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QUASIREFLECTIONS AND QUASIFACTORIZATIONS

by

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Contents

Introduction

The study of *reflections* in abstract category theory is widespread, and has often been used to study the concrete notion of "completion of an object" that occurs in many fields of Mathematics, such as the Čech-Stone compactification of a Tychonoff space ([Čech 37]) or the completion of a uniform space ([Weil 38]). More recent work relating reflections to completions was published by Brümmer and Giuli [Brümmer Giuli 92], and in this thesis many of their ideas are extended to the more general setting of *quasireflections* [Bargenda 94].

In particular, one would like to view the well-known concept of an injective hull as a "completion", and this can be accomplished via a Galois correspondence between such hulls on one hand, and quasireflections on the other. Thus the theory of completion of objects can be extended to include many widely studied and significant examples, the most paradigmatic of which is the Mac Neille completion of a partially ordered set [Mac Neille 37]. These ideas are presented in chapters 1 and 2 of the present thesis.

Further, the widely accepted characterization of factorization structures for sources in terms of certain colimits (pushouts and cointersections) was successfully extended to a characterization of factorization structures relative to a subcategory in the PhD thesis of Vaclav Vajner ([Vajner 94]). In chapter 3 of this thesis, the characterization is further generalized to include quasifactorization structures relative to a subcategory. This result relates to the results of chapters 1 and 2 via an important result of Bargenda's, which proves a Galois correspondence between quasireflective subcategories and relative quasifactorization structures (proposition 3.7)

Notation

All results on categories used in this thesis are from [AHS 90]. Some notation not found in that book is convenient in the present context, specifically the following:

- If $\mathcal{U}$ is a class of morphisms in a category $\mathbf{X}$, the notation $\text{inj}(\mathcal{U})$ is used for the class of all $\mathcal{U}$-injective $\mathbf{X}$-objects,
- The class $\mathcal{U}^*$ denotes all $\mathcal{U}$-essential morphisms, similarly $\mathcal{U}_*$ denotes the class of all $\mathcal{U}$-coessential morphisms,
- A class $\mathcal{U} \subseteq \text{mor}(\mathbf{X})$ is called essential (respectively coessential) iff $\mathcal{U} = \mathcal{U}^*$ (respectively $\mathcal{U} = \mathcal{U}_*$).

Chapter 1

Quasireflective subcategories

In this chapter a generalised notion of "reflective subcategory" is introduced. This notion differs from the reflective case in that the uniqueness requirement of extensions is dropped. Similar generalisations due to Skula [Skula 74] and Tholen et al [Tholen 94] are both included in this more general notion, which is due to Hubertus Bargenda [Bargenda 94].

1.1 Definitions

Let $\mathbf{A}$ be a (full, isomorphism-closed)[1] subcategory of a fixed category $\mathbf{X}$. The following definition is well-known:

Definition 1.1 *A morphism $e : X \to Y$ is called* <u>*$\mathbf{A}$-extendible*</u> *iff*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "e"'] & & A \\ & Y \arrow[ur, dashed, "g"'] & \end{tikzcd}$$

for all $f : X \to A$, $A \in \mathbf{A}$, there exists g such that $ge = f$ In other words, any map from the domain of e to any $\mathbf{A}$-object has an extension along e.

Denote by <u>$\mathrm{ext}(\mathbf{A})$</u> *the class of all $\mathbf{A}$-extendible morphisms in $\mathbf{X}$.*

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[1] Henceforth all subcategories are assumed to be full and isomorphism-closed.

Definition 1.2 ([Porst 81]) *A morphism $e : X \to Y$ is called thick iff for any endomorphism $t : Y \to Y$, $(te = e \Rightarrow t$ iso).*

Denote by thick(**X**) *the class of all thick* **X***-morphisms.*

The following concept occured in [Börger Tholen 76], where it was defined under the name stable weak reflection.

Definition 1.3 ([Bargenda 94], also [Börger Tholen 76]) *A morphism $p : X \to Y$ in $\mathbf{X}$ is called an* **A***-quasireflection of X iff*

(i) $\mathrm{cod}(p) \in \mathbf{A}$,

(ii) $p \in \mathrm{ext}(\mathbf{A})$,

(iii) p is thick.

Denote by qrefl(**A**) *the class of all* **A***-quasireflections in* **X***.*

Note that quasireflections are unique up to isomorphism; see lemma 1.14.

A *is called a quasireflective subcategory of* **X** *if every object $X \in \mathbf{X}$ has an* **A***-quasireflection. In this case we will denote the* **A***-quasireflection by $q_X : X \to QX$.*

If further every $X \in \mathbf{X}$ has an **A***-quasireflection belonging to some class $\mathcal{U}$ of* **X***-morphisms, we say* **A** *is $\mathcal{U}$-quasireflective in* **X***.*

We now define a class of morphisms qfirm(**A**) which will play a central role in chapter 2:

Definition 1.4 *If* **A** *is a quasireflective subcategory of* **X** *then define*

(i) $\forall f : X \to Y \in \mathrm{mor}(\mathbf{X})\ Qf := \{h : QX \to QY : hq_X = q_Y f\}$,

(ii) $\mathrm{qfirm}(\mathbf{A}) := \{f : Qf \cap \mathrm{iso}(\mathbf{X}) \neq \emptyset\}$.

Lemma 1.5 *If* $\mathbf{A}$ *is a quasireflective subcategory of* $\mathbf{X}$*, then for all* $f \in \mathrm{mor}(\mathbf{X})$*,* $Qf \neq \emptyset$.

Proof: Suppose $f : X \to Y$ is an $\mathbf{X}$-morphism.

$$\begin{tikzcd} X \arrow[r, "q_X"] \arrow[d, "f"'] & QX \arrow[d, dashed, "h"] \\ Y \arrow[r, "q_Y"'] & QY \end{tikzcd}$$

Since q_X is $\mathbf{A}$-extendible by definition, and $\mathrm{cod}(q_Y f) \in \mathbf{A}$, there exists an extension h of $q_Y f$ along q_X, i.e. $hq_X = q_Y f$. But then $h \in Qf$, so $Qf \neq \emptyset$. □

Definition 1.6 ([Herrlich 93]) *A subcategory* $\mathbf{A}$ *of* $\mathbf{X}$ *is called* almost reflective *iff for every object* $X \in \mathbf{X}$*, there exists a morphism* $q_X : X \to A \in \mathrm{ext}(\mathbf{A})$ *with* $\mathrm{cod}(q_X) \in \mathbf{A}$*, and* $\mathbf{A}$ *is closed under the formation of retracts in* $\mathbf{X}$.

The following proposition shows that quasireflective subcategories as defined above are a reasonable generalization of reflective subcategories.

Proposition 1.7 ([Bargenda 94] and [Börger Tholen 76]) *Let* $\mathbf{A}$ *be a subcategory of* $\mathbf{X}$*. Then*

(i) $\mathbf{A}$ *reflective* $\Rightarrow$ $\mathbf{A}$ *quasireflective* $\Rightarrow$ $\mathbf{A}$ *almost reflective,*

(ii) $\mathbf{X}$ *has equalizers of pairs in* $\mathbf{A}$ $\Rightarrow$ *(*$\mathbf{A}$ *reflective in* $\mathbf{X}$ $\Leftrightarrow$ $\mathbf{A}$ *quasireflective and closed under equalizers in* $\mathbf{X}$*).*

The implications in (i) above are not in general reversible; for example, the category

$$\begin{tikzcd} A \arrow[r, "p"] \arrow[dr, "s"'] & B \arrow[loop right, "t"] \arrow[d, shift right, "q"'] \arrow[d, shift left, "r"] \\ & C \end{tikzcd}$$

where $qp = s = rp$, $tp = p$ but $t \neq \mathbf{1}_B$ has an almost reflective subcategory $\{B, C\}$ which is not quasireflective, since p is not thick.

It will be shown later in this chapter that the Mac Neille completion of a partially ordered set is a quasireflection, but complete posets are not reflective in the category **Pos**, so the first implication above is not reversible either.

1.2 A-extendible morphisms

In this section we prove some results on the class ext($\mathbf{A}$) of all $\mathbf{A}$-extendible morphisms. The following lemma is well-known and can be found for example in [Maranda 64].

Lemma 1.8 *For any class* $\mathcal{U} \subseteq \mathrm{mor}(\mathbf{X})$, *if* $\mathbf{A} \subseteq \mathrm{inj}(\mathcal{U})$ *then* $\mathcal{U} \subseteq \mathrm{ext}(\mathbf{A})$.

The following lemma is partly contained in [Börger Tholen 76], section 1.1.

Lemma 1.9 *If* $\mathbf{A}$ *is quasireflective in* $\mathbf{X}$ *then* $\mathrm{inj}(\mathrm{ext}(\mathbf{A})) = \mathbf{A}$.

PROOF: The inclusion $\mathbf{A} \subseteq \mathrm{inj}(\mathrm{ext}(\mathbf{A}))$ is well-known. Conversely, suppose $X \in \mathrm{inj}(\mathrm{ext}(\mathbf{A}))$.

$$\begin{tikzcd} X \arrow[r, "q_X"] \arrow[d, "\mathbf{1}_X"'] & QX \arrow[dl, dashed, "g"] \\ X & \end{tikzcd}$$

Since $q_X \in \mathrm{ext}(\mathbf{A})$, by injectivity of X we have $(\exists g)(g q_X = \mathbf{1}_X)$. Thus q_X is a section. Then $(q_X g) q_X = q_X \mathbf{1}_X = q_X$, but q_X is thick, so $q_X g$ is an iso. Hence q_X is a retraction and so is iso. Then $X \in \mathbf{A}$ as $\mathbf{A}$ is iso-closed. □

Corollary 1.10 *If* $\mathbf{A}$ *is quasireflective in* $\mathbf{X}$ *then* $\mathrm{ext}(\mathbf{A})$ *is the largest class* $\mathcal{U}$ *such that* $\mathrm{inj}(\mathcal{U}) = \mathbf{A}$.

The following result is also well-known:

Lemma 1.11 *(i)* ext($\mathbf{A}$) *is coessential,*

(ii) ext($\mathbf{A}$) *is compositive,*

(iii) sect($\mathbf{X}$) $\subseteq$ ext($\mathbf{A}$).

Note that (i) and (ii) together imply that ext($\mathbf{A}$) is closed under composition with isomorphisms (on both sides).

Theorem 1.12 *Let* $\mathbf{A}$ *be a (full, isomorphism-closed) quasireflective subcategory of* $\mathbf{X}$. *Then TFAE:*

(i) $f \in$ ext($\mathbf{A}$),

(ii) $Qf \subseteq$ sect($\mathbf{A}$),

(iii) $Qf \cap$ sect($\mathbf{A}$) $\neq \emptyset$.

Proof: (i)$\Rightarrow$(ii): Suppose f is $\mathbf{A}$-extendible and let $s \in Qf$.

$$\begin{tikzcd}
X \arrow[r, "q_X"] \arrow[d, "f"'] & QX \arrow[d, shift left, "s"] \\
Y \arrow[r, "q_Y"'] \arrow[ur, dashed, "p"] & QY \arrow[u, dashed, shift left, "t"]
\end{tikzcd}$$

Since $f \in$ ext($\mathbf{A}$), $(\exists p)(pf = q_X)$. Now as $q_Y \in$ ext($\mathbf{A}$), $(\exists t)(tq_Y = p)$. Hence $(ts)q_X = tq_Y f = pf = q_X$, and q_X is thick, so ts is an iso. Hence s is a section, so $Qf \subseteq$ sect($\mathbf{A}$).

(ii)$\Rightarrow$(iii): This is clear since $Qf \neq \emptyset$ by lemma 1.5.

(iii)$\Rightarrow$(i): Suppose $(\exists s \in Qf)(s$ is a section), say $ts = \mathbf{1}_{QX}$.

$$\begin{tikzcd}
X \arrow[rr, "q_X"] \arrow[dr, "g"] \arrow[dd, "f"'] & & QX \arrow[dl, dashed, "k"'] \arrow[dd, shift left, "s"] \\
& Z & \\
Y \arrow[rr, "q_Y"'] \arrow[ur, dashed, "m"] & & QY \arrow[uu, shift left, "t"]
\end{tikzcd}$$

Let $Z \in \mathbf{A}$, $g : X \to Z$. As $q_X \in \mathrm{ext}(\mathbf{A})$, $(\exists k)(kq_X = g)$. Define $m := ktq_Y$. Then $mf = ktq_Yf = ktsq_X = kq_X = g$. Hence $f \in \mathrm{ext}(\mathbf{A})$. □

1.3 Some useful results

The following results assume that $\mathbf{A}$ is a quasireflective subcategory of $\mathbf{X}$. Recall (see definition 1.3) that $\mathrm{qrefl}(\mathbf{A})$ is the class of all $\mathbf{A}$-quasireflections.

Lemma 1.13 *If* $f \in \mathrm{qrefl}(\mathbf{A})$ *and* $\mathrm{dom}(f) \in \mathbf{A}$ *then* f *is an isomorphism.*

PROOF:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "\mathbf{1}_A"'] & B \arrow[loop right, "fk"] \arrow[d, dashed, "k"] \\
 & A
\end{tikzcd}$$

As $A \in \mathbf{A}$, $(\exists k)(kf = \mathbf{1}_A)$. Then $fkf = f\mathbf{1}_A = f$, but f is thick, so fk is an iso. Hence f is a retraction and a section, and thus iso. □

Lemma 1.14 (Uniqueness of quasireflections) *If* $f, g \in \mathrm{qrefl}(\mathbf{A})$ *satisfy* $\mathrm{dom}(f) = \mathrm{dom}(g)$, *then there exists an isomorphism* p *such that* $pf = g$.

PROOF:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "g"'] & B \arrow[d, dashed, shift left, "p"] \\
 & C \arrow[u, dashed, shift left, "q"]
\end{tikzcd}$$

$(\exists p)(pf = g)$ as $f \in \mathrm{qrefl}(\mathbf{A})$, and $(\exists q)(qg = f)$ as $g \in \mathrm{qrefl}(\mathbf{A})$. So, $pqg = pf = g$ and $qpf = qg = f$, but f and g are thick, so qp and pq are isos. Thus p is iso, and $pf = g$. □

Lemma 1.15 $\mathrm{qfirm}(\mathbf{A})$ *is compositive and* $\mathrm{iso}(\mathbf{X}) \subseteq \mathrm{qfirm}(\mathbf{A})$.

PROOF: Suppose $f : X \to Y$ and $g : Y \to Z$ are both in $\mathrm{qfirm}(\mathbf{A})$. Choose isomorphisms $h : QX \to QY$ and $k : QY \to QZ$ in Qf and Qg respectively.

Then kh is an isomorphism, and $(kh)q_X = kq_Yf = q_Z(gf)$, so $kh \in Q(gf)$. Thus $gf \in \text{qfirm}(\mathbf{A})$.

Now suppose $h : X \to Y \in \text{iso}(\mathbf{X})$. There exists p such that $pq_X = q_Yh$ as q_X is a quasireflection, and there also exists q such that $qq_Y = q_Xh^{-1}$ as q_Y is a quasireflection. Then $(pq)q_Y = pq_Xh^{-1} = q_Yhh^{-1} = q_Y$, and $(qp)q_X = qq_Yh = q_Xh^{-1}h = q_X$, so since q_X and q_Y are thick, pq and qp are isos. Thus p is a section and a retraction, hence $p \in Qh \cap \text{iso}(\mathbf{X})$, so $h \in \text{qfirm}(\mathbf{A})$. □

Lemma 1.16 *If $f : X \to Y$ is both a thick map and a section, then f is an isomorphism.*

Proof: Since f is a section, there exists $g : Y \to X$ such that $gf = \mathbf{1}_X$. Then $(fg)f = f\mathbf{1}_X = f$, so fg is iso as f is thick. Thus f is a retraction too, hence iso. □

Lemma 1.17 *If $f : X \to Y$ is thick and $h : Y \to Z$ is an iso in $\mathbf{X}$, then $hf : X \to Z$ is thick.*

Proof: Suppose $t : Z \to Z$ is an endomorphism such that $thf = hf$. Let $k := h^{-1}th$. Then $kf = h^{-1}thf = h^{-1}hf = f$, so k is iso as f is thick. Hence t is iso, so hf is thick. □

Lemma 1.18 $\text{qrefl}(\mathbf{A}) \subseteq \text{qfirm}(\mathbf{A})$

Proof: Let $f : X \to Y \in \text{qrefl}(\mathbf{A})$. By lemma 1.14, there exists an isomorphism k such that $kq_X = f$. Also, by lemma 1.13 q_Y is an iso since $Y \in \mathbf{A}$. Define $h = q_Yk$. Then h is an iso, and $hq_X = q_Ykq_X = q_Yf$, so $f \in \text{qfirm}(\mathbf{A})$. □

Corollary 1.19

$$f \in \text{qrefl}(\mathbf{A}) \Leftrightarrow f \in \text{qfirm}(\mathbf{A}) \textit{ and } \text{cod}(f) \in \mathbf{A}$$

This result forms the basis of a characterization of the reflection morphisms in case **A** is a reflective subcategory of **X**:

Corollary 1.20 *If* **A** *is* reflective *in* **X** *then* f *is a reflection morphism iff* $f \in \text{qfirm}(\mathbf{A})$ *and* $\text{cod}(f) \in \mathbf{A}$.

Lemma 1.21 *Let* $\mathcal{U} \subseteq \text{mor}(\mathbf{X})$ *be closed under composition with isos. If* **A** *is* $\mathcal{U}$*-quasireflective in* **X** *then* $\text{qrefl}(\mathbf{A}) \subseteq \mathcal{U}$.

PROOF: Let $f \in \text{qrefl}(\mathbf{A})$. Now $(\exists g \in \text{qrefl}(\mathbf{A}))(g \in \mathcal{U})$ by definition of $\mathcal{U}$-quasireflective, with $\text{dom}(f) = \text{dom}(g)$. So, by lemma 1.14 $(\exists \text{ iso } h)(f = hg)$. Then since $g \in \mathcal{U}$ and $\mathcal{U}$ is closed under composition with isos, $f \in \mathcal{U}$. Thus $\text{qrefl}(\mathbf{A}) \subseteq \mathcal{U}$. □

Thus, whenever **A** is $\mathcal{U}$-quasireflective we may assume $q_X \in \mathcal{U}$ for every $X \in \mathbf{X}$, provided $\mathcal{U}$ is closed under composition with isos.

Lemma 1.22 $\text{qrefl}(\mathbf{A})$ *is compositive and closed under composition with isos.*

PROOF: Let $f \in \text{qrefl}(\mathbf{A})$, and suppose fg exists, where g is an isomorphism. $fg \in \text{ext}(\mathbf{A})$ by lemma 1.11, and $fg \in$ thick as g is iso. Thus $fg \in \text{qrefl}(\mathbf{A})$.

Now suppose h is an isomorphism and that hf exists. By lemma 1.11, $hf \in \text{ext}(\mathbf{A})$ and by lemma 1.17 $hf \in$ thick. Also, $\text{cod}(hf) \in \mathbf{A}$ as $\text{cod}(f) \in \mathbf{A}$ and **A** is iso-closed. Hence $hf \in \text{qrefl}(\mathbf{A})$.

We have shown that $\text{iso}(\mathbf{X}) \circ \text{qrefl}(\mathbf{A}) \subseteq \text{qrefl}(\mathbf{A})$ and that $\text{qrefl}(\mathbf{A}) \circ \text{iso}(\mathbf{X}) \subseteq \text{qrefl}(\mathbf{A})$. To show $\text{qrefl}(\mathbf{A})$ compositive, suppose that $u, v \in \text{qrefl}(\mathbf{A})$ and that uv exists. Then $\text{dom}(u) = \text{cod}(v) \in \mathbf{A}$, so by lemma 1.13 u is an isomorphism. By the above, $uv \in \text{qrefl}(\mathbf{A})$. Hence $\text{qrefl}(\mathbf{A})$ is closed under composition and under composition with isomorphisms. □

Lemma 1.23 $\text{qrefl}(\mathbf{A}) \subseteq \text{ext}(\mathbf{A})^*$*, and moreover* $\text{qrefl}(\mathbf{A})$ *is essential.*

Proof: Suppose $f \in \mathrm{qrefl}(\mathbf{A})$ and $q = gf \in \mathrm{ext}(\mathbf{A})$. Since $\mathrm{cod}(f) \in \mathbf{A}$, q is a factor of f, i.e. $(\exists h)(hq = f)$. Then $(hg)f = hq = f$, but f is thick, so hg

$$\begin{tikzcd}
\bullet \arrow[r, "f"] \arrow[dr, "q=gf"'] & \bullet \arrow[d, shift left, "g"] \\
& \bullet \arrow[u, shift left, dashed, "h"] \arrow[loop below, "t"]
\end{tikzcd}$$

is an iso. Hence g is a section. Thus by lemma 1.11, $g \in \mathrm{ext}(\mathbf{A})$. Hence $\mathrm{qrefl}(\mathbf{A}) \subseteq \mathrm{ext}(\mathbf{A})^*$. Now if in addition $q \in \mathrm{qrefl}(\mathbf{A})$, then $\mathrm{cod}(g) \in \mathbf{A}$, and if $tg = g$ then $tgf = gf$, but gf is thick, so t is iso. Hence g is thick. Thus $g \in \mathrm{qrefl}(\mathbf{A})$, so $\mathrm{qrefl}(\mathbf{A})$ is essential. □

Lemma 1.24 *If* $\mathbf{X}$ *has* $\mathcal{U}$*-injective hulls then* $\mathrm{inj}(\mathcal{U}) = \mathrm{inj}(\mathcal{U}^*)$.

Proof: $\mathrm{inj}(\mathcal{U}) \subseteq \mathrm{inj}(\mathcal{U}^*)$ since $\mathcal{U}^* \subseteq \mathcal{U}$.

For the reverse inclusion, suppose $X \in \mathrm{inj}(\mathcal{U}^*)$. Let $u : Y \to Z \in \mathcal{U}$ and $f : Y \to X \in \mathrm{mor}(\mathbf{X})$.

$$\begin{tikzcd}
Y \arrow[rr, "w"] \arrow[dr, "u"] \arrow[ddr, "f"'] & & W \arrow[ddl, dashed, "g"] \\
& Z \arrow[ur, dashed, "h"] \arrow[d, dashed, "k"] & \\
& X &
\end{tikzcd}$$

Let $w : Y \to W$ be the $\mathcal{U}$-injective hull of Y. Then $w \in \mathcal{U}^*$, so f has an extension g along w, i.e. $gw = f$. Also, $W \in \mathrm{inj}(\mathcal{U})$, so w has an extension h along $u \in \mathcal{U}$, i.e. $hu = w$. Defining $k = gh$ gives a morphism k satisfying $ku = ghu = gw = f$, so f has an extension along u. Thus $X \in \mathrm{inj}(\mathcal{U})$. □

1.4 Relation to injective hull structures

In this section a correspondence between quasireflective subcategories of $\mathbf{X}$ and $\mathcal{U}$-injective hull structures on $\mathbf{X}$, due to Hubertus W. Bargenda, is described.

Proposition 1.25 ([Bargenda 95], also [Adámek Rosický 94]) *For all $X \in \mathbf{X}$, any $\mathcal{U}$-injective hulls of X are determined up to isomorphism.*

Proof: Suppose $u : X \to A$ and $v : X \to B$ are $\mathcal{U}$-injective hulls of $X \in \mathbf{X}$.

$$\begin{tikzcd}
A \arrow[rr, dashed, "c"] \arrow[d, equal] & & B \arrow[d, equal] \\
A \arrow[rr, dashed, "a"'] & & B \arrow[llu, dashed, "b"'] \\
& X \arrow[lu, "u"] \arrow[ru, "v"'] &
\end{tikzcd}$$

As $B \in \mathrm{inj}(\mathcal{U})$ $(\exists a)(au = v)$. Now $a \in \mathcal{U}$ since $u \in \mathcal{U}^*$. Then since $A \in \mathrm{inj}(\mathcal{U})$, $(\exists b)(ba = \mathbf{1}_A)$. Now $bv = bau = u$, so $b \in \mathcal{U}$ as $v \in \mathcal{U}^*$. Thus $(\exists c)(cb = \mathbf{1}_B)$ as $B \in \mathrm{inj}(\mathcal{U})$. So b is a retraction and a section and consequently an iso, thus also a is an isomorphism, satisfying $au = v$. □

Corollary 1.26 *For any class $\mathcal{U}$ of $\mathbf{X}$-morphisms, $\mathcal{U}$-injective hulls are thick.*

Remarks:

1. The result that $\mathcal{U}$-injective hulls are thick was stated without proof by Porst, but only for classes $\mathcal{U}$ satisfying certain conditions [Porst 81, p 399–402].
2. Proposition 1.25 also generalizes proposition 9.19 in [AHS 90] (see also [AHS 90, remark 9.23], where the essential uniqueness of $\mathcal{U}$-injective hulls is claimed to hold for $\mathcal{U}$ subject to suitable assumptions).
3. Proposition 1.25 is contained in Exercise 4.d of [Adámek Rosický 94], where the name stable weak reflection is used.
4. Clearly if $\mathbf{X}$ has $\mathcal{U}$-injective hulls, then $\mathrm{inj}(\mathcal{U})$ is quasireflective in $\mathbf{X}$. Moreover the $\mathrm{inj}(\mathcal{U})$-quasireflections can be assumed to be $\mathcal{U}$-essential maps.
5. [Bargenda 94] If $\mathbf{A}$ is quasireflective in $\mathbf{X}$ then $\mathbf{X}$ has $\mathrm{ext}(\mathbf{A})$-injective hulls.

So there is a Galois correspondence, due to Hubertus W. Bargenda, between $\mathcal{U}$-injective hull structures on $\mathbf{X}$ and $\mathcal{U}$-quasireflective subcategories of $\mathbf{X}$: if every $X \in \mathbf{X}$ has a $\mathcal{U}$-injective hull, then $\mathbf{A} = \mathrm{inj}(\mathcal{U})$ is $\mathcal{U}$-quasireflective in $\mathbf{X}$, and given any $\mathcal{U}$-quasireflective subcategory $\mathbf{A}$ of $\mathbf{X}$, the corresponding class $\mathrm{ext}(\mathbf{A})$ gives rise to the required injective hulls in $\mathbf{X}$.

The correspondence is not bijective in general since for a fixed quasireflective subcategory $\mathbf{A}$ of $\mathbf{X}$, $\mathbf{X}$ has $\mathrm{qrefl}(\mathbf{A})$-injective hulls in addition to $\mathrm{ext}(\mathbf{A})$-injective hulls, but in many cases $\mathrm{qrefl}(\mathbf{A}) \neq \mathrm{ext}(\mathbf{A})$.

1.5 Examples

a) Injective hull structures: Remark 3 on page 15 together with the observation that the embeddings in any concrete category are a class of extensions shows that the injective objects in any construct with injective hulls form a quasireflective subcategory.

b) Any "Skula" quasireflection [Skula 74] is clearly a quasireflection. In fact the only difference is that instead of necessarily being thick maps, Skula quasireflections are required to *admit only identities*, meaning that if t is any endomorphism such that $tq_X = t$, where q_X is any Skula quasireflection, then t must be an identity morphism.

Skula gives the following examples which fail to be reflections:

1. The full subcategory of all complete spaces in the category **Unif** of all uniform spaces with uniformly continuous maps,
2. The full subcategory of all complete posets in the category **Pos** of all posets (partially ordered sets) and order-preserving functions.

An example of a quasireflection which is not a Skula quasireflection is the algebraic closure of a field, where the field extensions into closed fields certainly admit non-identity automorphisms.

c) The notion of quasireflection introduced in [Börger Tholen 76] is more specialized than that defined in this thesis. In particular, those authors drop the uniqueness requirement (as done here) but instead of requiring quasireflection maps to be thick, their condition states that any two extensions of a map along a quasireflection must be linked by an automorphism. Let $q_X : X \to QX$ be such a map.

$$\begin{tikzcd}
X \arrow[r, "q_X"] \arrow[dr, "q_X"'] & QX \arrow[loop above, "g"] \arrow[d, shift right, "t"'] \arrow[d, shift left, "\mathbf{1}_{QX}"] \\
& QX
\end{tikzcd}$$

Suppose $tq_X = q_X$. Also $\mathbf{1}_{QX}q_X = q_X$, so by assumption there is an automorphism g such that $t = \mathbf{1}_{QX}g$. Thus t is iso, hence q_X is thick and so is a quasireflection. A simple example of a quasireflection which is not captured by those authors' definition is the category

$$\begin{tikzcd}
A \arrow[r, "p"] \arrow[dr, "s"'] & B \arrow[d, shift right, "q"'] \arrow[d, shift left, "r"] \\
& C
\end{tikzcd}$$

where $q \neq r$, $qp = rp = s$ and the only automorphisms are the identities. The subcategory $\{B, C\}$ is clearly quasireflective but q and r are not linked by an automorphism.

Chapter 2

Firm morphism classes

The following chapter generalizes the notion of a (sub)firm class of morphisms defined in [Brümmer Giuli 92]. Many of the results on completions of objects obtained in that paper are also generalized here to the quasireflective case.

2.1 Definitions

Definition 2.1 *A $\mathcal{U}$-quasireflective subcategory* $\mathbf{A}$ *of* $\mathbf{X}$ *is called* <u>*subfirmly $\mathcal{U}$-quasireflective*</u> *iff every map f in $\mathcal{U}$ with codomain in* $\mathbf{A}$ *is a quasireflection, i.e. for every $f : X \to Y \in \mathcal{U}$, $Y \in \mathbf{A} \Rightarrow f \in \mathrm{qrefl}(\mathbf{A})$.*

There is an equivalent definition more along the lines of a subfirmly $\mathcal{U}$-reflective subcategory as introduced in [Brümmer Giuli 92], given by the following lemma:

Lemma 2.2 *A $\mathcal{U}$-quasireflective subcategory* $\mathbf{A}$ *of* $\mathbf{X}$ *is subfirmly $\mathcal{U}$-quasireflective iff every map $f : X \to Y$ in $\mathcal{U}$ with* $\mathrm{cod}(f) \in \mathbf{A}$ *has an extension h along q_X such that h is an isomorphism.*

PROOF: $\Leftarrow$: Suppose $\mathbf{A}$ is subfirmly $\mathcal{U}$-quasireflective. Let $f : X \to Y \in \mathcal{U}$, and $Y \in \mathbf{A}$.

$$\begin{tikzcd}
X \arrow[r, "q_X"] \arrow[dr, "f"'] & QX \arrow[d, shift left, "m"] \\
& Y \arrow[u, shift left, "n"]
\end{tikzcd}$$

By definition 2.1 $f \in \mathrm{qrefl}(\mathbf{A})$. Thus $(\exists n)(nf = q_X)$. Also as $Y \in \mathbf{A}$, $(\exists m)(mq_X = f)$. So $(nm)q_X = q_X$ and $(mn)f = f$, but f and q_X are thick, so both mn and nm are isos. Hence m is a retraction and a section, and thus an iso.

$\Rightarrow$: Let $f : X \to Y \in \mathcal{U}$ with $Y \in \mathbf{A}$. We assume that there is an isomorphism h such that $hq_X = f$. Then $f \in \mathrm{qrefl}(\mathbf{A})$ as $\mathrm{qrefl}(\mathbf{A})$ is closed under composition with isos (lemma 1.22). Hence $\mathbf{A}$ is subfirmly $\mathcal{U}$-quasireflective. □

Definition 2.3 *A class $\mathcal{U}$ of $\mathbf{X}$-morphisms is called subfirm iff*

(i) $\mathcal{U}$ is compositive,

(ii) there exists a subfirmly $\mathcal{U}$-quasireflective subcategory $\mathbf{A}$ of $\mathbf{X}$.

There is an alternative definition of subfirm given by the following lemma:

Lemma 2.4 *If $\mathcal{U} \subseteq \mathrm{mor}(\mathbf{X})$ is compositive and $\mathbf{A}$ is a $\mathcal{U}$-quasireflective subcategory of $\mathbf{X}$ then TFAE:*

(i) $\mathbf{A}$ is subfirmly $\mathcal{U}$-quasireflective in $\mathbf{X}$,

(ii) $\mathcal{U} \subseteq \mathrm{qfirm}(\mathbf{A})$.

PROOF: (i)$\Rightarrow$(ii): Let $u : X \to Y \in \mathcal{U}$. Since $\mathcal{U}$ is compositive $q_Y u \in \mathcal{U}$, and $QY \in \mathbf{A}$, so by lemma 2.2 there exists an isomorphism h such that $hq_X = q_Y u$. Thus $u \in \mathrm{qfirm}(\mathbf{A})$.

(ii)$\Rightarrow$(i): Let $f : X \to Y \in \mathcal{U}$ with $Y \in \mathbf{A}$. Then q_Y is an isomorphism, and there exists an isomorphism $h : QX \to QY$ by (ii). So $q_Y^{-1}h$ is an isomorphic extension of f along q_X, and by lemma 2.2 $\mathbf{A}$ is subfirmly $\mathcal{U}$-quasireflective. □

The condition $\mathcal{U} \subseteq \mathrm{qfirm}(\mathbf{A})$ will usually be used when working with subfirm classes.

Definition 2.5 *A class $\mathcal{U}$ of $\mathbf{X}$-morphisms is called firm iff there exists a quasireflective subcategory $\mathbf{A}$ of $\mathbf{X}$ with $\mathcal{U} = \mathrm{qfirm}(\mathbf{A})$.*

2.2 Characterisations of (sub)firm classes

The following theorem extends 1.4 of [Brümmer Giuli 92] to quasireflective subcategories:

Theorem 2.6 *Let $\mathcal{U} \subseteq \mathrm{mor}(\mathbf{X})$ be compositive and suppose $\mathbf{A}$ is $\mathcal{U}$-quasireflective in $\mathbf{X}$. TFAE:*

(i) $\mathbf{A}$ is subfirmly $\mathcal{U}$-quasireflective in $\mathbf{X}$,

(ii) $\mathbf{A} = \mathrm{inj}(\mathcal{U})$ and every $u \in \mathcal{U}$ with $\mathrm{cod}(u) \in \mathbf{A}$ is thick.

PROOF: (i)$\Rightarrow$(ii): Assume (i) holds. We firstly prove $\mathbf{A} = \mathrm{inj}(\mathcal{U})$. Let $A \in \mathbf{A}$, $f : X \to A$ any morphism into A, and $u : X \to Y \in \mathcal{U}$.

$$\begin{tikzcd}
X \arrow[rr, "q_X"] \arrow[dr, "f"] \arrow[dd, "u"'] & & QX \arrow[dl, dashed, "p"] \arrow[dd, dashed, "\text{iso}"', "g"] \\
& A & \\
Y \arrow[ur, dashed, "m"'] \arrow[rr, "q_Y"'] & & QY
\end{tikzcd}$$

As q_X is a quasireflection, $(\exists p)(pq_X = f)$. Now $q_Y \in \mathcal{U}$ as $\mathbf{A}$ is $\mathcal{U}$-quasireflective, and $\mathcal{U}$ is closed under composition, so $q_Y u \in \mathcal{U}$. By the subfirm property, $(\exists \text{ iso } g)(gq_X = q_Y u)$. Define $m := pg^{-1}q_Y$. Then $mu = pg^{-1}q_Y u = pg^{-1}gq_X = pq_X = f$. Thus m is an extension of f along $u \in \mathcal{U}$. Hence A is $\mathcal{U}$-injective. Thus $\mathbf{A} \subseteq \mathrm{inj}(\mathcal{U})$.

For the reverse inclusion, suppose $X \in \mathrm{inj}(\mathcal{U})$.

$$\begin{tikzcd}
X \arrow[r, "q_X"] \arrow[d, equal, "1_X"'] & QX \arrow[dl, dashed, "g"] \\
X &
\end{tikzcd}$$

Since $q_X \in \mathcal{U}$, $(\exists g)(gq_X = \mathbf{1}_X)$, and so $q_X g q_X = q_X \mathbf{1}_X = q_X$. Since q_X is a quasireflection, it is thick, so $q_X g$ is an isomorphism. Hence q_X is a section and a retraction, and so is iso. Thus $X \cong QX$ and since $\mathbf{A}$ is iso-closed, $X \in \mathbf{A}$. Hence $\mathbf{A} = \mathrm{inj}(\mathcal{U})$.

Secondly, let $u : X \to A \in \mathcal{U}$, where $A \in \mathbf{A}$. Then by (i) and lemma 1.14, there exists an iso $h : QX \to A$ with $u = hq_X$. Hence by lemma 1.17 u is thick.

(ii)⇒(i): Assume (ii) holds. To show $\mathbf{A}$ is subfirmly $\mathcal{U}$-quasireflective, let $f : X \to A \in \mathcal{U}$ with $A \in \mathbf{A}$.

$$\begin{tikzcd}
X \arrow[r, "q_X"] \arrow[dr, "f"'] & QX \arrow[loop right, "gp"] \arrow[d, dashed, shift left, "p"] \\
& A \arrow[u, dashed, shift left, "g"] \arrow[loop right, "pg"]
\end{tikzcd}$$

Now $(\exists p)(pq_X = f)$ as q_X is a quasireflection. Since $QX \in \mathbf{A} = \mathrm{inj}(\mathcal{U})$ by assumption, $(\exists g)(gf = q_X)$ as $f \in \mathcal{U}$. We have $(gp)q_X = gf = q_X$ and q_X is thick, so gp is iso. Also $(pg)f = pq_X = f$ and f is thick by (ii), so pg is an iso. Hence p is a retraction and a section, and thus an iso. So, $\mathbf{A}$ is subfirmly $\mathcal{U}$-quasireflective. □

Corollary 2.7 *Given any subfirm class* $\mathcal{U} \subseteq \mathrm{mor}(\mathbf{X})$ *there is exactly one subfirmly* $\mathcal{U}$*-quasireflective subcategory* $\mathbf{A}$ *of* $\mathbf{X}$*, namely* $\mathbf{A} = \mathrm{inj}(\mathcal{U})$.

Concerning the reverse assignment, see proposition 2.16. The next proposition is a generalization of 1.6 in [Brümmer Giuli 92]:

Proposition 2.8 *If* $\mathcal{U}$ *is closed under composition then TFAE:*

(i) $\mathrm{inj}(\mathcal{U})$ *is subfirmly* $\mathcal{U}$*-quasireflective,*

(ii) $\mathbf{X}$ *has enough* $\mathcal{U}$*-injectives and for all* $u \in \mathcal{U}$, $\mathrm{cod}(u) \in \mathrm{inj}(\mathcal{U}) \Rightarrow u$ *thick.*

PROOF: (i)⇒(ii): If (i) holds, then $\mathrm{inj}(\mathcal{U})$ is $\mathcal{U}$-quasireflective and so $\mathbf{X}$ has enough $\mathcal{U}$-injectives. Part (ii) now follows from theorem 2.6. (ii)⇒(i): Suppose (ii) holds. If $\mathrm{inj}(\mathcal{U})$ is $\mathcal{U}$-quasireflective we are done by theorem 2.6, so let $X \in \mathbf{X}$. Now $\exists Y_X \in \mathrm{inj}(\mathcal{U})$ and $y_X : X \to Y_X \in \mathcal{U}$ by (ii).

$$\begin{tikzcd} X \arrow[r, "y_X"] \arrow[dr, "f"'] & Y_X \arrow[d, dashed, "\bar{f}"] \\ & Z \end{tikzcd}$$

Let $f : X \to Z$ with $Z \in \mathrm{inj}(\mathcal{U})$. Since Z is $\mathcal{U}$-injective, f has an extension $\bar{f}$ along y_X. Then by (ii), y_X is thick. Thus y_X is an $\mathrm{inj}(\mathcal{U})$-quasireflection of X, and $y_X \in \mathcal{U}$. Hence $\mathrm{inj}(\mathcal{U})$ is $\mathcal{U}$-quasireflective in $\mathbf{X}$. □

There is an analogous result for subcategories:

Proposition 2.9 *If $\mathcal{U} \subseteq \mathrm{mor}(\mathbf{X})$ is compositive, then $\mathbf{A}$ is subfirmly $\mathcal{U}$-quasireflective iff $\mathbf{A} = \mathrm{inj}(\mathcal{U})$, $\mathbf{X}$ has enough $\mathcal{U}$-injectives and each $u \in \mathcal{U}$ with $\mathrm{cod}(u) \in \mathbf{A}$ is thick.*

PROOF: This is clear from theorem 2.6 and proposition 2.8. □

Proposition 2.10 *If $\mathbf{X}$ has $\mathcal{U}$-injective hulls and $\mathcal{U}$ is compositive then $\mathcal{U}^*$ is subfirm in $\mathbf{X}$, in particular $\mathcal{U}^* \subseteq \mathrm{qfirm}(\mathrm{inj}(\mathcal{U}))$.*

PROOF: By remark 3 on page 15, $\mathrm{inj}(\mathcal{U})$ is $\mathcal{U}^*$-quasireflective. It is obvious that $\mathrm{inj}(\mathcal{U})$ is subfirmly $\mathcal{U}^*$-quasireflective in $\mathbf{X}$. Now by lemma 1.24, $\mathrm{inj}(\mathcal{U}) = \mathrm{inj}(\mathcal{U}^*)$, so $\mathcal{U}^*$ is subfirm in $\mathbf{X}$. □

The proposition below extends some of 1.8 from [Brümmer Giuli 92]:

Proposition 2.11 *If $\mathcal{U}$ is subfirm, $\mathbf{A} = \mathrm{inj}(\mathcal{U})$ and $\mathcal{U}$ is compositive then (i)⇔(ii) below:*

(i) $\mathcal{U} \subseteq \mathrm{mono}(\mathbf{X})$,

(ii) qrefl($\mathbf{A}$) $\subseteq$ mono($\mathbf{X}$),

Proof: (i)$\Rightarrow$(ii): Suppose (i) holds. By lemma 1.21 qrefl(inj($\mathcal{U}$)) $\subseteq$ iso-closure of $\mathcal{U}$. Now suppose that f belongs to the iso-closure of $\mathcal{U}$, and that $gf = hf$. Then $f = luk$, where $u \in \mathcal{U}$ and $l, k \in$ iso($\mathbf{X}$). So, $gluk = hluk$. Since k is iso, $glu = hlu$. By (i) $u \in$ mono($\mathbf{X}$), so $gl = hl$. Thus $g = h$ as l iso. Hence $f \in$ mono($\mathbf{X}$).

(ii)$\Rightarrow$(i): Let $u : X \to Y \in \mathcal{U}$. Then $q_Y u \in \mathcal{U}$ as $q_Y \in \mathcal{U}$ and $\mathcal{U}$ is compositive. Also $\mathcal{U} \subseteq$ qfirm($\mathbf{A}$) by lemma 2.4, so $q_Y u \in$ qfirm($\mathbf{A}$) and cod($q_Y u$) $\in \mathbf{A}$. Thus by corollary 1.19, $q_Y u \in$ qrefl($\mathbf{A}$) $\subseteq$ mono($\mathbf{X}$). Hence $u \in$ mono($\mathbf{X}$). □

The next result improves 1.11 in [Brümmer Giuli 92]:

Theorem 2.12 *$\mathcal{U}$ is firm $\Leftrightarrow$ $\mathcal{U}$ is subfirm, coessential, and closed under composition with isos.*

Proof: $\Rightarrow$: Let $\mathcal{U}$ = qfirm($\mathbf{A}$) for a quasireflective subcategory $\mathbf{A}$.

$$\begin{tikzcd}
A \arrow[r, "q_A"] \arrow[d, "g"'] & QA \arrow[d, dashed, "j"] \arrow[dd, bend left=60, "k"', "\text{iso}"] \\
B \arrow[r, "q_B"] \arrow[d, "f"'] & QB \arrow[d, "\text{iso}"', "h"] \\
C \arrow[r, "q_C"] & QC
\end{tikzcd}$$

$\mathcal{U}$ is subfirm by lemma 2.4. Let $fg \in$ qfirm($\mathbf{A}$), with $f \in$ qfirm($\mathbf{A}$). We must show $g \in$ qfirm($\mathbf{A}$): Now Qf contains an iso h, and $Q(fg)$ contains an iso k. Let $j = h^{-1}k$. Then $jq_A = h^{-1}kq_A = h^{-1}q_C fg = h^{-1}hq_B g = q_B g$, so $j \in Qg$ and j iso. Thus $g \in$ qfirm($\mathbf{A}$) and therefore is coessential. By lemma 1.15 qfirm($\mathbf{A}$) is closed under composition with isos.

$\Leftarrow$: Let $\mathcal{U}$ be coessential and subfirm. Then by definition 2.3 $\mathcal{U}$ is compositive and there exists a $\mathcal{U}$-quasireflective subcategory $\mathbf{A}$ of $\mathbf{X}$. Then $\mathcal{U} \subseteq$ qfirm($\mathbf{A}$) by lemma 2.4.

$$\begin{tikzcd} X \arrow[r, "q_X"] \arrow[d, "f"'] & QA \arrow[d, "h"] \arrow[d, phantom, "\text{iso}"', shift right=3] \\ Y \arrow[r, "q_Y"'] & QB \end{tikzcd}$$

We show $\text{qfirm}(\mathbf{A}) \subseteq \mathcal{U}$: Let $f \in \text{qfirm}(\mathbf{A})$. Thus there exists an iso h such that $q_Y f = h q_X$. Hence $h q_X \in \mathcal{U}$ as $\mathcal{U}$ is closed under composition with isos, and $q_Y \in \mathcal{U}$, so since $\mathcal{U}$ is coessential, $f \in \mathcal{U}$. Therefore $\mathcal{U} = \text{qfirm}(\mathbf{A})$. □

Now a result similar to 3.3 in [Brümmer Giuli 92] is given: let $\mathcal{D}(\mathbf{A})$ be the class $\{f : Qf \text{ contains a retraction}\}$, where $\mathbf{A}$ is a quasireflective subcategory of $\mathbf{X}$. Then we have:

Proposition 2.13 *If* $\mathbf{A}$ *is a quasireflective subcategory of* $\mathbf{X}$ *then*

$$\text{qfirm}(\mathbf{A}) = \text{ext}(\mathbf{A}) \cap \mathcal{D}(\mathbf{A})$$

PROOF: Let $f \in \text{qfirm}(\mathbf{A})$. Then Qf contains an iso, so in particular Qf contains a retraction. Hence $f \in \mathcal{D}(\mathbf{A})$. Also, Qf contains a section, so by theorem 1.12, $f \in \text{ext}(\mathbf{A})$.

For the reverse inclusion, if $f \in \mathcal{D}(\mathbf{A}) \cap \text{ext}(\mathbf{A})$, then $f \in \text{ext}(\mathbf{A})$, so by theorem 1.12 $Qf \subseteq \text{sect}(\mathbf{X})$. But Qf contains a retraction, so $(\exists \text{ iso } h)(h \in Qf)$. Thus $f \in \text{qfirm}(\mathbf{A})$. □

The proposition below generalizes 3.1 in [Brümmer Giuli 92]:

Proposition 2.14 *Let* $\mathbf{A}$ *be a quasireflective subcategory of* $\mathbf{X}$. *Then* $\text{qrefl}(\mathbf{A})$ *is subfirm and* $\text{inj}(\text{qrefl}(\mathbf{A})) = \mathbf{A}$. *Moreover* $\text{qrefl}(\mathbf{A})$ *is the smallest iso-closed subfirm class* $\mathcal{U}$ *such that* $\text{inj}(\mathcal{U}) = \mathbf{A}$.

PROOF: By lemma 1.18, $\text{qrefl}(\mathbf{A}) \subseteq \text{qfirm}(\mathbf{A})$. Hence by lemma 2.4 $\mathbf{A}$ is subfirmly $\text{qrefl}(\mathbf{A})$-quasireflective in $\mathbf{X}$. By theorem 2.6, $\mathbf{A} = \text{inj}(\text{qrefl}(\mathbf{A}))$. Now suppose $\mathcal{U}$ is subfirm and $\text{inj}(\mathcal{U}) = \mathbf{A}$. To show that $\text{qrefl}(\mathbf{A}) \subseteq \mathcal{U}$, let $f \in \text{qrefl}(\mathbf{A})$.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[dr, "q_X"'] & A \arrow[d, dashed, "\text{iso}"] \\
& QX
\end{tikzcd}$$

Since $q_X \in \text{qrefl}(\mathbf{A})$, by lemma 1.14 $(\exists\ \text{iso})(q_X \circ \text{iso} = f)$. Now $q_X \in \mathcal{U}$ as $\mathbf{A}$ is $\mathcal{U}$-quasireflective, so since $\mathcal{U}$ is closed under composition with isos, $f \in \mathcal{U}$. □

The remaining results in this chapter extend theorem 3.2 from [Brümmer Giuli 92]:

Proposition 2.15 *Let* $\mathbf{A}$ *be a quasireflective subcategory of* $\mathbf{X}$. *Then* $\text{qfirm}(\mathbf{A})$ *is subfirm and* $\text{inj}(\text{qfirm}(\mathbf{A})) = \mathbf{A}$. *Moreover* $\text{qfirm}(\mathbf{A})$ *is the largest subfirm class* $\mathcal{U}$ *such that* $\text{inj}(\mathcal{U}) = \mathbf{A}$.

Proof: It is clear that $\mathbf{A}$ is subfirmly $\text{qfirm}(\mathbf{A})$-quasireflective, hence by theorem 2.6 $\text{inj}(\text{qfirm}(\mathbf{A})) = \mathbf{A}$. Now suppose $\mathbf{A}$ is subfirmly $\mathcal{U}$-quasireflective for some compositive class $\mathcal{U}$.

$$\begin{tikzcd}
X \arrow[r, "q_X"] \arrow[d, "u"'] & QX \arrow[d, dashed, "\text{iso}"', "k"] \\
Y \arrow[r, "q_Y"'] & QY
\end{tikzcd}$$

Let $u \in \mathcal{U}$. Now $q_Y \in \mathcal{U}$ and $u \in \mathcal{U}$, so $q_Y u \in \mathcal{U}$ as $\mathcal{U}$ is compositive. Thus by subfirmness $(\exists\ \text{iso}\ k)(kq_X = q_Y u)$. Hence $u \in \text{qfirm}(\mathbf{A})$. By lemma 1.15, $\text{qfirm}(\mathbf{A})$ is compositive and iso-closed. □

Proposition 2.16 *Given a quasireflective subcategory* $\mathbf{A}$ *of* $\mathbf{X}$, *the conglomerate* $\text{qsf}(\mathbf{A})$ *of all iso-closed subfirm classes* $\mathcal{U}$ *with* $\text{inj}(\mathcal{U}) = \mathbf{A}$ *is a complete lattice ordered by class inclusion. Moreover if* $\text{inj}(\mathcal{U}) = \mathbf{A}$ *and* $\mathcal{U}$ *is compositive and closed under composition with isos then* $\mathcal{U}$ *is subfirm iff* $\text{qrefl}(\mathbf{A}) \subseteq \mathcal{U} \subseteq \text{qfirm}(\mathbf{A})$.

Proof: By propositions 2.14 and 2.15 $\text{qrefl}(\mathbf{A})$ and $\text{qfirm}(\mathbf{A})$ are respectively the smallest and the largest members of $\text{qsf}(\mathbf{A})$. Suppose $\text{inj}(\mathcal{U}) = \mathbf{A}$. By propositions 2.14 and 2.15 if $\mathcal{U}$ is subfirm then $\text{qrefl}(\mathbf{A}) \subseteq \mathcal{U} \subseteq \text{qfirm}(\mathbf{A})$, and if $\text{qrefl}(\mathbf{A}) \subseteq \mathcal{U} \subseteq \text{qfirm}(\mathbf{A})$ then $\mathcal{U}$ is subfirm by lemma 2.4. If $\mathcal{U}_i$ is a family in $\text{qsf}(\mathbf{A})$, then clearly $\text{qrefl}(\mathbf{A}) \subseteq \bigcap_i \mathcal{U}_i \subseteq \text{qfirm}(\mathbf{A})$, and hence also $\text{inj}(\bigcap_i \mathcal{U}_i) = \bigcup_i \text{inj}(\mathcal{U}_i) = \mathbf{A}$. Since each $\mathcal{U}_i$ is compositive and closed under composition with isos, so is $\bigcap_i \mathcal{U}_i$. Hence $\bigcap_i \mathcal{U}_i \in \text{qsf}(\mathbf{A})$. Thus $\text{qsf}(\mathbf{A})$ has arbitrary meets and is bounded, which shows that $\text{qsf}(\mathbf{A})$ is a complete lattice. □

2.3 Examples

a) If $\mathcal{U}$ is (sub)firm in the sense of [Brümmer Giuli 92] then $\mathcal{U}$ is quasi(sub)firm, since reflective ⇒ quasireflective. Thus all of the examples presented in that paper are included in this more general setting.

b) The Mac Neille completion of a poset: If $\mathbf{A}$ is the full subcategory of **Pos** consisting of complete posets, then

- $\text{qfirm}(\mathbf{A})$ consists of the meet-and-join-dense embeddings.

PROOF: The essential embeddings are widely known to be the meet-and-join-dense embeddings. See for example [Banaschewski Bruns 67, Lemma 3, p 371] We know by proposition 2.10 that the essential embeddings are subfirm, so if we show that they are coessential we are done by theorem 2.12. Suppose $gf : X \to Y \to Z \in \text{emb}^*$ and $g \in \text{emb}^*$. Since embeddings are coessential, f is an embedding. Assume wlog that g, f are inclusion maps. To show f is meet-dense, let $b \in Y$. Then as gf is meet-dense $(\exists S \subseteq X)(b = \bigwedge_Z S)$. Then clearly $b = \bigwedge_Y S$, so f is meet-dense. Similarly, f is join-dense. Hence $f \in \text{emb}^*$. □

c) The divisible hull of an abelian group: If $\mathbf{A}$ is the full subcategory of divisible abelian groups in **Ab**, then

- $\text{qfirm}(\mathbf{A})$ consists of all embeddings $m : A \to B$ such that every nontrivial subgroup of B meets $m[A]$ nontrivially.

PROOF: The essential embeddings are precisely all embeddings $m : A \to B$ such that every nontrivial subgroup of B meets $m[A]$ nontrivially [AHS 90, 9.3(5), p 142]. Thus by proposition 2.10 they are subfirm. We show that they are coessential: Let $u \in$ emb* and $ug \in$ emb*$(A \xrightarrow{g} B \xrightarrow{u} C)$. g is an embedding since embeddings are coessential. Assume wlog that g, u are inclusions. Let $S \subseteq B$ be a nontrivial subgroup of B. Since u is 1-1, S is a nontrivial subgroup of C, and so S meets A nontrivially. Hence $g \in$ emb*. So, by theorem 2.12 qfirm($\mathbf{A}$) = emb*. □

Chapter 3

Quasifactorization structures

The main result of this chapter extends the well-known characterization of factorization structures (for sources) in terms of multiple pushouts, to the case of quasifactorization structures relative to a subcategory.

3.1 Quasifactorizations

Throughout this chapter $\mathbf{X}$ will be any abstract category and $\mathbf{A}$ will be a subcategory of $\mathbf{X}$, always full and isomorphism-closed. Recall that a source $(f_i : X \to A_i)_I$ in $\mathbf{X}$ is called A-structured iff $A_i \in \mathbf{A}$ for every $i \in I$.

Definition 3.1 ([Bargenda 94]) *A pair* $(\mathcal{E}, \mathbf{M})$ *comprising a class* $\mathcal{E}$ *of* $\mathbf{X}$*-morphisms and a conglomerate* $\mathbf{M}$ *of* $\mathbf{A}$*-structured sources in* $\mathbf{X}$ *is called an* $\mathbf{A}$*-relative quasifactorization structure on* $\mathbf{X}$ *iff*

(i) $\mathrm{iso}(\mathbf{X}) \circ \mathcal{E} \subseteq \mathcal{E}$ *and* $\mathbf{M} \circ \mathrm{iso}(\mathbf{X}) \subseteq \mathbf{M}$,

(ii) every $\mathbf{A}$*-structured source* $(f_i)_I$ *in* $\mathbf{X}$ *has an* $(\mathcal{E}, \mathbf{M})$*-factorization, i.e. there exists* $e \in \mathcal{E}$ *and* $(m_i)_I \in \mathbf{M}$ *such that* $f_i = m_i e$ *for every* $i \in I$,

(iii) $\mathbf{X}$ *has the* $(\mathcal{E}, \mathbf{M})$*-semidiagonalization property, i.e. given* $e \in \mathcal{E}$ *and* $(m_i)_I \in \mathbf{M}$*, along with an* $\mathbf{X}$*-morphism* g *and an* $\mathbf{A}$*-structured source* $(f_i)_I$

such that the outer rectangle of the diagram

$$\begin{tikzcd}
\bullet \arrow[r, "e"] \arrow[d, "g"'] & \bullet \arrow[d, "f_i"] \arrow[dl, dashed, "d"'] \\
\bullet \arrow[r, "m_i"'] & \bullet
\end{tikzcd}$$

commutes for every $i \in I$, there exists an $\mathbf{X}$-morphism d, called a semidiagonal, making the upper triangle commute, i.e. $de = g$.

(iv) $\mathcal{E}$ consists only of thick morphisms.

Remark: Every ($\mathbf{A}$-relative) factorization structure on $\mathbf{X}$ is an ($\mathbf{A}$-relative) quasifactorization structure on $\mathbf{X}$.

As pointed out by Hubertus W. Bargenda, many well-known results on factorizations extend to $\mathbf{A}$-relative quasifactorization structures. In particular, he proves the following result:

Proposition 3.2 ([Bargenda 94]) *Let $(\mathcal{E}, \mathbf{M})$ be an $\mathbf{A}$-relative quasifactorization structure on $\mathbf{X}$. TFAE:*

(i) $\mathbf{A}$ is $\mathcal{E}$-quasireflective in $\mathbf{X}$,

(ii) $\mathbf{A} = \mathrm{inj}(\mathcal{E}')$ for some $\mathcal{E}' \subseteq \mathcal{E}$,

(iii) $\mathbf{A}$ is closed in $\mathbf{X}$ under the formation of $\mathbf{M}$-sources.

The following two results extend some properties of relative factorization structures proved in [Vajner 94]:

Lemma 3.3 *Let $(\mathcal{E}, \mathbf{M})$ be an $\mathbf{A}$-relative quasifactorization structure on $\mathbf{X}$. For $e \in \mathcal{E}$ and $(m_i)_I \in \mathbf{M}$, if the diagram*

$$\begin{tikzcd}
\bullet \arrow[r, "e"] \arrow[d, "\mathbf{1}_\bullet"'] & \bullet \arrow[d, "m_i"] \arrow[dl, dashed, "d"'] \\
\bullet \arrow[r, "f_i"'] & \bullet
\end{tikzcd}$$

has a semidiagonal d, i.e. $de = \mathbf{1}_\bullet$, then e is an isomorphism and $(f_i)_I \in \mathbf{M}$.

PROOF: Since $de = \mathbf{1}_\bullet$, e is a section. But e is thick, so e is iso. Then $(f_i)_I = (m_i)_I \circ e \in \mathbf{M}$ as $\mathbf{M} \circ \mathrm{iso}(\mathbf{X}) \subseteq \mathbf{M}$. □

Proposition 3.4 *Let $(\mathcal{E}, \mathbf{M})$ be an $\mathbf{A}$-relative quasifactorization structure on $\mathbf{X}$. Then*

- *(i) $(\mathcal{E}, \mathbf{M})$-factorizations are "unique", in the sense that if $(e, (m_i)_I)$ and $(\bar{e}, (\bar{m}_i)_I)$ are two $(\mathcal{E}, \mathbf{M})$-factorizations of $(f_i)_I$, then there exists an isomorphism h such that $he = \bar{e}$,*
- *(ii) $\mathcal{E} \cap \mathbf{M} \subseteq \mathrm{iso}(\mathbf{A})$,*
- *(iii) $\mathbf{M}$ is compositive on the left,*
- *(iv) $\mathcal{E}$ determines $\mathbf{M}$ uniquely via the semidiagonalization property: An $\mathbf{A}$-structured source $(m_i)_I \in \mathbf{M}$ iff for each commuting square*

$$\begin{tikzcd}
\bullet \arrow[r, "e"] \arrow[d, "f"'] & \bullet \arrow[d, "g_i"] \arrow[dl, dashed, "d"'] \\
\bullet \arrow[r, "m_i"'] & \bullet
\end{tikzcd} \qquad (*)$$

with $e \in \mathcal{E}$, there exists a map d such that $de = f$.

PROOF: (i): Since $e \in \mathcal{E}$ and $(\bar{m}_i)_I \in \mathbf{M}$, there exists a semidiagonal h such that $he = \bar{e}$. Similarly there exists k such that $k\bar{e} = e$. Then $hk\bar{e} = he = \bar{e}$, and $khe = k\bar{e} = e$, so since e and $\bar{e}$ are thick, both hk and kh are isos. Hence h is the required isomorphism.

(ii): Let $f \in \mathcal{E} \cap \mathbf{M}$.

$$\begin{tikzcd}
\bullet \arrow[r, "f"] \arrow[d, "\mathbf{1}_\bullet"'] & \bullet \arrow[d, "\mathbf{1}_\bullet"] \arrow[dl, dashed, "d"'] \\
\bullet \arrow[r, "f"'] & \bullet
\end{tikzcd}$$

Then there exists d such that $df = \mathbf{1}_\bullet$, so f is a section. Since $f \in \mathcal{E}$, f is thick. Thus f is iso, but $\mathrm{cod}(f) \in \mathbf{A}$, so $f \in \mathrm{iso}(\mathbf{A})$ as $\mathbf{A}$ is iso-closed.

(iii): Let m and $(m_i)_I \in \mathbf{M}$ be such that $(m_i)_I \circ m$ exists. Let $(e, (n_i)_I)$ be an $(\mathcal{E}, \mathbf{M})$-factorization of $(m_i)_I \circ m$.

$$\begin{tikzcd}
A \arrow[rr, "e"] \arrow[d, "\mathbf{1}_A"'] & & D \arrow[d, "n_i"] \arrow[dll, dashed, "d"'] \arrow[dl, dashed, "k"] \\
A \arrow[r, "m"'] & B \arrow[r, "m_i"'] & C_i
\end{tikzcd}$$

Since $(m_i)_I \in \mathbf{M}$ there exists k such that $ke = m\mathbf{1}_A$. Then, since $m \in \mathbf{M}$ there exists d such that $de = \mathbf{1}_A$. Hence by lemma 3.3, $(m_i)_I \circ m \in \mathbf{M}$.

(iv): If $(m_i)_I \in \mathbf{M}$, clearly diagram (*) has a semidiagonal d. Conversely suppose $(m_i)_I$ has the given property (*). Let $(e, (n_i)_I)$ be an $(\mathcal{E}, \mathbf{M})$-factorization of $(m_i)_I$.

$$\begin{tikzcd}
\bullet \arrow[r, "e"] \arrow[d, "\mathbf{1}_\bullet"'] & \bullet \arrow[d, "n_i"] \arrow[dl, dashed, "d"'] \\
\bullet \arrow[r, "m_i"'] & \bullet
\end{tikzcd}$$

Then the outer square of the diagram commutes, and by condition (*) there exists d such that $de = \mathbf{1}_\bullet$. Thus by lemma 3.3, $(m_i)_I \in \mathbf{M}$. □

In light of (iv) above, the following well-known definition will prove to be useful for studying quasifactorization structures:

Definition 3.5 *If $\mathcal{E}$ is any class of $\mathbf{X}$-morphisms, then define $\mathcal{E}^{\downarrow}$ to be the conglomerate of all sources $(m_i)_I$ in $\mathbf{X}$ such that for any commuting square (*) with $e \in \mathcal{E}$, there exists a semidiagonal d, i.e. there exists an $\mathbf{X}$-morphism d such that $de = f$.*

Corollary 3.6 *If $(\mathcal{E}, \mathbf{M})$ and $(\mathcal{E}, \overline{\mathbf{M}})$ are two $\mathbf{A}$-relative quasifactorization structures on $\mathbf{X}$, then $\mathbf{M} = \overline{\mathbf{M}} = \mathcal{E}^{\downarrow} \cap \{\mathbf{A}\text{-structured } \mathbf{X}\text{-sources}\}$.*

Let $\mathbf{M}(\mathbf{A})$ be the full subcategory of $\mathbf{X}$-objects admitting $\mathbf{M}$-sources into $\mathbf{A}$. Another important result proved in [Bargenda 94], which relates relative quasifactorization structures to the work done in the previous chapters of this thesis, is the following:

Proposition 3.7 **([Bargenda 94])** *Let* $(\mathcal{E}, \mathbf{M})$ *be an* $\mathbf{A}$*-relative quasifactorization structure on* $\mathbf{X}$*. Then* $\mathbf{M}(\mathbf{A})$ *is the* $\mathcal{E}$*-quasireflective hull of* $\mathbf{A}$ *in* $\mathbf{X}$*. Moreover, there are (Galois) correspondences between injective hull structures, quasireflective subcategories and relative quasifactorization structures.*

3.2 Relative multiple quasipushouts

In order to characterize relativized quasifactorization structures for sources, a more general notion of multiple pushout is needed. To this end, the notion of an $\mathbf{A}$-relative $\mathcal{E}$-multiple pushout introduced in [Vajner 94, pages 62–65] is extended by dropping certain uniqueness and commutativity conditions.

First we recall some well-known notation: Given any source $(e_i : B \to B_i)_I$ and any morphism $f : B \to C$ in $\mathbf{X}$, a pair $(g, (h_i)_I)$ is called an upper bound for $(f, (e_i)_I)$ in $\mathbf{X}$ iff the diagram

$$\begin{tikzcd} B \arrow[r, "e_i"] \arrow[d, "f"'] & B_i \arrow[d, "h_i"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

commutes, i.e. for each $i \in I$, $h_i e_i = gf$.

If in addition $D \in \mathbf{A}$, $(g, (h_i)_I)$ is called an $\mathbf{A}$-valued upper bound for $(f, (e_i)_I)$.

Definition 3.8 *An upper bound* $(g, (h_i)_I)$ *for* $(f, (e_i)_I)$ *in* $\mathbf{X}$ *is called* $\mathbf{A}$*-extendible iff given any* $\mathbf{A}$*-valued upper bound* $(p, (q_i)_I)$ *for* $(f, (e_i)_I)$ *there exists a morphism* k *such that* $kg = p$*, i.e.*

$$\begin{tikzcd} B \arrow[r, "e_i"] \arrow[d, "f"'] & B_i \arrow[d, "h_i"] \arrow[ddr, "q_i"] & \\ C \arrow[r, "g"'] \arrow[drr, "p"'] & D \arrow[dr, dashed, "k"] & \\ & & A \end{tikzcd}$$

the lower right triangle in the above diagram commutes. (Note that we do not *require that $q_i = kh_i$ for all $i \in I$).*

Definition 3.9 *Let $\mathcal{E}$ be a class of morphisms in $\mathbf{X}$. Given any source $(e_i : B \to B_i)_I$ with $e_i \in \mathcal{E}$ for every $i \in I$, and any $\mathbf{X}$-morphism $f : B \to C$, an $\mathbf{A}$-extendible upper bound $(e, (f_i)_I)$ for $(f, (e_i)_I)$ is called an $\underline{\mathbf{A}\text{-relative } \mathcal{E}\text{-multiple quasipushout}}$ of $(e_i)_I$ along f iff*

(i) $e \in \mathcal{E}$,

(ii) given any $\mathbf{A}$-extendible upper bound $(p, (q_i)_I)$ for $(f, (e_i)_I)$ with $p \in \mathcal{E}$, there exists an $\mathbf{X}$-morphism s such that $e = sp$, i.e. the lower right triangle below commutes:

$$\begin{tikzcd}
B \arrow[r, "e_i"] \arrow[d, "f"'] & B_i \arrow[d, "f_i"'] \arrow[ddr, "q_i"] & \\
C \arrow[r, "e"'] \arrow[drr, "p"'] & P & \\
& & Q \arrow[ul, dashed, "s"']
\end{tikzcd}$$

(iii) e is thick.

In the case where f above is an identity morphism, an $\mathbf{A}$-relative $\mathcal{E}$-multiple quasipushout will be called an $\underline{\mathbf{A}\text{-relative } \mathcal{E}\text{-quasicointersection}}$ of the source $(e_i)_I$.

We say $\mathbf{X}$ has $\underline{\mathbf{A}\text{-relative } \mathcal{E}\text{-multiple quasipushouts}}$ if every source $(e_i : B \to B_i)_I$ with $e_i \in \mathcal{E}$ for every $i \in I$ has an $\mathbf{A}$-relative $\mathcal{E}$-multiple quasipushout along any $\mathbf{X}$-morphism $f : B \to C$.

Lemma 3.10 ("Uniqueness") *If $(e, (f_i)_I)$ and $(\bar{e}, (\bar{f}_i)_I)$ are both $\mathbf{A}$-relative $\mathcal{E}$-multiple quasipushouts of $(e_i)_I$ along f, then there exists an isomorphism k such that $\bar{e} = ke$.*

PROOF: Since $(\bar{e}, (\bar{f}_i)_I)$ is an $\mathbf{A}$-extendible upper bound, there exists h such that $e = h\bar{e}$. Similarly there exists k such that $\bar{e} = ke$. Then $kh\bar{e} = ke = \bar{e}$, and

$hke = h\bar{e} = e$, but e and $\bar{e}$ are thick, so kh and hk are isos. Thus k is an iso as required. □

Remark: Every $\mathbf{A}$-relative $\mathcal{E}$-multiple pushout as defined in [Vajner 94, pages 62–65] is also an $\mathbf{A}$-relative $\mathcal{E}$-multiple quasipushout. In particular, an $\mathbf{A}$-relative $\mathcal{E}$-cointersection is also an $\mathbf{A}$-relative $\mathcal{E}$-multiple quasicointersection.

3.3 Quasifactorizations via quasipushouts

The characterization theorem proved in this section generalizes both a result in [Vajner 94, Chapter 4, Theorem 1.7] which applies to relative factorization structures, and an earlier result in [AHS 90, Theorem 15.14] on factorization structures.

Theorem 3.11 *Let* $\mathbf{A}$ *be a subcategory of* $\mathbf{X}$, *and let* $\mathcal{E} \subseteq \mathrm{mor}(\mathbf{X})$ *be compositive. TFAE:*

(i) $\mathcal{E}$ *is the first part of an* $\mathbf{A}$*-relative quasifactorization structure on* $\mathbf{X}$,

(ii) $\mathbf{X}$ *has* $\mathbf{A}$*-relative* $\mathcal{E}$*-multiple quasipushouts.*

Proof: (i)⇒(ii): Suppose $(\mathcal{E}, \mathbf{M})$ is an $\mathbf{A}$-relative quasifactorization structure on $\mathbf{X}$. Let $(e_i : B \to B_i)_I$ be any source consisting only of $\mathcal{E}$-morphisms, and let $f : B \to C$ be any $\mathbf{X}$-morphism. Let $((f_j : C \to A_j), (g_{ij} : B_i \to A_j)_{i \in I})_{j \in J}$ be the collection of all $\mathbf{A}$-valued upper bounds for $(f, (e_i)_I)$. Since $(f_j)_J$ is an $\mathbf{A}$-structured source, choose an $(\mathcal{E}, \mathbf{M})$-factorization $(m_j \circ e : C \to P \to A_j)_J$ of $(f_j)_J$.

$$\begin{tikzcd}
B \arrow[rr, "e_i"] \arrow[dd, "f"'] & & B_i \arrow[dl, dashed, "d_i"'] \arrow[dd, "g_{ij}"] \\
& P \arrow[dr, "m_j"] & \\
C \arrow[ur, "e"] \arrow[rr, "f_j"'] & & A_j
\end{tikzcd}$$

Since $m_j(ef) = g_{ij}e_i$ for each $i \in I$ and each $j \in J$, for every $i \in I$ there exists a semidiagonal d_i such that $d_i e_i = ef$. Thus $(e, (d_i)_I)$ is an upper bound for $(f, (e_i)_I)$. Since for any **A**-valued upper bound $(p, (q_i)_I)$ for $(f, (e_i)_I)$ we must have $p = f_{j_0}$ for some $j_0 \in J$, clearly $(e, (d_i)_I)$ is an **A**-extendible upper bound for $(f, (e_i)_I)$ as $m_{j_0}e = f_{j_0} = p$.

$$\begin{tikzcd}
A_j & & B_i \arrow[ll, "g_{ij}"] \arrow[dl, "h_i"'] \arrow[dd, dashed, "d_i"] \\
& Q \arrow[ul, dashed, "p_j"'] \arrow[dr, dashed, "d"] & \\
C \arrow[uu, "f_j"] \arrow[ur, "\bar{e}"'] \arrow[rr, "e"'] & & P
\end{tikzcd}$$

Now if $(\bar{e} : C \to Q, h_i : B_i \to Q)$ is any **A**-extendible upper bound for $(f, (e_i)_I)$ with $\bar{e} \in \mathcal{E}$, then there exists, for each $j \in J$, an **X**-morphism $p_j : Q \to A_j$ such that $p_j\bar{e} = f_j$, since $(f_j, (g_{ij})_I)$ is an **A**-valued upper bound for $(f, (e_i)_I)$. Hence for each $j \in J$, $p_j\bar{e} = m_j e$. Since $(m_j)_J \in \mathbf{M}$ and $\bar{e} \in \mathcal{E}$, there exists by (i) a semidiagonal $d : Q \to P$ such that $d\bar{e} = e$. Since $e \in \mathcal{E}$ and e is thick by construction, is is clear that $(e, (d_i)_I)$ is an **A**-relative $\mathcal{E}$-multiple quasipushout of $(e_i)_I)$ along f.

(ii)⇒(i): Suppose **X** has **A**-relative $\mathcal{E}$-multiple quasipushouts. Let **M** be the conglomerate of all **A**-structured sources belonging to $\mathcal{E}^{\downarrow}$. Then **X** clearly has the $(\mathcal{E}, \mathbf{M})$-semidiagonalization property. To show that every **A**-structured source in **X** has an $(\mathcal{E}, \mathbf{M})$-factorization, let $(f_i : B \to A_i)_I$ be any **A**-structured source in **X**. Let $(e_j : B \to B_j)_J$ be the (possibly empty) source of all those morphisms $e_j \in \mathcal{E}$ for which, for all $i \in I$, there exists an **X**-morphism $f_{ij} : B_j \to A_i$ such that $f_i = f_{ij}e_j$. By (ii) let $(e : B \to P, (g_j)_J)$ be an **A**-relative $\mathcal{E}$-cointersection of $(e_j)_J$.

$$\begin{tikzcd}
B \arrow[rr, "e_j"] \arrow[dd, "\mathbf{1}_B"'] & & B_j \arrow[dl, "f_{ij}"] \arrow[dd, "g_j"] \\
& A_i & \\
B \arrow[ur, "f_i"] \arrow[rr, "e"'] & & P \arrow[ul, "m_i"']
\end{tikzcd}$$

For each $i \in I$ $(f_i, (f_{ij})_J)$ is an **A**-valued upper bound for $(\mathbf{1}_B, (e_i)_I)$, so there exists an **X**-morphism $m_i : P \to A_i$ such that $m_i e = f_i$. Thus $(m_i)_I \circ e$ is a factorization of $(f_i)_I$, $e \in \mathcal{E}$ and e is thick by construction. To show that $(m_i)_I \in \mathbf{M}$, suppose $\bar{e} \in \mathcal{E}$ and h, $(k_i)_I$ are **X**-morphisms such that $k_i \bar{e} = m_i h$ for every $i \in I$.

$$\begin{tikzcd}
G \arrow[r, "\bar{e}"] \arrow[d, "h"'] & H \arrow[dl, dashed, "d"'] \arrow[d, "k_i"] \arrow[ddr, "h'"] & \\
P \arrow[r, "m_i"] \arrow[drr, "e'"'] & A_i & \\
& & Q \arrow[ul, dashed, "p_i"]
\end{tikzcd}$$

By (ii), form the **A**-relative $\mathcal{E}$-multiple quasipushout $(e' : P \to Q, h' : H \to Q)$ of $(h, \bar{e})$. For any $i \in I$, since (m_i, k_i) is an **A**-valued upper bound for $(h, \bar{e})$, there exists $p_i : Q \to A_i$ such that $p_i e' = m_i$. As $e \in \mathcal{E}$ and $e' \in \mathcal{E}$, the composition $e'e \in \mathcal{E}$. Since for every $i \in I$, $p_i(e'e) = m_i e = f_i$, $e'e$ is an $\mathcal{E}$-morphism which factorizes $(f_i)_I$. Thus $e'e = e_{j_0}$ for some $j_0 \in J$, so for $x = g_{j_0} e'$, $xe = g_{j_0} e'e = g_{j_0} e_{j_0} = e$. Therefore x is iso as e is thick, so e' is a section. Since also e' is thick, e' is an iso. Setting $d := e'^{-1} h'$, we obtain a semidiagonal: $d\bar{e} = e'^{-1} h' \bar{e} = e'^{-1} e' h = h$. Thus $(m_i)_I \in \mathbf{M}$ as $(m_i)_I \in \mathcal{E}^{\downarrow}$ and $(m_i)_I$ is **A**-structured. □

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