  $E(X)$  will often be discrete.

4.1 Proposition.  $E(X)$  is discrete if and only if  $X$  is a  $T_D$ -space.

Proof. The topology of  $E(X)$  is induced by the uniformity  $\mathcal{P} \vee \mathcal{P}^{-1}$  where  $\mathcal{P}$  is the Pervin quasi-uniformity on  $X$ . Recalling that  $\mathcal{P}$  has the subbase  $\{S(A) : A \text{ open in } X\}$ , one sees that  $E(X)$  is discrete if and only if, for each  $x \in |X|$ , there exist finitely many open sets  $A_j$  and  $B_k$  of  $X$  such that

$$\{x\} = (\bigcap S(A_j) \cap \bigcap S(B_k)^{-1})[x],$$

that is, by [4 prop. 0.18],

$$\{x\} = (\bigcap S(A_j)[x]) \cap (\bigcap S(B_k)^{-1}[x]).$$

Now  $S(A_j)[x] = A_j$  if  $x \in A_j$ ,  $|X|$  if  $x \notin A_j$ ; and  $S(B_k)^{-1}[x] = |X|$  if  $x \in B_k$ ,  $|X| - B_k$  if  $x \notin B_k$ . Hence the condition reduces to each  $\{x\}$  being the intersection of an open and a closed set. This is equivalent to  $X$  being a  $T_D$ -space.

4.2 Remark. In the present context we can answer the question of THRON in [10 p. 108], whether  $T_D$  is preserved under infinite topological products. Clearly  $D$  is a  $T_D$ -space. Consider the space  $\mathbb{R}$  obtained by giving the real line the topology with base for open sets consisting of the open intervals of the form  $(-\infty, x)$ . Since  $\mathbb{R}$  is a countably based  $T_0$ -space, it can by 1.1 be embedded in the topological product  $D^{*\aleph_0}$ . But  $T_D$  is hereditary and  $\mathbb{R}$  is not a  $T_D$ -space. Hence  $D^{*\aleph_0}$  is not a  $T_D$ -space.

Note added in proof: Natural quasi-uniform analogues of the propositions (\*) and (†) of Section 2 have now been found by S. SALBANY [14].

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