The Influence of Structure Formation on the evolution of the Universe.

By

Obinna Umeh

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Supervisors:
Assoc. Prof. Chris Clarkson & Prof. G. F. R. Ellis

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Abstract

The next generation of telescopes will usher in a new era of precision cosmology capable of determining key parameters of a cosmological model to percent level and beyond. For this to be effective, the theoretical model must be understood to at least the same level of precision. A range of subtle physical spacetime effects remain to be explored theoretically, for example, the effect of backreaction on cosmological observables. A good understanding of this effect is paramount given that it is a consequence of any space-time theory of gravity. We provide a comprehensive study of this effect from the perspective of geometric averaging on a hyper-surface and averaging on the celestial sphere. We concentrate on Friedmann-Lemaitre-Robertson-Walker spacetime with small perturbation up to non-linear order . This enables us to quantify by how much this effect could change the standard model interpretation of the universe. We study in great detail key parameters of the standard model, Hubble rate, deceleration parameter and area distance. We find that the Hubble rate depends on the choice of definition of the Hubble rate and the spatial surface on which the average is performed. Within the ΛCDM model, the backreaction effect on the background Hubble rate is of order 1% at a scale of 100 Mpc, and much less on larger scales. We find that for the deceleration parameter adapted to observation, the perturbation theory gives divergent answers in the UV and corrections to the background are of order unity or more depending on the choice of UV cut-off. For the area distance, we identify a range of new lensing effects, which include: double-integrated and nonlinear integrated Sach-Wolfe contributions, transverse Doppler effects in redshift space distortions, lensing from the induced vector mode and gravitational wave backgrounds, in addition to lensing from the second-order potential and we also identify a new double-coupling between the density fluctuations integrated along the line of sight, and gradients in the density fluctuations coupled to transverse velocities along the line of sight. We conclude that the precision cosmology would be unsuccessful without the effect of backreaction being properly taking into account in parameter estimation. Also we need to re-think our theoretical approach to sub-horizon universe because un-renormalized perturbation theory appear not to be working.

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Most of the computations here were done with the help of the tensor algebra package xPert/xAct [1] and xPand which I developed in collaboration with Cyril Pitrou and Xavier Roy.

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Any opinion, findings and conclusions or recommendations expressed in this work are that of the author and therefore NITheP, SKA, NRF, Royal Society (UK) and UCT PhD package do not accept any liability with regard thereto.

Dedication

This work is dedicated to the loving memory

of my mother,

Late Mrs. Felicia Umeh

Declaration

This thesis is based on works done by the author under the supervision of Assoc. Prof. Chris Clarkson and Prof. George R. Ellis, of Department of Mathematics and Applied Mathematics, University of Cape Town. I benefitted from collaboration with Prof. Roy Maartens and Dr. Andreas Faltenbacher of Physics Department, University of the Western Cape and also Dr. Xavier Roy of Department of Mathematics and Applied Mathematics, University of Cape Town. The other collaborators include Prof. J. P Uzan and Cyril Pitrou of Institut d'Astrophysique de Paris, Université Pierre et Marie Curie, Paris, France and Dr. Julien Larena of Rhodes University, South Africa. The product of my collaboration with these people resulted in the following peer-reviewed papers on the topic of my thesis:

1. **Obinna Umeh**, Julien Larena and Chris Clarkson "The Hubble rate in averaged cosmology", JCAP **1103**, 029 (2011) [arXiv:1011.3959 [astro-ph.CO]].

2. Chris Clarkson , George Ellis , Julien Larena, and **Obinna Umeh** "Does the growth of structure affect our dynamical models of the universe? The averaging, backreaction and fitting problems in cosmology", Rept. Prog. Phys. **74**, 112901 (2011) [arXiv:1109.2314 [astro-ph.CO]].

3. Chris Clarkson and **Obinna Umeh** "Is backreaction really small within concordance", Class. Quant. Grav. **28**, 164010 (2011) [arXiv:1105.1886 [astro-ph.CO]].

4. Chris Clarkson, George F. R. Ellis, Andreas Faltenbacher, Roy Maartens, **Obinna, Umeh** and Jean Philippe Uzan, "(Mis-) Interpreting supernovae observations in a lumpy universe", Mon. Not. Roy. Astron. Soc. **426**, 1121 (2012) [arXiv:1109.2484 [astro-ph.CO]].

5. **Obinna Umeh**, Chris. Clarkson and Roy. Maartens, "Nonlinear general relativistic corrections to redshift space distortions, gravitational lensing magnification and

cosmological distances," arXiv:1207.2109 [astro-ph.CO].

6. Cyril Pitrou, Xavier Roy and **Obinna Umeh**, "xPand: An algorithm for perturbing homogeneous cosmologies," [arXiv:1302.6174 [astro-ph.CO]], http://www.xact.es/xpand or http://www2.iap.fr/users/pitrou/xpand.htm

Chapter 1 of this thesis is a review of the standard model of cosmology. Chapter 2 builds up materials on cosmological perturbation theory in a form we need them in the subsequent chapters, there are new results in this chapter but they are minor.

Chapter 3 includes reprints of some sections in number 2 and 4 of the already published papers listed above. Chapter 4 also includes a re-formated reprints of paper 1 above. The sections included here are only the sections I contributed most in the papers.

Chapter 5 and 6 are entirely new results of mine that are yet to be sent to a peer-reviewed journal for publication.

I hereby declare that this thesis has not been submitted, either in the same or different form, to this or any other university for a degree and that it represents my own work.

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Obinna Umeh

Contents

Chapter 1

Origin of Structures within the Standard Model

1.1 Introduction

The cosmological principle supported by 'Occam's razor' is one of the fundamental pillars of modern cosmology. It puts strong constraints on the type of spacetime and matter field that could describe the universe on large scales. The two key constraints are [2]:

- when the universe is averaged over sufficiently large scales, the mean motion of radiation and matter in the Universe with respect to all averaged observable properties are isotropic. This implies that every observable must be rotationally invariant on large scales.

- all fundamental observers experience the same history of the Universe, i.e. the same averaged observable properties, provided they set their clocks suitably. Such a Universe is called observer-homogeneous. This means that the statistical property of observable must be translationally invariant.

The biggest observational support for the isotropy constraint comes from the Cosmic Microwave Background radiation (CMB) [3, 4] and the ability to understand small angular scale physics of the CMB using the linear theory provides support for large scale homogeneity in the early universe.

The isotropy constraint requires that clocks be synchronised such that the metric tensor vanish, $g_{0i} = 0$ for the spacetime in comoving cordinates. The only metric that satisfies these constraints is the Friedmann-Lematre-Robertson-Walker-metric(FLRW) and it is given by

$$\mathrm{d}\,s^2 \;=\; -\mathrm{d}\,t^2 + a(t)^2\left[\mathrm{d}\,\chi^2 + f_K^2(\chi)\left(\mathrm{d}\,\theta^2 + \sin^2\theta\mathrm{d}\,\phi^2\right)\right]\,, \tag{1.1}$$

where

$$f_K(\chi) = \begin{cases} K^{-\frac{1}{2}} \sin\left(K^{\frac{1}{2}}\chi\right) & (K > 0) \\ \chi & (K = 0) \\ (-K)^{\frac{1}{2}} \sinh\left[(-K)^{\frac{1}{2}}\chi\right] & (K < 0) \end{cases},$$

with $f_K(\chi)$ and $K^{-\frac{1}{2}}$ having the dimensions of length and the constraints also require that the 4-velocity of the comoving observer be given by $u^a = (1/a, 0, 0, 0)$. If we define the radius of the 2-shpere as $f_K(\chi) = r$, then equation (1.1) has an alternative form,

$$\mathrm{d}\,s^2 = -\mathrm{d}\,t^2 + a^2(t)\left[\frac{\mathrm{d}\,r^2}{1 - Kr^2} + r^2(\mathrm{d}\,\theta^2 + \sin^2\theta \mathrm{d}\,\phi^2)\right]. \tag{1.2}$$

Here $a(t)$ is the scale factor, t is the cosmic time and the constant K is the mean-spatial curvature of the hyper-surface or the spatial surface. It could take a positive, zero, or a negative value, where each K value corresponds respectively to closed (elliptic), flat, or open universe (hyperbolic). Transformation to more user-friendly conformal time is achieved by rescaling $\mathrm{d}\,t = a\mathrm{d}\,\eta$. The evolution of the scale factor is determined by the Einstein equations of general relativity [5]

$$G_{ab} \equiv R_{ab} - \frac{1}{2}g_{ab}R = 8\pi G T_{ab} - \Lambda g_{ab}, \tag{1.3}$$

where R_{ab}, R, T_{ab}, and G are the Ricci tensor, Ricci scalar, energy momentum tensor, gravitational constant, respectively. Λ is the cosmological constant. The metric (1.2) admits a perfect fluid and it is given by the energy-momentum tensor

$$T_{ab} = (\rho + p)u_a u_b + p g_{ab}\,, \tag{1.4}$$

where ρ and p are the energy density and pressure of the universe. Contracted Bianchi identity requires that the divergence of T_{ab} vanishes, resulting in a conservation equation for the matter field,

$$\nabla_c T^{cb} = 0, \tag{1.5}$$

where ∇ denotes the covariant derivative associated with the metric g_{ab}. Computing the Einstein tensor G_{ab} from the metric given in equation (1.2) and substituting the energy-momentum tensor (1.4) into equation (1.3) gives a system of equations that govern the evolution of the scale factor and the energy density:

$$H^2 = \frac{8\pi G\rho}{3} - \frac{K}{a^2}, \tag{1.6}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3p\right), \tag{1.7}$$

$$\dot{\rho} = -3H\left(\rho + p\right), \tag{1.8}$$

where the first, second and the third equations are the Friedmann equation, the Raychudhuri equation and the energy conservation equation respectively. We have also defined the Hubble parameter as $H \equiv \dot{a}/a$. Assuming that matter in the universe could be characterized as a barotropic fluid, its equation of state is given by $p = w\rho$. If the universe is dominated by a non relativistic matter we have $w \simeq 0$, by a gas of relativistic particles, $w = 1/3$ and by a cosmological constant, $w = -1$. The evolution of the scale factor and the energy density in different universe scenarios is given in Table 1.1. In our universe, several species

	w	$\rho(a)$	$a(t)$	$a(\eta)$	$\eta_{\rm ini}$
Radiation dominated	1/3	a^{-4}	$t^{1/2}$	η	0
Matter dominated	0	a^{-3}	$t^{2/3}$	η^2	0
curvature dominated	-	a^{-2}	t	$e^{H_0\Omega_k^{1/2}\eta}$	$-\infty$
Λ dominated	-1	a^0	e^{Ht}	$-\eta^{-1}$	$-\infty$

Table 1.1: Possible FLRW solutions with one form of fluid dominating the energy budget. $\eta_{\rm ini}$ is the conformal time at the early time.

with different equations of state coexist, and it has become customary to characterize their relative contributions by the dimensionless parameters

$$\Omega_{(i)} \equiv \frac{8\pi G\rho_{(i)}}{3H_0^2}, \tag{1.9}$$

where the $\rho_{(i)}$ denotes the present energy densities of the various species (for example Ω_m stands of contribution from non-relativistic matter) and H_0 is today's Hubble parameter. The current best fit parameters for the observable universe is given in Table 1.2: Applying equation (1.9) to equations (1.6) and (1.8), the Hubble rate and the Raychaudhuri equation become

$$H^2(z) = H_0^2\left[(1+z)^4\Omega_{r,0} + (1+z)^3\Omega_{m,0} + (1+z)^2\Omega_{K,0} + \Omega_{\Lambda,0}\right], \tag{1.10}$$

$$q = \frac{1}{2}\left(\Omega_m + 2\Omega_r - 2\Omega_\Lambda\right), \tag{1.11}$$

where q is the declaration parameter and we have introduced the redshift z,which characterize the change of a signal emitted by a source with a 4-velocity u_s^a, e.g a galaxy (here denoted as 's') and the signal measured by an observer with 4-velocity u_o^a

$$(1+z) = \frac{(k_a u^a)_o}{(k_b u^b)_s} = \frac{\lambda_o}{\lambda_s} = \frac{a_0}{a} \, . \tag{1.12}$$

Here k^a is the photon 4-vector or a tangent vector to the photon geodesic. For $H(z_o) = H_0$ and using equation (1.10) the constant mean curvature may be defined in terms of energy density parameter for different specie

$$K = {H_0}^2 \left(\Omega_{r0} + \Omega_{m0} + \Omega_{\Lambda 0} - 1\right) , \tag{1.13}$$

where we have set the scale factor today to to unity ($a_0 = 1$). In a matter dominated universe, the dimensionless energy density parameter for the non-relativistic matter and the cosmological constant may be written as

$$\Omega_m(z) = \frac{8\pi G \rho_0 a^{-3}}{3\mathrm{H}^2(z)} = \frac{\Omega_{m0}(1+z)^3}{[(1+z)^4\Omega_{r,0} + (1+z)^3\Omega_{m,0} + (1+z)^2\Omega_{K,0} + \Omega_{\Lambda,0}]} , \tag{1.14}$$

$$\Omega_\Lambda(z) = \frac{\Lambda}{3\mathrm{H}^2(z)} = \frac{\Omega_\Lambda}{[(1+z)^4\Omega_{r,0} + (1+z)^3\Omega_{m,0} + (1+z)^2\Omega_{K,0} + \Omega_{\Lambda,0}]} . \tag{1.15}$$

1.1.1 Distances in Standard Cosmology

The comoving coordinate distance, χ, which characterizes the distance between two points measured along a path defined at the present cosmological time may be derived from equation (1.1)

$$\chi(z) = \frac{1}{a_0 H_0} \int_0^z \frac{\mathrm{d}\, z'}{\sqrt{\Omega_{m,0}(1+z')^3 + \Omega_{r,0}(1+z')^4 + \Omega_{\Lambda,0} + \Omega_{K,0}(1+z')^2}} , \tag{1.16}$$

The area distance and the luministy distance follow directly from equation (1.16) with some assumptions about the observed luminosity

$$D_L(z) = (1+z) f(\chi(z)) , \qquad D_A(z) = \frac{1}{(1+z)} f(\chi(z)) . \tag{1.17}$$

Notice the duality relation $D_L = (1+z)^2 D_A$. In general, the area distance is given by

$$D_A = \frac{1}{(1+z)H_0\sqrt{-\Omega_k}} \sin\left(\sqrt{-\Omega_K} \int_0^z \frac{\mathrm{d}\, z'}{h(z')}\right) , \tag{1.18}$$

where

$$h(z) = \frac{H(z)}{H_0} = \sqrt{[(1+z)^4\Omega_{r,0} + (1+z)^3\Omega_m + (1+z)^2\Omega_{K,0} + \Omega_{\Lambda,0}]} \,. \quad (1.19)$$

In an Einstein de Sitter universe ($\Omega_{m,0} = 1, \Omega_{r,0} = 0, \Omega_{\Lambda,0} = 0$), equation (1.18), may be integrated to give

$$D_A(z) = \frac{2}{H_0(1+z)}\left[1 - (1+z)^{-\frac{1}{2}}\right] \,. \quad (1.20)$$

Observationally the luminosity distance is given as a distance modulus

$$\mu_0 = m - M = 5\log\left(\frac{D_L}{\text{Mpc}}\right) \,, \quad (1.21)$$

where the apparent magnitude m and the absolute magnitude M are logarithmic measure of flux and luminosity respectively.

Class	Parameter	WAMP 7-year ML	Class	Parameter	WAMP 7-year ML
Primary	$100\Omega_b h^2$	2.270	Derived	σ_8	0.803
	$\Omega_c h^2$	0.1107		H_0	$71.4km/s/Mpc$
	Ω_Λ	0.738		Ω_b	0.0445
	n_s	0.969		Ω_c	0.217
	τ	0.086		$\Omega_m h^2$	0.1334
	$\Delta^2_{\mathcal{R}}(k_0)^c$	2.38×10^{-9}		$z_{\text{reion.}}$	10.3
				t_0	13.71 Gyr

Table 1.2: The summary of maximum likelihood cosmological parameters for the ΛCDM model. Here the superscript c stands for the estimation performed for $\Delta^2_{\mathcal{R}}(k) = k^3 P_{\mathcal{R}}(k)/(2\pi^2)$ with $k_0 = 0.002$ Mpc^{-1} and b represents the mean of the posterior distribution

1.1.2 The Inevitability of an ultra-rapid expansion

Although very successful in some aspects of its attempt to describe the universe, the standard Big Bang picture of the universe suffered from a couple of problems so much so that in less than four decades was almost unappealing despite its initial successes. The major reason for some of these problems is that for normal matter, only decelerating expansion is possible. An attempt to find a solution to some of the problems requires that the universe at a certain stage in its evolution underwent an ultra-rapid expansion. Some of they problems include

- *Horizon Problem:* This is a problem associated with the fact that different patches in the sky that are just entering the Hubble volume, $(aH)^{-1}$, do not have enough have time to interact in the past within the standard Big Bang theory, because the physical distance or the comoving particle horizon between them was very short. The comoving particle horizon is the maximum causal distance within which particles can exchange information at the speed of light, This implies that if this distance was smaller than the comoving Hubble radius, $(aH)^{-1}$ in the past, the two particles could never have communicated in the past.

Yet the CMB we observe today show that different patches in the sky appear to have had enough time to talk to each other at the time of last scattering when they were first emitted.

Mathematically, the maximum causal distance between particles is given by

$$\tau \equiv \int_0^t \frac{\mathrm{d}\,t'}{a(t')} = \int_0^a \frac{\mathrm{d}\,a}{Ha^2} = \int_0^a \mathrm{d}\,\ln a \left(\frac{1}{aH}\right), \tag{1.22}$$

For a perfect fluid dominated universe, the comoving Hubble rate depends on the equation of state parameter, w, according to $(aH)^{-1} = H_0^{-1} a^{\frac{1}{2}(1+3w)}$. In a standard Big Bang scenario, with the equation of state parameter, $w \gtrsim 0$, the fraction of the universe in causal contact also grows with time $\tau \propto a^{\frac{1}{2}(1+3w)}$. This shows that the size of the comoving horizon increases monotonically with time, hence the comoving horizon was very small in the past.

A simple fix to this problem looking at equation (1.22) will be to find a way to get the comoving Hubble radius, $(aH)^{-1}$, to shrink or keep H constant at the early time before decoupling, so that the particle could communicate and after the standard big bang evolution continues.

- *Flatness problem:* The universe we observe today appear to be well-described by a flat Euclidean space. However, according to the standard model where

$$1 - \Omega(a) = \frac{-K}{(aH)^2}, \tag{1.23}$$

this is unlikely, because the comoving Hubble radius, $(aH)^{-1}$, increases with time and equation. (1.23) says that the quantity $|\Omega - 1|$ must diverge with time. Thus the critical value $\Omega = 1$ or the condition for flatness within the standard model is an unstable fixed point. In fact the universe was flat even at the time of Big Bang Nucleosynthesis (BBN), $|\Omega(a_{\mathrm{BBN}}) - 1| \leq \mathcal{O}(10^{-16})$

In order to reconcile with the standard Big Bang cosmology, the comoving Hubble radius has to decrease at early time so that it drives the universe toward flatness.

- *The origin of large-scale structure in the universe:*

  We see structures (galaxies and clusters of galaxies) in the sky, even up to the largest possible scale. The general understanding is that these structures formed via gravitational instability from some seed perturbations. The origin of these seed perturbations was not predicted by the standard Big Bang model. Also the existence of these structures on large-scale is also unlikely within the standard model because the particle horizon was very small in the past.

  A fix to the problem of initial condition for structure formation will be to require that the comoving horizon decreases or that the Hubble rate remains constant at some early time, so that the required perturbations could be generated by a quantum mechanical process on the sub-horizon scale, which then exit the horizon once the Hubble radius becomes smaller than their comoving wavelength. At the superhorizon scale, they become classical perturbations that will in turn re-enter the horizon in the subsequent evolution of the universe to provide the seed fluctuations that would form the large-scale structure we see today through gravitational collapse [6].

- *Monopole Problem:*

  The early universe had more particle symmetries than what exist today. This implies that some of these symmetries were broken at some point during the evolution of the universe. Most particle Physics models predict that whenever a symmetry is broken, some exotic particles are created, for example Monopole, Domain walls, etc. Most of these particles evolve like normal matter (i.e protons) but with energy density significantly larger than that of the normal matter. This implies that if they were created large enough in the early universe, their presence would have forced the universe to re-collapse.

  But if there was a period of rapid expansion in the early universe, these particles would be redshifted away.

Finally, to solve all these problems, all that is needed is to find a mechanism that would effectively shrink the size of the comoving Hubble radius, $(aH)^{-1}$ during a brief period in the early time and returns the evolution back to the standard Big Bang model afterwards. One mechanism that elegantly realizes this is called cosmic Inflation.

1.2 A Brief Period of Accelerated Expansion

The key requirement from any mechanism that will solve the problems above is that the Hubble volume shrinks $\mathrm{d}\,(aH)^{-1}/\mathrm{d}\,t < 0$ and from equation (1.8), the Hubble volume is related to the accelerated expansion $\mathrm{d}\,(aH)^{-1}/\mathrm{d}\,t = -\ddot{a}/(aH)^2$, therefore the decrease in comoving Hubble radius implies an accelerated expansion $d^2a/dt^2 > 0$. Using the Ray-chaudhuri equation (1.8), an accelerated expansion puts a constraint on the equation of state of the required matter, $p < -\frac{1}{3}\rho$, thus, matter with a negative pressure is needed to achieve an accelerated expansion. A negative pressure matter violates the strong energy condition (SEC), but does not undermine the elegance of this mechanism because it does not violate causality for particle interactions.

The action of a single scalar field ϕ hereafter called the *inflaton*, minimally coupled to gravity is given by

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{1}{2}R + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - V(\phi)\right] = S_{\rm EH} + S_\phi\,. \tag{1.24}$$

where $S_{\rm EH}$ is the action of the gravitational field, the gravitational field is described by general relativity, and S_ϕ is the action of a scalar field with canonical kinetic term and a potential term. The potential $V(\phi)$ describes the self-interactions of the scalar field. The variation of the action, S_ϕ, with respect to the metric gives the energy-momentum tensor for the scalar field

$$T^{(\phi)}_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta S_\phi}{\delta g^{\mu\nu}} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left(\frac{1}{2}\partial^\sigma\phi\partial_\sigma\phi + V(\phi)\right). \tag{1.25}$$

and variation with respect to the inflation gives the equation of motion for ϕ

$$\frac{\delta S_\phi}{\delta\phi} = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\partial^\mu\phi) + \frac{dV}{d\phi} = 0\,, \tag{1.26}$$

It is possible to deduce the energy density and pressure associated with the energy-momentum tensor $T^{(\phi)}_{\mu\nu}$ by comparing equation (1.25) with the general expression for the energy-momentum tensor for a perfect fluid (equation (1.4))

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) + \frac{1}{2}(\vec{\nabla}\phi)^2, \tag{1.27}$$

$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi) - \frac{1}{6}(\vec{\nabla}\phi)^2\,. \tag{1.28}$$

Specializing to the case of a homogeneous field $\phi(t, \mathbf{x}) \equiv \phi(t)$, leads to an equation of state for ϕ

$$w_\phi \equiv \frac{p_\phi}{\rho_\phi} = \frac{\frac{1}{2}\dot{\phi}^2 - V}{\frac{1}{2}\dot{\phi}^2 + V}\,, \tag{1.29}$$

Equation (1.29) shows that an inflaton can lead to a shrinking Hubble sphere if the potential energy term, $V(\phi)$, dominates over the kinetic energy, $\frac{1}{2}\dot{\phi}^2$, that is whenever $V > \frac{1}{2}\dot{\phi}^2$, implying that a negative pressure ($w_\phi < 0$) dominates and expansion of the universe accelerates ($w_\phi < -1/3$).

1.2.1 Conditions for a Slow-Roll Inflation

The evolution of a homogeneous inflaton on an FLRW spacetime is determined by

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0\,, \tag{1.30}$$

$$H^2 = \frac{1}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right)\,, \tag{1.31}$$

$$\frac{\ddot{a}}{a} = -\frac{1}{6}(\rho_\phi + 3p_\phi) = H^2(1-\varepsilon)\,, \tag{1.32}$$

where

$$\varepsilon \equiv \frac{3}{2}(w_\phi + 1) = \frac{1}{2}\frac{\dot{\phi}^2}{H^2}\,. \tag{1.33}$$

Equation (1.32) is the acceleration equation for a universe dominated by a homogeneous inflaton field. For large $\dot{\phi}$, equation (1.30) shows that the inflaton experiences significant Hubble friction from the term $H\dot{\phi}$ slowing down the rate of expansion. The slow-roll parameter ε is related to the evolution of the Hubble parameter

$$\varepsilon = -\frac{\dot{H}}{H^2} = -\frac{d\ln H}{dN}\,, \tag{1.34}$$

where $\mathrm{d}N = H\mathrm{d}t$. To drive the universe through an accelerated expansion phase, ε must be less than one ($\varepsilon < 1$). As $p_\phi \to -\rho_\phi$, the spacetime tends to a de Sitter limit, which corresponds to $\varepsilon \to 0$. In this case, the potential energy dominates over the kinetic energy, $\dot{\phi}^2 \ll V(\phi)$. Inflation occurs if the inflaton is evolving slow enough that the potential energy dominates the kinetic energy, and $\ddot{\phi}$ is small enough to allow this slow-roll condition to be maintained for a sufficient period. Thus, inflation requires

$$\dot{\phi}^2 \ll V(\phi) \tag{1.35}$$

$$|\ddot{\phi}| \ll |3H\dot{\phi}| \;\&\; |\frac{\mathrm{d}V(\phi)}{\mathrm{d}\phi}|\,. \tag{1.36}$$

Accelerated expansion will only be sustained for a sufficiently long period of time if the second time derivative of ϕ is small enough and

this requires smallness of a second slow-roll parameter

$$\eta = -\frac{\ddot{\phi}}{H\dot{\phi}} = \varepsilon - \frac{1}{2\varepsilon}\frac{\mathrm{d}\,\varepsilon}{\mathrm{d}\,N}\,, \tag{1.37}$$

where the condition, $|\eta| < 1$ ensures that the fractional change of ε per e-fold is small. The number of e-folds before inflation ends is given by

$$\begin{aligned} N(\phi) &\equiv \ln\frac{a_{\mathrm{end}}}{a} \\ &= \int_{t}^{t_{\mathrm{end}}} H\mathrm{d}\,t = \int_{\phi}^{\phi_{\mathrm{end}}} \frac{H}{\dot{\phi}}\mathrm{d}\,\phi \approx \int_{\phi_{\mathrm{end}}}^{\phi} \frac{V}{V_{,\phi}}\mathrm{d}\,\phi\,. \end{aligned} \tag{1.38}$$

To solve the horizon and flatness problems requires that the total number of inflationary e-folds exceeds about 60,

$$N_{\mathrm{tot}} \equiv \ln\frac{a_{\mathrm{end}}}{a_{\mathrm{start}}} \gtrsim 60\,. \tag{1.39}$$

The precise value depends on the energy scale of inflation and on the details of reheating after inflation. The fluctuations observed in the CMB anisotropies are created by $N_{\mathrm{cmb}} \approx 40 - 60$ e-folds before the end of inflation.

1.2.2 Other Approaches to Inflation

There are other approaches towards realizing inflation in the early universe that deviate slightly from the general well-behaved canonical kinetic term and minimally coupled single field inflation, they include: Gauge Inflation [7–10] , Non-minimal coupling to gravity, Non-canonical kinetic term, for example DBI inflation etc. There are also alternatives to inflation: these are possibilities that could potentially lead to the type of universe we observed today that do not involve any kind of accelerated expansion by a scalar field minimally coupled to general relativity. They include: Ekyrotic/Cyclic Universe, Conformal Cyclic cosmology and a group of Modified gravity theories such as higher derivative gravity and a class of f(R,G,T) theories. Other groups with link to quantum gravity include: Horava-Lifzshit gravity (the Lorentz invariance is broken to retain high energy renormalizability), Brane Collision (String theory inspired approach where the present universe might be a product of high energy collision of two branes in the past), Supergravity inflation (most Supergravity theories live in higher dimensions, their dimensional reduction could provide a fundamental scalar field that drives the accelerated expansion of the universe), String or M-Theory inspired approach

(M-theory is a theory of quantum gravity, hence could naturally provide a consistent guideline not only on inflation but on the initial conditions of the universe [11, 12]), etc.

1.3 Quantum Origin of cosmological Structures

The density inhomogeneities we observed today were seeded by the initial vacuum quantum fluctuations of the inflaton in the sub-horizon region and were stretched out of the horizon where they got converted to classical cosmological perturbations before re-entry into the horizon [13, 14]. The detail of this scenario is given below:

1.3.1 Origin of Curvature Perturbations

The necessary part of the action for curvature perturbations, $\mathcal{R}$, is given by [15]

$$S = \frac{1}{2}\int \mathrm{d}\,\tau \mathrm{d}^{\,3}x \,\left[(v')^2 + (\partial_i v)^2 + \frac{z''}{z}v^2\right] \tag{1.40}$$

$v \equiv z\mathcal{R}$, with $z^2 \equiv a^2\frac{\dot{\phi}^2}{H^2} = 2a^2\varepsilon$, and we have switched to conformal time with a notation τ to avoid confusion with the slow-roll parameter. In Fourier space the equation of motion from (equation (1.40)), is given by

$$v_k'' + \left(k^2 - \frac{z''}{z}\right)v_k = 0\,. \tag{1.41}$$

Equation (1.41) resembles that of a harmonic oscillator with a time dependent mass. In de Sitter space ($z''/z = a''/a = 2\tau^2$), the general solution is given by

$$v_k = \sqrt{\frac{1}{2k}}\left(1 - \frac{i}{k\tau}\right)e^{-ik\tau}\,, \tag{1.42}$$

where we have selected the mode that matches the Minkowski vacuum, i.e Bunch-Davies vacuum. The power spectrum in quasi-de Sitter space time for the field $\psi_{\mathbf{k}}$ ($\hat{\psi}_{\mathbf{k}} \equiv a^{-1}\hat{v}_{\mathbf{k}}$) [6],

$$\langle\hat{\psi}_{\mathbf{k}}(\tau)\hat{\psi}_{\mathbf{k}'}(\tau)\rangle \to (2\pi)^3\delta(\mathbf{k}+\mathbf{k}')\frac{H^2}{2k^3}\,. \tag{1.43}$$

where we have taken the superhorizon limit, $|k\tau| \ll 1$. We have also used the generic quantization technique to arrive at equation 1.43. Equation. (1.43), allows to calculate the power spectrum of the curvature perturbation $\mathcal{R} = \frac{H}{\dot{\phi}}\psi$ at horizon crossing, $a(t_\star)H(t_\star) = k$,

$$\langle \mathcal{R}_{\mathbf{k}}(t)\mathcal{R}_{\mathbf{k}'}(t)\rangle = (2\pi)^3\delta(\mathbf{k}+\mathbf{k}')\frac{H_\star^2}{2k^3}\frac{H_\star^2}{\dot{\phi}_\star^2} = (2\pi)^3\delta(\mathbf{k}+\mathbf{k}')P_{\mathcal{R}}(k)\,, \tag{1.44}$$

From equation 1.44 dimensionless power spectrum may also be defined in terms of $P_{\mathcal{R}}(k)$

$$\Delta_{\mathcal{R}}^2(k) \equiv \frac{k^3}{2\pi^2}P_{\mathcal{R}}(k)\,. \tag{1.45}$$

1.3.2 Gravitational Wave: Tensor Perturbations

The action for tensor fluctuations is given by

$$S = \frac{M_{\rm pl}^2}{8}\int \mathrm{d}\,\tau \mathrm{d}\,x^3 a^2\left[(h'_{ij})^2 - (\partial_l h_{ij})^2\right]\,, \tag{1.46}$$

where h_{ij} a rank two trace-less tensor which denotes the gravitational waves and the prime $'$ denotes derivative with respect to the conformal time. h_{ij} may be expanded in Fourier space as.

$$h_{ij} = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\sum_{s=+,\times} q_{ij}^s(k)h_{\mathbf{k}}^s(\tau)e^{i\mathbf{k}\cdot\mathbf{x}}\,, \tag{1.47}$$

where $q_{ii} = k^i q_{ij} = 0$ and $q_{ij}^s(k)q_{ij}^{s'}(k) = 2\delta_{ss'}$. It is convenient to re-define, $v_{\mathbf{k}}^s \equiv \frac{a}{2}M_{\rm pl}h_{\mathbf{k}}^s$, so that the action becomes

$$S = \sum_s \frac{1}{2}\int \mathrm{d}\,\tau \mathrm{d}^3\mathbf{k}\left[(v_{\mathbf{k}}^{s\prime})^2 - \left(k^2 - \frac{a''}{a}\right)(v_{\mathbf{k}}^s)^2\right]\,, \tag{1.48}$$

After going through a similar quantization procedure in the scalar field case, the dimensionless power spectrum of the two polarization states of tensor fluctuations is given by

$$\Delta_{\rm T}^2 = 2\Delta_h^2(k) = \frac{2}{\pi^2}\frac{H_\star^2}{M_{\rm pl}^2}\,, \tag{1.49}$$

where $\Delta_{\rm T}^2$ stands for the dimensional power spectrum for two polarization states of the gravitational waves while $\Delta_h^2(k)$ is the dimensional power spectrum for a single polarization state.

1.3.3 Interesting Things for Observation

- *Tensor-to-Scalar ratio:*

  The tensor-to-scalar ratio provides a link to directly measure the energy scale of inflation. It relies on the link between the tensor power spectrum and the inflaton potential,

$\Delta_{\rm T}^2 \propto H^2 \approx V$, so that given the tensor-to-scalar ratio

$$r \equiv \frac{\Delta_{\rm T}^2(k)}{\Delta_{\mathcal{R}}^2(k)}\,, \tag{1.50}$$

and assuming a nearly scale-invariant scalar power spectrum, the height of the inflaton potential may be written in terms tensor-to-scalar ratio

$$V^{1/4} \sim \left(\frac{r}{0.01}\right)^{1/4} 10^{16}\,{\rm GeV}\,. \tag{1.51}$$

If the tensor-to-scalar ratio, $r \geq 0.01$, this implies that inflation happened at the GUT scale energies.

- *Number of e-folding:* The power spectra of the scalar and tensor fluctuations created by inflation are given by

$$\Delta_{\mathcal{R}}^2(k) \equiv \Delta_{\mathcal{R}}^2(k) = \frac{1}{8\pi^2}\frac{H^2}{M_{\rm pl}^2}\frac{1}{\varepsilon}\bigg|_{k=aH}\,, \tag{1.52}$$

$$\Delta_{\rm T}^2(k) \equiv 2\Delta_h^2(k) = \frac{2}{\pi^2}\frac{H^2}{M_{\rm pl}^2}\bigg|_{k=aH}\,, \tag{1.53}$$

tensor-to-scalar ratio evaluated at the horizon crossing $k = aH$ is related to one of the slow-roll parameters through

$$r \equiv \frac{\Delta_{\rm T}^2}{\Delta_{\mathcal{R}}^2} = 16\,\varepsilon_\star\,. \tag{1.54}$$

where

$$\varepsilon = -\frac{d\ln H}{dN}\,. \tag{1.55}$$

Hence, tensor-to-scalar ratio provides a constraint on the number of e-foldings a single inflaton field could generate.

- *Scale-Dependence of power spectrum*

The scale dependence of the spectra follows from the time-dependence of the Hubble parameter and is quantified by the spectral indices

$$n_{\mathcal{R}} - 1 \equiv \frac{d\ln\Delta_{\mathcal{R}}^2}{d\ln k}\,, \qquad n_{\rm T} \equiv \frac{d\ln\Delta_{\rm T}^2}{d\ln k}\,. \tag{1.56}$$

a short derivation shows that the spectral indices relate to the first order Hubble slow-roll parameters according to

$$n_{\mathcal{R}} - 1 = 2\eta_\star - 4\varepsilon_\star \,, \tag{1.57}$$

$$n_{\rm T} = -2\varepsilon_\star \,. \tag{1.58}$$

These imply that any deviation from perfect scale-invariance ($n_{\mathcal{R}} = 1$ and $n_{\rm T} = 0$) provides an indirect window to probe the inflationary dynamics as quantified by the parameters ε and η.

- *Inflationary Potential*

  In the slow-roll approximation the Hubble and potential slow-roll parameters are related as follows

$$\varepsilon \approx \epsilon_{\rm v} \,, \qquad \eta \approx \eta_{\rm v} - \epsilon_{\rm v} \,. \tag{1.59}$$

where $\epsilon_{\rm v} = V_{,\phi}$ and $\eta_{\rm v} = V_{,\phi\phi}$. It is possible to express the scalar and tensor spectra purely in terms of $V(\phi)$ and $\epsilon_{\rm v}$ (or $V_{,\phi}$)

$$\Delta^2_{\mathcal{R}}(k) \approx \frac{1}{24\pi^2}\frac{V}{M^4_{\rm pl}}\frac{1}{\epsilon_{\rm v}}\bigg|_{k=aH} \,, \qquad \Delta^2_{\mathcal{R}}(k) \approx \frac{2}{3\pi^2}\frac{V}{M^4_{\rm pl}}\bigg|_{k=aH} \,. \tag{1.60}$$

Using equation (1.59), we notice that the scalar spectral index is given by $n_{\mathcal{R}} - 1 = 2\eta^\star_{\rm v} - 6\epsilon^\star_{\rm v}$, while the tensor spectral index is given by $n_{\rm T} = -2\epsilon^\star_{\rm v}$, and the tensor-to-scalar ratio then becomes

$$r = 16\epsilon^\star_{\rm v} \,. \tag{1.61}$$

We see that single-field slow-roll models satisfy a consistency condition between the tensor-to-scalar ratio r and the tensor tilt $n_{\rm T}$

$$r = -8n_{\rm T} \,. \tag{1.62}$$

Within the slow-roll approximation, measurements of the scalar and tensor spectra relate directly to the shape of the potential $V(\phi)$, therefore, measurement of the amplitude and the scale-dependence of the cosmological perturbations encode information about the potential driving the inflationary expansion.

1.4 Non-Gaussian Signature in Cosmological Observables

There are various sources that could contribute a non-Gaussian signal to any cosmological observation. The major sources include:

- *Primordial non-Gaussianity*

  This is non-Gaussianity in the primordial curvature perturbtion $\mathcal{R}$ or the gravitational waves produced in the very early universe by inflation.

- *Second-order non-Gaussianity*

  This type of non-Gaussianities may be generated when perturbations which have exited the horizon, re-enters and due to non-linearity of gravitational field equations, they get amplified.

- *Secondary non-Gausssianity*

  Non-Gaussianity generated by 'late' time effect after recombination, for instance weak gravitational lensing. This is likely to be the dominant source of non-gauassianity in the large scale structure surveys.

- *Foreground non-Gaussianty*

  This form of non-Gaussianity mainly get imprinted in the CMB signal due to background Galactic and extra-Galactic sources.

To properly extract non-Gaussian signal from observation, it is crucial we understand properly the signatures of each source of non-Gaussianity on the observed signal. Our interest is on primordial non-Gaussianity and there are plausible mechanisms to generate it, they include:

- *Self-interaction of the inflaton field:* Self-interaction of the inflaton field generate non-Gaussianities on subhorizon scales which gets amplified through its coupling to the metric perturbations.

- *Multi-field inflation field:* In the multi-field case, non-Gaussianities are generated on superhorizon scales either during or after inflation. This is mainly due to non-linear coupling of the scalar field to gravitational field. Since they are generated on superhorizon scales, the mechanism involved is classical, which is opposite to non-Gassianity generated from self-interation of the inflation field.

- Subtle Mechanisms: Other subtle mechanisms that could produce a detectable amount of primordial non-Gaussianity include: the violation of the slow-roll condition, non-canonical kinetic energy term, deviation of the initial vacuum state from the Bunch-Davies vacuum.

For a single field inflation scenario, presence of non-Gaussianity will induce corrections to a Gaussian inflaton fluctuation and the gravitational potential Φ (or $\mathcal{R}_k$). This may be estimated by evaluating the coupling between Φ and the inflaton fluctuation at horizon crossing [16]

$$\begin{aligned}\Phi(\mathbf{x}) &= \frac{12\pi G}{5}\int_{\phi_0}^{\phi_0+\delta\phi}\left[\frac{\partial \ln H}{\partial\phi}\right]^{-1} d\phi\,, \\ &\approx \frac{12\pi G}{5}\left[\frac{\partial \ln H}{\partial\phi}\right]^{-1}\delta\phi + \frac{6\pi G}{5}\frac{\partial}{\partial\phi}\left[\frac{\partial \ln H}{\partial\phi}\right]^{-1}\delta\phi^2 \\ &\quad + \frac{2\pi G}{5}\frac{\partial^2}{\partial\phi^2}\left[\frac{\partial \ln H}{\partial\phi}\right]^{-1}\delta\phi^3 + \mathcal{O}(\delta\phi^4). \end{aligned} \quad (1.63)$$

The gravitational potential (or the Bardeen potential, Newtonian curvature) may also be expanded locally in real space according to

$$\Phi(\mathbf{x}) = \Phi_g(\mathbf{x}) + f_1\left(\Phi_g^2(\mathbf{x}) - \langle\Phi_g^2(\mathbf{x})\rangle\right) + f_2\Phi_g^3(\mathbf{x}) + \mathcal{O}(\Phi_g^4), \quad (1.64)$$

Comparing the leading order term in the expansion of equation (1.63) to the leading order term in equation (1.64), we notice immediately that $\Phi_g(\mathbf{x})$, carries Gaussian random fluctuations from $\delta\phi$. The higher order terms can then be written as

$$\begin{aligned} f_1 &= -\frac{5}{6}\frac{1}{8\pi G}\frac{\partial^2 \ln V}{\partial\phi^2}, \\ f_2 &= \frac{25}{54}\frac{1}{(8\pi G)^2}\left[2\left(\frac{\partial^2 \ln V}{\partial\phi^2}\right)^2 - \frac{\partial^3 \ln V}{\partial\phi^3}\frac{\partial \ln V}{\partial\phi}\right], \end{aligned} \quad (1.65)$$

where $V(\phi)$ is the inflaton potential and f_1 corresponds to the so-called f_{NL} parameter while f_2 corresponds to the g_{NL} parameter. Equation (1.64) may also be written in terms of the curvature perturbation ($\Phi = (3/5)\mathcal{R}_g$). Equation (1.63) may also be written in terms of the number of e-foldings. For a single field inflation, non-linearity parameters may be written in terms of the slow-roll parameters

$$f_{NL} = \frac{5}{6}(\eta - 2\epsilon) \tag{1.66}$$

$$g_{NL} = \frac{25}{54}\left(2\epsilon\eta - 2\eta^2 + \xi^2\right) \tag{1.67}$$

1.4.1 Primordial Curvature Bispectrum

Assuming statistical homogeneity and isotropy, the functional form of the bispectrum of the primordial gravitational potential is given by

$$\langle\Phi(\mathbf{k}_1)\Phi(\mathbf{k}_2)\Phi(\mathbf{k}_3)\rangle = (2\pi)^3\delta_D(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3)B_\Phi(k_1, k_2, k_3) \tag{1.68}$$

where $B_\Phi(k_1, k_2, k_3)$ is the spatial bispectrum of the gravitational potential. Its actual form depends on the details of the mechanism that generates the primordial fluctuations. The delta function in equation (1.68) enforces that the three wave number vectors form a closed triangle. Depending on the inflation models, the maximal signals arise from several triangle configurations. The shape and amplitude bispectrum is determined through the function $F^{\lambda_1\lambda_2\lambda_3}(k_1, k_2, k_3)(k_2/k_1)^2(k_3/k_1)^2$ which in most cases is plotted as a function of k_2/k_1 and k_3/k_1 for $k_3 \leq k_2 \leq k_1$. The limits on the magnitude of k_1, k_2, k_3 determine various shapes of the triangles, for example squeezed limit ($k_1 \approx k_2 \gg k_3$), elongated ($k_1 = k_2 + k_3$), folded ($k_1 = 2k_2 = 2k_3$), isosceles ($k_2 = k_3$), and equilateral ($k_1 = k_2 = k_3$). thus, specifying the shape of the non-Gaussianity is a powerful probe of the exact mechanism that lead to the primordial fluctuations [17, 18].

1.4.2 Shapes and Amplitude of Non-Gaussanity:Bispectrum

- *Local Shape:* Here the primordial gravitational potential Φ is generated by a local quadratic expansion in real space is given by equation (1.64). The dimensionless constant $f_{\rm NL}$ sets the magnitude/amplitude of the bispectrum and non-gaussianity is localized at a given point in real space and $f_{\rm NL} \to f_{\rm NL}^{\rm local}$ (a local type nonGaussianity). The first signal for the bispectrum comes from the lowest order terms of this kind $\langle\phi^{(1)}\phi^{(1)}\phi^{(2)}\rangle$. Using equations (1.68) and (1.64) and Wick's theorem, the local bispectrum is given by

$$B_\Phi(k_1, k_2, k_3) = f_{\rm NL}^{\rm local}F^{\rm local}(k_1, k_2, k_3), \tag{1.69}$$

where

$$\begin{aligned} F^{\text{local}}(k_1,k_2,k_3) &= 2\left[P_{\Phi_g}(k_1)P_{\Phi_g}(k_2)+P_{\Phi_g}(k_1)P_{\Phi_g}(k_3)+P_{\Phi_g}(k_2)P_{\Phi_g}(k_3)\right] \\ &= 2\Delta^2_{\mathcal{R}}\cdot\left(\frac{1}{(k_1k_2)^{3-(n_s-1)}}+\frac{1}{(k_1k_3)^{3-(n_s-1)}}+\frac{1}{(k_2k_3)^{3-(n_s-1)}}\right) \end{aligned} \quad (1.70)$$

where $P_\Phi(k)$ is the gravitational potential power spectrum, defined through

$$\langle\Phi_g(\mathbf{k}_1)\Phi_g(\mathbf{k}_2)\rangle \equiv \delta^3\big(\mathbf{k}_1+\mathbf{k}_2\big)P_{\Phi_g}(k_1)=\delta^3\big(\mathbf{k}_1+\mathbf{k}_2\big)\Delta_{\mathcal{R}}\cdot k^{-3+(n_s-1)} \quad (1.71)$$

with scalar amplitude $\Delta_{\mathcal{R}}$ and the spectral index n_s. The bispectrum of the local non-Gaussianity is largest in the squeezed limit: $k_3 \ll k_1 \approx k_2$ with the momenta ordered such that $k_3 \leq k_2 \leq k_1$. In slow-roll single scalar field inflation, $f_{\text{NL}}^{\text{local}}$ is relatively very small due to suppression by the slow-roll parameters [15]. There exits a consistency relation for a single field slow-roll inflation between local $f_{\text{NL}}^{\text{local}}$ and the spectral index, n_s

$$f_{\text{NL}}^{\text{local}} = \frac{5}{12}(1-n_s)\ , \quad (1.72)$$

for $n_s = 0.963$ we have $f_{\text{NL}}^{\text{local}} = 0.015$. $f_{\text{NL}}^{\text{local}}$ could be large in a multi-field inflationary case [19–26].

- *Equilateral Shape:* The equilateral shape bispectrum is given by [27]

$$\begin{aligned} F^{\text{eq}}(k_1,k_2,k_3) &= 6\Delta^2_{\mathcal{R}}\cdot\Bigg[\left(-\frac{1}{(k_1k_2)^{3-(n_s-1)}}-2\text{perm.}\right)-\frac{2}{(k_1k_2k_3)^{2-\frac{2}{3}(n_s-1)}} \\ &\quad +\frac{1}{k_1^{1-\frac{1}{3}(n_s-1)}k_2^{2-\frac{2}{3}(n_s-1)}k_3^{3-(n_s-1)}}+(5\text{ perm.})\Bigg]\ . \end{aligned} \quad (1.73)$$

This type of bispectrum is associated with inflationary models where scalar fields have non-canonical kinetic terms, for example Dirac-Born-Infeld (DBI) inflation [28, 29], where $f_{\text{NL}}^{\text{equil}} \propto -1/c_s^2$ for $c_s \ll 1$ with c_s being the effective sound speed of propagation of the scalar field fluctuations. The normalized equilateral primordial bispectrum $F^{\text{equil}}(k_1,k_2,k_3)(k_2/k_1)^2(k_3/k_1)^2$, is maximized at the equilateral limit, namely, $k_1 = k_2 = k_3$. At present $f_{\text{NL}}^{\text{eq}}$ is constrained by the CMB to be $-125 < f_{\text{NL}}^{\text{eq}} < 435$ (95% C.L.) [30].

- *Orthogonal type:* This bispectrum shape was constructed such that it is nearly orthogonal to both the local-type and equilateral-type non-Gaussianities [30].

$$F^{\mathrm{orthog}}(k_1,k_2,k_3) = 6\Delta^2_{\mathcal{R}}\left[\left\{-\frac{3}{(k_1k_2)^{(4-n_s)}}+2\text{ perms.}\right\}-\frac{8}{(k_1k_2k_3)^{\frac{2(4-n_s)}{3}}} + \left\{\frac{3}{k_1^{\frac{(4-n_s)}{3}}k_2^{\frac{2(4-n_s)}{3}}k_3^{(4-n_s)}}+5\text{ perms.}\right\}\right]. \quad (1.74)$$

This shape approximately represents the forms of non-Gaussianity that arises in a linear combination of the higher-derivative scalar-field interaction terms. Each contributes a form similar to the equilateral shape. Senatore et al. [30] showed based on an effective field theory approach to inflation [31], that a certain linear combination of similar equilateral shapes can lead to a distinct shape which is orthogonal to both the local and equilateral forms. The orthogonal bispectrum $F^{\mathrm{orthog}}(k_1,k_2,k_3)(k_2/k_1)^2(k_3/k_1)^2$ has a positive peak at the equilateral configuration, and negative valley along the elongated configurations.

- *Folded Shape:* This form of non-Gaussianity is generated when the choice of a vacuum deviates from an adiabatic Bunch-Davies vacuum state as initial state [32].

$$F^{\mathrm{fol.}}(k_1,k_2,k_3) = 6\Delta^2_{\mathcal{R}}\cdot\left(\frac{1}{(k_1k_2)^{3-(n_s-1)}}+(2\text{ perm.})+\frac{3}{(k_1k_2k_3)^{2-\frac{2}{3}(n_s-1)}} - \frac{1}{k_1^{1-\frac{1}{3}(n_s-1)}k_2^{2-\frac{2}{3}(n_s-1)}k_3^{3-(n_s-1)}}-(5\text{ perm.})\right). \quad (1.75)$$

The primordial bispectrum for this model is maximized by configurations with modes obeying the momentum configuration $k_2 \approx k_3 \approx k_1/2$. It is sometimes called flat shape.

1.4.3 Trispectrum

The trispectrum of $\Phi(\mathbf{k})$ is the connected part of the four point correlator in Fourier space,

$$\langle\Phi(\mathbf{k}_1)\Phi(\mathbf{k}_2)\Phi(\mathbf{k}_3)\Phi(\mathbf{k}_4)\rangle_c = (2\pi)^3T(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4)\delta_D(\mathbf{k}_{1234}), \quad (1.76)$$

where $T(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4)$ is the spatial trispectrum of the gravitational potential. Just like in the case of bispectrum, the delta function imposes a constraint on $\mathbf{k}_1\cdots\mathbf{k}_4$. We will consider

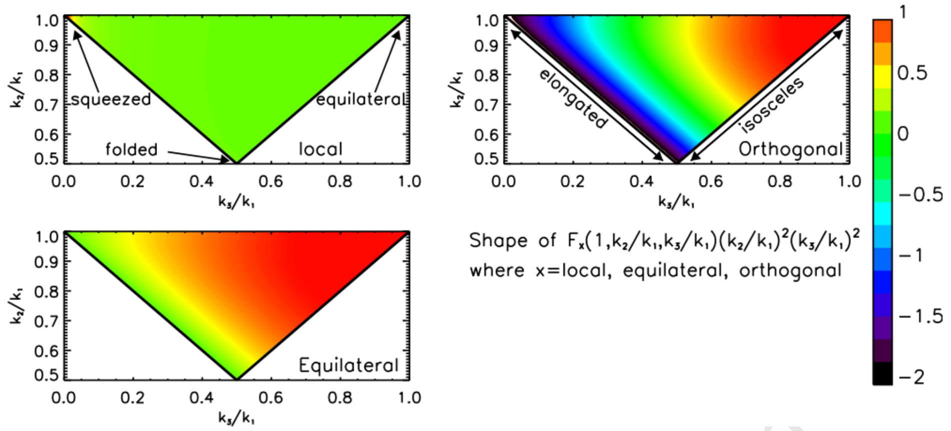

Figure 1.1: Shapes of Non-Gaussianity. The x & y coordinates are the rescaled momenta k_2/k_1 and k_3/k_1 respectively. Momenta are ordered such that $k_3/k_1 < k_2/k_1 < 1$ and satsify they the triangle inequality $k_2/k_1 + k_3/k_1 > 1$. Figure adapted from [6].

only the local form trispectrum and the leading order contribution is given by

$$\begin{aligned} T_\Phi(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4) &= 6g_{NL}P_\Phi(k_1)P_\Phi(k_2)P_\Phi(k_3) + (3 \text{ cylic}) \\ &\quad +2f_{NL}^2 P_\Phi(k_1)P_\Phi(k_2)\left[P_\Phi(|\mathbf{k_1}+\mathbf{k_3}|)+P_\Phi(|\mathbf{k_1}+\mathbf{k_4}|)\right] + (11 \text{ cylic}). \end{aligned} \tag{1.77}$$

In general, the coefficient of the trispectrum is not related to the non-linearity parameter especially for the multi-field inflationary models, hence the f_{NL}^2 in equation (1.78) is replaced with a new term, τ_{NL}

$$\begin{aligned} T_\Phi(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4) &= 6g_{NL}P_\Phi(k_1)P_\Phi(k_2)P_\Phi(k_3) + (3 \text{ cylic}) \\ &\quad +\frac{25}{18}\tau_{NL} P_\Phi(k_1)P_\Phi(k_2)\left[P_\Phi(|\mathbf{k_1}+\mathbf{k_3}|)+P_\Phi(|\mathbf{k_1}+\mathbf{k_4}|)\right] + (11 \text{ cylic}). \end{aligned} \tag{1.78}$$

where $\tau_{NL} = (6f_{NL}/5)^2$. We will not go into a detailed discussion of other shapes of trispectrum because they are not used later in this work.

1.5 Structure Formation via Gravitational Instability

The large scale structures we see on the night sky today, were seeded by the tiny quantum fluctuations created during during inflation. The wavelength of these fluctuations was comoving with the expansion of the universe, as a result they exited the horizon during inflation but re-enters at a later time as classical perturbations. It is these classical perturbations lead to the initial over-density that kick started the formation of the structures we see today on the night sky.

The well understood theory of formation in the universe relies on gravitational instability: the regions that contains more matter tend to attract more matter into its potential well unitil it collapses under its own weight. The distribution of inhomogeneities in the universe is normally expressed in terms of the density contrast

$$\delta(\mathbf{x},t) = \frac{\rho(\mathbf{x},t) - \bar{\rho}(t)}{\bar{\rho}(t)}. \tag{1.79}$$

In a universe that consists only of non-relativistic matter with $w = 0$, the background density is given by $\bar{\rho}(t) \propto 1/a(t)^3$. Rearranging equation (1.79) gives the total energy density content of the unverse universe as

$$\rho(\mathbf{x},t) = \bar{\rho}(t)[1 + \delta(\mathbf{x},t)]. \tag{1.80}$$

The evolution of matter density, $\rho(\mathbf{x},t)$, velocity field, $\mathbf{u}(\mathbf{x},t)$ and pressure, $p(\mathbf{x},t)$ is governed by three equations[1]: the continuity equation, which describes conservation of mass [20],

$$\left(\frac{\partial \rho}{\partial t}\right)_{\mathbf{x}} + \rho\, \nabla_{\mathbf{x}} \cdot \mathbf{u} = 0\,, \tag{1.81}$$

the Euler equation, which specifies conservation of momentum,

$$\left(\frac{\partial \mathbf{u}}{\partial t}\right)_{\mathbf{x}} + (\mathbf{u} \cdot \nabla_{\mathbf{x}})\, \mathbf{u} + \frac{\nabla_{\mathbf{x}} p}{\rho} = -\nabla_{\mathbf{x}} \Phi\,, \tag{1.82}$$

and the Poisson equation, which describes gravity in the Newtonian limit

$$\nabla^2_{\mathbf{x}} \Phi = 4\pi G \rho\,. \tag{1.83}$$

We transform to the comoving coordinates by setting $\mathbf{r} = \mathbf{x}/a(t)$ and define comoving time derivatives as

$$\frac{\mathrm{d}}{\mathrm{d}\,t} \equiv \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla\,. \tag{1.84}$$

Using equation (1.84) and and equation 1.79), we find an equation which describes the evolution of density fluctuations in the universe

$$\frac{d^2\delta}{dt^2} + 2H\frac{d\delta}{dt} = \frac{c_s^2}{a^2}\nabla^2_{\mathbf{r}}\delta + 4\pi G \bar{\rho}\, \delta \tag{1.85}$$

where $\nabla^2_{\mathbf{r}}$ is the Laplacian in comoving coordinates [20]

[1]Here we limit the discussion on structure formation to the Newtonian treatment[2], a general relativistic treatment in given chapter 2

We can gain more insight by expanding density contrast in Fourier space as plane waves

$$\delta(\mathbf{r},t)=\int\frac{d^3k}{(2\pi)^3}\exp[i(\mathbf{k}\cdot\mathbf{r}-\omega t)]\delta_k(t), \tag{1.86}$$

where $\mathbf{k}$ is the vector of spatial wave-numbers, and ω is the oscillation frequency. In terms of these Fourier modes, eqn. 1.85 becomes

$$\ddot{\delta}_k+2H\dot{\delta}_k-\delta_k(4\pi G\bar{\rho}-k^2c_s^2)=0. \tag{1.87}$$

Equation (1.87) describes the fluctuations of density modes with a "drag" term due to Hubble expansion ($2H\dot{\delta}_k$). According to equation (1.87), the rate of growth of perturbations δ_k depends on the balance between the gravitational force through $4\pi G\bar{\rho}$ and the pressure through $k^2c_s^2$. The scale $\lambda_J=2\pi/k_J$ where the two forces balance is called the Jeans length:

$$\lambda_J\equiv c_s\sqrt{\frac{\pi}{G\bar{\rho}}}. \tag{1.88}$$

Jean's length characterizes the scale above which pressure cannot halt gravitational collapse. Notice from equation 1.88, that Jean's length is directly proportional to sound speed $c_s=\left(\sqrt{\partial p/\partial\rho}\right)_S$ (The derivatives are taken at a constant entropy), and sound speed depends on partial variation in pressure which in ture makes the Jean's length sensitive to the equation of state of the total energy in the universe.

Now consider two important regimes in the early universe: radiation dominance and matter dominance. In the regime where radiation dominates the energy density, the equation of state $w=1/3$ leads to $c_s^2=c^2/3$. In a flat universe, the Jeans length in a radiation dominated era may be expressed

$$\lambda_J^{(r)}=\frac{\pi c\sqrt{8}}{3H}. \tag{1.89}$$

The Jeans length in a radiation dominated era is of the same order as the horizon scale, $\lambda_s\approx 2c/(H\sqrt{3})$. This implies that sub-horizon modes cannot collapse during the epoch when the universe was dominated by radiation energy density. In the matter-dominated era, there are two possibilities: if radiation and matter are coupled, then the pressure comes from the radiation component while the density is dominated by the matter component. This gives $c_s^2\sim c^2\Omega_r/\Omega_m\sim c^2(1+z)\Omega_{r,0}/\Omega_{m,0}$. Using the data from WMAP experiment [34], we find approximately $c_s\approx 10^{-2}c\sqrt{1+z}$.

If radiation and matter are decoupled, the pressure comes from the nonzero temperature of matter itself: $c_s^2\sim kT/m_p$, where m_p is the proton mass. If matter in the universe is in thermal equilibrium with the CMB, its temperature depends on the redshift according to $T \propto (1+z)$. Using observational constraints from the WMAP experiment, we find $c_s \approx 10^{-7}c\sqrt{1+z}$. Therefore the Jeans length during the matter-dominated epoch is given by

$$\lambda_J^{(m)} \approx \lambda_J^{(r)}\sqrt{1+z} \;\times \begin{cases} 10^{-2} & \text{before decoupling} \\ 10^{-7} & \text{after decoupling}\,, \end{cases} \tag{1.90}$$

where the units of Jeans length is in Mpc. The redshift of decoupling is approximately $z \sim 1100$ [6, 35]. This implies that prior to decoupling, growth below approximately sub-horizon scales is suppressed by pressure. After decoupling, the Jeans length shrinks by at least five orders of magnitude, allowing linear structure on this scale to form.

1.5.1 Structure Growth Beyond the Jean's Length

The rate of growth of structures on scales larger than the Jeans length may be understood by defining the linear growth factor $D(t)$ such that

$$\delta(\mathbf{r},t) = \delta_0(\mathbf{r})D(t). \tag{1.91}$$

We will neglect pressure at scales greater than the Jeans length. Using $\bar{\rho} = \Omega_m \rho_c$, it is possible to rewrite equation 1.85 as

$$\ddot{D} + 2H\dot{D} - \frac{3}{2}\Omega_m H^2 D = 0. \tag{1.92}$$

The general solution to equation (1.92), admit a growing mode and a decaying mode:

$$D(t) = A_1 D_1(t) + A_2 D_2(t). \tag{1.93}$$

In cosmology, we consider only the growing mode solution and the growth factor in a flat universe is given by [20]

$$D(a) \propto \int_0^a \frac{da'}{[a'H(a')]^3}, \tag{1.94}$$

where the normalization is usually chosen such that $D(a) = 1$ at the present day. A full relativistic treatment of structure formation is given in chapter 2.

1.6 The Aim of the Study

Cosmological Principle remains the bedrock of our current understanding of the universe even when it has not been thoroughly verified through rigorous mathematical and observational techniques. Our interpretation of parameters of a given cosmological model will have to be revised, if the Cosmological Principle turned out to be invalid. Given the stakes, especially now that we are headed towards a precision cosmology era, there is an urgent need to consistently and systematically verify the bedrock upon which estimation of all the fundamental and derived parameters of the ΛCDM model of the universe is based.

Recently there has been a great deal of efforts directed towards mapping out or identifying the regions of the universe where the cosmological principle could be valid [36]. This is motivated by the fact that the Universe is clearly not homogenous and isotropic on regions so far covered by observation (see figure 1.2), however, it is likely that on large enough scales ($\gtrsim 100h^{-1}$Mpc in ΛCDM), the distribution of matter may be assumed to be 'statistically homogeneous and isotropic'. In principle this implies that the small-scale inhomogeneities as shown in figure 1.2 may be considered as perturbations which have a statistical distribution that is independent of position and direction in space. This assumption is yet to gather sound observational or theoretical support.

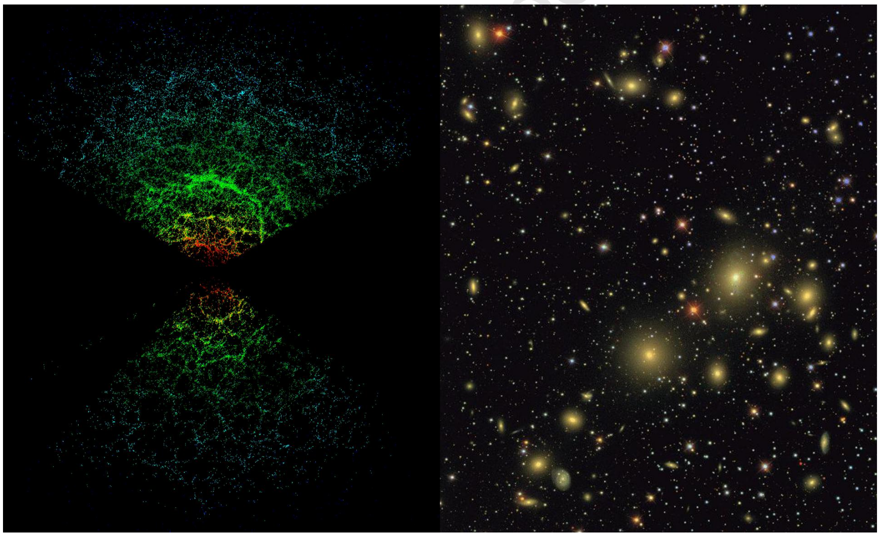

Figure 1.2: Distribution of galaxies in one slice of the sky from the Sloan Digital Sky Survey (SDSS). Different colors indicate different kinds of galaxies on the sky. We are located at the bottom vertex of the cone. There is obviously a rich structure of voids and filaments, The distribution thins out far away from us, simply because galaxies become fainter and harder to find and not that the structures are absent or homogeneously distributed. The statistical properties of the measured distribution of galaxies reveal important information about the structure and evolution of the late time universe.

One indirect support so far is from the early universe inflation. A single field inflationary theory based on FLRW spacetime predicts a relatively small level of density fluctuations on all scales, i.e. the primordial density power spectrum was close to scale-invariant. This prediction has been verified by the WMAP, QUIET, ACT experiments of the CMB [37–39]. The spectral index n_s, which quantifies the scale-dependence of the primordial power spectrum, is measured to be close to 0.96 [34], while a scale-invariant power spectrum has $n_s = 1$. Also the anisotropies in the temperature fluctuations of the CMB implies that the density fluctuations induce fluctuations in the metric, $\delta\Phi$, which are virtually independent of scale, and are of the order of $\delta\Phi/c^2 \sim 10^{-5}$ [6, 40]. The smallness of these perturbations implies that the FLRW metric on which they live was valid at the time when the primordial radiation was emitted, it does not imply that they remain small till today. The theory of structure formation in the universe has it that the small perturbations at early time did grow into the type of inhomogeneities we see today on night sky (see figure 1.2), which clearly do not have small level of density inhomogeneities on small scales.

The presence of non-vanishing levels of inhomogeneities on small scales compelled Ellis [41] to initiate a search for the best fit FLRW model of the universe given that the observable universe is not after-all homogenous. This exercise later evolved to be known as the 'averaging problem' in general relativity (more on this in chapter 5). Mathematically, averaging problem is a consequence of non-linearity of the equations of general relativity. The non-linearity ensures that the curvature ($G[g] \sim \partial\Gamma + \Gamma^2$) of an average/smooth spacetime does not commute with the average of the curvature of an inhomogeneous spacetime leading to a modification of equations of general relativity. The modified version is given by

$$\langle G[g]\rangle \sim 8\pi G\langle T\rangle - \mathcal{C} \tag{1.95}$$

where the additional term $\mathcal{C} = \langle\Gamma^2\rangle - \langle\Gamma\rangle^2$ is called the backreaction term or the correlation tensor [42–47]. Notice that the backreaction term $\mathcal{C}$ occupies the same position as the cosmological constant ($\Lambda g_{ab} = 8\pi G T_{ab} - G_{ab}$). Equation (1.95) and the coincidence problem in the standard cosmology are the two key factors that inspired works that tend to link the cosmological backreaction and dark energy problem. Equation (1.95) also tells us that whenever we fit a cosmological observation to a model based on the FLRW spacetime, there is an additional term that is not being taking into account.

Apart from the size of the backreaction term giving us information about the scale of homogeneity of the universe, it will give us information about the scale, where large scale interpretation of cosmological observables are likely to break down. *This thesis is devoted towards quantifying the magnitude of the backreaction term within the standard model of cosmology.* Specifically we will calculate the effect of backreaction on the Hubble rate, deceleration parameter and the distance-redshift relation.

Out approach to the problem is mainly theoretical but focused on the FLRW model of cosmology with special attention to the ΛCDM model and the Einstein-de Sitter model of the universe. More importantly, our spatial averaging formalism is based generalized Burchert formalism [48, 49]. In order to cover all physically possible options, we smooth observables on the hypersurface where the gravitational field is at rest and also on the hypersurface comoving with the fluid, this is mainly useful for fitting of Supernovae Ia (SN Ia) light curves [50–53] and the former being suitable for the CMB experiments [54, 55]. We cover all possible choices of definitions of averaged quantities that have been used in literature to enable us provide a comprehensive argument on the actual size of the backreaction term in equation (1.95).

The second approach we used in this study is based on the Kristian and Sachs series expansion [56, 57]. This is a series expansion of the distance-redshift relation in an arbitrary inhomogeneous spacetime, the results/predictions from this approach is considered physical because it is based on the past light cone and there is no hypothetical averaging hypersurface involved [58–60]. Our approach is to take the general expression for the distance-redshift relation from this formalism and fit it to the corresponding one from an FLRW spacetime. This is exactly how some cosmological observations are fitted to an FLRW model [61]. We expect the result of this approach to carry actual physical degrees of freedom because they free from gauge problems that cast doubts on the observational relevance of spatial averaging formalism [62].

The third approach used in this thesis involves calculating the area distance up to non-linear order and taking the all sky average after a full sky spherical harmonic decomposition. Since this does not involve any series approximation, its results are valid at all epochs where cosmological perturbation theory could be trusted.

Finally we provide a custom-made review of cosmological perturbation theory in chapter 2 to familiarize the reader to the symbols and notations used in the rest of the report.

Chapter 2

Post-Inflationary Growth of Large Scale Structures

2.1 Introduction

The general relativistic treatment of the evolution of structures in the universe involves solving the Einstein equations of general relativity order by order in perturbation theory. The Newtonian approach described in chapter one breaks in the neighborhood of the Hubble scale. Apart from providing a completion to the Newtonian treatment, general relativistic treatment allows the treatment of the evolution of gravitational waves and induced vector waves propagating on an FLRW background spacetime.

The most general metric for describing the evolution of classical perturbation is given by :

$$\mathrm{d}\,s^2 = a^2\left[-(1+2\Phi)\mathrm{d}\,\eta^2 + 2\omega_i \mathrm{d}\,x^i \mathrm{d}\,\eta + [a^2(1-2\Psi)\gamma_{ij} + h_{ij}]\mathrm{d}\,x^i \mathrm{d}\,x^j\right], \qquad (2.1)$$

where a is the scale factor of the universe. From here on, η is the conformal time and it is related to the cosmic time, t, according to $\mathrm{d}\,t = a\mathrm{d}\,\eta$.

Since our background is homogenous and isotropic with mean constant curvature, K, of the spatial surface, it is easier to perform a scalar-vector-tensor (SVT) decomposition of the perturbed variables on a homogenous background

$$\omega_i = \partial_i \hat{B} + \hat{B}_i, \qquad (2.2)$$

$$h_{ij} = 2\hat{H}_{ij} + \partial_i \hat{H}_j + \partial_j \hat{H}_i + 2\partial_i\partial_j \hat{H}, \qquad (2.3)$$

where $\hat{B}_i$, $\hat{H}_i$ and $\hat{H}_{ij}$ are transverse ($\partial^i \hat{H}_i = \partial^j \hat{B}_i = \partial^i \hat{H}_{ij} = 0$), and $\hat{H}_{ij}$ is traceless

($\hat{H}^i{}_j = 0$). There are four scalar degrees of freedom (Φ, Ψ, $\hat{B}$, $\hat{H}$), four vector degrees of freedom ($\hat{B}_i$, $\hat{H}_i$) and two tensor degrees of freedom ($\hat{H}_{ij}$). Fortunately at first order, different degrees of freedom do not mix, so they may be solved independently. The indices of the perturbed variables are lowered and raised by the background metric, i.e γ^{ij} and γ_{ij}. Because the spacetime itself is a dynamical field, the idea that the perturbation variables live on the background spacetime introduce some gauge problems. This problem is mitigated by considering two different spacetimes and performing a gauge transformation to relate quantities in the two spacetimes [63]. At first order in perturbation theory, it is straight-forward to identify gauge invariant quantities, however, at higher order, Nakamura has identified a clever way of identifying gauge invariant quantities at any order [64–66] and we provide a detailed description of this in the appendix. To be more precise, we will restrict our choice of gauge to conformal Newtonian gauge or Poison gauge.

2.2 Derivation of Einstein field equations

Our target now is to derive Einstein's field equations based on the metric in equation (2.1) for evolution of perturbations on a flat homogenous background. We focus on Einstein field equations in this form

$$G^a{}_b = 8\pi G\, T^a{}_b - \Lambda g^a{}_b, \tag{2.4}$$

where the energy-momentum tensor for perfect fluid is given in equation (1.4). Since our interest is to understand late-time universe, we will set pressure to zero. We will also make use of the background solutions given in chapter one.

We expand the matter variables, u^c and ρ that appear in the energy-momentum tensor up to second order around a homogeneous FLRW background. For matter 4-velocity we find

$$u^b = \frac{1}{a}\left(\delta^b_0 + \delta u^b + \frac{1}{2}\delta^2 u^b\right) . \tag{2.5}$$

where δu^b and $\delta^2 u^b$ stand for the first order and second order perturbations of u^b. Using the normalization condition for time-like particles $u^c u_c = -1$ and $u_c = g_{bc}u^b$, to solve for δu^b and $\delta^2 u^b$ and we find

$$
\begin{aligned}
u_0 &= a\left(-1-\Phi^{(1)}-\frac{1}{2}\Phi^{(2)}+\frac{1}{2}\left(\Phi^{(1)}\right)^2-\frac{1}{2}v_i^{(1)}v^{i\,(1)}-v^{i\,(1)}\mathrm{D}_i v^{(1)}-\frac{1}{2}\mathrm{D}_i v^{(1)}\mathrm{D}^i v^{(1)}\right), \\
u_i &= a\left(v_i^{(1)}+\omega_1^{(1)}+\frac{1}{2}\omega_i^{(2)}+\frac{1}{2}v_i^{(2)}-\omega_i^{(1)}\Phi^{(1)}-2v_i^{(1)}\Psi^{(1)}-2\Psi^{(1)}\mathrm{D}_i v^{(1)}\right. \\
&\quad\left.+\frac{1}{2}\mathrm{D}_i v^{(2)}+2h_{ij}^{(1)}v^{j\,(1)}+2h_i^{j\,(1)}\mathrm{D}_j v^{(1)}\right),
\end{aligned}
\tag{2.6}
$$

where we have decomposed the velocity perturbation $v^{i(r)}$ into a scalar (irrotational) and a vector (solenoidal) part, as

$$
v^{i(r)} = \mathrm{D}^i v^{(r)} + v^{i(r)}\,, \tag{2.7}
$$

with $\mathrm{D}_i v^{i(r)} = 0$ (divergence free).

The energy density ρ may also be split into a homogeneous background $\bar{\rho}(\eta)$ and a perturbation $\delta\rho(\eta, x^i)$ as follows

$$
\rho(\eta, x^i) = \bar{\rho}(\eta) + \delta\rho(\eta, x^i) = \bar{\rho}(\eta) + \delta^{(1)}\rho(\eta, x^i) + \frac{1}{2}\delta^{(2)}\rho(\eta, x^i)\,, \tag{2.8}
$$

where the perturbation has been expanded into a first and a second–order part in the second equality.

Substituting equations (2.6) and (2.154) in the expression for perturbed energy-momentum tensor (equation 1.4) gives at first order:

$$
\begin{aligned}
\delta T_0^0 &= -\delta\rho\,, &(2.9)\\
\delta T_0^i &= \bar{\rho}\left[\omega^{i\,(1)}-\left(\mathrm{D}^i v^{(1)}+v^{i\,(1)}\right)\right], &(2.10)\\
\delta T_i^0 &= \bar{\rho}\left(\mathrm{D}_i v^{(1)}+v_i{}^{(1)}\right), &(2.11)\\
\delta T_i^j &= 0\,, &(2.12)
\end{aligned}
$$

and at second order,we find

$$
\begin{aligned}
\delta^2 T_0{}^0 &= -2\bar{\rho}\mathrm{D}^i v^{(1)}\mathrm{D}_i v^{(1)}-\delta^2\rho, &(2.13)\\
\delta^2 T_i{}^0 &= \bar{\rho}\left(\mathrm{D}_i v^{(2)}+v^{(2)}{}_i-2\Phi^{(1)}\,\mathrm{D}_i v^{(1)}\right)+2\delta\rho\mathrm{D}_i v^{(1)}, &(2.14)\\
\delta^2 T_0{}^i &= \bar{\rho}\left(\omega^{i\,(2)}-\mathrm{D}^i v^{(2)}-v^{i\,(2)}-6\Phi^{(1)}\,\mathrm{D}^i v^{(1)}\right)-2\delta\rho\mathrm{D}^i v^{(1)}, &(2.15)\\
\delta^2 T_i{}^j &= 2\bar{\rho}\mathrm{D}_i v^{(1)}\mathrm{D}^j v^{(1)}\,. &(2.16)
\end{aligned}
$$

For the components of perturbed Einstein tensor at first order we find

$$\delta G^0_0 = -\frac{1}{a^2}\left[-6\mathcal{H}\Psi'^{(1)} + 2\nabla^2\Psi^{(1)} - 6\mathcal{H}^2\Phi^{(1)}\right], \tag{2.17}$$

$$\delta G^0_i = -\frac{1}{a^2}\left[2\mathrm{D}_i\Psi'^{(1)} + 2\mathcal{H}\mathrm{D}_i\Phi^{(1)} - \frac{1}{2}\nabla^2\omega_i^{(1)}\right], \tag{2.18}$$

$$\delta G^i_0 = \frac{1}{a^2}\left[2\mathrm{D}^i\Psi'^{(1)} + 2\mathcal{H}\mathrm{D}^i\Phi^{(1)} + \frac{1}{2}\left(-\nabla^2 + 4\mathcal{H}^2 - 4\mathcal{H}'\right)\omega^{i(1)}\right], \tag{2.19}$$

$$\begin{aligned}\delta G^j_i = \frac{1}{a^2}\Big\{&\mathrm{D}_i\mathrm{D}^j\left(\Psi^{(1)} - \Phi^{(1)}\right) + \left[-\nabla^2\Psi^{(1)} + 2\Psi''^{(1)} + 4\mathcal{H}\Psi'^{(1)}\right.\\ &\left.+2\mathcal{H}\Phi' + 4\mathcal{H}'\Phi^{(1)} + 2\mathcal{H}^2\Phi^{(1)} + \nabla^2\Phi^{(1)}\right]\gamma^j_i - \frac{1}{2}\left(\mathrm{D}_i\omega^{j\prime(1)} + \mathrm{D}^j\omega_i'^{(1)}\right)\\ &+\mathcal{H}\left(\mathrm{D}_j\omega^{j(1)} + \mathrm{D}^j\omega_i^{(1)}\right) + \frac{1}{2}\left(h_i^{j''(1)} + 2\mathcal{H}h^j_i - \nabla^2 h^j_i\right)\Big\}\end{aligned} \tag{2.20}$$

where $' = d/d\eta$, and $\mathcal{H} = a'/a$ is the conformal Hubble rate. At second order we will follow the approach of [66] to split the Einstein tensor into purely second order part and the part that contains product of two first order terms. Also in order to reduce complexity at non-linear order, we set

$$\omega_i^{(1)} = 0, \quad h_{ij}^{(1)} = 0. \tag{2.21}$$

when considering second order perturbations. The pure second order part of Einstein equation is given by

$$\delta^2 G_0{}^0 = -\frac{1}{a^2}\left\{-6\mathcal{H}\Psi'^{(2)} + 2\nabla^2\Psi^{(2)} - 6\mathcal{H}^2\Phi^{(2)}\right\}, \tag{2.22}$$

$$\delta^2 G_i{}^0 = -\frac{1}{a^2}\left(2\mathrm{D}_i\Psi'^{(2)} + 2\mathcal{H}\mathrm{D}_i\Phi^{(2)} - \frac{1}{2}\nabla^2\omega_i{}^{(2)}\right), \tag{2.23}$$

$$\delta^2 G_0{}^i = \frac{1}{a^2}\left\{2\mathrm{D}^i\Psi'^{(2)} + 2\mathcal{H}\mathrm{D}^i\Phi^{(2)} + \frac{1}{2}\left(-\nabla^2 + 4\mathcal{H}^2 - 4\mathcal{H}'\right)\omega_i{}^{(2)}\right\}, \tag{2.24}$$

$$\begin{aligned}\delta^2 G_i{}^j = \frac{1}{a^2}\Big[&\mathrm{D}_i\mathrm{D}^j\left(\Psi^{(2)} - \Phi^{(2)}\right) + \left\{\left(2\Psi''^{(2)} - \nabla^2\Psi^{(2)} + 4\mathcal{H}\Psi'^{(2)}\right)\right.\\ &\left.+\left(2\mathcal{H}\Phi'^{(2)} + 4\mathcal{H}'\Phi^{(2)} + 2\mathcal{H}^2\Phi^{(2)} + \nabla^2\Phi^{(2)}\right)\right\}\gamma_i{}^j - \frac{1}{2}\left[\mathrm{D}_i\omega'^{j(2)} + \mathrm{D}^i\omega_i'^{(2)}\right.\\ &\left.+2\mathcal{H}\left(\mathrm{D}_i\omega^{j(2)} + \mathrm{D}^j\omega_j{}^{(2)}\right)\right]\\ &+\frac{1}{2}\left(h_i''{}^j + 2\mathcal{H}h_i'{}^j - \nabla^2 h_i{}^j\right)\Big].\end{aligned} \tag{2.25}$$

According to the characterization of gauge restrictions by [64, 66–68], this part is gauge-invariant, while the part quadratic in first order terms is gauge variant and it is given by

$$\begin{aligned}
\delta^2 {G_0}^0\left[\Phi,\Phi\right] &= \frac{2}{a^2}\Big\{12\mathcal{H}\Psi'^{(1)}\left(\Psi^{(1)}-\Phi^{(1)}\right)-12\mathcal{H}^2\left(\Phi^{(1)}\right)^2 \\
&\quad -3\left(\mathrm{D}_k\Psi^{(1)}\mathrm{D}^k\Psi^{(1)}+(\Psi'^{(1)})^2\right)-8\Psi^{(1)}\nabla^2\Psi^{(1)}\Big\}, \qquad (2.26)\\
\delta^2 {G_0}^i\left[\Phi,\Phi\right] &= \frac{4}{a^2}\left\{2\mathcal{H}\mathrm{D}^i\Phi^{(1)}\left(\Psi^{(1)}-\Phi^{(1)}\right)+\Psi'^{(1)}\mathrm{D}^i\left(2\Psi^{(1)}-\Phi^{(1)}\right)+4\Psi^{(1)}\mathrm{D}^i\Psi'^{(1)}\right\}, \qquad (2.27)\\
\delta^2 {G_i}^0\left[\Phi,\Phi\right] &= \frac{4}{a^2}\left\{4\mathcal{H}\Phi^{(1)}\mathrm{D}_i\Phi^{(1)}-\Psi'^{(1)}\mathrm{D}_i\left(2\Psi^{(1)}-\Phi^{(1)}\right)-2\mathrm{D}_i\Psi'^{(1)}\left(\Psi^{(1)}-\Phi^{(1)}\right)\right\}, \qquad (2.28)\\
\delta^2 {G_i}^j\left[\Phi,\Phi\right] &= \frac{2}{a^2}\Big[\mathrm{D}_i\Phi^{(1)}\mathrm{D}^j\left(\Phi^{(1)}-\Psi^{(1)}\right)-\mathrm{D}_i\Psi^{(1)}\mathrm{D}^j\left(\Phi^{(1)}-3\Psi^{(1)}\right)+2\mathrm{D}_i\mathrm{D}^j\Phi^{(1)}\left(\Phi^{(1)}-\Psi^{(1)}\right) \\
&\quad +4\Psi^{(1)}\mathrm{D}_i\mathrm{D}^j\Psi^{(1)}+\Big\{\left(\Psi'^{(1)}\right)^2-2\Psi'^{(1)}\Phi'^{(1)}+4\Psi''^{(1)}\left(\Psi^{(1)}-\Phi^{(1)}\right)-8\mathcal{H}\Phi^{(1)}\Phi'^{(1)} \\
&\quad +8\mathcal{H}\Psi'^{(1)}\left(\Psi^{(1)}-\Phi^{(1)}\right)-4\left(2\mathcal{H}'+\mathcal{H}^2\right)\left(\Phi^{(1)}\right)^2 \\
&\quad -2\mathrm{D}_k\Psi^{(1)}\mathrm{D}^k\Psi^{(1)}-\mathrm{D}_k\Phi^{(1)}\mathrm{D}^k\Phi^{(1)}+2\left(\Psi^{(1)}-\Phi^{(1)}\right)\nabla^2\Phi^{(1)} \\
&\quad -4\Psi^{(1)}\nabla^2\Psi^{(1)}\Big\}\,{\gamma_i}^j\Big] \qquad (2.29)
\end{aligned}$$

For perfect fluids, Bardeen potential equals the gravitational potential at first order

$$\Psi^{(1)}=\Phi^{(1)}=\Phi. \qquad (2.30)$$

Implementing the constraint from equation (2.30) to the set of first order equations, the scalar part of the linear perturbation equations become

$$\begin{aligned}
\nabla^2\Phi-3\mathcal{H}\Phi'-3\left(\mathcal{H}^2\Phi\right) &= 4\pi Ga^2\delta\rho, \qquad (2.31)\\
\mathrm{D}_i\Phi'+\mathcal{H}\mathrm{D}_i\Phi &= -4\pi Ga^2\rho\mathrm{D}_i v^{(1)}, \qquad (2.32)\\
\Phi''+3\mathcal{H}\Phi'+2\mathcal{H}'\Phi+\mathcal{H}^2\Phi &= 0. \qquad (2.33)
\end{aligned}$$

On small scales or the Newtonian limit, Equation. (2.31) reduces to the Poisson equation for the gravitational potential Φ induced by the energy-density perturbation $\delta\rho$. Equation (2.32) relates the velocity potential to the gravitational potential and equation (2.33) is also called the Bardeen equation, it determines the evolution of primordial curvature perturbation. Using the equation of state for a barotropic fluid, we obtain the master equation for the scalar perturbation[69]:

$$\Phi''+3\mathcal{H}(1+c_s^2)\Phi'-c_s^2\nabla^2\Phi+2\mathcal{H}'\Phi+(1+3c_s^2)\mathcal{H}^2\Phi \;=\; 4\pi Ga^2\eta\mathcal{S}\,, \qquad (2.34)$$

where $\mathcal{S}$ encodes the perturbation of the entropy. Setting $\mathcal{S}=0$ describes adiabatic perturbation. The scalar perturbations are completely determined by this master equation. The evolution of vector and tensor perturbations at first order are determined by

$$\omega_i' + 2\mathcal{H}\omega_i = 0\,, \tag{2.35}$$
$$h_{ij}'' + 2\mathcal{H}h_{ij}' - \nabla^2 h_{ij} = 0\,. \tag{2.36}$$

We will study equations (2.31,2.32, 2.33, 2.35) and (2.36) in greater detail in the following subsections. Furthermore, we impose the vanishing anisotropic stress tensor constraint (2.30), on ${}^{(2)}G_a{}^b\,[\Phi,\Phi]$ and it simplifies further

$$\delta^2 G_0{}^0\,[\Phi,\Phi] = \frac{2}{a^2}\left\{3\mathrm{D}_k\Phi\mathrm{D}^k\Phi + 3\left(\Phi'\right)^2 + 8\Phi\nabla^2\Phi + 12\mathcal{H}^2\Phi^2\right\}, \tag{2.37}$$
$$\delta^2 G_i{}^0\,[\Phi,\Phi] = \frac{4}{a^2}\left(4\mathcal{H}\,\Phi\mathrm{D}_i\Phi - \Phi'\mathrm{D}_i\Phi\right), \tag{2.38}$$
$$\delta^2 G_0{}^i\,[\Phi,\Phi] = \frac{4}{a^2}\left(\Phi'\mathrm{D}^i\Phi + 4\Phi\mathrm{D}^i\Phi'\right), \tag{2.39}$$
$$\delta^2 G_i{}^j\,[\Phi,\Phi] = \frac{2}{a^2}\Big[2\mathrm{D}_i\Phi\,\mathrm{D}^j\Phi + 4\Phi\,\mathrm{D}_i\mathrm{D}^j\Phi - \big(3\mathrm{D}_k\Phi\,\mathrm{D}^k\Phi + 4\Phi\,\nabla^2\Phi + \left(\Phi'\right)^2 + 8\mathcal{H}\,\Phi\,\Phi' + 4\left(2\mathcal{H}' + \mathcal{H}^2\right)\Phi^2\big)\,\gamma_i{}^j\Big]. \tag{2.40}$$

Putting everything together at second order, $\delta^2 G_a{}^b + \delta^2 G_a{}^b\,[\Phi,\Phi] = \delta^2 T_a{}^b$, we find

$$3\mathcal{H}\,\Psi'^{(2)} - \nabla^2\Psi^{(2)} + 3\mathcal{H}^2\Phi^{(2)} + 4\pi G a^2\delta^2\rho = -\Gamma_0, \tag{2.41}$$
$$2\mathrm{D}_i\Psi'^{(2)} + 2\mathcal{H}\,\mathrm{D}_i\Phi^{(2)} - \frac{1}{2}\nabla^2\omega_i{}^{(2)} + 8\pi G a^2\bar{\rho}\left(\mathrm{D}_i v^{(2)} + v^{(2)}{}_i\right) = \Gamma_i, \tag{2.42}$$
$$\mathrm{D}_i\mathrm{D}^j\left(\Psi^{(2)} - \Phi^{(2)}\right) + \big\{2\Psi''^{(2)} - \nabla^2\Psi^{(2)} + 4\mathcal{H}\,\Psi'^{(2)} + \big(2\mathcal{H}\,\Phi'^{(2)} + 4\mathcal{H}'\Phi^{(2)} + 2\mathcal{H}^2\Phi^{(2)} + \nabla^2\Phi^{(2)}\big)\big\}\,\gamma_i{}^j \tag{2.43}$$
$$- \left[\mathrm{D}_{(i}\omega'^{j)} + 2\mathcal{H}\mathrm{D}_{(i}\omega^{j)}\right] + \frac{1}{2}\left(h_i''{}^j + 2\mathcal{H}\,h_i'{}^j - \nabla^2 h_i{}^j\right) = \Gamma_i{}^j,$$

where

$$\Gamma_0 \equiv 8\pi G a^2\bar{\rho}\mathrm{D}^i v\mathrm{D}_i v - 3\mathrm{D}_k\Phi\,\mathrm{D}^k\Phi - 3\left(\Phi'\right)^2 - 8\Phi\,\nabla^2\Phi - 12\mathcal{H}^2\Phi^2, \tag{2.44}$$
$$\Gamma_i \equiv -16\pi G a^2\delta\rho\mathrm{D}_i v + 12\mathcal{H}\,\Phi\,\mathrm{D}_i\Phi - 4\Phi\,\mathrm{D}_i\Phi' - 4\Phi'\,\mathrm{D}_i\Phi \tag{2.45}$$
$$\Gamma_i{}^j \equiv 16\pi G a^2\bar{\rho}\mathrm{D}_i v D^j v - 4\mathrm{D}_i\Phi\,\mathrm{D}^j\Phi - 8\Phi\,\mathrm{D}_i\mathrm{D}^j\Phi' + 2\big(3\mathrm{D}_k\Phi\,\mathrm{D}^k\Phi + 4\Phi\,\nabla^2\Phi + \left(\Phi'\right)^2 + 4\left(2\mathcal{H}' + \mathcal{H}^2\right)\left(\Phi\right)^2 + 8\mathcal{H}\,\Phi\,\Phi'\big)\,\gamma_i{}^j. \tag{2.46}$$

The vector and scalar parts of the perculair velocity may be obtained by taking the divergence of (2.42),

$$-2\mathrm{D}_i\Psi'^{(2)} - 2\mathcal{H}\,\mathrm{D}_i\Phi^{(2)} + \mathrm{D}_i\nabla^{-2}\mathrm{D}^k\Gamma_k = 8\pi G a^2\bar{\rho}\mathrm{D}_i v^{(2)}\,, \tag{2.47}$$

$$\frac{1}{2}\nabla^2\omega_i^{(2)} + \left(\Gamma_i - \mathrm{D}_i\nabla^{-2}\mathrm{D}^k\Gamma_k\right) = 8\pi G a^2\bar{\rho}{v^{(2)}}_i. \tag{2.48}$$

2.3 Solving Perturbed Einstein equations

2.3.1 Scalar Perturbations

At first order, scalar perturbations are described by the Bardeen equation (equation 2.33)

$$\Phi'' + 3\mathcal{H}\Phi' + a^2\Lambda\Phi = 0 = \ddot{\Phi} + 4H\dot{\Phi} + \Lambda\Phi\,. \tag{2.49}$$

The solution to the Bardeen equation that describes the growing mode may be written as:

$$\Phi(\eta, \mathbf{x}) = g(\eta)\Phi_0(\mathbf{x})\,, \tag{2.50}$$

where $\Phi_0(\boldsymbol{x})$ is the Bardeen potential today ($\eta = \eta_0,\ z = 0$) and $g(\eta)$ is the growth factor, which may be approximated, in terms of redshift [70, 71],

$$g(z) = \frac{5}{2}g_\infty\Omega_m(z)\left\{\Omega_m(z)^{4/7} - \Omega_\Lambda(z) + \left[1 + \frac{1}{2}\Omega_m(z)\right]\left[1 + \frac{1}{70}\Omega_\Lambda(z)\right]\right\}^{-1}, \tag{2.51}$$

and g_∞ is chosen so that $g(z = 0) = 1$. The Bardeen potential $\Phi(\eta, \mathbf{k})$ may be separated into primordial fluctuation and the transfer function $\Phi(\eta, \mathbf{k}) = T_\Phi(\eta, k)\Phi(\mathbf{k})$, where $T_\Phi(\eta, k)$ is the transfer function for Φ, it evolves the primordial fluctuation from early time to now. The power spectrum of the primordial fluctuations is given by

$$\langle\Phi(\mathbf{k})\Phi'(\mathbf{k})\rangle = \frac{2\pi^2}{k^3}\mathcal{P}_\Phi(k)\delta_D\left(\mathbf{k} + \mathbf{k}'\right) \tag{2.52}$$

where

$$\mathcal{P}(k) = \frac{4}{9}\Delta^2_{\mathcal{R}}(k_0)\left(\frac{k}{k_0}\right)^{n_S - 1} \tag{2.53}$$

where n_S and $\Delta_{\mathcal{R}}(k_0)$ are respectively the spectral index and the power spectrum of the primordial scalar field defined in chapter 1. Whenever the transfer function is written as $T_\Phi(\eta, k)$ in this work, it means that it could further be decomposed into time dependent part and the spatial part $T_\Phi(\eta, k) = g(\eta)T_\Phi(k)$. The transfer function for the matter density contrast, $T_\delta(k)$, is related to the transfer function for the Bardeen potential $T_\Phi(k)$ on small scales through

$$T_\delta(\eta, \mathbf{k}) = -\frac{2}{3}\left(\frac{k}{\mathcal{H}}\right)^2 T_\Phi(\eta, \mathbf{k}) \tag{2.54}$$

The most accurate transfer function with the right initial conditions may be generated with any Boltzmann code but some fairly good approximate fitting functions exit. The simplest fitting function approximation is given by [6]

$$T_\delta(k) \approx \begin{cases} 1 & k < k_{\rm eq} \\ (k_{\rm eq}/k)^2 & k > k_{\rm eq} \end{cases} . \tag{2.55}$$

This is not accurate enough because of the break in the spectrum at $k \approx k_{\rm eq}$. The transfer function for the cases of Adiabatic CDM, Adiabatic massive neutrinos (1 massive, 2 massless), and Isocurvature CDM is given respectively by Bardeen et al [72]

$$T_\delta(k) = \frac{\ln(1+2.34q)}{2.34q}\left[1 + 3.89q + (1.61q)^2 + (5.46q)^3 + (6.71q)^4\right]^{-1/4}, \tag{2.56}$$

$$T_\delta(k) = \exp\left(-3.9q - 2.1q^2\right), \tag{2.57}$$

$$T_\delta(k) = \left(1 + \left[15.0q + (0.9q)^{\frac{3}{2}} + (5.6q)^2\right]^{1.24}\right)^{\frac{1}{1.24}}, \tag{2.58}$$

where $q = k/\left(\Gamma h \,\mathrm{Mpc}^{-1}\right)$

$$\Gamma \equiv \Omega_m h \exp(-\Omega_b - \sqrt{2h}\,\Omega_b/\Omega_m)\,. \tag{2.59}$$

More accurate fitting function may be found in Eisenstein and Hu [73], but for the most accurate result one will have to solve the Einstein-Bltzmann system with correct initial conditions.

The evolution equation for the second-order Bardeen potential, Ψ , is obtained from the trace of the Einstein equations (2.44) (for details see[74]):

$$\Psi''^{(2)} + 3\mathcal{H}\Psi'^{(2)} + a^2\Lambda\Psi^{(2)} = S(\eta, \mathbf{k})\,, \tag{2.60}$$

where the source term, $S(\eta)$, is given by

$$S(\eta,\mathbf{k}) = g^2\Omega_m\mathcal{H}^2\Bigg[\frac{(f-1)^2}{\Omega_m}\Phi_0(\mathbf{k})^2 + 2\Bigg(2\frac{(f-1)^2}{\Omega_m} - \frac{3}{\Omega_m} + 3\Bigg) \tag{2.61}$$

$$\times\Bigg(\nabla^{-2}\left(\partial^i\Phi_0(\mathbf{k})\partial_i\Phi_0(\mathbf{k})\right) - 3\nabla^{-4}\partial_i\partial^j\left(\partial^i\Phi_0(\mathbf{k})\partial_j\Phi_0(\mathbf{k})\right)\Bigg)\Bigg]$$

$$+g^2\Bigg[\frac{4}{3}\left(\frac{f^2}{\Omega_m} + \frac{3}{2}\right)\nabla^{-2}\partial_i\partial^j\left(\partial^i\Phi_0(\mathbf{k})\partial_j\Phi_0(\mathbf{k})\right)$$

$$-\left(\partial^i\Phi_0(\mathbf{k})\partial_i\Phi_0(\mathbf{k})\right)\Bigg], \tag{2.62}$$

and

$$f(\eta) = \frac{d\ln D_+}{d\ln a} = 1 + \frac{g'(\eta)}{\mathcal{H}g(\eta)}, \tag{2.63}$$

where ∇^{-n} represent the inverse of the Laplacian operator and $D_+ = a(\eta)g(\eta)$. Using the traceless part of the Einstein equations,

$$\nabla^2\nabla^2\Psi^{(2)}(\eta,\mathbf{k}) = \nabla^2\nabla^2\Phi^{(2)}(\eta,\mathbf{k}) - 4g^2\nabla^2\nabla^2\Phi_0^2(\mathbf{k}) - \frac{4}{3}g^2\Big(\frac{f^2}{\Omega_m} + \frac{3}{2}\Big) \tag{2.64}$$

$$\times\Bigg[\nabla^2(\partial_i\Phi_0(\mathbf{k})\partial^i\Phi_0(\mathbf{k})) - 3\partial_i\partial^j(\partial^i\Phi_0(\mathbf{k})\partial_j\Phi_0(\mathbf{k}))\Bigg],$$

It is possible to disentangle the second order Bardeen potential, $\Psi^{(2)}$, and the second order gravitational potential, $\Phi^{(2)}$ [74],

$$\Psi^{(2)}(\eta,\mathbf{x}) = \left(B_1(\eta) - 2g(\eta)g_m - \frac{10}{3}(a_{\rm nl}-1)g(\eta)g_m\right)\Phi_0^2(\mathbf{x}) + \left(B_2(\eta) - \frac{4}{3}g(\eta)g_m\right)$$

$$\times\left[\nabla^{-2}\left(\partial^i\Phi_0(\mathbf{x})\partial_i\Phi_0(\mathbf{x})\right) - 3\nabla^{-4}\partial_i\partial^j\left(\partial^i\Phi_0(\mathbf{x})\partial_j\Phi_0(\mathbf{x})\right)\right] \tag{2.65}$$

$$+B_3(\eta)\nabla^{-2}\partial_i\partial^j(\partial^i\Phi_0(\mathbf{x})\partial_j\Phi_0(\mathbf{x})) + B_4(\eta)\partial^i\Phi_0(\mathbf{x})\partial_i\Phi_0(\mathbf{x}),$$

$$\Phi^{(2)}(\eta,\mathbf{x}) = \left(B_1(\eta) + 4g^2(\eta) - 2g(\eta)g_m - \frac{10}{3}(a_{\rm nl}-1)g(\eta)g_m\right)\Phi_0^2(\mathbf{x})$$

$$+\left[B_2(\eta) + \frac{4}{3}g^2(\eta)\left(e(\eta) + \frac{3}{2}\right) - \frac{4}{3}g(\eta)g_m\right]$$

$$\times\left[\nabla^{-2}\left(\partial^i\Phi_0(\mathbf{x})\partial_i\Phi_0(\mathbf{x})\right) - 3\nabla^{-4}\partial_i\partial^j\left(\partial^i\Phi_0(\mathbf{x})\partial_j\Phi_0(\mathbf{x})\right)\right]$$

$$+B_3(\eta)\nabla^{-2}\partial_i\partial^j(\partial^i\Phi_0\partial_j\Phi_0(\mathbf{x})) + B_4(\eta)\partial^i\Phi_0(\mathbf{x})\partial_i\Phi_0(\mathbf{x}), \tag{2.66}$$

where

$$B_i(\eta) = \mathcal{H}_0^{-2}\left(f_0 + 3\Omega_{0m}/2\right)^{-1}\tilde{B}_i(\eta) \tag{2.67}$$

with the following definitions

$$\tilde{B}_1(\eta) = \int_{\eta_m}^{\eta} d\tilde{\eta}\,\mathcal{H}^2(\tilde{\eta})(f(\tilde{\eta})-1)^2 C(\eta,\tilde{\eta})\,, \tag{2.68}$$

$$\tilde{B}_2(\eta) = 2\int_{\eta_m}^{\eta} d\tilde{\eta}\,\mathcal{H}^2(\tilde{\eta})\Big[2(f(\tilde{\eta})-1)^2 - 3 + 3\Omega_m(\tilde{\eta})\Big]C(\eta,\tilde{\eta})\,, \tag{2.69}$$

$$\tilde{B}_3(\eta) = \frac{4}{3}\int_{\eta_m}^{\eta} d\tilde{\eta}\left(e(\tilde{\eta}) + \frac{3}{2}\right)C(\eta,\tilde{\eta})\,, \tag{2.70}$$

$$\tilde{B}_4(\eta) = -\int_{\eta_m}^{\eta} d\tilde{\eta}\,C(\eta,\tilde{\eta})\,, \tag{2.71}$$

$$C(\eta,\tilde{\eta}) = g^2(\tilde{\eta})a(\tilde{\eta})\left[g(\eta)\mathcal{H}(\tilde{\eta}) - g(\tilde{\eta})\frac{a^2(\tilde{\eta})}{a^2(\eta)}\mathcal{H}(\eta)\right], \tag{2.72}$$

and finally $e(\eta) = f^2(\eta)/\Omega_m(\eta)$. The initial conditions for $\Psi^{(2)}$ are deduced from the conservation of $\mathcal{R}_k$ according to Mukhanov equation, such that during the matter domination epoch in the linear theory, $\mathcal{R}_k^{(1)}$ is related to Φ according to $\mathcal{R}_k^{(1)} = -5g_m\Phi_0/3$ and beyond the linear theory, the effect of non-gaussainity is incorporated through $\mathcal{R}_k^{(2)} = \frac{50}{9}a_{nl}g_m^2\Phi_0^2$, for further details on this please see [74].

On small scales, $[B_3(\eta)\nabla^{-2}\partial_i\partial^j(\partial^i\Phi_0\partial_j\Phi_0)+B_4(\eta)\partial^i\Phi_0\partial_i\Phi_0]$ dominates and it is responsible for the Rees-Sciama effect due to the non-linear evolution of the gravitational potentials. Equations (2.65) and (2.66) may be expanded in Fourier space:

$$\Psi^{(2)}(\eta,\mathbf{k}) = \int\frac{\mathrm{d}\,\mathbf{k}_1}{(2\pi)^3}\int\mathrm{d}\,\mathbf{k}_2\delta_D\left(k_1+\mathbf{k}_2-\mathbf{k}\right)\left[f_{\Psi^{(2)}}(\eta,\mathbf{k}_1,\mathbf{k}_2,\mathbf{k})\Phi(\mathbf{k}_1)\Phi(\mathbf{k}_2)\right], \tag{2.73}$$

$$\Phi^{(2)}(\eta,\mathbf{k}) = \int\frac{\mathrm{d}\,\mathbf{k}_1}{(2\pi)^3}\int\mathrm{d}\,\mathbf{k}_2\,\delta_D\left(k_1+\mathbf{k}_2-\mathbf{k}\right)\left[f_{\Phi^{(2)}}(\eta,\mathbf{k}_1,\mathbf{k}_2,\mathbf{k})\Phi(\mathbf{k}_1)\Phi(\mathbf{k}_2)\right], \tag{2.74}$$

where

$$
\begin{aligned}
f_{\Psi^{(2)}}(\eta,\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}) &= \Bigg[\left(B_1(\eta)-2g(\eta)g_m-\frac{10}{3}(a_{\rm nl}-1)g(\eta)g_m\right)+\left(B_2(\eta)-\frac{4}{3}g(\eta)g_m\right)\\
&\times\left[\frac{(\mathbf{k}_1\cdot\mathbf{k}_2)}{k^2}-\frac{3(\mathbf{k}_1\cdot\mathbf{k})\,(\mathbf{k}\cdot\mathbf{k}_2)}{k^4}\right]+B_3(\eta)\frac{(\mathbf{k}_1\cdot\mathbf{k})\,(\mathbf{k}\cdot\mathbf{k}_2)}{k^2}\\
&+B_4(\eta)\mathbf{k}_1\cdot\mathbf{k}_2\Bigg]T_\Phi(\eta_0,\mathbf{k}_1)T_\Phi(\eta_0,k_2|)\,, \qquad (2.75)\\
f_{\Psi^{(2)}}(\eta,\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}) &= \Bigg[\left(B_1(\eta)+4g^2(\eta)-2g(\eta)g_m-\frac{10}{3}(a_{\rm nl}-1)g(\eta)g_m\right)+\Bigg[B_2(\eta)\\
&+\frac{4}{3}g^2(\eta)\left(e(\eta)+\frac{3}{2}\right)-\frac{4}{3}g(\eta)g_m\Bigg]\times\left[\frac{(\mathbf{k}_1\cdot\mathbf{k}_2)}{k^2}-\frac{3(\mathbf{k}_1\cdot\mathbf{k})\,(\mathbf{k}\cdot\mathbf{k}_2)}{k^4}\right]\\
&+B_3(\eta)\frac{(\mathbf{k}_1\cdot\mathbf{k})\,(\mathbf{k}\cdot\mathbf{k}_2)}{k^2}+B_4(\eta)\mathbf{k}_1\cdot\mathbf{k}_2\Bigg]T_\Phi(\eta_0,k_1)T_\Phi(\eta_0,k_2)\,. \qquad (2.76)
\end{aligned}
$$

Here the second order Bardeen potential and the second order gravitational potentials equations (2.73) and (2.74) have been expressed as a convolution of two first order gravitational potentials.

We obtain the power spectrum of second gravitational potential by assuming statistical homogeneity and isotropy

$$
\langle\Psi^{(2)}(\eta,\mathbf{k})\Psi'^{(2)}(\eta,\mathbf{k})\rangle=\frac{2\pi^2}{k^3}\mathcal{P}_{\Psi^{(2)}}(\eta,k)\delta_D\left(\mathbf{k}+\mathbf{k}'\right)\,. \qquad (2.77)
$$

Substituting equation (2.73), we find

$$
\begin{aligned}
\langle\Psi^{(2)}(\eta,\mathbf{k})\Psi^{(2)}(\eta,\bar{\mathbf{k}})\rangle &= \int\frac{\mathrm{d}\,\mathbf{k}_1}{(2\pi)^3}f_{\Psi^{(2)}}(\eta,\mathbf{k}_1,\mathbf{k})\int\frac{\mathrm{d}\,\bar{\mathbf{k}}_1}{(2\pi)^3}f_{\Psi^{(2)}}(\eta,\bar{\mathbf{k}}_1,\bar{\mathbf{k}}) \qquad (2.78)\\
&\times\langle\Phi(\mathbf{k}_1)\Phi(|\mathbf{k}-\mathbf{k}_1|)\Phi(\bar{\mathbf{k}}_1|)\Phi(|\bar{\mathbf{k}}-\bar{\mathbf{k}}_1)\rangle\,.
\end{aligned}
$$

Using the Wicks theorem, we may split the four point function into the connected part and non-connected part. Non-connected part may be written in terms of the irreducible two-point function

$$\begin{aligned}\langle\Phi_1(\mathbf{k}_1)\Phi_1(\mathbf{k}_2)\Phi_1(\mathbf{k}_3)\Phi_1(\mathbf{k}_4)\rangle &= (2\pi)^6\Big[P_{\Phi_g}(k_1)P_{\Phi_g}(k_3)\delta_D(\mathbf{k}_1+\mathbf{k}_2)\delta_D(\mathbf{k}_3+\mathbf{k}_4) \\ & P_{\Phi_g}(k_3)P_{\Phi_g}(k_2)\delta_D(\mathbf{k}_1+\mathbf{k}_3)\delta_D(\mathbf{k}_2+\mathbf{k}_4) \\ & +P_{\Phi_g}(k_2)P_{\Phi_g}(k_1)\delta_D(\mathbf{k}_1+\mathbf{k}_4)\delta_D(\mathbf{k}_2+\mathbf{k}_3)\Big] \\ & +(2\pi)^3 T_\Phi(\mathbf{k_1},\mathbf{k_2},\mathbf{k_3},\mathbf{k_4})\delta_D(\mathbf{k_1}+\mathbf{k_2}+\mathbf{k_3}+\mathbf{k_4}),\end{aligned} \tag{2.79}$$

Comparing equation (2.77) to equation (2.78), we find the power spectrum for $\Psi^{(2)}$ may written as

$$\mathcal{P}_{\Psi^{(2)}}(\eta,k)=\mathcal{P}^{(\mathrm{NC})}_{\Psi^{(2)}}(\eta,k)+\mathcal{P}^{(\mathrm{C})}_{\Psi^{(2)}}(\eta,k)\,, \tag{2.80}$$

where

$$\begin{aligned}\mathcal{P}^{(\mathrm{NC})}_{\Psi^{(2)}}(\eta,k) &= \int\frac{k^3}{4\pi^5}\mathrm{d}\,\mathbf{k}_1\int_0^{4\pi}\mathrm{d}\,\Omega_{k_1}\Big[f_{\Psi^{(2)}}(\eta_1,\mathbf{k}_1,\bar{\mathbf{k}})\left(f_{\Psi^{(2)}}(\eta_2,\mathbf{k}_1,\mathbf{k})\right. \\ & \left.+f_{\Psi^{(2)}}(\eta_2,\mathbf{k}_1,\mathbf{k})\right)\Big]\frac{\mathcal{P}(|\mathbf{k}-\mathbf{k}_1|)}{|\mathbf{k}-\mathbf{k}_1|^3}\frac{\mathcal{P}(k_1)}{k_1}\end{aligned} \tag{2.81}$$

and the linear trispectrum contribution

$$\begin{aligned}\mathcal{P}^{(\mathrm{C})}_{\Psi^{(2)\prime}}(\eta,\mathbf{k}) &= \int\frac{\mathrm{d}^3k_1}{(2\pi)^3}\int\frac{\mathrm{d}^3k_2}{(2\pi)^3}f_{\Psi^{(2)}}(\eta_1,\mathbf{k}_1,\mathbf{k}-\mathbf{k}_1)f'_{\Psi^{(2)}}(\eta_1,\mathbf{k}_1,-\mathbf{k}-\mathbf{k}_2) \\ & \times T_\Phi(\mathbf{k}_1,\mathbf{k}-\mathbf{k}_1,\mathbf{k}_2,-\mathbf{k}-\mathbf{k}_2)\,.\end{aligned} \tag{2.82}$$

We repeat the same procedure for the power spectrum of $\Phi^{(2)}(\eta,\mathbf{k})$, which we define through

$$\langle\Phi^{(2)}(\eta,\mathbf{k})\Phi'^{(2)}(\eta,\mathbf{k})\rangle=\frac{2\pi^2}{k^3}\mathcal{P}_{\Phi^{(2)}}(\eta,k)\delta_D\left(\mathbf{k}+\mathbf{k}'\right). \tag{2.83}$$

After simplification, the power spectrum becomes

$$\mathcal{P}_{\Phi^{(2)}}(\eta,k)=\mathcal{P}^{(\mathrm{NC})}_{\Phi^{(2)}}(\eta,k)+\mathcal{P}^{(\mathrm{C})}_{\Phi^{(2)}}(\eta,k)\,, \tag{2.84}$$

where

$$\begin{aligned}
\mathcal{P}^{(\mathrm{NC})}_{\Phi^{(2)}}(\eta,k) &= \int \frac{k^3}{4\pi^5}\mathrm{d}\,\mathbf{k}_1 \int_0^{4\pi} \mathrm{d}\,\Omega_{k_1}\left[f_{\Phi^{(2)}}(\eta_1,\mathbf{k}_1,\bar{\mathbf{k}})\left(f_{\Phi^{(2)}}(\eta_2,\mathbf{k}_1,\mathbf{k})\right.\right. \qquad (2.85)\\
&\quad \left.\left.+f_{\Phi^{(2)}}(\eta_2,\mathbf{k}_1,\mathbf{k})\right)\right]\frac{\mathcal{P}(|\mathbf{k}-\mathbf{k}_1|)}{|\mathbf{k}-\mathbf{k}_1|^3}\frac{\mathcal{P}(k_1)}{k_1}\,,\\
\mathcal{P}^{(\mathrm{C})}_{\Phi^{(2)\prime}}(\eta,\mathbf{k}) &= \int \frac{\mathrm{d}^{\,3}k_1}{(2\pi)^3}\int \frac{\mathrm{d}^{\,3}k_2}{(2\pi)^3} f_{\Phi^{(2)}}(\eta_1,\mathbf{k}_1,\mathbf{k}-\mathbf{k}_1) f'_{\Phi^{(2)}}(\eta_1,\mathbf{k}_1,-\mathbf{k}-\mathbf{k}_2) \qquad (2.86)\\
&\quad \times T_\Phi(\mathbf{k}_1,\mathbf{k}-\mathbf{k}_1,\mathbf{k}_2,-\mathbf{k}-\mathbf{k}_2)\,.
\end{aligned}$$

Any observable calculated consistently up to second order in perturbation theory, would contain contribution from the cross-power spectrum between $\Psi^{(2)}$ and $\Phi^{(2)}$, therefore assuming statistical homogeneity and isotropy we define the cross-power spectrum between $\Phi^{(2)}$ and $\Psi^{(2)}$ as:

$$\langle \Phi^{(2)}(\eta,\mathbf{k})\Psi'^{(2)}(\eta,\mathbf{k})\rangle = \frac{2\pi^2}{k^3}\mathcal{P}_{\Phi^{(2)}\times\Psi^{(2)}}(\eta,k)\delta_D\left(\mathbf{k}+\mathbf{k}'\right)\,. \qquad (2.87)$$

Substituting equations (2.73) and (2.74), we find

$$\begin{aligned}
\langle \Phi^{(2)}(\eta,\mathbf{k})\Psi^{(2)}(\eta,\bar{\mathbf{k}})\rangle &= \int \frac{\mathrm{d}^{\,3}\mathbf{k}_1}{(2\pi)^3} f_{\Phi^{(2)}}(\eta,\mu_{k_1},\mathbf{k}_1,\mathbf{k})\int \frac{\mathrm{d}^{\,3}\bar{\mathbf{k}}_1}{(2\pi)^3} f_{\Psi^{(2)}}(\eta,\mu_{k_1},\bar{\mathbf{k}}_1,\bar{\mathbf{k}}) \qquad (2.88)\\
&\quad \times\langle \Phi(\mathbf{k}_1)\Phi(|\mathbf{k}-\mathbf{k}_1)\Phi(\bar{\mathbf{k}}_1)\Phi(|\bar{\mathbf{k}}-\bar{\mathbf{k}}_1)\rangle.
\end{aligned}$$

After comparing equation (2.87) and equation (2.88), we find the cross-power between $\Phi^{(2)}$ and $\Psi^{(2)}$ to be

$$\mathcal{P}_{\Phi^{(2)}\times\Psi^{(2)}}(\eta,k) = \mathcal{P}^{(\mathrm{NC})}_{\Phi^{(2)}\times\Psi^{(2)}}(\eta,k) + \mathcal{P}^{(\mathrm{C})}_{\Phi^{(2)}\times\Psi^{(2)}}(\eta,k) \qquad (2.89)$$

where

$$\begin{aligned}
\mathcal{P}^{(\mathrm{NC})}_{\Phi^{(2)}\times\Psi^{(2)}}(\eta,k) &= \int \frac{k^3}{4\pi^5}\mathrm{d}\,\mathbf{k}_1 \int_0^{4\pi} \mathrm{d}\,\Omega_{k_1}\Bigg[f_{\Phi^{(2)}}(\eta_1,\mu_{k_1},\mathbf{k}_1,\mathbf{k})\mathbf{f}_{\mathbf{\Psi}^{(2)}}(\eta_{\mathbf{2}},\mathbf{k}_{\mathbf{1}},\mathbf{k})\,, \qquad (2.90)\\
&\quad +f_{\Psi^{(2)}}(\eta_1,\mu_{k_1},\mathbf{k}_1,\mathbf{k})\mathbf{f}_{\mathbf{\Phi}^{(2)}}(\eta_{\mathbf{2}},\mu_{\mathbf{k_1}},\mathbf{k}_{\mathbf{1}},\mathbf{k}-\mathbf{k}_{\mathbf{1}})\Bigg]\frac{\mathcal{P}(|\mathbf{k}-\mathbf{k}_{\mathbf{1}}|)}{|\mathbf{k}-\mathbf{k}_{\mathbf{1}}|^{\mathbf{3}}}\frac{\mathcal{P}(\mathbf{k}_{\mathbf{1}})}{\mathbf{k}_{\mathbf{1}}}\,,\\
\mathcal{P}^{(\mathrm{C})}_{\Phi^{(2)}\times\Psi^{(2)}}(\eta,\mathbf{k}) &= \int \frac{\mathrm{d}^{\,3}k_1}{(2\pi)^3}\int \frac{\mathrm{d}^{\,3}k_2}{(2\pi)^3} f_{\Phi^{(2)}}(\eta_1,\mathbf{k}_1,\mathbf{k}-\mathbf{k}_1) f_{\Psi^{(2)}}(\eta_1,\mathbf{k}_1,-\mathbf{k}-\mathbf{k}_2) \qquad (2.91)\\
&\quad \times T_\Phi(\mathbf{k}_1,\mathbf{k}-\mathbf{k}_1,\mathbf{k}_2,-\mathbf{k}-\mathbf{k}_2)\,.
\end{aligned}$$

2.3.2 Vector Perturbations

At first order, for a dust dominated universe, the evolution equation for vector perturbation is given by

$$\omega_i' + 2\mathcal{H}\omega_i = 0\,. \tag{2.92}$$

Equation (2.92) can easily be solved, $\omega \sim a^{-2}$. This implies that metric vector perturbation becomes negligible as the universe expands. Also their power spectrum is not generated by any physically plausible inflationary model, hence we will not consider it any further, At second order, the vector perturbations may be sourced by the product of two first order scalar perturbations. It is most convenient to determine the induced vector perturbations from [75],

$$\omega_i^{(2)} = \frac{16}{3\mathcal{H}^2\Omega_m}\nabla^{-2}\Big\{\nabla^2\Phi\,\partial_i\left(\Phi' + \mathcal{H}\Phi\right)\Big\}^V\,. \tag{2.93}$$

where V indicates that we are only interested in the vector contribution. We expand the vector perturbation in Fourier space as

$$\omega_i(\mathbf{x},\eta) = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\left[\omega(\mathbf{k},\eta)e_i(\mathbf{k}) + \bar{\omega}(\mathbf{k},\eta)\bar{e}_i(\mathbf{k})\right]e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{2.94}$$

where the two orthonormal basis vectors $\mathbf{e}$ and $\bar{\mathbf{e}}$ are orthogonal to $\mathbf{k}$. $\omega(\mathbf{k},\eta)$ and $\bar{\omega}(\mathbf{k},\eta)$ are possible polarization states of the vector perturbation. In order to extract just the vector mode, we define an appropriate projection operator, $\mathcal{V}_i{}^l$,

$$\mathcal{V}_i{}^l\omega_l = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}e^{i\mathbf{k}\cdot\mathbf{x}}\left[e_i(\mathbf{k})e^l(\mathbf{k}) + \bar{e}_i(\mathbf{k})e^l(\mathbf{k})\right]\omega_l(\mathbf{k})\,. \tag{2.95}$$

Using the projector of vector perturbation (equation (2.95)) in equation (2.93) in Fourier space, we find

$$\omega^{(2)}(\eta,\mathbf{k}) = \int \frac{\mathrm{d}^3\mathbf{k}_1}{(2\pi)^3}\mathrm{d}^3\mathbf{k}_2\delta_D\left(k_1 + \mathbf{k}_2 - \mathbf{k}\right)f_{\omega^{(2)}}(\eta,\mu_k,\mathbf{k},\mathbf{k}_1)\Phi(\mathbf{k}_1)\Phi(\mathbf{k}_2)\,, \tag{2.96}$$

where

$$\begin{aligned} f_{\omega^{(2)}}(\eta,\mu_k,\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}) &= \frac{16i}{3\mathcal{H}^2\Omega_m}\sqrt{1-\mu_k^2}\Bigg[k_1\frac{(k_2\cdot\mathbf{k}_2)}{k^2}\left(T_\Phi'(\eta,k_1)T_\Phi(\eta,k_2)\right. \\ &\quad \left.+\mathcal{H}T_\Phi(\eta,k_1)T_\Phi(\eta,k_2)\right)\Bigg]\,, \end{aligned} \tag{2.97}$$

and we have made use of

$$e(\mathbf{k},\bar{\mathbf{k}}) \equiv e^i(k)\bar{k}_i = \bar{k}\sqrt{1-\mu_k^2}, \quad \mu_k \equiv \frac{\mathbf{k}\cdot\bar{\mathbf{k}}}{k\bar{k}}\,. \tag{2.98}$$

Assuming statistical homogeneity and isotropy, the power spectrum for the vector mode may be defined as

$$\left\langle \omega^{(2)*}\left(\mathbf{k},\eta\right)\omega^{(2)}\left(\bar{\mathbf{k}},\eta\right)\right\rangle = \frac{2\pi^2}{k^3}\delta^3\left(\mathbf{k}+\bar{\mathbf{k}}\right)\mathcal{P}_{\omega^{(2)}}(k,\eta)\,. \tag{2.99}$$

using equation (2.96) and Wicks theorem (equation (2.79)), we find

$$\begin{aligned}\left\langle \omega^{(2)*}\left(\mathbf{k},\eta\right)\omega^{(2)}\left(\bar{\mathbf{k}},\eta\right)\right\rangle &= \delta_D(\mathbf{k}+\bar{\mathbf{k}})\int \mathrm{d}\,k_1\int_{-1}^{1}\mathrm{d}\,\mu_k f_{\omega^{(2)}}(\eta,\mu_k,\mathbf{k},\mathbf{k}_1) \\ &\quad\times\left[f_{\omega^{(2)}}(\eta,\mu_k,\mathbf{k},\mathbf{k}_1)+f_{\omega^{(2)}}(\eta,\mu_k,\mathbf{k},|\mathbf{k}-\mathbf{k}_1|)\right]\frac{\mathcal{P}(|\mathbf{k}-\mathbf{k}_1|)}{|\mathbf{k}-\mathbf{k}_1|^3}\frac{\mathcal{P}(k_1)}{k_1^3}\,.\end{aligned} \tag{2.100}$$

Hence the induced power spectrum for the vector mode becomes

$$\mathcal{P}_{\omega^{(2)}}(k,\eta) = \mathcal{P}^{(\mathrm{NC})}_{\omega^{(2)}}(k,\eta) + \mathcal{P}^{(\mathrm{C})}_{\omega^{(2)}}(k,\eta)\,. \tag{2.101}$$

where

$$\mathcal{P}^{(\mathrm{NC})}_{\omega^{(2)}}(k,\eta) = \int \mathrm{d}\,k_1\int_{-1}^{1}\mathrm{d}\,\mu_k\mathcal{F}_{\omega^{(2)}}(\eta,k_1,|\mathbf{k}-\mathbf{k_1}|)\mathcal{P}(|\mathbf{k}-\mathbf{k}_1|)\mathcal{P}(k_1)\,, \tag{2.102}$$

$$\begin{aligned}\mathcal{P}^{(\mathrm{C})}_{\omega^{(2)}}(\eta,\mathbf{k}) &= \int\frac{\mathrm{d}^3k_1}{(2\pi)^3}\int\frac{\mathrm{d}^3k_2}{(2\pi)^3}f_{\omega^{(2)}}(\eta,\mathbf{k}_1,\mathbf{k}-\mathbf{k}_1)f_{\omega^{(2)}}(\eta,\mathbf{k}_1,-\mathbf{k}-\mathbf{k}_2) \\ &\quad\times T_\Phi(\mathbf{k}_1,\mathbf{k}-\mathbf{k}_1,\mathbf{k}_2,-\mathbf{k}-\mathbf{k}_2)\,,\end{aligned} \tag{2.103}$$

with

$$\begin{aligned}\mathcal{F}_{\omega^{(2)}}(\eta,k_1,|\mathbf{k}-\mathbf{k_1}|) &= \frac{k^3}{4\pi^5k_1|\mathbf{k}-\mathbf{k}_1|^3}f_{\omega^{(2)}}(\eta,\mu_k,\mathbf{k},\mathbf{k}_1)\left[f_{\omega^{(2)}}(\eta,\mu_k,\mathbf{k},\mathbf{k}_1)\right. \\ &\quad\left.+f_{\omega^{(2)}}(\eta,\mu_k,\mathbf{k},|\mathbf{k}-\mathbf{k}_1|)\right]\,.\end{aligned} \tag{2.104}$$

An alternative derivation of non-connected part of this result may be found in [75, 76].

The most power from the induced vector modes according figure 2.3.2 appear at the equality scale, and it has the same spectrum as Φ below this scale, but with $\lesssim$1% of the amplitude. Given that the induced vector mode has its largest power around equality, it might be possible to observe its imprint on Redshift-space distortion, Large-angle CMB, weak lensing, etc [76].

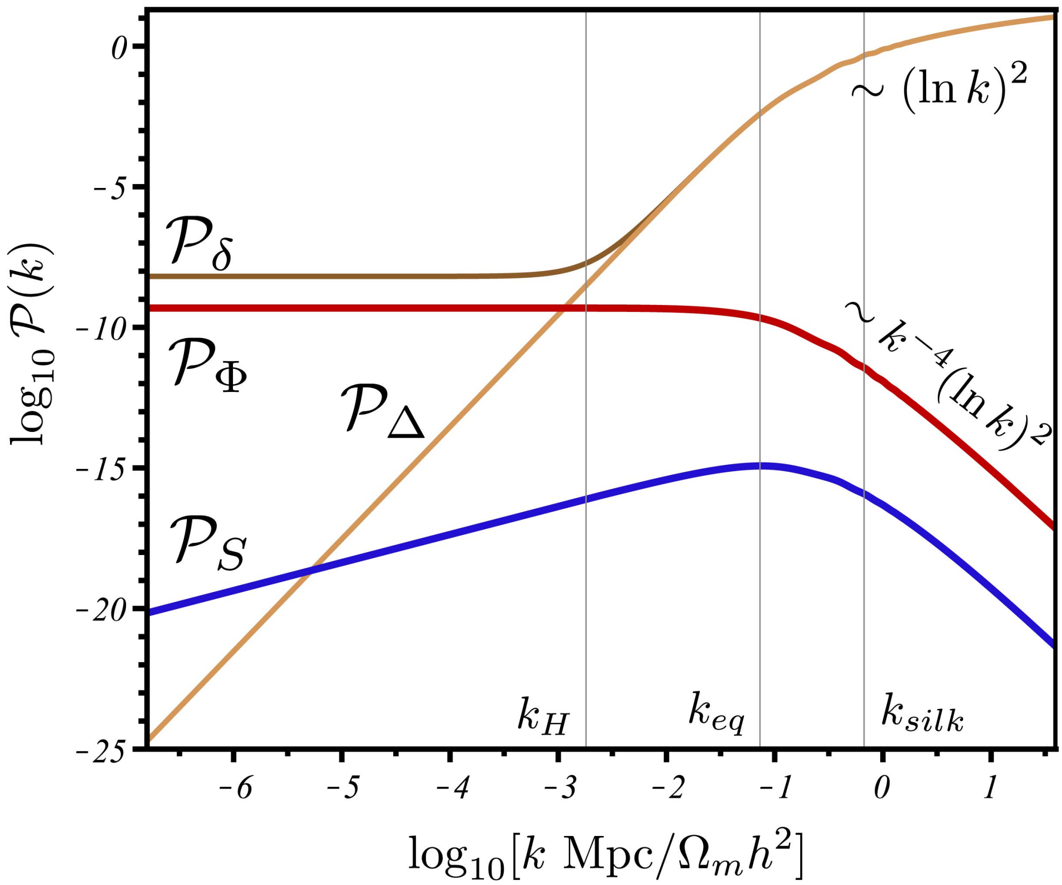

Figure 2.1: The power spectrum today of vector modes generated at second order by first order scalar perturbations. $\mathcal{P}_S \to \mathcal{P}_{\omega^{(2)}}$ is plotted along side power spectra of first-order Bardeen potential, the density perturbation $\delta = \delta\rho/\rho$ and also comoving density perturbation $\Delta = \delta - 3\mathcal{H}v^{(1)}$, and Φ. The computation did not include contribution from the primordial trispectrum. Figure credit [76].

2.3.3 Tensor Perturbations

At first order, the equation of motion describing the propagation of tensor perturbations is given by

$$h''_{ij} + 2\mathcal{H}h'_{ij} + k^2 h_{ij} = 0\,. \tag{2.105}$$

Equation (2.105) is simply a wave equation with a damping term $2\mathcal{H}\dot{h}_{ij}$, therefore, h_{ij} describes gravitational waves propagating on an FLRW background with the Hubble expansion tending to dampen its propagation. In Fourier space, tensor perturbation may be expanded as

$$h_{ij} = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \left[e^{+}_{ij}(\mathbf{k})h^{+}(k,\eta) + e^{\times}_{ij}(\mathbf{k})h^{\times}(k,\eta)\right] e^{i\mathbf{k}\cdot\mathbf{x}} \tag{2.106}$$

The power spectrum for the two polarization modes of h_{ij},

$$\langle h_s(\eta,\mathbf{k})h_{s'}(\eta',\mathbf{k},')\rangle = (2\pi)^3\delta^3(\mathbf{k}-\mathbf{k}')\delta_{ss'}\frac{1}{2}T_T(\eta,k)T_T(\eta',k)P_h(k), \tag{2.107}$$

where $e^s_{ij}(\hat{\vec{k}})$, $s = +, \times$, are transverse (with respect to $\hat{\vec{k}}$) and traceless polarization tensors normalized such that $e^s_{ij}e^{s'\,ij} = 2\delta^{ss'}$. We assume both polarizations to be independent of each other and to have equal power spectra: It is most convenient to expand the gravitational waves, in helicity basis [77]

$$h_{ij} = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \left[e^{(+2)}_{ij}(\mathbf{k})h^{(+2)}(k,\eta) + e^{(-2)}_{ij}(\mathbf{k})h^{(-2)}(k,\eta)\right] e^{i\mathbf{k}\cdot\mathbf{x}} \tag{2.108}$$

where we have defined helicity±2 polarization tensors, $h^{\pm2}$ and Fourier amplitudes

$$e^{\pm2}_{ij} \equiv e^{+}_{ij} \pm ie^{\times}_{ij} \tag{2.109}$$

$$h_{\mp2} \equiv \frac{1}{2}(h^{+} \pm ih^{\times}). \tag{2.110}$$

The corresponding power spectrun is given by

$$\langle h^{\pm}_s(\eta,\mathbf{k})h^{\pm}_{s'}(\eta',\mathbf{k},')\rangle = (2\pi)^3\delta^3(\mathbf{k}-\mathbf{k}')\delta_{ss'}\frac{1}{2}T_T(\eta,k)T_T(\eta',k)P_{h^{\pm2}}(k), \tag{2.111}$$

where $T_T(\eta,k)$ is the tensor transfer function. Note that $P_{h_{\pm2}}(k) = P_h(k)/8$. We define the power spectrum of the tensor perturbation as the sum of the power spectra for the two polarization states, $\Delta^2_h \equiv 2\Delta^2_{h^x}$, which may also be expressed in terms of tensor spectral index, $\Delta^2_h(k) = A_h(k_*)\left(\frac{k}{k_*}\right)^{n_T(k_*)}$. Δ^2_h and Δ_{h^x} are defined in chapter one and A_h is the amplitude of the tensor fluctuations.

At second order, for the late-time universe the propagation equation for tensor perturbation is given by [78]

$$\begin{aligned} h''_{ij} + 2\mathcal{H}h'_{ij} - \nabla^2 h_{ij} &= -16\Phi\partial_i\partial_j\Phi - 8\partial_i\Phi\partial_j\Phi \\ &+\frac{4}{\mathcal{H}^2\Omega_m}\left[\mathcal{H}^2\partial_i\Phi\partial_j\Phi + 2\mathcal{H}\partial_i\Phi\partial_j\Phi' + \partial_i\Phi'\partial_j\Phi'\right] \end{aligned} \tag{2.112}$$

We define a projection tensor for the source term in Fourier space as

$$\mathcal{T}_{ij}{}^{lm} S_{lm} = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \left[e_{ij}(\mathbf{k}) e^{lm}(\mathbf{k}) + \bar{e}_{ij}(\mathbf{k}) e^{lm}(\mathbf{k}) \right] S_{lm}(\mathbf{k}) \tag{2.113}$$

where $\mathcal{T}$ denotes the tensor projection [79] and $S_{lm}(\mathbf{k})$ is the Fourier transform of the source term, $S_{\ell m}(\mathbf{x}')$

$$S_{lm}(\mathbf{k}) = \int \frac{\mathrm{d}^3\mathbf{x}'}{(2\pi)^3} e^{-i\mathbf{k}\cdot x} S_{\ell m}(\mathbf{x}') \,. \tag{2.114}$$

Using these tools, we may re-write equation (2.112) in Fourier space as

$$h''(\eta, \mathbf{k}) + 2\mathcal{H} h'(\eta, \mathbf{k}) + k^2 h(\eta, \mathbf{k}) = S(\eta, \mathbf{k}) \tag{2.115}$$

where the source term is a convolution of two first-order scalar perturbations

$$\begin{aligned} \mathcal{S}(\eta, \mathbf{k}) \;\; = \;\; & \int \frac{\mathrm{d}^3\mathbf{k}_1}{(2\pi)^3} k^2 (1 - \mu_k^2) \Bigg[\frac{4}{\Omega_m} (6\Omega_m - 1)\, T_\Phi(\eta, k_1) T_\Phi(\eta, |k - k_1|) \\ & - \frac{4}{\mathcal{H}^2 \Omega_m} \left(2\mathcal{H} T'_\Phi(\eta, k_1) T_\Phi(\eta, |k - k|_1) \right. \\ & \left. + T'_\Phi(\eta, k_1) T'_\Phi(\eta, |k - k_1|) \right) \Bigg] \Phi(\mathbf{k}_1) \Phi(|\mathbf{k} - \mathbf{k}_1|) \,. \end{aligned} \tag{2.116}$$

where we have made use of

$$e(\mathbf{k}, \bar{\mathbf{k}}) \equiv e^{ij}(k) \bar{k}_i \bar{k}_j = \bar{k}[1 - \mu_k^2], \;\; \mu_k \equiv \frac{\mathbf{k} \cdot \bar{\mathbf{k}}}{k\bar{k}} \,. \tag{2.117}$$

A particular solution to equation (2.115) is using the Greeen function's approach and it is given by

$$h(\eta, \mathbf{k}) = \frac{1}{a(\eta)} \int \mathrm{d}\,\bar{\eta} g(\eta; \bar{\eta}, \mathbf{k}) a(\bar{\eta}) \mathcal{S}(\eta, \mathbf{k}) \,, \tag{2.118}$$

where the Green function $g(\eta, \mathbf{k})$ satisfies the following equation

$$g''(\eta, \mathbf{k}) + \left(k^2 - \frac{a''}{a} \right) g(\eta, \mathbf{k}) = \delta(\eta - \bar{\eta}) \,. \tag{2.119}$$

In the matter universe, equation (2.119) may be solved exactly $g(\eta; \bar{\eta}, k) = \sin[k(\eta - \bar{\eta})]/k$ and during the radiation domination we have $g(\eta; \bar{\eta}, k) = (\sin(k\bar{\eta}) \cos(k\eta) - \sin(k\bar{\eta}) \cos(k\eta))$.

It is more convenient to have equation (2.118) given as

$$h(\eta,\mathbf{k}) = \int \frac{\mathrm{d}^3\mathbf{k}_1}{(2\pi)^3}\mathrm{d}^3\mathbf{k}_2\delta_D\left(k_1+\mathbf{k}_2-\mathbf{k}\right) f_{h^{(2)}}(\eta,\mu_k,k_1,k_2,k)\Phi(\mathbf{k}_1)\Phi(\mathbf{k}_2)\,, \quad (2.120)$$

where

$$\begin{aligned} f_{h^{(2)}}(\eta,\mu_k,\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}) &= \frac{1}{a(\eta)}\int \mathrm{d}\,\eta g(\eta;\bar{\eta},\mathbf{k})a(\bar{\eta})k^2(1-\mu_k^2) \quad (2.121)\\ &\times\Bigg[\frac{4}{\Omega_m}\left(6\Omega_m-1\right)T_\Phi(\eta,k_1)T_\Phi(\eta,k_2)\\ &-\frac{4}{\mathcal{H}^2\Omega_m}\left(2\mathcal{H}T'_\Phi(\eta,k_1)T_\Phi(\eta,k_2)+T'_\Phi(\eta,k_1)T'_\Phi(\eta,k_2)\right)\Bigg]\,. \end{aligned}$$

Assuming statistical homogeneity and isotropy, the power spectrum of induced gravitational waves is given by

$$\langle h(\eta,\mathbf{k})h'(\eta,\bar{\mathbf{k}})\rangle = \frac{2\pi^2}{k^3}\mathcal{P}_h(\eta,k)\delta_D\left(\mathbf{k}+\bar{\mathbf{k}}\right)\,. \quad (2.122)$$

substituting equation (2.118) into equation (2.147) and performing the delta function integral we find

$$\begin{aligned} \langle h(\eta,\mathbf{k})h'(\eta,\bar{\mathbf{k}})\rangle &= \delta_D(k+\bar{k})\int\frac{\mathrm{d}^3\mathbf{k}_1}{(2\pi)^3}f_{h^{(2)}}(\eta_1,\mu_k,\mathbf{k}_1,\mathbf{k})\Bigg[f_{h^{(2)}}(\eta_1,\mu_k,\mathbf{k},\mathbf{k}_1) \quad (2.123)\\ &+f_{h^{(2)}}(\eta_1,\mu_k,\mathbf{k},\mathbf{k}-\mathbf{k_1})\Bigg]\frac{P(|\mathbf{k}-\mathbf{k}_1|)}{|\mathbf{k}-\mathbf{k}_1|^3}\frac{P(k_1)}{k_1^3}\,, \end{aligned}$$

where we have made use of Wicks theorem. Comparing equation (2.122) with equation (2.123), the second order power spectrum for the scalar induced gravitational waves becomes

$$\mathcal{P}_{h^{(2)}}(\eta,k) = \mathcal{P}^{(\mathrm{NC})}_{h^{(2)}}(\eta,k)+\mathcal{P}^{(\mathrm{C})}_{h^{(2)}}(\eta,k)\,, \quad (2.124)$$

where

$$\mathcal{P}^{(\mathrm{NC})}_{h}(\eta,k) = \int_0^\infty \mathrm{d}\,k_1\int_{-1}^1\mathrm{d}\,\mu\mathcal{P}_\Phi(|\mathbf{k}-\mathbf{k}_1|)\mathcal{P}_\Phi(k_1)\mathcal{F}(k,k_1,\mu_k,\eta)\,, \quad (2.125)$$

$$\begin{aligned} \mathcal{P}^{(\mathrm{C})}_{h^{(2)}}(\eta,\mathbf{k}) &= \int\frac{\mathrm{d}^3k_1}{(2\pi)^3}\int\frac{\mathrm{d}^3k_2}{(2\pi)^3}f_{h^{(2)}}(\eta_1,\mathbf{k}_1,\mathbf{k}-\mathbf{k}_1)f_{h^{(2)}}(\eta_1,\mathbf{k}_1,-\mathbf{k}-\mathbf{k}_2) \quad (2.126)\\ &\times T_\Phi(\mathbf{k}_1,\mathbf{k}-\mathbf{k}_1,\mathbf{k}_2,-\mathbf{k}-\mathbf{k}_2)\,, \end{aligned}$$

with

$$\mathcal{F}_h(k, k_1, \mu_k, \eta) \equiv \frac{k^3}{4\pi^5 k_1 |\mathbf{k} - \mathbf{k}_1|^3} f_{h^{(2)}}(\eta_1, \mu_k, \mathbf{k}_1, \mathbf{k}) \Big[f_{h^{(2)}}(\eta_1, \mu_k, \mathbf{k}, \mathbf{k}_1) + f_{h^{(2)}}(\eta_1, \mu_k, \mathbf{k}, \mathbf{k} - \mathbf{k_1}) \Big] . \quad (2.127)$$

Figure 2.2 shows the power spectrum of induced gravitational waves at three different epochs, notice that it also peaks in power around the equality scale, and is surprisingly larger than the primordial background on these scales.

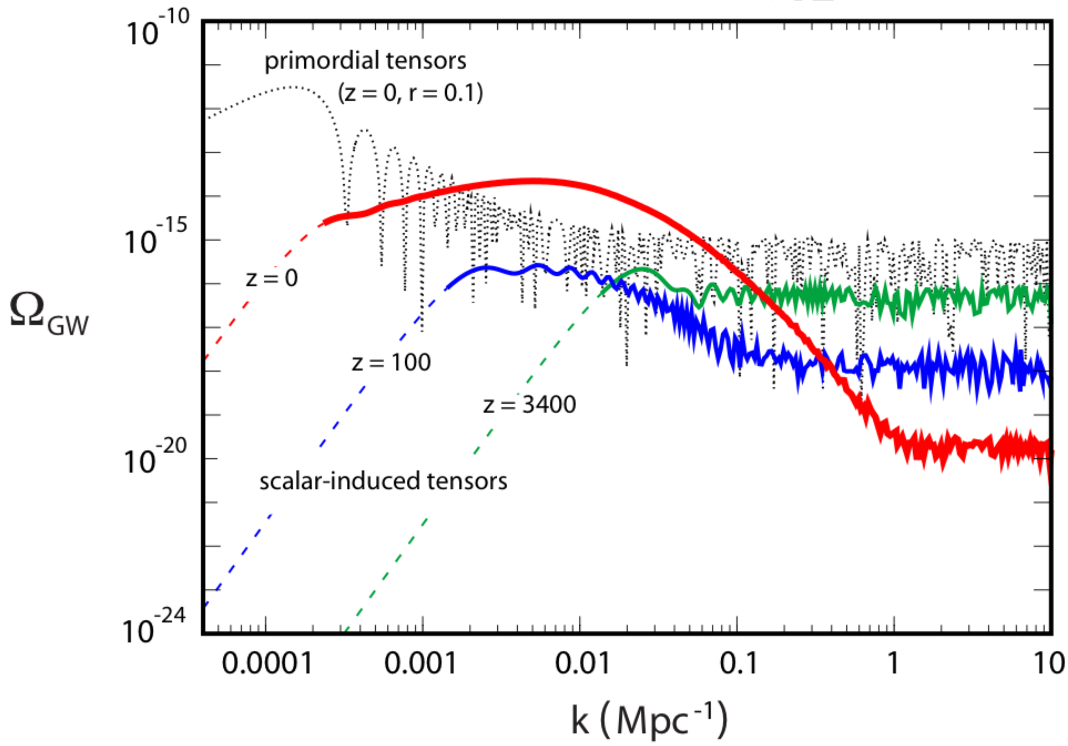

Figure 2.2: The power spectrum of tensor modes generated at second order by first order scalar perturbations (lower curves) plotted along side the scale invariant power spectrum of the primordial tensor modes (upper curves). The spectra of induced tensor modes are shown at three different epochs, $(1+z) = 3400$, 100 and 1. The contributions to induced power spectrum today ($(1+z) = 1$) cross the primordial counter-part at intermediate wavelength. The trispectrum contribution was not included in the computation. Figure credit [78].

2.4 Velocity Perturbations

The first-order velocity potential, v, is obtained by taking divergence of the off-diagonal component of the Einstein equation (equation 2.32),

$$v = -\frac{2}{3\mathcal{H}^2\Omega_m}\left(\mathcal{H}\Phi + \Phi'\right). \tag{2.128}$$

In Fourier space, equation (2.128) may be split into the primordial gravitational potential and transfer function parts

$$v(\eta, \mathbf{k}) = T_v(\eta, k)\Phi(\mathbf{k}), \tag{2.129}$$

where

$$T_v(\eta, k) = -\frac{2}{3\mathcal{H}^2\Omega_m}\left(\mathcal{H}T_\Phi(\eta, k) + T'_\Phi(\eta, k)\right). \tag{2.130}$$

The power spectrum for the velocity perturbation is given by

$$\langle v(\eta_1, \mathbf{k})v(\eta_2, \mathbf{k}')\rangle = \frac{2\pi^2}{k^2}T_v(\eta_1, k)T_v(\eta_2, k')\mathcal{P}_\Phi(k)\delta_D(\mathbf{k}+\mathbf{k}'). \tag{2.131}$$

At second-order, the velocity perturbations may be decomposed according to the field that sources the second order perturbation. For example, the velocity perturbation sourced by vector perturbation is given by

$$\mathcal{H}^2\Omega_m\left[v_i^{(2)} + \mathrm{D}_i v^{(2)}\right]^{(\omega)} = [\frac{1}{2}\nabla^2 - 3\mathcal{H}^2\Omega_m]\omega_i, \tag{2.132}$$

and the component sourced by second order gravitational potential and the Bardeen potential is given by

$$\mathcal{H}^2\Omega_m\left[v_i^{(2)} + \mathrm{D}_i v^{(2)}\right]^{(\Phi^{(2)})} = -2\partial_i[\mathcal{H}\Phi^{(2)} + \Psi^{(2)\prime}], \tag{2.133}$$

and finally, the contribution from terms quadratic in first order gravitational potential

$$\mathcal{H}^2\Omega_m\left[v_i^{(2)} + \mathrm{D}_i v^{(2)}\right]^{(\Phi^2)} = \frac{4}{\Omega_m}\left[(\Omega_m - 2)(\mathcal{H}\Phi\mathrm{D}_i\Phi + \Phi'\mathrm{D}_i\Phi) - (2+3\Omega_m)\Phi\mathrm{D}_i\Phi'\right]$$
$$-\frac{8}{9}\frac{1}{\mathcal{H}^2\Omega_m}\left(3\mathcal{H}\Phi'\mathrm{D}_i\Phi' - \nabla^2\Phi\mathrm{D}_i\Phi' - \mathcal{H}\nabla^2\Phi\mathrm{D}_i\Phi\right). \tag{2.134}$$

For simplicity, we let $V_i = \left[v_i^{(2)} + \mathrm{D}_i v^{(2)}\right]$ since in all our equations they appear in this combination. It is better to work in helicity basis, thus we decompose the velocity in real space as

$$V_i(\eta,\mathbf{x}) = \mu V_\parallel(\eta,\mathbf{x})n_{_i} + V_+(\eta,\mathbf{x})m_i^+ + V_-(\eta,\mathbf{x})m_i^- \,, \tag{2.135}$$

and in Fourier space, it may it may be decomposed as

$$V_i(\eta,\mathbf{x}) = \int \frac{\mathrm{d}^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \left[\mu V_\parallel(\eta,\mathbf{k})e_{3_i} + V_+(\eta,\mathbf{k})e_i^+ + V_-(\eta,\mathbf{k})e_i^-\right] \,, \tag{2.136}$$

we have aligned e_{3i} to our line of sight, such that $e_{3i}e^i_\pm = 0$ and $V_\pm = \mathbf{V}\cdot\mathbf{e}_\mp$. e_{3i} and n^i have similar properties. The line of sight component of the velocity perturbation may be given by

$$\begin{aligned} V_\parallel^{(2)}(\eta,\mathbf{k}) &= \int \frac{\mathrm{d}^3k_1}{(2\pi)^3}\Bigg[{}^{(\omega)}F_{V_\parallel^{(2)}}(\eta,\mu_k,\mathbf{k}_1,\mathbf{k}) + {}^{\Psi^{(2)}/\Phi^{(2)}}F_{V_\parallel}(\eta,\mu_k,\mathbf{k}_1,\mathbf{k}) \qquad (2.137)\\ &\quad + {}^{\Phi\times\Phi}F_{v_\parallel}(\eta,\mu_k,\mathbf{k}_1,\mathbf{k})\Bigg]\Phi(\mathbf{k}_1)\Phi(\mathbf{k}-\mathbf{k}_1)\,, \end{aligned}$$

where

$$\begin{aligned} {}^{\Psi^{(2)}/\Phi^{(2)}}F_{V_\parallel}(\eta,\mu_k,\mathbf{k}_1,\mathbf{k}) &= -\frac{2k}{3\Omega_m\mathcal{H}^2}\left[\mathcal{H}F_{\Phi^{(2)}}(\eta,\mu,\mathbf{k}_1,\mathbf{k}) + F'_{\Psi^{(2)}}(\eta,\mu,\mathbf{k}_1,\mathbf{k})\right] \qquad (2.138)\\ {}^{(S)}F_{V_\parallel^{(2)}}(\eta,\mu_k,\mathbf{k}_1,\mathbf{k}) &= \frac{1}{\Omega_m\mathcal{H}^2}\sqrt{\frac{1-\mu^2}{\mu^2}}\left[\frac{1}{6}k^2 - \mathcal{H}^2\Omega_m\right]F_{S^{(2)}}(\eta,\mu,k_1,\mathbf{k}) \qquad (2.139)\\ {}^{\Phi\times\Phi}F_{v_\parallel}(\eta,\mu_k,\mathbf{k}_1,\mathbf{k}) &= \frac{4}{(3\Omega_m^2\mathcal{H}^2)}\Big[(\Omega_m-2)\left(\mathcal{H}T_\Phi(k_1,\eta_1)T_\Phi(\eta_2,k_2)\right.\\ &\quad \left.+T'_\Phi(\eta_1,k_1)T_\Phi(\eta,k_2)\right) - (2+3\Omega_m)T_\Phi T_\Phi(\eta_2,k_2)\Big]\,,\\ &\quad -\frac{8}{9\Omega_m\mathcal{H}^2}\left[3\mathcal{H}T'_\Phi(\eta_1,k_1)T'_\Phi(\eta_2,k_2) - k^2T_\Phi(\eta_1,k_1)\right. \qquad (2.140)\\ &\quad \left.\times T_\Phi(\eta_2,k_2) - \mathcal{H}k^2T_\Phi(\eta_1,k_1)T_\Phi(\eta_2,k_2)\right] \end{aligned}$$

and the transverse velocity component may be given by

$$\begin{aligned} V_\perp^{(2)}(\eta,\mathbf{k}) &= \int \frac{\mathrm{d}^3k_1}{(2\pi)^3}\Bigg[{}^{(S)}F_{V_\perp^{(2)}}(\eta,\mu_k,\mathbf{k}_1,\mathbf{k}) + {}^{\Psi^{(2)}/\Phi^{(2)}}F_{V_\perp}(\eta,\mu_k,\mathbf{k}_1,\mathbf{k}) \qquad (2.141)\\ &\quad + {}^{\Phi\times\Phi}F_{v_\perp}(\eta,\mu_k,\mathbf{k}_1,\mathbf{k})\Bigg]\Phi(\mathbf{k}_1)\Phi(\mathbf{k}-\mathbf{k}_1)\,, \end{aligned}$$

where

$$
\begin{aligned}
{}^{\Psi^{(2)}/\Phi^{(2)}}F_{V_\perp}(\eta,\mu_k,\mathbf{k}_1,\mathbf{k}) &= -\frac{2k}{3\Omega_m\mathcal{H}^2}\sqrt{1-\mu^2}\,[\mathcal{H}F_{\Phi^{(2)}}(\eta,\mu,\mathbf{k}_1,\mathbf{k}) \\
&\quad +F'_{\Psi^{(2)}}(\eta,\mu,\mathbf{k}_1,\mathbf{k})]\ , && (2.142)\\
{}^{(S)}F_{V_\perp^{(2)}}(\eta,\mu_k,\mathbf{k}_1,\mathbf{k}) &= \frac{1}{\Omega_m\mathcal{H}^2}\left[\frac{1}{6}k^2-\mathcal{H}^2\Omega_m\right]F_{S^{(2)}}(\eta,\mu,k_1,\mathbf{k})\,, && (2.143)\\
{}^{\Phi\times\Phi}F_{v_\perp}(\eta,\mu_k,\mathbf{k}_1,\mathbf{k}) &= \frac{4}{(3\Omega_m^2\mathcal{H}^2)}\sqrt{1-\mu^2}\,[(\Omega_m-2)\left(\mathcal{H}T_\Phi(k_1,\eta_1)T_\Phi(\eta_2,k_2)\right. \\
&\quad \left.+T'_\Phi(\eta_1,k_1)T_\Phi(\eta,k_2)\right)-(2+3\Omega_m)T_\Phi T_\Phi(\eta_2,k_2)] \\
&\quad -\frac{8}{9\Omega_m\mathcal{H}^2}\left[3\mathcal{H}T'_\Phi(\eta_1,k_1)T'_\Phi(\eta_2,k_2)\right. \\
&\quad \left.-k^2T_\Phi(\eta_1,k_1)T_\Phi(\eta_2,k_2)-\mathcal{H}k^2T_\Phi(\eta_1,k_1)T_\Phi(\eta_2,k_2)\right] && (2.144)
\end{aligned}
$$

Equation (2.137) and equation (2.141) may be written in a more compact form

$$
V_{V^{(2)}_{\parallel/\perp}}(\eta,\mathbf{k}) = \int\frac{\mathrm{d}^3k_1}{(2\pi)^3}f_{V^{(2)}_{\parallel/\perp}}(\eta,\mu_k,\mathbf{k}_1,\mathbf{k})\Phi(\mathbf{k}_1)\Phi(\mathbf{k}-\mathbf{k}_1)\,, \tag{2.145}
$$

where

$$
f_{V^{(2)}_{\parallel/\perp}}(\eta,\mu_k,\mathbf{k}_1,\mathbf{k}) = \left[{}^{(S)}F_{V^{(2)}_{\parallel\perp}}(\eta,\mu_k,\mathbf{k}_1,\mathbf{k})+{}^{\Psi^{(2)}/\Phi^{(2)}}F_{V_{\parallel\perp}}(\eta,\mu_k,\mathbf{k}_1,\mathbf{k})\right. \left.+{}^{\Phi\times\Phi}F_{v_{\parallel\perp}}(\eta,\mu_k,\mathbf{k}_1,\mathbf{k})\right]. \tag{2.146}
$$

With the subscript of $V^{(2)}_{\parallel/\perp}$ indicating both the transverse and the parallel contributions. Should one be interested in the second order gravitational potential and the Bardeen potential, he will simply set $f_{V^{(2)}_{\parallel/\perp}}(\eta,\mu_k,\mathbf{k}_1,\mathbf{k})\to{}^{\Psi^{(2)}/\Phi^{(2)}}F_{V_\perp}(\eta,\mu_k,\mathbf{k}_1,\mathbf{k})$, a similar treatment applies to other effects. If we define the velocity power spectrum following the general prescription

$$
\langle V^{(2)}_{\parallel/\perp}(\eta,\mathbf{k})V^{(2)}_{\parallel/\perp}(\eta,\bar{\mathbf{k}}')\rangle = \frac{2\pi^2}{k^3}\mathcal{P}_{V^{(2)}_{\parallel/\perp}}(\eta,k)\delta_D\left(\mathbf{k}+\bar{\mathbf{k}}\right) \tag{2.147}
$$

The power spectrum for the velocity perturbation then becomes

$$
\mathcal{P}_{V^{(2)}_{\parallel/\perp}}(k,\eta) = \mathcal{P}^{(\mathrm{NC})}_{V^{(2)}_{\parallel/\perp}}(k,\eta)+\mathcal{P}^{(\mathrm{C})}_{V^{(2)}_{\parallel/\perp}}(k,\eta) \tag{2.148}
$$

where

$$\mathcal{P}^{(\mathrm{NC})}_{V^{(2)}_{\parallel/\perp}}(\eta,k) = \int_0^\infty \mathrm{d}\,k_1 \int_{-1}^{1} \mathrm{d}\,\mu \mathcal{P}_\Phi(|\mathbf{k}-\mathbf{k}_1|)\mathcal{P}_\Phi(k_1)\mathcal{F}_{V^{(2)}_{\parallel/\perp}}(k,k_1,\mu_k,\eta) \tag{2.149}$$

$$\mathcal{P}^{(\mathrm{C})}_{V^{(2)}_{\parallel/\perp}}(\eta,k) = \int \frac{\mathrm{d}^3k_1}{(2\pi)^3}\int \frac{\mathrm{d}^3k_2}{(2\pi)^3} f_{V^{(2)}_{\parallel/\perp}}(\eta_1,\mathbf{k}_1,\mathbf{k}-\mathbf{k}_1) f_{V^{(2)}_{\parallel/\perp}}(\eta_1,\mathbf{k}_1,-\mathbf{k}-\mathbf{k}_2) \tag{2.150}$$
$$\times T_\Phi(\mathbf{k}_1,\mathbf{k}-\mathbf{k}_1,\mathbf{k}_2,-\mathbf{k}-\mathbf{k}_2)$$

with

$$\mathcal{F}_{V^{(2)}_{\parallel/\perp}}(k,k_1,\mu_k,\eta) \equiv \frac{k^3}{4\pi^5 k_1|\mathbf{k}-\mathbf{k}_1|^3} f_{V_{\parallel/\perp}}(\eta_1,\mu_k,\mathbf{k}_1,\mathbf{k})\Big[f_{V_{\parallel/\perp}}(\eta_1,\mu_k,\mathbf{k},\mathbf{k}_1) \tag{2.151}$$
$$+f_{V_{\parallel/\perp}}(\eta_1,\mu_k,\mathbf{k},\mathbf{k}-\mathbf{k_1})\Big]$$

Evolution of velocity perturbation in the matter dominated universe is described by the following equations:

$$v' + \mathcal{H}v + \Phi = 0 \tag{2.152}$$
$${}^{(2)}v'^b + \mathrm{D}^{b(2)}v' + {}^{(2)}S'^b + \mathcal{H}\left({}^{(2)}S^b + {}^{(2)}v^b + {}^{(2)}\mathrm{D}^b v\right) + 2\Phi\mathrm{D}^b\Phi + \mathrm{D}^{b(2)}\Phi \tag{2.153}$$
$$+2\mathrm{D}_c\mathrm{D}^b v\mathrm{D}^c v - 4\Phi'\mathrm{D}^b v = 0$$

2.5 Matter Perturbations

As shown in equation (2.154), the energy density ρ may be split into a homogeneous background $\bar{\rho}(\eta)$ and a perturbation $\delta\rho(\eta, x^i)$ as follows

$$\rho(\eta,x^i) = \rho_0(\eta) + \delta^{(1)}\rho(\eta,x^i) + \frac{1}{2}\delta^{(2)}\rho(\eta,\mathbf{x})\,. \tag{2.154}$$

At first order, time-time part of the Einstein equations (equation (2.31)) gives the matter-density,

$$3\mathcal{H}\Phi' - \nabla^2\Phi + 3\mathcal{H}^2\Phi = -4\pi G a^2\delta\rho\,. \tag{2.155}$$

The matter density at second-order is also given by the time-time part Einstein equations (equation (2.42))

$$3\mathcal{H}\Psi^{(2)\prime} + 3\mathcal{H}^2\Phi^{(2)} - \nabla^2\Psi^{(2)} - 2\nabla^2\Phi^2 - 12\mathcal{H}^2\Phi^2 - 3\left(\Phi'\right)^2 + \mathrm{D}_i\Phi\mathrm{D}^i\Phi - 4\Phi\nabla^2\Phi$$
$$= -3\mathcal{H}^2\Omega_m\left(\frac{1}{2}\delta^{(2)} + v^{(1)2}\right), \tag{2.156}$$

where v is given in Section 2.4 Density contrast may be defined as $\delta(\mathbf{x}) \equiv \rho(\mathbf{x})/\bar{\rho} - 1$ and up to second order may be decomposed as [74, 80]

$$\delta(\eta, \mathbf{x}) = \delta^{(1)}(\eta, \mathbf{x}) + \delta^{(2)}(\eta, \mathbf{x}), \tag{2.157}$$

where

$$\delta^{(1)}(\eta, \mathbf{x}) = \frac{2}{3\Omega_m\mathcal{H}^2}\left[\nabla^2\Phi - 3\mathcal{H}\left(\Phi' + \mathcal{H}\Phi\right)\right], \tag{2.158}$$

$$\begin{aligned}
\delta^{(2)}(\eta, \mathbf{x}) = {} & \frac{1}{\Omega_m}\left[(f-1)^2 - \frac{2}{g^2}\frac{A'(\tau)}{\mathcal{H}} - \frac{2}{g^2}A(\tau) - 1\right]\Phi^2 - \frac{2}{\Omega_m g^2}\left[\frac{B_2'(\tau)}{\mathcal{H}}\right. \\
& \left. - \frac{4}{3}\frac{g'}{\mathcal{H}}g_{\rm in} + B_2(\tau) - \frac{4}{3}gg_{\rm in} + \frac{4}{3}g^2\left(e + \frac{3}{2}\right)\right]\alpha(\mathbf{x}, \tau) \\
& - \frac{2}{\Omega_m g^2}\frac{B_3'(\tau)}{\mathcal{H}}\nabla^{-2}\partial_i\partial^j(\partial^i\Phi\partial_j\Phi) - \frac{2}{\Omega_m g^2}\frac{B_4'(\tau)}{\mathcal{H}}\partial_i\Phi\partial^i\Phi \\
& + \frac{2}{3\Omega_m g^2}\left[B_1(\tau) - 2gg_{\rm in} - \frac{10}{3}(a_{\rm NL} - 1)gg_{\rm in} + 2g^2\right]\frac{\nabla^2\Phi^2}{\mathcal{H}^2} \\
& + \frac{2}{3\Omega_m g^2}\left[B_2(\tau) - \frac{4}{3}gg_{\rm in} - \frac{4}{3}\frac{f^2g^2}{\Omega_m} - g^2 - 3\mathcal{H}^2B_4(\tau)\right]\frac{\partial_i\Phi\partial^i\Phi}{\mathcal{H}^2} \\
& + \frac{2}{3\Omega_m g^2}\left[-3B_2(\tau) + 4gg_{\rm in} - 3\mathcal{H}^2B_3(\tau)\right]\frac{1}{\mathcal{H}^2}\nabla^{-2}\partial_i\partial^j(\partial^i\Phi\partial_j\Phi) \\
& + \frac{8}{3\Omega_m}\frac{\Phi\nabla^2\Phi}{\mathcal{H}^2} + \frac{2}{3\Omega_m g^2}\frac{B_3(\tau)}{\mathcal{H}^2}\partial_i\partial^j(\partial^i\Phi\partial_j\Phi) + \frac{2}{3\Omega_m g^2} \\
\times {} & \frac{B_4(\tau)}{\mathcal{H}^2}\nabla^2(\partial^i\Phi\partial_i\Phi),
\end{aligned} \tag{2.159}$$

With

$$A(\tau) \equiv B_1(\tau) - 2gg_{\rm in} - (10/3)(a_{\rm NL} - 1)gg_{\rm in}, \tag{2.160}$$
$$\alpha(\mathbf{x}) \equiv \nabla^{-2}\left(\partial^i\Phi\partial_i\Phi\right) - 3\nabla^{-4}\partial_i\partial^j\left(\partial^i\Phi\partial_j\Phi\right). \tag{2.161}$$

The density contrast given in equation (2.157) may be expanded in Fourier space,

$$\begin{aligned} \delta(\eta,\mathbf{k}) &= \delta^{(1)}(\eta,\mathbf{k}) + \frac{1}{2}\delta^{(2)}(\eta,\mathbf{k}) \\ &= \delta^{(1)}(\eta,\mathbf{k}) + \int \frac{d^3\mathbf{k}_1 d^3\mathbf{k}_2}{(2\pi)^3}\,\mathcal{K}_{\delta^{(2)}}(\eta,\mathbf{k}_1,\mathbf{k}_2)\delta^{(1)}(\eta,\mathbf{k})\delta^{(1)}(\eta,\mathbf{k})\delta_D(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k})\,. \end{aligned} \tag{2.162}$$

where the kernel $\mathcal{K}_\delta(\eta,\mathbf{k}_1,\mathbf{k}_2)$ is given as a sum of the traditional Newtonian terms and the contribution due to primordial non-Gaussainity [81, 82]

$$\mathcal{K}_{\delta^{(2)}}(\eta,\mathbf{k}_1,\mathbf{k}_2) = \mathcal{K}_\delta^N(\eta,\mathbf{k}_1,\mathbf{k}_2) + \frac{3}{2}\Omega_m\mathcal{H}^2 f_{\rm NL}(\eta,\mathbf{k}_1,\mathbf{k}_2)\frac{g_{\rm in}}{g(\eta)}\frac{k^2}{k_1^2k_2^2}\,, \tag{2.163}$$

where $k^2 \equiv |\mathbf{k}_1+\mathbf{k}_2|^2$ and $\mathcal{K}_\delta^N(\eta,\mathbf{k}_1,\mathbf{k}_2)$ is kernel for the second-order 'Newtonian' contribution, it is given by [81],

$$\mathcal{K}_\delta^N(\eta,\mathbf{k}_1,\mathbf{k}_2) = \frac{3}{4}\frac{\Omega_m}{g^2}\,\mathcal{H}^2\left[B_3(\eta)\frac{(\mathbf{k}\cdot\mathbf{k}_1)(\mathbf{k}\cdot\mathbf{k}_2)}{k_1^2k_2^2} + B_4(\eta)\frac{k^2(\mathbf{k}_1\cdot\mathbf{k}_2)}{k_1^2k_2^2}\right]. \tag{2.164}$$

The second term in equation (2.163) appears due to non-vanishing non-Gaussainity. In a flat matter-dominated (Einstein-de Sitter) universe, where $g(\eta)=1$ and $B_1(\eta)=B_2(\eta)=0$, while $B_3(\eta)\to(5/21)\eta^2$ and $B_4(\eta)\to-\eta^2/14$, equation (2.164) leads to the well known Newtonian kernel from Newtonian perturbation theory [83],

$$\mathcal{K}_\delta^N(\eta,\mathbf{k}_1,\mathbf{k}_2) \equiv \frac{5}{7} + \frac{2}{7}\frac{(\mathbf{k}_1\cdot\mathbf{k}_2)^2}{k_1^2k_2^2} + \frac{1}{2}\frac{\mathbf{k}_1\cdot\mathbf{k}_2\,(k_1^2+k_2^2)}{k_1^2k_2^2}\,. \tag{2.165}$$

The non-linearity parameter is then defined as a collection of terms that do not fit into the traditional Newtonian kernel

$$\begin{aligned} f_{\rm NL}(\eta,\mathbf{k}_1,\mathbf{k}_2) &= \left[\frac{5}{3}(a_{\rm NL}-1)+1-\frac{g}{g_{\rm in}}-\frac{1}{2}\frac{B_1(\eta)}{gg_{\rm in}}\right] - \frac{(k_1^2+k_2^2)}{k^2}\frac{g}{g_{\rm in}} \\ &+ \frac{(\mathbf{k}_1\cdot\mathbf{k}_2)}{k^2}\left[\frac{2}{3}e(\eta)\,\frac{g}{g_{\rm in}}+\frac{1}{2}\frac{g}{g_{\rm in}}+\frac{2}{3}-\frac{1}{2}\frac{B_2(\eta)}{gg_{\rm in}}+\frac{3}{2}\mathcal{H}^2\frac{B_4(\eta)}{gg_{\rm in}}\right] \\ &+ \frac{(\mathbf{k}\cdot\mathbf{k}_1)(\mathbf{k}\cdot\mathbf{k}_2)}{k^4}\left[\frac{3}{2}\mathcal{H}^2\frac{B_3(\eta)}{gg_{\rm in}}-2+\frac{3}{2}\frac{B_2(\eta)}{gg_{\rm in}}\right] \\ &- \frac{(\mathbf{k}\cdot\mathbf{k}_1)(\mathbf{k}\cdot\mathbf{k}_2)}{k^2}\frac{k_1^2+k_2^2}{k_1^2k_2^2}\frac{3}{2}\mathcal{H}^2 f(\eta)\frac{B_3(\eta)}{gg_{\rm in}} - (\mathbf{k}_1\cdot\mathbf{k}_2)\frac{k_1^2+k_2^2}{k_1^2k_2^2}\frac{3}{2}\mathcal{H}^2 f(\eta)\frac{B_4(\eta)}{gg_{\rm in}}\,. \end{aligned} \tag{2.166}$$

In order to obtain equations (2.165) and (2.166) in their present form, we have performed an expansion in orders of $(\mathcal{H}/k_{1,2})\ll 1$ up to terms $(\mathcal{H}/k_i)^2$.

The Fourier transform of galaxy two-point correlation up to non-linear order is defined as

$$\langle \delta(\eta, \mathbf{k})\delta(\eta, \mathbf{k}')\rangle = (2\pi)^3 P_\delta(k)\delta_D(\mathbf{k}+\mathbf{k}')\,, \tag{2.167}$$

where $P_\delta(k) = [P_{\delta,11}(\eta,\mathbf{k}) + 2P_{\delta,12}(\eta,\mathbf{k}) + P_{\delta,22}(\eta,\mathbf{k})]$. Using the Poisson equation or equation (2.158) for $(\mathcal{H}/k) \ll 1$, the gravitational potential, $\Phi(\eta,\mathbf{k})$ is related to the density field according to

$$k^2\Phi(\eta,\mathbf{k}) = 4\pi G a^2 \bar{\rho}\delta^{(1)}(\eta,\mathbf{k}) = \frac{3}{2}H_0^2\Omega_m(1+z)\delta^{(1)}(\eta,\mathbf{k})\,. \tag{2.168}$$

The time dependent part of the growing mode solution to the density perturbation equation is given by

$$D_+(a) = \frac{5}{2}\Omega_{m0}\frac{H(a)}{H_0}\int_0^a \frac{\mathrm{d}\,a'}{(a'H(a')/H_0)^3}\,, \tag{2.169}$$

where Ω_{m0} denotes the present value and $D_+(a)$ is normalized such that, $D(a) = D_+(a)/D_+(a = 1)$, so that $D(1) = 1$, where $D(a)$ is related to the time dependent part of the Bardeen's potential, $g(z) = (1+z)D(z)$,

$$\delta^{(1)}(\mathbf{k},\eta) = \frac{2}{3}\frac{k^2T(k)}{H_0^2\Omega_m}D(z)\Phi(\mathbf{k}) = \mathcal{M}(k)D(z)\Phi(\mathbf{k})\,. \tag{2.170}$$

Using equation (2.170) in equation (2.167) we find that the matter power spectrum at late-time up to first order may be given by

$$P_{\delta^{(1)},11}(k) = \frac{8\pi^2}{25}\frac{[D_+(a=1)]^2}{H_0^4\Omega_m^2}\Delta^2_{\mathcal{R}}(k_p)D^2(z)T^2(k)\left(\frac{k}{k_p}\right)^{n_S(k_p)+\frac{1}{2}\alpha_S\ln\left(\frac{k}{k_p}\right)}\,, \tag{2.171}$$

where α_S is the running index, k_p is a pivot wavenumber, it depends on the observation and n_S is the spectra tilt for the scalar curvature perturbation. The linear power spectrum for the density field may also be normalized by fixing σ_8. σ_8 is a r.m.s density fluctuation smoothed by the spherical top-hat window function of radius 8 Mpc/h, it is given by

$$\sigma_8^2 \equiv \int \mathrm{d}\,\ln k\frac{k^3P_{\delta,11}(k)}{2\pi^2}W^2(kR)\,, \tag{2.172}$$

where

$$W(kR) = 3\left[\frac{\sin(kR)}{(kR)^3} - \frac{\cos(kR)}{(kR)^2}\right], \quad (2.173)$$

with $R = 8$Mpc/h. The next leading order correction to density clustering according to equation (2.167) comes from non-Gaussian contribution, $P_{\delta,12}(k)$ in the form of bispectrum

$$P_{\delta,12}(\mathbf{k}) = \int \frac{d^3k_1}{(2\pi)^3}\mathcal{K}_{\delta^{(2)}}(\eta,\mathbf{k}_1,\mathbf{k}-\mathbf{q}_1)B_\delta(\mathbf{k}',\mathbf{k}_1,\mathbf{k}-\mathbf{k}_1). \quad (2.174)$$

The primordial bispectrum of a local shape (equation (1.68)) is given by

$$B_\Phi(z,k_1,k_2,k_3) = 2f_{NL}\left[P_{\Phi_g}(k_1)P_{\Phi_g}(k_2) + (2\text{perm})\right]. \quad (2.175)$$

The matter-density bispectrum is defined by evolving the primordial bispectrum (equation (2.175)) to the present time

$$B_\delta(z,k_1,k_2,k_3) = 2f_{NL}D^3(z)\mathcal{M}(k_1)\mathcal{M}(k_2)\mathcal{M}(k_3)\left[P_{\Phi_g}(k_1)P_{\Phi_g}(k_2) + (2\text{perm})\right] \quad (2.176)$$

so that the leading order non-Gaussian correction to matter-density power spectrum becomes

$$P_{\delta,12}(z,k) = 4f_{NL}D^3(z)\mathcal{M}(k)\int \frac{d^3k_1}{(2\pi)^3}\mathcal{M}(k_1)\mathcal{M}(|\mathbf{k}-\mathbf{k}_1|)\mathcal{K}_{\delta^{(2)}}(\eta,\mathbf{k}_1,\mathbf{k}-\mathbf{k_1}) \quad (2.177)$$
$$\times P_{\Phi_g}(k_1)\left[2P_{\Phi_g}(k) + P_{\Phi_g}(|\mathbf{k}-\mathbf{k_1}|)\right].$$

The first non-linear correction to the two point correlation for the density field comes from the product of two second order terms, i.e

$$\langle\delta^{(2)}(\mathbf{k})\delta^{(2)}(\mathbf{k}')\rangle \equiv (2\pi)^3P_{\delta,22}(\mathbf{k})\delta_D(\mathbf{k}-\mathbf{k}'). \quad (2.178)$$

where $P_{\delta,22}(\mathbf{k})$ depends on the product of four first order density contrast, hence we use Wicks theorem to simplify further

$$\begin{aligned}\langle\delta^{(1)}(\mathbf{k}_1)\delta^{(1)}(\mathbf{k}_2)\delta^{(1)}(\mathbf{k}_3)\delta^{(1)}(\mathbf{k}_4)\rangle = (2\pi)^6\Big[&P_\delta(k_1)P_\delta(k_3)\delta_D(\mathbf{k}_1+\mathbf{k}_2)\delta_D(\mathbf{k}_3+\mathbf{k}_4) \quad (2.179)\\ &P_\delta(k_3)P_\delta(k_2)\delta_D(\mathbf{k}_1+\mathbf{k}_3)\delta_D(\mathbf{k}_2+\mathbf{k}_4)\\ &+P_\delta(k_2)P_\delta(k_1)\delta_D(\mathbf{k}_1+\mathbf{k}_4)\delta_D(\mathbf{k}_2+\mathbf{k}_3)\Big]\\ &+(2\pi)^3T_\delta(\mathbf{k_1},\mathbf{k_2},\mathbf{k_3},\mathbf{k_4})\delta_D(\mathbf{k_1}+\mathbf{k_2}+\mathbf{k_3}+\mathbf{k_4}).\end{aligned}$$

Substituting equation (2.179) into equation (2.178), we find that it is more convenient to split $P_{\delta,22}(\mathbf{k})$ into the Gaussian part and non-Gaussian part

$$P_{\delta,22} = P_{\delta_G,22}(\mathbf{k}) + P_{\delta_{NG},22}(\mathbf{k}) \tag{2.180}$$

where

$$\begin{aligned} P_{\delta_G,22}(\mathbf{k} &= \int \frac{\mathrm{d}^3k_1}{(2\pi)^3} P_\delta(k_1) P_\delta(|\mathbf{k}-\mathbf{k}_1|)\mathcal{K}_{\delta^{(2)}}(\eta_1,\mu_k,\mathbf{k}_1,\mathbf{k}) \qquad (2.181)\\ &\times\left[\mathcal{K}_{\delta^{(2)}}(\eta_1,\mu_k,\mathbf{k},\mathbf{k}_1) + \mathcal{K}_{\delta^{(2)}}(\eta_1,\mu_k,\mathbf{k},\mathbf{k}-\mathbf{k_1})\right], \\ P_{\delta_{NG},22}(\mathbf{k}) &= \int \frac{\mathrm{d}^3k_1}{(2\pi)^3} \int \frac{\mathrm{d}^3k_2}{(2\pi)^3} \mathcal{K}_{\delta^{(2)}}(\eta\mathbf{k}_1,\mathbf{k}-\mathbf{k}_1)\mathcal{K}_{\delta^{(2)}}(\eta,\mathbf{k}_1,-k-k_2) \qquad (2.182)\\ &\times T_\delta(\mathbf{k}_1,\mathbf{k}-\mathbf{k}_1,\mathbf{k}_2,-\mathbf{k}-\mathbf{k}_2)\,, \end{aligned}$$

where the linear trispectrum, $T_\delta(\mathbf{k}_1,\mathbf{k}-\mathbf{k}_1,\mathbf{k}_2,-\mathbf{k}-\mathbf{k}_2)$, for the density field is defined in terms of the primordial trispectrum of curvature perturbation

$$\begin{aligned} T_\Phi(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4) &= 6g_{NL}P_{\Phi_g}(k_1)P_{\Phi_g}(k_2)P_{\Phi_g}(k_3) + (3\ \text{cylic}) \qquad (2.183)\\ &+\frac{25}{18}\tau_{NL}P_{\Phi_g}(k_1)P_{\Phi_g}(k_2)\left[P_{\Phi_g}(|\mathbf{k_1}+\mathbf{k_3}|) + P_{\Phi_g}(|\mathbf{k_1}+\mathbf{k_4}|)\right] + (11\ \text{cylic}). \end{aligned}$$

The matter-density trispectrum is then obtained by evolving the primordial trispectrum to the present time

$$\begin{aligned} T_\delta(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4) &= 6g_{NL}D^4(z)\mathcal{M}(k_1)\mathcal{M}(k_2)\mathcal{M}(k_3)\mathcal{M}(k_4)\Bigg[P_{\Phi_g}(k_1)P_{\Phi_g}(k_2)P_{\Phi_g}(k_3) \qquad (2.184)\\ &+(3\ \text{cylic})\Bigg] + \frac{25}{18}\tau_{NL}D^4(z)\mathcal{M}(k_1)\mathcal{M}(k_2)\mathcal{M}(k_3)\mathcal{M}(k_4)\\ &\times\Bigg[P_{\Phi_g}(k_1)P_{\Phi_g}(k_2)\left(P_{\Phi_g}(|\mathbf{k}_1+\mathbf{k}_3|) + P_{\Phi_g}(|\mathbf{k}_1+\mathbf{k}_4|)\right) + (11\ \text{cylic})\Bigg]. \end{aligned}$$

2.6 Coordinate Independent Description of the Universe

Our universe may also be conveniently described by some fluid variables which do not depend on a particular set of coordinates [84]. The key ingredient in the formulation is a 4-velocity vector, u^a of the fundamental observers

$$u^a = \frac{dx^a}{d\tau}, \quad u^a u_a = -1\,, \tag{2.185}$$

where τ is the proper time measured along the world lines of the observer, the 4-velocity is time-like from the normalization condition in equation (2.185). Every quantity has an interpretation in terms of observers comoving with u^a. The metric on the hypersurface orthogonal to u^a is given by

$$h_{ab} = g_{ab} + u_a u_b\,, \tag{2.186}$$

where g_{ab} is the spacetime metric. h_{ab} also serves as a projector of a given quantity onto the hypersurface and it satisfies the following conditions

$$h^a_c h^c_b = h^a_b, \;\; h^a{}_a = 3, \;\; h_{ab}u^b = 0. \tag{2.187}$$

Anti-symmetric tensors are projected using the alternating Levi-Civita tensor

$$\eta_{abcd} = 2u_{[a}\epsilon_{b]cd} - 2\epsilon_{ab[c}u_{d]}\,, \;\; \epsilon_{abc}\epsilon^{def} = 3!h_{[a}{}^d h_b{}^e h_{c]}{}^f\,. \tag{2.188}$$

One of the key features of this approach is the decomposition of covariant derivative of u^a into irreducible parts

$$\nabla_b u_a = -A_a u_b + \frac{1}{3}\Theta h_{ab} + \epsilon_{abc}\omega^c + \sigma_{ab}\,. \tag{2.189}$$

where A^a is the acceleration of the fundamental observers, Θ describes the expansion of the fluid lines or the hypersurface in the case of an FLRW spacetime, ω^c is the vorticity, it describes the twisting of the fluid flow lines and σ_{ab} is the shear. Apart from the expansion, every other quantity vanishes in the special case of a highly symmetric FLRW spacetime.

Proper time derivatives and the spatial derivative of a given tensor $J^{a\cdots}{}_{\cdots b}$ are defined as a component parallel to u^a and a component orthogonal to the observer respectively

$$\dot{J}^{a\cdots}{}_{\cdots b} = u^c\nabla_c J^{a\cdots}{}_{\cdots b}\,, \; \mathrm{D}_c J^{a\cdots}{}_{\cdots b} = h_c{}^d h^a{}_e \cdots h_b{}^f \nabla_d J^{e\cdots}{}_{\cdots f}\,. \tag{2.190}$$

Note that $\mathrm{D}_c h_{ab} = 0 = \mathrm{D}_d\epsilon_{abc}$, while $\dot{h}_{ab} = 2u_{(a}\dot{u}_{b)}$ and $\dot{\epsilon}_{abc} = 3u_{[a}\epsilon_{bc]d}\dot{u}^d$. The projected symmetric tracefree (PSTF) parts of vectors and rank-2 tensors are given by

$$V_{\langle a\rangle} = h_a{}^b V_b\,, \; S_{\langle ab\rangle} = \Big\{h_{(a}{}^c h_{b)}{}^d - \frac{1}{3}h^{cd}h_{ab}\Big\}S_{cd}\,. \tag{2.191}$$

The skew part of a projected rank-2 tensor is spatially dual to the projected vector, $S_a =$

$\frac{1}{2}\epsilon_{abc}S^{[bc]}$, and any projected rank-2 tensor may be decomposed

$$S_{ab} = \frac{1}{3}Sh_{ab} + \epsilon_{abc}S^c + S_{\langle ab\rangle}\,, \tag{2.192}$$

where $S = S_{cd}h^{cd}$. The technique for decomposing higher rank tensors is given in [85, 86]. The projected derivative D_a defines a covariant PSTF divergence, $\mathrm{div}V = \mathrm{D}^aV_a\,,\ \mathrm{div}S_a = \mathrm{D}^bS_{ab}$, and a covariant PSTF curl,

$$\mathrm{curl}\,V_a = \epsilon_{abc}\mathrm{D}^bV^c\,,\ \mathrm{curl}S_{ab} = \epsilon_{cd(a}\mathrm{D}^cS_{b)}{}^d\,. \tag{2.193}$$

The PSTF dynamical quantities describe the sources of the gravitational field: the (total) energy density $\rho = T_{ab}u^au^b$, isotropic pressure $p = \frac{1}{3}h_{ab}T^{ab}$, momentum density $q_a = -T_{\langle a\rangle b}u^b$, and anisotropic stress $\pi_{ab} = T_{\langle ab\rangle}$, where T_{ab} is the total energy-momentum tensor. The locally free gravitational field, i.e. the part of the spacetime curvature not directly determined locally by dynamical sources, is given by the Weyl tensor C_{abcd}. It splits into the PSTF gravito-electric and gravito-magnetic fields

$$E_{ab} = C_{acbd}u^cu^d\,,\quad H_{ab} = \frac{1}{2}\epsilon_{acd}C^{cd}{}_{be}u^e\,, \tag{2.194}$$

which provide a covariant description of tidal forces and gravitational radiation. By manipulating the Ricci and Bianchi identities,

$$\nabla_{[a}\nabla_{b]}u_c = R_{abcd}u^d, \quad \nabla^dC_{abcd} = -\nabla_{[a}\Big\{R_{b]c} - \frac{1}{6}Rg_{b]c}\Big\}, \tag{2.195}$$

one arrives at a set of fundamental evolution and constraint equations governing the covariant quantities. Einstein's equations are incorporated via the algebraic replacement of the Ricci tensor

$$R^{ab} = T^{ab} - \frac{1}{2}T_c{}^cg^{ab} + \Lambda g^{ab}\,. \tag{2.196}$$

The resulting equations, in fully nonlinear form and for a general source of the gravitational field, are:

Evolution Equations:

$$\dot{\rho} + (\rho + p)\Theta + \mathrm{div}q = -2A^a q_a - \sigma^{ab}\pi_{ab}\,, \tag{2.197}$$

$$\dot{\Theta} + \frac{1}{3}\Theta^2 + \frac{1}{2}(\rho + 3p) - \Lambda - \mathrm{div}A = -\sigma_{ab}\sigma^{ab} + 2\omega_a\omega^a + A_a A^a\,, \tag{2.198}$$

$$\dot{q}_{\langle a\rangle} + \frac{4}{3}\Theta q_a + (\rho + p)A_a + \mathrm{D}_a p + \mathrm{div}\pi_a = -\sigma_{ab}q^b + \epsilon_{abc}\omega^b q^c - A^b\pi_{ab}\,, \tag{2.199}$$

$$\dot{\omega}_{\langle a\rangle} + \frac{2}{3}\Theta\omega_a + \frac{1}{2}\mathrm{curl}A_a = \sigma_{ab}\omega^b\,, \tag{2.200}$$

$$\dot{\sigma}_{\langle ab\rangle} + \frac{2}{3}\Theta\sigma_{ab} + E_{ab} - \frac{1}{2}\pi_{ab} - \mathrm{D}_{\langle a}A_{b\rangle} = -\sigma_{c\langle a}\sigma_{b\rangle}{}^c - \omega_{\langle a}\omega_{b\rangle} + A_{\langle a}A_{b\rangle}\,, \tag{2.201}$$

$$\dot{E}_{\langle ab\rangle} + \Theta E_{ab} - \mathrm{curl}H_{ab} + \frac{1}{2}(\rho + p)\sigma_{ab} + \frac{1}{2}\dot{\pi}_{\langle ab\rangle} + \frac{1}{6}\Theta\pi_{ab} + \frac{1}{2}\mathrm{D}_{\langle a}q_{b\rangle} \tag{2.202}$$

$$= -A_{\langle a}q_{b\rangle} + 2A^c\epsilon_{cd(a}H_{b)}{}^d + 3\sigma_{c\langle a}E_{b\rangle}{}^c - \omega^c\epsilon_{cd(a}E_{b)}{}^d - \frac{1}{2}\sigma^c{}_{\langle a}\pi_{b\rangle c}$$

$$-\frac{1}{2}\omega^c\epsilon_{cd(a}\pi_{b)}{}^d\,,$$

$$\dot{H}_{\langle ab\rangle} + \Theta H_{ab} + \mathrm{curl}E_{ab} - \frac{1}{2}\mathrm{curl}\pi_{ab} = 3\sigma_{c\langle a}H_{b\rangle}{}^c - \omega^c\epsilon_{cd(a}H_{b)}{}^d \tag{2.203}$$

$$-2A^c\epsilon_{cd(a}E_{b)}{}^c - \frac{3}{2}\omega_{\langle a}q_{b\rangle} + \frac{1}{2}\sigma^c{}_{(a}\epsilon_{b)cd}q^d\,.$$

Constraint Equations:

$$\mathrm{div}\omega = A^a\omega_a\,, \tag{2.204}$$

$$\mathrm{div}\sigma_a - \mathrm{curl}\omega_a - \frac{2}{3}\mathrm{D}_a\Theta + q_a = -2\epsilon_{abc}\omega^b A^c\,, \tag{2.205}$$

$$\mathrm{curl}\sigma_{ab} + \mathrm{D}_{\langle a}\omega_{b\rangle} - H_{ab} = -2A_{\langle a}\omega_{b\rangle}\,, \tag{2.206}$$

$$\mathrm{div}E_a + \frac{1}{2}\mathrm{div}\pi_a - \frac{1}{3}\mathrm{D}_a\rho + \frac{1}{3}\Theta q_a = \epsilon_{abc}\sigma^b{}_d H^{cd} - 3H_{ab}\omega^b \tag{2.207}$$

$$+\frac{1}{2}\sigma_{ab}q^b - \frac{3}{2}\epsilon_{abc}\omega^b q^c\,,$$

$$\mathrm{div}H_a + \frac{1}{2}\mathrm{curl}\,q_a - (\rho + p)\omega_a = -\epsilon_{abc}\sigma^b{}_d E^{cd} - \frac{1}{2}\epsilon_{abc}\sigma^b{}_d\pi^{cd} \tag{2.208}$$

$$+3E_{ab}\omega^b - \frac{1}{2}\pi_{ab}\omega^b\,.$$

The energy and momentum conservation equations are the evolution equations (2.197) and (2.199). The dynamical quantities ρ, p, q_a, π_{ab} in the evolution and constraint equations (2.197)–(2.209) are the total quantities, with contributions from all dynamically significant particle species. We made extensive use of these equations in Chapters three, four and five.

2.7 Covariant equations in Perturbation theory

The irreducible quantities defined in equation (2.189) may be calculated in cosmological perturbation theory based on the line element given in equation (2.1). The easiest way to calculate these quantities is use the matter 4-velocity given in equation (2.6). Using the definition of expansion in terms of u^a, $\Theta \equiv h^{ab}\nabla_a u_b$, we find that in perturbation theory, Θ is given by

$$
\begin{aligned}
{}^{(0)}\Theta &= \frac{3\mathcal{H}}{a}\,, &(2.209)\\
{}^{(1)}\Theta &= -\frac{3\mathcal{H}\Phi}{a} - \frac{3\Psi'}{a} + \frac{\mathrm{D}_k\mathrm{D}^k v}{a}\,, &(2.210)\\
{}^{(2)}\Theta &= -\frac{3}{2}\left(\frac{\mathcal{H}^{(2)}\Psi}{a} + \frac{{}^{(2)}\Psi}{a}\right) + \frac{3}{a}\left(\Phi - 2\Psi\right)\Psi' + \frac{9}{2}\frac{\mathcal{H}\Phi^2}{a} + \frac{\mathrm{D}_k\mathrm{D}^{k(2)}v}{a} &(2.211)\\
&\quad + \frac{3\mathcal{H}}{2a}\mathrm{D}_k v\mathrm{D}^k v + \frac{\mathrm{D}_k v'\mathrm{D}^k v}{a} + \frac{\mathrm{D}_k\Phi\mathrm{D}^k v}{a} + \frac{\mathrm{D}_k\Psi\mathrm{D}^k v}{a}\,.
\end{aligned}
$$

The vorticity, which is defined covariantly as $W_{ab} \equiv h^c_a h^d_b \nabla_{[c}u_{d]}$, in perturbation theory, it is given by

$$
\begin{aligned}
{}^{(0)}\omega_{ij} &= 0\,, &(2.212)\\
{}^{(1)}\omega_{ij} &= 0\,, &(2.213)\\
{}^{(2)}\omega_{ij} &= \frac{1}{2}a\mathrm{D}_{[i}\omega_{j]} + \frac{1}{2}a\mathrm{D}_{[i}v_{j]} + 2a\mathrm{D}_{[i}v\mathrm{D}_{j]}\Psi\,. &(2.214)
\end{aligned}
$$

For the shearing of the matter fluid defined as $\sigma_{ab} \equiv h^c_a h^d_b \nabla_{(c}u_{d)} - \frac{1}{3}\Theta h_{ab}$, we find

$$
\begin{aligned}
{}^{(0)}\sigma_{ij} &= 0\,, &(2.215)\\
{}^{(1)}\sigma_{ij} &= a\mathrm{D}_i\mathrm{D}_j v - \frac{1}{3}a\gamma_{ij}\mathrm{D}_k\mathrm{D}^k v\,, &(2.216)\\
{}^{(2)}\sigma_{ij} &= \frac{1}{2}a^{(2)}h'_{ij} + \frac{1}{2}a\mathrm{D}_{(i}{}^{(2)}v_{j)} + \frac{1}{2}a\left(\mathrm{D}_i\mathrm{D}_j{}^{(2)}v - \frac{1}{3}\mathrm{D}_k\mathrm{D}^{k(2)}v\right) &(2.217)\\
&\quad -2a\Psi\left(\mathrm{D}_i\mathrm{D}_j v + \frac{1}{3}\delta_{ij}\mathrm{D}_k\mathrm{D}^k v\right)\,.
\end{aligned}
$$

Finally the covariant form of the acceleration is defined in terms of the 4-velocity $A_b = u^a \nabla_a u_b$ and in perturbation theory A_b is given by

$$
\begin{aligned}
{}^{(0)}A_i &= 0\,, &(2.218)\\
{}^{(1)}A_i &= \mathrm{D}_i\Phi + \mathcal{H}\mathrm{D}_i v + \mathrm{D}_i v'\,, &(2.219)\\
{}^{(2)}A_i &= \frac{1}{2}{}^{(2)}\left[\omega_i' + {}^{(2)}\omega_i\mathcal{H}\right] - 2\Phi\mathrm{D}_i\Phi + \frac{1}{2}\mathrm{D}_i{}^{(2)}\Phi + \frac{1}{2}\left(\mathcal{H}^{(2)}v_i + {}^{(2)}v_i'\right) &(2.220)\\
&\quad -\mathcal{H}\left(\Phi + 2\Psi\right)\mathrm{D}_i v - 2\Psi'\mathrm{D}_i v + \frac{1}{2}\left(\mathcal{H}\mathrm{D}_i{}^{(2)}v + \mathrm{D}_i{}^{(2)}v'\right)\\
&\quad - \left(\Phi + 2\Psi\right)\mathrm{D}_i v' + \mathrm{D}^k v\mathrm{D}_i\mathrm{D}_k v\,,
\end{aligned}
$$

In the subsequent chapters, we made use of these expressions without going into detail on how they were calculated.

2.8 Conclusion

We have described the evolution of post-inflationary perturbation from the time of decoupling to the present time where dust approximation is valid. This chapter provides us with the basic tools we will need to understand the key questions, this thesis was designed to address. We derived evolution equations for all the perturbed metric variables and provided a solution in each case in the form that would be most convenient for us to use them in the subsequent chapters. We will make extensive use of 1 + 3 covariant equations and the solution to the irreducible quantities in perturbation theory given in Section 2.6 in in chapter 5. The solution to the Bardeen equations and the power spectrum given in Section 2.3 will be used in chapter 5 and chapter 6.

Chapter 3

Evolution of Large Scale Structures on Average

3.1 Introduction

The world looks completely different depending on the scale of description chosen: for example a fluid looks wholly different on an every day scale (say 1 m) as opposed to atomic scale (say 10^{-11}m). The same is true for astronomy and cosmology. The background models in standard cosmology are the FLRW models, which ignore all local structural details. More realistic models include linearised perturbations about this background, in principle representing the largest scale growing structures down to the scale of clusters of galaxies (10's of Mpc) from early times to today; but they do not represent the non-linear smaller scale structures, such as our galaxy or the solar system, at later times. The universe looks far from homogeneous when viewed on any scale of from 1 AU to 1 Mpc, and only approaches statistical homogeneity past 100 Mpc (though even this is in dispute [87, 88]) – see figure. 3.1.

In order to describe the Universe on its largest scales, one has to make approximations that postulate or derive a high degree of symmetries for the metric of space-time. Practically, this means that one wants to calculate the large scale observables using a background geometry, i.e. a geometry that ignores, on average, the details present on small scales and that are not probed by the observables. Such a background is usually found in a FLRW solution to the Einstein Field Equations (EFE). This issue is referred to as the *fitting problem* [41]: what is the best-fit FLRW model to the lumpy Universe? In the standard concordance cosmology, the existence of this background is postulated, and no smoothing mechanism to obtain it from the real lumpy Universe is provided. Despite the success of this approach in fitting the observations there still remains the problem of properly defining the background geometry in relation to the real lumpy universe.

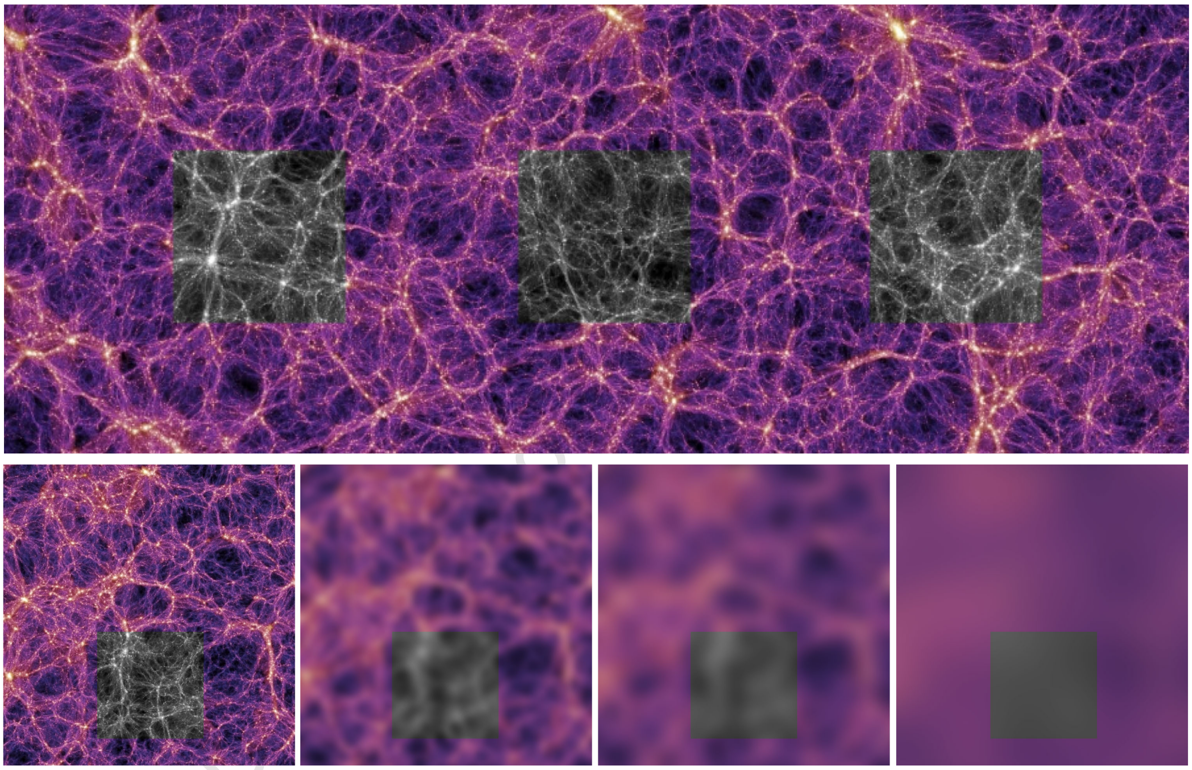

Figure 3.1: Structure in the Millennium simulation [89] (from [90]). Can we describe the universe as smooth on scales of order 150Mpc, shown here in the black and white boxes (top panel)? The averaging problem is shown in the bottom row: how do we go from left to right? Does this process give us corrections to the 'background', or is it the 'background' itself? How does it relate to the 'background' left at the end of inflation?

For example, which set of observers (worldlines) are associated to the background, i.e. are supposed to measure an homogeneous and isotropic Universe? It is clear that such a fitting procedure needs an explicit method of averaging or smoothing. This averaging can be performed with different techniques and on different quantities. For example, one can argue that homogeneity is a spatial property and average quantities that define particular spatial hypersurfaces, such as densities, pressures, or expansion rate of geodesics bundles. In that case, the background is defined by surfaces of constant density, pressure or expansion rate and the average quantities are used to fit the background. Another possibility is to fit the model via averaged observable relations such as the magnitude/redshift or number count/redshift relations; then the average has to be performed in some sense on the past-null cone of the observer. Of course, these procedures will, in general, give different results and the FLRW fitting model reconstructed will depend on which method has been used.

The construction of the background is a crucial issue: if the wrong background is compared to the data, it will imply the existence of a backreaction that may disappear if a better background is chosen. This emphasizes the fact that a gauge choice is always part of an averaging procedure. As such, all the approaches listed and commented on in what follows somehow intend to clarify the (usually unstated) way this is handled in the standard approach to cosmological modeling in most papers on cosmology.

We have three closely related but distinct problems to consider:

Averaging Coarse-graining of structure, such that small-scale effects are hidden to reveal large scale geometry and dynamics.

Backreaction Gravity gravitates, so local gravitational inhomogeneities may affect the cosmological dynamics. How this is calculated depends on the degree of coarse graining.

Fitting How do we appropriately fit an idealized model to observations made from one location in a lumpy universe, given that this 'background' does not in fact exist?

3.2 Averaging/Backreaction

Although we have distinguished between averaging and backreaction in the previous section, both effects are technically related. There are general relativistic effects, simply because they both vanish in the Newtonian limit [91]. In most cases, averaging leads to backreaction effect, as a result we will treat both effects under one subject heading providing distinction where necessary. The averaging/backreaction effects within a geometric theory of gravity arise mainly due to :

1. non-linearity of the equations of general relativity,
2. non-commutativity of the proper time derivative of an average defined on a lumpy space-time.

This effect is best illustrated by considering the relationship between the Einstein tensor and the Christoffel connection is given by $G[g] \sim \partial\Gamma + \Gamma^2$ and that the Christoffel connection relates to the metric as $\Gamma \sim \partial g$. Let us then consider N number of standard candles (i.e it could be Supernovae Ia (SN Ia) data sets that provides evidence for accelerated expansion) within a local lumpy volume V. If we measure the luminosity distances to a particular standard candle as d_i and the corresponding recession velocities as $v_i = cz_i$, where z_i is the redshift of a particular standard candle. The Hubble rate today may be calculated from this set-up as

$$H_0 \equiv \frac{1}{N}\sum_{i=1}^{N}\frac{v_i}{d_i}. \tag{3.1}$$

In the limit of a very large sample, one gets an average over the lumpy volume as $H_0 = \frac{1}{V}\int \frac{v}{d} dV$ [52, 92], where $V = \sqrt{g}$ (i.e the square-root of the determinant of the metric of the lumpy local universe). In relation to the Einstein tensor, this operation may be denoted as $\langle G[g]\rangle$. But we know that distances in cosmology is model dependent, as a result the traditional practice is to use the distance from a large scale smoothed FLRW space-time to fit to cosmological models, this operation may be denoted as $G[\langle g\rangle]$. The point here is that these two operations do not commute

$$G[\langle g\rangle] - \langle G[g]\rangle \sim \langle\Gamma^2\rangle - \langle\Gamma\rangle^2 \neq 0, \tag{3.2}$$

where we have made use of $G[\langle g\rangle]_{ab} = 8\pi G\langle T_{ab}\rangle$. The average general relativistic description of the universe will have an additional term as $\langle G[g]\rangle \sim 8\pi G\langle T\rangle - \mathcal{C}$, where $\mathcal{C} = \langle\Gamma^2\rangle - \langle\Gamma\rangle^2$ is called the backreaction term or the correlation tensor. Notice that the backreaction term $\mathcal{C}$ occupies the same position as the cosmological constant ($G_{ab} = 8\pi G T_{ab} - \Lambda g_{ab}$), a coincidence that boosted renewed interest in cosmological backreaction in relation to dark energy.

3.3 Non-perturbative Approach to Backreaction

One approach is to build a model of the universe 'bottom up'. Advocated by Buchert [43, 93, 94], Zalaletdinov [44, 46], Wiltshire [95–97], Räsänen [58, 59, 98], among others [47, 99–102], these approaches dispute the idea that one needs a background to work from: rather the background model and its dynamics should emerge as a large-scale approximation to a more detailed inhomogeneous model, which can be compared with a FLRW model for the universe assumed *ab initio* as in the standard approach. As explained in previous sections, the field equations for such a model may be expected to be different from those in a standard FLRW model: we want to understand that difference.

Non-perturbative approaches fall into two main categories. One are generic averaging formalisms, which aim to understand the nature of the backreaction terms in general, a bit like deriving and understanding the macroscopic Maxwell Equations. The other approach is to create fully relativistic inhomogeneous or even N-body models by use of simplifying assumptions; then, by comparing observables in these models with their averaged FLRW counterparts one can hope to quantify non-perturbatively the backreaction effect and the magnitude of the fitting problem. Let us consider each proposal in turn, with some of the main attempts in the literature.

3.3.1 Averaging Formalisms

Buchert's approach

Alongside early attempts by [103–107], Buchert [43, 93] builds on the Newtonian averaging by Buchert and Ehlers [42, 91] to provide a bare-bones approach to the problem, concentrating on averaging scalars on spatial hypersurfaces. The kinematic scalar equations for vorticity-free perfect fluid are averaged, to give evolution equations for the averaged expansion and shear scalars. Several authors [48, 59, 86, 93, 108, 109] have generalized this approach to any arbitrary space-time, but we will illustrate with the original Buchert proposal.

Let us assume a dust space-time, and observers and coordinates at rest with respect to the dust. The average of a scalar quantity S may be (non-covariantly) defined as simply its integral over a region of a spatial hypersurface $\mathcal{D}$ of constant proper time divided by the Riemannian volume:

$$\langle S(t,\boldsymbol{x})\rangle_{\mathcal{D}} = \frac{1}{V_{\mathcal{D}}}\int_{\mathcal{D}} \sqrt{\det h}\,\mathrm{d}^3x\; S(t,\boldsymbol{x}) \tag{3.3}$$

Taking the time derivative of Eq. (4.14) yields the commutation relation

$$[\partial_t\cdot, \langle\cdot\rangle_{\mathcal{D}}]S = \langle\Theta S\rangle_{\mathcal{D}} - \langle\Theta\rangle_{\mathcal{D}}\langle S\rangle_{\mathcal{D}}\ , \tag{3.4}$$

where Θ is the expansion of the dust, and we assume the domain is comoving with the dust. The dimensionless volume scale factor is defined as $a_{\mathcal{D}} \propto {V_{\mathcal{D}}}^{1/3}$, which ensures $\langle\Theta\rangle_{\mathcal{D}} = 3\partial_t \ln a_{\mathcal{D}}$. Then, the second derivative of the scale factor is given by the averaged Raychaudhuri equation:

$$3\frac{\ddot{a}_{\mathcal{D}}}{a_{\mathcal{D}}} + 4\pi G\langle\rho\rangle_{\mathcal{D}} = \Lambda + \mathcal{Q}_{\mathcal{D}}, \tag{3.5}$$

where $\mathcal{Q}_\mathcal{D} = \frac{2}{3}\left[\langle\Theta^2\rangle_\mathcal{D} - \langle\Theta\rangle^2_\mathcal{D}\right] - 2\langle\sigma^2\rangle_\mathcal{D}$ is the kinematic backreaction term and $\sigma^2 = \frac{1}{2}\sigma_{ab}\sigma^{ab}$ is the magnitude of the shear tensor. The *non-local* variance of the *local* expansion rate could potential act in similar way as the cosmological constant, causing the average expansion rate to speed up, even if the local expansion rate is slowing down. Even more tantalisingly, if this were the cause of the observed acceleration, the coincidence problem would be solved in the most natural way: as structure forms the variance in the expansion rate grows, as matter coalesces and virialises [110]. This is a truly remarkable possibility in moving from local to non-local quantities on a non-trivial geometry, and is the reason for the recent excitement in the averaging problem.

One can see how this counter-intuitive idea works as follows [111]: If the average scale factor of a universal domain, $a_\mathcal{D}$, can be written as a union of locally homogeneous and isotropic regions, each with its scale factor a_i, then the acceleration of the universal domain $\mathcal{D}$ is given by [53, 98, 112]:

$$a^2_\mathcal{D}\ddot{a}_\mathcal{D} = a_1^2\ddot{a}_1 + a_2^2\ddot{a}_2 + \cdots + \frac{2}{a^3_\mathcal{D}}\sum_{i\neq j} a_i^3 a_j^3 \left(\frac{\dot{a}_i}{a_i} - \frac{\dot{a}_j}{a_j}\right)^2 , \tag{3.6}$$

where a_i represents the locally defined scale factor in the i-th sub-region, and, $a_\mathcal{D} \equiv (a_1^3 + a_2^3 + \cdots)^{1/3}$. Acceleration of the universal domain, $\ddot{a}_\mathcal{D} > 0$, can easily be achieved, for example, for a two disjointed dust filled FLRW sub-region in which one might be expanding while the other is contracting at a time t, i.e $\dot{a}_1 = -\dot{a}_2$ (assuming same sized sub-regions $a \equiv a_1 = a_2$), one obtains $a^2_\mathcal{D}\ddot{a}_\mathcal{D} = 2a^3\left\{\frac{\ddot{a}}{a} + 4\left(\frac{\dot{a}}{a}\right)^2\right\} = \frac{7}{3}\kappa^2 a^3\rho > 0$. Here one easily obtain an acceleration for the universal domain, $\ddot{a}_\mathcal{D} > 0$, even when the two sub-regions are decelerating $\ddot{a}_1 < 0$, $\ddot{a}_2 < 0$; i.e all observers see only deceleration. However, it has been argued [62] that acceleration found in this toy model does not necessary imply that the physical universe is accelerating, since this model have not been shown to satisfy other rigorous observational tests. It also ignores problems due to matching conditions, adding scalars at different space-time points, and so on.

In Buchert's scheme, all tensor contributions appear as scalars in these equations, and are collected into unknown source terms. Of course, the system of scalar equations is not closed, so one has to make an ansatz about the effect of averaging the shear terms; so it is very difficult to say how big the backreaction effect is. One can derive an evolution equation for the averaged shear scalar; but that would be sourced by products of Weyl curvature tensors, amongst other things, and one quickly sees that the system of equations can never close. So the method of averaging only scalars reaches this limitation quickly. This feature can further be understood by considering the integrability condition:

$$\frac{1}{a_{\mathcal{D}}^6}\partial_t(\mathcal{Q}_{\mathcal{D}}a_{\mathcal{D}}^6) + \frac{1}{a_{\mathcal{D}}^2}\partial_t(\langle\mathcal{R}\rangle_{\mathcal{D}}\, a_{\mathcal{D}}^2) = 0 \tag{3.7}$$

where $\langle\mathcal{R}\rangle_{\mathcal{D}}$ is the average local curvature. This coupling between the curvature and the volume scale factor implies that if $\langle\mathcal{R}\rangle_{\mathcal{D}} \sim a_{\mathcal{D}}^{-2}$ as it is in FLRW cosmology, the kinematic backreaction term will scale as $\mathcal{Q}_{\mathcal{D}} \sim a_{\mathcal{D}}^{-6}$, which mimics the behavior of some kind of dark fluid.

Having said that, [113] has suggested that, if the so-called scalar curvature invariants can uniquely characterise any space-time, a scalar averaging scheme can work in general by averaging these invariants. He thus arrives at a complete, closed way of averaging space-time using only scalars. Whether it is practical remains to be seen.

The averaged quantities in Buchert's formalism do not have a clear observational meaning. Nevertheless, it is worth noticing that [58, 59] argues that in a statistically homogeneous and isotropic Universe, the average quantities are approximately the ones that describe observations along the past lightcone. It will be interesting to see if such an argument can be made rigorous.

Zalaletdinov's Macroscopic Gravity

A comprehensive approach to covariantly averaging tensors was initiated and explored by Zalaletdinov [44, 46] and later by others [47, 102]. It is a foundational attempt to average the complete set of Cartan structure equations, in order to define Einstein field equations for the averaged quantities. This approach is directly inspired by the way in which a macroscopic theory of electromagnetism can be obtained from the microscopic Lorentz-Maxwell theory [46]. Once averaged, the 'macroscopic field equations' resemble Einstein's, but with a source term – analogously to the polarization term in macroscopic Maxwell equations.

The core issue of averaging tensors covariantly is managed by using bi-local extensions of tensors, so that they transform as tensors at some point of interest, x, but as scalars in a neighbourhood of the point. Because of that, they can be averaged over that region Σ. The covariant space-time average is defined as [44]

$$\bar{T}_{ab}(x) = \frac{\int_\Sigma \mathcal{A}_a^{a'}(x,x')\mathcal{A}_b^{b'}(x,x')T_{a'b'}(x')\sqrt{-g(x')}d^4x'}{\int_\Sigma \sqrt{-g(x')}d^4x'} \tag{3.8}$$

where $\mathcal{A}_a^{a'}(x,x')$ is the bi-local transport operator. The backreaction generated by the smoothing procedure takes the form of a correlation tensor for the gravitational degrees of freedom which appears as an effective source in the Einstein field equations. Alternatives to this definition, and the resulting averaging scheme were initiated in [47].

Implementing this operation results in a completely covariant smoothing procedure provided the transport operators satisfy certain conditions. The result is formally independent of the averaging scale and as such can be seen as generating a universal description of the collective behaviour of local gravitational degrees of freedom when only their very large scale properties matter (much the same way as thermodynamics encompasses the collective behaviour of particles on very large scales compared to the particles themselves).

This approach makes an extensive use of transport of tensorial quantities along geodesics, but by using the natural parallel transport the metric is invariant so that no smoothed metric is obtained. Hence, it relies on a specific choice of bi-tensor that satisfies a set of differential equations and conditions, in order to imply the correct properties of the average. This bi-tensor is used to evaluate integrals on finite domains, and it is not clear how the formalism is affected by the choice of this bi-tensor [102]. Additionally his averaged Einstein equations rely on some "splitting rules" (see eqns (45) and (48) of [45]) which can be questioned (although they are consistent with an analysis of high frequency gravitational waves, see eqn (68) of the same paper).

In [114–116] it is shown that in a flat FLRW macroscopic background, the correlation tensor is of the form of a spatial curvature, while [117] showed that Zalaletdinov's MG reduces to Buchert's equations with corrections in an appropriate limit. It has further been employed to evaluate the backreaction effect in a perturbed FLRW model [117–119], but the requirement that a FLRW background exists makes this attempt fall under the category described in Sec. 3.4. While the amplitude of backreaction is similar to that obtained by simpler means, the details of using such a covariant approach might be important for an effective fluid description of perturbations at second-order, such as [120].

3.3.2 Other approaches

Another rigorous approach is based on the deformation of the spatial metric of initial data sets along its Ricci flow [121, 122]. In principle, this method is a nice, natural way of smoothing a space-time that can be linked to the standard renormalization group approach of effective field theories [103], but the non-linearity of the Ricci flow equations is a serious complication that can lead to the development of singularities along the flow; that makes its use in a cosmological context particularly difficult, but it might possibly be done using the surgery approach introduced by Perelman [123, 124]. A renormalisation group approach to coarse graining in the very early universe is given in [103, 125–127].

3.3.3 Model building Approach

Timescape Cosmology

Another very original viewpoint called by its instigator the Timescape cosmology, has been proposed and investigated by Wiltshire [95, 97, 128]. It is a brave but contentious attempt to seriously look at the status of binded regions of space-time and their interaction with an expanding cosmological model.

The idea is to separate the Universe into expanding, underdense regions whose boundary are overdense regions enclosing virialized regions (galaxies) such as the one we, as observers, live in. An average is then performed spatially (using Buchert's formalism [94]) to define a reference cosmic background. Interestingly, the amount of backreaction in these models is at most of order a few percent (when normalized as a fraction of the energy density) and is not solely responsible for explaining the apparent cosmic acceleration: the non standard effects principally come from the desynchronization of local clocks (in the virialized regions) with respect to cosmic clocks defined via the average background. Indeed, the gravitational redshift effects imply different ticking rates for clocks inside the voids and in the virialized regions. Wiltshire argues that the effect is cumulative when an average is performed to define the background clocks. A possible interpretation of the model is that the extra redshift effects change the observable relation in the effective FLRW background. As such, the model is not actually accelerating, but the extra redshift account fully for the dimming of supernovae because they appear to be at lower redshift than expected. A detailed discussion of the observational consequences of the Timescape cosmology can be found in [129, 130].

On the other hand, Wiltshire's proposal is more original since rather than simply viewing the problem as one of backreaction of structures on the overall global dynamics, it also recognises that the position of the observer (in virialized structures as opposed to voids) may be important to the fitting problem when the variance in local geometries becomes large. Nevertheless, it suffers from its own problems, among which one is the use of a pure two-zones model to describe the Universe, without any care for the proper junction conditions between the zones (this problem is emphasized in [131, 132]).

Swiss-Cheese Models

The Swiss-Cheese model consists of one or more spherically symmetric vacuum regions, each described by the Schwarzchild region metric, joined across spherical boundaries to an FLRW model [133–137]. This set-up represents a very natural way to model the part of the universe that we see, for example relationship between bubbles at the end of inflation, the boundary of a galaxy and intergalactic space, effect of the expansion of the universe on the motion of planets, etc.

The Swiss-cheese type construction can be adopted to study the effect of inhomogeneities on cosmological observations in a fully non-linear and relativistic manner. The algorithm commonly in use is to start with FLRW metric and cut out comoving holes and fill it with the Schwarzschild metric, while making sure through the matching conditions that the mass in the holes equals the mass that was there before. Hence, Swiss Cheese models do not affect the global dynamics of a lumpy universe, i.e there is no dynamical backreaction effect.

Another more interesting construction is to modify the homogeneous FLRW metric by the introduction of spherical regions of the spherically symmetric Lemaˆitre-Tolman-Bondi (LTB) dust space-time [138–146] and a quasi-spherical Szekeres model can also be introduced [147]. One has to ensure through the matching conditions that the spherical regions of inhomogeneities is comoving and mass compensating, to ensure that the LTB and FLRW regions evolve independently. Thus again there are no dynamical backreaction effects.

There are however significant observational differences from standard cosmology. These space-times model precisely the difference between Weyl and Ricci focussing of null geodesics. The null geodesics in the void regions are focussed only by shear induced by the Weyl tensor. One of the critical problems in this area is the relation between Ricci curvature and Weyl curvature, and Swiss-cheese models are a very interesting way to study this. We refer to other papers for a discussion of these observational effects: see [138–144, 146] for differing viewpoints.

Lindquist-Wheeler type models

All the preceding models relied on the hypothesis that a cosmological fluid can be employed to model the distribution of matter in the Universe. On the contrary, Lindquist-Wheeler type models are a genuine approach at constructing an expanding universe model out of locally static domains. It consists in modelling the Universe by approximately paving its compact spatial sections (topologically homomorphic to S^3) with Schwarzchild domains that stand for the static regions constituting the 'particles' of cosmology (such as galaxies) [148]. Of course, the matching cannot be exact, and shells have to be introduced at the boundaries between the cells. These boundaries then obey equations of motion that produce an overall expanding and recollapsing model that closely mimics a $k = 1$ FLRW Universe.

This approach fundamentally differs from a Swiss-Cheese model in that no reference to a FLRW metric is needed in addition to the static regions: the dynamical properties really emerge from the interaction between the static cells encoded in the motion of their boundaries. As such, the fluid approximation is not required and the fluid-like behaviour only appears when the dynamics is coarse-grained over the detailed, local, structure. An interesting point is that it leads to a solution that is only similar to the equivalent FLRW $k = +1$ solution, but different in the details (for example, the relation between the total mass and the maximum radius is modified).

It has recently been extended interestingly by Clifton and Ferreira [149, 150] in their Archipelagean cosmology. They showed that the optical properties of such a model are quite different from the ones of the "equivalent" FLRW model and can lead to a correction in the fitted value of Ω_Λ of order 10%. However, the solution is not, as such, self consistent: it is only an approximation that neglects the interaction between neighbouring cells, interaction that would result in the deformation of the geometry around each vertex and the appearance of anisotropies. But the approximation is plausible and is worth exploring because it is a neat way to explore the emergence of a collective expanding Universe formed by locally static regions. More recent progress in Lindquist-Wheeler type models my be found in [151, 152].

3.4 Backreaction in the standard model

All these approaches are enlightening, but do not yet result in detailed models that can be directly compared with precision cosmology observations. For that we need to turn to the standard perturbed FLRW models, which enable us to comprehensively investigate backreaction effects in the linear and weak non-linear regime.

3.4.1 Short Wavelength Approximation

The problem associated with the ill-definition of an average was examined by Green and Wald in [153], generalizing earlier work by Burnett [154]. They replaced averages by the notion of weak limits in the weak field approximation, and thereby obtained strong restrictions on back-reaction effects (in this context) through a mathematically precise point limit process. In this approximation, they show that if the small-scale motions of matter inhomogeneities are non-relativistic, the effect of small scale inhomogeneities on large scale dynamics can be written as an effective trace-free (radiation) stress energy tensor, and hence cannot lead to acceleration via a negative active gravitational mass.

However the degree to which this analysis captures the physically relevant degrees of freedom is questionable because of the nature of the ultra-local limiting process, which ignores all details of the actual clustering of matter; but we have seen above that whether backreaction effects are significant or not depends crucially on the nature of that clustering. Unlike the analysis in [120], the Green and Wald method is unable to analyze the interrelationships between different length scales, because it ignores all details of finite length scales. Furthermore, this analysis is based on the work in [154], which centers on handling a singular limit at the origin (a vanishing energy momentum tensor at all points away from the origin has a non-zero limit at the origin) arising through behaviour of the '$x\sin(1/x)$' variety. But that does not realistically represent any real physical matter distribution in cosmology. Indeed it seems likely that the crucial quantity representing this discontinuity (μ_{mnabcs} in [154], μ_{abcdef} in [153]) will vanish for any realistic matter distribution: the kind of ultra-local backreaction mechanism envisaged in these papers will not occur in physical reality. It is not generically the same as the backreaction due to averaging over finite volumes that is the concern of this paper,[1] although it might be a limit of such a mechanism in specific singular geometric circumstances.

3.4.2 Effective Fluid Approach

In order to avoid the divergence problems caused when δ exceeds unity, Baumann *et al* [120] have considered a reorganization of the perturbative expansion, using a coordinate based Euclidean smoothing to separate long and short wavelength modes. Further, as we have discussed, $v^2 \sim (\partial\Phi)^2 \sim \Phi$ in magnitude, and on small scales when $\delta \sim 1$ a natural expansion variable is v^2, provided each spatial derivative reduces the order by v. The field equations are linear in the matter variables, so there is no need to expand δ. Using this, averaging over suitable scales, one can derive effective pressure and densities which, when averaged, obey Newtonian-like equations [155] for the kinetic and potential energies on small scales and proving a viral theorem for local systems imbedded in the expanding universe. From this they argue that backreaction is always small, even in a model with no radiation era, and (in agreement with Green and Wald) cannot generate a negative active gravitational mass ($\rho + 3p \geq 0$ always) . In effect, the potential for a negative gravitational mass to be generated by Buchert's averaging process is vitiated because of the existence of local equilibrium states characterized by their virial theorem.

The main idea behind the effective fluid approach proposed in [120] is to re-write the Einstein equations into the background, forms linear in X, and those non-linear in X:

$$\bar{G}_{ab} + (G_{ab})^{\mathrm{L}}[X] + (G_{ab})^{\mathrm{NL}}[X^2] = T_{ab}\,. \tag{3.9}$$

and to assume that the background equations, $\bar{G}_{ab} = \bar{T}_{ab}$, and the linearized Einstein equations, $(G_{ab})^{\mathrm{L}} = (T_{ab})^{\mathrm{L}}$, are defined in the standard way. Then the Einstein equations may be written in a form that is very similar to the linear equations, $(G_{ab})^{\mathrm{L}} = (\tau_{ab} - \bar{T}_{ab})$, where the effective stress-energy pseudo-tensor τ_{ab} may then be defined as, $\tau_{ab} \equiv T_{ab} - (G_{ab})^{\mathrm{NL}}$. The second part in this process requires that the perturbation on the right hand side of the field

[1] The integrals in [153] are related to definition of weak limits, not to averaging over finite volumes.

equation be performed in orders of the peculiar velocity instead of density, since the density contrast is ill-defined at non-linear scales.

Then each linear term is split into short wavelength modes and the long wavelength modes as, $X = X_\ell + X_s$, and the non-linear splits as,

$$\langle fg \rangle_\Lambda = f_\ell g_\ell + \langle f_s g_s \rangle_\Lambda + \frac{1}{\Lambda^2} \nabla f_\ell \cdot \nabla g_\ell + \dots \ , \tag{3.10}$$

After smoothing, the effective energy momentum pseudo tensor becomes,

$$\langle \tau_{ab} \rangle_\Lambda = \langle \tau_{ab} \rangle^\ell + \langle \tau_{ab} \rangle^s + \langle \tau_{ab} \rangle^{\partial^2} \ . \tag{3.11}$$

The subscripts 's' and 'ℓ' denote the short wavelength and the long wavelength part of each term, and Λ is the cut-off for the effective theory and last term in equation 3.11 stands for high derivative terms that appear after smoothing . The tensor $\tau^a{}_b$ is conserved by virtue of the linearized Bianchi identity, and can be re-written into the form of a fluid with density, pressure and anisotropic stress,

$$\rho_{\text{eff}} = \overline{\langle \tau_{ab} \rangle^s} \tilde{u}^a_\ell \tilde{u}^b_\ell \ , 3p_{\text{eff}} = \overline{\langle \tau_{ab} \rangle^s} \gamma^{ab}_\ell \ , \Sigma_{<ij>\text{eff}} \ \approx \ \overline{\tau}_{<ij>}, \tag{3.12}$$

where $\tilde{u}^a$ is the renormalized matter 4-velocity and overline denotes ensemble average. It was shown in [120] that non-linear terms in τ_{ab} may be re-written in the form of the kinetic energy κ and the potential energy ω and can be organized as $3\bar{p}_{\text{eff}} = \bar{\rho}_m(2\kappa + \omega)$, and its equation of state becomes $\bar{w}_{\text{eff}} \equiv \frac{\bar{p}_{\text{eff}}}{\bar{\rho}_{\text{eff}}} = \frac{1}{3}(2\kappa + \omega)$. The point here is that virial scale decouples from the effective long wavelength expansion of the universe and effective pressure vanishes: $2\kappa + \omega = 0$. The more detailed version of this proof in [120] relies on the fact that within the sub-horizon region, one can safely ignore the expansion of the universe (so one can set $a = 1$) and that the smoothing domain is much larger than the size of the system. Setting $a = 1$ is equivalent to imposing a stationary orbit condition as was done in [156] for a general relativistic version of the virial theorem.

3.4.3 Buchert Averaging and the Standard model

The standard model of cosmology ignores all the complexity of smoothing the space-time and *assumes* that on 'large' scales (say larger than a few Mpc) we can model the universe as homogeneous and isotropic, with linear fluctuations describing structure propagating as smooth fields on this background. On smaller scales we can jump to Newtonian gravity, and model the universe as discrete particles in simulations. Because it is the only model we have where we can calculate anything realistically at all, it is provides a good test arena to study backreaction in detail.

We shall give a comprehensive overview of the issues involved in the subsequent chapters, which illustrate more about the backreaction effect on the Hubble rate, decceleration parameter and the area distance in the standard model.

3.5 Fitting Problem in Cosmological Observations

The process of averaging is one form of fitting of a large-scale smoothed-out model to a more realistic lumpy model of the universe [41]. In this more general process, one starts out a priori with a smoothed out exact FLRW model, represented by a metric $g_{ab}^{(\text{cos})}$, and uses some best-fitting process by which it is adapted to be a background model for the more realistic lumpy models on scales represented by metrics $g_{ab}^{(\text{local})}$, $g_{ab}^{(\text{gal})}$ or $g_{ab}^{(\text{lss})}$. This is the fitting process underlying the standard use of FLRW background models in cosmology.

Averaging is in some respects a fitting process, but does not necessarily correspond to any actual observational procedure. So a key issue is how this all relates to cosmological observations. One can do a null fitting, where one in effect averages astronomical observations. Here one will have to take into account not only the effect of averaging on the geometry and dynamics (as represented by (3.2) but also the effect of lumps on null geodesics and on observational relations.

In the real universe, as pointed out in [56, 157–159], observations take place via null geodesics lying in the empty space-time between galaxies, which are focused only by the curvature actually inside the beam, not the matter that would be there in a completely uniform model. The effect on the observational relations of introducing inhomogeneities into a given background space-time is twofold: it alters the redshift and it changes area distances.

The major modification to the redshift comes from the peculiar velocity, while the area distance is modified mainly by the gravitational lensing effect. Weinbergin 1976 [160] argues that on average, the luminosity distance or the apparent magnification produced by randomly distributed masses is exactly the same as that in a homogeneous universe of equal mean density. This is based on the assumption that photon number is conserved in an expanding universe, hence the magnification produced by the clumps is largely cancelled by the demagnification by the under-dense region. Most justification for the use of FLRW space-time approximation rely on this argument, so it is what exploring further. Consider a celestial sphere with a source at (comoving distance) $\chi = 0$ and observer telescope positioned at $\chi = \chi_{\text{obs}}$. The surface area of the celestial sphere is given by $4\pi D_A^2 = 4\pi a_{\text{obs}}^2(t)\chi_{\text{obs}}^2$, where in an exactly FLRW universe, $a_{\text{obs}}^2(t)$ is the cosmological scale factor and D_A is the general expression for area distance. If a source at a comoving cordinate $\chi = 0$ emits N number of photons, and the observer at, χ_{obs}, with a telescope of area A_T catches n number of photons, the ratio of the two photon numbers is given by

$$\frac{4\pi a_{\text{obs}}^2 \chi_{\text{obs}}^2}{A_T} = \frac{N}{n} \tag{3.13}$$

Now if we introduce matter into the sphere, the number of photon measured by the telescope will change due to gravitational lensing, however, if we impose the conservation of photon number, we immediately notice that since the area of the telescope is fixed, the area distance in a universe with matter must be equal to the area distance of an exact FLRW space-time.

Weinberg result (equation 3.13) is not scale independent, within the strong field region, caustics can lead to a non-vanshing gravitational lensing effect [161]. Also it has been shown in [162–165] that non-linear inhomogeneities of the form (δ^2) lead to more photon in an inhomogeneous universe than in a homogeneous universe without violating the conservation of photon number.

$$\frac{n_{\text{obs}}}{n_{\text{FLRW}}} = \frac{r_{\text{obs}}^2}{r_{\text{FLRW}}^2} \geq 1 \tag{3.14}$$

On super-Hubble scales, contribution from non-lninear inhomogeneities of the form δ^2 is almost vanishing, hence equation (3.13) remains valid. However, light from 'point sources' such as supernovae is observed with a beam width of order of the sources' size – typically less than 1 AU. Such a beam probes matter and curvature distributions that are very different from coarse-grained representations in N-body simulations, which are smoothed on scales much larger than 1 AU. The beam typically travels through unclustered dark matter and hydrogen with a mean density much less than the cosmic mean, and through dark matter mini-halos and hydrogen clouds. Large dark matter halos are rarely encountered directly and so are mainly experienced through their Weyl (tidal) curvature. How observations of many such beams averages this Weyl curvature into the Ricci curvature of the background is not understood.

These effects induce, in particular, a dispersion of the observed SNIa luminosities and hence an extra scatter in the Hubble diagram [166–169]. "Precision cosmology" within the standard approach could be compromised by the effects of lensing on the interpretation of SNIa data – and thus it is crucial to characterise the magnitude of these effects precisely.

3.6 Light Propagation on a General Spactime

From a theoretical point of view, the effects of matter inhomogeneities can be described by the geodesic deviation equation, which describes the evolution of a bundle of geodesics $x^a(\lambda, s)$, where λ is the affine parameter and s labels the geodesics. The past lightcones of the central observer are given by $w =$const, where w is the phase. Then $k_a = \partial_a w$, so these curves are irrotational null geodesics:

$$k^a k_a = 0,\ k^b \nabla_b k_a = 0,\ \nabla_{[a} k_{b]} = 0. \tag{3.15}$$

The connecting vector $\eta^a = \mathrm{d}x^a/\mathrm{d}s$ relates neighbouring geodesics with tangent vector $k^a = \mathrm{d}x^a/\mathrm{d}\lambda$ to an arbitrary reference geodesic of the bundle, $\bar{x}^a(\lambda) = x^a(\lambda, 0)$, giving the distance between neighbouring geodesics and hence the physical size and shape of the bundle as one follows it down into the past. The connecting vector can always be chosen such that $k^a \eta_a = 0$ and it evolves according to the geodesic deviation equation:

$$k^a k^b \nabla_a \nabla_b \eta^c = R^c{}_{dab} k^d k^a \eta^b. \tag{3.16}$$

This equation describes the change of shape of the bundle.

For fundamental observers with four-velocity u^a ($u^a u_a = -1$), the redshift is defined by

$$1 + z(\lambda) = \frac{(k_a u^a)_\lambda}{(k_b u^b)_0}, \tag{3.17}$$

where the past-directed photon four-momentum is

$$k^a = (1+z)(-u^a + e^a), \quad e^a u_a = 0,\, e^a e_a = 1. \tag{3.18}$$

Here e^a is the spatial direction of observation, and the spatial direction of propagation is $n^a = -e^a$. The affine parameter increases monotonically along each ray and coincides in an infinitesimal neighborhood of the observation point with the Euclidean distance in the rest frame of $u^a(0)$. Note that while it depends on the 4-velocity $u^a(0)$ of the observer, it does not depend on the 4-velocity $u^b(\bar{x}^a(s))$ of the observed source.

The screen space at each point along a ray is in the observer's rest space and orthogonal to the ray direction. It is spanned by unit vectors n_i^a ($i = 1, 2$), with $g_{ab} n_i^a n_j^b = \delta_{ij}$ and $n_i^a u_a = n_i^a k_a = 0$, that are parallel transported along the ray ($k^a \nabla_a n_i^b = 0$). We can choose the connecting vector to lie in the screen space, so that[2] $\eta^a = \eta_1 n_1^a + \eta_2 n_2^a$. By (3.16)

$$\frac{\mathrm{d}^2}{\mathrm{d}\lambda^2}\eta_i = \mathcal{R}_{ij}\eta^j, \tag{3.19}$$

where $\mathcal{R}_{ij} = R_{abcd}k^d k^c n_i^a n_j^d$ is the screen projection of the Riemann tensor. We write

$$\mathcal{R}_{ij} = \begin{pmatrix} \Phi_{00} & 0 \\ 0 & \Phi_{00} \end{pmatrix} + \begin{pmatrix} -\mathrm{Re}\,\Psi_0 & \mathrm{Im}\,\Psi_0 \\ \mathrm{Im}\,\Psi_0 & \mathrm{Re}\,\Psi_0 \end{pmatrix} \tag{3.20}$$

with

$$\Phi_{00} = -\frac{1}{2}R_{ab}k^a k^b, \quad \Psi_0 = -\frac{1}{2}C_{abcd}m^a k^b m^c k^d, \tag{3.21}$$

where $m^a \equiv n_1^a - \mathrm{i}n_2^a$. The Einstein equations give $R_{ab}k^a k^b = 8\pi G T_{ab}k^a k^b$, where T_{ab} is the total energy-momentum tensor,

$$T_{ab} = (\rho + p)u_a u_b + p g_{ab} + \pi_{ab} + q_a u_b + q_b u_a. \tag{3.22}$$

Here π_{ab} is the anisotropic stress and q_b is the momentum density. (For a perfect fluid $\pi_{ab} = 0 = q_a$; for more general fluids, we can always choose $q_b = 0$, corresponding to the frame where comoving observers see no momentum flux). Then we find

$$\Phi_{00}(\lambda) = -4\pi G[1 + z(\lambda)]^2 \left(\rho + p + 2q_a e^a + \pi_{ab}e^a e^b\right)\Big|_{\bar{x}^\alpha(\lambda)}. \tag{3.23}$$

Note that a cosmological constant Λ makes *no* contribution to Φ_{00}.

The linearity of (3.19) implies that

$$\eta^i(\lambda) = \mathcal{D}^i_j(\lambda)\,\frac{\mathrm{d}\eta^j}{\mathrm{d}\lambda}\Big|_{\lambda=0}, \tag{3.24}$$

where $\mathcal{D}^a_b$ is the Jacobi map. By (3.19), we have the Jacobi matrix equation

$$\frac{\mathrm{d}^2}{\mathrm{d}\lambda^2}\mathcal{D}^i_j = \mathcal{R}^i_k\mathcal{D}^k_j, \quad \eta^i(0) = 0, \quad \frac{\mathrm{d}\mathcal{D}^i_j}{\mathrm{d}\lambda}(0) = \delta^i_j. \tag{3.25}$$

This second-order linear equation can be rewritten as a first-order nonlinear equation:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\mathcal{S}^i_j + \mathcal{S}^i_k\mathcal{S}^k_j = \mathcal{R}^i_j, \tag{3.26}$$

[2]This is the Sachs basis, unique up to transformations $n_i^a \to r_{ij}(\alpha)n_j^a + p_i k^a$, where $r_{ij}(\alpha)$ is a rotation through angle α, and p_i are constants.

by defining the deformation matrix

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\mathcal{D}^i_j = \mathcal{D}^i_k \mathcal{S}^k_j. \tag{3.27}$$

The Jacobi map $\mathcal{D}^i_j$ or equivalently the deformation matrix $\mathcal{S}^i_j$ are the central quantities to describe the distortion of the geodesic bundle. The deformation matrix is usually decomposed as[3]

$$\mathcal{S}^j_i = \begin{pmatrix} \hat{\theta} & 0 \\ 0 & \hat{\theta} \end{pmatrix} + \begin{pmatrix} \hat{\sigma}_1 & \hat{\sigma}_2 \\ \hat{\sigma}_2 & -\hat{\sigma}_1 \end{pmatrix}, \tag{3.28}$$

which defines the optical scalars $\hat{\theta}$ (null expansion) and $\hat{\sigma} \equiv \hat{\sigma}_1 + \mathrm{i}\hat{\sigma}_2$ (null shear). These satisfy the Sachs equations [170]

$$\frac{\mathrm{d}\hat{\theta}}{\mathrm{d}\lambda} + \hat{\theta}^2 + |\hat{\sigma}|^2 = \Phi_{00}, \tag{3.29}$$

$$\frac{\mathrm{d}\hat{\sigma}}{\mathrm{d}\lambda} + 2\hat{\theta}\hat{\sigma} = \Psi_0, \tag{3.30}$$

$$\hat{\theta} \equiv \frac{1}{2}\nabla_i k^i, \quad |\hat{\sigma}^2| \equiv \frac{1}{2}\nabla_i k_j \nabla^i k^j - \hat{\theta}^2. \tag{3.31}$$

The evolution of a ray bundle can then be discussed in terms of *Ricci* focussing (Φ_{00}) and *Weyl* focussing (Ψ_0). The first is generated by matter inside the beam [see (3.23)] while the second derives from matter outside the beam, which can generate a non-vanishing Weyl tensor inside the beam. This distinction leads to the problem raised by Zel'dovich [171] and Feynman [172], and posed in terms of the curvature tensor by Bertotti [173]: if the matter of the universe is clustered in massive galaxies, the bundle propagates almost exclusively in vacuum, or at least in underdense regions, and is thus mostly subject only to the Weyl focussing; by contrast, the cosmological effect is modelled using a homogeneous fluid which generates only Ricci focussing (Weyl tensor vanishes in FLRW space-time). Dyer and Roeder [174, 175] (see also [176–178]) effectively reproduced Zel'dovich's idea and proposed an ansatz to model the propagation in regions with no intergalactic medium. Weinberg [179] disputed this model, arguing that multiple Weyl deflections by individual masses average to mimic the Ricci effect of a fluid with equal average density. Weinberg's argument has been disputed [161, 163]. Later work (e.g. [164, 165]) has not produced a definitive answer to the question, in particular for the case of the very narrow beams involved in SNIa observations.

In order to properly describe a thin geodesic bundle, we need to have a good description of the matter distribution on the scales of the extension of the bundle, and determine how the effect of the inhomogeneities average during the propagation of the bundle, with two main

[3] The null rotation $\hat{\omega}_{ab} = \nabla_{[a}k_{b]}$ vanishes by (3.15).

issues in mind: (1) determining the typical amplitude of the effect and (2) understanding why the description by a smooth universe seems to provide a good description and determine its validity.

3.6.1 Area distance

The Jacobi matrix can be diagonalized by rotations:

$$\mathcal{D}_i^j = r(-\alpha_1) \begin{pmatrix} D_+ & 0 \\ 0 & D_- \end{pmatrix} r(\alpha_2), \tag{3.32}$$

where the shape parameters $D_\pm$ are nonzero almost everywhere. Their absolute values give the semi-axes of the (elliptic) cross-section of the bundle. Once $D_\pm$ are fixed, the angles $\alpha_{1,2}$ are unique at all points where the bundle is non-circular.

The area distance or angular diameter distance is then defined as[4]

$$D_A(\lambda) = \sqrt{\det|\mathcal{D}(\lambda)|} = \sqrt{|D_+(v)D_-(\lambda)|}. \tag{3.33}$$

For a bundle converging at the observer, D_A relates the cross-sectional area A at the source to the opening solid angle at the observer. It depends on the 4-velocity of the observer, but not of the source. From (3.31), the null convergence is

$$\hat{\theta} = \frac{1}{\sqrt{A}} \frac{\mathrm{d}}{\mathrm{d}\lambda} \sqrt{A}, \tag{3.34}$$

and (3.29) becomes

$$\frac{\mathrm{d}^2 D_A}{\mathrm{d}\lambda^2} = -\left(|\hat{\sigma}|^2 - \Phi_{00}\right) D_A. \tag{3.35}$$

For $T_{ab}k^a k^b > 0$ we have $\Phi_{00} \leq 0$ by (3.23), so that $|\hat{\sigma}|^2 - \Phi_{00} \geq 0$. Thus

$$\frac{\mathrm{d}^2 D_A}{\mathrm{d}\lambda^2} \leq 0, \tag{3.36}$$

in any cosmological model, as long as the null energy condition holds, irrespective of the value of the cosmological constant. In order to compare to observations, we need the relation

[4]Note that the terminology 'angular diameter distance' has two interpretations: as an area angular diameter distance, as used here, and as a linear angular diameter distance which are D_+ and D_- [180].

between v and z. We have $\mathrm{d}z/\mathrm{d}v = \mathrm{d}(u^a k_b)/\mathrm{d}\lambda = k^b \nabla_b (u^a k_a) = k^a k^b \nabla_b u_a$. Now

$$\nabla_a u_b = \frac{1}{3}\Theta\,(g_{ab} + u_a u_b) + \sigma_{ab} + \omega_{ab} - u_a A_b, \tag{3.37}$$

where Θ is the expansion, σ_{ab} is the shear, ω_{ab} is the vorticity and A^a is the acceleration. In a universe containing CDM and baryons with four-velocity u^a, and Λ (where radiation is dynamically negligible), we have $A_b = 0$. By Eqs. (3.17) and (3.18), we obtain [181]

$$\frac{\mathrm{d}z}{\mathrm{d}\lambda} = (1+z)^2 \left(\frac{1}{3}\Theta + \sigma_{ab} e^a e^b - A_a e^a\right). \tag{3.38}$$

For any quantity X evaluated along the ray bundle

$$\frac{\mathrm{d}X}{\mathrm{d}\lambda} = (1+z)^2 H_{\|}(z, e^a) \frac{\mathrm{d}X}{\mathrm{d}z}, \tag{3.39}$$

where $H_{\|}$ is the observed expansion rate along the line of sight [181],

$$H_{\|}(z, e^a) = \frac{1}{3}\Theta + \sigma_{ab} e^a e^b - A_a e^b. \tag{3.40}$$

The observed expansion rate is made up of an isotropic expansion monopole, an acceleration dipole and a shear quadrupole.

The set of equations (3.30), (3.35) and (3.38) is the basis for analyzing the effect of inhomogeneities. There are four physical effects induced by inhomogeneities that need to be taken into account:

- *area distance modifications:* due to the difference between Ricci focussing (when the rays move through a uniform medium) and Weyl focussing (due to the tidal effects of nearby matter);
- *redshift adjustment:* due to the differences between the true redshift of a source and its redshift in a smoothed out model;
- *affine parameter distortions:* since inhomogeneities change the relation $v(z)$ (this is actually where Λ affects observational relations);
- *displacement of the light beam:* since the ray path is shifted sideways by inhomogeneities and so experiences different Weyl and Ricci terms at the same v because it is at a different space-time point.

3.6.2 From affine parameter to redshift dependence

The evolution equations (3.30), (3.35) for the null shear and angular distance are in terms of the unobservable affine parameter λ. We need to convert to the observed redshift, using (3.38). Using (3.39), we obtain

$$\begin{aligned} \frac{\mathrm{d}^2 z}{\mathrm{d}\lambda^2} &= k^a k^b k^c \nabla_a \nabla_b u_c \,, \\ &= -\frac{2}{3}(1+z)^3 H_\| \Theta - \frac{1}{3}(1+z)^3 k^c \nabla_c \Theta + (1+z) k^a k^c \nabla_c A_b \\ &\quad + (1+z)^2 H_\| k^b A_b - k^a k^b k^c \nabla_c \sigma_{ab}. \end{aligned} \tag{3.41}$$

The last term can be evaluated by expanding k^c with (3.18) and using $u^b u^c \nabla \sigma_{ab} = -\sigma_{ab} A^b$ and $u^b e^c \nabla \sigma_{ab} = -\sigma_{ab} e^c \nabla_c u^b$. It follows that for any quantity X,

$$\frac{\mathrm{d}^2 X}{\mathrm{d}\lambda^2} = (1+z)^4 H_\|^2 \frac{\mathrm{d}^2 X}{\mathrm{d}z^2} + (1+z)^3 Q \frac{\mathrm{d}X}{\mathrm{d}z}, \tag{3.42}$$

where

$$\begin{aligned} Q &= \frac{2}{3}\Theta H_\| - \frac{1}{3}\dot{\Theta} + A_a A^a + e^a \Big(-\frac{1}{3}\nabla_a \Theta + H_\| A_a - \dot{A}_a - u^b \nabla_a A_b + 2\sigma_{ab} A^b \Big) \\ &+ e^a e^b \Big(\frac{2}{3}\Theta \sigma_{ab} - \dot{\sigma}_{ab} - 2\sigma_a{}^c \sigma_{bc} - 2\omega_a{}^c \omega_{cb} + \nabla_a A_b \Big) - e^c e^b e^a \nabla_c \sigma_{ab} \,. \end{aligned} \tag{3.43}$$

This form of Q is completely general, for any space-time geometry and energy-momentum tensor, and independent of the field equations. It is convenient to write $H_\|^2$ and Q in terms of covariant multipoles, using a covariant generalization of a spherical harmonic expansion [181]. We expand in terms of the trace-free products $e^{\langle a} e^{b \rangle}$, $e^{\langle a} e^b e^{c \rangle}$ and $e^{\langle a} e^b e^c e^{d \rangle}$, and use the spatial covariant derivative D_a. Then we obtain [182]:

$$\begin{aligned} H_\|^2 &= \frac{1}{9}\Theta^2 + \frac{1}{3}A_a A^a + \frac{2}{15}\sigma_{ab}\sigma^{ab} - e^a \Big[\frac{2}{3}\Theta A_a + \frac{4}{5}A^b \sigma_{ab} \Big] \\ &\quad + e^{\langle a} e^{b \rangle} \Big[A_a A_b + \frac{2}{3}\Theta \sigma_{ab} + \frac{4}{7}\sigma^c{}_a \sigma_{bc} \Big] - 2 e^{\langle a} e^b e^{c \rangle} A_a \sigma_{bc} + e^{\langle a} e^b e^c e^{d \rangle} \sigma_{ab} \sigma_{cd}, \end{aligned} \tag{3.44}$$

and

$$
\begin{aligned}
Q \;=\;& \frac{4\pi G}{3}(\rho+3p) - \frac{1}{3}\Lambda + \frac{1}{3}\Theta^2 + \sigma_{ab}\sigma^{ab} - \frac{1}{3}\omega_{ab}\omega^{ab} + A_a A^a - \frac{2}{3}\mathrm{D}^a A_a \\
&+ e^a\left[\frac{1}{3}\mathrm{D}_a\Theta + \frac{2}{5}\mathrm{D}^b\sigma_{ab} + \dot{A}_a - \frac{4}{3}\Theta A_a - \frac{17}{5}A^b\sigma_{ab} - A^b\omega_{ab}\right] \\
&+ e^{\langle a}e^{b\rangle}\left[E_{ab} - 4\pi G\pi_{ab} + 2\Theta\sigma_{ab} + 3\sigma^c{}_a\sigma_{bc} + \omega_a{}^c\omega_{cb} + 2\omega_{ca}\sigma_b{}^c - 2\mathrm{D}_a A_b\right] \\
&+ e^{\langle a}e^b e^{c\rangle}\left[\mathrm{D}_a\sigma_{bc} - A_a\sigma_{bc}\right]
\end{aligned} \tag{3.45}
$$

where we also used the covariant evolution and constraint equations of GR (see [183]). Here $E_{ab} = C_{acbd}u^c u^d$ is the electric part of the Weyl tensor (generalizing the Newtonian tidal tensor). These expressions show clearly the covariant monopole and higher multipoles; for example, the octupole of Q is $\mathrm{D}_{\langle a}\sigma_{bc\rangle} - A_{\langle a}\sigma_{bc\rangle}$. Note that the monopole of $H_\parallel^2$ has contributions from the shear even though the monopole of $H_\parallel$ does not.

Finally we can rewrite the evolution equation (3.35) for the angular distance in terms of redshift:

$$
(1+z)^2 H_\parallel^2 \frac{\mathrm{d}^2 D_A}{\mathrm{d}z^2} + (1+z)\, Q \frac{\mathrm{d}D_A}{\mathrm{d}z} = -\left[4\pi G\Big(\rho + p + 2q_a e^a + \pi_{ab}e^a e^b\Big) + \frac{|\hat{\sigma}|^2}{(1+z)^2}\right] D_A .
$$

This is a completely general and nonlinear equation, valid in any space-time, with any matter content, where $H_\parallel^2$ and Q are given by equations (3.44) and (3.45).

3.7 Conclusion

We have discussed various approaches that have been developed to study the effect of backreaction in cosmology and we have also made a case on why averaging is very crucial in cosmology, especially in a precision cosmology era. We discussed extensively the general theory of light propagation since our only means of extracting information from an evolving universe is through the photons that arrive at the observer's telescope here on earth (Gravitational wave astronomy is still in its infancy and will not replace the traditional techniques that rely on light propagation).

In the subsequent chapters, we will be focussing attention on three key parameters of the standard model: the Hubble rate, deceleration parameter and area distance. We will study these parameters in great detail with the aim quantifying the actual effect of backreaction on them within the standard model. We hope that our approach to the problem will help resolve some of the discrepancies in literature relating to whether backreaction in cosmology is important or not.

Chapter 4

The Influence of Structure Growth on Expansion of the Universe

4.1 Introduction

The late time Universe is not perfectly homogeneous and isotropic, and the overdensities and voids that develop via gravitational collapse make it significantly inhomogeneous. As a result, the notion of a background, highly symmetric, geometry, that is at the core of the standard concordance model needs to be addressed carefully. Specifically, one would like to construct such an average model, suitable to describe the Universe on sufficiently large scales, as a coarse-grained version of the actual distribution of matter and energy in the Universe. In the last decade, this issue has attracted considerable attention in cosmology, in particular through the so-called averaging problem (see [94] and references therein), mostly because it is sometimes believed that it could provide an answer to the Dark Energy problem (see e.g. [98, 110, 184]). Even though this has not been shown to be the case, the physics of averaging are still be worth investigating; the parameters for cosmological concordance are quite sensitive to the backreaction effect – important in the era of precision cosmology.

One method for evaluating this backreaction lies within the standard cosmological model. That is, one can evaluate the backreaction from the perturbations which describe structure formation. At second-order in perturbation theory, this gives rise to corrections to the local Hubble flow. This idea was first investigated in an Einstein-de Sitter model in [184, 185], and followed up in more detail in [51, 52, 92]. This was extended to include the case of a cosmological constant in [86, 108, 186, 187] . On the face of it there appears to be some discrepancy between these results: While [51, 52, 92] found an important effect from backreaction, [62] found much smaller changes to the value of H_0. Our aim in this chapter is to reconcile these results, and present them in a unified framework.

Bearing this idea in mind, in this chapter, we will be using the averaging formalism as developed in [43, 48, 86] in order to estimate the corrections to the averaged local Hubble flow induced by the small scale inhomogeneities in the matter distribution. Such studies have been performed in the past, [51, 52, 86, 92, 186, 187], with different definitions for the Hubble rate, different slicing for averaging, and/or different approaches to perturbation theory. Specifically, on the one hand, [186, 187] defined the averaged Hubble flow in the longitudinal gauge by following the expansion of the coordinate grid adapted to the gauge; this is the expansion associated with the gravitational potential [86]. On the other hand, [51, 52, 86, 92] looked at the expansion of the matter fluid in the comoving synchronous gauge [51, 52, 92] and in the longitudinal gauge [86]. The results and the claims inferred from these results differ from one analysis to the other. To clarify the issue and evaluate precisely the corrections to the concordance model due to the backreaction effect, we propose to compare quantitatively the different definitions and results. We will discuss how a consistent second order treatment of the backreaction effect in perturbation theory changes in the value of the Hubble rate. We will also analyse the intrinsic variance created by the fluctuations that could affect the measurements of H_0.

This chapter is organized as follows: In Sections 4.2 and 4.3, we briefly recall the averaging formalism to be used, and discuss the definitions of the averaged Hubble rate in two different hypersurfaces of interest. In this section, we will also use the gauge invariant formalism developed in [109, 188] to calculate the averaged Hubble rate defined in the fluid frame. To the best of our knowledge, it is the first practical calculation making use of of this gauge invariant formalism to study backreaction effect. The discussion of our results is presented in Section 4.4 and a fitting formula for the variance of the Hubble rate will be given here. We also show that the two classes of definitions can be clearly distinguished. A brief comment on their relevance is also given. Finally, in Section 4.5, we draw some conclusions and discuss future works. We present in Section 4.6 detailed expressions of the various Hubble rates considered in this paper, at second oder in cosmological perturbation theory.

4.2 Equations of Motion

Buchert's averaging formalism [43, 93] (a similar averaging formalism was presented earlier in [104]) and its generalization to arbitrary coordinate systems [48, 59, 108, 187] rely on Einstein equations written in the Arnowitt-Deser-Misner form. Within this formalism, one considers a set of observers defined at each point of the spacetime manifold, and characterized by a unit 4-velocity field, n^a, that is everywhere timelike and future directed, i.e. $n^a n_a = -1$, with zero vorticity. This 4-velocity field induces a natural foliation of spacetime by a continuous set of space-like hypersurfaces locally orthogonal to n^a. The projection tensor field onto these hypersurfaces is defined as $h_{ab} = g_{ab} + n_a n_b$. The line element can then be written with respect to this foliation:

$$ds^2 = -(N^2 - N_i N^i)dt^2 + 2N_i dt dx^i + h_{ij} dx^i dx^j \ , \tag{4.1}$$

where we have introduced respectively the lapse function $N(x^a)$ and the shift 3-vector $N^i(x^a)$. The components of the 4-velocity of the fluid comoving with the coordinate grids is given in relation with the lapse and shift functions as

$n^a = \frac{1}{N}(1, -N^i)$, $n_a = N(-1, 0, 0, 0)$. It is orthogonal to the hypersurface h_{ab} . The intrinsic curvature of the hypersurfaces is given by $\mathcal{R} \equiv h^{ab}\mathcal{R}_{ab}$, where $\mathcal{R}_{ab}$ is the 3-Ricci curvature of the hypersurfaces and the extrinsic curvature (or second fundamental form): $K_{ab} \equiv -h^c_a h^d_b n_{c;d}$.

Here we will consider only the Hamiltonian constraint and the evolution equation for the metric of the spacelike hypersurface.(for the complete set of ADM decomposed Einstein equations see [48]).

$$(\partial_t - \mathcal{L}_{\Sigma_t})\, h_{ij} = -2NK_{ij} \ , \tag{4.2}$$

$$R + K^2 - K_{ij}K^{ij} = 16\pi\epsilon \ , \tag{4.3}$$

where $\epsilon = n^a n^b T_{ab}$ and T_{ab} is the energy momentum tensor defined to include the cosmological constant as $T_{ab} = (\rho + p)u_a u_b + (p + \Lambda/(8\pi G))g_{ab}$. u^a is time-like 4-velocity for the matter field normalized to $u^a u_a = -1$, it is related to n^a through

$$u^a = \gamma(n^a + v^a) \ , \text{ where } \gamma = \frac{1}{\sqrt{1 - v^a v_a}} \ . \tag{4.4}$$

The vector v^a is spacelike and it is orthogonal to n^a ($v^a n_a = 0$).

The non-local, free gravitational field is described by the Weyl tensor. Given a timelike vector this is split into electric and magnetic parts. For example, with respect to n^a these are

$$E^{(n)}_{ab} = C_{acbd} n^c n^d \quad \text{and} \quad H^{(n)}_{ab} = {}^*C_{acbd} n^c n^d, \tag{4.5}$$

where C_{abcd} is the Weyl tensor and ${}^*C_{abcd}$ is its dual. Analogous definitions exist for the vector field, u^a. This means that observers in the frame of the fluid and observers in the coordinate frame observe this electric-magnetic split differently (see [189] for the transformation relations between the two), analogously to boosted observers measuring different electric and magnetic parts of the electromagnetic field. In particular, in certain gravitational fields there may exist a special frame whereby one of these two components vanishes. For example, in so-called silent universes which are not conformally flat, there exists a preferred frame in which the magnetic part of the Weyl tensor is zero – such a frame may be considered the rest-frame of the gravitational field. In spacetimes where this is possible, it is unique as follows from the transformation laws in [189], and there exist (at least) two physical, well motivated, frames: the rest-frame of the fluid, and the rest-frame of the non-local gravitational field.

So far we have defined two different 4-velocities, which according to standard 1 + 3 decomposition of a covariant derivative of 4-vector, will imply defining respectively two expansion rates.

4.2.1 Decomposition of velocities

The covariant derivatives of the two 4−velocities, u^a and n^a, as well as the spacelike relative velocity v^a, may be invariantly decomposed with respect to the coordinate frame, n^a, (this corrects the decomposition presented in [48]; however the expression for the Hubble rate is not affected):

$$\nabla_a n_b = -n_a \dot{n}_b + \frac{1}{3}\xi h_{ab} + \Sigma_{ab} \ , \tag{4.6}$$

$$\begin{aligned}\nabla_a u_b =& -\gamma v^c \left(\gamma^2 \dot{v}_c + \dot{n}_c\right) n_a n_b - \gamma \left(\gamma^2 v^c \tilde{\nabla}_a v_c + \frac{1}{3}\xi v_a + \Sigma_{ac} v^c\right) n_b \\ & + \gamma n_a \left(\gamma^2 v^c \dot{v}_c v_b + \dot{n}_{\langle b \rangle} + \dot{v}_{\langle b \rangle}\right) + \frac{1}{3}\theta h_{ab} + \sigma_{ab} + \omega_{ab} \ ,\end{aligned} \tag{4.7}$$

$$\nabla_a v_b = -\dot{n}_c v^c n_a n_b - n_a \dot{v}_{\langle b \rangle} + \left(\frac{1}{3}\xi v_a + \Sigma_{ac} v^c\right) n_b + \frac{1}{3}\kappa h_{ab} + \beta_{ab} + W_{ab} \ , \tag{4.8}$$

where:

$$\xi \equiv h^{ab}\nabla_a n_b \ , \qquad \Sigma_{ab} \equiv h_a^c h_b^d \nabla_{(c} n_{d)} - \frac{1}{3}\xi h_{ab} \ ,$$

$$\theta \equiv h^{ab}\nabla_a u_b \ , \qquad \sigma_{ab} \equiv h_a^c h_b^d \nabla_{(c} u_{d)} - \frac{1}{3}\theta h_{ab} \ ,$$

$$\omega_{ab} \equiv h_a^c h_b^d \nabla_{[c} u_{d]} \ , \qquad \kappa \equiv h^{ab}\nabla_a v_b \ ,$$

$$\beta_{ab} \equiv h_a^c h_b^d \nabla_{(c} v_{d)} - \frac{1}{3}\kappa h_{ab} \ , \qquad W_{ab} \equiv h_a^c h_b^d \nabla_{[c} v_{d]} \ .$$

Where we have used the notation $\dot{A}_{a\cdots b} = n^c \nabla_c A_{a\cdots b}$, the angle brackets denote symmetric, trace free, and projected with respect to n_a. $\tilde{\nabla}$ denotes the spatially projected covariant derivative.

Here ξ and θ are the expansion rates, while κ is the divergence of the 3-velocity v^a; Σ_{ab}, σ_{ab}

and β_{ab} are the shear, while ω_{ab} and W_{ab} are the vorticity in the respective definitions. Every quantity defined here has a natural interpretation in terms of observers comoving with the fundamental 4−velocity n^a. Provided these definitions are unique and consistent, all related quantities have a direct physical or geometric meaning with respect to the fundamental 4−velocity n^a. Any difference between such 4-velocities will be of $\mathcal{O}(\epsilon)$ in perturbed FLRW case and will disappear in the FLRW limit [189] . Note that:

$$\theta - \gamma\xi = \gamma\kappa\left(1 + \frac{1}{3}\gamma^2 v^2\right) + \gamma^3 v^a v^b \beta_{ab}, \tag{4.9}$$

in FLRW limit, γ and v^2 are of the order $\mathcal{O}(\epsilon^2)$, hence $\theta \sim \xi$. The decomposition of the matter 4-velocity, u^a, is quite unusual, since it is with respect to n^a. One can calculate directly the normal acceleration, vorticity and shear and so on; for us the intrinsic expansion rate is important:

$$\Theta \;=\; \nabla_a u^a = \theta + \gamma v^a\left(\gamma^2 \dot{v}_a + \dot{n}_a\right). \tag{4.10}$$

The following relations between expansion rates will be used later:

$$\xi = \gamma^{-1}(\theta + \theta_B)\ , \tag{4.11}$$

$$\Sigma_{ij} = \gamma^{-1}(\sigma_{ij} + \sigma_{Bij})\ . \tag{4.12}$$

where $\theta_B \equiv -\gamma\kappa - \gamma^3 B$ and for the shear: $\sigma_{Bij} \equiv -\gamma\beta_{ij} - \gamma^3\left(B_{(ij)} - \frac{1}{3}Bh_{ij}\right)$. The tensor B_{ab} is defined as

$$B_{ab} \;\equiv\; \frac{1}{3}\kappa(v_a n_b + v_a v_b) + \beta_{ca}v^c n_b + \beta_{ca}v^c v_b + W_{ca}v^c n_b + W_{ca}v^c v_b\ , \tag{4.13}$$

and its trace is given by $B = \frac{1}{3}\kappa v^2 + \beta_{ab}v^a v^b$.

4.3 Averaged Hubble rates

In general, the average of a scalar quantity $S(t,x)$ may be defined as:

$$\langle S(t,x)\rangle_{\mathcal{D}} \equiv \frac{\int d^3x J S(t,x)}{\int d^3x J}, \tag{4.14}$$

where $J = \sqrt{h}$ is the square root of the determinant of the metric on the hypersurface orthogonal to n^a. The time derivative of Eq. (4.14) leads to a commutation relation [48]

$$[\partial_t \cdot, \langle\cdot\rangle_{\mathcal{D}}]S(t, x^i) = \langle N\xi S\rangle_{\mathcal{D}} - \langle N\xi\rangle_{\mathcal{D}} \langle S\rangle_{\mathcal{D}} \ , \tag{4.15}$$

as is usual in the averaging context.

There are different definitions of the averaged Hubble parameter $H_{\mathcal{D}}$ in the literature, and we would like to be able to compare them in the context of the standard model, up to second-order in cosmological perturbation theory. We shall employ the longitudinal gauge below in order to calculate averages in the concordance model, which fixes our coordinate frame n^a. In the longitudinal gauge the magnetic part of the Weyl tensor vanishes, and the electric part is a pure potential field in the absence of anisotropic stress [86] (see also Appendix 4.6), making this the rest-frame of the gravitational field, or Newtonian frame. In this sense, both n^a and u^a are physically well defined reference frames.

There are different local expansion rates:

- ξ: the expansion of the family of coordinate observers. In the longitudinal gauge we employ below, this is the rest-frame of the gravitational field.
- Θ: The expansion of the fluid, as observed in the fluid rest-frame.
- θ: The expansion of the fluid, as observed in the gravitational rest-frame.

When performing averaging, there are two spatial hypersurfaces of interest:

- $\langle\ \rangle_{\mathcal{D}}$: Averaging in the gravitational frame.
- $\langle\ \rangle_{\mathcal{F}}$: Averaging in the rest frame of the fluid.

Finally, when averaging expansion rates associated with the gravitational field, there is the issue of the time coordinate to use: we can associate the time coordinate t with the proper time of the 'averaged observers', which, when using n^a requires an extra factor of N in the expansion rate [86].

Definitions based on ξ

As argued in [186, 188], one can consider the evolution of the metric of the hypersurface:

$$\partial_t h_{ij} = \frac{2}{3} N h_{ij}\xi + 2N\Sigma_{ij} + D_i N_j + D_j N_i \tag{4.16}$$

and also assume that the dimensionless domain scale factor can be defined as $a_{\mathcal{D}} = \left(\frac{V_{\mathcal{D}}}{V_{\mathcal{D}}}\right)^{1/3}$ where

$V_{\mathcal{D}}$ is the volume of the domain. It is easy to show from equation (4.16), that

$$\begin{aligned} 3H_{\mathcal{D}} &= \frac{\partial_t V_{\mathcal{D}}}{V_{\mathcal{D}}} = \frac{1}{V_{\mathcal{D}}}\int_{\mathcal{D}} \left(N\xi + D_k N^k\right)\sqrt{h}d^3x \\ &= \left\langle N\xi + D_k N^k \right\rangle_{\mathcal{D}} . \end{aligned} \tag{4.17}$$

This definition describes the average expansion of the coordinate grid and says nothing directly about the matter field. It has been used in the recent literature for calculations in the longitudinal gauge [186, 187], in which case it can be interpreted as the expansion rate of the gravitational rest frame. We will find that this definition exhibits some interesting features, such as weak scale dependence of backreaction effects.

With this definition, the averaged Hamiltonian constraint (4.3) becomes:

$$\begin{aligned} 6H_{\mathcal{D}}^2 &= 16\pi G \left\langle N^2\gamma^2\left(\rho + v^2 p\right)\right\rangle_{\mathcal{D}} + 2\left\langle N^2\Lambda\right\rangle_{\mathcal{D}} - \left\langle N^2\mathcal{R}\right\rangle_{\mathcal{D}} - \mathcal{Q}_{\mathcal{D}} + \mathcal{P}_{\mathcal{D}} , \qquad (4.18) \\ \mathcal{Q}_{\mathcal{D}} &\equiv \frac{2}{3}\left\langle N^2\xi^2\right\rangle_{\mathcal{D}} - \frac{2}{3}\left\langle N\xi\right\rangle_{\mathcal{D}}^2 - 2\left\langle N^2\Sigma^2\right\rangle_{\mathcal{D}} , \\ \mathcal{P}_{\mathcal{D}} &\equiv \frac{4}{3}\left\langle N\xi\right\rangle_{\mathcal{D}}\left\langle D_k N^k\right\rangle_{\mathcal{D}} + \frac{2}{3}\left\langle D_k N^k\right\rangle_{\mathcal{D}} , \end{aligned}$$

where $\mathcal{Q}_{\mathcal{D}}$ is the usual backreaction term and $\mathcal{P}_{\mathcal{D}}$ is an additional backreaction term which arises because of the inclusion of the shift parameter N^k. This definition was used in [186, 187]. One can also choose to define the Hubble factor without the lapse function as $3H_{\mathcal{D}} = \langle\xi\rangle_{\mathcal{D}}$ and the corresponding averaged Hamiltonian constraint becomes

$$\begin{aligned} 6H_{\mathcal{D}}^2 &= 16\pi G \left\langle \gamma^2\left(\rho + v^2 p\right)\right\rangle_{\mathcal{D}} + 2\left\langle\Lambda\right\rangle_{\mathcal{D}} - \left\langle\mathcal{R}\right\rangle_{\mathcal{D}} - \mathcal{Q}_{\mathcal{D}}, \qquad (4.19) \\ \mathcal{Q}_{\mathcal{D}} &\equiv \frac{2}{3}\left\langle\xi^2\right\rangle_{\mathcal{D}} - \frac{2}{3}\left\langle\xi\right\rangle_{\mathcal{D}}^2 - 2\left\langle\Sigma^2\right\rangle_{\mathcal{D}} . \end{aligned}$$

Definitions based on Θ

Assuming all types of matter follow the same 4-velocity, the local expansion of the matter is given by Θ. If we average this on spatial surfaces orthogonal to u^a, we have $3H_{\mathcal{D}} = \langle\Theta\rangle_{\mathcal{F}}$. This definition is equivalent to that studied in [52, 92, 190], and is the same as the expansion of the coordinates if we choose the synchronous gauge. The equations in that case are well known and presented in [94]. This choice might be most natural for supernova observations, where the observer here on earth is assumed comoving with the source.

Definitions based on θ

A final definition of the expansion we consider is given by θ: the derivative of the matter observers worldline projected into the rest-space of the gravitational frame. This was introduced in [48, 86] as a way of recognising the fact the rest-frame of the matter before and after averaging are not the same. Hence, a useful definition of the average Hubble factor is: $3H_{\mathcal{D}} = \langle N\theta\rangle_{\mathcal{D}}$ .This will lead to the following averaged Friedmann's equation:

$$\begin{aligned} 6H_{\mathcal{D}}^2 &= 16\pi G\left(\left\langle\gamma^4N^2\rho\right\rangle_{\mathcal{D}} + \left\langle\gamma^2(\gamma^2-1)N^2p\right\rangle_{\mathcal{D}}\right) + 2\Lambda\left\langle N^2\gamma^2\right\rangle_{\mathcal{D}} \\ &\quad - \left\langle\gamma^2N^2\mathcal{R}\right\rangle_{\mathcal{D}} - \mathcal{Q}_{\mathcal{D}} + \mathcal{L}_{\mathcal{D}}\ , \\ \mathcal{Q}_{\mathcal{D}} &\equiv \frac{2}{3}\left(\left\langle(N\theta)^2\right\rangle_{\mathcal{D}} - \langle N\theta\rangle_{\mathcal{D}}^2\right) - 2\left\langle N^2\sigma^2\right\rangle_{\mathcal{D}}\ , \\ \mathcal{L}_{\mathcal{D}} &\equiv 2\left\langle N^2\sigma_B^2\right\rangle_{\mathcal{D}} - \frac{2}{3}\left\langle(N\theta_B)^2\right\rangle_{\mathcal{D}} - \frac{4}{3}\left\langle N^2\theta\theta_B\right\rangle_{\mathcal{D}}\ . \end{aligned} \tag{4.20}$$

where $\sigma^2 = \sigma^i_j\sigma^j_i/2$ and $\sigma_B^2 = \sigma^i_{Bj}\sigma^j_{Bi}/2 + \sigma_{ij}\sigma_B^{ij}$ had being adopted for simplification purposes. In the same vein, we can also consider a definition of average Hubble factor without scaling with lapse function as $3H_{\mathcal{D}} = \langle\theta\rangle_{\mathcal{D}}$, in this case the averaged Friedmann's equation becomes

$$\begin{aligned} 6H_{\mathcal{D}}^2 &= 16\pi G\left(\left\langle\gamma^4\rho\right\rangle_{\mathcal{D}} + \left\langle\gamma^2(\gamma^2-1)p\right\rangle_{\mathcal{D}}\right) + 2\Lambda\left\langle\gamma^2\right\rangle_{\mathcal{D}} - \left\langle\gamma^2\mathcal{R}\right\rangle_{\mathcal{D}} - \mathcal{Q}_{\mathcal{D}} + \mathcal{L}_{\mathcal{D}}\ , \\ \mathcal{Q}_{\mathcal{D}} &\equiv \frac{2}{3}\left(\left\langle(\theta)^2\right\rangle_{\mathcal{D}} - \langle\theta\rangle_{\mathcal{D}}^2\right) - 2\left\langle\sigma^2\right\rangle_{\mathcal{D}}\ , \\ \mathcal{L}_{\mathcal{D}} &\equiv 2\left\langle\sigma_B^2\right\rangle_{\mathcal{D}} - \frac{2}{3}\left\langle(\theta_B)^2\right\rangle_{\mathcal{D}} - \frac{4}{3}\langle\theta\theta_B\rangle_{\mathcal{D}}\ . \end{aligned} \tag{4.21}$$

Notice that the Friedmann part of the Buchert equations averaged on the comoving hypersurface may be recovered from the last two definitions in the limit where $v_i \to 0$, $\gamma \to 1$ and $\langle\theta\rangle_{\mathcal{D}} \to \langle\Theta\rangle_{\mathcal{F}}$ [94]. The equations above [4.18-4.21] contain the standard backreaction term $\mathcal{Q}_{\mathcal{D}}$ and additional backreaction term $\mathcal{L}_{\mathcal{D}}$. The additional $\mathcal{L}_{\mathcal{D}}$ term exist because of the non-varnishing peculiar velocity, v^a. In almost FLRW metric, its contribution will be subdominant $\mathcal{O}(\epsilon^2)$.

4.3.1 Spatial averaging of a perturbed FLRW model

The equations derived in Sec. 4.2 are not closed, but physical information can be extracted from them if we suppose that the Universe is well described by a perturbed FLRW background. We shall consider perturbations in the longitudinal (Poisson) gauge, where the metric may be written as

$$ds^2 = -\left(1 + 2\Phi + \Phi_2\right)dt^2 + a^2\left(1 - 2\Phi - \Psi_2\right)\delta_{ij}dx^i dx^j\ . \tag{4.22}$$

Here, the coordinates are chosen to coincide with the n^a frame such that $n_a = -N\partial_a t$, where the lapse function is $N = \left(1 + \Phi + \frac{1}{2}\Phi_2 - \frac{1}{2}\Phi^2\right)$. We have used the trace-free part of the momentum constraint to set: $\Psi_1 = \Phi_1 = \Phi$ (that is, there is no anisotropic stress at first-order). It was shown in [75, 76, 79, 80] that the vector and tensor modes induced by scalars are subdominant at second order, hence this line element is sufficiently accurate for our purposes. The expression for the peculiar velocity v^i is given in Chapter 2

In this framework, the average quantities on the hypersurface orthogonal to n^a can easily be expanded to second order in perturbation theory, so that one would rather evaluate Euclidean integrals (the metric or the determinant consists only of unit elements) instead of a Riemann integrals (the determinant depends on space time position) [184]:

$$\langle S\rangle_{\mathcal{D}} = S_0 + \langle S_1\rangle + \langle S_2\rangle + \frac{1}{J_0}\langle S_1 J_1\rangle - \frac{1}{J_0}\langle S_1\rangle\langle J_1\rangle . \tag{4.23}$$

where J_0 and J_1 respectively stand for the background and the first order piece of the square root of the determinant of the 3−metric, $\sqrt{h}$, while S_0, S_1 and S_2 are the background, first order and the second order component of any perturbed scalar on the hypersurface orthogonal to n^a. Note the important terms of the form $\langle\ \rangle\langle\ \rangle$ which appear due to the Riemann average – such terms do not appear if we average perturbations on the background only (as in [120] for example).

Frame switching

In order to perform spatial averages on the hypersurface comoving with the matter fluid, i.e. on the hypersurface orthogonal to u^a, while using the coordinate system of the longitudinal gauge presented in Eq. (5.136) we employ the technique developed in [188]. This will allow us to perform an average in a frame which is tilted with respect to the coordinates. We do this because the longitudinal gauge is well defined at second-order, and the solutions up to second-order are known in the case where the cosmological constant is non-zero [74].

Before applying the formalism of [188] to the particular case of interest here, we summarise it and generalise it for our purposes. When defining the average of a spacetime scalar there is considerable freedom in the definition, and this freedom can be used to switch from an average defined in one frame to that in another ([188] used it to define gauge-invariant averages). Consider defining the average of a quantity using a spacetime window function W_Ω:

$$\langle S\rangle_{A_0,r_0} = \frac{\int_{\mathcal{M}_4} d^4x\sqrt{-g}\, S\, W_\Omega(x)}{\int_{\mathcal{M}_4} d^4x\sqrt{-g}\, W_\Omega(x)} , \tag{4.24}$$

where a a suitable window function might be:

$$W_\Omega(x) = \delta(A(x) - A_0)H(r_0 - B(x)) . \tag{4.25}$$

In this definition of the window function, H is the Heaviside step function and $B(x)$ is a positive function of the coordinates with space-like gradient, $\nabla_a B$, and A is a suitable scalar field with time-like gradient, $\nabla_a A$, such that it takes on a constant value A_0 on the hyper-surface of interest. The scalar field A then defines the foliation of spacetime for averaging. The range of integration across the hyper-surface is specified by inserting a step-like definition of the spatial boundary using the function $B(x)$, which is then bounded by a constant positive value $r_0 > 0$.

It was argued in [109, 188] that one can consistently integrate out the coordinate time to define an average of the scalar field S on the hypersurface of constant A by performing a suitable change of coordinates that transforms the integration variable from $t \mapsto \tilde{t}$. This can be achieved by defining $t = f(\tilde{t}, x)$, where the function f is chosen to ensure that the scalar field S transforms as $S(f(\tilde{t}, x), x) = \tilde{S}(\tilde{t}, x)$. By the use of the Jacobian factor $\partial t/\partial\tilde{t}$, the 3-metric is also transformed from h_{ij} into another metric $\tilde{h}_{ij}$. The function f ensures that the scalar field $A(x,t)$ is homogeneous in the tilde frame: $A(f(\tilde{t}, x), x) = \tilde{A}(\tilde{t}, x) \equiv A^{(0)}(\tilde{t})$ (see [188] for details). Inserting this into Eq. (4.24), one finds:

$$\langle S \rangle_{A_0} = \frac{\int_{\Sigma_{A_0}} d^3x \tilde{J} \ \tilde{S}(t_0, x)}{\int_{\Sigma_{A_0}} d^3x \tilde{J}} \ , \tag{4.26}$$

where the tilde quantities are evaluated in the new coordinate system. According to [188], this represents a gauge invariant prescription for the average of a scalar object S on the hypersurface Σ_{A_0} of constant $A = A_0$.

In cosmological perturbation theory, the square root of the determinant of the metric $\tilde{g_{ab}}$ and the scalar field $\tilde{S}$ can be expanded to second order in perturbation theory to give the average of a scalar field in the new coordinate system:

$$\langle S \rangle_{A_0} = S_0 + \left\langle \tilde{S}_1 \right\rangle + \left\langle \tilde{S}_2 \right\rangle + \frac{1}{\tilde{J}_0} \left\langle \tilde{J}_1 \tilde{S}_1 \right\rangle - \frac{1}{\tilde{J}_0} \left\langle \tilde{S}_1 \right\rangle \left\langle \tilde{J}_1 \right\rangle , \tag{4.27}$$

where $\tilde{J}_0 = \sqrt{-\tilde{g}_0}$ and $\tilde{J}_1 = \sqrt{-\tilde{g}_1}$ are the background and the first order piece of the square root of the determinant of the metric $\tilde{g}_{ab}$ respectively. The metric $\tilde{g}$ is still a 4-dimensional metric. By making a gauge transformation [63] back to the original coordinates, we obtain:

$$
\begin{aligned}
\langle S\rangle_{A_0} \;=\; & S_0+\langle S_1\rangle+\langle S_2\rangle+\frac{1}{J_0}\langle J_1S_1\rangle-\frac{1}{J_0}\langle S_1\rangle\langle J_1\rangle \\
& -\frac{\dot{S}_0}{\dot{A}_0}\left[\langle A_1\rangle+\frac{1}{J_0}\langle J_1A_1\rangle+\langle A_2\rangle\right]+2\frac{\dot{S}_0}{\dot{A}_0^2}\left\langle A_1\dot{A}_1\right\rangle-\frac{1}{\dot{A}_0}\left[\left\langle A_1\dot{S}_1\right\rangle+\left\langle S_1\dot{A}_1\right\rangle\right] \\
& +\frac{1}{2}\left[\frac{\ddot{S}_0}{\dot{A}_0^2}-3\frac{\ddot{A}_0\dot{S}_0}{\dot{A}_0^3}+2\frac{\partial_t\left(\ln J_0\right)\dot{S}_0}{\dot{A}_0^2}\dot{S}_0\right]\left\langle A_1^2\right\rangle+\left[\frac{\ddot{A}_0}{\dot{A}_0^2}-\frac{\partial_t\left(\ln J_0\right)}{\dot{A}_0}\right]\langle S_1A_1\rangle \\
& +2\frac{\dot{S}_0}{J_0\dot{A}_0}\langle A_1\rangle\langle J_1\rangle-\left[\frac{\ddot{A}_0}{\dot{A}_0^2}-\frac{\partial_t\left(\ln J_0\right)}{\dot{A}_0}\right]\langle S_1\rangle\langle A_1\rangle-\left[\frac{\dot{S}_0\ddot{A}_0}{\dot{A}_0^3}+\frac{\partial_t\left(\ln J_0\right)\dot{S}_0}{\dot{A}_0^2}\right]\langle A_1\rangle^2 \\
& -\frac{\dot{S}_0}{\dot{A}_0^2}\langle A_1\rangle\left\langle\dot{A}_1\right\rangle+\frac{1}{\dot{A}_0}\langle S_1\rangle\left\langle\dot{A}_1\right\rangle . \qquad (4.28)
\end{aligned}
$$

Here $J_0=\sqrt{-g_0}$ and $J_1=\sqrt{-g_1}$ are the background and the first order piece of the square root of the determinant of the four dimensional metric g respectively. This is the major difference between this averaging prescription and the conventional one defined in equation (4.23). Once the scalar variable A is chosen to specify the averaging hypersurface, the above averaging prescription can easily be applied. Eq. (4.28) was first derived in [188], but the authors set the spatial average of a first order scalar quantity $\langle S_1\rangle$ to zero (see Eq. (3.10) in [188]) before performing the gauge transformation, thereby neglecting the terms of the form $\langle S_1\rangle\langle A_1\rangle$, $\langle S_1\rangle\left\langle\dot{A}_1\right\rangle$, etc, which are non-zero and are explicitly scale dependent at second order [86]. We have inserted them as they play an important role in the average of the Hubble rate.

To fix the definition of A in terms of the quantities of the perturbed FLRW background and at the same time fix the foliation of interest, we employ the technique used in [184]. This involves relating the scalar field A to the time, τ, measured by the observers with 4-velocity u^a comoving with the fluid: $u^0\partial_0+u^i\partial_i=\partial_\tau$. The scalar field A can be expanded to second order in perturbation theory, subject to the condition $\tilde{A}(t,x)=A_0(t)+A_1(t,x)+A_2(t,x)\equiv\tau$ [109] to give (using $u^a\nabla_a\tau=1$):

$$
(1-\Phi-\frac{1}{2}\Phi_2+\frac{3}{2}\Phi^2+v_1^kv_{1k})\partial_t\tilde{A}(t,x)+\frac{1}{a^2}(v_1^i+v_2^i)\partial_i\tilde{A}(t,x)=1\ . \qquad (4.29)
$$

We can now calculate the higher order A in terms of Φ_1 and Φ_2 of the perturbed FLRW background. This gives:

$$A_0(t) = t, \tag{4.30}$$
$$A_1(t,x) = \int_0^t \Phi_1 dt, \tag{4.31}$$
$$A_2(t,x) = \frac{1}{2}\int_0^t \Phi_2 dt - \frac{1}{2}\int_0^t \Phi_1^2 dt - \int_0^t v_1^k v_{1k} dt - \int_0^t \frac{1}{a^2} v_1^i \partial_i A_1 dt\ . \tag{4.32}$$

The average Hubble factor calculated using this prescription is given in the Appendix 4.6.

4.3.2 The ensemble average and the variance

With the tools developed in Section. 4.3.1, we have performed a consistent second order perturbative expansion of the Riemann average defined in Sec. 4.2.1 to obtain a corresponding Euclidean average. Given a specific realisation of a cosmology, we could now calculate spatial averages directly. Alternatively, we can calculate the ensemble average of a given spatial average which will tell us the expectation values of spatially averaged quantities. The ensemble-variance tell us how much we expect that to vary from one domain to another.

The ensemble average of a spatial average may be written as:

$$\overline{\langle X \rangle} = \frac{1}{V}\int d^3x W(x/R_{\mathcal{D}})\overline{X(\boldsymbol{x})}, \tag{4.33}$$

where the overbar denotes an ensemble average. We have specified the domain though the window function W. The Euclidean volume of the spatial domain of averaging $\mathcal{D}$ is then given by: $V = \int d^3x W(x/R_{\mathcal{D}})$ which in the case of a Gaussian window function which we mostly employ is $V = 4\pi R_{\mathcal{D}}^3 \int_0^\infty y^2 W(y) dy = (2\pi)^{3/2} R_{\mathcal{D}}^3$ for any $R_{\mathcal{D}}$. The inverse Fourier transform of this window function reads: $W(kR_{\mathcal{D}}) = \frac{1}{V}\int d^3x W(x/R_{\mathcal{D}})\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}$. The Fourier and the inverse Fourier transforms of any scalar quantity Φ are given as

$$\Phi(\boldsymbol{x}) = \frac{1}{(2\pi)^{3/2}}\int d^3k \Phi(\boldsymbol{k}) e^{i\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{4.34}$$
$$\Phi(\boldsymbol{k}) = \frac{1}{(2\pi)^{3/2}}\int d^3x \Phi(\boldsymbol{x}) e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}. \tag{4.35}$$

For statistically homogenous Gaussian random variables, we have: $\overline{\Phi(\boldsymbol{k})} = 0$, and the power spectrum of Φ is defined by

$$\overline{\Phi(\boldsymbol{k})\Phi(\boldsymbol{k}')} = \frac{2\pi^2}{k^3}\mathcal{P}_\Phi(k)\delta(\boldsymbol{k}+\boldsymbol{k}'). \tag{4.36}$$

Assuming scale-invariant initial conditions from inflation, this is given by

$$\mathcal{P}_\Phi(z,k) = \left(\frac{3\Delta_\mathcal{R}}{5g_\infty}\right)^2 g(z)^2 T(k)^2 \tag{4.37}$$

where $T(k)$ is the normalised transfer function, $\Delta^2_\mathcal{R}$ is the primordial power of the curvature perturbation [34], with $\Delta^2_\mathcal{R} \approx 2.41 \times 10^{-9}$ at a scale $k_{CMB} = 0.002\text{Mpc}^{-1}$ (for the definitions of $g(z)$ and g_∞, please see equation 2.51 in Appendix 4.6).

It is not difficult to notice from the equations displayed in the appendix that most of the terms we are dealing with are scalars which schematically appear in the form $\partial^m\Phi(x)\partial^n\Phi(x)$ where m and n represent the number of derivatives (not indices), such that $m+n$ is even so that there are no free indices. (For example, $\partial_i\Phi\partial^2\partial^i\Phi$ has $m=1$ and $n=3$.) Then from the results of [86], the ensemble average of these kind of terms, if a Gaussian window function is assumed, is given by:

$$\overline{\langle\partial^m\Phi(x)\partial^n\Phi(x)\rangle} = \frac{(-1)^{(m+3n)/2}}{2\pi^2}\int dk\, k^{m+n-1}k^3\mathcal{P}_\Phi(k). \tag{4.38}$$

Using $\Phi = g(t)\Phi_0(x)$, $g(t)$ being the growth suppression factor and $\Phi_0(x)$ the spatial dependent initial condition (see the Appendix), the terms that appear with a time derivative of the gravitational potential can be re-written to pull out the time component before evaluating the ensemble average:

$$\dot\Phi(t,x) = -\left(1+z\right)H\frac{d\ln g}{dz}\Phi(t,x). \tag{4.39}$$

For the details of the calculation of the ensemble average of the inverse laplacian appearing the second order Bardeen potential refer to [86].

The ensemble variance in the Hubble factor is given by

$$\text{Var}[H_\mathcal{X}] = \overline{H^2_\mathcal{X}} - \overline{H}^2_\mathcal{X}\ , \tag{4.40}$$

where $H_\mathcal{X}$ can be any definition of averaged expansion rate we are studying. With this definition, it is easy to see that pure second order contributions drop out of the variance, so that only terms that are quadratic in first order quantities remain.

4.4 Results and Discussion

We shall now investigate the expectation values of the different average Hubble rates, along with their variances. For this we will consider an Einstein-de Sitter model, and a standard concordance model. We shall use length scales intrinsic to the model as reference points for averaging: these scales are the equality scale, $k_{\rm eq}^{-1}$, and the Hubble scale, $k_{\rm H}^{-1}$:

$$\begin{aligned} k_{\rm eq} &\approx 7.46\times 10^{-2}\Omega_0 h^2 {\rm Mpc}^{-1}, \\ k_{\rm H} &\approx \frac{h}{3000}{\rm Mpc}^{-1}, \end{aligned} \tag{4.41}$$

where Ω_b and Ω_0 are the baryon and total matter contributions today and $H_0 = 100\,h\,{\rm kms}^{-1}{\rm Mpc}^{-1}$. We shall also use two models for comparison. The first is an Einstein-de Sitter model with $h = 0.7$ and 5% baryon fraction (WMAP [34] whose estimate as an energy density is given as $\Omega_b \approx 0.046$). This has $k_{\rm eq}^{-1} \simeq 27.9$Mpc. The other model we shall use is the concordance model with $\Omega_0 = 0.26, h = 0.7, f_{\rm baryon} = 0.175$ (this is the WMAP best fit [191]). The key length scales in these model are $k_{\rm eq}^{-1} \simeq 107.2$Mpc and the Hubble scale $k_{\rm H}^{-1} \simeq 4.3$Gpc.

To calculate the integrals we shall use the transfer functions presented in [73]. All lengths scales shown are in Mpc unless otherwise stated. Because some of the integrals have a logarithmic IR divergence, all k-integrals have an IR cut-off set at ten times the Hubble scale, it did not appear explicitly in any of our calculations. Moreover, the position of the IR cut-off does not affect the result, that is one can set the IR cut-off within this range $10 - 10^9$ and the results shown here will remain unchanged. Since the divergence is logarithmic, the result depends very mildly on where the cut-off is set.

4.4.1 Comparison between the different definitions

We can now turn to estimating and comparing the Hubble rates as well as their intrinsic variances as defined above consistently up to second order in perturbation theory. When determining the ensemble average of the Hubble rate, we shall consider two alternatives: a kinematical ensemble average given by $\overline{H}_{\mathcal{D}}$, and a dynamical one, which arises from taking the ensemble average of the Friedman equation: $\sqrt{\overline{H^2_{\mathcal{D}}}}$. We shall find that the difference between these two is large because the variance is large.

Fig. **??** presents the evolution of the averaged Hubble rate as functions of redshift for different definitions of Hubble rate in a ΛCDM and an EdS scenarios. Fig. **??** depicts the values of the same Hubble rates at $z = 0$ as functions of the averaging scale R_D, and Fig. 4.5 shows the scaling of their variances with the averaging scale R_D.

It is clear that the two types of Hubble rates defined here, i.e. those of the gravitational frame, and the ones defined in terms of the physical matter flow can be distinguished as far as the magnitude of their mean and variance are concerned.

First, the ones defined through the local expansion of the observers' worldlines, $\langle\xi\rangle_{\mathcal{D}}$ and $\langle N\xi\rangle_{\mathcal{D}}$ present a very small correction to the FLRW background Hubble rate, which is of

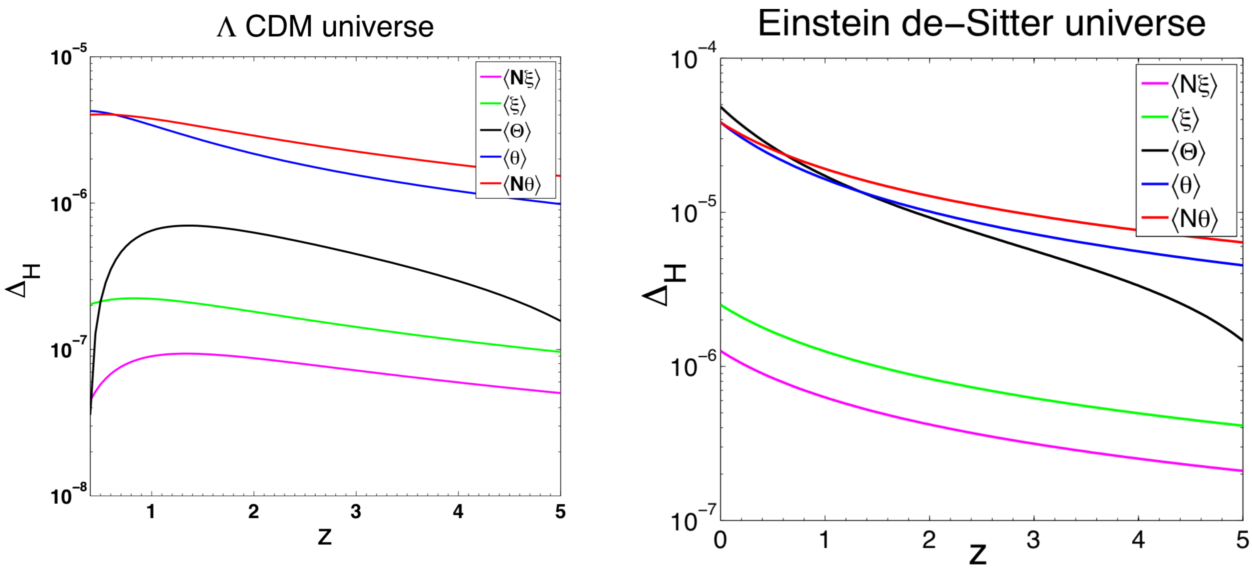

Figure 4.1: Fractional change to the background Hubble rate as a function of redshift for the different definitions of averaged Hubble rates under study. Here we have averaged at the equality scale. Here, $\Delta_H = (\overline{H}_{\mathcal{D}} - H_0)/H_0$.There is an overlap of some of the lines in the plot.

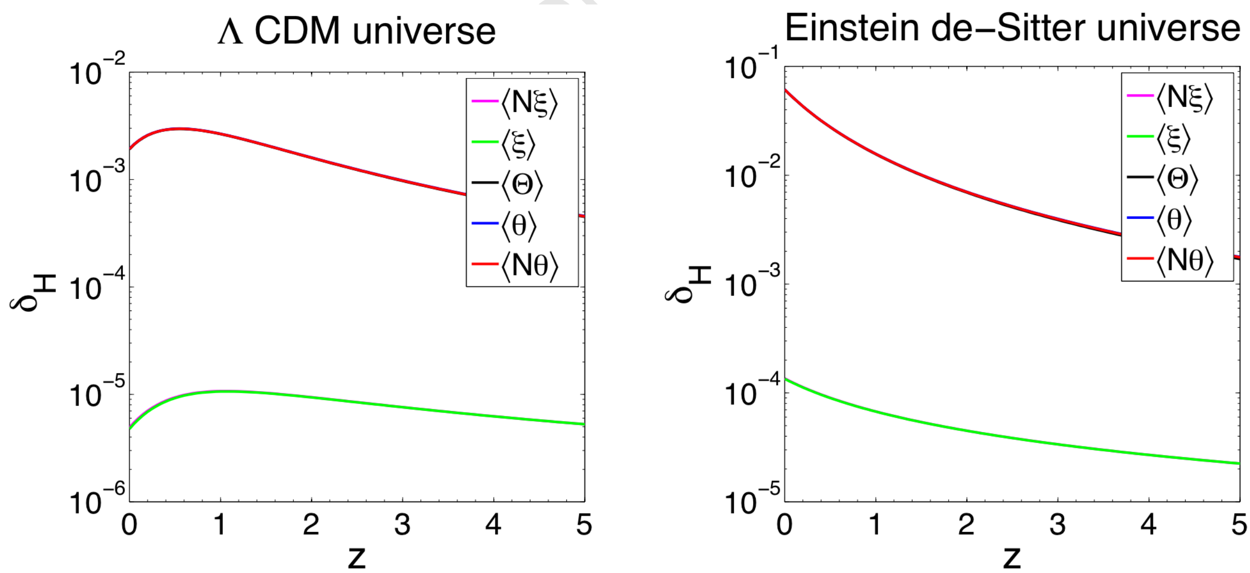

Figure 4.2: Fractional change to the background Hubble rate as a function of redshift for the different definitions of averaged Hubble rates under study. Here we have averaged at the equality scale. Here,$\delta_H = (\sqrt{\overline{H^2_{\mathcal{D}}}} - H_0)/H_0$. Again there is an overlap of some of the lines in the plot.

the order 10^{-5} for ΛCDM and 10^{-4} for an EdS scenario at 20 h^{-1}Mpc . Such a small effect was also reported from a study using numerical simulation [192]. Moreover, they appear to be scale independent when compared with the definition based on θ. The scale dependence is determined by terms with two angle brackets $\langle\Phi\rangle\langle\Phi\rangle$. For the definitions based on ξ, for example, the variance in $\langle\xi\rangle_{\mathcal{D}}$ or $\langle N\xi\rangle_{\mathcal{D}}$ is scale dependent but its scale dependence is suppressed when compared with the definitions based on θ because the dominant term in ξ definition is $\langle\Phi\rangle\langle\Phi\rangle$ which is of the order of 10^{-10} at 20 h^{-1}Mpc while in θ definition the dominant term is $\langle\partial^2\Phi\rangle\langle\partial^2\Phi\rangle$ which is of the order of 10^{-4} at 20 h^{-1}Mpc.

Second, the Hubble rates defined through the local expansion of the matter worldlines systematically present corrections to the background Hubble rate, which is two orders of magnitude bigger than the previous ones, and are indistinguishable from each other, except when the averaging scale is much larger than the equality scale. It is interesting to note that both the values of these averaged Hubble rates and their variances are indistinguishable up to scales of averaging of order >100 Mpc, after which they start to differ. This scale have been interpreted in a previous work [86] as naturally defining the scale of statistical homogeneity of the universe (note that this is the case even for EdS; so it is not simply the equality scale). Around the same scale the expansion rate of the gravitational frame becomes comparable with the others because the peculiar velocity tends to zero.

Finally, let us note that the results are consistent, for a pure CDM Universe, with those found on small scales in [52, 92]. This can be seen on Fig. 4.6.

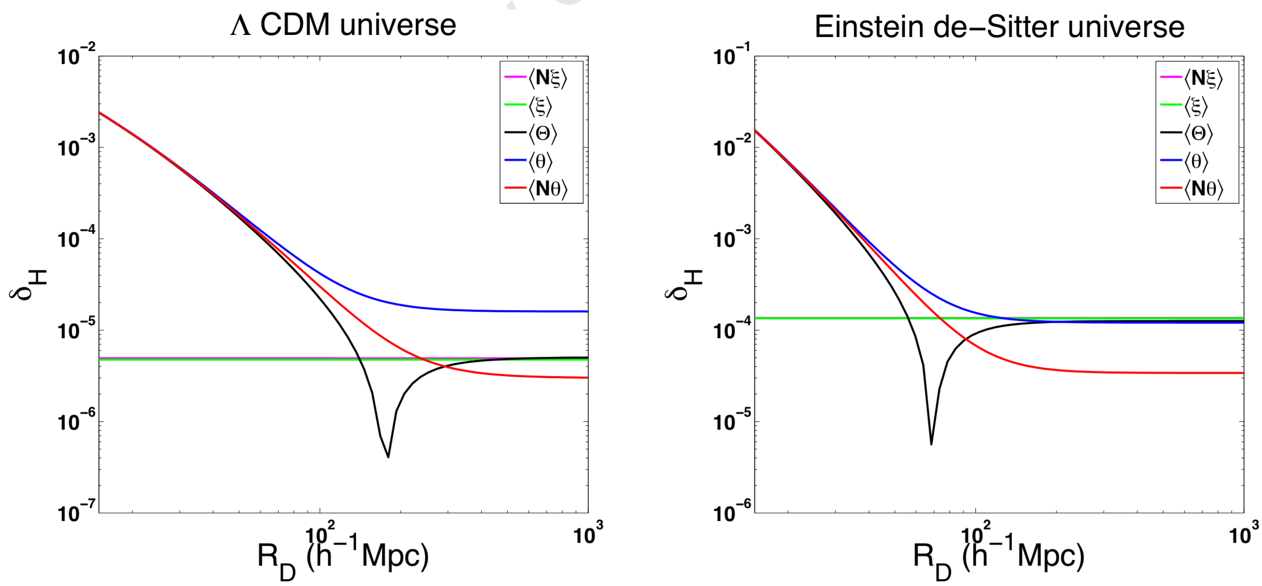

Figure 4.3: Fractional change to the background Hubble rate as a function of the averaging scale for the different definitions of averaged Hubble rates under study. In both models, $\langle\xi\rangle_{\mathcal{D}}$ and $\langle N\xi\rangle_{\mathcal{D}}$ almost coincide and are indistinguishable in the figure. Notice the turn-down in $\langle\Theta\rangle_{\mathcal{F}}$ definition in Einstein de Sitter Universe, this feature has been reported earlier in [52, 92].

This analysis shows that the averaged Hubble rates defined through the expansion of the

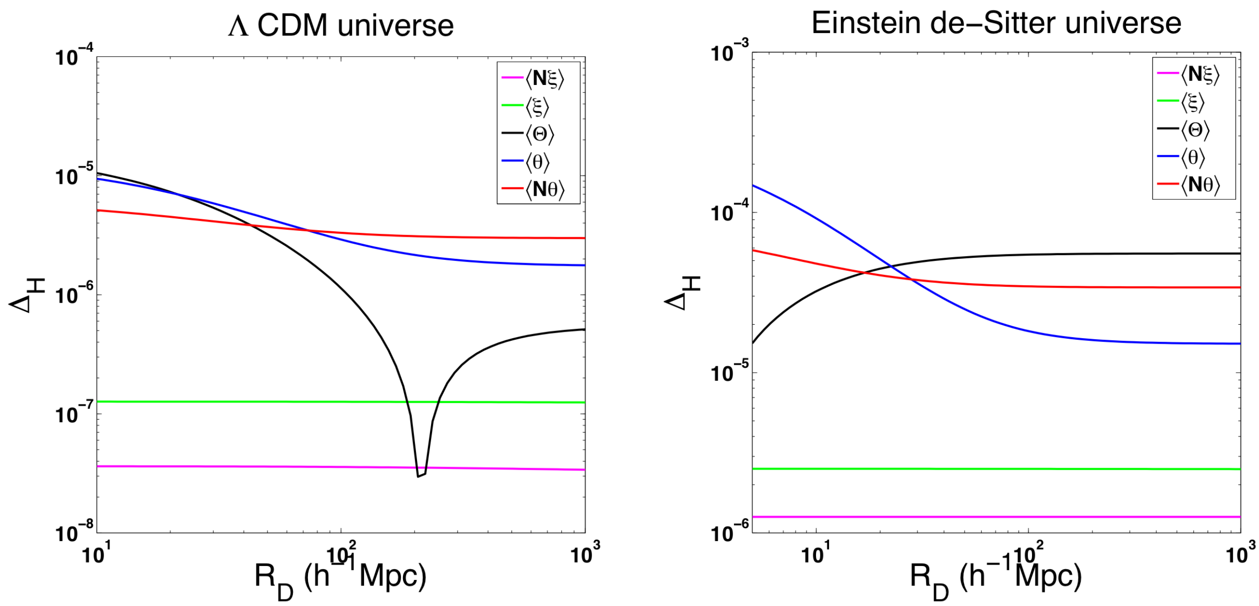

Figure 4.4: Fractional change to the background Hubble rate as a function of the averaging scale for the different definitions of averaged Hubble rates under study. In both models, $\langle \xi \rangle_{\mathcal{D}}$ and $\langle N\xi \rangle_{\mathcal{D}}$ almost coincide and are indistinguishable in the figure. Notice the turn-down in $\langle \Theta \rangle_{\mathcal{F}}$ definition in Einstein de Sitter Universe, this feature has been reported earlier in [52, 92].

Newtonian-like or gravitational frame, as in [186, 187], is not a good tracer of the expansion of the cosmic fluid, except beyond the homogeneity scale. The fluid frame is more relevant for local measurements since it is attached to the matter component of the Universe. The 'gravitational frame', as we have referred to it here, seems useful on much larger scales, which is the situation in [186, 187] in which it was first evaluated – their domain was the Hubble scale.

4.4.2 Fluctuations in the measurement of H_0

We would like to conclude this chapter by addressing the following questions:

- What is the physical relevance of the averaged Hubble rate and its variance?
- Can there be any signature of backreaction in the observations leading to the measurement of H_0?

First, let us note that on sufficiently small scales, such as scales smaller than ~ 100 Mpc, which are the standard scales at which the Hubble rate is evaluated, and in a statistically homogeneous and isotropic Universe, spatial averages are expected to be a good approximation of what happens along the past lightcone on which observations are made. Along the

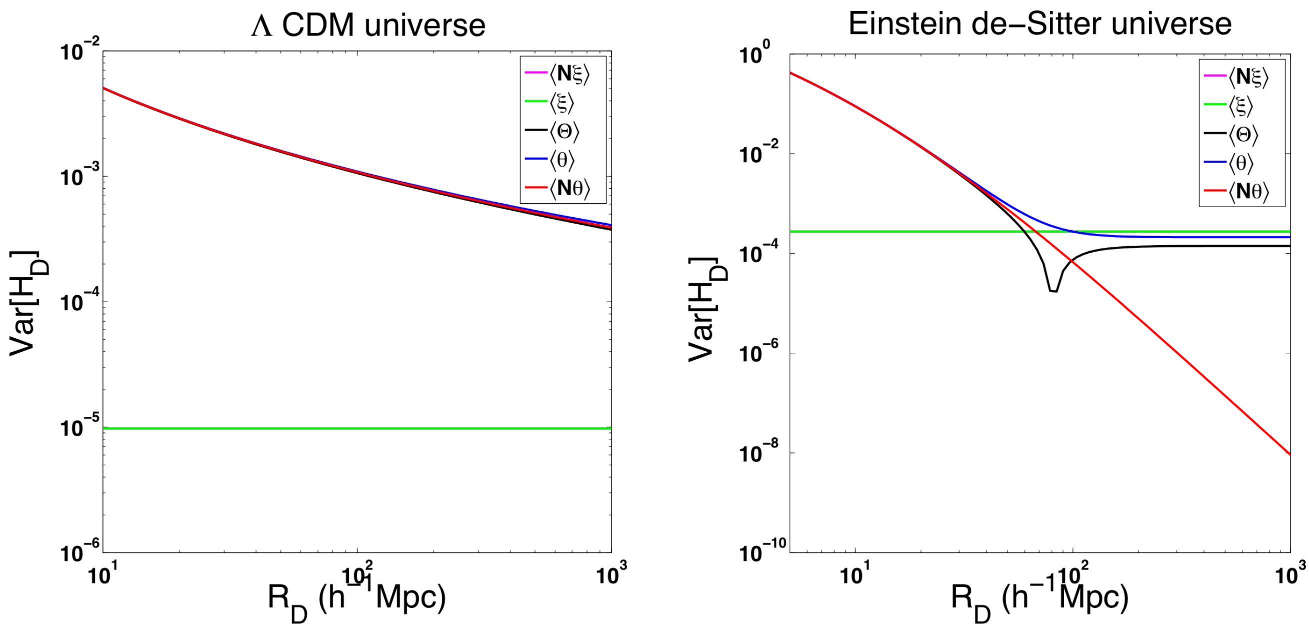

Figure 4.5: Variance in the fractional change to the background Hubble rate as a function of redshift for the different definitions of averaged Hubble rates under study.

past lightcone the monopole contribution to the Hubble rate, which is the one that remains once a full sky average has been performed, is exactly the covariant quantity $\Theta = \nabla_a u^a$ [170].

Hence, our estimate of $H_{\mathcal{D}}$ on a scale $R_{\mathcal{D}}$ can be interpreted as the average Hubble rate in a patch of the local Universe of size $R_{\mathcal{D}}$ as long as this size remains sufficiently small compared with the Hubble scale. Moreover the variance we calculated is the intrinsic dispersion on the measurement of H_0 that comes from the fluctuations in the peculiar velocity of the sources and gravitational potential. In a concordance cosmology, this dispersion appears small, of order 1% at a scale of 100 Mpc, and even less on larger scales, as can be seen on Fig. 4.6.

This is consistent with previous estimates that were based on an estimate of the first order velocity power spectrum [190, 193]. It is due to the fact that the pure second order terms cancel out consistently at second order in our expression of the variance, allowing only contributions of squares of first order quantities. As noted before, this is a similar effect to that found in [52, 92], where the calculations were made in the comoving synchronous gauge, for a pure CDM Universe. Our calculation of $\langle\Theta\rangle_{\mathcal{F}}$ using the gauge-invariant approach of [109] corresponds to a gauge-invariant version of the average expansion rate in the synchronous gauge.

To quantify the backreaction effect on the variance for a large class of cosmological models, we provide a fitting formula for the variance of the Hubble rate (defined via the flow of matter), $Var[H]$, that is accurate to a few percents across the scales of interest:

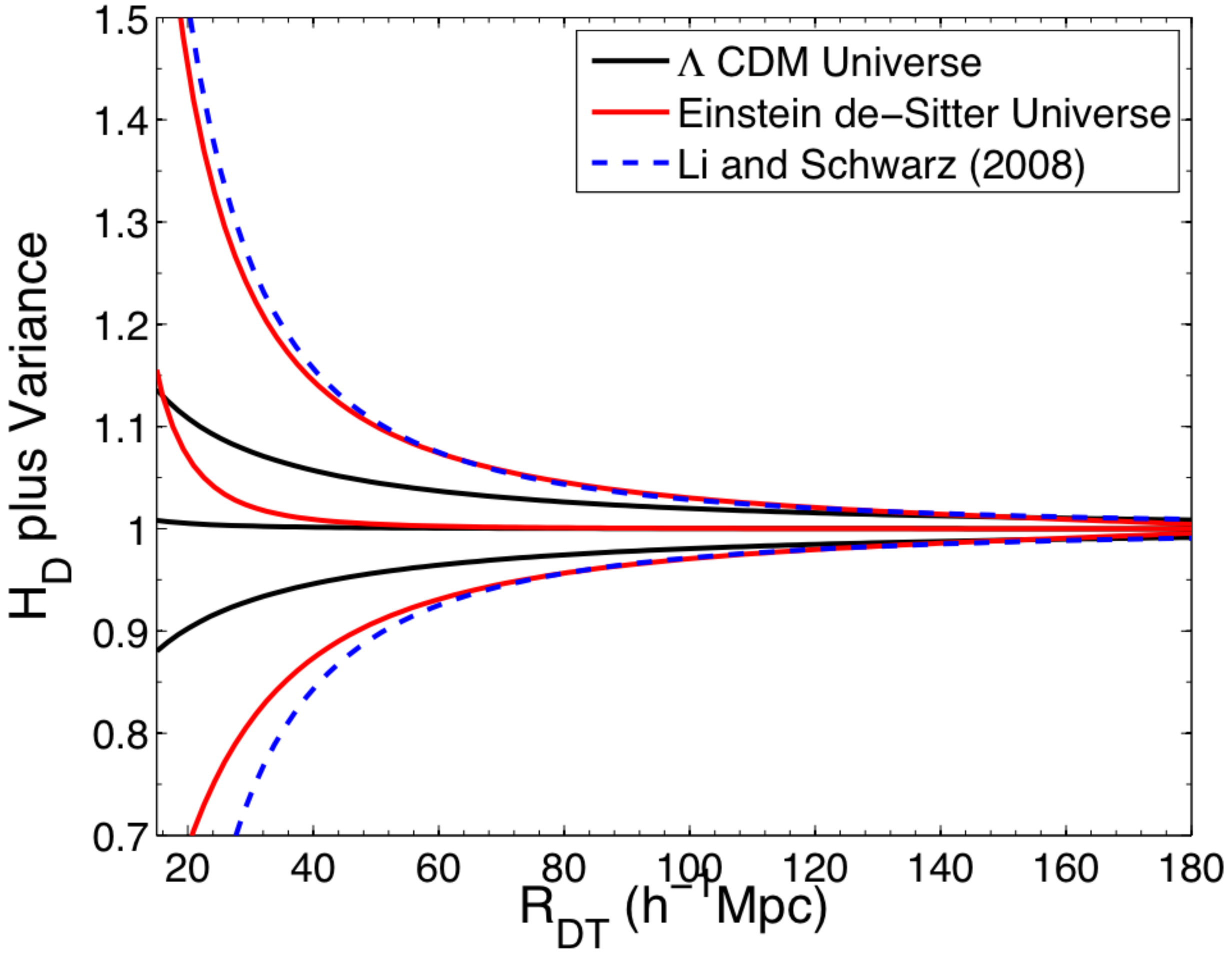

Figure 4.6: Dynamical Hubble rate $\langle\Theta\rangle_{\mathcal{F}}$ today, plus/minus the variance as a function of the averaging scale (normalized to the background Hubble rate), where we have used a top-hat window function to define our domain for comparison with [52, 92]. The blue curve represents the variance calculated in [52, 92]; differences for small domains are a consequence of the different transfer functions used here.

$$\begin{aligned} \ln Var[H] &= -43.61 + 46.0\Omega_m^{0.0293} - 0.7969 f_b^{0.0347} \\ &\quad + \lambda(\log_{10} R)^{\alpha} + \gamma \exp(-\beta(\log_{10} R)^2) \end{aligned} \quad (4.42)$$

where

$$\begin{aligned}
\lambda &= 10.32 - 9.084\Omega_m^{0.0469} - 3.611/f_b^{0.00497} \\
\gamma &= 1.309 - 2.355\Omega_m^{0.055} - 1.073 f_b^{2.1778} \\
\beta &= -1.805 + 3.260\Omega_m^{0.0279} - 0.7180 f_b^{0.665} \\
\alpha &= 1.222 + 0.0334\Omega_m^{3.635} + 0.0591 f_b^{0.3944}.
\end{aligned} \tag{4.43}$$

This formula gives the variance on the measurement of H_0, normalised to the value of H_0: Ω_m is the CDM density parameter, f_b the baryon fraction, and R the length characteristic of the survey, i.e. the distance to the farthest object (in units of h^{-1}Mpc). Note that this fitting formula is valid for a Gaussian window function. Top-hat window functions generically lead to a slight increase of the variance.

4.5 Conclusion

In this work, we have presented the first comparison of the different definitions of the averaged Hubble rate that can be found in the literature. We did this by calculating these various definitions consistently to second order in cosmological perturbation theory. Also for the first time we have calculated the average of the expansion rate using the formalism of [109], which the authors claim is gauge invariant at some limit. We have also found that the definitions that involve the flow of the dust matter component are consistent with each other at second order in cosmological perturbation theory, but differ significantly on small scales from the definition based one the expansion of the coordinate grids. In particular, we noticed the following features of the averaged Hubble rate:

- On small scales all definitions which involve the matter flow agree, and give a small sub-percent change to the background Hubble rate.
- The variance in the average of the expansion rate of the gravitational frame are very small and appear to exhibit weak scale dependence when compared with the definition involving the matter flow.
- On large scales (much larger than the equality scale) all definitions become scale invariant once their ensemble average is evaluated.
- The hypersurface used in averaging is not really important when computing the variance for perturbed FLRW model, the differences only show up on large scales and it is only noticeable in Einstein de Sitter models.

- Including N in the definition of the averaged expansion leaves a residual effect on large scales, and it tends to reduce the backreaction effect. The inclusion of the lapse function is made compulsory if one wants to keep the coordinate time t as the proper time in the averaged model (there is a discussion on this issue in [86]). But this is only one possible choice, since no-one knows how to explicitly construct the average model in this setting where only the scalars are averaged.

We have also derived the dispersion affecting the Hubble rate and arising from the peculiar velocities of the matter flow. We found an effect consistent with previous estimates from backreaction in the literature [52, 92], and our results are consistent with effects evaluated previously [86, 190].

We close with a comment on the origin of the scale dependence of the various quantities. The scale dependence we have found here comes only from 'non-connected' terms such as $\langle\Phi\rangle\langle\partial^2\Phi\rangle$ since the domain size factors out of all other terms (for details on how this type of terms appear see [86]). Non-connected terms only arise when we perform averages in the spacetime itself, which many authors on backreaction have stressed is important. It is interesting to note that these terms (i.e those ones involving laplacian of a gravitational potential, for example $\langle\Phi\rangle\langle\partial^2\Phi\rangle$) do not appear if we treat perturbations as fields propagating on the background, and calculate average quantities only with respect to the background geometry i.e., if we perform a Euclidean average and not a Riemannian one [120].

4.6 Some Follow-up Equations

We present the expressions for different definitions of the Hubble rate calculated consistently up to second order in cosmological perturbation theory. The superscript determines the quantity that has been averaged to define the average Hubble rate.

$$H_{\mathcal{D}}^{N\xi} = H - \left\langle \dot{\Phi} \right\rangle - 3\left\langle \dot{\Phi} \right\rangle \langle \Phi \rangle + \left\langle \Phi\dot{\Phi} \right\rangle - \frac{1}{2}\langle \Psi_2 \rangle \tag{4.44}$$

$$H_{\mathcal{D}}^{\xi} = H - \left\langle \dot{\Phi} \right\rangle - H\langle \Phi \rangle - 3\left\langle \dot{\Phi} \right\rangle \langle \Phi \rangle + 2\left\langle \Phi\dot{\Phi} \right\rangle - 3H\langle \Phi \rangle^2 + \frac{9}{2}\langle \Phi^2 \rangle - \frac{1}{2}\left\langle \dot{\Psi}_2 \right\rangle - \frac{1}{2}H\langle \Phi_2 \rangle \tag{4.45}$$

$$\begin{aligned} H_{\mathcal{D}}^{N\theta} &= H - \left\langle \dot{\Phi} \right\rangle - 3\left\langle \dot{\Phi} \right\rangle \langle \Phi \rangle + 2\left\langle \dot{\Phi}\Phi \right\rangle - \frac{1}{2}\left\langle \dot{\Psi}_2 \right\rangle + \frac{(1+z)}{6}\partial_k v_2^k - \frac{2(1+z)^2}{9\Omega H^2}\left[\left\langle \partial^2\dot{\Phi} \right\rangle\right. \\ &\left. + H\langle \partial^2\Phi \rangle\right] + \frac{(1+z)^2}{\Omega^2 H^3}\left[\frac{8}{9}H\left(1+\frac{\Omega}{2}\right)\left\langle \partial_k\dot{\Phi}\partial^k\Phi \right\rangle + \frac{4}{9}H^2(1+\Omega)\langle \partial_k\Phi\partial^k\Phi \rangle\right. \\ &\left. - \frac{4}{9}\left\langle \partial_k\dot{\Phi}\partial^k\dot{\Phi} \right\rangle\right] + \frac{2(1+z)^2}{3\Omega H^2}\left[\frac{2}{3}\left\langle \Phi\partial^2\dot{\Phi} \right\rangle + \frac{2}{3}H\langle \Phi\partial^2\Phi \rangle - \left\langle \partial^2\dot{\Phi} \right\rangle \langle \Phi \rangle - H\langle \partial^2\Phi \rangle \langle \Phi \rangle\right] \end{aligned} \tag{4.46}$$

$$\begin{aligned} H_{\mathcal{D}}^{\theta} &= H - \left\langle \dot{\Phi} \right\rangle - H\langle \Phi \rangle - 3\left\langle \dot{\Phi} \right\rangle \langle \Phi \rangle + 2\left\langle \Phi\dot{\Phi} \right\rangle - \frac{1}{2}\left\langle \dot{\Psi}_2 \right\rangle - H\langle \Phi_2 \rangle - 3H\langle \Phi \rangle^2 \\ &- \frac{2(1+z)^2}{9\Omega H^2}\left[\left\langle \partial^2\dot{\Phi} \right\rangle + H\langle \partial^2\Phi \rangle\right] + \frac{9}{2}H\langle \Phi^2 \rangle + \frac{(1+z)}{6}\langle \partial_k v^k \rangle \\ &+ \frac{2(1+z)^2}{9\Omega^2 H^3}\left[\left\langle \partial_k\dot{\Phi}\partial^k\dot{\Phi} \right\rangle + 2H\left\langle \partial_k\dot{\Phi}\partial^k\Phi \right\rangle + H^2\langle \partial_k\Phi\partial^k\Phi \rangle\right] \\ &+ \frac{2(1+z)^2}{3\Omega H^2}\left[\left\langle \partial_k\dot{\Phi}\partial^k\Phi \right\rangle - \left\langle \partial^2\dot{\Phi} \right\rangle \langle \Phi \rangle + \left\langle \Phi\partial^2\dot{\Phi} \right\rangle\right] + \frac{2(1+z)^2}{3\Omega H^2}\left[H\langle \partial_k\Phi\partial^k\Phi \rangle\right. \\ &\left. - H\langle \partial^2\Phi \rangle \langle \Phi \rangle + H\langle \Phi\partial^2\Phi \rangle\right] \end{aligned} \tag{4.47}$$

$$\tag{4.48}$$

$$\begin{aligned} H_{\mathcal{F}}^{\Theta} &= H - \left\langle \dot{\Phi} \right\rangle - \langle \Phi \rangle H\left[1 + \frac{3}{2}\Omega_m H g_I\right] - \frac{2(1+z)^2}{9\Omega_m H^2}\left[\left\langle \partial^2\dot{\Phi} \right\rangle + H\langle \partial^2\Phi \rangle\right] - 2\left\langle \Phi\dot{\Phi} \right\rangle \\ &+ \frac{(1+z)}{6}\langle \partial_k v_2^k \rangle - \frac{1}{2}\left[\left\langle \dot{\Psi}_2 \right\rangle - H\langle \Phi_2 \rangle\right] - 3H\langle \Phi \rangle^2\left[(1+g_I H) + \frac{3}{2}\Omega_m H g_I(1 - H g_I)\right] \\ &- 3\left\langle \dot{\Phi} \right\rangle \langle \Phi \rangle [1 + H g_I] + \frac{9}{2}\langle \Phi^2 \rangle \Omega_m H\left[1 - \frac{2}{3}H g_I \Omega_m\left(1 - \frac{3}{2}H g_I\right)\right] \\ &+ \frac{2(1+z)^2}{27\Omega^2 H^3}\left[\left\langle \partial_k\dot{\Phi}\partial^k\dot{\Phi} \right\rangle + 2H\left(1 + \frac{9}{2}\Omega_m\right)\left\langle \partial_k\dot{\Phi}\partial^k\Phi \right\rangle + H^2(1+9\Omega_m)\langle \partial_k\Phi\partial^k\Phi \rangle\right] \\ &+ \frac{2(1+z)^2}{3\Omega_m H^2}\left[\left(1 - \frac{2}{3}H g_I\right)\left\langle \Phi\partial^2\dot{\Phi} \right\rangle + \left(1 - \frac{2}{3}H g_I\right)H\langle \Phi\partial^2\Phi \rangle - \left(1 - \frac{2}{3}H g_I\right)\right. \\ &\left. \times\left\langle \partial^2\dot{\Phi} \right\rangle \langle \Phi \rangle - \left(1 - \frac{2}{3}H g_I\right)H\langle \partial^2\Phi \rangle \langle \Phi \rangle - \Omega_m H^2 g_I\langle \Phi\partial^2\Phi \rangle\right] + \frac{3}{2}\langle A_2 \rangle H^2\Omega_m g_I, \end{aligned} \tag{4.49}$$

where $g_I = \frac{1}{g(t)}\int_0^t g(t')dt'$.

Chapter 5

The Influence of Structure Growth on Acceleration of the Universe

Within the past few years, there has been great improvements and breakthroughs on attempts to map the universe on different scales. The most notable among them include the WMAP [34], ACT [194], Supernovae Cosmology [195] and QUIET [39] experiments. These experiments have generated large volumes of data on the origin and late-time evolution of structures in the universe, this success has made our dream of a precision cosmology more realizable.

However, in order to fit these sets of data consistently to a theoretical model of the universe, we are compelled to introduce two new forms of matter with a questionable origin within any particle physics model. These two forms of matter are called dark matter and dark energy and the model that needs both forms of matter to explain cosmological observation is called the ΛCDM model. In this model we need over over 70% of dark energy and 23% of dark matter to explain why the present universe is expanding at an accelerating pace and how structure formation takes place.

Apart from the fact that ΛCDM model introduced unknown forms of matter into the equation, it could not also explain why the accelerating phase of expansion started about a billion years ago. These unanswered questions opened up avenues for investigation into the very foundations and assumptions made during construction of the ΛCDM model. One of the key basic assumptions at the heart of the ΛCDM model is that the FLRW space-time on which this model is built is the right space-time based in the cosmological principle that describes the universe on very large scale. Less than two decades before the dark energy crisis, Ellis et al [41] pointed out that smoothing observations over the sky in an inhomogeneous mode might lead to a significantly different cosmology than the usual one based on an FLRW space-time, which is spatially smooth on very large scale (this is known as the fitting problem).

One of the key motivations for this study is the place of the background FLRW space-time assumption within general relativity, this is important because the space-time itself is a dynamical quantity in general relativity and inhomogeneities that live on it evolve. General relativity predicts that their evolution must leave some backreaction imprints on the space-time. Another key motivation that is not always mentioned is the fact that some of the cosmological observations (e.g. SN I) that we are fitting to a cosmological model probe small angle curvature that is vanishing on an FLRW space-time and in the standard approach. Other motivations include the fact that backreaction is a prediction of general relativity, hence detecting its effect in a cosmological observation would be a breakthrough on its own.

Because of the coincidence problem, some works [43, 91, 93, 98, 110, 184, 196–199] have explored the possibility that the introduction of dark energy in the ΛCDM model might be a misunderstanding of the model. There is a possibility that backreaction effects might be the reason for the accelerated expansion, however, many in the community disagree with this notion [62, 108, 120, 153, 200–212]. Other moderate views [49, 51, 52, 86, 90, 92, 104, 213] claim that the effect of backreaction from structure formation is not large enough to explain the present day accelerated expansion, however, their effect is large enough to be taken into proper consideration during cosmological parameter estimation. Some members of the community [62, 120, 153] who use averaging formalism that do not calculate any measurable quantity like the deceleration parameter claim that backreaction will have a negligible effect on the measurement of present day cosmological parameters. These disagreements make the backreaction effect on cosmological observables an area open to further exploration.

In this chapter we focus attention solely on the deceleration parameter. This is not only because it is central to the determination of accelerated expansion of the universe, it also incorporates all the effects of backreaction within the Buchert formalism, i.e it is a proper time derivative of another averaged physical quantity, thereby incorporating the effect of non-commutativity of proper time derivative of an averaged quantity with the average of a proper time derivative of a quantity. This effect is totally vanishing for the Hubble rate.

We will approach the problem from two different observationally motivated perspectives:

- we will study the deceleration parameter using the Buchert formalism [43] on arbitrary space-time, this is to enable us to understand the effect of the choice of spatial slicing on the deceleration parameter.

- we will use the Kristian and Sachs approach [170] to perform a null fitting of the lumpy universe to the background FLRW spacetime. We will then quantify the all-sky average of the distance-redshift relation, because this quantity is central to the determination of local acceleration. In principle the result from this approach corresponds approximately to what is observed [61].

This chapter is organized as follows: in section 5.1, we present the equations of general relativity in the AMD form. We introduce fluid variables and re-write the equations of motion in terms of the fluid variables in section 5.2. We perform the spatial averaging of these in section 5.3 for different possible definitions of the deceleration parameter. We study the definitions of the deceleration parameter that corresponds to the formalism used by the group that reports negligible backreaction effect in section 5.4. We present the covariant relations valid on any spacetime adapted to the past light cone in section 5.5. In section 5.6 we provide a technique for obtaining an FLRW best fit model from observation in inhomogeneous universe. A description of our standard cosmological model is provided in section 5.7. Results in the form of plots and discussion are presented in section 5.8. Out final conclusion is given in section 5.9. All the definitions of the deceleration parameter calculated in perturbation theory are given in section 5.10.

5.1 Equations of Motion in the ADM form

The original Burchert averaging formalism is based on a spacetime manifold describable in synchronous coordinates [121, 122, 214], This has be generalized for an arbitrary coordinate system (see [48, 59, 109, 187] for different versions of the generalization). We shall utilise the formalism of [48] with some modifications and corrections. Consider a foliation of spacetime by a continuous set of spacelike hypersurfaces ($t = constant$) with normal vector $n_a = \nabla_a t$. The vector field n_a is timelike $n^a n_a = -1$ and it must have zero vorticity. The projection tensor field onto the hypersurfaces is defined as $h_{ab} = g_{ab} + n_a n_b$. Consider also a family of reference timelike world lines with tangent vector u^a that are in general not orthogonal to these surfaces (we will choose them to be the fluid flow lines, see below)

The general line element with respect to the continuous set of spacelike hypersurfaces orthogonal to n^a may be written as:

$$ds^2 = -\left(N^2 - N_i N^i\right) dt^2 + 2N_i dt dx^i + h_{ij} dx^i dx^j, \tag{5.1}$$

where we have introduced respectively the lapse function $N(x^a)$ and the shift 3-vector $N^i(x^a)$ such that the the shift vector relates the normals n^a to the reference field u^a and the lapse function relates proper time along the normals to the coordinate time t. The relation

$$n^a = \frac{1}{N}\left(1, -N^i\right) \ , n_a = N\left(-1, 0, 0, 0\right) \ . \tag{5.2}$$

The intrinsic curvature of the hypersurfaces is given by $\mathcal{R} \equiv h^{ab}\mathcal{R}_{ab}$, where $\mathcal{R}_{ab}$ is the 3-Ricci curvature of the hypersurfaces. The extrinsic curvature (or second fundamental form) is $K_{ab} \equiv -h^c_a h^d_b n_{c;d}$. The projected covariant three-derivative on the spatial hypersurfaces of any tensor field $t^{a...c}{}_{d...f}$ is defined as

$$\tilde{\nabla}_d t^{a...c}{}_{\bar{a}...\bar{c}} : \ = \ h^e_d h^a_{a'} ... h^c_{c'} h^{a''}_{\bar{a}} ... h^{c''}_{\bar{c}} \nabla_e t^{a'...c'}{}_{a''...c''} \tag{5.3}$$

and the time derivatives along the normal is given by

$$\partial_t f \ = \ \dot{f} = n^a \nabla_a f. \tag{5.4}$$

The Einstein equations written in the ADM form are given by

$$\frac{1}{N}\partial_t K \ = \ \mathcal{R} + K^2 - 4\pi G\left(3\epsilon - S\right) - 3\Lambda - \frac{1}{N}\tilde{\nabla}_k\tilde{\nabla}^k N + \frac{1}{N}N^k\tilde{\nabla}_k K \ , \tag{5.5}$$

$$\mathcal{R} \ = \ 16\pi\epsilon + K_{ij}K^{ij} - K^2 + 2\Lambda \ , \tag{5.6}$$

$$\partial_t h_{ij} \ = \ -2NK_{ij} + 2\tilde{\nabla}_{(j}N_{i)} \ , \tag{5.7}$$

where $K = g^{ab}K_{ab}$ is the trace of the second fundamental form, $\epsilon = n^a n^b T_{ab}$ is the energy density relative to the normal (not to the reference field), $S_{ij} = h^a{}_i h^b{}_j T_{ab}$ is the stress energy tensor, and $J_i = -T_{ab}n^a h^b{}_j$ is the momentum density relative to the normal (not to the reference field). These quantities are defined in terms of the energy-momentum tensor T_{ab}, which is given by (for a perfect fluid),

$$T_{ab} = (\rho + p)u_a u_b + p g_{ab}, \tag{5.8}$$

where ρ and p are the energy density and pressure defined with respect to the fluid four velocity u^a. The energy conservation equation holds, $\nabla_a T^{ab} = 0$ due to Bianchi identities. The matter four velocity u^a (or the reference field) is time-like and it is normalized such that $u^a u_a = -1$. u^a is related to n^a through

$$u^a = \gamma(n^a + v^a), \ \text{where} \ \gamma = 1/\sqrt{1 - v^a v_a} \ . \tag{5.9}$$

where v^a is the peculiar velocity. It is spacelike and orthogonal to n^a ($v^a n_a = 0$). The quantities of the energy-momentum tensor defined with respect to n^a and those defined with respect to u^a are related according to

$$\begin{aligned}\epsilon &= \gamma^2(p+\rho)-p, &(5.10)\\ S_{ij} &= \gamma^2\rho v_i v_j + p(h_{ij}+\gamma^2 v_i v_j), &(5.11)\\ J_j &= \gamma^2(\rho+p)v_j. &(5.12)\end{aligned}$$

For further details on ADM decomposition and fluid variables see [48, 215, 216].

5.2 Equation of motion in terms fluid quantities

It is more convenient to have the set of equations in section 5.1 (i.e equations (5.5 ,5.6 ,5.7) written in terms of fluid variables only, rather than the second fundamental form. To achieve this, we decompose the covariant derivatives of the two 4−velocities, u^a and n^a, as well as the peculiar velocity v^a, with respect to the normal frame, n^a.

$$\begin{aligned}\nabla_a n_b &= -n_a\dot{n}_b + \frac{1}{3}\xi h_{ab} + \Sigma_{ab}\ , &(5.13)\\ \nabla_a u_b &= -\gamma v^c\left(\gamma^2\dot{v}_c+\dot{n}_c\right)n_a n_b - \gamma\left(\gamma^2 v^c\tilde{\nabla}_a v_c + \frac{1}{3}\xi v_a + \Sigma_{ac}v^c\right)n_b + \gamma n_a\left(\gamma^2 v^c\dot{v}_c v_b + \dot{n}_{\langle b\rangle}\right.\\ &\left. +\dot{v}_{\langle b\rangle}\right) + \frac{1}{3}\theta h_{ab} + \sigma_{ab} + \omega_{ab}\ , &(5.14)\\ \nabla_a v_b &= -\dot{n}_c v^c\, n_a n_b - n_a\dot{v}_{\langle b\rangle} + \left(\frac{1}{3}\xi v_a + \Sigma_{ac}v^c\right)n_b + \frac{1}{3}\kappa h_{ab} + \beta_{ab} + W_{ab}\ ,\end{aligned}$$

where the following definitions have been made:

$$\begin{aligned}&\xi \equiv h^{ab}\nabla_a n_b\ , \qquad \Sigma_{ab} \equiv h^c_a h^d_b \nabla_{(c}n_{d)} - \frac{1}{3}\xi h_{ab}\ ,\\ &\theta \equiv h^{ab}\nabla_a u_b\ , \qquad \sigma_{ab} \equiv h^c_a h^d_b \nabla_{(c}u_{d)} - \frac{1}{3}\theta h_{ab}\ ,\\ &\omega_{ab} \equiv h^c_a h^d_b \nabla_{[c}u_{d]}\ , \qquad \kappa \equiv h^{ab}\nabla_a v_b\ ,\\ &\beta_{ab} \equiv h^c_a h^d_b \nabla_{(c}v_{d)} - \frac{1}{3}\kappa h_{ab}\ , \qquad W_{ab} \equiv h^c_a h^d_b \nabla_{[c}v_{d]}\ .\end{aligned}$$

We have used the notations $\langle ...\rangle$ i.e the angle brackets at the subscript or the superscript to denote projected (onto the hypersurface orthogonal to n^a) symmetric trace-free tensor, $(...)$ for symmetric tensor, $[...]$ for antisymmetric tensors and $\tilde{\nabla}$ denotes the spatially projected covariant derivative. Other definitions we have made include ξ which is the expansion of the observer geodesics as seen by the observer himself, θ is expansion of the fluid geodesics as seen by the observer characterized by n^a, while κ is the divergence of the 3-velocity v^a

with respect to n^a. We have three quantities that could be interpreted as shear with respect to n^a: Σ_{ab}, σ_{ab} and β_{ab}, while ω_{ab} and W_{ab} are the vorticity in the respective definitions. We reiterate that every quantity we have defined has a natural interpretation in terms of observers comoving with n^a and these quantities are unique and consistent. As we shall see later, the Buchert equations are recovered in the limit $n^a = u^a$, which corresponds to the coordinate observers comoving with the fluid with a vanishing peculiar velocity. In a multi-fluid universe, this is not a unique choice because observers do not necessary need to be comoving with the fluid, hence it is important to explore other physically plausible possibilities. With respect to the foliation we have adopted, the following transformation relations exist:

$$\theta = \gamma\xi + \gamma\kappa\left(1 + \frac{1}{3}\gamma^2 v^2\right) + \gamma^3 v^a v^b \beta_{ab}, \tag{5.15}$$

$$\Sigma_{ab} = \gamma^{-1}\sigma_{ab} - \beta_{ab} - \gamma^2\left(B_{(ab)} - \frac{1}{3}Bh_{ab}\right), \tag{5.16}$$

$$W_{ab} = \gamma^{-1}\omega_{ab} - \gamma^2 B_{[ab]}, \tag{5.17}$$

where the tensor B_{ab} is defined as [48]

$$B_{ab} \equiv \frac{1}{3}\kappa(v_a n_b + v_a v_b) + \beta_{ca}v^c n_b + \beta_{ca}v^c v_b + W_{ca}v^c v_b \ , \tag{5.18}$$

$$B = \frac{1}{3}\kappa v^2 + \beta_{ab}v^a v^b. \tag{5.19}$$

with B defined as the trace of B_{ab}. To reduce clutter, we make the following definitions

$$\theta_B \equiv -\gamma\kappa - \gamma^3 B \tag{5.20}$$

$$\sigma_{Bij} \equiv -\gamma\beta_{ij} - \gamma^3\left(B_{(ij)} - \frac{1}{3}Bh_{ij}\right), \tag{5.21}$$

so that we obtain important relations between ξ and θ and between $\Sigma_{ij}$and σ_{ij} [48],

$$\xi = \gamma^{-1}(\theta + \theta_B) \ , \tag{5.22}$$

$$\Sigma_{ij} = \gamma^{-1}(\sigma_{ij} + \sigma_{Bij}) \ . \tag{5.23}$$

These relations make it easier to present the Einstein equations in terms of any quantity of interest. For certain cosmological observation like the Supernova Cosmology Project (SCP), the four velocity of the fluid and that of the observer are on average equal but it is more instructive to treat it separately. Thus we decompose the covariant derivative of u^a, in its frame. This choice coincides with the synchronous coordinate system (the original Buchert equations were derived in this set of coordinate system). In terms of the metric it corresponds to setting $N = 1$ and $N^k = 0$. The irreducible decomposition of the covariant derivative of the fluid velocity u^a in its frame is given by

$$\nabla_a u_b = -u_a u'_b + \frac{1}{3}\Theta h^{\mathcal{F}}_{ab} + \omega^{(u)}_{ab} + \sigma^{(u)}_{ab} , \tag{5.24}$$

where $u^a\nabla_a = \partial_\tau =' $ and Θ, $\omega^{(u)}_{ab}$, $\sigma^{(u)}_{ab}$ have their obvious meaning with respect to this foliation and $h^{\mathcal{F}}_{ab}$ is the metric of the continuous set of hyper-surfaces orthogonal to u^a. There exists a relationship between quantities defined on both hypersurfaces, for example, the expansion

$$\Theta = \theta + \gamma v^a \left(\gamma^2 \dot{v}_a + \dot{n}_a\right) , \tag{5.25}$$

and similar relations hold for the normal acceleration, vorticity and the shear see [189]. Using the relations we have defined in equation (5.15), it is now easy to have Einstein equations (5.5) written in terms of the fluid quantities:

$$\gamma^2\mathcal{R} = 16\pi G\gamma^2\epsilon + 2\Lambda\gamma^2 - \frac{2}{3}\theta^2 - \frac{2}{3}\theta_B^2 - \frac{4}{3}\theta\theta_B + 2\sigma^2 + 2\sigma_B^2 \tag{5.26}$$

$$\partial_t\theta = -4\pi G\gamma N\left(\epsilon + S\right) + 3N\gamma\Lambda + \frac{\dot{\gamma}}{\gamma}\left(\theta + \theta_B\right) - \frac{N^k D_k\gamma}{\gamma}\left(\theta + \theta_B\right) + N^k\left(D_k\theta + D_k\theta_B\right) - \frac{2N}{\gamma}\left(\sigma_B^2 + \sigma^2\right) - \dot{\theta}_B - \frac{N\theta^2}{3\gamma} - \frac{N\theta_B^2}{3\gamma} - \frac{2N\theta\theta_B}{3\gamma} + \gamma D_k D^k N \tag{5.27}$$

$$\partial_t h_{ij} = \frac{2}{3}N h_{ij}\gamma^{-1}(\theta + \theta_B) + 2N\gamma^{-1}(\sigma_{ij} + \sigma_{Bij}) + D_i N_j + D_j N_i \tag{5.28}$$

where $\sigma^2 = \sigma^i_j\sigma^j_i/2$ and $\sigma_B^2 = \sigma^i_{Bj}\sigma^j_{Bi}/2 + \sigma_{ij}\sigma^{ij}_B$. In the Buchert limit , $i.eN \to 1, N^k \to 0, v \to 0$, the Hamiltonian constraint and the evolution equation equations become

$$\mathcal{R}_{(u)} = 16\pi G\rho - \frac{2}{3}\Theta^2 + 2\sigma^2_{(u)}, \tag{5.29}$$

$$\partial_\tau\Theta = -4\pi G(\rho + 3p) - \frac{\Theta^2}{3} - 2\sigma^2_{(u)} \tag{5.30}$$

respectively. The Einstein equations expressed in terms of these kinematical quantities, i.e. equations. (5.26), (5.28) and (5.28), offer more freedom to define the Hubble rate and other quantities in different ways other than the special choice investigated by Buchert [43] and many others (that is equation (5.29)).

5.3 Possible Definitions and Averaged Scalar equations

The average of any scalar field ψ may be defined simply as its integral over a region of a spatial hypersurface $\mathcal{D}$ of constant proper time divided by the Riemannian volume (we follow mainly the approach of [43]):

$$\langle\psi\rangle_{\mathcal{D}} \equiv \frac{1}{V_{\mathcal{D}}}\int_{\mathcal{D}} J d^3x\, \psi(t, x^i) \ , \tag{5.31}$$

where $J = \sqrt{h}$ is the square root of the determinant of the metric of the hypersurface. As described in [86] the definition of the average given above should be understood as choosing coordinates which occur in both the rough and smooth spacetimes. In particular, we can choose our coordinates in the rough spacetime with the hope that they become the ones we want in the smooth spacetime after a smoothing process. It is worth noting that a 'covariant and gauge invariant' version of the equation (5.31) proposed in[109] is simply an illustration that equation (5.31) could be obtained from a scalar average defined on the full spacetime. The time derivative of equation (5.31) shows that spatial averaging does not commute with the time derivative:

$$\partial_t \langle\psi\rangle_{\mathcal{D}} - \langle\partial_t\psi\rangle_{\mathcal{D}} = \left\langle\psi\Gamma^i_{i0}\right\rangle_{\mathcal{D}} - \langle\psi\rangle_{\mathcal{D}}\left\langle\Gamma^i_{i0}\right\rangle_{\mathcal{D}} \equiv \mathcal{C}_1, \tag{5.32}$$

where $\Gamma^i_{i0} = h^{ij}\partial_t h_{ij}$ is the Christoffel connection associated with any arbitrary hypersurface. The size of the non-commutative term, $\mathcal{C}_1$, depends on the choice of the hypersurface through the expansion of the hypersurface, $\nabla_b n^b$. The definition of the Hubble rate does not incorporate $\mathcal{C}_1$, but the deceleration parameter does. Using equation (5.31) and (5.32), we may now consider different possible definitions of the Hubble rate and the deceleration parameter.

5.3.1 Inhomogeneous lapse function definition

Following [48], we define an average Hubble rate based on the expansion of the fluid as seen by the observer attached to the coordinate grid

$$H_{\mathcal{D}} \equiv \frac{1}{3}\langle N\theta\rangle_{\mathcal{D}} = \frac{1}{3V_{\mathcal{D}}}\int_{\mathcal{D}} J d^3x\, N\theta \ . \tag{5.33}$$

We call the definition of the Hubble rate and deceleration parameter with a multiplicative factor of the lapse function *inhomogeneous lapse function* because it is multiplied by the lapse function to ensure that the time parameter remains the same on all scales [49]. With this choice, the relationship between the scale factor $a_{\mathcal{D}}$ and the volume expansion $V_{\mathcal{D}}$ becomes highly non-trivial [48], while the averaged scalar part of Einstein equations becomes:

$$6H_{\mathcal{D}}^2 = 16\pi G\left(\left\langle\gamma^2N^2\epsilon\right\rangle_{\mathcal{D}}\right) + 2\Lambda\left\langle N^2\gamma^2\right\rangle_{\mathcal{D}} - \left\langle\gamma^2N^2\mathcal{R}\right\rangle_{\mathcal{D}} - \mathcal{Q}_{\mathcal{D}} + \mathcal{L}_{\mathcal{D}}, \quad (5.34)$$

$$3\frac{\partial_t^2 a_{\mathcal{D}}}{a_{\mathcal{D}}} = -4\pi G\left\langle N^2\gamma(\epsilon+S)\right\rangle_{\mathcal{D}} + \Lambda\left\langle N^2\gamma\right\rangle_{\mathcal{D}} + \mathcal{Q}_{\mathcal{D}} + \mathcal{P}_{\mathcal{D}} + \mathcal{K}_{\mathcal{D}} + \mathcal{F}_{\mathcal{D}} - \mathcal{L}_{\mathcal{D}}\ ,$$

where different back reaction terms may be defined as:

$$\mathcal{Q}_{\mathcal{D}} \equiv \frac{2}{3}\left(\left\langle(N\theta)^2\right\rangle_{\mathcal{D}} - \langle N\theta\rangle_{\mathcal{D}}^2\right) - 2\left\langle N^2\sigma^2\right\rangle_{\mathcal{D}}\ , \quad (5.35)$$

$$\mathcal{L}_{\mathcal{D}} \equiv 2\left\langle N^2\sigma_B^2\right\rangle_{\mathcal{D}} - \frac{2}{3}\left\langle(N\theta_B)^2\right\rangle_{\mathcal{D}} - \frac{4}{3}\left\langle N^2\theta\theta_B\right\rangle_{\mathcal{D}} \quad (5.36)$$

$$\mathcal{P}_{\mathcal{D}} \equiv \langle\theta\partial_t N\rangle_{\mathcal{D}} + \left\langle\gamma N D_k D^k N\right\rangle_{\mathcal{D}} \quad (5.37)$$

$$\begin{aligned}\mathcal{K}_{\mathcal{D}} \equiv\ & \left\langle N\gamma^{-1}\dot{\gamma}\theta\right\rangle_{\mathcal{D}} + \left\langle N\gamma^{-1}\dot{\gamma}\theta_B\right\rangle_{\mathcal{D}} - \left\langle N\gamma^{-1}\theta_B\right\rangle_{\mathcal{D}}\langle N\theta\rangle_{\mathcal{D}} + \left\langle D_kN^kN\theta\right\rangle_{\mathcal{D}} \quad (5.38)\\ & - \left\langle D_kN^k\right\rangle_{\mathcal{D}}\langle N\theta\rangle_{\mathcal{D}} + \left\langle NN^kD_k\gamma(\theta+\theta_B)\right\rangle_{\mathcal{D}} + \left\langle NN^kD_k\theta\right\rangle_{\mathcal{D}}\\ & + \left\langle NN^kD_k\theta_B\right\rangle_{\mathcal{D}} - \langle N\partial_t\theta_B\rangle_{\mathcal{D}}\end{aligned}$$

$$\begin{aligned}\mathcal{F}_{\mathcal{D}} \equiv\ & \frac{2}{3}\left\langle N^2\theta^2(\gamma^{-1}-1)\right\rangle_{\mathcal{D}} - 2\left\langle N^2\sigma^2(\gamma^{-1}-1)\right\rangle_{\mathcal{D}} \quad (5.39)\\ & - \langle N\theta\rangle_{\mathcal{D}}\left\langle N\theta(\gamma^{-1}-1)\right\rangle_{\mathcal{D}}\\ & -\frac{1}{3}\left\langle N^2\theta_B^2(\gamma^{-1}+2)\right\rangle_{\mathcal{D}} + \frac{1}{3}\left\langle N^2\theta\theta_B(\gamma^{-1}-4)\right\rangle_{\mathcal{D}} - 2\left\langle N^2\sigma_B^2(\gamma^{-1}-1)\right\rangle_{\mathcal{D}}. \quad (5.40)\end{aligned}$$

Here $\mathcal{Q}_{\mathcal{D}}$ is the usual kinematical backreaction term, $\mathcal{P}_{\mathcal{D}}$ is the dynamical backreaction term, $\mathcal{K}_{\mathcal{D}}$, $\mathcal{L}_{\mathcal{D}}$ and $\mathcal{F}_{\mathcal{D}}$ are backreaction terms due to relativistic tilting of the observer frame from the matter frame. There exist integrability conditions relating the two sets of equations and the energy density conservation equation, we will not be needing them here(see [48] for further details).

The averaged set of equations (i.e. equation (5.34 and 5.35) may also be written in the form of an FLRW background Friedmann and Raychaudhuri equation. In this case the background energy density and pressure are replaced with the effective energy density and effective pressure

$$H_D^2 = \frac{8\pi G}{3}\rho_{\rm eff}\ , \quad (5.41)$$

$$\frac{\ddot{a}_D}{a_D} = -\frac{4\pi G}{3}\left(\rho_{\rm eff} + p_{\rm eff}\right)\ . \quad (5.42)$$

The corresponding equation of state may also be obtained assuming a barotropic fluid equation of state as $w_{\rm eff} = p_{\rm eff}/\rho_{\rm eff}$, where have made the following definitions:

$$\rho_{\rm eff} = \langle\epsilon\rangle_{\mathcal{D}} - \frac{1}{16\pi G}\left(\left\langle\gamma^2 N^2 \mathcal{R}_{\mathcal{D}}\right\rangle_{\mathcal{D}} + \mathcal{Q}_{\mathcal{D}} - \mathcal{L}_{\mathcal{D}}\right) \tag{5.43}$$

$$p_{\rm eff} = \left\langle N^2\gamma\left(\epsilon + S\right) - N^2\epsilon\right\rangle_{\mathcal{D}} - \frac{1}{16\pi G}\left(3\mathcal{Q}_{\mathcal{D}} - 3\mathcal{L}_{\mathcal{D}} + 4\mathcal{P}_{\mathcal{D}} + 4\mathcal{F}_{\mathcal{D}} + 4\mathcal{K}_{\mathcal{D}} - \left\langle\gamma^2 N^2 \mathcal{R}\right\rangle_{\mathcal{D}}\right) \tag{5.44}$$

The deceleration parameter $q_{\mathcal{D}}$ may be constructed from the averaged Hubble rate and average Raychaudhuri equation,

$$q_{\mathcal{D}}^{N\theta} = -\frac{1}{H_{\mathcal{D}}^2}\frac{\ddot{a}_{\mathcal{D}}}{a_{\mathcal{D}}}, \tag{5.45}$$

where $\ddot{a}_{\mathcal{D}}/a_{\mathcal{D}}$ is given by the averaged Raychaudhuri equation (5.35) and $H_{\mathcal{D}}^2$ is given by equation (5.35). This definition of decceleration parameter $q_{\mathcal{D}}$ describes the deceleration of *averaged fluid flow lines projected to the hypersurface orthogonal to the normals* or the deceleration of the fluid congruence averaged on the hypersurface of the observers. Note the emphasis because we are going to consider other kinds of definition of the deceleration parameter.

5.3.2 Homogeneous Lapse function definition

We consider a definition of average Hubble rate where the lapse function is set to unity in a conformal metric or to the square of the scale factor in a general FLRW spacetime. The average Hubble rates in this case becomes

$$H_{\mathcal{D}} = \frac{1}{3}\langle\theta\rangle. \tag{5.46}$$

The corresponding averaged Friedmann equation and acceleration equation become:

$$6H_{\mathcal{D}}^2 = 16\pi G\left\langle\gamma^2\epsilon\right\rangle_{\mathcal{D}} + 2\left\langle\Lambda\gamma^2\right\rangle_{\mathcal{D}} - \left\langle\gamma^2\mathcal{R}\right\rangle_{\mathcal{D}} - \mathcal{Q}_{\mathcal{D}} + \mathcal{L}_{\mathcal{D}} \tag{5.47}$$

$$3\frac{\partial_t^2 a_{\mathcal{D}}}{a_{\mathcal{D}}} = -4\pi G\left\langle N\gamma(\epsilon+S)\right\rangle_{\mathcal{D}} + \left\langle\Lambda\gamma\right\rangle_{\mathcal{D}} + \mathcal{Q}_{\mathcal{D}} + \mathcal{K}_{\mathcal{D}} + \mathcal{P}_{\mathcal{D}} + \mathcal{F}_{\mathcal{D}} - \mathcal{L}_{\mathcal{D}}\ , \tag{5.48}$$

with their corresponding backreaction terms defined as follows:

$$\mathcal{Q}_\mathcal{D} \equiv \frac{2}{3}\left(\left\langle(\theta)^2\right\rangle_\mathcal{D} - \langle\theta\rangle_\mathcal{D}^2\right) - 2\left\langle\sigma^2\right\rangle_\mathcal{D} \;, \tag{5.49}$$

$$\mathcal{L}_\mathcal{D} \equiv 2\left\langle\sigma_B^2\right\rangle_\mathcal{D} - \frac{2}{3}\left\langle(\theta_B)^2\right\rangle_\mathcal{D} - \frac{4}{3}\langle\theta\theta_B\rangle_\mathcal{D}\,, \tag{5.50}$$

$$\mathcal{P}_\mathcal{D} \equiv \left\langle\theta D_k N^k\right\rangle_\mathcal{D} + \left\langle\gamma D_k D^k N\right\rangle_\mathcal{D}\,, \tag{5.51}$$

$$\begin{aligned}\mathcal{K}_\mathcal{D} \equiv\; & -\left\langle N\gamma^{-1}\theta_B\right\rangle_\mathcal{D}\langle\theta\rangle_\mathcal{D} - \left\langle D_k N^k\right\rangle_\mathcal{D}\langle\theta\rangle_\mathcal{D} + \left\langle\partial_t\gamma\gamma^{-1}\theta\right\rangle_\mathcal{D} + \left\langle\partial_t\gamma\gamma^{-1}\theta_B\right\rangle_\mathcal{D} \\ & -\left\langle N^k\gamma^{-1}D_k\gamma(\theta+\theta_B)\right\rangle_\mathcal{D} + \left\langle N^k(D_k\theta + D_k\theta_B)\right\rangle_\mathcal{D} - \langle\partial_t\theta_B\rangle_\mathcal{D}\,, \end{aligned}\tag{5.52}$$

$$\begin{aligned}\mathcal{F}_\mathcal{D} \equiv\; & \frac{2}{3}\left\langle\theta^2(N\gamma^{-1}-1)\right\rangle_\mathcal{D} + \frac{1}{3}\left\langle\theta\theta_B\left(N\gamma^{-1}-4\right)\right\rangle_\mathcal{D} - \frac{1}{3}\left\langle\theta_B^2(N\gamma^{-1}+2)\right\rangle_\mathcal{D} \\ & -\left\langle\theta\left(N\gamma^{-1}-1\right)\right\rangle_\mathcal{D}\langle\theta\rangle_\mathcal{D} - 2\left\langle\sigma^2\left(N\gamma^{-1}-1\right)\right\rangle_\mathcal{D} - 2\left\langle\sigma_B^2\left(N\gamma^{-1}-1\right)\right\rangle_\mathcal{D}\,.\end{aligned}$$

The physical interpretation of each of the backreaction terms is similar to the case of inhomogeneous lapse function, hence we will not repeat it here. An effective fluid interpretation leads to equations (5.41) and (5.42), with the effective energy density and pressure are given by

$$\rho_\text{eff} = \epsilon - \frac{1}{16\pi G}\left(\left\langle\gamma^2\mathcal{R}_\mathcal{D}\right\rangle_\mathcal{D} + \mathcal{Q}_\mathcal{D} - \mathcal{L}_\mathcal{D}\right), \tag{5.53}$$

$$p_\text{eff} = \langle N\gamma\,(\epsilon+S) - \epsilon\rangle_\mathcal{D} - \frac{1}{16\pi G}\left(3\mathcal{Q}_\mathcal{D} - 3\mathcal{L}_\mathcal{D} + 4\mathcal{F}_\mathcal{D} + 4\mathcal{K}_\mathcal{D} - \left\langle\gamma^2\mathcal{R}\right\rangle_\mathcal{D}\right). \tag{5.54}$$

The decceleration parameter $q_\mathcal{D}$ associated with this definition is given by

$$q_\mathcal{D}^\theta = -\frac{1}{H_\mathcal{D}^2}\frac{\ddot{a}_\mathcal{D}}{a_\mathcal{D}}\,, \tag{5.55}$$

with $\ddot{a}_\mathcal{D}/a_\mathcal{D}$ and $H_\mathcal{D}^2$ given by equations (5.48) and (5.47) respectively.

5.3.3 Comoving Observer definition

This is a case where the observer is comoving with the fluid, that is the observers and coordinates are at rest with respect to the fluid. The average of a scalar quantity is defined on the spatial hypersurface orthogonal to u^a and not n^a, this is possible only if the fluid is irrotational. In our case it is possible since we are considering on the post-recombination perturbation where dust approximation is valid. The metric on the hypersurface is denoted as $\tilde{h}_{ab}$. The average Hubble rate is defined as

$$\langle 3H\rangle_\mathcal{D} \equiv \frac{1}{V_\mathcal{F}}\int_\mathcal{F}\sqrt{\tilde{h}}\,\Theta d^3x \;, \tag{5.56}$$

In this limit we have a trivial relationship with the dimensionless volume expansion of the domain $a_{\mathcal{D}} \propto {V_{\mathcal{F}}}^{1/3}$ [43]. This limit have been widely studied [52, 92, 94, 110, 190]. It has been argued that it is the most suited for the supernova observations but may not be suitable for the CMB observation [54, 55]. Here the proper time is given by $u^a\nabla_a = \partial_\tau$. The corresponding averaged equations may be deduced from equations (5.34) and (5.35) by setting $\langle S(t,\boldsymbol{x})\rangle_{\mathcal{D}} \to \langle S(\tau,\boldsymbol{x})\rangle_{\mathcal{F}}$, $N \to 1, N^k \to 0, v \to 0$, so that

$$3\left(\frac{\partial_\tau a_{\mathcal{F}}}{a_{\mathcal{F}}}\right)^2 = \langle\rho\rangle_{\mathcal{F}} + \Lambda - \frac{1}{2}\left[\mathcal{Q}_{\mathcal{F}} + \langle\mathcal{R}\rangle_{\mathcal{F}}\right] , \tag{5.57}$$

$$3\frac{\partial_\tau^2 a_{\mathcal{F}}}{a_{\mathcal{F}}} = -\frac{1}{2}\langle\rho\rangle_{\mathcal{F}}\Lambda + \mathcal{Q}_{\mathcal{F}}, \tag{5.58}$$

where

$$\mathcal{Q}_{\mathcal{F}} = \frac{2}{3}\left[\langle\Theta^2\rangle_{\mathcal{F}} - \langle\Theta\rangle_{\mathcal{F}}^2\right] - 2\langle\sigma^2\rangle_{\mathcal{F}} . \tag{5.59}$$

Here $\mathcal{Q}_{\mathcal{F}}$ is the kinematic backreaction term and $\sigma^2 = \frac{1}{2}\sigma_{ab(u)}\sigma^{ab}_{(u)}$ is the magnitude of the shear tensor of the fluid. The effective energy density and pressure are,

$$\langle\rho\rangle^{\text{eff}}_{\mathcal{F}} = \langle\rho\rangle_{\mathcal{F}} - \frac{1}{16\pi G}\mathcal{Q}_{\mathcal{F}} - \frac{1}{16\pi G}\mathcal{R}_{\mathcal{F}}, \tag{5.60}$$

$$\langle p\rangle^{\text{eff}}_{\mathcal{F}} = -\frac{1}{16\pi G}\mathcal{Q}_{\mathcal{F}} + \frac{1}{48\pi G}\mathcal{R}_{\mathcal{F}}. \tag{5.61}$$

The associated decceleration parameter $q_{\mathcal{F}}$ for averaged fluid hypersurface becomes

$$q_{\mathcal{F}} = -\frac{1}{H^2_{\mathcal{F}}}\frac{\partial_\tau^2 a_{\mathcal{F}}}{a_{\mathcal{F}}} \tag{5.62}$$

where $\ddot{a}_{\mathcal{F}}/a_{\mathcal{F}}$ is given by the averaged Raychaudhuri equation 5.57.

5.3.4 Averaged deceleration parameter

A coordinate independent description of the local volume expansion for a comoving observer is given by

$$\Theta = 3H = \nabla_a u^a = 3u^a\nabla_a \ln l = 3\frac{\partial_\tau l}{l} , \tag{5.63}$$

where u^a is the comoving local fluid 4-velocity with length scale [84]. The expansion rate obeys the usual Friedmann and Raychaudhuri equations

$$\Theta^2 = 3\left[\rho + \Lambda + \sigma^2 - \frac{1}{2}{}^3\mathcal{R}\right], \tag{5.64}$$

$$\partial_\tau\Theta = -\frac{1}{3}\,\Theta^2 - \frac{1}{2}\,(\rho + 3p) + \Lambda - 2\,\sigma^2 + 2\,\omega^2 + A_a A^a + \mathrm{D}_a A^a. \tag{5.65}$$

A local deceleration parameter may be defined as [54]

$$q_\Theta = -1 - 3\frac{\partial_\tau\Theta}{\Theta^2} \tag{5.66}$$

It is also possible to smooth Θ and the deceleration parameter, q_Θ, on the hypersurface orthogonal to u^a as

$$\left\langle q^\Theta \right\rangle_{\mathcal{F}} = -1 - 3\left\langle \frac{\partial_\tau\Theta}{\Theta^2} \right\rangle_{\mathcal{F}}, \tag{5.67}$$

where $\left\langle q^\Theta \right\rangle_{\mathcal{F}}$ is the average deceleration parameter or the dimensionless measure of change of the Hubble rate over a period of time, while the other definitions based on the Buchert formalism describe the deceleration of the averaged hypersurface or the change over a time period of the Hubble rate of a smooth hypersurface. In other words, the definition $\left\langle q^\Theta \right\rangle_{\mathcal{F}}$ gives an average information about the deceleration/acceleration of a rough/inhomogeneous domain while $q_{\mathcal{F}}$ describes the deceleration/acceleration of a properly smoothed domain.

In summary, we will study the following set of definitions of averaged deceleration parameter,

$$\substack{\text{Deceleration}\\ \text{parameters}} = \left\{\left\langle q^{N\theta}\right\rangle_{\mathcal{D}}, \left\langle q^{\theta}\right\rangle_{\mathcal{D}}, \langle q\rangle_{\mathcal{F}}, \left\langle q^{\Theta}\right\rangle_{\mathcal{F}}\right\}. \tag{5.68}$$

5.4 CMB Observer Definition

The set of field equations given in section 5.1 (i.e equations (5.5 ,5.6,5.7) may be written in terms of the quantities defined with respect to the observer attached to the coordinate grid. That is quantities defined in the frame of the observer with 4−velocity n^a, these definition are related to the ones considered in [54, 55, 108, 187]. Ishibashi and Wald [62] and other advocates [120, 153] of smallness of backreaction effect maintain that observational quantities defined in this way is the only sensible ones, since they are free from any UV problem and do not contain more than two spatial derivatives in line with the symmetry of general relativity.

Replacing the second fundamental form in equations (5.5 ,5.6,5.7) with the corresponding fluid equivalent, we find

$$\begin{aligned}
\mathcal{R} &= 2\Sigma^2 16\pi - \frac{2}{3}\xi^2 + G\gamma^2\left(\rho + v^2 p\right) + 2\Lambda\gamma^2 && (5.69)\\
\dot{\xi} &= -4\pi G\left[NP\left(3 + 2\gamma^2 v^2\right) + \gamma^2 N\rho\left(1 + v^2\right)\right] - N\Lambda - \frac{1}{3}N\xi^2 && (5.70)\\
&\quad -2N\Sigma^2 + D_i D^i N\\
\partial_t h_{ij} &= \frac{2}{3}N h_{ij}\xi + 2N\Sigma_{ij} + D_i N_j + D_j N_i && (5.71)
\end{aligned}$$

The commutation relation relates to the expansion of the hypersurface

$$\partial_t \langle\phi\rangle_{\mathcal{D}} - \langle\partial_t\phi\rangle_{\mathcal{D}} = \langle\phi\xi\rangle_{\mathcal{D}} - \langle\phi\rangle_{\mathcal{D}}\langle\xi\rangle_{\mathcal{D}} \qquad (5.72)$$

5.4.1 $N\xi$ definition

As argued in [186, 188], the evolution of the metric of the hypersurface equation 5.71 relates the dimensionless domain scale factor, $a_{\mathcal{D}} = (V_{\mathcal{D}}/V_{\mathcal{D}})^{1/3}$, to the volume of the domain, $V_{\mathcal{D}}$. With this relationship, it is then straight-forward to define the domain Hubble rate

$$\begin{aligned}
3H_{\mathcal{D}}^{N\xi} &= 3\frac{\dot{a}_{\mathcal{D}}}{a_{\mathcal{D}}} = \frac{\partial_t V_{\mathcal{D}}}{V_{\mathcal{D}}} = \frac{1}{V_{\mathcal{D}}}\int_{\mathcal{D}}\left(N\xi + D_k N^k\right)\sqrt{h}d^3x\,, && (5.73)\\
&= \left\langle N\xi + D_k N^k\right\rangle_{\mathcal{D}}\,. && (5.74)
\end{aligned}$$

Equation (5.74) describes the expansion of the coordinate grids, it says nothing about the matter field living on the spacetime and it does not capture the effect of peculiar velocity. Using equation (5.74), the averaged Hamiltonian constraint (5.69) and the corresponding averaged Raychaudhuri equation (5.70) become

$$\begin{aligned}
6H_{\mathcal{D}}^2 &= 16\pi G\left\langle N^2\gamma^2\left(\rho + v^2 p\right)\right\rangle_{\mathcal{D}} + \left\langle 2N^2\Lambda\right\rangle_{\mathcal{D}} - \left\langle N^2\mathcal{R}\right\rangle_{\mathcal{D}} - \mathcal{Q}, && (5.75)\\
3\frac{\partial_t^2 a_{\mathcal{D}}}{a_{\mathcal{D}}} &= -4\pi G\left\langle N^2 p\left(3 + 2\gamma^2 v^2\right) + \gamma^2 N^2\rho\left(1 + v^2\right)\right\rangle_{\mathcal{D}} - \Lambda\langle N\rangle_{\mathcal{D}}\mathcal{Q} + \mathcal{L}, && (5.76)
\end{aligned}$$

where the less complicated backreaction terms become,

$$\begin{aligned}
\mathcal{Q} &\equiv \frac{2}{3}\left\langle N^2\xi^2\right\rangle_{\mathcal{D}} - \frac{2}{3}\langle N\xi\rangle_{\mathcal{D}}^2 - 2\left\langle N^2\Sigma^2\right\rangle_{\mathcal{D}}\,, && (5.77)\\
\mathcal{L} &\equiv \langle\xi\partial_t N\rangle_{\mathcal{D}} + \left\langle N\xi D_k N^k\right\rangle_{\mathcal{D}} - \langle N\xi\rangle_{\mathcal{D}}\left\langle D_k N^k\right\rangle_{\mathcal{D}} + \left\langle N D_k D^k N\right\rangle_{\mathcal{D}}\,. && (5.78)
\end{aligned}$$

The decceleration parameter $q_{\mathcal{D}}$ associated with this definition is given by

$$q_{\mathcal{D}}^{N\xi} = -\frac{1}{H_{\mathcal{D}}^2}\frac{\ddot{a}_{\mathcal{D}}}{a_{\mathcal{D}}}\,, \tag{5.79}$$

with $\ddot{a}_{\mathcal{D}}/a_{\mathcal{D}}$ and $H_{\mathcal{D}}^2$ given by equations (5.75) and (5.76) respectively.

5.4.2 ξ Definition

The other possible definition that would help us understand better the role of lapse function in this formulation is $3H_{\mathcal{D}} = \langle\xi\rangle_{\mathcal{D}}$. The averaged Friedman and the Raychaudhuri equations associated with this choice become

$$\begin{aligned} 6H_{\mathcal{D}}^2 &= 16\pi G\left\langle\gamma^2\left(\rho + v^2 p\right)\right\rangle_{\mathcal{D}} + 2\langle\Lambda\rangle_{\mathcal{D}} - \langle\mathcal{R}\rangle_{\mathcal{D}} - \mathcal{Q}_{\mathcal{D}}\,, &(5.80)\\ 3\frac{\partial_t^2 a_{\mathcal{D}}}{a_{\mathcal{D}}} &= -4\pi G\left\langle Np\left(3+2\gamma^2 v^2\right) + \gamma^2 N\rho\left(1+v^2\right)\right\rangle_{\mathcal{D}} - \langle N\Lambda\rangle_{\mathcal{D}} &(5.81)\\ &\quad +\mathcal{Q}_{\mathcal{D}} + \mathcal{L}_{\mathcal{D}} + \mathcal{F}_{\mathcal{D}}, \end{aligned}$$

where the corresponding backreaction terms become

$$\begin{aligned} \mathcal{Q}_{\mathcal{D}} &\equiv \frac{2}{3}\left\langle\xi^2\right\rangle_{\mathcal{D}} - \frac{2}{3}\langle\xi\rangle_{\mathcal{D}}^2 - 2\left\langle\Sigma^2\right\rangle_{\mathcal{D}} &(5.82)\\ \mathcal{L} &\equiv \left\langle\xi D_k N^k\right\rangle_{\mathcal{D}} - \langle\xi\rangle_{\mathcal{D}}\left\langle D_k N^k\right\rangle_{\mathcal{D}} + 2\left\langle D_k D^k N\right\rangle_{\mathcal{D}} &(5.83)\\ \mathcal{F}_{\mathcal{D}} &\equiv \left\langle\xi^2\left(N-1\right)\right\rangle_{\mathcal{D}} - \langle\xi\rangle_{\mathcal{D}}\langle\xi(N-1)\rangle_{\mathcal{D}} - \frac{1}{3}\left\langle\xi^2\left(N-1\right)\right\rangle_{\mathcal{D}} - 2\left\langle\Sigma^2\left(N-1\right)\right\rangle_{\mathcal{D}} &(5.84) \end{aligned}$$

The decceleration parameter $q_{\mathcal{D}}$ associated with this definition is given by

$$q_{\mathcal{D}}^{\xi} = -\frac{1}{H_{\mathcal{D}}^2}\frac{\ddot{a}_{\mathcal{D}}}{a_{\mathcal{D}}}\,, \tag{5.85}$$

with $\ddot{a}_{\mathcal{D}}/a_{\mathcal{D}}$ and $H_{\mathcal{D}}^2$ given by equations (5.80) and (5.81) respectively. As we shall see, this set of definitions, $\{q_{\mathcal{D}}^{N\xi}, q_{\mathcal{D}}^{\xi}\}$, is well behaved even in the finite UV, and the correction to the background value lies within the expected range from cosmological perturbation theory [49].

All the deceleration parameters we have defined so far are based on a given domain and it is very tedious and complicated to calculate $\ddot{a}_{\mathcal{D}}/a_{\mathcal{D}}$ in perturbation theory from the respective averaged Raychaudhuri equations. This difficulty might be mitigated if the definition of the averaged Hubble rate is given in the form $3H_{\mathcal{D}} = \langle S\rangle = 3\dot{a}_{\mathcal{D}}/a_{\mathcal{D}}$, where S stands for any definition of average Hubble rate. With the Hubble rate in this form, it is then easier to derive a simplified version of averaged Raychaudhuri equation by simply taking a proper time derivative of $H_{\mathcal{D}}$ and using the commutation relation (5.32),

$$3\frac{\ddot{a}_{\mathcal{D}}}{a_{\mathcal{D}}} = \partial_t \langle S\rangle_{\mathcal{D}} + \frac{1}{3}\langle S\rangle_{\mathcal{D}}^2 \quad (5.86)$$

$$= \langle \partial_t S\rangle_{\mathcal{D}} + \langle SN\xi\rangle_{\mathcal{D}} + \left\langle S\partial^k N_k\right\rangle_{\mathcal{D}} - \langle S\rangle_{\mathcal{D}}\langle N\xi\rangle_{\mathcal{D}} - \langle S\rangle_{\mathcal{D}}\left\langle \partial^k N_k\right\rangle_{\mathcal{D}} + \frac{1}{3}\langle S\rangle_{\mathcal{D}}^2 \quad (5.87)$$

where $\xi = \gamma^{-1}(\theta + \theta_B)$. It is much easier to handle equation (5.87) than the full averaged Raychaudhuri equation.

5.5 Fitting Observation to an FLRW Model

The spatially averaged quantities described in sections 5.3 and 5.4 come with a significant drawback: even for moderately sized domains, they are unobservable because they are averaged on spatial hypersurfaces, whereas we only have observational access to our past lightcone. Rasanen [59] argues that there appears to be a relationship between the spatially averaged quantities and those defined on the past lightcone, but this link has not been made explicit. Therefore our aim is to fit quantities calculated on the lumpy past lightcone of an observer to a homogeneous and isotropic FLRW spacetime in a fashion similar to what is done observationally.

In a general spacetime, the distance-redshift relation is the key relation used to access the spacetime geometry. The angular-diameter distance d_A and luminosity distance, d_L, are functions of direction on the observers' sky, as well as redshift. That is, we can write $d_L = d_L(z; e^a) = (1+z)^2 d_A(z; e^a)$, and each line of sight gives rise to a different distance-redshift relation, each of which may be a very complicated, multivalued, function. These functions can be expanded in terms of a spherical harmonic expansion (see Appendix B):

$$d_A(z; e^a) = \sum_{\ell=0}^{\infty} D_{A_\ell}(z) e^{\langle A_\ell\rangle} \quad (5.88)$$

where the PSTF tensors $D_{A_\ell}(z)$ describe the spherical harmonic moments of the distance-redshift relation. It is the monopole $[D(z)]$ of this relation that is normally fitted to observables as the higher multipoles are assumed to be insignificant (i.e., $D_{A_\ell}(z) \ll D(z)$ for $\ell \geq 1$). In the context of our discussions of backreaction, effects of backreaction will show up if $D(z)$ differs from the background value.

The general equation that determes distances from redshifts may be derived from the Sachs optical equations. For a past pointing null geodesic, k^a, with spatial direction e^a at the observer u^a ($e^a e_a = 1$, $e_a u^a = 0$), the area distance to a source at redshift z is determined by [182]

$$(1+z)^2 H_{\|}(z)\frac{\mathrm{d}}{\mathrm{d}\,z}\left[(1+z)^2 H_{\|}(z)\frac{\mathrm{d}}{\mathrm{d}\,z}d_A(z)\right] = -\left[4\pi G\,(1+z)^2\,(\rho+p) + |\hat{\sigma}|^2\right] d_A(z), \quad (5.89)$$

where

$$\mathrm{H}_{\|}(z) = \Theta/3 - A_a e^a + \sigma_{ab}e^a e^b, \quad (5.90)$$

and $\hat{\sigma}$ is the null shear of k^a. We have assumed that the matter content is a perfect fluid as seen by the observer. An alternative approach to finding the distance-redshift relation is to use the Kristian and Sachs approach [170]. This performs a local Taylor series expansion of distances and redshift in terms of the affine parameter along the null ray. This gives the general relations (for details of the derivation see [57, 170, 217])

$$z = [K^c K^d \nabla_d u_c]_0 d_A + \frac{1}{2}\left[K^c K^d K^e \nabla_e \nabla_d u_c\right]_0 d_A^2 + \mathcal{O}(d_A)^3, \quad (5.91)$$

$$d_A = \frac{z}{[K^c K^d \nabla_c u_d]_0}\left\{1 - \left[\frac{1}{2}\frac{K^c K^d K^e \nabla_e \nabla_d u_c}{\left(K^c K^d \nabla_c u_d\right)^2}\right]_0 z + \mathcal{O}(z^2)\right\}, \quad (5.92)$$

$$d_L = \frac{z}{[K^c K^d \nabla_d u_c]_0}\left\{1 + \frac{1}{2}\left[4 - \frac{K^c K^d K^e \nabla_e \nabla_d u_c}{\left(K^c K^d \nabla_c u_d\right)^2}\right]_0 z + \mathcal{O}(z^2)\right\}, \quad (5.93)$$

where K^a is a past pointing null vector given by

$$K^a = -u^a + e^a, \quad (5.94)$$

where e^a is a normalised spacelike vector orthogonal to u^a: $e^a e_a = 1$; $u_a e^a = 0$; e^a simply defines the direction of the photon relative to u^a, which we have chosen to point down the past lightcone. The associated null geodesics are given by $k^a = (1+z)K^a$.

We shall make use of these equations to define the deceleration parameter from the observational point of view, which we can then contrast with the definitions which are given by smoothing spatially on a given hypersurface (see equation 5.68). Within the standard cosmology, cosmological parameters like the Hubble rate, and deceleration parameter are well defined based on the background metric. These cosmological parameters can be evaluated today, at t_0, by taking a Taylor series expansion of the scale factor $a(t)$

$$a(t) = a_0\left[1 + H_0\,(t-t_0) + \frac{1}{2}\,q_0\,H_0^2\,(t-t_0)^2 + O([t-t_0]^3)\right]. \quad (5.95)$$

Similarly, we can calculate the important observational quantities of interest, such as the relation between the angular-diameter distance and redshift $d_A(z)$:

$$z = H_0 d_A + \frac{1}{2}(3+q_0)(d_A H_0)^2 + \mathcal{O}(d_A H_0)^3 . \tag{5.96}$$

$$d_A(z) = \frac{z}{H_0}\left[1 - \frac{1}{2}(3+q_0) z + \mathcal{O}(z)^2\right], \tag{5.97}$$

$$d_L(z) = \frac{z}{H_0}\left[1 + \frac{1}{2}[1-q_0] z + \mathcal{O}(z)^2\right] . \tag{5.98}$$

It is certainly debatable whether these series expansions have much relation to the distance-redshift relation in the real universe. (In particular, this can't be analytic!) In certain directions the relations will be multivalued as a light beam passes through collapsing regions [1]. However, we shall be primarily concerned with the spherical harmonic expansion of the distance-redshift relation, where such multi-valued problems should average out.

5.5.1 General spacetime Observables

In an arbitrary spacetime, the terms in the Kristian and Sachs series expansions may be decomposed using the 1 + 3 covariant approach (see the Appendix B) as

$$K^a K^b \nabla_a u_b = \frac{1}{3}\Theta - A_a e^a + \sigma_{ab} e^a e^b , \tag{5.99}$$

$$\begin{aligned} K^a K^b K^c \nabla_a \nabla_b u_c = {} & \frac{1}{6}(\rho + 3p) + \frac{1}{3}\Theta^2 - \frac{1}{3}\Lambda - \frac{2}{3}\omega_a\omega^a + \sigma_{ab}\sigma^{ab} + A_a A^a - \frac{2}{3}\tilde{\nabla}_a A^a \\ & + e^a \left[\frac{1}{3}\mathrm{D}_a\Theta + \frac{2}{5}\mathrm{D}_b\sigma^b_a + \dot{A}_a - \frac{4}{3}\Theta A_a - \frac{7}{5}A^b\sigma_{ab} - \epsilon_{abc}A^b\omega^c\right] \\ & + e^{\langle a}e^{b\rangle}\left[E_{ab} - \frac{1}{2}\pi_{ab} + 2\Theta\sigma_{ab} + \omega_a\omega_b + 3\sigma^c{}_a\sigma_{bc} + 2\epsilon_{acd}\omega^c\sigma_b{}^d \right. \\ & \left. - 2\mathrm{D}_a A_b\right] - e^{\langle a}e^b e^{c\rangle}\left[A_a\sigma_{bc} - \mathrm{D}_a\sigma_{bc}\right] \end{aligned} \tag{5.100}$$

For details on the derivation and moment decomposition of the results above see [60] and Appendix B.

The Projected Symmetric Trace-Free tensors (PSTF) $e^{\langle A_\ell \rangle}$ in these expression are a covariant representation of the spherical harmonics when evaluated at a given point in spacetime (see Appendix B). One could consider defining some 'generalised' observables such as the Hubble rate as the leading term of the distance-redshift relation, which gives:

$$H_0^{\mathrm{obs}} = \sum_{\ell=0}^{2} \mathcal{H}_{A_\ell} e^{A_\ell} = [K^a K^b \nabla_a u_b]_0 = \frac{1}{3}\Theta - e^a A_a + e^a e^b \sigma_{ab} \tag{5.101}$$

[1] S. Räsänen, private communication.

where the PSTF tensors $\mathcal{H}_{A_\ell}$ are the spherical harmonic moments of the generalised Hubble rate $H_0^{\rm obs}$. It is easy to identify the moments in this case as,

$$\mathcal{H} = \frac{1}{3}\Theta, \quad \mathcal{H}_a = A_a \;\text{ and }\; \mathcal{H}_{ab} = \sigma_{ab}\,, \tag{5.102}$$

and understood to be evaluated at the observer. The square of $H_0^{\rm obs}$, which we shall require below, may also be decomposed as (for details of the moment decomposition, see appendix B)

$$\begin{aligned}
\left(H_0^{\rm obs}\right)^2 &= \left(K^aK^b\nabla_a u_b|_0\right)^2 && (5.103)\\
&= \frac{1}{9}\Theta^2 + \frac{1}{3}A_aA^b + \frac{2}{15}\sigma_{ab}\sigma^{ab} - e^{\langle a\rangle}\left(\frac{2}{3}A_a\Theta + \frac{4}{5}A^b\sigma_{ab}\right) && (5.104)\\
&\quad + e^{\langle a}e^{b\rangle}\left(A_aA_b + \frac{4}{7}\sigma_a^c\sigma_{bc} + \frac{2}{3}\Theta\sigma_{ab}\right)\\
&\quad - 2e^{\langle a}e^be^{c\rangle}\left(A_a\sigma_{bc}\right) + e^{\langle a}e^be^ce^{d\rangle}\left(\sigma_{ab}\sigma_{cd}\right)\\
&= \Xi + e^{\langle a\rangle}\Xi_a + e^{\langle a}e^{b\rangle}\Xi_{ab} + e^{\langle a}e^be^{c\rangle}\Xi_{abc} + e^{\langle a}e^be^ce^{d\rangle}\Xi_{abcd}\,, && (5.105)
\end{aligned}$$

where

$$\begin{aligned}
\Xi &= \frac{1}{9}\Theta^2 + \frac{1}{3}A_aA^b + \frac{2}{15}\sigma_{ab}\sigma^{ab}\,, && (5.106)\\
\Xi_a &= -\frac{2}{3}A_a\Theta - \frac{4}{5}A^b\sigma_{ab}\,, && (5.107)\\
\Xi_{ab} &= A_{\langle a}A_{b\rangle} + \frac{4}{7}\sigma_{\langle a}{}^c\sigma_{b\rangle c} + \frac{2}{3}\Theta\sigma_{ab}\,, && (5.108)\\
\Xi_{abc} &= -2A_{\langle a}\sigma_{bc\rangle}\,, && (5.109)\\
\Xi_{abcd} &= \sigma_{\langle ab}\sigma_{cd\rangle}\,. && (5.110)
\end{aligned}$$

Note that each moment Ξ_{A_ℓ} does not imply a corresponding squared moment of $H_0^{\rm obs}$, as it represents the coefficient of $e^{\langle A_\ell\rangle}$ in the $\left(H_0^{\rm obs}\right)^2$ expansion. Again, these are understood to be evaluated at the observer.

We would like to define a generalised deceleration parameter with the requirement that it reduces to FLRW case in appropriate limit. So we must consider relating the FLRW $(3+q_0)H_0^2$ to $K^aK^bK^c\nabla_a\nabla_b u_c$. The simplest way was given in [60] as

$$(3+q_0^{\rm obs})\mathcal{H}^2 = [K^aK^bK^c\nabla_a\nabla_b u_c]_0\,. \tag{5.111}$$

However, it is more consistent to define $q_0^{\rm obs}$ via:

$$(3+q_0^{\rm obs})(H_0^{\rm obs})^2 = [K^aK^bK^c\nabla_a\nabla_b u_c]_0\,. \tag{5.112}$$

Then, writing

$$q_0^{\rm obs} = \sum_{\ell=0}^{\infty} \mathcal{Q}_{A_\ell} e^{A_\ell}, \tag{5.113}$$

where the PSTF tensors $\mathcal{Q}_{A_\ell}$ are the multipole moments of the observational deceleration parameter. Using the relations described in the appendix B, the full decomposition of the deceleration parameter becomes,

$$\begin{aligned}
&e^{A_\ell}\Big[\Xi\mathcal{Q}_{\langle A_\ell\rangle} + \Xi_{\langle a_\ell}\mathcal{Q}_{A_{\ell-1}\rangle} + \Xi_{\langle a_{\ell-1}a_\ell}\mathcal{Q}_{A_{\ell-2}\rangle} + \Xi_{\langle a_{\ell-2}a_{\ell-1}a_\ell}\mathcal{Q}_{A_{\ell-3}\rangle} \\
&+\Xi_{\langle a_{\ell-3}a_{\ell-2}a_{\ell-1}a_\ell}\mathcal{Q}_{A_{\ell-4}\rangle} + \frac{(\ell+1)}{(2\ell+3)}\Xi^b\mathcal{Q}_{b\langle A_\ell\rangle} + \frac{2\ell}{(2\ell+3)}\mathcal{Q}_{d\langle A_{\ell-1}}\Xi^d{}_{a_\ell\rangle} \\
&+\frac{3(\ell-1)}{(2\ell+3)}\mathcal{Q}_{d\langle A_{\ell-2}}\Xi^d{}_{a_{\ell-1}a_\ell\rangle} + \frac{4(\ell-2)}{(2\ell+3)}\mathcal{Q}_{d\langle A_{\ell-3}}\Xi^d_{a_{\ell-2}a_{\ell-1}a_\ell\rangle} + \frac{(\ell+2)(\ell+1)}{(2\ell+3)(2\ell+5)}\mathcal{Q}_{ed\langle A_\ell\rangle}\Xi^{ed} \\
&+\frac{3\ell(\ell+1)}{(2\ell+3)(2\ell+5)}\mathcal{Q}_{ef\langle A_{\ell-1}}\Xi^{ef}{}_{a_\ell\rangle} + \frac{6\ell(\ell-1)}{(2\ell+5)(2\ell+3)}\mathcal{Q}_{mn\langle A_{\ell-2}}\Xi^{mn}{}_{a_{\ell-1}a_\ell\rangle} \\
&+\frac{(\ell+3)(\ell+2)(\ell+1)}{(2\ell+7)(2\ell+5)(2\ell+3)}\mathcal{Q}_{edfA_\ell}\Xi^{edf} + \frac{\ell(\ell+1)(\ell+2)}{(2\ell+7)(2\ell+5)(2\ell+3)}\mathcal{Q}_{mno\langle A_{\ell-1}}\Xi^{mno}{}_{a_\ell\rangle} \\
&+\frac{(\ell+4)(\ell+3)(\ell+2)(\ell+1)}{(2\ell+9)(2\ell+7)(2\ell+5)(2\ell+3)}\mathcal{Q}_{mnopA_\ell}\Xi^{mnop}\Big] \\
&= \left[\frac{1}{6}(\rho+3p) - \frac{1}{3}\Lambda - \frac{2}{3}\tilde{\nabla}_bA^b + \frac{3}{5}\sigma_{ba}\sigma^{ba} - \frac{2}{3}\omega_b\omega^b\right] \\
&+e^b\left[-\frac{2}{3}\Theta A_b - \dot{A}_b + A^a\sigma_{ba} - \frac{1}{3}\mathrm{D}_b\Theta - \frac{2}{5}\mathrm{D}_a\sigma^a_b\right] \\
&+e^be^a\Big[-2\mathrm{D}_{\langle b}A_{a\rangle} - 3A_{\langle b}A_{a\rangle} + E_{\langle ba\rangle} - \frac{1}{2}\pi_{\langle ba\rangle} + \frac{2}{7}\sigma_{c\langle b}\sigma_{a\rangle}{}^c \\
&-\frac{1}{2}\Theta\sigma_{ab} + \omega_{\langle b}\omega_{a\rangle} + \epsilon_{\langle acd}\omega^c\sigma^d_{b\rangle}\Big] + e^be^ae^c\left[-5A_{\langle b}\sigma_{ac\rangle} - \mathrm{D}_{\langle b}\sigma_{ac\rangle}\right] - 3e^be^ae^ce^d\sigma_{\langle ba}\sigma_{cd\rangle}\,.
\end{aligned} \tag{5.114}$$

This equation tells us how the monopole, dipole, and higher multipoles of the deceleration parameter are sourced by the kinematics of the observers' congruence and the gravitational field. The individual multipoles of Eq. (5.114) may be found as:

$$\begin{aligned}
\mathcal{Q} &= \frac{1}{\Xi}\left\{\left[\frac{1}{6}(\rho+3p)-\frac{1}{3}\Lambda-\frac{2}{3}\mathrm{D}_bA^b+\frac{3}{5}\sigma_{ba}\sigma^{ba}-\frac{2}{3}\omega_b\omega^b\right]-\frac{1}{3}\Xi_b\mathcal{Q}^b\right. \qquad (5.115)\\
&\left.-\frac{2}{15}\Xi_{cd}\mathcal{Q}^{cd}-\frac{2}{35}\Xi_{mno}\mathcal{Q}^{mno}-\frac{8}{315}\Xi_{mnop}\mathcal{Q}^{mnop}\right\},\\
\mathcal{Q}_a &= \frac{1}{\Xi}\left[\left(-\frac{2}{3}\Theta A_b-\dot{A}_a+A^b\sigma_{ba}-\frac{1}{3}\mathrm{D}_a\Theta-\frac{2}{5}\mathrm{D}_b\sigma^b_a\right)-\Xi_a\mathcal{Q}-\frac{2}{5}\mathcal{Q}^d\Xi_{da}\right. \qquad (5.116)\\
&-\frac{2}{5}\Xi^b\mathcal{Q}_{ba}-\frac{6}{35}\mathcal{Q}^{gf}\Xi_{fga}-\frac{6}{35}\Xi_{cd}\mathcal{Q}^{cd}_a-\frac{8}{105}\mathcal{Q}_{mnoa}\Xi^{mno}\\
&\left.-\frac{2}{105}\mathcal{Q}_{mno}\Xi^{mno}{}_a-\frac{8}{231}\mathcal{Q}_{mnopa}\Xi^{mnop}\right],\\
\mathcal{Q}_{\langle ab\rangle} &= \frac{1}{\Xi}\left[-2\mathrm{D}_{\langle b}A_{a\rangle}-3A_{\langle b}A_{a\rangle}+E_{\langle ba\rangle}-\frac{1}{2}\pi_{\langle ba\rangle}+\frac{2}{7}\sigma_{c\langle b}\sigma_{a\rangle}{}^c-\frac{1}{2}\Theta\sigma_{ab}\right. \qquad (5.117)\\
&+\omega_{\langle b}\omega_{a\rangle}+\epsilon_{\langle acd}\omega^c\sigma^d_{b\rangle}-\Xi_{\langle ab\rangle}\mathcal{Q}-\Xi_{\langle a}\mathcal{Q}_{b\rangle}-\frac{4}{7}\mathcal{Q}^d{}_{\langle a}\Xi_{|d|b\rangle}-\frac{3}{7}\Xi^c\mathcal{Q}_{c\langle ab\rangle}\\
&-\frac{3}{7}\mathcal{Q}^c\Xi_{c\langle ab\rangle}-\frac{4}{21}\mathcal{Q}_{mn\langle ab\rangle}\Xi^{mn}-\frac{2}{7}\mathcal{Q}^{gf}{}_{\langle a}\Xi_{|fg|b\rangle}\\
&\left.-\frac{20}{231}\mathcal{Q}_{mno\langle ab\rangle}\Xi^{mno}-\frac{4}{21}\Xi^{de}{}_{\langle ab\rangle}\mathcal{Q}_{de}-\frac{8}{231}\mathcal{Q}_{mnoa}\Xi^{mno}{}_b-\frac{40}{1001}\mathcal{Q}_{defg\langle ab\rangle}\Xi^{defg}\right],
\end{aligned}$$

$$\begin{aligned}
\mathcal{Q}_{\langle abc\rangle} &= \frac{1}{\Xi}\left[\left[-5A_{\langle b}\sigma_{ac\rangle}-\mathrm{D}_{\langle b}\sigma_{ac\rangle}\right]-\Xi_{\langle abc\rangle}\mathcal{Q}-\Xi_{\langle ab}\mathcal{Q}_{c\rangle}-\Xi_{\langle a}\mathcal{Q}_{bc\rangle}\right. \qquad (5.118)\\
&-\frac{4}{9}\Xi^d\mathcal{Q}_{d\langle abc\rangle}-\frac{6}{9}\mathcal{Q}_{d\langle ab}\Xi^d{}_{c\rangle}-\frac{6}{9}\mathcal{Q}_{d\langle a}\Xi^d{}_{bc\rangle}\\
&-\frac{20}{99}\mathcal{Q}_{de\langle abc\rangle}\Xi^{de}-\frac{4}{11}\mathcal{Q}_{mn\langle ab}\Xi^{mn}{}_{c\rangle}-\frac{40}{429}\mathcal{Q}_{mno\langle abc\rangle}\Xi^{mno}-\frac{20}{429}\mathcal{Q}_{mno\langle ab}\Xi^{mno}{}_{c\rangle}\\
&\left.-\frac{4}{9}\mathcal{Q}_d\Xi^d{}_{\langle abc\rangle}-\frac{4}{11}\mathcal{Q}_{de\langle a}\Xi^{de}{}_{bc\rangle}-\frac{56}{1287}\mathcal{Q}_{defg\langle abc\rangle}\Xi^{defg}\right],\\
\mathcal{Q}_{\langle abcd\rangle} &= \frac{1}{\Xi}\left[-3\sigma_{\langle ab}\sigma_{cd\rangle}-\Xi_{abcd}\mathcal{Q}-\Xi_{\langle a}\mathcal{Q}_{bcd\rangle}-\Xi_{\langle ab}\mathcal{Q}_{cd\rangle}-\Xi_{\langle abc}\mathcal{Q}_{d\rangle}\right. \qquad (5.119)\\
&-\frac{5}{11}\Xi^e\mathcal{Q}_{e\langle abcd\rangle}-\frac{8}{11}\mathcal{Q}_{e\langle abc}\Xi^e{}_{d\rangle}-\frac{9}{11}\mathcal{Q}_{e\langle ab}\Xi^e{}_{cd\rangle}-\frac{30}{143}\mathcal{Q}_{fg\langle abcd\rangle}\Xi^{fg}\\
&-\frac{8}{143}\mathcal{Q}_{fgh\langle abc}\Xi^{fgh}{}_{d\rangle}-\frac{14}{143}\mathcal{Q}_{efg\langle abcd\rangle}\Xi^{efg}-\frac{8}{11}\mathcal{Q}_{m\langle a}\Xi^m{}_{bcd}\\
&\left.-\frac{60}{143}\mathcal{Q}_{ef\langle abc}\Xi^{ef}{}_{d\rangle}-\frac{72}{143}\mathcal{Q}_{mn\langle ab}\Xi^{mn}{}_{cd\rangle}-\frac{112}{2431}\mathcal{Q}_{mnop\langle abcd\rangle}\Xi^{mnop}\right],
\end{aligned}$$

while for $\ell \geq 5$ we have

$$
\begin{aligned}
\mathcal{Q}_{\langle A_\ell\rangle} = -\frac{1}{\Xi}\Big\{&\Xi_{\langle a_\ell}\mathcal{Q}_{A_{\ell-1}\rangle} + \Xi_{\langle a_{\ell-1}a_\ell}\mathcal{Q}_{A_{\ell-2}\rangle} + \Xi_{\langle a_{\ell-2}a_{\ell-1}a_\ell}\mathcal{Q}_{A_{\ell-3}\rangle} \\
&+\Xi_{\langle a_{\ell-3}a_{\ell-2}a_{\ell-1}a_\ell}\mathcal{Q}_{A_{\ell-4}\rangle} + \frac{(\ell+1)}{(2\ell+3)}\Xi^b\mathcal{Q}_{b\langle A_\ell\rangle} + \frac{2\ell}{(2\ell+3)}\mathcal{Q}_{d\langle A_{\ell-1}}\Xi^d{}_{a_\ell\rangle} \\
&+\frac{3(\ell-1)}{(2\ell+3)}\mathcal{Q}_{d\langle A_{\ell-2}}\Xi^d{}_{a_{\ell-1}a_\ell} + \frac{4(\ell-2)}{(2\ell+3)}\mathcal{Q}_{d\langle A_{\ell-3}}\Xi^d_{a_{\ell-2}a_{\ell-1}a_\ell} \\
&+\frac{(\ell+2)(\ell+1)}{(2\ell+3)(2\ell+5)}\mathcal{Q}_{ed\langle A_\ell\rangle}\Xi^{ed} + \frac{3\ell(\ell+1)}{(2\ell+3)(2\ell+5)}\mathcal{Q}_{ef\langle A_{\ell-1}}\Xi^{ef}{}_{a_\ell\rangle} \\
&+\frac{6\ell(\ell-1)}{(2\ell+5)(2\ell+3)}\mathcal{Q}_{mn\langle A_{\ell-2}}\Xi^{mn}{}_{a_{\ell-1}a_\ell\rangle} + \frac{(\ell+3)(\ell+2)(\ell+1)}{(2\ell+7)(2\ell+5)(2\ell+3)}\mathcal{Q}_{edfA_\ell}\Xi^{edf} \\
&+\frac{\ell(\ell+1)(\ell+2)}{(2\ell+7)(2\ell+5)(2\ell+3)}\mathcal{Q}_{mno\langle A_{\ell-1}}\Xi^{mno}{}_{a_\ell\rangle} \\
&+\frac{(\ell+4)(\ell+3)(\ell+2)(\ell+1)}{(2\ell+9)(2\ell+7)(2\ell+5)(2\ell+3)}\mathcal{Q}_{mnopA_\ell}\Xi^{mnop}\Big\}
\end{aligned}
\tag{5.120}
$$

which relates the ℓ'th multipole to the $(\ell-4)$'th and $(\ell+4)$'th, as well as those in between.

Notice the build up of a hierarchy due to coupling between multipoles is similar to the Boltzmann hierarchy in Kinetic theory [189]. The coupling between multipoles here is algebraic and linear, so can, in principle, be solved through Gauss reduction. However, it is impossible to isolate a single harmonic in this way for a general spacetime.

Neglecting the coupling to lower and higher multipoles and also assuming a universe dominated by dust, the dipole moment of the deceleration parameter is sourced only by $\mathcal{Q}_a \sim \frac{5}{3}\mathrm{D}_a\Theta$ (we have used one of the constraint equations to substitute for the divergence of shear), it gives information about the covariant gauge invariant description of velocity perturbations. There are papers [218, 219], for example that have detected a weak directional dependence in the deceleration parameter with SNIa data sets. The result given in equation (5.114) clearly points towards the kind of information that could be obtained from such data set and their physical implication. For instance, equation (5.116) says that within the standard model that the source of non-vanishing dipole of the deceleration parameter is the velocity perturbation along the line of sight.

Similarly the quadrupole moment depends on the shear and the electric part of the Weyl tensor, $\mathcal{Q}_{ab} \sim E_{ab} + \frac{9}{7}\sigma_{c\langle a}\sigma_{b\rangle}{}^c$, clearly pointing to the role of tidal force on activating the quadrupole moment of the observed deceleration parameter.

5.6 Fitting Problem: Almost FLRW observables

To fit observables in the real universe to an FLRW spacetime, we have to average over the observers' sky to eliminate the direction dependence [200, 220]. Any discrepancy from the background caused by the spacetime being almost-FLRW will show up in the distance-redshift relations we will discuss in this section as a systematic shift to the monopole. Exactly how this happens depends on how we define the monopole, which depends on the particular observables we compare.

Here we calculate the monopole of deceleration parameter in a manner similar in principle to how it is calculated from observation in a real universe, i.e we fit the distance-redshift relation obtained using the Kristian and Sachs approach (equations 5.915.925.93) to the distance-redshift relation from the FLRW spacetime (equations 5.96,5.97,5.98). The deceleration parameter obtained in this manner forms yet another set of definitions of deceleration parameter we will be comparing to each other in the subsequent sections.

To be more explicit, we will be matching the FLRW definition for the redshift, z (equation (5.96) to the Kristian and Sachs expression for redshift in a general spacetime (equation 5.91) and the FLRW expression for angular-diameter distance, d_A, is matched to the Kristain and Sachs definition for the same quantity (equation 5.92). We do the same thing for the luminosity distance, d_L. We denote all sky average by $\langle \cdots \rangle_\Omega$.

- **Redshift based definition**

  Matching the $\langle z(d_A)\rangle_\Omega$ equation, i.e., comparing equations (5.91) and (5.96) order by order in area distance d_A expansion and using equations (5.99, 5.101, 5.103) and taking the all sky average, we find

$$H_0^z = \left\langle K^a K^b \nabla_a u_b \right\rangle_\Omega = \frac{1}{3}\Theta = \mathrm{H}\,, \tag{5.121}$$

$$q_0^z = \frac{\left\langle K^a K^b K^c \nabla_a \nabla_b u_c \right\rangle_\Omega}{\langle K^a K^b \nabla_a u_b \rangle_\Omega^2} - 3\,. \tag{5.122}$$

- **Angular-diameter distance based definition**

  Considering the $\langle d_A(z)\rangle_\Omega$ equations for the general and the FLRW spacetime, i.e., matching equations (5.92) and (5.97) order by order in redshift z expansion, and then using equations (5.99, 5.101, 5.103), we find

$$\left(H_0^{d_A}\right)^{-1} = \left\langle \left(K^a K^b \nabla_a u_b\right)^{-1} \right\rangle_\Omega , \tag{5.123}$$

$$q_0^{d_A} = H_0^{d_A} \left\langle \frac{K^a K^b K^c \nabla_a \nabla_b u_c}{(K^a K^b \nabla_a u_b)^3} \right\rangle_\Omega - 3 . \tag{5.124}$$

- **Luminosity distance based definition**

  Similarly, considering the $\langle d_L(z) \rangle_\Omega$ equations, i.e equations (5.93) and (5.98) and performing all sky average, we find:

$$(H_0^{d_L})^{-1} = \left\langle \left(K^a K^b \nabla_a u_b\right)^{-1} \right\rangle_\Omega , \tag{5.125}$$

$$q_0^{d_L} = 1 - H_0^{d_L} \left[\left\langle \frac{4}{K^a K^b \nabla_a u_b} \right\rangle_\Omega - \left\langle \frac{K^a K^b K^c \nabla_a \nabla_b u_c}{(K^a K^b \nabla_a u_b)^3} \right\rangle_\Omega \right] . \tag{5.126}$$

  The $q_0^{d_L}$ definition was studied in [197, 200, 201]. Clearly, these definitions reduce to the same as defined via the angular-diameter distance, and we shall denote them both with a d superscript.

The difficulty in defining monopole quantities via $d(z)$ is apparent even for H_0. Consider the case of geodesic observers. Writing

$$\frac{1}{H_0^{\rm obs}} = \frac{1}{\mathrm{H} + \sigma_{ab} e^a e^b} = \sum_{\ell=0}^{\infty} \xi_{A_\ell} e^{A_\ell}, \tag{5.127}$$

so that $\xi^{-1} = H_0^d$. We then find the hierarchy

$$\mathrm{H}\, \xi + \frac{2}{15} \xi_{ab} \sigma^{ab} = 1 , \tag{5.128}$$

$$\mathrm{H}\, \xi_a + \frac{2}{5} \xi^b \sigma_{ab} + \frac{6}{35} \xi_{abc} \sigma^{bc} = 0 , \tag{5.129}$$

and for $\ell \geq 2$,

$$\mathrm{H}\, \xi_{A_\ell} + \xi_{\langle A_{\ell-2}} \sigma_{a_{\ell-1} a_\ell \rangle} + \frac{2\ell}{2\ell+3} \xi_{c \langle A_{\ell-1}} \sigma^c{}_{a_\ell \rangle} + \frac{(\ell+1)(\ell+2)}{(2\ell+3)(2\ell+5)} \xi_{cd A_\ell} \sigma^{cd} = 0 . \tag{5.130}$$

Note that the odd and even multipoles decouple here.

It is striking how much more complicated it is to evaluate the $d(z)$ relation in spherical harmonics than the equivalent $z(d)$ relation. On a sphere of fixed *distance*, the $z(d)$ relation expands the redshift in spherical harmonics, while on a sphere of fixed *redshift* the $d(z)$ relation expands the distance in spherical harmonics.

For an almost FLRW spacetime we can simplify the expressions by neglecting powers of non-FLRW terms. Furthermore, we are interested in a irrotational spacetime with cosmological constant, so that $A_a = \omega_a = \mathcal{H}_a = \mathcal{H}_{abc} = 0$ in our expressions above. As backreaction primarily takes place at second-order in perturbation theory, we shall need to consider the above expressions up to quadratic terms in the shear. Consider first inverse powers of $H_0^{\rm obs} = K^a K^b \nabla_a u_b$ which regularly appear. Expanding in powers of the dimensionless shear $\tilde\sigma_{ab} = \sigma_{ab}/\mathrm{H}$ we have

$$(H_0^{\rm obs})^{-n} = \mathrm{H}^{-n}\left[1 - n\tilde\sigma_{ab}e^a e^b + \frac{1}{2}n(n+1)(\tilde\sigma_{ab}e^a e^b)^2 + \mathcal{O}(\tilde\sigma^3)\right]. \qquad (5.131)$$

With this we then find

$$H_0^{d_A} = H_0^{d_L} = \mathrm{H}\left[1 - \frac{2}{15}\frac{\sigma_{ab}\sigma^{ab}}{\mathrm{H}^2} + \mathcal{O}(\tilde\sigma^4)\right]. \qquad (5.132)$$

For the three deceleration parameters we have

$$q_0^z = \frac{1}{\mathrm{H}^2}\left[\frac{1}{6}\rho - \frac{1}{3}\Lambda + \sigma_{ab}\sigma^{ab}\right], \qquad (5.133)$$

$$q_0^d = \frac{1}{\mathrm{H}^2}\left[\frac{1}{6}\rho - \frac{1}{3}\Lambda + \left(\frac{1}{18}\rho - \frac{1}{9}\Lambda + \frac{4}{5}\mathrm{H}^2\right)\frac{\sigma_{ab}\sigma^{ab}}{\mathrm{H}^2} - \frac{6}{15}\frac{\sigma_{ab}E^{ab}}{\mathrm{H}}\right] + \mathcal{O}(\tilde\sigma^3), \qquad (5.134)$$

$$\mathcal{Q} = \frac{1}{\mathrm{H}^2}\left[\frac{1}{6}\rho - \frac{1}{3}\Lambda + \left(\frac{1}{15}\rho - \frac{2}{15}\Lambda + \frac{9}{15}\mathrm{H}^2\right)\frac{\sigma_{ab}\sigma^{ab}}{\mathrm{H}^2} - \frac{4}{15}\frac{\sigma_{ab}E^{ab}}{\mathrm{H}}\right] + \mathcal{O}(\tilde\sigma^3). \qquad (5.135)$$

These expressions for $\{q_0^z, q_0^d, \mathcal{Q}\}$ form the basis of our evaluation of the deceleration parameter from an observational point of view.

5.7 Covariant quantities from perturbed FLRW model

The equations derived in Section 5.3 are not closed and the one derived in section 5.5 does not carry any significant quantitive information. However, we may extract predictive information from them if we suppose that the universe is well described by a perturbed FLRW background. The perturbed FLRW spacetime in the longitudinal (Poisson) gauge may be written as

$$ds^2 = -\left[1 + 2\Phi + \Phi^{(2)}\right]dt^2 - aV_i dx^i dt + a^2\left[(1 - 2\Psi - \Psi^{(2)})\gamma_{ij} + h_{ij}\right]dx^i dx^j, \qquad (5.136)$$

where the coordinates coincide with the n^a frame such that $n_a = -N\partial_a t$, the lapse function is given by $N = \left(1 + \Phi + \frac{1}{2}\Phi^{(2)} - \frac{1}{2}\Phi^2\right)$, the shift parameter $N_i = aV_i$ [2] and the metric of the hypersurface $g_{ij} = a^2\left[(1 - 2\Psi - \Psi^{(2)})\gamma_{ij} + h_{ij}\right]$. V_i and h_{ij} are the induced vector modes and tensor modes respectively[79, 80], but we will neglect them for simplicity. The first-order scalar perturbations are given by Φ, Ψ and the second-order part by $\Phi^{(2)}, \Psi^{(2)}$ are needed for a consistency. In this gauge we have the metric in its Newtonian-like form, which we may think of as the local rest-frame of the gravitational field because it is the frame in which the magnetic part of the Weyl tensor vanishes when vectors and tensors are ignored [86]. One could also choose a metric in synchronous coordinate such that u^a coincides with $u_a = -\partial_a \tau$ [51, 52]. It is important to note that the choice of coordinate does not affect the results if one considers only the physical quantities and appropriately identify a map between the averaging hypersurface in both coordinates [49, 188].

The peculiar velocity v^i may be expanded to second order and is given by:

$$v_i = \frac{1}{2a^2}\partial_i(2v_1 + v_2). \tag{5.137}$$

As usual, the background Friedmann's equation and the deceleration parameter for the pure dust and positive cosmological constant universe are given by:

$$\begin{aligned} H(z)^2 &= H_0^2\left[\Omega_0(1+z)^3 + 1 - \Omega_0\right], \\ q(z) &= -\frac{1}{H^2}\frac{\ddot{a}}{a} = -1 + \frac{1+z}{H(z)}\frac{dH}{dz} = -1 + \frac{3}{2}\Omega_m(z) \end{aligned}$$

respectively, where

$$\Omega_m(z) = \frac{\Omega_0(1+z)^3}{\left[\Omega_0(1+z)^3 + 1 - \Omega_0\right]^{1/2}} \tag{5.138}$$

For a single fluid with zero pressure and no anisotropic stress $\Psi = \Phi$, and Φ obeys the 'master' equation

$$\ddot{\Phi} + 4H\dot{\Phi} + \Lambda\Phi = 0\,. \tag{5.139}$$

For a ΛCDM universe the solution in time to this equation has Φ constant until Λ becomes important, and then starts to decay as Λ suppresses the growth of structure on all scales by about a factor of 2. We write it as $\Phi(t, \boldsymbol{x}) = g(t)\Phi_0(\boldsymbol{x})$ where $g(t)$ is the growing solution to Eq. (5.139) normalised to $g = 1$ today (we can use $g_\infty = g(t = 0) \approx \frac{1}{5}(3 + 2\Omega_m^{-0.45})$ as a very good approximation to its early time value). The time derivative of the Φ may be expressed

[2] We use 'V' for the vector mode in the metric in this chapter because we have already used 'ω' for the vorticity. We hope that this switch will not lead to problems for the reader since we set the vector mode to zero in this chapter

in terms of redshift, z, growing mode and Φ as

$$\dot{\Phi}(t,x) = -\left(1+z\right) H \frac{d\ln g}{dz}\Phi(t,x). \tag{5.140}$$

This relation is important especially when taking the ensemble average. The master equation (5.139) has no scale dependence, all of which comes from the initial conditions – usually a nearly scale-invariant Gaussian spectrum from frozen quantum fluctuations during inflation – and subsequent evolution during the radiation era. Evolution during the radiation era suppresses wavelengths which enter the Hubble radius compared to those which remain larger than it until the matter era begins. Consequently, in Fourier space, assuming scale invariant initial conditions from inflation, the power spectrum of Φ, $\mathcal{P}_\Phi$, is independent of scale for modes larger than the equality scale, $k_{eq} = \sqrt{2\Omega_m z_{eq}} H_0 \approx 0.07\Omega_m h^2\,\mathrm{Mpc}^{-1}$. A dimensionless transfer function describes the loss of power in the case of zero baryons (adapted from [73]):

$$T(k) = \frac{\ln\left(2e + 0.134\,\kappa\right)}{\ln\left(2e+0.134\,\kappa\right) + \left[0.079 + \frac{4.06}{1+4.66\,\kappa}\right]\kappa^2} \tag{5.141}$$

where $\kappa = k/k_{eq}$. This is unity for $\kappa \ll 1$ and $\sim (\ln\kappa)/\kappa^2$ for $\kappa \gg 1$. The change in behaviour at the equality scale is important for backreaction because it is the modes larger than the equality scale which are primarily responsible for any backreaction at all in terms such as the Hubble rate. In essence, the equality scale determines the size of the backreaction effect in such quantities [60].

All first-order quantities can be derived from Φ; for example off-diagonal component of the Einstein equation at first order leads to,

$$v_i^{(1)} = -\frac{2}{3aH^2\Omega_m}\partial_i\left(\dot{\Phi} + H\Phi\right), \tag{5.142}$$

which governs the peculiar velocity between the matter flow and the rest-frame of the gravitational field. Meanwhile, the gauge-invariant density perturbation is

$$\delta = \frac{\delta\rho}{\rho} = \frac{2}{3H^2\Omega_m}\left[a^{-2}\partial^2\Phi - 3H\left(\dot{\Phi}+H\Phi\right)\right]. \tag{5.143}$$

At second order, the perturbed energy density is given by

$$\begin{aligned}\kappa^2\delta^2\rho \;=\;& \frac{2}{a^2}\partial^2\Psi^{(2)} - 6H\dot{\Psi}^{(2)} - 6H^2\Phi^{(2)} + 24H^2\Phi^2 + 6\dot{\Phi}^2 + \frac{16}{a^2}\Phi\partial^2\Phi \\ & -\frac{8}{3a^2H^2\Omega_m}\left[H^2\left(1-\frac{9}{4}\Omega_m\right)\partial^k\Phi\partial_k\Phi + 2H\partial^k\Phi\partial_k\dot{\Phi} + \partial^k\dot{\Phi}\partial_k\dot{\Phi}\right]\end{aligned} \tag{5.144}$$

and the Laplacian of the perturbed velocity:

$$\begin{aligned} 3aH^2\Omega_m^2\,\partial^2 v^{(2)} &= -2\Omega_m\left(\partial^2\dot{\Psi}^{(2)} + H\partial^2\Phi^{(2)}\right) \\ &+4H(\Omega_m-2)\Phi\partial^2\Phi - 4(\Omega_m+2)\dot{\Phi}\partial^2\Phi - 4(3\Omega_m+2)\Phi\partial^2\dot{\Phi} - \frac{8}{H}\dot{\Phi}\partial^2\dot{\Phi} \\ &+4H(\Omega_m-2)\partial^k\Phi\partial_k\Phi - 16(\Omega_m+1)\partial^k\Phi\partial_k\dot{\Phi} - \frac{8}{H}\partial^k\dot{\Phi}\partial_k\dot{\Phi} \\ &+\frac{8}{3a^2H^2}\left[H\partial^2\Phi\;\partial^2\Phi + \partial^2\Phi\;\partial^2\dot{\Phi} + H\partial^k\Phi\;\partial^2\partial_k\Phi + \partial^k\dot{\Phi}\;\partial^2\partial_k\Phi\right] \end{aligned} \tag{5.145}$$

The full expression in terms of the perturbed metric variable, Φ, for each of the definitions of the deceleration parameter considered in this chapter is given in section 5.10.

5.7.1 Moving from Riemannian to Euclidean Average

The hypersurface orthogonal to n^a coincides with the spatial surfaces of the metric in the longitudinal gauge, which could be described as the gravitational rest-frame in this case. The Riemannian average $\langle ...\rangle_{\mathcal{D}}$ may be expanded in terms of the Euclidean average $\langle ...\rangle$ defined on the background spatial slices,

$$\langle\Upsilon\rangle_{\mathcal{D}} = \int_{\mathcal{D}}\sqrt{g}\,\mathrm{d}^3x\,\Upsilon \Big/ \int_{\mathcal{D}}\sqrt{g}\,\mathrm{d}^3x, \tag{5.146}$$

as:

$$\langle\Upsilon\rangle_{\mathcal{D}} = \Upsilon^{(0)} + \langle\Upsilon^{(1)}\rangle + \langle\Upsilon^{(2)}\rangle + 3\left[\langle\Upsilon^{(1)}\rangle\langle\Phi\rangle - \langle\Upsilon^{(1)}\Phi\rangle\right], \tag{5.147}$$

where $\Upsilon^{(0)}$, $\Upsilon^{(1)}$ and $\Upsilon^{(2)}$ denote respectively the background, first order and second order parts of the scalar function $\Upsilon = \Upsilon^{(0)} + \Upsilon^{(1)} + \Upsilon^{(2)}$. Note the important term in square brackets contains an additional backreaction contribution, which appears because we are averaging on a physical spacetime. This additional contribution is absent in the Baumann et al [120] calculation because they smooth on the background space-time. In our case it introduces a domain size dependence to the average quantity.

To connect with Buchert's comoving synchronous coordinate choice where average, $\langle\cdots\rangle_{\mathcal{F}}$ is defined on the hypersurfaces orthogonal to u^a i.e the local rest frame of dust, in this case we have instead [49, 60, 188]

$$\begin{aligned} \langle\Upsilon\rangle_{\mathcal{F}} &= \Upsilon^{(0)} + \left\langle\Upsilon^{(1)}\right\rangle - g_I\dot{\Upsilon}^{(0)}\langle\Phi\rangle + \left\langle\Upsilon^{(2)}\right\rangle + (1-3Hg_I)\left[\langle\Upsilon^{(1)}\Phi\rangle - \left\langle\Upsilon^{(1)}\right\rangle\langle\Phi\rangle\right] \\ &-g_I\left\langle\Phi\dot{\Upsilon}^{(1)}\right\rangle + g_I^2\left[3H\dot{\Upsilon}^{(0)2} + \frac{1}{2}\ddot{\Upsilon}^{(0)}\right]\left\langle\Phi^2\right\rangle + 3g_I(1-g_IH)\dot{\Upsilon}^{(0)}\langle\Phi\rangle^2 \\ &-\frac{1}{2}\dot{\Upsilon}^{(0)}\int^t \mathrm{d}t'\left[\left\langle\Phi^{(2)}\right\rangle - \left\langle\Phi^2\right\rangle - 2\left\langle v_i^{(1)}v_{(1)}^i\right\rangle - 2g_Ia^{-2}\left\langle v_{(1)}^i\partial_i\Phi\right\rangle\right], \end{aligned} \tag{5.148}$$

where $g_I = \frac{1}{g(t)}\int^t g(t')\mathrm{d}\,t'$. In each of these definitions of spatial average and also for the quantities defined on the past light cone, we still need to calculate the ensemble average in order to make sense of the statistical fluctuations Φ. The ensemble average of a spatial average may be defined as:

$$\overline{\langle X(\boldsymbol{x})\rangle} = \frac{1}{V}\int d^3x W(x/R_{\mathcal{D}})\overline{X(\boldsymbol{x})}\,, \tag{5.149}$$

and the corresponding ensemble variance is given by

$$\mathrm{Var}[X(\boldsymbol{x})] = \overline{X(\boldsymbol{x})^2} - \overline{X(\boldsymbol{x})}^2\,. \tag{5.150}$$

where the overbar denotes an ensemble average. We have specified the domain through the window function W, for the quantities on the past light cone, the domain is vanishing. The Euclidean volume of the spatial domain of averaging $\mathcal{D}$ is then given by: $V = \int d^3xW(x/R_{\mathcal{D}})$ which in the case of a Gaussian window function which we employ is $V = 4\pi R_{\mathcal{D}}^3\int_0^\infty y^2W(y)dy = (2\pi)^{3/2}R_{\mathcal{D}}^3$ for any $R_{\mathcal{D}}$. The Fourier transform of any given scalar is given as

$$\Phi(\boldsymbol{x}) = \frac{1}{(2\pi)^{3/2}}\int \mathrm{d}^3k\,\Phi(\boldsymbol{k})\,e^{i\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{5.151}$$

The initial conditions for Φ in terms of its correlations, which appear in the form of an ensemble average or expectation value is provided by the Inflationary models: a Gaussian distribution for example, the 2-point correlation function which is defined in real space is just (up to a factor of ℓ)

$$C(\boldsymbol{x},\boldsymbol{x}') = \overline{\Phi(\boldsymbol{x})\Phi(\boldsymbol{x}')}\,, \tag{5.152}$$

The higher correlation functions are obtained in terms of the two point function using the Wicks theorem, once the Gaussainity assumption is imposed on Φ. The two point function relates to the power spectrum as:

$$\overline{\Phi(\boldsymbol{k})\Phi(\boldsymbol{k}')} = \frac{2\pi^2}{k^3}\mathcal{P}_\Phi(k)\delta(\boldsymbol{k}+\boldsymbol{k}')\,. \tag{5.153}$$

where we impose a reality condition $\Phi(\boldsymbol{k}') = \Phi(-\boldsymbol{k})$. For the scale-invariant initial conditions from single field inflation, this is given by

$$\mathcal{P}_\Phi(t,k) = \left(\frac{3\Delta_{\mathcal{R}}}{5g_\infty}\right)^2 g(t)^2T(k)^2\,, \tag{5.154}$$

where $\Delta^2_{\mathcal{R}}$ is the primordial power of the curvature perturbation [34], with $\Delta^2_{\mathcal{R}} \approx 2.41\times10^{-9}$ at a scale $k_{CMB} = 0.002\mathrm{Mpc}^{-1}$.

5.7.2 How to evaluate spatial/ensemble average

To evaluate our various definitions of deceleration parameter we could use a realisation of Φ given an inflationary model. Alternatively, we can assume a spectrum for Φ and evaluate the statistics of the definition in question. This allows us to calculate the expectation value of each deceleration parameter as well as their ensemble variance, in terms of integrals over the power spectrum of Φ multiplied by powers of k. The reason we must go to second-order now becomes clear when we calculate the expectation value of an average: for Gaussian perturbations from inflation, the ensemble average of Φ is zero, which implies – assuming ergodicity – that when averaged on the background over a very large (strictly, infinite) domain they are zero too.

Let us estimate the approximate behaviour of each type of terms which appears. The relations for determining the scaling behaviour for the backreaction terms are ($n+m$ is even)

$$\overline{\tilde{\partial}^m \Phi \, \tilde{\partial}^n \Phi} \;=\; \frac{(-1)^{(m+3n)/2}}{(aH)^{n+m}} \int_0^\infty \mathrm{d}\, k \, k^{m+n-1} \mathcal{P}_\Phi(k)\,. \tag{5.155}$$

The inverse Laplacian term in $\Phi^{(2)}$ satisfies $\overline{\langle \partial^{-2}\partial_i\partial^j(\partial^i\Phi_0\partial_j\Phi_0)\rangle} = \frac{1}{3}\overline{\langle \partial^i\Phi_0\partial_j\Phi_0\rangle}$ [86]. We also have that, since Φ is statistically homogeneous and isotropic [86]

$$\overline{\partial^2\Phi^{(2)}} = \overline{\partial^2\Psi^{(2)}} = 0, \quad \overline{\partial_k v^{k\,(2)}} = 0\,, \tag{5.156}$$

which means that all the potentially large terms in the second-order deceleration parameter don't contribute to the expectation value (see below). For the non-connected terms we have

$$\overline{\left\langle \tilde{\partial}^m \Phi \right\rangle \left\langle \tilde{\partial}^n \Phi \right\rangle} = \frac{(-1)^{(m+3n)/2}}{(aH)^{n+m}} \int_0^\infty \mathrm{d}\, k \, k^{m+n-1} W(kR_D)^2 \mathcal{P}_\Phi(k), \tag{5.157}$$

where W is an appropriate window function specifying the domain. Typically this will become a delta-function as the domain tends to infinity (e.g., if the window function in real space is a Gaussian of width $R_{\mathcal{D}}$, then in Fourier space it is a Gaussian of width $1/R_{\mathcal{D}}$, centred at $k = 0$). Note that the connected terms have no dependence on the domain size or shape at all, and that the domain dependence arises from the non-connected terms – these in turn come from using the Riemannian volume element (see the paragraph below equation (5.147) for further details). The integral can be written as

$$\frac{1}{H^{n+m}} \int_0^\infty \mathrm{d}\, k\, k^{m+n-1} \mathcal{P}_\Phi(k) = \left(\frac{3\Delta_{\mathcal{R}}}{5 g_\infty}\right)^2 \left(\frac{k_{eq}}{k_H}\right)^{m+n} \int_0^\infty \mathrm{d}\, \kappa\, \kappa^{m+n-1} T(\kappa)^2\,. \tag{5.158}$$

Here, $k_H = H_0^{-1}$ is the wavenumber of the mode entering the Hubble volume today, and

$k_{eq}/k_H = \sqrt{2\Omega_m z_{eq}} \sim 40$. Using $z_{eq} \approx 2.4 \times 10^4 \Omega_m h^2$ implies the important relation

$$\Delta_{\mathcal{R}} \left(\frac{k_{eq}}{k_H}\right)^2 \approx 2.4\, \Omega_m^2 h^2 \,. \tag{5.159}$$

For pure CDM and a scale-invariant initial spectrum, the integral behaves as, replacing $\int_0^\infty \mapsto \int_{\kappa_{\rm IR}}^{\kappa_{\rm UV}}$ where necessary:

$$\int_0^\infty \mathrm{d}\,\kappa\,\kappa^{m+n-1} T(\kappa)^2 \approx \begin{cases} -\ln(\kappa_{\rm IR}) & \text{for} \quad m+n=0 \\ 3.9 & \text{for} \quad m+n=2 \\ F(\kappa_{\rm UV}) & \text{for} \quad m+n=4 \end{cases} \tag{5.160}$$

The function F is roughly $F(x) \sim 0.44\, x^{2.14}$ for $1 \lesssim x \lesssim 10$, $\sim 70x^{-0.1}(\log_{10} x)^{4.75}$ for $x \gg 1$, and approaches $\sim 53 \ln^3 x$ as $x \to \infty$. For integrals with the window function inside, $W(\kappa k_{eq} R_{\mathcal{D}})$, we can roughly replace $\kappa_{\rm UV} \mapsto 1/R_{\mathcal{D}} k_{eq}$, though this depends on the details of the window function used. Combining the above equations allows us to calculate reasonably precisely the size of each term at second-order.

5.8 Results and Discussion

We shall consider two models: the Einstein-de Sitter with $h = 0.7$ and estimated 5% of the total energy density as baryon energy density $\Omega_b \approx 0.046$ and the ΛCDM model with $\Omega_\Lambda = 0.26, h = 0.7$ and $f_{\rm baryon} = 0.175$ (these are best fit WMAP7 results [34]). Both models have the Hubble scale at $k_{\rm H}^{-1} \simeq 4.3$Gpc. We shall adopt the Silk scale, $k_{\rm silk}^{-1}$, the equality scale, $k_{\rm eq}^{-1}$, and the Hubble scale, $k_{\rm H}^{-1}$ as our reference scales. These scales are related to the baryon density today Ω_b and total matter contributions today Ω_0 as [73]

$$\begin{aligned} k_{\rm silk} &\approx 1.6\, \left(\Omega_b h^2\right)^{0.52} \left(\Omega_0 h^2\right)^{0.73} \left[1 + \left(10.4\Omega_0 h^2\right)^{-0.95}\right] \mathrm{Mpc}^{-1}, \\ k_{\rm eq} &\approx 7.46 \times 10^{-2} \Omega_0 h^2 \mathrm{Mpc}^{-1}, \quad \text{and} \quad k_{\rm H} = \frac{h}{3000}\mathrm{Mpc}^{-1}, \end{aligned}$$

for $H_0 = 100\, h\, \mathrm{kms}^{-1}\mathrm{Mpc}^{-1}$. In relation to the two models under consideration, these length scales are approximately for the Einstein de Sitter universe $k_{\rm eq}^{-1} \simeq 27.9$Mpc and $k_{\rm silk}^{-1} \simeq 6.0$Mpc and for the ΛCDM universe $k_{\rm eq}^{-1} \simeq 107.2$Mpc and the Slik scale $k_{\rm silk}^{-1} \simeq 11.5$Mpc. For the sake of completeness the ΛCDM model has just six fundamental parameters and they are given by [34]: the physical baryon density, $\Omega_b h^2 = 0.026 \pm 0.00053$, physical dark matter density, $\Omega_c h^2 = 0.1123 \pm 0.0038$, dark energy density, $\Omega_\Lambda = 0.728^{+0.015}_{-0.016}$, scalar spectral index, $n_s = 0.963 \pm 0.012$, curvature fluctuation amplitude, $\Delta^2_{\mathcal{R}} = 2.44^{+0.088}_{-0.097} \times$

$10^{-9}, k_0 = 0.002\text{Mpc}^{-1}$ and reionization optical depth, $\tau = 0.087 \pm 0.014$. The Hubble constant, deceleration parameter and age of the universe, are derived from these fundamental parameters. These parameters were determined without consistently taking into account the effect of backreaction. It would be interesting to see by how much they would change if backreaction is taking in account.

We handle the Infra-red divergence that appear in the $k-$integrals by setting a cut-off at $L = 10$, i.e at ten times the Hubble scale. As a general rule in this paper, we will display the graphs for both models side by side where the graph on the Left Hand Side (LHS) will represent the ΛCDM universe while on the Right hand Side (RHS) is for Einstein de Sitter (EdS) universe. This rule will always apply even when the plots are not titled for clarity purposes unless.

5.8.1 Backreaction is negligibly small

Baumann et al [120], Ishibashi and Wald [62] and Green and Wald [153, 221] have consistently argued that the effect of backreaction within general relativity is negligibly small. Their analysis is based on the original Isaacson's idea [222, 223] who considers a non-linear perturbation of Einstein tensor as a form of effective fluid, say τ_{eff}. τ_{eff} contains terms which have at most two spatial derivative. Within this formalism, the size of the amplitude of τ_{eff} compared to the background energy-momentum tensor determines whether backreaction is large or small.

The deceleration parameters associated to this formalism are $\{q_{\mathcal{D}}^{N\xi}, q_{\mathcal{D}}^{\xi}\}$. The connection between $\{q_{\mathcal{D}}^{N\xi}, q_{\mathcal{D}}^{\xi}\}$ and this formalism is made clearer using the ADM formalism [215, 216]. The general expression for $\{q_{\mathcal{D}}^{N\xi}, q_{\mathcal{D}}^{\xi}\}$ in perturbation theory is given in equations (5.167,5.168)

The first type of term that contributes to this definition is Φ^2 and it is nominally tiny, $\mathcal{O}(10^{-10})$ and the term that is primarily responsible for setting the fundamental amplitude of the backreaction is $\overline{\left\langle \Phi\tilde{\partial}^2\Phi \right\rangle} \propto (k_{eq}/k_H)^2$. It is indeed quite small,

$$\frac{\overline{\langle \Phi\partial^2\Phi \rangle}}{\Omega_m H_0^2} \sim \Delta_{\mathcal{R}}^2 \frac{k_{eq}^2}{\Omega_m k_H^2} \sim \Delta_{\mathcal{R}}^2 \frac{T_{eq}}{T_0} \sim 10^{-6} \tag{5.161}$$

for the concordance model. (The overall effect is somewhat larger than this due to the contribution of several such terms.) This gives sub-percent changes to $\{q_{\mathcal{D}}^{N\xi}, q_{\mathcal{D}}^{\xi}\}$ from backreaction. The backreaction contribution in this case is domain size independent because the terms such as $\langle\Phi\rangle^2$ that would introduce the dependence is sub-dominant.

In figure 5.1, we plot the amplitude of backreaction defined by subtracting the FLRW background value for the deceleration parameter, q_0 from the spatial average definition,

$\Delta q = q_{\mathcal{D}} - q_0$ as a function of redshift. The difference between $q_{\mathcal{D}}^{N\xi}$ and $q_{\mathcal{D}}^{\xi}$ is also negligibly

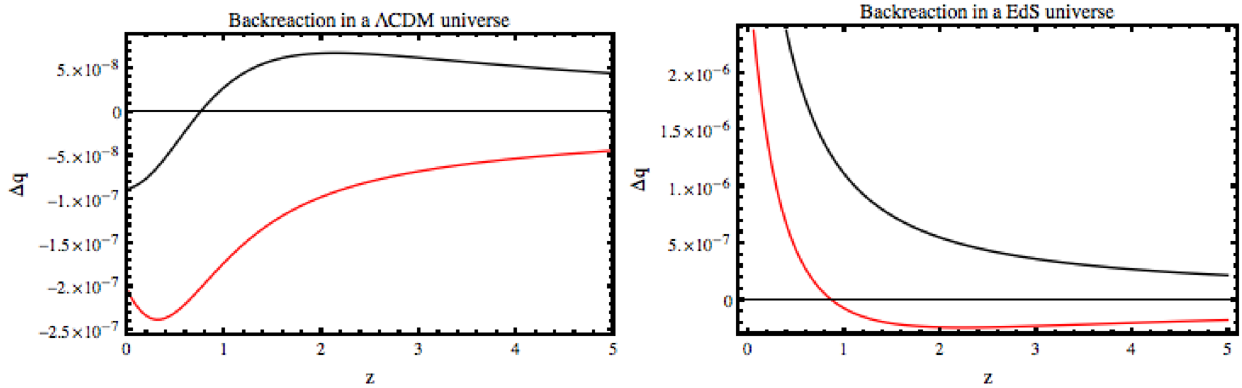

Figure 5.1: Backreaction effect in the deceleration parameters $\Delta q = q_{\mathcal{D}} - q_0$ as a function of redshift for $\{q_{\mathcal{D}}^{N\xi}, q_{\mathcal{D}}^{\xi}\}$ set of definitions. The 'red' plot is for $q_{\mathcal{D}}^{N\xi}$ and the 'black' for $q_{\mathcal{D}}^{\xi}$

small. In figure 5.2, we show the plot of $\{q_{\mathcal{D}}^{N\xi}, q_{\mathcal{D}}^{\xi}\}$ plus or minus associated ensemble variance defined in equation (5.150). There is no noticeable difference within the level of our accuracy between $q_{\mathcal{D}}^{N\xi}$ and $q_{\mathcal{D}}^{\xi}$ in a ΛCDM universe, while there is a shift in the epoch of importance in an Einstein-de Sitter universe.

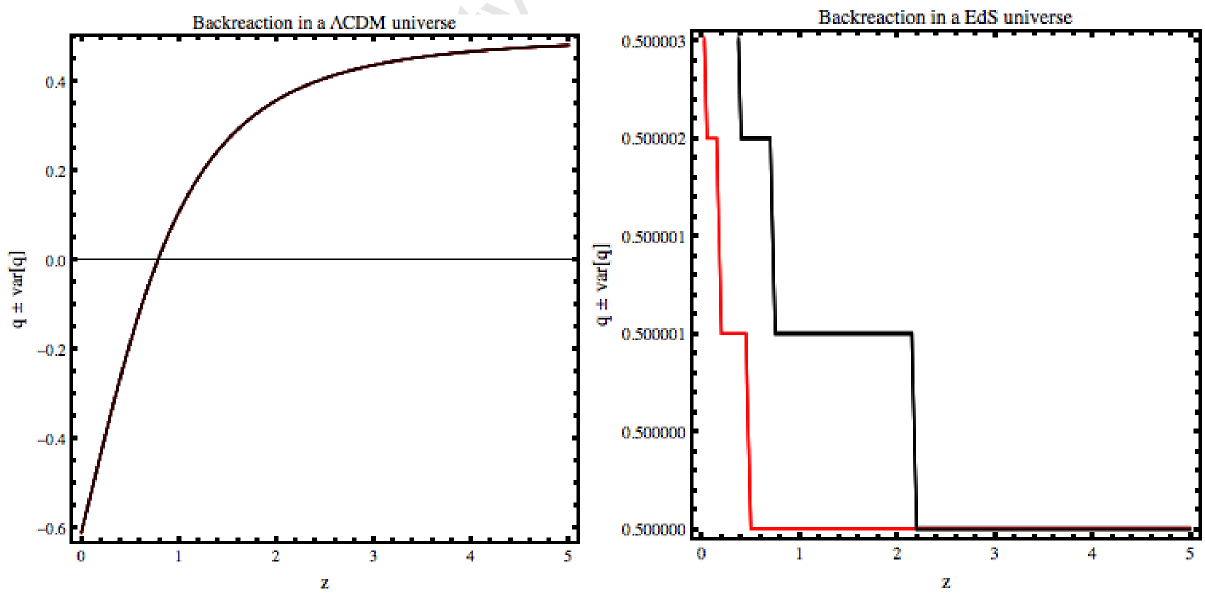

Figure 5.2: Deceleration parameter plus or minus variance in $\{q_{\mathcal{D}}^{N\xi}, q_{\mathcal{D}}^{\xi}\}$ as a function of redshift. The 'red' plot is for $q_{\mathcal{D}}^{N\xi}$ and the 'black' for $q_{\mathcal{D}}^{\xi}$

This set of definitions measures the acceleration/deceleration of an averaged hypersurface as seen by the observer attached to the coordinate grids. or it may also be understood as measuring the acceleration/deceleration of averaged congruence of the observer at rest in the gravitational frame. It differs from acceleration/deceleration measured by the observer comoving with the matter field through a relative velocity between the two frames. This definition is most suitable for the CMB observers [54, 55] since the effect of the relative motion of our galaxy is subtracted out.

5.8.2 The need for renormalization of the background

We would like to understand the major differences between the deceleration parameter defined using the Buchert formalism, $\langle q \rangle_{\mathcal{F}}$ and spatial average of a local definition of a local deceleration parameter, $\left\langle q^{\Theta} \right\rangle_{\mathcal{F}}$. The $\langle q \rangle_{\mathcal{F}}$ and $\left\langle q^{\Theta} \right\rangle_{\mathcal{F}}$ have many things in common: they are defined with respect to an observer comoving with the fluid or observer at rest with the fluid. the Riemannian averages for both quantities are evaluated on the same spatial surface and both definitions may be most suitable for the SN I observations [51, 52, 224].

However, the key difference between the two definitions of a deceleration parameter lies on whether the hypersurface on which they are defined is smooth or not. $\langle q \rangle_{\mathcal{F}}$ describes the acceleration/deceleration of an averaged/smooth hypersurface, while $\left\langle q^{\Theta} \right\rangle_{\mathcal{F}}$ describes the averaged deceleration/acceleration of a rough hypersurface.

The full analytical expressions for both definitions in terms of the perturbed metric variables are given in equations (5.172) and (5.171). For both definitions the dominant terms are those terms with with four spatial derivative operators, however, they appear in different format in both definitions. For $\langle q \rangle_{\mathcal{F}}$, the four gradient terms appear as non-connected term, $\langle \partial^2 \Phi \rangle_{\mathcal{D}}^2$ and according to equation (5.157), the $k-$integrals are regulated in the UV by the window function which has a natural physical interpretation within the Buchert averaging formalism. For $\left\langle q^{\Theta} \right\rangle_{\mathcal{F}}$, the four gradient terms appear as a connected term $\langle (\partial^2 \Phi)^2 \rangle_{\mathcal{D}}$ and according to equation (5.172), the dominant contribution after evaluating the ensemble averages, may be given by

$$\overline{\langle q_{\Theta} \rangle_{\mathcal{F}}} \approx -1 + \frac{3}{2}\Omega_m + \Omega_m^4 h^4 \left[-0.24(1-\Omega_m)^{3.94} + 0.66\Omega_m^{0.37} \right] F(\kappa_{\rm UV}) \qquad (5.162)$$

where we have replaced $\Delta_{\mathcal{R}} \left(k_{eq}/k_H \right)^2$ using equation (5.159) and the coefficient in square brackets are reasonable empirical estimates for $\Omega_m \gtrsim 0.1$ (accurate to a % or less). Since the ensemble average of $\left\langle q^{\Theta} \right\rangle_{\mathcal{F}}$ is UV dependent, we have to find a technique to handle it.

The traditional approach usually employed to remove UV divergences in Physics is to introduce a cut-off on some sufficiently high energy scales. For cosmology, this cut-off is set by the physical size of the universe after inflation [79, 197, 225]. However, It was shown in [60] that introduction of a cut-off of this kind makes its contribution difficult to interpret, for example inserting an inflationary inspired cut-off, this term gives:

$$\Omega_m^4 h^4 \left[-0.24(1-\Omega_m)^{3.94} + 0.66\Omega_m^{0.37}\right] F(\kappa_{\rm UV}) \sim 10^5 \tag{5.163}$$

where $F(\kappa_{UV}) = \int_{k_{IR}}^{k_{UV}} dk k^3 T(k)^2$ and a dark matter transfer function was used.

Here, we will handle the UV divergences by smoothing our perturbation $\Phi(\boldsymbol{x})$ in real space before taking the ensemble average. This is motivated by the fact that the linear theory is only accurate up to some scale and under-estimates power below this scale, hence it makes sense to smooth away structures on small scales before we apply our ensemble averaging procedure. The smoothing operation involves replacing Φ with a weighted average over nearby points using [226]

$$\Phi_{\mathcal{S}}(\boldsymbol{x}) = \frac{1}{V_{\mathcal{S}}} \int \mathrm{d}^3 x' \, W(|\boldsymbol{x}' - \boldsymbol{x}| R_{\mathcal{S}}) \, \Phi(\boldsymbol{x}'), \tag{5.164}$$

for simplicity we make use of the same type of window function we used in calculating the ensemble average, but we separate the smoothing length scale $R_{\mathcal{S}}$ from the averaging length scale $R_{\mathcal{D}}$. In Fourier space this amounts to replacing $\Phi(k)$ everywhere with $W(kR_{\mathcal{S}})\Phi(k)$, and $\mathcal{P}_\Phi$ with $W(kR_{\mathcal{S}})^2\mathcal{P}_\Phi$.

We present in figures 5.3 and 5.4 the dependence on the smoothing scale of the averaged local definition of deceleration parameter. We use the Buchert definition of deceleration parameter as reference. There is a significant difference between the two definitions especially

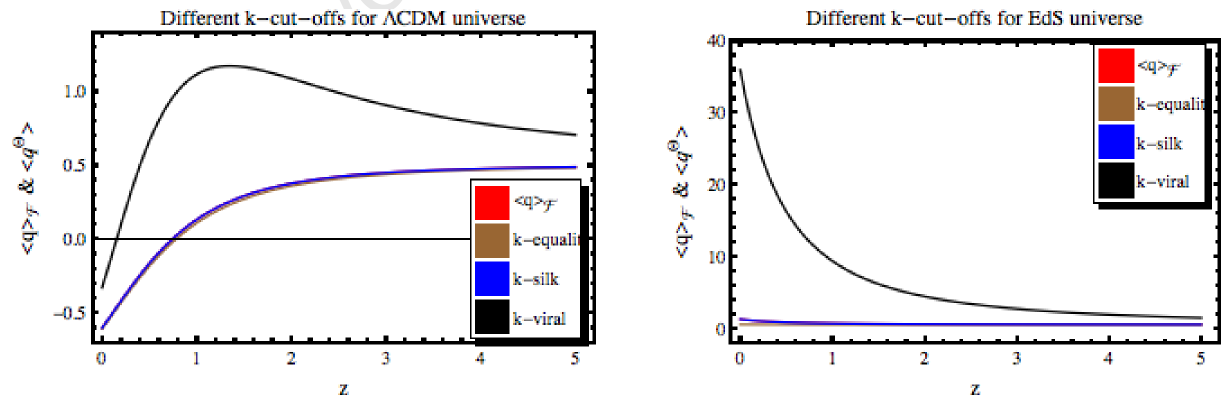

Figure 5.3: The deceleration parameter as a function of redshift for $\langle q \rangle_{\mathcal{F}}$ and $\langle q^\Theta \rangle_{\mathcal{F}}$. Separate plots illustrate the dependence of $\langle q^\Theta \rangle_{\mathcal{F}}$, on the smoothing scale. We have set our domain size at the Hubble scale, k_H^{-1}.

when the smoothing scale is on the virial scale, $k_{\rm virial}^{-1}$ (i.e around $1\,h^{-1}$Mpc. Both definitions only tend to agree when $\langle q^\Theta \rangle_{\mathcal{F}}$ is smoothed at or above the Silk scale $k_{\rm Silk}^{-1}$ or conservatively above $10\,h^{-1}$Mpc scale.

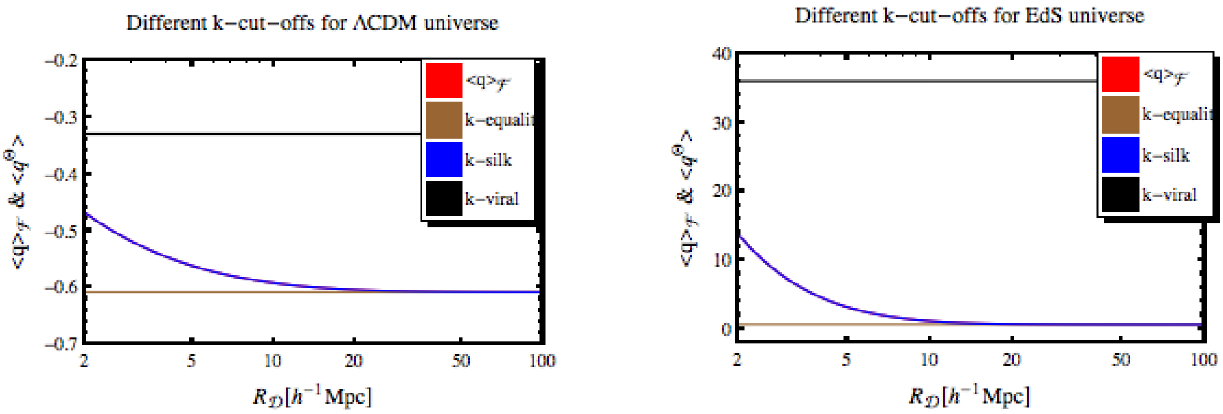

Figure 5.4: We plot $\langle q \rangle_{\mathcal{F}}$ and $\langle q^{\Theta} \rangle_{\mathcal{F}}$ as a function of the domain size. There is a significant suppression of the backreaction effect in the ΛCDM universe relative to the Einstein de Sitter universe.

Setting the domain size $R_{\mathcal{D}}$ at Hubble scale, k_H^{-1}, equality, k_{eq}^{-1} or even at the Silk scale k_{Silk}^{-1} in figures 5.3 and 5.4 does not make any difference. There could be more than 100% effect in an Einstein de Sitter universe if the smoothing scale is set at few h^{-1}Mpc. Both definition of the deceleration parameter agree when we smooth on the same scale as the domain size, that is setting $R_S = R_{\mathcal{D}}$.

One important thing we noticed in figures 5.3 and 5.4 is that when we set our smoothing scale equal to the size of our averaging domain. the $k-$integrals immediately become regularized by the window function just the same way the Buchert definition is regulated by the window function. This seems to suggest that the background spacetime, which in principle is smooth looses its smoothness property on scales where the $k-$integrals diverge.

However, it has been suggested [120] based on an effective field interpretation of cosmology that the problematic non-linear scale could be integrated out to recover the effective long wavelength dynamics of the universe. This approach suggests that the small sub-horizon scales are virialised and hence they decouple from the effective evolution of the long wavelength universe. We have seen from figures 5.3 and 5.4 that effective decoupling can only occur if the Virial scale, k_{virial}^{-1} is beyond $10\,h^{-1}$Mpc.

On the other hand, decoupling of the virialised structures does not imply that virialised structures do not participate in gravitational lensing [227], hence there is need to find a way to renormalize the divergent integrals or the background spacetime in general.

5.8.3 Domain averaged deceleration parameters

The following definitions of a deceleration parameter are based on spatial averaging:

$$\begin{matrix}\text{Deceleration}\\ \text{parameters}\end{matrix} = \left\{ \left\langle q^{N\theta} \right\rangle_{\mathcal{D}}, \left\langle q^{\theta} \right\rangle_{\mathcal{D}}, \left\langle q \right\rangle_{\mathcal{F}}, \left\langle q^{\Theta} \right\rangle_{\mathcal{F}} \right\}. \tag{5.165}$$

The analytical expression in cosmological perturbation theory in each case is presented in section 5.10. Apart from mild dependence on domain size, $\left\langle q^{\Theta} \right\rangle_{\mathcal{F}}$ depends strongly on the smoothing scale as discussed above, so for the ease of comparison we set $R_S = R_{\mathcal{D}}$ unless otherwise stated.

Firstly, we present the result of the backreaction effect on the deceleration parameter, $\Delta q = \langle q \rangle_{\mathcal{D}/\mathcal{F}} - q_o$ ($q_o$is the background value), for the ΛCDM universe and the Einstein de Sitter universe in figures 5.5 and 5.6. The effect of backreaction is relatively small in a ΛCDM

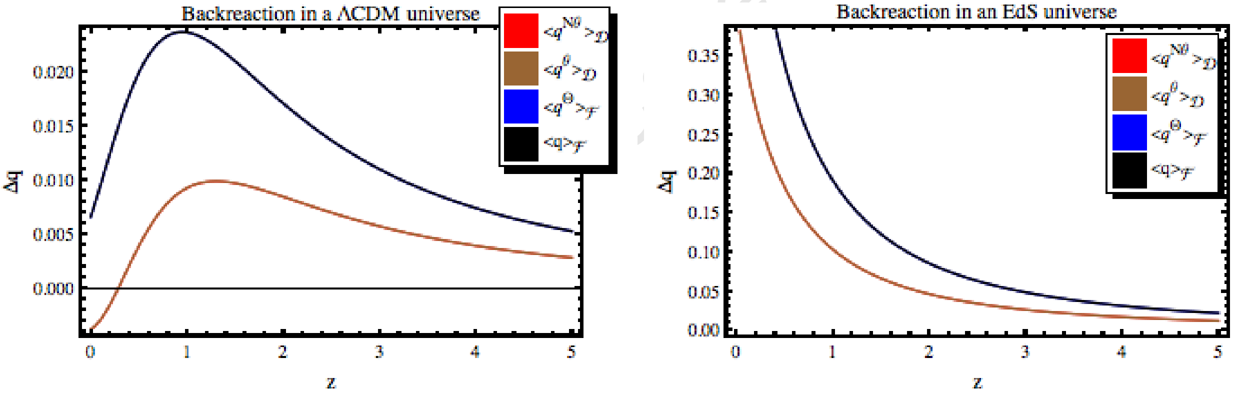

Figure 5.5: Backreaction effect in the deceleration parameter $\Delta q = q_{\mathcal{D}/\mathcal{F}} - q_0$, as a function of redshift. The domain size, $R_{\mathcal{D}}$ is set at the Silk scale, k_{silk}^{-1} .

universe at all redshifts while in an Einstein de Sitter universe, it is large near the redshift of today and it reduces to a few percent at high redshift. The smallness of the backreaction effect in the ΛCDM universe shown in figure 5.5 may be related to the interpretation by Wigglez collaboration that fewer large scale structures in a ΛCDM universe are responsible for smallness of gravitational lensing effect and this constitute evidence for dark energy [194].

Next we calculate the ensemble variance for spatial average deceleration parameters and the results is plotted the averaged deceleration parameter as a reference. The results are shown in figures 5.7 and 5.8. In figure 5.7, there is a large variance in an Einstein de Sitter universe at small redshift and it decreases gradually as we go towards higher redshift. This is contrary to the case of the ΛCDM universe where the variance is almost vanishing at small redshift and grows towards a higher value at high redshift. It is clear from figure 5.8 that for the large domain sizes, the backreaction effect disappears and the profiles of all the

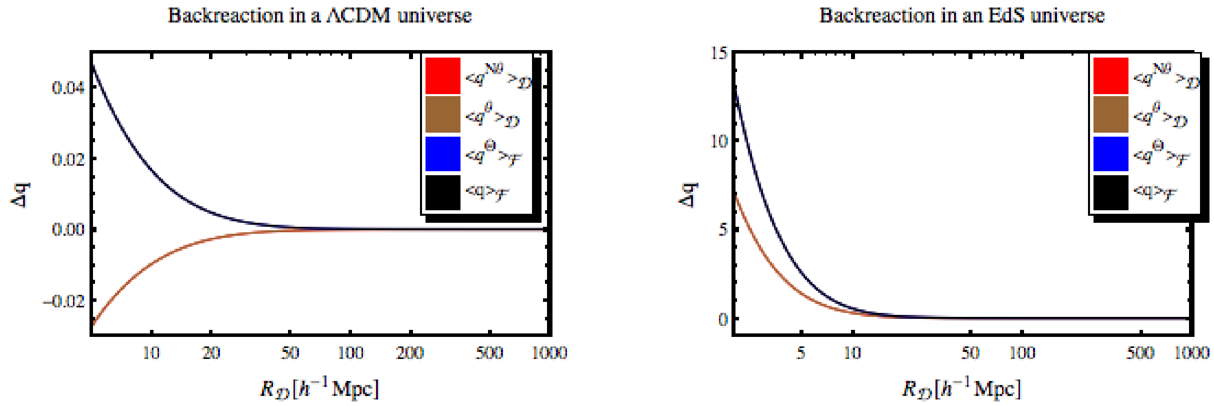

Figure 5.6: Backreaction effect in spatial averaged deceleration parameters $\Delta q = q_{\mathcal{D}/\mathcal{F}} - q_0$, as a function of the domain size $R_{\mathcal{D}}$. The difference associated with the choice of averaging hypersurface clearly shows up in the ΛCDM universe, with curves for averaging hypersurface orthogonal to n^a pointing in opposite direction to that of the hypersurface orthogonal to u^a just below the domain size of about 30 h^{-1}Mpc. In the Einstein de Sitter universe, there is no change in direction but there are obvious differences on scales less than 10 h^{-1}Mpc.

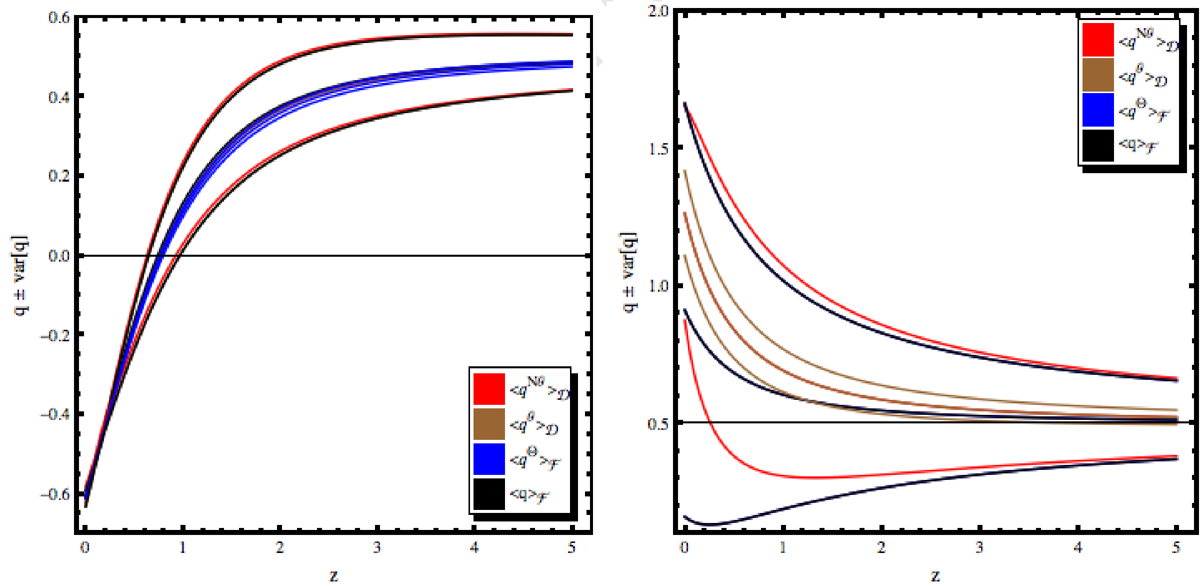

Figure 5.7: Deceleration parameter plus or minus variance, plotted as a function of redshift, z, for $\{\langle q^{N\theta}\rangle_{\mathcal{D}}, \langle q^{\theta}\rangle_{\mathcal{D}}, \langle q\rangle_{\mathcal{F}}, \langle q^{\Theta}\rangle_{\mathcal{F}}\}$.

definitions take the background value. One could argue from here that when one is interested in the large scale dynamics (up to say 100 h^{-1}Mpc) that the choice of slicing does not really matter for the deceleration parameter, but it is different for the Hubble rate [49].

In summary, figure 5.6 shows that except for very small sized domains ($< 30\,h^{-1}$Mpc), the backreaction effect is negligibly small in a ΛCDM universe irrespective of whether the

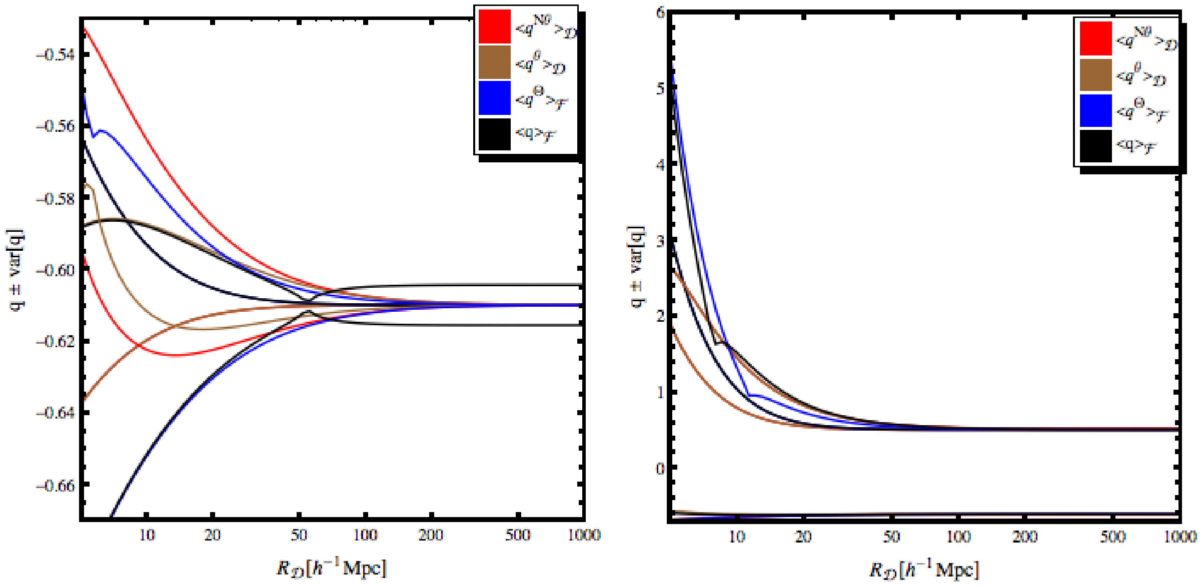

Figure 5.8: Spatial Averaged deceleration parameters $(\{\langle q^{N\theta}\rangle_{\mathcal{D}}, \langle q^{\theta}\rangle_{\mathcal{D}}, \langle q\rangle_{\mathcal{F}}, \langle q^{\Theta}\rangle_{\mathcal{F}}\})$ plus or minus associated variance plotted as a function of the domain size, $R_{\mathcal{D}}$.

deceleration parameter defined in terms of the 4-velocity of the fluid is observed by the observer at rest with the fluid or by the observer attached to the coordinate grids. Li et al [51] interprets this as a breakdown of the cosmological perturbation theory in this regime. In the Einstein-de Sitter universe, the backreaction effect is also small except when the domain size is less than $10\,h^{-1}$ Mpc. In this case it could be more than 100%.

5.8.4 Observables on the past null cone

One obvious disadvantage of the spatially averaged deceleration parameters is that they do not have an immediate physical interpretation, because they are averaged on the spatial domain while we have access to observation only through our past null cone. The set of definitions of deceleration parameters that is more aligned to observation is given by

$$\text{Deceleration parameter} = \left\{\mathcal{Q}, q_0^{d_L}, q_0^{z}\right\} \tag{5.166}$$

In this set of definitions, there is no dependence on the domain size or redshift because they correspond to the deceleration parameter obtained by averaging over the entire sky with respect to the observer located here and now.

The analytical expressions for $\left\{\mathcal{Q}, q_0^{d_L}, q_0^{z}\right\}$ in perturbation theory is presented in the section5.10. These definitions are also not insulated from the menace of the UV divergent integrals arising from the four gradient terms $\partial^2\Phi$. Hence we use the smoothing technique described in section 5.8.2 to handle the divergences.

In figure 5.9 , we present the dependence of the backreaction effect on the smoothing scale.

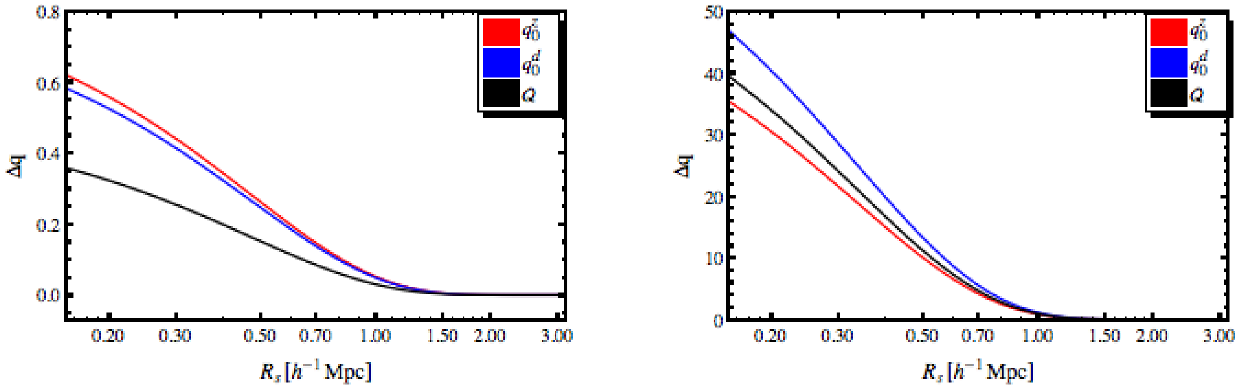

Figure 5.9: Backreaction effect from $\left\{\mathcal{Q}, q_0^{d_L}, q_0^z\right\}$ plotted as a function of the smoothing scale, R_S. Here the smoothing scale, R_S axis is log scale

Even for any physically motivated definitions of the deceleration parameter, the actual size of backreaction depends on the smoothing scale. Setting the smoothing scale at few $\sim 1\,h^{-1}$Mpc which corresponds to modes that freeze out of large scale expansion leads to more than 35% change to the background deceleration parameter in a ΛCDM universe and it is hundreds of percent more in an Einstein-de Sitter universe.

Recent analysis of data from the Wigglez experiment [36] that surveyed of over 200 000 blue galaxies in a cosmic volume of $\sim 1\ h^{-3}$Gpc^3 puts the homogeneity scale at 70 ± 5 h^{-1}Mpc at $z \sim 0.4$. Using this as a natural cut-off scale, leads to about 10% correction to the background deceleration parameter in a ΛCDM universe and a lot more in an Einstein-de Sitter universe.

We compute the variance in $\left\{\mathcal{Q}, q_0^{d_L}, q_0^z\right\}$ as function of the smoothing scale in figure 5.10.

Smoothing on a much smaller scales leads to a very large backreaction effect. The deceleration parameter q_0^z obtained from fitting the distance-redshift relation, could give zero if we smoothed on scales of about $1\,h^{-1}$Mpc for the ΛCDM universe. This is an effect expected of a Virialized region but the vanishing of the variance at that scale doesn't make sense.

In general for both universes irrespective of the fitting approach adopted, any smoothing scale less than $10\,h^{-1}$Mpc will leave a distinguishable effect on the observed deceleration parameter. Also we notice by comparing equations (5.134,5.135,5.135) with figures 5.9 and 5.10 that the larger the coefficient of the shear term, the less is the effect of structures or

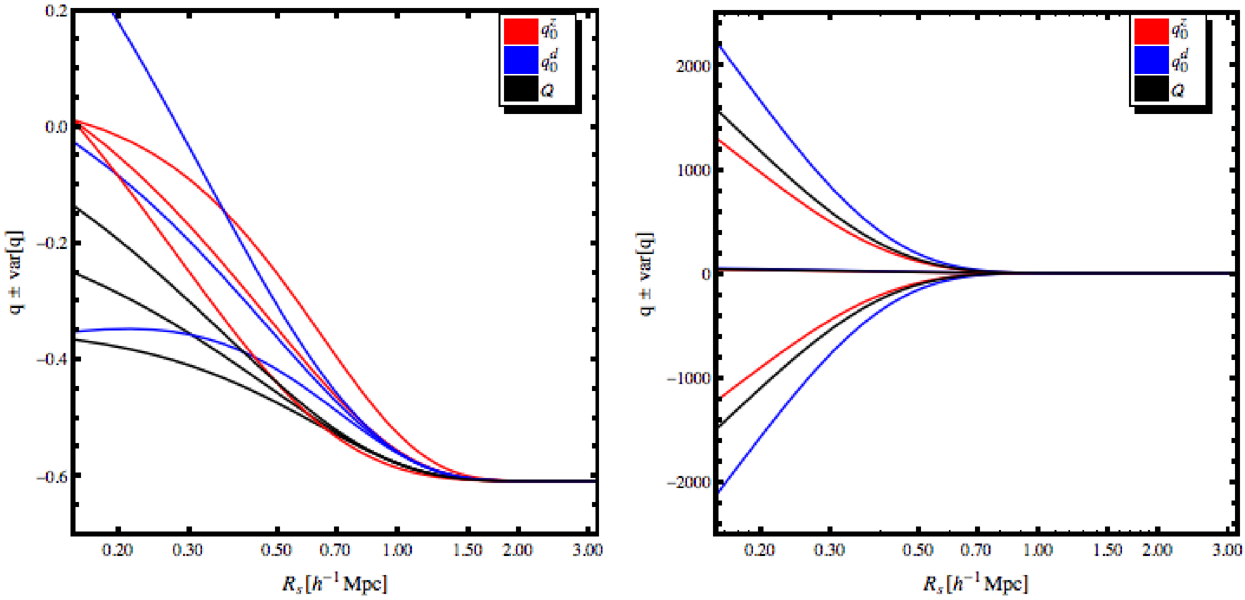

Figure 5.10: Observed deceleration parameter plus or minus variance, plotted as a function of smoothing scale, R_S.

backreaction on the deceleration parameter. This illustrates how matter shear can affect the rate of expansion of the universe.

Finally, in figure 5.11 we compare the smoothing scale, R_S dependence of a spatially averaged local deceleration parameter with $\left\{\mathcal{Q}, q_0^{d_L}, q_0^z\right\}$.

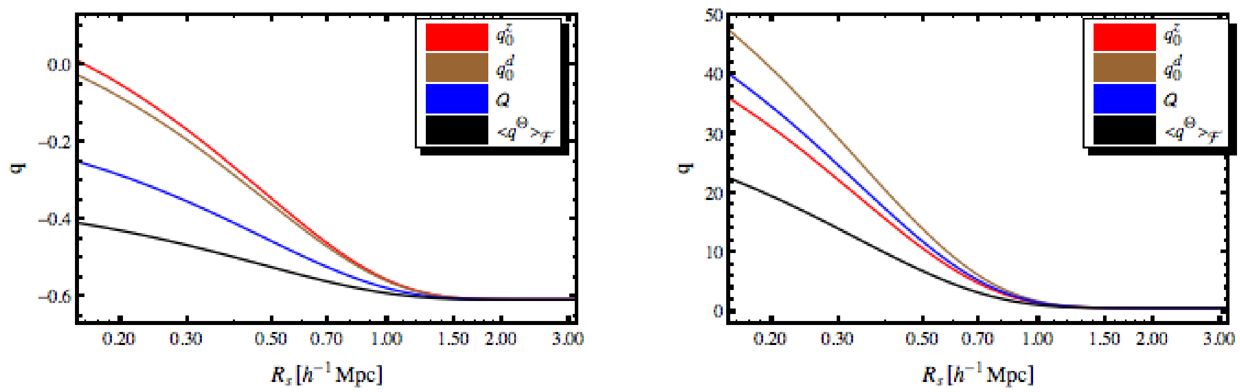

Figure 5.11: The observed deceleration parameter and averaged deceleration parameter as a function of the smoothing scale, R_S. We set the domain size for $\langle q_{\mathcal{D}}^{\Theta} \rangle_{\mathcal{F}}$ equal to the smoothing scale.

According to figure 5.11, spatial averaging tends to drive the deceleration parameter close to its FLRW background value.

5.9 Conclusion

We have discussed in this chapter examples of where the backreaction problem in the standard model of cosmology is *almost* negligible, not *necessarily* negligible and *excessively* large on some scales. The three key examples we have discussed are the cases where the deceleration parameter which gives a dimensionless measure of the second time derivative of the scale factor could be obtained from: the scale factor of an averaged hypersurface where the gravitational field is at rest, the scale factor of an averaged hypersurface where the matter field is at rest and from the best fit scale factor from an inhomogeneous universe. In the case where the inhomogeneous distance-redshift relation is fit to a distance-redshift relation from an FLRW spacetime and the the case where a local deceleration parameter is averaged on a hypersurface orthogonal to the fluid, the cosmological perturbation theory, does not give sensible answers because of the UV problem.

The backreaction terms in the $\{q_{\mathcal{D}}^{N\xi}, q_{\mathcal{D}}^{\xi}\}$, terms of the form $\Phi\tilde{\partial}^2\Phi$, are also the largest ones which appear in the lhs of the Einstein Field Equations, because the Einstein tensor has at most two derivatives of the metric in it. Re-formulation of backreaction as an effective fluid in [120], these terms are also the largest terms that appear. The induced gravitational waves are also of this order (see figure 2.2 of Chapter 2). As argued in [60] that backreaction is small simply because of a very *small* hierarchy of scales between the Hubble scale at equality and the Hubble scale today (they are only a factor of 50 apart in comoving terms). In this evaluation of backreaction, then, what happens on scales smaller than the equality scale is of little relevance.

In Buchert's interpretation of backreaction, where $q_{\mathcal{D}}$ describes the deceleration of the average scale factor, the divergence is neatly controlled by the domain size, and consequently has an elegant and straightforward interpretation, although it is not observable. This also appears to be robust against various possible gauge effects such as choice of the averaging hypersurface [48, 49]. The fact that there is a significant difference between the 'cosmological' deceleration and one which is smoothed on scales of a few Mpc, say, is expected since that is the scale where the Hubble flow kicks in.

On the other hand, definitions of the deceleration parameter – which do not depend on Riemannian averaging – reveal significant problems. Consider q_Θ and $\left\{\mathcal{Q}, q_0^{d_L}, q_0^{z}\right\}$ – the local deceleration parameter defined relative to the dust observers, and the observed one defined through the distance-redshift relation. If we calculate the expectation value of either of these we get *enormous* terms (see figure 5.11). Consider cutting off at a scale suggested by either a scale associated with the end of inflation, or from the dark matter suppression scale, which is around pc scales. Then we have $\kappa_{\rm UV} \sim 1\,{\rm pc}^{-1}/(100\,{\rm Mpc})^{-1} \sim 10^8$. Our divergent integral then gives $F(10^8) \sim 10^5$ where we have assumed a purely dark matter transfer function, and there is extra suppression from baryons on small scales, but this only reduces it by an order of magnitude for 20% baryons.

One can reformulate the UV cut-off as a smoothing of the first-order potential to give something like the same interpretation one might give to Buchert's $q_{\mathcal{D}}$, but this is rather ad hoc in this context – for $\langle q_\Theta \rangle_{\mathcal{D}}$, can kind of spatial averaging has already been done. Instead it maybe tells us that smoothing or renormalization is necessary order by order in perturbation theory especially on small scales. That is, before constructing second-order perturbation theory, one *necessarily* must smooth structure below a certain scale. But why? Does this imply that the very notion of ergodicity needs to be made Riemannian: should ensemble averages be replaced by *Riemannian* spatial averages [224], rather than spatial averages on the background? This might be a more cleaner way to go where one might use Ricci flow equation to connect the averaged evolution of local curvatures [121–124, 228]. An extended discussion on this may be found in [60].

5.10 Deceleration parameter from perturbation theory

We give here the full expression for deceleration parameters calculated from a perturbed spacetime in the Poison gauge (equation 5.136).

$$\begin{aligned}\langle q^{\xi} \rangle_{\mathcal{D}} &= -1 + \frac{3}{2}\Omega - \left[(2+3\hat{g}) - 3\Omega(1+\hat{g})\right]\langle\Phi\rangle - 3\left[\left(3 + \frac{14}{3}\hat{g} + \hat{g}^2\right)\right. \\ &\left. -\Omega\left(4 + \frac{11}{4}\hat{g} + \frac{3}{2}\hat{g}^2\right)\right]\langle\Phi\rangle^2 + \frac{1}{2}\left[15 + 4\hat{g} - 3\hat{g}^2 - (24+9\hat{g})\Omega\right]\langle\Phi^2\rangle \\ &-\frac{1}{6}\left[\langle\tilde{\partial}^2\Phi^{(2)}\rangle - \langle\tilde{\partial}^2\Psi^{(2)}\rangle\right] - \frac{3}{2H}\langle\dot{\Psi}^{(2)}\rangle(1-\Omega) \\ &-\langle\Phi^{(2)}\rangle\left(1 - \frac{3}{2}\Omega\right) + \frac{2}{9\Omega}\left[1 + 2\hat{g} + \hat{g}^2 + \frac{21}{4}\Omega\right]\langle\tilde{\partial}_k\Phi\tilde{\partial}^k\Phi\rangle + \frac{4}{3}\langle\Phi\tilde{\partial}^2\Phi\rangle\end{aligned} \tag{5.167}$$

$$\begin{aligned}\langle q^{N\xi} \rangle_{\mathcal{D}} &= -1 + \frac{3}{2}\Omega - \left[3(1-\Omega) + \hat{g}(4-3\Omega)\right]\langle\Phi\rangle - \frac{3}{2}(1-\Omega)\left(\langle\Phi^{(2)}\rangle + \langle\Psi^{(2)}\rangle\right) \\ &+\left[\left(9 + 8\hat{g} + \frac{1}{2}\hat{g}^2\right) - 3\Omega\left(3 + \hat{g} + \frac{1}{2}\hat{g}^2\right)\right]\langle\Phi^2\rangle - 3\left[\left(3 + 6\hat{g} + 2\hat{g}^2\right)\right. \\ &\left.-\Omega\left(3 + 5\hat{g} + 2\hat{g}^2\right)\right]\langle\Phi\rangle^2 - \frac{1}{2H}\langle\dot{\Phi}^{(2)}\rangle - \frac{1}{6}\left[\tilde{\partial}^2\Phi^{(2)} - \tilde{\partial}^2\Psi^{(2)}\right] \\ &+\frac{1}{9\Omega}\left[\left(1 + 2\hat{g} + \hat{g}^2\right) + \frac{21}{4}\Omega\right]\langle\tilde{\partial}_k\Phi\tilde{\partial}^k\Phi\rangle + \frac{4}{3}\langle\Phi\tilde{\partial}^2\Phi\rangle\end{aligned} \tag{5.168}$$

$$
\begin{aligned}
\langle q^{N\theta}\rangle_{\mathcal{D}} &= -1+\frac{3}{2}\Omega_m+3\left[\hat{g}\left(\Omega_m-\frac{4}{3}\right)+\left(\frac{\Omega_m}{2}-3\right)\right]\langle\Phi\rangle+\frac{1}{\Omega_m}\left[\Omega_m\left(1+\frac{2}{3}\hat{g}\right)\right.\\
&\left.-\frac{4}{9}\left(1+\hat{g}\right)\right]\left\langle\tilde{\partial}^2\Phi\right\rangle+\left[12\left(1+\hat{g}\right)-3\Omega_m\left(\frac{7}{4}+\hat{g}+\frac{1}{2}\hat{g}\right)\right]\left\langle\Phi^2\right\rangle\\
&+\left[-\left(9+18\hat{g}+5\hat{g}\right)+\Omega_m\left(\frac{9}{2}+12\hat{g}+\frac{9}{2}\hat{g}^2\right)\right]\langle\Phi\rangle^2+\left[\frac{1}{3}\left(11+16\hat{g}+6\hat{g}^2\right)\right.\\
&\left.+\frac{\Omega_m}{3}\left(8+14\hat{g}-6\hat{g}^2\right)\right]\langle\Phi\rangle\left\langle\tilde{\partial}^2\Phi\right\rangle-\frac{3}{2H}(1+\Omega_m)\left\langle\dot{\Psi}^{(2)}\right\rangle\\
&-\frac{3}{2}\left(1-\frac{1}{2}\Omega_m\right)\left\langle\Phi^{(2)}\right\rangle-\frac{1}{2H}\left\langle\dot{\Phi}^{(2)}\right\rangle+\frac{1}{6}\left(\tilde{\partial}^2\Psi^{(2)}-\tilde{\partial}^2\Phi^{(2)}\right)\\
&+\frac{1}{6}\left\langle\tilde{\partial}_k v^k_{(2)}\right\rangle(1-3\Omega_m)-\frac{1}{6H}\left\langle\tilde{\partial}_k\dot{v}^k_{(2)}\right\rangle+\frac{1}{\Omega_m}\left[-\Omega_m\left(1+\frac{4}{3}\hat{g}\right)\right.\\
&\left.+\frac{4}{3}\left(1+\frac{2}{3}\hat{g}-\frac{1}{3}\hat{g}^2\right)\right]\left\langle\Phi\tilde{\partial}^2\Phi\right\rangle+\frac{1}{\Omega_m}\left[-\frac{\Omega_m^2}{6}\left(11+12\hat{g}\right)\right.\\
&\left.+\frac{\Omega_m}{9}\left(5-2\hat{g}-7\hat{g}^2\right)+\frac{4}{9}\left(1+2\hat{g}+\hat{g}^2\right)\right]\left\langle\tilde{\partial}_k\Phi\tilde{\partial}^k\Phi\right\rangle\\
&+\frac{1}{\Omega_m^2}\left[-\frac{16}{81}\left(1+2\hat{g}+\hat{g}^2\right)+\frac{2\Omega_m}{27}\left(5+8\hat{g}+3\hat{g}^3\right)\right]\left\langle\tilde{\partial}^2\Phi\right\rangle^2 \qquad (5.169)
\end{aligned}
$$

$$
\begin{aligned}
\langle q^{\theta}\rangle_{\mathcal{D}} &= -1+\frac{3}{2}\Omega_m-3(1+\hat{g})(1-\Omega_m)\langle\Phi\rangle+\frac{1}{\Omega_m}\left[\Omega_m(1+2\hat{g})-\frac{4}{9}(1+\hat{g})\right] \qquad (5.170)\\
&\times\left\langle\tilde{\partial}^2\Phi\right\rangle+\left[\left(15+16\hat{g}-\frac{3}{2}\hat{g}^2\right)-3\Omega_m\left(\frac{9}{2}+2\hat{g}\right)\right]\left\langle\Phi^2\right\rangle+\left[-3\left(5+6\hat{g}+\hat{g}^2\right)\right.\\
&\left.+\frac{9}{2}\Omega_m\left(3+6\hat{g}+\hat{g}^2\right)\right]\langle\Phi\rangle^2-\frac{3}{2H}(1-\Omega_m)\left(\left\langle\Phi^{(2)}\right\rangle H-\left\langle\dot{\Psi}^{(2)}\right\rangle\right)\\
&-\frac{1}{6}\left(\left\langle\tilde{\partial}^2\Phi^{(2)}\right\rangle-\left\langle\tilde{\partial}^2\Psi^{(2)}\right\rangle\right)+\frac{1}{6}\left\langle\tilde{\partial}^k v_k^{(2)}\right\rangle(1-3\Omega_m)\\
&-\frac{1}{6}\left\langle\tilde{\partial}^k\dot{v}_k^{(2)}\right\rangle+\frac{1}{\Omega_m}\left[\frac{\Omega_m}{3}\left(17+20\hat{g}+6\hat{g}^2\right)-\frac{2}{9}\left(16+23\hat{g}-7\hat{g}^2\right)\right]\langle\Phi\rangle\left\langle\tilde{\partial}^2\Phi\right\rangle\\
&+\frac{1}{\Omega_m}\left[-2\Omega_m(1+\hat{g})+\frac{2}{9}(8+5\hat{g}-3\hat{g}^2)\right]\left\langle\Phi\tilde{\partial}^2\Phi\right\rangle+\frac{1}{\Omega_m^2}\left[-\frac{\Omega_m^2}{6}(11+12\hat{g})\right.\\
&\left.+\frac{\Omega_m}{9}\left(5-2\hat{g}-7\hat{g}^2\right)+\frac{4}{9}\left(1+2\hat{g}+\hat{g}^2\right)\right]\left\langle\tilde{\partial}_k\Phi\tilde{\partial}^k\Phi\right\rangle\\
&+\frac{1}{\Omega_m^2}\left[\frac{2\Omega_m}{27}\left(5+8\hat{g}+3\hat{g}^2\right)-\frac{16}{81}\left(1+2\hat{g}+\hat{g}^2\right)\right]\left\langle\tilde{\partial}^2\Phi\right\rangle^2
\end{aligned}
$$

$$
\begin{aligned}
\langle q\rangle_{\mathcal{F}} \; = \; & -1+\frac{3}{2}\Omega_m-3\left[(1+\hat{g})(1-\Omega_m)+\frac{3}{2}\Omega_m g_I H\right]\langle\Phi\rangle+\left[1+\frac{2}{3}\hat{g}-\frac{4}{9\Omega_m}(1+\hat{g})\right]\langle\tilde{\partial}^2\Phi\rangle \\
& +\left\{3\left(4+\hat{g}-\frac{1}{2}\hat{g}^2\right)+9g_I H\left[(1+\hat{g})(1-\Omega_m)+\frac{3}{4}g_I H\right]\right. \\
& \left.-\frac{27}{2}g_I^2H^2\Omega_m^2\left(\frac{5}{4}-\Omega_m\right)\right\}\langle\Phi^2\rangle-3\left\{(4+5\hat{g})+\frac{3}{2}\Omega_m(3+4\hat{g}+\hat{g}^2)\right. \\
& -3g_I H\left[(1+\hat{g})\left(1-\frac{1}{2}\Omega_m^2\right)-\Omega_m(1+2\hat{g}+4g_I H)\right] \\
& \left.-\frac{27}{2}\Omega_m^2 g_I^2 H^2\left(1-\frac{1}{2}\Omega_m\right)\right\}\langle\Phi\rangle^2 \\
& -\frac{1}{\Omega_m^2}\left[\frac{1}{2}\Omega_m^2(3+4\hat{g})-\frac{4}{9}\Omega_m(2+\hat{g}-\hat{g}^2)-\frac{4}{27}(1+2\hat{g}+\hat{g}^2)\right]\langle\tilde{\partial}_k\Phi\tilde{\partial}^k\Phi\rangle \\
& +\frac{1}{3}\left\{(17+20\hat{g}+6\hat{g}^2)-\frac{2}{3\Omega_m}(10+14\hat{g}+4\hat{g}^2)\right. \\
& \left.+g_I H\left[(1-2\hat{g})-\frac{4}{3\Omega_m}(1+2\hat{g})-\Omega_m(2+\hat{g})\right]\right\}\langle\Phi\rangle\langle\tilde{\partial}^2\Phi\rangle \\
& -\frac{3}{2H}(1-\Omega_m)\left(H\langle\Phi^{(2)}\rangle+\langle\dot{\Psi}^{(2)}\rangle\right)+\frac{1}{6}a(2-3\Omega_m)\langle\tilde{\partial}_k v_2^k\rangle+\frac{1}{6}(\langle\tilde{\partial}^2\Phi^{(2)}\rangle \\
& +\frac{9}{8}\Omega_m H\int^t \mathrm{d}\,t'\left[\langle\Phi^{(2)}\rangle-\frac{1}{2}\langle\Phi^2\rangle-\langle v_1^k v_{1k}\rangle-\frac{g_I H}{a}\langle v_1^k\tilde{\partial}_k\Phi\rangle\right] \\
& -\frac{2}{27\Omega_m}\left[\frac{2}{3}(1+2\hat{g}+\hat{g}^2)-\Omega_m(5+8\hat{g}+3\hat{g}^3)\right]\langle\tilde{\partial}^2\Phi\rangle^2 . \qquad (5.171)
\end{aligned}
$$

Note that there are no connected $(\partial^2\Phi)^2$ terms in all the definitions of the deceleration parameter derived so far.

First the deceleration parameter associated with the fluid expansion rate Θ:

$$\begin{aligned}\langle q_\Theta\rangle_{\mathcal{F}} &= -1+\frac{3}{2}\Omega_m-3(1+\hat{g})(1-\Omega_m)\Phi+\left[(1+\frac{2}{3}\hat{g})-\frac{4}{9\Omega_m}(1+\hat{g})\right]\tilde{\partial}^2\Phi \\ &-3\hat{g}\left(4+\frac{3}{2}\hat{g}\right)(1-\Omega_m)\,\Phi^2-\frac{3}{2H}(1-\Omega_m)(H\Phi^{(2)}+\dot{\Psi}^{(2)}) \\ &+\frac{a}{6H}(1-3\Omega_m)\left(H\tilde{\partial}_k v^k_{(2)}-\tilde{\partial}_k\dot{v}^k_{(2)}\right)-\frac{1}{6}\left(\tilde{\partial}^2\Phi^{(2)}-\tilde{\partial}^2\Psi^{(2)}\right) \\ &+\frac{1}{9\Omega_m}\left[3\Omega_m(11+14\hat{g}+6\hat{g}^2)-(16+36\hat{g}+2\hat{g}^2)\right]\Phi\tilde{\partial}^2\Phi \\ &-\frac{1}{27\Omega_m^2}\left[\frac{27}{6}\Omega_m^2(11+12\hat{g})-24\Omega(1-\hat{g})-4(1+2\hat{g}+\hat{g}^2)\right]\tilde{\partial}_k\Phi\tilde{\partial}^k\Phi \\ &-\frac{4}{27\Omega_m^2}(1+2\hat{g}+\hat{g}^2)\tilde{\partial}^k\Phi\tilde{\partial}_k\tilde{\partial}^2\Phi \\ &+\frac{2}{27\Omega_m}\left[\Omega_m(5+8\hat{g}+3\hat{g}^2)-\frac{8}{3}(1+2\hat{g}+\hat{g}^2)\right]\tilde{\partial}^2\Phi\tilde{\partial}^2\Phi\,.\end{aligned} \tag{5.172}$$

The deceleration parameter an observer would measure from the all-sky average of the redshift-distance relation:

$$\begin{aligned}q^{d_z} &= -1+\frac{3}{2}\Omega_m-3(1+\hat{g})(1-\Omega_m)\Phi+\left[(1+\frac{2}{3}\hat{g})-\frac{4}{9\Omega_m}(1+\hat{g})\right]\tilde{\partial}^2\Phi \qquad (5.173)\\ &-3\hat{g}\left(4+\frac{3}{2}\hat{g}\right)(1-\Omega_m)\,\Phi^2-\frac{3}{2H}(1-\Omega_m)(H\Phi^{(2)}+\dot{\Psi}^{(2)}) \\ &-\frac{1}{2}a\Omega_m\tilde{\partial}_k v^k_{(2)}-\frac{a}{3H}\tilde{\partial}_k\dot{v}^k_{(2)}-\frac{1}{3}\tilde{\partial}^2\Phi^{(2)}+\frac{1}{6}\tilde{\partial}^2\Psi^{(2)} \\ &+\frac{1}{9\Omega_m}\left[3\Omega_m(10+14\hat{g}+6\hat{g}^2)-8(2+5\hat{g}+3\hat{g}^2)\right]\Phi\tilde{\partial}^2\Phi \\ &-\frac{1}{\Omega_m^2}\left[\frac{1}{6}\Omega_m^2(13+12\hat{g})-\frac{4}{9}\Omega_m(2-\hat{g}-12\hat{g}^2)-\frac{4}{27}(1+2\hat{g}+\hat{g}^2)\right]\tilde{\partial}_k\Phi\tilde{\partial}^k\Phi \\ &+\frac{4}{27\Omega_m^2}(1+2\hat{g}+\hat{g}^2)\tilde{\partial}_i\tilde{\partial}_j\Phi\tilde{\partial}^i\tilde{\partial}^j\Phi-\frac{8}{27\Omega_m^2}(1+2\hat{g}+\hat{g}^2)\tilde{\partial}^k\Phi\tilde{\partial}^2\tilde{\partial}_k\Phi \\ &+\frac{2}{27\Omega_m}\left[\Omega_m(5+8\hat{g}+3\hat{g}^2)-4(1+2\hat{g}+\hat{g}^2)\right]\tilde{\partial}^2\Phi\tilde{\partial}^2\Phi\end{aligned}$$

$$\begin{aligned}
q_0^{d_L} \;=\;& -1+\frac{3}{2}\Omega_m-3(1+\hat{g})(1-\Omega_m)\Phi+\frac{1}{\Omega_m}\left[\Omega_m(1+2\hat{g})-\frac{4}{9}(1+\hat{g})\right]\tilde{\partial}^2\Phi \qquad (5.174)\\
&-\frac{3}{2H}(1+\Omega_m)\left(\Phi^{(2)}H-\dot{\Psi}^{(2)}\right)+\frac{1}{2}\left(1+\frac{3}{2}\Omega_m\right)\tilde{\partial}_k v^{(2)k}+\frac{1}{6}\left(\tilde{\partial}^2\Phi^{(2)}-\partial^2\Psi^{(2)}\right)\\
&-(1-\Omega_m)\left(12\hat{g}+\frac{9}{2}\hat{g}^2\right)\Phi^2+\frac{1}{\Omega_m}\left[\frac{\Omega_m}{3}\left(10+14\hat{g}+6\hat{g}^2\right)-\frac{8}{9}\left(2+5\hat{g}+3\hat{g}^2\right)\right]\Phi\tilde{\partial}^2\Phi\\
&+\frac{1}{\Omega^2}\left[\frac{4}{27}\left(1+2\hat{g}+\hat{g}^2\right)+\frac{4}{9}\Omega_m\left(2-\hat{g}-3\hat{g}^2\right)\right]\tilde{\partial}_k\Phi\tilde{\partial}^k\Phi\\
&+\frac{1}{\Omega_m^2}\Phi\left[\frac{2\Omega_m}{27}\left(5+8\hat{g}+3\hat{g}^2\right)-\frac{88}{405}\left(1+2\hat{g}+\hat{g}^2\right)\right]\tilde{\partial}^2\Phi\tilde{\partial}^2\\
&-\frac{8}{27\Omega_m^2}\left(1+2\hat{g}+\hat{g}^2\right)\tilde{\partial}^k\Phi\tilde{\partial}^2\partial_k\Phi-\frac{4}{45\Omega_m^2}\left(1+2\hat{g}+\hat{g}^2\right)\tilde{\partial}^k\tilde{\partial}^l\Phi\tilde{\partial}_k\tilde{\partial}_l\Phi
\end{aligned}$$

$$\begin{aligned}
\mathcal{Q} \;=\;& -1+\frac{3}{2}\Omega_m-3(1+\hat{g})(1-\Omega_m)\Phi+\frac{1}{\Omega_m}\left[\Omega_m(1+2\hat{g})-\frac{4}{9}(1+\hat{g})\right]\tilde{\partial}^2\Phi \qquad (5.175)\\
&-\frac{3}{2H}(1+\Omega_m)\left(\Phi^{(2)}H-\dot{\Psi}^{(2)}\right)+\frac{1}{2}\left(1+\frac{3}{2}\Omega_m\right)\tilde{\partial}_k v^{(2)k}+\frac{1}{6}\tilde{\partial}^2\Psi^{(2)}\\
&-(1-\Omega_m)\left(12\hat{g}+\frac{9}{2}\hat{g}^2\right)\Phi^2+\frac{1}{\Omega_m}\left[2\left(2+\frac{7}{3}\hat{g}+\hat{g}^2\right)-\frac{16}{9}\Omega_m\left(1+2\hat{g}+\hat{g}^2\right)\right]\Phi\tilde{\partial}^2\Phi\\
&+\frac{1}{\Omega^2}\left[-\Omega_m^2\left(\frac{3}{2}+2\hat{g}\right)+\frac{4}{9}\Omega_m\left(2+\hat{g}-\hat{g}^2\right)+\frac{4}{27}\left(1+2\hat{g}+\hat{g}^2\right)\right]\tilde{\partial}_k\Phi\tilde{\partial}^k\Phi\\
&+\frac{1}{\Omega_m^2}\left[\frac{2\Omega_m}{5}\left(1+\frac{44}{27}\hat{g}+\frac{17}{27}\hat{g}^2\right)-\frac{104}{405}\left(1+2\hat{g}+\hat{g}^2\right)\right]\tilde{\partial}^2\Phi\tilde{\partial}^2\Phi\\
&+\frac{1}{\Omega_m^2}\left[-\frac{4\Omega_m}{45}\left(1+2\hat{g}+\hat{g}^2\right)+\frac{44}{135}\left(1+2\hat{g}+\hat{g}^2\right)\right]\tilde{\partial}^k\tilde{\partial}^l\Phi\tilde{\partial}_k\tilde{\partial}_l\Phi
\end{aligned}$$

where $\hat{g}=\dot{g}/gH$ and $\tilde{\partial}=\frac{\partial}{aH}$.

Chapter 6

The Influence of Structure Growth on the Area Distance

6.1 Introduction

Area distance, $D_A(z)$, or up to a factor of redshift the luminosity distance, $D_L(z)$, are the major tools for all model fitting and parameter estimation in modern cosmology. They play a crucial role in the determination of the current value of all cosmological parameters. The level of precision to be achieved through cosmological observations depend on 'how well' we understand these tools in an inhomogeneous spacetimes: for example the peak in temperature anisotropy correlation of the CMB is very sensitive to the area distance to the last scattering surface (LSS), apparent magnitude of a typical supernova event is very sensitive to the luminosity distance. Therefore, any refinement in observational and data analysis technique, requires that we also improve on our theoretical understanding of these tools, which invloves the challenges non-linear effects pose to our interpretation of the background cosmology.

It is obvious that the first theoretical refinement will be to properly calculate these tools from a physical space-time on which light propagates rather than continue to use the result from a fictitious FLRW background space-time, especially in the era of precision cosmology. Other important aspects to understand are: does the monopole of area distance on a physical space-time corresponds to the FLRW background value at any scale? How are we going to re-interpret cosmology on scales where the monopole of the area distance on a physical space-time does not correspond to the FLRW background prediction? It is also important to point out that some of these issues might be resolved in few years from now given that significant effort have been directed towards measuring the area distance from Bayron Acoustic Oscillation, (BAO) [229, 230].

Light propagation in inhomogeneous space-times gives rise to modification of area distance due to the difference between Ricci focussing (when the rays move through a uniform medium) and Weyl focussing (due to the tidal effects of nearby matter). It leads to adjustments in redshift due to the differences between the true redshift of a source and its redshift in a smoothed out model, It also leads to distortion of the affine parameter, since inhomogeneities change the relation $\lambda(z)$, and it may be observed through displacement the light beam experiences, i.e shifting sideways of the ray path by inhomogeneities, hence it experiences different Weyl and Ricci terms at the same λ because it is at a different spacetime point.

The problem of quantifying the effects of light propagation in a universe with an inhomogeneous distribution of structures was first addressed, independently by Zel'dovich [171] and Feynman [172]. Zel'dovich introduced the empty-beam approximation to enable him study the problem of light rays propagating in vacuum. His approach was later extended to the case of a partially-filled beam in a series of papers by Dyer and Roeder [175–178, 231] and others [166, 173] . These results later came to be known as the Dyer–Roeder approximation. An extension of the approximation to incorporate inhomogeneities experienced by the Hubble rate and the deceleration parameter was recently suggested in [182] and there are pieces of evidence that the extended version fits observation much better [232] .

The first perturbative approach (i.e. with light propagating in a perturbed FLRW spacetime) towards calculating the area distance from a perturbed spacetime was initiated by Sasaki [233], his computation was based on the Sachs equations and was limited to first order in the Synchronous gauge. Pyne and Birkinshaw [234] derived the area distance in Poisson gauge at first order in perturbation theory. Bonvin and Durrer [203] focussed on scalar perturbation in Poisson gauge, including all relativistic effects. First order treatment is very insufficient towards understanding the propagation of light, recent study has shown that it underestimates the distortion of images and magnification of some images, because of gravitational lensing when light rays passes through a void [235]

The first second order attempt was reported in [197], the authors used the technique developed by Sasaki [233] and they focussed only on dark matter dominated universe. The most recent attempt at this problem was reported in [236], they focused also on dark matter dominated universe. Their approach is slight different from [197], they worked in Poisson gauge transformed into light-cone coordinates through a general coordinate transformation. One key drawback of this approach is that one will need a reference result to be able to interpret his/her result.

Here we present the first general and consistent second order calculation of the area distance in Poisson gauge valid for use in different standard cosmological models. The only approximation made in our derivation is that we neglect all tensors and vectors at first order because they are very small compared to the density contribution at first order. However, we allow second order vectors and tensors which are sourced by the product of first order density fluctuations because their amplitude is a percent of the first order density fluctuation [76, 79] and it is even greater than the contribution from density perturbation in the neighborhood of the equality scale [78]. Furthermore, we do not use a particular gravitational theory in the derivation, thereby expanding the range of application of equations we report here.

This chapter is organized as follows: we provide general covariant evolution equations for null shear, null expansion and the area distance in Section 6.2, we derive and solve the photon geodesic equations in perturbation theory in Section 6.3 and calculate the physical redshift in terms of the perturbed metric variables in Section 6.4. The area distance in perturbation theory is derived and solved order by order in Section 6.5. We provide a general non-linear framework for expressing the area distance in terms of the physical redshift of the source in Section 6.6. We summarized all our perturbation theory results in Section 6.7 and calculate two-point correlation function for various modes that contribute to the area distance in Section 6.8. We provide various formalisms for performing full sky spherical harmonic decomposition in Section 6.9. We provide further details on how to boost our result to frame of the observer with a non-vanishing relative velocity and how to 'renormalize' the background in spherical harmonic space in Section 6.10 and calculate backreaction effect on key observables in Section 6.12. We conclude in Section 6.11 and provide further technical details on subtle mathematical tools used in this chapter in Section 6.13

We use hat to denote quantities living on the physical spacetime so that quantities on the conformal spacetime[1] will have no hat. Background quantities will be denoted by an over-bar and perturbed quantities apart from the metric variables have a delta in front, for example $\delta D_A(z)$ stands for the first order part of the area distance and $\delta^2 D_A(z)$ stands for the second piece.

6.2 Covariant Description of Area Distance

First we consider a beam of light characterized by a tangent vector k^a (we also refer to k^a as photon 4-vector) pointing in the past of the light cone. The past null cone has a constant phase hypersurface, $\hat{S}$ and the tangent vector is defined as a covariant derivative of the phase

[1]The area distance is invariant under conformal transformation with a conformal weight of one, this offers an additional reduction in complexity of the calculation since one could do most of the computation on the conformal space-time and multiply by the necessary conformal factor at the end of the calculation.

$\hat{k}^a = g^{ab}\nabla_b \hat{S}$. k^a satisfies the null constraint and it is geodesic:

$$\hat{k}_a \hat{k}^a = 0, \quad \hat{k}^b \hat{\nabla}_b \hat{k}^a = 0, \tag{6.1}$$

The vector $\hat{k}^a$ may be decomposed with respect to an observer with 4-velocity u^a into parallel and orthogonal components.

$$\hat{k}^a = \frac{\mathrm{d}\,\hat{x}^a}{\mathrm{d}\,\lambda}, \quad \hat{k}^a = (-\hat{u}_b \hat{k}^b)\,(\hat{u}^a \pm \hat{n}^a)\ . \tag{6.2}$$

The plus or minus sign in equation (6.2) determines the direction of propagation of the light rays and the direction of observation is characterized by vector, $\hat{n}^a$. The positive in equation (6.2) stands for the future null cone while the negative sign indicates that observation is taking place on the past null cone. For an observer here on earth has observational access on the past null cone

$$\hat{k}^a = (-\hat{u}_b \hat{k}^b)\,(\hat{u}^a - \hat{n}^a) = \hat{E}\,(\hat{u}^a - \hat{n}^a)\ , \tag{6.3}$$

where the photon energy is defined as $\hat{E} = -\hat{u}_b \hat{k}^b$. The screen space projection tensor may be defined as

$$\hat{N}_{ab} \equiv \hat{g}_{ab} + \hat{u}_a \hat{u}_b - \hat{n}_a \hat{n}_b = \hat{h}_{ab} - \hat{n}_a \hat{n}_b \tag{6.4}$$

and it satisfies the following relations

$$\hat{N}^a{}_a = 2, \quad \hat{N}_{ac}\hat{N}^c_b = \hat{N}_{ab}, \quad \hat{N}_{ab}\hat{k}^b = 0\,. \tag{6.5}$$

The metric, $\hat{N}_{ab}$, projects tensors into a two-dimensional screen space orthogonal to the null vector $\hat{k}^a$ and n^a. It also projects the space-time covariant derivatives onto angular derivatives on the screen space:

$$\hat{t}^{a..c}{}_{b...d} = \hat{N}^a{}_e \hat{N}^f{}_b ... \hat{N}^e{}_g \hat{N}^h{}_d \hat{T}^{e...g}{}_{f...h}\ , \tag{6.6}$$

$$\hat{\nabla}_{\perp a} \hat{t}_{b...c} = \hat{N}^d{}_a \hat{N}^e{}_b ... \hat{N}^f{}_c \nabla_d \hat{T}_{e...f}\ , \tag{6.7}$$

$$\hat{t}_{||} = \hat{n}^a \cdots n^f\, \hat{T}_{a\cdots f}\ , \tag{6.8}$$

$$\hat{\mathrm{D}}_{||} \hat{t}_{b...f} = n^a \hat{\nabla}_a \hat{t}_{b...f}\ , \tag{6.9}$$

where $\nabla_{\perp a}$ denotes derivative on the screen space and $\hat{\mathrm{D}}_{||}$ is the radial derivative along the line of sight. Tensors with piped subscripts denote the radial component of that tensor and those with 'bot sign' as subscripts stand for the angular component of the tensor . The anti-symmetric tensor of the screen space is related to that of the Levi-Civita tensor on the hypersurface and also to the Levi-Civita tensor of the full space time according to,

$$\hat{\varepsilon}_{ab} = \hat{\varepsilon}_{abd}\hat{n}^{d} = \hat{\eta}_{cdab}\hat{u}^{c}\hat{n}^{d} = \varepsilon_{[ab]} \,. \tag{6.10}$$

Also the screen space alternating tensor satisfies $\varepsilon_{ab}\varepsilon^{cd} = 2\hat{N}_{[a}{}^{c}\hat{N}_{b]}{}^{d}$ and it is invariant under parallel transport along the direction of k^a,

$$\hat{\nabla}_{\perp a}\hat{\varepsilon}_{bc} = 0 = \frac{\mathrm{d}\,\hat{\varepsilon}_{bc}}{\mathrm{d}\,\hat{\lambda}} \qquad \text{and} \qquad \hat{\nabla}^{a}_{\perp}\hat{N}_{ab} = 0\,. \tag{6.11}$$

The full irreducible decomposition of the covariant derivative of the photon vector is given by,

$$\hat{\nabla}_{b}\hat{k}_{a} = \frac{1}{2}\hat{N}_{ab}\hat{\theta} + \hat{\Sigma}_{ab} + \hat{\Omega}_{ab}\,, \tag{6.12}$$

where the newly introduced quantities are defined in terms of the photon vector as,

$$\begin{aligned}
\hat{\Sigma}_{\langle ab\rangle} &\equiv \hat{N}_{(a}{}^{c}\hat{N}_{b)}{}^{d}\hat{\nabla}_{c}\hat{k}_{d} - \frac{1}{2}\hat{\theta}N_{ab}\,, \\
\hat{\Omega}_{ab} &\equiv \hat{N}_{[a}{}^{c}\hat{N}_{b]}{}^{d}\hat{\nabla}_{c}\hat{k}_{d}\,, \\
\hat{\theta} &\equiv \frac{1}{2}\hat{g}^{ab}\hat{N}_{a}{}^{c}\hat{N}_{b}{}^{d}\hat{\nabla}_{a}\hat{k}_{b}\,,
\end{aligned} \tag{6.13}$$

where $\hat{\theta}$ describes the expansion of the null congruence as projected unto the screen space, $\hat{\Sigma}_{ab}$ describes the shear effect on the null congruence, while $\hat{\Omega}_{ab}$ encodes the information about the twisting of the photon geodesics, this quantity is zero because k^a is a gradient of the phase. The acceleration is set to zero because k^a is geodesic. The covariant derivative of the observer 4-velocity $\hat{u}^a$ in our current notation is decomposed as:

$$\hat{\nabla}_{b}\hat{u}_{a} = -\hat{A}_{a}u_{b} + \frac{1}{3}\hat{\Theta}\hat{h}_{ab} + \hat{\epsilon}_{abc}\hat{\omega}^{c} + \hat{\sigma}_{ab}\,, \tag{6.14}$$

where the basic kinematical information about the spacetime is encoded in the volume expansion rate, $\hat{\Theta}$, of the hypersurface orthogonal to u^a [2], 4-acceleration $\hat{A}_a$, vorticity (twisting), $\hat{\omega}_{ab}$ and shear $\hat{\sigma}_{ab}$. In terms of these variables, the geodesic equation (i.e equation (6.1))

[2] the vorticity is required to vanish on this hypersurface[84]

reduces to the equations for the photon energy and observational direction

$$\frac{\mathrm{d}\,\hat{E}}{\mathrm{d}\,\hat{\lambda}} = -\hat{E}^2\left[\frac{1}{3}\hat{\Theta} - \hat{A}_a n^a + (\hat{\sigma}_{ab}\hat{n}^a\hat{n}^b\right], \tag{6.15}$$

$$\frac{\mathrm{d}\,\hat{n}_{\|}}{\mathrm{d}\,\hat{\lambda}} = \hat{E}\left[2\hat{\sigma}_{\|} - \hat{\omega}_{\|}\right], \tag{6.16}$$

$$\frac{\mathrm{d}\,\hat{n}^i_{\perp}}{\mathrm{d}\,\lambda} = \hat{E}\left[-A^i + n^a\sigma^i_a + n^a\omega^i_a\right], \tag{6.17}$$

where $\frac{\mathrm{d}}{\mathrm{d}\,\hat{\lambda}} = \hat{k}^a\hat{\nabla}_a$, which stands for derivative with respect to the affine parameter and we normalize the photon energy at the obersever as $\hat{E}_o = -\hat{u}_a\hat{k}^a|_o = 1$. The change of signal emitted by a source moving with a 4-velocity $\hat{u}^a_s$, e.g a galaxy (here denoted as 's'), and measured by an observer with 4-velocity $\hat{u}^a_o$ is related through the redshift,

$$(1+\hat{z}) = \frac{\lambda_0}{\lambda_s} = \frac{(-\hat{k}_a\hat{u}^a)_s}{(-\hat{k}_b\hat{u}^b)_o} = \frac{\hat{E}_s}{\hat{E}_o}\,. \tag{6.18}$$

Using equation (6.15), the redshift propagation equation becomes,

$$\frac{\mathrm{d}\,\hat{z}}{d\hat{\lambda}} = -(1+\hat{z})^2\left[\frac{1}{3}\hat{\Theta} - \hat{A}_a n^a + \sigma_{ab}n^a n^b\right]. \tag{6.19}$$

This is a general non-perturbative, coordinate independent representation of the propagation equation for the observed redshift. The propagation equations for the null shear and the null expansion are given by,

$$\frac{\mathrm{d}\,\hat{\theta}}{\mathrm{d}\,\hat{\lambda}} = -2\hat{\Sigma}^2 - \frac{1}{2}\hat{\theta}^2 - \hat{R}_{ab}\hat{k}^a\hat{k}^b\,, \tag{6.20}$$

$$\frac{d\hat{\Sigma}_{\langle ab\rangle}}{\mathrm{d}\,\hat{\lambda}} = -\hat{\Sigma}_{ab}\hat{\theta} + \hat{N}_{\langle e}{}^a\hat{N}_{f\rangle}{}^c\hat{R}_{abcd}\hat{k}^c\hat{k}^d\,. \tag{6.21}$$

$$\tag{6.22}$$

where $\hat{\Sigma}^2 = \Sigma_{\langle ab\rangle}\Sigma^{\langle ab\rangle}/2$. The propagation equation for shear is obtained by projecting the Ricci identity with $\hat{N}_{\langle e}{}^a\hat{N}_{f\rangle}{}^c$ and that of the expansion is obtained by contracted the Ricci identity. At this point one may use the Einstein equations to replace the Ricci tensor in equation (6.20), however, we did not approach the problem this way. The invariant area of the screen space, A, is related to the null expansion $\hat{\theta}$ and also to the Area distance $\hat{D}_A$ according to

$$\frac{1}{\sqrt{\hat{A}}}\frac{d\sqrt{\hat{A}}}{d\hat{\lambda}} = \frac{d\ln\hat{D}_A}{d\hat{\lambda}} = \frac{1}{2}\hat{\theta}\,. \tag{6.23}$$

Substituting equation (6.23) in equation (6.20), we obtain a second order differential equation for area distance.

$$\frac{\mathrm{d}^2\hat{D}_A}{\mathrm{d}\,\lambda^2} = -\left[\frac{1}{2}\hat{R}_{ab}\hat{k}^a\hat{k}^b + 2\hat{\Sigma}^2\right]D_A. \tag{6.24}$$

Our job in the subsequent sections is to calculate equations (6.21, 6.20) and (6.24) in perturbation theory. Equation 6.24 may also be written directly in terms of the physical redshift as shown in chapter 3 and in [182].

6.3 Geodesics Equation in Perturbation theory

The conformal transformation $\hat{g}_{\mu\nu} \to g_{\mu\nu}$ maps the null geodesic equation of the physical spacetime $\hat{g}_{\mu\nu}$ to a null geodesic on a perturbed Minkowski space time $g_{\mu\nu}$ with the affine parameter associated with each metric, transformaing as $\mathrm{d}\,\hat{\lambda} \to \mathrm{d}\,\lambda = a^{-2}\mathrm{d}\,\hat{\lambda}$, so that the photon 4-vector transforms as $\hat{k}^b = a^2k^b$. Hence the photon energy transforms as $\hat{E} = -\hat{u}_b\hat{k}^b = -a\,u_bk^b = aE$. This transformation makes it possible to calculate the redshift and the area distance from a conformally flat metric. The null geodesic equation is also invariant under conformal transformation provided that the null constraint condition, $k^bk_b = 0$, is satisfied. The time and space components of the perturbed geodesic equation at first order are given by:

$$\frac{\mathrm{d}\,\delta k^0}{\mathrm{d}\,\lambda} = -2n^k\mathrm{D}_k\Phi\,, \tag{6.25}$$

$$\frac{\mathrm{d}\,\delta k^i}{\mathrm{d}\,\lambda} = -\left[2n^i\Phi' - 2n^kn^i\mathrm{D}_k\Phi + 2\mathrm{D}^i\Phi\right]\,, \tag{6.26}$$

here $'$ stands for derivative with respect to conformal time. These equations are separable and can easily be solved:

$$\delta k^0_s = \delta k^0_o + 2\left(\Phi_s - \Phi_o\right) - 2\int_{\lambda_o}^{\lambda_s}\Phi'\mathrm{d}\,\lambda\,, \tag{6.27}$$

$$\delta k^i_s = \delta k^i_o - 2n^i\left(\Phi_s - \Phi_o\right) - 2\int_{\lambda_o}^{\lambda_s}\mathrm{D}^i\Phi\mathrm{d}\,\lambda\,. \tag{6.28}$$

The spatial part of δk^i may further be split into parallel component and the perpendicular component with respect to n^i

$$\delta k_{\|s} = \delta k_{\|} - 2\left(\Phi_s - \Phi_o\right)\,, \tag{6.29}$$

$$\delta k_{\perp s} = \delta k_{\perp io} - 2\int_{\lambda_o}^{\lambda_s}\nabla_{\perp i}\Phi\mathrm{d}\,\lambda\,. \tag{6.30}$$

The spatial position may also be calculated from $\mathrm{d}\,x^a/\lambda = \hat{k} = k^a + \delta k^a$,

$$\frac{\mathrm{d}\,x^i}{\mathrm{d}\,\lambda} = -n_o^i - 2n^i \int_{\lambda_o}^{\lambda} \Phi' \mathrm{d}\,\lambda - 2\int_{\lambda_o}^{\lambda} \nabla_{\perp}^i \Phi \mathrm{d}\,\lambda \,. \tag{6.31}$$

After performing an integration by parts, the final solution becomes

$$x^i(\lambda) = x_o^i - (\lambda - \lambda_o) n_o^i - 2n^i \int_{\lambda_o}^{\lambda} (\lambda_s - \lambda)\Phi' \mathrm{d}\,\lambda - 2\int_{\lambda_o}^{\lambda} (\lambda_s - \lambda) \nabla_{\perp}^i \Phi \mathrm{d}\,\lambda \,. \tag{6.32}$$

At second order, the time component of the geodesic equation becomes

$$\begin{aligned} \frac{\mathrm{d}\,\delta^2 k^0}{\mathrm{d}\,\lambda} &= 4\left(\delta k^j \left(\Phi' n_j + \mathrm{D}_j \Phi\right) - \delta k^0 \left(\Phi' - n^i \mathrm{D}_i \Phi\right)\right) \\ &\quad + \left[h'_{ij} n^i n^j - n^j n^i \mathrm{D}_i \omega_j^{(2)} + \left({}^{(2)}\Phi' - {}^{(2)}\Psi'\right) - 2n^k \mathrm{D}_k {}^{(2)}\Phi + 8\Phi n^k \mathrm{D}_k \Phi\right] \\ &\quad + 2\delta k^0 \delta k' - 2\delta k^j \mathrm{D}_j \delta k^0 \,. \end{aligned} \tag{6.33}$$

We simplify the $\delta k^j \times \Phi$ coupling as follows:

$$\begin{aligned} & 4\left(\delta k^j \left(\Phi' n_j + \mathrm{D}_j \Phi\right) - \delta k^0 \left(\Phi' - n^i \mathrm{D}_i \Phi\right)\right) \\ =\;& -8\frac{\mathrm{d}\,\Phi}{\mathrm{d}\,\lambda}(\Phi - \Phi_o) + 16\left(\frac{\mathrm{d}\,\Phi}{\mathrm{d}\,\lambda} - \Phi'\right)\int_{\lambda_o}^{\lambda} \Phi' \mathrm{d}\,\lambda - 8\nabla_{\perp i}\Phi \int_{\lambda_o}^{\lambda} \nabla_{\perp}^i \Phi \mathrm{d}\,\lambda \,. \end{aligned} \tag{6.34}$$

For $\delta k^0 \times \delta k$ non-linear terms, it is more helpful to convert the conformal time to affine parameter so that we can use the first order geodesic equation to simplify further

$$2\delta k^0 \delta k' - 2\delta k^i \mathrm{D}_i \delta k^0 = -4\delta k^0 n^i \mathrm{D}_i \Phi + 2\delta k^0 n^i \mathrm{D}_i \delta k^0 - 2\delta k^i \mathrm{D}_i \delta k^0 \tag{6.35}$$

$$\begin{aligned} &= 8\left(\left(\Phi' + \Phi'_o\right) - \left(n^k \mathrm{D}_k \Phi_o + \int_{\lambda_o}^{\lambda} \Phi'' \mathrm{d}\,\lambda\right)\right)(\Phi - \Phi_o) \\ &\quad + 8(\nabla_{\perp k}\Phi - \nabla_{\perp k}\Phi_o)\int_{\lambda_o}^{\lambda} \nabla_{\perp}^k \Phi \mathrm{d}\,\lambda + 8\left(\Phi' - \frac{\mathrm{d}\,\Phi}{\mathrm{d}\,\lambda}\right)\int_{\lambda_o}^{\lambda} \Phi' \mathrm{d}\,\lambda \\ &\quad - 8\int_{\lambda_o}^{\lambda} \nabla_{\perp k}\Phi \mathrm{d}\,\lambda \int_{\lambda_o}^{\lambda} \nabla_{\perp}^k \Phi \mathrm{d}\,\lambda \,. \end{aligned} \tag{6.36}$$

Putting everything back together and re-arranging some of the terms, the solution to equation (6.33) becomes ,

$$\begin{aligned}\delta^2 k_s^0 &= \delta^2 k_o^0 + 2\left(\Phi_s^{(2)} - \Phi_o^{(2)}\right) + n^b\left(\omega_b|_s - \omega_b|_o\right) - 8\left(\Phi_s^2 - \Phi_o^2\right) + 8\Phi_s\left(\Phi_s - \Phi_o\right) \\ &+ \int_{\lambda_s}^{\lambda_o}\left[h'_\parallel - \omega'_\parallel - \left({}^{(2)}\Phi' + {}^{(2)}\Psi'\right) + 8\Phi\Phi'\right]\mathrm{d}\,\lambda + 8\int_{\lambda_o}^{\lambda_s}(\Phi_s - \Phi)\Phi'\mathrm{d}\,\lambda \\ &-8\nabla_{\perp k}\Phi_o\int_{\lambda_o}^{\lambda_s}(\lambda_s - \lambda)\nabla^k\Phi\mathrm{d}\,\lambda + 8\int_{\lambda_o}^{\lambda_s}\left((\Phi' - n^k\mathrm{D}_k\Phi_o) + (\Phi'_o - \int_{\lambda_o}^{\lambda}\Phi''(\lambda_1)\mathrm{d}\,\lambda_1\right) \\ &\times(\Phi - \Phi_o)\mathrm{d}\,\lambda - 8\int_{\lambda_o}^{\lambda_s}\Phi'\int_{\lambda_o}^{\lambda}\Phi'(\lambda_1)\mathrm{d}\,\lambda_1\mathrm{d}\,\lambda \\ &-8\int_{\lambda_o}^{\lambda_s}\int_{\lambda_o}^{\lambda}\nabla_{\perp k}\Phi(\lambda_1)\mathrm{d}\,\lambda_1\int_{\lambda_o}^{\lambda}\nabla_\perp^k(\lambda_2)\mathrm{d}\,\lambda_2\mathrm{d}\,\lambda\,.\end{aligned} \tag{6.37}$$

6.4 Observed Redshift in Perturbation theory

From the definition of the observed redshift in equation (6.18), we may expand E up to second order:

$$(1+\hat{z}) = \frac{\hat{E}}{\hat{E}_0} = \frac{k^a u_a|_S}{k^b u_b|_o} = \frac{\left(E_s + \delta E_s + \frac{1}{2}\delta^2 E_s\right)}{\left(E_o + \delta E_o + \frac{1}{2}\delta^2 E_o\right)}\,. \tag{6.38}$$

For $\delta E/E \ll 1$, we expand in power series

$$\begin{aligned}(1+\hat{z}) &= \frac{E_s}{E_o} + \frac{1}{E_o}\left(\delta E_s - \frac{\delta E_o E_s}{E_o}\right) + \frac{1}{E_o}\left((\delta E_o)^2\frac{E_s}{E_o^2} - \frac{1}{2}\delta^2 E_o\frac{E_s}{E_o}\right. \\ &\left. - \frac{\delta E_o \delta E_s}{E_o} + \frac{1}{2}\delta^2 E_s\right)\,.\end{aligned} \tag{6.39}$$

Expanding the definition of photon energy, $\hat{E} = k^a u_a$ in perturbation theory up to second order we find,

$$\begin{aligned}\hat{E} &= E + \left(E(\Phi + \mathrm{D}_\parallel v) - \delta k^a u_a\right) + E\left(\frac{1}{2}\omega_\parallel^{(2)} - \frac{1}{2}\left(\Phi^2 - \Phi^{(2)}\right) - 2\Phi\mathrm{D}_\parallel v\right. \\ &+\frac{1}{2}\left(v_\parallel^{(2)} + \mathrm{D}_\parallel v^{(2)}\right) + \left.\frac{1}{2}\left(\mathrm{D}_\parallel v\mathrm{D}_\parallel v + \nabla_{\perp k}v\nabla_\perp^k v\right)\right) \\ &-\frac{1}{2}\delta^2 k^0 - \delta k^0\Phi - \delta k^i\mathrm{D}_i v\,.\end{aligned} \tag{6.40}$$

The 3-d covariant/partial derivative of the perculair velocity is decomposed into the transverse part and the parallel part with respect to the line of sight $\mathrm{D}_i v = \mathrm{D}_\| v n_i + \nabla_{\perp i} v$

$$\mathrm{D}_i v \mathrm{D}^i v = \mathrm{D}_\| v \mathrm{D}_\| v + \nabla_{\perp k} v \nabla_\perp^k v \,. \tag{6.41}$$

We set the perturbed photon vector at the observer to zero [233], so that the energy at the observer simplifies to

$$\begin{aligned}\hat{E}_o &= E_o + \left(E_o(\Phi_o + \mathrm{D}_\| v_o)\right) + E_0 \left(\frac{1}{2}\omega^{(2)}_{\|_o} - \frac{1}{2}\left(\Phi_o^2 - \Phi_s^{(2)}\right) - 2\Phi_o \mathrm{D}_\| v_o \right. \\ &\quad \left. + \frac{1}{2}\left(v^{(2)}_{\|o} + \mathrm{D}_\| v^{(2)}_o\right) + \frac{1}{2}\left(\mathrm{D}_\| v_o \mathrm{D}_\| v_o + \nabla_{\perp k} v_o \nabla_\perp^k v_o\right)\right) .\end{aligned} \tag{6.42}$$

At first order, the perturbed redshift is given by

$$\frac{\delta z_s}{(1+\bar{z})} = \left(\mathrm{D}_\| v_s - \mathrm{D}_\| v_o\right) - (\Phi_s - \Phi_o) + 2\int_{\lambda_o}^{\lambda_s} \Phi' \mathrm{d}\,\lambda \tag{6.43}$$

and at second order we find

$$\frac{\delta^2 z_s}{(1+\bar{z})} = \text{Doppler} + \text{SW} + \text{Doppler} \times \text{SW} + \text{ISW} \,, \tag{6.44}$$

where the pure Doppler term consists of pure second-order contributions, and quadratic first-order peculiar velocity terms, which include the transverse Doppler effect, and a new coupling between source and observer:

$$\begin{aligned}&\text{Doppler} = v^{(2)}_{\|s} - v^{(2)}_{\|o} + \mathrm{D}_\| v^{(2)}_s - \mathrm{D}_\| v^{(2)}_o + \mathrm{D}_\| v_s \mathrm{D}_\| v_s + \nabla_{\perp k} v_s \nabla_\perp^k v_s \\ &- \mathrm{D}_\| v_o \mathrm{D}_\| v_o - \nabla_{\perp k} v_o \nabla_\perp^k v_o + \mathrm{D}_\| v_o \mathrm{D}_\| v_s \,.\end{aligned} \tag{6.45}$$

The SW terms also contain pure and mixed contributions, which couple the potential at the source and observer:

$$\text{SW} = -\Phi_s^{(2)} + \Phi_o^{(2)} + 3\Phi_s^2 - \Phi_o^2 - 2\Phi_s\Phi_o - 16\Phi_o\Phi_o'(\lambda_s - \lambda_o) \,. \tag{6.46}$$

There is a coupling between the Doppler terms at source and observer with the potential:

$$\text{Doppler} \times \text{SW} = 4(\Phi_o \mathrm{D}_\| v_o - \Phi_s \mathrm{D}_\| v_s) - 2(\Phi_o \mathrm{D}_\| v_s - \Phi_s \mathrm{D}_\| v_o) \,. \tag{6.47}$$

The ISW effect at second order is much more complicated, and consists of several contributions:

$$\mathrm{ISW} \;=\; \mathrm{ISW}^{(2)} + \mathrm{ISW}\times\mathrm{SW} + \mathrm{ISW}\times\mathrm{Doppler} + \mathrm{Integrated\ ISW}\,, \tag{6.48}$$

where we have a pure second-order ISW contribution:

$$\mathrm{ISW}^{(2)} = \int_{\lambda_o}^{\lambda_s}\left[(\Phi^{(2)\prime} + \Psi^{(2)\prime}) + \omega_\parallel' - h_\parallel' - 8\Phi\Phi'\right]\mathrm{d}\,\lambda\,. \tag{6.49}$$

We have defined $h_\parallel = n^i n^j h_{ij}$ as the radial part of the tensor mode. Then we have the first-order ISW effect crossed with SW, ISW and Doppler terms:

$$\begin{aligned}\mathrm{ISW}\times\mathrm{SW} &= \int_{\lambda_o}^{\lambda_s}\Big\{16\Phi_o'\Phi - 12(\Phi_s-\Phi_o)\Phi' + 8(\lambda_s-\lambda)\nabla_{\perp i}\Phi_o\nabla_\perp^i\Phi \qquad (6.50)\\ &\quad +8(\lambda_s-\lambda_o)\Phi_o\Phi''\Big\}\mathrm{d}\,\lambda\\ \mathrm{ISW}\times\mathrm{Doppler} &= 4\left(\mathrm{D}_\parallel v_s - \mathrm{D}_\parallel v_o\right)\int_{\lambda_o}^{\lambda_s}\Phi'\mathrm{d}\,\lambda\,. \qquad (6.51)\end{aligned}$$

We have the double-integrated SW terms:

$$\begin{aligned}\mathrm{Integrated\ ISW} &= 8\int_{\lambda_o}^{\lambda_s}\Phi'\int_{\lambda_o}^{\lambda}\Phi'(\lambda_1)\mathrm{d}\,\lambda_1\mathrm{d}\,\lambda - 8\int_{\lambda_o}^{\lambda_s}\Phi\int_{\lambda_o}^{\lambda}\Phi''(\lambda_1)\mathrm{d}\,\lambda_1\mathrm{d}\,\lambda \qquad (6.52)\\ &+8\int_{\lambda_o}^{\lambda_s}\int_{\lambda_o}^{\lambda}\nabla_{\perp k}\Phi(\lambda_1)\mathrm{d}\,\lambda_1\int_{\lambda_o}^{\lambda}\nabla_\perp^k(\lambda_2)\mathrm{d}\,\lambda_2\mathrm{d}\,\lambda\,.\end{aligned}$$

These new effects contribute to the nonlinear redshift space distortions.

$$\begin{aligned}\mathrm{ISW} &= \mathrm{ISW}^{(2)} + \mathrm{ISW}\times\mathrm{SW} + \mathrm{ISW}\times\mathrm{Doppler} + \mathrm{Integrated\ ISW} \qquad (6.53)\\ &= \int_{\lambda_o}^{\lambda_s}\left[(\Phi^{(2)\prime}+\Psi^{(2)\prime}) + \omega_\parallel' - h_\parallel' - 8\Phi\Phi'\right]\mathrm{d}\,\lambda + \int_{\lambda_o}^{\lambda_s}\Big\{16\Phi_o'\Phi - 12(\Phi_s-\Phi_o)\Phi'\\ &\quad +8(\lambda_s-\lambda)\nabla_{\perp i}\Phi_o\nabla_\perp^i\Phi + 8(\lambda_s-\lambda_o)\Phi_o\Phi''\Big\}\mathrm{d}\,\lambda + 4\left(\mathrm{D}_\parallel v_s - \mathrm{D}_\parallel v_o\right) \qquad (6.54)\\ &\quad \times\int_{\lambda_o}^{\lambda_s}\Phi'\mathrm{d}\,\lambda + 8\int_{\lambda_o}^{\lambda_s}\Phi'\int_{\lambda_o}^{\lambda}\Phi'(\lambda_1)\mathrm{d}\,\lambda_1\mathrm{d}\,\lambda - 8\int_{\lambda_o}^{\lambda_s}\Phi\int_{\lambda_o}^{\lambda}\Phi''(\lambda_1)\mathrm{d}\,\lambda_1\mathrm{d}\,\lambda\\ &\quad +8\int_{\lambda_o}^{\lambda_s}\int_{\lambda_o}^{\lambda}\nabla_{\perp k}\Phi(\lambda_1)\mathrm{d}\,\lambda_1\int_{\lambda_o}^{\lambda}\nabla_\perp^k(\lambda_2)\mathrm{d}\,\lambda_2\mathrm{d}\,\lambda\,.\end{aligned}$$

Finally, the observed redshift of a source to second order is

$$(1+\hat{z}_s) = (1+\bar{z})\left\{1+\delta z+\frac{1}{2}\delta^2 z\right\}, \tag{6.55}$$

where $\bar{z}$ is the redshift in the background.

6.5 Area Distance in Perturbation theory

Under a conformal transformation, the area distance D_A transform as $\hat{D}_A = aD_A$, hence the area distance in the physical universe is related to that on a perturbed Minkowski background through a scale factor. This relationship simplifies most of the calculations that follows.

6.5.1 Background

On the background, peforming the conformal transformation for the area distance D_A, $\hat{D}_A = aD_A$ equation (6.24), simplifies to

$$\frac{\mathrm{d}^2 D_A}{\mathrm{d}\,\lambda^2} = 0, \tag{6.56}$$

with a solution $D_A = C_1 + \lambda C_2$. Implementing the standard initial conditions for the area distance $D_A(0) = 0$ and $\mathrm{d}\,D_A/\mathrm{d}\,\lambda = 1$ [2], we immediately find that $D_A = \lambda = (\lambda - \lambda_o)$. If one had solved this on an FLW background spacetime without the transformation of the photon vector $\hat{k}^b = a^{-2}k^b$ and the area distance $\hat{D}_A = aD_A$, one would wind up with the same result but multiplied by the scale factor. On the FLRW flat background, the area distance is

$$\hat{D}_A = \frac{1}{(1+z)}\int_{z_o}^{z_s}\frac{\mathrm{d}\,z}{(1+z)\mathcal{H}} = a(\lambda-\lambda_o)\,. \tag{6.57}$$

6.5.2 First Order Contribution

Using the background equations and the time component of the first order perturbed photon vector, the area distance at first order in perturbation theory is given by

$$\frac{\mathrm{d}^2\delta D_A}{\mathrm{d}\,\lambda^2} = -2\frac{\mathrm{d}\,D_A}{\mathrm{d}\,\lambda}n^b\mathrm{D}_b\Phi - D_A\left[\Phi'' - 2n^b\mathrm{D}_b\Phi' + \mathrm{D}_b\mathrm{D}^b\Phi\right], \tag{6.58}$$

$$= 2\left(\frac{\mathrm{d}\,\Phi}{\mathrm{d}\,\lambda} - \Phi'\right) - D_A\left(\frac{\mathrm{d}^2\Phi}{\mathrm{d}\,\lambda^2} + \nabla_\perp^2\Phi\right) = g_1(\lambda)\,. \tag{6.59}$$

In the second equality, we have made use of the relations (see Section 6.13 for a discussion on this)

$$\mathrm{D}_b\mathrm{D}^b\Phi = \nabla^2_\perp\Phi + n^b n^c \mathrm{D}_b\mathrm{D}_c\Phi = \nabla^2_\perp\Phi - \frac{\mathrm{d}^2\Phi}{\mathrm{d}\lambda^2} + 2\frac{\mathrm{d}\Phi'}{\mathrm{d}\lambda} - \Phi'' , \tag{6.60}$$

$$n^b\mathrm{D}_b\Phi = \Phi' - \frac{\mathrm{d}\Phi}{\mathrm{d}\lambda} . \tag{6.61}$$

We have also set $\mathrm{d}D_A/\mathrm{d}\lambda = 1$ following the background result. The homogeneous solution to equation (6.59) is the same as that of equation (6.56), hence for the particular solution, after calculating the Wronskian (W=1), is given by

$$\delta D_A = \int_{\lambda_o}^{\lambda_s}\int_{\lambda_o}^{\lambda} g(\lambda')\mathrm{d}\lambda'\mathrm{d}\lambda = \int_{\lambda_o}^{\lambda_s}(\lambda_s - \lambda)g_1(\lambda)\mathrm{d}\lambda . \tag{6.62}$$

Then after several integrations by parts and further simplification, we arrive at

$$\frac{\delta D_A}{D_A(s)} = \Bigg[-\Phi_s - 3\Phi_o + \frac{4}{(\lambda_s-\lambda_o)}\int_{\lambda_o}^{\lambda_s}\Phi\mathrm{d}\lambda - 2\int_{\lambda_o}^{\lambda_s}\frac{(\lambda_s-\lambda)}{(\lambda_s-\lambda_o)}\Phi'\mathrm{d}\lambda - \int_{\lambda_o}^{\lambda_s}\frac{(\lambda_s-\lambda)(\lambda-\lambda_o)}{(\lambda_s-\lambda_o)}\nabla^2_\perp\Phi\mathrm{d}\lambda\Bigg] . \tag{6.63}$$

We have also made use of the background solution. It happens that equation (6.63) is also proportional to the weak lensing convergence, κ, at first order in cosmological perturbation theory. Our result is in agreement with [237].

6.5.3 Second Order Contribution

At second order, things become more complicated because we have products of gravitational potential and the area distance and the shear contributing:

$$
\begin{aligned}
\frac{\mathrm{d}\,\delta^2 D_A}{\mathrm{d}\,\lambda^2} \;=\;& \mathrm{D}_i\delta D_A\left(n^i\Phi' + \mathrm{D}^i\Phi\right) - 4\left(\delta k^i - n^i\delta k^0\right)\mathrm{D}_i\left[\frac{\mathrm{d}\,\delta D_A}{\mathrm{d}\,\lambda}\right] \\
& -4n^i\mathrm{D}_i\Phi\left(2n^j\mathrm{D}_j\delta D_A + \frac{\mathrm{d}\,\delta D_A}{\mathrm{d}\,\lambda}\right) + \frac{\mathrm{d}\,D_A}{\mathrm{d}\,\lambda}\left(h'_{ij}n^in^j\right. \\
& +4\delta k_i\left(\Phi' n^i + \mathrm{D}_i\Phi\right) - 4\delta k^0\left(\Phi' + n^k\mathrm{D}_k\Phi\right) + \left(\Phi^{(2)\prime} - \Psi^{(2)\prime}\right) \\
& \left.+8\Phi n^k\mathrm{D}_k\Phi - 2n^k\mathrm{D}_k\Phi^{(2)} - n^in^j\mathrm{D}_i\omega_j\right) - D_A\left[\frac{1}{2}\mathrm{h}''_{ij}n^in^j - a^5\delta\Sigma_{ij}\delta\Sigma^{ij}\right. \\
& +2\Phi'^2 + 8\Phi\Phi'' + \Psi^{(2)\prime\prime} - 2n^k\Phi'\mathrm{D}_k\Phi - 8n^k\Phi\mathrm{D}_k\Phi' - 2n^k\mathrm{D}_k\Psi^{(2)\prime} \\
& +4\Phi\mathrm{D}_k\mathrm{D}^k\Phi + \frac{1}{2}\left(\mathrm{D}_k\mathrm{D}^k\Phi^{(2)} + \mathrm{D}_k\mathrm{D}^k\Psi^{(2)}\right) - \frac{1}{2}n^in^j\mathrm{D}_i\omega'_j + 2n^in^j\mathrm{D}_i\Phi\mathrm{D}_j\Phi \\
& +4n^in^j\Phi\mathrm{D}_i\mathrm{D}_j\Phi - \frac{1}{2}n^in^j(\mathrm{D}_i\mathrm{D}_j\Phi^{(2)} - \mathrm{D}_i\mathrm{D}_j\Psi^{(2)}) + \frac{1}{2}n^i\mathrm{D}_k\mathrm{D}^k\omega_i \\
& -\frac{1}{2}n^in^j\mathrm{D}_k\mathrm{D}^k\mathrm{h}_{ij} - 2\delta k^0\left(\Phi'' + 2n^k\mathrm{D}_k\Phi' - 2\mathrm{D}_k\mathrm{D}^k\Phi\right) \\
& \left.+2\delta k^i\left(\Phi'' n_i + 2\mathrm{D}_i\Phi' - n_i\mathrm{D}_k\mathrm{D}^k\Phi\right)\right]\,. \qquad (6.64)
\end{aligned}
$$

When we substitute the first order photon vector and do some re-organization and simplifications, we find

$$
\begin{aligned}
\frac{\mathrm{d}^2\delta^2 D_A}{\mathrm{d}\,\lambda^2} \;=\;& -2\delta D_A\left[\Phi'' - 2n^i\mathrm{D}_i\Phi' + \mathrm{D}_i\mathrm{D}^i\Phi\right] + 4\mathrm{D}^i\delta D_A\left[\mathrm{D}_i\Phi - 2n_in^c\mathrm{D}_c\Phi\right] \\
& -4\frac{\mathrm{d}\,\delta D_A}{\mathrm{d}\,\lambda}n^i\mathrm{D}_i\Phi + 8\mathrm{D}_i\left[\frac{\mathrm{d}\,\delta D_A}{\mathrm{d}\,\lambda}\right]\left(2n^i(\Phi - \Phi_o) + \int_{\lambda_o}^{\lambda}\mathrm{D}^i\Phi\mathrm{d}\,\lambda - n^i\int_{\lambda_o}^{\lambda}\Phi'\mathrm{d}\,\lambda\right) \\
& -\frac{\mathrm{d}\,D_A}{\mathrm{d}\,\lambda}\left[-h'_{ij}n^in^j + 8n^i\Phi'\int_{\lambda_o}^{\lambda}\mathrm{D}_i\Phi\mathrm{d}\,\lambda - 8\Phi'\int_{\lambda_o}^{\lambda}\Phi'\mathrm{d}\,\lambda + 16\Phi\Phi' - 16\Phi_o\Phi'\right. \\
& -(\Phi^{(2)} + \Psi^{(2)}) + 8\mathrm{D}_i\Phi\int_{\lambda_o}^{\lambda}\mathrm{D}^i\Phi - 8n^i\mathrm{D}_i\Phi\int_{\lambda_o}^{\lambda}\Phi'\mathrm{d}\,\lambda \\
& \left.+8n^i\Phi\mathrm{D}_i\Phi - 16\Phi_on^i\mathrm{D}_i\Phi + 2n^i\mathrm{D}_i\Phi^{(2)} + n^in^j\mathrm{D}_i\omega_j\right] \qquad (6.65)
\end{aligned}
$$

$$
\begin{aligned}
&-D_A\left[\frac{1}{2}h''_{ij}n^in^j - a^5\delta\Sigma_{ij}\delta\Sigma^{ij} + 2\Phi'^2 - 4\Phi''\left(n^i\int_{\lambda_o}^{\lambda}\mathrm{D}_i\Phi\mathrm{d}\,\lambda - \int_{\lambda_o}^{\lambda}\Phi'\mathrm{d}\,\lambda\right)\right.\\
&+8\Phi_o\Phi'' + \Psi^{(2)''} - 4n^i\Phi'\mathrm{D}_i\Phi - 8\mathrm{D}_i\Phi'\left(\int_{\lambda_o}^{\lambda}\mathrm{D}^i\Phi\mathrm{d}\,\lambda - n^i\int_{\lambda_o}^{\lambda}\Phi'\mathrm{d}\,\lambda\right)\\
&-24n^i\Phi\mathrm{D}_i\Phi' + 16n^i\Phi_o\mathrm{D}_i\Phi' - 2n^i\mathrm{D}_i{}^{(2)}\Psi' - 4\mathrm{D}_k\mathrm{D}^k\Phi\left(\int_{\lambda_o}^{\lambda}\Phi'\mathrm{d}\,\lambda - n^i\int_{\lambda_o}^{\lambda}\mathrm{D}_i\Phi\right)\\
&+12\Phi\mathrm{D}_i\mathrm{D}^i\Phi - 8\Phi_o\mathrm{D}_i\mathrm{D}^i\Phi + \frac{1}{2}\left(\mathrm{D}_i\mathrm{D}^i\Phi^{(2)} + \mathrm{D}_i\mathrm{D}^i\Psi^{(2)}\right) - \frac{1}{2}n^in^j\mathrm{D}_i\omega'_j\\
&+2n^in^j\mathrm{D}_i\Phi\mathrm{D}_j\Phi + 4n^in^j\Phi\mathrm{D}_i\mathrm{D}_j\Phi - \frac{1}{2}n^in^j\left(\mathrm{D}_i\mathrm{D}_j\Phi^{(2)} - \mathrm{D}_i\mathrm{D}_j\Psi^{(2)}\right)\\
&\left.+\frac{1}{2}n^i\mathrm{D}_k\mathrm{D}^k\omega_i - \frac{1}{2}n^in^j\mathrm{D}_k\mathrm{D}^kh_{ij}\right].
\end{aligned}
$$

Splitting the covariant derivative on the hyper-surface into the radial component and the angular component, equation (6.65) simplifies greatly. (The details of the decomposition is given in Section 6.13)

$$
\begin{aligned}
\frac{\mathrm{d}^2\delta^2D_A}{\mathrm{d}\,\lambda^2} &= 4\delta D'_A\frac{\mathrm{d}\,\Phi}{\mathrm{d}\,\lambda} + 8\frac{\mathrm{d}\,\delta D'}{\mathrm{d}\,\lambda}\left(\Phi-\Phi_o\right) - 4\frac{\mathrm{d}\,\delta D_A}{\mathrm{d}\,\lambda}\Phi' + 8\nabla_{\perp b}\left[\frac{\mathrm{d}\,\delta D_A}{\mathrm{d}\,\lambda}\right]\int_{\lambda_o}^{\lambda}\nabla_\perp^b\Phi \qquad (6.66)\\
&+4\nabla_{\perp b}\Phi\nabla_\perp^b\delta D_A - 2\delta D_A\left(\frac{\mathrm{d}^2\Phi}{\mathrm{d}\,\lambda^2} + \nabla_\perp^2\Phi\right)\\
&-\frac{\mathrm{d}\,D_A}{\mathrm{d}\,\lambda}\left[-\frac{\mathrm{d}\,\omega_{||}}{\mathrm{d}\,\lambda} + \omega'_{||} - h'_{||} - 2\frac{\mathrm{d}^{(2)}\Phi}{\mathrm{d}\,\lambda} + 16\frac{\mathrm{d}\,\Phi}{\mathrm{d}\,\lambda}\Phi - 8\frac{\mathrm{d}\,\Phi}{\mathrm{d}\,\lambda}\Phi_o - 8\Phi\Phi' + {}^{(2)}\Phi'\right.\\
&\left.+{}^{(2)}\Psi' + 8\nabla_{\perp b}\Phi\int_{\lambda_o}^{\lambda}\nabla_\perp^b\Phi\right] - D_A\left[\frac{1}{2}\frac{\mathrm{d}^2\omega_{||}}{\mathrm{d}\,\lambda^2} - \frac{1}{2}\frac{\mathrm{d}\,\omega'_{||}}{\mathrm{d}\,\lambda} - \frac{1}{2}\frac{\mathrm{d}^2\,h_{||}}{\mathrm{d}\,\lambda^2} + \frac{\mathrm{d}\,h_{||}}{\mathrm{d}\,\lambda}\right.\\
&-a^5\delta\Sigma_{ij}\delta\Sigma^{ij} + 2\left(\frac{\mathrm{d}\,\Phi}{\mathrm{d}\,\lambda}\right)^2 + 4\frac{\mathrm{d}^2\Phi}{\mathrm{d}\,\lambda^2}\Phi - 8\Phi\frac{\mathrm{d}\,\Phi'}{\mathrm{d}\,\lambda} + 4\Phi_o\frac{\mathrm{d}^2\Phi}{\mathrm{d}\,\lambda^2} + 8\Phi_o\Phi'' + \frac{\mathrm{d}^{2(2)}\Psi}{\mathrm{d}\,\lambda^2}\\
&\left.-8\nabla^b\Phi'\int_{\lambda_o}^{\lambda}\nabla_{\perp b}\Phi + \frac{1}{2}\left(\nabla_\perp^2\omega_{||} - \nabla_\perp^2\mathrm{h}_{||}\right) + 4\Phi_o\nabla_\perp^2\Phi + \frac{1}{2}\left(\nabla_\perp^2\Phi + \nabla_\perp^2\Psi\right)\right],\\
&= g_2(\lambda)\,. \qquad (6.67)
\end{aligned}
$$

It is more instructive to split up into scalar, vector and tensor perturbations. The contribution to the area distance at second from scalar perturbation only is given by

$$
\begin{aligned}
\frac{\mathrm{d}^2\delta^2 D_A}{\mathrm{d}\lambda^2} &= -\frac{\mathrm{d}D_A}{\mathrm{d}\lambda}\left[-2\frac{\mathrm{d}\,{}^{(2)}\Phi}{\mathrm{d}\lambda} + 16\frac{\mathrm{d}\Phi}{\mathrm{d}\lambda}\Phi - 8\frac{\mathrm{d}\Phi}{\mathrm{d}\lambda}\Phi_o - 8\Phi\Phi' + {}^{(2)}\Phi' + {}^{(2)}\Psi' \right. \qquad (6.68)\\
&\left. +8\nabla_{\perp b}\Phi\int_{\lambda_o}^{\lambda}\nabla_\perp^b\Phi\right] - D_A\left[-a^5\delta\Sigma_{ij}\delta\Sigma^{ij} + 2\left(\frac{\mathrm{d}\Phi}{\mathrm{d}\lambda}\right)^2 + 4\frac{\mathrm{d}^2\Phi}{\mathrm{d}\lambda^2}\Phi - 8\Phi\frac{\mathrm{d}\Phi'}{\mathrm{d}\lambda}\right.\\
&\left. +4\Phi_o\frac{\mathrm{d}^2\Phi}{\mathrm{d}\lambda^2} + 8\Phi_o\Phi'' + \frac{\mathrm{d}^{2(2)}\Psi}{\mathrm{d}\lambda^2} - 8\nabla^b\Phi'\int_{\lambda_o}^{\lambda}\nabla_{\perp b}\Phi + 4\Phi_o\nabla_\perp^2\Phi + \frac{1}{2}\left(\nabla_\perp^2\Phi + \nabla_\perp^2\Psi\right)\right]\\
&+4\delta D'_A\frac{\mathrm{d}\Phi}{\mathrm{d}\lambda} + 8\frac{\mathrm{d}\delta D'}{\mathrm{d}\lambda}(\Phi - \Phi_o) - 4\frac{\mathrm{d}\delta D_A}{\mathrm{d}\lambda}\Phi' + 8\nabla_{\perp b}\left[\frac{\mathrm{d}\delta D_A}{\mathrm{d}\lambda}\right]\int_{\lambda_o}^{\lambda}\nabla_\perp^b\Phi\\
&+4\nabla_{\perp b}\Phi\nabla_\perp^b\delta D_A - 2\delta D_A\left(\frac{\mathrm{d}^2\Phi}{\mathrm{d}\lambda^2} + \nabla_\perp^2\Phi\right),\\
&= gS_2(\lambda)\,. \qquad (6.69)
\end{aligned}
$$

The product of the first order area distance and the gravitational potential will be evaluated in detail in Section 6.13. The contribution from vector perturbations is given by

$$
\begin{aligned}
\frac{\mathrm{d}^2\delta^2 D_A}{\mathrm{d}\lambda^2} &= \frac{\mathrm{d}D_A}{\mathrm{d}\lambda}\left[\frac{\mathrm{d}\omega_{||}}{\mathrm{d}\lambda} - \omega'_{||}\right] - \frac{D_A}{2}\left[\frac{\mathrm{d}^2\omega_{||}}{\mathrm{d}\lambda^2} - \frac{\mathrm{d}\omega'_{||}}{\mathrm{d}\lambda} - \frac{1}{(\lambda-\lambda_o)^2}\omega_\| + \nabla_\perp^2\omega_{||}\right], \qquad (6.70)\\
&= gV_2(\lambda)\,. \qquad (6.71)
\end{aligned}
$$

By comparing with the first order result, the last term in equation (6.70), i,e the Laplacian of the vector perturbation may be interpreted as a contribution to the lensing effect from induced vector perturbations. For tensor perturbations we find

$$
\begin{aligned}
\frac{\mathrm{d}^2\delta^2 D_A}{\mathrm{d}\lambda^2} &= \frac{\mathrm{d}D_A}{\mathrm{d}\lambda}h'_{||} + D_A\left[\frac{1}{2}\frac{\mathrm{d}^2 h_{||}}{\mathrm{d}\lambda^2} - \frac{\mathrm{d}\,h_{||}}{\mathrm{d}\lambda} - \frac{3}{(\lambda-\lambda_o)^2}h_\| + \frac{1}{2}\nabla_\perp^2\mathrm{h}_{\,||}\right], \qquad (6.72)\\
&= gT_2(\lambda)\,. \qquad (6.73)
\end{aligned}
$$

Notice also the contribution to the gravitational lensing from induced gravitational waves. At second order, the shear contribution to the area distance is non-vanishing but it contributes as a square of contribution from the first order piece. Hence we calculate this from the equation (6.21),

$$
\frac{\mathrm{d}\,\delta\hat{\Sigma}_{ij}}{\mathrm{d}\,\hat{\lambda}} + 2\mathcal{H}\delta\hat{\Sigma}_{ij} = -\frac{2}{a^2}\left(N_i{}^e N_j{}^f - \frac{1}{2}N_{ij}N^{ef}\right)\mathrm{D}_e\mathrm{D}_f\Phi\,. \qquad (6.74)
$$

Without loss of generality, we set the shear at the observer $\delta\hat{\Sigma}_{bco} = 0$, then shear at first order becomes

$$\delta\hat{\Sigma}_{bc}(s) \;=\; -\frac{2}{a^2}\int_{\lambda_o}^{\lambda_s} \mathrm{d}\,\lambda \nabla_{\langle i}\nabla_{j\rangle}\Phi\,. \tag{6.75}$$

The general solution to equation (6.68) is given by

$$\delta^2 D_A \;=\; \int_{\lambda_o}^{\lambda_s}\int_{\lambda_o}^{\lambda_s} gS_2(\lambda')\mathrm{d}\,\lambda'\mathrm{d}\,\lambda = \int_{\lambda_o}^{\lambda_s}(\lambda_s-\lambda)gS_2(\lambda)\mathrm{d}\,\lambda\,. \tag{6.76}$$

In the second equality we have performed one of the integrations using an iterated integration technique for double integrals.

- **Scalar Perturbations:** We find for scalar perturbations after several integration by parts and simplifications,

$$\begin{aligned}
\frac{\delta^2 D_A}{D_A} \;=\; & -(\Psi_s^{(2)}+\Psi_o^{(2)}) - 2\Phi_o^{(2)} - 4\Phi_o(\Phi_s+\Phi_o) \\
& + \int_{\lambda_o}^{\lambda_s}\frac{(\lambda_s-\lambda)}{(\lambda_s-\lambda_o)}\Bigg[4\delta D_A'\frac{\mathrm{d}\,\Phi}{\mathrm{d}\,\lambda} + 8\frac{\mathrm{d}\,\delta D'}{\mathrm{d}\,\lambda}(\Phi-\Phi_o) - 4\frac{\mathrm{d}\,\delta D_A}{\mathrm{d}\,\lambda}\Phi' \\
& +8\nabla_{\perp b}\left[\frac{\mathrm{d}\,\delta D_A}{\mathrm{d}\,\lambda}\right]\int_{\lambda_o}^{\lambda}\nabla_\perp^b\Phi(\lambda_1)\mathrm{d}\,\lambda_1 + 4\nabla_{\perp i}\Phi\nabla_\perp^i\delta D_A - 2\delta D_A\left(\frac{\mathrm{d}^2\Phi}{\mathrm{d}\,\lambda^2}+\nabla_\perp^2\Phi\right)\Bigg]\mathrm{d}\,\lambda \\
& - \int_{\lambda_o}^{\lambda_s}\frac{(\lambda_s-\lambda)}{(\lambda_s-\lambda_o)}\left[\left(\Psi^{(2)'}+\Phi^{(2)'}\right)\right]\mathrm{d}\,\lambda + \frac{1}{(\lambda_s-\lambda_o)}\int_{\lambda_o}^{\lambda_s}\left[2\left(\Phi^{(2)}+\Psi^{(2)}\right)\right]\mathrm{d}\,\lambda \\
& -2\int_{\lambda_o}^{\lambda_s}\frac{(\lambda_s-\lambda)(\lambda-\lambda_o)}{(\lambda_s-\lambda_o)}\left[\left(\frac{\mathrm{d}\,\Phi}{\mathrm{d}\,\lambda}\right)^2 + 2\Phi\frac{\mathrm{d}^2\Phi}{\mathrm{d}\,\lambda^2} - 4\Phi\frac{\mathrm{d}\,\Phi'}{\mathrm{d}\,\lambda} + 2\left[2\Phi_o\Phi''\right.\right. \\
& \left.\left.+\Phi_o\nabla_\perp^2\Phi\right]\right]\mathrm{d}\,\lambda + 8\int_{\lambda_o}^{\lambda_s}\frac{(\lambda_s-\lambda)}{(\lambda_s-\lambda_o)}\Phi\Phi'\mathrm{d}\,\lambda + \frac{8}{(\lambda_s-\lambda_o)}\int_{\lambda_o}^{\lambda_s}\left[2\Phi_o\Phi-\Phi^2\right]\mathrm{d}\,\lambda \\
& +8\int_{\lambda_o}^{\lambda_s}\frac{(\lambda_s-\lambda)(\lambda-\lambda_o)}{(\lambda_s-\lambda_o)}\nabla_{\perp k}\Phi'\int_{\lambda_o}^{\lambda}\nabla_\perp^k\Phi(\lambda_1)\mathrm{d}\,\lambda_1\mathrm{d}\,\lambda \\
& -8\int_{\lambda_o}^{\lambda_s}\frac{(\lambda_s-\lambda)}{(\lambda_s-\lambda_o)}\nabla_{\perp k}\Phi\int_{\lambda_o}^{\lambda}\nabla_\perp^k\Phi(\lambda_1)\mathrm{d}\,\lambda_1\mathrm{d}\,\lambda \\
& +4\int_{\lambda_o}^{\lambda_s}\frac{(\lambda_s-\lambda)(\lambda-\lambda_o)}{(\lambda_s-\lambda_o)}a(\lambda)\left[\int_{\lambda_o}^{\lambda}\nabla_{\langle i}\nabla_{j\rangle}\Phi(\lambda_1)\mathrm{d}\,\lambda_1\int_{\lambda_o}^{\lambda}\nabla^{\langle i}\nabla^{j\rangle}\Phi(\lambda_2)\mathrm{d}\,\lambda_2\right]\mathrm{d}\,\lambda \\
& -\frac{1}{2}\int_{\lambda_o}^{\lambda_s}\frac{(\lambda_s-\lambda)(\lambda-\lambda_o)}{(\lambda_s-\lambda_o)}\left[\left(\nabla_\perp^2\Phi^{(2)}+\nabla_\perp^2\Psi^{(2)}\right)\right]\mathrm{d}\,\lambda\,.
\end{aligned} \tag{6.77}$$

- **Vector Pertubations:**

$$\begin{aligned}\frac{\delta^2 D_A}{D_A} &= -\frac{1}{2}(\omega_{||s} + 3\omega_{||o}) + \frac{2}{(\lambda_s - \lambda_o)}\int_{\lambda_o}^{\lambda_s} \omega_{||}\mathrm{d}\,\lambda + \frac{1}{2}\int_{\lambda_o}^{\lambda_s} \omega'_{||}\mathrm{d}\,\lambda - 2\int_{\lambda_o}^{\lambda_s} \frac{(\lambda_s - \lambda)}{(\lambda_s - \lambda_o)}\omega'_{||}\mathrm{d}\,\lambda \\ &+ \int_{\lambda_o}^{\lambda_s} \frac{(\lambda_s - \lambda)}{(\lambda_s - \lambda_o)(\lambda - \lambda_o)}\omega_{\|} - \frac{1}{2}\int_{\lambda_o}^{\lambda_s} \frac{(\lambda_s - \lambda)(\lambda - \lambda_o)}{(\lambda_s - \lambda_o)}\nabla^2_{\perp}\omega_{||}\mathrm{d}\,\lambda\,. \end{aligned} \tag{6.78}$$

- **Tensor Perturbations:**

$$\begin{aligned}\frac{\delta^2 D_A}{D_A} &= \frac{1}{2}(h_{||s} + h_{||o}) - \frac{1}{(\lambda_s - \lambda_o)}\int_{\lambda_o}^{\lambda_s} h_{||}\mathrm{d}\,\lambda - \frac{1}{2}\int_{\lambda_o}^{\lambda_s} h'_{||}\mathrm{d}\,\lambda + 2\int_{\lambda_o}^{\lambda_s} \frac{(\lambda_s - \lambda)}{(\lambda_s - \lambda_o)}h'_{||}\mathrm{d}\,\lambda \\ &-3\int_{\lambda_o}^{\lambda_s} \frac{(\lambda_s - \lambda)}{(\lambda_s - \lambda_o)(\lambda - \lambda_o)}h_{||}\mathrm{d}\,\lambda + \frac{1}{2}\int_{\lambda_o}^{\lambda_s} \frac{(\lambda_s - \lambda)(\lambda - \lambda_o)}{(\lambda_s - \lambda_o)}\nabla^2_{\perp}h_{||}\mathrm{d}\,\lambda\,. \end{aligned} \tag{6.79}$$

We remind the reader that $\nabla^2_{\perp}$ is related to angular derivative ∇^2_{Ω} through

$$\nabla^2_{\perp}X = \frac{1}{(\lambda - \lambda_o)^2}\nabla^2_{\Omega}X + \frac{2}{(\lambda - \lambda_o)}\partial_{\|}X\,. \tag{6.80}$$

We chose to use $\nabla^2_{\perp}$ because the equations appear more compact in the representation.

For quick understanding of which physical event is coming into play at second order, we may re-write the entire second order contribution as

$$\begin{aligned}\frac{\delta^2 D_A}{\bar{D}_A} &= \text{Boundary} + \text{2nd-order ISW} + \text{2nd-order lensing} + \text{integrated shear} \\ &+\delta D_A \times \Phi + (\text{1st-order potential})^2\,, \end{aligned} \tag{6.81}$$

where the boundary terms are the corrections to the distance given by differences between the metric potentials at the source and observer:

$$\begin{aligned}\text{Boundary} &= -\frac{1}{2}(\omega_{||s} + 3\omega_{||o}) + \frac{1}{2}(h_{||s} + h_{||o}) - (\Psi^{(2)}_s + \Psi^{(2)}_o + 2\Phi^{(2)}_o) \\ &-4\Phi_o(\Phi_s + \Phi_o)\,. \end{aligned} \tag{6.82}$$

The integrated pure second-order effects come in two parts. First the ISW terms are integrals over the metric potentials along the line of sight:

$$\begin{aligned}\mathcal{O}(2)\ \text{ISW} = \frac{1}{(\lambda_s - \lambda_o)}\int_{\lambda_o}^{\lambda_s} \Big[2\Phi^{(2)} + 2\Psi^{(2)} + 2\omega_{\|} - h_{\|} \quad (\lambda - \lambda_s)(\Psi^{(2)\prime} + \Phi^{(2)\prime}) \\ +\frac{1}{2}(-3\lambda_s - \lambda_o + \lambda)\omega'_{\|} + (2\lambda_s + \lambda_o - 3\lambda)h'_{\|}\Big]\mathrm{d}\,\lambda\,. \end{aligned} \tag{6.83}$$

Then we have the second-order lensing terms:

$$\mathcal{O}(2)\text{ lensing} = -\frac{1}{2}\int_{\lambda_o}^{\lambda_s}\frac{(\lambda_s-\lambda)(\lambda-\lambda_o)}{(\lambda_s-\lambda_o)}\nabla_\perp^2\left[\Phi^{(2)}+\Psi^{(2)}+\omega_\parallel-h_\parallel\right]\mathrm{d}\,\lambda\,. \quad (6.84)$$

The shear of the matter along the line of sight gives an integrated contribution at second-order:

$$\begin{aligned}&\text{integrated shear} && (6.85)\\ &= 4\int_{\lambda_o}^{\lambda_s}\frac{a(\lambda)(\lambda_s-\lambda)(\lambda-\lambda_o)}{(\lambda_s-\lambda_o)}\left[\int_{\lambda_o}^{\lambda}\nabla_{\langle i}\nabla_{j\rangle}\Phi(\lambda_1)\mathrm{d}\,\lambda_1\int_{\lambda_o}^{\lambda}\nabla^{\langle i}\nabla^{j\rangle}\Phi(\lambda_2)\mathrm{d}\,\lambda_2\right]\mathrm{d}\,\lambda\,.\end{aligned}$$

we have the terms which are integrals over the first-order potential:

$$\begin{aligned}(\text{potential})^2 &= -2\int_{\lambda_o}^{\lambda_s}\frac{(\lambda_s-\lambda)(\lambda-\lambda_o)}{(\lambda_s-\lambda_o)}\left[\left(\frac{\mathrm{d}\,\Phi}{\mathrm{d}\,\lambda}\right)^2+2\Phi\frac{\mathrm{d}^{\,2}\Phi}{\mathrm{d}\,\lambda^2}-4\Phi\frac{\mathrm{d}\,\Phi'}{\mathrm{d}\,\lambda}+4\Phi_o\Phi''\right.\\ &\left.+2\Phi_o\nabla_\perp^2\Phi\right]\mathrm{d}\,\lambda-8\int_{\lambda_o}^{\lambda_s}\frac{(\lambda_s-\lambda)}{(\lambda_s-\lambda_o)}\left[\nabla_{\perp k}\Phi-(\lambda-\lambda_o)\nabla_{\perp k}\Phi'\right]\left[\int_{\lambda_o}^{\lambda}\nabla_\perp^k\Phi(\lambda_1)\mathrm{d}\,\lambda_1\right]\mathrm{d}\,\lambda\\ &+\frac{8}{(\lambda_s-\lambda_o)}\int_{\lambda_o}^{\lambda_s}\left[(\lambda_s-\lambda)\Phi\Phi'-\Phi^2-2\Phi_o\Phi\right]\mathrm{d}\,\lambda\,. \quad (6.86)\end{aligned}$$

Next comes a contribution from an integrated coupling between the first-order distance, coupled to the potential

$$\begin{aligned}\delta D_A\times\Phi &= \int_{\lambda_o}^{\lambda_s}\frac{(\lambda_s-\lambda)}{(\lambda_s-\lambda_o)}\left[8\frac{\mathrm{d}\,\delta D_A'}{\mathrm{d}\,\lambda}(\Phi-\Phi_o)-4\frac{\mathrm{d}\,\delta D_A}{\mathrm{d}\,\lambda}\Phi'\right. \quad (6.87)\\ &+8\nabla_{\perp i}\left(\frac{\mathrm{d}\,\delta D_A}{\mathrm{d}\,\lambda}\right)\int_{\lambda_o}^{\lambda}\nabla_\perp^i\Phi(\lambda_1)\mathrm{d}\,\lambda_1+4\delta D_A'\frac{\mathrm{d}\,\Phi}{\mathrm{d}\,\lambda}+4\nabla_{\perp i}\Phi\nabla_\perp^i\delta D_A\\ &\left.-2\delta D_A\left(\frac{\mathrm{d}^{\,2}\Phi}{\mathrm{d}\,\lambda^2}+\nabla_\perp^2\Phi\right)\right]\mathrm{d}\,\lambda\,.\end{aligned}$$

This coupling integrates density fluctuations twice along the line of sight and is a key new effect for nonlinear lensing. Further simplification of equation (6.87) is given in the section 6.13, however, the dominant terms on small scale is given by

$$\begin{aligned}\delta D_A\times\Phi &\approx -2\int_{\lambda_o}^{\lambda_s}\frac{(\lambda_s-\lambda)}{(\lambda_s-\lambda_o)}\left[2\nabla_\perp^i\Phi\int_{\lambda_o}^{\lambda}(\lambda_s-\lambda)(\lambda-\lambda_o)\nabla_{\perp i}\nabla_\perp^2\Phi(\lambda_1)\mathrm{d}\,\lambda_1\right. \quad (6.88)\\ &-4\int_{\lambda_o}^{\lambda}(\lambda-\lambda_o)\nabla_{\perp i}\nabla_\perp^2\Phi(\lambda_2)\mathrm{d}\,\lambda_2\int_{\lambda_o}^{\lambda}\nabla_\perp^i\Phi(\lambda_1)\mathrm{d}\,\lambda_1\\ &\left.-\nabla_\perp^2\Phi\int^{\lambda}(\lambda_s-\lambda)(\lambda-\lambda_o)\nabla_\perp^2\Phi(\lambda_1)\mathrm{d}\,\lambda_1\right]\mathrm{d}\,\lambda.\end{aligned}$$

Finally the area distance in a perturbed FLRW spacetime is given by

$$\hat{D}_A = a(\lambda)(\lambda_s - \lambda_o)\left[1 + \frac{\delta D_A}{\bar{D}_A} + \frac{1}{2}\frac{\delta^2 D_A}{\bar{D}_A}\right] = \frac{(\lambda_s - \lambda_o)}{(1+\bar{z})}\left[1 + \frac{\delta D_A}{\bar{D}_A} + \frac{1}{2}\frac{\delta^2 D_A}{\bar{D}_A}\right], \quad (6.89)$$

where we have used the background relation between the redshift and the scalar factor $a(\lambda) = 1/(1+\bar{z})$.

6.6 Observables on the Physical Spacetime

The area distance in a perturbed FLRW space time up to second order in perturbation theory given by equation (6.89) is not observable because:

1. $\hat{D}_A$ depends on the background redshift which not observable.
2. The integration is over background photon paths which photons do not see.
3. The $\hat{D}_A$ in equation (6.89) is calculated with respect to the observer with a vanishing peculiar velocity.

Thus to obtain the correct physical area distance we start by expanding the background scalar factor as a function of physical affine parameter distance (radial distance) of the source through a Taylor series expansion:

$$a(\lambda) = a(\lambda_s)\left[1 + \mathcal{H}_s\delta\lambda + \frac{1}{2}\mathcal{H}_s\,\delta^2\lambda + \frac{1}{2}\left[\frac{\mathrm{d}\,\mathcal{H}_s}{\mathrm{d}\,\lambda} + \mathcal{H}_s^2\right](\delta\lambda)^2 + \mathcal{O}(\delta^3\lambda)\right], \quad (6.90)$$

where we have performed the Taylor series expansion around the position of the physical position of the source and we have also made the following replacements

$$\mathcal{H} = \frac{1}{a_s}\frac{\mathrm{d}\,a_s}{\mathrm{d}\,\lambda}, \qquad \frac{\mathrm{d}^2 a}{a\mathrm{d}\,\lambda} = \left(\frac{\mathrm{d}\,\mathcal{H}}{\mathrm{d}\,\lambda} + \mathcal{H}^2\right), \quad \hat{\lambda} = \lambda + \delta\lambda + \frac{1}{2}\delta^2\lambda. \quad (6.91)$$

Inverting equation (6.55) and then expanding in power series, the resulting equation becomes after using equation (6.90) ,

$$\frac{1}{(1+z_s)} = \frac{a(\hat{\lambda}_s)}{a(\hat{\lambda}_o)}\left[1 + (\mathcal{H}_s\delta\lambda - J_1) + \left(\frac{1}{2}\mathcal{H}_s\delta^2\lambda - \frac{1}{2}J_2 + J_1^2 - \mathcal{H}_s J_1\delta\lambda \right.\right. \quad (6.92)$$
$$\left.\left. + \frac{1}{2}\left(\frac{\mathrm{d}\,\mathcal{H}}{\mathrm{d}\,\lambda} + \mathcal{H}^2\right)(\delta\lambda)^2\right) + \mathcal{O}(\delta^3\lambda)\right],$$

Thus from equation (6.92), the background scale factor may now be re-defined in terms of the physical redshift according to

$$\frac{1}{(1+z_s)} = \frac{a(\hat{\lambda}_s)}{a(\hat{\lambda}_o)}. \qquad (6.93)$$

As we shall see in the subsequent section that this re-definition corresponds to performing the spherical harmonics expansion in a sphere of constant physical redshift. To ensure consistency of equation (6.92) given the re-definition in equation (6.93), It immediately requires that the following relations are satisfied order by order

$$\delta\lambda = \frac{J_1}{\mathcal{H}_s}, \qquad \delta^2\lambda = \frac{1}{\mathcal{H}_s}\left[J_2 - J_1^2\left(1 + \frac{\mathcal{H}'}{\mathcal{H}^2}\right)\right]. \qquad (6.94)$$

Thus using the scale factor in equation (6.90) and $\lambda = \hat{\lambda} - \delta\lambda - \frac{1}{2}\delta^2\lambda$, we find the area distance to second order to be

$$\begin{aligned} D_A &= a(\hat{\lambda}_s)(\hat{\lambda}_s - \hat{\lambda}_o)\Bigg\{1 + \left[\frac{\delta D_A}{D_A} + \left(1 - \frac{1}{\mathcal{H}_s(\hat{\lambda}_s - \hat{\lambda}_o)}\right)J_1\right] \qquad (6.95)\\ &+ \left[\frac{1}{2}\frac{\delta^2 D_A}{D_A} + \left(\frac{1}{2}J_2 + \frac{\delta D_A}{D_A}J_1\right)\left(1 - \frac{1}{\mathcal{H}_s(\hat{\lambda}_s - \hat{\lambda}_o)}\right)\right.\\ &\left.+\frac{1}{2}J_1^2\left(\frac{\mathcal{H}_s'}{\mathcal{H}^3(\hat{\lambda} - \hat{\lambda}_o)} - \frac{1}{\mathcal{H}(\hat{\lambda} - \hat{\lambda}_o)}\right)\right]\Bigg\}. \end{aligned}$$

Using the distance duality relation, $D_L = (1+z_s)^2\hat{D}_A$, the luminosity distance becomes

$$\begin{aligned} \hat{D}_L &= (1+z_s)(\hat{\lambda}_s - \hat{\lambda}_o)\Bigg\{1 + \left[\frac{\delta D_A}{D_A} + \left(1 - \frac{1}{\mathcal{H}_s(\hat{\lambda}_s - \hat{\lambda}_o)}\right)J_1\right] \qquad (6.96)\\ &+ \left[\frac{1}{2}\frac{\delta^2 D_A}{D_A} + \left(\frac{1}{2}J_2 + \frac{\delta D_A}{D_A}J_1\right)\left(1 - \frac{1}{\mathcal{H}_s(\hat{\lambda}_s - \hat{\lambda}_o)}\right)\right.\\ &\left.+\frac{1}{2}J_1^2\left(\frac{\mathcal{H}_s'}{\mathcal{H}^3(\hat{\lambda} - \hat{\lambda}_o)} - \frac{1}{\mathcal{H}(\hat{\lambda} - \hat{\lambda}_o)}\right)\right]\Bigg\}. \end{aligned}$$

The integrals in equations (6.337)and (6.96) are performed over the background affine parameter, at first order, it corresponds to the physical affine parameter under Bohm approximation, however, at non-linear level, the integration measure for the first order terms will need to change. This requires that we replace the background measure at first order

with

$$\mathrm{d}\,\lambda = \mathrm{d}\,\hat{\lambda} - \left(\frac{J_1\mathcal{H}'}{\mathcal{H}^2} - \frac{1}{\mathcal{H}}\frac{\mathrm{d}\,J_1}{\mathrm{d}\,\hat{\lambda}}\right)\mathrm{d}\,\hat{\lambda}, \qquad \text{where,} \quad \frac{\mathrm{d}\,J_1}{\mathrm{d}\,\hat{\lambda}} = \frac{\mathrm{d}\,\mathrm{D}_{\|}v}{\mathrm{d}\,\lambda} - \frac{\mathrm{d}\,\Phi}{\mathrm{d}\,\lambda} + 2\Phi'\,, \tag{6.97}$$

Now we boost the area distance we have calculated to the frame of the observer with non-vansihing peculiar velocity. This boosting introduces an additional Doppler term through the relation between the solid angle in both frames

$$\frac{\mathrm{d}\,\Omega}{\mathrm{d}\,\hat{\Omega}} = \left(\frac{k_c u^c}{\hat{k}_c\hat{u}^c}\right) = \frac{E}{\hat{E}}\,. \tag{6.98}$$

The area distance depends on the solid angle according to

$$D_A = \sqrt{\frac{\mathrm{d}\,A_o}{\mathrm{d}\,\Omega}}\,. \tag{6.99}$$

From equation (6.99), the area distance in both frames are related

$$\hat{D} = \left(\frac{\hat{E}}{E}\right) D_A\,, \tag{6.100}$$

where $\hat{E}/E = 1 - n^a v_a + \frac{1}{2}v_a v^a$. Expanding $\hat{E}/E$ in perturbation theory gives

$$\frac{\hat{E}}{E} = 1 - n^k\mathrm{D}_k v_s - n^k\left(v^{(2)}_{ks} + \mathrm{D}_k v^{(2)}_s\right) + (\Phi_s - \Phi_o)n^k\mathrm{D}_k v_s + \frac{1}{2}\mathrm{D}^i v_s \mathrm{D}_i v_s\,, \tag{6.101}$$

where we have substituted for δn^i using

$$\delta n^i = -n^i\left(\Phi_s - \Phi_o\right) - 2n^i\int_{\lambda_o}^{\lambda_s} n^k\mathrm{D}_k\Phi\mathrm{d}\,\lambda + 2\int_{\lambda_o}^{\lambda_s}\mathrm{D}^i\Phi\mathrm{d}\,\lambda\,. \tag{6.102}$$

For the general non-linear result, we need to re-express the first order contribution so that the integral could be performed on a physical spacetime instead of the background spacetime. This is a very complicated task and we have deferred the full details to Section 6.13. At first order the area distance is given by

$$\begin{aligned}\hat{D}_A(z_s,\lambda_s) &= a(\hat{\lambda}_s)(\hat{\lambda}_s-\hat{\lambda}_o)\left[1+\frac{1}{\mathcal{H}_s(\hat{\lambda}_s-\hat{\lambda}_o)}\partial_{\|}v_s-\left(1-\frac{1}{\mathcal{H}_s(\hat{\lambda}_s-\hat{\lambda}_o)}\right)\partial_{\|}v_o\right.\\ &-\left(2-\frac{1}{\mathcal{H}_s(\hat{\lambda}_s-\hat{\lambda}_o)}\right)\Phi_s+\left(2-\frac{1}{\mathcal{H}_s(\hat{\lambda}_s-\hat{\lambda}_o)}\right)\Phi_o\\ &+2\left(1-\frac{1}{\mathcal{H}_s(\hat{\lambda}_s-\hat{\lambda}_o)}\right)\int_{\hat{\lambda}_o}^{\hat{\lambda}_s}\Phi'\mathrm{d}\,\lambda+\frac{4}{(\lambda_s-\lambda_o)}\int_{\lambda_s}^{\lambda_o}\Phi\mathrm{d}\,\lambda\\ &\left.-2\int_{\lambda_s}^{\lambda_o}\frac{(\lambda_s-\lambda)}{(\lambda_s-\lambda_o)}\Phi'\mathrm{d}\,\lambda-\int_{\lambda_s}^{\lambda_o}\frac{(\lambda_s-\lambda)(\lambda-\lambda_o)}{(\lambda_s-\lambda_o)}\nabla_{\perp}^2\Phi\mathrm{d}\,\lambda\right]. \qquad (6.103)\end{aligned}$$

The luminosity distance may be calculated in the same way using the duality relation.

6.7 Recap of Important Equations

We would like to re-arrange the equations for the observed redshift and area distance we have calculated into a form which will be most suitable for expansion in spherical harmonics. First the observed redshift of the source is given by,

$$(1+\hat{z}_s)=\frac{a(\lambda_o)}{a(\lambda_s)}\left[1+\frac{\delta z}{(1+\bar{z})}+\frac{1}{2}\frac{\delta^2 z}{(1+\bar{z})}\right]=(1+\bar{z})\left[1+J_1+\frac{1}{2}J_2\right], \qquad (6.104)$$

where

$$J_1 = \left(\partial_{\|}v_s-\partial_{\|}v_o\right)+(\Phi_s-\Phi_s)+2\int_0^{\chi_s}n^k\mathrm{D}_k\Phi\mathrm{d}\,\chi \qquad (6.105)$$

and $\chi=(\lambda-\lambda_o)$. We split J_2 according to the mode of perturbation. This is same as in the CMB calculation, where the magnetic quantum number, m characterizes the mode of perturbation. For example $m=0,1,2$ denote respectively the Scalar, (S), Vector, (V) and Tensor, (T), perturbations, hence we split J_2 in line with this characterization.

$$J_2=j_2^{\mathrm{S}}+j_2^{\mathrm{V}}+j_2^{\mathrm{T}}+j_2^{\mathcal{O}(1)\times\mathcal{O}(1)_{\|}}+j_2^{\mathcal{O}(1)\times\mathcal{O}(1)_{\perp}}, \qquad (6.106)$$

where $\mathcal{O}(1)\times\mathcal{O}(1)_{\|}$ stands for the product of two first order terms that has its contribution from structures that lie along the line of sight and $\mathcal{O}(1)\times\mathcal{O}(1)_{\perp}$ stands for the product of two first order terms that has its contribution from structures moving transverse to the line of sight.

From the previous section, the J's are given by

$$
\begin{aligned}
j_2^{\rm S} &= \left(v_{\|s}^{(2)} - v_{\|o}^{(2)}\right) + \left(\mathrm{D}_\| v_s^{(2)} - \mathrm{D}_\| v_o^{(2)}\right) + \left(\Psi_s^{(2)} - \Psi_o^{(2)}\right) \\
&\quad + \int_0^{\chi_s} \left(\partial_\|{}^{(2)}\Phi + \partial_\|{}^{(2)}\Psi\right) \mathrm{d}\,\chi\,, && (6.107)\\
j_2^{\rm V} &= \left(v_{\|s}^{(2)} - v_{\|o}^{(2)}\right) + \left(\mathrm{D}_\| v_s^{(2)} - \mathrm{D}_\| v_o^{(2)}\right) + \left(\omega_{||s} - \omega_{||o})\right) + \int_0^{\chi_s} \partial_\|\omega_{||}\mathrm{d}\,\chi\,, && (6.108)\\
j_2^{\rm T} &= -\left(h_{||s} - h_{||o})\right) - \int_0^{\chi_s} \partial_\| h_{||}\mathrm{d}\,\chi\,, && (6.109)\\
j_2^{\mathcal{O}(1)\times\mathcal{O}(1)_\perp} &= \left(\nabla_{\perp k} v_s \nabla_\perp^k v_s\right) - \left(\nabla_{\perp k} v_o \nabla_\perp^k v_o\right) + 8\int_0^{\chi_s} (\chi_s - \chi)\nabla_{\perp k}\Phi_o \nabla^k \Phi \mathrm{d}\,\chi && (6.110)\\
&\quad + 8\int_0^{\chi_s}\int_0^{\chi} \nabla_{\perp k}\Phi(\chi_1)\mathrm{d}\,\chi_1 \int_0^{\chi} \nabla_\perp^k(\chi_2)\mathrm{d}\,\chi_2 \mathrm{d}\,\chi\,,\\
j_2^{\mathcal{O}(1)\times\mathcal{O}(1)_\|} &= \Big[\left(\mathrm{D}_\| v_s \mathrm{D}_\| v_s\right) - \left(\mathrm{D}_\| v_o \mathrm{D}_\| v_o\right) + \mathrm{D}_\| v_o \mathrm{D}_\| v_s + \left(3\Phi_s^2 - \Phi_o^2\right) - 2\Phi_s\Phi_o && (6.111)\\
&\quad -2\Phi_o\left(\mathrm{D}_\| v_s - 2\mathrm{D}_\| v_o\right) + 2\Phi_s\left(\mathrm{D}_\| v_o - 2\mathrm{D}_\| v_s\right) + 4\left(\mathrm{D}_\| v_s - \mathrm{D}_\| v_o\right)\int_0^{\chi_s}\Phi'\mathrm{d}\,\chi\\
&\quad +8\int_{\chi_s}^0 \Phi\Phi'\mathrm{d}\,\chi - 8\int_0^{\chi_s}(\Phi_s - \Phi)\Phi'\mathrm{d}\,\chi - 4\int_0^{\chi_s}(\Phi_s - \Phi_o)\Phi'\mathrm{d}\,\chi\\
&\quad -8\int_0^{\chi_s}\left((\Phi' - n^k\mathrm{D}_k\Phi_o) + (\Phi_o' - \int_0^{\chi}\Phi''(\chi_1)\mathrm{d}\,\chi_1\right)(\Phi - \Phi_o)\mathrm{d}\,\chi\\
&\quad +8\int_0^{\chi_s}\Phi'\int_0^{\chi}\Phi'(\chi_1)\mathrm{d}\,\chi_1\mathrm{d}\,\chi\Big]\,.
\end{aligned}
$$

The area distance at first order in the form most convenient for later use is given by

$$
\begin{aligned}
\frac{\delta D_A}{D_A(s)} &= \Big[-\Phi_s - \Phi_o + \frac{2}{\chi_s}\int_0^{\chi_s}\Phi\mathrm{d}\,\chi - 2\int_0^{\chi_s}\frac{(\chi_s - \chi)}{\chi_s}\partial_\|\Phi\mathrm{d}\,\chi && (6.112)\\
&\quad -\int_0^{\chi_s}\frac{(\chi_s - \chi)\chi}{\chi_s}\nabla_\perp^2\Phi\mathrm{d}\,\chi\Big]\,.
\end{aligned}
$$

At second order , we also split the area according the type of perturbation with an additional term coming from the product of the area distance at first order and gravitational potential at first order,

$$
\begin{aligned}
\frac{\delta^2 D_A}{D_A} &= \frac{\delta^2 D_A}{D_A}^{\rm S} + \frac{\delta^2 D_A}{D_A}^{\rm V} + \frac{\delta^2 D_A}{D_A}^{\rm T} + \frac{\delta^2 D_A}{D_A}^{\Phi\times\Phi} && (6.113)\\
&\quad + \frac{\delta^2 D_A}{D_A}^{\nabla_{\perp i}\Phi\nabla_\perp^i\Phi} + \frac{\delta^2 D_A}{D_A}^{\rm Shear} + \frac{\delta^2 D_A}{D_A}^{\delta D_A\times\Phi}\,.
\end{aligned}
$$

where

$$\frac{\delta^2 D_A}{D_A}^{\rm S} = -(\Psi_s^{(2)} + \Phi_o^{(2)}) + \frac{1}{\chi_s}\int_0^{\chi_s} (\Phi^{(2)} + \Psi^{(2)})\,\mathrm{d}\,\chi - \int_0^{\chi_s} \frac{(\chi_s - \chi)}{\chi_s} \Big(\partial_\| \Psi^{(2)} + \partial_\| \Phi^{(2)}\Big)\,\mathrm{d}\,\chi - \frac{1}{2}\int_0^{\chi_s} \frac{(\chi_s-\chi)\chi}{\chi_s}\left(\nabla_\perp^2 \Phi^{(2)} + \nabla_\perp^2 \Psi^{(2)}\right)\mathrm{d}\,\chi\,, \tag{6.114}$$

$$\frac{\delta^2 D_A}{D_A}^{\rm V} = \int_0^{\chi_s} \frac{(\chi_s-\chi)}{\chi_s\chi}\omega_\|\mathrm{d}\,\chi + \frac{1}{2}\int_0^{\chi_s}\partial_\|\omega_\|\mathrm{d}\,\chi - 2\int_0^{\chi_s}\frac{(\chi_s-\chi)}{\chi_s}\partial_\|\omega_\|\mathrm{d}\,\chi - \frac{1}{2}\int_0^{\chi_s}\frac{(\chi_s-\chi)\chi}{\chi_s}\nabla_\perp^2\omega_\|\mathrm{d}\,\chi\,, \tag{6.115}$$

$$\frac{\delta^2 D_A}{D_A}^{\rm T} = -\frac{1}{2}h_{||o} - 3\int_0^{\chi_s}\frac{(\chi_s-\chi)}{\chi_s\chi}h_\|\mathrm{d}\,\chi + \frac{1}{\chi_s}\int_0^{\chi_s} h_\|\mathrm{d}\,\chi - \frac{1}{2}\int_0^{\chi_s}\partial_\| h_\|\mathrm{d}\,\chi + 2\int_0^{\chi_s}\frac{(\chi_s-\chi)}{\chi_s}\partial_\| h_\|\mathrm{d}\,\chi + \frac{1}{2}\int_0^{\chi_s}\frac{(\chi_s-\chi)\chi}{\chi_s}\nabla_\perp^2 h_\|\mathrm{d}\,\chi\,, \tag{6.116}$$

$$\frac{\delta^2 D_A}{D_A}^{\Phi\times\Phi} = -2\int_0^{\chi_s}\frac{(\chi_s-\chi)\chi}{\chi_s}\left[\left(\frac{\mathrm{d}\,\Phi}{\mathrm{d}\,\chi}\right)^2 + 2\Phi\frac{\mathrm{d}^{\,2}\Phi}{\mathrm{d}\,\chi^2} - 4\Phi\frac{\mathrm{d}\,\Phi'}{\mathrm{d}\,\chi}\right]\mathrm{d}\,\chi + 8\int_0^{\chi_s}\frac{(\chi_s-\chi)}{\chi_s}\Phi\Phi'\mathrm{d}\,\chi - 4\int_0^{\chi_s}\frac{(\chi_s-\chi)\chi}{\chi_s}\left[2\Phi_o\Phi'' + \Phi_o\nabla_\perp^2\Phi\right]\mathrm{d}\,\chi + \frac{8}{\chi_s}\int_0^{\chi_s}\left[2\Phi_o\Phi - \Phi^2\right]\mathrm{d}\,\chi - 4\Phi_o(\Phi_s+\Phi_o)\,, \tag{6.117}$$

$$\frac{\delta^2 D_A}{D_A}^{\nabla_{\perp i}\Phi\nabla_\perp^i\Phi} = 8\int_0^{\chi_s}\frac{(\chi_s-\chi)\chi}{\chi_s}\nabla_{\perp k}\Phi'\int_0^{\chi}\nabla_\perp^k\Phi(\chi_1)\mathrm{d}\,\chi_1\mathrm{d}\,\chi - 8\int_0^{\chi_s}\frac{(\chi_s-\chi)}{\chi_s}\nabla_{\perp k}\Phi\int_0^{\chi}\nabla_\perp^k\Phi(\chi_1)\mathrm{d}\,\chi_1\mathrm{d}\,\chi\,, \tag{6.118}$$

$$\frac{\delta^2 D_A}{D_A}^{\rm Shear} = 4\int_0^{\chi_s}\frac{a(\chi)(\chi_s-\chi)\chi}{\chi_s}\left[\int_0^{\chi}\nabla_{\langle i}\nabla_{j\rangle}\Phi(\chi_1)\mathrm{d}\,\chi_1 \int_0^{\chi}\nabla^{\langle i}\nabla^{j\rangle}\Phi(\chi_2)\mathrm{d}\,\chi_2\right]\mathrm{d}\,\chi\,, \tag{6.119}$$

$$\frac{\delta^2 D_A}{D_A}^{\delta D_A\times\Phi} = \int_0^{\chi_s}\frac{(\chi_s-\chi)}{\chi_s}\left[4\delta D_A'\frac{\mathrm{d}\,\Phi}{\mathrm{d}\,\chi} + 8\frac{\mathrm{d}\,\delta D'}{\mathrm{d}\,\chi}(\Phi-\Phi_o) - 4\frac{\mathrm{d}\,\delta D_A}{\mathrm{d}\,\chi}\Phi' + 8\nabla_{\perp b}\left[\frac{\mathrm{d}\,\delta D_A}{\mathrm{d}\,\chi}\right]\int_0^{\chi}\nabla_\perp^b\Phi(\chi_1)\mathrm{d}\,\chi_1 + 4\nabla_{\perp i}\Phi\nabla_\perp^i\delta D_A - 2\delta D_A\left(\frac{\mathrm{d}^{\,2}\Phi}{\mathrm{d}\,\chi^2} + \nabla_\perp^2\Phi\right)\right]\mathrm{d}\,\chi\,.$$

Other terms needed for further simplification of equation (6.120) are given by

$$\begin{aligned}\delta D_A &= -\chi(\Phi+\Phi_o)+2\int_0^{\chi_s}\Phi\mathrm{d}\,\chi-2\int_0^{\chi_s}(\chi_s-\chi)\partial_\parallel\Phi\mathrm{d}\,\chi \\ &\quad -\int_0^{\chi_s}(\chi_s-\chi)\chi\nabla_\perp^2\Phi\,, \qquad (6.120)\\ \nabla_{\perp i}\delta D_A &= -\chi(\nabla_{\perp i}\Phi+\nabla_{\perp i}\Phi_o)+2\int_0^{\chi_s}\nabla_{\perp i}\Phi\mathrm{d}\,\chi-2\int_0^{\chi_s}(\chi_s-\chi)\nabla_{\perp i}\partial_\parallel\Phi\mathrm{d}\,\chi \\ &\quad -\int_0^{\chi_s}(\chi_s-\chi)\chi\nabla_{\perp i}\nabla_\perp^2\Phi\,, \qquad (6.121)\\ \frac{\mathrm{d}\,\delta D_A}{\mathrm{d}\,\chi} &= (\Phi+\Phi_o)-\chi\frac{\mathrm{d}\,\Phi}{\mathrm{d}\,\chi}-2(\chi_s-\chi)\partial_\parallel\Phi-(\chi_s-\chi)\chi\nabla_\perp^2\Phi\,, \qquad (6.122)\\ \nabla_{\perp i}\left[\frac{\mathrm{d}\,\delta D_A}{\mathrm{d}\,\chi}\right] &= (\nabla_{\perp i}\Phi+\nabla_{\perp i}\Phi_o)-\chi_s\frac{\mathrm{d}\,\nabla_{\perp i}\Phi}{\mathrm{d}\,\chi}-2(\chi_s-\chi)\nabla_{\perp i}\partial_\parallel\Phi \qquad (6.123)\\ &\quad -(\chi_s-\chi)\chi\nabla_{\perp i}\nabla_\perp^2\Phi\,.\end{aligned}$$

6.8 Full Sky Spherical Harmonic Expansion

The equations we want to expand in spherical harmonics depend on metric variables that we have properly decomposed into irreducible forms on the screen space. Having these variables in this form makes it easier for one to expand them in spherical harmonics without much further work. For example, the scalars in our equations may easily be expanded in spherical harmonics as follows:

$$X(\eta,\chi,\mathbf{n}) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}X_{\ell m}(\eta,\chi)Y_{\ell m}(\mathbf{n})\,, \qquad (6.124)$$

where $X=\{\Phi,\Phi',\Phi^{(2)},\Psi^{(2)},\omega_\parallel,h_\parallel\}$ and η is the conformal time on an FLRW background. Also FLRW background, $\eta=\lambda$ and χ is the radial distance which also corresponds to the affine parameter distance, $\chi=(\lambda-\lambda_o)$. In the literature, some workers use r instead of χ, here they represent the same thing. The moment of the scalar $X(\eta,\chi)$ may be obtained by multiplying both sides by $Y_{\ell m}(\mathbf{n})$ and using the orthogonality relation:

$$X_{\ell m}(\eta,\chi)=\int\mathrm{d}\,\Omega X(\eta,\chi,\mathbf{n})Y^*_{\ell m}(\mathbf{n})\,. \qquad (6.125)$$

The multipoles of each of these fields may be evaluated in terms of the spherical Bessel functions by first expanding them in Fourier space and using the Rayleigh's formula. For example any scalar field, say Φ may be expanded in Fourier space according to

$$\Phi(\mathbf{x}) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\Phi(k)e^{i\mathbf{k}\cdot\mathbf{x}}, \qquad \Phi(\mathbf{k}) = \int \mathrm{d}^3x\Phi(\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}}. \tag{6.126}$$

Assuming simple scalar plane wave propagating towards us, its waveform may be expressed in terms of the spherical Bessel functions using the famous Rayleigh's formula,

$$e^{ik_ix^i} = e^{-ik_i\cdot n^i(\lambda-\lambda_o)} = \sum_\ell(2\ell+1)(-i)^\ell j_\ell(k(\lambda-\lambda_o))Y^*_{\ell m}(\mathbf{k})Y_{\ell m}(\mathbf{n}), \tag{6.127}$$

where we have made use of equation (6.32). In most places we will not include the summation sign ($\sum_\ell$) in the equation to reduce clutter. The reader is hereby advised to bear in mind that the summation sign exists where appropriate. The spherical harmonics in different bases may be added using addition formula for spherical harmonics

$$P_\ell(\mathbf{k}\cdot\mathbf{n}) = Y^*_{\ell m}(\mathbf{k})Y_{\ell m}(\mathbf{n}), \tag{6.128}$$

where $P_\ell(\hat{\mathbf{k}}\cdot\mathbf{n})$ is the associated Legendre polynomial, which is approximately related to the spherical harmonics according to

$$P_\ell(\mathbf{k}\cdot\mathbf{n}) = \sqrt{\frac{4\pi}{(2\ell+1)}}Y_{\ell 0}(\mathbf{n}). \tag{6.129}$$

Here we have set the magnetic quantum number, m, to zero, this is equivalent to aligning the angle between **k** and **n** with the angle between **n** and z-axis . For the calculation of the monopole and the two-point correlation functions that we have in mind, this approximation is valid since a scalar plane wave propagating along the z-axis is symmetric with respect to ϕ rotations due to the axial symmetry. However, the gravitational waves and the induced vector field are polarized along x- and y-axis. This polarization activates a particular mode after integration over the entire sky. This polarization will contain a spherical harmonic expansion as $Y_{\ell m,\pm1}(\mathbf{n})$ for vectors and $Y_{\ell\pm2}(\mathbf{n})$ for the gravitational waves, indicating the mode each perturbations activates. The approximation may no longer be appropriate when calculating the bispectrum or trispectrum because one is required to average over all possible directions of the azimuthal angles. Substituting equations (6.129) and (6.128) into equation (6.127), we find

$$e^{ik\mu\chi} = \sum_\ell \sqrt{4\pi(2\ell+1)}(-i)^\ell j_\ell(k\chi)Y_{\ell 0}(\mathbf{n}), \tag{6.130}$$

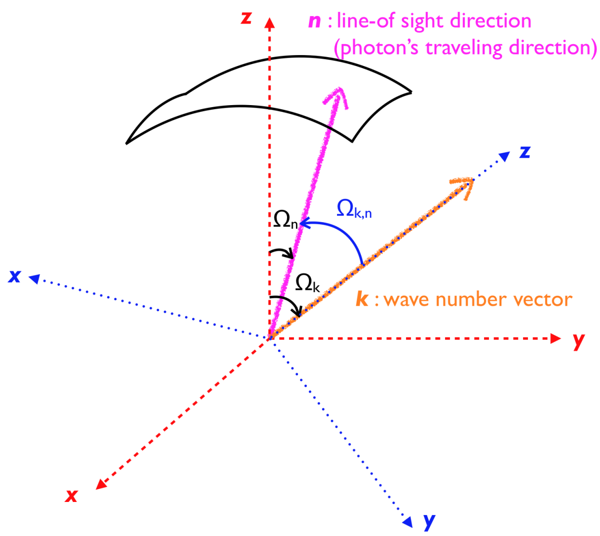

Figure 6.1: The direction of the line of sight indicating the angle each choice makes with the z-axis. Credit:[238]

where we have made use of $x^i = x^i_o - n^i(\lambda - \lambda_o)$, with $x^i_o = 0$. We also decompose the Fourier vector k^i in a helicity basis and align the radial component with n^i

$$k^i = \mu k n^i + k\sqrt{\frac{1-\mu^2}{2}}\left(e^{-i\xi}m^i_+ + e^{i\xi}m^i_-\right). \quad (6.131)$$

Here μ is the cosine of the angle between k^i and n^i, $m^i_\pm$ is the helicity basis and ξ is the rotation angle in helicity basis. The mathematical properties of $m^i_\pm$ are listed at the appendix. In the sections to come, we will see that it carries spin degrees of freedom,

$$\mathbf{m}^{(\pm)} \cdot \mathbf{k} = \sqrt{\frac{1-\mu^2}{2}} e^{\pm i\xi}, \quad n^i k_i = \mu k. \quad (6.132)$$

we also decompose the direction vector in helicity basis $\mathbf{e}$ in Fourier space

$$\mathbf{n} = \mu\hat{\mathbf{k}} + \sqrt{\frac{1-\mu^2}{2}}\left(e^{-i\phi}\,\mathbf{e}^+ + e^{i\phi}\,\mathbf{e}^-\right), \quad (6.133)$$

so that

$$\mathbf{e}^{(\pm)} \cdot \mathbf{n} = \sqrt{\frac{1-\mu^2}{2}} e^{\pm i\phi} \quad \text{and} \quad m^i_\pm \cdot e^\mp_i = (1 \pm \mu) e^{\pm i\xi} e^{\pm i\phi}, \quad (6.134)$$

here ϕ is the rotation angle in $e_i^{\mp}$ basis in Fourier space. With all these tools, we are now in a position to calculate the multipoles of the terms in equation (6.124) and the associated two-point correlation function.

- **Pure Scalar field**

  For a pure scalar field, the computation is straight forward, we only need to expand in Fourier space and use the Rayleigh's formula

$$\Phi_{\ell m} = \int \Phi(\eta, x^i) Y^*_{\ell m} \,\mathrm{d}\,\Omega_{\mathbf{n}}\,, \tag{6.135}$$

$$= \sum_{\ell=0}^{\infty} \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \Phi(\mathbf{k}) T_\Phi(\eta, k) i^\ell \sqrt{4\pi(2\ell+1)} j_\ell(k\chi) \delta_{m,0}\,, \tag{6.136}$$

  The associated two point correlation function for the first order $\Phi_{\ell m}$ is given by

$$C_\ell = \frac{1}{(2\ell+1)} \langle \Phi_{\ell_1 m_1} \Phi_{\ell_2 m_2} \rangle\,, \tag{6.137}$$

$$= \frac{2}{\pi} \int k^2 \mathrm{d}\, k T_\Phi(k, \eta) T'_{\Phi'}(k', \eta) P_\Phi(k) |j_\ell(k\chi)|^2\,, \tag{6.138}$$

  where we have made use of

$$\langle \Phi(\mathbf{k}, \eta) \Phi^*(\mathbf{k}', \eta) \rangle = (2\pi)^3 \delta^3(\mathbf{k} - \mathbf{k}') T_\Phi(k, \eta) T'_{\Phi'}(k', \eta) P_\Phi(k) \tag{6.139}$$

  and set $P_\Phi(k) = |\Phi(k)|^2$.

  At second order, we need to use the results of chapter 2 to finish the calculation for $\Phi^{(2)}$, and $\Psi^{(2)}$. For example using equation (6.135) for $\Phi^{(2)}$ we find,

$$C_\ell^{(2)} = \frac{1}{(2\ell+1)} \langle \Phi^{(2)}_{\ell_1 m_1} \Phi^{(2)}_{\ell_2 m_2} \rangle = \int \frac{\mathrm{d}\, k}{k} \int \frac{\mathrm{d}\,\Omega_k}{(2\pi)^3} \mathcal{P}_\Phi^{(2)}(\eta, k) |j_\ell(k\chi)|^2\,, \tag{6.140}$$

  where $\mathcal{P}_\Phi^{(2)}(\eta, k)$ is given in chapter 2 (equation (2.89)). The $C_\ell^{(2)}s$ for $\Psi^{(2)}$ and cross-correlation function for $\Phi^{(2)}$, and $\Psi^{(2)}$ is given by the same formula with an appropriate change of power spectrum as given in chapter 2.

- **A psudo-scalar from a Vector $\omega_{\|\ell m}$**

  Vector perturbation is expanded in Fourier space according to

$$\omega_i(\mathbf{x}, \eta) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \left[\omega(\mathbf{k}, \eta) e_i(\mathbf{k}) + \bar{\omega}(\mathbf{k}, \eta) \bar{e}_i(\mathbf{k})\right] e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{6.141}$$

where the two polarization states of the vector field are denoted by two orthonormal basis vectors $\mathbf{e}$ and $\bar{\mathbf{e}}$. They are orthogonal to $\mathbf{k}$. It is better to work in helicity basis, so we combine the orthonormal basis according to

$$\mathbf{e}^{\pm} = \frac{1}{\sqrt{2}}\left(\mathbf{e}_1 \pm i\mathbf{e}_2\right) . \tag{6.142}$$

Using equation (6.133), we obtain the angular dependence of $\omega_{\|}$

$$\omega_{\|\ell,m} = \frac{1}{(2\pi)^3}\int\int d^3\mathbf{k}\left[\omega^{\pm}(\mathbf{k},\eta)\sqrt{\frac{1-\mu^2}{2}}e^{\pm i\phi}\right]e^{i\mathbf{k}\cdot\mathbf{x}}Y^*_{\ell,m}\mathrm{d}\,\Omega\,, \tag{6.143}$$

$$= \frac{i^{\ell}\sqrt{4\pi(2\ell+1)}}{\Delta_1(2\pi)^3}\int\int d^3\mathbf{k}\mp\omega^{\pm}(\mathbf{k},\eta)j_{\ell}(k\chi)Y_{1\pm1}Y_{\ell,0}Y^*_{\ell,m}\mathrm{d}\,\Omega\,. \tag{6.144}$$

In the second line, we have made use of table 6.1 to convert the angle to spherical harmonics. The two spherical harmonics were then added using Clebsh-Gordon relation and the result is given by

$$\frac{1}{\Delta_1}Y_{1\pm1}(\mathbf{n})Y_{\ell m}(\mathbf{n}) = \left[-\sqrt{\frac{(\ell-1\mp m)(\ell\mp m)}{2(2\ell+1)(2\ell-1)}}Y_{(\ell-1)m\pm1}(\mathbf{n})\right. \tag{6.145}$$

$$\left.+\sqrt{\frac{(\ell\pm m+1)(\ell\pm m+2)}{2(2\ell+3)(2\ell+1)}}Y_{(\ell+1)m+\pm1}(\mathbf{n})\right].$$

In the approximation that we are working, we sometimes switch notations $Y_{\ell m}(\mathbf{k}\cdot\mathbf{n}) \to Y_{\ell m}(\mathbf{n}) \to Y_{\ell m}$ because they mean the same thing in the limit $m \to 0$. Putting everything together, we find

$$\omega_{\|\ell,m} = \sum_{\ell=0}^{\infty}\sqrt{4\pi(2\ell+1)}\sqrt{\frac{\ell'(\ell+1)}{2}}\int\frac{d^3\mathbf{k}}{(2\pi)^3}\mp\omega^{\pm}(\mathbf{k},\eta)i^{\ell+1}\frac{j_{\ell}(k\chi)}{k\chi}\delta_{m,\pm1} \tag{6.146}$$

In the simplification above, we have used one of the recursion relations for the spherical Bessel functions given in the appendix B . The associated two point correlation function becomes

$$C_{\ell} = \frac{1}{(2\ell+1)}\langle\omega_{\|\ell_1,m_1}\omega_{\|\ell_2,m_2}\rangle\,, \tag{6.147}$$

$$= \frac{(\ell+1)!}{(\ell-1)!}\int\frac{\mathrm{d}\,k}{k}\int\frac{\mathrm{d}\,\Omega_k}{(2\pi)^3}\mathcal{P}^{(2)}_{\omega}(\eta,k)\left|\frac{j_{\ell}(k\chi)}{k\chi}\right|^2. \tag{6.148}$$

The power spectrum $\mathcal{P}^{(2)}_{\omega}(\eta,k)$ is given in chapter 2 (equation (2.101)).

- **Psudo-scalars from a tensor**

The Fourier mode of the tensor perturbation, h_{ij} is given by :

$$h_{ij}(\mathbf{x},\eta) = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \left[q^{+}_{ij}(\mathbf{k})h^{+}(\mathbf{k},\eta) + q^{\times}_{ij}(\mathbf{k})h^{\times}(\eta,\mathbf{k})\right] , \qquad (6.149)$$

where the two polarization tensors q_{ij} and $\bar{q}_{ij}$ are expressed in orthonormal basis vectors $\mathbf{e}$ and $\bar{\mathbf{e}}$ and they are orthogonal to $\mathbf{k}$,

$$\begin{aligned} q^{+}_{ij}(\mathbf{k}) &= \frac{1}{\sqrt{2}}\left[e_i(\mathbf{k})e_j(\mathbf{k}) - \bar{e}_i(\mathbf{k})\bar{e}_j(\mathbf{k})\right], \\ \bar{q}^{\times}_{ij}(\mathbf{k}) &= \frac{1}{\sqrt{2}}\left[e_i(\mathbf{k})\bar{e}_j(\mathbf{k}) + \bar{e}_i(\mathbf{k})e_j(\mathbf{k})\right]. \end{aligned} \qquad (6.150)$$

where $q^{s}_{ij}(\mathbf{k})$, with $s = +, \times$, are transverse traceless polarization tensors, they are normalized such that $q^{s}_{ij}q^{s'\,ij} = 2\delta^{ss'}$. The polarization tensor, $q^{s}_{ij}(\mathbf{k})$ may also be given in helicity basis, $q^{\pm1}_{i}$ and $q^{\pm2}_{ij}$ and they transform with a phase $e^{\pm i\phi}$, and $e^{\pm 2i\phi}$ respectively under rotations around $\mathbf{k}$ with angle ϕ. Assuming that both polarization states are independent and have equal power spectra, we obtain

$$\langle h_s(\eta,\mathbf{k})h_{s'}(\eta',\mathbf{k},')\rangle = (2\pi)^3\delta^3(\mathbf{k}-\mathbf{k}')\delta_{ss'}T_h(\eta,k)T_h(\eta',k)P_h(k) , \qquad (6.151)$$

where $T_h(\eta, k)$ is the tensor transfer function, and the primordial tensor power spectrum is denoted as $P_h(k)$. Using equation (6.133) we decompose the polarization states into angles and use Table 6.1, to transform the angles into spherical harmonics

$$\begin{aligned} h_{\|\ell m} &= \int \frac{d^3k}{(2\pi)^3}\left[h^{(\pm2)}(k,\eta)\frac{(1-\mu^2)}{2}e^{\pm 2i\phi}\right] e^{ik\mu\chi}Y^{*}_{\ell m}\,d\,\Omega\,, & (6.152)\\ &= \frac{i^{\ell}\sqrt{4\pi(2\ell+1)}}{\Delta_2}\int \frac{d^3k}{(2\pi)^3}h^{(\pm)}(k,\eta)j_\ell(k\chi)Y_{2\pm2}Y_{\ell,0}Y^{*}_{\ell m}\,d\,\Omega. & (6.153) \end{aligned}$$

Now we use the Cledsh-Gordon rule to add up

$$\frac{1}{\Delta_2}Y_{2\pm2}(\mathbf{k}\cdot\mathbf{n})Y_{\ell m}(\mathbf{k}\cdot\mathbf{n}) \tag{6.154}$$
$$= \left[\sqrt{\frac{(\ell\mp m-3)(\ell-2\mp m)(\ell-1\mp m)(\ell\mp m)}{(2\ell-1)^2(2\ell-3)(2\ell+1)}}Y_{(\ell-2)m\pm2}(\mathbf{k}\cdot\mathbf{n}) - \frac{\sqrt{4(\ell\pm m+2)(\ell\pm m+1)(\ell\mp m+1)(\ell\mp m)}}{(2\ell+3)(2\ell-1)}Y_{\ell m\pm2}(\mathbf{k}\cdot\mathbf{n}) + \sqrt{\frac{(\ell+1\pm m)(\ell+2\pm m)(\ell+3\pm m)(\ell+4\pm m)}{(2\ell+3)^2(2\ell+1)(2\ell+5)}}Y_{(\ell+2)m\pm2}(\mathbf{k}\cdot\mathbf{n})\right].$$

using one of the recursion relations for the spherical Bessel functions listed in the appendix, we find

$$h_{\|\ell m} = \sum_{\ell=0}^{\infty} i^\ell\sqrt{4\pi(2\ell+1)}\sqrt{\frac{(\ell'+2)!}{(\ell'-2)!}}\int\frac{\mathrm{d}^3k}{(2\pi)^3}h^{(\pm)}(k,\eta)\frac{j_\ell(k\chi)}{(k\chi)^2}\delta_{\pm2,m}. \tag{6.155}$$

The two point correlration function for $h_\|$ is given by

$$C_\ell = \frac{(\ell'+2)!}{(\ell'-2)!}\int\int\frac{\mathrm{d}^3k_1}{(2\pi)^3}\frac{\mathrm{d}^3k_2}{(2\pi)^3}\langle h^{(\pm)}(\mathbf{k}_1,\eta)h'^{(\pm)}(\mathbf{k}_2,\eta)\rangle\frac{j_{\ell_1}(k_1\chi)}{(k_1\chi)^2}\frac{j_{\ell_2}(k_2\chi)}{(k_2\chi)^2}\delta_{\pm2,m_1} \tag{6.156}$$

In terms of the the primordial gravitational waves at first order, we find using equation (6.151)

$$C_\ell = \frac{1}{(2\ell+1)}\langle h_{\|\ell_1 m_1}h'_{\|\ell_2,m_2}\rangle\,, \tag{6.157}$$
$$= \frac{2}{\pi}\frac{(\ell+2)!}{(\ell-2)!}\int k^2\mathrm{d}\,kT_{h^\pm}(\eta,k)T_{h'^\pm}(\eta',k)P_{h^\pm0}(k)\frac{j_{\ell_1}(k\chi)}{(k\chi)^2}\frac{j_{\ell_2}(k\chi)}{(k\chi)^2}\,. \tag{6.158}$$

At second order, we simply replace the power spectrum in equation (6.156) with the induced power spectrum for gravitational waves at second order given in chapter 2 (equation (2.124)),

$$C_\ell = \frac{(\ell+2)!}{(\ell-2)!}\int\frac{\mathrm{d}\,k}{k}\int\frac{\mathrm{d}\,\Omega_k}{(2\pi)^3}\mathcal{P}_{h^{(2)}}(\eta,k)\frac{j_\ell(k\chi)}{(k\chi)^2}\frac{j_\ell(k\chi)}{(k\chi)^2}\,. \tag{6.159}$$

Spherical Harmonic	In terms of μ	Normalized	
${}_0Y_{10}(\mathbf{k}\cdot\mathbf{n})$	$\sqrt{\frac{3}{4\pi}}\mu$	$\Delta_1\mu$	
${}_{\pm1}Y_{10}(\mathbf{k}\cdot\mathbf{n})$	$\pm e^{\pm i\zeta}\sqrt{\frac{3}{4\pi}}\sqrt{\frac{(1-\mu^2)}{2}}$	$\pm iN_1e^{\pm i\zeta}\sqrt{\frac{3}{4\pi}}\sqrt{\frac{(1-\mu^2)}{2}}$	
${}_0Y_{1\pm1}(\mathbf{k}\cdot\mathbf{n})$	$\mp e^{\pm i\phi}\sqrt{\frac{3}{4\pi}}\sqrt{\frac{(1-\mu^2)}{2}}$	$\mp\Delta_1e^{\pm i\phi}\sqrt{\frac{(1-\mu^2)}{2}}$	
${}_{\pm2}Y_{20}(\mathbf{k}\cdot\mathbf{n})$	$\frac{1}{2}e^{\pm2i\zeta}\sqrt{\frac{15}{8\pi}}(1-\mu^2)$	$\Delta_2e^{\pm2i\zeta}\frac{1}{2}(1-\mu^2)$	
${}_0Y_{20}(\mathbf{k}\cdot\mathbf{n})$	$\frac{1}{2}\sqrt{\frac{5}{4\pi}}(3\mu^2-1)$	$-N_2\frac{1}{2}(3\mu^2-1)$	
${}_0Y_{2\pm2}(\mathbf{k}\cdot\mathbf{n})$	$\sqrt{\frac{15}{8\pi}}\frac{(1-\mu^2)}{2}e^{\pm2i\phi}$	$\Delta_2\frac{(1-\mu^2)}{2}e^{\pm2i\phi}$	
$Y_{3\pm1}(\mathbf{k}\cdot\mathbf{n})$	$\pm\frac{1}{2}e^{i\phi}\sqrt{\frac{21}{8\pi}}(1-5\mu^2)\sqrt{\frac{1-\mu^2}{2}}$	$\pm\frac{1}{2}\Delta_3e^{i\phi}(1-5\mu^2)\sqrt{\frac{1-\mu^2}{2}}$	
${}_{\pm1}Y_{1\pm1}(\mathbf{k}\cdot\mathbf{n})$	$\pm\frac{1}{2}\sqrt{\frac{3}{4\pi}}e^{\pm i(\zeta+\phi)}(\mu\mp1)$	$\pm\frac{1}{2}\Delta_1e^{\pm i(\zeta+\phi)}(\mu\mp1)$	
${}_{\pm1}Y_{20}(\mathbf{k}\cdot\mathbf{n})$	$e^{\pm i\zeta}\sqrt{\frac{15}{8\pi}}\mu\sqrt{1-\mu^2}$	$\frac{1}{\Delta_2}e^{\pm i\zeta}\mu\sqrt{1-\mu^2}$	
${}_{\pm2}Y_{2\pm2}(\mathbf{k}\cdot\mathbf{n})$	$\frac{1}{4}e^{\pm2i(\zeta+\phi)}\sqrt{\frac{5}{4\pi}}(\mu\mp1)^2$	$-\frac{1}{2}N_2e^{\pm2i(\zeta+\phi)}\frac{(\mu\mp1)^2}{2}$	
${}_{\pm2}Y_{2\pm1}(\mathbf{k}\cdot\mathbf{n})$	$\frac{1}{2}e^{\pm i(2\zeta+\phi)}\sqrt{\frac{5}{2\pi}}\sqrt{\frac{1-\mu^2}{2}}(\mu\mp1)$	$\Delta_2e^{\pm i(2\zeta+\phi)}\sqrt{\frac{1-\mu^2}{2}}(\mu\mp1)$	
${}_{\pm1}Y_{22}(\mathbf{k}\cdot\mathbf{n})$	$\mp\frac{1}{\sqrt{2}}e^{\pm i(\zeta+2\phi)}\sqrt{\frac{5}{4\pi}}\sqrt{\frac{1-\mu^2}{2}}(\mu\mp1)$	$\pm\frac{1}{2}N_2e^{\pm i(\zeta+2\phi)}\sqrt{\frac{1-\mu^2}{2}}(\mu\mp1)$	

Table 6.1: Spin Weighted Spherical harmonic Transformation table. The normalization was done using $\Delta_\ell^{-1}=\frac{4\pi\ell!}{(2\ell+1)!!}$, $N_\ell=i^\ell\sqrt{\frac{(2\ell+1)}{4\pi}}$.

6.9 Methods of Spherical Harmonics Expansion

There are two ways a full sky spherical expansion could be implemented:

- The first is called Total Angular Momentum (TAM) approach: it involves using the Clebsh-Gordon relation to add the spin component to the orbital angular momentum to obtain the total angular momentum which is measurable. This approach is most useful when a non-zero spin object is involved, because it leads naturally to a situation where possible coupling of the fields with different parities could occur.

- The second approach called the ð formalism approach: it involves invoking the ð formalism to convert all the angular derivatives on the screen space to ð. The resulting ðs will then act on the spherical harmonic by raising or lowering its spin degrees of freedom. The major advantage of this approach is that it becomes easier to apply the Limber approximation to the resulting equations.

The major difference between the two formalisms appears when the expansion in Fourier space expansion is performed. If the expansion in Fourier space is done first to handle the spatial derivatives, then we have TAM approach, the contribution from the Fourier space in this case is called the spin component. However, if we convert the real space angular derivatives first to ð before expanding the result in Fourier space, the ð formalism results. Analytically, both the results are differ in some cases but we have not yet verified if the difference remains after numerical calculation. Due to space constraint, we will give only the result from ð formalism.

6.9.1 ð formalism approach

With respect to the covariant derivatives on the screen space, it is possible to define respectively, the spin raising and lowering operators ð and $\bar{ð}$ using the complex basis $m_{\pm}$. For example the conversion of angular derivatives acting on $X_{A_{|s|}}$ to ð is given by

$$ð_{\pm|s|}X = -(m_{+}^{c}\nabla_{\Omega c}X_{A_{|s|}})m_{\pm}^{A_{|s|}}, \qquad (6.160)$$
$$\bar{ð}_{\pm|s|}X = -(m_{-}^{c}\nabla_{\Omega c}X_{A_{|s|}})m_{\pm}^{A_{|s|}}, \qquad (6.161)$$

where the minus signs are conventional. In our notation, we have for example, a typical scalar field of the form $X = X(\eta, \mathbf{x})$ has derivative operators of the form $\nabla_{\perp i}$ on the screen space acting on it. So we have to first re-write the derivative operators as follows $\nabla_{\perp i} = \frac{1}{\chi}\nabla_{\Omega i}$ and $\nabla_{\perp}^{2} = \frac{1}{\chi^{2}}\nabla_{\Omega}^{2} + \frac{2}{\chi}\partial_{\|}$, before converting them into raising and lowering operators. For instance, the second contribution to the redshift and the area distance contain terms of the form $\nabla_{\perp i}\Phi\nabla^i_\perp\Phi$, we handle such terms like this

$$m^i_+\nabla_{\perp i}X = -\frac{1}{\chi}\eth X\,,\ \ m^i_-\nabla_{\perp i}X = -\frac{1}{\chi}\bar{\eth}X\,. \tag{6.162}$$

The laplacian or $\nabla^2_\perp$ that appear in several places in the expression is converted to $\eth\bar{\eth}$

$$m^i_+m^-_j\nabla_{\perp i}\nabla^j_\perp X = \frac{1}{\chi^2}\nabla^2_\Omega X + \frac{2}{\chi}\partial_\| X = \frac{1}{\chi^2}\eth\bar{\eth}X + \frac{2}{\chi}\partial_\| X\,. \tag{6.163}$$

For a scalar field with $s = 0$, the spin raising and lowering operators commute.

$$m^i_-m_{+j}\nabla_{\Omega i}\nabla^j_\Omega X = m^i_+m_{-j}\nabla_{\Omega i}\nabla^i_\Omega X = \bar{\eth}\eth X = \eth\bar{\eth}X\,. \tag{6.164}$$

For the contribution of from the shear, both indices are symmetric and trace-free. Also $N^{ij}m^i_+m^j_+ = 0$, which leads to

$$m^i_+m^j_+\nabla_{\perp\langle i}\nabla_{\perp j\rangle}X = \frac{1}{\chi^2}\eth^2X\,,\ m^i_-m^j_-\nabla_{\perp\langle i}\nabla_{\perp j\rangle}X = \frac{1}{\chi^2}\bar{\eth}^2X\,. \tag{6.165}$$

Some terms in $\delta D_A\times\Phi$ have a three derivatives contribution and they are handled as follows:

$$\begin{aligned} m^k_+\nabla_{\perp k}\left(m^i_-m_{+j}\nabla_{\perp i}\nabla^j_\perp X\right) &= -\frac{1}{\chi}\eth\left(\frac{1}{\chi^2}\nabla^2_\Omega X + \frac{2}{\chi}\partial_\| X\right)\,, &(6.166)\\ &= -\frac{1}{\chi^3}\eth\eth\bar{\eth}X - \frac{2}{\chi^2}\partial_\|\eth X\,, &(6.167)\end{aligned}$$

The terms in the bracket were evaluated before the derivative is taken. After the conversion of angular derivatives to $\eth$s, we then expand every scalar quantity in spherical harmonics. The Laplacians appearing in the expression are immediately replaced with

$$\eth\bar{\eth}X = X_{\ell m}(\eta,r)\eth\bar{\eth}Y_{\ell m}(\mathbf{n}) = -\ell(\ell+1)X_{\ell m}(\eta,r)Y_{\ell m}(\mathbf{n}) \tag{6.168}$$

and at second order, we implement the following;

$$\begin{aligned} \eth X_1\bar{\eth}X_2 &= X_{1_{\ell m}}(\eta,r_1)X_{2_{\ell m}}(\eta,r_2)\eth Y_{1_{\ell m}}\bar{\eth}Y_{2_{\ell m}}\,, &(6.169)\\ &= -\left[\ell_1(\ell_1+1)\right]^{\frac{1}{2}}\left[\ell_2(\ell_2+1)\right]^{\frac{1}{2}}X_{1_{\ell m}}(\eta,r_1)X_{2_{\ell m}}(\eta,r_2)_{+1}Y_{1_{\ell m}-1}Y_{2_{\ell m}}\,, &(6.170)\\ \eth\eth X_1\bar{\eth}\bar{\eth}X_2 &= X_{1_{\ell m}}(\eta,r_1)\eth\eth Y_{1_{\ell m}}X_{2_{\ell m}}(\eta,r_2)\bar{\eth}\bar{\eth}Y_{2_{\ell m}}\,, &(6.171)\\ &= \sqrt{\frac{(\ell_1+2)!}{(\ell_1-2)!}}\sqrt{\frac{(\ell_2+2)!}{(\ell_2-2)!}}X_{\ell m}(\eta,r_1)X_{\ell m}(\eta,r_2)_{+2}Y_{1_{\ell m}-2}Y_{2_{\ell m}}\,. &(6.172)\end{aligned}$$

Applying all that we have described above to equation (6.247), we find:

$$\begin{aligned}\frac{\delta D_A}{D_A(s)} &= \sum_{\ell=0}^{\infty}\sqrt{4\pi(2\ell+1)}\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\Phi(\mathbf{k})i^\ell\left[-T_\Phi(\eta,k)j_\ell(k\chi_s)\right.\\ &+\frac{2}{\chi_s}\int_0^{\chi_s}T_\Phi(\eta,k)j_\ell(k\chi)\mathrm{d}\,\chi-4\int_0^{\chi_s}\frac{(\chi_s-\chi)}{\chi_s}T_\Phi(\eta,k)kj'_\ell(k\chi)\mathrm{d}\,\chi\\ &\left.-\ell(\ell+1)\int_0^{\chi_s}\mathrm{d}\,\chi\frac{(\chi_s-\chi)}{\chi_s\chi}T_\Phi(\eta,k)j_\ell(k\chi)\right]Y_{\ell 0}(\mathbf{n}).\end{aligned} \tag{6.173}$$

At second order perturbation theory, we split the area distance as follows:

$$\begin{aligned}\frac{\delta^2 D_A}{D_A} &= \frac{\delta^2 D_A}{D_A}^{\mathrm{S}}+\frac{\delta^2 D_A}{D_A}^{\mathrm{V}}+\frac{\delta^2 D_A}{D_A}^{\mathrm{T}}\\ &+\frac{\delta^2 D_A}{D_A}^{\Phi\times\Phi}+\frac{\delta^2 D_A}{D_A}^{\nabla_{\perp i}\Phi\nabla^i_\perp\Phi}+\frac{\delta^2 D_A}{D_A}^{\mathrm{Shear}}+\frac{\delta^2 D_A}{D_A}^{\delta D_A\times\Phi}.\end{aligned} \tag{6.174}$$

First we present the relativistic second order terms result

$$\frac{\delta^2 D_A}{D_A}=\frac{\delta^2 D_A}{D_A}^{\mathrm{S}}+\frac{\delta^2 D_A}{D_A}^{\mathrm{V}}+\frac{\delta^2 D_A}{D_A}^{\mathrm{T}}, \tag{6.175}$$

where

$$\begin{aligned}\frac{\delta D_A}{D_A}^{\mathrm{S}} &= \sum_{\ell=0}^{\infty}\sqrt{4\pi(2\ell+1)}\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\mathcal{K}\left[\Phi(\mathbf{k_1},\mathbf{k_2}\right]i^\ell\\ &\quad\Delta^{(2)}_{\mathrm{S}}(\chi_s,\chi,\mu_{k_1},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}))\delta_{m,0}Y_{\ell m}\,,\end{aligned} \tag{6.176}$$

$$\begin{aligned}\frac{\delta^2 D_A}{D_A}^{\mathrm{V}} &= \sum_{\ell=0}^{\infty}\sqrt{4\pi(2\ell+1)}\sqrt{\frac{\ell'(\ell+1)}{2}}\int\frac{d^3\mathbf{k}}{(2\pi)^3}\mathcal{K}\left[\Phi(\mathbf{k_1},\mathbf{k_2}\right]i^{\ell+1}\\ &\quad\Delta^{(2)}_{\mathrm{V}}(\chi_s,\chi,\mu_{k_1},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}))\delta_{m\pm1}Y_{\ell m}\,,\end{aligned} \tag{6.177}$$

$$\begin{aligned}\frac{\delta^2 D_A}{D_A}^{\mathrm{T}} &= \sum_{\ell=0}^{\infty}i^\ell\sqrt{4\pi(2\ell+1)}\sqrt{\frac{(\ell'+2)!}{(\ell'-2)!}}\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\mathcal{K}\left[\Phi(\mathbf{k_1},\mathbf{k_2}\right]\\ &\quad\Delta^{(2)}_{\mathrm{T}}(\chi_s,\chi,\mu_{k_1},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}))\delta_{m\pm2}Y_{\ell m}\,.\end{aligned} \tag{6.178}$$

We have made used of the following notations in the simplification

$$\mathcal{K}[\Phi(\mathbf{k}_1,\mathbf{k}_2)]=\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\int\frac{\mathrm{d}^3\mathbf{k}_2}{(2\pi)^3}\Phi(\mathbf{k}_1)\Phi(\mathbf{k}_2)(2\pi)^3\delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k})\,. \tag{6.179}$$

and

$$\begin{aligned}
\Delta^{(2)}_{\rm S}(\chi_s,\chi,\mu_{k_1},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k})) &= \Bigg[- f_{\Psi^{(2)}}(\eta,\mu_{k_1},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}) j_\ell(k\chi_s) \\
&-2\int_0^{\chi_s} \frac{(\chi_s-\chi)}{\chi_s}\left(f_{\Phi^{(2)}}(\eta,\mu_{k_1},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k})\right.\\
&\left.+f_{\Psi^{(2)}}(\eta,\mu_{k_1},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k})\right) j'_\ell(k\chi){\rm d}\,\chi\\
&+\frac{1}{\chi_s}\int_0^{\chi_s}\left(f_{\Phi^{(2)}}(\eta,\mu_{k_1},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k})+f_{\Psi^{(2)}}(\eta,\mu_{k_1},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k})\right) j_\ell(k\chi){\rm d}\,\chi\\
&-\frac{(\ell(\ell+1))}{2}\int_0^{\chi_s}\frac{(\chi_s-\chi)}{\chi_s\chi}\left(f_{\Phi^{(2)}}(\eta,\mu_{k_1},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k})\right.\\
&\left.+f_{\Psi^{(2)}}(\eta,\mu_{k_1},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k})\right) j_\ell(k\chi){\rm d}\,\chi\Bigg],
\end{aligned} \tag{6.180}$$

$$\begin{aligned}
\Delta^{(2)}_{\rm V}(\chi_s,\chi,\mu_{k_1},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k})) &= \mp\Bigg[-3\int_0^{\chi_s}\frac{(\chi_s-\chi)}{\chi_s} f_{\omega^{(2)}}(\eta,\mu_{k_1},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k})k\left(\frac{j_\ell(k\chi)}{k\chi}\right)'{\rm d}\,\chi\\
&+\frac{1}{2}\int_0^{\chi_s} f_{\omega^{(2)}}(\eta,\mu_{k_1},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k})k\left(\frac{j_\ell(k\chi)}{k\chi}\right)' d\chi\\
&+\int_0^{\chi_s} f_{\omega^{(2)}}(\eta,\mu_{k_1},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k})\frac{(\chi_s-\chi)}{\chi_s\chi}\left(\frac{j_\ell(k\chi)}{k\chi}\right) d\chi\\
&-\frac{\ell(\ell+1)}{2}\int_0^{\chi_s} f_{\omega^{(2)}}(\eta,\mu_{k_1},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k})\frac{(\chi_s-\chi)}{\chi_s\chi}\left(\frac{j_\ell(k\chi)}{k\chi}\right){\rm d}\,\chi\Bigg],
\end{aligned} \tag{6.181}$$

$$\begin{aligned}
\Delta^{(2)}_{\rm T}(\chi_s,\chi,\mu_{k_1},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k})) &= \Bigg[\frac{1}{\chi_s}\int_0^{\chi_s} f_{h^{(2)}}(\eta,\mu_{k_1},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k})\left(\frac{j_\ell(k\chi)}{(k\chi)^2}\right){\rm d}\,\chi\\
&+3\int_0^{\chi_s}\frac{(\chi_s-\chi)}{\chi_s} f_{h^{(2)}}(\eta,\mu_{k_1},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k})k\left(\frac{j_\ell(k\chi)}{(k\chi)^2}\right)'{\rm d}\,\chi\\
&-\frac{1}{2}\int_0^{\chi_s} f_{h^{(2)}}(\eta,\mu_{k_1},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k})k\left(\frac{j_\ell(k\chi)}{(k\chi)^2}\right)'{\rm d}\,\chi\\
&-3\int_0^{\chi_s} f_{h^{(2)}}(\eta,\mu_{k_1},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k})\frac{(\chi_s-\chi)}{\chi_s\chi}\left(\frac{j_\ell(k\chi)}{(k\chi)^2}\right){\rm d}\,\chi\\
&-\frac{\ell(\ell+1)}{2}\int_0^{\chi_s}\frac{(\chi_s-\chi)}{\chi_s\chi} f_{h^{(2)}}(\eta,\mu_{k_1},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k})\left(\left(\frac{j_\ell(k\chi)}{(k\chi)^2}\right)\right){\rm d}\,\chi\Bigg].
\end{aligned} \tag{6.182}$$

For terms quadratic in two first order gravitational potentials, we multiply two spherical harmonics using the Clebsh-Gordon rule:

$$
\begin{aligned}
{}_{s_1}Y_{\ell_1,m_1\,s_2}Y_{\ell_2,m_2} &= \sqrt{\frac{(2\ell_1+1)(2\ell_2+1)(2\ell_3+1)}{4\pi}}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ m_1 & m_2, & -m\end{pmatrix}\\
&\quad\times\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ -s_1 & -s_2, & -s\end{pmatrix}{}_sY_{\ell_3,m}\,, && (6.183)\\
&= {}_{(s_1,s_2,s)}\mathcal{G}_{\ell_1\ell_2\ell_3}{}^{m_1m_2m}{}_sY_{\ell_3m}\,, && (6.184)
\end{aligned}
$$

where the brackets stands for the well-known $3jm$-symbol. In this case we are using ℓ_3 to denote j the total angular momentum. This is to avoid a clash with the spherical Bessel functions. The particular cases of importance are

$$
\begin{aligned}
{}_2Y_{\ell_1,0\,-2}Y_{\ell_2,0} &= \sqrt{\frac{(2\ell_1+1)(2\ell_2+1)(2\ell_3+1)}{4\pi}}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ 0 & 0, & 0\end{pmatrix}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ 2 & -2, & 0\end{pmatrix}Y_{\ell_3,0}\\
&= {}_{(2,-2,0)}\mathcal{G}_{\ell_1\ell_2\ell_3}{}^{000}Y_{\ell_3,0} && (6.185)\\
{}_1Y_{\ell_1,0\,-1}Y_{\ell_2,0} &= -\sqrt{\frac{(2\ell_1+1)(2\ell_2+1)(2\ell_3+1)}{4\pi}}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ 0 & 0, & 0\end{pmatrix}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ 1 & -1, & 0\end{pmatrix}Y_{\ell_3,0}\\
&= -{}_{(1,-1,0)}\mathcal{G}_{\ell_1\ell_2\ell_3}{}^{000}Y_{\ell_3,0} && (6.186)\\
{}_0Y_{\ell_1,0\,0}Y_{\ell_2,0} &= \sqrt{\frac{(2\ell_1+1)(2\ell_2+1)(2\ell_3+1)}{4\pi}}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ 0 & 0, & 0\end{pmatrix}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ 0 & 0, & 0\end{pmatrix}Y_{\ell_3,0}\\
&= {}_{(0,0,0)}\mathcal{G}_{\ell_1\ell_2\ell_3}{}^{000}Y_{\ell_3,0}. && (6.187)
\end{aligned}
$$

where we have replaced the 3j Wigner symbols with equivalent Gaunt integral notation in the last line.

$$
\frac{\delta^2 D_A}{D_A} = \frac{\delta^2 D_A}{D_A}^{\Phi\times\Phi} + \frac{\delta^2 D_A}{D_A}^{\nabla_{\perp i}\Phi\nabla^i_\perp\Phi} + \frac{\delta^2 D_A}{D_A}^{\text{Shear}} + \frac{\delta^2 D_A}{D_A}^{\delta D_A\times\Phi}. \qquad (6.188)
$$

$$\frac{\delta^2 D_A}{D_A}^{\Phi\times\Phi} = \sum_{\ell=0}^{\infty} i^{\ell_1+\ell_2}\sqrt{4\pi(2\ell_3+1)}\mathcal{K}[\Phi(\mathbf{k}_1)\Phi(\mathbf{k}_2)] W_{\Delta^{\Phi\times\Phi}}(\eta,\chi,k_1,k_2))\delta_{m,0}Y_{\ell m}\,, \tag{6.189}$$

$$\frac{\delta^2 D_A}{D_A}^{\nabla_{\perp i}\Phi\nabla^i_\perp\Phi} = \sum_{\ell=0}^{\infty} i^{\ell_1+\ell_2}\sqrt{4\pi(2\ell_3+1)}\mathcal{K}[\Phi(\mathbf{k}_1)\Phi(\mathbf{k}_2)] W^{\nabla_{\perp i}\Phi\nabla^i_\perp\Phi}_{\Delta_{D_A}}(\eta,k_1,k_2,k)Y_{\ell 0}\,, \tag{6.190}$$

$$\frac{\delta^2 D_A}{D_A}^{\text{Shear}} = \sum_{\ell=0}^{\infty} i^{\ell_1+\ell_2}\sqrt{4\pi(2\ell_3+1)}\mathcal{K}[\Phi(\mathbf{k}_1)\Phi(\mathbf{k}_2)] W_{\Delta_{D_A}}{}^{\text{Shear}}(\eta,k_1,k_2,k)\delta_{m,0}Y_{\ell m}\,, \tag{6.191}$$

where

$$\mathcal{K}[\Phi(\mathbf{k}_1)\Phi(\mathbf{k}_2)] = \int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\int\frac{\mathrm{d}^3\mathbf{k}_2}{(2\pi)^3}\Phi(\mathbf{k}_1)\Phi(\mathbf{k}_2) \tag{6.192}$$

For $\delta D_A\times\Phi$ term, it makes more sense to split it into parts with a contribution along the line of sight and the part orthogonal to the line of sight,

$$\delta D_A\times\Phi = \delta D_A\times\Phi_\parallel + \delta D_A\times\Phi_\perp\,, \tag{6.193}$$

where

$$\delta D_A\times\Phi_\parallel = \sum_{\ell=0}^{\infty}\sqrt{4\pi(2\ell_3+1)}\mathcal{K}[\Phi(\mathbf{k}_1)\Phi(\mathbf{k}_2)]i^{\ell_1+\ell_2} W_{\Delta_{\delta D_A\times\Phi_\parallel}}(\eta,k_1,k_2,k)\delta_{m,0}Y_{\ell m}\,, \tag{6.194}$$

$$\delta D_A\times\Phi_\perp = \sum_{\ell=0}^{\infty}\sqrt{4\pi(2\ell_3+1)}\mathcal{K}[\Phi(\mathbf{k}_1)\Phi(\mathbf{k}_2)]i^{\ell_1+\ell_2} W_{\delta D_A\times\Phi_\perp}(\eta,k_1,k_2,k)\delta_{m,0}Y_{\ell m}. \tag{6.195}$$

with

$$W_{\Delta^{\Phi\times\Phi}}(\eta,\chi,k_1,k_2)) = \sqrt{\frac{4\pi(2\ell_1+1)(2\ell_2+1)}{(2\ell_3+1)}}\,{}_{(0,0,0)}\mathcal{G}_{\ell_1\ell_2\ell_3}{}^{000}\mathcal{W}_{\Delta^{\Phi\times\Phi}}(\eta,\chi,k_1,k_2))\,, \quad (6.196)$$

$$\begin{aligned}\mathcal{W}_{\Delta^{\Phi\times\Phi}}(\eta,\chi,k_1,k_2)) = {}& \Bigg[-2\int_0^{\chi_s}\frac{(\chi_s-\chi)\chi}{\chi_s}\Bigg(\big(T_\Phi(\eta_1,k_1)j'_{\ell_1}(k_1\chi)\big)\\ &\times\big(T_\Phi(\eta_1,k_2)j'_{\ell_2}(k_2\chi)\big)+2T_\Phi(\eta_1,k_1)j''_{\ell_1}(k_1\chi)T_\Phi(\eta_2,k_2)j_{\ell_2}(k_2\chi_2)\\ &-4T'_\Phi(\eta_1,k_1)j'_{\ell_1}(k_1\chi)T_\Phi(\eta_2,k_2)j_{\ell_2}(k_2\chi)\Bigg)\mathrm{d}\,\chi\\ &+8\int_0^{\chi_s}\frac{(\chi_s-\chi)}{\chi_s}T'_\Phi(\eta_1,k_1)T_\Phi(\eta_2,k_2)j_{\ell_1}(k_1\chi)j_{\ell_2}(k_2\chi)\mathrm{d}\,\chi\\ &-\frac{8}{\chi_s}\int_0^{\chi_s}T_\Phi(\eta_1,k_1)T_\Phi(\eta_2,k_2)j_{\ell_1}(k_1\chi)j_{\ell_2}(k_2\chi)\mathrm{d}\,\chi\Bigg]\,, \end{aligned} \quad (6.197)$$

$$\begin{aligned}W_{\Delta_{D_A}}^{\nabla_{\perp i}\Phi\nabla^i_\perp\Phi}(\eta,k_1,k_2,k) = {}& \sqrt{\frac{4\pi(2\ell_1+1)(2\ell_2+1)}{(2\ell_3+1)}}\,{}_{(1,-1,0)}\mathcal{G}_{\ell_1\ell_2\ell_3}{}^{000} \quad (6.198)\\ &\times\sqrt{\frac{(\ell_1+1)!}{(\ell_1-1)!}\frac{(\ell_2+1)!}{(\ell_2-1)!}}\mathcal{W}_{\Delta_{D_A}}^{\nabla_{\perp i}\Phi\nabla^i_\perp\Phi}(\eta,k_1,k_2,k)\,,\end{aligned}$$

$$\begin{aligned}\mathcal{W}_{\Delta_{D_A}}^{\nabla_{\perp i}\Phi\nabla^i_\perp\Phi}(\eta,k_1,k_2,k) = {}& 8\int_0^{\chi_s}\frac{(\chi_s-\chi)}{\chi_s\chi}\,(T'_{\Phi'}(\eta_2,k_2)j_{\ell_2}(k_2\chi_2)\\ &\times\int_0^{\chi}\frac{1}{\chi_1}T_\Phi(\chi_1,k_1)j_{\ell_1}(k_1\chi_1)\mathrm{d}\,\chi_1\mathrm{d}\,\chi-T_\Phi(\eta_2,k_2)j_{\ell_2}(k_2\chi_2)\\ &\int_0^{\chi}\frac{1}{\chi_1}T_\Phi(\chi_1,k_1)j_{\ell_1}(k_1\chi_1)\mathrm{d}\,\chi_1\mathrm{d}\,\chi\Big)\,,\end{aligned} \quad (6.199)$$

$$\begin{aligned}W_{\Delta_{D_A}}{}^{\mathrm{Shear}}(\eta,k_1,k_2,k) = {}& \sqrt{\frac{4\pi(2\ell_1+1)(2\ell_2+1)}{(2\ell_3+1)}}\,{}_{(2,-2,0)}\mathcal{G}_{\ell_1\ell_2\ell_3}{}^{000} \quad (6.200)\\ &\left[\sqrt{\frac{(\ell_1+2)!}{(\ell_1-2)!}\frac{(\ell_2+2)!}{(\ell_2-2)!}}\,,i^{\ell_1+\ell_2}\mathcal{W}_{\Delta_{D_A}}{}^{\mathrm{Shear}}(\eta,k_1,k_2,k)\right]\,,\end{aligned}$$

$$\begin{aligned}\mathcal{W}_{\Delta_{D_A}}{}^{\mathrm{Shear}}(\eta,k_1,k_2,k) = {}& 4\int_0^{\chi_s}\frac{(\chi_s-\chi)\chi}{\chi_s}a(\chi)\left[\int_0^{\chi}\frac{1}{\chi_1^2}T_\Phi(\chi_1,k_1)j_{\ell_1}(k_1\chi_1)\mathrm{d}\,\chi_1 \right. \quad (6.201)\\ &\left.\times\int_0^{\chi}\frac{1}{\chi^2}T_\Phi(\chi_2,k_2)j_{\ell_2}(k_2\chi_2)\mathrm{d}\,\chi_2\right]\mathrm{d}\,\chi\,,\end{aligned}$$

$$W_{\Delta_{\delta D_A \times \Phi_\parallel}}(\eta, k_1, k_2, k) = \sqrt{\frac{4\pi(2\ell_1+1)(2\ell_2+1)}{(2\ell_3+1)}}\,{}_{(0,0,0)}\mathcal{G}_{\ell_1\ell_2\ell_3}{}^{000} \times \left[\mathcal{W}_{\Delta_{\delta D_A \times \Phi_\parallel}}(\eta, k_1, k_2, k)\right], \quad (6.202)$$

$$\mathcal{W}_{\Delta_{\delta D_A \times \Phi_\parallel}}(\eta, k_1, k_2, k) = [4F_1'(\chi, k_1)G_2(\chi_2, k_2) + 8F_2'(\chi, k_1)G_1(\chi_2, k_2) -4F_2(\chi, k_1)G_1'(\chi_2, k_2) - 2F_1(\chi, k_1)\left(G_3(\chi_2, k_2) + G_5(\chi_2, k_2)\right)] , \quad (6.203)$$

$$W_{\delta D_A \times \Phi_\perp}(\eta, k_1, k_2, k) = \sqrt{\frac{4\pi(2\ell_1+1)(2\ell_2+1)}{(2\ell_3+1)}}\sqrt{\frac{(\ell_1+1)!}{(\ell_1-1)!}\frac{(\ell_2+1)!}{(\ell_2-1)!}}\,i^{\ell_1+\ell_2} \quad (6.204)$$

$$_{(1,-1,0)}\mathcal{G}_{\ell_1\ell_2\ell_3}{}^{000}\left[\mathcal{W}_{\delta D_A \times \Phi_\perp}(\eta, k_1, k_2, k)\right], \quad (6.205)$$

$$\mathcal{W}_{\delta D_A \times \Phi_\perp}(\eta, k_1, k_2, k) = \left[8F_4(\chi, k)\int_0^\chi G_4(\chi_1, k_1)\mathrm{d}\,\chi_1 + 4G_4(\chi, k_1)F_3(\chi, k_2)\right] \quad (6.206)$$

We have made the following definitions for clarity.

$$G_1(\chi, k) \equiv T_\Phi(\eta, k) j_\ell(k\chi)\,, \tag{6.207}$$

$$G_2(\chi, k) \equiv T_\Phi(\eta_1, k) j'(k\chi)\,, \tag{6.208}$$

$$G_3(\chi, k) \equiv T_\Phi(\eta, k) j''_\ell(k_2\chi)\,, \tag{6.209}$$

$$G_4(\chi, k) \equiv \left[T_\Phi(\eta, k)\frac{j_\ell\,(k\chi)}{\chi}\right]\,, \tag{6.210}$$

$$G_5(\chi, k) \equiv T_\Phi(\eta, k)\left[\frac{\ell(\ell+1)}{\chi^2} j(k\chi) + \frac{2}{\chi} j'(k\chi)\right]\,, \tag{6.211}$$

$$\begin{aligned} F_1(\chi, k) \equiv& \left[-\chi(T_\Phi(\eta, k) j_\ell(k\chi) + 2\int_0^{\chi_s} T_\Phi(\eta, k) j_\ell(k\chi)\mathrm{d}\,\chi - 4\int_0^{\chi_s} (\chi_s - \chi)\right. \\ & \left. \times T_\Phi(\eta, k) k j'_\ell(k\chi)\mathrm{d}\,\chi - \ell(\ell+1)\int_0^{\chi_s}\frac{(\chi_s - \chi)}{\chi} T_\Phi(\eta, k) j_\ell(k\chi)\right]\,, \end{aligned} \tag{6.212}$$

$$\begin{aligned} F_2(\chi, k) \equiv& \left[(T_\Phi(\eta, k) j_\ell(k\chi) - \chi T_\Phi(\eta_1, k) j'(k\chi) - 4(\chi_s - \chi) k T_\Phi(\eta_1, k) j'(k\chi)\right. \\ & \left. -\ell(\ell+1)\frac{(\chi_s - \chi)}{\chi} T_\Phi(\eta, k) j_\ell(k\chi)\right]\,, \end{aligned} \tag{6.213}$$

$$\begin{aligned} F_3(\chi, k) \equiv& \left[-\chi(T_\Phi(\eta, k) j_\ell(k\chi) + 2\int_0^{\chi_s} T_\Phi(\eta, k) j_\ell(k\chi)\mathrm{d}\,\chi - 4\int_0^{\chi_s} (\chi_s - \chi)\right. \\ & \left. \times T_\Phi(\eta, k) k j'_\ell(k\chi)\mathrm{d}\,\chi - \ell(\ell+1)\int_0^{\chi_s}\frac{(\chi_s - \chi)}{\chi} T_\Phi(\eta, k) j_\ell(k\chi)\right]\,, \end{aligned} \tag{6.214}$$

$$\begin{aligned} F_4(\lambda, k) \equiv& \left[(T_\Phi(\eta, k) j_\ell(k\chi) - \chi T_\Phi(\eta_1, k) j'(k\chi) - 4(\chi_s - \chi) k T_\Phi(\eta_1, k) j'(k\chi)\right. \\ & \left. -\ell(\ell+1)\frac{(\chi_s - \chi)}{\chi} T_\Phi(\eta, k) j_\ell(k\chi)\right]. \end{aligned} \tag{6.215}$$

It is more instructive to isolate the dominant terms in contribution from $\delta D_A \times \Phi$.

$$\begin{aligned} \delta D_A \times \Phi \approx& \sum_{\ell=0}^{\infty}\sqrt{4\pi(2\ell_3+1)}\mathcal{K}[\Phi(\mathbf{k}_1)\Phi(\mathbf{k}_2)] i^{\ell_1+\ell_2}\left[W_{\delta D_A\times\Phi_\parallel}(\chi_s, \chi, k_1, k_2)\right. \\ & \left. + W_{\delta D_A\times\Phi_\perp}(\chi_s, \chi, k_1, k_2)\right]\delta_{m,0}\,, \end{aligned} \tag{6.216}$$

where

$$W_{\delta D_A\times\Phi_\parallel}(\chi_s,\chi,k_1,k_2) = (\ell_2(\ell_2+1))(\ell_1(\ell_1+1))^{\frac{3}{2}}\int_0^{\chi_s}\frac{(\chi_s-\chi)}{\chi_s} \times\Bigg[\Bigg[-\frac{1}{\chi}T_\Phi(\eta_2,k_2)j_{\ell_2}(k_2\chi_2)\int_0^{\lambda}\frac{(\chi_s-\chi_1)}{\chi_1^2}T_\Phi(\eta_1,k_1)j_{\ell_1}(k_1\chi_1)\mathrm{d}\,\lambda_1 -4\int_0^{\lambda}\frac{1}{\chi^2}T_\Phi(\lambda_2,k_2)j_{\ell_2}(k_2\chi_2)\mathrm{d}\,\lambda_2\int_0^{\lambda}\frac{1}{\chi}T_\Phi(\lambda_1,k_1)j_{\ell_1}(k_1\chi_1)\mathrm{d}\,\lambda_1\Bigg] \times\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ 0 & 0, & 0\end{pmatrix}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ 1 & -1, & 0\end{pmatrix}\Bigg], \tag{6.217}$$

$$W_{\delta D_A\times\Phi_\perp}(\chi_s,\chi,k_1,k_2) = -\frac{(\ell_1(\ell_1+1))(\ell_2(\ell_2+1))}{\chi^2}T_\Phi(\eta_2,k_2)j_{\ell_2}(k_2\chi_2)\int^{\lambda}\frac{(\chi_s-\chi_1)}{\chi_1} \times T_\Phi(\lambda_1,k_1)j_{\ell_1}(k_1\chi_1)\mathrm{d}\,\lambda_1\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ 0 & 0, & 0\end{pmatrix}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ 0 & 0, & 0\end{pmatrix}\mathrm{d}\,\chi\,. \tag{6.218}$$

The dominant terms are those terms with the highest number of angular derivatives, because they each contribute $1/\chi^n$ and $\left(\sqrt{\ell(\ell+1)}\right)^n$ to the overall integral, where n is the number of screen space derivative.

Multipoles of the observed redshift

At first order, we expand the expression for the redshift in spherical harmonics

$$J_1 = \sum_{\ell=0}^{\infty}\sqrt{4\pi(2\ell+1)}i^\ell\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\Phi(\mathbf{k})W_{J_1}(\chi,k)\delta_{m,0}Y_{\ell m}\,, \tag{6.219}$$

where

$$W_{J_1}(\chi,k) = kT_V(\eta_s,k)j'_\ell(k\chi_s)+T_\Phi(\eta_s,k)j_\ell(k\chi_s)+2\int_0^{\chi_s}T_\Phi(\eta,k)kj'_\ell(k\chi)\mathrm{d}\,\chi \tag{6.220}$$

and we have separated the radial velocity into the primordial part and its transfer function, $V_{\|_s}(\eta,k) = T_V(\eta,k)\Phi(\mathbf{k})$. Because we have a non-vanishing tensor and vector contribution at second order, the total contribution to the peculiar velocity comes from more than two sources: the second order scalars, i.e gravitational potentials at second order , which we define by its contribution with an associated transfer function as $v_\parallel^{(2)}(\eta,\mathbf{k}) = {}^{(\Phi/\Psi)}F_{V_\parallel}(\eta,\mu,\mathbf{k}_1,\mathbf{k}_2,\mathbf{k})\Phi^{(2)}(\mathbf{k})$. The contribution sourced by the vector perturbation is given by $v_{\perp i} = {}^{(\omega)}F_{V_\parallel}(\eta_s,\mu_k\mathbf{k}_1,\mathbf{k}_2,\mathbf{k})\omega_i$. Tensor contributions at second order do not source the peculiar velocity. A third set of contribution comes from the product of gravitational potentials at first order, which we split into the longitudinal and transverse contributions.

$$j_2^{\mathrm{S}} = \sum_{\ell=0}^{\infty} \sqrt{4\pi(2\ell+1)} i^{\ell} \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \mathcal{K}\left[\Phi(\mathbf{k}_1, \mathbf{k}_2)\right] \times W_{j^{(2)}\mathrm{S}}(\eta, \mu_k, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}) \delta_{m0} Y_{\ell m} \,, \tag{6.221}$$

$$j_2^{\mathrm{V}} = \sum_{\ell=0}^{\infty} \sqrt{4\pi(2\ell+1)} \sqrt{\frac{(\ell+1)!}{(\ell-1)!}} i^{\ell+1} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \mathcal{K}\left[\Phi(\mathbf{k}_1, \mathbf{k}_2)\right] \times W_{j^{(2)}\mathrm{V}}(\eta, \mu, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}) \delta_{m,\pm 1} Y_{\ell m} \,, \tag{6.222}$$

$$j_2^{\mathrm{T}} = \sum_{\ell=0}^{\infty} \sqrt{4\pi(2\ell+1)} \sqrt{\frac{(\ell'+2)!}{(\ell'-2)!}} i^{\ell} \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \mathcal{K}\left[\Phi(\mathbf{k}_1, \mathbf{k}_2)\right] \times W_{j^{(2)}\mathrm{T}}(\eta, \mu, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}) \delta_{m,\pm 2} Y_{\ell m} \,, \tag{6.223}$$

$$j_{2\perp}^{\mathcal{O}(1)\times\mathcal{O}(1)_{\perp}} = \sum_{\ell=0}^{\infty} \sqrt{4\pi(2\ell_3+1)} i^{\ell_1+\ell_2} \mathcal{K}[\Phi(\mathbf{k}_1)\Phi(\mathbf{k}_2)] \times W_{j_{2\perp}^{\mathcal{O}(1)\times\mathcal{O}(1)_{\perp}}}(\chi_s, \chi, k_1, k_2) \delta_{m,0} Y_{\ell_3 m} \,, \tag{6.224}$$

$$j_{2\parallel}^{\mathcal{O}(1)\times\mathcal{O}(1)_{\parallel}} = \sum_{\ell=0}^{\infty} \sqrt{4\pi(2\ell_3+1)} i^{\ell_1+\ell_2} \mathcal{K}[\Phi(\mathbf{k}_1)\Phi(\mathbf{k}_2)] \times W_{2\parallel}^{\mathcal{O}(1)\times\mathcal{O}(1)_{\parallel}}(\chi_s, \chi, k_1, k_2) \delta_{m,0} Y_{\ell_3 m} \,, \tag{6.225}$$

where

$$W_{j^{(2)}\mathrm{S}}(\eta, \mu, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}) = \Bigg[{}^{\Phi/\Psi}F_{\mathrm{V}_{\parallel}}(\eta, \mu, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}) j_{\ell}(k\chi_s) + f_{\Phi^{(2)}}(\eta, \mu_{k_1}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}) j_{\ell}(k\chi_s) + \int_0^{\chi_s} \left(f_{\Phi^{(2)}}(\eta, \mu_{k_1}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}) + f_{\Psi^{(2)}}(\eta, \mu_{k_1}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}) \right) j'_{\ell}(k\chi) \mathrm{d}\,\chi \Bigg] \,, \tag{6.226}$$

$$W_{j_2^{(2)}\mathrm{V}}(\eta, \mu, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}) = \mp \left[{}^{(S)}F_{\mathrm{V}_{\parallel}}(\eta_s, \mu_k, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}) \frac{j_{\ell}(k\chi_s)}{\chi_s} + \int_0^{\chi_s} k f_{\omega^{(2)}}(\eta, \mu_k, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}) \left(\frac{j_{\ell}(k\chi_s)}{k\chi_s} \right)' \mathrm{d}\,\chi \right] \,, \tag{6.227}$$

$$W_{j_2^{(2)}\mathrm{T}}(\eta, \mu, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}) = \left[-f_{h^{(2)}}(\eta, \mu_{k_1}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}) \frac{j_{\ell}(k\chi_s)}{(k\chi_s)^2} - \int_0^{\chi_s} k f_{h^{(2)}}(\eta, \mu_{k_1}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}) \left(\frac{j_{\ell}(k\chi)}{(k\chi)^2} \right)' \mathrm{d}\,\chi \right] \,, \tag{6.228}$$

$$
\begin{aligned}
W_{j_{2\perp}^{\mathcal{O}(1)\times\mathcal{O}(1)_\perp}}(\eta,\mu,k_1,k_2) &= \sqrt{\frac{4\pi(2\ell_1+1)(2\ell_2+1)}{(2\ell_3+1)}}\,{}_{+1,-1,0}\mathcal{G}^{000}_{\ell_1\ell_2\ell_3} \\
&\times\sqrt{\left(\frac{(\ell_1+1)!}{(\ell_1-1)!}\frac{(\ell_2+1)!}{(\ell_2-1)!}\right)}\mathcal{W}_{j_{2\perp}^{\mathcal{O}(1)\times\mathcal{O}(1)_\perp}}(\eta,\mu,k_1,k_2)\,,
\end{aligned} \tag{6.229}
$$

$$
\begin{aligned}
\mathcal{W}_{j_{2\perp}^{\mathcal{O}(1)\times\mathcal{O}(1)_\perp}}(\eta,\mu,k_1,k_2) &= \left\{\left[T_{v_\parallel}(\eta_s,k_1)\frac{j_{\ell_1}(k_1\chi_1)}{\chi}T_{v_\parallel}(\eta_s,k_2)\frac{j_{\ell_2}(k_2\chi_2)}{\chi_2}\right]\right. \\
&+8\int_0^{\chi_s}\int_0^{\chi}\left[T_\Phi(\eta_1,k_1)\frac{j_{\ell_1}(k_1\chi_1)}{\chi}\mathrm{d}\,\chi_1\right. \\
&\left.\left.\int_0^{\chi}T_\Phi(\eta,k_2)\frac{j_{\ell_2}(k_2\chi_2)}{\chi_2}\mathrm{d}\,\chi_2\right]\mathrm{d}\,\chi\right\}\,,
\end{aligned} \tag{6.230}
$$

$$
\begin{aligned}
W^{\mathcal{O}(1)\times\mathcal{O}(1)_\parallel}_{j_{2\parallel}}(\eta,\mu,k_1,k_2) &= \sqrt{\frac{4\pi(2\ell_1+1)(2\ell_2+1)}{(2\ell_3+1)}}\,{}_{000}\mathcal{G}^{000}_{\ell_1\ell_2\ell_3} \\
&\times\mathcal{W}^{\mathcal{O}(1)\times\mathcal{O}(1)_\parallel}_{2\parallel}(\eta,\mu,k_1,k_2)\,,
\end{aligned} \tag{6.231}
$$

$$
\begin{aligned}
\mathcal{W}^{\mathcal{O}(1)\times\mathcal{O}(1)_\parallel}_{j_{2\parallel}}(\eta,\mu,\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}) &= \left[\left[k_1T_{v_\parallel}(\eta_s,k_1)k_2T_{v_\parallel}(\eta_s,k_2)j'_{\ell_1}(k_1\chi_s)\,j'_{\ell_2}(k_2\chi_s)\right]\right. \\
&+3T_\Phi(\eta_s,k_1)T_\Phi(\eta_s,k_2)\left[j_{\ell_1}(k_1\chi_s)j_{\ell_2}(k_2\chi_s)\right] \\
&-4T_\Phi(\eta_s,k_1)j_{\ell_1}(k_1\chi_s)T_{v_\parallel}(\eta_s,k_2)j'_{\ell_2}(k_2\chi_s) \\
&+4k_2T_{v_\parallel}(\eta_s,k_2)j'_{\ell_2}(k_2\chi_s)\int_0^{\chi_s}T_\Phi(\eta,k_1)j_{\ell_1}(k_1\chi)\mathrm{d}\,\chi \\
&-12\int_0^{\chi_s}T_{\Phi_s}(\eta_s,k_1)T'_{\Phi'}(\eta,k_2)j_{\ell_1}(k_1\chi_s)j_{\ell_2}(k_2\chi)\mathrm{d}\,\chi \\
&+8\int_0^{\chi_s}T_\Phi(\eta_1,k_1)T'_{\Phi'}(\eta_2,k_2)j_{\ell_1}(k_1\chi)j_{\ell_2}(k_2\chi)\mathrm{d}\,\chi \\
&+8\int_0^{\chi_s}\int_0^{\chi}T_\Phi(\eta,k_2)T''_{\Phi''}(\eta,k_1)j_{\ell_1}(k_1\chi_1)j_{\ell_2}(k_2\chi)\mathrm{d}\,\chi_1\mathrm{d}\,\chi \\
&\left.+8\int_0^{\chi_s}T'_{\Phi'}(\eta,k_2)\int_0^{\chi}T'_{\Phi'}(\eta_1,k_1)j_{\ell_1}(k_1\chi_1)j_{\ell_2}(k_2\chi)\mathrm{d}\,\chi_1\mathrm{d}\,\chi\right]\,.
\end{aligned} \tag{6.232}
$$

It is important to recall that equations (6.222-6.226) are simply equation (6.106) expanded in spherical harmonics using ð formalism.

6.9.2 Total Angular Momentum Approach:Normal modes

This approach was first introduced to the study of cosmic microwave background anisotropy in [239]. Its importance to the analysis of physics of clustering of large scale structures was recently sketched in [240]. It has a lot of advantages since it deals with the total angular momentum $j = \ell + s$, rather than the orbital angular angular momentum alone, this makes a lot difference in cases where non-zero spin object exists. Recall that the orbital angular momentum and the the spin component are not independently observable, only the total angular momentum is.

In the CMB study and also in the weak lensing studies, this approach shows how the $B-$ mode and the $E-$ mode of any polarization tensor couple in a clear and elegant fashion. Also it is much easier to establish correspondence between STF moment formalism [241–243] and the spherical harmonic formalism using this approach [244, 245].

The basic tool of this formalism is the Clebsh-Gordon addition rule for shperical harmonics. It is used to add the spin and orbital angular momentum components, before the recursion relations for the spherical Bessel functions is used to put the resulting expressions in a more compact form. In most cases, the spin component is associated with the spatial derivative operator on the spacetime and it is extracted by first expanding the fields (e.g gravitational potentials) in Fourier space as

$$\mathcal{X}(x^i)_{\ell m} = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \mathcal{X}_{\ell m}(k,\eta) e^{i k_i x^i} . \tag{6.233}$$

The resulting Fourier vector k^i associated with a spatial derivative operator is then decomposed in a helicity bases with its $z-$ direction aligned to the direction of the line of sight n^i,

$$k^i = \mu k n^i + k\sqrt{\frac{1-\mu^2}{2}} \left(e^{-i\zeta} m_+^i + e^{i\zeta} m_-^i\right) . \tag{6.234}$$

Using table 6.1, the result of the Fourier space decomposition may now be converted into the spin weighted spherical harmonics

$$X(x^i, \mathbf{n}) = \sum_{\ell=s} \sum_{m=-\ell}^{\ell} {}_{\pm s}\mathcal{X}(x^i)_{\ell m \pm s} Y_{\ell m}(\mathbf{n}) , \tag{6.235}$$

The equation (6.235) may be inverted by multiplying both sides by the conjugate of ${}_{\pm s}Y_{\ell m}$ and integrating over all sky

$${}_{\pm s}\mathcal{X}(x^i)_{\ell m} = \int X(x^i, \mathbf{n})_{\pm s} Y^*_{\ell m} \mathrm{d}\,\Omega . \tag{6.236}$$

Putting everything together we recover the general normal mode function introduced in [239] for the spherical harmonic decomposition of CMB polarization and anisotropy.

$$\mathcal{X}(x^i, n^j) = \sum_{\ell m} \int \frac{\mathrm{d}^3 k}{(2\pi)^3} {}_{\pm s}\mathcal{X}_{\ell m}(k,\eta) G^{\mathcal{X}}_{\ell m}(k, x^i, n^j) \tag{6.237}$$

where $\mathcal{X}$ stand for any field to be expanded in spherical harmonics and

$$G^{\mathcal{X}}_{\ell m}(k, x^i, n^j) \;=\; \frac{1}{N_\ell} e^{i k_i x^i} {}_{\pm s}Y^{\ell m}(\mathbf{n})\,, \tag{6.238}$$

with 's' indicating the spin of the fields. The normalization $N_\ell \equiv i^\ell \sqrt{\frac{(2\ell+1)}{4\pi}}$ is chosen to ensure agreement with the standard Lengendre polynomials or recursion relations for the spherical Bessel function. It also helps to ensure agreement with the multipole formalism [246]. The correspondence with the multipole formalism could not easily be established with the ð formalism.

We will now give a general prescription on how we move from from angular derivatives on the screen space to the spin weighted harmonics. The first thing is to re-express the angular derivatives that appear in the expression for the area distance into their 3D equivalent after multiplying by the appropriate helicity bases vectors. Then the field say a typical scalar field $X = X(\eta, x^i)$ is then expanded in Fourier space and the vector k^i as the case may be is decomposed in helicity bases. Once in helicity bases, it is easier to convert the cosine of the angle between k^i and n^iinto spherical harmonics using table 6.1. The common derivatives of $X(\eta, x^i)$ that appear in the expression for the area distance are handled as follows:

$$\begin{aligned} m^i_\pm m^j_\pm \nabla_{\perp\langle i}\nabla_{\perp j\rangle} X &= m^i_\pm m^j_\pm \mathrm{D}_i \mathrm{D}_j X = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} i^2 m^i_\pm m^j_\pm k_i k_j X(\mathbf{k}) \\ &= -\int \frac{\mathrm{d}^3 k}{(2\pi)^3} (k_\mp)^2 X(\mathbf{k}) = -\frac{1}{2}\int \frac{\mathrm{d}^3 k}{(2\pi)^3} (1-\mu^2) k^2 X(\mathbf{k}) e^{\pm 2i\zeta}\,, \end{aligned} \tag{6.239}$$

$$\begin{aligned} m^i_- m^j_- \nabla_{\perp\langle i}\nabla_{\perp j\rangle} X &= m^i_- m^j_- \mathrm{D}_i \mathrm{D}_j X = i^2 m^i_- m^j_- \int \frac{\mathrm{d}^3 k}{(2\pi)^3} k_i k_j X(\mathbf{k}) \\ &= -\int \frac{\mathrm{d}^3 k}{(2\pi)^3} k_-^2 X(\mathbf{k}) = -\frac{1}{2}\int \frac{\mathrm{d}^3 k}{(2\pi)^3} (1-\mu^2) k^2 X(\mathbf{k}) e^{-2i\zeta}\,, \end{aligned} \tag{6.240}$$

$$
\begin{aligned}
m^i_\pm \nabla_i X &= i m^i_\pm \mathrm{D}_i X = i m^i_\pm \int \frac{\mathrm{d}^3k}{(2\pi)^3} k_i X(\mathbf{k}) \qquad (6.241)\\
&= i \int \frac{\mathrm{d}^3k}{(2\pi)^3} k_\mp X(\mathbf{k}) = i \int \frac{\mathrm{d}^3k}{(2\pi)^3} k \sqrt{\frac{(1-\mu^2)}{2}} e^{\pm i\zeta} X(\mathbf{k}) \,,\\
\nabla^2_\perp X &= \mathrm{D}_b \mathrm{D}^b X - n^b n^c \mathrm{D}_b \mathrm{D}_c X = \int \frac{\mathrm{d}^3k}{(2\pi)^3} \left(-k_i k^i + \mu^2 k^2\right) X(\mathbf{k}) \qquad (6.242)\\
&= -\frac{1}{2} \int \frac{\mathrm{d}^3k}{(2\pi)^3} k^2 (3\mu^2 - 1) X(\mathbf{k}) \,,
\end{aligned}
$$

$$
\begin{aligned}
m^i_\pm \nabla_{\perp i} \nabla^2_\perp X &= i m^i_\pm \mathrm{D}_i \left(\mathrm{D}_b \mathrm{D}^b X - n^b n^c \mathrm{D}_b \mathrm{D}_c X\right) \qquad (6.243)\\
&= i \int \frac{\mathrm{d}^3k}{(2\pi)^3} k^3 \frac{1}{2} \sqrt{\frac{(1-\mu^2)}{2}} \left((1 \mp 5\mu)\right) e^{\pm i\zeta} X(\mathbf{k}) \,,\\
n^i \mathrm{D}_i X &= i \int \frac{\mathrm{d}^3k}{(2\pi)^3} n^i k_i X(\mathbf{k}) = i \int \frac{\mathrm{d}^3k}{(2\pi)^3} \mu k X(\mathbf{k}). \qquad (6.244)
\end{aligned}
$$

In the subsequent sections, we are going to make use these equations without specifying explicitly the steps involved.

First order Perturbations

Let us now apply this technique to equation (6.63). We will go through the calculation in steps at first order to familiarize the reader with the formalism and at second order we present the result with less details. First the expansion of the first area distance in spherical harmonics is given by

$$
\frac{\delta D_A}{D_A(s)} = \frac{\delta D_A}{D_A(s)}_{\ell m} Y_{\ell m} \,, \qquad (6.245)
$$

where the multipole moment may be given by

$$
\frac{\delta D_A}{D_A(s)}_{\ell m} = \int \mathrm{d}\,\Omega \frac{\delta D_A}{D_A(s)} Y^*_{\ell m} = \int \mathrm{d}\,\Omega \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \frac{\delta D_A}{D_A(s)}(\eta, \mathbf{k}) Y^*_{\ell m} \qquad (6.246)
$$

Substituting equation (6.63) for $\delta D_A/D_A(s)$ we find

$$\begin{aligned}\frac{\delta D_A}{D_A(s)}_{\ell m} &= \int \mathrm{d}\,\Omega \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \left[-\Phi_s(\eta_s,\mathbf{k}) - \Phi_o(\eta_o,\mathbf{k}) + \frac{2}{\chi_s}\int_0^{\chi_s} \Phi(\eta,k)\mathrm{d}\,\chi \right. \\ & -2\int_0^{\chi_s} \frac{(\chi_s+\chi)}{\chi_s} i\mu k \Phi(\eta,k)\mathrm{d}\,\chi \\ & \left. +\int_0^{\chi_s} \frac{(\chi_s+\chi)\chi}{\chi_s} k^2(3\mu^2-1)\Phi(\eta,k) \right] Y^*_{\ell m}\,.\end{aligned}$$

We may now replace $e^{i\mathbf{k}\cdot\mathbf{x}}$ with

$$e^{i\mathbf{k}\cdot\mathbf{x}} = \sum_{\ell m} \sqrt{4\pi(2\ell+1)} i^\ell j_\ell(k(\chi) Y_{\ell 0}(\mathbf{n})\,. \tag{6.247}$$

Next we convert the cosine of the angle between k^i and n^i, μ into spherical harmonics using Table 6.1

$$\begin{aligned}\frac{\delta D_A}{D_A(s)}_{\ell m} &= \int \mathrm{d}\,\Omega \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \sum_{\ell m} \sqrt{4\pi(2\ell+1)} i^\ell j_\ell \Big[-\Phi(\eta_s,k)Y_{\ell 0}(\mathbf{n}) - \Phi_(\eta_o,k)Y_{\ell 0}(\mathbf{n}) \\ & +\frac{2}{\chi_s}\int_0^{\chi_s} \Phi(\eta, kY_{\ell 0}(\mathbf{n})\mathrm{d}\,\chi + 2\int_0^{\chi_s} \frac{(\chi_s+\chi)}{\chi_s} k\Phi(\eta,k)\frac{1}{N_1} Y_{10}Y_{\ell 0}(\mathbf{n})\mathrm{d}\,\chi \\ & -\int_0^{\chi_s} \frac{(\chi_s+\chi)\chi}{\chi_s} k^2\Phi(\eta,k)\frac{2}{N_2}Y_{20}Y_{\ell 0}(\mathbf{n}) \Big] Y^*_{\ell m}\,.\end{aligned} \tag{6.248}$$

We now use the Clebsh-Gordon addition rule (see appendix B for details) to sum up the spherical harmonics, for example

$$\begin{aligned}-\frac{2}{N_2}Y_{20}(\mathbf{k})Y_{\ell m}(\mathbf{n})j_\ell &= \left[\sqrt{\frac{((\ell-1)^2-m^2)(\ell^2-m^2)}{(2\ell-1)^2(2\ell-3)(2\ell+1)}} j_{(\ell-2)} \right. \\ & +\frac{2\sqrt{(\ell(\ell+1)-3m^2)^2}}{(2\ell-1)(2\ell+3)} j_\ell \\ & \left. +\sqrt{\frac{((\ell+1)^2-m^2)((\ell+2)^2-m^2)}{(2\ell+1)(2\ell+3)^2(2\ell+5)}} j_{(\ell+2)} \right] Y_{\ell m}(\mathbf{n})\,.\end{aligned} \tag{6.249}$$

For scalars with $m = 0$, equation (6.249) may be simplified further

$$\begin{aligned}\sqrt{4\pi(2\ell+1)}i^\ell\frac{2}{N_2}Y_{20}(\mathbf{k})Y_{\ell 0}(\mathbf{n})j_\ell &= \left[\frac{\ell(\ell-1)}{(2\ell-1)(2\ell+1)}j_{(\ell-2)}(x)\right. \qquad (6.250)\\ &\quad -\frac{2\ell(\ell+1)}{(2\ell-1)(2\ell+3)}j_\ell(x)\\ &\quad \left.+\frac{(\ell+1)(\ell+2)}{(2\ell+1)(2\ell+3)}j_{(\ell+2)m}(x)\right]Y_{\ell 0}(\mathbf{n})\\ &= \sqrt{4\pi(2\ell+1)}\left[3j''_\ell(x)+j_\ell(x)\right]Y_{\ell 0}(\mathbf{n})\,. \qquad (6.251)\end{aligned}$$

In the second line we have used one of the recursion relations given in appendix B to finally reduced to expression to a more compact form. Putting equation (6.250) back to equation (6.248), the area distance in terms of the spherical Bessel function at first order becomes

$$\begin{aligned}\frac{\delta D_A}{D_A(s)}_{\ell m} &= \sqrt{4\pi(2\ell+1)}\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\Phi(\mathbf{k})i^\ell\left[-T_\Phi(\eta_s,k)j_\ell(k\chi_s)\right. \qquad (6.252)\\ &\quad +\frac{2}{\chi_s}\int_0^{\chi_s}T_\Phi(\eta,k)j_\ell(k\chi)\mathrm{d}\,\chi+2\int_0^{\chi_s}\frac{(\chi_s+\chi)}{\chi_s}kT_\Phi(\eta,k)j'_\ell(k\chi)\mathrm{d}\,\chi\\ &\quad \left.-\int_0^{\chi_s}\frac{(\chi_s+\chi)\chi}{\chi_s}k^2T_\Phi(\eta,k)\left(3j''_\ell(k\chi)+j_\ell(k\chi)\right)\right]\delta_{m,0}\,.\end{aligned}$$

The derivative on $j(k\chi)$ is with respect to the argument and we have set the multipoles at the observer position to zero following [203].

For future convenience, the area distance at first order may be written in a more compact form

$$\frac{\delta D_A}{D_A(s)} = \sqrt{4\pi(2\ell+1)}\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\Phi(\mathbf{k})i^\ell W_{\delta D_A/D_A}(\chi_s,\chi,k)\delta_{m,0}Y_{\ell m} \qquad (6.253)$$

where

$$\begin{aligned}W_{\delta D_A/D_A}(\chi_s,\chi,k) &= \left[-T_\Phi(\eta_s,k)j_\ell(k\chi_s)+\frac{2}{\chi_s}\int_0^{\chi_s}T_\Phi(\eta,k)j_\ell(k\chi)\mathrm{d}\,\chi\right. \qquad (6.254)\\ &\quad +2\int_0^{\chi_s}\frac{(\chi_s+\chi)}{\chi_s}kT_\Phi(\eta,k)j'_\ell(k\chi)\mathrm{d}\,\chi\\ &\quad \left.+\int_0^{\chi_s}\frac{(\chi_s+\chi)\chi}{\chi_s}k^2T_\Phi(\eta,k)\left(3j''_\ell(k\chi)+j_\ell(k\chi)\right)\right]\end{aligned}$$

Second Order Perturbations

We split the the area distance at second order into components just as in the case of the ð formalism

$$\frac{\delta^2 D_A}{D_A} = \frac{\delta^2 D_A}{D_A}^{\mathrm{S}} + \frac{\delta^2 D_A}{D_A}^{\mathrm{V}} + \frac{\delta^2 D_A}{D_A}^{\mathrm{T}} + \frac{\delta^2 D_A}{D_A}^{\Phi\times\Phi} + \frac{\delta^2 D_A}{D_A}^{\nabla_{\perp i}\Phi\nabla^i_\perp\Phi} + \frac{\delta^2 D_A}{D_A}^{\mathrm{Shear}} + \frac{\delta^2 D_A}{D_A}^{\delta D_A\times\Phi}. \tag{6.255}$$

We follow the same procedure as in the first order case with details on each step omitted. First we give the expressions for pure second order scalars, vectors and tensors

$$\frac{\delta^2 D_A}{D_A}^{\mathrm{S}} = \sum_{\ell=0}^{\infty} i^\ell \sqrt{4\pi(2\ell+1)} \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \mathcal{K}[\Phi(\mathbf{k}_1)\Phi(\mathbf{k}_2)] W_{\Delta^{(2)}\mathrm{S}}(\chi,\mathbf{k},\mathbf{k}_1,\mathbf{k}_2)\delta_{m,0} \tag{6.256}$$

$$\frac{\delta^2 D_A}{D_A}^{\mathrm{V}} = \mp\sum_{\ell=0}^{\infty} i^{\ell+1} \sqrt{4\pi(2\ell+1)} \sqrt{\frac{\ell'(\ell+1)}{2}} \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \mathcal{K}[\Phi(\mathbf{k}_1)\Phi(\mathbf{k}_2)] \times W_{\Delta^{(2)}\mathrm{V}}(\chi,\mathbf{k},\mathbf{k}_1,\mathbf{k}_2)\delta_{m,\pm1} \tag{6.257}$$

$$\frac{\delta^2 D_A}{D_A}^{\mathrm{T}} = \sum_{\ell=0}^{\infty} i^{\ell} \sqrt{4\pi(2\ell+1)} \sqrt{\frac{(\ell'+2)!}{(\ell'-2)!}} \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \mathcal{K}[\Phi(\mathbf{k}_1)\Phi(\mathbf{k}_2)] \times W_{\Delta^{(2)}\mathrm{T}}(\chi,\mathbf{k},\mathbf{k}_1,\mathbf{k}_2)\delta_{m,\pm2} \tag{6.258}$$

where

$$
\begin{aligned}
W_{\Delta^{(2)}\mathrm{S}}(\chi,k) &= \Big[-f_{(\Phi+\Psi)}(\eta,\mu_{k_1},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k})j_\ell(k\chi_s) &(6.259)\\
&+\frac{1}{\chi_s}\int_0^{\chi_s} f_{(\Phi+\Psi)}(\eta,\mu_{k_1},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k})j_\ell(k\chi)\mathrm{d}\,\chi\\
&-\int_0^{\chi_s}\frac{(\chi_s+\chi)}{\chi_s}kf_{(\Phi+\Psi)}(\eta,\mu_{k_1},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k})j'_\ell(k\chi)\mathrm{d}\,\chi\\
&-\frac{1}{2}\int_0^{\chi_s}\frac{(\chi_s+\chi)\chi}{\chi_s}k^2 f_{(\Phi+\Psi)}(\eta,\mu_{k_1},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k})\left(3j''_\ell(k\chi)+j_\ell(k\chi)\right)\mathrm{d}\,\chi\Big]\\
W_{\Delta^{(2)}\mathrm{V}}(\chi,k) &= \left[\frac{1}{2}\int_0^{\chi_s} kf_{\omega^{(2)}}(\eta,\mu_{k_1},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k})\left(\frac{j_\ell(k\chi)}{(k\chi)}\right)'\mathrm{d}\,\chi\right. &(6.260)\\
&-2\int_0^{\chi_s}\frac{(\chi_s+\chi)}{\chi_s}kf_{\omega^{(2)}}(\eta,\mu_{k_1},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k})\left(\frac{j_\ell(k\chi)}{(k\chi)}\right)'\mathrm{d}\,\chi\\
&+\int_0^{\chi_s}\frac{(\chi_s+\chi)}{\chi_s\chi}f_{\omega^{(2)}}(\eta,\mu_{k_1},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k})\frac{j_\ell(k\chi)}{(k\chi)}\\
&\left.-\frac{1}{2}\int_0^{\chi_s}\frac{(\chi_s+\chi)\chi}{\chi_s}f_{\omega^{(2)}}(\eta,\mu_{k_1},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k})k^2\left(3\left(\frac{j_\ell(k\chi)}{(k\chi)}\right)''+\frac{j_\ell(k\chi)}{(k\chi)}\right)\mathrm{d}\,\chi\right]\\
W_{\Delta^{(2)}\mathrm{T}}(\chi,k) &= \left[\frac{1}{\chi_s}\int_0^{\chi_s} f_{h^{(2)}}(\eta,\mu_{k_1},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k})\frac{j_\ell(k\chi)}{(k\chi)^2}\mathrm{d}\,\chi\right. &(6.261)\\
&-\int_0^{\chi_s} kf_{h^{(2)}}(\eta,\mu_{k_1},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k})\left(\frac{j_\ell(k\chi)}{(k\chi)^2}\right)'\mathrm{d}\,\chi\\
&+2\int_0^{\chi_s}\frac{(\chi_s+\chi)}{\chi_s}kf_{h^{(2)}}(\eta,\mu_{k_1},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k})\left(\frac{j_\ell(k\chi)}{(k\chi)^2}\right)'\mathrm{d}\,\chi\\
&-3\int_0^{\chi_s}\frac{(\chi_s+\chi)}{\chi_s\chi}f_{h^{(2)}}(\eta,\mu_{k_1},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k})\frac{j_\ell(k\chi)}{(k\chi)^2}\\
&\left.+\frac{1}{2}\int_0^{\chi_s}\frac{(\chi_s+\chi)\chi}{\chi_s}k^2 f_{h^{(2)}}(\eta,\mu_{k_1},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k})\left(3\left(\frac{j_\ell(k\chi)}{(k\chi)^2}\right)''+\frac{j_\ell(k\chi)}{(k\chi)^2}\right)h_{\|\ell m}\mathrm{d}\,\chi\right]
\end{aligned}
$$

with

$$f_{(\Phi+\Psi)}(\eta,\mu_{k_1},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}) = f_{\Phi^{(2)}}(\eta,\mu_{k_1},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}) + f_{\Psi^{(2)}}(\eta,\mu_{k_1},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}) \tag{6.262}$$

For terms quadratic in the first order gravitational potentials, we Use the Clebash-Gordon summation technique just as in the case of the ð formalism to add the two spherical harmonics. We provide only the necessary steps involved in the derivation without commenting much on each case

$$
\begin{aligned}
[\Phi\Phi]_{\ell m} &= 4\pi\sqrt{(2\ell_1+1)(2\ell_2+1)}\int\frac{\mathrm{d}^3\mathbf{k}_1}{(2\pi)^3}\int\frac{\mathrm{d}^3\mathbf{k}_2}{(2\pi)^3}\Phi(\mathbf{k}_1)\Phi(\mathbf{k}_2)i^{\ell_1+\ell_2} && (6.263)\\
&\quad T_\Phi(\eta_1,k_1)T_\Phi(\eta_2,k_2)\times\left[j_{\ell_1}(k_1\chi_1)j_{\ell_2}(k_2\chi_2)\right]Y_{\ell_1,0}Y_{\ell_2,0}\\
&= \sqrt{4\pi(2\ell_3+1)}(2\ell_1+1)(2\ell_2+1)\int\frac{\mathrm{d}^3\mathbf{k}_1}{(2\pi)^3}\int\frac{\mathrm{d}^3\mathbf{k}_2}{(2\pi)^3}\Phi(\mathbf{k}_1)\Phi(\mathbf{k}_2)i^{\ell_1+\ell_2} && (6.264)\\
&\quad\times T_\Phi(\eta_1,k_1)T_\Phi(\eta_2,k_2)\left[j_{\ell_1}(k_1\chi_1)j_{\ell_2}(k_2\chi_2)\right]\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ 0 & 0, & 0\end{pmatrix}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ 0 & 0, & 0\end{pmatrix}\\
&= 4\pi\sqrt{(2\ell_1+1)(2\ell_2+1)}_{000}\mathcal{G}^{000}_{\ell_1\ell_2\ell_3}\int\frac{\mathrm{d}^3\mathbf{k}_1}{(2\pi)^3}\int\frac{\mathrm{d}^3\mathbf{k}_2}{(2\pi)^3}\Phi(\mathbf{k}_1)\Phi(\mathbf{k}_2)i^{\ell_1+\ell_2} && (6.265)\\
&\quad\times T_\Phi(\eta_1,k_1)T_\Phi(\eta_2,k_2)\left[j_{\ell_1}(k_1\chi_1)j_{\ell_2}(k_2\chi_2)\right].
\end{aligned}
$$

We now present the final results for terms quadratic in Φ in a more compact form

$$
\begin{aligned}
\frac{\delta^2 D_A}{D_A}^{\Phi\times\Phi} &= \sum_{\ell=0}^{\infty} i^{\ell_1+\ell_2}\sqrt{4\pi(2\ell_3+1)}\mathcal{K}[\Phi(\mathbf{k}_1)\Phi(\mathbf{k}_2)] && (6.266)\\
&\quad\times W_{\Delta^{\Phi\times\Phi}}(\eta,\chi,k_1,k_2))\delta_{m,0}Y_{\ell_3 m}\,,\\
\frac{\delta^2 D_A}{D_A}^{\nabla_{\perp i}\Phi\nabla^i_\perp\Phi} &= \sum_{\ell=0}^{\infty} i^{\ell_1+\ell_2}\sqrt{4\pi(2\ell_3+1)}\mathcal{K}[\Phi(\mathbf{k}_1)\Phi(\mathbf{k}_2)] && (6.267)\\
&\quad\times W^{\nabla_{\perp i}\Phi\nabla^i_\perp\Phi}_{\Delta_{D_A}}(\eta,k_1,k_2)\delta_{m,0}Y_{\ell_3 m}\,,\\
\frac{\delta^2 D_A}{D_A}^{\text{Shear}} &= \sum_{\ell=0}^{\infty} i^{\ell_1+\ell_2}\sqrt{4\pi(2\ell_3+1)}\mathcal{K}[\Phi(\mathbf{k}_1)\Phi(\mathbf{k}_2)] && (6.268)\\
&\quad\times W_{\Delta_{D_A}}{}^{\text{Shear}}(\eta,k_1,k_2)\delta_{m,0}Y_{\ell_3 m}\,,\\
\delta D_A\times\Phi_\parallel &= \sum_{\ell=0}^{\infty} i^{\ell_1+\ell_2}\sqrt{4\pi(2\ell_3+1)}\mathcal{K}[\Phi(\mathbf{k}_1)\Phi(\mathbf{k}_2)] && (6.269)\\
&\quad\times W_{\Delta_{\delta D_A\times\Phi_\parallel}}(\eta,k_1,k_2)\delta_{m,0}Y_{\ell_3 m}\,,\\
\delta D_A\times\Phi_\perp &= \sum_{\ell=0}^{\infty} i^{\ell_1+\ell_2}\sqrt{4\pi(2\ell_3+1)}\mathcal{K}[\Phi(\mathbf{k}_1)\Phi(\mathbf{k}_2)] && (6.270)\\
&\quad\times W_{\delta D_A\times\Phi_\perp}(\eta,k_1,k_2)\delta_{m,0}Y_{\ell_3 m}\,,
\end{aligned}
$$

where

$$W_{\Delta^{\Phi\times\Phi}}(\eta,\chi,k_1,k_2) = \sqrt{\frac{4\pi(2\ell_1+1)(2\ell_2+1)}{(2\ell_3+1)}}{}_{(0,0,0)}\mathcal{G}_{\ell_1\ell_2\ell_3}{}^{000} \mathcal{W}_{\Delta^{\Phi\times\Phi}}(\eta,\chi,k_1,k_2))\,, \tag{6.271}$$

$$W^{\nabla_{\perp i}\Phi\nabla^i_\perp\Phi}_{\Delta_{D_A}}(\eta,k_1,k_2,k) = \sqrt{\frac{4\pi(2\ell_1+1)(2\ell_2+1)}{(2\ell_3+1)}}{}_{(1,-1,0)}\mathcal{G}_{\ell_1\ell_2\ell_3}{}^{000} \times\sqrt{\frac{\ell_1(\ell_1+1)}{2}\frac{\ell_2(\ell_2+1)}{2}}\mathcal{W}^{\nabla_{\perp i}\Phi\nabla^i_\perp\Phi}_{\Delta_{D_A}}(\eta,k_1,k_2,k)\,, \tag{6.272}$$

$$W_{\Delta_{D_A}}{}^{\mathrm{Shear}}(\eta,k_1,k_2,k) = \sqrt{\frac{4\pi(2\ell_1+1)(2\ell_2+1)}{(2\ell_3+1)}}{}_{(2,-2,0)}\mathcal{G}_{\ell_1\ell_2\ell_3}{}^{000} \times\left[\sqrt{\frac{(\ell_1+2)!}{(\ell_1-2)!}\frac{(\ell_2+2)!}{(\ell_2-2)!}}\mathcal{W}_{\Delta_{D_A}}{}^{\mathrm{Shear}}(\eta,k_1,k_2,k)\right], \tag{6.273}$$

$$W_{\Delta_{\delta D_A\times\Phi_\parallel}}(\eta,k_1,k_2) = \sqrt{\frac{4\pi(2\ell_1+1)(2\ell_2+1)}{(2\ell_3+1)}}{}_{(0,0,0)}\mathcal{G}_{\ell_1\ell_2\ell_3}{}^{000} \times\left[\mathcal{W}_{\Delta_{\delta D_A\times\Phi_\parallel}}(\eta,k_1,k_2,k)\right], \tag{6.274}$$

$$W_{\delta D_A\times\Phi_\perp}(\eta,k_1,k_2,k) = \sqrt{\frac{4\pi(2\ell_1+1)(2\ell_2+1)}{(2\ell_3+1)}}\sqrt{\frac{\ell_1(\ell_1+1)}{2}\frac{\ell_2(\ell_2+1)}{2}} \times i^{\ell_1+\ell_2}{}_{(1,-1,0)}\mathcal{G}_{\ell_1\ell_2\ell_3}{}^{000}\left[\mathcal{W}_{\delta D_A\times\Phi_\perp}(\eta,k_1,k_2,k)\right], \tag{6.275}$$

with

$$\mathcal{W}_{\Delta^{\Phi\times\Phi}}(\eta,\chi,k_1,k_2)) = \Bigg[-2\int_0^{\chi_s}\frac{(\chi_s-\chi)\chi}{\chi_s}\Bigg(\left(T_\Phi(\eta_1,k_1)j'_{\ell_1}(k_1\chi)\right) \left(T_\Phi(\eta_1,k_2)j'_{\ell_2}(k_2\chi)\right)+2T_\Phi(\eta_1,k_1)j''_{\ell_1}(k_1\chi)T_\Phi(\eta_2,k_2)j_{\ell_2}(k_2\chi_2) -4T'_\Phi(\eta_1,k_1)j'_{\ell_1}(k_1\chi)T_\Phi(\eta_2,k_2)j_{\ell_2}(k_2\chi)\Bigg)\mathrm{d}\,\chi +8\int_0^{\chi_s}\frac{(\chi_s-\chi)}{\chi_s}T'_\Phi(\eta_1,k_1)T_\Phi(\eta_2,k_2)j_{\ell_1}(k_1\chi)j_{\ell_2}(k_2\chi)\mathrm{d}\,\chi -\frac{8}{\chi_s}\int_0^{\chi_s}T_\Phi(\eta_1,k_1)T_\Phi(\eta_2,k_2)j_{\ell_1}(k_1\chi)j_{\ell_2}(k_2\chi)\mathrm{d}\,\chi\Bigg], \tag{6.276}$$

$$\mathcal{W}_{\Delta_{D_A}}^{\nabla_{\perp i}\Phi\nabla^i_{\perp}\Phi}(\eta,k_1,k_2,k) = 8\left[-\int_0^{\chi_s}\frac{(\chi_s+\chi)\chi}{\chi_s}\left(T'_\Phi(\eta_1,k_1)k_1\frac{j_{\ell_1}(k_1\chi_1)}{(k_1\chi)}\int_0^{\lambda}T_\Phi(\eta,k_2)k_2\frac{j_{\ell_2}(k_2\chi_2)}{(k_2\chi_2)}\mathrm{d}\,\lambda_2\right)\mathrm{d}\,\lambda+\int_0^{\chi_s}\frac{(\chi_s+\chi)}{\chi_s}\left(T_\Phi(\eta_1,k_1)k_1\frac{j_{\ell_1}(k_1\chi_1)}{(k_1\chi)}\int_0^{\lambda}T_\Phi(\eta,k_2)k_2\frac{j_{\ell_2}(k_2\chi_2)}{(k_2\chi_2)}\mathrm{d}\,\lambda_2\right)\mathrm{d}\,\lambda\right], \tag{6.277}$$

$$\mathcal{W}_{\Delta_{D_A}}{}^{\mathrm{Shear}}(\eta,k_1,k_2,k) = 4\int_0^{\chi_s}\frac{(\chi_s+\chi)\chi}{\chi_s}\left\{a(\chi)\int_0^{\chi}T_\Phi(\eta_1,k_1)k_1^2\frac{j_{\ell_1}}{(k_1\chi_1)^2}\mathrm{d}\,\chi_1\int_0^{\chi}T_\Phi(\eta_2,k_2)k_2^2\frac{j_{\ell_2}}{(k_2\chi_2)^2}\mathrm{d}\,\chi_2\right\}, \tag{6.278}$$

$$\mathcal{W}_{\Delta_{\delta D_A\times\Phi_\parallel}}(\eta,k_1,k_2,k) = 4M'_1(\lambda,k_1)G_2(\lambda_2,k_2)+8M'_2(\lambda,k_1)G_1(\lambda_2,k_2) -4M_2(\lambda,k_1)G'_1(\lambda_2,k_2)-2M_1(\lambda,k_1)\left(G_3(\lambda_2,k_2)+G_6(\lambda_2,k_2)\right), \tag{6.279}$$

$$\mathcal{W}_{\delta D_A\times\Phi_\perp}(\eta,k_1,k_2,k) = 8M_4(\lambda,k)\int_0^{\lambda}G_4(\lambda_1,k_1)\mathrm{d}\,\lambda_1+4G_4(\lambda,k_1)M_3(\lambda,k_2). \tag{6.280}$$

Just as in the case of ð formalism, we have made the following definitions for clarity sake

$$
\begin{aligned}
G_6(\lambda,k) &= k^2 T_\Phi(\eta,k)\left[3j''_\ell(k\chi)+j_\ell(k\chi)\right], && (6.281)\\
M_1(\lambda,k) &= \Bigg[-T_\Phi(\eta,k)j_\ell(k\chi)+2\int_0^{\chi_s}T_\Phi(\eta,k)j_\ell(k\chi)\mathrm{d}\,\chi && (6.282)\\
&\quad -2\int_{\chi_o}^{\chi_s}(\chi_s+\chi)kT_\Phi(\eta,k)j'_\ell(k\chi)\mathrm{d}\,\chi\\
&\quad -\int_0^{\chi_s}(\chi_s+\chi)\chi k^2T_\Phi(\eta,k)\left(3j''_\ell(k\chi)+j_\ell(k\chi)\right)\Bigg],\\
M_2(\lambda,k) &= \left[T_\Phi(\eta,k)j_\ell(k\chi)-\chi\left(T_\Phi(\eta_1,k)j'(k\chi)\right)\right. && (6.283)\\
&\quad -2(\chi_s+\chi)kT_\Phi(\eta,k)j'_\ell(k\chi)\\
&\quad \left.-(\chi_s+\chi)\chi k^2T_\Phi(\eta,k)\left(3j''_\ell(k\chi)+j_\ell(k\chi)\right)\right],\\
M_3(\lambda,k) &= \Bigg[-\chi T_\Phi(\eta,k_2)k_2\frac{j_{\ell_2}(k_2\chi_2)}{(k_2\chi_2)} && (6.284)\\
&\quad +2\int_0^{\chi_s}T_\Phi(\eta,k_2)k_2\frac{j_{\ell_2}(k_2\chi_2)}{(k_2\chi_2)}\mathrm{d}\,\lambda\\
&\quad -2\int_0^{\chi_s}(\chi_s+\chi)T_\Phi(\eta,k_2)k_2^2\left(\frac{j_{\ell_2}(k_2\chi_2)}{(k_2\chi_2)}\right)'\mathrm{d}\,\chi\\
&\quad -\int_0^{\chi_s}(\chi_s+\chi)\chi T_\Phi(\eta,k)k_2^3\left(5\left(\frac{j_\ell(k_1\chi_2)}{(k_1\chi_2)}\right)''+\frac{j_\ell(k_2\chi_1)}{(k_2\chi_2)}\right)\mathrm{d}\,\lambda\Bigg],\\
M_4(\lambda,k) &= \Bigg[T_\Phi(\eta,k_2)k_2\frac{j_{\ell_2}(k_2\chi_2)}{(k_2\chi_2)} && (6.285)\\
&\quad -\chi\left(T_\Phi(\eta,k_2)k_2\left(\frac{j_{\ell_2}(k_2\chi_2)}{(k_2\chi_2)}\right)'\right)\\
&\quad -2(\chi_s+\chi)T_\Phi(\eta,k_2)k_2^2\left(\frac{j_{\ell_2}(k_2\chi_2)}{(k_2\chi_2)}\right)'\\
&\quad -(\chi_s+\chi)\chi T_\Phi(\eta,k)k_2^3\left(5\left(\frac{j_\ell(k_1\chi_2)}{(k_1\chi_2)}\right)''+\frac{j_\ell(k_2\chi_1)}{(k_2\chi_2)}\right)\Bigg].
\end{aligned}
$$

Multipoles of the redshift

The correction to the background redshift at first order is given by

$$
j_{1\ell} = \sqrt{4\pi(2\ell+1)}i^\ell\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\Phi(\mathbf{k})W_{J_1}, \qquad (6.286)
$$

where we have used $V_{\|s}(\eta,k) = T_V(\eta,k)\Phi(\mathbf{k})$ and

$$W_{J_1}(\chi,k) = \Bigg[kT_V(\eta_s,k)j'_\ell(k\chi_s) + T_\Phi(\eta_s,k)j_\ell(k\chi_s) + 2\int_0^{\chi_s} T_\Phi(\eta,k)kj'_\ell(k\chi)\mathrm{d}\,\chi\,.\Bigg] \tag{6.287}$$

At second order, we present first the contribution from pure second order terms

$$j^{\mathrm{S}}_{2\,\ell} = \sum_{\ell=0}^{\infty}\sqrt{4\pi(2\ell+1)}i^\ell\int\frac{\mathrm{d}^{\,3}\mathbf{k}}{(2\pi)^3}\mathcal{K}\left[\Phi(\mathbf{k}_1,\mathbf{k}_2)\right]W_{j^{(2)}\mathrm{S}}(\eta,\mu_k,\mathbf{k}_1,\mathbf{k}_2,\mathbf{k})\delta_{m0}\,, \tag{6.288}$$

$$j^{\mathrm{V}}_{2} = \mp\sum_{\ell=0}^{\infty}\sqrt{4\pi(2\ell+1)}i^{\ell+1}\int\frac{d^3\mathbf{k}}{(2\pi)^3}\mathcal{K}\left[\Phi(\mathbf{k}_1,\mathbf{k}_2)\right]W_{j^{(2)}\mathrm{V}}(\eta,\mu,\mathbf{k}_1,\mathbf{k}_2,\mathbf{k})\delta_{m,\pm1}\,, \tag{6.289}$$

$$j^{\mathrm{T}}_{2} = \sum_{\ell=0}^{\infty}\sqrt{4\pi(2\ell+1)}i^\ell\int\frac{\mathrm{d}^{\,3}\mathbf{k}}{(2\pi)^3}\mathcal{K}\left[\Phi(\mathbf{k}_1,\mathbf{k}_2)\right]W_{j^{(2)}\mathrm{T}}(\eta,\mu,\mathbf{k}_1,\mathbf{k}_2,\mathbf{k})\delta_{m,\pm2}\,, \tag{6.290}$$

where

$$W_{j^{(2)}\mathrm{S}}(\eta,\mu,\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}) = \Bigg[k^{\Phi/\Psi}F_{V_\|}(\eta,\mu,\mathbf{k}_1,\mathbf{k}_2,\mathbf{k})j_\ell(k\chi_s) + f_{\Phi^{(2)}}(\eta,\mu_{k_1},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k})j_\ell(k\chi_s) + \int_0^{\chi_s}k\left(f_{(\Phi^{(2)}+\Psi^{(2)})}(\eta,\mu_{k_1},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k})\right)j'_\ell(k\chi)\mathrm{d}\,\chi\Bigg] \tag{6.291}$$

$$W_{j_2^{(2)}\mathrm{V}}(\eta,\mu,\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}) = \sqrt{\frac{\ell'(\ell+1)}{2}}\Bigg[k^{(S)}F_{V_\|}(\eta_s,\mu_k\mathbf{k}_1,\mathbf{k}_2,\mathbf{k})\frac{j_\ell(k\chi_s)}{k\chi_s} + \int_0^{\chi_s}kf_{\omega^{(2)}}(\eta,\mu_k\mathbf{k}_1,\mathbf{k}_2,\mathbf{k})\left(\frac{j_\ell(k\chi_s)}{k\chi_s}\right)'\mathrm{d}\,\chi\Bigg] \tag{6.292}$$

$$W_{j_2^{(2)}\mathrm{T}}(\eta,\mu,\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}) = \sqrt{\frac{(\ell'+2)!}{(\ell'-2)!}}\Bigg[-f_{h^{(2)}}(\eta,\mu_{k_1},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k})\frac{j_\ell(k\chi_s)}{(k\chi_s)^2} - \int_0^{\chi_s}kf_{h^{(2)}}(\eta,\mu_{k_1},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k})\left(\frac{j_\ell(k\chi)}{(k\chi)^2}\right)'\mathrm{d}\,\chi\Bigg] \tag{6.293}$$

For terms quadratic in first order gravitational potential we find

$$j_{2\perp}^{\mathcal{O}(1)\times\mathcal{O}(1)_\perp} = \sqrt{4\pi(2\ell_3+1)}i^{\ell_1+\ell_2}\mathcal{K}[\Phi(\mathbf{k}_1)\Phi(\mathbf{k}_2)]W_{j_{2\perp}^{\mathcal{O}(1)\times\mathcal{O}(1)_\perp}}(\chi_s,\chi,k_1,k_2)\delta_{m,0}\,, \tag{6.294}$$

$$j_{2\|}^{\mathcal{O}(1)\times\mathcal{O}(1)_\|} = \sqrt{4\pi(2\ell_3+1)}i^{\ell_1+\ell_2}\mathcal{K}[\Phi(\mathbf{k}_1)\Phi(\mathbf{k}_2)]W_{2\|}^{\mathcal{O}(1)\times\mathcal{O}(1)_\|}(\chi_s,\chi,k_1,k_2)\delta_{m,0}\,, \tag{6.295}$$

where

$$W_{j_{2\perp}^{\mathcal{O}(1)\times\mathcal{O}(1)_\perp}}(\eta,\mu,\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}) = \sqrt{\frac{4\pi(2\ell_1+1)(2\ell_2+1)}{(2\ell_3+1)}}{}_{+1,-1,0}\mathcal{G}^{000}_{\ell_1\ell_2\ell_3} \times\sqrt{\left(\frac{\ell_1(\ell_1+1)}{2}\frac{\ell_2(\ell_2+1)}{2}\right)}\mathcal{W}_{j_{2\perp}^{\mathcal{O}(1)\times\mathcal{O}(1)_\perp}}(\eta,\mu,\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}) \quad (6.296)$$

$$W^{\mathcal{O}(1)\times\mathcal{O}(1)_\parallel}_{j_{2\parallel}}(\eta,\mu,\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}) = \sqrt{\frac{4\pi(2\ell_1+1)(2\ell_2+1)}{(2\ell_3+1)}}{}_{000}\mathcal{G}^{000}_{\ell_1\ell_2\ell_3} \times\mathcal{W}^{\mathcal{O}(1)\times\mathcal{O}(1)_\parallel}_{2\parallel}(\eta,\mu,\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}) \quad (6.297)$$

with

$$\mathcal{W}_{j_{2\perp}^{\mathcal{O}(1)\times\mathcal{O}(1)_\perp}}(\eta,\mu,\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}) = \Bigg\{\left[k_1T_{v_\parallel}(\eta_s,k_1)\frac{j_{\ell_1}(k_1\chi_1)}{(k_1\chi)}k_2T_{v_\parallel}(\eta_s,k_2)\frac{j_{\ell_2}(k_2\chi_2)}{(k_2\chi_2)}\right] +8\int_0^{\chi_s}\int_0^{\chi}\left[k_1T_\Phi(\eta_1,k_1)\frac{j_{\ell_1}(k_1\chi_1)}{(k_1\chi)}\mathrm{d}\,\chi_1 \times\int_0^{\chi}k_2T_\Phi(\eta,k_2)\frac{j_{\ell_2}(k_2\chi_2)}{(k_2\chi_2)}\mathrm{d}\,\chi_2\right]\mathrm{d}\,\chi\Bigg\}, \quad (6.298)$$

$$\begin{aligned}\mathcal{W}^{\mathcal{O}(1)\times\mathcal{O}(1)_\parallel}_{j_{2\parallel}}(\eta,\mu,\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}) = &\Big[\Big[k_1T_{v_\parallel}(\eta_s,k_1)k_2T_{v_\parallel}(\eta_s,k_2)j'_{\ell_1}(k_1\chi_s)\,j'_{\ell_2}(k_2\chi_s)\Big] \\ &+3T_\Phi(\eta_s,k_1)T_\Phi(\eta_s,k_2)\left[j_{\ell_1}(k_1\chi_s)j_{\ell_2}(k_2\chi_s)\right] \\ &-4T_\Phi(\eta_s,k_1)j_{\ell_1}(k_1\chi_s)T_{v_\parallel}(\eta_s,k_2)j'_{\ell_2}(k_2\chi_s) \\ &+4k_2T_{v_\parallel}(\eta_s,k_2)j'_{\ell_2}(k_2\chi_s)\int_0^{\chi_s}T_\Phi(\eta,k_1)j_{\ell_1}(k_1\chi)\mathrm{d}\,\chi \\ &-12\int_0^{\chi_s}T_{\Phi_s}(\eta_s,k_1)T'_{\Phi'}(\eta,k_2)j_{\ell_1}(k_1\chi_s)j_{\ell_2}(k_2\chi)\mathrm{d}\,\chi \\ &+8\int_0^{\chi_s}T_\Phi(\eta_1,k_1)T'_{\Phi'}(\eta_2,k_2)j_{\ell_1}(k_1\chi)j_{\ell_2}(k_2\chi)\mathrm{d}\,\chi \\ &-4\int_0^{\chi_s}\Phi_s(\eta_s,k_1)\Phi'(\eta_2,k_2)\mathrm{d}\,\chi \\ &+8\int_0^{\chi_s}\int_0^{\chi}T_\Phi(\eta,k_2)T''_{\Phi''}(\eta,k_1)j_{\ell_1}(k_1\chi_1)j_{\ell_2}(k_2\chi)\mathrm{d}\,\chi_1\mathrm{d}\,\chi \\ &+8\int_0^{\chi_s}T'_{\Phi'}(\eta,k_2)\int_0^{\chi}T'_{\Phi'}(\eta_1,k_1)j_{\ell_1}(k_1\chi_1)j_{\ell_2}(k_2\chi)\mathrm{d}\,\chi_1\mathrm{d}\,\chi\Big]\end{aligned} \quad (6.299)$$

6.9.3 Summary of both methods

Finally we bring together all the second order corrections to the observed redshift

$$\begin{aligned} J_2 &= J_2^{\rm S} + J_2^{\rm V} + J_2^{\rm T} + J_2^{\mathcal{O}(1)\times\mathcal{O}(1)_\parallel} + J_2^{\mathcal{O}(1)\times\mathcal{O}(1)_\perp}\,, \qquad (6.300)\\ &= \sum_{\ell=0}^{\infty}\sqrt{4\pi(2\ell+1)}\int\frac{{\rm d}^3\mathbf{k}}{(2\pi)^3}\mathcal{K}\left[\Phi(\mathbf{k}_1,\mathbf{k}_2)\right]\Bigg[i^\ell W_{j^{(2)}{\rm S}}(\eta,\mu_k,\mathbf{k}_1,\mathbf{k}_2,\mathbf{k})\delta_{m,0} \qquad (6.301)\\ &\mp i^{\ell+1}W_{j^{(2)}{\rm V}}(\eta,\mu,\mathbf{k}_1,\mathbf{k}_2,\mathbf{k})\delta_{m,\pm1} + i^\ell W_{j^{(2)}{\rm T}}(\eta,\mu,\mathbf{k}_1,\mathbf{k}_2,\mathbf{k})\delta_{m,\pm2}\Bigg]\\ &+i^{\ell_1+\ell_2}\mathcal{K}[\Phi(\mathbf{k}_1)\Phi(\mathbf{k}_2)]\Bigg[W_{j_{2\perp}^{\mathcal{O}(1)\times\mathcal{O}(1)_\perp}}(\eta,\mu,\mathbf{k}_1,\mathbf{k}_2)\delta_{m,0}\\ &+W_{2\parallel}^{\mathcal{O}(1)\times\mathcal{O}(1)_\parallel}(\eta,\mu,\mathbf{k}_1,\mathbf{k}_2)\delta_{m,0}\Bigg]\,,\\ J_2 &= \sum_{\ell=0}^{\infty}\sqrt{4\pi(2\ell+1)}\int\frac{{\rm d}^3\mathbf{k}}{(2\pi)^3}\mathcal{K}\left[\Phi(\mathbf{k}_1,\mathbf{k}_2\right]K_{z_s}^{(2)}(\chi,\mu_k,k_1,k_2,k)\,. \qquad (6.302)\end{aligned}$$

In the last equality, we have used

$$\int\frac{{\rm d}^3\mathbf{k}}{(2\pi)^3}\int\frac{{\rm d}^3\mathbf{k}_2}{(2\pi)^3}\Phi(\mathbf{k}_1)\Phi(\mathbf{k}_2) = \int\frac{{\rm d}^3\mathbf{k}}{(2\pi)^3}\mathcal{K}\left[\Phi(\mathbf{k}_1,\mathbf{k}_2)\right]\,. \qquad (6.303)$$

This is possible since the integral on the LHS does not depend on $\mathbf{k}$, hence we can perform the k-integral any time. The kernel for all the second order corrections becomes

$$\begin{aligned} K_{z_s}^{(2)}(\chi,\mu,k_1,k_2,k) &= i^\ell\,W_{j^{(2)}{\rm S}}(\eta,\mu_k,k_1,k_2,k)\delta_{m,0}\mp i^\ell W_{j^{(2)}{\rm Vector}}(\eta,\mu,k_1,k_2,k)\delta_{m,\pm1} \qquad (6.304)\\ &+i^\ell\,W_{j^{(2)}{\rm Tensor}}(\eta,\mu,k_1,k_2,k)\delta_{m,\pm2} + i^{\ell_1+\ell_2}\,W_{j_{2\perp}^{\mathcal{O}(1)\times\mathcal{O}(1)_\perp}}(\eta,\mu,k_1,k_2)\delta_{m,0}\\ &+i^{\ell_1+\ell_2}\,W_{2\parallel}^{\mathcal{O}(1)\times\mathcal{O}(1)_\parallel}(\eta,\mu,k_1,k_2)\delta_{m,0}\,.\end{aligned}$$

This is may be written in a form most useful for extracting the monopole

$$\begin{aligned}K^{(2)}_{z_s}(\chi,\mu,k_1,k_2,k) &= i^{\ell}W_{j^{(2)}\mathrm{S}}(\eta,\mu_k,k_1,k_2,k)\delta_{m,0} \\ &\mp i^{\ell+1}W_{j^{(2)}\mathrm{Vector}}(\eta,\mu,k_1,k_2,k)\delta_{m,\pm1} + i^{\ell}W_{j^{(2)}\mathrm{Tensor}}(\eta,\mu,k_1,k_2,k) \\ &\times\delta_{m,\pm2} + \sqrt{\frac{4\pi(2\ell_1+1)(2\ell_2+1)}{(2\ell_3+1)}}\Bigg[i^{\ell_1+\ell_2}{}_{(+1,-1,0)}\mathcal{G}^{000}_{\ell_1\ell_2\ell_3} \\ &\times\sqrt{\left(\frac{\ell_1(\ell_1+1)}{2}\frac{\ell_2(\ell_2+1)}{2}\right)}\mathcal{W}_{j^{\mathcal{O}(1)\times\mathcal{O}(1)_\perp}_{2\perp}}(\eta,\mu,k_1,k_2)\delta_{m,0} \\ &+ i^{\ell_1+\ell_2}{}_{(000)}\mathcal{G}^{000}_{\ell_1\ell_2\ell_3}\mathcal{W}^{\mathcal{O}(1)\times\mathcal{O}(1)_\parallel}_{2\parallel}(\eta,\mu,k_1,k_2)\delta_{m,0}\Bigg]. \end{aligned} \tag{6.305}$$

6.10 Boosting to the Observer frame: Doppler Effect

The area distance, we have calculated needs to be boosted to the frame of the observer moving with a 4-velocity with a non-vanishing relative velocity, as we explained in Section 6.6 and it is most convenient to do it at this stage of calculation. The multipoles of the Doppler term expanded up to second order are given by

$$\frac{\hat{E}}{\tilde{E}} = 1 + \frac{\delta\hat{E}}{\tilde{E}} + \frac{\delta^2\hat{E}}{\tilde{E}}, \tag{6.306}$$

where

$$\frac{\delta\hat{E}}{\tilde{E}} = -\sum_{\ell=0}^{\infty}\sqrt{4\pi(2\ell+1)}i^{\ell}\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\Phi(\mathbf{k})kT_v(\eta,k)j'_\ell(k\hat{\chi}_o), \tag{6.307}$$

$$\begin{aligned}\frac{\delta^2\hat{E}}{\tilde{E}} &= \sum_{\ell=0}^{\infty}\sqrt{4\pi(2\ell+1)}\Bigg[\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\mathcal{K}\left[\Phi(\mathbf{k}_1,\mathbf{k}_2)\right] \\ &\left(i^{\ell}k^{\Phi/\Psi}F_{V_\parallel}(\eta,\mu,\mathbf{k}_1,\mathbf{k}_2,\mathbf{k})j'_\ell(k\chi_o) \mp \sqrt{\frac{\ell'(\ell+1)}{2}}i^{\ell+1(S)}F_{v_\perp}(\eta,\mu_K,\mathbf{k}_1,\mathbf{k})\frac{j_\ell(k\chi_o)}{k\chi_o}\right)Y_{\ell,0} \\ &-\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\mathcal{K}\left[\Phi(\mathbf{k}_1,\mathbf{k}_2)\right]i^{\ell_2+\ell_1}W_{D_A\mathrm{Doppler}}(\chi_1,\chi_2,k_1k_2)Y_{\ell,0}\Bigg], \end{aligned} \tag{6.308}$$

where

$$
\begin{aligned}
& W_{D_A\text{Doppler}}(\chi_1,\chi_2,k_1k_2) \qquad (6.309)\\
= & \sqrt{\frac{4\pi(2\ell_1+1)(2\ell_2+1)}{(2\ell_3+1)}}\Bigg[{}_{000}\mathcal{G}^{000}_{\ell_1\ell_2\ell_3}\mathcal{W}1_{D_A\text{Doppler}}(\chi_{1o},\chi_{2o},k_1k_2)\\
& +\sqrt{\left(\frac{\ell_1(\ell_1+1)}{2}\frac{\ell_2(\ell_2+1)}{2}\right)}_{(1,-1,0)}\mathcal{G}^{000}_{\ell_1\ell_2\ell_3}\mathcal{W}2_{D_A\text{Doppler}}(\chi_{1o},\chi_{2o},k_1k_2)\Bigg],\\
& \mathcal{W}1_{D_A\text{Doppler}}(\chi_{1o},\chi_{2o},k_1k_2) \qquad (6.310)\\
= & \Bigg[\Big(T_\Phi(\eta_o,k_1)j_{\ell_1}(k_1\chi_o)\,k_2T_{v_\parallel}(\eta_o,k_2)j'_{\ell_2}(k_2\chi_o)\Big)\\
& +\frac{1}{2}\Big(k_1T_{v_\parallel}(\eta_o,k_1)j'_{\ell_1}(k_1\chi_o)\,k_2T_{v_\parallel}(\eta_o,k_2)j'_{\ell_2}(k_2\chi_o)\Big)\Bigg],\\
& \mathcal{W}2_{D_A\text{Doppler}}(\chi_{1o},\chi_{2o},k_1k_2) \qquad (6.311)\\
= & \frac{1}{2}\left[k_1T_{v_\parallel}(\eta_o,k_1)\frac{j_{\ell_1}(k_1(\lambda_1-\lambda_o))}{(k_1(\lambda-\lambda_o))}k_2T_{v_\parallel}(\eta_o,k_2)\frac{j_{\ell_2}(k_2(\lambda_2-\lambda_o))}{(k_2(\lambda_2-\lambda_o))}\right].
\end{aligned}
$$

6.10.1 Photons do not travel on background spacetime

Cosmological perturbation theory assumes a reference background where the perturbation variables live, we have calculated the area distance/luminosity distance based on this understanding. However, photons we observed today do not travel on the background spacetime, so they need not be integrated along the background spacetime but rather in the real spacetime. Also the redshift of large scale structures we see on the night sky is not the background redshift. For the interpretation of a particular cosmological observable like the area distance, the background approximation will remain valid at first order perturbation theory because the Bohm approximation is valid. This is also supported by the Gaussianity assumption. Bohm approximation breaks down at the order preceding the highest order under consideration, so we need to consistently re-write all our observables to depend on the physical photon path and redshift of the source. We treat them one after the other in the following:

- *First order area distance*

  Since we are going to calculate observables up to second order, we have to change our integration measure at first order for $\hat{D}_A$ so that we integrate along the physical photon path rather than on the background spacetime. The part of equation (6.247) we need to switch $\chi \to \hat{\chi}$ is just $W_{\delta D_A/D_A}(\chi_s,\chi,k)$ and the result of expanding it in

Taylor series is given by

$$W_{\delta D_A/D_A}(\chi_s,\chi,k) \;=\; W_{\delta D_A/D_A}(\hat{\chi}_s,\hat{\chi},k)+\partial_\chi W_{\delta D_A/D_A}(\chi_s,\chi,k)\bigg|_{\hat{\chi}}\frac{J_1}{\mathcal{H}_s}\,. \tag{6.312}$$

Substituting this in equation (6.247) leads to

$$\begin{aligned}\frac{\delta D_A}{D_A(s)}{}_{\ell 0}(\chi\to\hat{\chi}) \;&=\; \sqrt{4\pi(2\ell+1)}\Bigg[i^\ell\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\Phi(\mathbf{k})W_{\delta D_A/D_A}(\hat{\chi}_s,\hat{\chi},k)\delta_{m,0} &(6.313)\\ &+i^{\ell_2+\ell_1}\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\mathcal{K}\left[\Phi(\mathbf{k}_1,\mathbf{k}_2)\right]W_{\delta D_A/D_A\mathrm{Distort}}(\chi,k_1,k_2)\delta_{m,0}\Bigg]Y_{\ell_3 m}\,,\end{aligned}$$

where

$$\begin{aligned}W_{\delta D_A/D_A\mathrm{Distort}}(\chi,k_1,k_2) \;&=\; \sqrt{\frac{4\pi(2\ell_1+1)(2\ell_2+1)}{(2\ell_3+1)}}{}_{000}\mathcal{G}^{000}_{\ell_1\ell_2\ell_3}\\ &\quad\mathcal{W}_{\delta D_A/D_A\mathrm{Distort}}(\chi,k_1,k_2)\,,\\ \mathcal{W}_{\delta D_A/D_A\mathrm{Distort}}(\chi,k_1,k_2) \;&=\; \frac{1}{\mathcal{H}_s}\partial_\chi W_{\delta D_A/D_A}(\chi,k_2)\bigg|_{\hat{\chi}}W_{j_1}(\chi,k_1)\,. &(6.314)\end{aligned}$$

There is also a more complicated first principle approach to this (see Section 6.13).

- *First order redshift*

We now evaluate the correction to the redshift at first order when we switch the integration photon path from the background spacetime to the physical spacetime . The part that contains the background photon path or the background redshift is $W_{J_1}(\chi,k)$ and when expanded in Taylor series we find

$$W_{J_1}(\chi,k) \;=\; W_{J_1}(\hat{\chi},k)+\partial_\chi W_{J_1}(\chi,k)\bigg|_{\hat{\chi}}\frac{J_1}{\mathcal{H}_s}\,, \tag{6.315}$$

Implementing this in equation (6.286) we find

$$\begin{aligned}j_{1\ell}(\chi\to\hat{\chi}) \;&=\; \sqrt{4\pi(2\ell+1)}\Bigg[i^\ell\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\Phi(\mathbf{k})W_{J_1}(\hat{\chi},k)\delta_{m,0} &(6.316)\\ &+i^{\ell_2+\ell_1}\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\mathcal{K}\left[\Phi(\mathbf{k}_1,\mathbf{k}_2)\right]W_{j_1\mathrm{Distort}}(\chi,k_1,k_2)\delta_{m,0}\Bigg]Y_{\ell_3 m}\,,\end{aligned}$$

where

$$W_{j_1\text{Distort}}(\chi, k_1, k_2) = \sqrt{\frac{4\pi(2\ell_1+1)(2\ell_2+1)}{(2\ell_3+1)}}{}_{000}\mathcal{G}^{000}_{\ell_1\ell_2\ell_3}\mathcal{W}_{j_1\text{Distort}}(\chi, k_1, k_2) \quad (6.317)$$

$$\mathcal{W}_{j_1\text{Distort}}(\chi, k_1, k_2) = \frac{1}{\mathcal{H}_s}\partial_\chi W_{j_1}(\chi, k_2)\Big|_{\hat{\chi}} W_{j_1}(\chi, k_1)\,. \quad (6.318)$$

- *First order Doppler term*

  We need to also expand $n^k \mathrm{D}_k v_s$ in a Taylor series so the we evaluate all the field along the perturbed geodesic,

$$\mathrm{D}_\| v(\chi_s) = \mathrm{D}_\| v(\hat{\chi}_s) + \frac{J_{1s}}{\mathcal{H}_s}\left(\partial_\| v(\hat{\eta})'|_s\right)\,. \quad (6.319)$$

  We now expand the additional term in spherical harmonics

$$\frac{J_{1s}}{\mathcal{H}_s}\left(\partial_\| v(\hat{\eta})'|_s\right) = \sqrt{4\pi(2\ell_3+1)}\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\mathcal{K}\left[\Phi(\mathbf{k}_1,\mathbf{k}_2)\right]i^{\ell_2+\ell_1} \times\left[W_{\lambda\mathrm{DS}}(\hat{\chi}, k_1k_2)\right]Y_{\ell_3,0}\,, \quad (6.320)$$

  where

$$W_{\mathrm{DSD}}(\hat{\chi}, k_1k_2) = \sqrt{\frac{4\pi(2\ell_1+1)(2\ell_2+1)}{(2\ell_3+1)}}{}_{000}\mathcal{G}^{000}_{\ell_1\ell_2\ell_3}\mathcal{W}_{\mathrm{DSD}}(\hat{\chi}, k_1k_2)\,, \quad (6.321)$$

$$\mathcal{W}_{\mathrm{DSD}}(\hat{\chi}, k_1k_2) = \frac{1}{\mathcal{H}_s}k_1\partial_\chi(T_{v_\|}(\eta, k_1)j_\ell(k_1\chi))|_s W_{J_1}(\chi_{2s}, k_2)\,. \quad (6.322)$$

6.11 Backreaction effects on Key Observables

The standard model of cosmology assumes that the FLRW spacetime appears when the universe with real structures is smoothed on very large scale. This implies that, if we take the monopole of the area distance or redshift we have calculated, we will recover an FLRW spacetime because the monopole is the all sky average of the cosmological observable of interest.

- *Observed Redshift*

  We want to verify this claim for the observed redshift. Observed redshift calculate up to second order in perturbation theory is given by

$$
\begin{aligned}
(1+\hat{z}_s) &= (1+\bar{z}_s)\left[1+\sum_{\ell=0}^{\infty}\sqrt{4\pi(2\ell+1)}\left[i^{\ell}\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\Phi(\mathbf{k})W_{J_1}(\chi,k)\delta_{m0}\right.\right. && (6.323)\\
&\left.\left.+\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\mathcal{K}\left[\Phi(\mathbf{k_1},\mathbf{k_2}\right]K^{(2)}_{z_s}(\chi,\mu_k,k_1,k_2,k)\right]Y_{\ell m}\right],
\end{aligned}
$$

where

$$
\begin{aligned}
W_{J_1}(\lambda,k) &= \left[kT_V(\eta_s,k)j'_\ell(k\hat{\chi}_s)+T_\Phi(\eta_s,k)j_\ell(k\hat{\chi}_s)\right. && (6.324)\\
&\left.+2\int_0^{\chi_s}T_\Phi(\eta,k)kj'_\ell(k\hat{\chi})\mathrm{d}\,\hat{\chi}\right],\\
K^{(2)}_{z_s}(\chi,\mu,k_1,k_2,k) &= i^{\ell}W_{j^{(2)}\mathrm{S}}(\eta,\mu_k,k_1,k_2,k)\delta_{m,0}\mp i^{\ell+1}W_{j^{(2)}\mathrm{V}}(\eta,\mu,k_1,k_2,k)\delta_{m,\pm1}\\
&+i^{\ell}W_{j^{(2)}\mathrm{T}}(\eta,\mu,k_1,k_2,k)\delta_{m,\pm2}+i^{\ell_1+\ell_2}\sqrt{\frac{4\pi(2\ell_1+1)(2\ell_2+1)}{(2\ell_3+1)}}\\
&\left[{}_{(+1,-1,0)}\mathcal{G}^{000}_{\ell_1\ell_2\ell_3}\sqrt{\left(\frac{\ell_1(\ell_1+1)}{2}\frac{\ell_2(\ell_2+1)}{2}\right)}\right.\\
&\mathcal{W}^{\mathcal{O}(1)\times\mathcal{O}(1)_\perp}_{j_{2\perp}}(\eta,\mu,k_1,k_2,k)\delta_{m,0}+{}_{(000)}\mathcal{G}^{000}_{\ell_1\ell_2\ell_3}\\
&\left.\mathcal{W}^{\mathcal{O}(1)\times\mathcal{O}(1)_\parallel}_{2\parallel}(\eta,\mu,k_1,k_2,k)\delta_{m,0}+\mathcal{W}_{j_1\mathrm{Distort}}(\chi,k_1,k_2)\right]. && (6.325)
\end{aligned}
$$

Notice that we have included the redshift distortion term in the definition of the second order redshift kernel in equation (6.325). We quantify the effect of inhomogeneity on the observed redshift by subtracting off the FLRW background contribution

$$
\begin{aligned}
\Delta_z &= \sum_{\ell=0}^{\infty}\sqrt{4\pi(2\ell+1)}\left[i^{\ell}\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\Phi(\mathbf{k})W_{J_1}(\chi,k)\delta_{m0}\right. && (6.326)\\
&\left.+\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\mathcal{K}\left[\Phi(\mathbf{k_1},\mathbf{k_2}\right]K^{(2)}_{z_s}(\chi,\mu_k,k_1,k_2,k)\right]Y_{\ell m}.
\end{aligned}
$$

Assuming that the primordial gravitational potential is Gaussian, the monopole of the backreaction effect on the observed redshift is given by

$$
\Delta_{z_{00}} = \sum_{\ell=0}^{\infty}\sqrt{4\pi(2\ell+1)}\left[\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\mathcal{K}\left[\Phi(\mathbf{k_1},\mathbf{k_2}\right]K^{(2)}_{z_s\mathrm{Mono}}(\chi,\mu_k,k_1,k_2,k)\right], \quad (6.327)
$$

where

$$\begin{aligned}
&K^{(2)}_{z_s\mathrm{Mono}}(\chi,\mu_k,k_1,k_2,k) \qquad (6.328)\\
&= \Bigg[i^{\ell} W_{j^{(2)}\mathrm{S}}(\eta,\mu_k,k_1,k_2,k)\delta_{m,0} \mp i^{\ell+1} W_{j^{(2)}\mathrm{Vector}}(\eta,\mu,k_1,k_2,k)\delta_{m,\pm1}\\
&+ i^{\ell} W_{j^{(2)}\mathrm{Tensor}}(\eta,\mu,k_1,k_2,k)\delta_{m,\pm2}\Bigg]_{\ell=0}\\
&+(2\ell_1+1)\left[i^{\ell_1+\ell_2}\sqrt{\left(\frac{\ell_1(\ell_1+1)}{2}\frac{\ell_2(\ell_2+1)}{2}\right)}\mathcal{W}_{j_{2\perp}^{\mathcal{O}(1)\times\mathcal{O}(1)_\perp}}(\eta,\mu,k_1,k_2,k)\delta_{m,0}\right.\\
&\left.+\, i^{\ell_1+\ell_2}\mathcal{W}_{2\|}^{\mathcal{O}(1)\times\mathcal{O}(1)_\|}(\eta,\mu,k_1,k_2,k)\delta_{m,0} + \mathcal{W}_{j_1\mathrm{Distort}}(\chi,k_1,k_2)\delta_{m,0}\right]\\
&\times(-1)^{-m_1-s_1}\delta_{\ell_1,\ell_2}\delta_{m_1,-m_2}\delta_{s_1,-s_2}\,.
\end{aligned}$$

The ensemble average of equation (6.327) ($\bar{\Delta}_{z_0 0}$) should vanish if the correction to the background redshift is negligible. However, the parallel and the transversal contribution of the peculiar velocity terms are unlikely to vanish especially on small scales. A recent paper [235] on anti-lensing claims that the peculiar velocity terms could account for up to 10% change to the weak lensing convergence on small angular scales. This claim from weak lensing studies gives us a clue what to expect. Part of this effect has been dicussed in [182] as the physical redshift distortion effect. It is an important effect for parameter estimation in precision cosmology since all our observations depend on physical redshift.

- *Radial distance*

  As shown in equation (6.94) the physical path travelled by the photons that arrive at our telescope may be expressed in terms of the background photon travel path plus small deviations up to second order in perturbation theory

$$\hat{\lambda} \;=\; \lambda+\delta\lambda+\frac{1}{2}\delta^2\lambda\,. \qquad (6.329)$$

We may now substitute for $\delta\lambda$ and $\delta^2\lambda$ using the appropriate terms in equation (6.94)

$$\hat{\lambda} \;=\; \lambda+\frac{J_1}{\mathcal{H}_s}+\frac{1}{\mathcal{H}_s}\left[J_2-J_1^2\left(1+\frac{\mathcal{H}'}{\mathcal{H}^2}\right)\right]\,, \qquad (6.330)$$

where the difference between the background affine parameter and the physical affine parameter is known as the affine parameter distortion effect [182]. The multipoles of the affine parameter distortion effect is defined through a spherical harmonic expansion

$$\Delta_\lambda(\chi, \mathbf{n}) = \frac{\hat{\lambda} - \lambda}{\lambda} = \sum_{\ell=0} \sum_{m=-\ell}^{\ell} \Delta_{\lambda\ell m}(\chi, r) Y_{\ell m}(\mathbf{n}) \,, \tag{6.331}$$

where the multipoles becomes

$$\begin{aligned}\Delta_{\lambda\ell m}(\chi, r) &= \sum_{\ell=0}^{\infty} \sqrt{4\pi(2\ell+1)} \frac{1}{\mathcal{H}_s} \left[\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} i^\ell \Phi(\mathbf{k}) W_{J_1}(\lambda, k) + \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \mathcal{K}\left[\Phi(\mathbf{k}_1, \mathbf{k}_2)\right] \right. \\ &\left. \times i^{\ell_2+\ell_1} \left(K^{(2)}_{z_s}(\chi, \mu_k, k_1, k_2, k) - W_{j_1^2}(\chi, k_1, k_2, k) \left(1 + \frac{\mathcal{H}'}{\mathcal{H}^2}\right) \right) \right] \,, \end{aligned} \tag{6.332}$$

with

$$W_{j_1^2}(\chi, k_1, k_2, k) = \sqrt{\frac{4\pi(2\ell_1+1)(2\ell_2+1)}{(2\ell_3+1)}} {}_{000}\mathcal{G}^{000}_{\ell_1\ell_2\ell_3} \mathcal{W}_{j_1^2}(\chi, k_1, k_2) \,, \tag{6.333}$$

$$\mathcal{W}_{j_1^2}(\chi, k_1, k_2, k) = W_{J_1}(\chi_1, k_1) W_{J_1}(\chi_2, k_2) \,. \tag{6.334}$$

Assuming primordial Gaussainity, the all sky average of $\Delta_{\lambda\ell m}(\chi, r)$ becomes

$$\begin{aligned}\Delta_{\lambda 00}(\chi, r) &= \frac{1}{\mathcal{H}_s} \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \mathcal{K}\left[\Phi(\mathbf{k}_1, \mathbf{k}_2)\right] i^{\ell_2+\ell_1} \left(K_{z_s} \mathrm{Mono}^{(2)}(\chi, \mu_k, k_1, k_2, k) \right. \\ &\left. -(2\ell_1+1)\mathcal{W}_{j_1^2}(\mathrm{Mono}(\chi, k_1, k_2, k) \left(1 + \frac{\mathcal{H}'}{\mathcal{H}^2}\right) \right) \\ &\times (-1)^{-m_1-s_1} \delta_{\ell_1,\ell_2} \delta_{m_1,-m_2} \delta_{s_1,-s_2} \,. \end{aligned} \tag{6.335}$$

where

$$\begin{aligned}\mathcal{W}_{j_1^2}\mathrm{Mono}(\chi, k_1, k_2, k) &= W_{J_1}(\chi_1, k_1) W_{J_1}(\chi_2, k_2)(-1)^{-m_1-s_1} \\ &\quad \delta_{\ell_1,\ell_2} \delta_{m_1,-m_2} \delta_{s_1,-s_2}\end{aligned} \tag{6.336}$$

with $K_{z_s}\mathrm{Mono}^{(2)}(\chi, \mu_k, k_1, k_2, k)$ has already been defined. Equation (6.335) carries the affine parameter distortion effect discussed in [182].

- *Distance-Redshift relation*

The general expression for the area distance up to second order is given by

$$\begin{aligned}\hat{D}_A &= a(\hat{\lambda}_s)\hat{\chi}_s \left\{1 + \left[\frac{\delta D_A}{D_A} + \left(1 - \frac{1}{\mathcal{H}_s \chi_s}\right) J_1 + \frac{\delta \hat{E}}{\tilde{E}}\right]\right. \\ &+ \left[\frac{1}{2}\frac{\delta^2 D_A}{D_A} + \left(\frac{1}{2} J_2 + \frac{\delta D_A}{D_A} J_1\right)\left(1 - \frac{1}{\mathcal{H}_s \chi_s}\right)\right. \\ &\left.\left. + \frac{1}{2} J_1^2 \left(\frac{\mathcal{H}'_s}{\mathcal{H}^3 \hat{\chi}} - \frac{1}{\mathcal{H}\hat{\chi}}\right) + \frac{\delta^2 \hat{E}}{\tilde{E}}\right]\right\}. \end{aligned} \quad (6.337)$$

The product of first order area distance-first order redshift and first order redshift-first order redshift terms may be evaluated at this stage and they are given by

$$J_1 \frac{\delta D_A}{D_A} = \sum_{\ell=0}^{\infty} \sqrt{4\pi(2\ell_3+1)} \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \mathcal{K}\left[\Phi(\mathbf{k}_1, \mathbf{k}_2)\right] i^{\ell_2+\ell_1} \times W_{j_1 \frac{\delta D_A}{D_A}}(\chi, k_1, k_2) Y_{\ell_3 0}\,, \quad (6.338)$$

$$J_1^2 = \sum_{\ell=0}^{\infty} \sqrt{4\pi(2\ell_3+1)} \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \mathcal{K}\left[\Phi(\mathbf{k}_1, \mathbf{k}_2)\right] i^{\ell_2+\ell_1} \times W_{j_1^2}(\chi, k_1, k_2) Y_{\ell_3 0}\,, \quad (6.339)$$

$$\frac{\delta^2 D_A}{D_A} = \sum_{\ell=0}^{\infty} \sqrt{4\pi(2\ell+1)} \int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^3} \mathcal{K}\left[\Phi(\mathbf{k_1}, \mathbf{k_2})\right] \times K^{(2)}(\chi_1, \chi_2, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}) Y_{\ell 0}\,, \quad (6.340)$$

where

$$W_{J_1 \frac{\delta D_A}{D_A}}(\chi, k_1, k_2)) = \sqrt{\frac{4\pi(2\ell_1+1)(2\ell_2+1)}{(2\ell_3+1)}}\, {}_{000}\mathcal{G}^{000}_{\ell_1\ell_2\ell_3} \mathcal{W}_{J_1 \frac{\delta D_A}{D_A}}(\chi, k_1, k_2)\,, \quad (6.341)$$

$$\mathcal{W}_{J_1 \frac{\delta D_A}{D_A}}(\chi, k_1, k_2) = W_{\delta D_A/D_A}(\chi_s, \chi, k_1) W_{J_1}(\lambda, k_2)\,, \quad (6.342)$$

$$W_{J_1^2}(\chi, k_1, k_2) = \sqrt{\frac{4\pi(2\ell_1+1)(2\ell_2+1)}{(2\ell_3+1)}}\, {}_{000}\mathcal{G}^{000}_{\ell_1\ell_2\ell_3} \mathcal{W}_{J_1^2}(\chi, k_1, k_2, k)\,, \quad (6.343)$$

$$\mathcal{W}_{J_1^2}(\chi, k_1, k_2) = W_{J_1}(\chi_1, k_1) W_{J_1}(\chi_2, k_2)\,. \quad (6.344)$$

$$
\begin{aligned}
&K^{(2)}(\chi_1,\chi_2,k_1,k_2,k)\\
&= W_{\Delta^{(2)}\mathrm{S}}(\chi,\mathbf{k})\delta_{m,0} \mp iW_{\Delta^{(2)}\mathrm{Vector}}(\chi,\mathbf{k})\delta_{m,\pm1} + W_{\Delta^{(2)}\mathrm{Tensor}}(\chi,\mathbf{k})\delta_{m,\pm2}\\
&+\sqrt{\frac{4\pi(2\ell_1+1)(2\ell_2+1)}{(2\ell_3+1)}}\Bigg[{}_{(0,0,0)}\mathcal{G}_{\ell_1\ell_2\ell_3}{}^{000} i^{\ell_1+\ell_2}\mathcal{W}_{\Delta^{\Phi\times\Phi}}(\eta,\chi,k_1,k_2))\delta_{m,0} \qquad (6.345)\\
&+{}_{(1,-1,0)}\mathcal{G}_{\ell_1\ell_2\ell_3}{}^{000}\sqrt{\frac{\ell_1(\ell_1+1)}{2}\frac{\ell_2(\ell_2+1)}{2}}\, i^{\ell_1+\ell_2}\mathcal{W}^{\nabla_{\perp i}\Phi\nabla^i_\perp\Phi}_{\Delta_{D_A}}(\eta,k_1,k_2,k)\delta_{m,0}\\
&+{}_{(2,-2,0)}\mathcal{G}_{\ell_1\ell_2\ell_3}{}^{000}\sqrt{\frac{(\ell_1+2)!}{(\ell_1-2)!}\frac{(\ell_2+2)!}{(\ell_2-2)!}}\, i^{\ell_1+\ell_2}\mathcal{W}_{\Delta_{D_A}}{}^{\mathrm{Shear}}(\eta,k_1,k_2,k)\delta_{m,0}\\
&+{}_{(0,0,0)}\mathcal{G}_{\ell_1\ell_2\ell_3}{}^{000} i^{\ell_1+\ell_2}\mathcal{W}_{\Delta_{\delta D_A\times\Phi_\parallel}}(\eta,k_1,k_2,k)\delta_{m,0}\\
&+\sqrt{\frac{\ell_1(\ell_1+1)}{2}\frac{\ell_2(\ell_2+1)}{2}}\,{}_{(1,-1,0)}\mathcal{G}_{\ell_1\ell_2\ell_3}{}^{000} i^{\ell_1+\ell_2}\mathcal{W}_{\delta D_A\times\Phi_\perp}(\eta,k_1,k_2,k)\delta_{m,0}\Bigg].
\end{aligned}
$$

Substituting everything in equation (6.337) we find

$$
\begin{aligned}
\hat{D}_A &= a(\hat\chi)\hat\chi_s\Bigg[1+\sum_{\ell=0}^{\infty}\sqrt{4\pi(2\ell+1)}\Bigg(i^\ell\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\Phi(\mathbf{k})\mathcal{K}_{\Delta_{D_A^{(1)}}}(\chi_s,k) \qquad (6.346)\\
&\quad +\int\frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^3}\mathcal{K}\left[\Phi(\mathbf{k_1},\mathbf{k_2})\right]\mathcal{K}_{\Delta_{D_A^{(2)}}}(\chi_s,k_1,k_2,k)\,,
\end{aligned}
$$

where we define the following kernels

$$
\begin{aligned}
\mathcal{K}_{\Delta_{D_A^{(1)}}}(\chi_s,k) &= \Bigg[W_{\delta D_A/D_A}(\hat\chi_s,\hat\chi,k)+\left(1-\frac{1}{\mathcal{H}_s\chi_s}\right)W_{J_1}(\hat\chi_s,\hat\chi,k) \qquad (6.347)\\
&\quad -kT_v(\eta_s,k)j'_\ell(k\hat\chi_s)\Bigg]\,,
\end{aligned}
$$

$$
\begin{aligned}
&\mathcal{K}_{\Delta_{D_A^{(2)}}}(\chi_s,\mathbf{k_1},\mathbf{k_2},\mathbf{k}) \qquad (6.348)\\
&= \left[\frac{1}{2}K^{(2)}(\chi_1,\chi_2,\mathbf{k_1},\mathbf{k_2},\mathbf{k}) + \frac{1}{2}K^{(2)}_{z_s}(\chi_s,\mu_k,\mathbf{k_1},\mathbf{k_2},\mathbf{k})\left(1-\frac{1}{\mathcal{H}_s\chi_s}\right)\right.\\
&-i^{\ell\Phi+\Psi}F_{V_\parallel}(\eta,\mu,\mathbf{k}_1,\mathbf{k}_2,\mathbf{k})kj'_\ell(k\chi_s)\\
&\pm\sqrt{\frac{\ell'(\ell+1)}{2}}i^{\ell+1(S)}F_{v_\perp}(\eta,\mu_K,\mathbf{k}_1,\mathbf{k})\frac{j_\ell(k\chi_s)}{k\chi_s}\\
&+\sqrt{\frac{4\pi(2\ell_1+1)(2\ell_2+1)}{(2\ell_3+1)}}\Big[{}_{000}\mathcal{G}^{000}_{\ell_1\ell_2\ell_3}\mathcal{W}_{\delta D_A/D_A(\mathrm{Distort})}(\chi,k_1,k_2)\\
&+\left(1-\frac{1}{\mathcal{H}_s\chi_s}\right){}_{000}\mathcal{G}^{000}_{\ell_1\ell_2\ell_3}\mathcal{W}_{J_1}\lambda(\mathrm{Distort})(\hat{\chi},k_1k_2)\\
&-{}_{000}\mathcal{G}^{000}_{\ell_1\ell_2\ell_3}\mathcal{W}_{\mathrm{DSD}}(\hat{\chi},k_1k_2)+{}_{000}\mathcal{G}^{000}_{\ell_1\ell_2\ell_3}\mathcal{W}_{j_1\frac{\delta D_A}{D_A}}(\chi,k_1,k_2,k)\left(1-\frac{1}{\mathcal{H}_s\chi_s}\right)\\
&+\frac{1}{2}{}_{000}\mathcal{G}^{000}_{\ell_1\ell_2\ell_3}\mathcal{W}_{J_1^2}(\chi,k_1,k_2,k)\left(\frac{\mathcal{H}'_s}{\mathcal{H}^3\hat{\chi}_s}-\frac{1}{\mathcal{H}\hat{\chi}_s}\right)\\
&+{}_{000}\mathcal{G}^{000}_{\ell_1\ell_2\ell_3}\mathcal{W}1_{D_A\mathrm{Doppler}}(\chi_{1s},\chi_{2s},k_1k_2)\\
&\left.+\sqrt{\left(\frac{\ell_1(\ell_1+1)}{2}\frac{\ell_2(\ell_2+1)}{2}\right)}{}_{(1,-1,0)}\mathcal{G}^{000}_{\ell_1\ell_2\ell_3}\mathcal{W}2_{D_A\mathrm{Doppler}}(\chi_{1s},\chi_{2s},k_1k_2)\right].
\end{aligned}
$$

The new curly kernels we have defined may be interpreted as transfer functions for the observed area distance. for example:

- $\mathcal{K}_{\Delta_{D_A^{(1)}}}(\chi_s,k)$ is the transfer function of the observed area distance at first order in perturbation theory.
- $\mathcal{K}_{\Delta_{D_A^{(2)}}}(\chi_s,\mathbf{k_1},\mathbf{k_2},\mathbf{k})$ is an angular dependent transfer function for second order contribution to the area distance.

The backreaction effect on the area distance, may be quantified, if we define the background component of the area distance as $\bar{D}_A = a(\hat{\chi})\hat{\chi}_s$, so that we may expand the difference in spherical harmonics

$$\Delta(z_s,\mathbf{n}) = \frac{\hat{D}_A-\bar{D}_A}{\bar{D}_A} = \sum_{\ell=0}^{\ell'}\Delta_{\ell m}(z_s)Y_{\ell m}(\mathbf{n})\,. \qquad (6.349)$$

Substituting equation (6.346) into equation (6.349), we find

$$\Delta_{\ell m}(z_s,\mathbf{n}) = \sum_{\ell=0}^{\infty}\sqrt{4\pi(2\ell+1)}\Bigg(i^{\ell}\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\Phi(\mathbf{k})\mathcal{K}_{\Delta_{D_A^{(1)}}}(\chi_s,k) + \int\frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^3}\mathcal{K}\left[\Phi(\mathbf{k_1},\mathbf{k_2})\right]\mathcal{K}_{\Delta_{D_A^{(2)}}}(\chi_s,k_1,k_2,k)\Bigg). \quad (6.350)$$

Assuming primordial Gaussainity, the monopole of the distance-redshift relation becomes

$$\Delta_{00}(z_s,\mathbf{n}) = \sqrt{4\pi}\int\frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^3}\mathcal{K}\left[\Phi(\mathbf{k_1},\mathbf{k_2})\right]\mathcal{K}_{\mathrm{Mono}\Delta_{D_A^{(2)}}}(\chi_s,k_1,k_2,k)\,, \quad (6.351)$$

where

$$\begin{aligned}
&\mathcal{K}_{\mathrm{Mono}\Delta_{D_A^{(2)}}}(\chi_s,k_1,k_2,k)\\
&= \Bigg[\frac{1}{2}K^{(2)}_{\mathrm{Mono}}(\chi_1,\chi_2,\mathbf{k_1},\mathbf{k_2},\mathbf{k}) + \frac{1}{2}K^{(2)}_{z_s\mathrm{Mono}}(\chi_s,\mu_k,\mathbf{k_1},\mathbf{k_2},\mathbf{k})\left(1-\frac{1}{\mathcal{H}_s\chi_s}\right) \quad (6.352)\\
&-i^{\ell\Phi+\Psi}F_{V_\parallel}(\eta,\mu,\mathbf{k}_1,\mathbf{k}_2,\mathbf{k})kj'_\ell(k\chi_s)\Big|_{\ell=0} + (2\ell_1+1)\Bigg[\mathcal{W}_{\delta D_A/D_A(\mathrm{Distort})}(\chi,k_1,k_2)\\
&+\left(1-\frac{1}{\mathcal{H}_s\chi_s}\right)\mathcal{W}_{J_1}\lambda(\mathrm{Distort})(\hat{\chi},k_1k_2) - \mathcal{W}_{\mathrm{DSD}}(\hat{\chi},k_1k_2)\\
&+\mathcal{W}_{j_1\frac{\delta D_A}{D_A}}(\chi,k_1,k_2,k)\left(1-\frac{1}{\mathcal{H}_s\chi_s}\right) + \frac{1}{2}\mathcal{W}_{J_1^2}(\chi,k_1,k_2,k)\left(\frac{\mathcal{H}'_s}{\mathcal{H}^3\hat{\chi}_s}-\frac{1}{\mathcal{H}\hat{\chi}_s}\right)\\
&+\mathcal{W}1_{D_A\mathrm{Doppler}}(\chi_{1s},\chi_{2s},k_1k_2)\\
&+\sqrt{\left(\frac{\ell_1(\ell_1+1)}{2}\frac{\ell_2(\ell_2+1)}{2}\right)}\mathcal{W}2_{D_A\mathrm{Doppler}}(\chi_{1s},\chi_{2s},k_1k_2)\Bigg]\\
&\times(-1)^{-m_1-s_1}\delta_{\ell_1,\ell_2}\delta_{m_1,-m_2}\delta_{s_1,-s_2}\Bigg],
\end{aligned}$$

with

$$
\begin{aligned}
&K^{(2)}_{\text{Mono}}(\chi_1,\chi_2,k_1,k_2,k) \qquad (6.353)\\
&= \Bigg[\mathcal{K}\left[\Phi(\mathbf{k_1},\mathbf{k_2}\right] W_{\Delta^{(2)}\text{S}}(\chi,\mathbf{k})\delta_{m,0} \mp \mathcal{K}\left[\Phi(\mathbf{k_1},\mathbf{k_2}\right] W_{\Delta^{(2)}\text{Vector}}(\chi,\mathbf{k})\delta_{m,\pm1}\\
&+\mathcal{K}\left[\Phi(\mathbf{k_1},\mathbf{k_2}\right] W_{\Delta^{(2)}\text{Tensor}}(\chi,\mathbf{k})\delta_{m,\pm2}\Bigg]_{\ell=0}\\
&+(2\ell_1+1)\Bigg[\mathcal{K}\left[\Phi(\mathbf{k_1},\mathbf{k_2}\right] i^{\ell_1+\ell_2}\mathcal{W}_{\Delta^{\Phi\times\Phi}}(\eta,\chi,k_1,k_2))\delta_{m,0}\\
&+\mathcal{K}\left[\Phi(\mathbf{k_1},\mathbf{k_2}\right]\sqrt{\frac{\ell_1(\ell_1+1)}{2}\frac{\ell_2(\ell_2+1)}{2}}\, i^{\ell_1+\ell_2}\mathcal{W}^{\nabla_{\perp i}\Phi\nabla^i_\perp\Phi}_{\Delta_{D_A}}(\eta,k_1,k_2,k)\delta_{m,0}\\
&+\mathcal{K}\left[\Phi(\mathbf{k_1},\mathbf{k_2})\right]\sqrt{\frac{(\ell_1+2)!}{(\ell_1-2)!}\frac{(\ell_2+2)!}{(\ell_2-2)!}}\, i^{\ell_1+\ell_2}\mathcal{W}_{\Delta_{D_A}}{}^{\text{Shear}}(\eta,k_1,k_2,k)\delta_{m,0}\\
&+\mathcal{K}\left[\Phi(\mathbf{k_1},\mathbf{k_2}\right] i^{\ell_1+\ell_2}\mathcal{W}_{\Delta_{\delta D_A\times\Phi_\parallel}}(\eta,k_1,k_2,k)\delta_{m,0}\\
&+\mathcal{K}\left[\Phi(\mathbf{k_1},\mathbf{k_2}\right]\sqrt{\frac{\ell_1(\ell_1+1)}{2}\frac{\ell_2(\ell_2+1)}{2}}\, i^{\ell_1+\ell_2}\mathcal{W}_{\delta D_A\times\Phi_\perp}(\eta,k_1,k_2,k)\delta_{m,0}\Bigg]\\
&\times(-1)^{-m_1-s_1}\delta_{\ell_1,\ell_2}\delta_{m_1,-m_2}\delta_{s_1,-s_2}\,.
\end{aligned}
$$

and the $K^{(2)}_{z_s\text{Mono}}(\chi_s,\mu_k,\mathbf{k_1},\mathbf{k_2},\mathbf{k})$ term is given in equation (6.328)

6.12 Conclusion

We have presented for the first time the detailed distance-redshift relation to second order. We have identified key new terms which govern gravitational lensing magnification for large over-densities – and under-densities where it has recently been shown that the linear lensing terms do not capture the full relativistic signal [235]. In addition we have presented new effects which contribute to redshift space distortions. These are:

- *Nonlinear Doppler effect*

  This comes in several forms. The radial parts of the scalar and vector second-order velocities contribute in the same way as at first order. Then the terms $\mathcal{O}(v^2)$ reveal the transverse Doppler contribution in the cosmological context. While small, these give the potential to measure transverse velocities through redshift space distortions.

- *Nonlinear density coupling* From (6.216), when the first-order $\delta D_A(z_s)$ is substituted, we have a product of the first-order lensing term and gradients of the gravitational potential, which gives the dominant contribution to second-order lensing. By (6.216) this may be approximated as:

$$
\begin{aligned}
&W_{\delta D_A\times\Phi_\parallel}(\chi_s,\chi,k_1,k_2) &&(6.354)\\
&= (\ell_2(\ell_2+1))\,(\ell_1(\ell_1+1))^{\frac{3}{2}}\int_0^{\chi_s}\frac{(\chi_s-\chi)}{\chi_s}\\
&\times\Bigg[\Bigg[-\frac{1}{\chi}T_\Phi(\eta_2,k_2)j_{\ell_2}(k_2\chi_2)\int_0^{\lambda}\frac{(\chi_s-\chi_1)}{\chi_1^2}T_\Phi(\eta_1,k_1)j_{\ell_1}(k_1\chi_1)\mathrm{d}\,\lambda_1\\
&-4\int_0^{\lambda}\frac{1}{\chi^2}T_\Phi(\lambda_2,k_2)j_{\ell_2}(k_2\chi_2)\mathrm{d}\,\lambda_2\int_0^{\lambda}\frac{1}{\chi}T_\Phi(\lambda_1,k_1)j_{\ell_1}(k_1\chi_1)\mathrm{d}\,\lambda_1\Bigg]\\
&\times\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ 0 & 0, & 0\end{pmatrix}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ 1 & -1, & 0\end{pmatrix}\Bigg]\\
&W_{\delta D_A\times\Phi_\perp}(\chi_s,\chi,k_1,k_2) &&(6.355)\\
&= -\,(\ell_1(\ell_1+1))\,(\ell_2(\ell_2+1))\int_0^{\chi_s}\frac{(\chi_s-\chi)}{\chi_s}\frac{1}{\chi^2}T_\Phi(\eta_2,k_2)j_{\ell_2}(k_2\chi_2)\\
&\times\int^{\lambda}\frac{(\chi_s-\chi_1)}{\chi_1}T_\Phi(\lambda_1,k_1)j_{\ell_1}(k_1\chi_1)\mathrm{d}\,\lambda_1 &&(6.356)\\
&\times\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ 0 & 0, & 0\end{pmatrix}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ 0 & 0, & 0\end{pmatrix}\mathrm{d}\,\chi
\end{aligned}
$$

This double-integrated term has some very significant contributions. The first two terms couple transverse derivatives of the density fluctuations to the transverse velocity integrated along the line of sight. These offer the potential to measure transverse velocities as these terms can be very significant. The final term in this integral is a coupling between all the density fluctuations along the line of sight, and when the density contrast is $\mathcal{O}(1)$ can easily be comparable to the main first-order lensing term. Note that similar terms appear in the integrated shear term (6.200). We can estimate the magnitude of these terms as follows. Assuming a density contrast with a top-hat profile we can evaluate equation (6.355). This forms the second-order lensing convergence $\kappa^{(2)}$. Comparing to the standard first-order convergence κ we find, provided the distance to the source is much larger than the width of the inhomogeneity,

$$\kappa^{(2)} = C\frac{\chi_s}{\chi}\kappa^2, \tag{6.357}$$

where χ is the distance to the lens and C is a constant of order $\mathcal{O}(1)$. Thus, for configurations where $\kappa \gtrsim 0.1$ we can expect changes to the magnification $\mathcal{O}(10\%)$. We re-iterate that we have not used the Einstein field equations in our derivation, so this estimate should hold into the weakly non-linear regime containing significant over-densities.

While the other terms in the distance-redshift expansion are generally small, they offer a rich variety of new physical effects to be understood. In particular, there are many terms which contribute to an ISW-type of effect, but now it involves integrals over the scalar, vector and tensor potentials at second-order, as well as the first-order potential squared, together with its radial and tangential derivatives. Furthermore, we have also identified several instances of double integrated SW terms in both the redshift space distortions and in the distance-redshift relation. These may be important in further refining our understanding of dark energy.

A final important relativistic effect which can be probed with the formalism presented above relates to the interpretation of the background model itself. Measurements of distances of supernovae, for example, are fitted to the background model via an all-sky average of the distance-redshift relation. This includes the monopole of the second-order corrections presented here, which may include non-trivial corrections to the background. There has been considerable debate as to the size of these corrections – see [247] for a review. A closely related monopole contribution was estimated in [248] and found to be small, although others [60] claim this quantity diverges in the UV. It is an important open problem to dermine this quantity.

6.13 Appendix:Some Basic Decomposition rules and Techniques of Integration used

For vectors and tensor that appear in the calculation, we decompose them as follows:

$$\omega^a = \omega_{||} n^a + \omega^a_{\perp}, \tag{6.358}$$

$$v^a = v_{||} n^a + v^a_{\perp}, \tag{6.359}$$

$$\mathrm{h}_{ab} = \mathrm{h}_{||}\left(n_a n_b - \frac{1}{2}N_{ab}\right) + 2\mathrm{h}_{\perp(a}n_{b)} + \mathrm{h}_{\perp ab}. \tag{6.360}$$

Each of the screen space projected term is defined according to

$$\omega_{||} \equiv \omega_a n^a\,, \qquad \omega_{\perp a} \equiv N^b_a \omega_b\,, \qquad v_{||} \equiv v_a n^a \tag{6.361}$$
$$v_{\perp a} \equiv N^b_a v_b\,, \qquad \mathrm{h}_{||} \equiv n^a n^b \mathrm{h}_{ab},\,, \qquad \mathrm{h}_{\perp a} \equiv N_a{}^b n^c \mathrm{h}_{bc}$$
$$\mathrm{h}_{\perp ab} \equiv \mathrm{h}_{\langle ab\rangle} \equiv \left(N_{(a}{}^c N_{b)}{}^d - \frac{1}{2}N_{ab}N^{cd}\right)\mathrm{h}_{cd}.$$

We decompose the covariant/partial derivatives on the background tangent space according to [249, 250]

$$\mathrm{D}_a\Phi = n_a\partial_{||}\Phi + \nabla_{\perp a}\Phi = \Phi' n_a - \frac{\mathrm{d}\,\Phi}{\mathrm{d}\,\lambda}n_a + \nabla_{\perp a}\Phi \tag{6.362}$$

$$\mathrm{D}_a\omega_{\perp b} = n_a\partial_{||}\omega_{\perp b} - \frac{1}{\chi}\omega_a n_b + \nabla_{\perp a}\omega_{\perp b} \tag{6.363}$$

$$= \omega'_{\perp b}n_a - \frac{\mathrm{d}\,\omega_{\perp b}}{\mathrm{d}\,\lambda}n_a - \frac{1}{\chi}\omega_a n_b + \nabla_{\perp a}\omega_{\perp b} \tag{6.364}$$

$$\mathrm{D}_a v_{\perp b} = n_a\partial_{||}v_{\perp b} - \frac{1}{\chi}v_a n_b + \nabla_{\perp a}v_{\perp b} \tag{6.365}$$

$$= v'_{\perp b}n_a - \frac{\mathrm{d}\,v_{\perp b}}{\mathrm{d}\,\lambda}n_a - \frac{1}{\chi}v_a n_b + \nabla_{\perp a}v_{\perp b} \tag{6.366}$$

$$\mathrm{D}_a\mathrm{h}_{\perp bc} = n_a\partial_{||}\mathrm{h}_{\perp bc} - \frac{2}{\chi}\mathrm{h}_{a(c}n_{b)} + \nabla_{\perp a}\mathrm{h}_{\perp bc} \tag{6.367}$$

$$= \mathrm{h}'_{\perp bc}n_a - \frac{\mathrm{d}\,\mathrm{h}_{\perp bc}}{\mathrm{d}\,\lambda}n_a - \frac{2}{\chi}\mathrm{h}_{a(c}n_{b)} + \nabla_{\perp a}\mathrm{h}_{\perp bc} \tag{6.368}$$

Also for double spatial derivatives we have

$$\mathrm{D}_a\mathrm{D}_b\Phi = \mathrm{D}_b\partial_{||}\Phi n_a + \mathrm{D}_b\nabla_{\perp a}\Phi \tag{6.369}$$

$$= n_a n_b\partial^2_{||}\Phi + 2n_{(a}\nabla_{\perp b)}\partial_{||}\Phi + \nabla_{\perp a}\nabla_{\perp b}\Phi \tag{6.370}$$

(6.371)

where we have used the identity

$$\mathrm{D}_a n_b = (h_{ab} - n_a n_b) = \frac{1}{(\lambda - \lambda_o)}(\delta_{ab} - n_a n_b) = \frac{1}{(\chi)}(\delta_{ab} - n_a n_b) \tag{6.372}$$

This implies that with respect to the derivative on the 2-sphere we have

$$\nabla_{\perp a}\Phi = \frac{1}{\chi}\nabla_{a\Omega}\Phi \tag{6.373}$$

$$\nabla_{\perp a}\nabla_{\perp b}\Phi = \frac{1}{\chi}(\delta_{ab} - n_a n_b)\,\partial_{||}\Phi + \frac{1}{\chi^2}\nabla_{\Omega a}\nabla_{\Omega b}\Phi \tag{6.374}$$

$$\nabla^2_{\perp}\Phi = \frac{1}{\chi^2}\nabla^2_{\Omega}\Phi + \frac{2}{\chi}\partial_{||}\Phi \tag{6.375}$$

we made use of the relation to replace redial derivatives

$$n^k \mathrm{D}_k \Phi = \partial_{||}\Phi = \Phi' - \frac{\mathrm{d}\,\Phi}{\mathrm{d}\,\lambda} \tag{6.376}$$

$$n^k n^l \mathrm{D}_k \mathrm{D}_l \Phi = \partial_{||}^2 \Phi = \frac{\mathrm{d}^2\Phi}{\mathrm{d}\,\lambda^2} - 2\frac{\mathrm{d}\,\Phi'}{\mathrm{d}\,\lambda} + \Phi'' \tag{6.377}$$

The projected trace-free partial derivative of a scalar is defined as

$$\nabla_{\perp\langle i}\nabla_{\perp j\rangle}\Phi = \left(N_i{}^e N_j{}^f - \frac{1}{2}N_{ij}N^{ef}\right)\mathrm{D}_e\mathrm{D}_f\Phi = \nabla_{\Omega\langle i}\nabla_{\Omega j\rangle}\Phi \tag{6.378}$$

For the integrals we used severally the basic rule for simplifying double iterated integrals

$$\int_{\lambda_o}^{\lambda_s}\int_{\lambda_o}^{\lambda} A(\lambda)\mathrm{d}\,\lambda = \int_{\lambda_o}^{\lambda_s}(\lambda_s - \lambda)A(\lambda)\mathrm{d}\,\lambda \tag{6.379}$$

At first order in perturbation theory, we used for example

$$\int_{\lambda_o}^{\lambda_s}(\lambda_s - \lambda)(\lambda - \lambda_o)\frac{\mathrm{d}^2\Phi}{\mathrm{d}\,\lambda^2}\mathrm{d}\,\lambda = (\lambda_s - \lambda_o)(\Phi_s + \Phi_o) - 2\int_{\lambda_o}^{\lambda_s}\Phi\mathrm{d}\,\lambda \tag{6.380}$$

$$\int_{\lambda_o}^{\lambda_s}(\lambda_s - \lambda)(\lambda - \lambda_o)\frac{\mathrm{d}\,\Phi}{\mathrm{d}\,\lambda}\mathrm{d}\,\lambda = \int_{\lambda_o}^{\lambda_s}(\lambda - \lambda_o)\Phi - \int_{\lambda_o}^{\lambda_s}(\lambda_s - \lambda)\Phi\mathrm{d}\,\lambda \tag{6.381}$$

$$= (\lambda_s - \lambda_o)\int_{\lambda_o}^{\lambda_s}\Phi\mathrm{d}\,\lambda - 2\int_{\lambda_o}^{\lambda_s}(\lambda_s - \lambda)\Phi\mathrm{d}\,\lambda. \tag{6.382}$$

$$\int_{\lambda_o}^{\lambda}(\lambda - \lambda_o)\frac{\mathrm{d}^2\Phi}{\mathrm{d}\,\lambda^2}\mathrm{d}\,\lambda = (\lambda - \lambda_o)\frac{\mathrm{d}\,\Phi}{\mathrm{d}\,\lambda} - (\Phi - \Phi_o) \tag{6.383}$$

$$\int_{\lambda_o}^{\lambda}(\lambda - \lambda_o)\frac{\mathrm{d}\,\Phi}{\mathrm{d}\,\lambda}\Phi\mathrm{d}\,\lambda = \frac{1}{2}(\lambda - \lambda_o)\Phi^2 - \frac{1}{2}\int_{\lambda_o}^{\lambda}\Phi^2\mathrm{d}\,\lambda. \tag{6.384}$$

Using the following results which were obtained by performing several integration by parts

$$\int_{\lambda_o}^{\lambda_s}(\lambda_s - \lambda)(\lambda - \lambda_o)\frac{\mathrm{d}^2 h_{\|}}{\mathrm{d}\,\lambda^2}\mathrm{d}\,\lambda = (\lambda_s - \lambda_o)(h_{\|_s} + h_{\|_o}) - 2\int_{\lambda_o}^{\lambda_s}h_{\|}\mathrm{d}\,\lambda \tag{6.385}$$

$$\int_{\lambda_o}^{\lambda_s}(\lambda_s - \lambda)(\lambda - \lambda_o)\frac{\mathrm{d}\,h'_{\|}}{\mathrm{d}\,\lambda}\mathrm{d}\,\lambda = \int_{\lambda_o}^{\lambda_s}(\lambda_s - \lambda)h'_{\|}\mathrm{d}\,\lambda - \int_{\lambda_o}^{\lambda_s}(\lambda - \lambda_o)h'_{\|}\mathrm{d}\,\lambda \tag{6.386}$$

$$\int_{\lambda_o}^{\lambda}(\lambda - \lambda_o)\frac{\mathrm{d}\,h_{\|}}{\mathrm{d}\,\lambda}\mathrm{d}\,\lambda = (\lambda - \lambda_o)h_{\|} - \int_{\lambda_o}^{\lambda}h_{\|}\mathrm{d}\,\lambda \tag{6.387}$$

The differentiation over a generic integral sign is performed according to

$$\frac{\mathrm{d}}{\mathrm{d}\,a}\int_{\phi(1)}^{\varphi(a)} f(x,a)\mathrm{d}\,x \quad = \quad f(\varphi(a),a)\frac{\mathrm{d}\,\varphi(a)}{\mathrm{d}\,a} - f(\psi,a)\frac{\mathrm{d}\,\phi(a)}{\mathrm{d}\,a} + \int_{\phi(a)}^{(\varphi(a)} \frac{\mathrm{d}}{\mathrm{d}\,a} f(x,a)\mathrm{d}\,x$$

and when the independent variable is the upper limit of the integral we use

$$\frac{\mathrm{d}}{\mathrm{d}\,a}\int_{b}^{a} f(x)\mathrm{d}\,x = f(a) \tag{6.388}$$

$$\frac{\mathrm{d}}{\mathrm{d}\,a}\int_{b}^{a} f(x)\mathrm{d}\,x = -f(b) \tag{6.389}$$

Chapter 7

Conclusion and Future Work

7.1 Conclusion

Averaging, even though it implicitly defines the FLRW space-time on which standard model of cosmology is built, it is not properly understood within general relativity. This is mainly due to non-linearity of the Einstein field equations and other equations that describe physical events on cosmological scales.

In Chapter 3, we showed how non-linear nature of the field equations leads to extra terms in the field equations after averaging and these extra terms are of the same size as a typical non-linear term the equation would have without any form of averaging. In fact in Chapter 5 we showed that these extra terms could lead to a cancellation of terms that diverge in the region of high over-density in Buchert formalism. This seem to suggest a strong connection as we shall see later between averaging and renormalization [251] required to make matter power spectrum which also diverge in high over-density region sensible.

The importance of averaging in cosmology have so far been downplayed because, the reach of our current cosmological observations especially that of the cosmic microwave background experiment can easily be explain using linearized equations only [34, 36–39, 191, 194, 252, 253]. It is important to re-iterate that the validity of linear approximation is only possible because the physics responsible for emission of the radiation and possible imprint of primordial inhomogeneities it carries took place on sub-horizon scales at early time. If we fast-forward to the late-time universe, this is no longer valid.

This is clearly evident from the analysis of the SDSS data for Baryon Acoustic Oscillation (BAO) [229, 230]. The analysis indicates that linear approximation is beginning to fall part especially when one approaches Hubble scale ($k_H \sim H^{-1}$). In fact in order to make sense of the SDSS data for BAO analysis, one is compelled to go beyond linear approximation to be able to reasonably account for the physics of matter power spectrum or the velocity distribution of galaxies at late-time [230].

Given the subtle connections we highlighted earlier between the renormalization required to cure the matter power spectrum today at non-linear level of its UV problem and the role spatial averaging plays as a renormalization technique, we are compelled to say that now is the most suitable time to begin talking about averaging problem as a problem that must be properly understood if we must reap a maximum benefit from planned big experiments of like EUCLID and SKA.

We discussed in Chapter 5 and 3, the large variety of ways that this problem is being investigated. Some argue it is simply irrelevant because it is suppressed by many orders of magnitude below the background model which consequently holds genuine physical significance. Others take the opposite view and contend that it holds the key to the whole dark energy issue. More moderate views prevail between these two extremes, pointing out there may be significant effects in intermediate scales of cosmological interest. It is also fair to point to studies where simplified but fully non-linear spacetimes have observational properties significantly different from their spatially averaged counterparts. And it is not clear that perturbation theory is genuinely convergent, and that it is a well behaved study tool to use for the evolution of the late-time universe.

This is important because small density contrast at early time does not guarantee a small inhomogeneity at late-time. Given the UV behavior of the four gradient term, does the perturbation theory properly converged at second order, where claims for tiny backreaction are made? At the very least, these considerations surely tell us that it is important to understand the averaging,backreaction and fitting problems to see what if any effects there may have a distinct cosmological signature. The point we make here is that there *are* some scales where backreaction may be important: such scale is not the largest scales relevant to the cosmic acceleration, but others where precision cosmology is significant. It is fair to argue that the effects we describe here is important at any scale where non-linear matter power spectrum is important, i.e you can't use non-linear power spectrum without taking into account the corresponding effect of backreaction from non-linear structures.

Finally, the results of Chapter 6 is expected to finally settle most of the 'size of the backreaction effect' issues because it is based on observables only and it is central to all cosmological analysis. A preliminary study by [236, 248] indicates that the k-integrals converge which is in agreement with our results in Chapter 6. They reported a backreaction effect of the order 10^{-3} in a ΛCDM universe. This is a much bigger effect when compared to the prediction of studies based on the effective field approach, who claim that backreaction effect is small [62, 153] (see also figure 5.1) and it is of the same order as the prediction of studies based on the Buchert formalism (see figure 5.5 for the deceleration parameter) and smaller than the predictions based on the Kristian and Sachs formalism (see figure 5.9). It is also important to note that although the the k-integrals converge, there is a subtle *short distance* problem lurking around which was not investigated in [236, 248]. It is likely that the k-integral problem now manifests as short distance problem.

7.2 Future Works

- *The role of* $(\partial^2\Phi)^2$ *at second-order*

  How should such terms that appear in equations **??** be dealt with? Do they signify that something unphysical has been calculated, or do they signify a breakdown in perturbation theory on scales in which they are extra-ordinarily large? These terms also appear in the expression for the area distance which do not involve general relativity. We noticed that when decomposed on a celestial sphere, the $k-$divergence associated with this term move to a short distance divergence after Limber approximation have been used. Does this signify a natural break-down of our perturbation theory. If this is a break-down, how shall we proceed? Should perturbations be smoothed on a given scale at each order before calculating the next, which can then go to smaller scales? Or should we abandon perturbation theory entirely.

- *Non-Gaussian perturbations*

  Many of the key results we discussed here, rely on a Gaussian initial spectrum. What happens for more general initial conditions? Should a small tilt in the spectral index lead to a large backreaction effect. How would backreaction effect affect the proposed measurement of bispectrum from large scale galaxy clustering [254–257]

- *Relativistic simulations.*

  Is a fully general relativistic simulation out of the question? This would have to be attempted very differently from current N-body simulations, but could be attempted on the expectation of improving computing power. Rather than work with point particles, one could start with the ADM equations with different fluids, plus perturbations laid down early. A simulation could consist of perturbative modes up to a maximum wavenumber, rather than N particles of a given mass. The resolution would be determined by the maximum wavenumber, so simulations would start big and get smaller as computing power improved.

- *Averaging observables*

The role of observables and the lightcone is still very unclear in the averaging scenario. Observable quantities such as distances are usually implicitly sky-averaged quantities. While it is not clear what an averaged lightcone might mean, sky-averages of observables such as the redshift-distance relation have a reasonably clear meaning, even in a very inhomogeneous universe. One way to understand the connection better will be to compare the results of light-cone averaging given in [236, 248] for the distance-redshift relation with our sky-averaged result for the same observable given in Chapter 6 .

- *Numerical Calculation*

  The Chapter 6 contains complicated and scary looking formulas for most people and it makes little or no sense to an average observational cosmologist to whom it is intended for. Hence the next thing we plan to do is to have the analytical results in Chapter 6 calculated numerically and a possible friendly-looking and easy to use fitting formula generated.

Appendix A

Introduction to Cosmological Perturbation theory

A.1 Gauge Problem at second order

General covariance i.e. the requirement that physics should be independent of a particular choice of coordinates system characterizing the space time is different from gauge invariance, the later refers to fixing the degrees of freedom of gauge fields introduced to describe the dynamics of the spacetime on a different background spacetime. Gauge problem is about fixing the relevant dynamical degrees of freedom for the irreducible decompositions of the gauge fields introduced to a physical system.

At first order in cosmological perturbation theory, Bardeen [258] showed that this is possible, while at second order this problem has not been convincingly resolved. In general, gauge problem in cosmology refers to a mathematical difficulty associated with the redundant degrees of freedom from gauge fields introduced to describe a perturbed dynamical spacetime which must satisfy general covariance.

One way to deal with this problem is to choose a physically motivated gauge and calculate physical quantities base on it. For example, one could choose the Newtonian gauge because it provides a better approximation to the Newtonian limit of a relativistic theory. Although for density perturbation well inside the well inside the horizon ($k \gg H/a$), there are no gauge ambiguities. The choice of gauge becomes important becomes important when the distance scale between two points is of the order of the Hubble scale.

The other solution to the gauge problem is due to Nakamura [68], it involves decomposing every tensor defined on the manifold into gauge invariant and gauge variant parts. However, this solution is a generalization of the traditional approach in a more rigorous way.

The concept of gauge transformation in cosmological perturbation theory was properly articulated by [63]. The idea is to consider a $4+1$ dimensional manifold $\mathcal{N} = \mathcal{M} \times \mathbb{R}$, with the dimension of the manifold equal to 4 and $\lambda = \mathbb{R}$. The projection $\mathcal{P}_\lambda$ on the manifold $\mathcal{N}$ defines the sub-manifolds $\mathcal{M}_0 = \mathcal{N}|_{\lambda=0}$ and $\mathcal{M}_\lambda = \mathcal{N}|_{\mathbb{R}=\lambda}$ This set-up defines a foliation of $\mathcal{N}$ into a collection of four dimensional sub-manifold $\mathcal{M}_\lambda$ with distinct λ. Each of these elements is diffeomorphic to the physical space-time $\mathcal{M}$ and the background space-time $\mathcal{M}_0$. Normally a vector field X is defined on $\mathcal{N}$ to ensure that perturbations are continuous such that $(\mathcal{M}, \bar{g}_{ab})$ and $(\mathcal{M}_0, \bar{g}_{ab})$ are connected by a continuous curve along the component of the space-time slicing $\mathbb{R}$. This vector field induces a one parameter family of diffeomorphisms or an exponential map, $\phi(\lambda, .)$, that maps $\phi(0, p \in \mathcal{P}_0(\mathcal{N})) = p \in \mathcal{P}_0(\mathcal{N})$ along the integral curves to $\phi\left(\lambda, p \in \mathcal{P}_0(\mathcal{N})\right) = q \in \mathcal{P}_\lambda(\mathcal{N})$. Notice that the choice of this gauge vector field X is arbitrary and then should not have a physical meaning, and this is of course the well known *gauge freedom or gauge choice.*

This exponential map induces a transport along the flow for tensors living on the tangent bundle, which is determined by the push-forward $\phi_{\star\lambda}$ and the pull-back $\phi^\star_\lambda$ associated with an element ϕ_λ of the group of diffeomorphisms.

The pull-back $\phi^\star_\lambda$ is related to the vector field,X, by Taylor expansion of any tensor T ,[63]

$$\phi^\star_{X,\lambda}(T) = \sum_{k=0}^{k=\infty} \frac{\lambda^k}{k!} \mathcal{L}^k_X T, \tag{A.1}$$

The expansion of equation (A.1) on $\mathcal{P}_0(\mathcal{N})$ provides a way to compare a tensor field on $\mathcal{P}_\lambda(\mathcal{N})$ to the corresponding one on the background space-time $\mathcal{P}_0(\mathcal{N})$. The perturbation of an tensor is defined as:

$$_XT_\lambda \equiv T_0 + \sum_{k=1}^{k=\infty} \frac{\lambda^k}{k!} \mathcal{L}^k_X T\Big|_{\mathcal{P}_0(\mathcal{N})}, \tag{A.2}$$

where $T_0 \equiv \mathcal{L}^0_X T|_{\mathcal{P}_0(\mathcal{N})}$ is the background value and the n-th order perturbation of a physical tensor field is given by $_XT^{(n)} \equiv \mathcal{L}^n_X T|_{\mathcal{P}_0(\mathcal{N})}$.

If we consider two different gauge choices X and Y with generators given by X_{η^a} and Y_{η^a} and we also suppose that they have different tangential components to each $\mathcal{M}_\lambda$. This gauge transformation from X to Y could be given by an exponential map or by the diffeomorphism [63], $\phi_{X\to Y,\lambda} = (\phi_{X,\lambda})^{-1}(\phi_{Y,\lambda}) : \mathcal{M}_0 \to \mathcal{M}_0$,This induces a pull-back that carries the tensor perturbation, $\Delta_X T_\lambda$, of the gauge choice X, to another tensor perturbation $\Delta_Y T_\lambda$, of the gauge choice Y. This point displacement by $\phi_{X\to Y,\lambda}$ is carried along a sequence of vector fields ξ_n, that generate nth parameter family of metrics. To second order, the so-called knight-diffeomorphism is given by

$$\phi^\star_{Y,\lambda}(T) = \phi^\star_{X\to Y,\lambda}\phi^\star_{X,\lambda}(T) = \phi^\star_{X,\lambda}(T) + \lambda\mathcal{L}_{\xi_1}\phi^\star_{X,\lambda}(T) + \frac{\lambda^2}{2!}(\mathcal{L}_{\xi_2} + \mathcal{L}^2_{\xi_1})\phi^\star_{X,\lambda}(T) + \ldots$$

The knight vector fields ξ_1, ξ_2 are related to the gauge vector fields X and Y by $\xi_1 = Y - X$ and $\xi_2 = [X, Y]$. Using equation (A.2), and order by order identification in λ, the transformation rules up to second order perturbation theory is obtained.

$$\begin{aligned} {}_YT^{(1)} - {}_XT^{(1)} &= \mathcal{L}_{\xi_1} T_0, \\ {}_YT^{(2)} - {}_XT^{(2)} &= 2\mathcal{L}_{\xi_1}\, {}_XT^{(1)} + (\mathcal{L}_{\xi_2} + \mathcal{L}^2_{\xi_1}) T_0 \,. \end{aligned} \tag{A.3}$$

The transformation rules (equation (A.3)), tells us that a tensor T is gauge-invariant up to n-th order if it satisfies $\mathcal{L}_\xi\, {}_XT^{(r)} = 0$ for any vector field ξ and any $r \leq n$. This implies that a tensor is gauge-invariant up to order n if and only if T_0 and all its perturbations of order lower than n either vanish, or are constant scalars, or are combinations of Kronecker deltas with constant coefficients.

A.1.1 Gauge invariant combination at second order

At first order, there are clever ways of selecting gauge invariant terms [69, 258, 259], however at second order, such methods are not trivial. Here, we will follow the method proposed by [66] and then relate it to the traditional method later on. The procedure goes like, given a metric, g_{ab} of a physical spacetime $\mathcal{M}_\lambda$ with a pull back on the background $\mathcal{M}_0$, we can expand this metric in *any gauge* to any other in perturbation theory, for example, to second order we have,

$$\phi^\star_{X,\lambda}(g_{ab}) = g_{ab} + \lambda\, {}_Xh_{ab} + \frac{\lambda^2}{2}\, {}_Xl_{ab} + \mathcal{O}(\lambda). \tag{A.4}$$

Then for tensor h_{ab} characterizing the first order perturbation in Eq. (A.4), we will suppose that there exist a tensor $\tilde{h}_{ab}$ and a vector X^a_ξ such that the tensor h_{ab} may be decomposed as

$$h_{ab} \equiv \tilde{h}_{ab} + \mathcal{L}_X g_{ab}, \tag{A.5}$$

we then impose that $\tilde{h}_{ab}$ and X^a_ξ satisfy the transformation ${}_Y\tilde{h}_{ab} - {}_X\tilde{h}_{ab} = 0$, ${}_YX^a_\xi - {}_XX^a_\xi = \xi^a_{(1)}$, where $\tilde{h}_{ab}$ and X^a_ξ can easily be identified as the gauge invariant and gauge variant part of the parent tensor h_{ab}. This can be interpreted as the formalization of the traditional way of combining gauge invariant variables as we shall see later. For second order metric perturbation, l_{ab} from Eq.A.4, may be decomposed it

$$l_{ab} \equiv \tilde{l}_{ab} + 2\mathcal{L}_X h_{ab} + (\mathcal{L}_Y - \mathcal{L}^2_X) g_{ab} \tag{A.6}$$

However, Nakamura [66, 67, 260] showed that by defining a new variable as $\hat{L}_{ab} \equiv \tilde{l}_{ab} + 2\mathcal{L}_X h_{ab} + \mathcal{L}^2_X g_{ab}$ that the transformed tensor may be put in the form of the first order transformation rule as ${}_Y\hat{L}_{ab} - {}_X\hat{L}_{ab} = \mathcal{L}_\sigma g_{ab}$, with $\sigma^a = \xi^a_{(2)} + \left[\xi_{(1)}, X_\xi\right]^a$, hence may be decomposed as

$$\hat{L}_{ab} \equiv \tilde{l}_{ab} + \mathcal{L}_Y g_{ab}, \tag{A.7}$$

where ${}_Y\tilde{l}_{ab} - {}_X\tilde{l}_{ab} = 0$, ${}_YY^a_\xi - {}_XY^a_\xi = \sigma^a = \xi^a_{(2)} + \left[\xi_{(1)}, X_\xi\right]^a$. The ability to put the second order transformation in the form of the first order, tells us that we can use at second order the same method used to define gauge invariant variables at first order. For an arbitrary tensor, the transformation is defined as

$$\begin{aligned} {}^{(1)}T &\equiv {}^{(1)}T - \mathcal{L}_X T_0 & \text{(A.8)}\\ {}^{(2)}T &\equiv {}^{(2)}T - 2\mathcal{L}_X^{(1)} T - \left(\mathcal{L}_Y - \mathcal{L}_X^2\right) T_0 & \text{(A.9)}\end{aligned}$$

which may be decomposed into gauge invariant and gauge variant part as

$$\begin{aligned} {}^{(1)}T &\equiv {}^{(1)}\tilde{T} + \mathcal{L}_X T_0 & \text{(A.10)}\\ {}^{(2)}T &\equiv {}^{(2)}\tilde{T} + 2\mathcal{L}_X^{(1)} T + \left(\mathcal{L}_Y - \mathcal{L}_X^2\right) T_0 & \text{(A.11)}\end{aligned}$$

A.1.2 Gauge Invariant variables and Cosmological perturbation

The most general metric for an almost FLRW universe is given by

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -(1+2\Phi)\mathrm{d}t^2 + 2a\omega_i \mathrm{d}x^i \mathrm{d}t + [a^2(1-2\Psi)\delta_{ij} + h_{ij}]\mathrm{d}x^i \mathrm{d}x^j, \tag{A.12}$$

where a is the scale factor of the universe and transformation to a conformal metric is achieve by setting $\mathrm{d}t = a\mathrm{d}\eta$ with $\eta =$ conformal time . We may perform a scalar-vector-tensor (SVT) decomposition as

$$\begin{aligned} \omega_i &= \partial_i B + B_i, & \text{(A.13)}\\ h_{ij} &= 2H_{ij} + \partial_i E_j + \partial_j E_i + 2\partial_i\partial_j E, & \text{(A.14)}\end{aligned}$$

where B_i, E_i and H_{ij} are transverse ($\partial^i E_i = \partial^j B_i = \partial^i H_{ij} = 0$), and H_{ij} is traceless ($H^i_j = 0$). There are four scalar degrees of freedom (Φ, Ψ, B, E), four vector degrees of freedom (B_i, E_i) and two tensor degrees of freedom (H_{ij}). The indices of the perturbed variables are lowered and raised by the spatial section of the background metric and its inverse, i.e δ^{ij} and δ_{ij}. To include the effect of non-linearity, we need to expand up to second-order, $W = W^{(1)} + \frac{1}{2}W^{(2)}$ (we have assumed that $W^{(0)}$ vanishes). We will follow the notations of [245] and split the vector fields ξ_1^μ and ξ_2^μ as follows

$$\xi_1{}^o = T^{(1)}, \quad \xi_1^i = \partial^i L^{(1)}, \quad \xi_2{}^o = T^{(2)}, \quad \xi_2^i = \partial^i L^{(2)} + L^{i(2)}\,, \tag{A.15}$$

with $\partial_i L^{i(2)} = 0$.

The gauge transformation of first-order terms in the metric (A.12) are given in the table below: Here prime denotes a derivative w.r.t. to conformal time η, and where $\mathcal{H} \equiv a'/a$.

Table A.1: First order gauge transformation of the metric

First order scalar perturbations	First order vector perturbations
$\Phi^{(1)} \to \Phi^{(1)} + T'^{(1)} + \mathcal{H}T^{(1)}$	$B_{\rm i}^{(1)} \to B_{\rm i}^{(1)} - L_{\rm i}'^{(1)}$
$B^{(1)} \to B^{(1)} - T^{(1)} + L'^{(1)}$	$E_{\rm i}^{(1)} \to E_{\rm i}^{(1)} + L_{\rm i}'^{(1)}$
$\Psi^{(1)} \to \Psi^{(1)} - \mathcal{H}T^{(1)}$	First order tensor perturbation
$E^{(1)} \to E^{(1)} + L^{(1)}$	$H_{\rm ij}^{(1)} \to H_{\rm ij}^{(1)}$

Note that the first-order tensorial modes are automatically gauge invariant. In cosmology, the tradition way of constructing gauge invariant variables at first order is to set the redundant degrees of freedom to zero [258, 259]. It is conventional to neglect the vector mode at first order.

Following [258], we can define a set of gauge invariant variables at first order by setting $T^{(1)} = B^{(1)} - E'^{(1)}$, and $L^{(1)} = -E^{(1)}$, this leads to a gauge invariant combinations.

$$\Phi^{(1)} \;\to\; \hat{\Phi}^{(1)} \equiv \Phi^{(1)} + \mathcal{H}\left(B^{(1)} - E'^{(1)}\right) + \left(B^{(1)} - E'^{(1)}\right)' \tag{A.16}$$

$$\Psi^{(1)} \;\to\; \hat{\Psi}^{(1)} \equiv \Psi^{(1)} - \mathcal{H}\left(B^{(1)} - E'^{(1)}\right) \tag{A.17}$$

$$B^{(1)} \;\to\; 0 \tag{A.18}$$

$$E^{(1)} \;\to\; 0 \tag{A.19}$$

$$H_{\rm ij}^{(1)} \;\to\; H_{\rm ij}^{(1)}. \tag{A.20}$$

When substituted into GR, we immediately see that the redundant modes drop out, we can also set $B^{(1)}$ and $E^{(1)}$ to zero and obtain the perturbed Einstein equations only as a function of gauge-invariant variables..

In relation to Nakamura's terminology of decomposing the perturbed metric into a gauge-invariant part and a gauge variant part , $B^{(1)}$ and $E^{(1)}$ are the two gauge variant variables of the metric perturbation while $\hat{\Phi}^{(1)}$ and $\hat{\Psi}^{(1)}$ are the gauge-invariant part.

At second order, the perturbations transform as follows:

Table A.2: Second-order gauge-invariant variables

Second order scalar perturbations	Second order vector perturbations
$\Phi^{(2)} \to \Phi^{(2)} + T'^{(2)} + \mathcal{H}T^{(2)} + S_\Phi$	$B_i^{(2)} \to B_i^{(2)} - L_{\rm i}^{'(2)} + S_{B_i}$
$B^{(2)} \to B^{(2)} - T^{(2)} + L'^{(2)} + S_B$	$E_i^{(2)} \to E_i^{(2)} + L_{\rm i}^{(2)} + S_{E_{\rm i}}$
$\Psi^{(2)} \to \Psi^{(2)} - \mathcal{H}T^{(2)} + S_\Psi$	Second order tensor perturbations
$E^{(2)} \to E^{(2)} + L^{(2)} + S_E$	$H_{\rm ij}^{(2)} \to H_{\rm ij}^{(2)} + S_{H_{\rm ij}}$

The notations $S_\Phi, S_B, S_\Psi, S_E, S_{B_{\rm i}}, S_{E_{\rm i}}, S_{H_{\rm ij}}$ represent terms quadratic in first order perturbations and decomposed gauge vector field, they are given later in the appendix.

The second order transformation rules are much more complicated than their first-order counterparts. However, Nakamura suggest we make we make a transformation to put them in the form of first order transformation by defining $\hat{L}_{ab} \equiv \tilde{l}_{ab} + 2\mathcal{L}_X h_{ab} + \mathcal{L}_X^2 g_{ab}$, which leads ${}_Y\hat{L}_{ab} - {}_X\hat{L}_{ab} = \mathcal{L}_\sigma g_{ab}$, with $\sigma^a = \xi^a_{(2)} + \left[\xi_{(1)}, X_\xi\right]^a$. With this transformation, we will be dealing with σ^a instead of $\xi_{(1)}$ as in first order treatment and it also if we we implement our first order gauge choice[1] , in our case the Newtonian gauge X_ξ, i.e $X_\xi \to \xi_{\to\text{New. gau}}$.

Table A.3: Transformed Second-order variables

Second order scalar perturbations	Second order vector perturbations
$\Phi_{\hat{L}} \equiv \Phi^{(2)} + S_\Phi(\xi^{(1)}_{\to\text{New.gau}})$	$B_{\hat{L}_{\rm i}} \equiv B_{\rm i}^{(2)} + S_{B_{\rm i}}(\xi^{(1)}_{\to\text{New.gau}})$
$B_{\hat{L}} \equiv B^{(2)} + S_B(\xi^{(1)}_{\to\text{New.gau}})$	$E_{\hat{L}_{\rm i}} \equiv E_{\rm i}^{(2)} + S_{E_{\rm i}}(\xi^{(1)}_{\to\text{New.gau}})$
$\Psi_{\hat{L}} \equiv \Psi^{(2)} + S_\Psi(\xi^{(1)}_{\to\text{New.gau}})$	Second order tensor perturbations
$E_{\hat{L}} \equiv E^{(2)} + S_E(\xi^{(1)}_{\to\text{New.gau}})$	$H_{\hat{L}_{\rm ij}} \equiv H_{\rm ij}^{(2)} + S_{H_{\rm ij}}(\xi^{(1)}_{\to\text{New.gau}})$

With these transformations gauge-invariant combination at second order can easily be constructed as

$$
\begin{aligned}
\hat{\Phi}^{(2)} &\equiv \Phi_{\hat{L}} + \left(B_{\hat{L}} - E'_{\hat{L}}\right)' + \mathcal{H}\left(B_{\hat{L}} - E'_{\hat{L}}\right) \\
\hat{\Psi}^{(2)} &\equiv \Psi_{\hat{L}} - \mathcal{H}\left(B_{\hat{L}} - E'_{\hat{L}}\right) \\
\hat{B}^{(2)}_{\rm i} &\equiv B_{\hat{L}_{\rm i}} - E'_{\hat{L}_{\rm i}} \\
\hat{H}^{(2)}_{\rm ij} &\equiv H_{\hat{L}_{\rm ij}}.
\end{aligned} \tag{A.21}
$$

Enforcing the condition $X_\xi \to \xi_{\to\text{New. gau}}$ with Nakamura formalism is equivalent to trans-

[1]The important clause here is that we will construct gauge invariant variables at second order easily if we choose at second order the same gauge we chose at first order

forming quantities in the Newtonian gauge all the way to second order, since it basically transforms B, E and $E^{\rm i}$ into a null value up to second order. As we can see that performing the transformation (below) defined by $\xi^{(2)}_{\to {\rm New.gau}}$ in Table A.3 will lead to the same result as in Eq. (A.21).

$$\begin{aligned} T^{(2)}_{\to{\rm New.gau}} &= B^{(2)} - E^{'(2)} + S_B\left(\xi^{(1)}_{\to{\rm New.gau}}\right) - S^{'}_E\left(\xi^{(1)}_{\to{\rm New.gau}}\right) \\ L^{(2)}_{\to{\rm New.gau}} &= -E^{(2)} - S_E\left(\xi^{(1)}_{\to{\rm New.gau}}\right) \\ L^{{\rm i}(2)}_{\to{\rm New.gau}} &= -E^{{\rm i}(2)} - S_{E^{\rm i}}\left(\xi^{(1)}_{\to{\rm New.gau}}\right)\,. \end{aligned} \tag{A.22}$$

A.1.3 Source terms in second order transformation

The perturbation variables in the decomposition (A.12) are extracted as follows

$$\begin{aligned} \Phi^{(n)} &= -\frac{1}{2a^2} g^{(n)}_{zz}, \\ \Psi^{(n)} &= -\frac{1}{4a^2}{\rm P}_v{}^{\rm ij} g^{(n)}_{\rm ij}, \\ B^{(n)} &= \frac{1}{a^2}{\rm P}_s{}^{\rm i} g^{(n)}_{z{\rm i}}, \\ B^{(n)}_{\rm i} &= \frac{1}{a^2}{\rm P}_{v}{}_{\rm i}^{\rm j} g^{(n)}_{z{\rm j}}, \\ E^{(n)} &= \frac{1}{4a^2}\left(\Delta\Delta\right)^{-1}\left(3\partial^{\rm i}\partial^{\rm j} - \Delta\delta^{\rm ij}\right) g^{(n)}_{\rm ij}, \\ E^{(n)}_n &= \frac{1}{a^2}{\rm P}_{v}{}_n^{\rm l}{\rm P}_s{}^{\rm k}\left(\delta^{\rm i}_{\bf k}\delta^{\rm j}_{\rm l} - \frac{1}{3}\delta_{\rm kl}\delta^{\rm ij}\right) g^{(n)}_{\rm ij} \\ H^{(n)}_{mn} &= \frac{1}{2a^2}{\rm P}_{v}{}_m^{\rm k}{\rm P}_{v}{}_n^{\rm l}\left(\delta^{\rm i}_{\rm k}\delta^{\rm j}_{\rm l} - \frac{1}{3}\delta_{\rm kl}\delta^{\rm ij}\right) g^{(n)}_{\rm ij}\,, \end{aligned} \tag{A.23}$$

where $n = 1, 2$ is the order and where we have used the definitions

$$ {\rm P}_s{}^{\rm i} \equiv \Delta^{-1}\partial^{\rm i}\,, \qquad {\rm P}_v{}^{\rm ij} \equiv \delta^{\rm ij} - \Delta^{-1}\partial^{\rm i}\partial^{\rm j}\,. \tag{A.24}$$

Using this method we can read the source terms defined in equation (**??**), which are quadratic in the gauge change variables T, L and the perturbation variables $\Phi, \Psi, B, E, E_{\rm ij}$

$$\begin{aligned} S_\Phi &= T\left(T'' + 5\mathcal{H}T' + (\mathcal{H}' + 2\mathcal{H}^2)T + 4\mathcal{H}\Phi + 2\Phi'\right) + \partial_{\rm i}L'\partial^{\rm i}\left(T - 2B - L'\right) \quad \text{(A.25)} \\ &\quad + T'\left(2T' + 4\Phi\right) + \partial_{\rm i}L\partial^{\rm i}\left(T' + \mathcal{H}T + 2\Phi\right) \\ S_\Psi &= -T\left(\mathcal{H}T' + (\mathcal{H}' + 2\mathcal{H}^2)T - 2\Psi' - 4\mathcal{H}\Psi\right) - \partial_{\rm i}\left(\mathcal{H}T - 2\Psi\right)\partial^{\rm i}L \\ &\quad -\frac{1}{2}\left(\delta^{\rm ij} - \Delta^{-1}\partial^{\rm i}\partial^{\rm j}\right)\mathcal{X}_{\rm ij} \end{aligned} \tag{A.26}$$

$$
\begin{aligned}
S_B &= \mathrm{P}_s{}^{\mathrm{i}} \mathcal{X}_{\mathrm{i}} && \text{(A.27)}\\
S_{B_{\mathrm{i}}} &= \mathrm{P}_{v_{\mathrm{i}}}^{\ \mathrm{j}} \mathcal{X}_{\mathrm{j}} && \text{(A.28)}\\
S_E &= (\Delta\Delta)^{-1}\left(\frac{3}{2}\partial^{\mathrm{i}}\partial^{\mathrm{j}} - \frac{1}{2}\Delta\delta^{\mathrm{ij}}\right)\mathcal{X}_{\mathrm{ij}} && \text{(A.29)}\\
S_{E_n} &= 2\mathrm{P}_{v_n}^{\ \mathrm{l}}\mathrm{P}_s{}^{\mathrm{k}}\left(\delta^{\mathrm{i}}_{\mathrm{k}}\delta^{\mathrm{j}}_{\mathrm{l}} - \frac{1}{3}\delta_{\mathrm{kl}}\delta^{\mathrm{ij}}\right)\mathcal{X}_{\mathrm{ij}} && \text{(A.30)}\\
, S_{H_{mn}} &= \mathrm{P}_{v_n}^{\ \mathrm{l}}\mathrm{P}_{v_m}^{\ \mathrm{k}}\left(\delta^{\mathrm{i}}_{\mathrm{k}}\delta^{\mathrm{j}}_{\mathrm{l}} - \frac{1}{3}\delta_{\mathrm{kl}}\delta^{\mathrm{ij}}\right)\mathcal{X}_{\mathrm{ij}}\,, && \text{(A.31)}
\end{aligned}
$$

with

$$
\begin{aligned}
\mathcal{X}_{\mathrm{i}} \equiv\ & \{T'\partial_{\mathrm{i}}(2B + L' - T) + 2\mathcal{H}T\partial_{\mathrm{i}}(2B + L' - T)\\
&+\partial^{\mathrm{j}}L'\left[2\partial_{\mathrm{i}}\partial_{\mathrm{j}}L + 2\left(\mathcal{H}T - 2\Psi\right)\delta_{\mathrm{ij}} + 4\left(H_{\mathrm{ij}} + \partial_{\mathrm{i}}\partial_{\mathrm{j}}E\right)\right]\\
&+\partial^{\mathrm{j}}\partial_{\mathrm{i}}L\partial_{\mathrm{j}}\left(2B + L' - T\right) + \partial^{\mathrm{j}}L\partial_{\mathrm{j}}\partial_{\mathrm{i}}\left(2B + L' - T\right)\\
&+\partial_{\mathrm{i}}T(-4\Phi - 2T' - 2\mathcal{H}T) + T\partial_{\mathrm{i}}(2B' + L'' - T')\}\ , && \text{(A.32)}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{X}_{\mathrm{ij}} \equiv\ & \{\partial_{\mathrm{j}}\left(2B + L' - T\right)\partial_{\mathrm{i}}T + T\partial_{\mathrm{i}}\partial_{\mathrm{j}}(L' + 2\mathcal{H}L)\\
&+\partial_{\mathrm{i}}\partial^{\mathrm{k}}L\left[2\partial_{\mathrm{k}}\partial_{\mathrm{j}}L + 4\partial_{\mathrm{k}}\partial_{\mathrm{j}}E + 4H_{\mathbf{kj}} + (2\mathcal{H}T - 4\Psi)\delta_{\mathbf{kj}}\right]\\
&+T\left(2H'_{\mathrm{ij}} + 2\partial_{\mathrm{i}}\partial_{\mathrm{j}}E' + 4\mathcal{H}H_{\mathrm{ij}} + 4\mathcal{H}\partial_{\mathrm{i}}\partial_{\mathrm{j}}E\right)\\
&+\partial^{\mathrm{k}}L\partial_{\mathrm{k}}\left(\partial_{\mathrm{i}}\partial_{\mathrm{j}}L + 2H_{\mathrm{ij}} + 2\partial_{\mathrm{i}}\partial_{\mathrm{j}}E\right)\}\,. && \text{(A.33)}
\end{aligned}
$$

As for the matter perturbation variables, the source terms in the transformation rules are

$$
\begin{aligned}
S_\rho &= T(\bar\rho''T + \bar\rho'T' + 2\delta\rho') + \partial^{\mathrm{i}}L\partial_{\mathrm{i}}(2\delta\rho + \bar\rho'T) && \text{(A.34)}\\
S_P &= T(\bar P''T + \bar P'T' + 2\delta P') + \partial^{\mathrm{i}}L\partial_{\mathrm{i}}(2\delta P + \bar P'T) && \text{(A.35)}\\
S_V &= \mathrm{P}_{s\mathrm{i}}\left[\mathcal{H}T\partial^{\mathrm{i}}(L' - 2V) + T\partial^{\mathrm{i}}(2V' - L'') + \partial^{\mathrm{j}}(L' - 2V)\partial_{\mathrm{j}}\partial^{\mathrm{i}}L\right.\\
&\qquad\left. + L^{\mathrm{j}}\partial_{\mathrm{j}}\partial^{\mathrm{i}}(2V - L') + \partial^{\mathrm{i}}L'\left(\mathcal{H}T + T' + 2\Phi\right)\right] && \text{(A.36)}\\
S_{\tilde V^{\mathrm{k}}} &= \mathrm{P}_{v\,\mathrm{i}}^{\ \mathrm{k}}\left[\mathcal{H}T\partial^{\mathrm{i}}(L' - 2V) + T\partial^{\mathrm{i}}(2V' - L'') + \partial^{\mathrm{j}}(L' - 2V)\partial_{\mathrm{j}}\partial^{\mathrm{i}}L\right.\\
&\qquad\left. + L^{\mathrm{j}}\partial_{\mathrm{j}}\partial^{\mathrm{i}}(2V - L') + \partial^{\mathrm{i}}L'\left(\mathcal{H}T + T' + 2\Phi\right)\right] && \text{(A.37)}\\
S_{\pi^{\mathrm{ij}}} &= 2T\left(\pi^{\mathrm{ij}}\right)' + 2\partial^{\mathrm{k}}L\partial_{\mathrm{k}}\pi^{\mathrm{ij}} - 2\pi^{\mathbf{ik}}\partial_{\mathrm{k}}\partial^{\mathrm{j}}L - 2\pi^{\mathbf{jk}}\partial_{\mathrm{k}}\partial^{\mathrm{i}}L\,. && \text{(A.38)}
\end{aligned}
$$

A.2 Statistics and Fourier Transform

- *Continous Fourier transform*

  In our chosen convention, the Fourier transform of a real space spin-zero field, $f(\mathbf{x})$ is defined as

$$f(\mathbf{x}) \equiv \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} f(\mathbf{k}) \,, \tag{A.39}$$

and the inverse Fourier transform is defined as

$$f(\mathbf{k}) \equiv \int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}} f(\mathbf{x}) \,, \tag{A.40}$$

where $f(\mathbf{k})$ is the Fourier mode function of $f(\mathbf{x})$. The integration extend to entire space. A Fourier transform followed by an inverse Fourier transform defines a delta function

$$f(\mathbf{k}) = \int_{-\infty}^{\infty} \mathrm{d}^3x \left(\int_{-\infty}^{\infty} \frac{\mathrm{d}^3k'}{(2\pi)^3} f(\mathbf{k}') e^{i\mathbf{k}'\cdot\mathbf{x}} \right) e^{-\mathbf{i}\mathbf{k}\cdot\mathbf{x}} \tag{A.41}$$

$$\tag{A.42}$$

Hence

$$\delta_D(\mathbf{x} - \mathbf{x}') \equiv \int_{-\infty}^{\infty} \frac{\mathrm{d}^3x}{(2\pi)^3} e^{\mathbf{i}\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} \tag{A.43}$$

- *Convolution Theorem*

The Fourier transformation of product of two functions $f(\mathbf{x})$ and $g(\mathbf{x})$ is defined as

$$h(\mathbf{k}) = \int \mathrm{d}^3x f(\mathbf{x}) g(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}} \tag{A.44}$$

$$= \int \frac{\mathrm{d}^3k_1}{(2\pi)^3} \int \mathrm{d}^3k_2 f(\mathbf{k}_1) g(\mathbf{k}_2) \delta_D \left(k_1 + \mathbf{k_2} - \mathbf{k}\right) \tag{A.45}$$

This process may be extended to any number of fields, $f_1\mathbf{x}$, $f_2\mathbf{x}_2 \ldots f_n(\mathbf{x})$

$$h(k) = \int \frac{\mathrm{d}^3k_1}{(2\pi)^3} \int \frac{\mathrm{d}^3}{(2\pi)^3} \cdots \int \mathrm{d}^3k_n f_1(\mathbf{k}_1) \cdots f_n(\mathbf{k}_n) \delta_D \left(\mathbf{k} - \sum_{i=1}^{n} k_i \right) \tag{A.46}$$

The N-point correlation function for $f(\mathbf{x})$ are averages over the ensemble of the products $f(\mathbf{x}_1)f(\mathbf{x}_2)...f(\mathbf{x}_n)$ where $\mathbf{x}_1$, $\mathbf{x}_2$, ..., $\mathbf{x}_n$ represent different points in space:

$$\langle f(\mathbf{x}_1) f(\mathbf{x}_2) ... f(\mathbf{x}_n) \rangle \equiv \int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} \cdots \frac{d^3k_n}{(2\pi)^3} e^{i(\mathbf{k}_1\cdot\mathbf{x}_1 + \mathbf{k}_2\cdot\mathbf{x}_2 + ... + \mathbf{k}_n\cdot\mathbf{x}_n)} \langle f(\mathbf{k}_1) f(\mathbf{k}_2) ... f(\mathbf{k}_n) \rangle \,. \tag{A.47}$$

Thus, the correlation functions in real space may be studied via the correlation functions in momentum space.

- *Statistical homogeneity* The probability distribution function associated with $f(\mathbf{x})$ is statistical homogeneous if the N-point correlation function in real space are invariant under translations in space, i.e.

$$\langle f(\mathbf{x}_1+\mathbf{d})f(\mathbf{x}_2+\mathbf{d})...f(\mathbf{x}_n+\mathbf{d})\rangle = \langle f(\mathbf{x}_1)f(\mathbf{x}_2)...f(\mathbf{x}_n)\rangle\,, \tag{A.48}$$

where $\mathbf{d}$ is some vector in real space establishing the amount of spatial translation. In momentum space, the statistical homogeneity is enforced through Dirac delta function:

$$\langle f(\mathbf{k}_1)f(\mathbf{k}_2)...f(\mathbf{k}_n)\rangle \equiv (2\pi)^3\delta^3(\mathbf{k}_{12...n})M_f(\mathbf{k}_1,\mathbf{k}_2,...,\mathbf{k}_n)\,. \tag{A.49}$$

where, $\mathbf{k}_{12...n}$ means $\mathbf{k}_1+\mathbf{k}_2+...+\mathbf{k}_n$, and $M_f(\mathbf{k}_1,\mathbf{k}_2,...,\mathbf{k}_n)$ is called the $(N-1)$-spectrum, For example, $N=2$ is called the power spectrum, $N=3$ Bispectrum, etc. Statistical homogeneity assumption in Cosmology is protected by Ergodic theorem.

- *Statistical isotropy* The field $f(\mathbf{x})$ is statistical isotropic if its probability distribution function is invariance under spatial rotations, that is the N-point correlation function satisfies, .

$$\langle f(\tilde{\mathbf{x}}_1)f(\tilde{\mathbf{x}}_2)...f(\tilde{\mathbf{x}}_n)\rangle = \langle f(\mathbf{x}_1)f(\mathbf{x}_2)...f(\mathbf{x}_n)\rangle\,, \tag{A.50}$$

where $\tilde{\mathbf{x}}_i = \mathcal{R}\ \mathbf{x}_i$, $\mathcal{R}$ is the rotation matrix. The $(N-1)$-spectrum must satisfy the condition

$$M_f(\tilde{\mathbf{k}}_1,\tilde{\mathbf{k}}_2,...,\tilde{\mathbf{k}}_n) = M_f(\mathbf{k}_1,\mathbf{k}_2,...,\mathbf{k}_n)\,, \tag{A.51}$$

For the two lowest spectrum, we have

$$M_f(\mathbf{k}_1,\mathbf{k}_2) \equiv P_f(\mathbf{k}_1,\mathbf{k}_2) = P_f(k)\,, \tag{A.52}$$

$$M_f(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) \equiv B_f(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) = B_f(k_1,k_2,k_3)\,, \tag{A.53}$$

where $k=|\mathbf{k}_1|=|\mathbf{k}_2|$.

- *Gaussianity*

If $f(\mathbf{x})$is a statistically Gaussian field, the first three lowest Gaussian field, satisfy the following

$$\langle f(\mathbf{k}_1)f(\mathbf{k}_2)\rangle = (2\pi)^3\delta^3(\mathbf{k}_{12})P_f(\mathbf{k}_1)\,, \tag{A.54}$$

$$\langle f(\mathbf{k}_1)f(\mathbf{k}_2)f(\mathbf{k}_3)\rangle = 0\,, \tag{A.55}$$

$$\begin{aligned}\langle f(\mathbf{k}_1)f(\mathbf{k}_2)f(\mathbf{k}_3)f(\mathbf{k}_4)\rangle &= \langle f(\mathbf{k}_1)f(\mathbf{k}_2)\rangle\langle f(\mathbf{k}_3)f(\mathbf{k}_4)\rangle \\ &\quad +\langle f(\mathbf{k}_1)f(\mathbf{k}_3)\rangle\langle f(\mathbf{k}_2)f(\mathbf{k}_4)\rangle \\ &\quad +\langle f(\mathbf{k}_1)f(\mathbf{k}_4)\rangle\langle f(\mathbf{k}_2)f(\mathbf{k}_3)\rangle \qquad (A.56) \\ &= (2\pi^6)\delta^3(\mathbf{k}_{12})\delta^3(\mathbf{k}_{34})P_f(\mathbf{k}_1)P_f(\mathbf{k}_3) \qquad (A.57) \\ &\quad +\text{two permutations}\,,\end{aligned}$$

The N-point correlators, with N odd, are zero, while those with$N$ even, may all be written in terms of 2-point correlators.

- *Non-Gaussianity*

If $f(\mathbf{k})$ is non-gaussian, the connected terms that appear in higher order N-point correlators are non-zero

$$\langle f(\mathbf{k}_1)f(\mathbf{k}_2)\rangle = (2\pi)^3\delta^3(\mathbf{k}_{12})P_f(\mathbf{k}_1)\,, \tag{A.58}$$

$$\begin{aligned}\langle f(\mathbf{k}_1)f(\mathbf{k}_2)f(\mathbf{k}_3)\rangle &= \langle f(\mathbf{k}_1)f(\mathbf{k}_2)f(\mathbf{k}_3)\rangle_c \qquad (A.59) \\ &= (2\pi)^3\delta^3(\mathbf{k}_{123})B_f(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)\,, \qquad (A.60)\end{aligned}$$

$$\begin{aligned}\langle f(\mathbf{k}_1)f(\mathbf{k}_2)f(\mathbf{k}_3)f(\mathbf{k}_4)\rangle &= \langle f(\mathbf{k}_1)f(\mathbf{k}_2)f(\mathbf{k}_3)f(\mathbf{k}_4)\rangle_c \\ &\quad +\langle f(\mathbf{k}_1)f(\mathbf{k}_2)\rangle\langle f(\mathbf{k}_3)f(\mathbf{k}_4)\rangle \\ &\quad +\langle f(\mathbf{k}_1)f(\mathbf{k}_3)\rangle\langle f(\mathbf{k}_2)f(\mathbf{k}_4)\rangle \\ &\quad +\langle f(\mathbf{k}_1)f(\mathbf{k}_4)\rangle\langle f(\mathbf{k}_2)f(\mathbf{k}_3)\rangle \qquad (A.61) \\ &= (2\pi)^3\delta^3(\mathbf{k}_{1234})T_f(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4)\,+ \\ &\quad (2\pi^6)\delta^3(\mathbf{k}_{12})\delta^3(\mathbf{k}_{34})P_f(\mathbf{k}_1)P_f(\mathbf{k}_3) \\ &\quad +\text{two permutations}\,, \qquad (A.62)\end{aligned}$$

and so on. In the above, $B_f(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)$ and $T_f(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4)$ are called the connected bispectrum and trispectrum of f.

Appendix B

Symmetric-Trace Tensors and Spherical Harmonics

B.1 Spherical Harmonics and Symmetric tensors

An observer moving with 4−velocity u^a at position x^i, in a direction n^a on the unit sphere measures the luminosity of a distant supernova or galaxy. The direction vector is space-like $n^a n_a = 1$ and orthogonal to u^a, $n^a u_a$ The vector n^a may be given in terms of an orthnormal tertrad frame:

$$n^a(\theta, \phi) = (0, \sin\theta \sin\phi, \sin\theta \cos\phi, \cos\theta) \tag{B.1}$$

Any observable measured by the observer may be expanded in the spherical harmonics, for example, the spherical harmonics expansion of an observable f is given by

$$f = \sum_{l=0}^{\infty} f_l = \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} A_l^m(x^i) Y_l^m(\theta, \phi) \tag{B.2}$$

where $Y_l^m(\theta, \phi)$ are the surface spherical harmonic, it depends only on the angle θ and ϕ. There is a $1-1$ correspondence between all symmetric trace-free tensors of rank ℓ and the spherical harmonics of order ℓ, [241, 243, 261, 262] This implies that all symmetric-trace-free tensors of rank ℓ generates an irreducible representation of the weight ℓ and dimension $(2\ell + 1)$ of the group of rotations $SO(3)$ or the $SU(2)$. The dimension $(2\ell + 1)$ is due to summation over all degrees of freedom association with the magnetic quantum number. Also $Y_{\ell,m}(\theta, \phi)$ or a spin weighted spherical harmonic or spin zero is related to the associated Legendre polynomial, $P^{\ell m}(\cos\theta)$[246, 261]:

$$
\begin{aligned}
Y^{\ell m} &= (-1)^m \frac{(2\ell+1)(\ell-m)!}{4\pi(\ell+m)!}\, e^{im\phi} P^{\ell m}(\cos\theta), &\text{(B.3)}\\
&= (-1)^m \frac{(2\ell+1)(\ell-m)!}{4\pi(\ell+m)!}\, (n^{i\phi}\sin\theta)^m &\text{(B.4)}\\
&\sum_{j=0}^{[(\ell-m)/2]} \frac{(-1)^j(2\ell-2j)!}{2^\ell j!(\ell-j)!(\ell-m-2j)!}(\cos\theta)^{\ell-m-2j}\\
&\forall\ m \geq 0.
\end{aligned}
$$

The conjugate of $Y^{\ell m}$ is given by $Y^{\ell m} = (-1)^m Y^{\ell|m|*} \quad \forall \;\; m \leq 0$. The spherical harmonic, $Y^{\ell m}$ is related to the direction vector, n^{A_ℓ} according to

$$
Y^{\ell m}(\theta,\phi) = \mathcal{Y}^{\ell m}_{A_\ell} n^{A_\ell}, \tag{B.5}
$$

The definition of $\mathcal{Y}^{\ell m}_{A_\ell}$ follows from equation (B.5) by substituting

$$
n^1 + in^2 = e^{i\phi}\sin\theta \qquad n^3 = \cos\theta \tag{B.6}
$$

so that

$$
\begin{aligned}
\mathcal{Y}^{\ell m}_{A_\ell} &= (-1)^m \frac{(2\ell+1)(\ell-m)!}{4\pi(\ell+m)!} \sum_{j=0}^{[(\ell-m)/2]} \frac{(-1)^j(2\ell-2j)!}{2^\ell j!(\ell-j)!(\ell-m-2j)!} &\text{(B.7)}\\
&\times \prod_{k=0}^{m}\left(h^1{}_{(a_k} + ih^2{}_{(a_k}\right) \prod_{p=m+1}^{\ell-2j} h^3{}_{a_p} \prod_{q=1}^{j}(h^b{}_{a_{2q-1+\ell-2j}} h_{ba_{2q+\ell-2j})}).
\end{aligned}
$$

Because of the validity of the mapping between the spherical harmonics, $Y_{\ell m}$ and the direction vector, n^A, one may also perform a harmonic expansion of f in this form

$$
f = \sum_{l=0}^{\infty} F_\ell \tilde{n}^{A\ell} = F + F_a n^a + F_{ab} n^a n^b + F_{abc} n^a n^b n^c + F_{abcd} n^a n^b n^c n^d + \cdots \tag{B.8}
$$

where the spherical harmonic coefficients $F_{A\ell}$ are symmetric, trace-free tensors orthogonal to u^a:

$$
F_{A_\ell} = F_{(A_\ell)}\,, \quad F_{A_\ell ab} h^{ab} = 0\,, \quad \tau_{A_\ell a} u^a \tag{B.9}
$$

Round bracketts "$(\cdots)$" denote the symmetric part of a set of indices, angle bracketts "$\langle\cdots\rangle$" the (orthogonally-) Projected Symmetric Trace-Free (PSTF) part of the indices : $F_{A_\ell} =$

$F_{\langle A_\ell \rangle}$. We introduced the shorthand notation using the compound index $A_\ell = a_1 a_2 \cdots a_\ell$.

Given a symmetric tensor $F_{a \cdots z}$, one may construct a trace-free tensor out of it by subtracting off the trace of the tensor $F_{a \cdots z}$. For a projected symmetric tensor $F_{(a_1 a_2 \cdots a_\ell)}$, the trace $T[F_{(a_1 a_2 \cdots a_\ell)}]$ of such a tensor is given by

$$T[F_{(a_1 a_2 \cdots a_\ell)}] \;=\; \sum_{k=0}^{[\frac{l}{2}]} b_{\ell,k} \left(h_{a_1 a_2 \ldots} h_{a_{2k-1} a_{2k}} F^{(k)}_{a_{2k+1} \cdots a_\ell} + \cdots (p_{\ell,k} \text{terms}) \right) \tag{B.10}$$

where $F^{(k)}_{a_{2k+1} \cdots a_\ell}$ is the kth trace of $F_{a_1 \ldots a_\ell}$,

$$b_{\ell,k} = \frac{(-1)^k}{\prod_{m=1}^{k} \{ n + 2 [\ell - (k-m) - 2] \}}, \;\; p_{\ell,k} = \frac{\ell!}{(\ell - 2k)! 2^k k!}, \tag{B.11}$$

and the sum in the brackets extends over all p_{lk} different permutations of the indices and n is the number of dimensions of the projected hypersurface. The trace-free part of any tensor $F_{(a_1 a_2 \cdots a_\ell)}$ is then constructed by subtracting all the traces as [244, 246, 263, 264]:

$$F_{\langle a_1 a_2 \cdots a_\ell \rangle} \;=\; F_{(a_1 a_2 \cdots a_\ell)} - \sum_{k=0}^{[\frac{l}{2}]} b_{\ell,k} \left(h_{a_1 a_2 \ldots} h_{a_{2k-1} a_{2k}} F^{(k)}_{a_{2k+1} \cdots a_\ell} + \cdots (p_{\ell,k} \text{terms}) \right) , \tag{B.12}$$

in three dimensions, i.e $n = 3$, we may then expand as follows

$$\begin{aligned} F_{\langle a_1 a_2 \cdots a_\ell \rangle} \;=\; & F_{(a_1 a_2 \ldots a_\ell)} - \frac{1}{2\ell - 1} h_{\{a_1 a_2} F_{a_2 a_2 a_3 a_4 \cdots a_\ell \}} \\ & + \frac{1}{(2\ell - 3)(2\ell - 1)} h_{\{a_1 a_2 a_3 a_4} F_{a_2 a_2 a_4 a_4 a_5 a_6 \cdots a_l \}} \\ & - \cdots \frac{(-1)^{\ell/2}}{\prod_{m=1}^{\frac{\ell}{2}} [\ell + 2m - 1]} h_{a_1 a_2 \cdots a_l} F_{a_2 a_2 \cdots a_\ell a_\ell} \end{aligned} \tag{B.13}$$

the last is replaced by $h_{\{a_1 a_2 \cdots a_{2\frac{\ell}{2}}} F_{a_2 a_2 \cdots a_{\frac{\ell}{2}} a_{\frac{\ell}{2}} a_\ell \}}$ for odd number of indices. We have also used another shorthand notation $F_{\{a_1 a_2 \cdots a_k\}} = k! F_{(a_1 a_2 \cdots a_k)}$. So for any rank tensor ℓ or combination of tensors $\sigma_{ab} \sigma_{cd}$ that appear after the covariant derivative decomposition , we use Eq. (B.13) to extract the trace and the trace-free part of the tensor.

To illustrate how these relations work, let us consider the decomposition of $e^a e^b e^c e^d \sigma_{ab} \sigma_{cd}$ into trace and trace-free part. We will be a bit more general by considering $e^{A_{\ell+2}} G_{(ab} F_{A_\ell)}$, at the end of the process we set $\ell = 2$ to recover the case of interest. First we use Eq. B.12 to split the rank four tensor into a rank two and a rank zero tensor. This is not the end of the process because the rank two tensor i.e the second term on RHS of Eq. B.12 still contains a trace part, so we use Eq.B.13 on the rank two tensor to extract the trace part as .

$$F_{c(A_{\ell-1}}G^{c}_{a_\ell)} = F_{c\langle A_{\ell-1}}G^{c}_{a_\ell\rangle} + \frac{(\ell-1)}{(2\ell-1)}F_{cd(A_{\ell-2}}G^{cd}h_{a_{\ell-1}a_\ell)} \tag{B.14}$$

After some simplification this process leads to

$$e^{A_{\ell+2}}F_{(A_\ell}G_{ab)} = e^{A_{\ell+2}}\left[F_{\langle A_\ell}G_{ab\rangle} + \frac{2\ell}{2\ell+3}F_{c\langle A_{\ell-1}}G^{c}{}_{a_\ell\rangle}h_{ab} + \frac{\ell(\ell-1)}{(4\ell^2-1)}G_{cd}F^{cd}{}_{(A_{\ell-2}}h_{a_{\ell-1}a_\ell}h_{ab)}\right] \tag{B.15}$$

Then for the special case of interest ($e^a e^b e^c e^d \sigma_{ab}\sigma_{cd}$) we set $\ell = 2$ and replace F and G with σ to obtain

$$e^a e^b e^c e^d \sigma_{(ab}\sigma_{cd)} = e^a e^b e^c e^d \sigma_{\langle ab}\sigma_{cd\rangle} + \frac{4}{7}e^a e^b \sigma^{c}_{\langle a}\sigma_{b\rangle c} + \frac{2}{15}\sigma_{ab}\sigma^{ab}. \tag{B.16}$$

B.2 Spin Weighted Harmonics

On a three dimensional Euclidean hypersurface orthogonal to the fundamental observer with the 4-velocity u^a, we may introduce orthornormal triad $(n^i_r, n^i_\theta, n^i_\phi)$, such that the direction n^a on a unit sphere ($n^a n_a = 1, n^a u_a = 0$) is given by

$$n^a(\theta,\phi) = (0, \sin\theta\sin\phi, \sin\theta\cos\phi, \cos\theta)\,. \tag{B.17}$$

In most cases, it is convenient to introduce in place of (n^i_θ, n^i_ϕ), the complex vector m_+ and its complex conjugate m_- in the two-dimensional Euclidean section of the three dimensional hypersurface with coordinates (θ,ϕ) . The spin weighted spherical harmonics is related to the Wigner D matrices according to

$$D^{\ell}{}_{-ms}(\theta,\phi,-\psi) = (-1)^m\sqrt{\frac{4\pi}{(2\ell+1)}}\,{}_sY_{\ell m}(\theta,\phi)e^{is\psi} \tag{B.18}$$

It is possible to define to two complex vector field on a unit sphere by combining two triads, n^i_θ, n^i_ϕ):

$$m^i_\pm \equiv \frac{1}{\sqrt{2}}\left(n^i_\theta \mp i\, n^i_\phi\right) = \frac{1}{\sqrt{2}}\begin{pmatrix}\cos\theta\cos\phi \pm i\sin\phi \\ \cos\theta\sin\phi \mp i\cos\phi \\ -\sin\theta\end{pmatrix}. \tag{B.19}$$

Using the Pauli basis, it is straight forward to establish the following properties

$$m^i_\pm m_{\pm i} = 0, \quad m^i_\pm m_{\mp i} = 1 \quad m^i_\pm n_i = 0 \quad N^{ij} m_{\pm j} = m^i_\pm, \tag{B.20}$$

where m is defined up to a phase factor ψ

$$m_\pm \equiv \frac{1}{\sqrt{2}} \left(n^i_\theta \mp i\, n^i_\phi \right) \to e^{i\psi} m_\pm \tag{B.21}$$

In general a quantity of spin-weight s defined this manifold as ${}_s\eta$ is said to be of spin-weight s if under the transformation in equation (B.21), it transforms as ${}_s\eta \to {}_s\eta e^{is\psi}$. Just like in the STF harmonics, the symmetric trace-free part of a tensor with a rank-$|s|$ is given by $\eta_{A_s} \equiv \eta_{a_1 \dots a_s}$, where $a_1 \dots a_s$ is a collection of indices. For $s \geq 0$, η_{A_s} is decomposed in the complex basis according to

$$\eta^{A_s} \equiv \frac{1}{2^{|s|}} \left({}_{+s}\eta\, m_-^{A_s} + {}_{-s}\eta\, m_+^{A_s} \right), \tag{B.22}$$

Using equation (B.20), the inverse relation becomes

$${}_{-s}\eta = \eta_{A_s} m_+^{A_s}, \qquad {}_{+s}\eta = \eta_{A_s} m_-^{A_s} \tag{B.23}$$

With respect to the covariant derivatives on the screen space, it is possible to define respectively, the spin raising and lowering operators $\eth$ and $\bar{\eth}$ using the complex basis $m_\pm$. For example the the covariant derivatives of $\eta_{A_{|s|}}$ is given by

$$\eth_{\pm|s|}\eta = -(m^c_+ \nabla_{\perp c} \eta_{A_{|s|}}) m_\pm^{A_{|s|}}, \tag{B.24}$$

$$\bar{\eth}_{\pm|s|}\eta = -(m^c_- \nabla_{\perp c} \eta_{A_{|s|}}) m_\pm^{A_{|s|}}, \tag{B.25}$$

where the minus signs are conventional. In a spherical polar coordinate system, we have $m^a_\pm \nabla_a m^b_\pm = \cot\theta\, m^b_\pm$ and $m^a_\mp \nabla_a m^b_\pm = -\cot\theta\, m^b_\pm$ and the spin raising and lowering operators

$$\eth\, {}_s\eta = -\sin^s\theta \left[\frac{\partial}{\partial\theta} + \frac{i}{\sin\theta}\frac{\partial}{\partial\phi} \right] \sin^{-s}\theta\, {}_s\eta \tag{B.26}$$

$$\bar{\eth}\, {}_s\eta = -\sin^{-s}\theta \left[\frac{\partial}{\partial\theta} - \frac{i}{\sin\theta}\frac{\partial}{\partial\phi} \right] \sin^{s}\theta\, {}_s\eta. \tag{B.27}$$

The symmetric trace-free tensor of any rank, decomposed in this complex basis may be expanded in spin weighted spherical harmonics, $Y_{\ell m}(\mathbf{n})$

$${}_{\pm s}A(\mathbf{n}) = \sum_{lm} a^A_{\ell m}\, {}_{\pm s}Y_{\ell m}(\mathbf{n}), \tag{B.28}$$

where $a^A_{\ell m}$ is the spin weighted spherical harmonic moments. $a^A_{\ell m}$ may be calculated from equation (B.28) by multiplying both sides with the conjugate of $Y^*_{\ell m}(\mathbf{n})$ and integrating over the entire sky

$$a^A_{\ell m} = \int {}_{\pm s}A(\mathbf{n})\,[{}_{\pm s}Y_{\ell m}(\mathbf{n})]^*\,d\Omega \tag{B.29}$$

(B.30)

For a non-zero spin object, i.e $s \neq 0$, $a^A_{\ell m}$ is further decomposed into the electric E-part and the magnetic B- part

$$a^A_{\ell m} = a^{AE}_{\ell m} + i a^{AB}_{\ell m}. \tag{B.31}$$

The E- part and B- part may be extraction using

$$a^{AE}_{\ell m} = \frac{1}{2}\left(a^A_{\ell m} + a^{A*}_{\ell m}\right), \qquad a^{AB}_{\ell m} = \frac{1}{2i}\left(a^A_{\ell m} - a^{A*}_{\ell m}\right). \tag{B.32}$$

The E- part is invariant under parity transformation and the B- part changes signs under parity transformation

$$a^A_{\ell m} \to (-1)^{\ell+s} a^{A*}_{\ell -m}, \quad a^{AE}_{\ell m} \to (-1)^{\ell+s} a^{AE}_{\ell -m}, \quad a^{AB}_{\ell m} \to -(-1)^{\ell+s} a^{AB}_{\ell -m}. \tag{B.33}$$

The spin-s weighted harmonics is given by

$$\begin{aligned} {}_sY_{\ell m}(\theta,\phi) &= \left[\frac{2\ell+1}{4\pi}\frac{(\ell+m)!(\ell-m)!}{(\ell+s)!(\ell-s)!}\right]^{1/2} (\sin\theta/2)^{2\ell} \sum_r \begin{pmatrix} \ell-s \\ r \end{pmatrix} \begin{pmatrix} \ell+s \\ r+s-m \end{pmatrix} \\ &\quad \times (-1)^{\ell-r-s} e^{im\phi} (\cot\theta/2)^{2r+s-m}. \end{aligned} \tag{B.34}$$

The group theory rotation matrix $\mathcal{D}^\ell_{-s,m}(\phi,\theta,\psi)$, is related to the spin weighted harmonics according to

$$\mathcal{D}^\ell_{-s,m}(\phi,\theta,\psi) = \sqrt{4\pi/(2\ell+1)}\,{}_sY_{\ell m}(\theta,\phi)e^{-is\psi} \tag{B.35}$$

represents rotations by the Euler angles (ϕ,θ,ψ). By virtue of their relation to the rotation matrices, the spin harmonics satisfy: the compatibility relation with spherical harmonics, ${}_0Y^m_\ell = Y^m_\ell$; the conjugation relation ${}_sY^{m*}_\ell = (-1)^{m+s}{}_{-s}Y^{-m}_\ell$; the orthonormality relation, the completeness relation, the parity relation, and the generalized addition relation.

$$\int d\Omega \; ({}_sY_\ell^{m*}) \; ({}_sY_\ell^m) = \delta_{\ell,\ell'}\delta_{m,m'} \,, \tag{B.36}$$

$$\sum_{\ell,m} [{}_sY_\ell^{m*}(\theta,\phi)] \, [{}_sY_\ell^m(\theta',\phi')] = \delta(\phi-\phi')\delta(\cos\theta - \cos\theta') \,, \tag{B.37}$$

$${}_sY_\ell^m \to (-1)^\ell {}_{-s}Y_\ell^m \text{ and} \tag{B.38}$$

$$\sum_m {}_{s_1}Y_\ell^{m*}(\theta',\phi') {}_{s_2}Y_\ell^m(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}} {}_{s_2}Y_\ell^{-s_1}(\beta,\alpha) e^{-is_2\gamma} \,, \tag{B.39}$$

respectively. These relations follow from the group multiplication property of rotation matrices which relates a rotation from (θ',ϕ') through the origin to (θ,ϕ) with a direct rotation in terms of the Euler angles (α,β,γ). The addition of two angular momentum states is handled using the Clebsch-Gordan relation,

$$\begin{aligned} {}_{s_1}Y_{\ell_1}^{m_1} {}_{s_2}Y_{\ell_2}^{m_2} &= \sqrt{\frac{(2\ell_1+1)(2\ell_2+1)}{4\pi(2\ell+1)}} \sum_{\ell,m,s} \langle \ell_1,\ell_2;m_1,m_2|\ell_1,\ell_2;\ell,m\rangle \\ &\quad \times \langle \ell_1,\ell_2;-s_1,-s_2|\ell_1,\ell_2;\ell,-s\rangle \; {}_sY_\ell^m \,. \end{aligned} \tag{B.40}$$

We may use the spin-raising/lowering operators to define spin-weighted spherical harmonics through

$${}_sY_{\ell m} = \sqrt{\frac{(\ell-|s|)!}{(\ell+|s|)!}} \begin{cases} \eth^s \, Y_{\ell m}, & s \geq 0 \\ (-1)^s \bar{\eth}^{|s|} \, Y_{\ell m}, & s < 0. \end{cases} \tag{B.41}$$

The action of raising/lowering operator on $Y_{\ell m}$ is given by

$${}_sY_{\ell m}^* = (-1)^{m+s} {}_{-s}Y_{\ell m} \tag{B.42}$$

$$\eth_s Y_{\ell m} = [(\ell-s)(\ell+s+1)]^{\frac{1}{2}} \, {}_{s+1}Y_{\ell m} \tag{B.43}$$

$$\bar{\eth}_s Y_{\ell m} = -[(\ell+s)(\ell-s+1)]^{\frac{1}{2}} \, {}_{s-1}Y_{\ell m} \tag{B.44}$$

$$\bar{\eth}\eth_s Y_{\ell m} = -(\ell-s)(\ell+s+1) {}_sY_{\ell m} \tag{B.45}$$

for double derivatives

$$\eth\eth_s Y_{\ell m} = [(\ell-s)(\ell+s+1)] \, {}_{s+2}Y_{\ell m} \tag{B.46}$$

$$\bar{\eth}\bar{\eth}_s Y_{\ell m} = [(\ell+s)(\ell-s+1)] \, {}_{s-2}Y_{\ell m} \tag{B.47}$$

$$\tag{B.48}$$

Acting with the spin-lowering operator on ${}_{+s}A$ and vice versa then yields

$$\bar{\eth}^s{}_sA(\mathbf{n}) = \sum_{\ell m} a^A_{\ell m}\sqrt{\tfrac{(\ell-s)!}{(\ell+s)!}}\bar{\eth}^s\eth^s Y_{\ell m}(\mathbf{n}) = \sum_{\ell m} a^A_{\ell m}(-1)^s\sqrt{\tfrac{(\ell+s)!}{(\ell-s)!}}Y_{\ell m}(\mathbf{n}) \quad \text{(B.49)}$$

$$\eth^s{}_{-s}A(\mathbf{n}) = \sum_{lm} a^A_{lm}(-1)^s\sqrt{\tfrac{(l-s)!}{(l+s)!}}\eth^s\bar{\eth}^s Y_{lm}(\mathbf{n}) = \sum_{lm} a^A_{lm}\sqrt{\tfrac{(l+s)!}{(l-s)!}}Y_{lm}(\mathbf{n}). \quad \text{(B.50)}$$

We thus have

$$a^A_{\ell m} = \int {}_{\pm s}A(\mathbf{n})\left[{}_{\pm s}Y_{\ell m}(\mathbf{n})\right]^* d\Omega \quad \text{(B.51)}$$

$$= \sqrt{\frac{(\ell-|s|)!}{(l+|s|)!}}(-1)^s\int\left[\bar{\eth}^s\ {}_{+s}A(\mathbf{n})\right]Y^*_{lm}(\mathbf{n})d\Omega \quad \text{(B.52)}$$

$$= \sqrt{\frac{(\ell-|s|)!}{(\ell+|s|)!}}\int\left[\eth^s\ {}_{-s}A(\mathbf{n})\right]Y^*_{\ell m}(\mathbf{n})d\Omega. \quad \text{(B.53)}$$

This ensures that the harmonics moments $a^A_{\ell m}$ act as a scalar on the sky, i.e it is invariant under a rotation of the coordinate system around $\mathbf{n}$.

The Wigner $3-\ell m$ sympbols are related to the standard Clebsch-Gordan coefficient $\langle \ell_1, m_1, \ell_2, m_2|j_3, -m_3\rangle$ through

$$\begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2, & -m \end{pmatrix} = \frac{(-1)^{\ell_1-\ell_2-m_3}}{\sqrt{(2\ell_3+1)}}\langle \ell_1, m_1, \ell_2, m_2|j_3, -m_3\rangle \quad \text{(B.54)}$$

and it may be expanded in terms of orbital angular momentum

$$\begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2, & -m \end{pmatrix} = (-1)^{\ell_1-m_1}\delta_{m_1+m_2,-m_3} \quad \text{(B.55)}$$
$$\times\left[\frac{(\ell_1+\ell_2-\ell_3)!(\ell_1+\ell_3-\ell_2)!(\ell_2+\ell_3-\ell_1)!(\ell_3+m_3)!(\ell_3-m_3)!}{(\ell_1+\ell_2+\ell_3+1)!(\ell_1+m_1)!(\ell_1-m)!(\ell_2+m_2)!(\ell_2-m_2)!}\right]^{\frac{1}{2}}$$
$$\times\sum_{k\geq 0}\frac{(-1)^k}{k!}\left[\frac{(\ell_2+\ell_3+m_1-k)!(\ell_1-m_1+k)!}{(\ell_3-\ell_1+\ell_2-k)!(\ell_3-m_3-k)!(\ell_1-\ell_2+m_3+k)!}\right]$$

The sum runs over all values of k for which the arguments inside the factorials are non-negative.

One special case of interest is the case of zero total angular momentum

$$\begin{pmatrix} \ell_1 & \ell_2 & 0 \\ m_1 & m_2, & 0 \end{pmatrix} = \langle \ell_1, m_1, \ell_2, m_2 | 0, 0 \rangle \quad \text{(B.56)}$$

$$= \frac{(-1)^{\ell_1 - m_1}}{\sqrt{2\ell_1 + 1}} \delta_{\ell_1,\ell_2} \delta_{m_1,-m_2} \quad \text{(B.57)}$$

(B.58)

and for the spin degrees of freedom

$$\begin{pmatrix} \ell_1 & \ell_2 & 0 \\ -s_1 & -s_2, & 0 \end{pmatrix} = \langle \ell_1, -s_1, \ell_2, -s_2 | 0, 0 \rangle \quad \text{(B.59)}$$

$$= \frac{(-1)^{\ell_1 - s_1}}{\sqrt{2\ell_1 + 1}} \delta_{\ell_1,\ell_2} \delta_{s_1,-s_2} \quad \text{(B.60)}$$

(B.61)

also

$$\begin{pmatrix} \ell_1 & \ell_2 & 0 \\ 0 & 0, & 0 \end{pmatrix} = \langle \ell_1, 0, \ell_2, 0 | 0, 0 \rangle \quad \text{(B.62)}$$

$$= \frac{(-1)^{\ell_1}}{\sqrt{2\ell_1 + 1}} \delta_{\ell_1,\ell_2} \quad \text{(B.63)}$$

At some stage in the calculation, we added two angular momentum states using the Wigner $3-\ell m$ symbol in the form given below:

$$_{s_1}Y_{\ell_1,m_1\,s_2}Y_{\ell_2,m_2} = \sqrt{\frac{(2\ell_1+1)(2\ell_2+1)(2\ell_3+1)}{4\pi}} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2, & -m \end{pmatrix} \times \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ -s_1 & -s_2, & -s \end{pmatrix} {}_sY_{\ell,m}\,. \quad \text{(B.64)}$$

This is how we extract our monopole $\ell_3 = 0$

$$\begin{pmatrix} \ell_1 & \ell_2 & 0 \\ m_1 & m_2, & 0 \end{pmatrix} \begin{pmatrix} \ell_1 & \ell_2 & 0 \\ -s_1 & -s_2, & 0 \end{pmatrix} \quad \text{(B.65)}$$

$$= \langle \ell_1, m_1, \ell_2, m_2 | 0, 0 \rangle \langle \ell_1, -s_1, \ell_2, -s_2 | 0, 0 \rangle\,,$$

$$= \frac{(-1)^{2\ell_1 - m_1 - s_1}}{(2\ell_1 + 1)} \delta_{\ell_1,\ell_2} \delta_{m_1,-m_2} \delta_{s_1,-s_2} \quad \text{(B.66)}$$

(B.67)

This is how we extract the dipole

$$\begin{pmatrix} \ell_1 & \ell_2 & 1 \\ m_1 & m_2, & 0 \end{pmatrix} = (-1)^{\ell_1-m_1}\delta_{m_1+m_2,0}\left[\frac{2m_1\delta_{\ell_1,\ell_2}}{\sqrt{(2\ell_1+2)(2\ell_1+1)(2\ell_1)}} + \delta_{\ell_1,\ell_2+1}\sqrt{\frac{(\ell_1+m_1)(\ell_1-m_1)}{\ell_1(2\ell_1+1)(2\ell_1-1)}} - \delta_{\ell_1+1,\ell_2}\sqrt{\frac{(\ell_2-m_2)(\ell_2+m_2)}{(2\ell_2+1)(2\ell_2-1)}}\right] \quad (B.68)$$

$$\begin{pmatrix} \ell_1 & \ell_2 & 1 \\ m_1 & m_2, & \pm 1 \end{pmatrix} = (-)^{\ell_1-m_1}\delta_{m_1+m_2,\mp 1}\left[\pm\delta_{\ell_1,\ell_2}\sqrt{\frac{(\ell_1\mp m_1)(\ell_1\mp m_2)}{\ell_1(2\ell_1+2)(2\ell_1+1)}} + \delta_{\ell_1,\ell_2+1}\sqrt{\frac{(\ell_1\mp m_1)(\ell_1\pm m_2)}{2\ell_1(2\ell_1+1)(2\ell-1)}} + \delta_{\ell_1+1,\ell_2}\sqrt{\frac{(\ell_2\mp m_2)(\ell_2\pm m_1)}{2\ell_2(2\ell_2+1)(2\ell_2-1)}}\right] \quad (B.69)$$

B.2.1 Recursion Relations

We made use of some of the following recursion relations at some stage in the derivation.

$$\frac{j_\ell(x)}{x} = \frac{1}{2\ell+1}\left[j_{\ell-1}(x)+j_{\ell+1}(x)\right], \quad (B.70)$$

$$\frac{j_\ell(x)}{x^2} = \frac{j_{\ell+2}(x)}{(2\ell+3)(2\ell+1)} + \frac{2j_{\ell(x)}}{(2\ell+3)(2\ell-1)} + \frac{j_{\ell-2}(x)}{(2\ell+1)(2\ell-1)} \quad (B.71)$$

$$j'_\ell(x) = \frac{1}{2\ell+1}\left[\ell j_{\ell-1}(x) - (\ell+1)j_{\ell+1}(x)\right] \quad (B.72)$$

$$j''_\ell(x) = \frac{\ell(\ell-1)}{(2\ell+1)(2\ell-1)}j_{\ell-2}(x) - \frac{2\ell^2+2\ell-1}{(2\ell-1)(2\ell+3)}j_\ell(x) + \frac{(\ell+1)(\ell+2)}{(2\ell+1)(2\ell+3)}j_{\ell+2}(x) \quad (B.73)$$

$$j'''_\ell(x) = \frac{\ell(\ell-1)(\ell-2)}{(2\ell+1)(2\ell-1)(2\ell-3)}j_{\ell-3}(x) - \frac{3\ell(\ell^2-2)}{(2\ell+1)(2\ell-3)(2\ell+3)}j_{\ell-1}(x) + \frac{3(\ell+1)(\ell(\ell+2)-1)}{(2\ell+1)(2\ell-1)(2\ell+5)}j_{\ell+1}(x) - \frac{(\ell+1)(\ell+2)(\ell+3)}{(2\ell+1)(2\ell+3)(2\ell+5)}j_{\ell+3}(x) \quad (B.74)$$

$$\left(\frac{j_\ell(x)}{x}\right)'' = \frac{(\ell+2)(\ell+3)}{(2\ell+1)(2\ell+3)}\frac{j_{\ell+2}}{x} - \frac{2\ell^2+2\ell-3}{(2\ell+3)(2\ell-1)}\frac{j_\ell}{x} + \frac{(\ell-1)(\ell-2)}{(2\ell+1)(2\ell-1)}\frac{j_{\ell-2}}{x} \quad (B.75)$$

$$\left(\frac{j_\ell(x)}{x}\right)'' = \frac{(\ell+2)(\ell+3)}{(2\ell+1)(2\ell+3)(2\ell+5)}j_{\ell+3} - \frac{\ell^2+\ell-3}{(2\ell+5)(2\ell+1)(2\ell-1)}j_{\ell+1} - \frac{\ell^2+\ell-3}{(2\ell+3)(2\ell+1)(2\ell-3)}j_{\ell-1} + \frac{(\ell-1)(\ell-2)}{(2\ell+1)(2\ell-1)(2\ell-3)}j_{\ell-3} \quad (B.76)$$

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