

UNIVERSITY OF CAPE TOWN

DOCTORAL THESIS

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# Compound Lévy Random Bridges And Credit Risky Asset Pricing

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*A thesis submitted in fulfillment of the requirements*

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Department of Mathematics and Applied Mathematics

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# Declaration of Authorship

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UNIVERSITY OF CAPE TOWN

# *Abstract*

Faculty of Science

Department of Mathematics and Applied Mathematics

Doctor of Philosophy

## **Compound Lévy Random Bridges And Credit Risky Asset Pricing**

by Dennis Chinemerem IKPE

In this thesis, we study Random Bridges of a certain class of Lévy processes and their application to credit risky asset pricing. In the first part, we construct the compound random bridges (CLRBS) and analyze some tools and properties that make them suitable models for information processes. We focus on the Markov property, dynamic consistency, measure changes and increment distributions. Thereafter, we study their applications in credit risky asset pricing. We generalize the information based credit risky asset pricing framework to incorporate prematurity default possibilities. Lastly we derive closed-form expressions for default trend and intensities for a randomly timed cash flow with a CLRB as the background partial information process. We obtain analytical expressions for specific CLRBS. The second part looks at application of stochastic filtering in the current information based asset pricing framework. First, we formulate our information-based framework as a filtering problem under incomplete information. We derive the Kalman-Bucy filter in one dimension for bridges of Lévy processes with a given finite variance.



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*Dedicated to God Almighty*



# Chapter 1

## Introduction and Summary

The purpose of this thesis is to generalize the information based asset pricing theory of Brody, Hughston and Macrina in [23] and study the applications of stochastic filtering in the framework. This generalization enables us to deal with models in which default may occur before the maturity, which according to our knowledge has not been addressed so far in the literature. It is usually the case in mathematical finance that when confronted with the challenge of generalizing an existing asset pricing model one has to be clear on certain modeling issues to avoid too much practical and computational complications. In the particular area of credit risk modeling one often faces questions of the following kind: which additional asset classes are going to be incorporated? how does one ensure that existing models can be recovered from the extended model? what mathematical technologies exist for such generalization? and so on. All these questions guide the direction and extent to which one goes in generalizing an asset pricing model. In this work, great effort is made to ensure that our model is flexible enough and maintains the *tractability* property in general so as to handle practical computational issues in a reasonable fashion. In the first part of this thesis, we generalize the information based credit risky asset pricing framework to incorporate prematurity default possibilities. We first construct the information process as a conditioned stochastic process and then connect two credit risky modeling approaches by this construction where we consider the application of our model to credit derivative products. According to our knowledge this work has not been done by any researcher. Lastly, we look at application of stochastic

filtering in the current information based asset pricing framework. First, we formulate credit risky asset pricing in our information-based framework as a filtering problem under incomplete information. We then present the Kalman-Bucy filter algorithm for a special case of the information process. The contents of this thesis are adapted in part from a preliminary study which was published in [45]. In particular, chapter two in which we detail the relevant background theory and properties of conditioned stochastic process with applications in the information based framework and which form the basis for the developments in sub-sequent chapters contains research associated with [45]. Once this has been completed we move onto the main subject of this thesis, that of the theory of Lévy random bridges with applications in credit risky asset pricing. We begin chapter three by introducing Lévy bridges as a conditioning on the space of Lévy processes with continuous distribution. The key result here is that as we extend a fixed conditioning argument to a random conditioning, the resulting bridge processes (which we call compound Lévy random bridges) possess important properties required of an information model in the information based asset pricing framework. The results on the Markov and dynamic consistency properties were summarized in *Propositions 3.2.3* and *3.2.4* respectively. This generalization allows us to model credit risky assets with pre-maturity default possibilities in subsequent chapters. In chapter four we then use the compound Lévy random bridges (CLRBs) constructed in the previous chapter as information process in pricing credit risky bonds and their derivatives in the setting where default can occur before the maturity of the bond. The main tools here are the Markov property, joint conditional distribution of the CLRB and its occupation time and martingale arguments. We conclude this chapter by looking at information based corridor bond options. We show that in the case of corridor bond with a constant face value, the estimates of the corresponding call option price can be calculated in closed-form for Brownian and Gamma bridge information processes. Chapter five looks at the default *trend* and *intensity* calculations in our framework. We start by proving the cornerstone of the intensity based credit risk valuation - a pricing formula based on the *trend* of the

information model, this time using occupation time default definition. To use the pricing formula for price estimation in the current partial information framework requires computing the estimates of the default trend and intensity under the CLRB information. The main result here is the derivation of simple expressions for these estimates. These results are summarized in *Theorem 5.4.1*. To conclude this chapter we analyze the *credit spread* term structure for a credit risky zero coupon bond. Chapter 6 deals with the part of this thesis where we look at application of stochastic filtering in the current information based asset pricing framework. This is carried out by following the standard innovations process approach. We derive the famous Kalman-Bucy filter (*Theorem 6.4.2*) in one dimension for bridges of Lévy processes with a given finite variance. We conclude this chapter by looking at numerical simulation.

There are many possible avenues for future research to extend the work of this thesis. One of the potential criticism of the extended BHM framework is that the information process may be difficult to calibrate due to lack of specifics on what real market data the process represents. A study focused on calibration and empirical issues in this framework would be very useful. Another limitation of the results presented in this study is that the pricing mechanism is constructed under the martingale assumption on the resulting price process. However, the assumption of deterministic rates may not be consistent with the market. Therefore, an extension of the proposed model that allows for stochastic interest rates is very desirable. The pricing expressions for certain specification of the information process involve integrals which cannot be evaluated analytically. A thorough numerical analysis, in particular, investigating the prices of exotic products produced by specific CLRB information models, is also very desirable. We did not conduct such a study in the present work, due to its already quite extensive length. Finally, one can examine the problem of pricing other path dependent financial instruments, such as the barrier or American options using a CLRB as the information process.



# Chapter 2

## Preliminaries and Motivation

In this chapter we introduce some important results with relevant background theory and properties of conditioned stochastic process and their applications which will be required in sub-subsequent chapters. This sets the stage and provide the motivation and preliminary context leading to the main results of this work. Once this has been completed we move onto the main subject of this thesis, that of the theory of Lévy random bridges with applications in credit risky asset pricing.

### 2.1 Credit Risky Asset Pricing

Credit risky asset pricing models are classified into two: structural and reduced form models. In this introductory section, we are interested in the information-based perspective of the two modeling viewpoints. In particular, we show how the models compare with respect to information and demonstrate the importance of time-dependent revelation of the information in both model design and computational flexibility. The whole point here is to illustrate the need for a unifying information based framework for credit risky asset pricing, which we address in subsequent chapters.

### 2.1.1 Structural Versus Reduced Form Models

Credit risk modelling basically investigates the valuation of a contractual agreement(asset) between a credit issuer and an investor. Credit risk market provides a platform for smooth execution of this form of financial transaction. Structural models approach this valuation problem by reference to the value of the firm or entity in need of the credit while reduced form models ignore the firm value but rather specify exogenous market variables that affect the value of the asset. Throughout this section, we consider a continuous time model in the time period  $[0, T]$ . Let  $(\Omega, \mathcal{G}, \mathbb{Q}, (\mathcal{G}_t : t \in [0, T]))$  denote a filtered probability space on  $[0, T]$  satisfying the usual conditions. In what follows we describe the valuation problem in each of these approaches with emphasis on the assumptions on the information set  $(\mathcal{G}_t : t \in [0, T])$  available to the market. We consider without loss of generality, a generic asset in the form of a defaultable zero coupon bond which promises to pay one dollar at the maturity date of the contract,  $\bar{T} \in (0, T]$ . Let us denote the price of a defaultable zero coupon bond at time  $t \leq T$  by  $B(t, \bar{T})$ . In addition, we assume the default free zero coupon bonds are traded in the market and we denote the deterministic default free spot interest rate by  $r_t$ . These assumptions ensure that the market is arbitrage free hence there exist an equivalent probability measure  $\mathbb{Q}$ , such that discounted bond prices are martingale with respect to the information set  $(\mathcal{G}_t : t \in [0, T])$ .

### Structural models

Structural models were first introduced in [63], our focus in this section however, shall be kept on the information based perspective on structural models as described in [47]. The main idea is that the information set  $\{\mathcal{G}_t\}_{0 \leq t \leq T}$  observed by the market participants contains the filtration  $\{\mathcal{F}_t\}_{0 \leq t \leq T}$  generated by the firms asset value. Denoting the firm's asset value by  $V_t$ , we then have  $\mathcal{F}_t = \sigma(V_s : s \leq t) \subset \mathcal{G}_t$ . Let the firm value be given by

an adapted exponential Lévy process:

$$V_t = \exp(L_t) \quad (2.1)$$

The classical Merton's model in [63] corresponds to the case where the Lévy process is a Brownian motion with drift. That is  $L_t = (\mu - \sigma^2/2)t + \sigma W_t$ , with  $\mu \in \mathbb{R}$ ,  $\sigma > 0$  and  $W$  is a standard Brownian motion. However, with some technical consideration, the case of an arbitrary Lévy process can be specified.

Now, under the above set up, we consider the valuation problem for a simple zero coupon bond paying a notional value of 1 at the maturity time  $\bar{T}$  and default can only occur at time  $\bar{T}$ . Hence, default occurs only if  $V_{\bar{T}} \leq 1$  and the probability of default is given by  $\mathbb{Q}(V_{\bar{T}} \leq 1)$ . The time  $t$  value of the bond is given by

$$B(t, \bar{T}) = e^{-\int_t^{\bar{T}} r_s ds} \mathbb{E}_{\mathbb{Q}}(\min(V_{\bar{T}}, 1)) \quad (2.2)$$

The expression in the right hand side of (2.2) can be solved explicitly for certain specification of the Lévy process  $L$ . In particular for the Black-Scholes and Merton case, we have

$$B(t, \bar{T}) = e^{-r(\bar{T}-t)} N(d_2) + V_t N(-d_1) \quad (2.3)$$

where  $N(\cdot)$  is the cumulative standard normal distribution function,  $d_1 = [\log(V_t) + (r + \sigma^2/2)(\bar{T} - t)] / \sigma \sqrt{(\bar{T} - t)}$  and  $d_2 = d_1 - \sigma \sqrt{(\bar{T} - t)}$ . The computation leading to the expression in (2.3) is obtained by application of standard option pricing formula as the right-hand side of (2.2) is the payoff of an European call option on the firm's value at time  $\bar{T}$  with strike price 1.

The fixed maturity structural model of Black-Scholes and Merton was later generalized to allow default prior to time  $\bar{T}$ . In this extended setting, default happens if the asset's value hits a prespecified (possibly stochastic) default barrier  $D$ . Then the market information set is given by  $\mathcal{F}_t = \sigma(V_s, D_s : s \leq t)$ . Here the default time becomes a

random variable which we define as the first hitting time of the barrier given by

$$\tau = \inf \{t > 0 : V_t \leq D_t\} \quad (2.4)$$

We assume that the default barrier is paid at the maturity time  $\bar{T}$ .

Given the above first passage time default definition as in the expression (2.4), the value of the zero-coupon bond at time  $t$  on the event  $\{t \leq \tau\}$  becomes

$$B(t, \bar{T}) = e^{-\int_t^{\bar{T}} r_s ds} \mathbb{E}_{\mathbb{Q}} (1_{\{\tau \leq \bar{T}\}} D_{\tau} + 1_{\{\tau > \bar{T}\}}) \quad (2.5)$$

If both the interest rate  $r_t$  and default barrier  $D$  are constant, then on the event that default has not occurred at time  $t$  we have

$$B(t, \bar{T}) = e^{-r(\bar{T}-t)} [D\mathbb{Q}(\tau \leq \bar{T}) + (1 - \mathbb{Q}(\tau \leq \bar{T}))] \quad (2.6)$$

Analogously, if the firm value is specified by a Geometric Brownian Motion (GBM), the expression in (2.6) can be evaluated explicitly (see [77, 32] for more details). Various extensions of this generalization can be found in the literature. For example formulation of more complex liability structures was considered in [74].

To summarize, the information based perspective for structural models is premised on the assumption that the information set of market participant is generated by continuous observation of both the firm's asset and liability processes. This often leads to a predictable default time which cannot be verified by empirical study. Next we describe reduced form models from information-based perspective.

## Reduced form models

In this section, we review reduced form models with regards to assumptions on the information available to the market participants. Reduced form models were first introduced in [48] with subsequent contributions in [50, 24]. The main idea of this modeling approach is that the filtration of the market participants is generated by the default time  $\tau$  and a state process  $L_t$ . The default time is modeled in an exogenous fashion as a stopping time<sup>1</sup> generated by the indicator process  $N_t = 1_{\{\tau \leq t\}}$  with an intensity  $\lambda_t$  which depends on the state variable  $L_t$ . That is  $\mathcal{F}_t = \sigma(\tau, L_s : s \leq t) \subset \mathcal{G}_t$ . The process  $N$  is often referred to as a Cox process. If the state variable is modeled by a diffusion process then conditional on the filtration  $\sigma(L_s : s \leq \bar{T})$  generated by the state variable over the entire time horizon  $[0, \bar{T}]$ , the conditioned point process  $\mathbb{Q}(N_t | \sigma(L_s : s \leq \bar{T}))$  is Poisson with intensity  $\lambda_t(L_t)$ , see [17]. The case of more general Lévy process was considered in [7] and for general semimartingales in [10].

Now, let us consider under the reduced form approach the valuation of the defaultable zero coupon bond described in the previous section. First we assume that the payoff on the event of default is given by a stochastic process  $\delta_t$ , which we call the recovery rate. The market information set is then augmented to include the information provided by the continuous observation of the recovery rate, i.e.  $\mathcal{F}_t = \sigma(\tau, L_s, \delta_s : s \leq t)$ . To be consistent with the previous section, we assume the recovery rate is paid at the maturity time  $\bar{T}$  of the debt obligation.

The probability of default prior to time  $\bar{T}$  under the current formulation is given by

$$\mathbb{Q}(\tau \leq \bar{T}) = \mathbb{E}_{\mathbb{Q}}(\mathbb{E}_{\mathbb{Q}}(N_{\bar{T}} = 1 | \sigma(L_s : s \leq \bar{T}))) \quad (2.7)$$

The expression in the right-hand side of (2.7) can be evaluated explicitly for certain

<sup>1</sup>A stopping time  $\tau$  a positive random variable such that  $\forall t \in [0, \bar{T}]$  the event  $\{\tau \leq t\} \in \mathcal{F}_t$ . If there exist a sequence of stopping times  $(\tau_k)_{k \geq 1}$  such that  $\tau_k \leq \tau \forall k$  and  $\lim_{k \rightarrow \infty} \tau_k = \tau$ , then we say that  $\tau$  is predictable. A stopping time  $\tau$  is said to be totally inaccessible if for every predictable stopping time  $\hat{\tau}$ ,  $\mathbb{Q}(\omega : \tau(\omega) = \hat{\tau}(\omega) < \infty) = 0$ . (See [47])

specification of the state variable. The value of the debt at time  $t$  on the event  $\{\tau \leq t\}$  is given by

$$B(t, \bar{T}) = e^{-\int_t^{\bar{T}} r_s ds} \mathbb{E}_{\mathbb{Q}} \left( [1_{\{\tau \leq \bar{T}\}} \delta_{\tau} + 1_{\{\tau > \bar{T}\}}] \right) \quad (2.8)$$

For constant intensity and recovery rate processes, the expression in (2.8) leads to explicit evaluation of the default probability as studied in [10, 50]. To summarize, in the reduced form credit risk models the information set observed by the market is specified through exogenous market parameters. This is the key distinction from the structural models where the market information set is endogenously specified through the firm's assets and liabilities. The information based perspective provides a platform to relate the two modeling frameworks through filtration enlargement<sup>2</sup> or reduction rather than the usual case of accessibility and inaccessibility of the default time. In particular, it was shown in [47] that reducing the market information set from more to less transforms a structural model into a reduced form model with inaccessible default time and vice versa. In other words, reduced form models do not consider directly the actual cause of the default. Finally it is important to note that the information set in both structural and reduced form models as described above is assumed to be known from inception. In the next section, we review credit risk models under time dependent revelation of the information set.

### 2.1.2 Information Time-Dependency and Credit Risk Models

It is assumed in both structural and reduced form credit risk models as described in the previous sections that the market information is fixed from inception. What this means is that time-dependent revelation of investor information plays no role in the model definition. An alternative credit risk modeling framework was introduced by Brody, Hughston and Macrina in [23, 57] based on time dependency of market information.

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<sup>2</sup>See chapter VI in [47] for introduction to the theory of enlargement of filtration and [17, 74, 18] for applications in credit risk modeling

The authors analyzed the "role of time-dependent revelation of investor information". In this section we present a brief description of this asset pricing theory. The objective is to outline the important ideas behind the framework so that the reader will easily follow the applications of conditioned stochastic processes to the theory in subsequent chapters as market information processes .

We fix a time horizon,  $[0, T]$ ,  $T > 0$  on the probability space  $(\Omega, \mathcal{F}, \mathbb{Q})$ . We consider a deterministic interest rate  $\{r_t > 0\}$  then the time  $s$  price of a zero-coupon bond that pays 1 unit of currency at maturity  $t$ , is given by

$$P_{st} = \exp\left(-\int_s^t r_u du\right), \quad (s \leq t). \quad (2.9)$$

For any  $t < T$ , the time  $t$  price of a cash flow  $H_T$  occurring at time  $T$  is given by

$$H_{tT} = P_{tT} \mathbb{E}(H_T | \mathcal{F}_t),$$

where  $\{\mathcal{F}_t\}$  is the filtration generated by the information process . The *market information* represents the information available to the market participants at the current time  $t$ . We assume that there exists an information process  $\{I_{tT}\}_{0 \leq t < T}$  (possibly multi-dimensional) which generates the market filtration (i.e.  $\mathcal{F}_t = \sigma(\{I_{sT}\}_{0 \leq s \leq t})$ ). The challenge to the analyst is to construct an appropriate model for  $\{I_{tT}\}_{0 \leq t < T}$  for an asset with a fixed time cash flow  $H_T = h(X_T)$  for some function  $h(x)$  and market factor  $X_T$ , called the  $X$ -factor. In their initial paper, Brody et al. [23] considered a finite time horizon,  $T < \infty$  and used a *heuristic* approach to construct a model of the market information process. The Brownian bridge and Gamma bridge information processes of Brody, Hughston and Macrina, [23] were first obtained by this approach. Brownian bridge information process is modelled explicitly as

$$I_{tT} = \alpha t H_T + \beta_{tT}, \quad I_{TT} = H_T,$$

where the process  $\{\beta_{tT}\}_{0 \leq t \leq T}$  is a standard Brownian bridge and the constant  $\alpha$  denotes the rate of information arrival. Similarly, the Gamma bridge information process is modelled explicitly as

$$I_{tT} = \gamma_{tT} H_T, \quad (0 \leq t \leq T),$$

where  $\{\gamma_{tT}\}_{0 \leq t \leq T}$  is a Gamma bridge process starting at 0 and ending at 1. See [23, 57] for more explanations on this approach. More recently, Hoyle et al. [27] used a different approach to construct the market information process, which we refer to as the *probabilistic* approach. The view in Hoyle's paper is that the task of modeling the evolution of market information can be reduced to that of specifying the law of the information process. In this approach, the authors work on a finite time horizon and model the information process  $I_{tT}$  as a *Lévy Random Bridge* (LRB). The LRBs are obtained by the specification of the law  $LRB_C([0, T], \{f_t\}, \nu)$  or  $LRB_D([0, T], \mathcal{Q}_t, \mathcal{P})$  for the cases of continuous densities and discrete marginal probability mass respectively. For example, the Brownian bridge information process corresponds to the case where  $\{f_t\}$  is the density process of a standard Brownian motion and  $\nu$  is the marginal law of the terminal value of a Brownian motion. The emphasis here is that the information is not considered as given rather the information availability is viewed as an "emergent" phenomenon. In the present study, we extend this framework by constructing the market information via a conditioning approach. Specifically, we model the market information process by a random conditioning argument. Motivated by this conditioning argument, we further generalize the information based approach to include randomly-timed cash flows. So far in the literature, the discussion on this topic has avoided incorporation of pre-maturity default times. The assumption in the previous treatments is that "default" of credit risky bonds simply imply a failure of the bond issuer to meet debt obligation at the maturity date of the bond. Several extensions of the framework till date has maintained this view (See, [23, 27]). Here, we provide explicit construction of the information process through a conditioning on the space of a certain subclass of Lévy process which includes the Wiener space. The key benefit for modeling the information process as a

conditioned stochastic process is that this approach provides for the modeling framework be generalized to include other functionals of the market factor than its value at a fixed future date. For instance, we can consider a case where the signal is in the form of the knowledge of the distribution of first hitting time of the market factor to a certain level in a given period. In the subsequent chapters, we shall use this approach to construct price processes for more general class of assets. It was mentioned in [27] that the Brownian bridge information process is identical in law to a conditioned Brownian motion. In the next section we define the conditioned stochastic process and construct the Brownian bridge information process explicitly as a conditioned stochastic process.

## 2.2 Construction of Market Information by Conditioning

### 2.2.1 Introduction

The information based asset pricing theory of Brody, Hughston and Macrina in [23], presents researchers with the important problem of modeling the flow of market information. This problem involves finding a suitable class of models for the information regarding the cash flow of wider class of assets in the financial market, where issues of tractability and computational complexities are appropriately taken into consideration. In the information based asset pricing framework, an asset is associated with a sequence of random cash flows. The price of the asset is given as the sum of the discounted conditional expectations of the cash flows. The conditional expectation is taken with respect to the filtration generated by the information process. The complexity and tractability of the conditional expectation depends on (i) nature of the cash flow (ii) the law of the information process. Therefore, modelling the information process in this framework involves specifying its law and /or the nature of the cash flow. In this section, we model the nature of the cash flow by a random conditioning on a filtered probability

space and define the information process as the unique stochastic process associated with the conditioning. Then we derive the law of the resulting information process using techniques from classical theory of conditioned stochastic differential equations and general theory of Markovian bridges. Following the simple Brownian bridge and Gamma bridge models of market information by Brody et al in [23], Hoyle et al in [27] extended the market information model to a wider class of models called Lévy Random Bridges (LRBs). In both [23] and [27], the authors postulate the existence of a market information process and obtain closed form expression for prices of European style contracts. In particular the model for market information was given explicitly in [23] while in [27] it was given by the specification of the density of a Lévy process and an *a priori* law for the asset terminal cash flow in which case the information process corresponds to the unique stochastic process whose terminal distribution matches the specified *a priori* distribution of the cash flow. These contributions provide the required background and foundation for the construction of more general market information models which are not only attractive due to their tractability but for practical purposes are suitable as market information models for a wider class of assets, different level of investors and broader financial markets. More specifically, there is need to study market information models that are appropriate for pricing assets with more complex cash flow structures. For example cash flows of credit risky assets with prematurity default, cash flows of futures on realized volatility of an asset and cash flows of american style derivative products. To incorporate situations as described above into the information based asset pricing, we model the information process in the present work as the unique solution of a special class of stochastic differential equation associated with a conditioning. We refer these processes as Compound Lévy Random Bridges (CLRBs). To lay the foundation for explicit mathematical construction and analysis of CLRBs as well as their subsequent application as market information model we dedicate the remainder of this chapter to the following: In subsection 2, we introduce Conditioned Stochastic Process (CSPs). First we present important results from the theory of conditioned

stochastic differential equations (CSDEs), see [6, 31]. Then we define a Conditioned Stochastic Process as the unique solution to a CSDE. We only consider CSPs which are semi-martingale and Markovian. The emphasis is on the properties of CSPs that make them appropriate for the modeling of the dynamics of a market information process. We also verify the bridge and Markov properties of certain class of CSPs. In subsection 3, we model the market information by conditioning. We established that the Brownian bridge information process can be constructed as conditioned stochastic process. Precisely, we re-derive the Brownian bridge information process of [23] as a CSP associated with conditioning of the marginal law of a Brownian motion for a specified initial condition. In conclusion, we derive expression for the price of European option and a binary credit risky bond. Throughout the rest of the chapter, we fix a probability space  $(\Omega, \mathbb{Q}, \mathcal{F})$  and assume that all processes and filtration under consideration are adapted to it. We consider the time horizon,  $[0, T], T \in (0, \infty]$  and assume that all stochastic processes take values in  $\mathbb{R}$ .

## 2.2.2 Conditioned Stochastic Process

In this section, we define conditioned stochastic processes (CSPs) in the Wiener space and present important results on the characterization of their laws. We further investigate the Markov property as well as their conditional terminal distributions. These properties are useful in the application of CSPs as information models in the information based asset pricing framework.

We consider the space  $(C_\infty, \{\mathcal{F}_t\}_{t \geq 0}, \{X_t\}_{t \geq 0}, \mathbb{Q})$  on the constant time horizon,  $T \in (0, \infty]$ , where  $C_\infty$  is the space of continuous functions  $\mathbb{R}_+ \rightarrow \mathbb{R}$ . For  $T > 0$ ,  $C_T$  will denote the space of continuous functions  $[0, T] \rightarrow \mathbb{R}$ ,  $\{X_t\}_{t \geq 0}$  is the coordinate process defined by  $X_t(\omega) = \omega(t)$  for each path  $\omega$  and  $\mathbb{Q}$  is the Wiener measure. We want to construct the information process,  $I_{tT} = F(H_T, \eta_{tT})$ , which generates a filtration  $\{\mathcal{F}_t^I\}_{t \geq 0}$ .  $H_T$  is an  $\mathcal{F}_T$ -measurable random variable with *a priori* law  $\mu$  and  $\eta_{tT}$  is the noise term.

The process  $\{I_{tT}\}$  is defined as the conditioned stochastic process associated with the conditioning,  $(T, Y, \mu)$ , where the terminal value  $Y$  of the process  $\{X_t\}$  is forced to assume the law  $\mu$ .

**Definition 2.2.1.** *A conditioning on the Wiener space is a triplet  $(T, Y, \mu)$  with the following properties;*

1. *there exists a jointly measurable process  $\Lambda_t^y$ ,  $0 \leq t < T; y \in \mathbb{R}$  such that for any bounded and  $\mathcal{F}_t$ -measurable random variable  $Z$  and  $\mathbb{Q}_Y$  (where  $\mathbb{Q}_Y$  is the law of  $Y$  under  $\mathbb{Q}$ ) -a.s.  $y \in \mathbb{R}$*

$$\mathbb{E}(Z|Y = y) = \mathbb{E}(\Lambda_t^y Z) \quad (2.10)$$

2. *Supp  $\mu \subset$  Supp  $\mathbb{Q}_Y$  and  $L^1(\mathbb{R}, \mathbb{Q}_Y) \subset L^1(\mathbb{R}, \mu)$ .*

The interpretation of the triplet  $(T, Y, \mu)$  is as follows:  $T \in (0, \infty]$  corresponds to the period of time on which the conditioning is made.  $Y$  is a  $\mathcal{F}_T$ -measurable random variable with values in  $\mathbb{R}$  and differentiable in Malliavin sense. It represents a functional of the trajectory being conditioned.  $\mu$  is a probability measure on the Borel  $\sigma$ -algebra  $\mathcal{B}(\mathbb{R})$  corresponding to the actual conditioning.

The following theorem characterizes the law of a conditioned stochastic process for a given conditioning.

**Theorem 2.2.1.** *There exists a unique probability measure,  $\mathbb{Q}^\mu$  such that*

- *if  $Z : (C_T; \mathcal{F}_T) \rightarrow (\mathbb{R}; \mathcal{B}(\mathbb{R}))$  is a bounded random variable then*

$$\mathbb{E}_\mu(Z|Y) = \mathbb{E}(Z|Y) \quad (2.11)$$

- *the law of  $Y$  under  $\mathbb{Q}^\mu$  is precisely  $\mu$ .*

$\mathbb{Q}^\mu$  is given by the formula: For  $A \in \mathcal{F}_T$ :

$$\mathbb{Q}^\mu(A) = \int_{\mathbb{R}} \mathbb{Q}(A|Y = y)\mu(dy). \quad (2.12)$$

The proof of the above theorem can be found in ([6], p. 119).

Given a conditioning  $(T, Y, \mu)$  on a Wiener space, we define the associated conditioned stochastic process as follows;

**Definition 2.2.2.** Let a standard Brownian motion  $\{w_t\}_{0 \leq t < T}$  be defined on a filtered probability space  $(\Omega, \{\mathcal{H}_t\}_{0 \leq t < T}, \mathbb{Q})$ , the process

$$I_{tT} = \int_0^t \left[ \frac{\int_{\mathbb{R}} \alpha_s^y \Lambda_s^y \mu(dy)}{\int_{\mathbb{R}} \Lambda_s^y \mu(dy)} \right]_{s \leq t} ds + w_t, \quad t < T \quad (2.13)$$

is called a conditioned stochastic process (CSP) associated with the conditioning  $(T, Y, \mu)$ .

Here,  $\alpha_t^y$  is a measurable process such that:

1. For  $\mathbb{Q}_Y - a.s.$ ,  $y \in \mathbb{R}$ ,  $\alpha_t^y$  is  $\mathcal{F}_t$ -predictable
2. For  $\mathbb{Q}_Y - a.s.$ ,  $y \in \mathbb{R}$  and for  $0 \leq t < T$

$$\mathbb{Q} \left( \int_0^t \alpha_s^y ds < +\infty \right) = 1.$$

3. For  $\mathbb{Q}_Y - a.s.$ ,  $y \in \mathbb{R}$  and for  $0 \leq t < T$

$$\langle \Lambda^y, X \rangle_t = \int_0^t \alpha_s^y \Lambda_s^y ds.$$

**Lemma 2.2.1.** The probability law of the conditioned stochastic process (CSP),  $\{I_{tT}\}_{0 \leq t < T}$ , associated with the conditioning  $(T, Y, \mu)$  is given by

$$\mathbb{Q}^\mu(I_{tT} \in dx) = \int_{y=-\infty}^{y=\infty} \mathbb{Q}(I_{tT} \in dx|Y = y)\mu(dy). \quad (2.14)$$

*Proof.* This is a direct consequence of theorem (2.2.1). □

## Markov Property

It is desirable in the information based asset pricing framework for the model of market information to possess the Markov property. In this section, we show that CSPs are Markov processes. This simplifies subsequent calculations in the determination of asset price dynamics. As we shall see, the Markov property of the CSPs follows from the independent increments of the driving Wiener process. The independent increment property of Lévy processes makes the extension of this approach to a general Lévy models a viable endeavor. This we shall consider in chapter three below.

**Proposition 2.2.1.** *Given a conditioning  $(T, Y, \mu)$ , the associated Conditioned Stochastic Process  $\{I_{tT}\}_{0 \leq t < T}$  is a Markov process.*

*Proof.* Let  $\mathcal{F}_t^I$  denote the filtration generated by the process,  $\{I_{tT}\}_{0 \leq t < T}$ . Then we need to verify that

$$\mathbb{E}[h(I_{tT})|\mathcal{F}_s^I] = \mathbb{E}[h(I_{tT})|I_{sT}] \quad (2.15)$$

for any bounded measurable function  $h(x)$  and for all  $s, t$  such that  $0 \leq s \leq t < T$ .

It suffices to show that

$$\mathbb{E}[h(I_{tT})|I_{sT}, I_{s_1T}, I_{s_2T}, \dots, I_{s_nT}] = \mathbb{E}[h(I_{tT})|I_{sT}] \quad (2.16)$$

for any collection of times  $t, s, s_1, s_2, \dots, s_n$  such that  $0 \leq s_n \leq \dots \leq s_2 \leq s_1 \leq s \leq t < T$ .

First we note that the conditioned process,  $\{I_{tT}\}_{0 \leq t < T}$  can be expressed in the form

$$I_{tT} = \int_0^t H(s, (X_u)_{u \leq s}) ds + \omega_t \quad (2.17)$$

where  $H$  is a predictable function such that for all  $t < T$

$$\frac{\int_{\mathbb{R}} \alpha_s^y \Lambda_s^y \mu(dy)}{\int_{\mathbb{R}} \Lambda_s^y \mu(dy)} = H(t, (X_s)_{s \leq t}). \quad (2.18)$$

We know that the increments of a standard Brownian motion  $\omega_s - \omega_{s_1}, \omega_{s_1} - \omega_{s_2}, \dots, \omega_{s_{n-1}} - \omega_{s_n}$  are independent for  $s_n \leq s_{n-1} \leq \dots \leq s_2 \leq s_1 \leq s$ . Then it follows that

$$\begin{aligned} \mathbb{E}[h(I_{tT}) | I_{sT}, I_{s_1T}, I_{s_2T}, \dots, I_{s_nT}] &= \mathbb{E}[h(I_{tT}) | I_{sT}, I_{sT} - I_{s_1T}, \dots, I_{s_{n-1}T} - I_{s_nT}] \\ &= \mathbb{E}[h(I_{tT}) | \omega_s, \omega_s - \omega_{s_1}, \dots, \omega_{s_{n-1}} - \omega_{s_n}] \\ &= \mathbb{E}[h(I_{tT}) | I_{sT}], \end{aligned}$$

since  $I_{tT}$  and  $I_{sT}$  are independent of  $\omega_s - \omega_{s_1}, \omega_{s_1} - \omega_{s_2}, \dots, \omega_{s_{n-1}} - \omega_{s_n}$ .  $\square$

## Conditional Terminal Law

In this section, we want to show that  $\mathcal{F}_s^I$ -conditional law (density) of the terminal value  $I_{TT} = \lim_{t \rightarrow T} I_{tT}$  exists. Given a conditioning  $(T, Y, \mu)$ , we write the conditional law of  $Y$  under  $\mathbb{Q}$  as  $\mathbb{Q}_Y(Y \in dy | \mathcal{F}_t^I) = p_t(x, y) dy$ . Let  $\mu_s$  denote the  $\mathcal{F}_s^I$ -conditional law of  $I_{TT}$ . We have  $\mu_0(A) = \mu(A)$ . Then for  $s > 0$ , it follows from theorem 2.2.1 and equation (3.25) that

$$\mu_s(z; dy) = \frac{\Lambda_s^y \mu(dy)}{\int_{-\infty}^{\infty} \Lambda_s^y \mu(dy)}. \quad (2.19)$$

The  $\mathcal{F}_t^I$ -conditional  $k$ th moment of the terminal value,  $I_{TT}$  is given by

$$\int_{-\infty}^{\infty} |y|^k \mu_t(x; dy).$$

If

$$\int_{-\infty}^{\infty} |y|^k \mu(dy) < \infty \quad (2.20)$$

then the  $\mathcal{F}_t^I$ -conditional  $k$ th moment of  $I_{TT}$  is finite.

**Proposition 2.2.2.** *Let  $M_k(I_{tT}) = \int_{-\infty}^{\infty} |y|^k \mu_t(x; dy)$ . If (2.20) holds for  $k \in \mathbb{Z}$ , then  $M_k(I_{tT})$  is a martingale with respect to  $\mathcal{F}_t^I$ .*

*Proof.* We want to show that under the probability  $\mathbb{Q}^\mu$ ,

$$\mathbb{E} (M_k(I_{tT}) | \mathcal{F}_s^I) = M_k(I_{sT}).$$

Note that for  $s \leq t$ ,  $\mathcal{F}_s^I \subseteq \mathcal{F}_t^I$ .

Then using the tower property of conditional expectation we have

$$\begin{aligned} \mathbb{E} (M_k(I_{tT}) | \mathcal{F}_s^I) &= \mathbb{E} (\mathbb{E} (|y|^k | \mathcal{F}_t^I) | \mathcal{F}_s^I) \\ &= \mathbb{E} (|y|^k | \mathcal{F}_s^I) \text{ (by the tower property)} \\ &= \mathbb{E} (|y|^k | I_{sT}) \\ &= M_k(I_{sT}) \end{aligned}$$

as required. □

### 2.2.3 Market Information Model by Conditioning

#### Conditioned Brownian Motion

Perhaps the best way to further illustrate the intuition that market information process as described in the information based asset pricing framework can be modelled by a conditioning in the Wiener space is by way of example. Thus the objective in this section is to construct the Brownian bridge information process of Brody et al. [23] as a conditioned stochastic process. Specifically, we obtain the explicit form of the process as a solution of a conditioned stochastic differential equation associated with the conditioning of the marginal law of a standard Brownian motion. The intention is that

this simple application of the theory of conditioned stochastic differential equations will motivate further research in the theory of CSDEs associated with the conditioning of the marginal law of other forms of Lévy processes (e.g. Gamma and Stable-half processes) which in turn will pave way for further application of the theory to the construction of wider class of market information process. Recall the Brownian bridge information process given by

$$Y_{tT} = \alpha t X_T + \beta_{tT}, \quad Y_{TT} = X_T, \quad (2.21)$$

with  $\alpha$  as constant, denoting the rate of information flow to market participants,  $X_T$  as the payoff of the asset at the terminal point  $T$  and  $\beta_{tT}$  as the Brownian bridge process on the interval  $[0, T]$ , such that  $\beta_{0,T} = \beta_{TT} = 0$ .

**Remark 2.2.1.** We know that  $\beta_{tT}$  has mean zero and the covariance of  $\beta_{sT}$  and  $\beta_{tT}$  is  $\frac{s(T-t)}{T}$ .

We consider the filtered probability space  $(\Omega, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{Q})$  on which a standard  $(\mathcal{F}_t, \mathbb{Q})$  Brownian motion  $\{\omega_t\}_{0 \leq t \leq T}$  is defined. The expression for the CSDE associated with the conditioning of marginal law of a standard Brownian motion is given in (2.13).

**Proposition 2.2.3.** The market information process defined by (2.21) is a Conditioned Stochastic Process associated with the conditioning of the marginal law of a standard Brownian motion, with the initial condition  $Y_{0T} = 0$ .

*Proof.* Let  $(T, \omega_T, \nu)$  denote the conditioning of the marginal law of a standard Brownian motion.  $\omega_T$  is the value at  $T$  of a standard Brownian motion.  $\nu$  is a probability measure such that for  $x \in \mathbb{R}$ ,

$$\int_{\mathbb{R}} x^2 \nu(dx) < \infty$$

and the law of  $\omega_T$  in  $\mathbb{Q}$  is  $\nu$ . Let  $\alpha_t^y = \frac{y-x}{T-t}$ , then from (2.17) we obtain;

$$dI_{tT} = \frac{\int_{-\infty}^{+\infty} \left( \frac{x-I_{tT}}{T-t} \right) e^{\frac{x^2}{2T} - \frac{(x-I_{tT})^2}{2(T-t)}} \nu(dx)}{\int_{-\infty}^{+\infty} e^{\frac{x^2}{2T} - \frac{(x-I_{tT})^2}{2(T-t)}} \nu(dx)} dt + d\omega_t, \quad t < T, \quad I_{0T} = 0. \quad (2.22)$$

Now, let us choose  $\nu(dx) = \frac{e^{-(x-mT)^2/2T}}{\sqrt{2\pi T}} dx$ , then Eq. (2.22) becomes

$$dI_{tT} = mdt + d\omega_t.$$

The corresponding solution associated with the initial value,  $I_{0T} = 0$  is

$$I_{tT} = mt + \omega_t, \quad t < T. \quad (2.23)$$

Recall that the Brownian bridge,  $\beta_{tT}$  can be transformed into a Brownian motion with drift as

$$\beta_{tT} = \omega_t - \frac{t}{T}\omega_T.$$

Hence, Eq. (2.23) can be re-written as

$$I_{tT} = \left(m + \frac{\omega_T}{T}\right)t + \beta_{tT}. \quad (2.24)$$

Now, setting  $m + \frac{\omega_T}{T} = \alpha X_T$  (where  $\alpha$  is constant and  $X_T$  a random variable) gives the first expression for the information process in (2.21). To conclude, we show that

$$I_{TT} = \lim_{t \rightarrow T} I_{tT} = X_T.$$

Indeed, observe that for  $\alpha = \frac{1}{T}$  as in [23], and from (2.24),

$$I_{TT} = \lim_{t \rightarrow T} I_{tT} = mT + \omega_T = X_T$$

as required. □

The interpretation of the above proposition is that the Brownian bridge market information process can be constructed explicitly by the specification of an appropriate conditioning,  $(T, Y, \mu)$  on the Wiener space. This corresponds to the situation where the functional is the law of a coordinate process and the conditioning is the specification

of the *a priori* law of the cash flow such that the associated CSP provides the required information about the *a posteriori* law of the Cash flow. In some real life applications, it may be easier to specify a conditioning on other functionals of the coordinate process than its law. For example, its quadratic variation or first hitting time of a level. This is particularly desirable when the payoff (cash flow) is in the form of such a functional. For instance, when the *a priori* law of the cash flow is more difficult to determine compared to that of its quadratic variation or its first hitting time of a level. In this case, the market information regarding a contingent claim on the asset at a future time can be obtained by an appropriate conditioning on the functional. In the following sections we price European option on a credit risky discount bond with a conditioned stochastic process as the market information process.

## 2.2.4 Applications to Credit Risky Assets Pricing

### European Options

In this section we focus on a market with a single factor <sup>3</sup>denoted by  $X_T$ . We work in continuous time. The asset is modelled by a random cash flow  $H_T = h(X_T)$  occurring at time  $T$ . For simplicity we consider contingent cash flow of the form  $h(x) = x$ . We assume that  $X_T$  is an integrable random variable with *a priori* probability law  $\mu$ . The market information regarding  $X_T$  is provided by the process  $\{I_{tT}\}_{0 \leq t < T}$ . The information process  $I_{tT}$  is the unique conditioned stochastic process associated with the conditioning  $(T, Y, \mu)$ .  $Y$  is a functional of a coordinate process on the Wiener space. Using the Markov property of the information process, the time  $t < T$  price of the cash flow is given by

$$H_{tT} = P_{tT} \mathbb{E}_\mu[X_T | I_{tT}] \quad (2.25)$$

<sup>3</sup>The case of multiple cash flow follows analogously in which case  $N$  cash flows,  $H_{T_1}, H_{T_2}, \dots, H_{T_N}$ , are to be received at dates,  $T_1 \leq T_2 \leq \dots \leq T_N$  respectively. For each date  $T_j$  and each  $X$ -factor,  $X_{T_j}^i$  we have the conditioning  $(T_j, Y_j, \mu_j)$  where  $Y_j$  is a  $\mathcal{B}(\mathbb{R}) \rightarrow \mathbb{R}$  random functional,  $\mu_j$  is some probability measures on  $\mathcal{B}(\mathbb{R})$ .

where  $P_{tT}$  is the discount factor as defined in (2.9). The  $\mathcal{F}_t^I$ -conditional law of  $X_T$  is given in (2.19) as

$$\mu_t(x; dy) = \frac{\Lambda_t^y \mu(dy)}{\int_{-\infty}^{\infty} \Lambda_t^y \mu(dy)}. \quad (2.26)$$

Then we obtain,

$$H_{tT} = P_{tT} \int_{-\infty}^{\infty} y \mu_t(x; dy). \quad (2.27)$$

In the case of conditioning of the marginal law of a standard Brownian motion we have

$$\Lambda_t^y = \sqrt{\frac{T}{T-t}} \exp \left[ \frac{y^2}{2T} - \frac{(y-x)^2}{2(T-t)} \right]. \quad (2.28)$$

In addition, if  $\mu(dy)$  admits a density  $\rho(y)$  then the  $\mathcal{F}_t^I$ -conditional law of the cash flow becomes

$$\mu_t(x; dy) = \frac{\Lambda_t^y \rho(y) dy}{\int_{-\infty}^{\infty} \Lambda_t^y \rho(y) dy}. \quad (2.29)$$

Now, we want to determine the price of European option written on  $H_{tT}$ . The time  $s$ , ( $0 \leq s \leq t$ ) price of a  $t$ -maturity put option on  $H_{tT}$  with strike  $K$  is given by

$$C_{st} = P_{st} \mathbb{E}[(K - H_{tT})^+ | I_{sT}]. \quad (2.30)$$

Using results from the previous sections we have

$$\begin{aligned} C_{st} &= P_{st} \mathbb{E}_\mu \left[ (K - P_{tT} \mathbb{E}_\mu[H_T | I_{tT}])^+ | I_{sT} \right] \\ &= P_{st} \mathbb{E}_\mu \left[ \left( \int_{-\infty}^{\infty} (K - P_{tT} y) \mu_t(x; dy) \right)^+ | I_{sT} \right] \\ &= P_{st} \mathbb{E}_\mu \left[ \frac{1}{\int_{-\infty}^{\infty} \Lambda_t^y \mu(dy)} \left( \int_{-\infty}^{\infty} (K - P_{tT} y) \Lambda_t^y \mu(dy) \right)^+ | I_{sT} \right]. \end{aligned} \quad (2.31)$$

Define the process  $D_t$  by

$$D_t = \int_{-\infty}^{\infty} \Lambda_t^y \mu(dy) \quad (2.32)$$

It can be shown easily that  $\{D_t\}$  is a density process and that under  $\mathbb{Q}$ ,  $D_t^{-1}$  is a martingale. Then the expression for the option price becomes,

$$C_{st} = \frac{P_{st}}{D_s} \mathbb{E}_{\mathbb{Q}} \left[ \left( \int_{-\infty}^{\infty} (K - P_{tT}y) \Lambda_t^y \mu(dy) \right)^+ \middle| I_{sT} \right] \quad (2.33)$$

$$= \frac{P_{st}}{D_s} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} (K - P_{tT}y) \Lambda_t^y \mu(dy) \right)^+ \mu_s(z; dy). \quad (2.34)$$

Let the set  $A_t$  be defined by

$$A_t = \left\{ x \in (-\infty, \infty) : \int_{-\infty}^{\infty} (K - P_{tT}y) \Lambda_t^y \mu(dy) > 0 \right\}^4 \quad (2.35)$$

the expression for the put option price becomes

$$C_{st} = P_{st} \int_{-\infty}^{\infty} \int_{x \in A_t} (K - P_{tT}y) \mu_s(z; dy) \mu_{st}(dx; y) \quad (2.36)$$

where  $\mu_{st}(x; y) = \frac{\mu_t(x; y)}{D_s}$ .

Notice that  $H_{tT}$  can be written as

$$H_{tT} = \zeta(t, I_{tT}) \quad (2.37)$$

then the set  $A_t$  becomes

$$A_t = \{I \in (-\infty, \infty) : \zeta(t, I) < K\}.$$

The break even market information  $I_{tT}^*$  is such that  $A_t = \{x : x \in (-\infty, I_{tT}^*)\}$ , where  $\zeta$  is some deterministic function.

If the information is generated by the conditioned stochastic process associated with the conditioning of the marginal law of a Brownian motion  $\{B_t\}$ , then  $\Lambda_t^y$  is given by (2.28) and the functional  $Y$  is defined by  $Y = B_T$ . In this case  $\mu_{st}(x; y)$  is normal with

<sup>4</sup> Note that  $\Lambda_t^y$  is a function of  $x$ . See (2.28).

mean  $\bar{Z}(y) = \frac{T-t}{T-s}I_{sT} + \frac{t-s}{T-s}y$  and variance  $\sigma^2 = \frac{t-s}{T-s}(T-t)$ . Then the inner integral in (2.36) can be written as

$$\int_{-\infty}^{I_t^*} \mu_{st}(dx; y) = \mathcal{N}\left(\frac{I_t^* - \bar{Z}(y)}{\sigma}\right)$$

where  $\mathcal{N}(x)$  is the standard normal cumulative distribution function.

The option price becomes

$$C_{st} = \int_{-\infty}^{\infty} \left[ (P_{st}K - P_{sT}y) \mathcal{N}\left(\frac{I_t^* - \bar{Z}(y)}{\sigma}\right) \mu_s(z; dy) \right]. \quad (2.38)$$

## Binary Credit Risky Bond

Consider a credit risky bond that pays  $H_T \in [h_0, h_1]$ ,  $h_0 < h_1$  at maturity time  $T$ . The bond pays a principle of  $h_1$  when there is no default but a recovery amount of  $h_0$  in the case of a partial default. The bond price  $H_{tT}$  at time  $t$  is given by

$$H_{tT} = P_{tT} \mathbb{E}_\mu(H_T | I_{tT}).$$

$I_{tT}$  is the conditioned stochastic process associated with the conditioning  $(T, Y, \mu)$ . From (2.27) the expression for the bond price becomes

$$\begin{aligned} H_{tT} &= P_{tT} \int_{-\infty}^{\infty} y \mu_t(x; dy) \\ &= P_{tT} \left[ \sum_{i=0}^1 h_i \mu_t(x; h_i) \right] \\ &= P_{tT} \left[ \frac{h_0 \Lambda_t^{h_0} \mu(h_0) + h_1 \Lambda_t^{h_1} \mu(h_1)}{\Lambda_t^{h_0} \mu(h_0) + \Lambda_t^{h_1} \mu(h_1)} \right]. \end{aligned}$$

Assume that *a priori*  $\mu(h_0) = \rho_0$  and  $\mu(h_1) = \rho_1 = (1 - \rho_0)$ . Then we have

$$H_{tT} = P_{tT} \left[ \frac{h_0 \Lambda_t^{h_0} \rho_0 + h_1 \Lambda_t^{h_1} \rho_1}{\Lambda_t^{h_0} \rho_0 + \Lambda_t^{h_1} \rho_1} \right]. \quad (2.39)$$

In addition, when the information is generated by the conditioned stochastic process associated with the conditioning of the marginal law of a Brownian motion  $\{B_t\}$ , then  $\Lambda_t^y$  is given by (2.28) and the functional  $Y$  is defined by  $Y = B_T$ . Particularly, the function  $\zeta(t, I_t)$  is given by

$$\zeta(t, I_t) = P_{tT} \left[ \frac{h_0 \Lambda^{h_0}(I_t) \rho_0 + h_1 \Lambda^{h_1}(I_t) \rho_1}{\Lambda^{h_0}(I_t) \rho_0 + \Lambda^{h_1}(I_t) \rho_1} \right].$$

Then the equation  $\zeta(t, x) = K$  can be solved explicitly for  $x$  using the appropriate initial condition.

The price of a put option on  $H_{tT}$  is then given by

$$C_{st} = \sum_{i=0}^1 \left[ (P_{st}K - P_{sT}h_i) \mathcal{N} \left( \frac{I_t^* - \bar{Z}(h_i)}{\sigma} \right) \mu_s(z; h_i) \right]. \quad (2.40)$$



# Chapter 3

## Compound Lévy Random Bridges

### 3.1 Introduction

Sequel to the conditioning argument in the previous chapter, we introduce in this chapter a class of processes that we call Compound Lévy Random Bridges (CLRBs). We begin by introducing Lévy bridges as a conditioning on the space of Lévy processes with continuous distribution. The key result here is that as we extend the fixed conditioning argument to a random conditioning, the resulting bridge processes (which we call compound Lévy random bridges) possess important properties required of an information process in the information based asset pricing framework. This generalization allows us to model credit risky assets with pre-maturity default possibilities in the chapters that follow.

We fix a time horizon  $[0, T]$ ,  $T > 0$  and a probability space  $(\Omega, \mathcal{F}, \mathbb{Q})$ . Recall that  $\mathcal{C}[0, T]$  denotes the class of Lévy processes  $\{L_t\}$  with continuous density  $p_t : \mathbb{R} \rightarrow \mathbb{R}_+$  for every  $t \in (0, T]$ . A CLRB is identical in law to a Lévy process conditioned to have a prespecified joint law at an independent random time. This definition is motivated by the construction of the information process as a conditioned stochastic process as well as the need for suitable information models for credit risky assets with randomly-timed cash flows. Later we shall use CLRBs as information processes in the information-based credit risky asset pricing framework. In what follows, we set the stage by presenting

important results on Lévy processes and their bridges. Thereafter we define and analyze key properties of Compound Lévy Random Bridges through a generalized notion of Lévy bridges.

### 3.1.1 Lévy Processes

In this section, we summarize a few well established results about Lévy processes. Extensive analysis of these results with detailed proofs can be found in [8] and [21]. A Lévy process is a stochastic process with stationary and independent increments which starts with the value zero. It is important to always assume a weak form of continuity for Lévy processes popularly known as stochastic continuity. A non-decreasing Lévy process is called a *subordinator*. Subordinators are important processes, they are particularly useful for the construction of Lévy processes from Brownian motion through a stochastic-time change. We denote a Lévy process by  $\{L_t\}$  and its *characteristic exponent*,  $\psi : \mathbb{R} \rightarrow \mathbb{C}$  is defined by

$$\mathbb{E}[e^{i\alpha L_t}] = \exp(-t\psi(\alpha)), \alpha \in \mathbb{R}. \quad (3.1)$$

The law of a Lévy process is characterized by the characteristic exponent. The explicit form of the characteristic exponent is given through the Lévy-Khintchine formula:

$$\psi(\alpha) = ib\alpha + \frac{1}{2}\sigma^2\alpha^2 + \int_{-\infty}^{\infty} (1 - e^{ix\alpha} + ix\alpha\mathbb{1}_{\{|x|<1\}}) \mu(dx) \quad (3.2)$$

where  $b \in \mathbb{R}$ ,  $\alpha > 0$  and  $\mu$  is called the *Lévy measure* on  $\mathbb{R} \setminus \{0\}$  satisfying

$$\int_{-\infty}^{\infty} (1 \wedge |x|^2) \mu(dx) < \infty. \quad (3.3)$$

In this thesis, we focus on a particular subclass of Lévy processes defined as follows;

**Definition 3.1.1.** Let  $\{L_t\}_{0 \leq t \leq T}$  be a Lévy process, then we write  $\{L_t\} \in \mathcal{C}[0, T]$  if the density of  $L_t$  exists for every  $t \in (0, T]$ .

Let  $p_t(z)$  denotes the density of  $L_t$  for some  $\{L_t\} \in \mathcal{C}[0, T]$ . then we have that  $p_t : \mathbb{R} \rightarrow \mathbb{R}_+$  and  $\mathbb{Q}[L_t \in dz] = p_t(z)dz$ .

To develop things fully we assume a bit more. We suppose there is a second process  $\{\hat{L}_t\} \in \mathcal{C}[0, T]$  in duality with  $\{L_t\}$  with respect to the Lebesgue measure. This means that the density of  $\hat{L}_t$  denoted by  $\hat{p}_t$  for some  $\{\hat{L}_t\} \in \mathcal{C}[0, T]$  is related to  $p_t(z)$  by

$$\int_{-\infty}^{\infty} h(z)p_{T-t}(-z)g(z)dz = \int_{-\infty}^{\infty} \hat{p}_t(z)h(z)g(z)dz \quad (3.4)$$

for every  $t \in [0, T]$  and positive Borel functions  $h$  and  $g$ . It is known that definition 3.1.1 together with (3.4) imply that the density function  $p_t(z)$  satisfies the Chapman-Kolmogorov identity

$$p_{t+s}(y) = \int_{-\infty}^{\infty} p_t(z)p_s(y-z)dz \quad (3.5)$$

for every  $s, t \in [0, T]$  and  $y, z \in \mathbb{R}$ .

### 3.1.2 Lévy Bridges

A bridge of any stochastic process is a process derived from fixing at inception the value of the process at a fixed future time. In a more general setting, bridges of Markov processes were constructed and analyzed in [65]. Our focus in this section is on the bridges of any Lévy process which belongs to the class  $\mathcal{C}[0, T]$ , for any  $T > 0$ . Particularly, we are interested in the Markov property of these bridges, hence we have the following;

**Proposition 3.1.1.** *The bridges of processes in  $\mathcal{C}[0, T]$  are Markov processes with transition densities*

$$p_{t+s}^{x,T}(z; y) = \frac{p_s(y-z)p_{T-(t+s)}(x-y)}{p_{T-t}(x-z)} \quad (3.6)$$

for  $s, t \geq 0$  such that  $s + t < T$ .

*Proof.* We want to show that the process  $\{L_t\} \in \mathcal{C}[0, T]$  is a Markov process if we know *a priori* that  $L_T = x$  for some constant  $x$  such that  $0 < p_T(x) < \infty$ . That is, we need to

show that

$$\mathbb{Q}[L_{t+s} \leq y | L_t = z, L_{t+s_1} = x_1, \dots, L_{t+s_m} = x_m, L_T = x] = \mathbb{Q}[L_{t+s} \leq y | L_{t+s_m} = x_m, L_T = x] \quad (3.7)$$

for all  $m \in \mathbb{N}_+$ ,  $(x_1, \dots, x_m, y) \in \mathbb{R}^{m+1}$  and  $0 \leq s_1 < \dots < s_m < s$ . Let us write

$$L_{s_i} = L_{t+s_i} - L_{t+s_{i-1}} \quad (3.8)$$

$$l_i = x_i - x_{i-1} \quad (3.9)$$

for  $1 \leq i \leq m$ , where  $s_0 = 0$  and  $x_0 = 0$ . Then by the independent increment property of  $\{L_t\}$  we have:

$$\begin{aligned} & \mathbb{Q}[L_{t+s} \leq y | L_t = z, L_{t+s_1} = x_1, \dots, L_{t+s_m} = x_m, L_T = x] \\ &= \mathbb{Q}[L_{t+s} - L_{t+s_m} \leq y - x_m | L_{s_1} = l_1, \dots, L_{s_m} = l_m, L_T - L_{t+s_m} = x - x_m] \\ &= \mathbb{Q}[L_{t+s} - L_{t+s_m} \leq y - x_m | L_T - L_{t+s_m} = x - x_m] \\ &= \mathbb{Q}[L_{t+s} - L_{t+s_m} \leq y - x_m | L_T - L_{t+s_m} = x - x_m, L_{t+s_m} = x_m] \\ &= \mathbb{Q}[L_{t+s} \leq y | L_T = x, L_{t+s_m} = x_m]. \end{aligned} \quad (3.10)$$

Now it remains to verify the formula for the transition densities. Let us write  $\{L_{t,T}^{(x)}\}$  for the  $\{L_t\}$ -bridge to the value  $x \in \mathbb{R}$  at time  $T$ . Since  $0 < p_T(x) < \infty$ , we have by Bayes theorem that

$$\begin{aligned} \mathbb{Q}[L_{t+s,T}^{(x)} \in dy | L_{t,T}^{(x)} = z] &= \mathbb{Q}[L_{t+s} \in dy | L_t = z, L_T = x] \\ &= \frac{\mathbb{Q}[L_{t+s} \in dy, L_T \in dx | L_t = z]}{\mathbb{Q}[L_T \in dx | L_t = z]} \\ &= \frac{p_s(y-z)p_{T-(t+s)}(x-y)}{p_{T-t}(x-z)} \end{aligned} \quad (3.11)$$

for  $s, t > 0$  such that  $t + s < T$ . □

### 3.1.3 Generalized Lévy Bridges

In this section we introduce the notion of generalized bridges of some Lévy process  $\{L_t\} \in \mathcal{C}[0, T]$ . This generalization is based on the extension of the conditioning argument presented in chapter 1 to a wider class of processes and Lévy functionals. We can interpret a Lévy bridge of the previous section as a process obtained by conditioning a Lévy process  $\{L_t\}$  to arrive at some fixed value  $x$  at a fixed future time  $T$ . In other words the Lévy bridge is obtained by the conditioning  $(T, Y, \mu)$  where  $Y$  corresponds to the terminal value  $L_T$  of some Lévy process  $\{L_t\} \in \mathcal{C}[0, T]$  and  $\mu$  is the dirac measure which assigns full mass to the singleton  $\{L_T = x\}$ . This representation is useful for a natural generalization of bridges of Lévy processes. The next question in the Lévy bridge folklore would be what process is obtained by conditioning the terminal random variable  $L_T$  on a more general marginal law  $\mu(dx)$  rather than to a fixed value  $x$ . This question was addressed in [27] through the construction of Lévy Random Bridges (LRBs). The authors showed that a Lévy Random Bridge coincides with a Lévy process conditioned to have a specified marginal law at the terminal point. Consider a Lévy process  $\{L_t\} \in \mathcal{C}[0, T]$ , its finite dimensional densities is given by

$$\mathbb{Q}[L_{t_1} \in dz_1, \dots, L_{t_n} \in dz_n, L_T \in dx] = \prod_{i=1}^n [p_{t_i - t_{i-1}}(z_i - z_{i-1}) dz_i] p_{T - t_n}(x - z_n) \quad (3.12)$$

for every  $n \in \mathbb{N}_+$ ,  $0 < t_i < \dots < t_n \leq T$  and  $(z_1, \dots, z_n) \in \mathbb{R}$ . Now suppose that the random variable  $X$  has law  $\mu$ , then we have:

$$\begin{aligned} \mathbb{Q}[L_{t_1} \in dz_1, \dots, L_{t_n} \in dz_n, L_T \in dx | L_T = X] &= \mathbb{Q}[L_{t_1} \in dz_1, \dots, L_{t_n} \in dz_n | L_T = x] \mu(dx) \\ &= \frac{p_{T - t_n}(x - z_n)}{p_T(x)} \prod_{i=1}^n [p_{t_i - t_{i-1}}(z_i - z_{i-1}) dz_i] \mu(dx). \end{aligned} \quad (3.13)$$

The above fixed time conditioning prompts the following questions:

- (a) what process  $\{L_{sT}\}$  is obtained by prescribing a probability measure  $\mu(dt)$  for a random time  $T'$  on  $[0, T]$  such that conditional on the event  $\{T' = t\}$  the process  $\{L_{sT}\}$  splits into two independent pieces  $\{L_{sT}\}_{0 \leq s \leq t}$  and  $\{L_{sT}\}_{t \leq s \leq T}$  whose conditional distributions are those of standard Lévy bridges  $\{L_{st}^0\}_{0 \leq s \leq t}$  and  $\{L_{sT}^0\}_{t \leq s \leq T}$  over the time intervals  $[0, t]$  and  $[t, T]$  respectively.
- (b) what process  $\{L_{sT}\}$  is obtained by choosing a joint probability measure  $\mu(dx, dt)$  on  $\mathbb{R} \times [0, T]$  for the random space-time variables  $(X, T')$  such that conditional on the event  $\{X = x, T' = t\}$ , the process  $\{L_{sT}\}$  is a splicing of a Lévy bridge  $\{L_{st}^x\}_{0 \leq s \leq t}$  and its dual-bridge  $\{\hat{L}_{sT}^x\}_{t \leq s \leq T}$  over the time intervals  $[0, t]$  and  $[t, T]$  respectively.

The processes in (a) and (b) were constructed in [65] for general Markov processes. In the next section, we use a conditioning approach to define a class of processes christened Compound Lévy Random Bridges (CLRb) with properties as described in (a) and (b) above for some Lévy process  $\{L_t\} \in \mathcal{C}[0, T]$ . We further analyze other properties of CLRb that make them suitable as information processes for credit risky assets with prematurity default possibilities.

## 3.2 Compound Lévy Random Bridges

In this section we define CLRb as Lévy processes jointly conditioned at some random space-time point. This proves useful in the subsequent analysis of important results.

**Definition 3.2.1.** A joint conditioning on  $[0, T] \times \mathbb{R}$  is defined by the triplet  $(Y, T', \nu)$

- (i)  $T' \in ]0, T]$  is an independent random time corresponding to the random interval on which a conditioning is made.
- (ii)  $Y$  is a certain functional of a Lévy process  $\{L_s\}_{0 \leq s \leq T} \in \mathcal{C}[0, T]$

(iii)  $\nu$  is a joint probability measure on  $\mathbb{R} \times [0, T]$  corresponding to the actual conditioning.

**Definition 3.2.2.** We say that a probability measure  $\mathbb{Q}^\nu$  is associated with a joint conditioning  $(Y, T', \nu)$  if the following are satisfied.

(i) If  $Z$  is a bounded  $\mathcal{F}_T$  measurable random variable taking values in  $\mathbb{R}$ , with the Borel  $\sigma$ -algebra,  $\mathcal{B}(\mathbb{R})$  then there exists a jointly measurable process  $\gamma_s^{y,t}; \forall s < t; y \in \mathbb{R}$  satisfying

$$\mathbb{E}_{\mathbb{Q}^\nu}(Z|(Y, T')) = \mathbb{E}_{\mathbb{Q}^\nu}(\gamma_s^{y,t}Z)$$

(ii) The joint law of  $(Y, T')$  under  $\mathbb{Q}^\nu$  is  $\nu$ .

Let  $\{L_s\}_{0 \leq s \leq T}$  be a Lévy process in  $\mathcal{C}[0, T]$ . We call  $\ell_s^z = \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_0^s \mathbb{1}_{\{|L_s|=z+\epsilon\}} ds \quad s \geq 0$  Lévy's local time of  $\{L_s\}_{0 \leq s \leq T}$  at  $z$ . The (positive, non-decreasing) occupation time process of  $L$  is given by

$$A_s = \int_0^s \mathbb{1}_{\{|L_u| < z\}} du. \quad (3.14)$$

Let  $\tau(s) = \inf\{t : \ell_t^z > s\}$  be the right-continuous inverse of  $\ell$ , then we define the time-changed process  $Y_s = A_{\tau(s)} = \int_0^{\tau(s)} \mathbb{1}_{\{|L_s| < z\}} ds$ . The following definition characterizes a subclass of conditioned Lévy process of interest.

**Definition 3.2.3.** We say that a process  $\{L_{sT}\}_{0 \leq s \leq T}$  has law  $\text{CLRB}([0, T], \{f_s(t, y)\}, \nu)$  if the following are satisfied;

(i)  $(A_{TT}, L_{TT})$  has joint law  $\nu$ .

(ii) There exists a Lévy process  $\{L_s\} \in \mathcal{C}[0, T]$ , such that  $(A_s, L_s)$  has joint density  $f_s(t, y)$  for all  $s, t \in (0, T]$  and  $y \in \mathbb{R}$ .

(iii)  $\nu$  concentrates mass where  $f_T(t, y)$  is positive and finite, that is  $0 < f_T(t, y) < \infty$

(iv) For every  $k \in \mathbb{N}_+$ , every  $s_1 < \dots < s_k < T$ , every  $(y_1, \dots, y_k) \in \mathbb{R}^k$ , and  $\nu$  - a.e.t,  $z$ , we have

$$\mathbb{Q}[L_{s_1, T} \leq y_1, \dots, L_{s_k, T} \leq y_k | A_{TT} = t, L_{TT} = z] = \mathbb{Q}[L_{s_1} \leq y_1, \dots, L_{s_k} \leq y_k | A_T = t, L_T = z]$$

**Remark 3.2.1.** If  $z = \infty$  then  $A_T = T$  (a.s) and  $\nu$  becomes a marginal law which can be specified as  $\nu(dz) = f(z)dz$ , the associated Compound Lévy Random Bridge coincides with the Lévy Random Bridges (LRBs) studied in [27]. In this case,  $\mathbb{Q}$  is the measure associated with the fixed time conditioning  $(Y, T, \nu)$ , where the functional  $Y$  is the terminal value of a Lévy process  $\{L_s\}_{0 \leq s \leq T} \in \mathcal{C}[0, T]$ . On the other hand and of particular interest in our application to pricing of credit risky assets with random cash flow, if  $L_T$  is fixed, then  $\nu$  can be specified as  $\nu(A_{TT} \in dt) = f(t)dt$ , the associated Compound Lévy Random Bridge coincides with the inverse of the Lévy Random Bridges (LRBs) in the sense of a fixed point in space  $z \in \mathbb{R}$  at a random time.

### 3.2.1 The Markov Property

The objective in this section is to prove the Markov property of Compound Lévy Random Bridges. To this end we first state and prove some important preliminary results leading to the statement of the Markov property.

**Proposition 3.2.1.** For the fixed time point  $T$  and  $\nu$  given by  $\nu(dz) = f(z)\mu(dz)$  the process  $\{L_{sT}\}_{0 \leq s < T}$  has the Markov property with transition densities;

$$\mathbb{Q}(L_{sT} \in dy | L_{uT} = x) = \frac{\int_{\mathbb{R}} \gamma_u^{z, T} f(z) \mu(dz)}{\int_{\mathbb{R}} \gamma_s^{z, T} f(z) \mu(dz)} p_{s-u}(y - x) dy \quad (3.15)$$

$$\mathbb{Q}(L_{TT} \in dz | L_{uT} = x) = \frac{\gamma_u^{z, T} f(z) \mu(dz)}{\int_{\mathbb{R}} \gamma_u^{z, T} f(z) \mu(dz)} \quad (3.16)$$

for  $0 \leq u < s < T$

*Proof.* To prove that  $\{L_{sT}\}$  has the Markov property for the case of  $\nu(dz) = f(z)\mu(dz)$ , it is sufficient to show that for all  $k \in \mathbb{N}_+$ ,  $(y_1, \dots, y_k, y) \in \mathbb{R}^{k+1}$  and all  $0 \leq s_1 < \dots < s_k < s \leq T$ ,

$$\mathbb{Q}(L_{sT} \leq y | L_{s_1T} = y_1, \dots, L_{s_kT} = y_k) = \mathbb{Q}(L_{sT} \leq y | L_{s_kT} = y_k)$$

For  $s = T$ , we obtain by Bayes theorem that

$$\mathbb{Q}(L_{TT} \leq z | L_{s_1T} = y_1, \dots, L_{s_kT} = y_k) = \frac{\gamma_{s_k}^{z,T} \nu(dz)}{\int_{\mathbb{R}} \gamma_{s_k}^{z,T} \nu(dz)}$$

Now we consider the case  $s < T$ . From proposition 3.1.1 we have

$$\mathbb{Q}(L_s \leq y | L_{s_1} = y_1, \dots, L_{s_k} = y_k, L_T = z) = \mathbb{Q}(L_s \leq y | L_{s_k} = y_k, L_T = z) \quad (3.17)$$

Then the Markov property of  $\{L_{sT}\}_{0 \leq s < T}$  follows from definition 3.2.1. The form of the transition law in (6.4) follows from (3.6).  $\square$

To prove the Markov property of the general compound Lévy random bridges (that is with conditioning at a random time), we first state important result specifying the law of CLRBs in terms of splicing distribution of the conditioned Lévy process,  $\{L_{sT}\}_{0 \leq s < T}$ . To be succinct and for the purpose of our financial application in this thesis, we assume that the joint density  $\{f_s\}$  is of the form  $f_s(t, y) = \rho(t, y)p_s(y)$ , where  $\rho(t, y)$  is some joint probability density on  $(0, T] \times \mathbb{R}$  and  $\{p_s\}$  is the density of a Lévy process,  $L_s \in \mathcal{C}[0, T]$  for every  $s \in [0, T]$ .

**Proposition 3.2.2.** *For any Lévy process  $\{L_s\}_{0 \leq s \leq T} \in \mathcal{C}[0, T]$ , the law  $CLRB([0, T], \{f_s\}, \nu)$  is absolutely continuous with the law of an associated Lévy bridge and the Radon-Nikodym derivative is given by  $\int_0^T g(z, s) dA_s$ , where,  $g(z, s) = \rho(z, s) \left[ \frac{p_T(z)}{p_s(y)p_{T-s}(y, z)} \right]$ .*

*Proof.* Let  $F$  be a positive,  $\mathcal{F}_T$ -measurable function on  $\Omega$ . We want to show that for the process  $L = \{L_{sT}\}_{0 \leq s < T}$

$$\mathbb{Q}(F(L)) = \mathbb{Q}_{0,z}^T(FJ) \quad (3.18)$$

where  $J = \int_0^T g(L_s, s) dA_s$ . Now using condition (i) and (iv) in definition 3.2.3 and lemma 3.1.1 we have for a borel function  $h \geq 0$  on  $\mathbb{R} \times [0, T]$ ,

$$\begin{aligned} \mathbb{Q}(FLh(L_{TT}, A_{TT})) &= \int_0^T \int_{\mathbb{R}} dz ds \rho(z, s) \mathbb{Q}_{0,y}^s \mathbb{Q}_{y,z}^{T-s}(F) h(z, s) \\ &= \int_0^T \int_{\mathbb{R}} dz ds \left[ \frac{p_s(y) p_{T-s}(y, z)}{p_T(z)} \right] g(z, s) \mathbb{Q}_{0,y}^s \mathbb{Q}_{y,z}^{T-s}(F) h(z, s) \\ &= \mathbb{Q}_{0,z}^T(F \bar{J}) \end{aligned} \quad (3.19)$$

where  $\bar{J} = \int_0^T h(L_s, s) g(L_s, s) dA_s$ . Taking  $h = 1$  completes the proof.  $\square$

**Corollary 3.2.1.** For any Lévy process  $\{L_s\}_{0 \leq s \leq T} \in \mathcal{C}[0, T]$  the law  $CLRB([0, T], \{f_s\}, \nu_z)$  is absolutely continuous with the law of an associated Lévy bridge and the Radon-Nikodym derivative is given by  $\int_0^T g(s) dl_s^z$ , where,  $g(s) = \rho(s) \left[ \frac{p_T(z)}{p_s(y) p_{T-s}(y, z)} \right]$ ,  $l_s^z = \int_0^s 1_{\{z\}}(L_u) dl_u^z$ , the local time at  $z$  up to time  $s$ .

*Proof.* The statement follows from proposition (3.2.2) applied to the specific functional of a Levy process  $\{L_s\}_{0 \leq s \leq T} \in \mathcal{C}[0, T]$ , namely; its local time  $l_s^z = \int_0^s 1_{\{z\}}(L_u) dl_u^z$  at  $z$  up to time  $s$ .  $\square$

**Proposition 3.2.3.** Suppose  $\mathbb{Q}(0 < \int_0^T g(z, s) dA_s < \infty) = 1$ , then the process  $\{L_{sT}\}_{0 \leq s < T}$  with law  $CLRB([0, T], \{f_s\}, \nu)$  has the Markov property and

$$\mathbb{Q}(h(L_{TT}, A_{TT}) | L_{sT} = y) = \frac{\int_0^s h(z, r) g(y, r) dA_{rT}}{\int_0^T g(y, r) dA_{rT}} \quad (3.20)$$

for some Borel function  $h \geq 0$  on  $\mathbb{R} \times [0, T]$  and  $\mathbb{Q}(0 < \int_0^T g(y, r) dA_{rT} < \infty) = 1$ .

*Proof.* Let  $\hat{J} = \int_0^T g(L_{sT}, s) dA_{sT}$ . Note that  $A_{sT}$  is measurable over the completion of  $\sigma(\{L_{sT}\})$ , since  $A_s$  is  $\mathcal{F}_s$ -measurable. Then by (3.19) we have

$$\mathbb{Q}(\hat{J} = 0) = \mathbb{Q}_{0,z}^T(J; J = 0) = 0$$

and

$$\mathbb{Q}(\hat{J} > k) = \mathbb{Q}_{0,z}^T(J; J > k) \rightarrow 0 \text{ as } k \rightarrow \infty$$

since  $\mathbb{Q}_{0,z}^T(J) = \int_0^T \int_{\mathbb{R}} \nu(dz; ds) = 1$ . Then  $\mathbb{Q}(0 < \hat{J} < \infty) = 1$ . The Markov property and (3.20) then follow from (3.19).  $\square$

### 3.2.2 Dynamic Consistency

In this section, we show that the CLRBs possess the dynamic consistency property. This property means that the process  $\{\eta_s\}_{0 \leq s \leq T}$  defined by  $\eta_{sT} = L_{sT} - L_{uT}$  for  $0 \leq u \leq s \leq T$  is a CLRB for a fixed  $u$  and  $L_{uT}$ .

Moreover,

$$\mathbb{Q}(G(\{L_{rT}\})_{u \leq r \leq T} | \mathcal{F}_s^\eta) = \mathbb{Q}(G(\{L_{rT}\})_{u \leq r \leq T} | \mathcal{F}_s^L)$$

where  $G$  is any measurable function and

$$\mathcal{F}_s^\eta = \sigma(L_{uT}, \{\eta_r\}_{u \leq r \leq s})$$

**Proposition 3.2.4.** Fix a time point  $u < T$  and  $L_{uT}$ , the process  $\{\eta_{sT}\}$  has law  $CLRB([u, T], \{f_s\}, \nu^*)$  is absolutely continuous with the law of an associated Lévy bridge and the Radon-Nikodym derivative is given by  $\int_u^T g^*(z, s) dA_s$ . where,  $g^*(z, s) = \rho^*(z, s) \left[ \frac{p_T(z)}{p_u(x, y) p_{T-u}(y, z)} \right]$  and  $\rho^*(z, s) = \rho(z - L_{uT}, s - A_{uT})$ .

*Proof.* Let  $\nu * (dz, ds) = \nu_u(dz, ds; L_{uT})$  so that  $f^*(z, s) = f(z - L_{uT}, s - A_{uT})$ . Now, for any given  $L_{uT} = x$  and any measurable function  $F$ , we have

$$\begin{aligned} \mathbb{Q}(F(\eta)) &= \mathbb{Q}^{\nu^*}(F(\eta)) \\ &= \mathbb{Q}^{\nu_u}(F(\eta)) \\ &= \mathbb{Q}^{\nu_u}(F(L_{sT} - L_{uT})) \\ &= \mathbb{Q}_{x,z}^{T-u}(FJ). \end{aligned}$$

Then the result follows from proposition (3.2.3) and the observation that  $\mathbb{Q}^{\nu^*}(0 < \int_u^T g^*(z, s) dA_s < \infty) = 1$ . □

### 3.2.3 Change of Measure

In this section, we show that there exists a measure under which a process with law  $CLRB([0, T], \{f_s\}, \nu)$  is a Levy process such that  $(A_{sT}, L_{sT})$  has a joint density  $f_s(t, x)$ .

**Proposition 3.2.5.** *Let  $\mathbb{L}$  be defined by*

$$\frac{d\mathbb{L}}{d\mathbb{Q}^{\nu^*}} \Big|_{\mathcal{F}_s^L} = \hat{J}_s(\mathbb{R} \times T; L_{sT}, A_{sT})^{-1} \quad (3.21)$$

for  $s \in [0, T]$ . Then  $\mathbb{L}$  is a probability measure and under  $\mathbb{L}$ , the process  $\{L_{sT}\}_{0 \leq s < T}$  is a Levy process such that  $(A_{sT}, L_{sT})$  has a joint density  $f_s(t, x)$ .

*Proof.* We want to show that  $\hat{J}_s(\mathbb{R} \times [0, T]; L_{sT}, A_{sT})$  is an  $\mathbb{L}$  martingale (w.r.t. the filtration generated by  $\{L_{sT}, A_{sT}\}$ ). Writing  $\hat{J} = \hat{J}_s(\mathbb{R} \times T, L_{sT}, A_{sT})$ , we have for  $0 \leq r < s$

$$\begin{aligned}
\mathbb{E}_{\mathbb{L}}[\hat{J}_s | \mathcal{F}_r^{\mathbb{L}}] &= \mathbb{E}_{\mathbb{L}} \left[ \int_0^T \int_{-\infty}^{\infty} \frac{f_s(x, t') f_{T-s}(z-x; t-t')}{f_T(z, t)} \nu(dz, dt) | \mathcal{F}_r^{\mathbb{L}} \right] \\
&= \mathbb{E}_{\mathbb{L}} \left[ \int_0^T \int_{-\infty}^{\infty} \frac{\rho(x, t') p_s(x) \rho(z-x; t-t') p_{T-s}(z-x)}{\rho(z, t) p_T(z)} \nu(dz, dt) | \mathcal{F}_r^{\mathbb{L}} \right] \\
&= \mathbb{E}_{\mathbb{L}} \left[ \int_0^T ds \int_{-\infty}^{\infty} dz \frac{p_s(y) p_{T-s}(z-y)}{p_T(z)} g(z, t; L_{sT}) | \mathcal{F}_r^{\mathbb{L}} \right] \\
&= \mathbb{E}_{\mathbb{L}} \left[ \int_0^T \int_{-\infty}^{\infty} \frac{\rho(x, t') p_s(x - L_{rT} + L_{rT}) \rho(z-x; t-t') p_{T-s}(z - L_{rT} - (x - L_{rT}))}{\rho(z, t) p_T(z)} \right. \\
&\quad \left. \dots \nu(dz, dt) | L_{rT} \right] \\
&= \int_{-\infty}^{\infty} \int_0^T \int_{-\infty}^{\infty} \frac{\rho(x, t') p_r(L_{rT}) p_{s-r}(x - L_{rT}) \rho(z-x; t-t') p_{T-s}(z - L_{rT} - y)}{\rho(z, t) p_T(z)} \\
&\quad \dots dy \nu(dz, dt) \\
&= \int_0^T \int_{-\infty}^{\infty} \frac{\rho(x, t') p_r(L_{rT}) \rho(z-x; t-t')}{\rho(z, t) p_T(z)} \nu(dz, dt) \int_{-\infty}^{\infty} p_{T-s}(z - L_{rT} - y) p_{s-r}(y) dy \\
&= \int_0^T \int_{-\infty}^{\infty} \frac{f_r(L_{rT}, A_{rT}) f_{T-r}(z - L_{rT}, t - A_{rT})}{f_T(z, t)} \nu(dz, dt) \\
&= \hat{J}_r
\end{aligned} \tag{3.22}$$

Observe that  $\hat{J}_0 = 1$ , therefore, a probability measure  $\mathbb{L}^{crb}$  can be defined by the Radon-Nikodým derivative

$$\frac{d\mathbb{L}^{crb}}{d\mathbb{L}} | \mathcal{F}_s^{\mathbb{L}} = \hat{J}_s \text{ for } 0 \leq s \leq T \tag{3.23}$$

As noted in the previous section,  $\mathbb{L}(0 < \hat{J}_s < \infty) = 1$ , hence  $\mathbb{L}^{crb}$  is equivalent to  $\mathbb{L}$  for  $s \leq T$ . Then we can compute the joint law of  $(L_{sT}, A_{sT})$  under  $\mathbb{L}^{crb}$  explicitly as

$$\begin{aligned}
L^{crb}[L_{sT} \in dy, A_{sT} \in dt' | \mathcal{F}_r^L] &= \mathbb{E}_{\mathbb{L}^{crb}}[\mathbb{1}_{\{L_{sT} \in dy, A_{sT} \in dt'\}} | \mathcal{F}_r^L] \\
&= \hat{J}_r^{-1} \mathbb{E}_{\mathbb{L}}[\hat{J}_s \mathbb{1}_{\{L_{sT} \in dy\}} | (L_{rT}, A_{rT})] \\
&= \hat{J}_r^{-1} \int_0^T ds \int_{-\infty}^{\infty} \frac{f_s(z, t) f_{T-s}(z - L_{sT}, t - A_{sT})}{f_T(s, t)} p_{rs}(y - L_{rT}) dy \\
&= \frac{\hat{J}_s(\mathbb{R} \times T; L_{sT}, A_{sT})}{\hat{J}_r(\mathbb{R} \times T; L_{rT}, A_{rT})} p_{rs}(y - L_{rT}) dy
\end{aligned} \tag{3.24}$$

where  $p_{rs}(y - L_{rT})$  is given by  $p_{rs}(y - L_{rT}) = \frac{p_{s-r}(y - L_{rT}) p_{T-s}(z - y)}{p_{T-r}(z - L_{rT})}$ .  $\square$

### 3.2.4 Conditional Terminal Joint Distributions

We recall that  $\{\mathcal{F}_s^L\}$  denotes the filtration generated by  $\{L_{sT}\}$ . Let  $\nu_r$  be the joint  $\{\mathcal{F}_r^L\}$ -conditional law of the functional  $(A_{TT}, L_{TT})$ . We have  $\nu_0 = \nu(dz, ds)$ . Then using results from previous section we obtain

$$\nu_r(dz, dt) = \frac{\psi_r(dz, dt; L_{rT}, A_{rT})}{\psi_r(\mathbb{R}, [0, T]; L_{rT}, A_{rT})} \tag{3.25}$$

where  $\psi_r(dz, dt; x, t') = \frac{f_{T-r}(z-x, t-t')}{f_T(z, t)} dz dt$ . When the *a priori*  $k$ th joint moment of the  $(L_{TT}, A_{TT})$  is finite then the  $\{\mathcal{F}_r^L\}$ -conditional  $k$ th moment  $M_r(k)$  of  $(L_{TT}, A_{TT})$  is finite and is given by

$$M_r(k) = \int_0^T \int_{-\infty}^{\infty} [h(z, t)]^k \nu_r(dz, dt) \tag{3.26}$$

for some positive Borel function  $h$  and  $k \in \mathbb{N}_+$ .

# Chapter 4

## Information Based Default Risk Valuation

### 4.1 Introduction

In this Chapter, we use the Compound Lévy Random Bridges (CLRBs) constructed in chapter two as information model for an asset with randomly-timed cash flow in the Brody-Hughston and Macrina (BHM) sense. The main tools here are the Markov property, the joint conditional distribution of the CLRB and its occupation time and martingale arguments. We conclude this chapter by looking at information based corridor bond options. We show that in the case of corridor bond with a constant face value, the estimates of the corresponding call option price can be calculated in closed-form for Brownian and Gamma bridge information processes. The exposition in this chapter provides a platform to compare different information based approaches to credit risky asset pricing. In the chapter that follow, we shall interpret CLRBs as partial market information process in valuing credit risky asset via the intensity based approach. This serves to provide a unified framework for modeling credit risky assets with partial market information. Before detailing the full description of the BHM pricing frameworks and its extension through CLRB information, we first outline its key features.

- The underlying assumption is that cash flows occur at random times (this idea was pointed out in [23])

- We assume the following functional form for a single cash flow occurring at a random time  $\tau$ ,

$$H_\tau = g(X, \tau), \quad g \geq 0$$

- It is assumed that the *a priori* probability distribution of the cash flow is known at inception
- The information process is constructed as a CLRB
- The market information is modelled as the filtration generated by a CLRB
- With this information, the market can value risky bonds and credit derivatives as conditional expectation of the cash flow (or its functional)

#### 4.1.1 Extension of the BHM Approach

The information based asset pricing theory of Brody, Hughston and Macrina as introduced in [23] is primarily designed as important framework for pricing and hedging of credit risky assets. In particular, the framework takes into consideration the issues of default risk in the market as fundamental in its formulation. This renders the theory very suitable for pricing credit sensitive assets. The main component of the framework involves finding a suitable class of models for the information regarding the cash flow of the assets in such a way that issues of tractability and computational complexities are appropriately taken into consideration. To meet these requirements, the literature on this topic has so far avoided incorporation of prematurity default times in the framework. For example, "default" of credit risky bonds was defined in the framework to simply imply a failure of the bond issuer to meet debt obligation at the maturity date of the bond. Several extensions of the framework till date have been based on this assumption(see, [23, 57, 22, 27]).

In the information based asset pricing framework, an asset is associated with a sequence of random cash flows. Default on each cash flow occurs only at the maturity dates of the debt obligation. The price of the asset is given as the sum of the discounted conditional expectations of the cash flows. The conditional expectation is taken with respect to the filtration generated by the information process. The complexity and tractability of the conditional expectation only depends on (i) nature of the cash flow (ii) the law of the information process. As such modeling the information process in the information based asset pricing framework is akin to specifying the law of the information process and/or the nature of the cash flow with no emphasis on prematurity default probabilities. The Brownian bridge and Gamma bridge models of market information by Brody et al in [23] as well as Hoyle et al's Lévy Random Bridge(LRB) model of market information in [27] are both based on the assumption that default can only occur at the maturity date of the debt obligation. The authors postulate the existence of a market information process and obtain closed form expression for prices of European style contracts. A new class of information processes were proposed in [45] named conditioned stochastic processes (CSP). The key idea in modeling the information process as a conditioned stochastic process is that the information content about the asset cash flow can be generalized to include other functionals of the market factor process than its value at a fixed future date. This class of information models allows for incorporation of prematurity default times in the information based asset pricing framework. In this present work, we consider a case of a randomly-timed cash flow (say  $h(Z, \tau)$ , where  $Z$  is some functional of a market factor process and  $\tau$  is an independent random time). We assume that market participants have at inception an *a priori* joint distribution for  $Z$  and  $\tau$ .

In what follows, we use the Compound Lévy Random Bridges (CLRBs) constructed in the previous sections as model for the market information process. This represents a new approach to credit risky discount bond pricing in the information based framework. The current approach allows for prematurity default as the market information

process takes into consideration the random nature of cash flow occurrence. We derive expression for the price of such credit risky assets and derivatives written on them. Throughout the remainder of this chapter we fix a probability space  $(\Omega, \mathbb{Q}, \mathcal{F})$  and assume that all processes and filtration under consideration are adapted to it. We consider the time horizon  $[0, T], T \in (0, \infty]$  and assume that all stochastic processes take values in  $\mathbb{R}$ .

## 4.2 Compound Lévy Information

### 4.2.1 The Setup

In this section, we describe the setup for the incorporation of random default times into the information based framework for pricing credit risky assets. Default risk modeling investigates an entity (corporations, banks, individuals) that borrows money under a pre-specified contractual agreement by taking into consideration a possibility of the entity to fail to meet all the terms of the contractual obligations. The classical assumption in the information based asset pricing framework as emphasized in the previous section is that default is allowed to occur only at the maturity date of the contractual arrangement in the form of a failure of the borrower to repay the borrowed funds in full. This assumption enables the information based model avoid issues of intensity of default times usually associated with credit risk modeling. In this new approach, the price process of a credit risky asset is given by the conditional expectation (under a martingale measure) of the asset future cash flows with respect to the filtration generated by the information process. The key purpose of this section is to extend the BHM information-based framework by incorporating pre-maturity default possibilities. To do this, we model the market information by specifying the law of a Compound Lévy Random Bridge  $CLRB([0, T], \{f_t\}, \mu)$ . In particular, with the information process so defined we are able to derive the dynamics of credit risky assets prices with the possibility

of default on or before maturity of the debt obligation, hence relaxing the classical fixed default time assumption.

We consider the subclass of Lévy processes  $\{L_t\} \in \mathcal{C}[0, T]$  on the filtered probability space  $(\Omega, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{Q})$  as defined in the previous chapters. The emphasis will be on the specific form of the information set  $\{\mathcal{F}_t : t \in [0, T]\}$  that allows for the modeling of default risk in random time under the information based asset pricing framework. Specifically, we consider the process  $I_{tT}$  such that the market information  $\mathcal{F}_t$  corresponds to the filtration  $\sigma(I_{sT}, s \leq t)$ , generated by  $I_{tT}$ . This is in line with the type of information process studied in [58] where the signal component is interpreted as the information held by the inside trader in the market and the noise component is the contribution of the noise trader to the overall information held by market maker. However, our interest here is to explicitly construct a specific class of similar kind of information processes and show how they apply to credit risky asset pricing under the information based framework. In particular, we model market information process as being identical in law with a Lévy process conditioned on a random measure  $\mu(dz, dt)$ . We take  $\mu$  as the *a priori* joint law of  $Z$  and  $\tau$ . The question of how to estimate the joint distribution  $\mu$  is an important one and thus is reserved for later consideration.

### 4.2.2 The Model

The starting point in the pricing of credit risky assets is the specification of the way default event is modeled. The major distinction between the various approaches for pricing and hedging defaultable assets lies on the structure of the default times. In the classical structural approach, the default time is defined as the first time the firm's value process crosses a pre-specified barrier level. In the reduced form approach on the contrary, the default time is defined in an exogenous fashion through an intensity process, which represents the conditional probability of default occurring over a very small interval (See [17, 77, 39]). As mentioned in the previous section, the information

based asset pricing framework attempts to avoid the need for intensity process for the reduced form models and the value process for the structural models by simplifying the default structure. In the information based framework for asset pricing literature, default is fixed to a possible occurrence at time  $T$  only. Here, we consider a situation where prematurity default is possible.

Consider a credit risky asset with a single cash flow (e.g. a credit risky zero coupon bond)  $H_T$  at maturity, time  $T \in (0, \infty]$ . We assume  $H_T$  is an integrable random variable with *a priori* probability law  $\mu$ . We assume that the market for the credit risky assets is arbitrage free, hence, there exists an equivalent probability measure  $\mathbb{Q}$  such that all discounted asset prices are martingales with respect to the information set  $\{\mathcal{F}_t : t \in [0, T]\}$ . The time  $t \leq T$  price of this asset denoted by  $H_{tT}$  is given by

$$H_{tT} = P_{tT} \mathbb{E}_{\mathbb{Q}}(H_T | \mathcal{F}_t) \quad (4.1)$$

where  $\mathcal{F}_t = \sigma(I_{sT}, s \leq t)$  and the discount factor,  $P_{tT}$  is as defined in the previous sections. When the *a priori* law  $\mu$  is given as the marginal law of a Brownian motion, the resulting information process gives a close form solution to (4.1) and the asset price dynamics is given by

$$dH_{tT} = r_t H_{tT} + \frac{P_{tT} \text{Var}[H_T | I_{tT}]}{T - t} d\omega_t \quad (4.2)$$

where  $\{\omega_t\}$  is a  $\mathbb{Q}$  - standard Brownian motion. This is the original information based asset pricing model of Brody, Hughston and Macrina in [23, 27] which is based on the assumption that default occurs only at the maturity time  $T$ .

### 4.2.3 Information-Based Default specification

Next we relax the fixed default time assumption to extend the Brody-Hughston and Macrina model to a situation where default prior to time  $T$  is possible .

In the traditional structural default model, default is specified as the first passage time given by

$$\tau_D = \inf\{t > 0 : A_t \leq D\} \quad (4.3)$$

where  $\{A_t\}_{0 \leq t \leq T}$  is a stochastic process representing the asset's future cash flow process and  $D$  is the *default barrier*, a time-invariant random variable independent of  $A$ . Let  $M$  denotes the running minimum of asset process defined by  $M_t = \min\{A_s : 0 \leq s \leq t\}$ , then default on a bond contingent on the asset's future cash flow is said to occur if  $A$  falls to a level  $D < A_0$ . First-passage time default specification does not isolate the information regarding the default state of the credit risky asset from that generated by the issuing firm's actual value process. This means that it does not distinguish between the time at which a firm enters bankruptcy and the time at which it is either liquidated or reorganized: default occurs at the instance a firm's assets are too low according to some criterion, and is immediately liquidated. This definition of default no longer reflects economic reality of the present information age: bankruptcy laws now allow the use of different sources and form of information (including rumors, innuendos, social media status updates) to specify default of a credit risky asset. An alternative default specification that reflects current advancement in information technology is the occupation time default definition. For a factor process  $\{X_t\}_{0 \leq t \leq T}$ , define the occupation time process inside a band  $[l, u]$ , in the time interval  $[0, t]$  by

$$\ell_t = \int_0^t \mathbb{1}_{\{l \leq X_s \leq u\}} ds. \quad (4.4)$$

If  $l = 0$ , then  $\ell_t$  is a measure of the amount of time the process  $\{X_t\}$  spends on or below the barrier  $u$  during the time period  $[0, t]$ . We note that  $\ell_T$  is the amount of time spent inside the band in the interval  $[0, T]$ . We define the default time as the first instant the process  $X$  has spent a pre-specified amount of time  $\theta \in [0, T[$  inside the band  $[l, u]$ :

$$\tau_\theta = \inf\{t \geq 0 : \ell_t \geq \theta\}. \quad (4.5)$$

The market information set is then given by the filtration,

$$\{\mathcal{G}_t = (\mathcal{F}_t \cap \{t \leq T\} \vee \{\tau_\theta = t\})\}_{t \geq 0}, \quad (4.6)$$

where  $\mathcal{F}_t$  is the sigma algebra generated by an information process  $\{I_{tT}\}$  with law  $CLRB([0, T], \{f_t\}, \mu)$ . The following limiting relations illustrates that the extended-BHM model that we develop here converges to the Hoyle, Hughston and Macrina's LRB model (see [27]). When  $\theta \uparrow T$ , the following limiting properties hold:

$$\begin{aligned} \lim_{\theta \uparrow T} \mathbb{1}_{\{\tau_\theta \geq T\}} &= \lim_{\theta \uparrow T} \mathbb{1}_{\{\ell_T \leq \theta\}} = 1 \text{ a.s.} \\ \lim_{\theta \uparrow T} \mathbb{1}_{\{\tau_\theta < T\}} &= \lim_{\theta \uparrow T} \mathbb{1}_{\{\ell_T > \theta\}} = 0 \text{ a.s.} \end{aligned}$$

Next, we note the following limiting relations between occupation times and first hitting times: When  $\theta \downarrow 0$  we obtain;

$$\lim_{\theta \downarrow 0} \mathbb{1}_{\{\tau_\theta \geq T\}} = \lim_{\theta \downarrow 0} \mathbb{1}_{\{\ell_T \leq \theta\}} = \mathbb{1}_{\{\tau_\theta \geq T\}} \text{ a.s} \quad (4.7)$$

$$\lim_{\theta \downarrow 0} \mathbb{1}_{\{\tau_\theta < T\}} = \lim_{\theta \downarrow 0} \mathbb{1}_{\{\ell_T \leq \theta\}} = \mathbb{1}_{\{\tau_\theta < T\}} \text{ a.s} \quad (4.8)$$

$$\lim_{\theta \downarrow 0} L_{\tau_\theta} \mathbb{1}_{\{\tau_\theta < T\}} = \lim_{\theta \downarrow 0} L_{\tau_\theta} \mathbb{1}_{\{\ell_T > \theta\}} = L_T \mathbb{1}_{\{\tau_\theta < T\}} \text{ a.s} \quad (4.9)$$

In a structural sense, the relations in (4.7) - (4.9) show that extended- BHM model relates to the classical Brody-Hughston-Macrina model in the same manner Black-Cox credit risk model (see [61, 32, 47]) compares with Merton model (see [61, 63]) with regards to assumptions on default time and payoff upon default. Precisely, the Black-Scholes and Merton model has default only occurring on the maturity time  $T$ . This was then generalized in the Black-Cox model to allow default prior to time  $T$ , where the default time becomes a random variable.

## Information Based Recovery Rules

We shall consider the following recovery rules in the current setup;

- If default does not occur on or before time  $T$ , the promised claim  $H$  is paid in full at time  $T$ .
- If default occurs at time  $T$  or prior to time  $T$ , then either (a) an amount  $\hat{H}(X_T)$  is paid at the maturity date  $T$  or (b) an amount  $H(X_{\tau_\theta})$  is paid at time  $t = \tau_\theta$ .

In view of the above recovery rule, we define in general, a randomly-timed cash flow structure by the collection;

$$V = (H_T, \hat{H}_T, C, \tau_\theta).$$

We assume that market participants observe the state of a factor process  $X$  (e.g the state when the issuing entity is in a deep financial distress) with a corresponding random cash flow  $V$  only at the time  $t = \tau_\theta$  or at the maturity time  $T$ .

## Some Technical Assumptions

In this set up, we assume that the filtration  $\mathbb{G} = (\mathcal{G}_t)_{0 \leq t \leq T}$ , is rich enough to support the following objects:

- i. the interest rate process  $\{r_t\}$
- ii. the process  $X$  and the occupation time process  $\{\ell_t\}$ , which jointly drive the randomly-observable cash flow  $h(X, \tau_\theta)$ .
- iii. the terminal cash flow  $H$  and the recovery amount  $\hat{H}$ .

We further assume that the random variables  $H(X_T)$  and  $\hat{H}(X_T)$  are  $\mathcal{G}_T$ -measurable.

### 4.3 Risk-Neutral Pricing

In this section, we present the pricing formula for a credit risky asset with general cash flow structure defined by

$$V = (H_T, \hat{H}_T, C, \tau_\theta).$$

We consider an arbitrage free financial market modeled by the specification of the probability space  $(\Omega, \mathcal{G}, \mathbb{Q})$ . The filtration  $\{\mathcal{G}_t\}$  is assumed to be generated by the information process  $\{I_{tT}\}$ . The probability measure  $\mathbb{Q}$  is taken to be the risk-neutral measure. We only consider the case of information process with continuous state space. The case of an information process with a discrete state space will be the subject of another study. We recall that  $P_{tT}$  denotes the discount factor given by

$$P_{st} = \exp\left(-\int_s^t r_u du\right), \quad (s \leq t).$$

We introduce the jump process  $N_t = \mathbb{1}_{\{\tau_\theta \leq t\}}$  and let  $U_t$  denote the cash flow received by the owner of the credit risky asset with payoff  $V$ . We write

$$H(T) = H_T \mathbb{1}_{\{\tau_\theta > T\}} + \hat{H}_T \mathbb{1}_{\{\tau_\theta \leq T\}}$$

**Definition 4.3.1.** *The cash flow process  $U$  of a credit risky asset whose payoff  $V = (H_T, \hat{H}_T, C, \tau_\theta)$  with maturity time  $T$  is given by*

$$U_t = H(T) \mathbb{1}_{\{t \geq T\}} + \int_0^t (1 - N_s) dC_s + \int_0^t H(X_s) dN_s \quad (4.10)$$

**Lemma 4.3.1.** *The process  $\{U_t\}_{t \geq 0}$  is of finite variation*

**Remark 4.3.1.** *We remark that if default occurs at some point  $t$ , the intermediate cash flow  $C_t - C_{t-}$ , which is due to be paid at this time will not be received by the holder of the risky asset.*

Let us write  $\tau_\theta \wedge t = \min(\tau_\theta, t)$  then we have

$$\int_0^t H(X_s) dN_s = H(X_{\tau_\theta \wedge t}) \mathbb{1}_{\{\tau_\theta \leq t\}} = H(X_{\tau_\theta}) \mathbb{1}_{\{\tau_\theta \leq t\}}$$

Next we define the price (ex-dividend)  $H(t, T)$  of a credit risky asset. The intuition here is that at any time  $t$ , the random variable  $H(t, T)$  represents the present value of all the total cash flows associated with the payoff  $V$ .

**Definition 4.3.2.** *The price process of a credit risky asset with payoff  $V = (H_T, \hat{H}_T, C, \tau_\theta)$  is given by*

$$H(t, T) = P_{tT} \mathbb{E}_{\mathbb{Q}} \left( \int_0^T dU_s | \mathcal{G}_t \right) \quad (4.11)$$

$$H(T, T) = 0. \quad (4.12)$$

where the filtration,  $\mathcal{G}_t$  is given by (6.14). We note that the random variable  $\left( \int_0^T dU_s \right)$  is  $\mathcal{G}_{\tau_\theta}$ -measurable ( $\mathcal{G}_{\tau_\theta} = \sigma(I_{sT}, s \leq \tau_\theta \leq t)$ ) but not  $\mathcal{G}_t$ -measurable for  $t < \tau_\theta$ .

### 4.3.1 Defaultable Zero-Coupon Bond

We consider here the valuation of a defaultable zero coupon bond with the possibility of default occurring prior to the maturity of the bond. In line with our current formulation, a defaultable zero coupon bond corresponds to a credit risky asset with payoff  $V = (H_T, \hat{H}_T, 0, \tau_\theta)$ . The contingent cash flow process becomes

$$\hat{U}_t = H(T) \mathbb{1}_{\{t \geq T\}} + \int_0^t H(X_s) dN_s.$$

Then the arbitrage free price  $H(t, T)$  of such a bond is given by

$$H(t, T) = P_{tT} \mathbb{E}_{\mathbb{Q}} \left( \int_0^T d\hat{U}_s | \mathcal{G}_t \right) \quad (4.13)$$

**Example 4.3.1** (HHM Model.). *If we assume a priori that  $\tau_\theta \equiv T$  the payoff  $V = (H_T, \hat{H}_T, 0, T)$  corresponds to the contingent cash flow of a defaultable zero coupon bond in Hoyle, Hughston and Macrina's LRB model (see [27]).*

The integral in (4.13) can be represented in a functional form as  $h(X_T, \ell_T) \geq 0$ , then we have by the strong Markov property of information process and the additive property of occupation time process;

$$H(t, T) = P_{tT} \mathbb{E}_{\mathbb{Q}} [h(Z, \ell_T) | (I_{tT}, \ell_{tT})] \quad (4.14)$$

The conditional expectation in (4.14) can be evaluated by proposition 3.2.3 as

$$H(t, T) = P_{tT} \int_0^T \int_{-\infty}^{+\infty} (h(z, s) \mu_t(dz, ds)) \quad (4.15)$$

The  $\mathcal{G}_t$ -conditional joint distribution  $\mu_t(dz, ds)$  is given by (3.25) as

$$\mu_t(dz, ds) = \frac{\psi_t(dz, ds; L_{tT}, \ell_{tT})}{\psi_t(\mathbb{R}, [0, T]; L_{tT}, \ell_{tT})}$$

where  $\psi_t(dz, ds; x, t') = \frac{f_{T-t}(z-x, s-t')}{f_T(z, s)} dz ds$ . If  $\mu$  admits a joint density  $\rho(z, s)$ , the  $\mathcal{G}_t$ -conditional joint density of  $(X_T, \ell_T)$  exists and is given by

$$\rho_t(z, s) = \frac{f_{T-t}(z - L_{tT}, s - \ell_{tT}) \rho(z, s)}{\psi_t(\mathbb{R}, [0, T]; L_{tT}, \ell_{tT}) f_T(z, s)} \quad (4.16)$$

In particular, the choice

$$\rho(z, s) = \frac{f_T(z, s) c(\mathbb{R}, T)}{f_t(L_{tT}, \ell_{tT}) f_{T-t}(z - L_{tT}, s - \ell_{tT})}, \quad (4.17)$$

where  $c(\mathbb{R}, T) = \int_0^T \int_{-\infty}^{\infty} \left[ \frac{f_t(L_{tT}, \ell_{tT}) f_{T-t}(z - L_{tT}, s - \ell_{tT})}{f_T(z, s)} \right] dz dt$ , gives

$$\rho_t(z, s) = \frac{1}{f_t(L_{tT}, \ell_{tT})} \quad (4.18)$$

for each  $t \in (0, T]$ .

**Example 4.3.2.** [*Lévy Local Time Information*] Suppose  $z \in \mathbb{R}$ , we define the local time  $\ell_t = \ell_t^z$  at  $z$  of a Lévy process  $\{L_t\} \in \mathcal{C}[0, T]$  as  $\ell_t = \int_0^t \mathbb{1}_{\{z\}}(L_s) d\ell_s$ . This is a continuous additive functional of the Lévy process. Let the market information process be given by a CLRB  $\{L_{tT}\}$  with law  $CLRB([0, T], \{f_t\}, \mu)$ . In this case  $\mu$  is the a priori joint law of  $(X_T, \ell_T)$ , the terminal value of the factor process and its total local time at  $z$  on the interval  $[0, T]$  given by  $\ell_T = \int_0^T \mathbb{1}_{\{z\}}(X) d\ell_s$ . Thus, the default time is defined in terms of the local time of the factor process as

$$\tau_\theta = \inf\{t : \ell_t \geq \theta\}, \quad \text{for some } \theta \in [0, T].$$

The price of a zero coupon bond on the event  $\{t \leq \tau_\theta\}$  and based on Lévy local time information is given by (4.15) as

$$H(t, T) = P_{tT} \frac{\int_0^T \int_{-\infty}^{\infty} \frac{h(z, s) f_{T-t}(z - L_{tT}, s - \ell_{tT}) \rho(z, s)}{f_T(z, s)} dz ds}{\int_0^T \int_{-\infty}^{\infty} \frac{f_{T-t}(z - L_{tT}, s - \ell_{tT})}{f_T(z, s)} dz ds}. \quad (4.19)$$

Then from (4.17), we obtain

$$H(t, T) = P_{tT} \int_0^T \int_{-\infty}^{\infty} \frac{h(z, s)}{f_t(L_{tT}, \ell_{tT})} dz ds \quad (4.20)$$

For a particular case of Brownian local time at zero, we obtain from (4.20) that

$$H(t, T) = P_{tT} \int_0^T \int_{-\infty}^{\infty} \frac{h(z, s)}{\frac{|L_{tT}| + \ell_{tT}}{\sqrt{2\pi t^3}} \exp\left(-\frac{(|L_{tT}| + \ell_{tT})^2}{2t}\right)} dz ds \quad (4.21)$$

## 4.4 Information Based Occupation-Time Derivatives

In this section we examine the pricing problem for occupation time derivatives with continuous time monitoring under the information based asset pricing framework. The payoff of these options depends on a cash flow occurring at a fixed future date as well

as on the time spent by an economic indicator process ( e.g. an index or credit rating) inside a band or below a given level. This type of financial options includes hurdle or switch derivatives, corridor and Parisian derivatives and range notes. In particular, we examine the case of the corridor bond, where the coupon is proportional to the time spent inside a given band and the corridor option that guarantees a minimum coupon (see [37, 44]). The structure of their payoff is similar to that of the *information derivatives* introduced in [23].

#### 4.4.1 The corridor bond option

Let us define  $x$  as the level of an economic factor at the current time  $t$ , and  $\mathcal{A}(x, T, t; a, b)$  the time spent by the factor process inside the band  $[a, b]$ , in the time interval  $[t, T]$ . With continuous time monitoring, we write:

$$\mathcal{A}(x, T, t; a, b) = \int_t^T \mathbb{1}_{\{a < X_s < b\}} ds$$

where  $\mathbb{1}_{\{a < X_s < b\}}$  is the indicator function of the set  $[a, b]$ . If  $a = 0$ , then we are considering the time spent below the level  $b$ . A *corridor bond* pays at time  $T$  the amount:

$$H(X_T) \times \frac{\mathcal{A}(x, T, t; a, b)}{T - t}$$

where  $H$  is an borel measurable positive function. Corridor bonds are classified (albeit in a somewhat interpretation ) as a credit risky asset. Let  $B(t, T)$  denote the time  $t$  price of the bond with maturity  $T$  under the current information based asset pricing framework, we obtain,

$$B(t, T) = P_{tT} \mathbb{E} \left[ H \times \frac{\mathcal{A}(x, T, t; a, b)}{T - t} \middle| \mathcal{F}_t \right] \quad (4.22)$$

Using the occupation time definition and notation in (4.4) we obtain,

$$\mathcal{A}(x, T, t; a, b) = \ell_T - \ell_t, \quad \forall t \in [0, T]$$

The bond price then becomes;

$$B(t, T) = \frac{P_{tT}}{T-t} \mathbb{E} [H(\ell_T - \ell_t) | \mathcal{F}_t] \quad (4.23)$$

where the filtration  $\mathcal{F}_t$  is that generated by the market information process with law  $CLRB([0, T], \{f_t\}, \mu)$  and  $\mu(dz, ds)$  is the *a priori* joint law of  $X_T$  and  $\ell_T$ . Writing  $\xi = T - t$  and  $\Delta \ell_t = \ell_T - \ell_t$  then the bond price formula in (4.23) can be expressed as

$$B(t, \xi) = \frac{P_{tT}}{\xi} \mathbb{E} [H \Delta \ell_t | \mathcal{F}_t] \quad \text{for all } t \in [0, T) \quad (4.24)$$

The expectation term in (4.24) can be evaluated explicitly using techniques in the previous chapters. Precisely, by proposition 3.2.3 we have

$$B(t, \xi) = \frac{P_{tT}}{\xi} \int_0^\xi \int_0^\infty xs \mu_t(dz, ds) \quad \text{for all } t \in [0, T) \quad (4.25)$$

If  $X_T$  is fixed then  $H$  plays the role of the nominal (constant) value of the bond, hence the pricing expression in (4.25) becomes

$$B(t, \xi) = \frac{P_{tT}}{\xi} H \int_0^\xi s \mu_t(z, ds) \quad \text{for all } t \in [0, T) \quad (4.26)$$

corresponding to the form of the pricing formula of *Lévy local time information* in (4.19). The integrals in (4.25) are straightforward to approximate numerically (some care must be taken though in dealing with double indicator functions usually associated with joint densities of Lévy processes and their occupation times).

Now we consider the price of an European option (corridor option) on a corridor

bond. The corridor option promises a fixed amount  $K$  (the strike) at the expiry, so the payoff for a unit nominal is given by:

$$\max\left(\frac{\mathcal{A}(x, T, t; a, b)}{T - t}; K\right).$$

Just as corridor bonds can be viewed as credit risky bonds, corridor options are analogous to credit default swaps. A digital corridor option pays at time  $T$  the amount  $N \times \mathbb{1}_{\mathcal{A}(x, T, t; a, b) > K}$ , that is a fixed amount  $N$  if the occupation time is greater than  $K$ . The time- $t$  price of a  $T$ -maturity (*call*) corridor option on  $\mathcal{A}(x, T, t; a, b)$  for a value of  $N$  dollars is

$$\begin{aligned} C_t(\xi, K) &= \frac{P_{tT}}{\xi} [N \mathbb{E} [(\mathcal{A}(x, T, t; a, b) - K)^+ | \mathcal{F}_t]] & (4.27) \\ &= \frac{P_{tT}}{\xi} [N [\mathbb{E} [(\Delta \ell_t - K)^+ | \mathcal{F}_t]]] \\ &= \frac{P_{tT}}{\xi} [N \mathbb{E} [(\Delta \ell_t | \mathcal{F}_t)] - K(1 - \mu_t(z; 0))] \\ &= \frac{P_{tT}}{\xi} \left[ N \int_K^\xi s \mu_t(z; ds) - K(1 - \mu_t(z; 0)) \right]. \end{aligned}$$

For a fixed terminal value  $X_T = z$ , we write  $\rho(z, s) = \rho(s)$ . If we choose  $\rho(s) = \frac{f_T(z, s)c(z, T)}{f_t(L_{tT}, \ell t T) f_{T-t}(z - L_{tT}, s - \ell t T)}$ , where  $c(z, T) = \int_0^T \left[ \frac{f_t(L_{tT}, \ell t T) f_{T-t}(z - L_{tT}, s - \ell t T)}{f_T(z, s)} \right] dt$ , as in example 4.3.2, we obtain (after straightforward algebraic manipulations) that

$$C_t(\xi, K) = \frac{P_{tT} N}{\xi} \left[ \frac{\xi}{f_t(L_{tT}, \ell t T)} - K \right] \quad (4.28)$$

For the case of a Brownian motion we have

$$f_t(x, s) = \Pi(s, \xi, x - z, z) \mathbb{1}_{\{x \geq z\}} + (\Pi(s, \xi, 0, 2z - x) + \delta(\xi) \Lambda_z(\xi, x)) \mathbb{1}_{\{x < z\}} \quad (4.29)$$

where  $\delta$  is the Dirac delta function and

$$\begin{aligned} \Pi(q, r, u, v) &= \frac{1}{\pi} \left( \frac{qv + ru}{(u + v)^2 \sqrt{uv}} \right) \exp \left( -\frac{q^2}{2v} - \frac{r^2}{2u} \right) + \sqrt{\frac{2}{\pi}} \left( \frac{1}{u + v} \right)^{\frac{3}{2}} \\ &\quad \times \left( 1 - \frac{(r - q)^2}{u + v} \right) \exp \left( -\frac{(r - q)^2}{2(u + v)} \right) \Phi \left( \frac{-qu - rv}{\sqrt{uv(u + v)}} \right) \end{aligned}$$

with  $\Phi$  denoting the standard normal CDF and

$$\Lambda_z(\xi, x) = \frac{1}{\sqrt{2\pi\xi}} \exp \left( -\frac{x^2}{2\xi} \right) - \frac{1}{\sqrt{2\pi\xi}} \exp \left( -\frac{(2x - z)^2}{2\xi} \right).$$



# Chapter 5

## Default Intensity, Trend and Extended-BHM Model

### 5.1 Introduction

This chapter gives the analysis of default intensity and trend in the extended Brody-Houghston-Macrina(BHM) model, with emphasis on the default *trend* and *intensity* calculations. The intention is to present key results to illustrate the important connections between the extended BHM information based model and classical reduced form models for credit risky assets. We start by proving the cornerstone of the intensity based credit risk valuation - a pricing formula based on the *trend* of the information model, this time using occupation time default definition. To use the pricing formula for price estimation in the current partial information framework requires computing the estimates of the default trend and intensity under the CLRB information. The main result here is the derivation of simple expressions for these estimates. To conclude this chapter we analyze the *credit spread* term structure for a credit risky zero coupon bond.

The incomplete information framework provides a common perspective on the BHM and intensity-based approaches to analyzing the price dynamics of credit risky assets. The classical reduced form approach for credit risk modeling is presented in the previous sections, where we describe its connection to the structural credit risk models through the assumptions on the information available to the market participants. The key feature of reduced form models is the specification of a default event with respect to information availability. For a given default definition, results in [39] shows how

market information characterizes all reduced form models. However, the emphasis in the previous discussions on reduced form models has concentrated on *information availability* to the market participants. Consequentially, serious limitations on the application of the existing theories for valuation and risk management of broader asset classes still exist. More precisely, the issues of computational complexities inherent in intensity based, reduced form models are yet to be fully addressed. In this present work we show the correspondence between extended BHM approach and the classical intensity based framework in such a way as to address the inherent computational problems in default intensity calculations. In what follows, we present an intensity based model through an information-based default specification. We derive the general expressions for the default process, default intensity and credit risky price processes. Our main contribution is presented in theorem 5.4.1 which gives a closed-form expression for the trend and intensity of the extended BHM model as a reduced form default model with occupation time default specification. Finally, we conclude with an application in the valuation of a credit risky zero coupon bond.

## 5.2 Default Specification

The starting point of the incomplete information approach to credit risk modeling is the specification of the default event. The key task here is to define a default event that leads to a consistent incomplete information model for credit risky assets. We consider a general probability space  $(\Omega, \mathbb{G}, \{\mathcal{G}_t\}_{t \geq 0}, \mathbb{Q})$  and the time horizon,  $[0, T]$ , where  $\mathbb{G}$  denotes the reference filtration representing the information held by the market participants. The measure,  $\mathbb{Q}$  is the risk-neutral measure under which discounted asset prices are martingale. Throughout this chapter, we shall adopt the following definition for generalized default model found in [39] ;

**Definition 5.2.1.** *A CLRB default model is specified by a pair  $(\tau, \mathbb{F})$  consisting of a  $\mathbb{G}$ -stopping time  $\tau$  designating the time of default and a filtration  $\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0} \subseteq \mathbb{G}$  generated by a CLRB*

information process.

Let  $N$  denote the default indicator process defined by

$$N_t = \mathbb{1}_{\{t \geq \tau\}}. \quad (5.1)$$

This implies that,  $N$  is zero before default and jumps to one at default. For the case of a portfolio of  $n \in \mathbb{N}$  credit risky assets, we have a default state vector,  $N = \{N_{t,1}, \dots, N_{t,n}\}$  with  $N_{t,i} = \mathbb{1}_{\{t \geq \tau_i\}}$ ,  $i = 1, \dots, n$ . For the purpose of simplifying the current setup, we will reserve the case of portfolio of assets for later treatment.

In this incomplete information default model  $(\tau, \mathbb{F})$ , the CLRB filtration  $\mathbb{F}$  is too crude to accurately infer the time to default: Mathematically, we say that  $\tau$  is not an  $\mathbb{F}$ -stopping time. Thus, the market filtration  $\mathbb{G}$  is generated by the sigma-algebras

$$\mathcal{G}_t = \{B \in \mathcal{G} : \exists B_t \in \mathcal{F}_t, B \cap \{\tau > t\} = B_t \cap \{\tau > t\}\} \quad (5.2)$$

for a given CLRB information process. Since the filtration  $\mathbb{G}$  contains  $\mathbb{F}$  we have that  $\tau$  is a  $\mathbb{G}$ -stopping time, in which case we say that  $\tau$  is inaccessible. This implies that  $\mathbb{Q}[\tau = T'] = 0$  for all predictable stopping times  $T'$ . We recall the definitions in the previous chapter of first-passage default time  $\tau_D$  and occupation time default specification  $\tau_\theta$  in equations (4.3) and (4.5) respectively. Default models of the form  $(\tau_D, \mathbb{F})$  was extensively analyzed in [39] as a generalized representation of classical reduced-form credit risk models based on incomplete information. More recently, default models of the form  $(\tau_\theta, \mathbb{F})$  has been used to analyze structural credit risk models, (see [61] and cited references). In what follows we analyze the extended BHM as an intensity based credit risk model of the form  $(\tau_\theta, \mathbb{F})$  for a specific class of information process and using a version of default model definition found on page 2291 in [39]. Now we write the default indicator process as  $N_t := \mathbb{1}_{\{\tau_\theta \leq t\}}$ . It is known that  $N$  is nondecreasing, hence a submartingale in  $\mathbb{G}$ . This implies that  $\mathbb{E}_{\mathbb{Q}}[N_t | \mathcal{G}_s] \geq N_s$  for  $s \leq t$ . Then Doob-Meyer

decomposition tells us that, there exists a unique nondecreasing process  $\hat{K}^\theta$  starting at zero such that the process  $N - \hat{K}^\theta$  is a  $\mathbb{G}$ -martingale.

The following proposition found in [52] allows us to obtain  $\hat{K}^\theta$  as the *trend* stopped at  $\tau_\theta$ .

**Proposition 5.2.1.** *The process  $N - \mathcal{A}^{\tau_\theta}$  is a martingale in the market filtration  $\mathbb{G}$ .*

**Definition 5.2.2.** *A credit risky asset is a triplet  $(H, T, \tau_\theta)$  with default possibility at some random time  $\tau_\theta$ , where  $H$  is  $\mathcal{G}_T$ -measurable random variable and  $T$  is the maturity date.*

We consider the following payoff structure for  $(H, T, \tau_\theta)$  which depends on its default state at the maturity date  $T$ . The asset pays at time  $T$ :

$$U_T = H \mathbb{1}_{\{H \geq \beta_1, \tau_\theta > T\}}. \quad (5.3)$$

The price  $C_t$  at time  $t \leq T$  defined by the conditional expectation under a martingale measure is given by

$$C_t = P_{tT} \mathbb{E}[H(1 - N_T) | \mathcal{G}_t],$$

The following result allows us to obtain the price expression in terms of the trend.

**Proposition 5.2.2.** *Let  $T$  be a fixed time point and  $X$  a  $\mathcal{G}_T$ -measurable random variable. Assume that the trend  $\mathcal{A}$  is a continuous process, suppose the path of the process  $\{Y_t\}_{0 \leq t \leq T}$  given by*

$$Y_t = \mathbb{E}[X e^{\mathcal{A}_t - \mathcal{A}_T} | \mathcal{G}_t],$$

*is continuous at  $\tau_\theta$  almost surely, then for each  $t < \tau_\theta$  we have*

$$\mathbb{E}[X(1 - N_T) | \mathcal{G}_t] = \mathbb{E}[X e^{\mathcal{A}_t - \mathcal{A}_T} | \mathcal{G}_t]$$

*almost surely  $t \leq T$ .*

*Proof.* Let  $W_t = \mathbb{E}[Xe^{-A_T}|\mathcal{G}_t]$ , then we have  $Y_t = e^{A_t}W_t$  for each  $t \leq T$ . The dynamics of  $Y$  is given as  $dY_t = e^{A_t}dW_t + Y_{t-}d\mathcal{A}_t$ . Let  $\Delta L = L_t - L_{t-}$  denotes the jump of some càdlàg process  $L$  at time  $t$ . Define  $Z_t = (1 - N_t)Y_t$ , then we get by the product rule,

$$\begin{aligned} dZ_t &= -Y_{t-}dN_t + (1 - N_{t-})dY_t + \Delta(1 - N_{t-})\Delta Y_t \\ &= (1 - N_{t-})e^{A_t}dW_t - Y_{t-}(dN_t - (1 - N_{t-})d\mathcal{A}_t) \\ &= (1 - N_{t-})e^{A_t}dW_t - Y_{t-}(dN_t - d\mathcal{A}_t^{\tau_\theta}) \end{aligned}$$

Thus,

$$Z_T = Z_t + \int_t^T (1 - N_{s-})e^{A_s}dW_s - \int_s^T Y_{s-}d(N_s - \mathcal{A}_s^{\tau_\theta})$$

We know from Proposition 5.2.1 that  $N - \mathcal{A}_t^{\tau_\theta}$  is a  $\mathbb{G}$ -martingale. We also note that  $\{W_t\}_{0 \leq t \leq T}$  is a  $\mathbb{G}$ -martingale and that  $(1 - N_{s-})e^{A_s}$  and  $Y_{s-}$  are bounded and predictable, which then implies that  $Z$  is a  $\mathbb{G}$ -martingale, hence  $Z_t = Y_t(1 - N_t) = \mathbb{E}[Z_T|\mathcal{G}_t] = \mathbb{E}[X(1 - N_T)|\mathcal{G}_t]$   $\square$

Proposition 5.2.2 provides a generalized expression for conditional default probabilities and the price process of a credit risky asset in the market filtration  $\mathbb{G}$  whenever the trend is continuous.

**Corollary 5.2.1.** *If the trend  $\mathcal{A}$  is a continuous process and the path of the process  $Y_{0 \leq t \leq T}$  is such that*

$$Y_t = P_{tT}\mathbb{E}[He^{A_t - A_T}|\mathcal{G}_t], \quad t \leq T$$

*is almost surely continuous at  $\tau_\theta$ , then for each  $t < \tau_\theta$  we have  $C_t = Y_t$  and*

$$\rho^\theta(t, \xi) = 1 - \mathbb{E}[e^{A_t - A_T}|\mathcal{G}_t],$$

*almost surely.*

The statement of corollary 5.2.1 implies that the credit risky asset with cashflow  $U_{\{\tau_\theta \wedge T\}}$  can be valued as an asset with cashflow  $H$  using an adjusted numéraire  $P_{tT} \exp(\mathcal{A}_t)$ . This is a central idea in most intensity-based credit risk models (see [47]). Further details on the above results can be found in [39], but we stress that these results are essentially known and have been used extensively in first passage default models.

Next we identify the extended-BHM model as an incomplete information model of the form  $(\tau_\theta, \mathbb{F})$ , where  $\tau_\theta$  is defined as in (5.4) and the information described by the model filtration is the compound Lévy information generated by the information process  $\{L_{tT}\}_{0 \leq t < T}$  with law  $CLRB([0, T], \{f_t\}, \nu)$  as defined and analyzed in chapter two. Unlike the standard BHM information based model, the current intensity based version does not rely on a fixed default time specification rather default is viewed as an "emergent" occurrence and the conditional survival probability  $S$  is modeled explicitly using the information described by the model filtration. In proposition 5.4.1 we characterize the trend for the compound Lévy information, and estimate prices of credit risky assets using the generalized price expression in corollary 5.2.1. We recall the occupation time default definition from previous chapter. We consider the occupation time process

$$\ell_t = \int_0^t \mathbb{1}_{\{\beta_1 \leq X_s \leq \beta_2\}} ds, \quad t \in [0, T] \quad (5.4)$$

corresponding to the amount of time the process  $\{X_t\}_{0 \leq t < T}$  generating the market filtration  $\mathbb{G}$  spends in the band  $[\beta_1, \beta_2]$ ,  $\beta_1, \beta_2 \in \mathbb{R}$ . Let  $\theta \in (0, T]$  be given, then we introduce the model default time based on the occupation time of the information process as

$$\tau_\theta = \inf\{t > 0 : \ell_t \geq \theta\} \quad (5.5)$$

Formally,  $\tau_\theta$  is the first time that the market information concerning the asset cashflow  $H$  has continued to be unfavourable for at least  $\theta$  units of time. The variables  $\beta_1, \beta_2$  and  $\theta$  are parameters of the default process that could be estimated from market

data. The CLRFB filtration  $\mathbb{F}$  is generated the information process  $\{L_{tT}\}_{0 \leq t < T}$  with law  $CLRFB([0, T], \{f_t\}, \nu)$ , where  $\nu$  is the *a priori* joint law of  $H$  and  $\ell_T$ . We note that the default time  $\tau_\theta$  is not an  $\mathbb{F}$ -stopping time, hence default is totally inaccessible in  $\mathbb{G}$ .

### 5.3 Trend, Intensity and CLRFB Information

In this section, we present the trend and intensity definitions for the CLRFB information. Later, we derive explicit expressions for the trend and intensity of default based on these definitions. Let us consider the *conditional survival probability* at time  $t \in ]0, T[$  denoted by  $S_t^\theta$ . By definition we have that

$$S_t^\theta = \mathbb{E}[1 - N_t | \mathcal{F}_t] = \mathbb{Q}[\tau_\theta > t | \mathcal{F}_t]. \quad (5.6)$$

The conditional survival probability  $S_t^\theta$  provides a useful expression involving the default indicator process  $N$  and the CLRFB filtration  $\mathbb{F}$ . We impose that there is always a possibility that the asset defaults, so we get for every  $t > 0$ , that  $S_t^\theta > 0$  almost surely and  $\mathbb{E}[S_t^\theta] > 0$ .

**Definition 5.3.1.** We call a process  $\gamma^\theta$  the intensity of the CLRFB default model  $(\tau_\theta, \mathbb{F})$  if  $\gamma^\theta$  is bounded, nonnegative and  $\mathbb{F}$ -predictable such that for every  $t \geq 0$

$$S_t^\theta = \exp\left(-\int_0^t \gamma_s^\theta ds\right) \quad (5.7)$$

almost surely.

Definition 5.3.1 allows us to model the conditional survival probability directly in terms of the information described by the filtration generated by a CLRFB. The process  $S$  is a supermartingale in the CLRFB filtration  $\mathbb{F}$ : for  $s < t$ , we have

$$S_s^\theta \geq \mathbb{E}[S_t^\theta | \mathcal{F}_s].$$

Here again, the Doob-Meyer decomposition theorem applies and we have that a unique nondecreasing and  $\mathbb{F}$ -predictable process  $K$  starting at zero can be found such that the process  $S + K$  is guaranteed to be an  $\mathbb{F}$ -martingale. We say that  $K$  is the compensator to  $S$ .

**Definition 5.3.2.** *The trend  $\mathcal{A}$  of a CLRB default model  $(\tau_\theta, \mathbb{F})$  is defined by the Stieltjes integral*

$$\mathcal{A}_t = \int_0^t \frac{dK_s^\theta}{S_{s-}^\theta}, \quad (5.8)$$

where  $S_{t-}^\theta = \lim_{s \uparrow t} S_s^\theta$  and  $S_{0-}^\theta = 1$ .

We note that the trend is nondecreasing and  $\mathbb{F}$ -predictable. The following model definition based on the trend will be useful in subsequent analysis of asset price processes.

**Definition 5.3.3.** *We say that a CLRB default model  $(\tau_\theta, \mathbb{F})$  is intensity-based if there exists a bounded, nonnegative and  $\mathbb{F}$ -predictable process  $\gamma^\theta$  such that for every  $t \geq 0$*

$$\mathcal{A}_t = \int_0^t \gamma_s^\theta ds \quad (5.9)$$

*almost surely. The process  $\gamma^\theta$  is called the intensity of the model  $(\tau_\theta, \mathbb{F})$ .*

It was demonstrated in [39] that definition 5.3.1 is stronger than 5.3.3 for first passage time intensity based models. In the next section we use the weak version for the occupation time default definition to estimate prices of credit risky assets for various specification of market information processes whose laws are CLRBs.

## 5.4 Trend and Credit Risky Assets Prices

Given the above setup, we describe the connection between the *trend* of default process and its  $\mathbb{F}$ -compensator. Then we show how the credit risky asset prices implied by the trend can be parameterized in terms of the information contents of the filtration

$\mathbb{F}$ . Specifically, we characterize the trend in terms of the value of the information process observed at the current time  $t$  and its occupation time within a given band on the interval  $[0, t]$ . Finally, we use the resulting expression to estimate prices of credit risky assets.

If the filtration  $\mathbb{F}$ , generated by a CLRB information model coincides with the market information  $\mathbb{G}$  we have that  $\tau_\theta$  is an  $\mathbb{F}$ -stopping time, then the *conditional survival probability*  $S = 1 - N$ , in which case the assumption  $S_t^\theta > 0$  is no longer valid. Moreover, if  $\tau_\theta$  is predictable, then the default indicator  $N$  is its own  $\mathbb{G}$ -compensator. Throughout the remainder of this section, we shall consider the nontrivial case where  $\tau_\theta$  is not an  $\mathbb{F}$ -stopping time. The following theorem is the main result of this chapter. It gives explicit link between the analytic properties of the *trend* and those of the *conditional survival probability*  $S$  resulting to a straightforward estimation formula for the trend with the conditional survival probability as the argument.

**Theorem 5.4.1.** *Suppose the parameters  $\beta_1, \beta_2$  and  $\theta$  are  $\mathcal{F}_0$ -measurable but the variables  $\ell_T$  and  $H$  are never  $\mathcal{F}_t$ -measurable. Assuming the market participants form a prior on  $\ell_T$  and  $H$  with joint distribution function  $\nu$  on  $[0, T] \times [0, \infty)$  such that the model filtration  $\mathcal{F}_t$  is generated by the process  $\{L_{sT}, s \leq t\}, t \in [0, T]$  with law  $CLRB([0, T], f_t, \nu)$ . Then, for each  $t \in [0, T]$  we have*

$$S_t^\theta = 1 - \int_0^\theta \int_{\beta_1}^{\beta_2} \nu_t(dz, ds) \text{ almost surely.} \quad (5.10)$$

The trend  $\mathcal{A}$  is continuous and for each  $t \in [0, T]$  we obtain

$$\mathcal{A}_t = -\log(S_t^\theta). \quad (5.11)$$

*Proof.* By definition  $L_{0T} = 0$  and  $\nu_0(dz, ds) = \nu(dz, ds)$ , we have for  $\theta \in (0, T], S_0^\theta = 1$ . Let  $V(t; \cdot, \cdot)$  denote the functional form of the  $\mathcal{F}_t$ -joint conditional distribution of  $H$  and  $\ell_T$ . The expression in (5.10) follows from (3.25) in section 3.2.4 and item (iv) in definition

3.2.3. Precisely, we get

$$\begin{aligned}
 S_t^\theta &= 1 - \mathbb{Q}(L_{tT} \leq H, \ell_{tT} \leq \ell_t | \mathcal{F}_t) \\
 &= 1 - \mathbb{E}(V(t, H, \ell_t) | \mathcal{F}_t) \\
 &= 1 - \int_{\beta_1}^{\beta_2} \int_0^\theta V(t, x, y) d\mathbb{Q}(H \leq x, \ell_t \leq y | \mathcal{F}_t) \\
 &= 1 - \int_{\beta_1}^{\beta_2} \int_0^\theta \nu_t(dz, ds) \quad (\text{using (3.25)})
 \end{aligned} \tag{5.12}$$

Since  $V(t; x, y) < 1$  by assumption, we get  $S_t^\theta > 0$ . In addition, noting that for all  $\beta_1 < x < \beta_2$  and  $0 < y < \theta$  the process  $V(t; x, y)$  is continuous and monotone, we have that  $S^\theta$  is continuous and monotone. Then  $K^\theta = 1 - S^\theta$  and  $\mathcal{A}$  is given by (5.10). We remark that since  $S^\theta$  is continuous, the  $\mathbb{G}$ -compensator  $\mathcal{A}^{\tau_\theta}$  to  $N$  is also continuous, verifying that the default time  $\tau_\theta$  is a totally inaccessible  $\mathbb{G}$ -stopping time.  $\square$

Next we establish a connection between the probabilistic properties of the default time  $\tau_\theta$  and the properties of the *credit spread* term structure from a CLRB information. The credit spread  $\Gamma^\theta(t, \xi)$  on a credit risky zero coupon bond is the difference between the yield at time  $t$  of the bond with zero recovery and that on a credit risk-free zero coupon bond, both with the same time to maturity  $\xi$ . We consider a constant interest rate and unit nominal value, then we get for every  $t < \tau_\theta$  that

$$\Gamma^\theta(t, \xi) = -\frac{1}{\xi} \log(1 - \rho^\theta(t, \xi)), \quad t \geq 0, \quad \xi > 0. \tag{5.13}$$

Where  $\rho^\theta(t, \xi)$  is the conditional default probability at time  $t$  for the term  $\xi$  defined by

$$\rho^\theta(t, \xi) = \mathbb{Q}[\tau_\theta \leq t + \xi | \mathcal{G}_t]. \tag{5.14}$$

The schedule at time  $t$  of the credit spread  $\Gamma^\theta(t, \xi)$  in terms of the time to maturities  $\xi \in (t, T]$  is called its *term structure*. The *short spread*  $\lim_{\xi \downarrow 0} \Gamma^\theta(t, \xi)$  at  $t$  is the additional

yield over the risk-free yield charged by market participants for taking credit risk over an infinitesimal period to maturity.

**Proposition 5.4.1.** *For a CLRB default model  $(\tau_\theta, \mathbb{F})$ , the limit  $\lim_{\xi \downarrow 0} (\frac{1}{\xi}) \rho^\theta(t, \xi)$  exists, it is positive and finite (almost surely). For every  $t < \tau_\theta$*

$$\lim_{\xi \downarrow 0} \Gamma^\theta(t, \xi) = \lim_{\xi \downarrow 0} \frac{1}{\xi} \rho^\theta(t, \xi), \text{ almost surely}$$

*Proof.* If for each  $t$  on the event  $\{t < \tau_\theta\}$ , the  $\lim_{\xi \downarrow 0} \frac{1}{\xi} \rho^\theta(t, \xi)$  exists and is finite almost surely, then we have that  $\rho^\theta(t, \xi) \rightarrow 0$  as  $\xi \rightarrow 0$  almost surely. The statement follows from (5.13) by Taylor's expansion.  $\square$

**Proposition 5.4.2.** *For a CLRB default model  $(\tau_\theta, \mathbb{F})$ , the limit  $\lim_{\xi \downarrow 0} (\frac{1}{\xi}) \rho^\theta(t, \xi)$  exists, it is positive and finite (almost surely). For every  $t < \tau_\theta$*

$$\lim_{\xi \downarrow 0} \Gamma^\theta(t, \xi) = \lim_{\xi \downarrow 0} \frac{1}{\xi} \rho^\theta(t, \xi) = \gamma_t^\theta, \text{ almost surely}$$

*Proof.* Let  $U \geq 0$  be a square integrable random variable, then we have from theorem 14 in [67] that

$$\mathbb{E}[U | \mathcal{G}_t] = \frac{1}{S_t^\theta} \mathbb{E}[U \mathbb{1}_{\{\tau_\theta > t\}} | \mathcal{F}_t] + U \mathbb{1}_{\{\tau_\theta \leq t\}} \quad t \geq 0 \quad (5.15)$$

almost surely. Setting  $U = \mathbb{1}_{\{\tau_\theta \leq t + \xi\}}$  in (5.15) we obtain,

$$\begin{aligned} \rho^\theta(t, \xi) &= \mathbb{Q}[\tau_\theta \leq t + \xi | \mathcal{G}_t] \\ &= \frac{1}{S_t^\theta} \mathbb{Q}[t < \tau_\theta \leq t + \xi | \mathcal{F}_t] + \mathbb{1}_{\{\tau_\theta \leq t\}} \end{aligned}$$

for each  $t \geq 0$  and  $\xi > 0$ . It follows then that on the set  $\{t < \tau_\theta\}$

$$\rho^\theta(t, \xi) = \frac{1}{S_t^\theta} \mathbb{E}[S_t^\theta - S_{t+\xi}^\theta | \mathcal{F}_t] = \mathbb{E} \left[ 1 - \exp \left( - \int_t^{t+\xi} \gamma_s^\theta ds \right) | \mathcal{F}_t \right].$$

Then the second equality follows from,

$$\begin{aligned}
\lim_{\xi \downarrow 0} \frac{1}{\xi} \rho^\theta(t, \xi) &= \lim_{\xi \downarrow 0} \frac{1}{\xi} \mathbb{E} \left[ \int_t^{t+\xi} \gamma_s^\theta ds + O \left( \int_t^{t+\xi} \gamma_s^\theta ds \right) \middle| \mathcal{F}_t \right] \text{ (Taylor's theorem)} \\
&= \lim_{\xi \downarrow 0} \frac{1}{\xi} \mathbb{E} \left[ \int_t^{t+\xi} \gamma_s^\theta ds \middle| \mathcal{F}_t \right] + \lim_{\xi \downarrow 0} \frac{1}{\xi} \mathcal{O}(\xi) \text{ (}\gamma^\theta \text{ is bounded)} \\
&= \lim_{\xi \downarrow 0} \mathbb{E} \left[ \frac{1}{\xi} \int_t^{t+\xi} \gamma_s^\theta ds \middle| \mathcal{F}_t \right] \\
&= \mathbb{E} \left[ \lim_{\xi \downarrow 0} \frac{1}{\xi} \int_t^{t+\xi} \gamma_s^\theta ds \middle| \mathcal{F}_t \right] \text{ (dominated convergence theorem)} \\
&= \mathbb{E} [\gamma_{t+}^\theta | \mathcal{F}_t] \\
&= \gamma_t^\theta \text{ (right continuity of } \gamma^\theta \text{)}
\end{aligned}$$

Lastly, the first equality follows from Proposition 5.4.1. □

Next we consider an application of the current intensity-based framework to the valuation of a simple credit risky binary discount bond. However, the application of our framework to other examples and more complicated credit sensitive securities such as credit risky coupon bonds, credit default swaps (CDS) or CDOs are left for future consideration.

**Example 5.4.1** (Binary Discount Bond). *A credit risky binary discount bond with maturity at time  $T$  pays a principal of  $h_1$  dollars, if there is no default and  $h_0$  in the event of default, where  $h_0 < h_1$ . We set the default band as  $[0, h_0]$ , and let  $\theta = t$ , for each  $t \in [0, T]$ . Writing*

$$\psi(t, x) = \int_0^t \int_0^{h_0} \nu_t(dz, ds), \quad (5.16)$$

then proposition 5.4.1 implies that  $\mathcal{A}_t = -\log(1 - \psi(t, L_{tT}))$  and we obtain from corollary 5.2.1, that *i*

$$\rho^\theta(t, \xi) = \frac{\psi(t + \xi, L_{tT}) - \psi(t, L_{tT})}{1 - \psi(t, L_{tT})}.$$

which is the required expression for the conditional default probability at time  $t < \tau^\theta$  and tenure  $\xi > 0$ .

The double integral in (5.16) can be approximated numerically for a any Lévy process  $\{L_t\} \in \mathcal{C}[0, T]$  with joint density  $\{f_t\}_{0 \leq t \leq T}$  such that  $\nu$  is the a priori law of  $\ell_T$  and  $\mathbb{1}_{[h_0, h_1]}(L_T)$  (see [42, 8] for relevant exact simulation techniques and Laplace transform approach via the Sturm-Liouville equation). For the particular case of a Brownian motion, the integrals can be evaluated in closed-form (see page 8 in [61]) with  $f_t(s, z)$  given by (4.29).



# Chapter 6

## Linear Filtering and Credit Risky Assets

### 6.1 Introduction

In this chapter, we employ stochastic filtering technique in the price estimation of credit risky assets under the information based asset pricing framework of Brody-Hughston and Macrina. This is carried out by following the standard innovations process approach. We derive the famous Kalman-Bucy filter in one dimension for bridges of Lévy processes with a given finite variance. We conclude this chapter by looking at numerical simulations.

The concept of filtering originated from control engineering and signal processing. Subsequently, filtering captured the attention of many researchers in various communities including those of statistics, economics and mathematical finance. In many applications, a filter is the term used to describe an algorithm that enables one to obtain recursively in time, good estimates of the state of a stochastic dynamical system based on partial observations of the system. Credit risk models with incomplete information have been considered in modern credit risk modelling literature. The contributions by [74, 17, 53, 47, 34] are concerned with the structural models, where the value of the assets or liabilities are not directly observable. Reduced form credit risk models with incomplete information has also been considered in [75, 16, 25, 33, 23, 22]. In the later

case, default intensities are driven by an unobservable factor process,  $\mathbf{X}$ . Given information about  $\mathbf{X}$ , the default times are conditionally independent random times and the investor information,  $(\mathcal{F}_t^I)$  is given by the default history of the portfolio, augmented by economic covariate. In [75, 16], the unobservable factor is modeled by a static random vector  $\mathbf{X}$ - called *frailty*, and the conditional distribution  $\pi_{\mathbf{X}|\mathcal{F}_t^I}$  is determined using Bayesian updating. In Duffie et al, [25] the unobservable factor,  $\mathbf{X}$  is modeled by an Ornstein-Uhlenbeck process. Frey et al in [33] extends this choice of models in two directions: first, the joint dynamics of the state process and the default indicator function is modeled by a fairly general jump-diffusion model. The second extension is on the information set; the investor information,  $\mathcal{F}_t^I$  includes theoretical prices of traded credit derivatives observed in additive noise in addition to the default history of the firms under consideration. In order to determine the conditional distribution,  $\pi_{\mathcal{F}_t^I}$  in this extended set up, the authors solve a nonlinear filtering problem with mixed observation of marked point processes of diffusion type and with common jumps of point process and a state process.

In [23, 22], a new class of reduced-form credit risk models under the incomplete information modelling framework was developed. This modeling approach completely abandoned the need for an intensity based approach, instead the cash flows of the debt obligation (for instance, coupon payment and principal repayment) are modelled by a collection of random variables and default is identified as the event of first such payment that fails to achieve the terms specified in the contract. For a particular cash flow and default specification, the authors derive an exact expression for the bond price process. For the particular case of a defaultable discount bond, admitting two possible payouts, they further derive exact expression for the value of an option on the bond. The noisy observation represents partial information (rather than full information) regarding the bond payout, then the bond price dynamics is given by the conditional expectation of the bond payout under a martingale measure with respect to the current information set. Since the bond payout is unobservable at times before the maturity or a

future random default time, the computation of the conditional expectation results in a filtering problem which we solve here using martingale representation and innovation approach to stochastic filtering.

The key intuition behind the information-based asset pricing framework as described in the previous chapters (see [23, 22, 60, 43, 78] for more details and applications) is that the price process of an asset should be regarded as an *emergent phenomenon*. What this means is that the price process is an output of (rather than an input into) various decisions regarding possible transactions in the asset. These decisions are in turn induced by the flow of information to market participant. To put this in another way, the view in the information based asset pricing framework is that it is unsatisfactory to simply fix the market filtration and assume that asset price processes are adapted to it, without indicating the nature of the information, which the background filtration represents. Thus, this framework is based on modeling the flow of *market information*. This information is that concerning the values of the future cash flows associated with the given asset. In addition to the contributions of this framework to the general theory of asset pricing, it also offers a very important computational insight - application of stochastic filtering techniques in estimating asset prices. In what follows we present the information based credit risky bond pricing as a stochastic filtering problem and derive Kalman-Bucy filter for the conditional distribution of the asset cash flow given the market information. This also serves to illustrate the application of stochastic filtering techniques for more general class of credit risky asset model.

## 6.2 Filtering and Credit Risky Asset Pricing

### 6.2.1 The Set Up

In this section, we give the basic setup for representing the information-based asset pricing as a stochastic filtering problem. Our emphasis is based on the construction of

a Kalman filter for the price process of a generalized credit risky asset model. We refer anyone interested in a detailed study of filtering theory to the chapters 3 and 5 in [5, 80] and the references in them.

We consider the probability space,  $(\Omega, \mathcal{F}, \mathbb{P})$ , equipped with a filtration  $(\mathcal{F}_t)_{t \geq 0}$ . The filtering model consists of two processes: the *signal process* and the *observation process*. The observation process satisfies the stochastic differential equation.

$$dY_t = h(t, X_t)dt + dL_t \quad (6.1)$$

where  $\{L_t\}$  is a Lévy process as in definition 3.2.1, ( see [5] for the regularity conditions on  $h$ ).

The signal process is an  $\mathcal{F}_t$ - adapted process  $X = \{X_t, t \geq 0\}$  which takes values in  $\mathbb{R}$ .

The aim of filtering theory is to find the conditional distribution of the signal process given observations to date. That is, how to compute

$$\pi_t(\phi) := \mathbb{E}(\phi(X_t) | \mathcal{F}_t^y) \forall t \in [0, T]. \quad (6.2)$$

or alternatively, the unnormalised conditional distribution

$$\rho_t(\phi) := \mathbb{E}_{\mathbb{P}} \left[ \phi(X_t) \tilde{Z}_t | \mathcal{F}_t^y \right] \quad (6.3)$$

where,  $\mathcal{F}_t^y$  is the filtration generated by the observation process,  $Y_t$  and  $\phi$  is a reasonably well defined function.

and

$$\tilde{Z}_t = \frac{1}{Z_t}$$

with

$$Z_t = \exp \left( - \sum_{i=1}^m \int_0^t h^i(X_s) dL_s - \frac{1}{2} \sum_{i=1}^m \int_0^t h^i(X_s)^2 ds \right)$$

The evolution equation for the process  $\pi$  was studied extensively in the sixties and early

seventies (see [49], [76], [82]). Recent studies in this area include [20], [19] and [12], with emphasis on the construction of evolution equation for a finite dimensional version of the process  $\pi$ . The main result of the earlier studies in filtering theory is summarized in theorems 6.2.1 and 6.2.2, describing the infinite dimensional evolution equations for the filtering process.

For more practical purposes, the filtering problem has been particularly studied in the literature for two representations of the signal process. The first representation is the case where the signal process  $X$ , is modeled as a diffusion process. That is  $X$  is defined to be the solution of the SDE;

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t, \quad (6.4)$$

where  $B$  is a  $d$ -dimensional Brownian motion,  $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$ ,  $\sigma : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$  and  $h : \mathbb{R}^m \rightarrow \mathbb{R}^m$  are bounded and Lipschitz continuous.

**Theorem 6.2.1** (Zakai equation). *The unnormalized conditional distribution of the signal process  $\{X_t\}$  in (6.4) satisfies*

$$\rho_t(\phi) = \pi_0(\phi) + \int_0^t \rho_s(\mathcal{A}_s \phi) ds + \int_0^t \rho_s(h' \phi) dY_s, \forall t \geq 0, \forall \phi \in \mathcal{D}(\mathcal{A}) \quad (6.5)$$

where  $\mathcal{L}$  is the infinitesimal generator of  $X$ , given by

$$\mathcal{A} = \sum_{i=1}^d b^i \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i=1}^d (\sigma \sigma^T)^{ij} \frac{\partial^2}{\partial x_i \partial x_j}$$

where  $\phi$  is any real valued and twice differentiable function.

**Theorem 6.2.2** (Kushner-Stratonovich). *The normalized conditional distribution of the process,  $\{X_t\}$  in (6.4) satisfies the Kushner-Stratonovich equation*

$$\pi_t(\phi) = \pi_0(\phi) + \int_0^t \pi_s(\mathcal{A}_s\phi)ds + \int_0^t (\pi_s(h'\phi) - \pi_s(h')\pi_s(\phi)) (dY_s - \pi_s(h)ds). \quad (6.6)$$

Detailed proof and analysis of the above theorems can be found in [5], [56], [2], [80]. These theorems provide practical methods by which the filtering problem (both linear & non-linear) may be approached today. However, the optimal filters in theorems (6.2.1) and (6.2.2) are infinite-dimensional in the case of a general signal and the observation processes. A detailed theoretical study of the associated filtering equations for the case of a finite state Markov process can be found in [5]. More recently, the construction of optimal finite dimensional filters or an equivalent problem of finding a signal process and observation equation that will result to an optimal finite-dimensional filter has caught the attention of many researchers. Examples in this line of research are the assumed density filters, [62] and differential geometry approach to non-linear filtering, (see [20],[19], and [13]). Approximations to the solution of the Zakai's equation do exist. These include, the extended Kalman filter, the unscented Kalman filter and Particle filter (see [26]). Although, the construction of finite state Markov-chain signal processes has been extensively studied in the literature (see [5], [19], [20]), its application to real world problems remains an interesting research endeavor. One interesting application of finite state Markov-chain filtering approach to credit risk modelling can be found [33].

In this chapter, we are particularly interested in the solution of the filtering problem for a specific case where the signal process is driven by some Lévy process  $\{L_t\} \in \mathcal{C}[0, T]$ <sup>1</sup> while the observation process is driven by an independent Lévy bridge  $\{L_{tT}\}$ . This gives a direct application of finite dimensional filters in asset pricing under the information based asset pricing framework. However, our approach can be applied in

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<sup>1</sup>we say that a Lévy process  $\{L_t\} \in \mathcal{C}[0, T]$  if for each  $t \in (0, T]$ ,  $L_t$  has a density  $p_t$ .

the development of computational technique for a more general class of information processes in the information based credit risky asset pricing framework.

### 6.3 Filtering and Information Based Credit Risky Assets

In this section, we employ linear filtering techniques to estimate the price of a credit risky asset where the information process is driven by the bridge of a certain subclass of Lévy processes. Specifically, we use the Kushner-Stratonovich equation to deduce the Kalman-Bucy Filter for the price of a credit risky asset of which the observed information concerning the asset cash flow is obscured by a Lévy bridge noise that vanishes at some fixed finite time  $T$ . The filter gives the expression for the conditional expectation of the random cash flow given the market information at time  $t \leq T$ . The market information contains the actual cash flow and a Lévy bridge noise term. As mentioned in the previous section, application of linear filtering in credit-risk modeling has been studied to a reasonable extent with emphasis on the classical structural and reduced form approaches to credit risk modelling. In particular, the focus has been on credit risk modelling under incomplete information. In this thesis, we extend the filtering application to a new class of reduced form credit risk models. In what follows, we formulate the linear filtering problem within the context of Lévy processes.

Consider the following stochastic differential equation (SDE) in  $\mathbb{R}$ :

$$dX_t = \lambda_t X_t dt + dL_t \quad (6.7)$$

where  $\lambda$  is a locally bounded left continuous real valued function and  $\{L_t\} \in \mathcal{C}[0, T]$  is a Lévy process such that  $L_t$  has density  $f_t$  for every  $t \in (0, T]$ . We assume that the initial random variable  $Y_0$  is  $\mathcal{F}_0$ -measurable. Hence the SDE (6.7) has a unique solution (see [21], page 363)  $X = \{X_t\}$ ,  $t \in (0, T]$  taking values in  $\mathbb{R}$ . Precisely, the variation of

constant formula gives

$$X_t = \exp\left(\int_0^t \lambda_s ds\right) X_0 + \int_0^t \exp\left(\int_u^t \lambda_s ds\right) dL_u, \quad (6.8)$$

for all  $t \in (0, T]$ . We call  $\{X_t\}$  the *unobservable process*. Now let  $\sigma_t$  and  $\eta_t$  be locally bounded left continuous functions taking values in  $\mathbb{R}$ . We further assume that  $\eta_t^{-1}$  exist. Then the SDE

$$dY_{tT} = \sigma_t X_t dt + \eta_t dL_{tT} \quad (6.9)$$

with the initial condition  $Y_{0T} = 0$  (a.s.) has a unique càdlàg solution  $Y = \{Y_{tT}\}$   $t \in (0, T]$  taking values in  $\mathbb{R}$ . The noise term  $L_{tT}$  is the standard bridge of a Lévy process  $\{\hat{L}_t\} \in \mathcal{C}[0, T]$  independent of  $\{L_t\}$  (See chapter one for more detail). We denote the density of  $\hat{L}_t$  by  $\hat{f}_t$  for every  $t \in (0, T]$ . The process  $\{Y_{tT}\}$ ,  $t \in (0, T]$  contains the observed partial market information regarding the payout of a credit risky asset. The *unobservable process*  $\{X_t\}$  represents the economic factor process which determines the value of the payout of the credit risky asset. The terminal value  $X_T$  of the factor process is observable at time  $T$  through the realization of the actual cash flow of the credit risky asset. However, at any time  $t < T$  the factors  $X_t$  and  $X_T$  are not observable by the market participants, only the noisy *market information* generated by the process  $\{Y_{tT}\}$ ,  $t < T$  is available. Let the market filtration be denoted by  $\mathcal{F}^M$ . We have that the problem of computing values of credit risky assets under partial information about the factor process amounts to that of computing conditional expectation of the following form

$$\pi_t(Y_{tT}) = \mathbb{E}(h(X_t) | \mathcal{F}_t^M), \quad (6.10)$$

which is a linear filtering problem for a Lévy process with continuous density, given a conditioned Lévy observation process.

### 6.3.1 Kalman Filter Setup

Given the introductory background in the previous sections about linear filtering and information based approach to valuing credit risky assets we now proceed to present the Kalman filter framework under the information based pricing of credit-risky discount bonds.

In the information based asset pricing theory, the value process  $\{B_{tT}(X_t)\}$ ,  $t \in [0, T)$  for a credit risky discount bond with payout  $h(X_T)$  at maturity time  $T$  can be written as

$$B_{tT}(X_t) = P_{tT}H_{tT}(X_t) \quad (6.11)$$

where  $h$  is a generic (bounded) function and  $H_{tT}(X_t)$  is the condition expectation of the bond payout with respect to the full-information  $\mathcal{F}$  (the filtration generated by the unobservable factor process) given by

$$H_{tT}(X_t) = \mathbb{E}[h(X_T)|\mathcal{F}_t]. \quad (6.12)$$

Therefore, conditional on the information regarding the state of the economy up to the current time, the distribution of the process  $\{X_t\}$  is an important ingredient in estimating the value of the credit-risky discount bond at time  $t < T$ .

The main objective of the current filtering framework is to compute the market price of a credit risky discount bond,  $\hat{B}_{tT}$ , given by,

$$\hat{B}_{tT} = P_{tT}\hat{H}_{tT} \quad (6.13)$$

with

$$\hat{H}_{tT} := \mathbb{E}[H_{tT}(X_t)|\mathcal{F}_t^{\mathbb{M}}] = \mathbb{E}[H_{tT}(X_t)|\mathcal{F}_t^{\mathbb{Y}}] = \int h(x)\pi_t(dx)$$

Recall that market participants cannot observe  $X_t$  nor  $L_{tT}$  but the process  $Y_{tT}$  (in the form of rumors, innuendos, etc) is observable. Hence we have the following inclusion

property

$$\mathcal{F}_t^{\mathbb{M}} = \mathcal{F}_t^{\mathbb{Y}} \subsetneq \mathcal{F}_t^{\mathbb{X}, \mathbb{L}} = \mathcal{F}_t.$$

Naturally, the computation of  $\hat{H}_{tT}$  leads to a filtering problem. In particular, we have by the tower property of conditional expectation and the martingale property of the unobservable process that

$$\hat{H}_{tT} = \mathbb{E}[h(X_T)|\mathcal{F}_t^{\mathbb{Y}}] = \mathbb{E}[\mathbb{E}[h(X_T)|\mathcal{F}_t]| \mathcal{F}_t^{\mathbb{Y}}] = \mathbb{E}[H_{tT}(X_t)|\mathcal{F}_t^{\mathbb{Y}}]$$

where the  $\mathcal{F}_t$ -measurable random variable  $H_{tT}(\cdot)$  is the full-information value of the conditional expectation of the bond payout as derived in the previous section.

We consider the simple terminal cash flow of the form  $h(X_T) = X_T$ . The unobservable factor process is given by (6.7) and the observable process by (6.9). The deterministic and bounded function  $\sigma_t$  in (6.9) represents the rate at which the true value of factor is revealed. If  $\sigma_t = 0, \forall t < T$ , then  $Y$  carries no information, on the other hand, with large  $\sigma_t$  the factor can be observed with high precision before the maturity date of the bond. The above state space representation captures the important basic assumptions of the information based credit risky asset pricing framework; the unobservable process is consistent with the assumption of a factor process with observable terminal value which determines the payout of a credit risky asset and the observable process has the required Markov property of a market information process. The Markov property of the observable process can easily be established using a similar procedure to that used in the previous section to verify the Markov property of Conditioned Lévy processes in chapter two. In what follows, we present the filtering equations to compute the conditional distribution of the factor process which in turn is used to determine the value of the discount bond at time point  $t < T$ .

As usual, all our processes are defined on a complete probability space,  $(\Omega, \mathbb{F}, \mathbb{Q})$  and we work on the continuous time horizon  $[0, T]$  with  $T$  corresponding to the point of

maturity of the discount bond. We recall that in order to determine the conditional expectation of the credit-risky bond payout,  $h(X_T)$  at  $t < T$ , we first need to work out the estimate of  $H_{tT}(X_t)$  given the market information up to the current time  $t$ . Mathematically, we need to compute;

$$\hat{H}_{tT} = H_{tT}(\hat{X}_t) \quad (6.14)$$

where  $\hat{X}_t = \mathbb{E}(X_t | \mathcal{F}_t^Y)$ . We also recall that  $\mathcal{C}[0, T]$  denotes the space of Lévy process with continuous density  $\{f_t\}_{0 \leq t \leq T}$ . Let  $L^2(\Omega, \mathbb{F}, \mathbb{Q}; \mathbb{R})$  denote the space of real valued square integrable random variables. We write  $\mathcal{L}[X, T]$  for the closure in  $L^2(\Omega, \mathbb{F}, \mathbb{Q}; \mathbb{R})$  of all linear combination  $c_0 + c_1 X_{t_1} + c_2 X_{t_2} + \dots + c_k X_{t_k}; 0 \leq t_i \leq T, c_0 \in \mathbb{R}$ , for any  $L \in \mathcal{C}[0, T]$  such that  $X_t$  has density  $f_t$  for every  $t \in (0, T]$ . The following proposition is needed in order to find a representation of processes in  $\mathcal{L}[L, T]$  in terms of functions in  $L^2(\Omega, \mathbb{F}, \mathbb{Q}; \mathbb{R})$  (see detailed proof in [27] page 12).

**Proposition 6.3.1.** *Let  $\mathbb{L}$  be defined by*

$$\frac{d\mathbb{L}}{d\mathbb{Q}} \Big|_{\mathcal{F}_t} = \psi_t(\mathbb{R}; Y_{tT})^{-1} \quad (6.15)$$

for  $t \in [0, T)$ . Then  $\mathbb{L}$  is a probability measure, under  $\mathbb{L}$ , the process  $\{Y_{tT}\}_{0 \leq t < T}$  is a Lévy process with density  $\{f_t\}$ , where  $\psi_t(dz; y) = \frac{f_{T-t}(z-y)}{f_T(z)} dz$ .

We are now in a position to present a representation found in [11] for functions in  $\mathcal{L}[Y, T]$ .

**Lemma 6.3.1.**  $\mathcal{L}[Y, T] = \{c_0 + \int_0^T g(t) dY_{tT} | g \in \mathcal{H}_2(T), c_0 \in \mathbb{R}\}$ , where  $\mathcal{H}_2(T)$  denotes the Hilbert space on  $[0, T]$ .

*Proof.* Let the right hand side be denoted by  $\mathcal{H}[Y, T]$ . Then we want to show the following;

- (i)  $\mathcal{H}[Y, T] \subset \mathcal{L}[Y, T]$

(ii)  $\mathcal{H}[Y, T]$  contains all linear combination of the form  $c_0 + c_1 Y_{t_1 T} + c_2 Y_{t_2 T} + \dots + c_k Y_{t_k T}; 0 \leq t_i < T$

(iii)  $\mathcal{H}[Y, T]$  is closed in  $L^2(\Omega, \mathbb{F}, \mathbb{Q}; \mathbb{R})$

and the result follows.

i) If  $g$  is continuous, then using dyadic intervals and limits in  $L^2(\Omega, \mathbb{F}, \mathbb{Q}; \mathbb{R})$ , we have that  $\int_0^T g(t) dY_{tT}$  is of the form  $c_1 Y_{t_1 T} + c_2 Y_{t_2 T} + \dots + c_k Y_{t_k T}; 0 \leq t_i < T$  (see [11], Lemma 6.2.4 for more details). If  $g$  is not continuous we can approximate with simple functions.

ii) Let  $0 = t_0 \leq t_1 < \dots < t_k \leq t_{k+1} = T, c_i := c'_{i-1} - c'_i, c_0 = 0$ , we write  $\Delta Y_T(j) := Y_{t_{j+1}T} - Y_{t_j T}$ , then we obtain

$$\begin{aligned} \sum_{i=1}^k c_i Y_{t_i T} &= \sum_{j=0}^{k-1} c'_j \Delta Y_T(j) \\ &= \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} c'_j dY_{tT} \\ &= \int_0^T \left( \sum_{j=0}^{k-1} \mathbb{1}_{\{t_j, t_{j+1}\}}(t) c'_j \right) dL_{tT} \end{aligned}$$

iii) Let  $H_n(T) = \left\{ \int_0^T g_n(t) dY_{tT}, n \in \mathbb{N} \right\}$  be a sequence which converges in  $L^2(\Omega, \mathbb{F}, \mathbb{Q}; \mathbb{R})$  to say  $H(T)$ . From Lemma 2.3.1 in [11] we have

$$\begin{aligned} C_2 \int_0^T |g_n(t) - g_m(t)|^2 dt &\leq \mathbb{E} \left[ \left| \int_0^T (g_n(t) - g_m(t)) dL_{tT} \right|^2 \right] \\ &\leq C_1 \int_0^T |g_n(t) - g_m(t)|^2 dt \end{aligned} \quad (6.16)$$

for  $C_1, C_2 > 0$ . It follows that  $\{g_n, n \in \mathbb{N}\}$  is Cauchy in  $\mathcal{H}_2(T)$  and so converges to  $g(t)$ . Taking limits as  $n \rightarrow \infty$  gives

$$\begin{aligned} C_2 \int_0^T |g_n(t) - g(t)|^2 dt &\leq \mathbb{E} \left[ \left| \int_0^T (g_n(t) dY_{tT} - H(T)) \right|^2 \right] \\ &\leq C_1 \int_0^T |g_n(t) - g(t)|^2 dt. \end{aligned} \quad (6.17)$$

We also have

$$\begin{aligned} C_2 \int_0^T |g_n(t) - g(t)|^2 dt &\leq \mathbb{E} \left[ \left| \int_0^T (g_n(t) - g(t)) dY_{tT} \right|^2 \right] \\ &\leq C_1 \int_0^T |g_n(t) - g(t)|^2 dt. \end{aligned} \quad (6.18)$$

Then the result follows from the uniqueness of limits in  $L^2(\Omega, \mathbb{F}, \mathbb{Q}; \mathbb{R})$  and we obtain

$$H(T) = \int_0^T g(t) dY_{tT}$$

□

Let  $\mathcal{P}_Y$  denote the projection from  $L^2(\Omega, \mathbb{F}, \mathbb{Q}; \mathbb{R})$  onto  $\mathcal{L}[Y, T]$ . Then we have for  $0 \leq t \leq T$ , that  $\hat{X}_t$  is the projection of  $X_t$  onto  $\mathcal{L}[Y, T]$ . That is  $\hat{X}_t = \mathcal{P}_Y(X_t)$ . This implies that the random variable,  $\bar{X}_t = X_t - \hat{X}_t = (I - \mathcal{P}_Y)X_t$  and hence is orthogonal to  $\mathcal{L}[Y, T]$ . The main question that we want to answer becomes; given the observations  $\{Y_{sT}\}_{0 \leq s \leq t}$ , defined through (6.9), what is the best estimate (*in mean square sense*)  $\hat{X}_t$  of the unobservable  $X_t$ .

## The Innovation Process

The innovation process  $\{Z_{tT}\}$  is defined by

$$Z_{tT} = Y_{tT} - \int_0^t \sigma_s \hat{H}_{sT} ds$$

then we have

$$dZ_{tT} = \sigma_t(X_t - \hat{X}_t)dt + \eta_t dL_{tT}, \quad 0 \leq t \leq T \quad (6.19)$$

**Corollary 6.3.1.** *There exists a probability measure under which the process  $\{Z_{tT}\}_{0 \leq t < T}$  is a Lévy process with density  $\{f_t\}$ .*

*Proof.* This is a direct consequence of proposition 6.3.1. □

The following Lemma will be used in determining the first and second moments of the innovation process.

**Lemma 6.3.2.**

$$\mathbb{E}[\hat{H}_{tT}] = \mathbb{E}[H_{tT}], \quad \forall 0 \leq t \leq T.$$

*Proof.* Using corollary 6.3.1, the result follows from Lemma 2.3.4 in [11]. □

Let a process  $\{M_{tT}\}_{0 \leq t \leq T}$  be defined by

$$dM_{tT} = \eta_0 + \eta_t dZ_{tT} \quad (6.20)$$

Then the following proposition holds;

**Proposition 6.3.2.**

$$\mathcal{L}[M, T] = \mathcal{L}[Y, T]$$

*Proof.*

$$\begin{aligned}
c_0 + c_1 M_{t_1 T} + \dots + c_k M_{t_k T} &= c_0 + c_1 (\eta_0 + \eta_{t_1} Z_{t_1 T}) + \\
&\quad \dots + c_k (\eta_0 + \eta_{t_k} Z_{t_k T}) \\
&= c_0 + \sum_{i=1}^k c_i \eta_0 + c_1 \eta_{t_1} \left( Y_{t_1 T} - \int_0^{t_1} \sigma_s \hat{H}_s ds \right) \\
&\quad + \dots + c_k \eta_{t_k} \left( Y_{t_k T} - \int_0^{t_k} \sigma_s \hat{H}_s ds \right) \\
&= c_0 + \sum_{i=1}^k c_i \eta_{t_i} Y_{t_i T} + \sum_{i=1}^k c_i \eta_0 - \sum_{i=1}^k c_i \eta_{t_i} \int_0^{t_i} \sigma_s \hat{H}_s ds \\
&\subset \mathcal{L}[Y, T]
\end{aligned}$$

□

The reverse inclusion follows similar argument.

The next corollary allows us to use the technology of theorem 6.2.2 to derive the associated filtering equations for the case where the observation noise is given by a Lévy Bridge process.

**Corollary 6.3.2.**

$$\mathcal{L}[M, T] = \left\{ c_0 + \int_0^T g'(t) dZ_{tT} \mid g' \in \mathcal{H}(T), c_0 \in \mathbb{R} \right\}$$

*Proof.* This is a direct consequence of Proposition 6.3.2 and Lemma 6.3.1

□

In what follows we derive explicit continuous time linear filters for  $H_{tT}$ .

## 6.4 Kalman-Bucy Filter for Lévy Bridge Noise

In this section we describe the linear Kalman filter for Lévy bridge noise processes. To assist with the construction, we start with a general specification of the model for a

Lévy process in the class  $\mathcal{C}[0, T]$ . Two specific examples of processes in  $\mathcal{C}[0, T]$  are the Brownian and Gamma bridges. The unobservable process is given by (6.7) while on the other hand the observation process is given by (6.9). These representations, coupled with the results of the previous sections allow us to construct explicit expressions for the transition and measurement update equations for conditional distribution of the unobservable random cash flows at each time  $t < T$ . Our method uses the orthogonal projection approach for linear filters in [11] and the measure change of Proposition 3.2.5 in chapter two.

Let  $q(X_t|Y_{tT})$  denote the conditional density of the unobservable process  $X_t$  given the filtration generated by the observation process  $Y_{tT}$ . We define the prior process estimate as

$$\hat{X}_{t|t-1} = \mathbb{E}(X_t|Y_{t-1T}). \quad (6.21)$$

This represents the estimate of  $X_t$  at time  $t - 1$  before the current time  $t$  observation. Similarly, we define the posterior estimate as

$$\hat{X}_{t|t} = \mathbb{E}(X_t|Y_{tT}). \quad (6.22)$$

The posterior estimate represents the estimate at time  $t$  after the observation has been made. In the same manner, we denote the corresponding estimation errors as  $e_{t|t-1} = X_t - \hat{X}_{t|t-1}$  and  $e_t = X_t - \hat{X}_t$ . Then the estimate of the variances are given by

$$S_{t|t-1} = \mathbb{E}[e_{t|t-1}^2], \quad S_{t|t} = \mathbb{E}[e_t^2] \quad (6.23)$$

for  $0 \leq t < T$ . The computation of the above means and variances requires the corresponding transition and measurement conditional densities  $q(X_t|Y_{t-1T})$  and  $q(X_t|Y_{tT})$  respectively. We determine this through the transition and measurement update steps in the Kalman filter equations. The transition update step gives

$$\begin{aligned}
q(X_t|Y_{t-1T}) &= \int q(X_t|X_{t-1}, Y_{t-1T})q(X_{t-1}|Y_{t-1T})dX_{t-1} \\
&= \int q(X_t|X_{t-1})q(X_{t-1}|Y_{t-1T})dX_{t-1} \\
&= \int q(X_{t-1}|Y_{t-1T})dX_{t-1}
\end{aligned} \tag{6.24}$$

On the other, the observation update step gives

$$q(X_t|Y_{tT}) = \frac{q(Y_{tT}|X_t)q(X_t|Y_{t-1T})}{\int q(Y_{tT}|X_t)q(X_t|Y_{t-1T})dX_t} \tag{6.25}$$

The transition update is based on the Chapman-Kolmogorov identity and the Markov property of the observation process while the observation step follows from the Bayes rule. Additional information on the derivation can be found on page 195 in [49].

The following lemma gives a representation of  $\hat{X}_t$  from theorem 6.2.2 more explicitly.

**Lemma 6.4.1.** *For all  $0 \leq t \leq T$ , we have*

$$\hat{X}_t = \hat{X}_0 + \int_0^t \exp\left(\int_s^t \lambda_u du\right) V_s \sigma_s \Sigma_s^{-1} dM_{sT} \tag{6.26}$$

where

$$V_s = \mathbb{E}[(X_s - \hat{X}_s)^2] \tag{6.27}$$

represents the mean square error and

$$\Sigma_s = \mathbb{E}[M_{tT}M_{sT}]$$

*Proof.* Let  $J(s, t) = \frac{\partial}{\partial s} \frac{\mathbb{E}[X_t M_{sT}]}{\mathbb{E}[X_s^2]}$  for  $s \leq t$ . Then from Proposition 6.3.2 above and Lemma 2.3.8 in [11] we have

$$\hat{X}_t = c_0(t) + \int_0^t J(s, t) dM_{tT}, \quad c_0(t) \in \mathbb{R} \tag{6.28}$$

Now, taking expectations we get from Lemma 6.3.2,

$$c_0(t) = \mathbb{E}[\hat{X}_t] = \mathbb{E}[X_t].$$

from (6.19) we have

$$M_{sT} = \int_0^s \eta_r \sigma_r (X_r - \hat{X}_r) dr + \int_0^s \eta_r L_{rT}$$

since  $X$  and  $L_{tT}$  are independent we obtain

$$\mathbb{E}[X_t M_{sT}] = \int_0^s \mathbb{E}[X_t (X_r - \hat{X}_r)] \sigma_r \eta_r dr. \quad (6.29)$$

Using Lemma 2.2.1 in [11] we have

$$\mathbb{E}[X_r (X_r - \hat{X}_r)] = V_r \quad (6.30)$$

Substituting (6.30) into (6.29) we get

$$\mathbb{E}[X_t M_{sT}] = \int_0^s \int_r^s \exp(\lambda_u du) V_r \sigma_r \eta_r dr$$

then

$$\frac{\partial}{\partial s} \mathbb{E}[X_t M_{sT}] = \int_r^s \exp(\lambda_u du) V_s \sigma_s \eta_s$$

and so

$$J(s, t) = \int_r^s \exp(\lambda_u du) V_s \sigma_s \eta_s \Sigma_s^{-1} \quad (6.31)$$

and the result follows.  $\square$

**Remark 6.4.1.** The process  $\{V_s\}_{0 \leq s \leq T}$  is a supermartingale since it can be expressed as the difference between a martingale and a submartingale. That is  $V_s = \mathbb{E}[X_s^2] - (\hat{X}_s)^2$ .

Next we show that the mean square error,  $V_s$  satisfies a deterministic Reccatti equation.

**Theorem 6.4.1.**

$$\frac{dV_t}{dt} = 2\lambda_t V_t - \left( \frac{V_t \sigma_t \eta_t}{\Sigma_t} \right)^2.$$

*Proof.* We have

$$\mathbb{E}[(X_t - \hat{X}_t)(X_t - \hat{X}_t)] = \mathbb{E}[X_t^2] - \mathbb{E}[\hat{X}_t^2],$$

then from (6.28) and ito's isometry,

$$\begin{aligned} V_t &= \mathbb{E}[X_t^2] - (\mathbb{E}[X_t])^2 - \int_0^t J(s, t)^2 \Sigma_s ds \\ &= S_t - \int_0^t J(s, t)^2 \Sigma_s ds - \mathbb{E}[X_t] \mathbb{E}[X_t] \end{aligned} \quad (6.32)$$

where  $S_t = (\mathbb{E}[X_t])^2$ . Then Lemma 2.4.1 in [11] implies that

$$S_t = \exp\left(2 \int_0^t \lambda_s ds\right) \mathbb{E}[X_0^2] + \int_0^t \exp\left(2 \int_s^t \lambda_u du\right) ds$$

upon differentiating we have

$$\frac{dS_t}{dt} = 2\lambda_t S_t$$

similarly, we have

$$\frac{d}{dt} \mathbb{E}[X_t] \mathbb{E}[X_t] = 2\lambda_t (\mathbb{E}[X_t])^2$$

then

$$\begin{aligned} \frac{dV_t}{dt} &= \frac{dS_t}{dt} - J(t, t)^2 \Sigma_t - \int_0^t J(s, t) \Sigma_t \frac{\partial}{\partial s} J(s, t) ds \\ &\quad - \int_0^t \frac{\partial}{\partial s} J(s, t) \Sigma_t J(s, t) ds - 2\lambda_t \mathbb{E}[X_t]^2 \\ &= 2\lambda_t S_t - V_t \sigma_t \eta_t \Sigma_t \eta_t \sigma_t V_t \\ &\quad - 2\lambda_t \int_0^t J(s, t) \Sigma_t J(s, t) ds - 2\lambda_t \mathbb{E}[X_t]^2 \\ &= 2\lambda_t V_t - V_t \sigma_t \eta_t \Sigma_t \eta_t \sigma_t V_t \end{aligned}$$

as required. □

Finally, we are now in a position to find the Stochastic Differential Equation satisfied by  $\hat{X}_t$ .

**Theorem 6.4.2.** For all  $(0 \leq t < T)$

$$d\hat{X}_t = \lambda_t \hat{X}_t dt + \frac{V_t \sigma_t}{\Sigma_t} \eta_t \left[ dY_{tT} - \sigma_t \hat{X}_t dt \right] \quad (6.33)$$

with initial condition  $X_0 = \mathbb{E}[X_0]$  and  $V_0 = \mathbb{E}[(X_0 - \mathbb{E}[X_0])^2]$

*Proof.* from (6.28) we have

$$\hat{X}_t = c_0(t) + \int_0^t J(s, t) dM_{sT}, \quad c_0(t) = \mathbb{E}[X_t]$$

then it follows that

$$d\hat{X}_t = \frac{dc_0(t)}{dt} dt + J(s, t) dM_{tT} + \left( \int_0^t \frac{\partial}{\partial t} J(s, t) dM_{sT} \right) dt$$

so by (6.31) and the fact that  $\frac{dc_0(t)}{dt} = \frac{d}{dt} \mathbb{E}[X_t] = c_0(t)$ , we get

$$\begin{aligned} d\hat{X}_t &= \frac{dc_0(t)}{dt} dt + V_t \sigma_t \eta_t \Sigma_t^{-1} dM_{sT} + (J(s, t) dM_{tT}) dt \\ &= \frac{dc_0(t)}{dt} dt + (\hat{X}_t - c_0(t)) dt + V_t \sigma_t \eta_t \Sigma_t^{-1} dM_{tT} \\ &= \hat{X}_t dt + V_t \sigma_t \eta_t \Sigma_t^{-1} dM_{tT} \end{aligned}$$

Using (6.20) we have

$$\begin{aligned} dM_{tT} &= \eta_t dZ_{tT} \\ &= \eta_t [\sigma_t (X_t - \hat{X}_t) dt + dL_{tT}] \\ &= \eta_t [\sigma_t X_t dt + dL_{tT} - \sigma_t \hat{X}_t dt] \\ &= \eta_t [dY_{tT} - \sigma_t \hat{X}_t dt] \end{aligned}$$

from where we conclude that

$$d\hat{X}_t = \hat{X}_t dt + V_t \sigma_t \eta_t \Sigma_t^{-1} \eta_t [dY_{tT} - \sigma_t \hat{X}_t dt]$$

□

**Example 6.4.1.** *In the next section, we present a numerical example for the case of a Brownian bridge noise.*

### 6.4.1 Numerical Simulation

The objective in this section is to present a simple numerical implementation of the Kalman filter algorithm of the previous sections. We simulate the price of a credit risky bond for given maturities under the information based pricing framework. The example we present here is based on the basic assumptions of the information based asset price modelling framework; the financial market is modelled with the specification of a probability space,  $(\Omega, \mathcal{F}, \mathbb{Q})$  on which the filtration  $\{\mathcal{F}_t\}_{0 \leq t < \infty}$  is constructed. The probability measure is understood to be the risk-neutral measure and the filtration  $\{\mathcal{F}\}$  is the market filtration, which is the only observable information to market participants. We assume a deterministic default-free system of interest rate (discount bond), with the discount functions  $\{P_{tT}\}_{0 \leq t < \infty}$  written in the form

$$P_{tT} = \frac{P_{0T}}{P_{0t}}, \quad t \leq T$$

where  $0 < P_{0t} \leq 1$ ,  $\lim_{t \rightarrow \infty} P_{0t} = 0$  and  $P_{TT} = 1$ . Under this assumption we have that if the integrable random variable  $H$  represents a cash flow occurring at time  $T$ , which depends on the value of an economic factor process  $\{X_t\}$  at time  $T$ , then its price at time  $t < T$  is given by

$$H_{tT} = P_{tT} \mathbb{E}[H | \mathcal{F}_t] \tag{6.34}$$

We assume  $X$  is has the following form

$$X_t = X_0 - \int_0^t X_s ds + B_t \quad (6.35)$$

where  $\{B_t\}$  is a Brownian motion and  $X_0$  is a standard Gaussian random variable which is independent of  $(B_t, t > 0)$ . For simplicity, we further assume that  $H = X_T$ . The factor process is not observable at time  $t < T$ . From (6.34), the value of the cash flow at this earlier time is then given by,

$$H_{tT} = P_{tT} \mathbb{E}[X_T | \mathcal{F}_t] \quad (6.36)$$

Notice that  $H_{TT} = X_T$  by definition. The market participants observe partial information concerning the bond cash flow of the form

$$dY_{tT} = \alpha \frac{t}{T} X_t dt + d\beta_{tT} \quad (6.37)$$

for  $t \in [0, T]$ , where  $\alpha$  is a constant and  $\{\beta_{tT}\}$  is a standard Brownian bridge process independent of  $\{B\}$ . Clearly,  $Y$  is a random bridge process (see [27], page 19). The parameter  $\alpha$  controls the information "content" of  $\{Y_{tT}\}$ . In other words,  $\alpha$  governs the speed of convergence of the filter estimates. This is evident in the figures that follow. Precisely, we investigate the values of the result in (6.33) as the price estimates of a credit risky bond for different values of  $\alpha$ . Each figure compares the error estimates of the observation and the Kalman filter to the true bond prices.

## 6.4.2 Random Bridge Observation of a Brownian Motion

In this simple example we look at the filtering of a mean-reverting Brownian motion (6.35) from a *Brownian random bridge* observations given by (6.37) with the last observation to occur at time  $T$  (to simplify things, we take  $T = 1$ ). In order to proceed we will make use of the commonly used Euler approximation (see [55], page 305) in simulating

the paths of the unobservable factor process as well as the filter estimates. For the observation process we employ the bridge approximation of Lévy process algorithm on page 259 in [81].

Now we have for the filter described in Theorems 6.4.1 and 6.4.2 that  $V_t$  has the form of a *Bernoulli's equation*,

$$\frac{dV_t}{dt} = 2V_t - \alpha^2 V_t^2, \text{ where } V_0 = 1 \quad (6.38)$$

which has the following solution for each  $t \in (0, T]$

$$V_t = \frac{2 \exp(2t)}{\alpha^2 [\exp(2t) - 1] + 2}$$

Then we compute the filter estimate using (6.33) which gives,

$$d\hat{X}_t = \hat{X}_t dt + V_t \alpha [dY_{tT} - \alpha \hat{X}_t dt]$$

To simulate the unobservable process, we use the Euler scheme which ultimately amounts to simulating the increment of a standard Brownian motion as follows

$$B_{t+\Delta t} - B_t \approx \sqrt{\Delta t} \epsilon$$

where  $\epsilon \sim \mathcal{N}(0, 1)$ , the density of a standard normal random variable.

We simulate the observation process using the following method

$$Y_T(i\Delta t) = \frac{1}{2} [Y_T((i-1)\Delta t) + Y_T((i+1)\Delta t)] + \sqrt{\frac{\Delta t}{2}} \epsilon \quad (6.39)$$

where  $\Delta t = \frac{T}{2^m}$ ,  $i = 1, \dots, 2^{m-1}$  and we assume that  $Y_T(0) = X_0$  and  $Y_T(m\Delta t) = \sqrt{T}\epsilon$ . This method simulates the sample path  $Y_T(i\Delta t)$  on the time horizon  $[0, T]$  as follows: First, we set  $Y_T(0) = 0$  and  $Y_T(T) = \sqrt{T}\epsilon$ . Then set the midpoint  $Y_T(\frac{T}{2})$  using 6.39.

Next, we find the midpoints for  $[Y_T(0), Y_T(\frac{T}{2})]$  and  $[Y_T(\frac{T}{2}), Y_T(\frac{T}{4})]$  i.e.  $Y_T(\frac{T}{4})$  and  $Y_T(\frac{3T}{4})$  respectively. Repeat the procedure for  $m - 2$  more times. Intuitively, this method can potentially increase the accuracy of the simulations as it takes into account the random bridge representation of the observation process and also reduces its effective dimension<sup>2</sup>. We discretize the interval  $[0, T]$  by setting the step size  $\Delta t = 0.01$  corresponding to the inter-arrival times of the observations. and for generate 100,000 Monte-Carlo simulations.

In all the figures below, we have chosen the following values: Initial default probability is 10%. Bond maturity is ten years, discount rate is at 5%. These simulations illustrate the effect on the default probability of the bond when the information flow rate is increased from a low rate ( $\alpha = 0.2$ ) up to a high rate ( $\alpha = 5$ ). For a given information flow rate, each figure shows the difference in the default probability based on the simulated noise information in comparison to its Kalman filter estimate. Figure 1.1 corresponds to the situation where the investors information set contains very little useful knowledge ( $\alpha = 0.2$ ) about the true a posteriori default probability of the bond. In this case Kalman filter gives a weak estimate of the default probability resulting to high estimation errors. In figure 1.3 on the other hand, the information set contains a considerable amount of useful information ( $\alpha = 2$ ). In this scenario, the Kalman filter performance improved to a great extent, resulting in the low error estimates of the a posteriori default probabilities. With high information flow rate ( $\alpha = 4$  and 5) in figures 1.4 and 1.5 respectively, the trajectories for the Kalman filter error estimates become highly deterministic. This behavior is expected because if the simulated information process contains good enough amount of useful information regarding the future cash flow of the bond, the Kalman filter estimate of default probability converges to the deterministic interest rate system  $P_{tT}$  as defined in the previous section.

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<sup>2</sup>It should be very desirable to conduct a formal comparative study to verify this hypothesis.

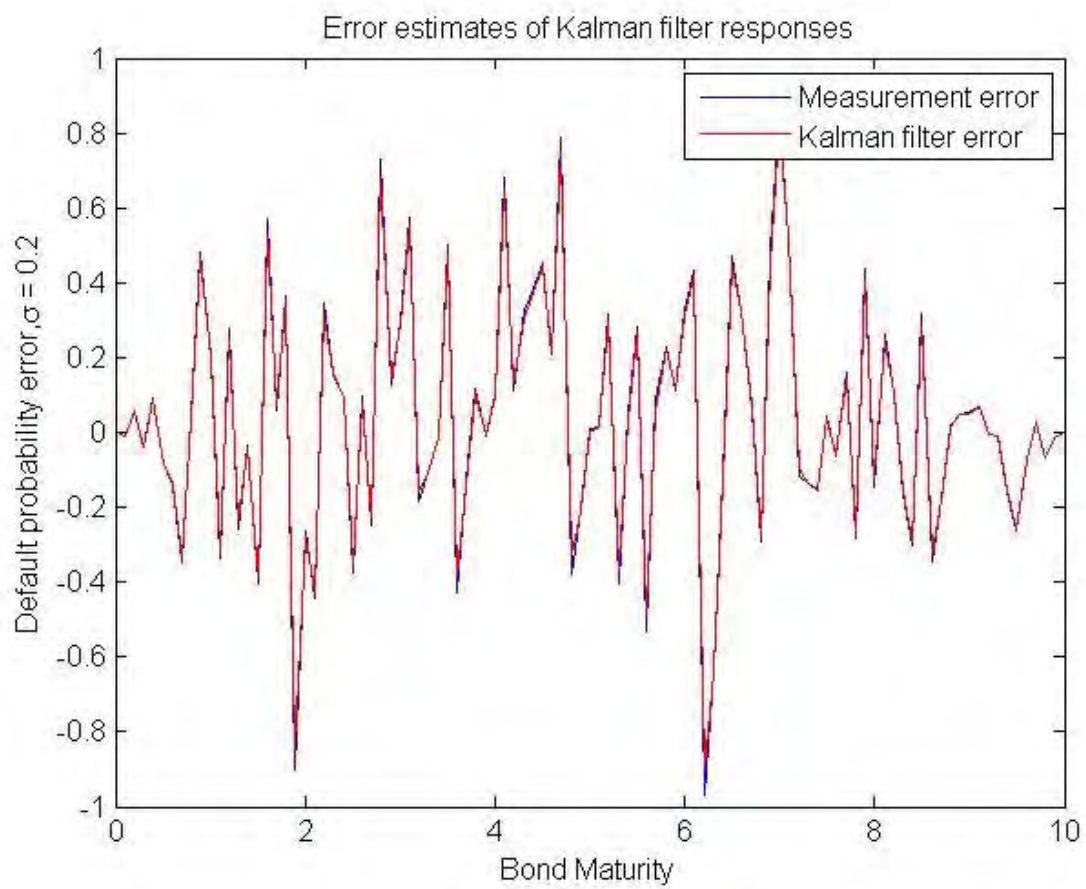


FIGURE 6.1: Default probability response:  $\alpha = 0.2$ . Initial default probability is 10%. Bond maturity is ten years, discount rate is at 5%, and information flow rate is 0.2.

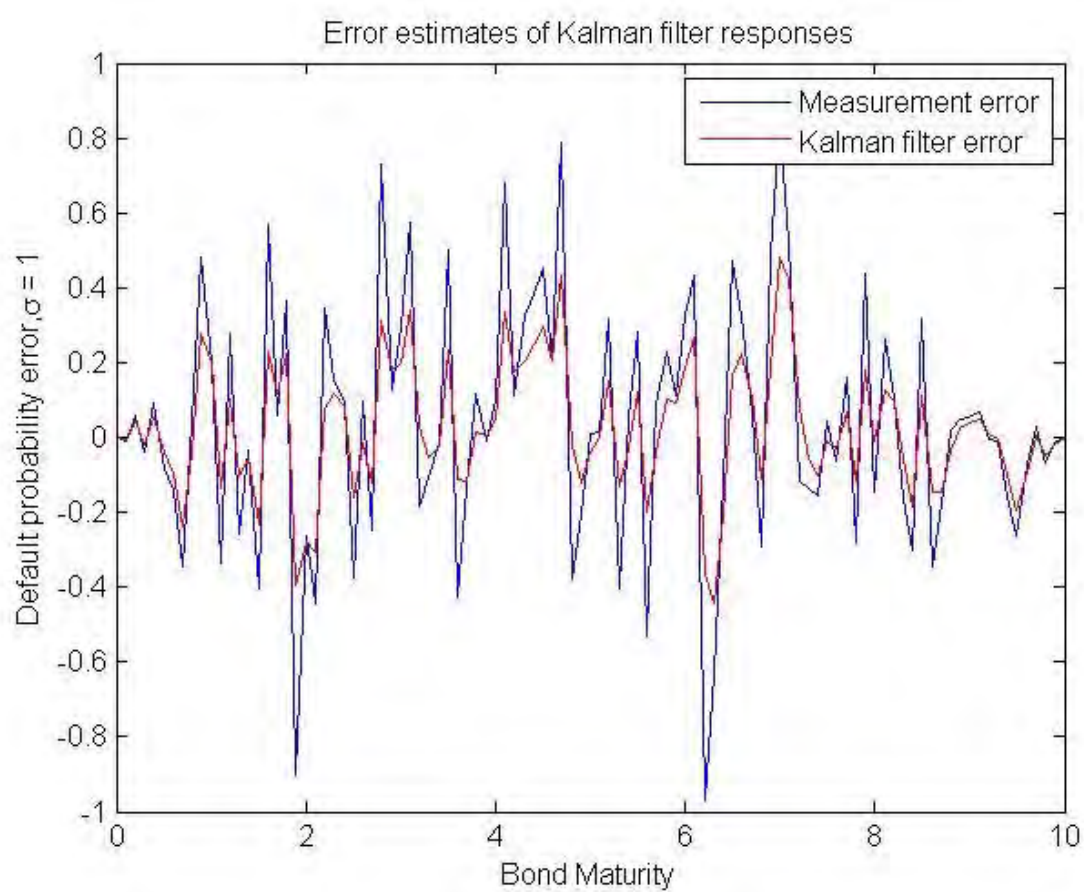


FIGURE 6.2: Default probability response:  $\alpha = 1$ . Initial default probability is 10%. Bond maturity is ten years, discount rate is at 5%, and information flow rate is 1.

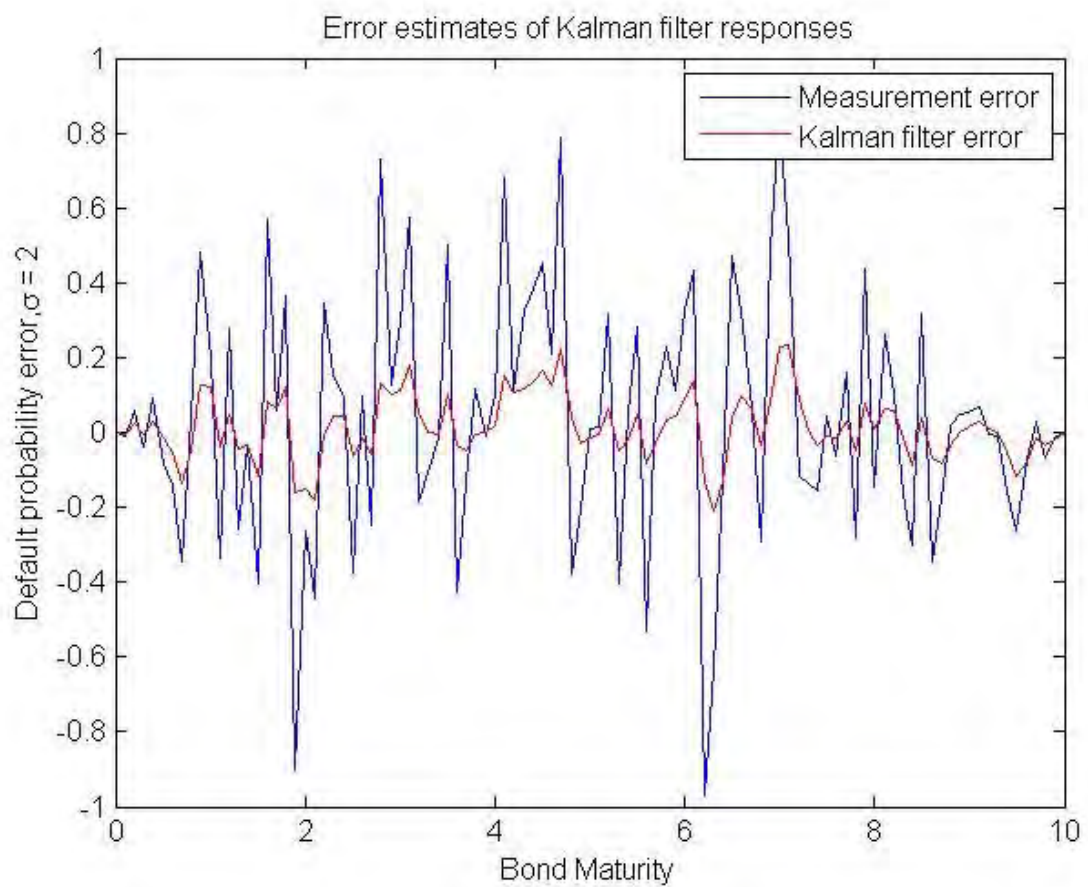


FIGURE 6.3: Default probability response:  $\alpha = 2$ . Initial default probability is 10%. Bond maturity is ten years, discount rate is at 5%, , and information flow rate is 2.

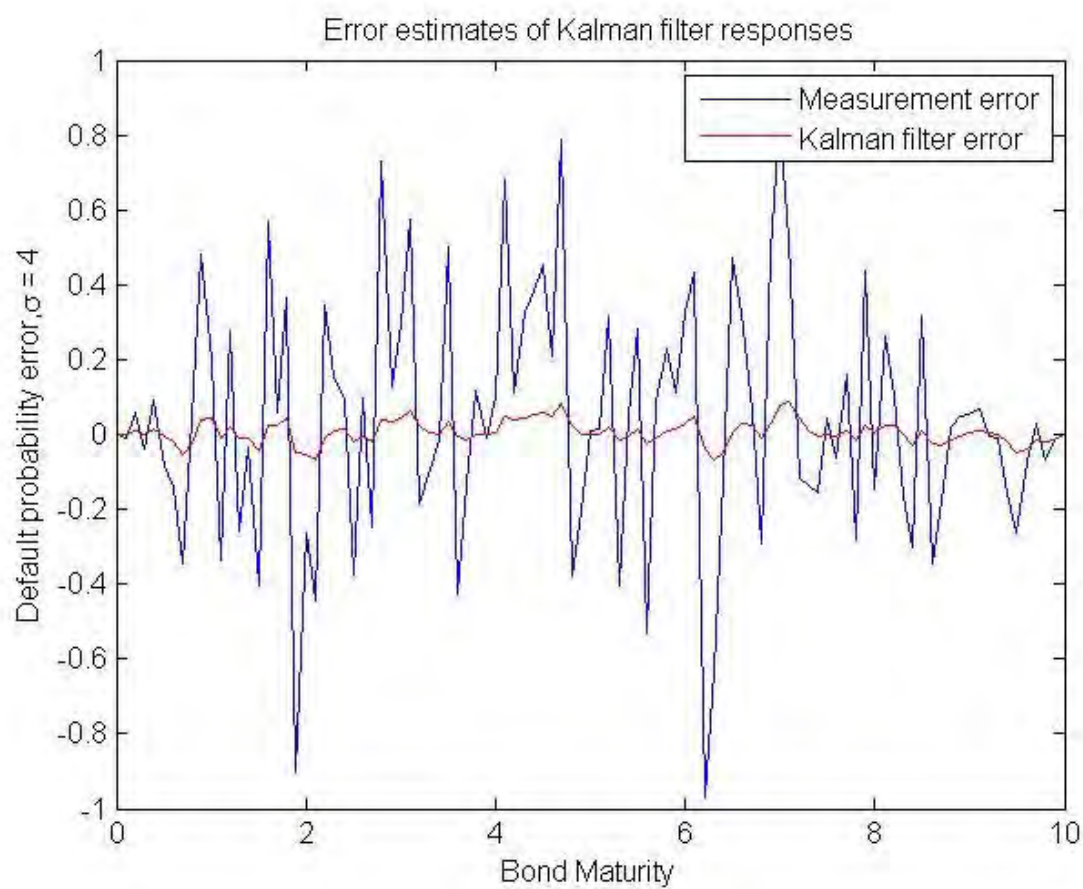


FIGURE 6.4: Default probability response:  $\alpha = 4$ . Initial default probability is 10%. Bond maturity is ten years, discount rate is at 5%, and information flow rate is 4.

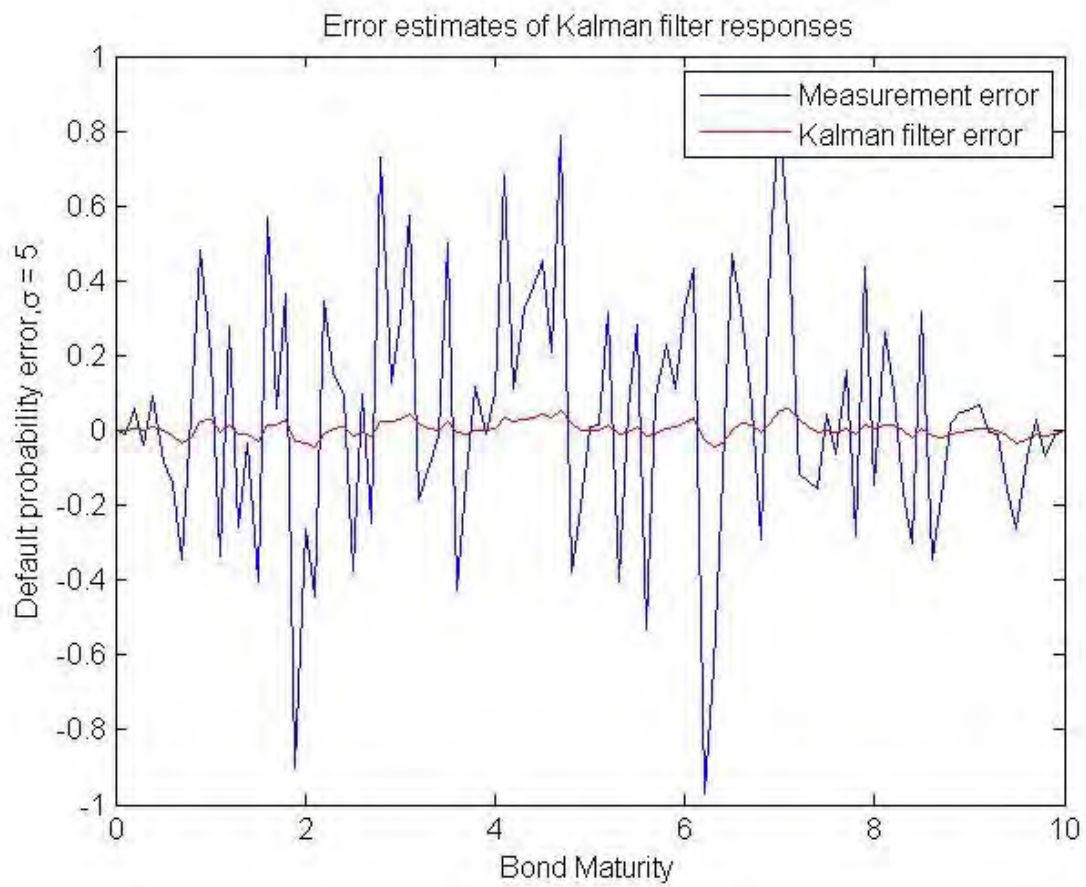


FIGURE 6.5: Default probability response:  $\alpha = 5$ . Initial default probability is 10%. Bond maturity is ten years, discount rate is at 5%, and information flow rate is 5.



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