Thesis for the degree of Master of Science

1+1+2 covariant approach to Gravitational Lensing in $f(R)$ Gravity

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Declaration

The work presented in this thesis is partly based on collaborations with my supervisor Prof. Peter Dunsby together with Dr. Rituparno Goswami (both of Department of Mathematics and Applied Mathematics, University of Cape Town), and Dr. Sante Carloni (Institut d'Estudis Espacials de Catalunya, Spain).
Part of the work has been published in the paper *1+1+2 covariant formalism in* $f(R)$ *Gravity: Spherically symmetric static solutions* (submitted to Physical Review D). Sections (4.4, 4.5 and 4.6), are based on it.
I herby declare that the presented thesis has not previously been submitted to this or any other university for a degree and that it represents my own work.

Anne Marie Nzioki

Abstract

In this thesis, we develop the 1+1+2 formalism, a technique originally devised for General Relativity, to treat spherically symmetric spacetimes in for fourth order theories of gravity. Using this formalism, we derive equations for a static and spherically symmetric spacetime for general $f(R)$ gravity. We apply these master eqautions to derive some exact solutions, which are used to gain insight on Birkhoff's theorem in this framework. Additionally, we derive a covariant form of the lensing angle for a specific spherically symmetric solution in $f(R) = R^n$ gravity.

Keywords: $f(R)$ gravity, spherically symmetric solutions, Birkhoff's theorem, deflection angle

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Conventions and Abbreviations

Sign conventions

Signature:	$[-,+,+,+]$.
Natural units:	$\hbar = c = k = 8\pi G = 1$.
Latin indices:	0, 1, 2, 3.

Sign conventions follow Ellis [10] and Ellis and van Elst [11].
For a tensor $T^{a..b}{}_{c..d..e..f}$ we have:

symmetrization:	$T_{(ab)} = \frac{1}{2}(T_{ab} + T_{ba})$,
antisymmetrization:	$T_{[ab]} = \frac{1}{2}(T_{ab} - T_{ba})$,

over the indexes of the tensor. The symbol ∇ represents the usual covariant derivative and ∂ corresponds to partial differentiation.
The Riemann tensor is defined by

$$R^a{}_{bcd} = \Gamma^a{}_{bd,c} - \Gamma^a{}_{bc,d} + \Gamma^e{}_{bd}\Gamma^a{}_{ce} - \Gamma^e{}_{bc}\Gamma^a{}_{de} \ ,$$

and $\Gamma^a{}_{bd}$ are the Christoffel symbols (i.e. symmetric in the lower indices), defined by

$$\Gamma^a_{bd} = \frac{1}{2} g^{ae} \left(g_{be,d} + g_{ed,b} - g_{bd,e} \right) \ .$$

The Ricci tensor is obtained by contracting the *first* and the *third* indices

$$R_{ab} = g^{cd} R_{acbd} \ .$$

Abbreviations

GL	Gravitational lensing
GR	General relativity
VLBI	very long baseline interferometry
QSO	Quasars
LRS	Local rotational symmetry
CMBR	Cosmic Microwave Background Radiation
DE	Dark energy
CDM	Cold dark matter
FLRW	Friedmann-Lemaitre Robertson Walker
SPSTF	screen-space projected symmetric and trace-free
SEC	Strong energy condition
WEC	Weak energy condition

Chapter 1

Introduction

The influence of gravity on light was considered by Einstein three years before completing his theory of relativity [1], though the possibility had been explored earlier on (see [2] for the history). The first observational evidence of bending of light was provided Eddington et al [3] by observing light rays grazing the sun surface during a total solar eclipse. The deflection of light rays had been predicted in GR and this experiment provided the first experimental verification the theory. Eddington was to later on predict that multiple images can occur when two stars are aligned suitably, with one of the stars acting as the deflector in between the other star and the observer [4]. This deflection of light by a massive body is referred to as gravitational lensing (GL).

A good theory of gravity principally develops with the agreement of the theory with observations. GL provides one such test for the underlying theory of gravity and has had important astrophysical and cosmological consequences for GR. Experiments using very long baseline interferometry (VLBI) have been performed to measure the gravitational deflection of the position of quasars (QSOs) [5] during the time they are eclipsed by the Sun that give results in agreement with that of the GR value of 1.75". GL studies have been used to constrain matter distribution of the universe (and other cosmological parameters) to about 30% of the energy density of the universe, $\Omega_m \sim 0.26$ [6]. Also, GL of QSOs has placed a limit on the cosmological constant (the dark energy component attributed to the accelerated expansion of the universe) at around 70% of the universe, $\Omega_\Lambda \sim 0.74$ and therefore these tests should be confronted with changes in the background geometry within GR context and/or modifications in the underlying theory of gravity itself. The results mentioned above are based on GR and are model dependent. The relative success of the results in the GR context demonstrates that the theory of gravity considered may be effectively constrained through these tests.

Since the lensing effect is dependent on the underlying theory of gravity, considering a modified theory of gravity would result in deviations from the standard expression of the deflection angle and is worth investigating. The theory of gravity must account for the bending and distribution of light, corresponding to null geodesics and solutions to the geodesic equations. These equations require the knowledge of a metric, and in the context of GR, the Schwarzschild metric represents the gravitational field outside a static spherically symmetric mass distribution like a non-rotating star or black hole. Therefore, spherically symmetric solutions are needed as a generalization to modify gravity from GR and in order to check the compatibility of these theories through experimental tests of gravity like the bending of light. GL in fourth order theories of gravity has already been considered in [7, 8]

In this thesis, we use the covariant method to find propagation equations for variables in a spherically symmetric static spacetime as applied to fourth order gravity theory. The determined solutions are then used to obtain the general expression of the deflection angle in such theories.

1.1 Outline of the thesis

In chapter 2, a review of fourth order gravity is provided, with a focus on the $f(R)$ theory of gravity. The chapter ends with a presentation of the field equations for $f(R)$ gravity.

Chapter 3 serves as a review to the covariant approach. We give an outline of the 1+3 covariant approach based on [9–11], where a time-like congruence is defined such that the spacetime is split relative to a timelike vector field and an orthogonal 3-space hypersurface. The propagation, evolution, energy and constraint equations are defined, showing how the kinematical and dynamical quantities change along the congruence. Since light rays move along null geodesics, we apply the 1+3 approach to the geometry of these geodesics, where the screen-space is defined as a two-dimensional plane orthogonal to the null geodesic tangent vector. The 1+3 analysis is then extended further to obtain a 1+1+2 decomposition in which the three spatial degrees of freedom are further decomposed relative to a spatial vector, providing a natural foundation for studying spherically symmetric spacetimes. A detailed discussion of the 1+1+2 formalism presented here can be seen in [59, 61], from which most of the material presented in section 3.3 has been drawn

In chapter 4, we restrict the analysis to locally rotational symmetric (LRS) spacetimes and consider in particular the LRS class II spacetime that's characterized by a vanishing vorticity. The spacetime is constrained further by including spherical symmetry in order to determine a set of general results in $f(R)$ gravity and to admit spherically symmetric solutions. Spherical symmetry implies that the vector and tensor 1+1+2 variables vanish, leaving only a set of scalar variables. After imposing vacuum and static conditions, this reduces the set of propagation equations to four master equations which provide a way of obtaining covariant results for spherically symmetric systems in $f(R)$ gravity. We choose a proper radial coordinate system to be able to express the master equations as full differential equations that can be solved. In this coordinate system, the scalar variables that constitute the equations can be expressed as relations for the general metric coefficients of the general spherically symmetric static metric. We present exact solutions in the spherically symmetric spacetime by specializing to R^n form of $f(R)$ theory and imposing various conditions.

In chapter 5, in analogy to the 1+3 null geodesic presentation, the 1+1+2 splitting of the null vector is constructed. The resulting lensing variables and their propagation equations comprise the scalar variables that define the spherically symmetric spacetime propagation equations. The general expression of the deflection angle that is characterized by the lensing variables is defined and it's form obtained for a spherically symmetric solution of R^n gravity.

Chapter 6 summarizes the thesis and contain conclusions and an outlook for further work.

Chapter 2

$f(R)$ Gravity

2.1 Introduction

Modifications of General Relativity (GR) were proposed by Herman Weyl [13] as early as 1918, just three years after Einstein completed his theory. Weyl's motivation stemmed from a desire to obtain a unified field theory and proposed modifications to GR by including higher order invariants in its action. He extended the geometrical representation of GR to account not just for gravitational but also electromagnetic fields. In 1921, Arthur Eddington, also began to consider fourth order theories of gravity and went on to publish *The Mathematical Theory of Relativity* [14] in 1923 containing his work on generalized versions of Weyl's theory. Since then there has been a great number of proposed higher order theories of gravity that propose modification of the Einstein-Hilbert action.

Early motivation to consider modified theories of gravity was primarily due limitations when considering strong gravity regimes important on scales close to the Planck length. Utiyama and DeWitt [15], showed that fourth order theories are renormalizable at the one-loop quantum level and results from string theory or quantum corrections have also shown that the effective low energy gravitational action admits higher order invariants. Fourth order gravity considerations were consequently limited to the early universe and provided for example, a nice geometrical explanation for inflation [16, 17] in cosmology. Studies were also relevant in attempts to avoid cosmological singularities like that of the Big Bang in GR and astrophysical singularities like black holes [18] (see [19] for a detailed review of $f(R)$ theories of gravity). More recently however, the corrections to GR have been introduced to accommodate recent observations and more so to account for the "dark" sector of the universe. Studies of weak gravitational lensing, the cosmic microwave background radiation (CMBR) and supernovae surveys, [6, 20–24], indicate that the energy density budget of the universe is 5% ordinary matter (baryons, radiation and neutrinos), 25% dark matter and 70% dark energy.

• Dark matter is an unknown form of matter responsible for the gravitational clumping of galaxies, galaxy clusters and large-scale structures that is yet to be detected. It is an idea pioneered by Zwicky [25] from studies of the Coma cluster and also from studies of individual galaxies by [26, 27]. The effect is seen on extragalactic scales and is inferred from a discrepancy in the galactic velocity rotation curves, which are the main kinematic characteristics of galaxies, based on Newtonian dynamics. A rapid Keplarian $1/\sqrt{r}$ decline in the curves towards the exterior of the galaxies is expected but observations of detected matter (visible stars and gas clouds) shows otherwise.

• Dark energy is a term coined for an unknown form of undetectable energy to accommodate the recently observed accelerated expansion of the Universe. Gravity dynamics is dominated by the dark energy which is identified with a cosmological constant Λ or dynamically as a cosmologically evolving scalar field (quintessence).

Thus, in order to get a consistent picture, the observations are fitted by a simple cosmological model based on GR coupled to standard matter (baryons, neutrinos and radiation) and the two dark components, dark energy and dark matter. If we retain General Relativity as the theory of the gravitational action, the best fit model is called the ΛCDM model (*concordance model*). This model is spatially flat and dominated by cold dark matter (CDM) and Dark Energy (DE) in the form of an effective cosmological constant. Although CDM candidates have not yet been directly detected, there are strong theoretical arguments based on the standard model of particle physics that seem to suggest that CDM has a non-gravitational origin. The cosmological constant and coincidence problems together with the fact that there are no convincing DE candidates, seem to suggest that the Concordance model is incomplete, and despite enormous effort over the past few years, this problem remains one of the greatest puzzles in contemporary physics.

A solution to circumnavigate the problems addressed above might be to consider the alternative explanation that on large scales, the validity of the current theory of gravity fails and a modification to the theory is needed. A wide number of modified theories of gravity have been proposed to explain the late-time acceleration of the Universe without the need for dark energy. These include scalar-tensor theories, Dvali-Gabadadze-Porrati (DGP) braneworld model [28], TeVeS (Tensor-Vector-Scalar) [29] and brane-world gravity [30]. One of the theoretical proposals that has received a considerable amount of attention recently, is focused on *fourth order gravity*, in which the standard Hilbert-Einstein action is modified by replacing the Ricci scalar R with the function $f(R)$. These fourth order theories propose a geometric origin for DE and lead naturally to cosmologies which admit a DE like era [31–35] without the introduction of any additional cosmological fields and are even able to account for the rotation curves for spiral galaxies without the need for dark matter [36].

The feature of fourth order gravity that suggests that these cosmologies can give rise to a phase of accelerated expansion is considered to be an important footprint of DE. In particular, dynamical systems analysis shows that for Friedmann-Lemaitre Robertson Walker (FLRW) models there exist classes of fourth order theories which admit a transient decelerated expansion phase that is important for structure growth, followed by one with an accelerated expansion rate [38]. These cosmic evolutions therefore mimic the standard ΛCDM cosmic history.

Using the 1+3 covariant approach [9–12], the theory of linear perturbations for these models has been developed which provide important features that differentiate the structure growth in fourth order theories from the GR results [37,38]. It was found that the evolution of density perturbations is determined by a fourth order differential equation rather than a second order one, leading to a richer behaviour of the pertubation dynamics. Also, the perturbations are found to be scale-dependent on the equation of state of standard matter as opposed to the scale-invariant GR perturbations and thirdly, the growth of large density perturbations in these theories are found to also occur in backgrounds in which the expansion rate increases with time. Despite these results, there are still some important open problems to be addressed with these theories. Of particular interest is the degree to which the physics of fourth order gravity is consistent with *both* cosmological and solar system scales. There has been considerable debate about the short-scale behavior of higher order over the past few years leading to much work on the Newtonian and post-Newtonian limits of these theories [39–42]. Indeed, measurements coming from weak field limit tests like the bending of light, the perihelion shift of planets and frame dragging experiments represent critical tests for any theory of gravity. Fundamental to confronting such tests with fourth order gravity is the existence of physically viable spherically symmetric solutions in these theories. This will be the major theme of the thesis.

There are three versions of $f(R)$ gravity, that is, the *standard metric* (or second order) f(R) gravity, *Palatini* (or first order) $f(R)$ gravity and the *metric-affine* $f(R)$ gravity. In the standard metric formalism, variation of the action is with respect to the metric whilst the Palatini formalism is such that the metric and the connection are taken to be independent variables and hence the action is varied with respect to both of them. The metric-affine $f(R)$ gravity uses the Palatini variation but without the assumption that the matter action is independent of the connection (see [19] for a detailed review). Both the metric and Palatini formalisms lead to the same field equation for an action whose Lagrangian is linear in R. However, for a more general action, the resultant field equations depend on the variatinal formalism used. We adapt the standard metric formalism in this thesis.

2.2 General equations for fourth order gravity

In a completely general context, a fourth order theory of gravity is obtained by adding a linear combination of the quadratic terms R^2, $R_{ab}R^{ab}$, $R_{abcd}R^{abcd}$ to the Einstein-Hilbert action

$$\mathcal{A} = \int dx^4 \sqrt{-g}\left[\frac{1}{2}(R-2\Lambda) + \mathcal{L}_m\right], \tag{2.1}$$

where $\mathcal{L}_m$ represents the matter contribution and Λ is the usual cosmological constant. We use the linear combination ($\mathcal{G} = R^2 - 4R_{ab}R^{ab} + R_{abcd}R^{abcd}$) (Gauss-Bonnet theorem) of the quadratic terms and bear in mind that the functional derivative of this expression vanishes and hence does not affect the field equations. We can use this symmetry to rewrite $R_{abcd}R^{abcd}$ in terms of the other two and consequently the action for fourth order theory of gravity can be written as [43]:

$$\mathcal{A} = \frac{1}{2}\int d^4x\sqrt{-g}\left[c_0 R + c_1 R^2 + c_2 R_{ab}R^{ab} - 2\Lambda + \mathcal{L}_m\right]. \tag{2.2}$$

Furthermore, if the spacetime is highly symmetric, particularly in homogeneous and isotropic spacetimes, then the variation of the term $R_{ab}R^{ab}$ can always be rewritten in terms of the variation of R^2 [44, 45]. It then follows that the "effective" fourth-order Lagrangian for this highly symmetric spacetime only contains powers of R and we can, with out loss of generality write the action as

$$\mathcal{A} = \frac{1}{2}\int d^4x\sqrt{-g}\left[f(R) - 2\Lambda + \mathcal{L}_m\right]. \tag{2.3}$$

Varying the action with respect to the metric gives the following field equations:

$$f'G_{ab} = f'\left(R_{ab} - \frac{1}{2}g_{ab}R\right) = T^m_{ab} + \frac{1}{2}(f - Rf') + \nabla_b\nabla_a f' - g_{ab}\nabla_c\nabla^c f',$$

where f' denotes the derivatives of the function with respect the Ricci Scalar and T^m_{ab} is the matter energy momentum tensor (EMT). The equations reduce to the standard Einstein field equations with non-zero cosmological constant when $f(R) = R$.

It is convinient to write (2.4) in the form [38]

$$G_{ab} = \left(R_{ab} - \frac{1}{2}g_{ab}R\right) = \tilde{T}^m_{ab} + T^R_{ab} = T^{tot}_{ab}. \tag{2.4}$$

Here the two components of the total stress energy tensor are defined as

$$\tilde{T}^m_{ab} = \frac{T^m_{ab}}{f'} \tag{2.5}$$

and

$$T^R_{ab} = \frac{1}{f'}\left[\frac{1}{2}(f - Rf') + \nabla_b\nabla_a f' - g_{ab}\nabla_c\nabla^c f'\right]. \tag{2.6}$$

These components of the total stress energy tensor can be considered to represent two effective "fluids" [35]: the *curvature "fluid"* (associated with T^R_{ab}) and the *effective matter "fluid"* (associated with $\tilde{T}^m_{ab}$), and this allows us to adapt more easily techniques originally devised for GR from the 1+3 approach to study a wide range of problems in $f(R)$ gravity.

Chapter 3

$f(R)$ Covariant methods

3.1 1+3 Covariant decomposition of higher order gravity

The 1+3 covariant approach, initially developed by Ehlers and Ellis [9–11], is based on a 1+3 threading of the spacetime manifold w.r.t. a timelike congruence. In this way spacetime is split into a timelike and a orthogonal three-dimensional spacelike hypersurface. Instead of the metric, it employs kinematic and dynamical variables, the energy momentum tensor of the fluid, the Ricci identities for the dynamics, and the gravito-magnetic parts of the Weyl tensor to describe nature, all of which relate to observational quantities [46].

The Einstein field equations relate the energy momentum tensor to curvature. The geometry is determined by the Riemann curvature tensor and it's covariant derivative and thus any evaluation is frame dependent. We begin the analysis with a suitable choice of frame, i.e., one corresponding to the 4-velocity u^a of an observer in spacetime. A natural choice of frame for u^a includes; the *energy frame* u^a_E, which is defined to be a timelike eigenvector of the stress energy tensor T_{ab}, the *matter frame* u^a_N, that is derived from the particle flux vector N_a and the *entropy frame* u^a_S, defined by the entropy flux vector S_a [38]. The point is to simplify the calculations by way of restructuring the equations for example by choosing the energy frame, the flux of matter is zero [47] or by choosing the matter frame such that $u^a = u^a_{dust}$ or the CDM frame, then in this frame both matter flux and acceleration are zero.

Equation (2.4) allows us to define two effective fluid where the physical components are only the standard matter components, while the curvature fluid is a mathematical construction displaying additional gravitational degrees of freedom. Choosing a frame corresponding to the total matter/curvature fluid would be physically unmatchable to observations as the energy conditions of the curvature fluid and effective standard matter are not necessarily satisfied [48] and thermodynamics of the curvature fluid is just pathological. The most natural choice of frame is therefore the one of associated with standard matter $u^a = u^a_m$. This choice is also physically motivated by the fact that the real observers are attached to galaxies and these galaxies follow the standard matter geodesics.

The non-intersecting timelike worldlines of these *fundamental observers* comoving with the cosmological fluid form a congruence in spacetime representing the average motion of matter at each point. The properties of the 4-velocity are given by

$$u^a = \frac{dx^a}{d\tau}, \quad u_a u^a = -1, \tag{3.1}$$

where τ is the proper time along the worldline of any fundamental observer. This vector field u^a provides a timelike threading for the spacetime.

3.1.1 Kinematics

The derivation of the kinematical quantities can be found once the frame has been chosen. Given u^a, the unique *projection tensor*

$$h_{ab} = g_{ab} + u_a u_b \tag{3.2}$$

projects into the rest space orthogonal to u^a and satisfies

$$h_{ab}u^b = 0 \ , \quad h^c{}_a h^b{}_c = h^b{}_a \ , \quad h^a{}_a = 3. \tag{3.3}$$

The effective *volume element* for the rest space of the comoving observer is given by

$$\varepsilon_{abc} = \eta_{abcd} u^d, \tag{3.4}$$

where η_{abcd} is the 4-dimensional volume element ($\eta_{0123} = \sqrt{|\det g_{ab}|}$). Since η_{abcd} is totally skew-symmetric $\eta_{abcd} = \eta_{[abcd]}$ and it follows that the spatial volume element satisfies the following identities

$$\begin{aligned}
\varepsilon_{abc}u^c &= 0 \ , \\
\varepsilon_{abc} &= \varepsilon_{[abc]} \ , \\
\varepsilon^{abc}\varepsilon_{efg} &= 3! h^{[a}{}_e h^b{}_f h^{c]}{}_g \ , \\
\varepsilon^{abc}\varepsilon_{afg} &= 2h^{[b}{}_f h^{c]}{}_g \ , \\
\varepsilon^{abc}\varepsilon_{abg} &= 2h^c{}_g \ , \\
\varepsilon^{abc}\varepsilon_{abc} &= 3!.
\end{aligned} \tag{3.5}$$

Two derivatives can be defined: the vector u^a is used to define the *covariant time derivative* (denoted by a dot) for any tensor $\dot{T}^{a..b}{}_{c..d}$ along the observers' worldlines defined by

$$\dot{T}^{a..b}{}_{c..d} = u^e \nabla_e T^{a..b}{}_{c..d} \ , \tag{3.6}$$

and the tensor h_{ab} is used to define the fully orthogonally *projected covariant derivative* D for any tensor $T^{a..b}{}_{c..d}$,

$$D_e T^{a..b}{}_{c..d} = h^a{}_f h^p{}_c ... h^b{}_g h^q{}_d h^r{}_e \nabla_r T^{f..g}{}_{p..q} \ , \tag{3.7}$$

with total projection on all the free indices. Angle brackets to denote orthogonal projections of vectors and the orthogonally *projected symmetric trace-free* PSTF part of tensors:

$$V^{\langle a\rangle} = h^a{}_b V^b \ , \ T^{\langle ab\rangle} = \left[h^{(a}{}_c h^{b)}{}_d - \frac{1}{3}h^{ab}h_{cd}\right] T^{cd}. \tag{3.8}$$

In analogy to vector analysis in three dimensions, we introduce the covariant spatial divergence and curl that generalizes these Newtonian operators to curved spacetimes [49, 50].

$$\begin{aligned} \text{div } V = D^a V_a \ , \qquad & (\text{ div } T)_a = D^b T_{ab} \\ \text{curl } V_a = \varepsilon_{abc} D^b V^c \ , \qquad & \text{curl } T_{ab} = \varepsilon_{cd\langle a} D^c T_{b\rangle}{}^d. \end{aligned} \tag{3.9}$$

The variation of the velocity with position and time is of interest here and therefore we consider it's covariant derivative into its irreducible parts:

$$\nabla_a u_b = D_a u_b - u_a \dot{u}_b. \tag{3.10}$$

We split the spatial change of the 4-velocity further into its symmetric and anti-symmetric parts and the symmetric part further into the trace and trace-free part:

$$\begin{aligned} \nabla_a u_b &= D_{\langle a} u_{b\rangle} + D_{[a} u_{b]} + \frac{1}{3}\Theta h_{ab} - u^c u_a \nabla_c u_b \\ &= \sigma_{ab} + \omega_{ab} + \frac{1}{3}\Theta h_{ab} - u_a \dot{u}_b, \end{aligned} \tag{3.11}$$

where the trace, Θ is the *expansion scalar* (volume expansion) represents the rate of expansion of the fluid defined as

$$\Theta = D^a u_a \ , \tag{3.12}$$

It can be used to define a representative length scale S along the observers worldline via

$$H = \frac{\dot{S}}{S} = \frac{1}{3}\Theta \ , \tag{3.13}$$

in a homogenous and isotropic universe (in a FLRW universe model) where H is the Hubble parameter. The *shear tensor*, σ_{ab}, is the symmetric trace-free part of the spatial change of the 4-velocity defined as

$$\sigma_{ab} = D_{\langle a}u_{b\rangle} \ , \tag{3.14}$$

where

$$\sigma_{ab} = \sigma_{(ab)} \ , \ \sigma_{ab}u^b = 0 \ , \ \sigma^a{}_a = 0. \tag{3.15}$$

The shear tensor determines the distortion arising in the matter flow, leaving the volume invariant. The shear magnitude is expressed as

$$\sigma^2 = \frac{1}{2}\sigma^{ab}\sigma_{ab} \geq 0 \text{ and } \sigma^2 = 0 \Leftrightarrow \sigma_{ab} = 0. \tag{3.16}$$

The anti-symmetric *vorticity tensor*, ω_{ab}, describes the rotation of matter relative to a non-rotating frame. It is defined as

$$\omega_{ab} = D_{[a}u_{b]} \ , \tag{3.17}$$

with

$$\omega_{ab} = \omega_{[ab]} \ , \ \omega_{ab}u^b = 0. \tag{3.18}$$

The vorticity tensor may also be represented by the *vorticity vector* ω^a, where

$$\begin{aligned} \omega^a &= \frac{1}{2}\eta^{abcd}\, u_d\, \omega_{bc} = \frac{1}{2}\varepsilon^{abc}\omega_{bc} = \frac{1}{2} \text{ curl } u^a \ \Leftrightarrow \ \omega_{ab} = \varepsilon_{abc}\omega^c \\ \text{and so } \omega^a u_a &= 0 = \omega^{ab}u_a, \quad \omega^a\omega_{ab} = 0. \end{aligned} \tag{3.19}$$

The vorticity magnitude is

$$\omega^2 = \frac{1}{2}\omega^a\omega_a = \omega^{ab}\omega_{ab} \geq 0 \quad \text{and} \quad \omega = 0 \ \Leftrightarrow \ \omega_a = 0 \ \Leftrightarrow \ \omega_{ab} = 0. \tag{3.20}$$

Finally $\dot{u}_b = u^c\nabla_c u_b$ is the *relativistic acceleration vector*, which represents the degree to which the matter moves under forces other than gravity (a free-falling observer has vanishing acceleration in her rest-frame i.e., moves under gravity and inertia alone).

3.1.2 Curvature tensor

Any given vector field u^a must obey the Ricci identity defined by the relation (*Ricci identity*)

$$2\nabla_{[a}\nabla_{b]}u_c = R^d{}_{abc}\, u_d, \tag{3.21}$$

where $R^d{}_{abc}$ is the *Riemann curvature*, a measurement of the curvature of the spacetime manifold, and is defined as

$$R^d{}_{abc} = \Gamma^d{}_{ac,b} - \Gamma^d{}_{ab,c} + \Gamma^e{}_{ac}\Gamma^d{}_{eb} - \Gamma^e{}_{ab}\Gamma^d{}_{ec} \ , \tag{3.22}$$

and $\Gamma^a{}_{bd}$ are the connection coefficients known as Christoffel symbols (i.e. symmetric in the lower indices), defined by

$$\Gamma^a_{bd} = \frac{1}{2}g^{ae}\left(g_{be,d} + g_{ed,b} - g_{bd,e}\right). \tag{3.23}$$

The curvature tensor (3.22) has the following symmetry properties

$$R_{abcd} = R_{[ab][cd]} = R_{cdab} \qquad R^d{}_{[abc]} = 0, \tag{3.24}$$

and satisfies the *Bianchi identity*

$$\nabla_{[e}\, R_{ab]cd} = 0. \tag{3.25}$$

It follows from the symmetry properties of the curvature tensor that it possesses only two independent contractions. One obtains the *Ricci tensor* $R_{ab} = R^c{}_{acb} = R_{ba}$ on the first contraction of the curvature tensor, and a further contraction yields the *Ricci scalar* (or *curvature scalar*) $R = R_a{}^a$.

The Bianchi identity (3.25) can be contracted once to give

$$\nabla_a R^a{}_{bcd} = \nabla_c R_{bd} - \nabla_d R_{bc} \tag{3.26}$$

A further contraction gives *twice-contracted Bianchi identities*:

$$\nabla_a R^a{}_c + \nabla_b R^b{}_c - \nabla_c R = 0 \tag{3.27}$$

when projected parallel and perpendicular to u^a, lead to the evolution equations (3.61)-(3.64).

Taking the trace of field equation (2.4) and substituting for the trace of the EMT $T^{tot} \equiv T^{a(tot)}{}_a = 3p^{tot} - \mu^{tot}$, gives an expression for the Ricci scalar:

$$R = \mu^{tot} - 3p^{tot}. \tag{3.28}$$

Using this expression in the field equation and replacing the EMT with its general form (3.38) yields

$$R_{ab} = \frac{1}{2}\left(\mu^{tot} + 3p^{tot}\right)u_a u_b + \frac{1}{2}\left(\mu^{tot} - p^{tot}\right)h_{ab} + 2u_{(a}q^{tot}_{b)} + \pi^{tot}_{ab}, \tag{3.29}$$

the 1+3 split of the Ricci tensor.

The Riemann curvature tensor can be decomposed into a trace part and a trace-free part, the Ricci tensor and the *Weyl tensor* $C^d{}_{abc}$, respectively. The Weyl tensor is defined by the equation

$$C_{abcd} = R_{abcd} - \frac{1}{2}\left(R_{ac}g_{bd} - R_{ad}g_{bc} + R_{bd}g_{ac} - R_{bc}g_{ad}\right) + \frac{1}{6}Rg_{abcd}, \tag{3.30}$$

where $g_{abcd} = g_{ac}g_{bd} - g_{ad}g_{bc}$.

It follows from the definition that the Weyl tensor has the symmetry properties (3.24) of the Riemann tensor and is trace-free on all indices.

$$C^{c}{}_{acb} = 0. \tag{3.31}$$

The 1+3 decomposition of the Weyl tensor $C^{d}{}_{abc}$ can be split relative to u^a into *'electric'* and *'magnetic' Weyl curvature* parts according to:

$$E_{ab} = C_{abcd}u^b u^d \quad \rightarrow \quad E^{a}{}_{a} = 0,\ E_{ab} = E_{(ab)},\ E_{ab}u^b = 0, \tag{3.32}$$

$$H_{ab} = \frac{1}{2}\varepsilon_{ade}C^{de}{}_{bc}u^c \quad \rightarrow \quad H^{a}{}_{a} = 0,\ H_{ab} = H_{(ab)},\ H_{ab}u^b = 0. \tag{3.33}$$

The Weyl tensor may thus be written as

$$C_{abcd} = C^{E}_{abcd} + C^{H}_{abcd}, \tag{3.34}$$

where

$$C^{E}_{abcd} = \left(g_{abpq}g_{cdrs} - \eta_{abpq}\eta_{cdrs}\right)u^p u^r E^{qs},$$
$$C^{H}_{abcd} = -\left(\eta_{abpq}g_{cdrs} + g_{abpq}\eta_{cdrs}\right)u^p u^r H^{qs}. \tag{3.35}$$

The magnetic and electric parts of the Weyl tensor represent the 'free gravitational field', enabling gravitational action at a distance (tidal forces, gravitational waves), and influence the motion of matter and radiation through the geodesic deviation equation for timelike and null vectors, respectively [51]– [53].

If the unit vector field u^a is hypersurface-orthogonal to the spacelike 3-surfaces then the congruence vorticity vanishes. The intrinsic 3-curvature tensor of these 3-spaces ${}^{(3)}R_{abc}{}^{d}$ is defined the three-dimensional version of the Ricci identity

$$2D_{[a}D_{b]}V_c = {}^{(3)}R_{abc}{}^{d}V_d \tag{3.36}$$

for any vector field V_a in the 3-spaces with $V_a u^a = 0$. The 3-curvature tensor is related to the Riemann curvature tensor of the spacetime by the *Gauss equation* for u^a [54]:

$${}^{(3)}R_{abcd} = (R_{abcd})_{\perp} - K_{ac}K_{bd} + K_{bc}K_{ad}, \tag{3.37}$$

where ${}^{(3)}R_{abcd}$ is the *3-curvature tensor*, $\perp$ means projection with h_{ab} on all indices and K_{ab} is the *extrinsic curvature.*

3.1.3 Energy momentum tensor for $f(R)$ gravity

The choice of the frame also allows for an irreducible decomposition of the stress *energy momentum tensor* (EMT). In a general frame, the total effective EMT can be decomposed relative to u_a by splitting it up into parts parallel and orthogonal to u_a as follows:

$$T^{tot}_{ab} = \mu^{tot} u_a u_b + q^{tot}_a u_b + u_a q^{tot}_b + p^{tot} h_{ab} + \pi^{tot}_{ab}; \tag{3.38}$$

where μ^{tot}, p^{tot}, q^{tot}_a and π^{tot}_{ab} denote the total effective *energy density* relative to u^a, *isotropic pressure*, *energy flux* (momentum density) relative to u^a and PSTF *anisotropic stress*, respectively, and the following properties hold:

$$\begin{aligned} q^{tot}_a u^a = 0\ , \quad \pi^{tot}_{ab} u^b &= 0\ , \quad \pi^{tot}_{ab} = \pi^{tot}_{(ab)}\ , \quad \pi^{a(tot)}{}_a = 0, \\ q^{tot}_a &= q^{tot}_{\langle a \rangle}\ , \quad \pi^{tot}_{ab} = \pi^{tot}_{\langle ab \rangle}. \end{aligned} \tag{3.39}$$

This decomposition can be applied to our effective EMTs. Relative to u^a we obtain

$$\mu^{tot} = T_{ab} u^a u^b = \tilde{\mu}^m + \mu^R, \quad p^{tot} = \frac{1}{3}(T_{ab} h^{ab}) = \tilde{p}^m + p^R, \tag{3.40}$$

$$q^{tot}_a = -T_{bc} u^c h^b_a = \tilde{q}^m_a + q^R_a, \quad \pi^{tot}_{ab} = T_{cd} h^c{}_{\langle a} h^d{}_{b \rangle} = \tilde{\pi}^m_{ab} + \pi^R_{ab}, \tag{3.41}$$

with

$$\tilde{\mu}^m = \frac{\mu^m}{f'}, \quad \tilde{p}^m = \frac{p^m}{f'}, \quad \tilde{q}^m_a = \frac{q^m_a}{f'}, \quad \tilde{\pi}^m_{ab} = \frac{\pi^m_{ab}}{f'}. \tag{3.42}$$

The physical behaviour of the matter present i.e. the relativistic energy, momentum and stresses associated with a matter field are represented in general by the EMT T^m_{ab} whereby, p^m, is induced by the random thermal motions, q^m_a is such that energy might be transmitted by heat conduction and that energy will carry a momentum (or is a heat conduction term, the *energy flux*, in the instantaneous rest frame) and π^m_{ab} is due to processes such as viscosity. These quantities are related by the *equation of state* to capture the physics. In addition, the description standard matter and radiation in the universe is such that at least one of the following conditions is obeyed:

1. The weak energy condition (WEC)

$$T^m_{ab} u^a u^b \geq 0, \tag{3.43}$$

for all timelike vectors u^c or for a perfect fluid,

$$\mu^m = T^m_{ab}u^a u^b \geq 0 \text{ and } \mu^m + p^m \geq 0, \tag{3.44}$$

is to hold for homogeneous and isotropic spacetime matter distribution. The expression $\mu + p \geq 0$ requires that the energy density is positive and implies that matter will tend to move in the direction of a pressure gradient applied to it.

2. The strong energy condition (SEC)

$$T^m_{ab}u^a u^b \geq \frac{1}{2}T^{mc}{}_c u^d u_d, \tag{3.45}$$

for all timelike vectors u^c or for a perfect fluid,

$$\mu^m + p^m \geq 0 \text{ and } \mu^m + 3p^m \geq 0, \tag{3.46}$$

stating that gravity is attractive, with the expression $\mu^m + 3p^m \geq 0$ being equivalent to positive the gravitational mass density of matter.

3. The speed of sound c_s must be less than the speed of light (i.e. no signal can be sent faster than light.)

$$0 \leq c_s^2 \leq 1 \qquad \Leftrightarrow \qquad 0 \leq (\partial p/\partial \mu)_{s=\text{const}} \leq 1, \tag{3.47}$$

to guarantee local stability of matter (lower bound) and causality (upper bound).

In this case however, the curvature fluid and the effective matter do not necessarily satisfy the WEC (3.44) [48]. This relation is the key hypothesis which allows the timelike vectors u^a_E, u^a_N, u^a_S to exist and is, in general, a very reasonable assumption [38]. The violation of this condition means that the *matter energy frame* u^a_m is the natural choice of frame and in this case it is chosen to be comoving with standard matter that is assumed to be a barotropic fluid with *equation of state* $p = w\mu$.

Now *curvature energy momentum tensor* is given as

$$T^R_{ab} = \frac{1}{f'}\left[\frac{1}{2}g_{ab}(f - Rf') + \nabla_b\nabla_a f' - g_{ab}\nabla^c\nabla_c f'\right], \tag{3.48}$$

and the derivative terms can be decomposed into time and spatial parts resulting in the curvature EMT taking the form

$$\begin{aligned} T^R_{ab} &= \frac{1}{f'}[\frac{1}{2}g_{ab}(f - Rf') + D_b D_a f' - \left(\sigma_{ab} + \omega_{ab} + \frac{1}{3}\Theta h_{ab}\right)(\dot{f'}) - u_a(\dot{\nabla_b f'}) \\ &\quad - u_b(\dot{\nabla_a f'}) - u_b u^e u_a(\dot{\nabla_e f'}) - g_{ab}\left[D^2 f' - \Theta(\dot{f'}) - (\ddot{f'}) + \dot{u}_c(\nabla^c f')\right]]. \end{aligned} \tag{3.49}$$

In this way, the *curvature thermodynamical quantities* $\mu^R = T^R_{ab}u^a u^b$, $p^R = \frac{1}{3}T^R_{ab}h^{ab}$, $\pi^R_{ab} = T^R_{cd}h^c{}_{\langle a}h^d{}_{b\rangle}$ and $q^R_a = -T^R_{bc}h^b{}_a u^c$ can be written in terms of (1+3) variables as:

$$\mu^R = \frac{1}{f'}[\frac{1}{2}(Rf' - f) + f'''D^aRD_aR + f''D^2R - \Theta f''\dot{R}]; \tag{3.50}$$

$$p^R = \frac{1}{f'}[\frac{1}{2}(f - Rf') - \frac{2}{3}f''D^2R - \frac{2}{3}f'''D^aRD_aR + \frac{2}{3}\Theta f''\dot{R} + f'''\dot{R}^2 + f''\ddot{R} - \dot{u}_c(\nabla^c f')]; \tag{3.51}$$

$$\pi^R_{ab} = \frac{1}{f'}\left[f'''D_{\langle a}RD_{b\rangle}R + f''D_{\langle a}D_{b\rangle}R - \sigma_{ab}(\dot{f'})\right]; \tag{3.52}$$

$$q^R_a = -\frac{1}{f'}\left[f'''\dot{R}D_aR + f''D_a\dot{R} - \frac{1}{3}\Theta f''D_aR\right]. \tag{3.53}$$

The field equation (2.4) can be rewritten as

$$R_{ab} = T^{tot}_{ab} - \frac{1}{2}g_{ab}T^{tot}, \tag{3.54}$$

and from this we obtain the *trace equation*

$$R^a{}_a = R = -T^{tot} = -(\widetilde{T}^m + T^R)\ . \tag{3.55}$$

Given that T^R is expressed as

$$T^R = g^{ab}T^R_{ab} = \frac{1}{f'}\left[2(f - Rf') - 3\left(f''D^2R + f'''D^aRD_aR - f'''\dot{R}^2 - f''\ddot{R} + \dot{u}_c\nabla^c f' - f''\theta\dot{R}\right)\right],$$

and that $\widetilde{T}^m_{ab}$

$$\widetilde{T}^m = g^{ab}\frac{1}{f'}T^m_{ab} = \frac{1}{f'}\left(3p^m - \mu^m\right), \tag{3.56}$$

the trace equation now takes the form

$$Rf' - 2f + \mu^m - 3p^m = -3f''D^2R - 3f'''D^aRD_aR + 3f'''\dot{R}^2 + 3f''\ddot{R} - 3\dot{u}_c\nabla^c f' + 3f''\theta\dot{R}. \tag{3.57}$$

The expression for the *curvature trace equation* is thus

$$Rf' - 2f = -3f''D^2R - 3f'''D^aRD_aR + 3f'''\dot{R}^2 + 3f''\ddot{R} - 3\dot{u}_c\nabla^c f' + 3f''\theta\dot{R}. \tag{3.58}$$

3.1.4 Energy Momentum Conservation

The conservation properties of these effective fluids are given by the equation $T^{tot;b}_{ab}$. This property reveals that if standard matter is conserved the total fluid is also conserved even though the curvature fluid may in general possess offdiagonal terms [14, 55, 56], that is, the effective stress energy tensor T^{tot}_{ab} is always divergence free if $T^{m;b}_{ab} = 0$. When applied to the individual effective tensors, the conservation equations read

$$\tilde{T}^{m;b}_{ab} = \frac{T^{m;b}_{ab}}{f'} - \frac{f''}{f'^2} T^{m}_{ab} R^{;b}; \tag{3.59}$$

$$T^{R;b}_{ab} = \frac{f''}{f'^2} T^{m}_{ab} R^{;b}. \tag{3.60}$$

It follows that the individual effective fluids are not conserved but exchange energy and momentum [38].

It is worth noting here that the standard matter still follows the usual conservation equations $T^{m;b}_{ab} = 0$. Also, the fluids with T^R_{ab} and $\tilde{T}^m_{ab}$ defined above are *effective* and consequently can admit features that one would normally consider un-physical for a standard matter field. Thus all the thermodynamical quantities associated with the curvature defined previously should be considered *effective* and not bounded by matter constraints. Standard matter maintains its thermodynamical properties, regardless, represented by the Lagrangian $\mathcal{L}_m$.

The evolution equations for $\dot{\mu}^m, \dot{\mu}^m, \dot{q}^m$ and $\dot{q}^R$ are obtained by substituting for the general form of the EMT (3.38) into $T^{tot;b}_{ab}$, giving

$$\dot{\mu}^m + D^a q^R_a = -\Theta\left(\mu^m + p^m\right) - 2\dot{u}^a q^m_a - \sigma^{ab}\pi^m_{ba}, \tag{3.61}$$

$$\dot{\mu}^R + D^a q^R_a = -\Theta\left(\mu^R + p^R\right) - 2\dot{u}^a q^R_a - \sigma^{ab}\pi^R_{ba} + \mu^m \frac{f''\dot{R}}{f'^2} \tag{3.62}$$

for the component parallel to u^a. The evolution equation for $\dot{\mu}^m$ is the *matter energy conservation equation* and determines the rate of change of relativistic energy μ along the fundamental world lines. The component orthogonal to u^a is

$$\dot{q}^m_{\langle a\rangle} + D_a p^m + D^b \pi^m_{ab} = -\frac{4}{3}\Theta q^m_a - \sigma^b{}_a q^m_b - (\mu^m + p^m)\dot{u}_a - \dot{u}^b\,\pi^m_{ab} - \varepsilon^{bc}{}_a \omega_b\, q^m_c, \tag{3.63}$$

$$\dot{q}^R_{\langle a\rangle} + D_a p^R + D^b \pi^R_{ab} = -\frac{4}{3}\Theta q^R_a - \sigma^b{}_a q^R_b - (\mu^R + p^R)\dot{u}_a - \dot{u}^b \pi^R_{ab} - \varepsilon^{bc}{}_a \omega_b q^R_c + \mu^m f'' \frac{D_a R}{f'^2}. \tag{3.64}$$

Here, the $\dot{q}^m$ equation gives the *momentum conservation equation* determining the acceleration caused by various pressure contributions. When we consider a perfect fluid case, the conservation equations for standard matter reduce to

$$\dot{\mu}^m = -\Theta\left(\mu^m + p^m\right), \tag{3.65}$$

$$D^a p^m = -\left(\mu^m + p^m\right)\dot{u}_a, \tag{3.66}$$

showing for (3.65), that $(\mu^m + p^m)$ is the initial mass density and also governs the conservation of energy and for (3.66), a relation connecting the acceleration $\dot{u}_a$ to μ^m and p^m.

3.1.5 Dynamics

We now look at the dynamical relations for an arbitrary spacetime in the 1+3 formulation. This spacetime may be completely characterized by the irreducible set of geometrical quantities,

$$\{\Theta, \sigma_{ab}, \omega_{ab}, \dot{u}^a, E_{ab}, H_{ab}\}, \quad (3.67)$$

together with the irreducible set of thermodynamic variables,

$$\mu, p, q^a, \pi_{ab}, \quad (3.68)$$

provided an equation of state which relates the thermodynamic variables is prescribed. The propagation and constraint equations for the above covariant variables can be obtained from the field equations (2.4) and its associated integrability conditions, resulting in propagation and constraint equations.

Propagation equations

The first set of propagation equations arises from contracting (3.21) with u^a the Ricci identity (3.21) with the fundamental timelike vector field u^a and then substituting from the first covariant derivative of u^a (3.11) and the field equation (2.4). The propagation equations are obtained by separating out the result into trace, symmetric trace-free, and skew symmetric parts:

1. The expansion propagation equation (*generalized Raychaudhuri equation*):

$$\dot{\Theta} + \frac{1}{3}\Theta^2 + \left(2\sigma^2 - 2\omega^2\right) - D^a\dot{u}_a - \dot{u}_a\dot{u}^a + \frac{1}{2}\left(\tilde{\mu}^m + 3\tilde{p}^m\right) = \frac{1}{2}\left(\mu^R + 3p^R\right), \quad (3.69)$$

   is the *basic equation of gravitational attraction.*

2. The *shear propagation equation*:

$$\dot{\sigma}_{\langle ab\rangle} + \frac{2}{3}\Theta\sigma_{ab} - D_{\langle a}\dot{u}_{b\rangle} + \sigma_{c\langle a}\sigma_{b\rangle}{}^c - \dot{u}_{\langle a}\dot{u}_{b\rangle} + (E_{ab} - \frac{1}{2}\tilde{\pi}^m_{ab}) + \omega_{\langle a}\omega_{b\rangle} = -\frac{1}{2}\pi^R_{ab}, \quad (3.70)$$

   shows how the tidal gravitational field E_{ab} directly induces anisotropic shear.

3. The *vorticity propagation equation*:

$$\dot{\omega}^{\langle a\rangle} + \frac{2}{3}\Theta\omega^a - \frac{1}{2}\mathrm{curl}\,\dot{u}^a - \sigma^a{}_b\omega^b = 0. \quad (3.71)$$

The once-contracted Bianchi identities (3.26) give an additional pair of propagation equations and a further pair of constraint equations when covariantly decomposed. The propagation equations are

4. the *Gravito-electric* ($\dot{E}$) *propagation equation*:

$$\begin{aligned}
& \dot{E}_{\langle ab\rangle} + \Theta\left(E_{ab} + \frac{1}{6}\tilde{\pi}^m_{ab}\right) - \mathsf{curl}\, H_{ab} + \frac{1}{2}\left(\tilde{\mu}^m + \tilde{p}^m\right)\sigma_{ab} - 2\dot{u}^c\varepsilon_{cd\langle a}H_{b\rangle}{}^d - 3\sigma^c{}_{\langle a}E_{b\rangle c} \\
+ & \frac{1}{2}\sigma^c{}_{\langle a}\tilde{\pi}^m_{b\rangle c} + \dot{u}_{\langle a}\tilde{q}^m_{b\rangle} + \frac{1}{2}\dot{\tilde{\pi}}^m_{\langle ab\rangle} + \frac{1}{2}D_{\langle a}\tilde{q}^m_{b\rangle} - \frac{1}{2}\omega^c\varepsilon^d{}_{c\langle a}\tilde{\pi}^m_{b\rangle d} + \frac{1}{2}\omega^c\varepsilon^{cd\langle a}E_{b\rangle}{}^d \\
= & \frac{1}{2}\omega^c\varepsilon^d{}_{c\langle a}\pi^R_{b\rangle d} - \frac{1}{2}\dot{\pi}^R_{\langle ab\rangle} - \frac{1}{2}\sigma^c{}_{\langle a}\pi^R_{b\rangle c} - \frac{1}{2}\left(\mu^R + p^R\right)\sigma_{ab} - \Theta\frac{1}{6}\pi^R_{ab} - \frac{1}{2}D_{\langle a}q^R_{b\rangle} - \dot{u}_{\langle a}q^R_{b\rangle},
\end{aligned} \tag{3.72}$$

5. and the *Gravito-magnetic* ($\dot{H}$) *propagation equation*:

$$\begin{aligned}
\dot{H}_{\langle ab\rangle} + \mathsf{curl}\, E_{ab} \quad & + \quad \Theta H_{ab} - 3\sigma_{c\langle a}H_{b\rangle}{}^c - \omega^c\varepsilon_{cd\langle a}H_{b\rangle}{}^d + 2\dot{u}^c\varepsilon_{cd\langle a}E_{b\rangle}{}^d - \frac{1}{2}\mathsf{curl}\,\tilde{\pi}^m_{ab} - \frac{3}{2}\omega_{\langle a}\tilde{q}^m_{b\rangle} \\
-\frac{1}{2}\sigma^c{}_{\langle a}\varepsilon_{b\rangle c}\tilde{q}^m_d \quad & = \quad \frac{1}{2}\mathsf{curl}\,\pi^R_{ab} + \frac{3}{2}\omega_{\langle a}q^R_{b\rangle} + \frac{1}{2}\sigma^c{}_{\langle a}\varepsilon_{b\rangle c}q^R_d,
\end{aligned} \tag{3.73}$$

respectively. These equations describe gravitational radiation arises by taking the time derivative of the equations, yielding a wave equation for E_{ab} as well as H_{ab}.

Constraint equations

The constraint equations are obtained by first orthogonally projecting Ricci identity (3.21) and then:

1. the *shear constraint* is obtained by contracting on the indices b and c of (3.21):

$$D^b\sigma_{ab} - \frac{2}{3}D_a\Theta + \mathsf{curl}\,\omega_a + 2\varepsilon^{abc}\dot{u}_b\omega_c + \tilde{q}^m_a = -q^R_a; \tag{3.74}$$

relating the heat flux q^m_a to the spatial inhomogeneity of the expansion and also to the spatial gradients of vorticity and shear;

2. the *vorticity constraint* is obtained by multiplying (3.21) with $\varepsilon^{ab}c$ (3.21):

$$D^a\omega_a - \dot{u}^a\omega_a = 0; \tag{3.75}$$

3. the *H constraint* is obtained by multiplying (3.21) with ε^{abf} and taking the PSTF part:

$$H^{ab} + 2\dot{u}_{\langle a}\omega_{b\rangle} + D_{\langle a}\omega_{b\rangle} - \mathsf{curl}\,\sigma_{ab} = 0; \tag{3.76}$$

characterizing the magnetic Weyl tensor as being constructed from the vorticity distortion and the curl of the shear.

The constraint equations derived from the once-contracted Bianchi identities (3.26) are the

4. *Gravito-electric (divE) divergence*:

$$D^b\left(E_{ab}+\frac{1}{2}\tilde{\pi}^m_{ab}\right)-\frac{1}{3}D_a\tilde{\mu}^m+\frac{1}{3}\Theta\tilde{q}^m_a-\frac{1}{2}\sigma^b{}_a\tilde{q}^m_b-3\omega_bH^{ab}-[\sigma,H]_a+\frac{3}{2}[\omega,\tilde{q}^m]_a\,;$$
$$=\frac{1}{2}\sigma^b{}_aq^R_b-\frac{1}{2}D^b\pi^R_{ab}+\frac{1}{3}D_a\mu^R-\frac{1}{3}\Theta q^R_a-\frac{3}{2}\left[\omega,q^R\right]_a, \tag{3.77}$$

where the spatial gradient of the energy density acts as a source,

5. and the *Gravito-magnetic (divH) divergence*:

$$D^bH_{ab}+(\tilde{\mu}^m+\tilde{p}^m)\,\omega_a+3\omega^b\left(E_{ab}-\frac{1}{6}\tilde{\pi}^m_{ab}\right)+\frac{1}{2}\mathrm{curl}\,\tilde{q}^m_a+[\sigma,E]_a+\frac{1}{2}[\sigma,\tilde{\pi}^m]_a$$
$$=-\left(\mu^R+p^R\right)\omega_a+\frac{1}{2}\pi^R_{ab}\omega^b-\frac{1}{2}\mathrm{curl}\,q^R_a-\frac{1}{2}\left[\sigma,\pi^R\right]_a, \tag{3.78}$$

with the fluid vorticity acting as the source.

The first equation shows how scalar modes are coupled to the non-zero divergence of the electric Weyl tensor, while the latter shows how vector (vorticity) modes are coupled to the non-zero divergence of the magnetic Weyl tensor.
The final set of equations are derived from the twice-contracted Bianchi identities (3.27) whereby projecting parallel and orthogonal to u^a gives the matter and curvature conservation equations as given in (3.61)–(3.64).

3.1.6 Derivatives and Commutators

Covariant derivatives do not commute unless spacetime is flat. This is a manifestation of spacetime curvature derived from the Ricci identities for spacetime scalars f, vectors V^a and second-rank tensors Z^{ab}, respectively [57]:

$$\nabla_{[a}\nabla_{b]}f = 0, \tag{3.79}$$
$$2\nabla_{[a}\nabla_{b]}V^c = R_{ab}{}^c{}_dV^d \tag{3.80}$$
$$2\nabla_{[a}\nabla_{b]}Z^{cd} = -R_{ab}{}^{ec}Z_e{}^d-R_{ab}{}^{ed}Z^c{}_e. \tag{3.81}$$

The 3-space commutator relations orthogonal to the congruence u^a, follow by successively writing out the 3-commutators explicitly and then using the Ricci identities (3.79)–(3.81), the irreducible splitting (3.11) of $\nabla_a u_b$ and the generalized Gauss equation (3.37). The following relations in the orthogonal 3-space are obtained for scalar functions f :

$$D_{[a}D_{b]}f = \omega_{ab}\dot{f}, \tag{3.82}$$

$$D_a\dot{f} - (D_a f)^{\cdot}_{\perp} = -\dot{u}_a\dot{f} + \left(\frac{1}{3}\Theta h_a{}^b + \sigma_a{}^b + \omega_a{}^b\right) D_b f; \tag{3.83}$$

for the 3-vectors V^a, where $(V^a u_a = 0)$:

$$2D_{[a}D_{b]}V^c = 2\omega_{ab}\dot{X}^{\langle c\rangle} -^{(3)} R_{abs}{}^c V^s, \tag{3.84}$$

$$\begin{aligned} D_a\dot{V}_b - (D_a V_b)^{\cdot}_{\perp} &= -\dot{u}_a\dot{V}_{\langle b\rangle} + \left(\frac{1}{3}\Theta h_a{}^c + \sigma_a{}^c + \omega_a{}^c\right)(V_c\dot{u}_b + D_c V_b) \\ &\quad -H_a{}^d\varepsilon_{dbc}V^c - \frac{1}{2}h_{ab}q_c V^c + \frac{1}{2}V_a q_b; \end{aligned} \tag{3.85}$$

and for the second-rank tensors Z_{ab}, where u^a $(Z_{ab}\,u^a = 0 = Z_{ab}\,u^b)$:

$$2D_{[a}D_{b]}Z^{cd} = 2\omega_{ab}\left(\dot{Z}^{cd}\right)_{\perp} -^{(3)} R_{abs}{}^c Z^{sd} -^{(3)} R_{abs}{}^d Z^{cs}, \tag{3.86}$$

$$\begin{aligned} D_a\dot{Z}_{bc} - (D_a Z_b c)^{\cdot}_{\perp} &= \left(\frac{1}{3}\Theta h_a{}^d\sigma_a{}^d + \omega_a{}^d\right)(\dot{u}_b Z_{dc} + \dot{u}_c Z_{bd} + D_d Z_{bk}) \\ &\quad + \left[h_{a[e}q_{b]} - \varepsilon_{ebd}H_a{}^d\right] Z^e{}_c + \left[h_{a[e}q_{c]} - \varepsilon_{ecd}H_a{}^d\right] Z^e_b \\ &\quad -\dot{u}_a\left(\dot{Z}_{bc}\right)_{\perp}. \end{aligned} \tag{3.87}$$

3.2 1+3 Covariant approach to null geodesics [1]

3.2.1 Kinematics

To obtain the bending angle due to a gravitational field source, the fact that light rays propagate as null geodesics must be considered. We now apply the reasoning given in section 3.1 of this chapter to null geodesics characterized by a the family of null curves, $x^a(\nu)$. ν is the affine parameter on the ray, which describes the geodesics by a fixed direction of its tangent vector k^a, that lies orthogonal to a two-dimensional screen-space. The components of the tangent vector are

$$k^a = \frac{dx^a}{d\nu}(\nu), \tag{3.88}$$

and it obeys:

$$k^a k_a = 0\ , \tag{3.89}$$

[1]Unless otherwise stated hereafter, the notation of the thermodynamical quantities as μ, p, q_a and π_{ab} will refer to their "total" quantity i.e., the total of the effective matter and curvature fluid thermodynamic terms.

. From here in this section, we use $\frac{d}{d\nu}$ to define the derivative along the ray as $\frac{d}{d\nu} = k^a\nabla_a$. From (3.89) it follows that

$$\frac{dk^a}{d\nu} = k^b\nabla_b k^a = 0 \ , \tag{3.90}$$

but, since k^a is a gradient, we have that $\nabla_b k^a = \nabla_a k^b$ and this implies

$$k^b\nabla_a k^b = 0. \tag{3.91}$$

The light propagation vector k^a is received by the observer from the direction determined by the unit spatial vector n^a:

$$n^a n_a = 1 \ , \quad n^a u_a = 0. \tag{3.92}$$

The null tangent vector k^a can be split into a timelike and spacelike component as

$$k^a = E(u^a + n^a), \tag{3.93}$$

where $E \equiv -u_a k^a$ and since $u_a k^a$ is the *circular frequency* ω of the electromagnetic wave, E can be interpreted as the *energy associated with the ray.* The propagation equation for E along the ray can be determined by differentiating E as given in (3.93)

$$\frac{dE}{d\nu} = -E^2\left(\frac{1}{3}\Theta + \sigma_{ab}n^a n^b + \dot{u}_a n^a\right). \tag{3.94}$$

Here Θ and σ is the isotropic expansion (or contraction) and the shear due to the matter inside the beam of light rays, respectively, and not the flow along the null geodesics.

Since the hypersurface orthogonal to null vector k contains k, then the projection onto a locally orthogonal space now has to be defined differently. We therefore define a second null vector l_a, obeying the following properties:

$$l^a l_a = 0 \ , \quad k^a l_a = -1 \ , \tag{3.95}$$

together with

$$\frac{dl^a}{d\nu} = k^b\nabla_b l^a = 0 \ . \tag{3.96}$$

The *screen-space projection tensor* may now be defined as

$$\tilde{h}_{ab} \equiv g_{ab} + 2k_{(a}l_{b)} \ , \quad \tilde{h}^a{}_a = 2 \ , \quad \tilde{h}_{ac}\tilde{h}^c{}_b = \tilde{h}_{ab} \ , \quad \tilde{h}_{ab}k^b = 0 \ , \tag{3.97}$$

which projects vectors and tensors onto a two-dimensional screen-space orthogonal to the light propagation vector k^a.

Tensorial and vector objects on the screen-space will be denoted by a tilde

$$\tilde{V}^a = \tilde{h}^c{}_b V^b \ , \quad \tilde{T}^{a..c}{}_{b..d} = \tilde{h}^a{}_e \tilde{h}^f{}_b..\tilde{h}^c{}_g \tilde{h}^h{}_d T^{e..g}{}_{f..h} \ , \tag{3.98}$$

and the derivatives by

$$\tilde{D}_a T_{b..c} = \tilde{h}^d{}_a \tilde{h}^e{}_b..\tilde{h}^f{}_c \nabla_d T_{e..f}. \tag{3.99}$$

The decomposition for a *screen-space projected symmetric and trace-free* (SPSTF) tensor will be denoted as

$$\tilde{A}_{\langle ab\rangle} = \tilde{h}^c{}_{(a}\tilde{h}^d{}_{b)}A_{cd} - \frac{1}{2}\tilde{A}^c{}_c h_{ab}. \tag{3.100}$$

The *2-volume element of the screen-space* is defined as

$$\tilde{D}_a \tilde{S}_{bc} = 0 = \frac{d\tilde{S}_{bc}}{d\nu} \tag{3.101}$$

and is invariant under parallel transport in the direction of k^a, such that

$$\tilde{S}_{ab} \equiv \eta_{cdab} u^c n^d = \tilde{S}_{[ab]}, \tag{3.102}$$

where η_{abcd} is the spacetime permutator.
The covariant derivative of the null vector k^a can be decomposed into

$$\nabla_b k_a = \frac{1}{2}\tilde{h}_{ab}\tilde{\Theta} + \tilde{\sigma}_{ab} + \tilde{\omega}_{ab} + \tilde{X}_a k_b + \tilde{Y}_b k_a + \lambda k_a k_b, \tag{3.103}$$

where

$$\begin{aligned} \tilde{X}_a &= \frac{1}{E} n^d \nabla_d k_a, \\ \tilde{Y}_b &= \frac{1}{E} n^c \nabla_b k_c, \\ \lambda &= -\frac{1}{E^2} n^c n^d \nabla_d k_c, \end{aligned} \tag{3.104}$$

and $\tilde{\Theta}$, $\tilde{\sigma}_{ab}$, $\tilde{\omega}$ are the expansion, shear and rotation of the family of null curves respectfully. The hydrodynamical geometry interpretation here is applied to images of an object projected onto the 2-surface screen space orthogonal to the family. It also follows that $\tilde{X}_a k^a = \tilde{Y}_a k^a = 0$ and $\tilde{\sigma}^a_a = 0$.

Equation (3.103) may be rewritten as

$$\nabla_b k_a = \frac{1}{2}\tilde{h}_{ab}\tilde{\Theta} + \tilde{\sigma}_{ab} + \tilde{S}_{ab}\tilde{\omega} + \tilde{V}_{(a}k_{b)} + \tilde{W}_{[a}k_{b]} + \lambda k_a k_b, \tag{3.105}$$

where $\tilde{V}_a = \tilde{X}_a + \tilde{Y}_a$ and $\tilde{W}_a = \tilde{X}_a - \tilde{Y}_a$.

3.2.2 Dynamics

The Ricci identity for the light propagation vector k^a is

$$2\nabla_{[c}\nabla_{d]}k_a = R_{abcd}k^b. \tag{3.106}$$

Propagation equations

By contracting (3.106) with k^d and using the fact that k^a is a geodesic we get

$$R_{abcd}k^b k^d + \frac{d}{d\nu}\left(\nabla_c k_a\right) + \nabla_a k^d \nabla_d k_a = 0; \tag{3.107}$$

from which we obtain the propagation equations along k^a.

1. The null expansion propagation equation (*null Raychaudhuri equation*):
   This is obtained by taking the trace of (3.107), by contracting on a and c to give
   $$\frac{d\tilde\Theta}{d\nu} = -E^2\left(\mu + p - 2q_a n^a + \pi_{ab}n^a n^b\right) - 2\left(\tilde\sigma^2 - \tilde\omega^2\right) - \frac{1}{2}\tilde\Theta^2. \tag{3.108}$$
   Assuming a vacuum case and $\omega = 0$ in (3.109), we see that the derivative of Θ along k^a is negative, that is, Θ decreases.

   Considering now the evolution of a bundle of rays with elliptical cross-section and area $\sqrt{A}$ such that $\sqrt{A}$ evolves according to $\frac{1}{\sqrt{A}}\frac{d\sqrt{A}}{d\nu} = \tilde\Theta$. We can transform (3.108) into
   $$\frac{d^2\sqrt{A}}{d\nu^2} = -\left[\frac{1}{2}E^2\left(\mu + p - 2q_a n^a + \pi_{ab}n^a n^b\right) + \left(\tilde\sigma^2 - \tilde\omega^2\right)\right]\sqrt{A}. \tag{3.109}$$
   This equation indicates the evolution of the light rays due to the matter inside the beam of light rays and is the *Optical Focussing Equation.* The terms in the parenthesis remain positive as long as the the strong energy condition is satisfied.

2. The *null shear propagation equation*:
   The shear down the null geodesics is obtained by taking the screen-space projected symmetric trace-free (SPSTF) part of (3.107) by multiplying through by $\tilde h_{\langle e}{}^a \tilde h_{f\rangle}{}^c$ and substituting in the subsequent expression for the SPSTF part of R_{abcd} from (3.34) to obtain
   $$\frac{d\tilde\sigma_{\langle ab\rangle}}{d\nu} = -\tilde\sigma_{ab}\tilde\Theta - 2E^2\left(\tilde E_{\langle ab\rangle} + \tilde H^c{}_{\langle a}\tilde S_{b\rangle c}\right). \tag{3.110}$$

3. The *null vorticity propagation equation*:
   The vorticity propagation equation is obtained from the projected anti-symmetric

part of (3.107) by multiplying through by the antisymmetric tensor $\tilde{S}^{ca}$ to give

$$\frac{d\tilde{\omega}}{d\nu} = -\tilde{\Theta}\tilde{\omega}. \tag{3.111}$$

4. The *energy propagation equation*:
We include the energy propagation equation

$$\frac{dE}{d\nu} = -E^2\left(\frac{1}{3}\Theta + \sigma_{ab}n^a n^b + \dot{u}_a n^a\right) \tag{3.112}$$

for completeness. Now the *cosmological redshift* is given by the formula

$$z = \frac{\omega_2}{\omega_1} = \frac{(u_a k^a)_2}{(u_a k^a)_1}, \tag{3.113}$$

where ω_1 and ω_2 are the frequencies connected to the change of phase of a light wave measured by two observers moving with the 4-velocities u_1^a and u_2^a, respectively. From the familiar definition of redshift for sources of light,

$$z = \frac{\lambda_o - \lambda_e}{\lambda_e} = \frac{\lambda_o}{\lambda_e} - 1,$$

where the subscripts e and o denote quantities calculated at the point of emission of the light ray and at the point of detection, respectively, and since $\lambda_o/\lambda_e = \omega_o/\omega_e$ then,

$$1 + z = \frac{\omega_e}{\omega_o} = \frac{(u_a k^a)_e}{(u_a k^a)_o}. \tag{3.114}$$

For small redshifts $z \sim (z << 1)$, then $z = d\lambda/\lambda$ and we now have

$$z = \frac{(u_a k^a)_e - (u_a k^a)_o}{(u_a k^a)_o} = \frac{d(u_a k^a)}{(u_a k^a)_o}. \tag{3.115}$$

From (3.89) and (3.93),

$$n_b k^b = -u_b k^b.$$

Thus, from the above expression and knowing that $[-(n_b k^b)_o]d\nu$ is the distance δl in the restspace travelled by the front of the light wave corresponding to a change of the affine parameter, i.e. distance between the light-source and the observer, we may write (3.112) as [58],

$$z = \left(\frac{1}{3}\Theta + \sigma_{ab}n^a n^b + n_a \dot{u}^b\right)_o \delta l. \tag{3.116}$$

From this, we see that rotation has no influence on redshift for $z << 1$. We also see that (3.112) is just the isotropic redshifting of photons due to expansion, and the anisotropic redshifting due the shear and acceleration.

Constraint equation

The constraint equation is obtained by multiplying the Ricci identity (3.106) through by $\tilde{h}^{ce}\tilde{h}^{ad}$ and substituting in the covariant derivative of the null tangent vector k^a from (3.103) giving

$$\begin{aligned} \tilde{D}_b\tilde{\sigma}^{ab} &= \frac{1}{2}\tilde{D}^a\tilde{\Theta} - \tilde{S}^{ab}\tilde{D}_b\tilde{\omega} + \frac{1}{2}\left(\tilde{\sigma}^{ab} - \frac{1}{2}\tilde{h}^{ab}\Theta\right)\left(\tilde{V}_b + \tilde{W}_b\right) \\ &+ \left(\tilde{h}^{ab}E_{bc} + \tilde{S}^{ab}H_{bc}\right)k^c + \frac{1}{2}E\left(\tilde{h}^{ab}\pi_{bc}n^c - \tilde{q}^a\right). \end{aligned} \tag{3.117}$$

3.3 1+1+2 Covariant approach

The 1+3 approach has proven to be a very useful technique in many aspects of astrophysics and relativistic cosmology. A natural extension to the 1+3 approach, optimized for problems which have spherical symmetry, is the 1+1+2 formalism developed recently by Clarkson and Barrett [59] and applied to the study of the generation of electromagnetic radiation by gravitational waves interacting with a strong magnetic field in the vicinity of a vibrating Schwarzschild black hole and of linear perturbations of locally rotationally symmetric (LRS) spacetimes [54, 60, 61].

3.3.1 Kinematics

In 1+3 approach, first we defined a timelike congruence by a timelike unit vector u^a which is split in the form $R \otimes V$, where R denotes the timeline along u^a and V is the 3-space perpendicular to u^a. As a natural extension of the 1+3 approach is the 1+1+2 approach in which we further split the 3-space V, by introducing the unit vector e^a orthogonal to u^a so that

$$e_a u^a = 0\ , \quad e_a e^a = 1. \tag{3.118}$$

The *projection tensor*

$$N_a{}^b \equiv h_a{}^b - e_a e^b = g_a{}^b + u_a u^b - e_a e^b\ , \quad N^a{}_a = 2 \tag{3.119}$$

then projects vectors orthogonal to e^a *and* the timelike vector u^a onto 2-surfaces referred to as the *sheet*, i.e. $e^a N_{ab} = 0 = u^a N_{ab}$.

In the 1+3 covariant approach to the study of null congruences (see section 3.2 of this chapter), the unit spatial vector, n^a was introduced, which has the properties $n^a n_a = 1$ and $u^a n_a = 0$. The projection tensor, $\tilde{h}^a{}_b \equiv h^a{}_b - n^a n_b$, which projects a vector or tensor into the 2-dimensional screen-space, orthogonal to the null tangent vector k^a, was also defined.

In the following analysis, the spatial vectors n^a and e^a do not coincide, but instead, if n^a is chosen to coincide with e^a, it immediately follows that $\tilde{h}^a{}_b = N^a{}_b$, so that the screen and sheet represent the same 2-dimensional surface [64].

A 3-vector, ψ^a, can be irreducibly split into a component along e^a and a sheet component Ψ^a, orthogonal to e^a i.e.

$$\psi^a = \Psi e^a + \Psi^a \ , \ \text{where} \ \ \Psi \equiv \psi^a e_a \ \ \text{and} \ \ \Psi^a \equiv N^{ab}\psi_b \ . \tag{3.120}$$

Similarly, a projected, symmetric, trace-free (PSTF) tensor, ψ_{ab}, can be split into scalar, vector and tensor part as follows:

$$\psi_{ab} = \psi_{\langle ab\rangle} = \Psi\left(e_a e_b - \frac{1}{2}N_{ab}\right) + 2\Psi_{(a}e_{b)} + \Psi_{ab} \ , \tag{3.121}$$

where

$$\begin{aligned} \Psi &\equiv e^a e^b \psi_{ab} = -N^{ab}\psi_{ab} \ , \\ \Psi_a &\equiv N_a{}^b e^c \psi_{bc} \ , \\ \Psi_{ab} &\equiv \psi_{\{ab\}} \equiv \left(N^c{}_{(a}N_{b)}{}^d - \frac{1}{2}N_{ab}N^{cd}\right)\psi_{cd} \ . \end{aligned} \tag{3.122}$$

The curly brackets denote the PSTF part of a tensor with respect to e^a. We also have

$$h_{\{ab\}} = 0 \ , \ N_{\langle ab\rangle} = -e_{\langle a}e_{b\rangle} = N_{ab} - \frac{2}{3}h_{ab} \ . \tag{3.123}$$

The sheet carries a natural 2-volume element, the alternating Levi-Civita 2-tensor defined by

$$\varepsilon_{ab} \equiv \varepsilon_{abc}e^c = \eta_{dabc}e^c u^d \ , \tag{3.124}$$

where ε_{abc} is the 3-space permutation symbol the volume element of the 3-space and η_{abcd} is just the spacetime permutator or the 4-volume. The following relations also hold

$$\varepsilon_{ab}e^b = 0 = \varepsilon_{(ab)} \ , \tag{3.125}$$

$$\varepsilon_{abc} = e_a\varepsilon_{bc} - e_b\varepsilon_{ac} + e_c\varepsilon_{ab} \ , \tag{3.126}$$

$$\varepsilon_{ab}\varepsilon^{cd} = N_a{}^c N_b{}^d - N_a{}^d N_b{}^c \ , \tag{3.127}$$

$$\varepsilon_a{}^c\varepsilon_{bc} = N_{ab} \ , \ \ \varepsilon^{ab}\varepsilon_{ab} = 2 \ . \tag{3.128}$$

With these definitions, it follows that any object in the 1+1+2 setting can be split into scalars and in the sheet, the 2-vectors and the PSTF 2-tensors.

Apart from the *'time'* (dot) derivative of an object (scalar, vector or tensor), which is the derivative along the timelike congruence u^a, we introduce two new derivatives which u^a defines for any object $\psi_{a..b}{}^{c..d}$:

$$\begin{aligned} \hat{\psi}_{a..b}{}^{c..d} &\equiv e^f \nabla_f \psi_{a..b}{}^{c..d} , \\ \delta_f \psi_{a..b}{}^{c..d} &\equiv N_a{}^f ... N_b{}^g N_h{}^c .. N_i{}^d N_f{}^j D_j \psi_{f..g}{}^{i..j} . \end{aligned} \quad (3.129)$$

The hat-derivative is the derivative along the e^a vector-field in the surfaces orthogonal to u^a [2]. The δ -derivative is the projected derivative on the sheet, with the projection on every free index.

We now decompose the covariant derivative of e^a orthogonal to u^a into it's irreducible parts:

$$\mathrm{D}_a e_b = e_a a_b + \frac{1}{2}\phi N_{ab} + \xi \epsilon_{ab} + \zeta_{ab} , \quad (3.130)$$

where

$$a_a \equiv e^c \mathrm{D}_c e_a = \hat{e}_a , \quad (3.131)$$

$$\phi \equiv \delta_a e^a , \quad (3.132)$$

$$\xi \equiv \frac{1}{2}\epsilon^{ab}\delta_a e_b , \quad (3.133)$$

$$\zeta_{ab} \equiv \delta_{\{a}e_{b\}} . \quad (3.134)$$

We see that travelling along e^a, ϕ represents the *expansion of the sheet*, ζ_{ab} is the *shear of* e^a (i.e., the distortion of the sheet) and a^a its *acceleration.* ξ can also be interpreted as the *vorticity* associated with e^a, so that it is a representation of the twisting' or rotation of the sheet.

The derivative of e^a in the direction of u^a given by,

$$\dot{e}_a = \mathcal{A} u_a + \alpha_a , \quad \text{where} \quad \mathcal{A} = e^a \dot{u}_a , \quad (3.135)$$

and α_a is the component lying in the sheet. The new variables a_a, ϕ, ξ, ζ_{ab} and α_a are fundamental objects of the spacetime and their dynamics give us information about the spacetime geometry. Essentially, they are treated on the same footing as the kinematical variables of u^a in the $1+3$ approach.

We split the 1+3 kinematical variables and Weyl tensors into the irreducible 1+1+2 set

[2]We note that the congruence u^a retains the primary importance it has in the 1+3 covariant approach. We choose to think of $\mathcal{A} \equiv u^a n^b{}_a u_b = u^a u^b{}_a n_b$ as the radial component of the acceleration of u^a, rather than the time component of $\dot{u}^a$

$\{\Theta, \mathcal{A}, \Omega, \Sigma, \mathcal{E}, \mathcal{H}, \mathcal{A}^a, \Omega^a, \Sigma^a, \mathcal{E}^a, \mathcal{H}^a, \Sigma_{ab}, \mathcal{E}_{ab}, \mathcal{H}_{ab}\}$ using equations (4.44) and (4.45) giving:

$$\dot{u}^a = \mathcal{A}e^a + \mathcal{A}^a \,, \quad (3.136)$$
$$\omega^a = \Omega e^a + \Omega^a \,, \quad (3.137)$$
$$\sigma_{ab} = \Sigma\left(e_a e_b - \frac{1}{2}N_{ab}\right) + 2\Sigma_{(a}e_{b)} + \Sigma_{ab} \,, \quad (3.138)$$
$$E_{ab} = \mathcal{E}\left(e_a e_b - \frac{1}{2}N_{ab}\right) + 2\mathcal{E}_{(a}e_{b)} + \mathcal{E}_{ab} \,, \quad (3.139)$$
$$H_{ab} = \mathcal{H}\left(e_a e_b - \frac{1}{2}N_{ab}\right) + 2\mathcal{H}_{(a}e_{b)} + \mathcal{H}_{ab} \,. \quad (3.140)$$

Similarly, the anisotropic fluid variables q_a and π_{ab} may be split as :

$$q_a = Qe^a + Q_a \,, \quad (3.141)$$
$$\pi_{ab} = \Pi\left[e_a e_b - \frac{1}{2}N_{ab}\right] + 2\Pi_{(a}e_{b)} + \Pi_{ab} \,. \quad (3.142)$$

3.3.2 Energy momentum tensor

In terms of 1+1+2 variables, the energy momentum tensor (3.38) reads explicitly as

$$T_{ab} = \mu u_a u_b + p h_{ab} + 2u_{(a}\left[Q n_{b)} + Q_{a)}\right] + \Pi\left(e_a\, e_b - \frac{1}{2}N_{ab}\right) + 2\Pi_{(a}e_{b)} + \Pi_{ab}, \quad (3.143)$$

where the decompositions (3.141) and (3.142) have been used.

The *curvature thermodynamical quantities* μ^R, p^R, π^R_{ab} and q^R_a can be written in terms of (1+1+2) variables as:

$$\mu^R = T^R_{ab}u^a u^b; \quad p^R = T^R_{ab}e^a e^b + T^R_{ab}N^{ab};$$
$$\pi^R_{ab} = T^R_{cd}h^c{}_{\langle a}h^d{}_{b\rangle}; \quad q^R_a = -T^R_{bc}h^b{}_a u^c. \quad (3.144)$$

3.3.3 Dynamics

In addition to the decomposition of the 1+3 equations, the dynamical equations for the covariant part of the derivative of e^a i.e. $\alpha_a,\ a_a,\ \phi,\ \xi,\ \epsilon$ and ζ_{ab} are required to determine completely the variables that characterize the spacetime i.e. the irreducible set of variables:

$$\{\Theta, \mathcal{A}, \Omega, \Sigma, \phi, \xi, \mathcal{E}, \mathcal{H}, \mathcal{A}^a, \Omega^a, \Sigma^a, \alpha^a, a^a, \mathcal{E}^a, \mathcal{H}^a; \Sigma_{ab}, \zeta_{ab}, \mathcal{E}_{ab}, \mathcal{H}_{ab}\}, \quad (3.145)$$

together with the irreducible set of thermodynamic variables,

$$\{\mu, p, Q, \Pi, Q_a, \Pi_{ab}\}. \quad (3.146)$$

The equations are obtained from the splitting the Ricci identity for e^a

$$R_{abc} \equiv 2\nabla_{[a}\nabla_{b]}e_c - R_{abcd}e^d = 0, \tag{3.147}$$

as 1+1+2 evolution equations along u^a and propagation equations along e^a. The constraint equations arise from both the splitting of the 1+3 constraint equations and suitable projections on R_{abc}. The complete set of 1+1+2 equations is available from Chris Clarkson [61] and for purposes of this thesis only a smaller set of equations governing the *locally rotationally symmetric* (LRS) spacetimes, will be needed.

3.3.4 Derivatives and Commutators

In terms of the 1+1+2 variables, the full covariant derivative of e^a is

$$\begin{aligned} \nabla_a e_b &= -\mathcal{A}u_a u_b - u_a \alpha_b + \left(\Sigma + \frac{1}{3}\Theta\right) e_a u_b + \left(\Sigma_a - \varepsilon_{ac}\Omega^c\right) u_b \\ &+ e_a a_b + \frac{1}{2}\phi N_{ab} + \xi\varepsilon_{ab} + \zeta_{ab} \ , \end{aligned} \tag{3.148}$$

and that of u^a

$$\begin{aligned} \nabla_a u_b &= -u_a(\mathcal{A}e_b + \mathcal{A}_b) + e_a e_b \left(\frac{1}{3}\Theta + \Sigma\right) + e_a \left(\Sigma_b + \varepsilon_{bc}\Omega^c\right) \\ &\quad + \left(\Sigma_a - \varepsilon_{ac}\Omega^c\right) e_b + N_{ab}\left(\frac{1}{3}\Theta - \frac{1}{2}\Sigma\right) + \Omega\varepsilon_{ab} + \Sigma_{ab} \ , \end{aligned} \tag{3.149}$$

which implies the useful relation

$$\hat{u}_a = \left(\frac{1}{3}\Theta + \Sigma\right) e_a + \Sigma_a + \varepsilon_{ab}\Omega^b. \tag{3.150}$$

The spatial covariant derivative of a scalar Ψ is defined as

$$\mathrm{D}_a\Psi = \hat{\Psi}e_a + \delta_a\Psi \ . \tag{3.151}$$

while for any vector Ψ^a orthogonal to both u^a and e^a (i.e. Ψ^a lies in the sheet), the various parts of its spatial derivative may be decomposed as follows [3] :

$$\mathrm{D}_a\Psi_b = -e_a e_b \Psi_c a^c + e_a \hat{\Psi}_{\bar{b}} - e_b \left[\frac{1}{2}\phi\Psi_a + \left(\xi\varepsilon_{ac} + \zeta_{ac}\right)\Psi^c\right] + \delta_a\Psi_b \ . \tag{3.152}$$

[3] Note that a bar on a particular index indicates that the vector or tensor lies in the sheet.

Similarly, for a tensor we have Ψ_{ab} (where $\Psi_{ab} = \Psi_{\{ab\}}$) :

$$\mathrm{D}_a\Psi_{bc} = -2e_a e_{(b}\Psi_{c)d}a^d + e_a\hat{\Psi}_{bc} - 2e_{(b}\left[\frac{1}{2}\phi\Psi_{c)a} + \Psi_{c)}{}^d\left(\xi\varepsilon_{ad} + \zeta_{ad}\right)\right] + \delta_a\Psi_{bc}\ . \quad (3.153)$$

We also include the derivatives that affect the projection tensor N_{ab} and the Levi-Civita tensor:

$$\begin{aligned}
\dot{N}_{ab} &= 2u_{(a}\dot{u}_{b)} - 2e_{(a}\dot{e}_{b)} = 2u_{(a}\mathcal{A}_{b)} - 2e_{(a}\alpha_{b)}\ ,\\
\hat{N}_{ab} &= -2e_{(a}a_{b)}\ ,\\
\delta_c N_{ab} &= 0\ ,\\
\dot{\varepsilon}_{ab} &= -2u_{[a}\varepsilon_{b]c}\mathcal{A}^c + 2e_{[a}\epsilon_{b]c}\alpha^c\ ,\\
\hat{\varepsilon}_{ab} &= 2e_{[a}\varepsilon_{b]c}a^c\ ,\\
\delta_c\varepsilon_{ab} &= 0\ . \quad (3.154)
\end{aligned}$$

The three derivatives defined so far, namely the dot derivative ($\dot{}$), the hat-derivative ($\hat{}$) and the delta-derivative (δ_a) do not commute and instead, when acting on a scalar ψ, they satisfy:

$$\begin{aligned}
\dot{\hat{\psi}} - \hat{\dot{\psi}} &= -\dot{u}\dot{\psi} + \left(\frac{1}{3}\Theta + \Sigma\right)\hat{\psi} + \left(\Sigma_a + \varepsilon_{ab}\Omega^b - \dot{e}_a\right)\delta^a\psi\ , \quad (3.155)\\
\delta_a\dot{\psi} - \left(\delta_a\psi\right)^{\cdot}_{\perp} &= -\dot{u}_a\dot{\psi} + \left(\dot{e}_a + \Sigma_a - \varepsilon_{ab}\,\Omega^b\right)\hat{\psi} + \left(\frac{1}{3}\Theta - \frac{1}{2}\Sigma\right)\delta_a\psi\\
&\quad + \left(\Sigma_{ab} + \Omega\,\varepsilon_{ab}\right)\delta^b\psi\ , \quad (3.156)\\
\delta_a\hat{\psi} - \left(\delta_a\psi\right)^{\hat{}}_{\perp} &= -2\varepsilon_{ab}\Omega^b\dot{\psi} + \hat{e}_a\dot{\psi} + \frac{1}{2}\phi\delta_a\psi + \left(\zeta_{ab} + \xi\,\varepsilon_{ab}\right)\delta^b\psi\ , \quad (3.157)\\
\delta_{[a}\delta_{b]}\psi &= \varepsilon_{ab}\left(\Omega\dot{\psi} - \xi\hat{\psi}\right). \quad (3.158)
\end{aligned}$$

where $\perp$ here denotes projection onto the sheet. Analogous relations for vectors and second-rank tensors hold but will not be needed in the later course of the thesis.

Chapter 4

1+1+2 description of spherically symmetric spacetimes in $f(R)$ gravity

4.1 1+1+2 LRS spacetimes

Locally rotationally symmetric (LRS) spacetimes posses continuous isotropy group at each point and hence a multi-transitive isometry group acting on the spacetime manifold [66]. These spacetimes exhibit locally (at each point) a unique preferred spatial direction, covariantly defined, for example, by either vorticity vector field or a non-vanishing non-gravitational acceleration of the matter fluids. The 1+1+2 formalism is therefore ideally suited for covariant description of these spacetimes, yielding a complete deviation in terms of invariant scalar quantities that have physical or direct geometrical meaning [54]. A detailed discussion of the covariant approach to LRS perfect fluid spacetimes can be seen in [66]. In deriving the lensing geometry in, we restrict ourselves to the case of spherically symmetric spacetimes.

4.1.1 Kinematics

The preferred spatial direction in the LRS spacetimes constitutes a local axis of symmetry and in this case e^a is just a vector pointing along the axis of symmetry and is thus called a 'radial' vector. LRS spacetimes are defined to be isotropic i.e. all observations are identical under rotations about it and in particular they are the same in all (spatial) directions orthogonal to that direction [67]. This allows for the vanishing of *all* 1+1+2 vectors and tensors, such that there are no preferred directions in the sheet. Thus, all the non-zero 1+1+2 variables are covariantly defined scalars. The full covariant derivative of the radial vector e^a (3.148) and 4-velocity u^a (3.149) reduce to

$$\nabla_a e_b = -\mathcal{A} u_a u_b + \left(\Sigma + \frac{1}{3}\Theta\right) e_a u_b + \frac{1}{2}\phi N_{ab} + \xi \varepsilon_{ab} \ , \quad (4.1)$$

and

$$\nabla_a u_b = -\mathcal{A} u_a e_b + e_a e_b \left(\frac{1}{3}\Theta + \Sigma\right) + N_{ab}\left(\frac{1}{3}\Theta - \frac{1}{2}\Sigma\right) + \Omega \varepsilon_{ab} \ , \quad (4.2)$$

respectively. We also find

$$\dot{e}_a = \mathcal{A} u_a \ , \quad (4.3)$$

is the evolution of e^a.
The kinematical quantities and Weyl tensors (3.136)-(3.142) reduce to

$$\dot{u}^a = \mathcal{A} e^a \ , \quad (4.4)$$
$$\omega^a = \Omega e^a \ , \quad (4.5)$$
$$\sigma_{ab} = \Sigma(e_a e_b - \frac{1}{2} N_{ab}) \ , \quad (4.6)$$
$$q_a^{tot} = Q^{tot} e^a \ , \quad (4.7)$$
$$\pi_{ab}^{tot} = \Pi^{tot}\left(e_a e_b - \frac{1}{2} N_{ab}\right) \ , \quad (4.8)$$
$$E_{ab} = \mathcal{E}(e_a e_b - \frac{1}{2} N_{ab}) \ , \quad (4.9)$$
$$H_{ab} = \mathcal{H}(e_a e_b - \frac{1}{2} N_{ab}) \ , \quad (4.10)$$

in an LRS spacetime.

4.1.2 Dynamics

The variables, $\{\mathcal{A}, \Theta, \phi, \xi, \Sigma, \Omega, \mathcal{E}, \mathcal{H}, \mu, p, \Pi, Q\}$ fully describe LRS spacetimes and are solved for in the 1+1+2 approach. The set of 1+1+2 evolution and/or propagation equations that these LRS (and thus any spherically symmetric) quantities satisfy, are obtained from the Bianchi and Ricci identities for the unit vector fields e^a and u^a and are given by: *Propagation* *and Evolution relations*

$$\begin{aligned}
\hat{\phi} &= -\frac{1}{2}\phi^2 + 2\xi^2 + \left(\frac{1}{3}\Theta + \Sigma\right)\left(\frac{2}{3}\Theta - \Sigma\right) \\
&- \frac{2}{3}\mu - \frac{1}{2}\Pi - \mathcal{E} \ , \qquad (4.11) \\
\hat{\xi} &= -\phi\xi + \left(\frac{1}{3}\Theta + \Sigma\right)\Omega \ , \qquad (4.12) \\
\hat{\Sigma} &= \frac{2}{3}\hat{\Theta} - \frac{3}{2}\phi\Sigma - 2\xi\Omega - Q \ , \qquad (4.13) \\
\hat{\Omega} &= (\mathcal{A} - \phi)\,\Omega \ , \qquad (4.14) \\
\hat{\mathcal{E}} &= \frac{1}{3}\hat{\mu} - \frac{1}{2}\hat{\Pi} - \frac{3}{2}\phi\left(\mathcal{E} + \frac{1}{2}\Pi\right) + \left(\frac{1}{2}\Sigma - \frac{1}{3}\Theta\right)Q + 3\Omega\mathcal{H} \ , \qquad (4.15) \\
\hat{\mathcal{H}} &= -\frac{3}{2}\phi\mathcal{H} - \left(3\mathcal{E} + \mu + p - \frac{1}{2}\Pi\right)\Omega - Q\xi \ . \qquad (4.16) \\
\dot{\phi} &= -\left(\Sigma - \frac{2}{3}\Theta\right)\left(\mathcal{A} - \frac{1}{2}\phi\right) + 2\xi\Omega + Q \ , \qquad (4.17) \\
\dot{\xi} &= \left(\frac{1}{2}\Sigma - \frac{1}{3}\Theta\right)\xi + \left(\mathcal{A} - \frac{1}{2}\phi\right)\Omega + \frac{1}{2}\mathcal{H} \ , \qquad (4.18) \\
\dot{\Omega} &= \mathcal{A}\xi + \Omega\left(\Sigma - \frac{2}{3}\Theta\right) \ , \qquad (4.19) \\
\dot{\Sigma} &= \frac{2}{3}\hat{\mathcal{A}} + \frac{1}{3}(2\mathcal{A} - \phi)\,\mathcal{A} - \left(\frac{2}{3}\Theta + \frac{1}{2}\Sigma\right)\Sigma - \frac{2}{3}\Omega^2 - \mathcal{E} + \frac{1}{2}\Pi \ , \qquad (4.20) \\
\dot{\mathcal{H}} &= -3\xi\mathcal{E} + \left(\Theta + \frac{3}{2}\Sigma\right)\mathcal{H} + \Omega Q + \frac{3}{2}\xi\Pi \ , \qquad (4.21) \\
\dot{\mu} &= -\hat{Q} - \Theta\,(\mu + p) - (\phi + 2\mathcal{A})\,Q - \frac{3}{2}\Sigma\Pi \ , \qquad (4.22) \\
\dot{Q} &= -\hat{p} - \hat{\Pi} - \left(\frac{3}{2}\phi + \mathcal{A}\right)\Pi - \left(\frac{4}{3}\Theta + \Sigma\right)Q - (\mu + p)\,\mathcal{A} \ , \qquad (4.23) \\
\hat{\mathcal{A}} &= \frac{1}{3}\dot{\Theta} + \dot{\Sigma} - \mathcal{A}^2 + \left(\frac{1}{3}\Theta + \Sigma\right)^2 + \frac{1}{6}(\mu + 3p) + \mathcal{E} - \frac{1}{2}\Pi \ , \qquad (4.24) \\
\hat{\mathcal{A}} &= \dot{\Theta} - (\mathcal{A} + \phi)\,\mathcal{A} + \frac{1}{3}\Theta^2 + \frac{3}{2}\Sigma^2 - 2\Omega^2 + \frac{1}{2}(\mu + 3p) \ , \qquad (4.25) \\
\dot{\mathcal{E}} &= -\frac{1}{2}\dot{\Pi} - \frac{1}{3}\hat{Q} + \left(\frac{3}{2}\Sigma - \Theta\right)\mathcal{E} - \frac{1}{2}\left(\frac{1}{3}\Theta + \frac{1}{2}\Sigma\right)\Pi \\
&+ \frac{1}{3}\left(\frac{1}{2}\phi - 2\mathcal{A}\right)Q + 3\xi\mathcal{H} - \frac{1}{2}(\mu + p)\,\Sigma \ , \qquad (4.26) \\
0 &= (2\mathcal{A} - \phi)\,\Omega - 3\xi\Sigma + \mathcal{H}. \qquad (4.27)
\end{aligned}$$

Commutation relations

$$\hat{\dot{\psi}} - \dot{\hat{\psi}} = -\mathcal{A}\dot{\psi} + \left(\frac{1}{3}\Theta + \Sigma\right)\hat{\psi}. \qquad (4.28)$$

4.2 1+1+2 LRS class II

This LRS class admits spherically symmetric solutions and is free of rotation, thus allowing for the vanishing of the variables Ω, ξ and $\mathcal{H}$, with the latter vanishing as a result of the constraint equation (4.27). The set of quantities that fully describe LRS class II spacetime are $\{\mathcal{A}, \Theta, \phi, \Sigma, \mathcal{E}, \mu, p, \Pi, Q\}$. The conditions defining this spacetime reduce the full LRS system of equations to:

Propagation equations

$$\hat{\phi} = -\frac{1}{2}\phi^2 + \left(\frac{1}{3}\Theta + \Sigma\right)\left(\frac{2}{3}\Theta - \Sigma\right) - \frac{2}{3}\mu - \frac{1}{2}\Pi - \mathcal{E}\ , \tag{4.29}$$

$$\hat{\Sigma} - \frac{2}{3}\hat{\Theta} = -\frac{3}{2}\phi\Sigma - Q\ , \tag{4.30}$$

$$\hat{\mathcal{E}} - \frac{1}{3}\hat{\mu} + \frac{1}{2}\hat{\Pi} = -\frac{3}{2}\phi\left(\mathcal{E} + \frac{1}{2}\Pi\right) + \left(\frac{1}{2}\Sigma - \frac{1}{3}\Theta\right)Q\ . \tag{4.31}$$

Evolution equations

$$\dot{\phi} = -\left(\Sigma - \frac{2}{3}\Theta\right)\left(\mathcal{A} - \frac{1}{2}\phi\right) + Q\ , \tag{4.32}$$

$$\dot{\Sigma} - \frac{2}{3}\dot{\Theta} = -\mathcal{A}\phi + 2\left(\frac{1}{3}\Theta - \frac{1}{2}\Sigma\right)^2 + \frac{1}{3}(\mu + 3p) - \mathcal{E} + \frac{1}{2}\Pi\ , \tag{4.33}$$

$$\begin{aligned}\dot{\mathcal{E}} - \frac{1}{3}\dot{\mu} + \frac{1}{2}\dot{\Pi} &= \left(\frac{3}{2}\Sigma - \Theta\right)\mathcal{E} + \frac{1}{4}\left(\Sigma - \frac{2}{3}\Theta\right)\Pi \\ &\quad + \frac{1}{2}\phi Q - \frac{1}{2}(\mu + p)\left(\Sigma - \frac{2}{3}\Theta\right)\ .\end{aligned} \tag{4.34}$$

Propagation/Evolution Equations

$$\dot{\mu} + \hat{Q} = -\Theta(\mu + p) - (\phi + 2\mathcal{A})Q - \frac{3}{2}\Sigma\Pi\ , \tag{4.35}$$

$$\dot{Q} + \hat{p} + \hat{\Pi} = -\left(\frac{3}{2}\phi + \mathcal{A}\right)\Pi - \left(\frac{4}{3}\Theta + \Sigma\right)Q - (\mu + p)\mathcal{A}\ , \tag{4.36}$$

$$\hat{\mathcal{A}} - \dot{\Theta} = -(\mathcal{A} + \phi)\mathcal{A} + \frac{1}{3}\Theta^2 + \frac{3}{2}\Sigma^2 + \frac{1}{2}(\mu + 3p)\ , \tag{4.37}$$

Commutation relations

$$\hat{\dot{\psi}} - \dot{\hat{\psi}} = -\mathcal{A}\dot{\psi} + \left(\frac{1}{3}\Theta + \Sigma\right)\hat{\psi}. \tag{4.38}$$

Since the vorticity vanishes, the unit vector field u^a is hypersurface-orthogonal to the space-like 3-surfaces whose intrinsic curvature can be calculated from the Gauss equation (3.37). With the additional constraint of the vanishing of the sheet distortion ξ, i.e., the sheet is a genuine 2-surface, the Gauss equation for e^a together with the 3-Ricci identities determine the 3-Ricci curvature tensor of the spacelike 3-surfaces orthogonal to u^a to be

$$^3R_{ab} = -\left[\hat{\phi} + \frac{1}{2}\phi^2\right] e_a e_b - \left[\frac{1}{2}\hat{\phi} + \frac{1}{2}\phi^2 - K\right] N_{ab}. \tag{4.39}$$

This gives the 3-Ricci-scalar as

$$^3R = -2\left[\frac{1}{2}\hat{\phi} + \frac{3}{4}\phi^2 - K\right], \tag{4.40}$$

where K is the *Gaussian curvature* of the sheet, $^2R_{ab} = KN_{ab}$. From this equation and (4.29) an expression for K is obtained in the form [54]:

$$K = \frac{1}{3}\mu - \mathcal{E} - \frac{1}{2}\Pi + \frac{1}{4}\phi^2 - \left(\frac{1}{3}\Theta - \frac{1}{2}\Sigma\right)^2. \tag{4.41}$$

From (4.29-4.34), the evolution and propagation equations of K can be determined as

$$\dot{K} = -\frac{2}{3}\left(\frac{2}{3}\Theta - \Sigma\right)K, \tag{4.42}$$

$$\hat{K} = -\phi K. \tag{4.43}$$

From equation (4.42), it follows that whenever the Gaussian curvature of the sheet is constant in time, then the shear is always proportional to the expansion: $\Sigma = \frac{2}{3}\Theta$.

4.3 Static spherically symmetric spacetimes in $f(R)$ gravity

A spherically symmetric static spacetime belongs to LRS class II. It is evident that the condition of staticity implies that the dot derivatives of all the quantities vanish. Furthermore the expansion also vanishes, as a non-vanishing expansion would imply that the timelike congruence would contract or expand in time, which is not possible in a static spacetime. Shear $\Sigma = 0$ vanishes by virtue of (4.38) and from equation (4.32) it follows that the heat flux Q also vanishes identically in these spacetimes.

Just like the vacuum case in General Relativity, we assume here the absence of standard matter. However, as we have already seen, the fourth order gravity field equations can be written as the field equations in GR in the presence of a '*curvature fluid*'. When the vacuum spherically symmetric static conditions are applied to $f(R)$, the covariant thermodynamic terms energy density μ^R, pressure p^R and anisotropic stress π^R due to the curvature fluid are still retained in the propagation equations. Hence the set of (1+1+2) equations which describe a static vacuum LRS II spacetime become

$$\hat{\phi} = -\frac{1}{2}\phi^2 - \frac{2}{3}\mu^R - \frac{1}{2}\Pi^R - \mathcal{E}\,, \tag{4.44}$$

$$\hat{\mathcal{E}} - \frac{1}{3}\hat{\mu}^R + \frac{1}{2}\hat{\Pi} = -\frac{3}{2}\phi\left(\mathcal{E} + \frac{1}{2}\Pi^R\right)\,, \tag{4.45}$$

$$0 = -\mathcal{A}\phi + \frac{1}{3}\left(\mu^R + 3p^R\right) - \mathcal{E} + \frac{1}{2}\Pi^R\,, \tag{4.46}$$

$$\hat{p}^R + \hat{\Pi}^R = -\left(\frac{3}{2}\phi + \mathcal{A}\right)\Pi^R - \left(\mu^R + p^R\right)\mathcal{A}\,, \tag{4.47}$$

$$\hat{\mathcal{A}} = -\left(\mathcal{A} + \phi\right)\mathcal{A} + \frac{1}{2}\left(\mu^R + 3p^R\right)\,, \tag{4.48}$$

Subtracting (4.44) and (4.46) in this case gives

$$\mathcal{E} = -\frac{1}{2}\hat{\phi} - \frac{1}{4}\phi^2 - \frac{1}{6}\mu^R + \frac{1}{2}p^R - \frac{1}{2}\mathcal{A}\phi, \tag{4.49}$$

and adding the two gives

$$\Pi^R = -\hat{\phi} - \frac{1}{2}\phi^2 - \mu^R - p^R + \mathcal{A}\phi. \tag{4.50}$$

From (4.49) and (4.50) we get the expression of $\mathcal{E}$ as

$$\mathcal{E} = \frac{1}{2}\Pi^R + \frac{1}{3}\mu^R + p^R - \mathcal{A}\phi \tag{4.51}$$

and its propagation equation from (4.49)

$$\hat{\mathcal{E}} = \frac{1}{2}\hat{\Pi}^R + \frac{1}{3}\hat{\mu}^R + \hat{p}^R - \hat{\mathcal{A}}\phi - \mathcal{A}\hat{\phi}. \tag{4.52}$$

Upon substituting (4.51) and (4.52) into (4.45), we obtain

$$\hat{\Pi}^R + \hat{p}^R = \hat{\mathcal{A}}\phi + \mathcal{A}\hat{\phi} - \frac{3}{2}\phi\left(\Pi^R + \frac{1}{3}\mu^R + p^R - \mathcal{A}\phi\right).$$

From this we see that (4.45) and (4.47) are simultaneous in $\hat{\Pi}^R + \hat{p}^R$, and from these two equations the following expression is obtained:

$$\begin{aligned} \hat{\mathcal{A}}\phi &= \frac{1}{2}\left(\mu^R + 3p^R\right)\phi - \left(\mathcal{A} + \phi\right)\mathcal{A}\phi, \\ \text{or } \hat{\mathcal{A}} &= -\left(\mathcal{A} + \phi\right)\mathcal{A} + \frac{1}{2}\left(\mu^R + 3p^R\right), \end{aligned} \tag{4.53}$$

which is just (4.48). So in eliminating $\mathcal{E}$, we find that (4.45) and (4.47) merely repeat the information in (4.48) and hence the useful equations are (4.44), (4.48).
Allowing for a further split of the 3-vector term, $\widetilde{\nabla}^a R$ which we may represent by the vector

x^a as

$$\tilde{\nabla}_a R = x_a = X e_a + X_a, \tag{4.54}$$

then by gnoring the vector term due to spherical symmetry, we get

$$\begin{aligned} \tilde{\nabla}^2 R = \tilde{\nabla}^a x_a &= \tilde{\nabla}^a (X e_a) \\ &= \hat{X} + X\phi. \end{aligned} \tag{4.55}$$

Applying the 1+1+2 decomposition and considering the above split for $\tilde{\nabla}^a$, then μ^R, p^R and π^R_{ab} now take the form,

$$\mu^R = \frac{1}{f'}\left(\frac{1}{2}(Rf' - f) + f''\hat{X} + f''X\phi + f'''X^2\right), \tag{4.56}$$

$$p^R = \frac{1}{f'}\left(\frac{1}{2}(f - Rf') - \frac{2}{3}f''\hat{X} - \frac{2}{3}f''X\phi - \frac{2}{3}f'''X^2 - \mathcal{A}f''X\right), \tag{4.57}$$

$$\pi^R_{ab} = \Pi^R\left(e_a e_b - \frac{1}{2}N_{ab}\right), \tag{4.58}$$

where

$$\begin{aligned} \hat{R} &= e^a\tilde{\nabla}_a R = e^a e_a X \\ &= X. \end{aligned} \tag{4.59}$$

The spatially projected scalar term Π^R of π^R_{ab} in can be obtained from it's definition (3.52) and the 1+1+2 breakdown (4.58) as

$$\Pi^R = e^a e^b \pi^R_{ab} = \frac{1}{f'}\left((e^d\hat{\tilde{\nabla}}_d f') - \hat{e}^d\tilde{\nabla}_d f' - \frac{1}{3}f''\tilde{\nabla}^2 R - \frac{1}{3}f'''\tilde{\nabla}^a R\tilde{\nabla}_a R\right), \tag{4.60}$$

and from $(e^d\hat{\tilde{\nabla}}_d f') = \hat{f}' = f'''\hat{R}^2 + f''\hat{\hat{R}}$ or $= f'''X^2 + f''\hat{X}$, and using (4.55), we obtain

$$\Pi^R = \frac{1}{f'}\left(\frac{2}{3}f'''X^2 + \frac{2}{3}f''\hat{X} - \frac{1}{3}f''X\phi\right) \tag{4.61}$$

(the acceleration term $\hat{e}^d$ has been ignored due to spherical symmetry).

To close the set of equations, we include the *trace equation* which in static spherical symmetric spacetime where upon considering 1+1+2 decomposition is expressed as

$$Rf' - 2f = -3f''\hat{X} - 3f''X\phi - 3f'''X^2 - 3\mathcal{A}f''X. \tag{4.62}$$

Now from (4.62), we obtain the equation for $\hat{X}$ as

$$\hat{X} = -\frac{1}{3}\frac{Rf'}{f''} + \frac{2}{3}\frac{f}{f''} - X\phi - \frac{f'''X^2}{f''} - X\mathcal{A}. \tag{4.63}$$

(4.50) gives $\hat{\phi}$ as

$$\hat{\phi} = -\frac{1}{2}\phi^2 + \frac{1}{3}R - \frac{2}{3}\frac{f}{f'} + \frac{f''}{f'}X\left(\phi + 2\mathcal{A}\right) + \mathcal{A}\phi, \tag{4.64}$$

and (4.48),

$$\hat{\mathcal{A}} = -\mathcal{A}^2 - \mathcal{A}\phi + \frac{1}{6}\frac{f}{f'} - \frac{1}{3}R - \frac{f''}{f'}X\mathcal{A}, \tag{4.65}$$

where we substitute for $\hat{X}$ from (4.63).

The equations (4.59), (4.63),(4.64) and (4.65), i.e.,

$$\hat{R} = X, \tag{4.66}$$

$$\hat{X} = -\frac{1}{3}\frac{Rf'}{f''} + \frac{2}{3}\frac{f}{f''} - X\phi - \frac{f'''X^2}{f''} - X\mathcal{A}, \tag{4.67}$$

$$\hat{\phi} = -\frac{1}{2}\phi^2 + \frac{1}{3}R - \frac{2}{3}\frac{f}{f'} + \frac{f''}{f'}X\left(\phi + 2\mathcal{A}\right) + \mathcal{A}\phi, \tag{4.68}$$

$$\hat{\mathcal{A}} = -\mathcal{A}^2 - \mathcal{A}\phi + \frac{1}{6}\frac{f}{f'} - \frac{1}{3}R - \frac{f''}{f'}X\mathcal{A}, \tag{4.69}$$

are a set of four coupled first order equations governing the spacetime in fourth order gravity and the static and vacuum spherically symmetric spacetime will be determined by any of the three non-zero scalar functions R, X, ϕ and $\mathcal{A}$ that define them.

We emphasize here that the above system of equations are in terms of the covariant quantities in 1+1+2 splitting and absolutely co-ordinate independent. Note that he system reduces to the second order system of GR in vacuum [59], if we put $f(R) = R$, $R = 0$ and $X = 0$ [1]. However as in the case of the Einstein equation, or any other fully covariant system of equations, the physics can be understood fully only if one chooses an observer. In the 1+3 approach this is done basically choosing a velocity field, but in the 1+1+2 framework this is not sufficient. One has to give also a particular form of '*radial*' co-ordinate. This in turn will define a specific form for the '*hat*' derivative. As we will see in the later sections there is a natural choice for this coordinate given by the geometry of our problem and we will use it to find exact spherically symmetric solutions for some specific $f(R)$ gravity models.

[1]The last two conditions are given by the fact that in GR the Ricci scalar is simply proportional to the matter density and the pressure of a fluid and becomes automatically zero in vacuum.

4.4 Covariant results for the spherically symmetric system

From the structure of (4.66-4.69) we can already deduce some important results for spherically symmetric static solutions in a general $f(R)$ gravity in absolute co-ordinate independent manner. These results are important because they can be used as guidelines to understand the behavior of any proposed $f(R)$ model in this setting and to design new ones.

1. Necessary condition for existence of solutions with a vanishing Ricci Scalar:
We impose the condition of class C^3 on the function $'f'$ with $R = 0$ implies,

$$|f'(0)| < +\infty\,, |f''(0)| < +\infty\,, |f'''(0)| < +\infty \quad . \tag{4.70}$$

We also impose the conditions

$$f(0) = 0, R = 0, \tag{4.71}$$

and note that the condition of vanishing of the Ricci scalar throughout the manifold automatically implies $X = 0$.
Now the two possibilities to consider are:

a. $f'(0) \neq 0$: In this case we see the system reduces to the following:

$$\hat{\phi} + \phi\left(\frac{1}{2}\phi - \mathcal{A}\right) = 0\,, \tag{4.72}$$

$$\hat{\mathcal{A}} + \mathcal{A}(\mathcal{A} + \phi) = 0\,, \tag{4.73}$$

We know from [54, 59] that this system has an unique solution which corresponds to the Schwarzschild metric in Schwarzschild co-ordinates. Birkhoff's theorem [2] states that as long spherical symmetry is maintained, the Schwarzschild metric is the only solution outside any mass distribution. The above system of equations then enables us to state a generalization of *Birkhoff's Theorem* in higher order gravity.

For all functions $f(R)$ which are of class C^3 at $R = 0$ and $f(0) = 0$ while $f'(0) \neq 0$, Schwarzschild solution is the only vacuum solution with vanishing Ricci scalar for a spherical symmetric matter distribution.

It is also interesting to note that the conditions $f' > 0$ and $f'' > 0$, guarantee the attractive nature of the gravitational interaction and the absence of tachyons [62], and the only solution with $R = 0$ is the Schwarzschild one. This shows a profound connection between this solution and the very nature of the gravitational interaction. The presence of this solution, can have interesting consequences on the validity of

[2] Of interest is that the theorem was actually discovered three years earlier by Norwegian physicist J.T. Jebsen and published in 1921 in the proceedings of the Swedish Academy of Sciences [65]

these models on the Solar System level. In particular if one concludes that the sun behaves very close to a Schwarzschild solution, the experimental data of the solar system would help constrain these models.

b. $f'(0) = 0, f(0) = 0$ [3]: In this case (4.66-4.69) are identically satisfied for all values of ϕ and $\mathcal{A}$ that guarantees $R = 0$ and hence $X = 0$. Hence for all models with $f'(0) = 0$, any solution with vanishing Ricci Scalar in GR would be a solution to the above system. This is interesting as it shows that gravity can in principle mimic other fields. For example the Reissner Nordstrom solution that represents the space time outside a spherically symmetric charged body, is a solution to the above system without any charge. Similarly, a static spherically symmetric solution for a perfect fluid interior with equation of state $p = 1/3\rho$ (for example Hajj-Boutros solution or the special case of Whittaker solution [68]) can be a solution of this system in the absence of any standard fluid.

The presence of solutions of type (*b*) shows that when the conditions given in paragraph (*a*) are not satisfied the Schwarzschild solution is not a unique static spherically symmetric solution. Such results hint towards a disproof of the general Birkhoff theorem in it's present form for fourth order gravity.

2. Necessary condition for existence of solutions with constant scalar curvature:
Solutions with constant Ricci scalar are characterized by the fact that $R = R_0 =$ *const.* and, as consequence, $X, \hat{X} = 0$. Imposing these conditions on (4.66-4.69) and supposing it to be at least of class C^3 in $R = R_0$ one obtains

$$f_0'\left[\hat{\phi} + \phi\left(\frac{1}{2}\phi - \mathcal{A}\right)\right] = \frac{1}{3}R_0 f_0' - \frac{2}{3}f_0\,, \tag{4.74}$$

$$f_0'\left[\hat{\mathcal{A}} + \mathcal{A}(\mathcal{A}+\phi)\right] = \frac{1}{6}f_0 - \frac{1}{3}R_0 f'\,, \tag{4.75}$$

$$0 = -R_0 f_0' + 2f_0\,, \tag{4.76}$$

where $f'(R_0) = f_0'$ *etc.* A first solution exists if

$$f_0' \neq 0\,, \quad f_0 \neq 0\,, \quad 2f_0 - R_0 f_0' = 0\,. \tag{4.77}$$

Instead in the case $f_0' \neq 0$, $f_0 = 0$ one obtains again the Schwarzschild solution ($R_0 = 0$). Finally another solution can be achieved if

$$f_0' = 0\,, \quad f_0 = 0\,, \quad R = R_0\,, \quad X,\, \hat{X} = 0\,, \tag{4.78}$$

[3] We note here that the case $f'(0) = 0, f(0) = 0$ is pathological if we consider that $f = 0$ corresponds to infinite gravitational coupling $G_{effective} = G/f$ and to a singular point of the field equations. This singularity also occurs in the widely used theory of a scalar field non-minimally coupled to the curvature [63]

is satisfied. As in the previous subsection, in this case also, any constant Ricci scalar solution in GR would identically be a solution to the system.
The relation (4.77) was already found by Barrow and Ottewill [69] in the cosmological context and later rediscovered in [70]. It relates the value of the constant Ricci scalar with the universal constants in the action. For example if we have the Lagrangian as $R - 2\Lambda$, which is the Lagrangian for GR with the cosmological constant, we must have, as well known, the relation $R_0 = 4\Lambda$.

3. The curious case of R^2 gravity:
As we have already explained, the condition for existence of solutions with covariantly constant scalar curvature connects the constant curvature with the universal constants of the Lagrangian. However this is not the case for $f(R) = KR^2$. In fact for this type of Lagrangian the third condition of (4.77) is identically satisfied. This means that we can have a constant curvature solution for any value of the curvature. Thus for R^2 gravity, the '*cosmological*' constant term in a Schwarzschild-dS/AdS spacetime becomes a local constant of integration just like the mass. Hence in this theory we can have two distant stars behaving like two different Schwarzschild-dS/AdS objects with different values of the constant, and just by studying the geodesic motion, it of a single test body around them, it is impossible to determine their mass uniquely.

4.5 Choosing a co-ordinate system and relation between the covariant variables and the metric

When considering spherical symmetry, it is convenient to express the hat derivative in terms of the proper radial coordinate r. The most natural way to this is to make the Gaussian curvature 'K' of the spherical sheets proportional to the inverse square of the radius $K = \frac{1}{r^2}$. In that case, this co-ordinate 'r' becomes the *area radius* of the sheets. This gives a geometrical definition to the '*hat*' derivative. As we have already seen, $\hat{K} = -\phi K$, therefore the most natural way to define the hat derivative of any scalar M would be

$$\hat{M} = \frac{dM}{d\nu} = \frac{1}{2}r\phi\frac{\partial M}{\partial r} + \hat{\tau}\dot{M}, \tag{4.79}$$

where τ denotes proper time and M is any scalar. For the static case the expression reduces to

$$\hat{M} = \frac{1}{2}\phi r\frac{dM}{dr}. \tag{4.80}$$

Hence in this coordinate the system of equations (4.66–4.69) becomes

$$\frac{1}{2}r\phi\frac{d\phi}{dr} = -\frac{1}{2}\phi^2 + \frac{1}{3}R - \frac{2}{3}\frac{f}{f'} + \frac{f''}{f'}X(\phi + 2\mathcal{A}) + \mathcal{A}\phi\,, \tag{4.81}$$

$$\frac{1}{2}r\phi\frac{d\mathcal{A}}{dr} = -\mathcal{A}^2 - \mathcal{A}\phi + \frac{1}{6}\frac{f}{f'} - \frac{1}{3}R - \frac{f''}{f'}X\mathcal{A}\,, \tag{4.82}$$

$$\frac{1}{2}r\phi\frac{dR}{dr} = X\,, \tag{4.83}$$

$$\frac{1}{2}r\phi\frac{dX}{\partial r} = -\frac{1}{3}\frac{Rf'}{f''} + \frac{2}{3}\frac{f}{f''} - X\phi - \frac{f'''X^2}{f''} - X\mathcal{A}\,. \tag{4.84}$$

In the r co-ordinate above, the most general spherically symmetric static metric is given by

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{4.85}$$

Now, from the properties of the four-velocity u^a and the radial vector e^a i.e. $u^a u_a = -1$ and $e^a e_a = 1$, we find that

$$u^t = \sqrt{A(r)}\,, \quad e^r = \sqrt{B(r)}\,, \tag{4.86}$$

Also, from the definitions of different covariant scalars we get

$$\mathcal{A} = -u^b u^a \nabla_b e_a = \frac{1}{2}\frac{1}{A\sqrt{B}}\frac{dA}{dr}; \tag{4.87}$$

$$\phi = N^b{}_a \nabla_b e^a = \frac{2}{r\sqrt{B}}. \tag{4.88}$$

R can be found in the usual way as a contraction of the Riemann tensor and X is derived from (4.83):

$$X = \frac{1}{r\sqrt{B}}\frac{dR}{dr}. \tag{4.89}$$

Thus we see that solutions to the equations (4.81–4.84), uniquely determine the metric for the spacetime.

4.6 Some exact solutions for R^n Gravity

In this section we present a few exact solutions for R^n gravity, in absence of standard matter. Specializing the choice of $f(R) = R^n$, equations (4.81-4.84) becomes

$$\begin{aligned}
\frac{1}{2}nr\phi\frac{d\phi}{dr}R^{n-1} &= \left(A - \frac{1}{2}\phi\right)\phi R^{n-1} + \frac{n-2}{3n}R^n \\
&\quad +(n-1)R^{n-2}X(\phi + 2\mathcal{A}) \ , \qquad (4.90)\\
\frac{1}{2}nr\phi\frac{d\mathcal{A}}{dr}R^{n-1} &= -(\mathcal{A}+\phi)\mathcal{A}R^{n-1} + \frac{1-2n}{6n}R^n \\
&\quad -(n-1)R^{n-2}X\mathcal{A} \ , \qquad (4.91)\\
\frac{1}{2}r\phi\frac{dR}{dr} &= X \ , \qquad (4.92)\\
\frac{1}{2}r\phi n(n-1)\frac{dX}{dr}R^{n-2} &= \frac{2-n}{3}R^n - X(\phi+\mathcal{A}) \\
&\quad -n(n-1)(n-2)R^{n-3}X^2 \, . \qquad (4.93)
\end{aligned}$$

4.6.1 Schwarzschild solution

Substituting $R = 0$, $dR/dr = 0$ in the above set of equations, we see that the equations are satisfied trivially provided that $n = 1, 2, > 3$. However since $R = 0$ is by itself a differential constraint involving ϕ and $\mathcal{A}$, then any ϕ and $\mathcal{A}$ that ensures a zero Ricci scalar would solve the system. The following solution

$$\phi = \frac{2}{r}\sqrt{1 - \frac{2m}{r}} \ , \ \mathcal{A} = \frac{m}{r^2}\left[1 - \frac{2m}{r}\right]^{-\frac{1}{2}} . \qquad (4.94)$$

with the equations (4.87) and (4.88) gives the usual Schwarzschild metric in (t, r, θ, ϕ) coordinates that has a zero Ricci scalar, hence the above solution is the solution of the system.

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4.6.2 A solution with constant non-zero Ricci scalar

As described before if we substitute $X = 0$, $R = R_0 \neq 0$ in the above system of equations then a solution is possible if and only if $n = 1$. In that case the solutions of the other two functions are

$$\phi = \frac{2}{r}\sqrt{1 - \frac{2m}{r} + \frac{R_0}{3}r^2} \ , \ \mathcal{A} = \frac{m + R_0 r^2}{r^2}\left[1 - \frac{2m}{r} + \frac{R_0}{3}r^2\right]^{-\frac{1}{2}} . \qquad (4.95)$$

This is the usual Schwarzschild-dS/AdS solution depending on the sign of R_0.

4.6.3 A solution with non-constant Ricci scalar vanishing at infinity

To find more non-trivial solutions of the above system of equations, we use a Schwarzschild like ansatz,

$$\phi = C_1 r^{\alpha} + C_2 r^{\beta}\ ,\ R = 1/r^{\gamma}\ (\gamma > 0), \tag{4.96}$$

such that the Ricci scalar vanishes at infinity. We use these ansatz in the system of equations and then algebraically solve for the powers and coefficients such that the system is identically satisfied. With this choice we get the following solution:

$$\mathcal{A} = \frac{-C(5-4n)r^{\frac{7-5n}{n-2}} + (2n-2)(2n-1)r^{\frac{5n-4n^2}{n-2}}}{2(2-n)r^{\frac{(2n-2)(2n-1)}{2-n}}\sqrt{\frac{(1+2n-2n^2)(7-10n+4n^2)\left(1+Cr^{-\frac{7-10n+4n^2}{2-n}}\right)}{(2-n)^2}}} \tag{4.97}$$

$$\phi = \frac{2}{r\sqrt{\frac{(1+2n-2n^2)(7-10n+4n^2)}{(2-n)^2\left(1+Cr^{-\frac{7-10n+4n^2}{2-n}}\right)}}} \tag{4.98}$$

The Ricci scalar for this solution is

$$R = \frac{6n(n-1)}{[2n(n-1)-1]r^2} \tag{4.99}$$

from which X is found as

$$X = -\frac{12n(n-1)}{[2n(n-1)-1]r^3\sqrt{\frac{(1+2n-2n^2)(7-10n+4n^2)}{(2-n)^2\left(1+Cr^{-\frac{7-10n+4n^2}{2-n}}\right)}}}. \tag{4.100}$$

Solving for the metric coefficients, we obtain

$$A(r) = r^{(2n-2)\frac{(2n-1)}{(2-n)}} + \frac{C}{r^{\frac{(5-4n)}{(2-n)}}}$$

$$\frac{1}{B(r)} = \frac{(2-n)^2}{(7-10n+4n^2)(1+2n-2n^2)}\left(1 + \frac{C}{r^{\frac{(7-10n+4n^2)}{(2-n)}}}\right) \tag{4.101}$$

This solution was originally found by Clifton [71]. The solution reduces to Schwarzschild for $n = 1$ and is valid for $n \neq 2$ and $n < (1+\sqrt{3})/2$, otherwise the metric has unphysical signature. However for $n \in (1, (1+\sqrt{3})/2)$ the Ricci scalar is negative and hence the action is only realvalued if n is an even rational number. That is in it's lowest form the numerator of the fractionis even. It is also interesting to note that in spite of the Ricci scalar vanishing at infinity this solution is *not* asymptotically flat.

Chapter 5

Gravitational lensing in fourth order gravity

This chapter derives the deflection angle for general $f(R)$ gravity and then for a particular R^n model. The initial of the chapter is drawn mostly from [64].

5.1 General Form of the Deflection Angle

The spatial vector, n^a, was introduced in chapter 3, which is a 3-vector in the direction of light propagation. Allowing for a further split of this vector with respect to e^a gives :

$$n^a = \kappa e^a + \kappa^a \ , \tag{5.1}$$

where κ is the *magnitude of the radial component* and κ^a is the *component lying in the 2-dimensional sheet*. The null tangent vector k^a can now be written as [64]

$$k^a = E(u^a + \kappa e^a + \kappa^a) \ . \tag{5.2}$$

The lensing geometry of a photon experiencing a deflection about the centre of symmetry can be obtained once κ and E have been determined.

Fig (5.1) traces the path of a photon in the presence of a gravitational field with k^a lying tangent to the null geodesic [1]. The two-dimensional sheet is represented as a sphere about a central point and the directions of the various vector components of the null vector are also included. For infinitesimal small angles of deflection $d\alpha$, the lensing geometry is represented by Fig (5.2), where the angle $d\alpha$ subtends an infinitesimal small displacement

[1]Note that the deflection seen in Fig (5.1) has been greatly exaggerated.

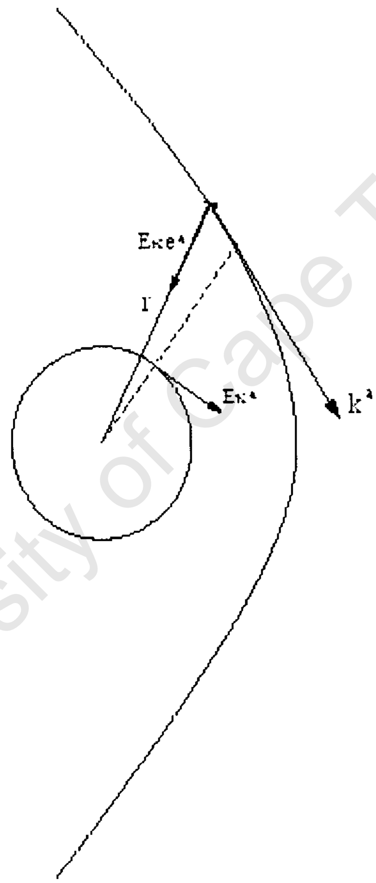

Figure 5.1: The lensing geometry of a photon in a spherically symmetric spacetime.

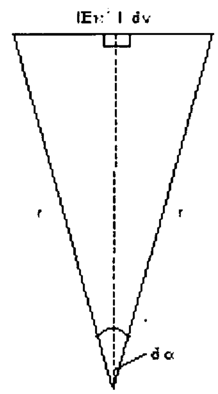

Figure 5.2: Geometry of the deflection angle. Figures taken from [64]

of the photon's path, given by $|E\kappa^a|$ and can be obtained from Fig (5.2) as

$$d\alpha = \frac{1}{r}\,|E\kappa^a|\,d\nu\ , \tag{5.3}$$

Integrating 5.3 gives the total deflection angle as

$$\alpha = \int_{\nu_1}^{\nu_2} \frac{1}{r}\,|E\kappa^a| d\nu - \alpha_0\ , \tag{5.4}$$

where α_0 is the integration constant. This is a completely general form of the *scalar deflection angle* which can be applied to any spherically symmetric spacetime. The deflection angle α (5.4) may be rewritten as

$$\alpha = \int_{\nu_1}^{\nu_2} \frac{1}{r}\,|E|\,\sqrt{\kappa^a\kappa_a} d\nu - \alpha_0\ , \tag{5.5}$$

where the magnitude of κ^a (i.e. $|\kappa^a|$) is just $\sqrt{\kappa^a\kappa_a}$. Substituting for k^a (5.2) into the null condition, $k_a k^a = 0$, gives a restriction on κ^a :

$$\kappa_a\kappa^a = 1 - \kappa^2\ , \tag{5.6}$$

to give the general deflection angle in the form [64]:

$$\alpha = \int_{\nu_1}^{\nu_2} \frac{1}{r}\,|E|\,\sqrt{1-\kappa^2} d\nu - \alpha_0\ . \tag{5.7}$$

Thus, knowing the relations $E(\nu)$, $\kappa(\nu)$ and $r(\nu)$, for a given spherically symmetric spacetime, it is possible to find an explicit form of the deflection angle.

5.2 General Propagation Equations along k^a

Since each ray is parameterized by the affine parameter, ν, then the geodesic condition can be written as :

$$k^b\nabla_b k^a = \frac{\delta k^a}{\delta \nu} = (k^a)' = 0 \;, \tag{5.8}$$

where the prime derivative ($'$) denotes change along the ray (i.e. with respect to the affine parameter ν). Using the geodesic condition (5.8) we can derive propagation equations for the lensing variables E and κ in the direction of the null ray. Substituting for the null vector k^a (5.2) into (5.8) gives

$$\begin{aligned} k^b\nabla_b k^a &= \left[E' + E^2\kappa\mathcal{A} + E^2\kappa^2\left(\Sigma + \frac{1}{3}\Theta\right)\right]u^a + \left[E'\kappa + E^2\mathcal{A} + E^2\kappa\left(\frac{1}{3}\Theta + \Sigma\right) + E\kappa'\right]e^a \\ &\quad + \left[E' + \frac{1}{2}E^2\kappa\phi + E^2\left(\frac{1}{3}\Theta - \frac{1}{2}\Sigma\right)\right]\kappa^a + E^2\left(\Omega + \kappa\xi\right)\varepsilon^a{}_b\kappa^b + E\kappa^{a\prime} \\ &= 0, \end{aligned} \tag{5.9}$$

where spherical symmetry has been considered and the properties

$$k^b u_b = -E, \quad k^b e_b = E\kappa, \quad N^a{}_b k^b = E\kappa^a, \quad \varepsilon^a{}_b k^b = E\varepsilon^a{}_b\kappa^b \tag{5.10}$$

have been used. We also have used,

$$u_a' = E\mathcal{A}e_a + E\kappa\left(\frac{1}{3}\Theta + \Sigma\right)e_a + E\left(\frac{1}{3}\Theta - \frac{1}{2}\Sigma\right)\kappa_a + E\Omega\varepsilon_{ab}\kappa^b \;, \tag{5.11}$$

$$e_a' = E\mathcal{A}u_a + E\kappa\left(\Sigma + \frac{1}{3}\Theta\right)u_a + \frac{1}{2}E\phi\kappa_a + E\xi\varepsilon_{ab}\kappa^b \;, \tag{5.12}$$

which are obtained from (3.149) and (3.148), respectively.

When (5.9) is projected along the timelike direction (u_a), we obtain the general equation of E', as:

$$E' = -E^2\kappa\mathcal{A} - E^2\kappa^2\left(\Sigma + \frac{1}{3}\Theta\right) + u_a E\kappa^{a\prime}. \tag{5.13}$$

Knowing that $u_a\kappa^a = 0$, it follows that,

$$\begin{aligned} u_a\kappa^{a\prime} &= -\frac{1}{3}\Theta E + \frac{1}{2}\Sigma E + \frac{1}{3}\Theta\kappa^2 E - \frac{1}{2}\Sigma\kappa^2 E, \\ \Rightarrow u_a E\kappa^{a\prime} &= -\frac{1}{3}\Theta E^2 + \frac{1}{2}\Sigma E^2 + \frac{1}{3}\Theta\kappa^2 E^2 - \frac{1}{2}\Sigma\kappa^2 E^2. \end{aligned} \tag{5.14}$$

Using this, (5.13) can now be written as

$$E' = -E^2\kappa\mathcal{A} - \frac{3}{2}\Sigma\kappa^2E^2 - E^2\left(\frac{1}{3}\Theta - \frac{1}{2}\Sigma\right). \tag{5.15}$$

When (5.9) is projected along the radial direction (e^a), we obtain the general equation of κ',

$$E\kappa' = -E'\kappa - E^2\mathcal{A} - E^2\kappa\left(\frac{1}{3}\Theta + \Sigma\right) - e_aE\kappa^{a\prime}, \tag{5.16}$$

and as with the E' case, knowing $e_a\kappa^a = 0$, then,

$$\begin{aligned} e_a\kappa^{a\prime} &= -\frac{1}{2}E\phi\left(1-\kappa^2\right), \\ \Rightarrow e_aE\kappa^{a\prime} &= -\frac{1}{2}E^2\phi + \frac{1}{2}E^2\phi\kappa^2. \end{aligned} \tag{5.17}$$

Now (5.16) can be written as

$$E\kappa' = E^2\kappa^2\mathcal{A} + \frac{3}{2}\Sigma\kappa^3E^2 - \frac{3}{2}\Sigma E^2\kappa - E^2\mathcal{A} + \frac{1}{2}E^2\phi - \frac{1}{2}E^2\phi\kappa^2$$

or

$$\kappa' = E\left(1-\kappa^2\right)\left(\frac{1}{2}\phi - \mathcal{A} - \frac{3}{2}\Sigma\right), \tag{5.18}$$

which can be used in a general LRS spacetime.

The propagation of scalars along k^a in a spherically symmetric spacetime can be determined using [2]

$$X' = E\left(\dot{X} + \kappa\hat{X}\right), \tag{5.19}$$

where the dot term vanishes for static conditions.

5.3 Solutions for the lensing variables for R^n gravity

As stated previously, the only non-zero 1+1+2 variables in a vacuum spherically symmetric static spacetime are the scalars $\{\mathcal{A},\ \phi,\ R,\ X\}$ (and their derivatives $\{\hat{\mathcal{A}},\ \hat{\phi},\ \hat{R},\ \hat{X}\}$). Thus, the general propagation equations of E in (5.15) and κ in (5.18), in the direction of the ray, reduce to the form :

$$E' = -E^2\mathcal{A}\kappa\ , \tag{5.20}$$
$$\kappa' = E(1-\kappa^2)(\frac{1}{2}\phi - \mathcal{A})\ . \tag{5.21}$$

[2] Note that $\delta_a X = 0$ in a spherically symmetric spacetime

To find general solutions to these differential equations, we use the forms for $\mathcal{A}, \phi$ and X as given in (4.87), (4.88) and (4.100), respectively.

Using (5.19), the propagation equations for E (5.20) and κ (5.21) can be rewritten in terms of the hat derivative as

$$\begin{aligned} E' &= k^a \nabla_a E = E\kappa \hat{E} = -E^2 \mathcal{A}\kappa \,, && (5.22) \\ \kappa' &= k^a \nabla_a \kappa = E\kappa\hat{\kappa} = E\left(1-\kappa^2\right)\left(\frac{1}{2}\phi - \mathcal{A}\right) && (5.23) \end{aligned}$$

for a static spherically symmetric spacetime. We can then express the hat-derivative in terms of the radial parameter r:

$$\frac{1}{2} r\phi \frac{dE}{dr} = -E\mathcal{A} \,, \ \Rightarrow \frac{1}{E}\frac{dE}{dr} = -\frac{2\mathcal{A}}{r\phi}. \tag{5.24}$$

We specialize the spacetime geometry to R^n gravity and utilize the spherically symmetric solution (4.101). We substitute $\mathcal{A}$ (4.97) and ϕ (4.98) for this solution in the differential equation (5.24) in order to solve for E. This yields the solution:

$$E = \frac{r^{6+\frac{13}{n-2}+2n}\lambda_1}{\sqrt{r^{\frac{10n}{n-2}} + C r^{\frac{7+4n^2}{n-2}}}}, \tag{5.25}$$

valid for $n \neq 2$ and here λ_1 is an integration constant with respect to affine parameter ν. λ_1 can be determined by taking the limit of (5.25) as r tends to some r_*. We find

$$\lambda_1 = E_* r_*^{-\frac{2n^2+2n+1}{n-2}} \sqrt{r_*^{\frac{10n}{n-2}} + C r_*^{\frac{7+4n^2}{n-2}}} \,. \tag{5.26}$$

We are now able to solve for κ using the differential equation for κ

$$\begin{aligned} \frac{1}{2} r\phi \frac{d\kappa}{dr} &= \frac{\left(1-\kappa^2\right)}{\kappa}\left(\frac{1}{2}\phi - \mathcal{A}\right); \\ \Rightarrow \frac{\kappa}{\left(1-\kappa^2\right)}\frac{d\kappa}{dr} &= \frac{2}{r\phi}\left(\frac{1}{2}\phi - \mathcal{A}\right) \\ &= \frac{1}{r} - \frac{2\mathcal{A}}{r\phi} \\ &= \frac{1}{r} + \frac{1}{E}\frac{\partial E}{\partial r}. && (5.27) \end{aligned}$$

This equation can be integrated giving

$$\begin{aligned}\frac{1}{2}ln\left(1-\kappa^2\right) &= -\left(lnr + lnE + ln\lambda_2\right)\\ \Rightarrow \kappa^2 &= \left(1 - \frac{1}{r^2E^2\lambda_2}\right).\end{aligned} \tag{5.28}$$

where λ_2 is a constant of integration with respect to ν. To determine λ_2 we consider that at the point of closest approach [3] $r = r_0$, $\kappa = 0$ and so from (5.28)

$$\lambda_2 = \left[\frac{r_0^{\frac{-4n^2+10n-7}{n-2}} + C}{r_0^{\frac{3(2n-3)}{n-2}}}\right]\left[E_*^2\left(r_*^{\frac{-2(2n-1)(n-1)}{n-2}} + Cr_*^{\frac{5-4n}{n-2}}\right)\right]^{-1}. \tag{5.29}$$

5.4 The Deflection Angle

We now find the form of the deflection angle (5.7) by substituting the solutions for E (5.25) and κ (5.28) giving

$$\begin{aligned}\alpha &= \int_{\nu_1}^{\nu_2}\frac{1}{r}\left|E\right|\sqrt{1-\kappa^2}d\nu - \alpha_0\\ &= \int_{\nu_1}^{\nu_2}\frac{1}{r}\left|\frac{r^{6+\frac{13}{n-2}+2n}\lambda_1}{\sqrt{r^{\frac{10n}{n-2}} + Cr^{\frac{7+4n^2}{n-2}}}}\right|\sqrt{1 - 1 + \frac{r^{\frac{10n}{n-2}} + Cr^{\frac{7+4n^2}{n-2}}}{r^{14+\frac{26}{n-2}+4n}\lambda_1^2\lambda_2}}d\nu - \alpha_0\\ &= \int_{\nu_1}^{\nu_2}\frac{1}{r^2}\sqrt{1/\lambda_2}d\nu - \alpha_0\\ &= \int_{\nu_1}^{\nu_2}\frac{E_*}{r^2}MJd\nu - \alpha_0,\end{aligned} \tag{5.30}$$

where J is the *impact parameter* defined by

$$J = \left[\frac{r_0^{\frac{3(2n-3)}{n-2}}}{r_0^{\frac{-4n^2+10n-7}{n-2}} + C}\right]^{\frac{1}{2}}, \tag{5.31}$$

and M is the expression

$$M = r_*^{-\frac{2n^2+2n+1}{n-2}}\sqrt{r_*^{\frac{10n}{n-2}} + Cr_*^{\frac{7+4n^2}{n-2}}}.$$

A transformation relation between the affine parameter $d\nu$ and radial distance dr is now needed. We start by including the propagation equation for ϕ, 4.68 and the hat derivative

[3] $_0$ is the closest point that the ray would reach in the vicinity of the lensing object.

in the equation is converted to the prime derivative using the relation (5.19) to obtain:

$$\phi' = k^a \nabla_a \phi = E\kappa\hat{\phi} = E\kappa\left(\mathcal{A} - \frac{1}{2}\phi\right)\phi + E\kappa\left(\frac{1}{3}R - \frac{2}{3}\frac{f}{f'} + \frac{f''}{f'}X\left(\phi + 2\mathcal{A}\right)\right), \qquad (5.32)$$

which is

$$\phi' = E\kappa\left(\mathcal{A} - \frac{1}{2}\phi\right)\phi + E\kappa\left(\frac{n-2}{3n}R + (n-1)R^{-1}X\left(\phi + 2\mathcal{A}\right)\right), \qquad (5.33)$$

when extended to $f(R) = R^n$.

Substituting for the values of the scalar functions $\mathcal{A}$ (4.97), ϕ(4.98), R (4.99), X (4.100), E (5.25) and κ (5.28) into (5.33) results in:

$$\phi' = \frac{\left(2(n-2)^2 r^{\frac{5+2n^2}{n-2}} - C(n-2)(11-12n+4n^2)r^{\frac{12-10n+6n^2}{n-2}}\right)\sqrt{1 - \frac{r^{\frac{-2(n+1)(2n-3)}{n-2}}\left(r^2 + Cr^{\frac{(2n-1)(2n-3)}{n-2}}\right)}{\lambda_2\lambda_1^2}}}{\left(2n^2-2n-1\right)\left(4n^2-10n+7\right)\sqrt{r^{\frac{10n}{n-2}} + Cr^{\frac{7+4n^2}{n-2}}}} \qquad (5.34)$$

Differentiating the solution for ϕ (4.98) with respect to ν gives

$$\phi' = \frac{-\left(2(n-2)r^{\frac{10}{n-2}} - C\left(4n^2 - 12n + 11\right)r^{\frac{7+4n^2}{n-2}}\right)}{r^2(n-2)\sqrt{\frac{(2n^2-2n-1)(4n^2-10n+7)}{(n-2)^2\left(1+Cr^{\frac{4n^2-10n+7}{n-2}}\right)}}\left(\sqrt{r^{\frac{10n}{n-2}} + Cr^{\frac{7+4n^2}{n-2}}}\right)}r'. \qquad (5.35)$$

The transformation relation is obtained by equating equations (5.34) and (5.35) giving [4]

$$dr = E_* M\sqrt{\frac{(n-2)^2}{(1+2n-2n^2)(7-10n+4n^2)}}\left[r^{\frac{(4n^2-6n+2)}{n-2}} - J^2\left(r^{-2} + Cr^{\frac{(4n^2-12n+11)}{n-2}}\right)\right]^{\frac{1}{2}}$$

Using this transformation relation in (5.30) gives the deflection angle in the form

$$\alpha = 2\int_{r_0}^{r_*} L\frac{J}{r^2}\left[r^{\frac{(4n^2-6n+2)}{n-2}} - J^2\left(r^{-2} + Cr^{\frac{(4n^2-12n+11)}{n-2}}\right)\right]^{-\frac{1}{2}} dr - \pi, \qquad (5.36)$$

where L is

$$L = \sqrt{\frac{(1+2n-2n^2)(7-10n+4n^2)}{(n-2)^2}}.$$

The standard form of the deflection angle in GR as given in [72] is recovered here when $n = 1$ in (6.30).

[4] Note that same transformation relation is obtained when the propagation equation $\mathcal{A}'$ and derivative of $\mathcal{A}$ with respect to ν is used as in place of (5.34) and (5.35) respectively

5.5 Observables

We now analyze the behaviour of the deflection angle α by computing the deflection angle for different values of r_* (distance of closest approach), as well as for different values of r_0 (distance from the source) against n. We then plot the ratio α to α_{GR}, the GR case corresponding to $n = 1$, against n for the two cases as shown in Fig (5.3) and Fig (5.4) respectively. For this fiducial system, we choose the distance r_0 to be in units of the Schwarzschild radius.

The divergence of the curves in both plots is indicative of the deviation from the standard GR value. In Fig (5.3), the deflection angle is independent of the distance from the source r_* and in Fig (5.4), for a fixed n, the deflection angle varies for different values of r_0 such that for $n < 1$ there is more bending as the values of r_0 increases whereas the reverse happens for $n > 1$ where the more bending occurs as r_0 decreases in value. The dependence of α is therefore on the two parameters n and r_0.

More bending is thus expected for α different from GR with the value being greater for values of n beyond $n = 1$. The bending lessens with increase in n for $n < 1$ and increases as n increases for $n > 1$ but beyond $n = 1.23$ the function is undefined giving this upper limit to the physical region. R^n gravity therefore increases the deflection angle with respect to the standard GR result.

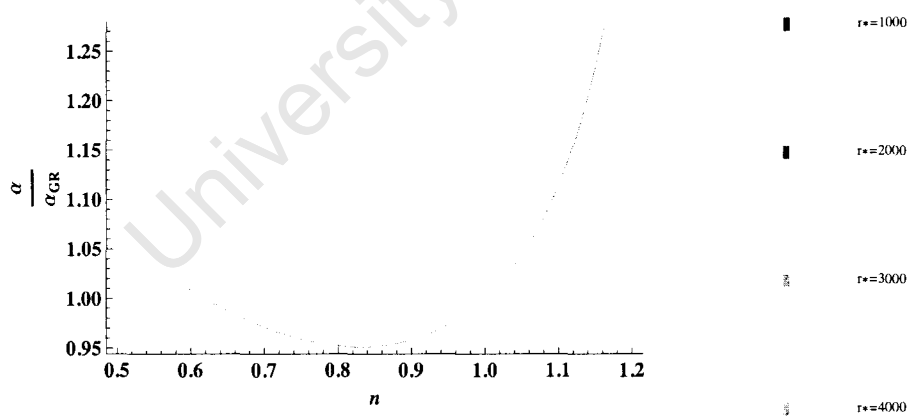

Figure 5.3: Plot of the bending angle α compared to the bending angle in general relativity against n corresponding to different values of r_*, distance from the source. The result shows that the bending angle value is independent of r_* as the plot remains the same when r_* is varied

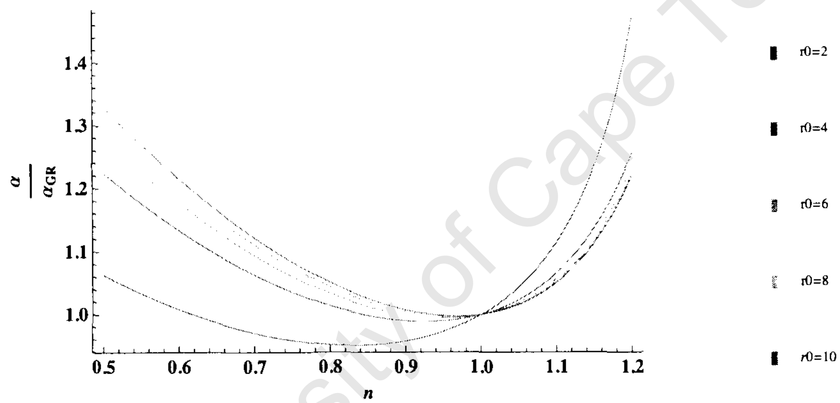

Figure 5.4: Plot of the bending angle α compared to the bending angle in general relativity against n corresponding to different values or r_0, distance of closest approach. The deflection angle varies for different values of r_0

Chapter 6

Summary and Conclusion

We have developed in this thesis the 1+1+2 formalism to treat spherically symmetric spacetimes in $f(R)$ theories of gravity. We derived general equations for these spacetimes for general $f(R)$ theories, presenting some general features that these equations create for the theories. Some exact solutions were also derived which were used to draw considerations on Birkhoff's theorem in this framework. Additionally, we derived a covariant form of the lensing angle for a specific spherically symmetric solution in $f(R) = R^n$ gravity.

Chapter 2 of the thesis gives an overview of fourth order theory of gravity and considers in particular the case of $f(R)$ theory of gravity. To adapt more easily techniques originally devised for GR from the 1+3 approach, the field equations are written in the form:

$$G_{ab} = \left(R_{ab} - \frac{1}{2}g_{ab}R\right) = \tilde{T}^{m}_{ab} + T^{R}_{ab} = T^{tot}_{ab}. \tag{6.1}$$

as with standard Einstein gravity but with two "effective fluids": the *curvature "fluid"* (associated with T^{R}_{ab}) and the *effective matter "fluid"* (associated with $\tilde{T}^{m}_{ab}$).

In chapter 3 we look at the 1+3 covariant approach by taking a fluid description of the universe as a congruence in spacetime. This entails a 1+3 threading of spacetime w.r.t a timelike congruence provided by the 4-velocity u^a and a projected orthogonal three-dimensional spacelike hypersurface. The kinematical quantities are determined from the variation of the 4-velocity and the matter description and the dynamics derived from the Ricci identity for u^a. The 1+3 covariant approach is then applied to the null congruence since null geodesics must be considered to obtain the bending angle. The null vector k^a lies orthogonal to a two-dimensional screen-space and tangent to the null geodesics. The propagations and constraint equations in this case are derived from the Ricci identities for k^a.

Finally in chapter 3 the 1+3 covariant spacetime decomposition is extended to a 1+1+2 decomposition by introducing the unit vector e^a orthogonal to u^a. In this approach, the three-dimensional space is split further into a direction along e^a and a projected 2-surface (the sheet), which is orthogonal to e^a and u^a. Tests like the bending of light, the perihelion shift of planets are critical for any theory of gravity and so viable spherically symmetric solutions in these theories are fundamental to confronting such tests with fourth order gravity. Since the 1+1+2 covariant approach is optimized for problems that have spherical symmetry, it is suited for a covariant description of these spacetimes.

The dynamical equations for the covariant part of the derivative of e^a are derived from the splitting of the Ricci identity for e^a and from the decomposition of the 1+3 equations. The variables that completely characterize the 1+1+2 spacetime are the irreducible set of variables:

$$\{\Theta, \mathcal{A}, \Omega, \Sigma, \phi, \xi, \mathcal{E}, \mathcal{H}, \mathcal{A}^a, \Omega^a, \Sigma^a, \alpha^a, a^a, \mathcal{E}^a, \mathcal{H}^a; \Sigma_{ab}, \zeta_{ab}, \mathcal{E}_{ab}, \mathcal{H}_{ab}\}, \tag{6.2}$$

together with the irreducible set of thermodynamic variables (for the total effective thermodynamic fluid),

$$\{\mu, p, Q, \Pi, Q_a, \Pi_{ab}\}. \tag{6.3}$$

In chapter 4, we consider the 1+1+2 approach in LRS spacetime in which the preferred direction constitutes a local axis of symmetry such that in this case e^a is a vector pointing along the axis. Since the LRS spacetimes are isotropic, this allows for the vanishing of all 1+1+2 vectors and tensors. The LRS spacetime is fully described by the variables:

$$\{\mathcal{A}, \Theta, \phi, \xi, \Sigma, \Omega, \mathcal{E}, \mathcal{H}, \mu, p, \Pi, Q\}, \tag{6.4}$$

that satisfy a set of 1+1+2 evolution, propagation and constraint equations. The spacetime is then restricted further to LRS class II spacetime, which is free of rotation and satisfied by the quantities:

$$\{\mathcal{A}, \Theta, \phi, \Sigma, \mathcal{E}, \mu, p, \Pi, Q\}. \tag{6.5}$$

A spherically symmetric static spacetime belongs to LRS class II and by imposing vacuum conditions within the $f(R)$ framework, the effective matter thermodynamic terms in the propagation equations vanish but the thermodynamic terms related to the curvature fluid are retained. The determined set of propagation equations for this static vacuum spherically symmetric spacetime in terms of 1+1+2 were found to be

$$\hat{R} = X, \tag{6.6}$$

$$\hat{X} = -\frac{1}{3}\frac{Rf'}{f''} + \frac{2}{3}\frac{f}{f''} - X\phi - \frac{f'''X^2}{f''} - X\mathcal{A}, \tag{6.7}$$

$$\hat{\phi} = -\frac{1}{2}\phi^2 + \frac{1}{3}R - \frac{2}{3}\frac{f}{f'} + \frac{f''}{f'}X\left(\phi + 2\mathcal{A}\right) + A\phi, \tag{6.8}$$

$$\hat{\mathcal{A}} = -\mathcal{A}^2 - \mathcal{A}\phi + \frac{1}{6}\frac{f}{f'} - \frac{1}{3}R - \frac{f''}{f'}X\mathcal{A}, \tag{6.9}$$

The static and vacuum spherically symmetric spacetime for general $f(R)$ gravity are determined by any of the three non-zero scalar functions R, X, ϕ and $\mathcal{A}$ that define the set of equations (6.6–6.9). These equations are coordinate independent and reduce to the second order system of GR in vacuum, if $f(R) = R$, $R = 0$ and $X = 0$.

Important results are deduced from the system of equations (6.6–6.9) for spherically symmetric static solutions in $f(R)$ gravity. It was found that if we impose the condition of class C^3 on the function $'f'$ with $R = 0$, then the condition for existence of solutions is that the system of equations is satisfied for all functions f such that:

$$|f'(0)| < +\infty \,, |f''(0)| < +\infty \,, |f'''(0)| < +\infty \quad , \tag{6.10}$$

and

$$f(0) = 0, R = 0. \tag{6.11}$$

On one hand imposing further, $f'(0) \neq 0$, reduces the system to a system with a unique solution which corresponds to the Schwarzschild metric in Schwarzschild co-ordinates. This then enables us to state a generalization of *Birkhoff's Theorem* in higher order gravity. us to state a generalization of *Birkhoff's Theorem* in higher order gravity.

For all functions $f(R)$ which are of class C^3 at $R = 0$ and $f(0) = 0$ while $f'(0) \neq 0$, Schwarzschild solution is the only vacuum solution with vanishing Ricci scalar for a spherical symmetric matter distribution.

On the other hand, imposing instead $f'(0) = 0, f(0) = 0$ ensures that (4.66-4.69) are identically satisfied for all values of ϕ and $\mathcal{A}$ that guarantees $R = 0$ and hence $X = 0$. Hence for all models with $f'(0) = 0$, any solution with vanishing Ricci Scalar in GR would be a solution to the above system. This shows that gravity can in principle mimic other fields. For example, the Reissner Nordstrom solution is a solution to the above system without any charge. Similarly, a static spherically symmetric solution for a perfect fluid interior with equation of state $p = 1/3\rho$ is also a solution of this system in the absence of the fluid. Gravity thus mimics the electromagnetic field in former case and the perfect fluid in latter. It is noted here that the fact that the Schwarzschild solution is not a unique static spherically symmetric solution hints towards a violation of Birkhoffs theorem in its present form [48] for these theories.

In the case for solutions with constant Ricci scalar, by imposing $R = R_0 = const$ and consequently $X, \hat{X} = 0$ on (6.6–6.9) for a class C^3 in $R = R_0$, one obtains:

$$f_0' \left[\hat{\phi} + \phi\left(\frac{1}{2}\phi - \mathcal{A}\right)\right] = \frac{1}{3}R_0 f_0' - \frac{2}{3}f_0\,, \tag{6.12}$$

$$f_0' \left[\hat{\mathcal{A}} + \mathcal{A}(\mathcal{A} + \phi)\right] = \frac{1}{6}f_0 - \frac{1}{3}R_0 f'\,, \tag{6.13}$$

$$0 = -R_0 f_0' + 2f_0. \tag{6.14}$$

Solutions in this case exist for

$$f_0' \neq 0, \quad f_0 \neq 0\,, \quad 2f_0 - R_0 f_0' = 0\,, \tag{6.15}$$

$$f_0' \neq 0, \quad f_0 = 0 \quad \Rightarrow \text{Schwarzschild solution} \;\; (R_0 = 0), \tag{6.16}$$

$$f_0' = 0, \quad f_0 = 0\,, \quad R = R_0\,, \quad X,\, \hat{X} = 0\,, \tag{6.17}$$

respectively. The relation (6.15) was already found by Barrow and Ottewill [69] in the cosmological context and later rediscovered in [70]. It relates the value of the constant Ricci scalar with the universal constants in the action.

On considering solutions for the case of $f(R) = KR^2$ it was found that the constant term in a Schwarzschild-dS/AdS spacetime becomes a local constant of integration just like the mass. Hence in this theory we can have two distant stars behaving like two different Schwarzschild-dS/AdS object with different values of the constant. Consequently, just by studying the geodesic motions around them, it is impossible to determine their mass uniquely.

In order to extract observable results, the 1+1+2 equations are further specialized by choosing a specific form of the radial coordinate. This is equivalent to the choice of an observer in the 1+1+2 formalism which, differing from the 1+3 case, requires not only the specification of a velocity field but also a specific spatial direction. The requirement that the Gauss curvature of the sheets has an inverse square dependence on the radius offers a natural choice for the co-ordinate 'r' to become the *area radius* of the sheets. This allows for a geometrical definition of the '*hat*' derivative as

$$\hat{M} = \frac{1}{2}r\phi\frac{dM}{\partial r}\,. \tag{6.18}$$

With this choice, given any $f(R)$-theory of gravity (and sufficient ingenuity), one can derive static and spherically symmetric solution(s) for this theory.

We use R^n-gravity as an example to derive some static and spherically symmetric solutions. In particular, using the Schwarzschild ansatz $\phi = \sqrt{C_1 r^{\alpha} + C_2 r^{\beta}}$, $R = C_3/r^{\gamma}$ ($\gamma > 0$) such that the Ricci scalar vanishes at infinity, the solution:

$$\mathcal{A} = \frac{-C(5-4n)r^{\frac{7-5n}{n-2}} + (2n-2)(2n-1)r^{\frac{5n-4n^2}{n-2}}}{2(2-n)r^{\frac{(2n-2)(2n-1)}{2-n}}\sqrt{\frac{(1+2n-2n^2)(7-10n+4n^2)\left(1+Cr^{-\frac{7-10n+4n^2}{2-n}}\right)}{(2-n)^2}}}, \quad (6.19)$$

$$\phi = \frac{2}{r\sqrt{\frac{(1+2n-2n^2)(7-10n+4n^2)}{(2-n)^2\left(1+Cr^{-\frac{7-10n+4n^2}{2-n}}\right)}}}, \quad (6.20)$$

$$R = \frac{6n(n-1)}{[2n(n-1)-1]r^2}, \quad (6.21)$$

$$X = -\frac{12n(n-1)}{[2n(n-1)-1]r^3\sqrt{\frac{(1+2n-2n^2)(7-10n+4n^2)}{(2-n)^2\left(1+Cr^{-\frac{7-10n+4n^2}{2-n}}\right)}}} \quad (6.22)$$

is obtained for which the lensing equation is later derived.

Solving for the metric coefficients for a general spherically symmetric static metric, $ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$ and from the metric definitions of different covariant scalars:

$$\mathcal{A} = \frac{1}{2A(r)}\frac{dA(r)}{dr}\sqrt{B(r)} \quad \text{and} \quad \phi = \frac{2}{r}\sqrt{B(r)}, \quad (6.23)$$

we obtain

$$A(r) = r^{(2n-2)\frac{(2n-1)}{(2-n)}} + \frac{C}{r^{\frac{(5-4n)}{(2-n)}}} \quad ; \quad \frac{1}{B(r)} = \frac{(2-n)^2}{(7-10n+4n^2)(1+2n-2n^2))}\left(1 + \frac{C}{r^{\frac{(7-10n+4n^2)}{(2-n)}}}\right) . \quad (6.24)$$

This solution was originally found by Clifton [71]. It reduces to Schwarzschild for $n = 1$ but despite the fact the Ricci scalar vanishes at infinity, the solution is *not* asymptotically flat.

In chapter 5 we apply the 1+1+2 decomposition of the null tangent vector

$$k^a = E(u^a + \kappa e^a + \kappa^a) , \quad (6.25)$$

where κ is the *magnitude of the radial component* and κ^a is the *component lying in the 2-dimensional sheet.* From lensing geometry in the figures (5.1) and (5.2) and properties of the null tangent vector k^a, the completely general form of the lensing angle is derived as

$$\alpha = \int_{\nu_1}^{\nu_2} \frac{1}{r} |E| \sqrt{1-\kappa^2} d\nu - \alpha_0 \ . \tag{6.26}$$

To find an explicit form for (6.26), the propagation equations along the ray for the lensing variables E and κ are determined which for R^n spacetime geometry took the form:

$$E' = -E^2 \mathcal{A} \kappa \ , \tag{6.27}$$

$$\kappa' = E(1-\kappa^2)(\frac{1}{2}\phi - \mathcal{A}) \ . \tag{6.28}$$

Since the prime derivative can be expressed in terms of the hat derivative as in the case of a scalar K, $K' = E\kappa\hat{K}$, utilizing this the equations can be rewritten as differential equations. To find solutions to these differential equations, we use the forms for $\mathcal{A}$, and ϕ as given in (6.19) and (6.20), respectively. The solutions found are

$$E = \frac{r^{6+\frac{13}{n-2}+2n}\lambda_1}{\sqrt{r^{\frac{10n}{n-2}} + Cr^{\frac{7+4n^2}{n-2}}}} \quad \text{and} \quad \kappa = \left(1 - \frac{1}{r^2 E^2 \lambda_2}\right)^{\frac{1}{2}} . \tag{6.29}$$

Substituting these solutions and the transformation relation between the affine parameter $d\nu$ and dr into the general expression of the deflection angle (6.26) results in

$$\alpha = 2\int_{r_0}^{r_*} L\frac{J}{r^2}\left[r^{\frac{(4n^2-6n+2)}{n-2}} - J^2\left(r^{-2} + Cr^{\frac{(4n^2-12n+11)}{n-2}}\right)\right]^{-\frac{1}{2}} dr - \pi, \tag{6.30}$$

form for the deflection angle for R^n gravity. It was found that more bending in the lensing angle occurs for R^n when compared to the GR case for a light ray passing close to a massive body. The physical region for the model used is limited to a value below $n = 1.34$ beyond which the theory breaks down for the equation.

6.1 Conclusion

In obtaining the covariant lensing angle and solutions in static spherically symmetric spacetimes using the 1+1+2 covariant approach, the power of the the covariant approach when compared to the metric method is demonstrated as the covariant decomposition of the spacetime yields physical and geometrical meaning of the covariant variables and thus manifests the underlying physics for better understanding.

From the system of equations (4.66–4.69) we deduced some important results for spherically symmetric static solutions in a general $f(R)$ gravity in absolute co-ordinate independent manner. The presence of a Schwarzschild solution in R^n gravity theories can have interesting consequences on the validity of these models on the Solar System level. In particular if one concludes that the sun behaves very close to a Schwarzschild solution, the experimental data of the solar system would help constraining these models. From our general considerations since more than one solution is generated. Our general considerations of the conditions $f' > 0$, $f'' > 0$ do not admit a unique solution and Birkhoffs theorem is violated. It is also interesting to note that the imposition of these conditions keeps the gravitational interaction at the same time at all times and prevents the appearance of tachyons, implying that the only $R = 0$ solution is the Schwarzschild one. This suggests a link between these conditions and the Birkhoff theorem, which definitely deserves a more detailed study.

In the lensing analysis, we found that more bending is expected for R^n gravity theories in comparison to GR and is dependent on the value of n and the value of distance of closest approach. This divergence can be used to obtain signatures for these theories of gravity. Additionally, of important astrophysical and cosmological consequence is finding observational results for the angle for these models. From experimental data, certain forms of the function f may ruled out or be constrained. Furthermore these results may play an important part in the covariant and gauge invariant perturbations of black hole solutions in higher order gravity. For more comparative results, the covariant application to lensing can be extended further, for example, it can be applied to studies of lensing of black holes and the CMB and also pertubations of spherically symmetric spacetimes [59, 61].

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