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A study of the phenomenon of conjoining terms in the simplification of algebraic expressions by Grade 8 learners at two secondary schools in the Western Cape

by

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Abstract

Mathematics education research on conjoining has been inconsistent with regard to the study of the phenomenon and the use of the term *conjoining*. This study aimed to identify justifications for the procedures implicated in producing instances of what we refer to as *aberrant conjoining* in early school algebra. The study took a rationalist view of knowledge acquisition, and used a computational approach to analyse data obtained through a written test and semi-structured interviews with a selection of research subjects. The rationalist research orientation used rests on the proposition that humans possess biologically endowed, core domain knowledge of number, as is evident in the results of experiments carried out on human infants as well as other animals.

Seventy-six Grade 8 students at two secondary schools in the Western Cape were given a test in which they were required to simplify ten algebraic expressions, all of which were sums. Eight students were selected for interviews on the basis that they had either displayed aberrant conjoining in their responses or that they had answered most items correctly.

The computational analysis identified four computational principles employed in instances of aberrant conjoining: the *use of type-specific computations*, the *treatment of constituents of terms as sets*, the *implicit use of string operations*, and the *use of addition-like monoids*. Furthermore, the biologically endowed cognitive operation, *merge*, is implicated as generative of the concatenation of the results of type-specific computations central to aberrant conjoining, and as the ground for the system of addition-like monoids used by students. The analysis also found the use of a *pedagogically exploited structure-preserving mapping between distributivity and indirect distributivity* that enables students to simplify algebraic expressions, but which *inadvertently contributes to the production of aberrant conjoining* in its reliance on the typographically legitimate conjoining of symbols to form written algebraic expressions. Finally, the description and analysis of aberrant conjoining in the literature is interrogated, and a more robust explication of the phenomenon is offered.

Key words: *conjoining*, *core domain knowledge*, *merge*, *simplification*, *algebraic expressions*, *computational analysis*, *distributivity*

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CHAPTER 1

INTRODUCTION

South African students continue to perform poorly on national and international standardised tests in mathematics. This remains a matter of great concern as many have acknowledged the importance of mathematics as a prerequisite for access to better employment opportunities. More importantly, the topic of algebra is highlighted as a pivotal component of the high school curriculum and a gatekeeper to future success within the subject (Knuth *et al*., 2005: 68). Additionally, student achievement in mathematics continues to be interrelated with class and 'race' in the South African context (Reddy *et al*. 2022, World Bank 2024). There have been many calls for reforms in mathematics education that provide more opportunities for students to gain a 'better conceptual understanding' of algebra, in the hope of improving overall student achievement in mathematics and thereby providing more equitable access to opportunities for future employment (Kaput, 1995; Kieran, 2006).

Despite the calls for educational reforms in the teaching and learning of algebra, students continue to find great difficulty with the acquisition of appropriate meanings of algebraic expressions and algebraic representations that are consistent with the axioms for the field of real numbers (Banerjee & Subramaniam, 2012; Booth, 1988; Falle, 2007; Hallagan, 2006; Pournara *et al*. 2016). Various reasons are offered in mathematics education research for the obstacles that students face when learning algebra, which has tried to identify the causes of such difficulties. Within this body of research, accounts of students' conceptualisations of algebraic expressions and the transformations of expressions spans a period of fifty years. Throughout that fifty year period, researchers and scholars have noted the occurrence of "conjoining" when students simplify algebraic expressions. The phenomenon has been identified in many schooling contexts and is a persistent student error that continues to occur across a range of ages and grade levels (Pournara *et al*., 2016: 9).

In this study the aim is to investigate the underlying reasons for the conjoining of algebraic terms by students. The different ways in which students conjoin terms are investigated and a more robust description of conjoining within the context of the simplification of simple algebraic expressions is proffered. The occurrence of conjoining in two different school contexts in South Africa is discussed and students' understandings of the processes employed in simplifying algebraic expressions are interrogated. The general framework of algebra which is being delivered through the Department of Basic Education approved textbooks on the topic of the simplification of simple algebraic expressions is also considered. Past research on the phenomenon of conjoining is reviewed and brought into relation with a different perspective by observing conjoining through the lens of cognitive science and higher mathematics. Specifically, contemporary cognitive science research is considered with respect to the ways in which humans carry out additive processes, as well as the concept of *merge*. A computational approach is used in analysing the syntax that students use by examining their externalisations of mathematical thought in speech and writing.

The following research questions are addressed:

- What are the computational procedures employed by Grade 8 students in the simplification of algebraic expressions when conjoining is observed?
- What are students' justifications for the computations they use, and how do those relate to the occurrences of conjoining in the simplification of algebraic expressions?
- How do student justifications relate to current research on genetically endowed cognitive structures regarding number, and how can the prevalence of conjoining be explained?

In the chapter that follows, research on various aspects of the simplification of algebraic expressions, the phenomenon of conjoining, and current number-related research in cognitive science, are reviewed.

CHAPTER 2

A REVIEW OF THE LITERATURE ON CONJOINING

2.1 Introduction

Many researchers have noted that conjoining is a common error that students make when simplifying algebraic expressions (Banerjee & Subramaniam 2012; Booth 1982; Booth 1988; Herscovics 1989; Liebenberg 1997; MacGregor & Stacey, 1997; Subramaniam, 2018). However, I am yet to find a stable description of conjoining. The term *conjoining* is not used uniformly in the literature. While a range of teaching strategies have produced varying degrees of success in improving students' ability to simplify algebraic expressions (see Addendum), Booth (1982:6) argues that the error of conjoining is consistently made and is not easily overcome through teaching. A longitudinal study done in South African schools shows that the error persists even as students move to higher grades (Pournara *et al.*, 2016: 9). It remains the case that students experience particular cognitive obstacles when simplifying algebraic expressions and that the conjoining error is as prevalent today as it was almost fifty years ago when Collis (1975) drew attention to it.

There are instances in which it is argued that conjoining may be valid (Subramaniam 2018:44-45): the statement $5 + 2a = 7a$ is true when $a = 1$. However, the transformation $(5 + 2a) \rightarrow 7a$, which is an instance of conjoining, is incorrect even when $a = 1$ since the computation entails a pseudo-operation inconsistent with the axioms of the field $(\mathbb{R}, +, \times)$.

A discussion of equivalence does not generally precede the teaching of the transformations for the simplification of algebraic expressions. As mentioned earlier, students are often not aware that the expression resulting from one or more algebraic transformations is an equivalent expression. Banerjee & Subramaniam (2012:357) note that evaluating expressions for equivalence was not very helpful in convincing students of the equivalence relation between successive expressions.

The conjoining error is broadly associated in the literature with five cognitive obstacles that students experience when simplifying algebraic expressions: (i) *misconceptions regarding concatenation*, (ii) *non-acceptance of lack of closure*, (iii) *name-process duality*, (iv) *misconceptions regarding the multiplication of terms and exponential notation*, and (v) *effects of the conventions of natural language*. I discuss each of the categories in the sections that follow.

2.2 Concatenation

The misconceptions entailing the concatenation of symbols in algebra has been identified as a problem by a number of researchers (Booth 1982; Booth 1988; Lim 2010; MacGregor & Stacey 1997). Concatenation can be loosely described as the operation of linking things together in a series or chain. In terms of algebraic expressions, lexical concatenation is used when alphanumeric symbols are arguments for multiplication. For example, 3 multiplied by x^2 multiplied by y is expressed as $3x^2y$. There are examples in arithmetic where concatenation means addition over the rational or real numbers rather than multiplication, as in the use of mixed number notation for fractions (e.g., $3\frac{5}{7}$). Furthermore, the place-value notion for numbers implies addition (Booth, 1988:24), as well as multiplication and exponentiation: e.g., $324 = 3\times10^2 + 2\times10^1 + 4\times10^0$. Even the notation of Roman numerals has been found to influence students' understandings of lexical concatenation. For example, where $h10$ is treated in the same way as VI, which means 5 + 1, so the transformation $h10 \rightarrow h + 10$ results (MacGregor & Stacey 1997:111). Chalouh & Herscovics (1988:39) report on instances where students erroneously transferred knowledge of lexical concatenation as it pertained to multiplication from algebra to multi-digit numbers, where, for example, 25 was seen as 2 multiplied by 5, resulting in a value 10 for the expression 25.

Concatenation also denotes addition in other contexts. MacGregor & Stacey (1997) suggest that *conventions from other areas* may influence students to conjoin terms in the same way as in chemistry. For example, CO_2 denotes the combination of one carbon atom bonded to two oxygen atoms. Thus, a carbon dioxide molecule consists of a carbon atom *added* to two oxygen atoms, in one sense of the term *add*. Lastly, Tall & Thomas (1991:2) suggest that *conventions from natural language* dictate that ab, when considered as a and b, will tend to be regarded as a plus b because *and* is treated as additive in many natural language uses of the conjunction rather than as multiplicative (which is how it is treated in mathematics and formal logic). Considering such findings, it has been suggested that the early omission of the multiplication sign may not be a useful strategy (Booth 1982:6). In fact, children who have never studied algebra exhibit a tendency to conjoin terms (Booth, 1988:25). This suggests that children have a natural tendency to conjoin terms, or items, in the context of addition.

2.3 Lack of closure

Prior learning in arithmetic also has an influence on children's ideas about the simplification of algebraic expressions. Collis (1974) describes the ability of a student to accept a statement of the form

$a + b$ as a result as *acceptance of lack of closure*. For Collis, 'closure' of an expression refers to an expression consisting of only one term. An 'open' algebraic expression, which is an instance of a 'lack of closure', is an expression that explicitly displays an operator symbol, such as addition: as in $2x + 3$, for example. According to Collis, many children are unable to accept the 'lack of closure' of algebraic expressions and are only able to view the algebraic expression 'operationally' and not 'structurally'. This means that students will want to "finish" open expressions by performing an operation like addition by conjoining terms (Tirosh, Even & Robinson, 1998). According to Collis, children expect algebra to follow the same expressive pattern as arithmetic, where the result of an algebraic computation is expected to be single-termed (Even 2003:39). The need for a single-termed result disrupts students' *acceptance of lack of closure* of an expression (Collis, 1974).

Instruction on viewing statements such as 5 + 3 = 8 in reverse order, i.e., as 8 = 3 + 5, may be limited in certain instances. For Booth, the lack of the perception of the equal sign as a "bidirectional" relation influences the student to view the statements 'operationally' (Booth, 1988:25). Students then feel the need to "do something" and consequently add or subtract terms when it is inappropriate to do so. Furthermore, many students see statements such as 3 + 5 operationally and not structurally. That is, they see the *process* "three plus five". However, 3 + 5 may also indicate a *name* for the "result when three is added to five". The duality of mathematical statements (both in arithmetic and algebraic contexts) creates additional confusion for students.

2.4 Name-process duality

A number of researchers have investigated the duality of mathematical expressions and statements (Davis 1975; Gray & Tall 2007; Sfard & Linchevski 1994). According to Davis (1975:18), an open algebraic expression has the duality of denoting both a *process* and *name* for an answer. Similarly, Sfard (1991:7-8) describes a process-product dilemma. Gray & Tall (1994: 121) propose the notion of a *procept* to describe the duality of expressions in terms of a *process* and a *concept* in mathematics. These notions all refer to the dichotomy in viewing mathematical statements 'structurally' or 'operationally'. Booth (1988:25) suggests that teachers should use appropriate language when reading aloud expressions like 3 + 5 by making use of statements such as "three add five", or, "the number that is five more than three". Moreover, students need to be made aware of the nature of equality and the fact the equal sign indicates a symmetric relation, since it indexes an equivalence relation. Kinzel (1999:441) also refers to improving instruction by explicitly explaining the ambiguity in mathematical notation. She suggests that the name-process dilemma may be overcome by a more flexible understanding of notation.

2.5 Exponent errors

Other common errors in the simplification of algebraic expressions involve the notation and procedures used in the multiplication of algebraic terms. Such errors are referred to in the literature as instances of an “exponent error” (Lim 2010; Pournara *et al*. 2016). Lim (2010:157) attributes the error to students’ confusions about the procedures for the multiplication and addition of terms. Pournara *et al*. (2016:7) also attributes this error to a confusion in the processes for multiplication (the law for multiplication of powers of the same base) and addition.

In their longitudinal study of a cohort of high school students in South Africa, Pournara *et al*. drew a distinction between an *exponent error* and a *conjoining error*. Figure 2.1 shows data on the students in Grade 9 to Grade 11 who have made ‘conjoining’ and ‘exponent’ errors. One cannot help but notice the similarity in the answers for these respective errors; e.g., $7ab$ and $8ab$ compared to $7a^2b$ and $8a^2b$. The responses $7a^2b$ and $8a^2b$ are not classified as conjoining errors but rather as exponent errors. The classification follows a similar categorisation of data reported in other studies (Lim 2010). The similarity in the answers would suggest that the distinction between the errors is not as clear cut as is portrayed by Pournara *et al*. (2016). This issue will be discussed in detail later.

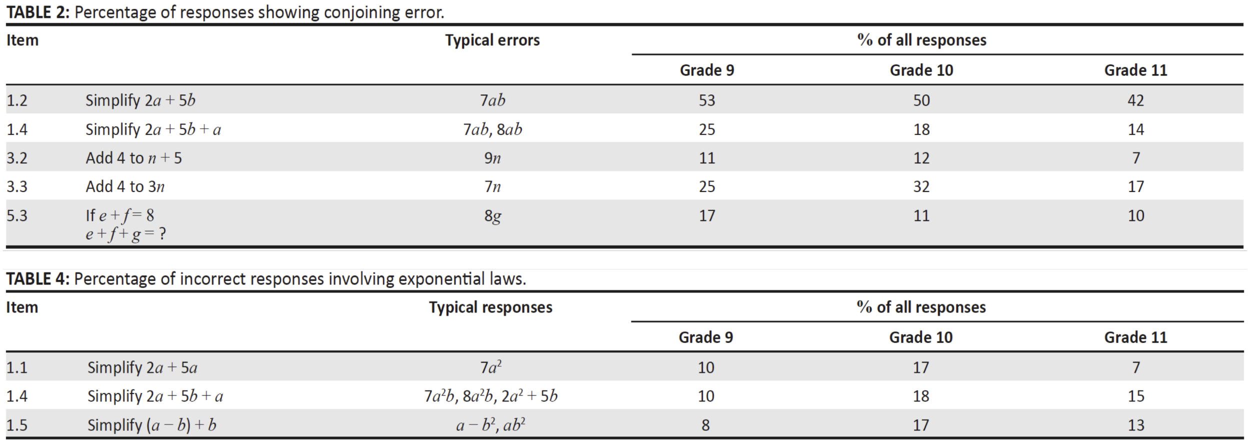

TABLE 2: Percentage of responses showing conjoining error.

Item		Typical errors	% of all responses		
			Grade 9	Grade 10	Grade 11
1.2	Simplify $2a + 5b$	$7ab$	53	50	42
1.4	Simplify $2a + 5b + a$	$7ab$, $8ab$	25	18	14
3.2	Add 4 to $n + 5$	$9n$	11	12	7
3.3	Add 4 to $3n$	$7n$	25	32	17
5.3	If $e + f = 8$ $e + f + g = ?$	$8g$	17	11	10

TABLE 4: Percentage of incorrect responses involving exponential laws.

Item		Typical responses	% of all responses		
			Grade 9	Grade 10	Grade 11
1.1	Simplify $2a + 5a$	$7a^2$	10	17	7
1.4	Simplify $2a + 5b + a$	$7a^2b$, $8a^2b$, $2a^2 + 5b$	10	18	15
1.5	Simplify $(a - b) + b$	$a - b^2$, ab^2	8	17	13

Figure 2.1: Extract from Pournara *et al*. (2016:5-7) illustrating their distinction between *conjoining errors* and *exponential errors*.

Some researchers have argued that exponent errors have not occurred. Liebenberg (1997:6) notes that there was no interference of the multiplication rules when the exponent was one, that is, when the expression was linear. Furthermore, only students with prior experience of multiplication rules exhibited the ‘exponent error’ (Liebenberg, 1997:5). Researchers have indicated that exponential notation is often misunderstood as multiplication. For example, y^5 is read as $5\times y$ (Booth, 1982: 6; Stacey & MacGregor, 1997b:112).

2.6 Conventions of natural language

Stacey & MacGregor (1997b:113) suggest that the use of language in the classroom may have an influence on students' interpretations of the notation. For example, "a^b described as a multiplied by itself b times", in response to which they offer the rephrasing: "the product of b factors, each factor having a value of a" (Stacey & MacGregor 1997b:113). Other notation errors regarding 'exponents' also occur. Lim (2010: 149) notes that students often confuse the placement of the exponent and associate it only with the coefficient of the term. For example, $3a^2 = 9a$ or $6a$.

2.7 Conjoining and ability

Some researchers correlate the tendency of students to conjoin terms and student ability (Falle 2007; Lim 2010). In Lim's (2010) study of the errors made by form 2 students when simplifying algebraic expressions, he argued that his data showed that students of low and medium ability tend to conjoin terms more frequently than do high ability students (Lim, 2010:157). He attributes the phenomenon to students misunderstanding algebraic expressions. Falle (2007) investigated the relation of ability to the tendency to conjoin terms in New South Wales, Australia. She also concluded that students in lower ability groups tend to conjoin terms more frequently than their peers (Falle, 2007:292). However, conjoining appeared to be a common error across all ability groups, except on one item in the test, viz., Item 6, $5a - 2b + 3a + 3b$. She states that the item most closely resembled the items in textbook exercises, and students were able to apply well-practiced procedures in order to solve it (Falle, 2007:291). Falle offers a few suggestions for improving student facility with simplifying algebraic expressions.

> The data discussed in this paper suggest that students of lower "ability" tend to conjoin terms more often than other students. However, a great number of reasonably successful students have a limited procedural understanding of algebraic techniques. Provided that they have only to deal with standard or familiar examples, they can do so. When challenged by examples requiring an understanding of ways in which mathematical meaning and mathematical structure are connected, they expose their reliance on visual cues (or oversimplified schemata) that prompt the exercise of a particular procedure. In order to provide students with a more comprehensive schema, students need to encounter a variety of forms of expression and to experience being able to write them in several ways without the meaning being altered. Perhaps the use of the instruction "to simplify"

> is too limiting. Asking students to rewrite expressions in many ways and discussing the mathematical usefulness of their responses may help students to attend to the structure and meaning of expressions and so develop their conceptual understanding (Falle 2007:293-294).

Interestingly, Falle (2007) also claims that high ability students were unable to provide justifications for the transformations. This may be because students are not able to verbalise their thought processes or because aspects of their knowledge of the topic may be tacit (Falle 2007:294). Furthermore, we are reminded by Erlwanger (1973) that a correct written answer does not necessarily imply mathematically sound procedures. For example, Liebenberg (1997:7) suggests that students may have been able to produce a correct answer to problems consisting of simple algebraic expressions (expressions in which the variable has an exponent of 1) by using a procedure whereby the variable x is merely attached to a numerical value calculated by the addition and subtraction of coefficients.

2.8 Conjoining and student age

The earlier literature on the topic of conjoining, mostly adopting a Piagetian framework, tends to relate students' tendencies to conjoin terms to their age (Collis 1971; Küchemann 1981; MacGregor & Stacey 1997). The researchers claimed that students are not able to think abstractly at younger ages and that most students are able to comprehend open algebraic expressions only at 15 years or older. Küchemann (1981b:313) noted that younger students conjoined terms more frequently than older students. Collis (1971) suggested that younger students are less able to think abstractly and that students reach a level of abstract thinking in only the later years of secondary schooling. Other researchers have, however, argued that age does not really affect the errors that students make regarding algebra (Booth, 1988; Knuth *et al*., 2005; Pournara *et al*., 2016; Rosnick, 1981). Pournara *et al*. (2016) have shown that the conjoining error persists in the higher grades of secondary schooling, in Grade 11, where students are of age 16 years or older. More generally, Rosnick (1981) has shown that students continue to have a poor understanding of variables even when they reach college. In contrast to the latter, there have been studies in which younger students have shown improved understanding of algebra through mediated teaching strategies (Banerjee & Subramaniam 2012; Knuth *et al.* 2005; Pegg & Redden 1990). Booth (1988) argues that age and experience in algebra do not affect students' difficulties in algebra and that similar errors are observed irrespective of age.

2.9 Concluding remarks

An early mention of conjoining is found in Booth (1982:5):

> This confusion over the conventions of recording algebraic answers possibly reflects "psychological" stumbling blocks which are not adequately addressed by the teaching process. For example, children often show a strong tendency to apply a "combining" or "putting together" model of addition (which they have possibly carried with them since early primary school days) to a variety of situations, perhaps regardless of applicability . […]
>
> That children do regard this "conjoining" in algebraic addition as a principle rather than producing it in a capricious or careless manner is perhaps illustrated by some of their explanations during interviews.

As we have seen, since then others have described conjoining very loosely as the "putting together of terms" (Banerjee & Subramaniam, 2012; Herscovics 1989; MacGregor & Stacey, 1997; Subramaniam, 2018). Lesson observations revealed that students tended to 'finish' algebraic expressions by conjoining terms. For example, students tended to perform transformations like $2x + 3 \rightarrow 5x$ or $2x + 3 \rightarrow 5$, i.e., '$2x + 3 = 5x$' or '$2x + 3 = 5$' (Tirosh, Even & Robinson 1998). Many teachers would recognise in this example one of the most frequent errors encountered in students' work. The term *conjoining error* is explicated as the incorrect concatenation of terms (Subramaniam 2018:45). It is important to note that many of the expressions discussed in the literature included only very simple linear algebraic expressions.

The observations of Booth (1982) has provided some of the backdrop for this investigation. The literature on conjoining has suggested many different conjectures and inferences with which to examine the responses of the students to the ten test items used in this investigation. The literature offers several different interpretations of the conjoining error, as was discussed in this chapter. Perhaps the most obvious of these offerings may be the misapplication of the rules and procedures for multiplication discussed in Section 2.5. In previous studies, there has been a tendency to separate different errors by referring to the misapplication of procedures for multiplication as an "exponent error" (Lim, 2010, Pournara *et al.*, 2016). This may have come about due to the lack of observationally and descriptively adequate treatments of what is referred to as conjoining. Due to the vagueness with which conjoining has previously been discussed, researchers may be inclined to separate certain errors when in fact they may result from the same general idea or principle. In the construction of this investigation, the aim was to identify the different ways in which students carried out procedures to simplify simple algebraic expressions, specifically with regard to the addition of algebraic terms. Additionally, we aim to identify the justifications for these procedures and how this translates into a conjoined response.

The various accounts of aberrant conjoining and the explanations for the occurrence of the error are inconsistent and often conflicting. In the next chapter, we outline a different perspective on conjoining, using findings from contemporary cognitive science. We describe the theoretical framework used to investigate the occurrences of conjoining within the simplification of simple algebraic expressions within this study. The framework accepts that humans are endowed with a core domain knowledge of numerical concepts. The semantic basis for the union of disjoint sets for the operation of addition is explained. Furthermore, the notion of *merge* as a primary cognitive resource employed in approaching the simplification of algebraic expressions is described. We outline the computational approach used to analyse the data and to substantiate why conjoining continues to occur so widely, despite prior research and strategies used to counteract this occurrence. This approach may provide an explanation for the pervasiveness of the conjoining phenomenon that plagues students of mathematics across different ability groups, grade levels and nations.

CHAPTER 3

THEORETICAL FRAMEWORK

3.1 Introduction: cognitive science and learning mathematics

There have been many who have studied the ways in which the human brain works when learning number and number concepts (e.g., Butterworth 1999; Dehaene 2011; Devlin 2000; Gallistel & King 2010; Gelman & Gallistel 1978; Gilmore, Göbel & Inglis 2018; Kimii 2004; Scholnick 1983; Spelke 2022). There have been studies suggesting that young babies are able to discern quantitative differences between collections of objects. According to researchers, the human brain is fitted by evolution with an ability to manipulate number-related data. Dehaene (2011:29) refers to the biological endowment that enables humans to perceive quantity as a "number sense" (Dehaene 2011:30).

> Our Western societies, ever since Euclid and Pythagoras, have placed mathematics at the pinnacle of human achievements. We view it as a supreme skill that either requires painful education, or comes as an innate gift. In many a philosopher's mind, the human ability for mathematics derives from our competence for language, so that it is inconceivable that an animal without language can count, much less calculate with numbers (Dehaene 2011:17).

However, various experiments have been carried out to substantiate the claim that other animals, too, possess a natural number sense (Dehaene 2011:3-29). Two classes of experiments are pertinent to the discussion. First, experiments like those carried out on rats have revealed that they have an approximate sense of number and an ability to compute with numerosities (like sequences of light flashes or sounds) in ways that are analogous to the addition of small, single digit numbers (Mechner 1958; Platt & Johnson 1971). Second, experiments like those carried out on chimpanzees demonstrating that they are able to relate symbols to concepts of number regarding the numerosity of sets of objects. Boysen taught her chimpanzee to carry out the (disjoint) union of sets of objects, which is analogous to natural number addition (Boysen & Berntson 1989; Boysen *et al*. 1996; Matsuzawa 1985; 2009).

Experiments on non-human animals, then, prompt us to ask questions concerning the number sense of humans. In Piagetian theories of learning, children learn through reflection on experience. The Piagetian constructivist-interactionist view of learning implicitly adopts the stance that the human baby has a blank slate before they are able to learn through experiences such as looking, feeling and hearing, despite Piaget's appeal to Kant's apriorism (Chomsky & Ronat 1998). The Piagetian sensorimotor phase of cognitive development refers to learning that occurs from birth to 2 years of age. Under this view of knowledge and knowledge growth, it would seem impossible for young babies to have any innate number knowledge. Experiments done in the Piagetian framework relied on the ability of children to respond with verbal indications or cues which occurs only after a certain age. However, the experiments that were carried out on nonhuman animals gave rise to the idea that perhaps there are ways to test whether pre-verbal babies have a sense of number (Dehaene 2011:31-36).

> One of the brain's specialized mental organs is a primitive number processor that prefigures, without quite matching it, the arithmetic that is taught in our schools. Improbable as it may seem, numerous animal species that we consider stupid or vicious, such as rats and pigeons, are actually quite gifted at calculation. They can represent quantities mentally and transform them according to some of the rules of arithmetic. The scientists who have studied these abilities believe that animals possess a mental module, traditionally called the "accumulator," that can hold a register of various quantities. […] The accumulator mechanism opens up a new dimension of sensory perception through which the cardinal of a set of objects can be perceived just as easily as their color, shape and position. This "number sense" provides animals and humans alike with a direct intuition of what numbers mean. (Dehaene 2011:xviii).

Experiments on babies rely on non-verbal cues to indicate interest in numerical changes. Starkey & Cooper (1980) showed that babies between 16 and 30 weeks old were able to recognise the difference between slides containing two dots and slides containing three dots. Babies were shown slides containing two dots repeatedly, then the slide was changed to a slide containing three dots. The interest of the babies was identified by measuring the time they had stared (referred to as "looking times") at the slides containing three dots (Starkey & Cooper 1980:1034). The experiment indicated that babies were able to recognise differences in the number of objects. There was some speculation about the validity of such experiments, one concern being that the babies could be responding to other stimuli such as the position of the dots and not number (Dehaene 2011:38). A later experiment by Strauss & Curtis (1981) used the same test but modified the slides to contain two or three objects in randomly situated positions. The same result was found: babies tended to stare at the slide showing a change in the number of objects longer, which indicated a recognition that there was a change in number (Strauss & Curtis 1981:1150). Bijeljac-Babic, Bertincini & Mehler (1993) showed that babies as young as four days old were able to recognise the numerical difference between words comprising of two syllables and words comprising of three syllables. They noted the difference by attending to the sucking action on the nipple by babies when breastfeeding. The babies sucking action slowed down when they heard two syllable words repeatedly but when three syllable were used their sucking rate increased (Bijeljac-Babic, Bertincini & Mehler 1993:713-714).

Experiments focused on babies' recognition of different numbers of objects or sounds. Can babies perform simple operations on quantities from a young age? In 1992, Wynn carried out an experiment which showed that babies were able to recognise the equivalent of 1 + 1 = 2. The experiment measured the puzzlement registered by babies when shown a mathematically incongruous situation. Babies were shown one mickey mouse doll which was then hidden behind a screen. The screen was lifted to reveal two mickey mouse dolls. The time the babies stared at this situation was measured in an alternative experiment where after habituated to a display showing a single doll, the doll was occluded by a screen and a second mickey mouse doll was added while the first was removed through a trap door. When the screen was lifted, only one mickey mouse doll was present. The babies stared longer at this situation which was taken as an indication that the contradicting situation piqued interest in the babies. That is, since they knew that the equivalent of 1 + 1 is not the equivalent of 1, they did not expect the presented situation (Wynn 1992). The experiment may have been influenced by the baby's knowledge of the position of the objects behind the screen, so Koechlin later revisited the experiment but placed the objects on a rotating turntable. The results showed that the babies were not confused by the changing location of the dolls but attended to the unexpected quantities once the screen was lifted (Koechlin *et al.* 1997). The experiment was carried out for the equivalents of 1 + 1 = 1 and 2 – 1 = 2. The experiments indicated that babies not only had an innate knowledge of quantity but also of correlates of the basic addition and subtraction.

The experiments on nonhuman animals and babies described here are examples of research that provide evidence of the existence of innate knowledge of quantity/number that is coded in the brain. Such knowledge systems are often referred to as *core domain knowledge*.[1] A core domain knowledge system is one that humans are biologically endowed, and so born with. Noncore domains of knowledge are systems of knowledge that are produced due to enculturation. The learning of the symbolic representation of number using the Arabic symbols is a *noncore domain knowledge* system that is gained through schooling and other mechanisms of cultural transmission. Over time, number concepts have been developed into cultural systems used to compute with and communicate mathematical ideas. There are times where the noncore numerical system works well to facilitate

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[1] See Spelke 2022 for a comprehensive recent synthesis of the wide range of cognitive scientific research on core domain knowledge and systems.

operations that align with the way in which the human brain computes. However, in other instances, there are points where noncore symbolic systems are not able to fully align with the way in which the core domain systems work. One such instance is the addition of algebraic terms and the simplification of algebraic expressions, which is discussed in some detail later.

Through the experiments of the type discussed earlier, we can draw the conclusion that humans possess an innate ability to not only perceive numerical data but also possess an inborn knowledge system that allows us to spontaneously carry out some basic arithmetic operations. Researchers now accept as fact that the semantic basis for the arithmetic operation of addition is the binary merging of collections of entities and counting (or measuring) the result of such a merger (Booth, 1989; Carey & Spelke, 1994; Dehaene 2011). The binary merging of sets of objects is routinely exploited to teach young children the process of symbolic addition in the early years of schooling. South African schools also use this method for introducing symbolic addition in the Foundation Phase and examples of the strategy can be found in many South African Department of Basic Education (DBE) workbooks for mathematics (Wüst, 2023), a Grade 1 example of which is shown in Figure 3.1.

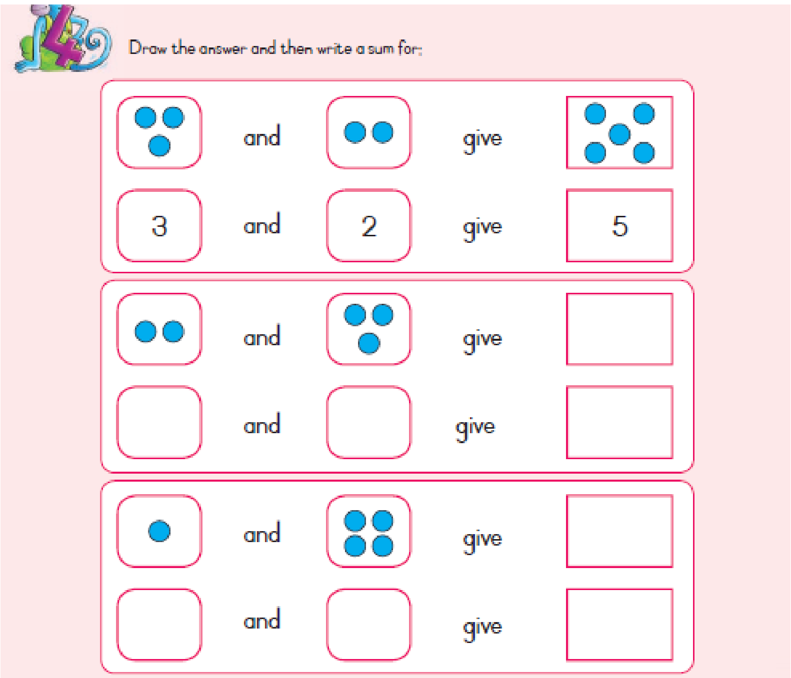
Draw the answer and then write a sum for:

Figure 3.1. Using sets and merging of sets to teach addition in Grade 1 (*Source*: DBE 2023:41).

3.2 Representations

The approach to understanding the mind as a representational "machine" is known as the *computational theory of the mind* (Pinker 1998). Gallistel & King (2010) outline a computational approach to understanding the way in which the brain uses representations as a central learning mechanism. Gallistel & King (2010:60) describe a representation as a structure-preserving mapping that holds between two systems: a *representing system* and a *represented system*. In the teaching of early elementary arithmetic, the representing system is the system of operations defined over natural numbers, and the represented system would be the system of intuitive operations defined over sets of objects. Referring to the example displayed in Figure 3.1, the represented system is disjoint union defined over the class of finite sets and the representing system is addition defined over the natural numbers, where the represented and representing systems are related in a structure-preserving manner by mappings that count the number of objects in a finite set.

In the teaching of school algebra, the representing system is the system of operations defined over a set of alphanumeric symbols and the represented system is the system of familiar arithmetic operations defined over the real numbers. Symbols represent aspects of the world (Gallistel & King 2010:60). In a representation, the mapping between the represented system and the representing system and the functions that occur between them must be structure preserving. Here, "the mappings from entities in the represented system to their symbols is such that functions defined on the represented entities are mirrored by functions of the same mathematical form between their corresponding symbols" (Gallistel & King, 2010:55).

Minimally, there are three criteria for the construction of a productive representation, namely, *causality*, *structure-preservation* and *efficaciousness*:

1. The mapping from entities in the represented system to their symbols in the representing system is *causal* (as, for example, when light reflected off an object in the world acts on sensory receptors in an eye causing neural signals that eventuate in a percept of the object).
2. The mapping is *structure-preserving*: The mapping from entities in the represented system to their symbols is such that functions defined on the represented entities are mirrored by functions of the same mathematical form between their corresponding symbols. Structure-preserving mappings are called homomorphisms.
3. Symbolic operations (procedures) in the representing systems are (at least sometimes) *behaviourally efficacious*: they control and direct appropriate behaviour within, or with respect to, the represented system. (Gallistel and King, 2010:60; italics in the original).

A homomorphism can be defined as follows: If $*$ and $\circ$ are binary operations defined over sets S and T, respectively, and f is a function from S to T, then f is a homomorphism if and only if $f(a * b) = f(a) \circ f(b)$ for all $a, b \in S$ (Krause, 1969:618). Referring to the example shown in Figure 3.1 once again, f is counting (#), which maps a set to its cardinality, $*$ is disjoint union ($\sqcup$) over discrete sets (S), and $\circ$ is addition (+) over natural numbers (T); a and b are the sets used in the example. We should, therefore, have $\#(a \sqcup b) = \#(a) + \#(b)$ for the representation to work. Clearly, $\#(a \sqcup b) = 5$ and $\#(a) + \#(b) = 3 + 2 = 5 = \#(a \sqcup b)$, so we have a homomorphism since $\#(a \sqcup b) = \#(a) + \#(b)$ and thus a valid representation. Figure 3.2 shows an object-and-arrow diagram description of the representation.

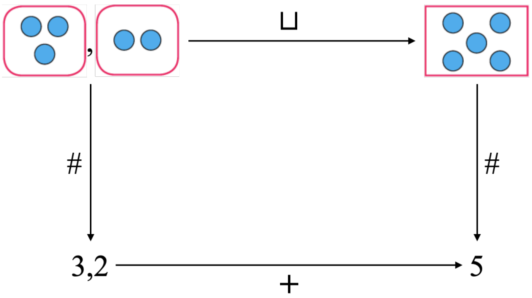

Figure 3.2: The system of mappings describing a representation of the type $\#(a \sqcup b) = \#(a) + \#(b)$ used in the example shown in Figure 3.1.

We see that counting maps disjoint union over the class of sets (SET) to addition over the natural numbers ($\mathbb{N}$) and so we can write $\#:(\mathrm{SET},\sqcup) \rightarrow (\mathbb{N},+)$, where $(\mathrm{SET},\sqcup)$ and $(\mathbb{N},+)$ are distinct computational structures. Table 3.1 lists the properties of $(\mathrm{SET},\sqcup)$ and $(\mathbb{N},+)$, from which it should be evident that the two structures have the same general features.

Table 3.1. A comparison of the structures $(\mathrm{SET},\sqcup)$ and $(\mathbb{N},+)$.

	Properties of $(\mathrm{SET},\sqcup)$	Properties of $(\mathbb{N},+)$
Closure:	$\forall X,Y \in \mathrm{SET},\ \exists Z \in \mathrm{SET}$ s.t. $X \sqcup Y = Z$.	$\forall a,b \in \mathbb{N},\ \exists c \in \mathbb{N}$ s.t. $a + b = c$.
Commutativity:	$\forall X,Y \in \mathrm{SET}$, $X \sqcup Y = Y \sqcup X$.	$\forall a,b \in \mathbb{N}$, $a + b = b + a$.
Associativity:	$\forall X,Y,Z \in \mathrm{SET}$, $X \sqcup (Y \sqcup Z) = (X \sqcup Y) \sqcup Z$.	$\forall a,b,c \in \mathbb{N}$, $a + (b + c) = (a + b) + c$.
Identity element:	$\exists \emptyset \in \mathrm{SET}$, s.t. $\forall X \in \mathrm{SET}$, $X \sqcup \emptyset = X = \emptyset \sqcup X$.	$\exists 0 \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N}$, $n + 0 = n = 0 + n$.

Moving on to the compositions described in algebraic expressions, there must exist mappings that take arithmetic operations defined over sets of numbers to operations defined over the alphanumeric symbols used in algebraic expressions. In other words, there must exist mappings that relate arithmetic operations to an algebraic notation in a manner that is *causal*, *structure-preserving*, and *behaviourally efficacious*, so that the relationship between operations over numbers and operations over alphanumeric symbols is a representation.

Such mappings are relatively easy to construct because all one needs to do is associate each of the non-constant components of arguments of operations with suitable variable markers (like a letter). One can go further by parametrising the constants in arguments, associating them with parameter markers (a letter or any other suitable symbol), as in Figure 3.3. The mapping VAR generalises a given composition of operations by substituting one or more components of arguments of operations by variable markers, while the mapping PAR generalises even further by substituting the constants with parameter marks, enabling a single expression to refer to an entire family of compositions that have a particular form.

$$\begin{tikzcd}
2\times3^2,5\times7^3 \arrow[r, "+"] \arrow[d, "\mathrm{VAR}"'] & 1733 \arrow[d, "\mathrm{VAR}"] \\
2a^2,5b^3 \arrow[r, "+"'] \arrow[d, "\mathrm{PAR}"'] & 2a^2+5b^3 \arrow[d, "\mathrm{PAR}"] \\
n\times a^p,m\times b^q \arrow[r, "+"'] & n\times a^p+m\times b^q
\end{tikzcd}$$

Figure 3.3. The generalisation of descriptions of compositions of operations by using variables and parameters to produce ‘algebraic expressions’.

The idea of a *variable* is key to the construction of an appropriate algebraic notation, which is approached in the South African Curriculum Assessment Policy Statement (CAPS) by way of “generalised arithmetic”, usually through the vehicle of having students detect and describe number patterns (sequences). Work on finite patterns is required by CAPS from Grade 1 and into secondary schooling (DBE 2011a; 2011b). Using alphanumeric symbols, a description of the general term of a finite patterned sequence produces expressions of the type typically used in high school algebra. The general term is a composition of operations that shows the compositional structure of every term of a given finite patterned sequence.[2] The arguments for the operations used in the general compositional structure of the terms are constructed by drawing from the set of natural numbers since a finite sequence is a bijection that has a subset of $\mathbb{N}$ of the type $\{1,2,3, \ldots, n\}$ as its domain. The notion of a *domain* is mentioned in the Intermediate Phase (IP) and Senior Phase (SP) CAPS, but it is rarely taught explicitly and so remains largely implicit in the teaching of patterns. When students are first introduced to the simplification of algebraic expressions in Grade 7 or 8, the notions of *domain* and *variable* tend not be explicated other than by ostension.

The term “variable” is used at times, but in a manner synonymous with the use of the term “letter”, to refer to alphabetic symbols which are thereby marked out from number symbols and operators as different types of object. The object types become important when analysing the work of students.

3.3 *Merge* and related computational structures

As discussed in Section 3.1, studies on infants indicate that humans have a biologically endowed ability that enables counting and elementary arithmetic, and which influences the ways in which students approach the simplification of algebraic expressions, including the student computations marked out as errors.

From the review of the literature one can argue that students experience a number of inter-related difficulties: (i) understanding algebraic symbols in a manner that is consistent with intended mathematical meanings of the notation; (ii) using mathematical equivalence in a task-appropriate manner; and (iii) constructing appropriate procedures to simplify algebraic expressions. The difficulty marked as (i) indicates that the representations constructed by students who fail to use algebraic notation appropriately are problematic if they are not consistent with the intended mathematics; (ii) is a consequence of (i) since the equivalences that are used establish relations derive from the constructed representations; and (iii) is a consequence of (i) and (ii) in the sense that the procedures constructed by students use inappropriate representations and chains of equivalences that are largely consistent with those representations. In other words, constructing inappropriate representations in

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[2] The phrase “patterned sequence” is used because, in general, a sequence need not exhibit any regularity. A function, f, from the set $A = \{1,2,3, \ldots,n\}$ to some other set, B, constitutes a sequence irrespective of whether $f(A)$, i.e. B, is patterned. The CAPS tends to focus predominantly on patterned sequences.

the context of processing algebraic expressions is foundational to the host of student computations that are categorised as errors in the literature.

One question that immediately arises is that of where the inappropriate representations used by students might derive from. When we consider the topic as realised in mathematics curricula, texts and teaching, it is difficult to hold to a position claiming that the computational peculiarities of students are taught to them. A more reasonable position to adopt is that the representations constructed and used by students who fail to understand algebraic notation result from the innate computational resources that they are biologically endowed with. A caveat is necessary at this point: the biologically endowed computational resources referred to here are general species properties and so are common to typical individuals. This does not, however, mean that variation does not exist across individuals, as is the case for all species properties.

3.3.1 *Merge*

To get at what in biological endowment is of interest to this study, I draw on Chomsky's discussion of a cognitive operation that he refers to as *Merge*, which is used to describe aspects of language and the externalisation of thought in spoken and written lexical systems. *Merge* is specified as follows:

> The simplest computational operation, embedded in some manner in every relevant computational procedure, takes objects X and Y already constructed and forms a new object Z. Call it Merge. The principle of Minimal Computation dictates that neither X nor Y is modified by Merge, and that they appear in Z unordered. Hence Merge(X,Y) = {X,Y}. (Chomsky 2015:16).

While *Merge* is used by Chomsky in the construction of his theory of language, the concept applies to computational systems in general (Chomsky 2015:10).[3] We can describe the general properties of *Merge* formally. Let *Merge* be denoted by the symbol μ, defined over a set that will be referred to abstractly as *objects*, denoted OBJECT. Note that *Merge* is a binary operation. Let the elements of OBJECT be denoted as Obj(A), Obj(B), ... ; i.e., OBJECT = {Obj(A), Obj(B), …}. Table 3.2 sets out the computational features of *Merge* in formal terms.

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[3] Note that for biolinguists language is a computational system. See Pinker (2007) for an accessible account of computational nature of human language.

Table 3.2. Properties of the computational structure (OBJECT,μ).

Closure:	$\forall\ \mathrm{Obj}(A), \mathrm{Obj}(B) \in \mathrm{OBJECT}, \mathrm{Obj}(A)\ \mu\ \mathrm{Obj}(B) = \mathrm{Obj}(C) \in \mathrm{OBJECT}.$
Commutativity:	$\forall\ \mathrm{Obj}(A), \mathrm{Obj}(B) \in \mathrm{OBJECT}, \mathrm{Obj}(A)\ \mu\ \mathrm{Obj}(B) = \mathrm{Obj}(B)\ \mu\ \mathrm{Obj}(A).$
Associativity:	$\forall\ \mathrm{Obj}(A), \mathrm{Obj}(B), \mathrm{Obj}(C) \in \mathrm{OBJECT},$ $(\mathrm{Obj}(A)\ \mu\ \mathrm{Obj}(B))\ \mu\ \mathrm{Obj}(C) = \mathrm{Obj}(A)\ \mu\ (\mathrm{Obj}(B)\ \mu\ \mathrm{Obj}(C)).$
Identity element:	$\exists\ \mathrm{Obj}(\ )$, called the *null object*, s.t. $\forall\ \mathrm{Obj}(N) \in \mathrm{OBJECT},$ $\mathrm{Obj}(\ )\ \mu\ \mathrm{Obj}(N) = \mathrm{Obj}(N) = \mathrm{Obj}(N)\ \mu\ \mathrm{Obj}(\ ).$

Chomsky (1988:169) argued that “we might think of the human number faculty as essentially an ‘abstraction’ from human language, preserving the mechanism of discrete infinity and eliminating the other special features of language”, pointing to a fundamental link between language and mathematics that has not been investigated in mathematics education, but which has been explored to some extent in cognitive science (e.g., Spelke 2022). In essence, Chomsky’s argument is that, given a single object to use as an argument, *Merge* applied recursively behaves like the successor function of Peano arithmetic, thereby generating more complex objects that are analogous to the natural numbers. In other words, we start with some given set containing Obj(N), and since *Merge* is binary we have to use $\{\mathrm{Obj}(N)\}$ for both of its arguments initially, and then as the second argument in each successive iteration of *Merge*. For this to work, *Merge* has to function like set theoretic *disjoint union* rather than union.

$$\{\mathrm{Obj}(N)\}$$

$$\mu{:}(\{\mathrm{Obj}(N)\},\{\mathrm{Obj}(N)\}) \rightarrow \{\ \mathrm{Obj}(N),\mathrm{Obj}(N)\};$$

$$\mu{:}(\{\mathrm{Obj}(N),\mathrm{Obj}(N)\},\{\mathrm{Obj}(N)\}) \rightarrow \{\mathrm{Obj}(N),\mathrm{Obj}(N),\mathrm{Obj}(N)\};$$

$$\mu{:}(\{\mathrm{Obj}(N),\mathrm{Obj}(N),\mathrm{Obj}(N)\},\{\mathrm{Obj}(N)\}) \rightarrow \{\mathrm{Obj}(N),\mathrm{Obj}(N),\mathrm{Obj}(N),\mathrm{Obj}(N)\}; \text{ and so forth.}$$

Note that while it is the case that each Obj(N) is identical to every other, disjoint union requires us to treat each instance of Obj(N) as distinct from every other. The cardinalities of the initial and output sets are 1, 2, 3, 4, …, which correspond to the natural numbers as generated by the Peano axioms.

3.3.2 Externalisation of computational hierarchies (compositions of operations)

When discussing the *basic property of language*, Chomsky draws attention to the cognitive interfaces for *thought* and *expression* (qua externalisation of thought) in a manner that is helpful when considering computational systems in general.

> Basic Property: each language provides an unbounded array of hierarchically structured expressions that receive interpretations at two interfaces, sensorimotor for externalization and conceptual-intentional for mental processes. (Chomsky 2015:4).

Here Chomsky refers to two cognitive interfaces that come into play in linguistic computation, but which apply to human computational systems in general, where the externalisation of thought is realised in speech, writing, gesture, or any other mode of expression. In mathematics, the composition of operations that constitute a computation is hierarchical while the externalised presentations of computations in standard algebraic notation are necessarily linear because the sensorimotor system responsible for externalisation is predisposed to produce linear sequences of sounds and written symbols.[4]

The hierarchical nature of a composition of operations can be rendered with greater fidelity in externalisation by using prefix rather than infix notion, but expressions that use prefix notation become relatively difficult to read when more than two or three operations are composed. For example, a fairly simple computation presented using infix notation, like $\frac{6+2\times4}{7}$, is rendered as ÷(+(6,×(2,4)),7) when using prefix notation. While the latter expression shows the hierarchy of computations unambiguously, it is more difficult to read than the former. Expressions like $\frac{6+2\times4}{7}$ necessarily require the use of stipulations concerning the order of operations (e.g., BODMAS)[5] to correctly realise the computational hierarchy of operations because the linear order of an infix expression results in ambiguity. Students who simply follow the left-right linear order of the infix expressions used in algebra often produce computational errors, like $\frac{6+2\times4}{7} = \frac{32}{7}$.

Figure 3.4 shows a diagrammatic representation of the hierarchy of operations in $\frac{6+2\times4}{7}$, which is expressed unambiguously when using prefix notation, ÷(+(6,×(2,4)),7).

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[4] Sign language used by deaf, or speech impaired individuals, disrupts the linearity of externalization somewhat because the syntax of signing uses both vertical and horizontal positioning as well as direction of the hands in space to construct signs.

[5] Other mnemonics that are used to stabilise computational hierarchy are BODMSA, BOMDAS, BEDMAS, BIDMAS, BUDMAS, PEDMAS, PEMDAS, POMDAS, PODMAS and iTAFF.

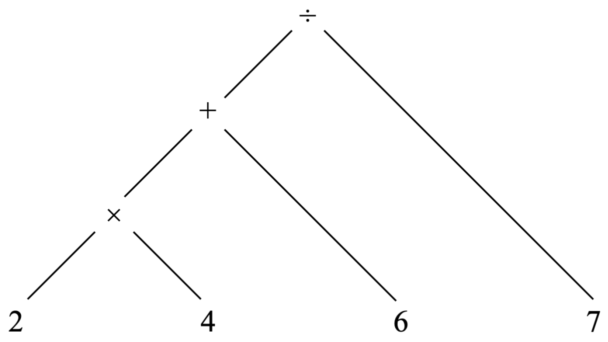
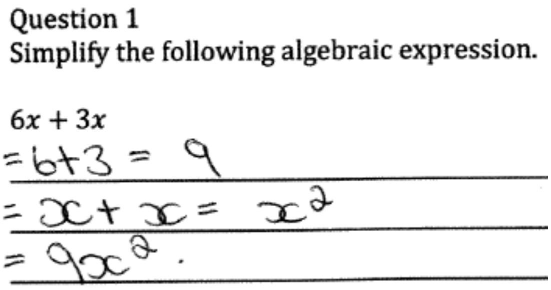


Figure 3.4. Hierarchy of operations in $\frac{6+2\times4}{7}$, expressed more explicitly in ÷(+(6,×(2,4)),7).

So, written arithmetic and algebraic expressions are externalisations of arithmetic/algebraic thought presented through the sensorimotor interface. The interpretation of received externalisations—like the algebraic expressions encountered in textbooks—at the conceptual-intentional interface is registered in the syntax of the processing of algebraic expressions. However, what becomes very clear in this study is that the compositional rules used to process the symbols that constitute expressions are often interpreted in ways different from what is intended by curricula, teachers and textbook authors.

Consider a student's solution to the problem shown in Figure 3.5, where the syntax used by the student does not agree with the intended syntax.



Question 1
Simplify the following algebraic expression.

$6x + 3x$

$= 6+3 = 9$

$= x + x = x^2$

$= 9x^2$.

Figure 3.5. A Grade 8 student's simplification of the expression "6*x* + 3*x*".

In general, teachers understand the syntax used in algebraic notation as intended. However, for many students who are just starting to use the notation there appears to be a divergence from what is intended in their processing of the externalisation of algebraic thought as relayed in written and spoken algebraic expressions.

An interesting series of questions arise when we examine the responses of students to the simplification of the algebraic expressions: "What is the syntax used by students to process externalisations of algebraic thought?" "What is the interpretation of algebraic expressions at the conceptual-intentional interface informing the particular syntax they use?" And, "Are there regularities in the use of aberrant syntax across students?"

In terms of what is proposed in the research literature, the student's solution shown in Figure 3.5 would most likely be interpreted as showing *addition* ($6 + 3 = 9$), an *exponent error* ($x + x = x^2$), and an instance of aberrant *conjoining* ($9 + x^2 = 9x^2$), generating the transformation $6x + 3x \rightarrow 9x^2$ (cf. Pournara *et al.* 2016). There are a number of problems with such an interpretation. First, there are implicit operations/mappings that are used but not captured in the analysis. For example, the sundering of terms along the lines of type is used ($6x \rightarrow 6,x$; $3x \rightarrow 3,x$) and the resulting objects are made available for further processing (numbers: 6 and 3; letters: x and x). Second, since the student is using a syntax different from that intended by the curriculum, one needs to be careful not to interpret the operations/mappings used by the student as aberrant instances of the use of intended operations/mappings. For example, when the student employs the transformation $x + x \rightarrow x^2$, one cannot simply claim that they are committing an exponent error because they are not necessarily working with exponents. Rather, it appears that the superscripted "2" is interpreted as a count of the number of xs that result from merging x and x. Further, the operation effecting the merging of x and x is something more along the lines of the disjoint union of sets, noting that addition and disjoint union are closely related: $\{x\} \sqcup \{x\} \rightarrow \{x',x''\}$.[6] Clearly, the cardinality of the set $\{x',x''\}$ is 2, and the expression "x^2" might more reasonably be interpreted as "there are two things of the type x in the expression $6x + 3x$, so two xs should be indicated in the solution". Hence, "$9x^2$". Additionally, the transformation $9 + x^2 \rightarrow 9x^2$ is a concatenation of the expressions 9 and x^2.

Since algebraic expressions are lists of alphanumeric symbols, what is referred to as the *conjoining of terms* is a way of combining lists, so a brief discussion of lists is required at this point.

3.3.3 Lists

Following Spivak (2014:120), given a set A, an ordered *list in A* is a pair (n,f) where $n \in \mathbb{N}$ is the *length of the list* and $f{:}n \rightarrow \{f(1), f(2), \dots, f(n)\}$. Such an ordered list is denoted $(n,f) = [f(1), f(2), \dots$

---

[6] Note that here x' and x'' are used to distinguish one x from the other because disjoint union is used. We have to use disjoint union rather than union because $\{x\} \cup \{x\} = \{x\}$, which is not what the student intended.

,$f(n)$] and the set of ordered lists in A is denoted LIST(A). The empty list is the list in A where $n = 0$, denoted [ ]. For any $a \in A$, the singleton list on a is the list [a]. Given any two lists on A, $L_1 = (n_1, f_1)$ and $L_2 = (n_2, f_2)$, define the *concatenation* of L_1 and L_2, denoted L_1++L_2, or ++(L_1,L_2) in prefix notation, to be the list (n_1+n_2, f_1++f_2) where f_1++f_2:($n_1 + n_2$) → A is given on $1 \leq i \leq n_1 + n_2$ by

$$(f_1{+}{+}f_2)(i) := \begin{cases} f_1(i) & \text{if } 1 \leq i \leq n_1, \\ f_2(i - n_1) & \text{if } n_1 + 1 \leq i \leq n_1 + n_2. \end{cases}$$

Note that since lists are taken to be ordered, [L_1,L_2] ≠ [L_2,L_1]. The concatenation of any list L with the empty list, [ ], is L.

If A is the set of alphanumeric symbols used to express the composition of operations that constitute an algebraic expression, then every algebraic expression is a list in A. For example, if we exclude spaces[7], the expression "6x + 3x" is the list [6,x,+,3,x], which can be viewed as a concatenation of suitable combination of lists constructed from the set of singleton lists {[ ],[6],[3],[x],[+]} ⊆ LIST(A). That is, the list [6,x,+,3,x], which corresponds to the expression "6x + 3x", is equivalent to any appropriate concatenation of lists, like: [6]++[x]++[+]++[3]++[x], [6,x]++[+]++[3,x], [6,x,+]++[3,x], [6,x]++[+,3,x], [6,x,+,3]++[x], [6] ++[x,+,3,x], …, [6,x,+,3,x]++[ ], [ ]++[6,x,+,3,x].

For the sake of brevity, not all possible concatenation configurations are presented here, but the reader should get the general idea. Note that while [6,x,+,3,x] ≠ [3,x,+,6,x] when we are considering ordered lists, this does not imply that $6x + 3x \neq 3x + 6x$ when considered with respect to values taken by x over some or other domain, like ℕ or ℝ.

In the solutions offered by some students in response to test items that contain terms of the form "*ab*" and "*ba*", the terms are treated as ordered lists only and not as products, so not as like terms. The general properties of (LIST,++) are set out in Table 3.3.

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[7] If we find that including spaces is necessary at times, the symbol 𝕓 will be used to indicate a space (blank), with [𝕓] as the singleton list derived from 𝕓. Note that the empty list, [ ], and the singleton list contain a space, [𝕓], are not identical.

Table 3.3. Properties of the computational structure (LIST,++).

Closure:	$\forall\, L_1, L_2$ lists made up of elements of the set of alphanumeric symbols, A, L_1++L_2 = $[L_1,L_2] \in$ LIST(A), the list made up of L_1 followed by L_2.
Associativity:	$\forall\, L_1, L_2, L_3$, lists made up of elements of the set of alphanumeric symbols, A, $(L_1$++$L_2)$++L_3 = L_1++$(L_2$++$L_3)$ = $[L_1,L_2,L_3] \in$ LIST(A).
Identity element:	$\exists$ [ ], called the *empty list*, s.t. $\forall\, L \in$ LIST(A), [ ]++L = L++[ ] = $L \in$ LIST(A).

3.3.4 Relating *Merge* to disjoint union, addition, and concatenation

When we consider the computational features of the structures (OBJ,μ), (SET,⊔), (ℕ,+) and (LIST,++), we see that they share a number of formal properties, viz., *closure*, *associativity* and having an *identity element* (see Table 3.4). Therefore, (OBJ,μ), (SET,⊔), (ℕ,+) and (LIST,++) are all *monoids* since monoids are algebraic structures that exhibit, at least, *closure*, *associativity* and *identity elements*.

Table 3.4. A comparison of the general features of (OBJ,μ), (SET,⊔), (ℕ,+) and (LIST,++)

Property	(OBJ,μ)	(SET,⊔)	(ℕ,+)	(LIST,++)
Closure:	Yes	Yes	Yes	Yes
Associativity:	Yes	Yes	Yes	Yes
Identity element:	Yes	Yes	Yes	Yes
Commutativity:	Yes	Yes	Yes	No
Inverse element:	No	No	No	No

Importantly, *Merge*, disjoint union (⊔), addition (+) and concatenation (++) are generative of monoids *of the same type*, which is one way of saying that when using (OBJ,μ), (SET,⊔), (ℕ,+) and (LIST,++) we are doing the same sort of thing in different, but related, computational contexts.

Figure 3.6 shows representations that relate *Merge* to disjoint union, addition and concatenation, which are all instances of combining two things in a manner that intuitively corresponds to addition, to produce a third thing of the same kind. It is, therefore, unsurprising that disjoint union, addition and concatenation might be invoked when an algebraic expression of the type $6x + 3x$ is encountered, since the explicit operator, +, is intuitively interpreted in terms of *Merge* at the conceptual-intentional interface.

$$\begin{tikzcd}
\mathrm{Obj}(A),\mathrm{Obj}(B) \arrow[r, "\mu"] \arrow[d, "\mathrm{SET}"'] & \mathrm{Obj}(C) \arrow[d, "\mathrm{SET}"] \\
X,Y \arrow[r, "\sqcup"'] & Z
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Obj}(A),\mathrm{Obj}(B) \arrow[r, "\mu"] \arrow[d, "\mathrm{NUM}"'] & \mathrm{Obj}(C) \arrow[d, "\mathrm{NUM}"] \\
m,n \arrow[r, "+"'] & p
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Obj}(A),\mathrm{Obj}(B) \arrow[r, "\mu"] \arrow[d, "\mathrm{LIST}"'] & \mathrm{Obj}(C) \arrow[d, "\mathrm{LIST}"] \\
L_1,L_2 \arrow[r, "++"'] & L_3
\end{tikzcd}$$

Figure 3.6. Structure-preserving mappings between *Merge* (μ) and each of disjoint union ($\sqcup$), addition ($+$), and concatenation ($++$).

The system of relations between *Merge*, disjoint union, addition and concatenation are shown in Figure 3.7.

$$\begin{tikzcd}[row sep=large, column sep=huge]
L_1,L_2 \arrow[r, "++"] \arrow[ddd, bend right=80, "\#"'] & L_3 \arrow[ddd, bend left=80, "\#"] \\
\mathrm{Obj}(A),\mathrm{Obj}(B) \arrow[r, "\mu"] \arrow[u, "\mathit{List}"] \arrow[d, "\mathit{Set}"'] \arrow[dd, bend right=40, "\mathit{Num}"'] & \mathrm{Obj}(C) \arrow[u, "\mathit{List}"'] \arrow[d, "\mathit{Set}"] \arrow[dd, bend left=40, "\mathit{Num}"] \\
X,Y \arrow[r, "\sqcup"'] \arrow[uu, bend left=50, "\mathit{List}(s)"] \arrow[d, "\#"'] & Z \arrow[uu, bend right=50, "\mathit{List}(s)"'] \arrow[d, "\#"] \\
m,n \arrow[r, "+"'] \arrow[uuu, bend left=65, "\mathit{List}(n)"] \arrow[u, bend left=40, "\mathit{Set}(n)"] & p \arrow[uuu, bend right=65, "\mathit{List}(n)"'] \arrow[u, bend right=40, "\mathit{Set}(n)"']
\end{tikzcd}$$

Figure 3.7. The system of relations between *Merge*, disjoint union, addition and concatenation.

The italicised labels of arrows in Figure 3.7 indicate *mappings* and not sets or classes, so are distinct from the sets/classes OBJ, SET, LIST and ℕ. The mappings shown in Figure 3.7 in terms of domains mapped to codomains are as follows:

μ:OBJ×OBJ → OBJ (generating a new object from the combination of two given objects);
⊔:SET×SET → SET (combining two given sets);
+: ℕ×ℕ → ℕ (adding two given natural numbers);
++:LIST×LIST → LIST (concatenating two given lists);
Num:OBJ → ℕ (contextualising the objects of interest as natural numbers);
Set:OBJ → SET (contextualising the objects of interest as sets);
List:OBJ → LIST (contextualising the objects of interest as lists);
#:SET → ℕ (computing the cardinality of a set);
#:LIST → ℕ (computing the length of a list);
Set(*n*):ℕ → SET (selecting a set of cardinality *n*);
List(*n*):ℕ → LIST (selecting a list of length *n*);
List(*s*):SET → LIST (treating a set as a list).

Counting (#) enables us to link disjoint union, addition and concatenation, fitting with the mappings from (OBJ,μ) to (SET,⊔),viz., (*Set*); to (ℕ,+), viz., (*Num*); and to (LIST,++), viz., (*List*). Of course, we can treat a set as a list of elements (*List*(*s*)), and so concatenating lists can be thought of as corresponding to disjoint union. Further, just as one can map a natural number, *n*, to a set (*Set*(*n*)), one can map a natural number, *n*, to a list consisting of *n* characters (*List*(*n*)).

3.4 Concluding remarks

In Chapter 2 and in Section 3.1 it was shown that there are many studies that support the notion that a *Merge*-like genetic endowment for quantitative computations exists in humans, indicating that young children, including pre-verbal infants, have proto-notions of addition before receiving instruction in arithmetic.

In this chapter, I have described the key concepts informing the generation and analysis of data. The work of Gallistel & King (2010) was used to describe the structure preserving mapping employed in the simplification of algebraic expressions between the represented and the representing systems. A computational approach as described in Davis (2010; 2013) is used to describe the objects, operations, and compositions of operations that are found in texts and teaching, as well as in student procedures and constructions. This approach together with the theoretical underpinnings of core domain knowledge and the concept of *Merge* is used to describe the student motivations for the conjoining of algebraic terms in the simplification of simple algebraic expressions. In the next chapter, I will describe the analytic procedures in some detail.

CHAPTER 4

ANALYTIC FRAMEWORK

4.1 Introduction

The goal of research is to provide a holistic picture and a depth of understanding about a phenomenon (Ary, 2006: 31). Qualitative research is routinely used to understand the meaning that is portrayed by participants in a particular social setting or context. Through a mixed-methods approach, we have gained a greater understanding of the conjoining phenomenon as it occurs within this context.

The study takes a rationalist position with regard to the acquisition of knowledge. The foundational propositions used are that humans are genetically endowed with core domain knowledge of number and some basic arithmetic. In contrast to an empiricist view, the position taken here is that what we know of the relevant innate knowledge structures must be considered when looking at the ways in which students encounter and approach the simplification of simple algebraic expressions. In empiricist approaches one would accept that the students learn conjoining primarily through experience, including exposure to the use of conjoining. In contradistinction to such a position, we accept that the pervasiveness of the conjoining phenomenon implies that there must be a cognitive predisposition which favours a conjoining response by students across pedagogic contexts, and which does not derive purely from pedagogic environments.

> Within an empiricist framework, one approaches the study of the body as a topic in the natural sciences, concluding that the body is constructed of varied and specialized organs which are extremely complex and genetically determined in their basic character, and that these organs interact in a manner which is also determined by human biology. On the other hand, empiricism insists that the brain is a tabula rasa, empty, unstructured, uniform at least as far as cognitive structure is concerned. I don't see any reason to believe that; I don't see any reason to believe that the little finger is a more complex organ than those parts of the human brain involved in the higher mental faculties; on the contrary, it is not unlikely that these are among the most complex structures in the universe. (Chomsky & Ronat, 1998: 81)

Within this framework, we are able to identify the computational bases for student procedures such that we can establish a clearer picture of the conjoining phenomenon. We have used a computational approach in order to examine the ways in which students simplify simple algebraic expressions. The computational approach allowed for a reasonably robust description of the procedures that were being employed and the objects that were being considered by the students.

4.2 Research design

A mixed-methods approach was used in analysing the data within this study. Both quantitative and qualitative methods were used to investigate students' responses in the simplification of algebraic expressions. Data was collected through a written test and an interview process.

4.2.1 Empirical context and participants

The participants included a total of seventy-six (*n*=76) Grade eight students from two public high schools in Cape Town, South Africa. The students were between 12 and 14 years old and were in their first year of high school. School A is a fee-paying school and is located in an affluent suburb in Cape Town. School B is a low fee-paying school and serves a community of lower socioeconomic status in Cape Town. The schools were selected because the researcher knew the principals of both schools and both schools were willing to participate in the study (convenience sampling). Twenty-seven students who participated in the study attended school A and forty-nine students attended School B.

Both schools follow the national curriculum (CAPS) and use DBE approved textbooks. The study was carried out during the third school term. At the time of the study, all students had been taught the simplification of algebraic expressions and they were assessed on the topic during their mid-year (June) examination. Two classes were selected from each school i.e., classes A1 and A2 from School A and classes B1 and B2 from School B. School A had five different Grade 8 mathematics teachers and teachers were selected based on their willingness to participate in the study. School B had only two Grade 8 Mathematics teachers and both teachers were willing to participate in the study. The four classes chosen were taught by different teachers. All four classes selected were of mixed ability and gender. The language of teaching and learning was English in all classes.

4.2.2 Data collection

The study consisted of two phases: a written test and an interview. The two phases were carried out over two days at each school. The written tests were carried out by the researcher during a single lesson of approximately forty minutes each. A test instrument containing ten algebraic expressions was constructed and used in the study. The Senior Phase curriculum did not provide an adequate description of the types of algebraic expressions that students in Grade 8 were required to simplify (see Addendum). Therefore, a selection of five DBE approved Grade 8 Mathematics textbooks were analysed. The selection of textbooks was done according to the availability of the textbooks. The textbooks used by the two schools were included in the sample of textbooks. The chapters covering the simplification of algebraic expressions were analysed regarding the construction of algebraic expressions that students were required to simplify. An analysis of the texts can be found in the Addendum. The analysis provided a starting point for the construction of the test items. It would be reasonable to believe that these would be the types of items that the students would have encountered. Furthermore, the textbooks were analysed regarding the presentation and framing of procedures for simplifying algebraic expressions. Here, the researcher observed the ways in which the topic was framed—specifically the ways in which "like terms" were defined—and also identified the justifications for the algebraic transformations employed when simplifying algebraic expressions.

A review of the chapters covering the simplification of algebraic expressions revealed a focus on the number of distinct variables in each expression, the degree of the expressions, the types of letters chosen, the number of distinct variables per term within an expression, and the collecting of like terms within the expression. Each of the occurrences the foci referred to were tabulated and analysed to find any patterns that arose as well as establishing at the frequency of types of items that students encounter. It is worth noting that many of the examples in the textbooks exhibit questions that are presented in a sequence or pattern with regard to the position of the like terms. For example, $2a + 5b + 4a - 7b$, or $2x^2 + 3x + 8 + 7x^2 + 10x + 2$, or $3x + 5 + 6x + 8$. The sequence of alternating like terms is a common occurrence and is something which students are accustomed to encounter in textbook exercises.

An assessment consisting of eight test items was constructed using the data obtained from the analysis of the DBE approved textbooks as well as examples from literature on the simplification of algebraic expressions (Demby, 1997; Liebenberg, 1997; Lim, 2010; Küchemann, 1978; Tirosh, Even & Robinson, 1998). The assessment was used in a pilot study which was conducted in order to check whether the items were appropriate (see Addendum). After examining the responses of students in the pilot study, the items were adapted, and two additional items were added in order to account for a greater number of combinations of the types of algebraic terms in the expressions. This resulted in the development of a test instrument consisting of ten items. The items targeted learners' understanding of the procedures for simplifying algebraic expressions of different forms. The general instruction for all ten items was: "Simplify the following algebraic expression". In a Grade 8 context, the instruction to simplify algebraic expressions means to write each term in its simplest form and to add or subtract any like terms within the expression. In the study, all individual terms within the expressions were already expressed in its simplest form. Therefore, the instruction referred only to students adding the like terms entailed in expressions. The research literature review revealed that subtraction introduced additional concerns and confusion for students. Therefore, a decision was taken to include only addition of terms in this study to avoid the interference of additional factors which would render the study more complex.

Table 4.1: The ten algebraic expressions that students were required to simplify

Item 1	$6x + 3x$
Item 2	$5a + 5b + a$
Item 3	$3n + 4$
Item 4	$4a^2 + 3a^2 + 7a$
Item 5	$4x^2 + 5x + 3x + 7x^2$
Item 6	$7kb + 4b + 3bk + 5kb + 4k$
Item 7	$3y^2 + 4y + 1 + 5y + 7y^2 + 8$
Item 8	$x + 5x + 2x$
Item 9	$9a^2r + 4r^2 + 5a^2 + 3ar^2 + 7a^2r$
Item 10	$6g^2 + 6k + 8t^3$

The researcher assessed and analysed each student's responses and noted occurrences of conjoining. The student responses were recorded and quantified by noting the frequency of each response. This allowed for a comparison of the different types of student productions and informed the selection of the students for the second phase of the study, the semi-structured interviews.

4.2.3 Semi-structured interviews

Two students from each class were selected to participate in a semi-structured interview. A total of eight students were interviewed. The interviews were conducted within two weeks of the day that the test was administered. Students were selected for an interview based on the prevalence of conjoining in their responses and their willingness to participate in the interview. Four students were selected on the basis that they had provided the mathematically correct responses. The other four students were selected on the basis that they provided mathematically incorrect conjoined responses. This sampling was done in order to identify the justifications for the procedures that lead to conjoining of terms and those that lead to mathematically correct responses. As a result, the eight students interviewed covered a range of responses. The duration of the interviews was between twenty and thirty-five minutes. Given the limitations imposed on this study with respect to scope, only a limited number of students could be interviewed.

Students were handed back their answer sheets for each of the ten items. The answer sheet had not been altered in any manner. The students were asked a set of predetermined questions designed to investigate the procedures that students used when simplifying the expressions. The questions were purposefully chosen to probe the students to explain their procedures when simplifying algebraic expressions. The researcher used the initial prompt, "So, tell me about this?". The students were supplied with a pen and a blank sheet of paper which they were encouraged to use when explaining their thinking. The interviews were video-recorded and the verbal exchanges were transcribed. The videos and associated transcriptions were then analysed to identify the ways in which students conceptualised the algebraic notation as well as the algebraic transformations employed in the process of simplifying algebraic expressions (See Addendum).

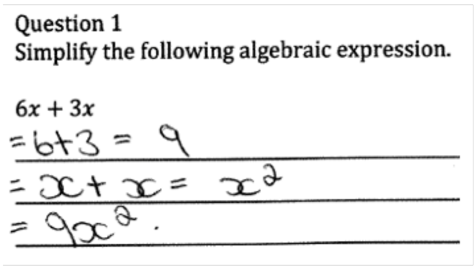
4.3 Protocol for the analysis of students' computational procedures

We have used the general framework developed by Davis (2010; 2013; 2016) to describe and analyse students' procedures for simplifying algebraic expressions. Davis's protocol for describing operations/operation-like manipulations, such that it relates to procedures followed by students within the context of this study, has been modified for my purposes.

Computational objects and operations are explicitly defined as they are presented in students' written work and explicated in interviews. That is, the semantic features and scope of the objects and operations used by students to constitute computational procedures are constructed through an analysis of the syntax of students' written responses and of their responses in interviews as it pertains to computation.

The students' procedures were represented as object-and-arrow diagrams. Each arrow indicates the operations or operation-like mappings used by students (detailed in Table 4.2); the arrows associate inputs (arguments; pre-images; domains) and outputs (values; images; codomains). Setting out the students' computational procedures in object-and-arrow diagrams enables us to: (i) reveal the computational syntax employed by students; (ii) describe the representations used by students; (iii) gain insight into the semantic bases of students; and (iv) compare different approaches to simplifying the algebraic expressions.

An example of a typical object-and-arrow diagram depicting the procedure followed by a student's response is shown in Figure 4.1.



Question 1
Simplify the following algebraic expression.

$6x + 3x$

= 6+3 = 9

= $x + x = x^2$

= $9x^2$.

$$\begin{tikzcd}
& 6x + 3x \arrow[d, "\mathrm{STR}"'] & \\
& /6x + 3x/ \arrow[d, "\mathrm{SUN}"'] & \\
& /6/, /x/, /+/, /3/, /x/ \arrow[dl, "\mathrm{SEL}(n)"'] \arrow[dr, "\mathrm{SEL}(l)"] \arrow[ddddd, "i_x\,[/x/]"] & \\
/6/, /3/ \arrow[d, "\mathrm{NUM}"'] & & /x/, /x/ \arrow[d, "\mathrm{SET}"] \\
6, 3 \arrow[d, "\Sigma"'] & & \{/x/_1, /x/_2\} \arrow[d, "\#"] \\
9 \arrow[d, "\mathrm{STR}"'] & & 2 \arrow[d, "\mathrm{SPS}"] \\
/9/ \arrow[dr, "i_x"'] & & /^2/ \arrow[dl, "i_x"] \\
& /9/, /x/, /^2/ \arrow[d, "++"'] & \\
& /9x^2/ &
\end{tikzcd}$$

Figure 4.1: A computational analysis of a student's procedure to simplify Item 1, resulting in $9x^2$.

Table 4.2: Descriptions of Operations and Operation-like Mappings

Operation	Domain	Codomain	Description
STR	Alphanumeric expressions	Alphanumeric expressions	STR takes an algebraic expression or term and returns a string of alphanumeric characters. For example, STR:($5x + 3$) $\rightarrow$/$5x + 3$/.
LIST(α)	SET	LIST	An ordered *list in A* is a pair (n,f) where $n \in \mathbb{N}$ is the *length of the list* and f:$n \rightarrow$ {f(1), f(2), … , f(n)}. Such an ordered list is denoted (n,f) = [f(1), f(2), … , f(n)] and the set of ordered lists in A is denoted LIST(A). LIST(x) returns a list of character strings, /μ/, derived from a discursive object, μ. A character string is a sequence of alphanumeric characters. Such strings are indicated by a pair of forward slashes: /α/. LIST($3x + 5x$) returns the alphanumeric string /$3x + 5x$/, consisting of the characters /3/,/x/,/+/,/5/,/x/.
++(/λ_1/, … ,/λ_n/)	LIST	LIST	++(/λ_1/, … ,/λ_n/) returns the concatenation of a list of strings (/λ_1/, … ,/λ_n/) to produce the alphanumeric string /$\lambda_1\lambda_2\ldots\lambda_n$/ where usually the numeric symbol is written before the letter. For example, ++(/9/,/x/) returns /$9x$/.
#({/l_i^n/})	Superscripted letters	Superscripted letters	#({/l_i^n/}) is the operation defined as the count on letters of the type l_i, where $n \in \mathbb{N}$ for the purposes of this project. That is, the operation acts on sets of symbols of the form l_i^n, where l_i is a letter and n is a superscript. n is treated as the cardinality of a set of letters, each element of which is a l_i. #({/l_i^n/}) returns /l_i^n/. For example, #(/x^2/) returns /x^2/. #({/l_1^a/,/l_1^b/,/l_1^c/, … ,/l_1^k/}) returns /$l_1^{a+b+c+\cdots+k}$/. For example, #(/x^2/,/x/,/x^3/,/x^2/) returns /x^{2+3+2}/ = /x^8/, or /$x^{2+1+3+2}$/ = /x^9/, depending on whether /x/ is treated as /x^0/ or as /x^1/. #({/l_1^a/,/l_2^b/,/l_1^c/,/l_3^k/}) returns /$l_1^{a+c}l_2^b l_3^k$/. For example, #(/x^2/,/y^2/,/z^3/,/x^3/) returns /$x^{2+3}y^2z^3$/ = /$x^5y^2z^3$/.
$\sum(\alpha_i)$	$\mathbb{N}$	$\mathbb{N}$	$\sum(\alpha_i)$ returns the sum of the α_i, where $\alpha_i \in \mathbb{N}$. This operation is applied to the numerical values as interpreted by the learners. For example, $\sum$ (2,5,8) returns 15.
$\cup$ (l_i)	Characters	Characters	$\cup$ (l_i) returns the concatenated union of the sets of all letters within a list of literal symbols. For example, $\cup$: (a; a; b; c; c; c) $\rightarrow$ *abc*. *The list is normally written alphabetically but this is not a requirement.

Table 4.2: Descriptions of Operations and Operation-like Mappings

Operation	Domain	Codomain	Description
$i_x[a]$	Alphanumeric symbols	Alphanumeric symbols	$i_x[a]$ indicates the inclusion of an element a into a list.
CMP	Letters of the alphabet	Letters of the alphabet	CMP is defined as the operation which computes and lists a single copy of each of the distinct letters present in a list of symbols. For example, CMP:(/*a*/, /*b*/, /*a*/) → /*a*/, /*b*/.
COM	Alphanumeric symbols	Alphanumeric Symbols	COM is an operation that 'completes' an algebraic expression by inserting a '1' as the coefficient of an algebraic term, or a superscript '1' as the exponent of an algebraic term. For example, COM:$(/5x + x + 7x/) \rightarrow /5x^1 + 1x^1 + 7x^1/$.
LIST(α)	Alphanumeric strings	Alphanumeric strings	An ordered *list in A* is a pair (n,f) where $n \in \mathbb{N}$ is the *length of the list* and $f{:}n \rightarrow \{f(1), f(2), \ldots, f(n)\}$. Such an ordered list is denoted $(n,f) = [f(1), f(2), \ldots, f(n)]$ and the set of ordered lists in A is denoted LIST(A). LIST(x) returns a list of alphanumeric strings, /μ/, derived from a given alphanumeric string, μ. Such strings are indicated by a pair of forward slashes: /α/. LIST($3x + 5x$) returns /3/,/x/,/+/,/5/,/x/.
NUM (/α/)	Numerical strings	ℕ	NUM (/α/) returns the numerical value of α, where α∈ℕ. The coefficients of all terms where natural numbers in this investigation. For example, NUM (/7/) returns 7.
SEL(μ,A)	A	Subset of A	SEL(μ,A) returns an object, according to an object-type, μ, from a collection of objects, A. SEL[n∈ℕ, (/3/,/x/,/+/, /5/,/x/)] returns (/3/,/5/). SEL [All terms containing x^2, (/$5x^2$/,/3x/,/$4x^2$/)] returns (/$5x^2$/,/$4x^2$/). SEL [All distinct letters , (/*a*/,/*a*/,/*b*/)] returns (/*a*/,/*b*/).
SPS	SET	ℕ	SPS returns a superscript that depicts the cardinality of a set containing letters of the same type.
STR	Alphanumeric symbols or expressions	Alphanumeric symbols or expressions	STR takes an algebraic expression or term and returns a string of alphanumeric characters. For example, STR:$(\ 5x + 3) \rightarrow /5x + 3/$.

Table 4.2: Descriptions of Operations and Operation-like Mappings

Operation	Domain	Codomain	Description
SUN(/λ/)	Alphanumeric strings	Alphanumeric strings	SUN(/λ/) sunders an alphanumeric string, /λ/, into a list of two or more alphanumeric strings (/λ_1/,…,/λ_n/), $n \geq 2$. SUN(/3*x*/) returns the list of alphanumeric strings (/3/,/*x*/), while SUN(/3*x* + 5*x*/) could return the list (/3*x*/,/+/,/5*x*/), or even the list (/3/,/*x*/,/+/,/5/,/*x*/) or any other combination of alphanumeric strings derivable from /3*x* + 5*x*/. Clearly the result of SUN(/λ/) is not unique, its output being contingent on the decision of the agent effecting the sundering.

4.4 Analytic Reliability

Reliability refers to the ability of the research method to produce consistent results when the same methods are used, under the same conditions, at different times, and by different analysists (Middleton 2024). Throughout the data collection process, reliability of the study was considered. The test was administered in the same way in each school and classroom, during a single mathematics lesson by the researcher, in the presence of the mathematics teacher. The instructions were projected on the board in each class (see Addendum). Each test item was printed on a separate sheet of paper and handed out individually. The answer sheets were collected after each item was attempted and then the next item was handed out. That was done to minimise the interference of previous items on responses to later items. Sufficient time was provided for each item: students were asked whether they were satisfied with their solution attempts before the answer sheets were collected. A trial run using students attending a school not involved in the study was conducted to check if the method used was appropriate for the purposes of the study. Furthermore, the research study was conducted within the third school term in both schools. There was, therefore, a standardisation of the conditions under which the test was administered in the participating schools.

Sampling was carefully considered. There was a deliberate sampling of students across different school contexts and different mathematic teachers. Thus, the sampling aids the quality of the research in that it provided a broad range of student experiences of simplifying algebraic expressions. The sampling of students for interviews was also done in a way that there was representation from each of the four classes. Two students from each of the four classes represented in this sample were chosen.

The interviews were conducted in the same manner with all participants. All interviews took place within two weeks of the test being written. Participants in the interviews were selected from all four classes. The interviews were semi-structured, using the same initial prompts and open-ended questions. There was thus a standardisation of the conditions of the interviews as well. Such a strategy strengthens the reliability of the research protocols. The process of interviewing students provides additional observational and descriptive adequacy because the justifications of their procedures offered by students are made explicit and serve as a triangulating resource.

Furthermore, the computational approach of the analysis of solution procedures as described by students was applied consistently to all solutions and solution justifications. The reliability of the method is displayed through the correlation of the same results being obtained when two different researchers applied the method. Additionally, the method also resulted in the same patterns in procedures being observed across different contexts (classes and schools). The object-and-arrow representations of solutions generates data enabling of increased descriptive adequacy and, consequently, increased explanatory adequacy.

4.5 Validity

Validity of research is the extent to which the results measure what they are supposed to measure (Middleton 2024). This study is concerned with the description of conjoining as it occurs in the computational activity of students. Published accounts of the conjoining phenomenon are considered and compared with the results of this study. In the survey of prior literature, it is evident that there are conflicting explanations for the occurrences of conjoining. In this study we aim to describe the processes that students employ in simplifying simple algebraic expressions.

The analysis of prior research on the topic of conjoining, the written responses of the students and the data obtained through the interviews have contributed to the conclusions drawn in this study. Here we should note that the *problem of reference* needs to be taken into account because we are dealing with mathematics in a pedagogic setting. Following Chomsky (2002) as he draws on Strawson (1951), it is important to stress that natural language has no reference function, by which is meant that words do not refer to things (ideas, objects, other words) in a denotative fashion. Rather, when humans use language we always have to ask what their uses of words refer to. In other words, it is not language that refers, but humans. Now, formal mathematics, formal logic and the various scientific specifications of objects and relations of concern to a scientific subfield do use words and symbolic resources denotatively. However, pedagogic communication, especially in schooling, uses natural language as the central resource to convey information concerning content, even in mathematics lessons. Research in mathematics education is, consequently, obliged to consider the problem of reference if it wishes to construct adequate descriptions and explanations of the phenomena it investigates. To that end, having students offer explanatory commentaries on their responses to test Items is a necessary validation requirement. To that end, triangulation of data is achieved through the interviews conducted with students (even if the number of interviews was necessarily limited). Additionally, incorporating students from different classes across different teachers and different schools enabled a reduction in potential biases in the generated data.

Furthermore, there were two researchers involved in the analysis of the data. This means that the collaborative effort in analysing the written responses and interview footage has resulted in investigator triangulation which further strengthens the arguments made within this dissertation. Lastly, the inclusion of different theories, different fields of study support the conclusions made in this study. Specifically, combining a cognitive scientific computational approach to human thought, mathematics education and a set of algebraic mathematical descriptive and analytic resources strengthens the claims made in this study. It also facilitates a more robust description of the phenomenon of conjoining as it appears in this study.

The constructions of the computational syntax used by students in each specific instance was obtained directly their proffered and clarified through the interviews. Thus, the semantic elements of the procedures employed in the simplification of algebraic expressions as described by the students themselves are depicted in this study. Careful consideration was taken with respect to the quality of the research presented here. Through ensuring the validity and reliability of the data collected we are strengthening the argument that is being made.

4.6 Ethical considerations

The research was conducted at schools, with students as research subjects, and ethical clearance therefore had to be obtained from the University of Cape Town. The Western Cape education department also provided clearance to conduct this study, as did the schools. The students were willing participant and written consent was obtained from their parents or guardians. Due to the nature of the study, participants were encouraged to describe their understanding of the problems. The correct procedures were always explained to students at the conclusion of interviews.

4.7 Concluding remarks

As described in Chapter 3, a methodology grounded in a computational theory of the mind uses computational resources to describe the way in which the mind works. Researchers in mathematics education have employed a computational approach to analysing data with respect to the way in which students understand mathematical processes and concepts (Davis 2010, 2013. 2016; Jaffer 2018; Wüst 2023). The mappings outlined in Table 4.2 will be used in the next section to describe and analyse the methods employed by Grade 8 learners in the simplification of algebraic expressions. This computational approach provides a way in which to describe the procedures and processes employed by the students in a way that facilitates a robust understanding of the phenomenon of conjoining within the scope of this study. In Chapter 5, an analysis of the data reveals the foundations for the justifications that students provide for the occurrences of conjoining. It becomes evident that there must exist an inbuilt knowledge system, i.e. core domain knowledge, that predisposes a response entailing conjoining.

CHAPTER 5

AN OVERVIEW OF THE TEST RESULTS

5.1 Introduction

The purpose of this investigation is to identify the computational procedures employed by Grade 8 students in the simplification of algebraic expressions. Furthermore, we would like to understand the justifications for the computations they use and how these computations result in a conjoined response. In this chapter, we present the results and a brief description of the student performance on the written test.

Recall that the investigation used a sample of 76 Grade 8 students attending two schools in the Western Cape. Twenty-seven students attended School A and forty-nine attended School B. The schools have varying class sizes, teachers, socio-economic statuses and resources. The sampling of the eight students to be interviewed was intentional and informed by the responses that students provided in the test. Four students who provided eight or more correct answers were selected to participate in an interview process—two students from School A and two students from School B. After identifying occurrences of conjoining in the responses of students, four additional students who had conjoined terms in most of the items were also selected to be interviewed. Again, two students from each school were selected.

5.2 Summary of student responses

For the purpose of this discussion, the correct solutions presented in the tables that follow observe the convention of alphabetical order and descending powers of terms. However, the simplified expressions in student solutions do not always present terms in alphabetical order or in order of descending powers of variables. Terms can, of course, be written in any order. For example, both of $6a + 5b$ and $5b + 6a$ are correct responses to Item 2, "$5a + 5b + a$". The responses to all questions by participating students were noted and recorded.

5.2.1 General remarks on student performances

Table 5.1 shows the totals for the numbers of students who have produced mathematically correct answers per Item. We see that there is a dramatic drop in computational competence from Item 1 (70%) to the rest of the Items (47% to 21%). However, Table 5.2 and Table 5.3 show the number of mathematically correct responses per Item per school, from which it is clear that the students attending School A perform significantly better than those attending School B.

Table 5.1 Summary of correct responses per Item, ordered from greatest to least success

Item	Expression Simplify the following algebraic expression.	Mathematically correct simplification	Number of correct responses (n = 76)	% correct
1	$6x + 3x$	$9x$	53	70%
8	$x + 5x + 2x$	$8x$	36	47%
10	$6g^2 + 6k + 8t^3$	$6g^2 + 6k + 8t^3$	31	41%
2	$5a + 5b + a$	$6a + 5b$	26	34%
7	$3y^2 + 4y + 1 + 5y + 7y^2 + 8$	$10y^2 + 9y + 9$	24	32%
9	$9a^2r + 4r^2 + 5a^2 + 3ar^2 + 7a^2r$	$5a^2 + 16a^2r + 3ar^2 + 4r^2$	21	28%
3	$3n + 4$	$3n + 4$	20	26%
6	$7kb + 4b + 3bk + 5kb + 4k$	$4b + 15bk + 4k$	19	25%
5	$4x^2 + 5x + 3x + 7x^2$	$11x^2 + 8x$	18	24%
4	$4a^2 + 3a^2 + 7a$	$7a^2 + 7a$	16	21%

Table 5.2 Summary of all mathematically correct responses per Item per school

Item	Expression Simplify the following algebraic expression.	School A (n = 27)		School B (n = 49)	
		No. of correct responses	% correct	No. of correct responses	% correct
1	$6x + 3x$	22	81%	31	63%
2	$5a + 5b + a$	21	78%	5	10%
3	$3n + 4$	16	59%	4	8%
4	$4a^2 + 3a^2 + 7a$	14	52%	2	4%
5	$4x^2 + 5x + 3x + 7x^2$	15	56%	3	6%
6	$7kb + 4b + 3bk + 5kb + 4k$	10	37%	9	18%
7	$3y^2 + 4y + 1 + 5y + 7y^2 + 8$	17	63%	8	16%
8	$x + 5x + 2x$	20	74%	16	33%
9	$9a^2r + 4r^2 + 5a^2 + 3ar^2 + 7a^2r$	17	63%	4	8%
10	$6g^2 + 6k + 8t^3$	17	63%	14	29%

The difference in performance across the schools is consistent with findings on South African student performances in mathematics assessments. For example, Figure 5.1 shows a graph indicating the differences in performance of Grade 9 mathematics students on TIMSS assessments across fee-paying and no-fee schools in South Africa. The performances of students attending fee-paying/no-fee schools correlates strongly with socio-economic status (relatively high/low SES), where students attending fee-paying schools are, generally, members of families with a higher socio-economic status than those attending no-fee schools. The mathematics education research literature offers a variety of sociological, political and economic explanations for the differences in performances with respect to social class, all of which is very well documented.

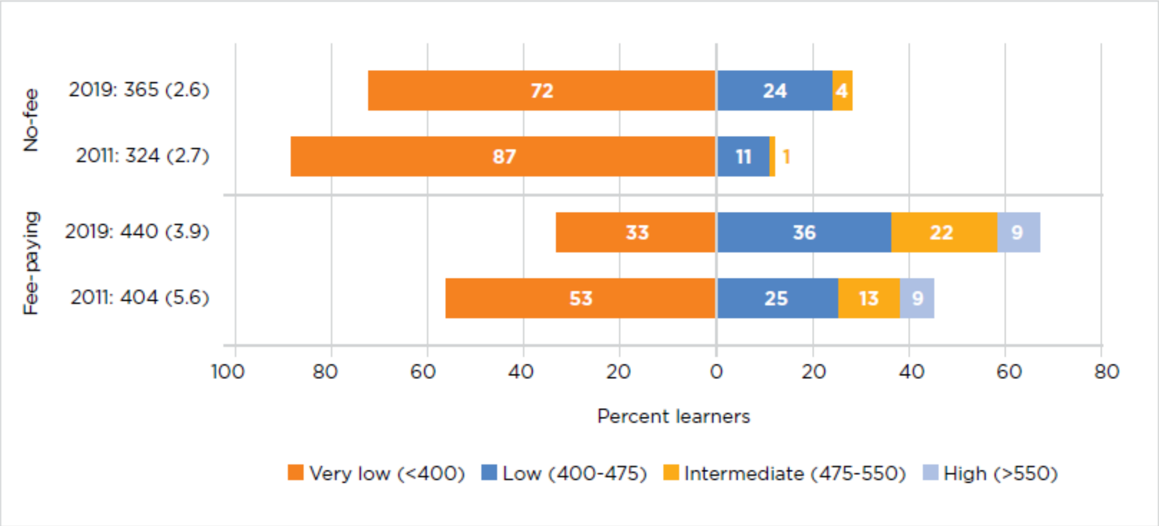


Figure 11: Trends in average mathematics scale score and percentage of learners reaching international benchmarks, by school fee status from 2011 to 2019

Source: Author's own calculations from TIMSS 2019 South African Grade 9 dataset.

Figure 5.1: Graph indicating the difference in performance in Mathematics of Grade 9 learners in South Africa along SES according the TIMSS Assessments (*Source*: Reddy *et al*. 2022:23).

Table 5.3 shows the differences in success per Item for School A and School B students, from the smallest to greatest differences for each Item.

Table 5.3 Differences in success per Item per school (%), ordered from least to greatest

Item	Expression Simplify the following algebraic expression.	School A ($n = 27$) No. of correct responses	 % correct	School B ($n = 49$) No. of correct responses	 % correct	% Diff. (A – B)
1	$6x + 3x$	22	81%	31	63%	18%
6	$7kb + 4b + 3bk + 5kb + 4k$	10	37%	9	18%	19%
10	$6g^2 + 6k + 8t^3$	17	63%	14	29%	34%
8	$x + 5x + 2x$	20	74%	16	33%	41%
7	$3y^2 + 4y + 1 + 5y + 7y^2 + 8$	17	63%	8	16%	47%
4	$4a^2 + 3a^2 + 7a$	14	52%	2	4%	48%
5	$4x^2 + 5x + 3x + 7x^2$	15	56%	3	6%	50%
3	$3n + 4$	16	59%	4	8%	51%
9	$9a^2r + 4r^2 + 5a^2 + 3ar^2 + 7a^2r$	17	63%	4	8%	55%
2	$5a + 5b + a$	21	78%	5	10%	68%

Table 5.3 and Table 5.4 show orderings of the test Items with respect to student success on Item, for each of the schools. Included in the tables are columns showing (i) the *number of terms*, and (ii) the *number of term types* in each of the expressions used in the items. The *number of terms* in an

expression is an index of the number of sums that need to be considered in the simplification of an expression, while the *number of term types* is an index of the least number of unlike terms. The *number of terms* and the *number of distinct term types* are features of an algebraic expression that potentially increase the *computational complexity* of an expression, if the latter is thought about in terms of the number of operations to be performed, or to be considered during a computation. For example, $5a + 5b + a$, used in Item 2, is composed of three terms and two term types. The expressions $6x + 3x$ and $3n + 4$ each consist of two terms, with the former consisting of a single term type and the latter of two term types.

Table 5.4 Success of students attending School A, ordered from most to least, per Item

Item	Expression Simplify the following algebraic expression.	# terms	# term types	School A (n = 27) No. of correct responses	% correct
1	$6x + 3x$	2	1	22	81%
2	$5a + 5b + a$	3	2	21	78%
8	$x + 5x + 2x$	3	1	20	74%
10	$6g^2 + 6k + 8t^3$	3	3	17	63%
7	$3y^2 + 4y + 1 + 5y + 7y^2 + 8$	6	3	17	63%
9	$9a^2r + 4r^2 + 5a^2 + 3ar^2 + 7a^2r$	5	4	17	63%
3	$3n + 4$	2	2	16	59%
5	$4x^2 + 5x + 3x + 7x^2$	4	2	15	56%
4	$4a^2 + 3a^2 + 7a$	3	2	14	52%
6	$7kb + 4b + 3bk + 5kb + 4k$	5	3	10	37%

Table 5.5 Success of students attending School B, ordered from most to least, per Item

Item	Expression Simplify the following algebraic expression.	# terms	# term types	School B (n = 49) No. of correct responses	% correct
1	$6x + 3x$	2	1	31	63%
8	$x + 5x + 2x$	3	1	16	33%
10	$6g^2 + 6k + 8t^3$	3	3	14	29%
6	$7kb + 4b + 3bk + 5kb + 4k$	5	3	9	18%
7	$3y^2 + 4y + 1 + 5y + 7y^2 + 8$	6	3	8	16%
2	$5a + 5b + a$	3	2	5	10%
3	$3n + 4$	2	2	4	8%
9	$9a^2r + 4r^2 + 5a^2 + 3ar^2 + 7a^2r$	5	4	4	8%
5	$4x^2 + 5x + 3x + 7x^2$	4	2	3	6%
4	$4a^2 + 3a^2 + 7a$	3	2	2	4%

We can now consider the performances of students with respect to the number of terms and the number of term types in greater detail.

In Table 5.6 and Table 5.7 we derive a measure of student success with respect to the number of terms in an expression, for School A and School B. The data is shown graphically in Chart 5.1 and Chart 5.2. Recall that n = 27 for School A.

Table 5.6 School A: Correct responses w.r.t. number of terms in expressions

# terms	# Items	# responses	# correct	% correct
2	2	54	38	70%
3	4	108	72	67%
4	1	27	15	56%
5	2	54	17	31%
6	1	27	17	63%

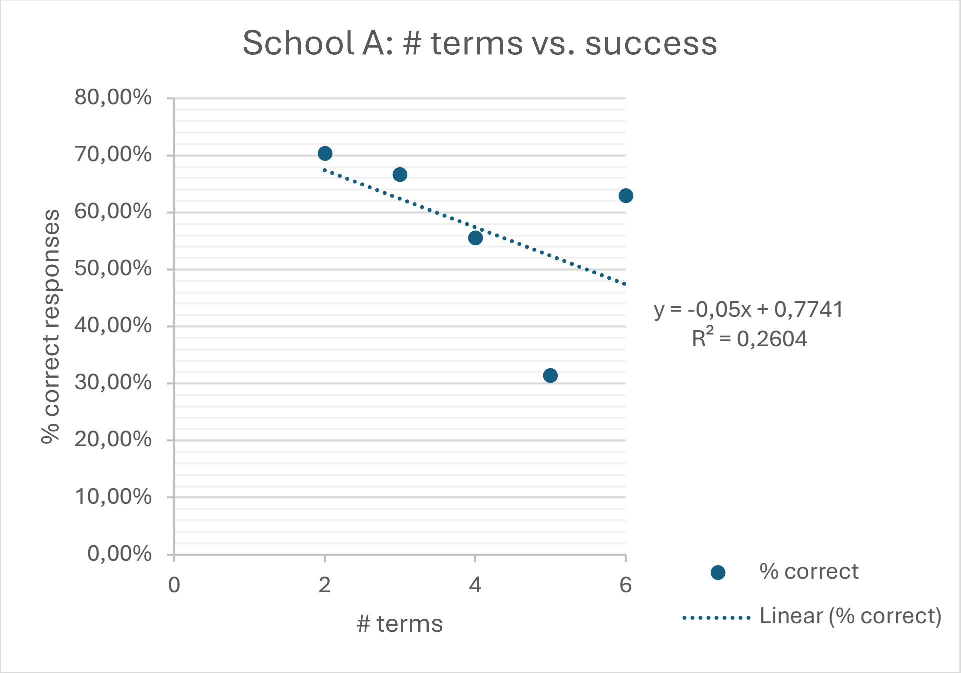


Chart 5.1: School A: student success on Items with respect number of terms in an expression

The dotted line in Chart 5.1 is the trend line for the data displayed in Table 5.6. Chart 5.1 suggests that the decline in student success is very close to a linear relationship for expressions consisting of 2 to 5 terms (y = -0.1278x + 1.0074; R² = 0.8858), while performances on Item 7 ($3y^2 + 4y + 1 + 5y + 7y^2 + 8$; 6 terms) diverges significantly from the general trend and needs to be examined more closely, which is addressed later.

Table 5.7 and Chart 5.2 show data describing the performances of School B students with respect to the number of terms in expressions. Recall that n = 49 for School B.

Table 5.7 School B: Correct responses w.r.t. number of terms in expressions

# terms	# Items	# responses	# correct	% correct
2	2	98	35	36%
3	4	196	37	19%
4	1	49	3	6%
5	2	98	9	9%
6	1	49	8	16%

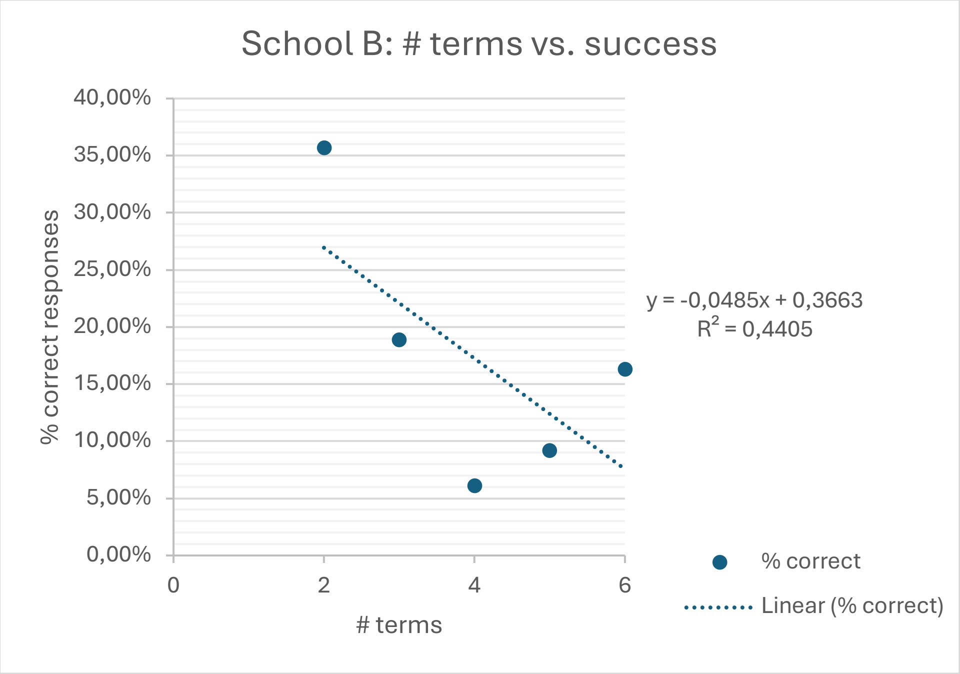


Chart 5.2: School A: student success on Items with respect number of terms in an expression

As was the case for School A, we see that the relations between performances on Items containing expression with 2, 3 and 4 terms is very close to linear (y = -0.0827x + 0.4883; R^2 = 0.886), with a divergence from that trend for Items with expressions of 5 and 6 terms (viz., Items 6 and 7). School B student performances on the latter is discussed later.

Overall, even though the performances of School B students are significantly weaker than those of School A students, the respective trends in the decline of success on Items other than outliers are very close to a linear relationships with a negative gradient. For both schools, student performances on Items containing 5 terms (Item 6 and Item 9), and those containing 6 terms (Item 7) demand further analytic consideration.

Turning now to a consideration of student performance with respect to the number of distinct term types in expressions, Table 5.8 and Chart 5.3 show that the decline in success for School A students is very close to linear for expressions having 1, 2 and 3 term types ($y = -0.1358x + 0.928$; $R^2 = 0.9231$), with a divergence from that trend for Items using expressions that have 4 term types. The divergence of the latter from the trend followed by performances on Items having 1, 2 and 3 term types indicates that performances on Item 9 (4 term types) should be studied more closely. ($n = 27$ for School A.)

Table 5.8 School A: Correct responses w.r.t. number of term types in expressions

# term types	# Items	# responses	# correct	% correct
1	2	54	44	81%
2	4	108	66	61%
3	3	81	44	54%
4	1	27	17	63%

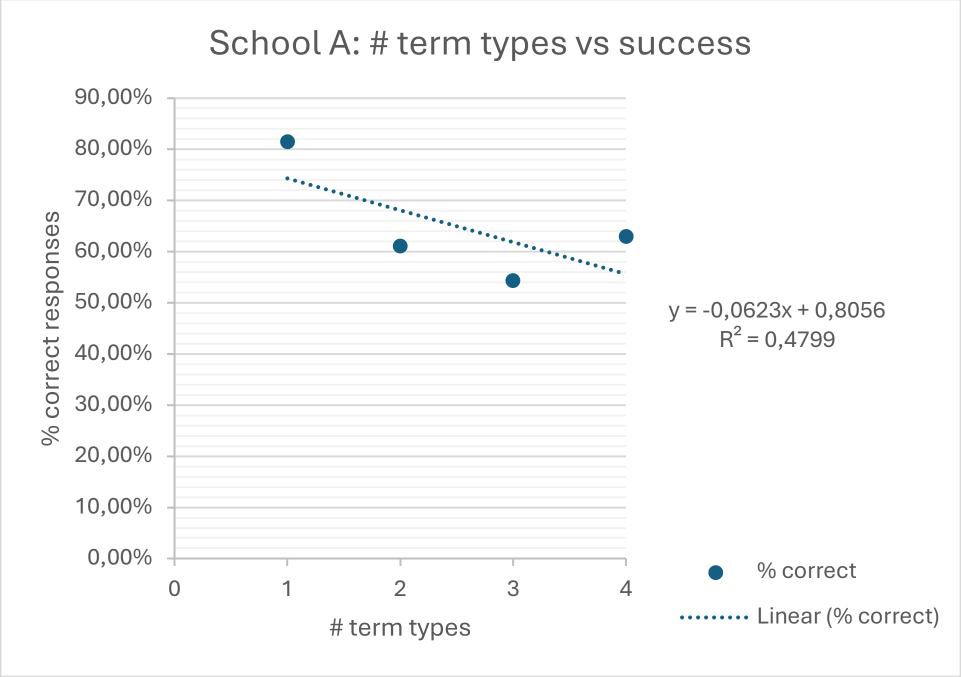


Chart 5.3: School A: student success on Items with respect number of term types in an expression

The results for School B student performances show a linear trend with respect to Items contains expressions using 1, 3 and 4 term types ($y = -0.1329x + 0.6118$; $R^2 = 0.9999$), with performances on Items using expressions having 2 term types diverging sharply from the trend (refer to Table 5.9 and Chart 5.4). The School B student performances on Items 2, 3, 4 an 5, each of which uses expressions having 2 term types will therefore need to be analysed in greater detail. ($n = 49$ for School B.)

Table 5.9 School B: Correct responses w.r.t. number of term types in expressions

# term types	# Items	# responses	# correct	% correct
1	2	98	47	48%
2	4	196	14	7%
3	3	147	31	21%
4	1	49	4	8%

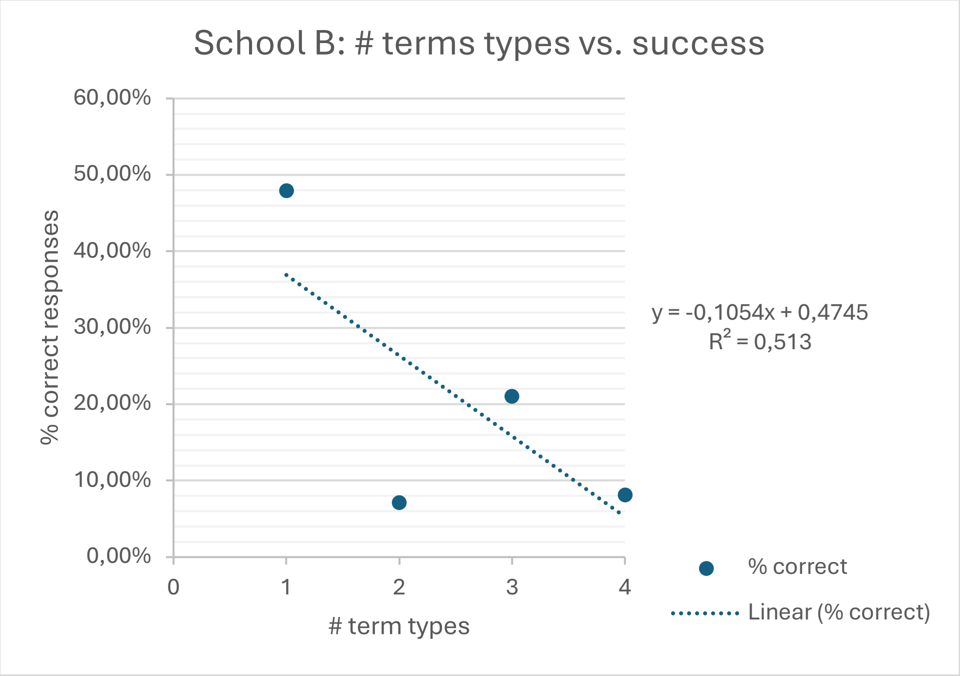


Chart 5.4: School B: student success on Items with respect number of term types in an expression

5.2.2 Student responses to selected Items

The ten test Items provided variation in the types of questions that Grade 8 students typically encounter. However, when looking at students' responses, we identified a series of Items that enabled us to identify the ways in which students produced aberrant conjoining. Given the limitations on space for a project such as this, it is not possible to report in detail on all student responses.

Table 5.10 shows students' responses to Item 1 ($6x + 3x$), which had the highest number of correct responses (70%). The most prevalent response to Item 1 was $9x$, which is mathematically correct. The proposed solution $9x^2$ appeared in 25% of the responses and is the second most prevalent response, apparently a concatenation of 9 and x^2, where the latter expression refers to the sum of the two xs in $6x + 3x$. Two students gave a response of $18x$, which could be achieved by multiplying the coefficients 6 and 3. However, one can be sure of the computations used by students only by engaging with them in an interview.

Table 5.10 Student responses to Item 1: $6x + 3x$

	All students (n = 76)	
Response	No. of responses	%
$9x$	53	70%
$9x^2$	19	25%
$18x$	2	3%
$3x^2$	1	1%
Incomplete	1	1%
Totals	76	100%

Given the reasonably good performances on Item 1, one could expect students to perform well on Item 8 ($x + 5x + 2x$) as well since the expression consists of a single term type, and the left to right sequence of transformations entailed in the solution is the same as that used to solve Item 1: $x + 5x + 2x \rightarrow 6x + 2x \longrightarrow 8x$. Surprisingly, only 47% of students solved Item 8 successfully (see Table 5.11), suggesting that the computations used by students to solve even simple problems like $6x + 3x$ are not what one might expect. The next most common response to Item 8, which is $8x^3$ (22%), appears to result from the kind of computation used to produce $9x^2$ as a solution to Item 1.

Table 5.11 Student responses to Item 8: $x + 5x + 2x$

	All students (n = 76)	
Response	No. of responses	%
$8x$	36	47%
$8x^3$	17	22%
$7x^3$	14	18%
$x + 7x$	4	5%
$10x$	1	1%
$3x$	1	1%
$8x^2$	1	1%
$-9x$	1	1%
Totals	76	100%

Table 5.12 shows the range of responses to Item 2, which uses a slightly more complex expression consisting of three terms and two distinct term types ($5a + 5b + a$). The predominant response was the mathematically correct response, $6a + 5b$ or $5b + 6a$, appearing in 34% of responses.

Table 5.12 Student responses to Item 2: $5a + 5b + a$

Response	All students (n = 76) No. of responses	%
$6a + 5b$ or $5b + 6a$	26	34%
$11a^2b$	13	17%
$10a^2b$ or $10ba^2$	11	14%
$11ab$	6	8%
$10ab$ or $10ba$	4	5%
$10a^3$	3	4%
$10a$	2	3%
$10aba$	2	3%
$6a^2 + 5b$	2	3%
$11aba$	1	1%
$5a + 5b$	1	1%
$15a$	1	1%
$a + 5b$	1	1%
$6a + 1a$	1	1%
$6a5b$	1	1%
$5a^2 + 5b$	1	1%
Totals	76	100%

Responses that would be classified in the literature as entailing instances of 'exponent errors' include $11a^2b, 10a^2b$, $10a^3$, $6a^2 + 5b$, and $5a^2 + 5b$ (cf. Lim 2010; Pournara *et al.* 2016). However, it appears that students who produce such solutions recognise superscripted numbers as counts of the occurrences of distinct letters populating expressions, and not as exponents proper. Responses like $11aba$ and $6a5b$ indicate that the coefficients of variables are often recognised as counts of the number of letters in manner similar to way in which superscripted numbers are used: 6 *a*s and 5 *b*s.

The responses that use 10 as a coefficient suggest that many students simply add the numerals that populate expressions. Twenty-two responses (29%) were of such a nature, but that is a lower bound because the students who used 11 as a coefficient might well have recruited the routinely used idea of the presence of a so-called 'invisible 1' when encountering a letter without a coefficient—in effect, reading the expression as $5a + 5b + 1a$ and adding 5 + 5 + 1.

Table 5.13 shows the range of responses to Item 5, which is an expression of the degree two.

Table 5.13 All student responses to Item 5: $4x^2 + 5x + 3x + 7x^2$

	All students (n = 76)	
Response	No. of responses	%
$19x^6$	28	37%
$11x^2 + 8x$ or $8x + 11x^2$	18	24%
$19x^4$	8	11%
$11x^4 + 8x^2$ or $8x + 11x^4$	3	4%
$19x^2$	3	4%
$11x^4 + 8x$ or $8x + 11x^4$	3	4%
$19x^8$	2	3%
$7x + 11x^2$	1	1%
$6x^4$	1	1%
$41x$	1	1%
$19x^{16}$	1	1%
$43a$	1	1%
$18x^6$	1	1%
$19x^3$	1	1%
17^6	1	1%
19^4	1	1%
19^8	1	1%
Incomplete	1	1%
Totals	76	100%

Note that a change in the degree of the expression increased the range of variation in responses. The predominant response of $19x^6$ was present in 37% of the responses. Solutions of the form ax^n were present in forty-five of the responses showing eight variations. The mathematically correct response of $11x^2 + 8x$ was present in only 24% of all responses. Responses that were somewhat similar to the mathematically correct response, like $11x^4 + 8x^2$ and $11x^4 + 8x,$ were also present. The computational analyses of the procedures discussed in the chapters that follow reveal how students generate such variations. An interesting, but limited, set of three responses are those that show an omission of the variable: 17^6, 19^4 and 19^8.

The expression used in Item 10 contained three unlike terms in three distinct variables which resulted in many different responses as seen in Table 5.14.

Table 5.14 All student responses to Item 10: $6g^2 + 6k + 8t^3$

Response	All students (*n* =76) No. of responses	%
$6g^2+6k+8t^3$	31	41%
$20g^2kt^3$	19	25%
$20gkt^5$	5	7%
$20gkt^6$	4	5%
$12g^2k+8t^3$	3	4%
$6g^26k8t^3$	2	3%
56gt	1	1%
554gkt	1	1%
$20g^5$	1	1%
$22g^6kt^3$	1	1%
$12g^2$	1	1%
20g	1	1%
$20gkt^4$	1	1%
$20g^2kt^2$	1	1%
$18gk^2+8t^3$	1	1%
36g+6k+512t	1	1%
$20g^2+k+t^3$	1	1%
12g+6k+24t	1	1%
Totals	76	100%

The predominant response was the mathematically correct response of $6g^2 + 6k + 8t^3$, which was present in 41% of all responses. Other common responses included variations of the form $20g^2kt^3$, $20gkt^5$or $20gkt^6$.

There are clear similarities present in the most common responses, suggesting that there is one or more underlying computational principles which most students are applying when simplifying algebraic expressions. A detailed computational analysis of the procedures employed by students will allow us to identify such principles. The interview transcripts were analysed in order to develop triangulated descriptions of the students' computational decisions. It was observed that there are two broad principles, or approaches, to simplifying algebraic expressions, which will be described in the next chapter.

5.3 Concluding remarks

In this chapter we have presented the results from the written tests. While students attending School A performed better than students attending School B, we also note that instances of aberrant conjoining were present in the responses of students of both schools. It appears that as the complexity of the Items increased, variation in the types of responses also increased.

How do the students generate such responses? What are the procedures they follow? And what are the justifications for such procedures? We were able to address the questions by conducting interviews with the students in order to identify the computational principles that result in the variation of aberrant conjoining seen in the results.

In Chapters 6 and 7 we present a discussion on the results introduced here and suggest explanations for how students process algebraic expressions to produce aberrant conjoining. Chapter 6 outlines general principles to the simplification of algebraic expressions that result in aberrant conjoining as well as the computational details of aberrant conjoining as manifested in this study.

CHAPTER 6

COMPUTATIONAL ANALYSIS OF RESULTS

6.1 Introduction

Since this study is concerned with understanding how students go about performing simplification computations, including their justifications of the computational procedures used, in this chapter we put to work the analytic computational resources set out in Chapter 4 in order to unpack the computational approaches employed by students. This allows us to provide a reasonably robust explanation of what is referred to as *conjoining* in the field.

The student procedures are described analytically through the use of object-and-arrow diagrams that show the details of the computational approaches to the simplification of typical Grade 8 algebraic expressions. Furthermore, through the interview process, we are able to triangulate our interpretations of the computations that result in the student responses to the Items.

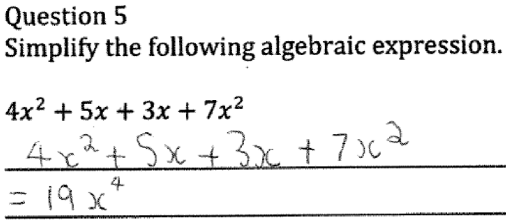
6.2 Computational principles employed in the production of aberrant conjoining

We observed four general principles employed by the Grade 8 students in the sample when simplifying algebraic expressions. The principles are applied either partially or in full to the expressions used in the test Items. A selection of responses that exemplify typical uses of the general computational principles are discussed analytically in the subsections that follow.

6.2.1 The use of type-specific computations

Consider the computational work of Student A2-20, a School A Grade 8 student. From their response to Item 5, the broad features of their procedure for simplifying $4x^2 + 5x + 3x + 7x^2$ are as follows:

> (1) He separates the coefficients from the letters and adds them: “I added, um, all of the numbers together. Four, plus five, plus three plus seven, which equals to nineteen”.
> (2) He then notes the existence of the letter, x, which is common to each term: “And then I added the variables which were all like, so I just got one x”.
> (3) He separates the superscripts from the letters and adds them: “and the power of the two plus the two”.
> (4) Finally, he concatenates the results of (1), (2) and (3) to produce $19x^4$: “it means nineteen plus, plus x um, plus to the power of four”.



Question 5
Simplify the following algebraic expression.

$4x^2 + 5x + 3x + 7x^2$

$4x^2 + 5x + 3x + 7x^2$

$= 19x^4$

Student A2-20: [00:13:33] Um, simplify the follow following algebraic expressions: four x squared plus five x plus three x plus seven x squared.

Researcher: [00:13:43] Okay, so what would that then tell me in the next line? What have you done in the next line?

Student A2-20: [00:13:51] I rewrote the sum first and wrote the answer.

Researcher: [00:13:57] Okay, and your answer. Can you read your answer to me?

Student A2-20: [00:14:00] Nineteen x to the power four.

Researcher: [00:14:02] Okay. And, um, can you explain how you got the nineteen?

Student A2-20: [00:14:08] I added, um, all of the numbers together. Four, plus five, plus three plus seven, which equals to nineteen. And then I added the variables which were all like, so I just got one x and the power of the two plus the two.

Researcher: [00:14:27] Okay. So the two plus the two …

Student A2-20: [00:14:30] Equals to the four.

Researcher: [00:14:30] And so nineteen x to the four. If I asked you what is nineteen x how do you say this?

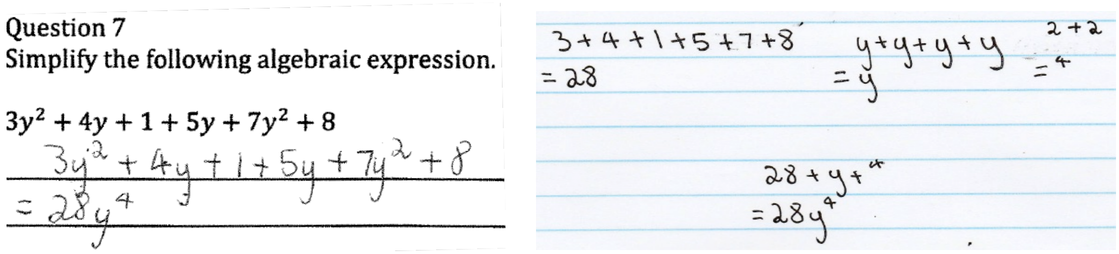
Student A2-20: [00:14:40] $19x$ to the power of four.

Researcher: [00:14:42] To the power of four, what does that mean?

Student A2-20: [00:14:47] Uh, it means nineteen plus, plus x um, plus to the power of four.

Student A2-20 describes their final answer of $19x^4$ as "nineteen plus, plus x um, plus to the power of four", which is a concatenation of 19, x and ^{4}, and is structurally similar to addition. Even though they use the phrase "to the power of", it is clear from their computations that they do not treat the superscripts as exponents as understood in the field of mathematics. In that regard, note additionally that since there are no superscripts associated with the second and third terms ($5x$ and $3x$, respectively) in the expression, Student A2-20 does not use the implied superscript, 1, for those terms.

Student A2-20 was also asked to discuss his response to Item 7 and was encouraged write down their working on a sheet of paper, an extract of which is shown in Figure 6.1. Note the difference in the way they write the coefficient values and superscript values in Figure 6.1. The working shows clearly that they distinguish between number objects (coefficients), letter objects (variables), and superscript objects (exponents) and that they processes the different object types separately by subjecting objects to what they take to be additive processes.


Question 7
Simplify the following algebraic expression.

$3y^2 + 4y + 1 + 5y + 7y^2 + 8$

$3y^2 + 4y + 1 + 5y + 7y^2 + 8$
$= 28y^4$

$3+4+1+5+7+8$
$= 28$

$y+y+y+y$
$= y$

$2+2$
$= 4$

$28 + y + 4$
$= 28y^4$

Figure 6.1: Student A2-20’s solution and explanatory notes on the procedure followed when simplifying Item 7, resulting in $28y^4$.

On examining Student A2-20’s responses to the rest of the test Items, we find that their procedure for generating solutions is used in a consistent manner, as is evident from both the interview transcript for Item 5 and the details regarding their response to Item 7.

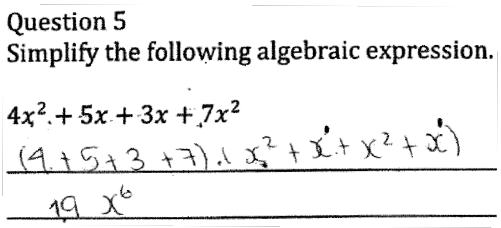
The work of Student A2-20 is a typical example of the use of a computational principle—viz., type-specific computation—used by many of the research subjects who produced solutions exhibiting aberrant conjoining.

6.2.2 The treatment of constituents of terms as sets

Student A1-03 is a Grade 8 student attending School A. The interview transcript presented here shows their explication of the ‘solution’ to Item 5, $4x^2 + 5x + 3x + 7x^2$. As was the case for Student A2-20, we observe terms being split apart in a type-specific manner: “I took all the, um, numbers and I left the variable, the letters, aside for so long and four plus five plus three plus seven, which got me to nineteen”. The work of Student A2-20 and Student A1-03 exhibit consistent and stable procedures for “putting together” (concatenating) terms but the two approaches show some differences.

Student A1-03 performs the transformation $(4x^2 + 5x + 3x + 7x^2) \rightarrow (4x^2 + 5x^1 + 3x^1 + 7x^2)$ to render explicit the superscript 1, which they take as implied by all subexpressions of the form ax. The superscripts associated with letters in algebraic expressions are viewed as counts of the number of letters: “It [i.e., x^6] means that there are six xs and you just making it easy for you to write it out”.

Student A1-03 describes the symbol x as a letter object, indicating that in their immediacy, letters are dislocated from the domain (of numbers) to which the notation intends them to refer. They ‘add’ the xs by counting the number of xs, implicitly using disjoint union to combine the sets of xs associated with each term. That is, $(x^2 + x^1 + x^2 + x^1)$ is implicitly mapped to $(\{x, x\} \sqcup \{x\} \sqcup \{x, x\} \sqcup \{x\})$ to give the set $\{x, x, x, x, x, x\}$, which is represented as x^6 in their use of the notation.[8] Here, the superscripts used in exponential notation are not understood as we expect in the field of mathematics, but rather as referring to the number of xs in a set. That is, the superscript is essentially a cardinality: the expression x^n denotes a set consisting of n xs.



Question 5
Simplify the following algebraic expression.

$4x^2 + 5x + 3x + 7x^2$

$(4+5+3+7).(x^2 + x^1 + x^2 + x^1)$

$19\ x^6$

Researcher: [00:11:40] Thank you. All right, so this next one is question five. So, it's almost the same like question four. So, can you read your answer to me, what's your answer?

Student A1-03: [00:12:04] I said nineteen x six.

Researcher: [00:12:08] Nineteen x six. And can you tell me, um, just again from the start, what you did to get to nineteen x six?

Student A1-03: [00:12:15] So, first I plussed all the, I took all the, um, numbers and I left the variable, the letters, aside for so long and four plus five plus three plus seven, which got me to nineteen. And then I did not put the plus sign there. So, that would have been plus. And then I just took all the letters. So, for here it's the same. The x has an exponent of one which I saw and then you just say x two, plus x, plus x two, plus x. So, x two, plus one, plus two, plus one, that got me to six. So, I just said nineteen x six.

Researcher: [00:13:03] And so what does x six mean to you?

Student A1-03: [00:13:06] It means that there are six xs and you just making it easy for you to write it out.

The responses to test Items show that many students view the superscript notation as an indication of the number of letters of a particular kind in an expression. Thirteen School A students (49%) used the computation that treats superscripts as cardinalities in response to at least one test Item. Forty-nine School B students (100%) used such a computation in response to at least one test Item. In total, sixty-two students (82%) used the computation.

6.2.3 The implicit use of string operations

The centrality of the use of concatenation in the production of responses exhibiting aberrant conjoining necessarily implies that (i) the terms of expressions are treated as composed of distinct type-specific objects that (ii) can be separated to form distinct collections which are (iii) processed

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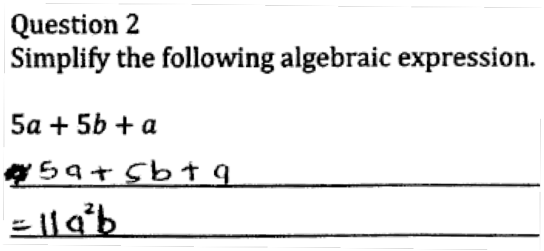
[8] Note that here we are using the set notation in a somewhat unconventional way. Ordinarily a set like $\{x, x\}$ is understood as $\{x\}$ because elements of sets are never repeated. However, for the student, all of the xs remain distinct entities when they are gathered together to form a set. One way of dealing with the problem is to use notation to tag each occurrence of x as distinct. For example, we could replace $(\{x, x\} \sqcup \{x\} \sqcup \{x, x\} \sqcup \{x\})$ with $(\{x_1, x_2\} \sqcup \{x_3\} \sqcup \{x_4, x_5\} \sqcup \{x_6\})$.

using type-specific computations, the results of which are (iv) combined to form new algebraic expressions.

In response to Item 2, Student B2-25, like the students discussed in Section 6.2.2 and Section 6.2.1, separate out the constituents of terms and perform type-specific computations.



Question 2
Simplify the following algebraic expression.

$5a + 5b + a$

$5a + 5b + a$

$= 11a^2b$

Researcher: [00:02:02] Okay so these two and this is different. How is that different?

Student B2-45: [00:02:06] Because the, there is a different variables in each sum.

Researcher: [00:02:14] Oh okay, so you see the variables are different. Okay. And here, can you tell me how you got to this answer?

Student B2-45: [00:02:23] I see in this question, I said five plus five is equal to ten plus an invisible one. So just say I said five plus five is ten plus one. Then I got 11. Then I put the variable. Then let's say *a*. Then I say *a* plus *a*. Then it's going to be a *a* squared. Then I put a *b*.

Researcher: [00:02:47] Okay. And you put it all next to each other? And so what does that mean? Like so what are you doing here? What's the operation?

Student B2-45: [00:02:58] I'm putting the, the capital letters in alphabetical, alphabetical order.

Researcher: [00:03:05] Good. And can you tell me what are you doing? Are you adding multiplying, dividing or subtracting?

Student B2-45: [00:03:11] Here, I'm adding miss.

Student B2-25's type-specific computations produce 11, a^2, and b as distinct entities, which are then concatenated to produce $11a^2b$. Even though Student B2-25 and their peers routinely refer to the final computation that they use as addition, $11a^2b$ cannot be the value that is produced from the sum $11 + a^2 + b$. Rather, the use of concatenation implies that 11, a^2, and b are treated as lists of character strings that are combined to arrive at the final answer.

The transformation $(11, a^2, b) \rightarrow 11a^2b$ can, of course, be the result of using multiplication recursively. However, Student B2-25 is clear that they are "adding". When asked, "Are you adding multiplying, dividing or subtracting?" they reply, "Here, I'm adding miss".

It is clear from the data that, in general, students fail to read terms of the type ax, where $a \in \mathbb{N}$ and x is a variable, as the product $a \times x$. Further, when computing with terms of the type ax^n, students who produce responses that show aberrant conjoining typically do not recognise the basic operation of exponentiation. So, both multiplication and exponentiation are generally absent as operations in the work of students who produce responses showing aberrant conjoining.

It appears to be the case that aberrant conjoining necessarily entails the use of concatenation as a computation, which is obvious in hindsight. How otherwise could students combine number objects, letter objects and superscript objects without using multiplication and exponentiation? Once we recognise that concatenation is a necessary computation in the production of aberrant conjoining we are obliged to conclude that at various moments in a computation the objects that constitute algebraic expressions have to be treated purely as lists of character strings since only the latter can serve as arguments (inputs) and values (outputs) for concatenation.

6.2.4 The use of addition-like monoids

Our discussion in the previous sections enables us to identify general computational structures used by students who produce responses exhibiting instances of aberrant conjoining, which are as follows:

1. *Addition defined over the set of natural numbers*, $(\mathbb{N}, +)$, which is used to sum coefficients of terms and, at times, the numbers that appear as superscripts (exponents).
2. *Disjoint union defined over the class of finite sets*, $(\text{SET},\sqcup)$, which is used as a resource for computing the number of letters believed to be implied by the superscripts (exponents) associated with letters (variables).
3. *Concatenation defined over the class of lists of character strings*, $(\text{LIST}, ++)$, which is used to combine the results of type-specific computations to constitute intermediate and final results of a computation aiming to simply an algebraic expression.

In Chapter 3 we showed that $(\mathbb{N}, +)$, $(\text{SET},\sqcup)$ and $(\text{LIST}, ++)$ share the same abstract computational structure. In other words, the structures $(\mathbb{N}, +)$, $(\text{SET},\sqcup)$ and $(\text{LIST}, ++)$ are the same kind of thing with respect to their computational characteristics. To emphasise the essential computational sameness of those structures, we showed that there exists monoid homomorphisms that enable us to construct structure-preserving mappings between all distinct pairs of $(\mathbb{N}, +)$, $(\text{SET},\sqcup)$ and $(\text{LIST}, + +)$. That is, there exist structure-preserving mappings $(\text{SET},\sqcup) \to (\mathbb{N}, +)$, $(\text{LIST}, ++) \to (\mathbb{N}, +)$, and $(\text{LIST}, + +) \to (\text{SET},\sqcup)$, as well as composites of those mappings that enable translations between $(\mathbb{N}, +)$, $(\mathrm{SET}, \sqcup)$ and $(\mathrm{LIST}, ++)$.[9]

What all of this means is that students who produce responses exhibiting aberrant conjoining use the same abstract computational structure for each of the type-specific computations that they employ, and that the results of their computations are brought together to produce the final result of a series of computations by means of $(\mathrm{LIST}, ++)$.

We discuss the computations that students use in more precise terms in the next section.

6.3 Detailed computational analyses of the computational procedures used by students

The following discussion uses the computational analytic approach outlined in Chapter 4 to describe and analyse the computations used by students to generate solutions, many of which show aberrant conjoining. We use object-and-arrow diagrams to display the computational mappings used in students' solutions. A selection of responses to four Items is discussed in this section: Items 1, 2, 5 and 10. Those Items have been selected because they cover a range of different types of algebraic expressions included in this study. The expression used in Item 1 is a simple linear algebraic expression consisting of two terms in one variable. Item 2 is an algebraic expression consisting of two distinct variables. Item 5 is a quadratic algebraic expression. Item 10 uses an expression consisting of three unlike terms in three distinct variables of varying degree.

The variation in the composition of the four expressions makes it possible to analyse the range of computational methods of the students in some detail, enabling some insight into the potential effects of changes in the initial conditions of the algebraic expressions on computational activity of students. Students' responses to all Items were analysed. However, each of the computations identified are adequately and completely described by using responses to Items 1, 2, 5 and 10, and show how the broad principles discussed in section 6.2 structure students' computations.

6.3.1 A computational analysis of procedures used in response to Item 1

When analysing Item 1 $(6x + 3x)$, we note from Table 5.1 that 70% of the student responses were classified as correct $(9x)$, which was the highest percentage of correct responses over all the Items. It must also be noted that the expression is in one variable and is linear.

Figure 6.2 and Figure 6.3 are object-and-arrow diagrams of the procedures that students followed in

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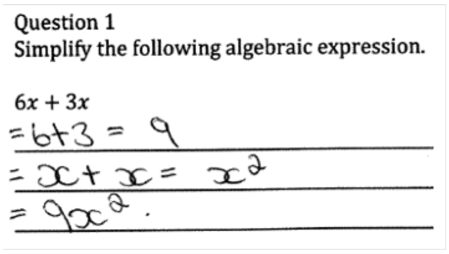
[9] Recall that monoids are algebraic structures that exhibit (i) closure, (ii) associativity, and (iii) identity elements.

order to simplify the expression $6x + 3x$. The diagrams show the operations and operation-like mappings as described in Table 4.2. Each arrow indicates an operation or operation-like mapping on the respective objects.

Figure 6.2 shows a chain of computations that produces an incorrect response, $9x^2$, to Item 1.



Question 1
Simplify the following algebraic expression.

$6x + 3x$
$= 6+3 = 9$
$= x + x = x^2$
$= 9x^2$.

$$\begin{tikzcd}
& 6x+3x \arrow[d, "\mathrm{STR}"'] & \\
& /6x+3x/ \arrow[d, "\mathrm{SUN}"'] & \\
& /6/, /x/, /+/, /3/, /x/ \arrow[dl, "\mathrm{SEL}(n)"'] \arrow[dr, "\mathrm{SEL}(l)"] \arrow[ddddd, "{i_x[/x/]}"] & \\
/6/, /3/ \arrow[d, "\mathrm{NUM}"'] & & /x/, /x/ \arrow[d, "\mathrm{SET}"] \\
6, 3 \arrow[d, "\Sigma"'] & & \{/x/_1, /x/_2\} \arrow[d, "\#"] \\
9 \arrow[d, "\mathrm{STR}"'] & & 2 \arrow[d, "\mathrm{SPS}"] \\
/9/ \arrow[dr, "i_x"'] & & /2/ \arrow[dl, "i_x"] \\
& /9/, /x/, /2/ \arrow[d, "++"'] & \\
& /9x^2/ &
\end{tikzcd}$$

Figure 6.2: A computational analysis Student B1-29's response to Item 1, resulting in $9x^2$.

We will describe the computations in some detail so that reader can get a feel for how the object-arrow-diagrams function analytically.[10] The detailed sequence of operations/mappings we employ to describe students' computations should not be read as a claim that they possess explicit understanding of such computational resources. While the analysis describes the computations explicitly, that is

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[10] We will not provide similar detailed explications of the rest of the object-and-arrow diagrams because of limitations on space.

something very different from students' subjective experiences of doing the computations, which remains largely automatic and intuitive to them.[11]

1. The student considers the expression as a character string:
   $\text{STR}: (6x + 3x) \rightarrow /6x + 3x/$.
2. The string is partitioned (sundered) to produce individual characters:
   $\text{SUN}: /6x + 3x/ \rightarrow (/6/,/x/,/+/,/3/,/x/)$.
3. The number objects and letter objects are selected in a type-specific manner and considered separately:
   $\text{SEL}(n): (/6/,/x/,/+/,/3/,/x/) \rightarrow (/6/,/3/)$; $\text{SEL}(l): (/6/,/x/,/+/,/3/,/x/) \rightarrow (/x/,/x/)$.
4. The number objects are treated as numbers, and the letter objects are treated as a set:
   $\text{NUM}: (/6/,/3/) \rightarrow (6,3)$; $\text{SET}: (/x/,/x/) \rightarrow \{/x/_1,/x/_2\}$.
5. The numbers are summed/added, and the cardinality of the set is computed:
   $\Sigma: (6,3) \rightarrow 9$; $\#: \{/x/_1,/x/_2\} \rightarrow 2$.
6. The numbers 9 and 2 are considered as a number object and a superscript object, respectively:
   $\text{STR}: 9 \rightarrow /9/$; $\text{SPS}: 2 \rightarrow /2/$.
7. /9/, /x/ and /^{2}/, the results of the intermediary computations, are selected for inclusion:
   $i_x: /9/ \rightarrow /9/$; $i_x[/x/]: (/6/,/x/,/+/,/3/,/x/) \rightarrow /x/$; $i_x: /2/ \rightarrow /2/$.
8. Finally, /9/, /x/ and /^{2}/ are combined (concatenated) to produce the student's answer:
   $++: (/9/,/x/,/2/) \rightarrow /9x^2/$.

The concatenation of the character strings /9/, /x/ and /^{2}/ is not understood in terms of multiplication and exponentiation, but rather as some form of addition. Note that what the student does is also not an application of the distributive property of the field $(\mathbb{R}, +, \times)$: $6x + 3x = (6 + 3)x = 9x$. They had no exposure to the factorisation of algebraic expressions at the time of the study.

Consider Figure 6.3, which shows the computational details of Student A2-20's solution to Item 1. Their procedure generates $9x$ as an answer—a mathematically correct response. That does not, however, necessarily mean that they used addition in relation to variables in a manner consistent with the field $(\mathbb{R}, +, \times)$. In an interview, Student A2-20 asserted that "x plus x is x". Their idea is that adding two or more copies of a letter object simply produces the letter as a value. In Chapter 4 we

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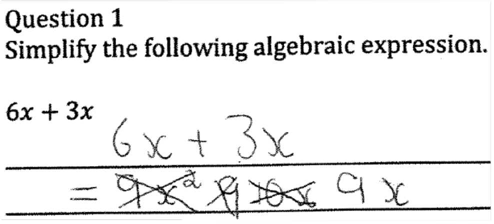
[11] It may be helpful to consider the analogous situation that pertains in our use of ordinary language. Individuals routinely compose coherent, grammatical utterances in speech and/or writing but, unless they have training in linguistics, they are unable to describe and explain linguistic computations they spontaneously use.

described such a mapping as CMP, which takes any list of identical letters as its argument and maps that to a value which is a single copy of the letter. In this instance, CMP: $(/x/,/x/) \rightarrow /x/$.



Question 1
Simplify the following algebraic expression.

$6x + 3x$

$6x + 3x$

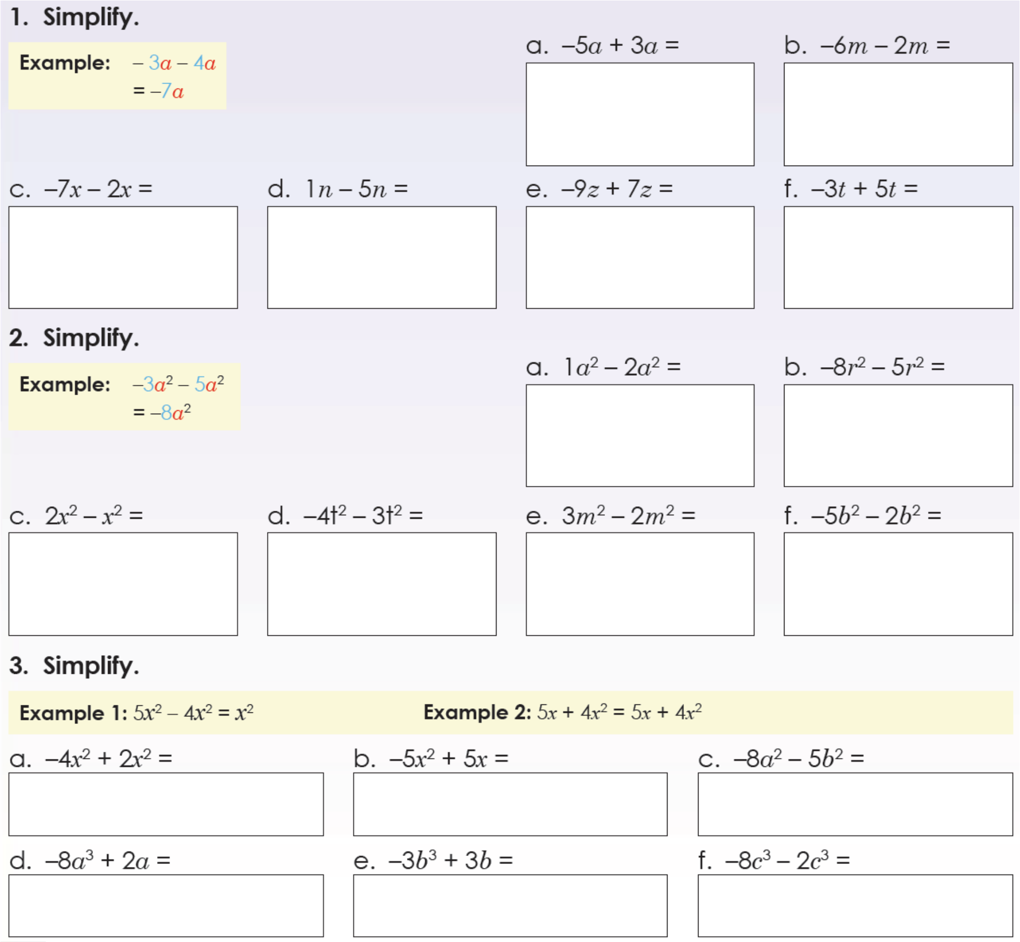
$= \cancel{9x^2}\ \cancel{9}\ \cancel{10x}\ 9x$

$$\begin{tikzcd}
 & 6x + 3x \arrow[d, "\mathrm{STR}"'] & \\
 & /6x + 3x/ \arrow[d, "\mathrm{SUN}"'] & \\
 & /6/, /x/, /+/, /3/, /x/ \arrow[dl, "\mathrm{SEL}(n)"'] \arrow[dr, "\mathrm{SEL}(l)"] & \\
/6/, /3/ \arrow[d, "\mathrm{NUM}"'] & & /x/, /x/ \arrow[ddd, "\mathrm{CMP}"] \\
6, 3 \arrow[d, "\Sigma"'] & & \\
9 \arrow[d, "\mathrm{STR}"'] & & \\
/9/ \arrow[dr, "i_x"'] & & /x/ \arrow[dl, "i_x"] \\
 & /9/, /x/ \arrow[d, "++"'] & \\
 & /9x/ &
\end{tikzcd}$$

Figure 6.3: A computational analysis of Student A2-20's response to Item 1, resulting in $9x$.

Twenty-one students (28%) across Schools A and B who responded to Item 1 with $9x$ did not respond to Item 8 correctly, which required them to simplify $x + 5x + 2x$. A small number of the errors appear to be simple addition errors, so we set those to one side. Three expressions offered as solutions to Item 8 by students are of interest: (i) $7x$; (ii) $7x^3$; and (iii) $8x^3$. Both (i) and (ii) show the addition of coefficients after sundering number objects and letter objects that constitute the terms $5x$ and $2x$. Since the first term of $x + 5x + 2x$, viz. x, does not show a concatenation of a number and a letter object, the students calculate $5 + 2 = 7$ rather than $1 + 5 + 2 = 8$. Further, since each of the three terms contain the letter object x, the students register that fact by means of the expression x^3. The strings $/7/$ and $/x^3/$ are then concatenated to, ultimately, produce $7x^3$ as a 'solution'. With (iii), the ‘solution’ $8x^3$ is produced by concatenating $/8/$ and $/x^3/$. The value 8 is generated by reading the first term of $x + 5x + 2x$ as including an ‘invisible 1’ as a number object (coefficient) and then calculating $1 + 5 + 2 = 8$ after sundering number objects and letter objects.

The use of the CMP mapping is of particular interest because it is a computation that enables the addition of like terms indirectly, without the use of the axiom that specifies the distribution of multiplication over addition in the field of real numbers, $(\mathbb{R}, +, \times)$. Students would need to be cognisant of the rules for identifying like as distinct from unlike terms. The specification of like and unlike terms as presented in DBE approved textbooks glosses like terms as those containing the same variables (letters) with the same exponents (superscripts). Grade 8 mathematics textbooks make mention of commutativity and associativity for adding like terms, but tend not to appeal to the distributivity of multiplication over addition, despite the fact that the Grade 8 curriculum *does* include distributivity: “Use commutative, associative and distributive laws for rational numbers and laws of exponents to: • Add and subtract like terms in algebraic expressions” (DBE 2011b: 92). However, when dealing with the addition of like terms, DBE texts, like the Grade 8 workbooks, do not refer to distributivity explicitly (see Figure 6.4).



1. Simplify.

Example: $-3a - 4a$
$= -7a$

a. $-5a + 3a =$

b. $-6m - 2m =$

c. $-7x - 2x =$

d. $1n - 5n =$

e. $-9z + 7z =$

f. $-3t + 5t =$

2. Simplify.

Example: $-3a^2 - 5a^2$
$= -8a^2$

a. $1a^2 - 2a^2 =$

b. $-8r^2 - 5r^2 =$

c. $2x^2 - x^2 =$

d. $-4t^2 - 3t^2 =$

e. $3m^2 - 2m^2 =$

f. $-5b^2 - 2b^2 =$

3. Simplify.

Example 1: $5x^2 - 4x^2 = x^2$

Example 2: $5x + 4x^2 = 5x + 4x^2$

a. $-4x^2 + 2x^2 =$

b. $-5x^2 + 5x =$

c. $-8a^2 - 5b^2 =$

d. $-8a^3 + 2a =$

e. $-3b^3 + 3b =$

f. $-8c^3 - 2c^3 =$

Figure 6.4: Extract from a Grade 8 DBE Workbook (*Source*: DBE (2024: 68)).

DBE approved Grade 8 textbooks, following the lead of the DBE, tend to present the simplification of algebraic expressions similarly. For example, Botha *et al.* (2013:113) declare that "Adding like terms is very much like adding whole numbers: $4 - 2 + 8 = 10$. Therefore $4a - 2a + 8a = 10a$", and then proceed to discuss the simplification of the expression $-5y + 4y + 6y$ in some detail, first pointing out that $-5y$, $4y$ and $6y$ are like terms, followed by:

$-5y + 4y + 6y \;= -5y + (4y + 6y)$ (Group $4y$ and $6y$ to make the addition easier.)
$= -5y + 10y$ ($4y + 6y = 10y$ because $4 + 6 = 10$)
$= 5y$ (Add $-5y$ to $10y$ in the same way as you would add -5 to 10.)

Had Botha *et al.* (2013) used distributivity explicitly, they could have presented the solution as follows:

$-5y + 4y + 6y = (-5 + 4 + 6)y$ (Since each term is the product of an integer and the variable y, we can use the distributive law.)
$= (-5 + 10)y$ (Using associativity, we can calculate $4 + 6 = 10$ first.)
$= 5y$ (Add $-5 + 10 = 5$.)

The manner in which the notions of *like* and *unlike terms* are used in Grade 8 texts, while clearly helpful, nevertheless facilitates the circumvention of the explicit use of distributivity in processing algebraic expressions, resulting in solution procedures that employ what we might refer to as *indirect distributivity*. The mappings that are enabling of indirect distributivity are those concerned with type-sensitive separation (SUN), the selection of number objects (SEL(*n*)) and of letter objects (SEL(*l*)), rendering numbers as character strings (STR), selecting a single letter from multiple occurrences of a particular letter (CMP), and concatenation $(++)$. However, it is CMP that is crucial to the production of a computational effect that is analogous to extracting a component of a term as a common factor (i.e., distributivity).

For indirect distributivity to function in a manner that generates results consistent with the axiom for distributivity in $(\mathbb{R}, +, \times)$, there must be a structure-preserving mapping (a homomorphism) that maps distributivity to indirect distributivity. In other words, the relation between distributivity and indirect distributivity is a pedagogical representation that facilitates the addition of like terms without having to explicitly employ distributivity. Following Brousseau (1997), we can say that such a mapping is an instance of *didactical transposition*.

Figure 6.5 is a rendition of the structure-preserving mapping between distributivity and indirect distributivity, using the inclusion mapping, i_x, as a mediating mapping.

$$\begin{tikzcd}
 & m\alpha + n\alpha \arrow[rrr, "i_x"] \arrow[ddddd, "\delta"'] & & & m\alpha + n\alpha \arrow[d, "\text{STR}"] & & \\
 & & & & /m\alpha + n\alpha/ \arrow[d, "\text{SUN}"] & & \\
 & & & & /m/,/\alpha/,/+/,/n/,/\alpha/ \arrow[dl, "\text{SEL}(n)"'] \arrow[dr, "\text{SEL}(l)"] & & \\
 & & & /m/,/n/ \arrow[d, "\text{NUM}"'] & & /\alpha/,/\alpha/ \arrow[ddd, "\text{CMP}"] & \\
 & & & m,n \arrow[d, "\Sigma"'] & & & \\
\text{DISTRIBUTIVITY} & (m+n)\alpha \arrow[dddd, "+"'] & & p \arrow[d, "\text{STR}"'] & & & \text{INDIRECT DISTRIBUTIVITY} \\
 & & & /p/ \arrow[dr, "i_x"'] & & /\alpha/ \arrow[dl, "i_x"] & \\
 & & & & /p/,/\alpha/ \arrow[d, "++"] & & \\
 & & & & /p\alpha/ \arrow[d, "\text{TRM}"] & & \\
 & p\alpha \arrow[rrr, "i_x"'] & & & p\alpha & &
\end{tikzcd}$$

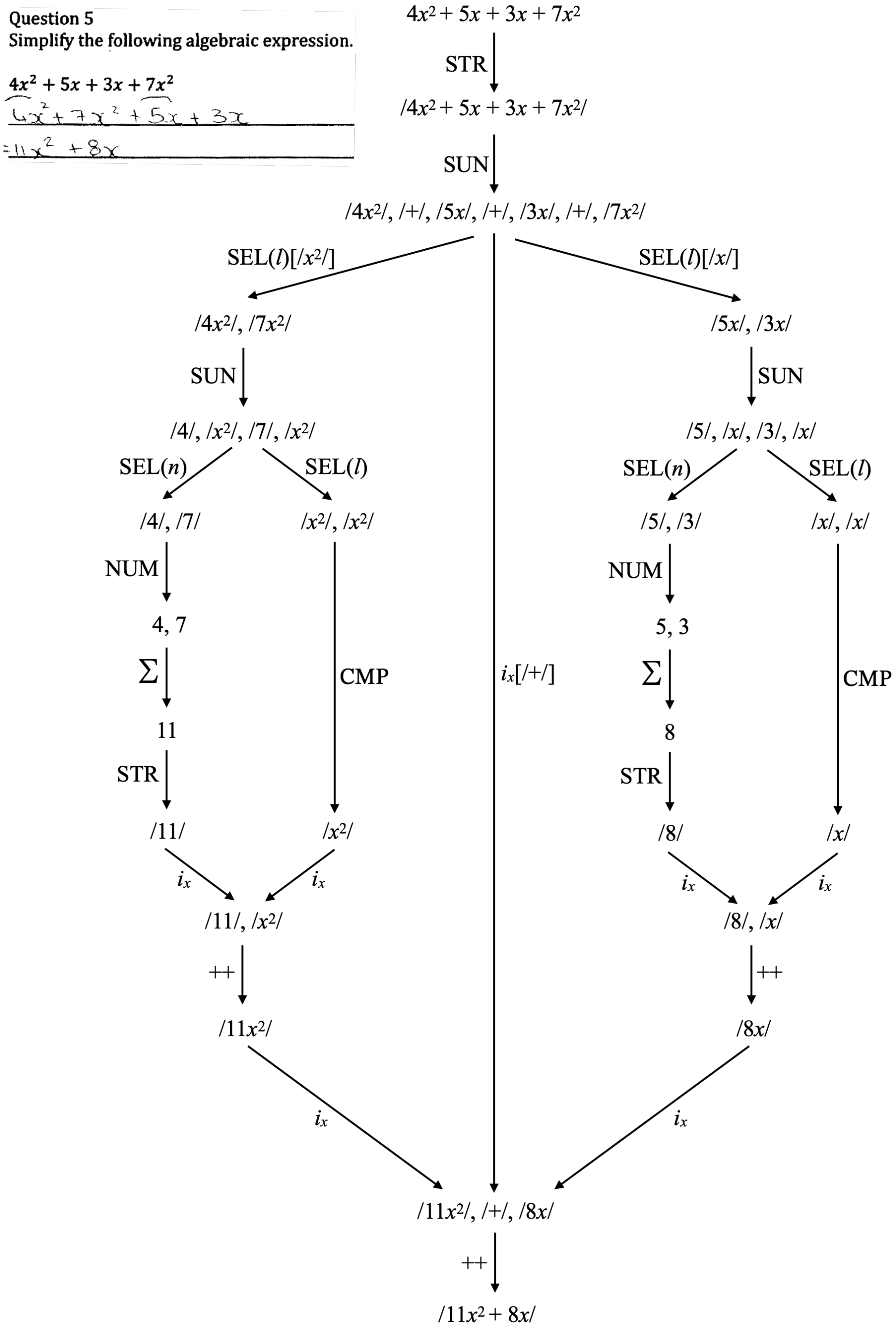
Figure 6.5: A structure-preserving mapping between distributivity and indirect distributivity, where $m, n, p \in \mathbb{R}$ are coefficients, $m + n = p$, and α is a variable defined over $\mathbb{R}$.

Note that the mapping TRM in Figure 6.5, which is used only here, simply indicates the treatment of an expression as a term.

6.3.2 A computational analysis of procedures used in response to Item 5

The computational analyses of responses to Item 5 show much more variation in the proposed solutions than those obtained for Item 1. Table 5.13 in Chapter 5 list all of the responses to Item 5,

$4x^2 + 5x + 3x + 7x^2$, along with their frequencies. The correct answer of $11x^2 + 8x$ (or, $8x + 11x^2$) was offered by only 24% student responses. Using evidence from the interviews, we are able to construct an object-and-arrow diagram illustrating the computations that students might employ to solve the problem, as in Figure 6.6.



Question 5

Simplify the following algebraic expression.

$4x^2 + 5x + 3x + 7x^2$

$4x^2 + 7x^2 + 5x + 3x$

$= 11x^2 + 8x$

Figure 6.6: A computational analysis of Student B2-42's procedure to simplify Item 5, resulting in $11x^2 + 8x$.

Students who are well-rehearsed in the use of such a procedure are able to produce mathematically correct responses, of which Figure 6.6 is an example. However, when students are asked to explain their working in interviews none of them provided satisfactory justifications.

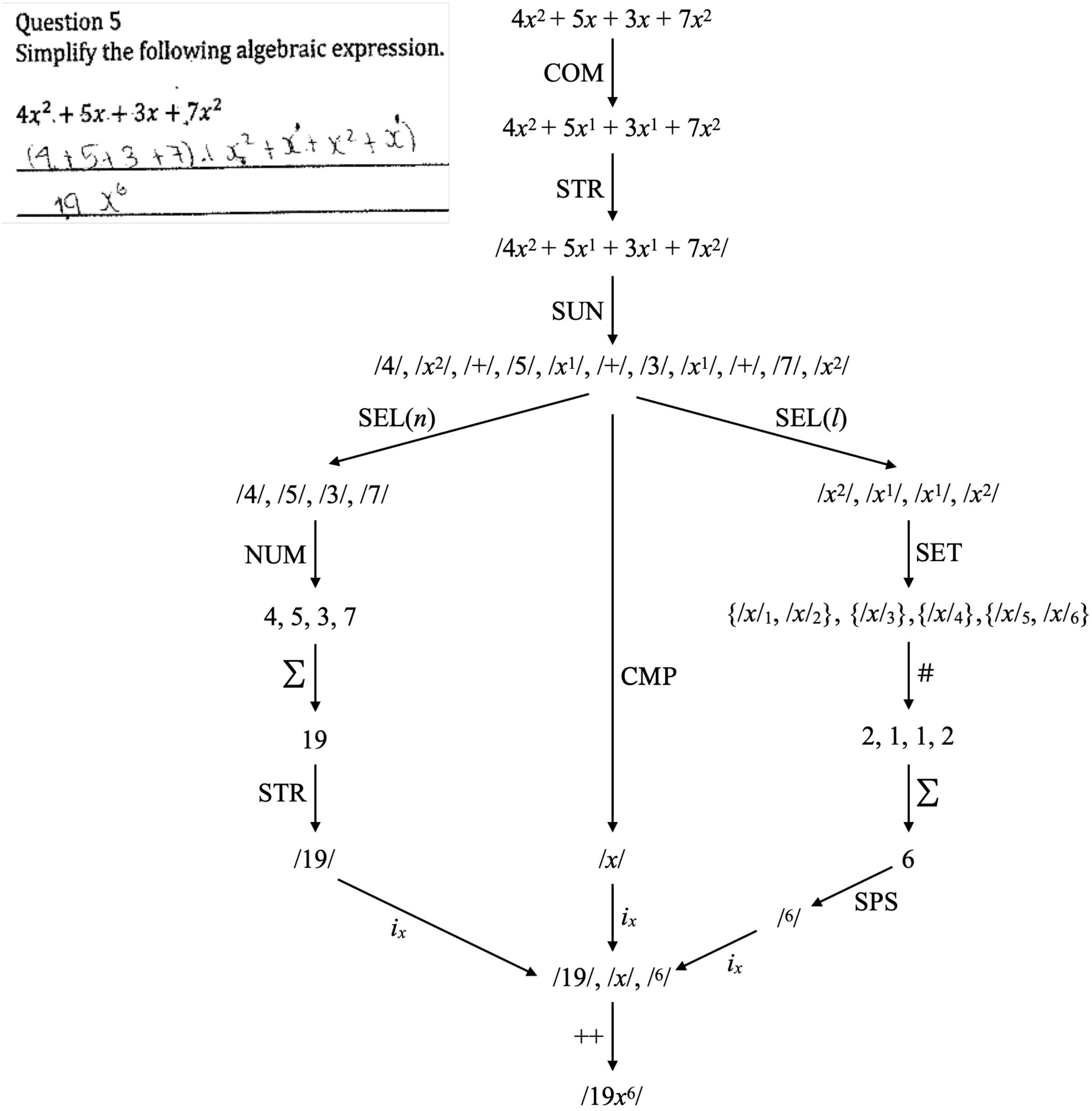
A student may well be aware of the criteria for recognising like terms and also be aware of the method for adding like terms so that their computations would lead to a mathematically correct response, a critical aspect of which is the comparison and selection of the exponents associated with variables. One could consider the variables and associated exponents as markers indicating which numerical values may be added. The method is reinforced by the popular "fruit salad" teaching method discussed in the literature review and found in DBE texts. For example, Task 30 of DBE (2024: 66), titled "Like terms: whole numbers", reminds students that "We can add '3 apples' and '4 apples', but we cannot add '3 apples' and '4 pears'". In the "fruit salad" teaching method, terms of algebraic expressions are to be treated as analogous to combinations of non-mathematical objects like apples and pears. Using the method requires students to mark like terms for the purpose of collecting together groups of coefficients to be added. Such a pedagogic resource enables students to produce correct mathematical outcomes without thinking about the mathematical meaning of algebraic terms, of the operation of adding like terms, and of the equivalence of antecedent and descendant expressions in a sequence of computations.

The expression $19x^6$ was the most prevalent response to Item 5, $4x^2 + 5x + 3x + 7x^2$, observed in 37% of student responses. Figure 6.7 and Figure 6.8 show typical variations in the procedures used by students in response to Item 5 that produce $19x^6$ as a result.

Figure 6.7 details an approach by students in which the superscripts are taken to denote the cardinality of a set containing a number of letters of the same kind—in this case, sets of xs—rather than exponents. Each term of type $/x^1/$ is mapped to a set containing a single element (SET), while each of those of the form $/x^2/$ is mapped to a set consisting of two elements (SET).[12] The cardinalities of the sets are computed by counting (#) and then summed (Σ), the result of which is taken to be a superscript object (SPS). CMP extracts a single copy of the letter x (variable). The type-specific results of the various computations are concatenated ($+\,+$) to produce a proposed solution of $19x^6$.

---

[12] Recall that, for the student, all of the xs remain distinct entities when they are collected together to form a set like $\{x, x\}$, which is not permitted in set notation. One way of dealing with the problem is to use notation to tag each occurrence of x as distinct. Hence the use of subscripts.



Question 5
Simplify the following algebraic expression.

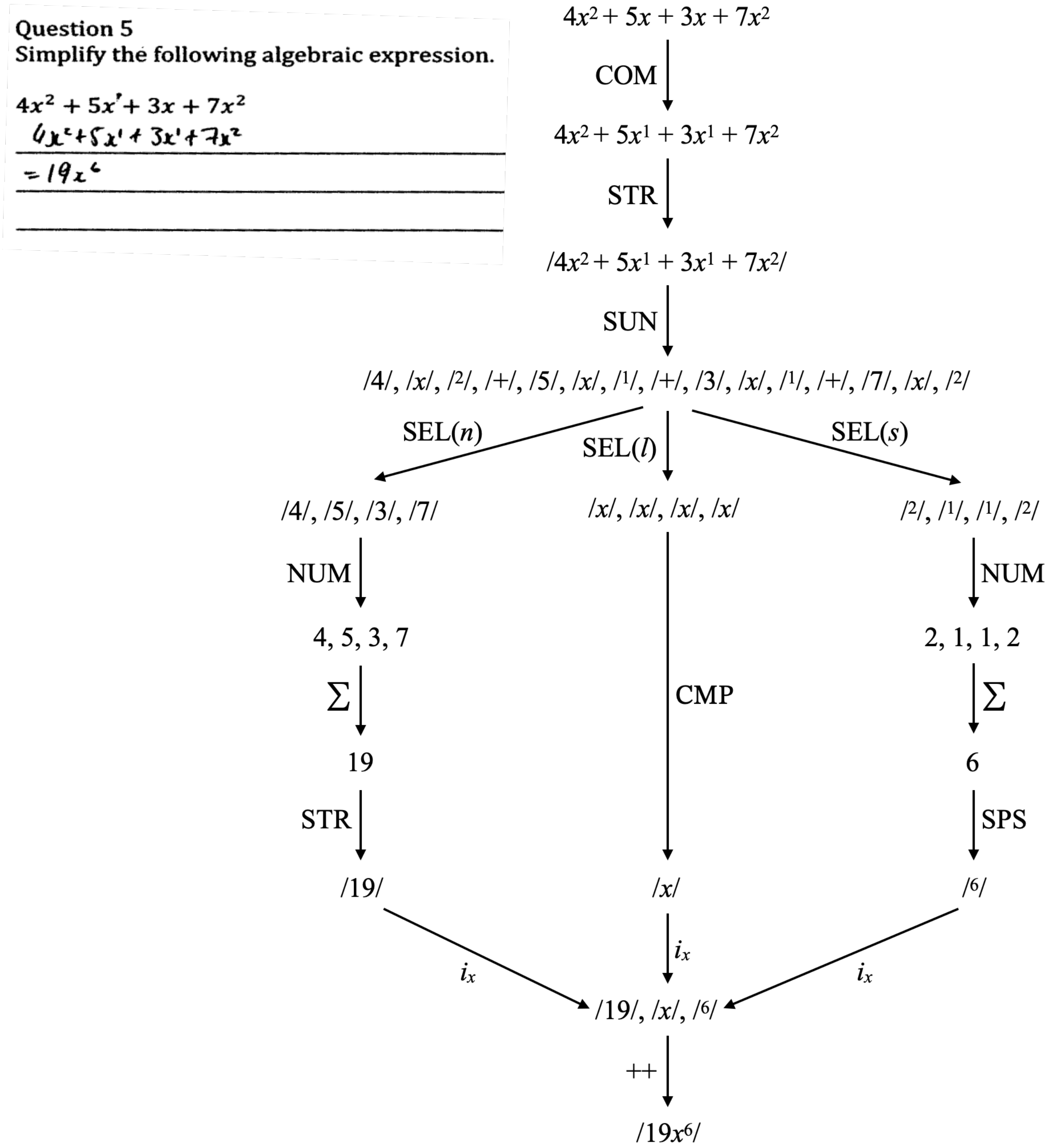
$4x^2 + 5x + 3x + 7x^2$

$(4+5+3+7)(x^2+x^1+x^2+x^1)$

$19x^6$

Figure 6.7: A computational analysis of Student A1-03's procedure to simplify Item 5, resulting in $19x^6$.

Figure 6.8 shows an alternate procedure that produces the result $19x^6$. The method follows a procedure similar to the approach taken in Figure 6.8. As before, the COM mapping 'completes' the expression by attributing an superscript of 1 to each of the x terms. The expression is then sundered into number objects, letter objects and superscript objects. The objects are acted on individually: (i) the numbers are summed to produce 19; (ii) the CMP mapping on the letters returns an x to be included in the final answer because x is the only letter present in this question; and (iii) the superscripts are added. The three results, /19/, /x/ and /^{6}/, are concatenated to produce $19x^6$ once again.



Question 5
Simplify the following algebraic expression.

$4x^2 + 5x^{\prime} + 3x + 7x^2$

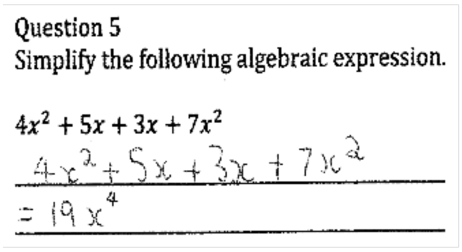
$4x^2 + 5x^1 + 3x^1 + 7x^2$

$= 19x^6$

Figure 6.8: A computational analysis of Student B1-18's procedure to simplify Item 5, resulting in $19x^6$.

A result of $19x^4$ occurred in 11% of the responses to $4x^2 + 5x + 3x + 7x^2$. Figure 6.9 details a student's procedure producing that result. The procedure is very similar to the procedure observed in Figure 6.8. There is one key difference in the procedures that would lead to the result $19x^4$: the student would not apply the COM mapping and would not include the superscript 1 implied by the expressions $5x$ and $3x$. Therefore, the procedure followed is essentially the same, but the superscripts /^{2}/ and /^{2}/ are added to result in a final superscript value of /^{4}/ in $19x^4$.

This is an interesting finding because it shows how certain responses may actually be the result of the same procedure with only one aspect being different. Therefore, there are regularities in the use of the computations employed across different students and schooling contexts despite variation. There are also similarities in the ways in which students understand and use the syntax of algebra that are not necessarily taught as they are prevalent across different classes, teachers and schools.



Question 5
Simplify the following algebraic expression.

$4x^2 + 5x + 3x + 7x^2$

$4x^2 + 5x + 3x + 7x^2$

$= 19x^4$

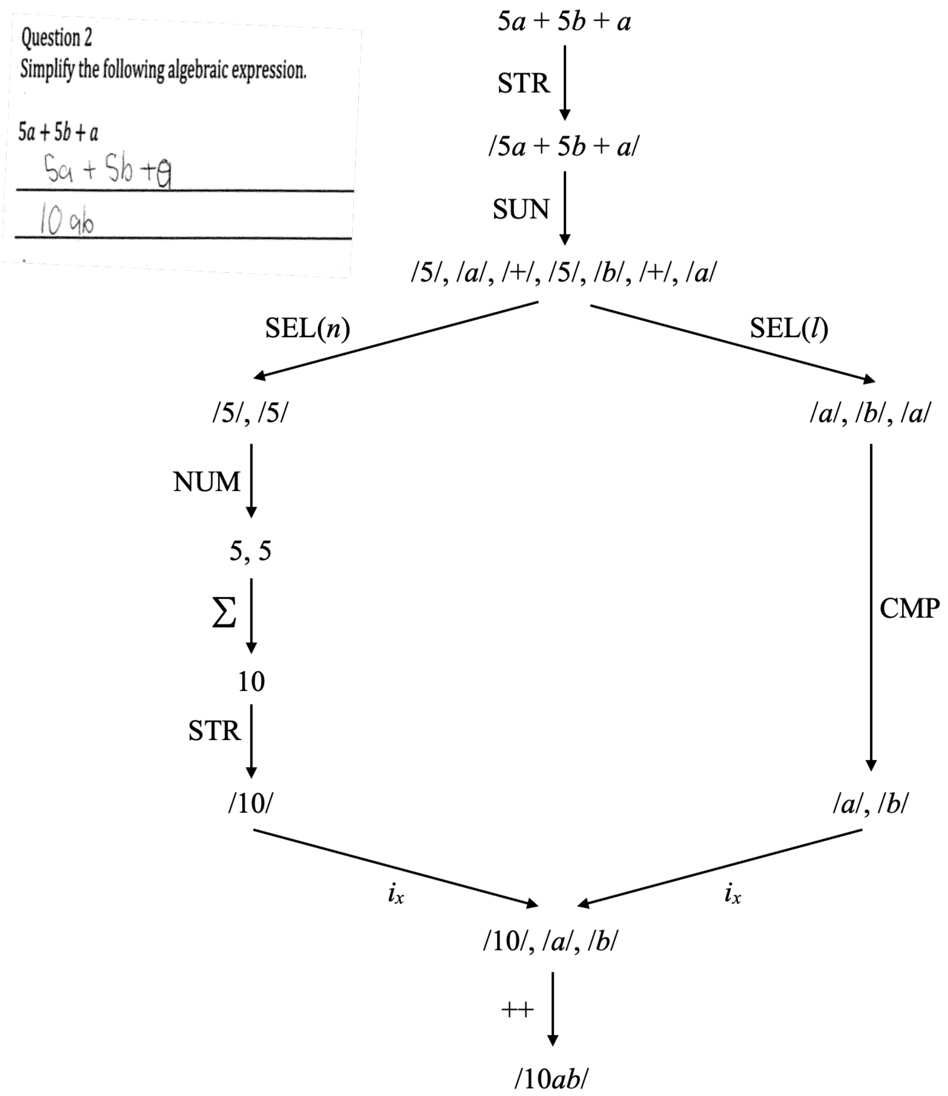
$$\begin{tikzcd}
 & 4x^2 + 5x + 3x + 7x^2 \arrow[d, "\mathrm{STR}"'] & \\
 & /4x^2 + 5x + 3x + 7x^2/ \arrow[d, "\mathrm{SUN}"'] & \\
 & /4/, /x/, /{}^2/, /+/, /5/, /x/, /+/, /3/, /x/, /+/, /7/, /x/, /{}^2/ \arrow[dl, "\mathrm{SEL}(n)"'] \arrow[d, "\mathrm{SEL}(l)"'] \arrow[dr, "\mathrm{SEL}(s)"] & \\
/4/, /5/, /3/, /7/ \arrow[d, "\mathrm{NUM}"'] & /x/, /x/, /x/, /x/ \arrow[ddd, "\mathrm{CMP}"] & /{}^2/, /{}^2/ \arrow[d, "\mathrm{NUM}"] \\
4, 5, 3, 7 \arrow[d, "\Sigma"'] & & 2, 2 \arrow[d, "\Sigma"] \\
19 \arrow[d, "\mathrm{STR}"'] & & 4 \arrow[d, "\mathrm{SPS}"] \\
/19/ \arrow[dr, "i_x"'] & /x/ \arrow[d, "i_x"] & /{}^4/ \arrow[dl, "i_x"] \\
 & /19/, /x/, /{}^4/ \arrow[d, "++"'] & \\
 & /19x^4/ &
\end{tikzcd}$$

Figure 6.9: A computational analysis of Student A2-20's procedure to simplify Item 5, resulting in $19x^4$.

6.3.3 A computational analysis of procedures used in response to Item 2

The computations described in the previous sections may also be applied to Items involving expressions of two or more variables. Once again, the interviews allowed for the identification of the different approaches taken by students.

Figure 6.10 is a description of Student A2-20's attempt at Item 2, that results in $10ab$. Here the CMP mapping recognises that there are two distinct letters (variables) and so one copy of each letter is included in the output of the mapping. There rest of the computations employed have been discussed in relation to responses to other Items and are not rehashed here. $10ab$ was offered as a result in 5% of all responses to Item 2.



Question 2
Simplify the following algebraic expression.

5a + 5b + a

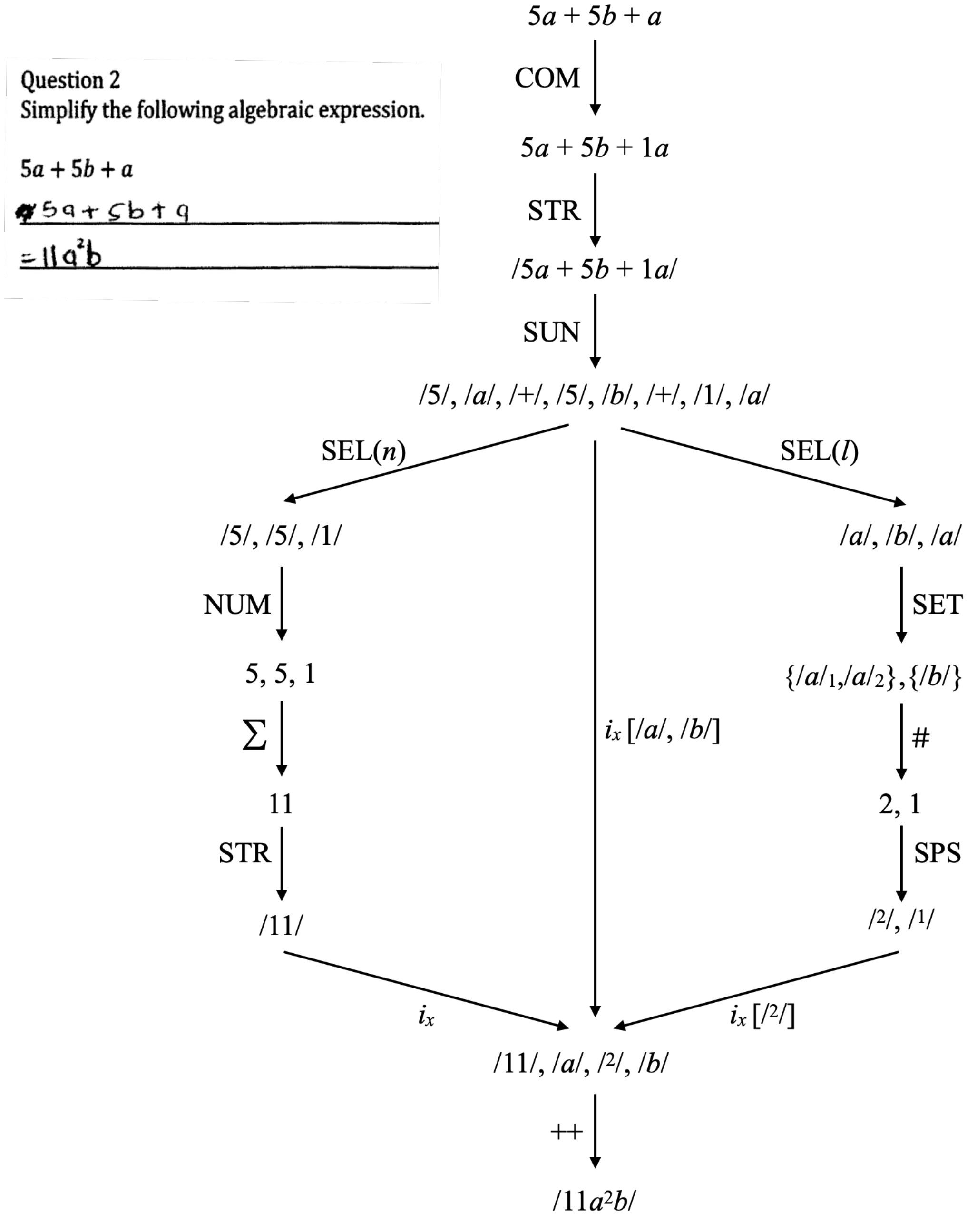
5a + 5b + a

10 ab

Figure 6.10: A computational analysis of Student A2-20's procedure to simplify Item 2, resulting in $10ab$.

The second most prevalent solution to Item 2 was $11a^2b$, one procedure for which is displayed in Figure 6.11, where Student B2-45 counts the number of as and bs (#) after forming sets of letters (SET). The SPS mapping results in the cardinality of the sets being denoted as superscripts, but it is only the superscript 2 that is selected for inclusion in the final result. Figure 6.11 also provides an example of some students' recognition of a coefficient of 1 for the term a, hence the computation

$5 + 5 + 1 = 11$. The idea of the presence of an “invisible one” is a questionable but common feature of school mathematics pedagogies.



Question 2
Simplify the following algebraic expression.

$5a + 5b + a$

5a + 5b + a

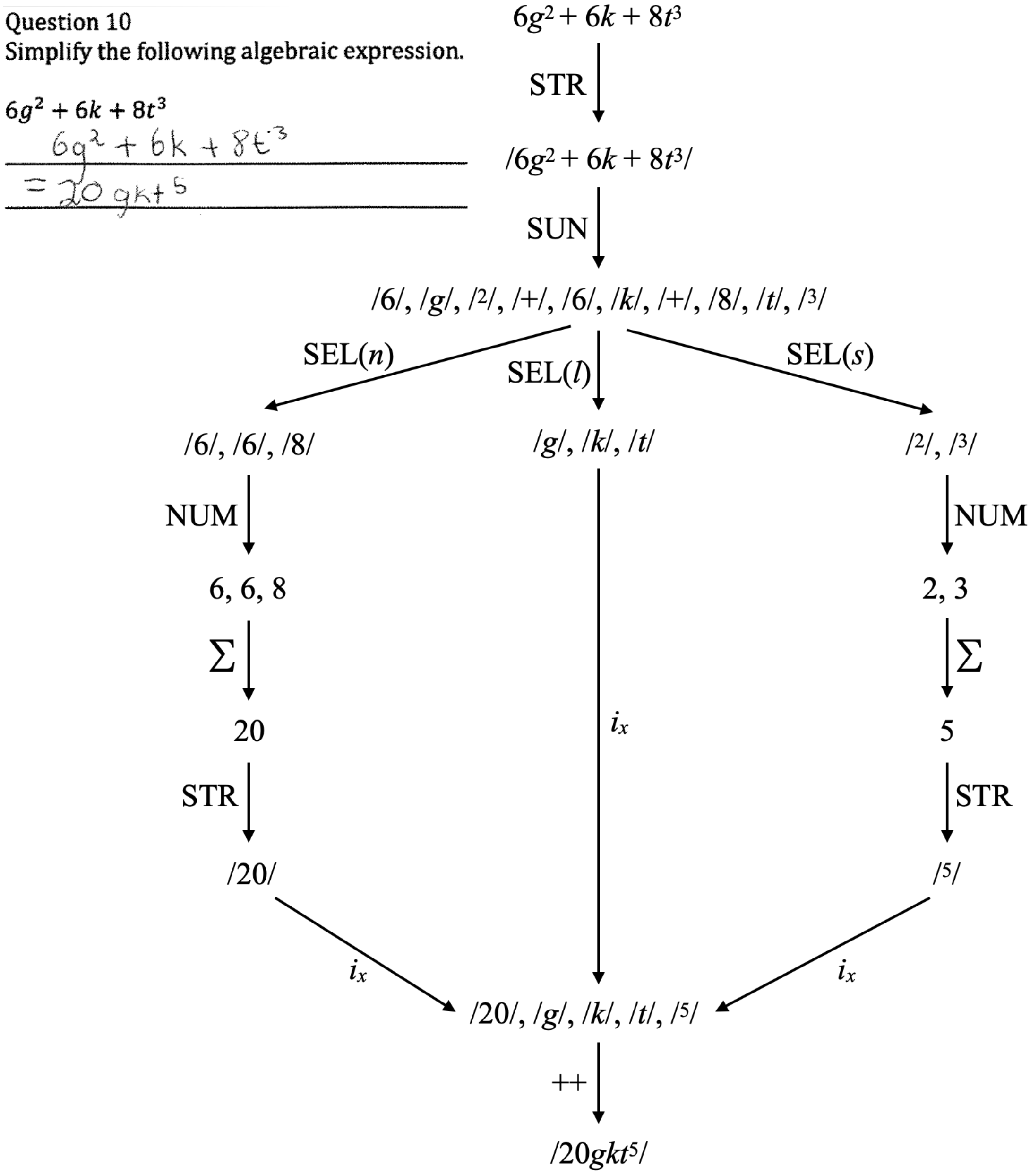
$= 11a^2b$

Figure 6.11: A computational analysis of Student B2-45’s procedure to simplify Item 2, resulting in $11a^2b$.

6.3.4 A computational analysis of procedures used in response to Item 10

The student responses to Item 10, $6g^2 + 6k + 8t^3$, are interesting for a few reasons. First, we see that a relatively large number of students (31 students; 41%) offered the correct response to Item 10. However, we have to take into account the fact that it is common practice for students to write down the initial expression as a first step when attempting to solve mathematics problems. Therefore, a number of the apparently correct responses to Item 10 are likely to be stalled attempts at solving the problem, especially in the cases of students who produced aberrantly conjoined responses to other Items that used expressions with terms of degree two or greater. In fact, 16 of the 31 students who provided correct responses to Item 10 (i.e., 52% of those students) show instances of aberrant conjoining in responses to one or more of Items 1 to 9, indicating that the true percentage of correct responses to Item 10 is much lower that 41% and could be as low as 20%.

Figure 6.12 shows Student A2-20's response to Item 10, demonstrating that the treatment of number objects, letter objects and superscript objects separately is applied consistently and is independent of the number and range of letters used.



Question 10
Simplify the following algebraic expression.

$6g^2 + 6k + 8t^3$

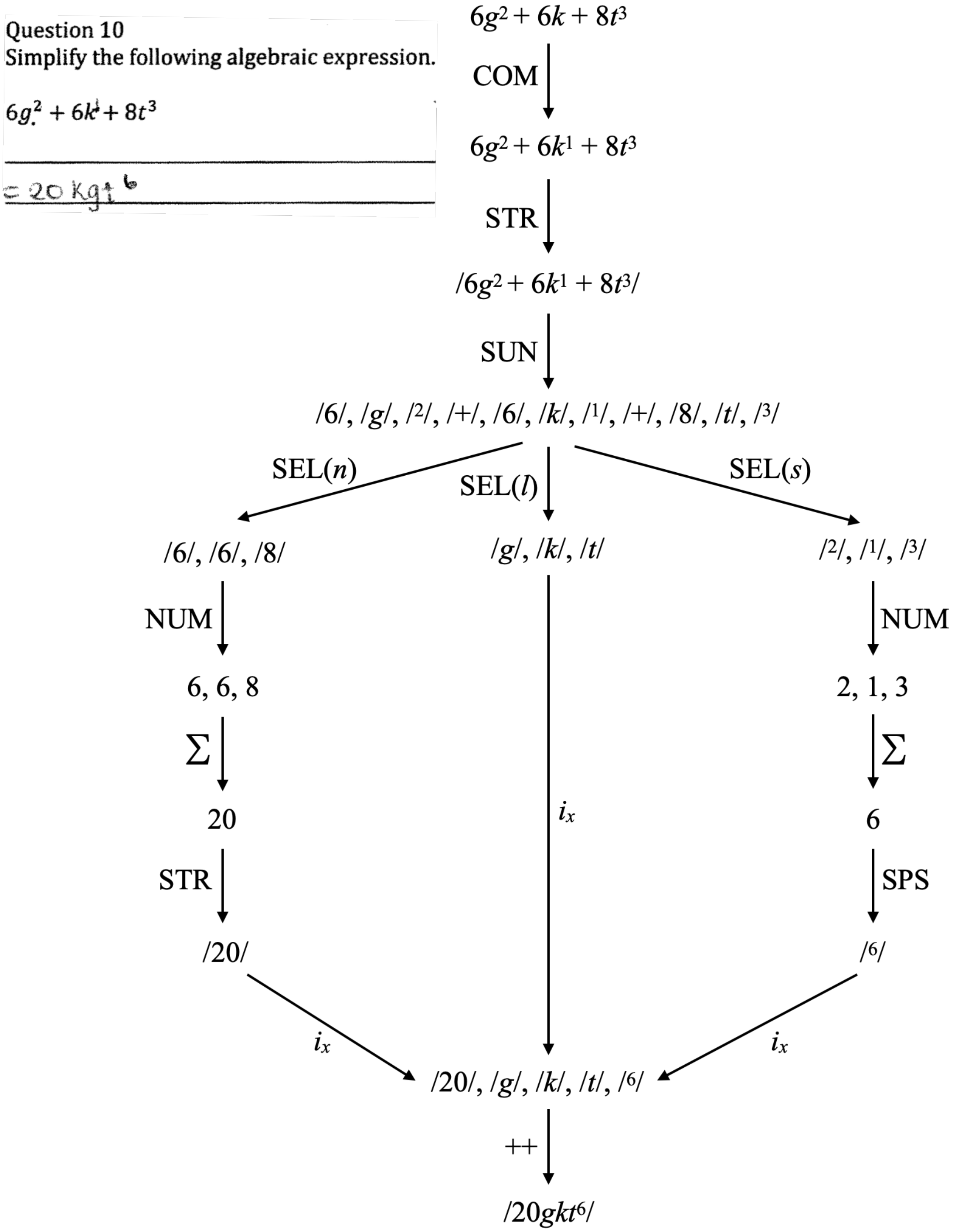
$6g^2 + 6k + 8t^3$

$= 20gkt^5$

Figure 6.12: A computational analysis of Student A2-20's procedure to simplify Item 10, resulting in $20gkt^5$.

In Item 10, the terms are all unlike terms and the student has continued to apply their general procedure for simplifying algebraic expressions, which results in aberrant conjoining.

Figure 6.12 shows the procedure used by Student B2-45, which a slight variation on that used by Student A2-20.



Question 10
Simplify the following algebraic expression.
$6g^2 + 6k^1 + 8t^3$
$= 20 Kgt^6$

Figure 6.13: A computational analysis of Student B2-45's procedure to simplify Item 10, resulting in $20kgt^6$.

Unlike Student A2-20, Student B2-25 recognises that a superscript of value 1 is to be associated with the letter object of the term $6k$. Such students tend to follow the same procedure but their final answer will have superscripts of 6 and not 5, as shown in Figure 6.13. A response of $20gkt^5$ was observed in 7% of all responses, while a response of $20gkt^6$ was observed in 5% of all responses.

6.4 Concluding remarks

The computational approach used to analyse the students' procedures of conjoining terms in the simplification of these expressions has allowed for the identification of four computational principles that contribute to the aberrant conjoining of terms. The analyses also revealed that the distributivity of multiplication over addition, which is necessary to 'add like terms', is replaced by indirect distributivity, implicitly guaranteed by the existence of a representation (structure-preserving mapping) relating distributivity and indirect distributivity in the work of students irrespective of the presence of aberrant conjoining.

Which justifications do students provide when carrying out procedures that lead to mathematically correct answers? What are the aspects of procedures that lead to mathematically correct answers? In the following section, we outline the justifications that students provide for the procedures that have resulted in mathematically correct responses. Additionally, an explication of the conjoining phenomenon as it has appeared in this study is proffered.

CHAPTER 7

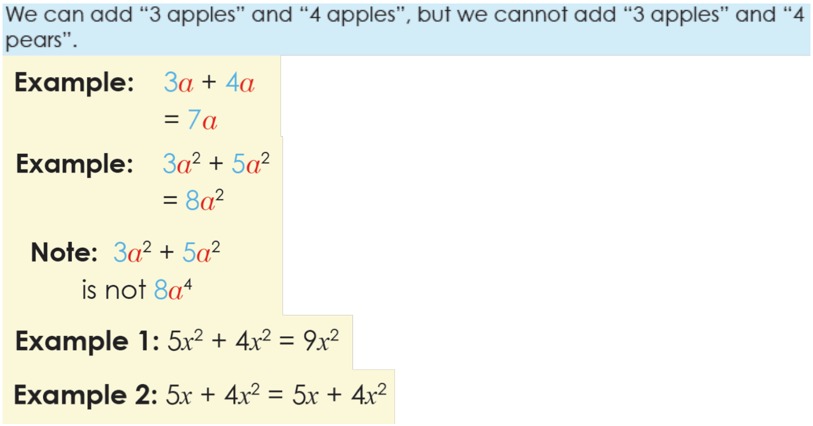
TOWARDS A ROBUST EXPLICATION OF ABERRANT CONJOINING

7.1 Introduction

The computational analyses conducted in Chapter 6 enabled the identification of four general principles used by students who use aberrant conjoining when simplifying algebraic expressions. In this chapter we consider student justifications of procedures that lead to mathematically correct responses as well. We offer further clarification on the phenomenon of aberrant conjoining as it emerged in the data. Lastly, we discuss the limitations of this study and suggest avenues for further investigation.

7.2 Students who produced mathematically correct responses using questionable procedures

Students who answered most Items correctly were interviewed in order to gain insights into their justifications of the computations and procedures used in generating mathematically correct responses. We observed that more successful students focus on criteria for distinguishing like from unlike terms in a manner that uses the composition of the original terms of an expression to regulate computational decisions rather than using type-sensitivity only to constitute type-specific sets. Their strategies were in line with the general way in which like terms and unlike terms are dealt with in DBE approved texts, where the letter objects and superscript objects act as signifiers indicating which terms are to be added. Recall that in Section 6.3.1 of Chapter 6 we showed that DBE approved texts implicitly employ the use of indirect distributivity to effect the addition of like terms. Figure 6.4 of Chapter 6 showed examples demonstrating the addition of like terms, using different colours to distinguish number objects (coefficients; blue) from letter objects (variables; red), extracts of which are shown here in Figure 7.1.



We can add “3 apples” and “4 apples”, but we cannot add “3 apples” and “4 pears”.

Example: $3a + 4a$
$= 7a$

Example: $3a^2 + 5a^2$
$= 8a^2$

Note: $3a^2 + 5a^2$
is not $8a^4$

Example 1: $5x^2 + 4x^2 = 9x^2$

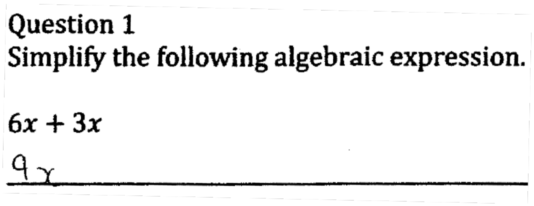
Example 2: $5x + 4x^2 = 5x + 4x^2$

Figure 7.1: Examples used in Task 30 of a Grade 8 DBE workbook (*Source*: DBE (2024: 66)).

Figure 7.1 does indicate that $3a^2 + 5a^2 \neq 8a^4$, but fails to provide a mathematical explanation of why that must necessarily be so. Further, the CAPS curriculum document (DBE 2011b: 93) warns against the misconception but does not explicitly show that distributivity might be used to obviate the problem: "$x^2 + x^2 = 2x^2$ **AND NOT** $2x^4$" (emphasis in the original).

Student B2-42 was the only student at School B to offer mathematically correct responses to all of the test Items. Student B2-42 was able to competently distinguish between like and unlike terms, and was aware of the rules for simplifying algebraic expressions.

> **Researcher:** [00:00:01] Okay. So, why don't you tell me about this first item?



Question 1
Simplify the following algebraic expression.

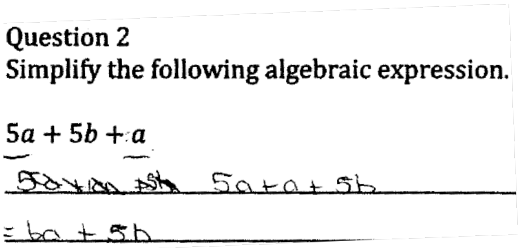
$6x + 3x$

$9x$

> **Student B2-42:** [00:00:12] In math class here by this item, they say if the variable, if this variable here and exponents are the same, they like terms then you can add them. But if it's multiplication you can, you add them and then exponents that will be there, you add them also. So, here's no exponent and the variables are the same. So, I can just add them and I'll get the answer.
>
> **Researcher:** [00:00:35] Okay. So how did you get the nine?
>
> **Student B2-42:** [00:00:38] I added the 3 and the 6 and then bring down the x, because in plus you don't add or subtract the exponents from the variables.
>
> **Researcher:** [00:00:47] Okay, so that's, that's the first one okay. Now, before we move on, what do you understand by simplify? What does that question actually tell you to do?
>
> **Student B2-42:** [00:00:55] To get the answer to the simplest form that you can get it in so that you can understand it better.
>
> **Researcher:** [00:01:00] Okay. And so how does this answer relate to the question? Like, like, what's the relationship between the $9x$ and the $6x$ plus the $3x$?
>
> **Student B2-42:** [00:01:14] The variable here stand in the place of a number that you don't know, that's why the variable is there.
>
> **Researcher:** [00:01:19] Okay.
>
> **Student B2-42:** [00:01:19] So, now when you add that [$6x + 3x$] and you get that answer [$9x$] this is actually multiplication. Like my Miss said now, so then if you find the answer, they call it solve for x like we're doing now. And then you will find the answer of x. And then that nine will be multiplied by x. And you'll get the final answer. But because now there is no multiplication you just get $9x$.

The use of CMP to effect indirect distributivity is clear in the work of Student B2-42: "I added the 3 and the 6 **and then bring down the x**, because in plus you don't add or subtract the exponents from the variables" (my emphasis).

> **Researcher:** [00:01:42] Okay. All right. Let's quickly see the next one. This one is a little bit different hey. What's the difference between this one [Item 2] and that one [Item 1]?



Question 2
Simplify the following algebraic expression.

$\underline{5a} + 5b + \underline{a}$

$5a + a + 5b$

$= 6a + 5b$

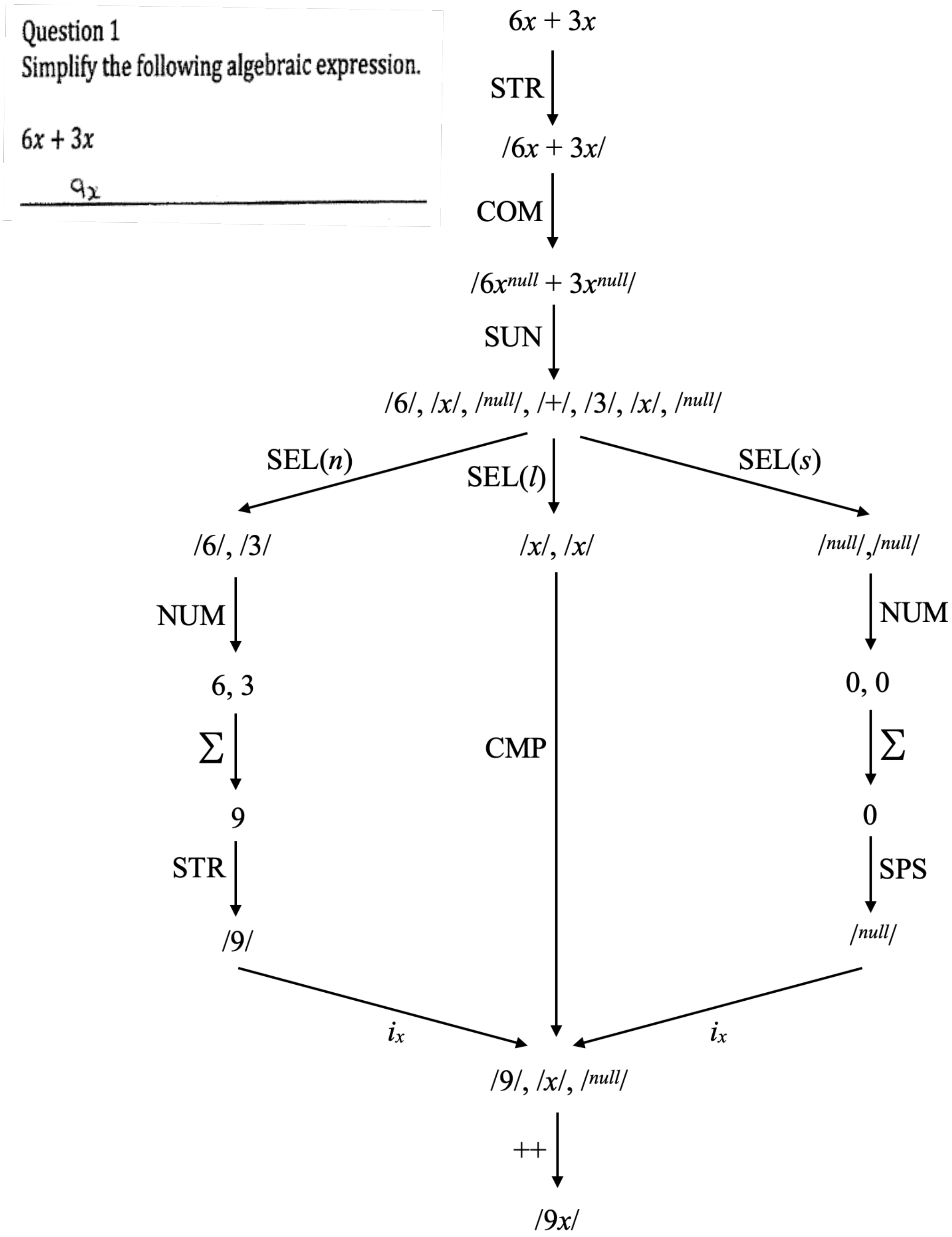
> **Student B2-42:** [00:01:52] In this one there's an unlike term. There's one that's different to the others. Now, just because there isn't a number in front doesn't mean it's the same because they say to make it to be like terms, the variable and the exponent will be the same so the number doesn't count. So, this five I add with that a and in front of this a there's an invisible one. And that $5a$ plus that a is $6a$. Then I just add the $5b$, because this is unlike term. We don't know what's there but that b and we can't add it with the a. If it was multiplication it would be different.
>
> **Researcher:** [00:02:24] Okay. And can you tell me why can't we add this?
>
> **Student B2-42:** [00:02:28] We can't add this [$6a + 5b$] because that b and that a they not the same. They're not the same variable. So we can't add them because we don't know what it is. It could be anything. So, we just write it like that because we don't know what it is.

Student B2-42 appears to have an incipient appreciation of variables defined over a domain, even though they articulate that intuition in a mathematically imprecise fashion: “Then I just add the $5b$, because this is unlike term. We don't know what's there but that b and we can't add it with the a. [...] that b and that a they not the same. They're not the same variable. So we can't add them because we don't know what it is. It could be anything”.

The case of Student B2-42 shows us that students who have a somewhat under-developed understanding of the addition of like terms can nevertheless produce mathematically correct answers by using the relation between variables and their exponents to regulate their perceptions of like and unlike terms.

However, we can also show how procedures that lead to aberrant conjoining when processing expressions that include non-linear terms could be applied to linear expressions and produce mathematically correct responses. While the approach of collecting the number of xs lead to an answer of $9x^2$ in a number of responses to Item 1, the approach of processing individual objects (number objects, letter objects and superscript objects) in a type-specific fashion could lead to an answer of $9x$.

Figure 7.2 demonstrates how the questionable method used by students in response to Item 5 could be applied to Item 1. I have used *null* to denote a null superscript (exponent). Here, students do not account for the superscript value of 1 that can be indicated for a term in a linear expression in the context of addition. The method results in the mathematically correct response of $9x$. Consequently, the result of 70% correct responses for Item 1 is probably misleading because the incorrect method could be applied to Item 1 and it would result in the correct response.



Question 1

Simplify the following algebraic expression.

$6x + 3x$

9x

Figure 7.2: A potential procedure to simplify Item 1, resulting in the correct answer of $9x$, using a null ‘exponent’.

Furthermore, the fact that only 47% of students were able to solve Item 8, $x + 5x + 2x$, correctly strengthens our suspicion that the result for Item 1, $6x + 3x$, (70%) is not a reliable indication of student competence. The computational analysis shown in Figure 7.2 could account for the vast disparity between the percentage of correct responses between linear expressions and non-linear expressions, specifically for those students who routinely used superscripts as cardinalities of sets made up of copies of associated letters.

Teachers do need to be aware of such subtle computational misconceptions and interrogate the methods used by students more carefully, even when the latter produce apparently correct responses to problems. Here, a correct response is achieved by use of a procedure that is grounded in a structure-preserving mapping between the represented system of addition over products of real numbers and the representing system of algebraic notation, as implied by Figure 6.5 of Chapter 6. However, the same procedure fails to produce a mathematically correct response in the simplification of expressions with non-linear terms.

7.2 A more robust explication of aberrant conjoining

The description of aberrant conjoining as the "incorrect putting together of terms" (Subramaniam 2018:45) can be expanded to include more detail on the various ways in which aberrant conjoining is generated. It may even be necessary to include mathematically correct responses in such discussions because a correct answer may be obtained through the application of a mathematically compromised principle or approach, as was demonstrated in Section 7.1.

The students use counting as a main resource in the simplification of algebraic expressions, whether counting the number of elements in a set containing letters, or adding the number objects (coefficients) and adding the superscript objects (exponents). The concatenation of number objects, letter objects and superscript objects in the final computation of procedures is consistently applied, which lends support to the proposition that aberrant conjoining is in part due to genetically endowed knowledge of number that predisposes such a response in the absence of appropriate mathematical knowledge. It appears that the students do not have access to the knowledge required to think about the addition of algebraic terms in the way that we would hope. Therefore, it is likely that they rely on an innate, hence intuitive, cognitive resource, *merge*, that predisposes aberrant conjoining in responses as seen in the examples presented. The various object-and-arrow computational analyses discussed clearly demonstrate how students view concatenation in a manner consistent with *merge* in the final conjoined response.

We cannot appeal to a student's *non-acceptance of lack of closure* (Collis 1974 in Chalouh & Herscovics, 1988) as a necessary justification for aberrant conjoining since it is often present in solution attempts where the final expression of a computation has been left 'open' (see Table 7.1).

Table 7.1: All instances of ‘open’ student responses that show aberrant conjoining.

Item	Expression	‘Open’ responses showing aberrant conjoining	#	Total
1	$6x + 3x$	–	0	0
2	$5a + 5b + a$	$5a^2 + 5b$	1	
		$6a^2 + 5b$	1	2
3	$3n + 4$	–	0	0
4	$4a^2 + 3a^2 + 7a$	$7a^4 + 7a$	5	5
5	$4x^2 + 5x + 3x + 7x^2$	$11x^4 + 8x$	3	
		$11x^4 + 8x^2$	4	7
6	$7kb + 4b + 3bk + 5kb + 4k$	$19kb^2 + bk + k$	1	
		$12kb^2 + bk^2$	1	
		$12k^2b^2 + 4b + 3bk + 4k$	2	
		$23kb^1b^1bk^1kb^1 + b$	1	
		$12k^2b^2 + 11bk$	1	
		$12kb^2 + bk^2$	1	
		$15kb^3 + 4b + 4k$	1	
		$12kb^2 + 11bk^2$	1	
		$15b3k3 + 4b + 4k$	1	
		$12kb2 + 3bk + 4k + 4b$	1	11
7	$3y^2 + 4y + 1 + 5y + 7y^2 + 8$	$19y^6 + 9$	3	
		$10y^4 + 9y^2 + 9$	2	
		$9y^4 + 1 + 8$	1	
		$10y^4 + 9y + 9$	3	
		$10y^8 + 9y^2 + 9$	1	
		$10y^2 + 18y$	1	11
8	$x + 5x + 2x$	–	0	0
9	$9a^2r + 4r^2 + 5a^2 + 3ar^2 + 7a^2r$	$34 + a^2r^2 + a^4 + 4r^2 + 3ar^2$	1	
		$11a^4r^2 + 4r^2 + 3ar^2 + 11a^4r$	1	
		$19a^5r^4 + 11a^4 + 4r^2$	1	
		$16a^2r + 4r^2 + 11a^4 + 3ar^2$	1	
		$16a^4r + 4r + 11a^4 + 3ar^2$	1	
		$16a^2r + 3ar^3 + 4r^2 + 11a^2$	1	
		$16a^4r + 11a^2 + 3ar^2 + 4r^2$	1	
		$16a^4r^2 + 4r^2 + 11a^4 + 3ar^2$	1	
		$27a^2r + 5a^2r$	1	
		$13a^4r + 4r^2 + 11a^2 + 3ar^2$	1	
		$19a^5r^4 + 4r^2 + 11a^4$	1	
		$27a^9 + 4r^2$	1	
		$20a^6 + 10a^3r^3 + 4r^2$	1	
		$17a^4 + 4r^2 + 11a^4$	1	
		$16a^4r + 3ar^3 + 4r^2 + 5a^2 + 6a^2$	1	
		$20a^4 + 14a^2r^2$	1	16
10	$6g^2 + 6k + 8t^3$	$12g2k + 8t3$	1	
		$18gk2 + 8t3$	1	
		$36g + 6k + 512t$	1	
		$20g2 + k + t3$	1	4
			Total	56

Given that there are fifty-six instances of ‘open’ solutions exhibiting aberrant conjoining in our data, it appears that many students are able to accept ‘lack of closure’ but still use computational methods that result in aberrant conjoining.

Returning to the discussion of the *exponent error*, discussed in Section 2.5, a few comments can be made. There are two broad principles employed by such students when dealing with superscripts. The first principle concerns the interpretation of superscripts associated with letters as an indicator of the cardinality of a set containing such letters. The second principle concerns the perception of superscripts as numbers defined by their spatial orientation and size (raised and in smaller typescript than coefficients).

Prior research has tended to distinguish ‘exponent errors’ from ‘conjoining errors’. However, our analysis suggests that where students interpret exponents as cardinal values of sets of identical letters, which they combine with other sets of the same kind, they are really producing an aberrant conjoining of variables. Pournara *et al.* (2016) argue that a computation like, for example, $2a + 5a = 7a^2$ is an instance of an ‘exponent error’. From our analytic take on such a computation we argue that each instance of a is implicitly mapped to the set $\{a\}$, followed by the disjoint union $\{a\} \sqcup \{a\} = \{a, a\}$, which has a cardinality of 2, which is in turn mapped to a^2. The computation $2a + 5a = 7a^2$ entails an ‘exponent error’ in only the superficial sense of the student not using the exponential notation as expected. More significantly, the student aberrantly conjoins the variables in the expression and indicates the result of such as a^2.

For some students, superscripted numbers are simply to be added, and what we have is students adding together numbers of a specific type. As before, such an interpretation is not consistent with an understanding of the superscripts as exponents. For example, Student A2-20 produced the computation $6g^2 + 6k + 8t^3 = 20gkt^5$, which they explained as follows:

> “I added six plus six, plus eight.” (Adding the coefficients/number objects.)
> “I added g plus k plus t.” (Concatenating the variables/letter objects.)
> “I added two plus three.” (Adding the explicit exponents/superscript objects.)

Students conceive of algebraic terms as a concatenation of its constituents, which *is* an appropriate interpretation from a typographical point of view. The constituents of a term are the coefficients (number objects), the variables (letter objects), and associated exponents (superscript objects).

Number objects, letter objects and superscript objects are *merged* to form a term. Following the foundational principle of *merge* as described by Chomsky (2015, 2023), it is unsurprising that many students interpret terms typographically rather than mathematically—that is, failing to recognise that terms of the type $6g^2$ are to be interpreted as composed by multiplication and exponentiation, and not by concatenation, $/6/{+}{+}/g/{+}{+}/^{2}/$. When such students are required to simplify algebraic expressions their first move is to *unmerge* each term, collect the results of unmerging in a type-specific manner, performing type-specific addition and/or addition-like computations, and finally merging the results of those computations (concatenation; ++).

Finally, as we demonstrated in Chapter 3, aberrant conjoiners are, essentially, using a system of inter-related additive monoids (addition over natural numbers; disjoint union over discrete sets; and concatenation of lists) which has the biologically endowed operation *merge* as its foundational monoid. This result is important because it begins to offer us explanatory insights into why it is that students produce the computations we have observed when they generate aberrantly conjoined responses to algebraic simplification.

7.3 Limitations to the study

One obvious limitation of the study is that the test was intentionally designed to focus on the way in which students simplify algebraic expressions that included only the addition of terms. The main reason for that choice was to provide greater clarity and eliminate the interference of the introduction of distractions, such as the subtraction of integer values, which is known to be a challenge for students. However, there are many similar investigations that have reported on algebraic expressions containing subtraction or negative coefficients. A few of these examples are shown in the Table 7.2.

Table 7.2: Examples of simplification including subtraction

	Source	Item and response
Example 1	Demby (1997:55)	$2x^2 - x - 5x^2$ $= 4x - x - 25x$ $= -22x$
Example 2	Liebenberg (1997:6)	$5x - x = 5$ $4x - x = 4$
Example 3	Liebenberg(1997:6)	$-3x + 2x = -5x$ $-6x + 3x = -11x$
Example 4	Lim (2015:149)	$-6a + 3a = -9a$

Example 2 shows how students would “take away” an x from a set and this would produce zero xs remaining in the set. Examples 3 and 4 demonstrate student difficulties with adding negative integers. The introduction of negative integer values and subtraction allows for more diversity in responses which was intentionally avoided in this study.

Given the restrictions on space, it was possible to interview only a small selection of students rather than the entire sample. No doubt more extensive interviews would have enabled us to gain deeper insights into the computational resources used by students.

Another limitation is that the study contained a relatively small sample size due to constraints with the involvement of students in a study of this nature. Furthermore, the introduction of two schools and four different teachers meant that the sample was quite diverse. This may be regarded as a strength or a weakness, though. The current study only focuses on the student responses and does not consider the teaching of the topic. The instances of aberrant conjoining were present in both schools across all four classes. Therefore, specific learning environments changed across the four classes, yet conjoining was observed in all contexts. Our argument is strengthened by the fact that the learning environments of the students differed across the four classes. Since learning environments differed, it would make sense that the phenomenon of conjoining is related to an innate knowledge of number, which is bound up with *merge*. The genetically endowered knowledge of number favours a conjoined response in a certain manner irrespective of teaching and learning contexts.

This study has revealed interesting aspects of the nature of aberrant conjoining and allowed us to consider ways to address the problem. A method of teaching that would counter the innate and overpowering urge that students have to merge terms must be considered. What does this finding mean for teaching and learning? How can this study help us to improve the teaching strategies employed when teaching the simplification of algebraic expressions? Our hope is that clarity on the motivations for aberrantly conjoining terms would provide guidance on how to approach the simplification of algebraic expressions within the classroom.

7.4 Concluding remarks

This study adopted a rationalist view of knowledge acquisition, and the analytic framework accepts that there exist a core domain knowledge structures concerning number that favour simple concatenation of objects. A computational approach was used to identify and analyse the objects and operations in circulation when students apply their own procedures for simplifying algebraic expressions.

Wynn's (1992) experiment with babies demonstrated that they possess innate knowledge of number that enables them to understand the addition of small quantities. Such innate knowledge of number influences the ways in which students approach the simplification of algebraic expressions. The position taken in this study is strengthened by the omnipresent and pervasive nature of the conjoining phenomenon. Students from different classes and school all exhibited similar answers to the questions. Therefore, it is unlikely that the phenomenon stems purely from misconceptions transmitted by teaching. In addition, Chomsky's (2015) conception of *merge* facilitates the formation of aberrant conjoining. Lastly, those who are successful at simplifying algebraic expressions focus on the way in which the algebraic terms are expressed. The students are acutely aware of the criteria for like and unlike terms as it is presented in the DBE approved textbooks, which enable them to use indirect distributivity to produce results consistent with axioms for the field of real numbers. Like and unlike terms are described by referring to the variables and exponents within a term. Therefore, student procedures are based on the syntactical features (the concatenation of coefficients, letters and superscripts) of the terms and not on the properties of operations described over the real numbers.

The omnipresence of the phenomenon of aberrant conjoining has intrigued many researchers. Booth (1981:30-31) suggests that the conjoining of algebraic terms is most likely a result of students applying "child-like" methods related to addition which is modelled as the joining of two or more sets in their early education. Since then, many researchers have presented arguments for the phenomenon of conjoining that range from reasons involving the non-acceptance of open expressions (failure to accept of lack of closure) to the transfer of meanings of the concatenation of symbols from other subject areas and, most commonly, the misapplication of the rules for the multiplication of algebraic terms.

The description of conjoining as it occurs within this study has been developed through a computational analysis of the procedures employed by students. The computational analysis allowed us to identify the objects and operations/mappings that have led to aberrant conjoined responses. The analysis has also highlighted the need to include a discussion of the distributivity of addition and multiplication in the teaching of the simplification of algebraic expressions. This needs to be made explicit in the teaching of the addition of like terms so that students do not circumvent the mathematical concepts involved in the simplification of algebraic expressions. Students are able to produce in a mathematically correct response through carrying out procedures that result in an indirect distributivity.

Through a computational analysis of students responses to test Items, supported by interviews, we were able to offer a more robust description of the phenomenon of aberrant conjoining. Gaining some clarity on how the responses entailing conjoining are generated may inform teaching strategies productively. It is hoped that the knowledge gained through this study will facilitate the process of investigating ways in which to teach the simplification of algebraic expressions. There is a need for teaching methods that allow students to be successful and overcome the strong urge to aberrantly conjoin terms. In this manner, the hope is that this research will contribute to the improvement and advancement of mathematics education.

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ADDENDUM

Contents

1. **Review of DBE approved textbooks** Table 1: Summary of the types of items in the simplification of algebraic expressions

Textbook	Chapter/ Unit page numbers	Exercises	The number of terms		The number of distinct variables		Exponent on single variables (excluding constant terms)		Multiplication of distinct variables within one term – number of variables		Exponents on terms consisting of more than one variable		Combinations of types of terms $(x, x^2, x^3, constants(k), ab, a, b, c, a^2b^2, c^2)$ (where $a, b, and\ c$ represent three distinct letters in alphabetical order)									
			Lower limit	Upper limit	Lower limit	Upper limit	Lower limit	Upper limit	Lower limit	Upper limit	Lower Limit	Upper limit	x,k	x,x^2	x,x^2,k	x,x^2,x^3,k	a,b,c	$ab,a,b,$	ab,a^2,b^2	ab,a^2,b^2,k	$*a^nb^n,ac,bc,$ $a,b,c\ a^2,b^2$	$*a^nb^nc^n,a,b,c,$ ab,bc,k
Solutions for All	Unit 7 105-118	Act. 7.4 Ex. 7.3 Check	2	8	1	3	1	3	2	3	1	2	✓		✓	✓	✓	✓				✓
	Unit 10 150-165	Getting started Check	4	7	1	3	1	3		2	1	1			✓	✓	✓				✓	
Clever – Keeping it simple	Topic 6 Pg 87-98	Ex 5	4	6	1	2*	1	2		2	1	2**			✓				✓	✓		
	Topic 8 Pg109-115	Ex1 Ex2	3	12	1	4***	1	3		2	1	2			✓	✓	✓	✓		✓		
Platinum	Topic 6 Pg58-65	Ex 6.3	2	6	1	3	1	2		2	1	3		✓			✓	✓			✓	
	Topic 8 Pg74-81	Ex 8.1 Ex 8.2 Ex 8.4 Revision	2	8	1	3	1	3	2	3		1	✓	✓		✓	✓				✓	✓

Textbook	Chapter/ Unit page numbers	Exercises	The number of terms		The number of distinct variables		Exponent on single variables (excluding constant terms)		Multiplication of distinct variables within one term – number of variables		Exponents on terms consisting of more than one variable		Combinations of types of terms $(x, x^2, x^3, constants(k)$, $(a^nb^nc^n, a, b, c, ab, bc, ac,\ a^2, b^2, c^2\ constants\ (k))$ (where $a, b, and\ c$ represent three distinct letters in alphabetical order)									
			Lower limit	Upper limit	Lower limit	Upper limit	Lower limit	Upper limit	Lower limit	Upper limit	Lower Limit	Upper limit	x, k	x, x^2	x, x^2, k	x, x^2, x^3, k	a, b, c	ab, a, b, k	$ab, a^2, b^2,$ a, b	ab, a^2, b^2, k	$*a^nb^n, ac, bc,$ $a, b, c\ a^2, b^2$ c^2	$*a^nb^nc^n, a, b, c,$ ab, bc, k
Shuter's Premier Mathematics	****Unit 6 Pg 50-57	Ex 4 Ex 6	3	9	1	3	1	2	2	3	1	3		✓			✓				✓	✓
	Unit 8 Pg66-74	Ex 2 Ex 3	4	9	1	4	1	2	2	3		1	✓		✓		✓	✓	✓			
Oxford Successful Mathematics	Chapter 5 pg 91-110	Unit 2 Ex 3 Unit 4 Ex 1, 2 & 3	2	9	1	3	1	3	1	2		1	✓	✓	✓	✓	✓	✓		✓		
	Chapter 6 pg 112-132	This chapter does not revisit the simplifications of simple algebraic expressions . The multiplication and division of terms is the main focus of this chapter. Interesting to note that there is a higher upper limit on the exponents of variables when the simplifications of algebraic expressions contain multiplication or division.																				

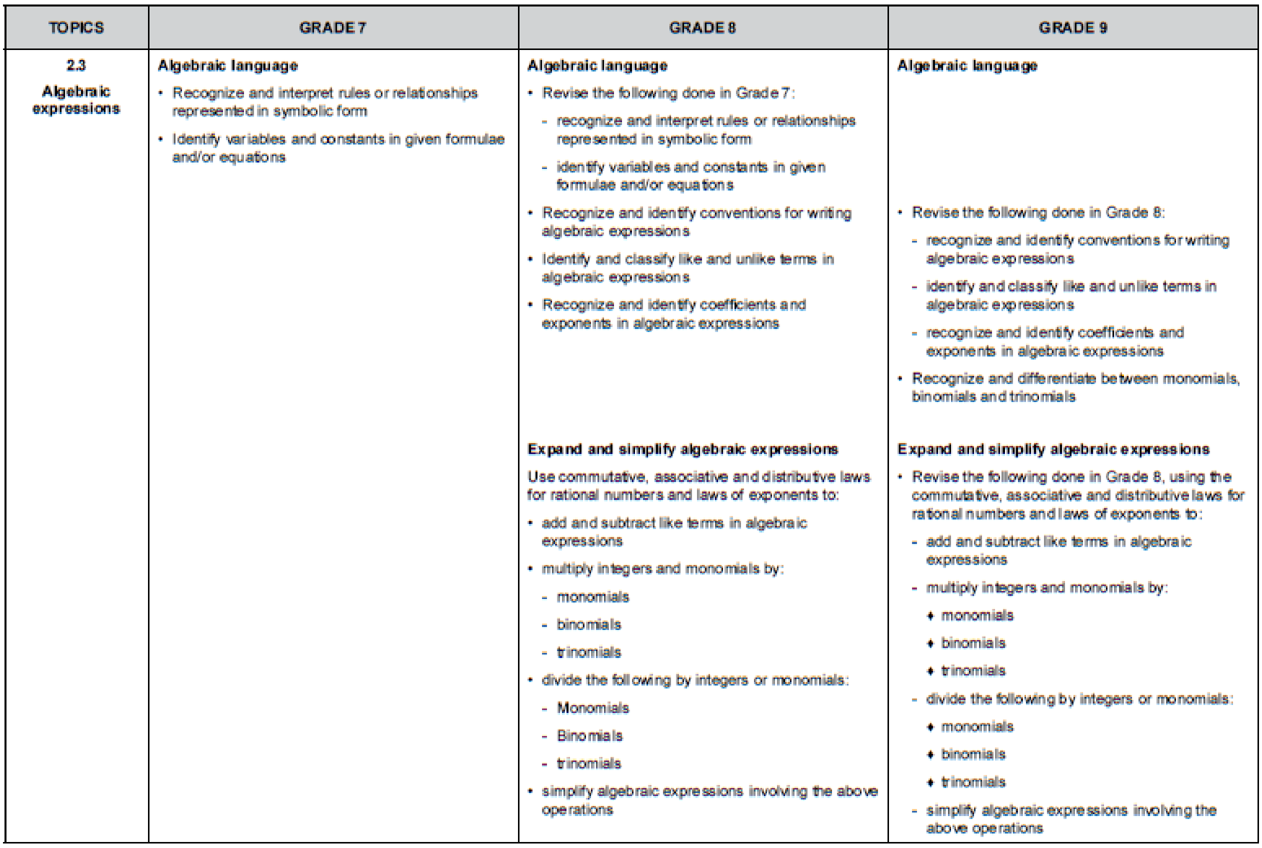
*The example on page 96 contains 3 distinct variables.

**The example on page 95 has a term $5y^3z$

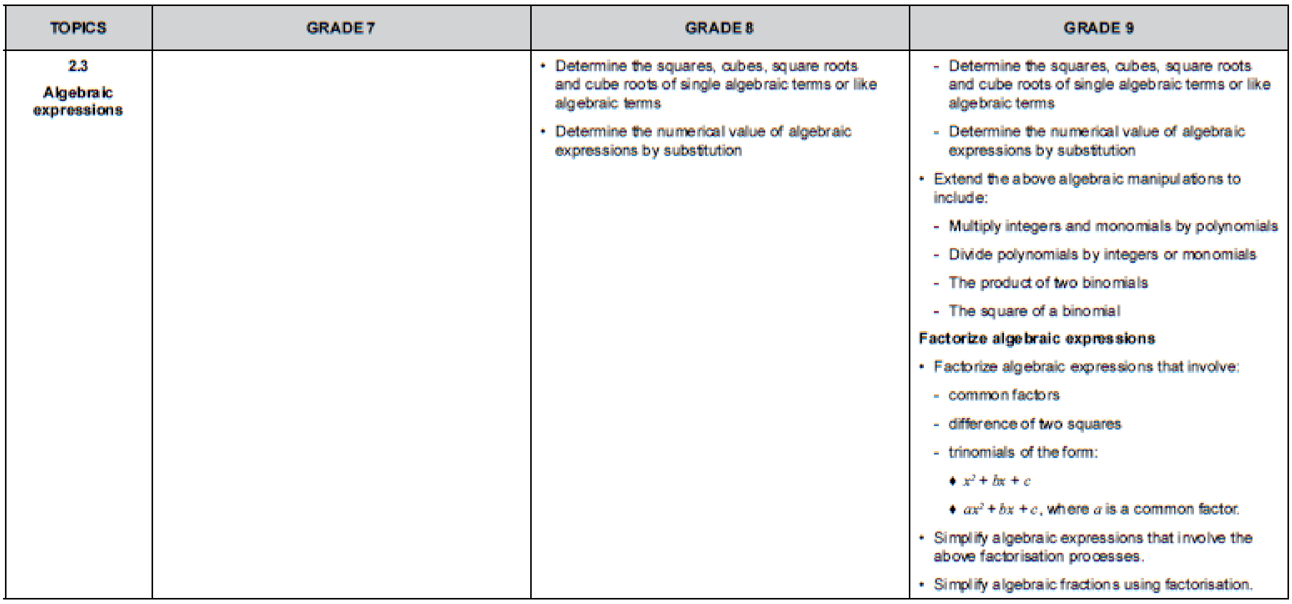
*** Ex. 2 2f is highlighted as being different from the rest of the items - $2ab^2 + 3c^2d - 4c^2d - 5ab^2$

**** no items containing constants

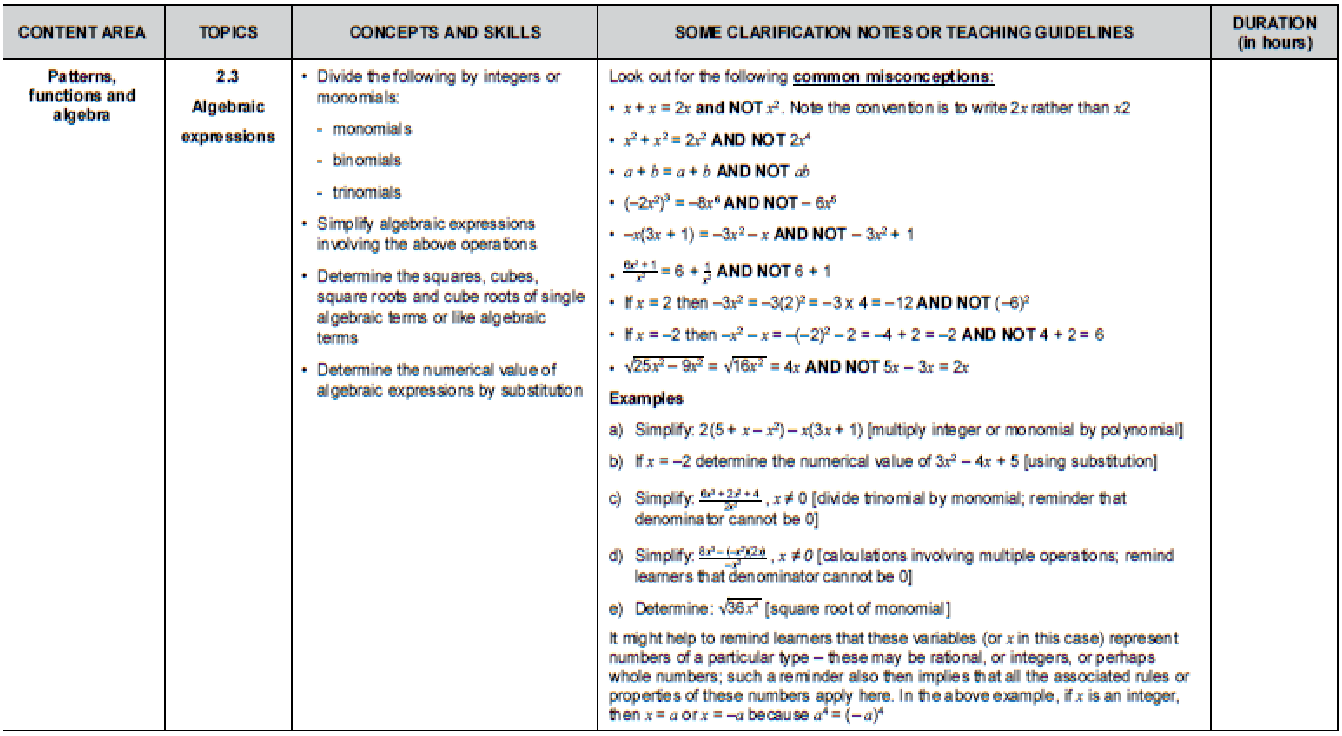
2. Outline of Algebraic Expressions from Grade 7-9 CAPS document



TOPICS	GRADE 7	GRADE 8	GRADE 9
2.3 **Algebraic expressions**	**Algebraic language** • Recognize and interpret rules or relationships represented in symbolic form • Identify variables and constants in given formulae and/or equations	**Algebraic language** • Revise the following done in Grade 7: - recognize and interpret rules or relationships represented in symbolic form - identify variables and constants in given formulae and/or equations • Recognize and identify conventions for writing algebraic expressions • Identify and classify like and unlike terms in algebraic expressions • Recognize and identify coefficients and exponents in algebraic expressions	**Algebraic language** • Revise the following done in Grade 8: - recognize and identify conventions for writing algebraic expressions - identify and classify like and unlike terms in algebraic expressions - recognize and identify coefficients and exponents in algebraic expressions • Recognize and differentiate between monomials, binomials and trinomials
		Expand and simplify algebraic expressions Use commutative, associative and distributive laws for rational numbers and laws of exponents to: • add and subtract like terms in algebraic expressions • multiply integers and monomials by: - monomials - binomials - trinomials • divide the following by integers or monomials: - Monomials - Binomials - trinomials • simplify algebraic expressions involving the above operations	**Expand and simplify algebraic expressions** • Revise the following done in Grade 8, using the commutative, associative and distributive laws for rational numbers and laws of exponents to: - add and subtract like terms in algebraic expressions - multiply integers and monomials by: ♦ monomials ♦ binomials ♦ trinomials - divide the following by integers or monomials: ♦ monomials ♦ binomials ♦ trinomials - simplify algebraic expressions involving the above operations



TOPICS	GRADE 7	GRADE 8	GRADE 9
2.3 **Algebraic expressions**		• Determine the squares, cubes, square roots and cube roots of single algebraic terms or like algebraic terms • Determine the numerical value of algebraic expressions by substitution	- Determine the squares, cubes, square roots and cube roots of single algebraic terms or like algebraic terms - Determine the numerical value of algebraic expressions by substitution • Extend the above algebraic manipulations to include: - Multiply integers and monomials by polynomials - Divide polynomials by integers or monomials - The product of two binomials - The square of a binomial **Factorize algebraic expressions** • Factorize algebraic expressions that involve: - common factors - difference of two squares - trinomials of the form: ♦ $x^2 + bx + c$ ♦ $ax^2 + bx + c$, where a is a common factor. • Simplify algebraic expressions that involve the above factorisation processes. • Simplify algebraic fractions using factorisation.



CONTENT AREA	TOPICS	CONCEPTS AND SKILLS	SOME CLARIFICATION NOTES OR TEACHING GUIDELINES	DURATION (in hours)
Patterns, functions and algebra	**2.3** **Algebraic expressions**	• Divide the following by integers or monomials: - monomials - binomials - trinomials • Simplify algebraic expressions involving the above operations • Determine the squares, cubes, square roots and cube roots of single algebraic terms or like algebraic terms • Determine the numerical value of algebraic expressions by substitution	Look out for the following **common misconceptions:** • $x + x = 2x$ **and NOT** x^2. Note the convention is to write $2x$ rather than $x2$ • $x^2 + x^2 = 2x^2$ **AND NOT** $2x^4$ • $a + b = a + b$ **AND NOT** ab • $(-2x^2)^3 = -8x^6$ **AND NOT** $-6x^5$ • $-x(3x + 1) = -3x^2 - x$ **AND NOT** $-3x^2 + 1$ • $\frac{6x^2+1}{x^2} = 6 + \frac{1}{x^2}$ **AND NOT** $6 + 1$ • If $x = 2$ then $-3x^2 = -3(2)^2 = -3 \text{ x } 4 = -12$ **AND NOT** $(-6)^2$ • If $x = -2$ then $-x^2 - x = -(-2)^2 - 2 = -4 + 2 = -2$ **AND NOT** $4 + 2 = 6$ • $\sqrt{25x^2 - 9x^2} = \sqrt{16x^2} = 4x$ **AND NOT** $5x - 3x = 2x$ **Examples** a) Simplify: $2(5 + x - x^2) - x(3x + 1)$ [multiply integer or monomial by polynomial] b) If $x = -2$ determine the numerical value of $3x^2 - 4x + 5$ [using substitution] c) Simplify: $\frac{6x^3 + 2x^2 + 4}{2x^2}$, $x \neq 0$ [divide trinomial by monomial; reminder that denominator cannot be 0] d) Simplify: $\frac{8x^3 - (-x^2)(2x)}{-x^2}$, $x \neq 0$ [calculations involving multiple operations; remind learners that denominator cannot be 0] e) Determine: $\sqrt{36x^4}$ [square root of monomial] It might help to remind learners that these variables (or x in this case) represent numbers of a particular type – these may be rational, or integers, or perhaps whole numbers; such a reminder also then implies that all the associated rules or properties of these numbers apply here. In the above example, if x is an integer, then $x = a$ or $x = -a$ because $a^4 = (-a)^4$	

3. Pilot Study – Development of the tests

Mathematics test

Instructions:

1. Show all your working when answering the following questions.
2. You may use a non-programmable calculator.
3. Write neatly and legibly.

Simplify the following algebraic expressions completely.

a) $3p + 2 + 5p + 4p + 11$
b) $4t + 3t + 2t$
c) $3x + 5x + 9y$
d) $7a + 8b + 4ab + 5a$
e) $4yx + 7xy + 3xy + 2y + 5x$
f) $5a^2 + 4a^2 + 2a + 7a^2$
g) $10c^2 + 7g^2 + 3c + 5c^2 + 10g^3$
h) $3a^2b^3 + 5a^3b2 + 7a^2b^3 + 4a^2b^2$

4. Additional information regarding the test instrument for this study

Instructions on test (This was shown on a slideshow during the lesson and read aloud.)

Slide 1:

- Answer all the questions
- Each question will be displayed for 2 min
- Write neatly and legibly
- Show all working
- You may use a non-programmable calculator
- Write in blue or black ink
- **Write your name and surname on each sheet**

Slide 2:

Simplify the following algebraic expressions.

Write the answers on the sheet provided. Show all working.

5. Tables of all responses per item for this study

Item One: $6x + 3x$

Responses	All students (*n* =76) No. of responses	%
9x	53	70
$9x^2$	19	25
18x	2	3
inc	1	1
$3x^2$	1	1
Totals	76	100

Item Two: $5a + 5b + a$

Responses	All students (*n* =76) No. of responses	%
$11a^2b$	13	17
$10a^2b$ or $10ba^2$	11	14
11ab	6	8
6a+5b or 5b+6a	26	34
10ab or 10ba	4	5
10a	2	3
$10a^3$	3	4
10aba	2	3
$6a^2+5b$	2	3
11aba	1	1
5a+5b	1	1
15a	1	1
a+5b	1	1
6a+1a	1	1
6a5b	1	1
$5a^2+5b$	1	1
Totals	76	100

Item Three : $3n + 4$

Responses	All students (*n* =76) No. of responses	%
7n or $7n^1$	53	70
3n+4 or 4+3n	20	26
"7n or 11n"	1	1
9n	1	1
7a	1	1
Totals	76	100

Item Four : $4a^2 + 3a + 7a$

Responses	All students (*n* =76) No. of responses	%
$14a^5$	31	41
$14a^4$	10	13
$14a^3$	4	5
$14a^2$	3	4
$7a^2+7a$ or $7a+7a^2$	16	21
14a	2	3
14^5	2	3
$14a^6$	1	1
28a	1	1
$4a^6$	1	1
$7a^4+7a$ or $7a+7a^4$	5	7
Totals	76	100

Item Five: $4x^2 + 5x + 3x + 7x^2$

Responses	All students (*n* =76) No. of responses	%
$19x^6$	28	37
$19x^4$	8	11
$11x^2+8x$ or $8x+11x^2$	18	24
$19x^2$	3	4
$19x^8$	2	3
$11x^4+8x^2$ or $8x^2+11x^4$	3	4
$7x+11x^2$	1	1
$6x^4$	1	1
41x	1	1
$19x^{16}$	1	1
43a	1	1
$18x^6$	1	1
$19x^3$	1	1
17^6	1	1
$11x^4+8x$ or $8x+11x^4$	3	4
19^4	1	1
19^8	1	1
inc.	1	1
Totals	76	100

Item Six: $7kb + 4b + 3bk + 5kb + 4k$

Responses	All students (*n* =76) No. of responses	%
$23k^4b^4$	10	13
15kb+4b+4k	19	25
23kb	9	12
12kb+4b+3bk+4k	8	11
$23k^3b^3$	3	4
$12k^2b^2+4b+3bk+4k$	2	3
20kb	2	3
12kb	1	1
$23kb^1b^1bk^1kb^1+b$	1	1
$23kb^3bbkk$	1	1
7kb+4b+3bk+5kb+4k	1	1
8kb	1	1
$23b^2k^3$	1	1
12kb+8bk	1	1
$12k^2b^2+11bk$	1	1
23kbkb	1	1
16kb+4b+4k	1	1
23b	1	1
$23k^4b^3$	1	1
12kb+4b+4k	1	1
$23kb^2b^1bk^1k^1$	1	1
$19kb^2+bk+k$	2	3

15kb3+4b+4k	2	3
$12kb^2+11bk^2$	1	1
4b+4k+3bk+11kb	1	1
$15b^3k^3+4b+4k$	1	1
4b+4k+3bk+13kb	1	1
$12kb^2+3bk+4k+4b$	1	1
Totals	76	100

Item Seven : $3y^2 + 4y + 1 + 5y + 7y^2 + 8$

Responses	All students (*n* =76)	
	No. of responses	%
$28y^6$	12	16
$10y^2+9y+9$	25	33
$28y^4$	4	5
$28y^5$	3	4
$19y^6+9$	3	4
$19y^6+9$	3	4
$29y^6$	3	4
$29y^2$	2	3
$10y^4+9y^2+9$	2	3
$29y^2$	2	3
$28y^2$	2	3
55y	1	1
$28y^2y^1y^1y^2$	1	1
28	1	1
$27y^6$	1	1
$(3y^2+7y^2)+(4y+5y)+(1+8)$	1	1
$19y^4+1+8$	1	1
$27y^4$	1	1
28y	1	1
$28y^3$	1	1
$10y^4+9y+9$	3	4
$10y^8+9y^2+9$	1	1
$10y^2+18y$	1	1
$34y^6$	1	1
Totals	76	100

Item Eight: $x + 5x + 2x$

Responses	All students (*n* =76)	
	No. of responses	%
8x	36	47
$8x^3$	17	22
$7x^3$	13	17

7x	5	7
10x	1	1
-9x	1	1
3x	1	1
$8x^2$	1	1
x+7x	1	1
Totals	76	100

Table 5 Item Ten: $6g^2 + 6k + 8t^3$

	All students (*n* =76)	
Responses	No. of responses	%
$20g^2kt^3$	19	25
$6g^2+6k+8t^3$	31	41
$20gkt^6$	4	5
$20gkt^5$	5	7
$6g^26k8t^3$	2	3
56gt	1	1
554gkt	1	1
$20g^5$	1	1
$22g^6kt^3$	1	1
$12g^2$	1	1
20g	1	1
$20gkt^4$	1	1
12g2k+8t3	3	4
20g2kt2	1	1
18gk2+8t3	1	1
36g+6k+512t	1	1
20g2+k+t3	1	1
12g+6k+24t	1	1
Totals	76	100

Item Nine: $9a^2r + 4r^2 + 5a^2 + 3ar^2 + 7a^2r$

	All students (*n* =76)	
Responses	No. of responses	%
$16a^2r+4r^2+11a^2+3ar^2$	19	25
$16a^2r+11a^2+3ar^2+4r$	2	3
$34a^9r^6$	6	8
$34ar^4ar^8$	1	1
$34+a^4r^2+a^4+4r^2+3ar^2$	1	1

34	2	3
$11a^4+4r^2+3ar^2+16a^4r$	1	1
$16a^2r^2+11a^2+7a^2r+4r^2$	1	1
$19a^5r^4+11a^4+4r^2$	1	1
$16a^2r+4r^2+11a^4+3ar^2$	1	1
$16a^4r+4r+11a^4+3ar^2$	1	1
$16a^2r+3ar^3+4r^2+11a^2$	1	1
$16a^4r+11a^2+3ar^2+4r^2$	1	1
80ar	1	1
$16a^4r^2+4r^2+11a^4+3ar^2$	1	1
$34a^8r^4$	2	3
$34a^2r^1r^2ar^2a^2r^1a^2$	1	1
$34ar^{12}$	1	1
$27a^2r+5a^2r$	1	1
$34a^5r^5$	1	1
$35ar^{10}$	1	1
$13a^4r+4r^2+11a^2+3ar^2$	1	1
$34a^{11}r^6$	1	1
$19a^5r^4+4r^2+11a^4$	1	1
$35a^{12}r$	1	1
$34a^6r^4$	1	1
$34ar^2$	1	1
$34a^8r^6$	2	3
$27a^8r$	1	1
$15a^2r+11a^2+4r^2+3ar^2$	1	1
$27a^9+4r^2$	1	1
$34a^3r$	1	1
$20a^6+10a^3r^3+4^{r2}$	1	1
$17a^4+4r^{2+}11a^4$	1	1
33a9r6	2	3
incomplete	1	1
34ar	1	1
$16a^4r+3ar^2+4r^2+5a^2+6a^2$	1	1
$34a^6r^5$	1	1
$34a^2r$	1	1
35a	1	1
$20a^2+14a^2r^2$	1	1
$19ar^6+4r^2+11a^4$	1	1
$24a^{10}r^6$	1	1
$19a^2r+4r^2+11a$	1	1
$16a^4r+4r^2+11a^4+3ar^2$	1	1
$34a^9r^5$	1	1
$34a^7r^3$	1	1
Totals	76	100

6. Tables showing all responses per item per school.

Table 1. Item one: $6x + 3x$

	School A ($n = 27$)		School B ($n = 49$)	
Responses	No. of responses	%	No. of responses	%
9x	22	81	31	63
$9x^2$	5	19	14	29
18x	0	0	2	4
3x2	0	0	1	2
inc	0	0	1	2
Totals	27	100	49	100

Table 3 Item three: $3n + 4$

	School A ($n = 27$)		School B ($n = 49$)	
Responses	No. of responses	%	No. of responses	%
7n or $7n^1$	11	41	42	86
3n+4 or 4+3n	16	59	4	8
"7n or 11n"	0	0	1	2
9n	0	0	1	2
7a	0	0	1	2
Totals	27	100	49	100

Table 2 Item two: $5a + 5b + a$

	School A ($n = 27$)		School B ($n = 49$)	
Responses	No. of responses	%	No. of responses	%
$11a^2b$	0	0	13	27
$10a^2b$ or $10ba^2$	2	7	9	18
11ab	0	0	6	12
6a+5b or 5b+6a	21	78	5	10
10ab or 10ba	2	7	2	4
10a	0	0	2	4
$10a^3$	1	4	2	4
10aba	0	0	2	4
$6a^2+5b$	1	4	1	2
11aba	0	0	1	2
5a+5b	0	0	1	2
15a	0	0	1	2
a+5b	0	0	1	2
6a+1a	0	0	1	2
6a5b	0	0	1	2
$5a^2+5b$	0	0	1	2
Totals	27	100	49	100

Table 5.4. Item four: $4a^2+3a^2+7a$

	School A (n = 27)		School B (n = 49)	
Responses	No. of responses	%	No. of responses	%
$14a^5$	4	15	27	55
$14a^4$	3	11	7	14
$14a^3$	1	4	3	6
$14a^2$	0	0	3	6
$7a^2+7a$ or $7a+7a^2$	14	52	2	4
$14a$	0	0	2	4
14^5	0	0	2	4
$14a^6$	0	0	1	2
$28a$	0	0	1	2
$4a^6$	0	0	1	2
$7a^4+7a$ or $7a+7a^4$	5	19	0	0
Totals	27	100	49	100

Table 5.5. Item five: $4x^2+5x+x+7x^2$

	School A (n = 27)		School B (n = 49)	
Responses	No. of responses	%	No. of responses	%
$19x^6$	2	7	26	53
$19x^4$	2	7	6	12
$11x^2+8x$ or $8x+11x^2$	15	56	3	6
$19x^2$	0	0	3	6
$19x^8$	0	0	2	4
$11x^4+8x^2$ or $8x^2+11x^4$	2	7	1	2
$7x+11x^2$	0	0	1	2
$6x^4$	0	0	1	2
$41x$	0	0	1	2
$19x^{16}$	0	0	1	2
$43a$	0	0	1	2
$18x^6$	0	0	1	2
$19x^3$	0	0	1	2
17^6	0	0	1	2
$11x^4+8x$ or $8x+11x^4$	3	11	0	0
19^4	1	0	0	0
19^8	1	0	0	0
inc.	1	4	0	0
Totals	27	100	49	100

Table 5.6. Item six: $7kb+4b+3bk+5kb+4k$

	School A (n = 27)		School B (n = 49)	
Responses	No. of responses	%	No. of responses	%
$23k^4b^4$	0	0	10	20
15kb+4b+4k	10	37	9	18
23kb	1	4	8	16
12kb+4b+3bk+4k	6	22	2	4
$23k^3b^3$	1	4	2	4
$12k^2b^2+4b+3bk+4k$	0	0	2	4
20kb	1	4	1	2
12kb	0	0	1	2
$23kb^1b^1bk^1kb^1+b$	0	0	1	2
$23kb^3bbkk$	0	0	1	2
7kb+4b+3bk+5kb+4k	0	0	1	2
8kb	0	0	1	2
$23b^2k^3$	0	0	1	2
12kb+8bk	0	0	1	2
$12k^2b^2+11bk$	0	0	1	2
23kbkb	0	0	1	2
16kb+4b+4k	0	0	1	2
23b	0	0	1	2
$23k^4b^3$	0	0	1	2
12kb+4b+4k	0	0	1	2
$23kb^2b^1bk^1k^1$	0	0	1	2
$19kb^2+bk+k$	1	4	1	2
$15kb^3+4b+4k$	2	7	0	0
$12kb^2+11bk^2$	1	4	0	0
4b+4k+3bk+11kb	1	4	0	0
$15b^3k^3+4b+4k$	1	4	0	0
4b+4k+3bk+13kb	1	4	0	0
12kb2+3bk+4k+4b	1	4	0	0
Totals	27	100	49	100

Table 5.7. Item seven: $3y^2+4y+1+5y+7y^2+8$

	School A (n = 27)		School B (n = 49)	
Responses	No. of responses	%	No. of responses	%
$28y^6$	3	11	9	18
$10y^2+9y+9$	17	63	8	16
$28y^4$	1	4	3	6
$28y^5$	0	0	3	6
$19y^6+^9$	0	0	3	6
$19y^6+^9$	0	0	3	6
29y6	0	0	3	6
$29y^2$	0	0	2	4
$10y^4+9y^2+9$	0	0	2	4
$29y^2$	0	0	2	4
$28y^2$	0	0	2	4
55y	0	0	1	2
$28y^2y^1y^1y^2$	0	0	1	2
28	0	0	1	2
$27y^6$	0	0	1	2
(3y2+7y2)+(4y+5y)+(1+8)	0	0	1	2
$19y^4+1+8$	0	0	1	2
$27y^4$	0	0	1	2
28y	0	0	1	2
$28y^3$	0	0	1	2
$10y^4+9y+9$	3	11	0	0
$10y^8+9y^2+9$	1	4	0	0
$10y^2+18y$	1	4	0	0
$34y^6$	1	4	0	0
Totals	27	100	49	100

Table 5.8. Item eight: $x+5x+2x$

	School A (n = 27)		School B (n = 49)	
Responses	No. of responses	%	No. of responses	%
$8x$	20	74	16	33
$8x^3$	1	4	16	33
$7x^3$	4	15	9	18
$7x$	1	4	4	8
$10x$	0	0	1	2
$-9x$	0	0	1	2
$3x$	0	0	1	2
$8x^2$	0	0	1	2
$x+7x$	1	4	0	0
Totals	27	100	49	100

Table 5.9. Item nine: $9a^2r+4r^2+5a^2+3ar^2+7a^2r+6a^2$

	School A (n = 27)		School B (n = 49)	
Responses	No. of responses	%	No. of responses	%
$16a^2r+4r^2+11a^2+3ar^2$	14	52	5	10
$16a^2r+11a^2+3ar^2+4r$	2	7	0	0
$34a^9r^6$	1	4	5	10
$34ar^4ar^8$	1	4	0	0
$34+a^4r^2+a^4+4r^2+3ar^2$	1	4	0	0
34	1	4	1	2
$11a^4+4r^2+3ar^2+16a^4r$	1	4	0	0
$16a^2r^2+11a^2+7a^2r+4r^2$	1	4	0	0
$19a^5r^4+11a^4+4r^2$	1	4	0	0
$16a^2r+4r^2+11a^4+3ar^2$	1	4	0	0
$16a^4r+4r+11a^4+3ar^2$	1	4	0	0
$16a^2r+3ar^3+4r^2+11a^2$	1	4	0	0
$16a^4r+11a^2+3ar^2+4r^2$	1	4	0	0
$80ar$	0	0	1	2
$16a^4r^2+4r^2+11a^4+3ar^2$	0	0	1	2
$34a^8r^4$	0	0	2	4
$34a^2r^1r^2ar^2a^2r^1a^2$	0	0	1	2
$34ar^{12}$	0	0	1	2
$27a^2r+5a^2r$	0	0	1	2
$34a^5r^5$	0	0	1	2
$35ar^{10}$	0	0	1	2
$13a^4r+4r^2+11a^2+3ar^2$	0	0	1	2
$34a^{11}r^6$	0	0	1	2
$19a^5r^4+4r^2+11a^4$	0	0	1	2
$35a^{12}r$	0	0	1	2
$34a^6r^4$	0	0	1	2

$34ar^2$	0	0	1	2
$34a^8r^6$	0	0	2	4
$27a^8r$	0	0	1	2
$15a^2r+11a^2+4r^2+3ar^2$	0	0	1	2
$27a^9+4r^2$	0	0	1	2
$34a^3r$	0	0	1	2
$20a^6+10a^3r^3+4r^2$	0	0	1	2
$17a^4+4r^2+11a^4$	0	0	1	2
$33a^9r^6$	0	0	2	4
incomplete	0	0	1	2
$34ar$	0	0	1	2
$16a^4r+3ar^2+4r^2+5a^2+6a^2$	0	0	1	2
$34a^6r^5$	0	0	1	2
$34a^2r$	0	0	1	2
$35a$	0	0	1	2
$20a^2+14a^2r^2$	0	0	1	2
$19ar^6+4r^2+11a^4$	0	0	1	2
$24a^{10}r^6$	0	0	1	2
$19a^2r+4r^2+11a$	0	0	1	2
$16a^4r+4r^2+11a^4+3ar^2$	0	0	1	2
$34a^9r^5$	0	0	1	2
$34a^7r^3$	0	0	1	2
Totals	27	100	49	100

Table 5.10. Item ten: $6g^2+6k+8t^3$

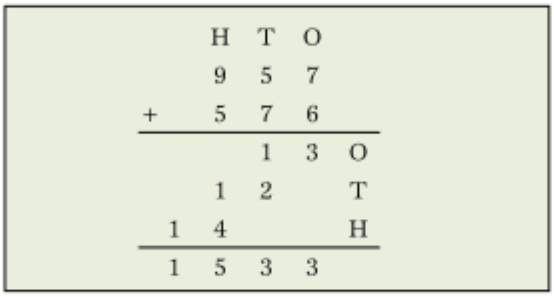
	School A (n = 27)		School B (n = 49)	
Responses	No. of responses	%	No. of responses	%
$20g^2kt^3$	2	7	17	35
$6g^2+6k+8t^3$	17	63	14	29
$20gkt^6$	0	0	4	8
$20gkt^5$	2	7	3	6
$6g^26k8t^3$	0	0	2	4
56gt	0	0	1	2
554gkt	0	0	1	2
$20g^5$	0	0	1	2
$22g6kt^3$	0	0	1	2
$12g^2$	0	0	1	2
20g	0	0	1	2
$20gkt^4$	0	0	1	2
$12g^2k+8t^3$	2	7	1	2
$20g^2kt^2$	0	0	1	2
$18gk^2+8t^3$	1	4	0	0
36g+6k+512t	1	4	0	0
$20g^2+k+t^3$	1	4	0	0
12g+6k+24t	1	4	0	0
Totals	27	100	49	100

7. Teaching strategies suggested in the literature

Teaching strategies for the simplification of algebraic expressions discussed in the literature

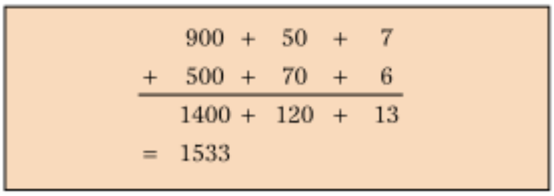
A traditional approach to teaching algebra in general is an emphasis on procedures and algorithms. Research shows that teaching approaches in algebra tend to be more procedural than structural (Hallagan, 2006:103). This approach to teaching algebra persists as the common approach amongst teachers, even though there has been extensive research that this is not helpful in developing students' understandings of algebraic concepts. "Transmitting algebraic rules by the teacher, memorizing them by students and practising them in a mechanical way is worthless" (Demby, 1997: 68). Literature also suggests that differences in approaches to teaching makes different mathematics (content) available for learning (Even & Kvatinsky, 2010:220) .

A very common approach to teaching the simplification of algebraic expressions involves the identification of "like and unlike" terms and the "collecting of like terms". The like terms are identified by observing the variables and exponents of individual terms within an expression. These terms are then "collected" in a procedure to simplify an algebraic expression. Students have difficulty with identifying like terms and do not recognise that terms such as 3ab and 4ba are in fact like terms (Lim, 2010:151). This error could be attributed to the fact that students are not aware of the commutativity of multiplication within the context of this problem, but it could also be attributed to the way in which students perceive the literal symbols. De Groot & Boyajian (2015) presented a teaching strategy that links the base-10 number system (generalised to other bases) to the concept of like and unlike terms to facilitate the underlying arithmetical understanding of the notions of like and unlike terms. The Figures below show the method using an example of $957 + 576$.



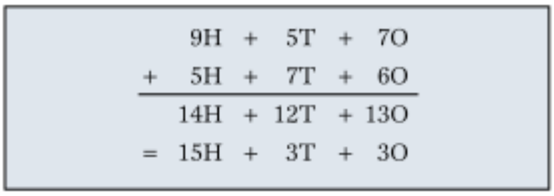
		H	T	O	
		9	5	7	
+		5	7	6	
			1	3	O
		1	2		T
	1	4			H
	1	5	3	3	

Fig. 1 Student competence with past experience establishes a foundation.



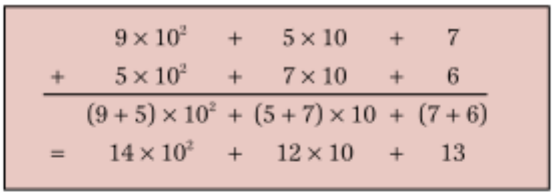
	900	+	50	+	7
+	500	+	70	+	6
	1400	+	120	+	13
=	1533				

Fig. 2 Expanded notation is used to calculate a sum.



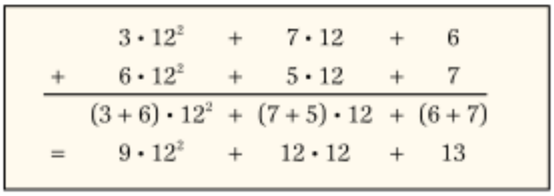
$$\begin{array}{rrcrcr} & 9H & + & 5T & + & 7O \\ + & 5H & + & 7T & + & 6O \\ \hline & 14H & + & 12T & + & 13O \\ = & 15H & + & 3T & + & 3O \end{array}$$

Fig. 3 Digits are named according to place value.



$$\begin{array}{rccccc} & 9\times10^2 & + & 5\times10 & + & 7 \\ + & 5\times10^2 & + & 7\times10 & + & 6 \\ \hline & (9+5)\times10^2 & + & (5+7)\times10 & + & (7+6) \\ = & 14\times10^2 & + & 12\times10 & + & 13 \end{array}$$

Fig. 4 Scientific notation and the distributive property support the transition to work with algebraic polynomials.

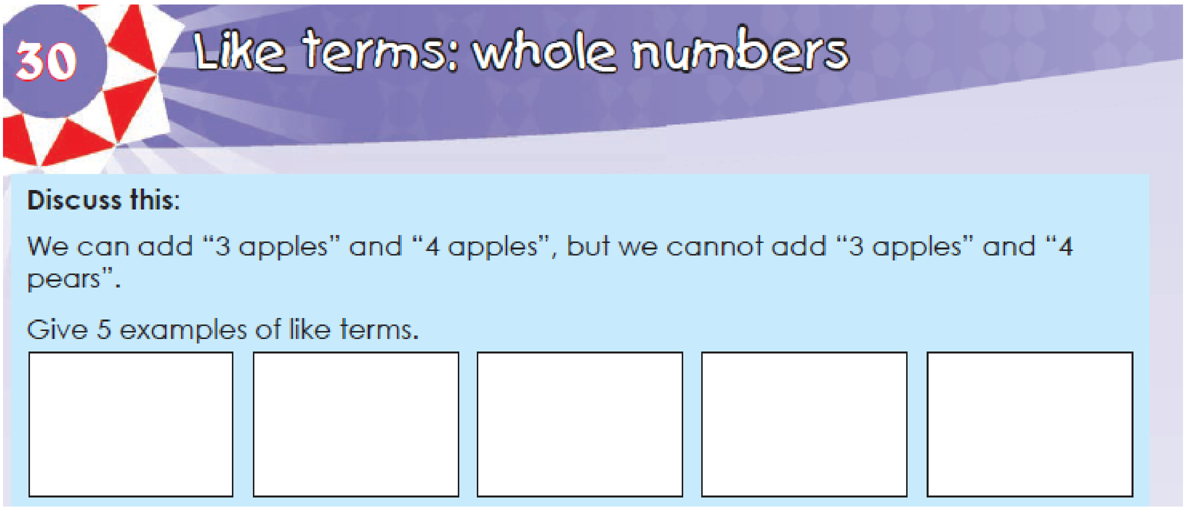


$$\begin{array}{rccccc} & 3\cdot12^2 & + & 7\cdot12 & + & 6 \\ + & 6\cdot12^2 & + & 5\cdot12 & + & 7 \\ \hline & (3+6)\cdot12^2 & + & (7+5)\cdot12 & + & (6+7) \\ = & 9\cdot12^2 & + & 12\cdot12 & + & 13 \end{array}$$

Fig. 5 Structure allows for generalization.

Figure 2.4: A teaching strategy highlighting like and unlike terms using base-10 place value system (De Groot & Boyajian, 2015: 509-510)

A common teaching strategy in the simplification of algebraic expressions is the “fruit salad” metaphor (Tirosh, Even & Robinson 1998:59). In this metaphor, students are told to perceive like and unlike terms as objects. For example, in the expression $2a + 4b + 3a$ students will be told to perceive the terms including the letter a as two apples and three apples which gives a result of 5 apples but that the four bananas may not be added to the apples. Pimm (1987 in Tirosh, Even & Robinson, 1998:60) suggests that the metaphor creates confusion and that it may lead students to believe that the variable a represents apples instead of the more appropriate “number of apples”. Subramaniam (2018: 45) has agreed with this finding. Furthermore, Booth (1988:26) suggests that this metaphor itself may lead students to justify the conjoining of terms with variables a and b because apples and bananas may be collected under the collective of fruits or “apples-and-bananas”. Despite the many negative views on using this method, it seems that it is still prevalent in teaching materials. Figure 2.5 shows how this is used in the DBE workbook for grade 8 students.



30 Like terms: whole numbers

Discuss this:

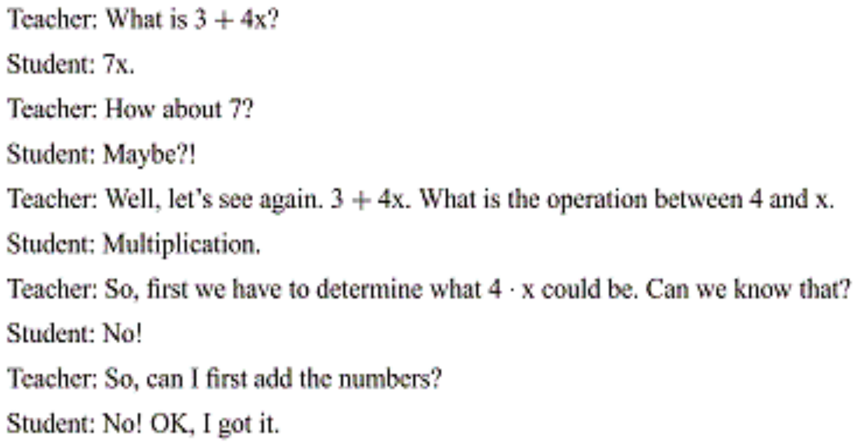
We can add “3 apples” and “4 apples”, but we cannot add “3 apples” and “4 pears”.

Give 5 examples of like terms.

Figure 2.5: An introduction to like terms from the DBE workbook for Grade 8 (DBE 2022: 66)

Researchers have looked at improving the teaching and learning of the simplification of algebraic expressions. An obvious course of action would be to include instruction on the properties of operations in arithmetic as well as the meaning of equivalence in arithmetic contexts. There are contrasting views on the effectiveness of this approach. In India, Subramaniam & Banerjee (2004) developed a teaching strategy that included modules on equivalence, the meaning of the equal sign and the parsing of terms in arithmetic contexts. The results of this teaching experiment were indicative but not conclusive in that students had improved their ability to recognise equivalent expressions but an improvement in the performance of students in other areas were not clearly observed (Subramaniam & Banerjee 2004: 127-128). Banerjee & Subramaniam (2012) showed that a deemphasis on the sequential rules of operations (i.e. BODMAS) and an increase in the focus on structure in arithmetic contexts improved student performance on the simplification of algebraic expressions. Furthermore, it is suggested that the knowledge of arithmetic contexts needs to be linked to the algebraic context for this approach to be beneficial. In contrast, Demby (1997:63-64) showed that the Polish curriculum which emphasised the properties of operations in arithmetic contexts, did not significantly improve the student’s ability to simplify algebraic expressions correctly.

Even though Subramaniam & Banerjee (2004) have shown that a deemphasis on BODMAS has been shown to be helpful, others have used and integrated BODMAS in the teaching of algebraic expressions (Tirosh, Even & Robinson, 1998; Subramaniam, 2018). Here, the order of carrying out operations is used to justify the transformations involved in simplifying algebraic expressions. In Tirosh, Even & Robinson (1998: 57-59), an experienced teacher uses this in her teaching as shown in the transcript below. Tirosh, Even & Robinson (1998) suggests that the explanation is not completely correct because it would mean that expressions such as $5x$ and $7x$ may not be added. Furthermore, students observed in the classroom still asked about the reasons why they could not conjoin terms (Tirosh, Even & Robinson, 1998: 58). Also, this method may emphasise an operational view of algebraic expressions, neglecting a structural perspective of the mathematical statements.



Teacher: What is 3 + 4x?

Student: 7x.

Teacher: How about 7?

Student: Maybe?!

Teacher: Well, let's see again. 3 + 4x. What is the operation between 4 and x.

Student: Multiplication.

Teacher: So, first we have to determine what 4 · x could be. Can we know that?

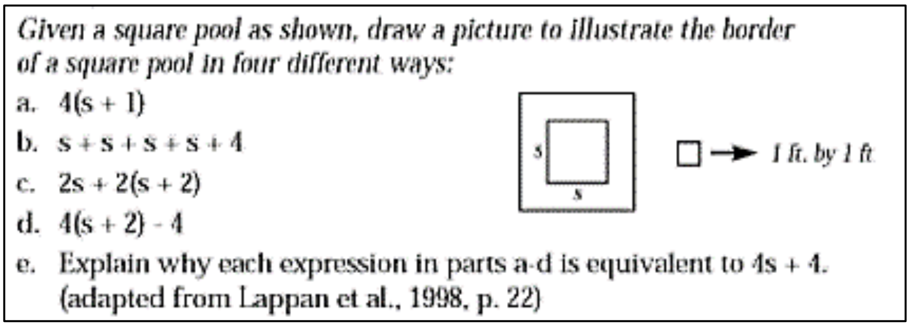
Student: No!

Teacher: So, can I first add the numbers?

Student: No! OK, I got it.

(Tirosh, Even & Robinson, 1998: 58)

Many have suggested the use of geometric representations to develop a better understanding of algebraic expressions and the transformations involved in the simplification of algebraic expressions (Chalouh & Herscovics, 1988a, 1988b; Hallagan, 2006; Demby, 1997). In the chapter titled "Teaching algebraic expressions in a meaningful way", Chalouh & Herscovics (1988b) devised a teaching strategy using the area of rectangles to enhance students' understandings of algebraic expressions. The example below illustrates how geometric descriptions of algebraic expressions may be used to facilitate student understanding of algebraic expressions. The students were first asked to calculate the number of 1-foot square tiles that could surround the border of a pool. Then they were asked to compare the different representations of the total number of tiles surrounding the pool.



Given a square pool as shown, draw a picture to illustrate the border of a square pool in four different ways:

a. 4(s + 1)

b. s + s + s + s + 4

c. 2s + 2(s + 2)

d. 4(s + 2) - 4

e. Explain why each expression in parts a-d is equivalent to 4s + 4. (adapted from Lappan et al., 1998, p. 22)

Figure 2.5: Geometric representation in teaching the simplification of algebraic expressions. Hallagan (2006: 109)

Lastly, there have also been attempts at integrating computer technologies into the teaching of algebra (Tabach, Hershkowitz & Arcavi, 2008). Tabach, Hershkowitz & Arcavi (2008) used spreadsheets to investigate the nature of algebraic expressions. The advantage of using such technology lies in the fact that the values of the variables may be changed and substituted with ease. This facilitates a better understanding of variable and thus improves the students understanding of algebraic expressions.

The teaching strategies presented above have been developed in an attempt to aid better understanding of the procedures and processes involved in the simplification of algebraic expressions. However, the common misconceptions in this very fundamental topic of algebraic expressions continue to be prevalent in mathematics classrooms. The common misconceptions and errors are outlined in the following section.

Implications for teaching and learning

The findings of this study suggest that students have an innate knowledge of number and numerical concepts such as addition. These innate knowledge systems provide the motivation for carrying out certain procedures. Furthermore, these knowledge structures are exploited when students learn operations in arithmetic. Considering the outcome of this study, there is a realisation that the conjoining of terms in the simplification of algebraic expressions may be as a result of an inborn characteristic that makes humans more susceptible to producing results such as the conjoining of terms. Dehaene (2011) suggests that findings of this nature can be used to improve teaching and learning if we consider the way in which these inborn characteristics inhibit or promote the successful learning of mathematics.

> I hope that the views I am defending here will eventually lead to improvements in teaching mathematics. A good curriculum would take into account the assets and limits of the learner's cerebral structure. To optimize the learning experiences of our children, we should consider what impact education and brain maturation have on the organization of mental representations. (Dehaene 2011:xxii)

If one considers what has been presented here, there is a need to develop a teaching strategy that will assist learners in overcoming this cognitive obstacle of merging algebraic terms within the context of the simplification of algebraic expressions. Curriculum documents, textbooks and teacher training should include some information on this obstacle and ways in which to assist students in overcoming it. However, it is not sufficient as this problem persists.

Perhaps a new way of defining like and unlike terms is necessary. Currently, the definitions both in the curriculum statements and DBE approved textbooks only refer to the syntactic features of like and unlike terms. More focus on the semantic meaning of like or unlike terms may prompt students to evaluate their unsound conjoined responses. This could be achieved through various methods such as the inclusion of visual representations and answer checking by the substitution of numerical values.

It is further noted, mentioned earlier in the literature review, that the notion of the equivalence of the transformed expressions should also be highlighted as an important aspect in the simplification of algebraic expressions. The literature also suggests that the concatenation of the coefficient and variable is introduced too soon and that leaving expressions with the multiplication symbol is perhaps one way to overcome this problem. For example , $3 \times x^2 + 4 \times x$.

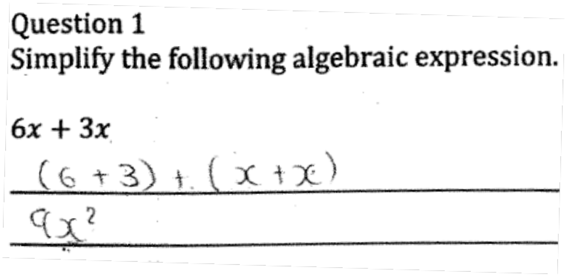
Lastly, it was noted that the methods used for simplifying expressions such as $6x + 3x = 9x$ provided the correct response but when the same method was applied to other items, it results in an incorrect response. Teachers should be aware of the choice of examples when introducing the topic. One should ensure that a variety of items are chosen and discussed. Textbooks should provide opportunities for discussion and variation in types of the algebraic expressions. Perhaps more time should be dedicated to learning how algebraic terms are constructed and the meanings of expressions. This is one of the introductory topics in Algebra. There can be many implications for students as they continue throughout the high school curriculum if this is not done well at the beginning. Imagine learning factorisation and not understanding the simplification of algebraic expressions or the construction of algebraic terms.

8. Interview transcripts

8.1 Student A1-03

Name: Student A1-03 (Part 1)

Researcher: [00:00:02] Okay, so Student A1-03, welcome to the interview. So, basically how this is set up is I will just go through what's the question, and then you're going to read out your answer and then tell me how you got there. Okay, so the first question, this is question one and it says simplify the following algebraic expression. So, before we start, what is an algebraic expression?



Question 1
Simplify the following algebraic expression.

$6x + 3x$

$(6 + 3) + (x + x)$

$9x^2$

Student A1-03: [00:00:30] Um, I think it's like letters and put together with variables.

Researcher: [00:00:37] Okay, so letters and variables put together. And if you read this question what is the question?

Student A1-03: [00:00:45] Simplify which means make it smaller.

Researcher: [00:00:48] Okay. And what's the first question?

Student A1-03: [00:00:53] Um, $6x$ plus $3x$.

Researcher: [00:00:55] And your answer, can you read your answer to me?

Student A1-03: [00:00:57] $9x$ squared.

Researcher: [00:00:59] Okay. So, can you tell me how have you thought about this and what was your method that you use to simplify?

Student A1-03: [00:01:06] So, first I put the two brackets, I mean put brackets and I put the variables together. Then I plus the um, the letters. And with that this made that squared because it's two and six plus three is nine.

Researcher: [00:01:22] So what are these called? [referring to the variables]

Student A1-03: [00:01:25] Numbers.

Researcher: [00:01:26] Numbers. So, and the variables what are the variables? Which are the variables?

Student A1-03: [00:01:35] The six and the three, the main numbers.

Researcher: [00:01:37] Okay. So the main numbers, those are your variables. And if you add this you get the…

Student A1-03: [00:01:43] Nine.

Researcher: [00:01:44] Okay. And then what happens here? These are the...?

Student A1-03: [00:01:48] Letters.

Researcher: [00:01:49] The letters. And so, can you tell me how you got from this to the, what did you say this was?

Student A1-03: [00:01:58] x squared.

Researcher: [00:01:59] How did you get there?

Student A1-03: [00:02:00] Um, so these two xs here. So, I just said x squared because it's two. And then I added two on top. An exponent.

Researcher: [00:02:09] And so I see here you've written a... [referring to the plus sign between the brackets]

Student A1-03: [00:02:11] Plus sign.

Researcher: [00:02:12] And can you tell me, um why there's a plus sign.

Student A1-03: [00:02:17] Um because it says $6x$ plus $3x$. And this is just taking the variables away. I split it up. So, that's the variables and that's the letters. And obviously the plus sign stays the same.

Researcher: [00:02:30] And how do we get to this last answer? So you take the nine that you got from six plus three and then...

Student A1-03: [00:02:40] The x squared. And then you put it together because it's plus.

Researcher: [00:02:44] And when you put it together, what does that answer mean to you? The $9x$ squared. What does that answer mean?

Student A1-03: [00:02:54] Um.

Researcher: [00:02:56] By putting it together, what do you mean? By taking the x squared and the nine and putting it together? What does that mean to you?

Student A1-03: [00:03:04] That's my answer for $6x$ plus $3x$.

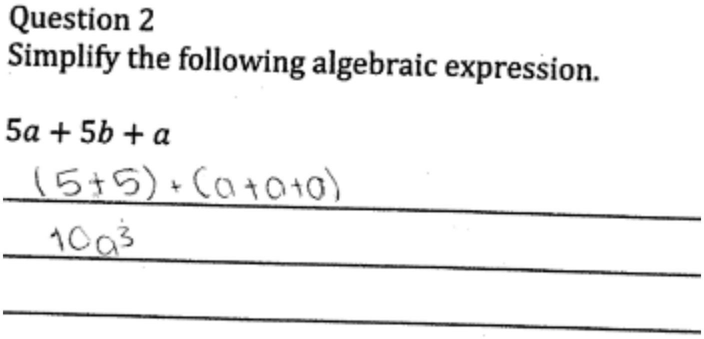
Researcher: [00:03:07] And do you, what are you doing, what operation are you doing when you, in this question?

Student A1-03: [00:03:16] BODMAS

Researcher: [00:03:17] Are you doing BODMAS? Okay, so let's talk about operations addition, subtraction, multiplication and division. We're just going to talk about those one. Can you tell me which operation you are doing in this question?

Student A1-03: [00:03:31] Um, so I'm doing addition, um brackets. And that's the only two.

Researcher: [00:03:39] Okay. All right. Okay. So, we're going to go quickly to the next one. We might come back to this one okay. At this next one. Can you have a look at it first and I'll read the question. It says question two, simplify the following algebraic expression. $5a$ plus $5b$, plus a. Okay. So can you look at what you've written. So, can you explain what you've done here in this in this question?



Question 2

Simplify the following algebraic expression.

$5a + 5b + a$

$(5+5)+(a+a+a)$

$10a^3$

Student A1-03: [00:04:13] So now that I look at it I actually did it wrong.

Researcher: [00:04:15] Okay. That's fine. You can rewrite the question there and you can do what you think you should have done. What do you think was wrong?

Student A1-03: [00:04:25] Um, I thought it was three as and then there was a b.

Researcher: [00:04:29] Okay.

Student A1-03: [00:04:33] So, I'm still gonna plus the five because it's like terms. So...

Researcher: [00:04:57] So you told me you still, you add the fives just like that because they are.

Student A1-03: [00:05:02] Like terms.

Researcher: [00:05:03] Okay. And what are like terms.

Student A1-03: [00:05:05] Um, like those are the same. So, if you have $10xy$ plus $11xy$ then those are like terms. So, because they both have the same variables. So, it's easy to add but if there was a, if this was a z this was a z here. That wouldn't be like terms because it's not the same letters.

Researcher: [00:05:34] Okay, and the fives here are, are these like terms or unlike terms?

Student A1-03: [00:05:41] Um the fives are like terms.

Researcher: [00:05:50] Okay, so what's the reason for these being like terms?

Student A1-03: [00:05:53] Um, because they the same number.

Researcher: [00:05:55] The same number. Okay. And so you've now, what do the brackets mean to you?

Student A1-03: [00:06:02] The brackets just makes it easier for me to separate the letters and to calculate because if I look at this, and if it was a whole long sum and the brackets is made it easier for me to work it out.

Researcher: [00:06:16] Okay. So to separate it. Okay. And so you've put the ... what did you say this was?

Student A1-03: [00:06:24] Like terms.

Researcher: [00:06:25] The like terms together. And you've put the...

Student A1-03: [00:06:29] They also like terms because they are the same..

Researcher: [00:06:31] They the same.

Student A1-03: [00:06:32] And this that's a b because it's the only b there. So that's just there.

Researcher: [00:06:39] And at the end here you've got your final answer. And your final answer is?

Student A1-03: [00:06:45] $10a$ squared b.

Researcher: [00:06:47] And you got the ten because of the...

Student A1-03: [00:06:49] Two fives which is like terms, plus two as and that is squared because there's two. And then you just add the b.

Researcher: [00:06:57] Okay, good. So, what does what does a squared mean to you?

Student A1-03: [00:07:02] a squared means, it means a plus a because there's two as.

Researcher: [00:07:07] So it's two a's. So, the two how would you say you get that two?

Student A1-03: [00:07:11] You plus this and you count how many letters they are.

Researcher: [00:07:17] Okay, and then you got that two. Yeah. All right. And so actually here it's basically what have you done there? You've basically also said if it was an a now, let's talk like that. If it was an a. So, you got this a and what is this answer again?

Student A1-03: [00:07:34] $10a$ cubed.

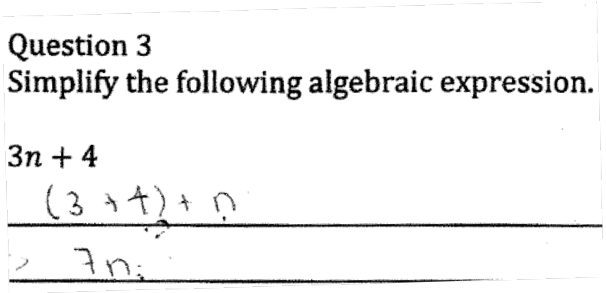
Researcher: [00:07:37] $10a$ cubed. And what is the a cubed mean then?

Student A1-03: [00:07:41] Um, it's almost the same like this. It's just that you plussing another and that is three as. So that's why you will get a cubed.

Researcher: [00:07:54] Okay, we're going to quickly go to the next one. So, this is question three. Unless you have anything else you want to add about this one.

Student A1-03: [00:08:06] No.

Researcher: [00:08:06] Okay. This is question three. Question three is three. How would you read this question?



Question 3

Simplify the following algebraic expression.

$3n + 4$

$(3 + 4) + n$

$= 7n$

Student A1-03: [00:08:12] Three *n* plus four.

Researcher: [00:08:15] Three *n* plus four. Okay. And so, I see you've used your method with your brackets again. And so, what did you put in the first bracket?

Student A1-03: [00:08:25] Um, so I put the numbers in the first bracket and it's three plus four because, and then I just took the n away and then I added it again at the end afterwards.

Researcher: [00:08:37] Okay, so you took the n away and put it there. And so, when you add these you get the seven and then what happens here at the end again?

Student A1-03: [00:08:47] Um, so you get three plus four which is seven. And then plus the n. So, you don't have to say seven plus n, you just put it together seven n.

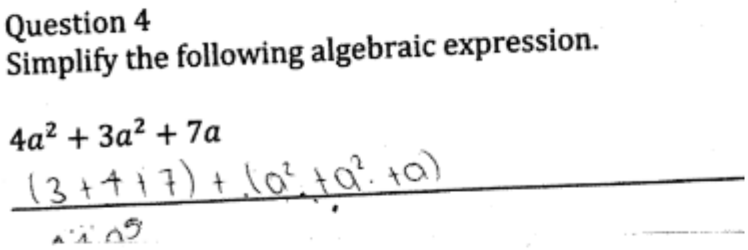
Researcher: [00:08:59] And when you put it together, what does seven n mean to you? When you put it together. So that actually means.

Student A1-03: [00:09:09] Um. Actually, I haven't thought of that.

Researcher: [00:09:14] Okay, so what have you written here? I didn't actually catch that. You wrote?

Student A1-03: [00:09:18] I did seven plus n because instead of you don't have to say seven plus n, you can just put it together and say $7n$.

Researcher: [00:09:31] Let's look at question four. So, question four is simplify the following expression. And it says how would you read this.



Question 4
Simplify the following algebraic expression.

$4a^2 + 3a^2 + 7a$

$(3+4+7)+(a^2.+a^2.+a)$

a^5

Student A1-03: [00:09:41] $4a$ squared plus $3a$ squared, plus $7a$.

Researcher: [00:09:46] And if we look at your method again, can you explain your method how did you do it?

Student A1-03: [00:09:51] So again, I use the um, numbers and I put it all together. And then I plussed the variables I mean the um, the letters and I just keep the exponent with the letters and a squared plus a squared, which will be a with a four and then you just have this one but still the same. So, then a plus a this is as a one of exponent of one. So you just plus four plus one which is five. That's how I got on to 14.

Researcher: [00:10:28] And in other words your 14 your, how do you say this?

Student A1-03: [00:10:34] a five.

Researcher: [00:10:37] Okay, so $14a$ five. This means, what is the a five mean to you?

Student A1-03: [00:10:44] The a five to me means that there are five as. So, they just they instead of writing $14aaaaa$ you just write $14a$ five.

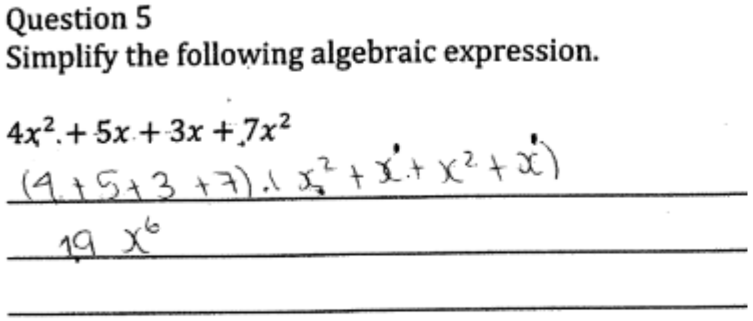
Researcher: [00:11:02] Okay. And when you write the $14a$ five, right. You've said that just like in the previous ones, you've said that you worked out your numbers, you got 14 and then, you basically, what have you done with the as you've...?

Student A1-03: [00:11:30] Plussed it together.

Researcher: [00:11:31] Yeah. And you've found that and then at the end?

Student A1-03: [00:11:35] I just put it together instead of saying 14 plus a five, I just said $14a$ five.

Researcher: [00:11:40] Thank you. All right, so this next one is question five. So, it's almost the same like question four. So, can you read your answer to me, what's your answer?



Question 5
Simplify the following algebraic expression.

$4x^2 + 5x + 3x + 7x^2$

$(4+5+3+7)(x^2+x+x^2+x)$

$19x^6$

Student A1-03: [00:12:04] I said 19x six.

Researcher: [00:12:08] 19x six. And can you tell me, um, just again from the start, what you did to get to 19x six?

Student A1-03: [00:12:15] So, first I plussed all the, I took all the, um, numbers and I left the variable, the letters aside for so long and four plus five plus three plus seven, which got me to nineteen. And then I did not put the plus sign there. So, that would have been plus. And then I just took all the letters. So, for here it's the same. The x has an exponent of one which I saw and then you just say x two, plus x, plus x two, plus x. So , x two, plus one, plus two, plus one, that got me to six. So, I just said nineteen x six.

Researcher: [00:13:03] And so what does x six mean to you?

Student A1-03: [00:13:06] It means that there are six xs and you just making it easy for you to write it out.

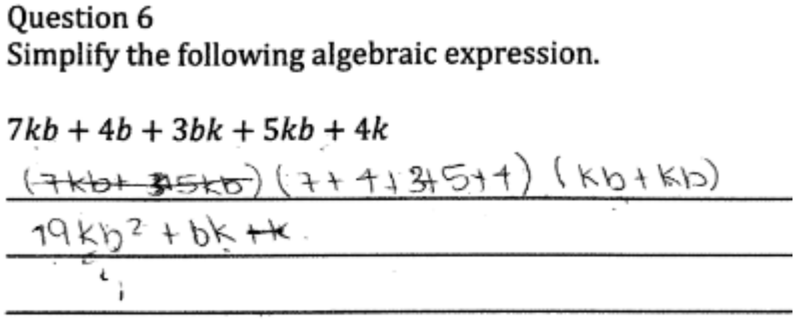
Researcher: [00:13:13] Okay, and you've stopped there, that's the end of your...?

Student A1-03: [00:13:18] Yeah.

Researcher: [00:13:18] So anything you want to say?

Student A1-03: [00:13:21] No.

Researcher: [00:13:21] Right, this is question six okay. So, question six says the following: Simplify the following algebraic expression. Can you read this to me?



Question 6
Simplify the following algebraic expression.

$7kb + 4b + 3bk + 5kb + 4k$

~~(7kb + 3 5kb)~~ $(7+4+3+5+4)(kb+kb)$

$19kb^2 + bk$ ~~+k~~

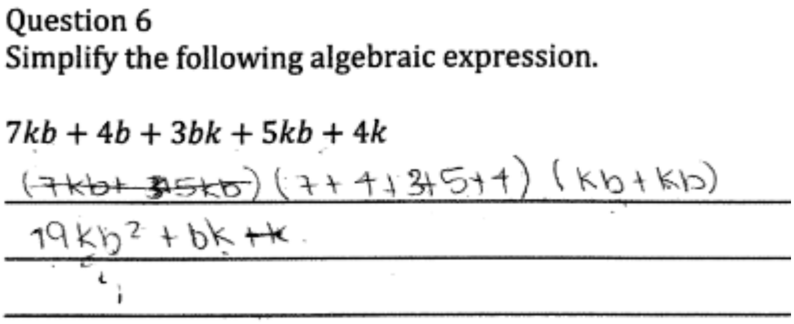
Student A1-03: [00:13:34] Seven kb, plus 4b, plus 3bk, plus 5kb , plus 4k.

Researcher: [00:13:40] Okay, and what have we done here in the second line?

Student A1-03: [00:13:46] So, I definitely know this one is wrong because I was very confused.

[The video recorder unexpectedly ended. The interview was continued in Part two.]
Name: Student A1-03 (Part 2)

[The video recorder stopped unexpectedly. This is a continuation of the interview.]



Question 6
Simplify the following algebraic expression.

$7kb + 4b + 3bk + 5kb + 4k$

(~~7kb+ 35kb~~) (7+4+3+5+4) (kb+kb)

$19kb^2 + bk$ ~~+k~~

Researcher: [00:00:01] And, it's recording. Okay. So, we're on question six and can you just start from the beginning again and explain what you did?

Student A1-03: [00:00:10] Um, so what I did is I took the numbers and I put it all together in one bracket so that I could differentiate between the two. Then, I didn't put a plus sign here, but the kb plus kb, those are like terms and so I put it together. Uh, then I got the answer of $19kb$ two. Why's it kb two, because there's two. It's the set of two. So, that's why there's two there. Um, and then I think I did this wrong.

Researcher: [00:00:38] Okay. So, then you had the bk plus a k. Okay, okay. So, you didn't put these bs the bk with the kb. Um, because of the order, it was different hey. You said? Right. So, um, let's quickly go from the beginning. So, we have the kb two because there are two. That's what you said. Now, let's quickly speak about k and kb. And, so if I write here, kb and bk these are.

Student A1-03: [00:01:16] Unlike terms.

Researcher: [00:01:17] Unlike terms and the reason for that?

Student A1-03: [00:01:19] Because the k is first and the k is second in this one. So that's why it's different.

Researcher: [00:01:24] Okay, so you think that they are different?

Student A1-03: [00:01:27] Yeah.

Researcher: [00:01:27] And, if we are comparing that k that you wrote there with the bk they also different?

Student A1-03: [00:01:34] Yes. Um, because the k is separate and the bk is like a pair. So, if you had to add that, actually you have to add it in bk squared.

Researcher: [00:01:47] Okay, so you just decided on the day that you're going to leave it like this?

Student A1-03: [00:01:52] Yes.

Researcher: [00:01:52] Okay. Um, so we were speaking about bk and kb. So, here's kb and add these bk and we spoke about um, how you told me that we can make the k a 1.

Student A1-03: [00:02:13] And the b two.

Researcher: [00:02:15] Okay. So, you said you can make k one and b two. So, let's write that down k is one and b is two. And then, I asked you if I give you a seven kb, Then that would mean if k is one and b is two then, what does that mean?

Student A1-03: [00:02:34] This means that seven hundred and twelve because the letters are just in place for the numbers.

Researcher: [00:02:41] Okay, good. And if I ask you next $7kb$ plus, let's keep it $5kb$. So, what would that be?

Student A1-03: [00:02:55] This would be seven hundred and twelve plus five hundred and twelve.

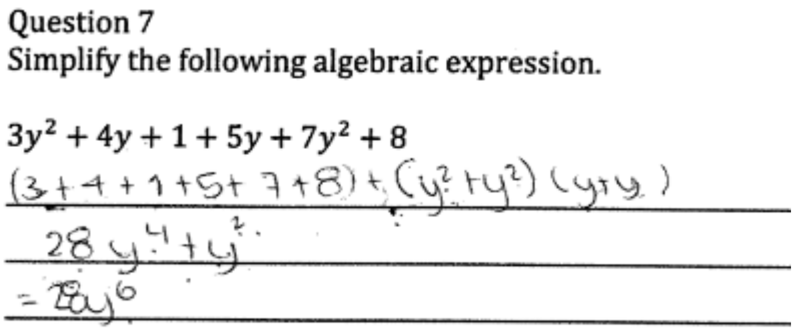
Researcher: [00:03:03] All right. So, it's the kb that we were discussing. Now, the other thing, let's look at this answer. If you read that part what is this going to be

Student A1-03: [00:03:16] $19kb$ squared.

Researcher: [00:03:17] Okay, and if I ask you what is the kb squared mean? What does it mean?

Student A1-03: [00:03:23] kb squared means that there are two of it. So, kb plus kb which is kb two because there are two here in the bracket.

Researcher: [00:03:32] All right. Thank you, I think we can go to the next one okay. So, let's look at this question. This one is question five. But let's actually go the other way around because everything is. Okay. Let's look at this. So, this is question seven. And question seven is about these terms here and can you read your final answer?



Question 7
Simplify the following algebraic expression.

$3y^2 + 4y + 1 + 5y + 7y^2 + 8$

$(3+4+1+5+7+8)+(y^2+y^2)(y+y)$

$28y^4+y^2$

$=28y^6$

Student A1-03: [00:03:59] $28y$ six.

Researcher: [00:04:00] Okay. So, this is $28y$ six and can you tell me how you've gone about answering this question?

Student A1-03: [00:04:07] So firstly, like what I did was I took all the numbers and I separated it from the letters and I just put it in its own bracket. Then, I plus it all together and then I plus the variables, the letters with the um, all the letters together. But, this one is like terms because they were squared and these aren't because they aren't squared. That's why I just said $28y$ four plus y two, because that became an exponent now because it was two. So that's why I just added the four and the two. And that gave me $28y$ six.

Researcher: [00:04:48] So, in other words, when you have $28y$ six you've got the 28 from the these from these numbers and the y six. What does that mean?

Student A1-03: [00:04:58] y six meaning that they are six ys, but instead of writing all of it out, you can just write y six.

Researcher: [00:05:04] Okay. And all six ys, they are what's happening to the six ys?

Student A1-03: [00:05:10] They just been, um. Simplified.

Researcher: [00:05:14] Simplified and when you say simplified, what does that mean in in maths?

Student A1-03: [00:05:19] Simplified makes it easier for you to write it out and work it out.

Researcher: [00:05:24] Okay and if I were to ask you, is there an operation that is happening with the ys, an operation you know, the four operations?

Student A1-03: [00:05:35] So there's addition happening with the two that I can.

Researcher: [00:05:39] Okay, over here. Okay. And where else?

Student A1-03: [00:05:44] Um, that's it. Addition.

Researcher: [00:05:46] Okay, so you say so, the ys. It's...

Student A1-03: [00:05:52] Addition.

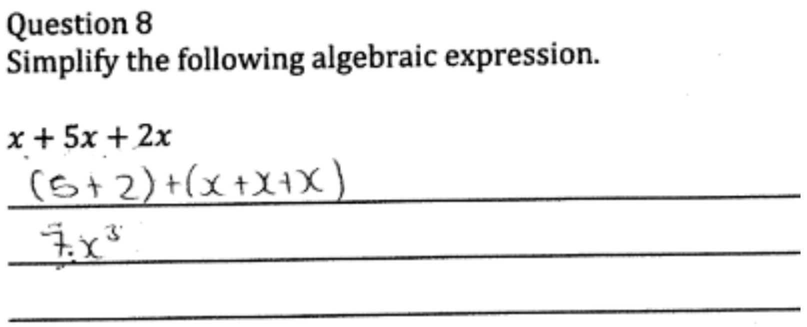
Researcher: [00:05:53] Okay. All right, so let's just look. So, this is question seven and let's ask you quickly about this from here to here. Right, so here you have your y two...?

Student A1-03: [00:06:19] To the power of four plus y to the power of two.

Researcher: [00:06:21] Okay, and there you've got bks and k and kb squared. So, here you've left it like this? But then you've put it together. So, what's the reason for you simplifying to that last step?

Student A1-03: [00:06:39] So before, before the ys were plus together because they are exponents, that's why they were like terms. These does not have exponents as I put it together but when I plus the two together this gave me four. And this became an exponent which would automatically be a like term because they both ys and they both have exponents so I just plussed the two.

Researcher: [00:07:04] All right. Let's go to the next one. Here we've got question eight. Okay, so can you explain what you're doing here?



Question 8
Simplify the following algebraic expression.

$x + 5x + 2x$

$(5+2)+(x+x+x)$

$7.x^3$

Student A1-03: [00:07:32] So the sum is x plus $5x$, plus $2x$. So, I just took the letters and the numbers and I separated it and I plus it all. So five plus two gave me seven and as you can see there's x, x and x. So, I just plus all together. And I got x three because there are three xs. So that made it easier for me to write it out.

Researcher: [00:07:55] And $7x$ three like you say. Um, what is the seven, what is the relation between the seven and the x three? What's the connection there?

Student A1-03: [00:08:13] Um, the seven is the base. Uh, and that is just so that the sum the gave the gave the sum. So, that would have been that automatically be seven if it had to be all xs, and that the two and the five wasn't there, then it would have just been x cubed.

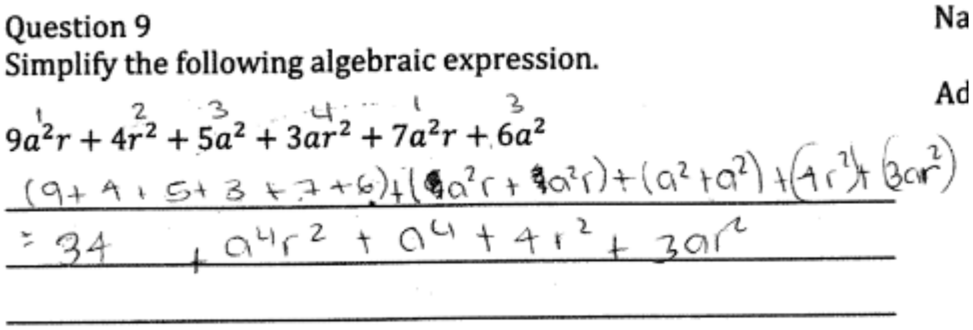
Researcher: [00:08:33] So, is there something happening between the seven and the x three.

Student A1-03: [00:08:37] No, they just plussed together and they become one.

Researcher: [00:08:40] So, you say plus together? So that means...

Student A1-03: [00:08:43] They become one expression.

Researcher: [00:08:45] Okay, all right. This one similar to the other one okay. So, I'm not going to say anything now you just going to tell me everything that you've done there okay. So, you go for it.



Question 9
Simplify the following algebraic expression.

Na

Ad

$\overset{1}{9a^2r} + \overset{2}{4r^2} + \overset{3}{5a^2} + \overset{4}{3ar^2} + \overset{1}{7a^2r} + \overset{3}{6a^2}$

$(9+4+5+3+7+6)+(a^2r+a^2r)+(a^2+a^2)+(4r^2)+(3ar^2)$

$=34 + a^4r^2 + a^4 + 4r^2 + 3ar^2$

Student A1-03: [00:09:00] So, firstly what I did was I passed all the numbers together and then what I did was I looked for like terms because the sum was long. I tried to take my time with it and look through if there was any like terms or anything. Because as you can see, the two is before the r anyways, after the r. So, I just numbered it according to the same,so this will be 123413. So those are the same and those are the same. So, I just did the, did what the numbers did. So, I just plussed all the numbers together and then I did the like terms for the letters and I got that. So, I left this this in brackets so I don't get confused because all in brackets makes it easier. Then I got 34 plus uh, a four r two, a four plus $4r$ two plus, $3ar$ two. So, that is the answer I got.

Researcher: [00:10:00] Can you explain how you got the a four r two from the second line?

Student A1-03: [00:10:08] So, what I did was I said a so the a has an exponent of two, so two plus two is four which gave me a four. And then, since they are two rs I just said a $4r$ two because there are two rs. And the as have two exponents on top, and I just plus the two so I got the four.

Researcher: [00:10:29] Now, if I give you this and I say $a\ two\ r$ right. And I say $ar\ two$. So, what is the difference between those two?

Student A1-03: [00:10:46] So, the difference between two is that the a has the exponent here and the r has the exponent here, which means that there are two rs and there are two as here. So, in this expression with this same here it has. So, it will be a aar. And this will be arr because there are two rs here and there are two as here.

Researcher: [00:11:10] And how are these connected to each other? Or they just are they just together. What is their connection to each other?

Student A1-03: [00:11:18] Um, they are sum. So, if there was a sum like $b\ two\ r$, then I would have plus the two because it has the same, then it would have been ab two r two.

Researcher: [00:11:35] Okay. Right. So, bs you've left.

Student A1-03: [00:11:39] Yeah.

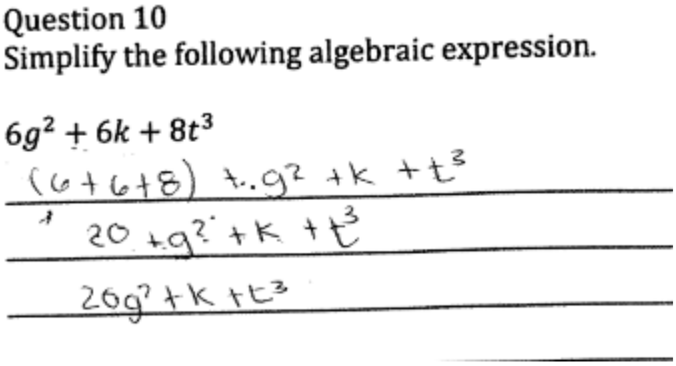
Researcher: [00:11:40] And then the a two here and the a two...

Student A1-03: [00:11:44] Gave me a four because two plus two is four.

Researcher: [00:11:47] And so what does a four mean to you?

Student A1-03: [00:11:50] A four means that there are four as and I just simplified it.

Researcher: [00:11:54] All right. Our last one, so this is the last one. Question ten and as you can see the question says simplify the following algebraic expression and the expression is, can you read the expression?



Question 10
Simplify the following algebraic expression.

$6g^2 + 6k + 8t^3$

$(6+6+8) + . g^2 + k + t^3$

$20 + g^2 + k + t^3$

$20g^2 + k + t^3$

Student A1-03: [00:12:08] $6g$ squared, plus $6k$, plus $8t$ cubed.

Researcher: [00:12:13] Okay, and then can you explain how you've done it?

Student A1-03: [00:12:16] So, again I just took all the numbers and I put it into one bracket. So, as you can see there aren't any like terms between the letters. So, I just left it as is and I said g two plus k plus t three. Then I said twenty plus g two plus k plus t three. And I just plus the 20 plus g squared, and that gave me $20g$ squared, plus k plus t three.

Researcher: [00:12:41] And you've left it like this?

Student A1-03: [00:12:42] Yes, because there wasn't any, because the k or the t aren't the same so I just didn't plus that at all. So, I just left it like that.

Researcher: [00:12:50] And like terms mean, what do like terms mean?

Student A1-03: [00:12:53] Like terms mean that they are the same. So like.

Researcher: [00:12:56] What's the same?

Student A1-03: [00:12:59] Um, so if it had to be $2ab$ plus $3ab$, that would have been a like term and in the would of just been ab squared and that would have been five. So, $5ab$ squared.

Researcher: [00:13:14] So what makes these like terms?

Student A1-03: [00:13:16] Because they are the same in letters.

Researcher: [00:13:20] Same letters. Yes, okay. All right and so that's question ten. All right, now I'm just going to ask you some general questions. Just to see how it goes, but let's just have a look back at our questions again and just go through. I think you explained this one quite well. Uh, this one we said you changed. Yes, this one is fine. That's fine. I think you've done an excellent job in explaining how you've thought about everything. Okay, so what I'm going to do now is I'm just going to ask you some general questions, and you can just, you know, any ideas that come to your mind, anything you think, you just say how you think of it. So, the first one is what does x squared mean?

Student A1-03: [00:14:10] So, x squared means that the x plus x.

Researcher: [00:14:15] And can you tell me what a squared b means?

Student A1-03: [00:14:21] a squared b means that there are two as and there's a b. So, that's why there's a squared because this a is there. So, that's a squared.

Researcher: [00:14:33] Now here you've written plus and here you've written them next to each other.

Student A1-03: [00:14:37] Yeah.

Researcher: [00:14:38] Is there a difference between the two?

Student A1-03: [00:14:41] Um no.

Researcher: [00:14:43] Okay. So what would you change?

Student A1-03: [00:14:45] That's been a plus ab.

Researcher: [00:14:51] Um, so if I gave you nine plus x squared, what would this be if you had to simplify this?

Student A1-03: [00:15:06] I would say this is $9x$ two.

Researcher: [00:15:13] And so let's look at our question here. So it says here $9x$ two. You got the nine, you got the x two from this and then you had $9x$ two. So what is $9x$ two mean to you? It means...?

Student A1-03: [00:15:27] That I, so if Mam had to give me this I would have gotten this answer because it's just easier. But if it had to come in the form like that which it did, then you would I would do it easier and slower and step by step to get to that answer.

Researcher: [00:15:42] Okay. So if I look at this and you have six plus three and your answer was.

Student A1-03: [00:15:50] Nine.

Researcher: [00:15:51] Nine, now can six plus three also be six three.

Student A1-03: [00:15:57] No.

Researcher: [00:15:58] No, because then this would be...

Student A1-03: [00:16:00] Sixty-three.

Researcher: [00:16:01] Okay, and so if I have x, plus x then what can that be? How else can I write x plus x?

Student A1-03: [00:16:15] You can write it x squared. Yeah.

Researcher: [00:16:23] Okay, now we're going to ask just like some comparison questions. So what's the difference between this and this...?

Student A1-03: [00:16:33] Okay, so the difference between this is that this is $2x$. This is the sum and there's only one of this. So, if there was an exponent there would have been $2x$ plus $2x$ plus so on. But for this you have x plus x which is two. The exponent shows that this shows how many of this. So, the exponent shows how many of the letter there are. So, if it's two then it would be x plus x. So that's the difference.

Researcher: [00:17:06] Now what is… Now I'm going to give you something okay. So, let's take a clean one. If I say xx plus x what is the answer. What would you simplify this. To simplify that. Okay. Now, if I said, um, x plus x plus x, what would that be? If you must simplify it. Okay. Now you learned in algebra about letters and variables and what do the letters and the variables represent?

Student A1-03: [00:17:48] Um, so the letters, so if this had to have to you don't always know. They don't always tell you what the letters are, but the letters represents a number.

Researcher: [00:17:59] Now let's choose a number for x okay.

Student A1-03: [00:18:01] Five.

Researcher: [00:18:01] Okay, five. I'm going to go with five? Okay. Now let's quickly say let's say what five plus five is. Okay. Now, if x was the number five and I wrote down x squared, this would be like saying x plus x because that's the number five right. Now what would x squared be if I write that down. So that would be five squared. Yeah. What's five squared?

Student A1-03: [00:18:33] Five squared would be five plus five.

Researcher: [00:18:37] Okay. And that is?

Student A1-03: [00:18:40] 10

Researcher: [00:18:40] Okay, now do you know your perfect squares? So, it's one times one is.

Student A1-03: [00:18:47] Two, two times two is four. Four times four is eight.

Researcher: [00:18:52] Are those the perfect squares?

Student A1-03: [00:18:54] Yay!

Researcher: [00:18:57] Yes. Okay. So what five squared?

Student A1-03: [00:19:02] Five squared is ten.

Researcher: [00:19:05] All right. So, let's quickly look at another one. So, what's the difference between x and x squared?

Student A1-03: [00:19:17] So, the difference is that this is an exponent and this is saying that there are two xs and this is just x.

Researcher: [00:19:33] Okay, let's look at the following. If I gave you, if I gave you this: x plus y and I tell you to simplify, what would you write? Okay. So now we've got xy, right.

Student A1-03: [00:19:57] Yeah.

Researcher: [00:19:58] So can you tell me, if we were now to take a number for x and y. Let's take easy numbers so what's a number for x.

Student A1-03: [00:20:07] Two.

Researcher: [00:20:08] And a number for y.

Student A1-03: [00:20:09] Three.

Researcher: [00:20:10] Okay, so if I get that then I get x plus y is two plus three which is.

Student A1-03: [00:20:18] Five.

Researcher: [00:20:19] And xy, what does that mean?

Student A1-03: [00:20:24] Oh.

Researcher: [00:20:25] What does xy mean?

Student A1-03: [00:20:28] Two plus three.

Researcher: [00:20:30] What do you think xy means? I mean it's all about...

Student A1-03: [00:20:35] Um. This, I would say that it. If you say x plus y we simplify it, just making this easier to see which is not the plus sign. So this would still be five if the letters would still okay. And if.

Researcher: [00:20:51] I wrote two three.

Student A1-03: [00:20:54] That wouldn't work because it's different because that's twenty plus three.

Researcher: [00:20:59] Yeah.

Researcher: [00:21:00] So it's different. Okay, so if we go back to this one here.

Researcher: [00:21:23] So if you go back here. Where's that one? And you look at this now, so do you still think this is still 712? 512 if it's kb and bk.

Student A1-03: [00:21:34] If it's kb and bk. Yeah. Um, no, because if this is kb and this is bk, then obviously means it swapped. So, this numbers would change.

Researcher: [00:21:47] Okay. So what would it be?

Student A1-03: [00:21:49] So then it would have been 521.

Researcher: [00:21:54] All right. So the last question, is why do you think we can add or.... let's just say that $6a$. So just lastly, if you add this what you get? If you simplify it?

Student A1-03: [00:22:19] So, I would plus the variables again, the letters the numbers again. And then, so because this is alone, it has an invisible one. So then you would say a three plus a which is equal to 11 a four.

Researcher: [00:22:46] All right. Thank you so much.

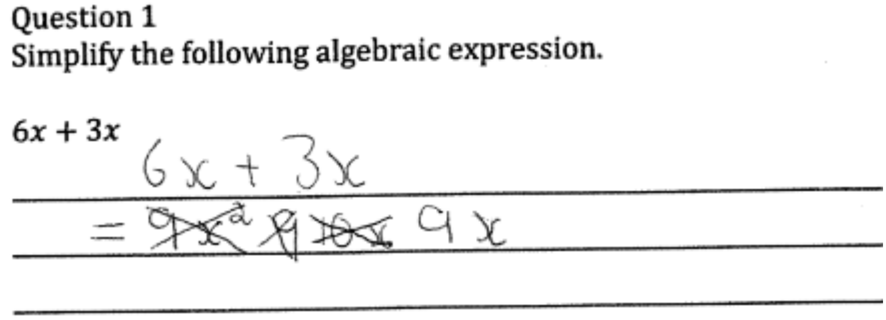
Student A1-03: [00:22:48] You welcome

Researcher: [00:22:48] I think we're going to stop there.

8.2 Student A2-20

Name: Student A2-20 (Part 1)

Researcher: [00:00:01] Um, yeah. So, let's just have a look here. That's fine. This is your answer sheet for question one and so we're just going to go through a couple of things. Um, you can have a look quickly and see what you've done. I'm just going to get the questions ready. Okay. So let's quickly start with question one. So if you look at that, can you tell me this question says simplify the following algebraic expression and can you read that question for me?



Question 1
Simplify the following algebraic expression.

$6x + 3x$

$6x + 3x$

$= \cancel{9x^2}\ \cancel{9}\ \cancel{10x}\ 9x$

Student A2-20: [00:00:53] Um, $6x$ plus $3x$.

Researcher: [00:00:56] And your answer, can you read that?

Student A2-20: [00:00:57] $9x$.

Researcher: [00:00:58] So, can you explain in detail how you got to your answer of $9x$?

Student A2-20: [00:01:05] I just added six plus three and got nine and x plus x equals...

Researcher: [00:01:13] Okay, so how have you decided to have an x there?

Student A2-20: [00:01:18] Uh, because, uh, the variables are the same.

Researcher: [00:01:22] Okay, so if it's the same, what does that mean when we are simplifying algebraic expressions?

Student A2-20: [00:01:29] Um.

Researcher: [00:01:31] You can write it. You can write over here if you want. Just not at the bottom of this, okay.

Student A2-20: [00:01:37] Um, it's just, it's easier to add because, um, I know if the terms aren't like, you can't add it either.

Researcher: [00:01:46] Okay, so if they are...?

Student A2-20: [00:01:49] Like, like terms.

Researcher: [00:01:51] So are these like terms?

Student A2-20: [00:01:53] Yes.

Researcher: [00:01:53] Okay, and what would be the opposite of like terms.

Student A2-20: [00:01:56] unlike terms.

Researcher: [00:01:57] Okay. So, these are like terms and the reason they are like terms is because...

Student A2-20: [00:02:01] They are the same. Um, uh, variables, letters.

Researcher: [00:02:06] Letters

Student A2-20: [00:02:07] Yeah.

Researcher: [00:02:07] Okay, so you said, add the six and the three, you got the nine, and then how did you decide that it's $9x$?

Student A2-20: [00:02:16] Oh, I said x plus x is x. So obviously three plus six and nine and x plus $6x$ okay.

Researcher: [00:02:25] And what does $9x$ mean?

Student A2-20: [00:02:29] Um, it's the answer.

Researcher: [00:02:31] Is it the answer, okay. So it's the answer, so if I gave you, I just wrote to $9x$ here and I asked you what does that mean. You must explain to me what does $9x$ mean in algebra?

Student A2-20: [00:02:53] I don't know how to explain.

Researcher: [00:02:57] Okay, what's the nine in relation to the x? How is that? What's the connection with the nine and the x?

Student A2-20: [00:03:06] Um.

Researcher: [00:03:14] Can you tell me what's the relation between this [the expression] and that $9x$?

Student A2-20: [00:03:21] There's nothing really to say, it was the answer to the question.

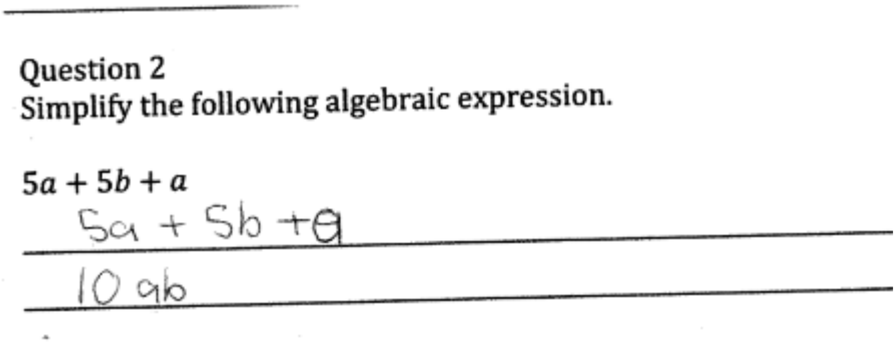
Researcher: [00:03:25] Okay, so the question and now we're just going to go really like, um, to a point where we, the question is $6x$ plus $3x$. So if you, if I asked you, there are four operations, which operation are you doing there?

Student A2-20: [00:03:44] Um, addition.

Researcher: [00:03:46] Addition. So $6x$ plus three, if we add that we get.

Student A2-20: [00:03:51] $9x$

Researcher: [00:03:52] Okay. Um, we might come back to this one. Okay. All right. Let's look at the next one. We've got something different here. So this is question two: Simplify the following algebraic expression. We've got $5a$, plus $5b$, plus a. Right, and what you've done there is you've written a line down okay. So basically what's this line? It's…



Question 2
Simplify the following algebraic expression.

$5a + 5b + a$

$5a + 5b + a$

$10\ ab$

Student A2-20: [00:04:20] a.

Researcher: [00:04:21] Okay and then you've gotten your answer of what's that. Can you read it?

Student A2-20: [00:04:26] $10ab$

Researcher: [00:04:27] Okay, so if you got $10ab$, can you explain how you got the ten?

Student A2-20: [00:04:31] Oh I added five, plus five, is equal to ten and then a plus b is equal to ab equals. I didn't have to add the other a.

Researcher: [00:04:41] And the reason why you don't have to add the a is because the other a?

Student A2-20: [00:04:46] You don't have to put the, um like terms together.

Researcher: [00:04:52] All right and so can you tell me what does ab mean?

Student A2-20: [00:05:01] Um, ab was the the answer to the, obviously a plus b...

Researcher: [00:05:09] So does this mean a plus b to you?

Student A2-20: [00:05:12] Yeah,

Researcher: [00:05:13] Yeah. So, if I ask you to just write down ab equals can you write it down. It's the same as for you...

Student A2-20: [00:05:25] a plus b

Researcher: [00:05:25] So, how does that look if you write it down. Okay. Right. And we have a ten here from the...

Student A2-20: [00:05:34] Five plus five.

Researcher: [00:05:35] Okay. So if I were to ask you what did you do first to get to the $10ab$. What did you do first?

Student A2-20: [00:05:43] I first added the numbers.

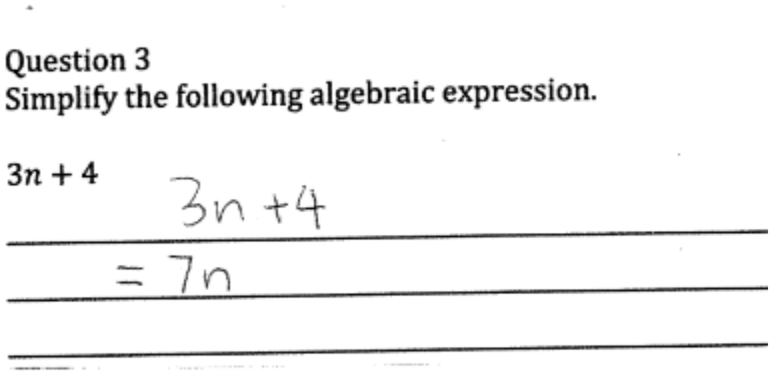
Researcher: [00:05:46] The numbers okay. And then.

Student A2-20: [00:05:50] And then I added the letters.

Researcher: [00:05:52] Okay. And you also looked for the....

Student A2-20: [00:05:54] Like terms and (inaudible).

Researcher: [00:05:59] All right. Let's look at the next one. So this one is...?



Question 3
Simplify the following algebraic expression.

$3n + 4$

$3n + 4$

$= 7n$

Student A2-20: [00:06:03] $3n$ plus four.

Researcher: [00:06:04] Yes. And your answer is...

Student A2-20: [00:06:06] $7n$.

Researcher: [00:06:07] Okay. So how did you get to that $7n$?

Student A2-20: [00:06:10] Added um $3n$ plus four. Well, first I added the numbers obviously. And since there was no other, term um, variable, I just...

Researcher: [00:06:20] Okay. So your answer is seven and you said you add the numbers first. Can I ask you what is $7n$ mean?

Student A2-20: [00:06:30] Um, it means the answer to that question.

Researcher: [00:06:37] And the question is asking you to do what?

Student A2-20: [00:06:43] To simplify...

Researcher: [00:06:44] To simplify and the operation that you are doing within this question is..

Student A2-20: [00:06:49] Addition.

Researcher: [00:06:50] Addition. So what does $7n$ mean? Your answer.

Student A2-20: [00:06:54] It means that for $3n$ plus four is equal to $7n$.

Researcher: [00:07:04] So, if I were to ask you to write out this in steps, could you write it out in some steps for me? What do you do first? And then you write that down.

Student A2-20: [00:07:14] I would say three plus four which is go to seven and then the n is just standing there. So, I would put it once because there isn't another variable to add to it.

Researcher: [00:07:28] So when you tell me that you put the n there right. So what are you doing when you're putting the n next to the seven?

Student A2-20: [00:07:37] I am adding a variable to it.

Researcher: [00:07:39] Okay, so can you tell me in your mind $7n$ is the same as...

Student A2-20: [00:07:46] $3n$ plus four.

Researcher: [00:07:49] If I ask you, what's $5n$?

Student A2-20: [00:07:56] Five plus n.

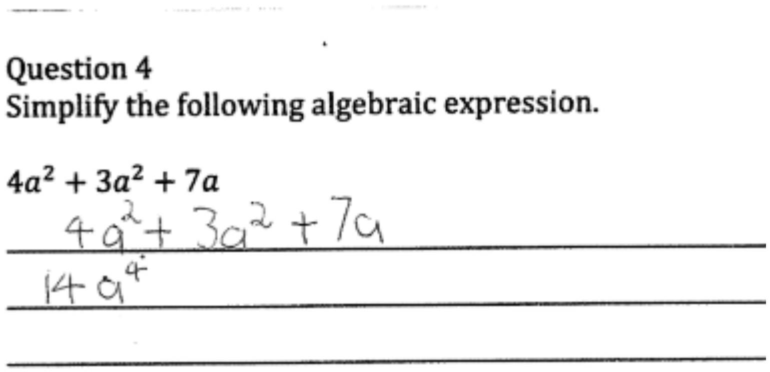
Researcher: [00:07:58] Okay. Is that five plus n? And if it's $7n$?

Student A2-20: [00:08:01] 7 plus n.

Researcher: [00:08:02] So can you write that quickly? Okay. So that would be five plus n and this one would be?

Student A2-20: [00:08:11] Seven plus n.

Researcher: [00:08:12] Okay. All right. So let's go to the next question. This is question three. Now we're on question four. Question four has the following okay. So I'm going to give you a clean sheet. Okay. Question four. It's $4a$ squared plus $3a$ squared plus $7a$. So can you tell me quickly can you read your answer for me?



Question 4
Simplify the following algebraic expression.

$4a^2 + 3a^2 + 7a$

$4a^2 + 3a^2 + 7a$

$14a^4$

Student A2-20: [00:08:43] $14a$ to the power of four.

Researcher: [00:08:45] Okay. So um, can you tell me how many, how you got to the, um, final answer, how many steps you took to get to the final answer?

Student A2-20: [00:09:02] First, I did the numbers, um, and I got 14, and then I added the variables, but they're all like, so it's just one a and then I added the, um, the to the power of two plus two is four.

Researcher: [00:09:17] Okay. So can I quickly see, if you were to write out the steps for this like you did with the previous one. Can you do that for this one quickly?

Student A2-20: [00:09:29] I'd say four plus (inaudible). And then I'll do. The. I don't know how you do squared. [referring to the exponents]

Researcher: [00:10:00] Okay. So these are your steps that you would have written out for this one. So it's first the numbers, that are these numbers that you refer to right 14, then. So what happened here? So is it because.....

Student A2-20: [00:10:22] Um, I'm feeling this is $3a$ because it's a plus, a plus, a but because it was a number, you already, I didn't know.

Researcher: [00:10:31] You just put it as...

Student A2-20: [00:10:32] Yeah.

Researcher: [00:10:33] So how how did you go from these three and choose to just put the a there?

Student A2-20: [00:10:41] Because they were all like terms...

Researcher: [00:10:43] Because they like terms and like terms to you means that they...

Student A2-20: [00:10:47] Um, that they should just be one or 2 or 3. Okay.

Researcher: [00:10:51] And how do you what's the meaning of like terms to you?

Student A2-20: [00:10:55] Um, that they are all the same or similar.

Researcher: [00:10:58] The same in what respect? In what regard?

Student A2-20: [00:11:02] Um, I can't change them.

Researcher: [00:11:09] Okay. So if you were to let's go, let's quickly just go back to this one. Okay. Are these like terms, in question two?

Student A2-20: [00:11:22] No.

Researcher [00:11:23] So why are these not like terms?

Student A2-20: [00:11:25] Because they are all different. They're aren't the same.

Researcher: [00:11:29] Okay. They aren't the same.

Student A2-20: [00:11:33] They are different.

Researcher: [00:11:38] Okay. Different in what? In what respect?

Student A2-20: [00:11:42] Um. This one doesn't have a number. First of all. This is $5a$ and this is $5b$. So they are all, some are different letters.

Researcher: [00:11:53] Okay. So they're not like terms. Okay. So, here when you say the a, plus a, plus a on this page then you get a because they are like terms. That's what you said. So how does this, how are they like terms?

Student A2-20: [00:12:07] Um, because they're all the same letter.

Researcher: [00:12:10] Okay. They're all the same letter and in this part here you got.

Student A2-20: [00:12:15] Four.[referring to the exponent]

Researcher: [00:12:16] From.

Student A2-20: [00:12:17] From two plus two.[referring to the exponents]

Researcher: [00:12:18] And those are what...

Student A2-20: [00:12:21] Those are the power of....

Researcher: [00:12:25] Okay, can I just ask you, what does this mean to you? [Researcher writes a^2 on the page]

Student A2-20: [00:12:31] a squared.

Researcher: [00:12:32] Okay, and what does that mean?

Student A2-20: [00:12:34] a plus a.

Researcher: [00:12:36] Okay, so write it down. I just want to get a sense of what it means to you.

Student A2-20: [00:12:42] a times a. All right.

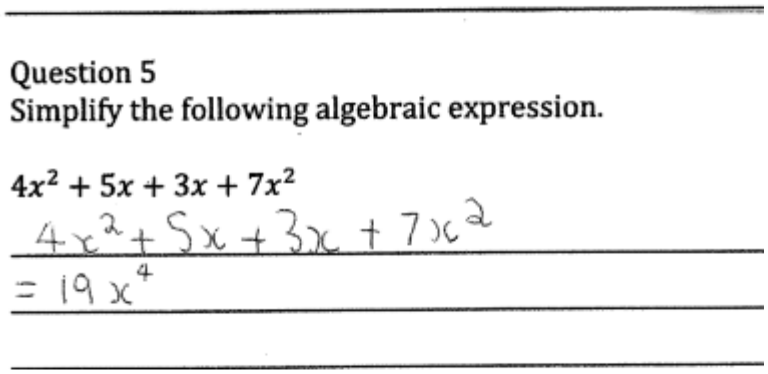
Researcher: [00:12:48] Do you think it's a times a?

Student A2-20: [00:12:50] Yes

Researcher: [00:12:50] Okay, so can you tell me what is a to the four? What does that mean to you?

Student A2-20: [00:13:00] a times, a times, a times, a.

Researcher: [00:13:05] Okay. All right, Let's go to the next one. Okay, so here we've got another similar case, right. This is question five and just to... Okay, so question five asks you to simplify the following algebraic expression. Can you read the question to me?



Question 5
Simplify the following algebraic expression.

$4x^2 + 5x + 3x + 7x^2$

$4x^2 + 5x + 3x + 7x^2$

$= 19x^4$

Student A2-20: [00:13:33] Um, simplify the follow following algebraic expressions: $4x$ squared plus $5x$ plus $3x$ plus $7x$ squared.

Researcher: [00:13:43] Okay, so what would that then tell me in the next line? What have you done in the next line?

Student A2-20: [00:13:51] I rewrote the sum first and wrote the answer.

Researcher: [00:13:57] Okay, and your answer. Can you read your answer to me?

Student A2-20: [00:14:00] $19x$ to the power four.

Researcher: [00:14:02] Okay. And, um, can you explain how you got the 19?

Student A2-20: [00:14:08] I added, um, all of the numbers together. Four, plus five, plus three plus seven, which equals to nineteen. And then I added the variables which were all like, so I just got one x and the power of the two plus the two.

Researcher: [00:14:27] Okay. So the two plus the two...

Researcher: [00:14:30] Equals to the four.

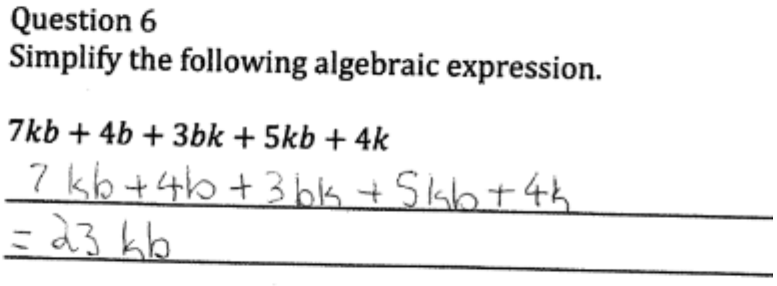
Student A2-20: [00:14:30] And so $19x$ to the four, if I asked you what is $19x$ how do you say this?

Student A2-20: [00:14:40] $19x$ to the power of four.

Researcher: [00:14:42] To the power of four, what does that mean?

Student A2-20: [00:14:47] Uh, it means 19 plus, plus x um, plus to the power of four.

Researcher: [00:14:56] Okay, so can you write it down? 19. Okay. All right, let's look at the next. Okay, we've got one, which is, you can read it. What's the question?



Question 6
Simplify the following algebraic expression.

$7kb + 4b + 3bk + 5kb + 4k$

$7 kb + 4b + 3bk + 5kb + 4k$

$= 23 kb$

Student A2-20: [00:15:30] Simplify the following algebraic expression. $7kb$ plus $4b$, plus $3bk$ plus $5kb$, plus 4k.

Researcher: [00:15:42] Okay. And, so what have you done here in this case?

Student A2-20: [00:15:46] I wrote out the sum first. And I wrote the answer.

Researcher: [00:15:51] So how did you get the 23?

Student A2-20: [00:15:53] I added seven plus four plus three plus five plus four.

Researcher: [00:15:58] Okay. And then you got?

Student A2-20: [00:16:01] twenty-three.

Researcher: [00:16:01] And how did you decide to put a k and a b?

Student A2-20: [00:16:05] Because the variables are all the same just in different order. So I wrote kb first because k was the first one that I added...

Researcher: [00:16:17] Okay. Um, can you tell me what in algebra, what do the letters stand for?

Student A2-20: [00:16:25] uh, they are placeholders.

Researcher: [00:16:27] For?

Student A2-20: [00:16:28] For, for numbers.

Researcher: [00:16:32] Okay. And, so what do the letters representing then?

Student A2-20: [00:16:38] Um. The numbers present. Um.

Researcher: [00:16:49] So, you said numbers. So, what do they represent? The numbers or the letters? So, if I gave you. Let's see, I gave you let's say kb. This is kb. Okay. So, firstly what does this mean kb? If it's written like that?

Student A2-20: [00:17:15] It's placeholders for numbers. So, I only wrote it as kb because of the order that I got it in.

Researcher: [00:17:23] Yeah. And does this mean anything, kb in maths?

Student A2-20: [00:17:31] It doesn't have any exact meaning.

Researcher: [00:17:33] So, you said it's placeholders for numbers. So, if I told you that, let's choose easy numbers okay. So, if I told you that k was two and b was three, then what would that be?

Student A2-20: [00:17:50] It would be five.

Researcher: [00:17:51] So kb then would actually mean what.

Student A2-20: [00:17:56] k plus b.

Researcher: [00:17:57] Okay. So, you said this would be five. So that means k plus b equals five. The two plus the three which is five, okay. So that's how you think about it. So, we've got here then. Now can you maybe tell me what $7kb$ would mean?

Student A2-20: [00:18:21] Oh seven plus k plus b.

Researcher: [00:18:24] Okay. So that would mean seven plus k plus b. Okay, and then that would be an answer of what if I use the numbers?

Student A2-20: [00:18:35] Oh, that would be um, depends if there's still placeholders. So, seven plus five.

Researcher: [00:18:43] That's five. So that will be.

Student A2-20: [00:18:47] Twelve.

Researcher: [00:18:52] All right. Do you think kb and bk are the same thing?

Student A2-20: [00:18:58] Oh, it depends on what the placeholder this is.

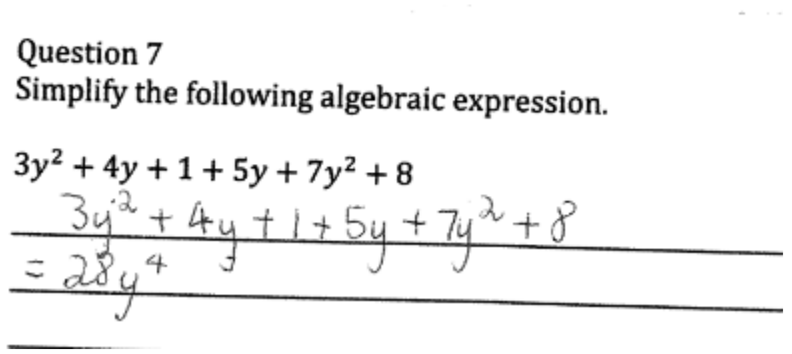
Researcher: [00:19:04] So if I say kb and bk within the same question, let's say that's added. And I tell you k is two and b is three, is kb and bk the same.

Student A2-20: [00:19:20] Um, they unlike.

Researcher: [00:19:22] They unlike they're not the same.

Student A2-20: [00:19:24] Because the order.

Researcher: [00:19:26] Because the order is not the same, so it's not the same. Okay. All right. I just want to go to the next question. We can just come back okay. Here, we've got another case okay and this one I just want you to explain from beginning to end how we would do this, but I'm going to quickly just start it off. Right. This is question seven. Question seven asks you to simplify the following algebraic expression. And you get this expression here. Okay. Can you read the question to me?



Question 7

Simplify the following algebraic expression.

$3y^2 + 4y + 1 + 5y + 7y^2 + 8$

$3y^2 + 4y + 1 + 5y + 7y^2 + 8$

$= 28y^4$

Student A2-20: [00:19:57] $3y$ squared, plus $4y$, plus one, plus $5y$, plus $7y$ squared, plus eight.

Researcher: [00:20:04] So, can you write out here the steps that you would have followed to get your final answer? Can you read that final answer for me?

Student A2-20: [00:20:14] $28y$ to the power of four.

Researcher: [00:20:17] So can you write that out, the steps that you would have followed? And you can explain why you're doing it.

Student A2-20: [00:20:32] Oh, I did the numbers first because, um, I always do the numbers first, and then I will do the letters again. But I would normally add the letters with the numbers so there wouldn't be, um, a placeholder. And to the power of that, I will add last when I get the answer.

Researcher: [00:21:06] Okay. So, first what, was that first step?

Student A2-20: [00:21:09] So, um, adding the numbers.

Researcher: [00:21:12] Okay And then...

Student A2-20: [00:21:14] Um, I got an answer of 28.

Researcher: [00:21:17] Okay, and then what did you do after that?

Student A2-20: [00:21:19] I would add the, um, um, variables.

Researcher: [00:21:23] So, can you write down how you would think about that?

Student A2-20: [00:21:29] Four ys.

Researcher: [00:21:37] Okay, and that will give you?

Student A2-20: [00:21:40] In this sense it will give me just y.

Researcher: [00:21:42] Okay, and then lastly, what's the last thing you've done?

Student A2-20: [00:21:46] Um, I added the value to the power of.

Researcher: [00:21:58] Okay, and then at the end, what do you do to get your final answer?

Student A2-20: [00:22:03] I'll add these together.

Researcher: [00:22:05] So, you're adding them? Okay, and your final answer there was...?

Student A2-20: [00:22:14] $28y$ to the power of four.

Researcher: [00:22:22] Okay. Right, so can I ask you something? Can I ask you, are these, like, terms, these ones?

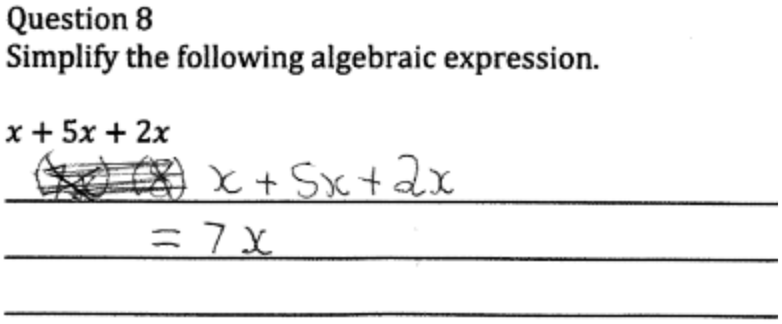
Student A2-20: [00:22:33] Oh, no. No, some of them have, um, squared.

Researcher: [00:22:37] Okay, so if it's $3y$ squared and your $4y$, you say these are not like terms. Okay, what does that mean to you? What's y squared compared to y?

Student A2-20: [00:22:52] y squared is y times y and that is y alone. So, they are unlike.

Researcher: [00:22:59] So they unlike okay. All right.

Student A2-20: [00:23:04] Um. We're going to come back to this. I just want to double check that everything's okay. This is question eight. Okay, and here we've got question eight which is simplify the following algebraic expression and we've got that expression right. So can you read that for me?



Question 8
Simplify the following algebraic expression.

$x + 5x + 2x$

$x + 5x + 2x$

$= 7x$

Student A2-20: [00:23:32] x plus five x, plus two x.

Researcher: [00:23:34] And your answer can you read that?

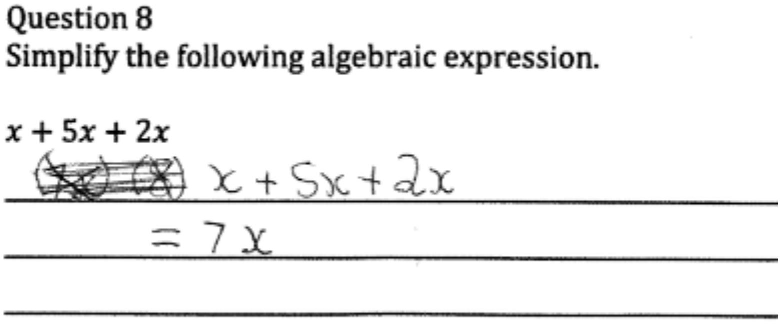
Student A2-20: [00:23:37] $7x$.

Researcher: [00:23:38] Okay. Now can you explain how you got the seven?

Student A2-20: [00:23:41] I first added on five plus two.

[The video recorder stopped unexpectedly. The interview was continued in part 2].
Name: Student A2-20 (Part 2)

[Video Recorder stopped unexpectedly. This is a continuation of the interview.]



Question 8
Simplify the following algebraic expression.

$x + 5x + 2x$

$x + 5x + 2x$

$= 7x$

Researcher: [00:00:00] Okay, this is question eight. I just want to see how you thought about question eight. The question is, can you read the question?

Student A2-20: [00:00:09] x plus five x, plus two x.

Researcher: [00:00:11] Okay, and then can you tell me how you've got that answer?

Student A2-20: [00:00:14] I added five plus two which gave me seven and then I added the variables which were all like so...

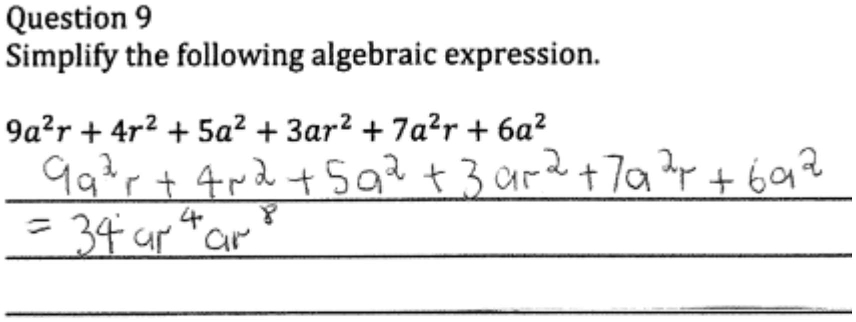
Researcher: [00:00:22] And because they like, you wrote down.

Student A2-20: [00:00:25] x

Researcher: [00:00:25] Okay, and if you have... So you wrote down x plus x, plus x, this is what you're thinking, and that gives you?

Student A2-20: [00:00:37] x (inaudible). But you first add the numbers.

Researcher: [00:00:40] So you first add the numbers, okay. All right. Let's look at question nine. Which nine, you've done the same hey? So, the thirty-four how did you get that?



Question 9
Simplify the following algebraic expression.

$9a^2r + 4r^2 + 5a^2 + 3ar^2 + 7a^2r + 6a^2$

$9a^2r + 4r^2 + 5a^2 + 3ar^2 + 7a^2r + 6a^2$

$= 34ar^4ar^8$

Student A2-20: [00:00:55] I added nine plus four, plus five, plus three, plus seven, plus six.

Researcher: [00:00:59] Okay. You got?

Student A2-20: [00:01:00] Gave me thirty-four.

Researcher: [00:01:01] And then you got?

Student A2-20: [00:01:03] ar to the power four, ar to the power of eight.

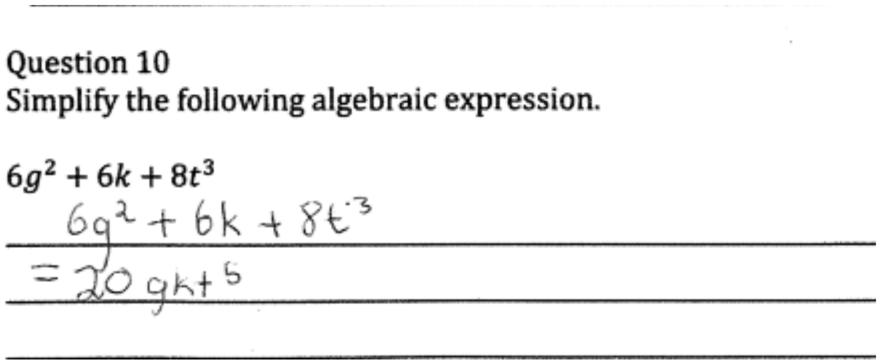
Researcher: [00:01:07] Okay, how did you get the first one?

Student A2-20: [00:01:10] Um here, the a comes first and because it's the first thing I added and the r there so I got ar. Then I added the power to the of, the one that was behind the last letter, which was the two. And I looked and I found another number behind the last letter. So I added this and I got four.

Researcher: [00:01:33] Okay and then...

Student A2-20: [00:01:36] I went again and I looked for the numbers that were the last letters. So, I got a it was again first. So, I got a and then r. And then I added two plus two plus two plus two.

Researcher: [00:01:53] Okay, good. Thank you. Right. Last question. Question ten. If you look at these terms you've got an answer of, can you read that?



Question 10
Simplify the following algebraic expression.

$6g^2 + 6k + 8t^3$

$6g^2 + 6k + 8t^3$

$= 20gkt^5$

Student A2-20: [00:02:03] $20gkt$ to the power of five.

Researcher: [00:02:07] Okay, now how did you get the 20?

Student A2-20: [00:02:10] I added six plus six, plus eight.

Researcher: [00:02:14] And how did you get the...[referring to variables]

Student A2-20: [00:02:16] I added g plus k plus t.

Researcher: [00:02:20] Okay. And the five?

Student A2-20: [00:02:21] I added two plus three.

Researcher: [00:02:23] And that's how you got that. Okay, all right, and are these like or unlike terms?

Student A2-20: [00:02:31] Um unlike.

Researcher: [00:02:33] So the last thing I want to do is just ask you to compare a few things. Okay, so if I gave you x squared and I gave you two x, what's the difference between them or are they the same?

Student A2-20: [00:02:52] Um, they are not the same.

Researcher: [00:02:54] You can write out what you think it is.

Student A2-20: [00:02:59] x squared is....x times x and $2x$ is two x.

Researcher: [00:03:08] Now would you write that down?

Student A2-20: [00:03:13] I would say two, two times x I guess.

Researcher: [00:03:19] Okay, and is there any other way you can write this.

Student A2-20: [00:03:22] Uh, two plus x

Researcher: [00:03:25] Two plus x.

Student A2-20: [00:03:27] Yeah.

Researcher: [00:03:28] So you saying those two are the same. Okay, and is there another way that we can get two x.

Student A2-20: [00:03:36] Um, no.

Researcher: [00:03:39] Okay. Now, I'm going to ask you, what's that answer?

Student A2-20: [00:03:45] It's x squared. No, that's, uh, two x.

Researcher: [00:03:49] Okay. So if I give you x plus x, plus x what's that.

Student A2-20: [00:03:56] $3x$.

Researcher: [00:03:57] Okay, and if I give you x times x times x?

Student A2-20: [00:04:02] Um.

Student A2-20: [00:04:03] x to the power of three.

Researcher: [00:04:05] Okay. So let's just go back to our questions here quickly. So here, if I have $4a$ squared plus $3a$ squared plus $7a$. Okay, you said you can add all of these numbers and you can also add the letters. Okay, but can you tell me are these like or unlike terms?

Student A2-20: [00:04:39] Unlike.

Researcher: [00:04:39] Unlike terms, which ones are unlike?

Student A2-20: [00:04:43] This one is unlike. [referring to the $7a$]

Researcher: [00:04:44] To those two. Okay, can you still add them?

Student A2-20: [00:04:47] Yeah. Uh. It depends on...

Researcher: [00:04:55] So what have you done there?

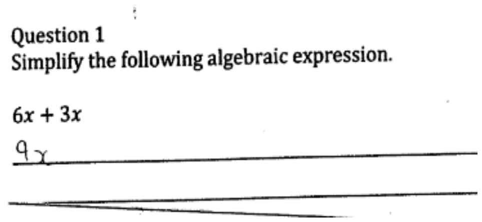
Student A2-20: [00:04:57] Well, it asked to simplify, so I just added the three.

Researcher: [00:05:02] Oh okay. So because it's simplify you've added it. But if you look at this and you say there unlike then.

Student A2-20: [00:05:10] You cancel it.

Researcher: [00:05:11] Okay. Cancel it. And the way you've added it is the way you've explained it to me okay. Okay, I'm going to say you can we can, uh, put this now as this interview.

8.3 Student B2-42



Question 1
Simplify the following algebraic expression.

$6x + 3x$

9x

Researcher: [00:00:01] Okay. So, why don't you tell me about this first question?

Student B2-42: [00:00:12] In math class here by this question, they say if the variable, if this variable here and exponents are the same, they like terms then you can add them. But if it's multiplication you can, you add them and then exponents that will be there, you add them also. So, here's no exponent and the variables are the same. So, I can just add them and I'll get the answer.

Researcher: [00:00:35] Okay. So how did you get the nine?

Student B2-42: [00:00:38] I added the 3 and the 6 and then bring down the x, because in plus you don't add or subtract the exponents from the variables.

Researcher: [00:00:47] Okay, so that's, that's the first one okay. Now, before we move on, what do you understand by simplify? What does that question actually tell you to do?

Student B2-42: [00:00:55] To get the answer to the simplest form that you can get it in so that you can understand it better.

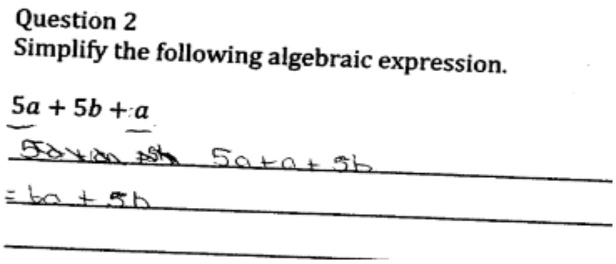
Researcher: [00:01:00] Okay. And so how does this answer relate to the question? Like. Like, what's the relationship between the $9x$ and the $6x$ plus the $3x$?

Student B2-42: [00:01:14] The variable here stand in the place of a number that you don't know, that's why the variable is there.

Researcher: [00:01:19] Okay.

Student B2-42: [00:01:19] So, now when you add that and you get that answer this is actually multiplication. Like my Miss said now, so then if you find the answer, they call it solve for x like we're doing now. And then you will find the answer of x. And then that nine will be multiplied by x. And you'll get the final answer. But because now there is no multiplication you just get $9x$.

Researcher: [00:01:42] Okay. All right. Let's quickly see the next one. This one is a little bit different hey. What's the difference between this one and that one?



Question 2

Simplify the following algebraic expression.

$\underline{5a} + 5b + \underline{a}$

~~5a+a+5b~~ 5a+a+5b

=6a+5b

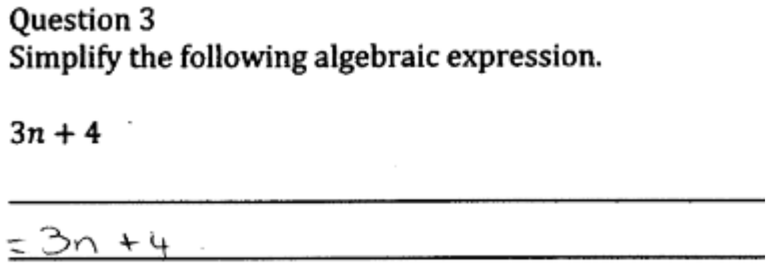
Student B2-42: [00:01:52] In this one there's an unlike term, there's one that's different to the others. Now, just because there isn't a number in front doesn't mean it's the same because they say to make it to be like terms, the variable and the exponent will be the same so the number doesn't count. So, this five I add with that a and in front of this a there's an invisible one. And that $5a$ plus that a is $6a$. Then I just add the $5b$, because this is unlike term. We don't know what's there but that b and we can't add it with the a. If it was multiplication it would be different.

Researcher: [00:02:24] Okay. And can you tell me why can't we add this?
0
Student B2-42: [00:02:28] We can't add this because that b and that a they not the same. They're not the same variable. So we can't add them because we don't know what it is. It could be anything. So, we just write it like that because we don't know what it is.

Researcher: [00:02:40] Okay. Good. Um let me see. I've got another question. Yeah. Okay. So you underlined...

Student B2-42: [00:02:47] The like terms.

Researcher: [00:02:48] The like terms. So you underline the like terms. That's fine, okay. That's perfect. Let's go to the next one. What happened in this question?



Question 3

Simplify the following algebraic expression.

$3n + 4$

=3n+4

Student B2-42: [00:03:00] This is both unlike terms. So, when it's unlike terms and there's nothing else you can do, you just write it the same way as it is.

Researcher: [00:03:07] Good. And then you can't.

Student B2-42: [00:03:08] You can't do anything.

Researcher: [00:03:09] Simplify that?

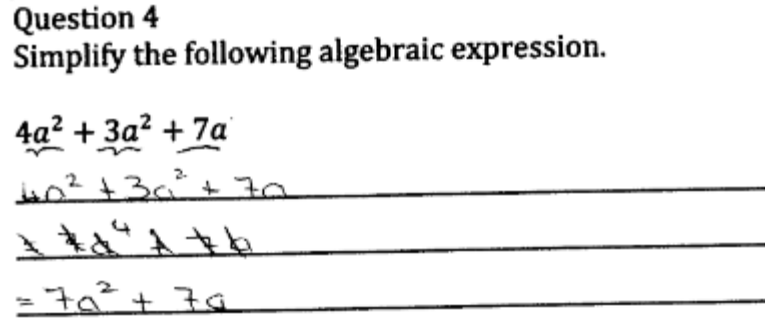
Student B2-42: [00:03:10] Unlike terms.

Researcher: [00:03:11] Okay. And this four what do you? What do you think about the four?

Student B2-42: [00:03:14] The four has no variable or exponent. So it's different to the three n. That's why they are unlike terms.

Researcher: [00:03:21] Okay. So that? Is that the same as that?

Student B2-42: [00:03:25] Yes. Because it's both unlike terms. You can't add it.



Question 4
Simplify the following algebraic expression.

$\underbrace{4a^2} + \underbrace{3a^2} + \underbrace{7a}$

$4a^2 + 3a^2 + 7a$

~~$= 7a^4 + 7b$~~

$= 7a^2 + 7a$

Now here's unlike terms and like terms. These two here have the same, same variable and exponent which makes them like terms. And this one, this one has a different exponent that's one. So you can't add that one.

Researcher: [00:03:41] Okay.

Student B2-42: [00:03:42] So you first take these two and you add them. And when you add that two you just add that. Because that exponent, that variable is different, you can't add the two. And that exponent is different. So you can't add them because they are unlike terms.

Researcher: [00:03:55] Okay good. So you decided you're going to stop there.

Student B2-42: [00:03:58] Yes.

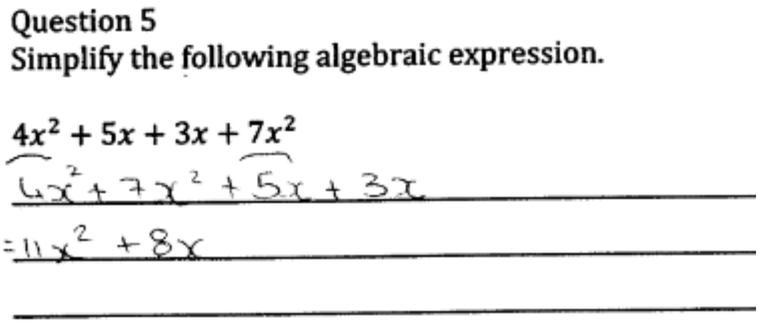
Researcher: [00:03:58] You're just going to add because these these what do you call these squiggly lines? Squiggly lines.

Student B2-42: [00:04:03] No that's just underline it differently because...

Researcher: [00:04:06] Because it's now like and unlike terms. Okay, and here you scratched that out hey.

Student B2-42: [00:04:14] That's a mistake.

Researcher: [00:04:15] Okay. Right now this one and this one are sort of.... Yeah.... What's the same with them?



Question 5
Simplify the following algebraic expression.

$4x^2 + 5x + 3x + 7x^2$

$4x^2 + 7x^2 + 5x + 3x$

$= 11x^2 + 8x$

Student B2-42: [00:04:24] This one, there's also with now exponents in this one because this one and that one is not the same. So, you put them together so you can understand it better than these two. You write them next to each other. Then you add these two and when you adding you don't add exponents or you don't do anything, you just keep it the same.

Researcher: [00:04:43] Yeah.

Student B2-42: [00:04:44] And then I add the two constants. And then I put down the variable with the same exponent. Then I add these two and put down the variable with the same exponent.

Researcher: [00:04:53] Okay.

Student B2-42: [00:04:55] But here, there's no exponents so here they not the same because they are different.

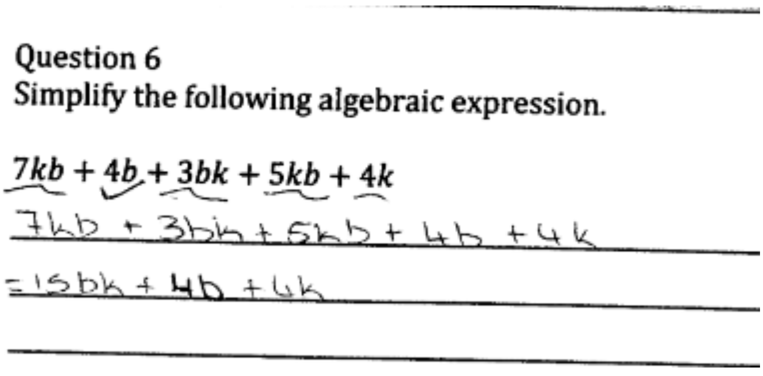
Researcher: [00:05:01] Okay. And this answer and this question is the answer of this?

Student B2-42: [00:05:08] Because these two are like to each other. So, we only add them. And these two are like but when I add them yeah they still unlike so I don't add them further from them.

Researcher: [00:05:17] So you stop there, you stop. And if I asked you this question and that answer, like, what's the relationship between that answer and this whole question?

Student B2-42: [00:05:30] Because these two here are the same, we can add them and then we will leave them just so because of their differences that they have to each other. So, then we can't add them any further.

Researcher: [00:05:41] Okay. Let's see this next one. Okay. Here, you made squiggly lines again. So which one's are the same?



Question 6
Simplify the following algebraic expression.

$7kb + 4b + 3bk + 5kb + 4k$

$7kb + 3bk + 5kb + 4b + 4k$

$= 15bk + 4b + 4k$

Student B2-42: [00:05:49] This one and that one, it doesn't matter what order the variables are in. It's still the same. Like my Miss said, if you find a variable and it's two different variables, then you must rather put in an alphabetical order to know it better.

Researcher: [00:06:05] So did you do that? No, but you kept it in your mind.

[00:06:07] I kept it like that and then here by the answer, I put in alphabetical order. Okay. And then these, these two, these three are like. And these two are unlike. So, I put now all my like terms together and I used three and that one and this one. I add these three like terms and I get my answer and I just bring down my variable.

Researcher: [00:06:27] Can you explain how you added that?

[00:06:30] I added seven and the three, that's ten plus and there's a $15k$. Adding the four b and the four k. Because they are unlike you can't do anything with them. So you just bring them down.

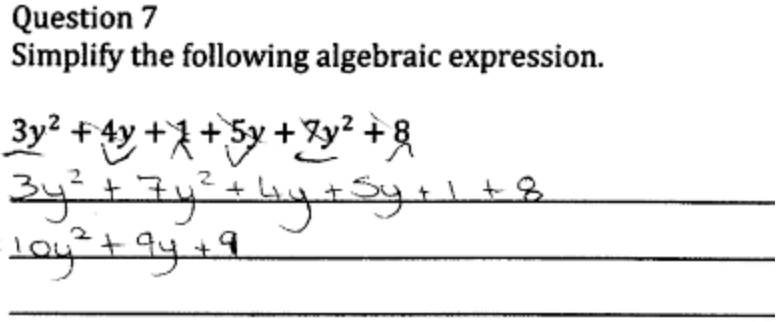
Researcher: [00:06:42] And you just leave it like that. So what makes it that the BK and the KB are like terms? What is the thing that makes it that they are the like terms?

Student B2-42: [00:06:53] The variables make them the same.

Researcher: [00:06:55] Because their variables are the same and...

Student B2-42: [00:06:57] And the variables are the same and the exponents are the same.

Researcher: [00:07:00] So, that's what makes them the like terms. Okay. All right. Now, we going to the next one. This one is a little different. But let's see.



Question 7
Simplify the following algebraic expression.

$3y^2 + 4y + 1 + 5y + 7y^2 + 8$

$3y^2 + 7y^2 + 4y + 5y + 1 + 8$

$= 10y^2 + 9y + 9$

Student B2-42: [00:07:11] Here, this one has two like terms. There's this one here and this one. I write them next to each other so I can understand it better. This one and that one. They all like terms. And the one and the eight, they are like terms. So, then I add the first two like terms with the exponent. And that's equal to ten y two. And then these two like terms with only y with no exponent. But they are the same I add them as nine y. And then because they have no variable and no exponent they are also the same. And then I add them and then I just leave them just so because these are all unlike terms so I can't add unlike terms, only like terms.

Researcher: [00:07:49] So now explain every single like little thing that you think when you go from those two to that one there.

Student B2-42: [00:07:56] These two here, in multiplication and division you do something with the exponents. So, it doesn't matter if it's unlike or like you can still multiply and divide them.

Researcher: [00:08:08] Yeah.

Student B2-42: [00:08:09] But so when there's an exponent and you multiplying and you add the exponents and you add it to the variable at the back, and when you dividing you, you minus the two exponents from each other and you keep the variable.

Researcher: [00:08:21] Okay, so that's why this addition is different from multiplication. And the three and the seven?

Student B2-42: [00:08:29] They are constant, so you don't worry about them. You only worry about the variable and the exponent

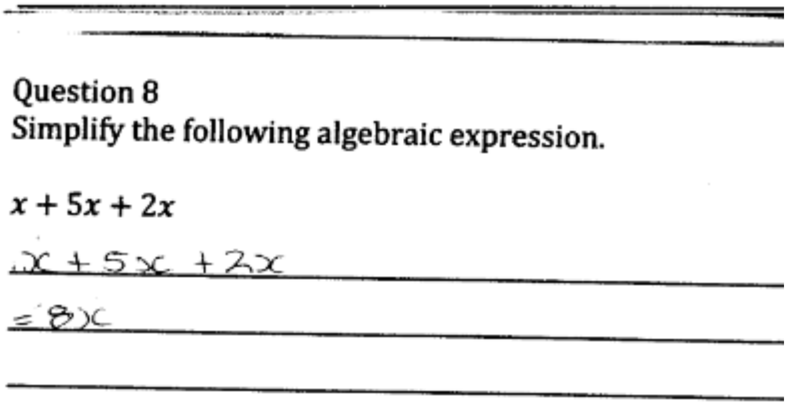
Researcher: [00:08:34] So, basically you are telling me the three the seven.

Student B2-42: [00:08:37] Is equal to the 10.

Researcher: [00:08:38] And then what happened with that? You just?

Student B2-42: [00:08:40] You just bring it down. Because you can't add exponents in addition.

Researcher: [00:08:44] Okay, that's good okay. Let's quickly let's quickly take a break because I just need to check that I've answered this question. So I just wanted you to explain that one okay. You ready? Okay. Let's just see that the camera is still on the right position. Okay. This one.



Question 8
Simplify the following algebraic expression.

$x + 5x + 2x$

$x + 5x + 2x$

$= 8x$

Student B2-42: [00:09:06] This is all like terms.

Researcher: [00:09:07] I threw this in here to check whether... So you know this is all like terms.

Student B2-42: [00:09:11] And in front of this x there's an invisible one there.

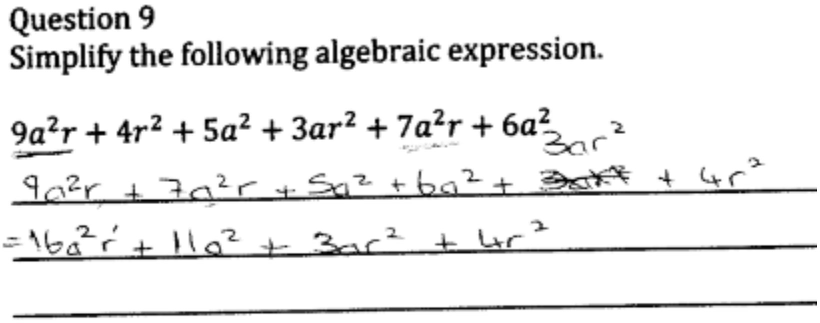
Researcher: [00:09:15] Okay.

Student B2-42: [00:09:15] So, that invisible one you add with the five and the two. So the five and the two make the seven and the one is eight x because these have the same variable and the same exponent. This one just doesn't have a constant. But it's still the same because you're only looking at the variable and the exponent.

Researcher: [00:09:34] Okay good. And so can you tell me you already answered this. You already answered this. Um, last time, how does this one relate to that question?

Student B2-42: [00:09:45] Here, you don't worry about the variables because they're all the same. They have the same base, the same variable. And I just add the constants and keep my variable and write it exactly like that.

Researcher: [00:09:59] Okay, Oh, yeah. I threw this one in just to see what would happen. Okay, so you underlined that one, there.



Question 9
Simplify the following algebraic expression.

$\underline{9a^2r} + 4r^2 + 5a^2 + 3ar^2 + 7a^2r + 6a^2$ $3ar^2$

$9a^2r + 7a^2r + 5a^2 + 6a^2 + \cancel{3ar^2} + 4r^2$

$= 16a^2r' + 11a^2 + 3ar^2 + 4r^2$

Student B2-42: [00:10:10] Now, I can look for my like terms here. And I found that the 9A2r and that one they are the same. They are both like terms. I write them next to each other. Then I look for other like terms and then I found the 5A and 6A2. And then these two are unlike terms they are not the same. Then you just write them there.

Researcher: [00:10:29] Okay now, why is this one not the same like that one?

Student B2-42: [00:10:34] Because this exponent here is by the r and that one is by the a. Okay, so if this was like a number there, that exponent would be for that number. And that would have been separate. But this one is together. So it's different.

Researcher: [00:10:49] Okay, so what's different? What's actually different?

Student B2-42: [00:10:53] If this was a number here like.

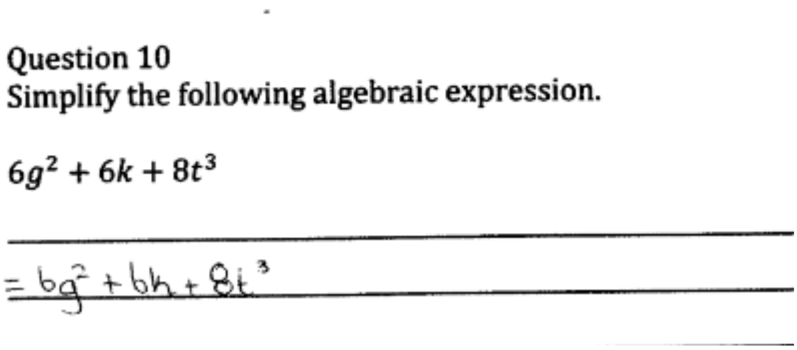
Researcher: [00:10:57] You can write.

Student B2-42: [00:10:59] If that was a number say now it was 98 because then that variable exponent would have been on the 98 and that r would have been separate. But if this was both numbers then would have been maybe like this. Then the exponents on the end, would be making that like more than that one. So that's different. Because exponents on the end.

Researcher: [00:11:20] Okay. Yeah.

Student B2-42: [00:11:21] And then that one is the same like that one there. But it just doesn't have an extra one. And I add my like terms and and then I stopped because I can't add any further.

Researcher: [00:11:32] And then that's, that's the end of this one. Okay. Right now this is the last one.



Question 10
Simplify the following algebraic expression.

$6g^2 + 6k + 8t^3$

$= 6g^2 + 6k + 8t^3$

Student B2-42: [00:11:38] This is all unlike terms okay. So I don't do anything. I just write it exactly the same because I can't do anything. All unlike.

Researcher: [00:11:46] And what makes them unlike terms.

Student B2-42: [00:11:47] Their variable and the exponents make them unlike terms. This one has a g and that one k two and that one has no exponent, but that one has a two and a three. So I can't do anything.

Researcher: [00:11:57] Okay.

Student B2-42: [00:11:57] Because the exponent and the variable are different. And I can't add anything.

Researcher: [00:12:04] Okay. Yeah. So that's perfect. Okay, so let me just see I just want to ask like two general questions just in general about how you understand the things. So when I asked you already what simplify means. Right. So if you have an algebraic expression and you simplify it you try to.

Student B2-42: [00:12:24] Understand it better okay.

Researcher: [00:12:26] Okay. Good. And then can you also tell me, um, your definition of like terms, what is your definition?

Student B2-42: [00:12:33] You don't worry about the constant in like terms. You just look at the variable and exponent. If the exponent and the variable are same, they are both like terms. Then you can add them or do whatever now and subtract. But if the variable and exponent are different, you can't do anything because the variable and exponent are different so I can't add them.

Researcher: [00:12:57] Okay. And like terms can you give me an example. They quickly just write like any example of two like terms.

Student B2-42: [00:13:04] Like terms if this is like terms now, because the variable and the exponents are the same, I can now continue in the sum and I will say it equals to six g. And now that's the final answer. But because these the variable and the exponents are the same I can add or minus.

Researcher: [00:13:36] Okay.

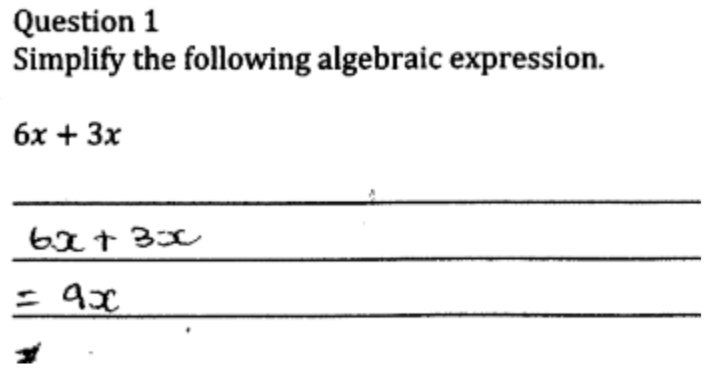
Student B2-42: [00:13:36] But if it was like this. And I minus or plus. I can't do anything now because yes, the variables are the same, but I mean, the exponents are the same, but the variables are different, so I can't do anything. So if they ask me this sum, I will just have to write it exactly the way it is because I can't go further.

Researcher: [00:14:02] Okay. Thank you so much. Um, let me just double check here. Everything done? Yes. Everything's done. I'm going to give you a little question quickly and just see what you make of it. So, um. That's supposed to be an X. And the question is to simplify. So these this question it's two of the same things. But what what would be the answer.

Student B2-42: [00:14:30] So, this is like terms. And in like terms you don't do anything to exponents or variables. So, I can just say now that my two constants I add them together and it's equal to ten x three.

Researcher: [00:14:43] Perfect. Okay. This was absolutely perfect. Thank you so much. Um, yes, I'm going to switch it off.

8.4 Student B2-45



Question 1
Simplify the following algebraic expression.

$6x + 3x$

$6x + 3x$

$= 9x$

Researcher: [00:00:00] Let me just double check. Okay. Cameras are rolling, so you have to speak a little louder because the, the microphone is a little soft. Okay. Okay. So basically what we have here is your answers to the questions okay. So what we're going to do is we're just going to go through all of them. And then you just tell me how you thought about it, what it is you did. And then I'll ask you questions in between okay. Okay. So the first thing is, let's just start by what simplifying means to you.

Student B2-45: [00:00:32] Simplify, I think simplify means you must work out your answer.

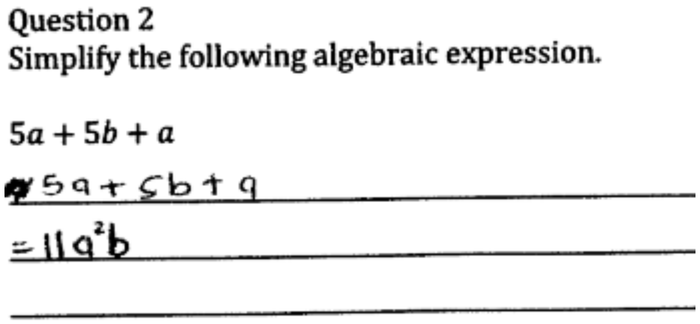
Researcher: [00:00:36] Okay. And then when you get your answer what's your answer in relation to the question?

Student B2-45: [00:00:44] So yeah, I think miss, to see if your answer is really correct or wrong.

Researcher: [00:00:51] Okay. Okay, good. All right. So this was the first question that I gave you with the x. And can you tell me how you got to that answer there?

Student B2-45: [00:01:01] Well I started by saying six plus three is equal to nine. So miss, oh my, um, my maths teacher told us if the variables are the same. So you must just write them.

Researcher: [00:01:17] Okay. So you said six plus three is nine and then you just put the x okay. Okay. Nice. Let's see. Anything else. That's fine. This one's fine. Right. Next one. Okay. This one is different to this one.



Question 2

Simplify the following algebraic expression.

$5a + 5b + a$

$5a + 5b + a$

$= 11a^2b$

Intercom: [00:01:40] Intentional act and youth. Youth leadership levels. Please report to Mr. Jay Slack during first week tomorrow. All cyclists and other learners interested in cycling please report to Mr.. That's room S6 during the first interval today. Thank you.

Researcher: [00:02:02] Okay so these two and this is different. How is that different?

Student B2-45: [00:02:06] Because the there is a different variables in each sum.

Researcher: [00:02:14] Oh okay, so you see the variables are different. Okay. And here, can you tell me how you got to this answer?

Student B2-45: [00:02:23] I see in this question, I said five plus five is equal to ten plus an invisible one. So just say I said five plus five is ten plus one. Then I got 11. Then I put the variable. Then let's say y. Then I say a plus a. Then it's going to be a a squared. Then I put a b.

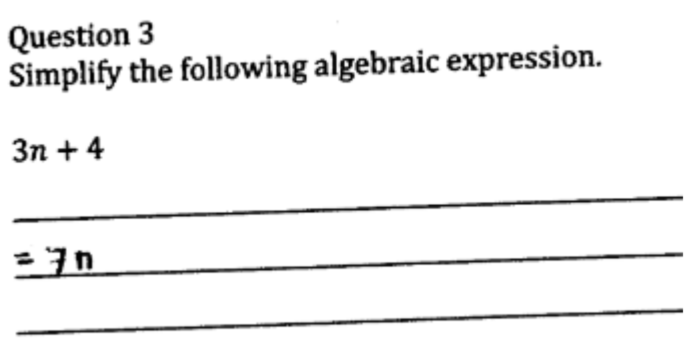
Researcher: [00:02:47] Okay. And you put it all next to each other? And so what does that mean? Like so what are you doing here? What's the operation?

Student B2-45: [00:02:58] I'm putting the, the capital letters in alphabetical, alphabetical order.

Researcher: [00:03:05] Good. And can you tell me what are you doing? Are you adding multiplying, dividing or subtracting.

Student B2-45: [00:03:11] Here, I'm adding miss.

Researcher: [00:03:12] You adding. Okay. And that's, that's what you do. Okay. Right. Next one. What's happening?



Question 3

Simplify the following algebraic expression.

$3n + 4$

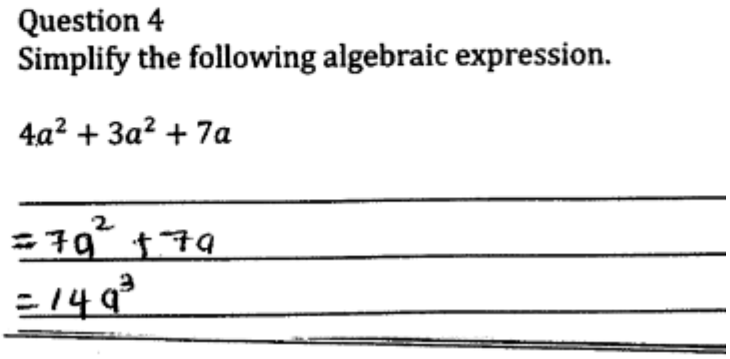
$= 7n$

Student B2-45: [00:03:19] I just need a little difficult because I didn't understand here. Okay, so I just said three plus four. Then I say one and then I say then. Yeah.

Researcher: [00:03:30] And then you just you put it together like that? And what does seven *n* mean if I tell you if I, let's just say I. Okay. I have to write here. I give you a seven n. What does that mean to you? If I just tell you that.[Researcher writes 7n]

Student B2-45: [00:03:49] It means, I think it's seven plus, then it's 7n, seven plus n.

Researcher: [00:03:55] Seven plus n? So this means seven plus n and three n plus four. Yes. The same. Okay. Okay. Thank you. Yeah. This one. You know two terms. Yeah. Okay. Next one. This one is different to all of these. Why?



Question 4
Simplify the following algebraic expression.

$4a^2 + 3a^2 + 7a$

$= 7a^2 + 7a$

$= 14a^3$

Student B2-45: [00:04:14] Because it just doesn't add a squares.

Researcher: [00:04:16] Yes, okay. Now I can see that you did something with the squares here. Can you tell me?

Student B2-45: [00:04:23] Here, I said because these are, these are unlike terms. So I just say four plus three. I just say four plus three, which is seven. Then I put the, the a square plus a square. So I just, I just did a square. Then I said plus seven a so I just take this down. Then I say seven plus seven. It's 14 plus two plus a plus a. Then is two, then two. There is invisible one. Just I just did one, I just said three.

Researcher: [00:05:03] So you're saying, firstly you first add these two to get that and then you just put the A there, that's a squared right.

Student B2-45: [00:05:13] They are unlike terms, they are like terms.

Researcher: [00:05:15] Because they like terms. Are they like or unlike terms?

Student B2-45: [00:05:18] They are like terms.

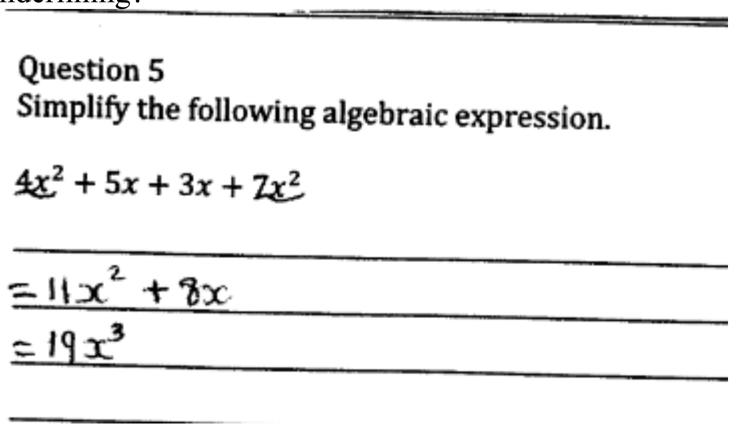
Researcher: [00:05:20] Like terms. So that's how you add like terms. And then these two you put the seven and the seven get the 14 and then those two. Yeah.

Student B2-45: [00:05:28] Because it's invisible one.

Researcher: [00:05:30] So how did you get the three? The two and the....

Student B2-45: [00:05:33] Invisible one.

Researcher: [00:05:35] Okay. Perfect. Thanks. Right now this one and this one are very similar hey? Because that one and that one both are squares. And I see you underlined, what is the underlining?



Question 5

Simplify the following algebraic expression.

$\underline{4x^2} + 5x + 3x + \underline{7x^2}$

$= 11x^2 + 8x$

$= 19x^3$

Student B2-45: [00:05:47] So, I did underline because our teacher told us if there are like terms and it was just underlined terms.

Researcher: [00:05:54] Yes. And then you first now?

Student B2-45: [00:05:56] The the first step (inaudible).

Researcher: [00:05:59] So the first step is almost the same like that? Okay, so you see the four and the seven...

Student B2-45: [00:06:04] And then it's going to be 11,then I said plus eight plus five x plus three x. Then it's eight x okay. So,then I just put 11 plus eight, then it was 19. Then I just say two plus invisible one. Then it's three, then okay 19x three okay.

Researcher: [00:06:22] Can I ask a question? So when you add the four x squared and the seven x squared why does this one stay x squared?

Student B2-45: [00:06:31] Because I like terms. They like terms.

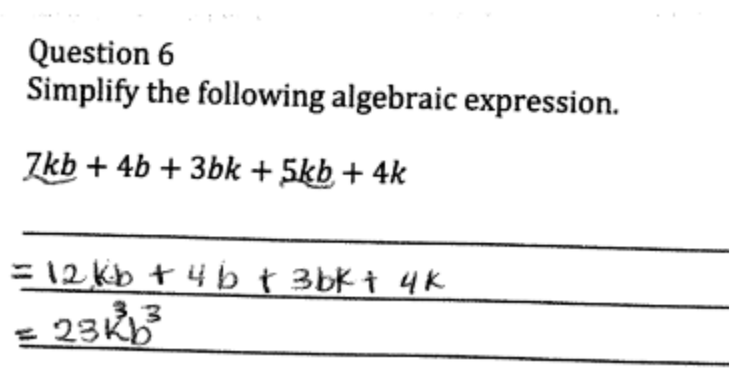
Researcher: [00:06:34] So you don't change the....

Student B2-45: [00:06:36] Yeah, you must not change the the variable.

Researcher: [00:06:39] okay, so you just put that x squared there. But here, when you add it, then you add the two and the one.

Student B2-45: [00:06:46] Yeah.

Researcher: [00:06:46] Okay. All right, so this one and that one was similar and then this one was a little bit different.



Question 6

Simplify the following algebraic expression.

$\underline{7kb} + 4b + 3bk + \underline{5kb} + 4k$

$= 12kb + 4b + 3bk + 4k$

$= 23k^3b^3$

Student B2-45: [00:06:52] Yeah.

Researcher: [00:06:53] Because the exponents.

Student B2-45: [00:06:55] Yeah, (inaudible) this one was just difficult. Now I just tried my best.

Researcher: [00:07:02] No but it's good because you still underlined there.

Student B2-45: [00:07:04] Yeah. So, here I said seven plus five, then I say 12kb plus because this is unlike terms, these numbers are unlike terms so I just put it there.

Researcher: [00:07:21] You just wrote it there.

Student B2-45: [00:07:22] Just I took I took the 4b down. Then I say three bk. Just I put it down. Then I said 4k, I just put it down. So I just say 12 plus four, plus three plus four. It is equal to 23. So, I just say okay because there are two k's. So it's like terms. So I just say k three. I just said k three, plus b three .

Researcher: [00:07:56] And how did you get the threes?

Student B2-45: [00:07:58] You 123.

Researcher: [00:08:00] Okay. And the other one?

Student B2-45: [00:08:02] I just got 123 because it's the first one.

Researcher: [00:08:05] Okay. Good. And then can you tell me this bk and kb you, you didn't add that because they...?

Student B2-45: [00:08:13] Are unlike terms. But it's like terms but they are in different positions.

Researcher: [00:08:20] Okay, so do you consider them like them or unlike terms?

Student B2-45: [00:08:24] Because I just consider them like terms. Because they just switch the the alphabet. The alphabet.

Researcher: [00:08:32] So, so why didn't you add the three k also? You did. You just wanted to also take it down there?

Student B2-45: [00:08:41] Yeah.

Researcher: [00:08:41] Okay. And um, so you tell me they the same. So what makes kb and bk the same?

Student B2-45: [00:08:49] Because the...

Researcher: [00:08:50] You can write if you want hey. Here's a pen.

Student B2-45: [00:08:56] So, I think I did this wrong. So Miss yeah, I just took that kb plus k is the same. Then they are in a different position. So just, just I just plussed the ks. Yeah. Then I just plussed the bs and the bs.

Researcher: [00:09:22] Quickly close it to just to make a little bit. Well, no we'll leave it. Um, let's quickly see. This is the next one. Right now I'm scared that there's going to be, is it interval now?

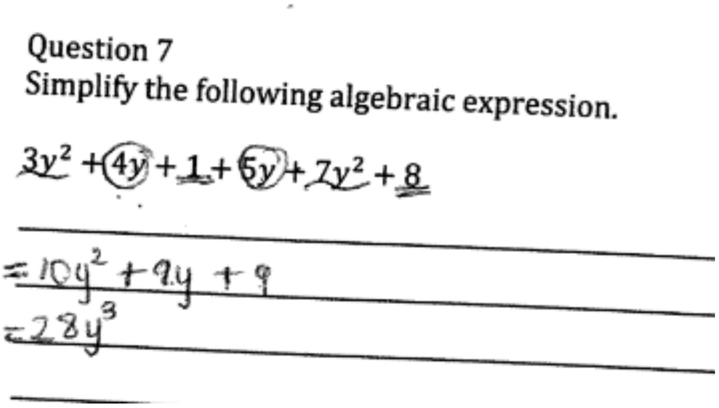
Student B2-45: [00:09:35] Yes, yes.

Intercom: [00:09:37] To go to the office immediately. Okay.

Researcher: [00:09:42] Okay. So what we can do is you can come here at the start of your next lesson, and then I'll take you late to class, and then I'll explain to you. Is that is that fine?

Student B2-45: [00:09:52] Yes.

Researcher: [00:09:52] Or, um, we can quickly finish now, if that's what you want? Okay. Okay. I'm just scared that the noise. Let's quickly finish it. Right. So this is the next one. And here we have Y's. And then these ones here are just.



Question 7

Simplify the following algebraic expression.

$\underline{3y^2} + 4y + \underline{1} + 5y + \underline{7y^2} + \underline{8}$

$= 10y^2 + 9y + 9$

$= 28y^3$

Student B2-45: [00:10:14] They're just numbers.

Researcher: [00:10:15] Okay. Numbers. What do you call those? Just these ones, just the numbers? Yeah. Just the numbers. So I see you circled the like terms hey and you underlined. And then that's the other one is. And then you added it like you said before. And then here what happened?

Student B2-45: [00:10:38] So Miss, here I just made like, ten, I just said three, I just said $3y$ two plus, plus five, I plus, plus $7y$ two then I got the answer. Which was ten. Because it was ten. So, I just put the x, y x, y two and y two. So I just put it here.

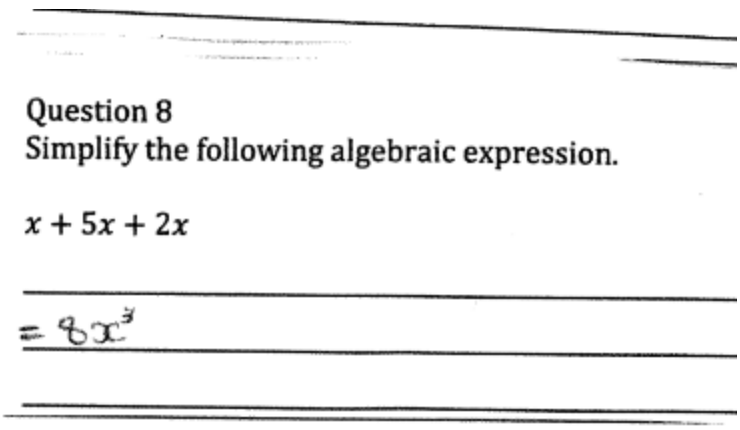
Researcher: [00:11:10] Okay.

Student B2-45: [00:11:10] Then, I said plus, so I just put $4y$, $4y$ plus five, plus five y. Which my answer was nine. Mine was nine y.

Researcher: [00:11:22] Okay.

Student B2-45: [00:11:24] Then I just, I just said one plus eight, which my answer was plus was equal to nine. So just say ten, plus nine, plus nine plus nine.18 plus 10. They just get 28.

Researcher: [00:11:43] Okay. And then, how did you get that one?



Question 8

Simplify the following algebraic expression.

$x + 5x + 2x$

$= 8x^3$

Student B2-45: [00:11:49] I just said this was invisible one, invisible I no. Yeah. Because they (inaudible) So I just, I just put them one by one.

Researcher: [00:12:01] And then that's how you got the three. Okay. Good. Okay. So, I'm sure that the, we just have to talk louder so that they can hear. Um, and at this one you go to eight.

Student B2-45: [00:12:15] Yeah.

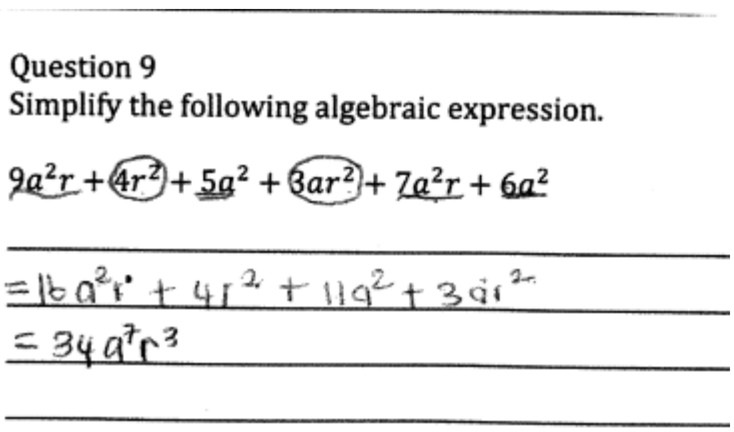
Researcher: [00:12:15] Because...

Student B2-45: [00:12:16] Because x supposed to be the invisible one by x. So I just said plus five x, plus two x. So, I just said this invisible one. So I just said one plus five with six, plus two. It's eight. So I just make so, just put the x and then so mustn't say x three. So I just put the x.

Researcher: [00:12:47] And here...

Student B2-45: [00:12:47] Oh, then oh, sorry. And then here there's the invisible one. Invisible one. Invisible one. So, I just said one, plus one, plus one which was three.

Researcher: [00:12:58] Okay. All right. Perfect. Okay, now this one was a bit tricky, but you did see the like terms. Okay, I see you underlined. So where are the like terms here? Just point them out.



Question 9

Simplify the following algebraic expression.

$9a^2r + 4r^2 + 5a^2 + 3ar^2 + 7a^2r + 6a^2$

$= 16a^2r + 4r^2 + 11a^2 + 3ar^2$

$= 34a^7r^3$

Student B2-45: [00:13:11] The like terms is y.

Researcher: [00:13:15] Um.

Student B2-45: [00:13:16] 9a square r plus 7 a square r.

Student B2-45: [00:13:23] *About 60 plus miles an hour. Plus three. Ah ah ah ah. So I just put it down. Just. I just put it down.Yeah. They said five because these are unlike them. So I just said five. Plus 6 is. Now just put the 3a down.

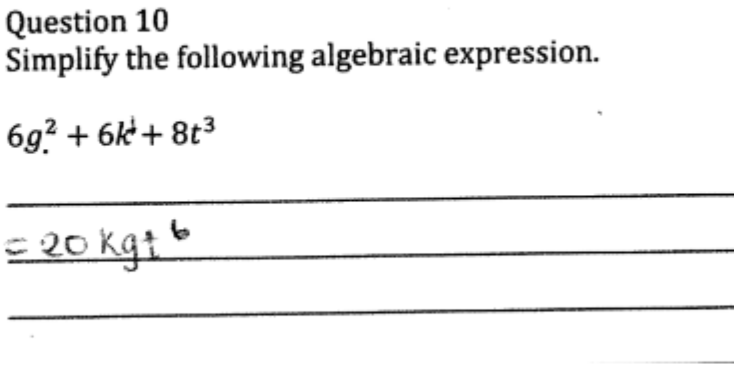
Researcher: [00:13:46] Okay.

Student B2-45: [00:13:48] *So, here I just plus 16 plus four, 16 plus 16 plus four plus plus three, three, four. Okay let's say A one, A two plus A two. Which is good. Got me four plus two, which got me seven. Then I say I just got to like the. Those one plus one because one. No musical performance. Yes, I just made the one that was supposed to be invisible.

Researcher: [00:14:25] Yeah.

Student B2-45: [00:14:25] So, I must just say one plus two is three. Plus one plus two, is three. Five. Yeah.

Researcher: [00:14:36] Okay, I see what you did okay. Last one. Explain this one.



Question 10

Simplify the following algebraic expression.

$6g^2 + 6k + 8t^3$

$= 20\,kgt^6$

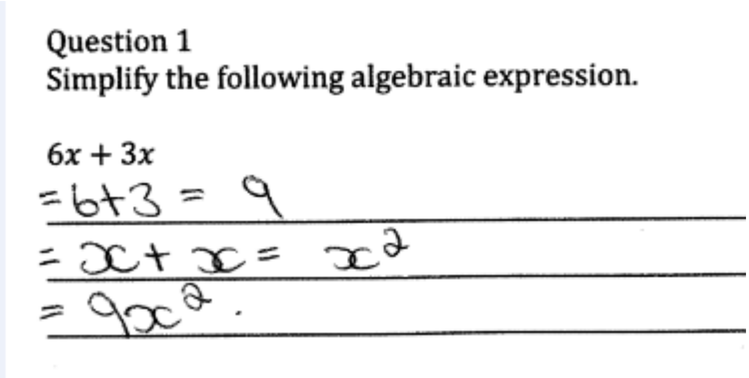
Student B2-45: [00:14:46] This one makes, it said 6k two plus two. Six k. Plus eight t three. So So, here I just said 6 plus 6. It's yeah. I just said 6 plus 6 which gave me 12. So 12, plus the eight.

Researcher: [00:15:18] Yeah.

Student B2-45: [00:15:20] Which gave me 20. So, I just calculate the variables first. And then I said, I put them in alphabetical order. So I just said, okay. then I said, two plus invisible one plus three, then two plus three is six. Yeah. Two plus three is four plus two, it's six.SO that's how I got the answer.

Researcher: [00:15:50] Okay, That's perfect. You see, I like that you wrote it down for me. It's actually, you know. Okay, so let me just double check that I asked all my questions that I wanted to ask you. Yes. Thank you for explaining so fully and for staying now five minutes at your break. I'm sure the microphone picked up your voice over the noise. Um, yeah. So, thank you very much. I'm going to... can I switch off quickly.

8.5 Student B1-29



Question 1
Simplify the following algebraic expression.

$6x + 3x$

$= 6+3 = 9$

$= x + x = x^2$

$= 9x^2$.

Researcher: [00:00:01] Okay, let's just double check that it's going. Yeah. All right, so this is your answer sheet here. And basically what you have is we've got these, um, questions here that you answered. Okay. So can you just tell me how you got to this answer? This first one?

Student B1-29: [00:00:25] Um, I just added six plus three and nine and then the x and plus the x, and then up to the like x squared, and then just. The $9x$ squared.

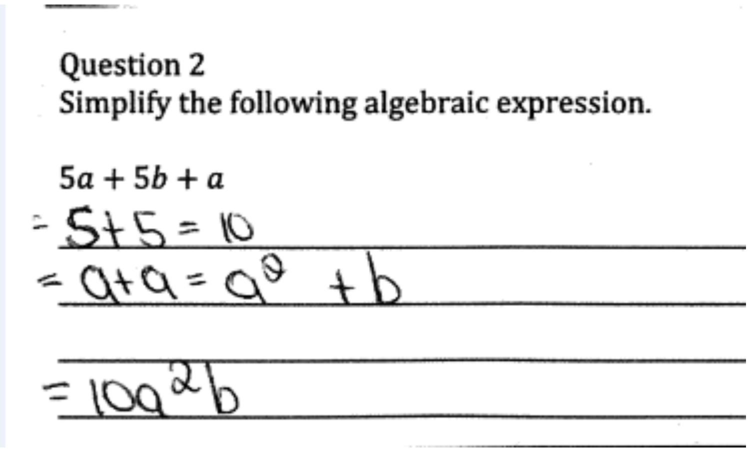
Researcher: [00:00:35] Okay. So x squared means that you have.

Student B1-29: [00:00:41] Two.

Researcher: [00:00:42] Two xs. Okay. And can you tell me um what can I ask you. The question says simplify. So what does that mean, when you look at that?

Student B1-29: [00:00:57] My friends still tell me that simplify means you must just write the answer. But I didn't know if I just saw the sum, I didn't still read the instruction.

Researcher: [00:01:06] Okay. And then you you 8B hey. Okay, let's see the next one. Okay. So this is the next one. So how's this question different from this question?



Question 2
Simplify the following algebraic expression.

$5a + 5b + a$

$= 5+5 = 10$

$= a+a = a^2 \quad + b$

$= 10a^2b$

Student B1-29: [00:01:21] It has two terms, unlike.

Researcher: [00:01:22] okay.

Student B1-29: [00:01:27] And it's like two. It's like unlike terms.

Researcher: [00:01:34] Okay, okay. So that's something new. So what in your words would you say unlike terms are?

Student B1-29: [00:01:41] Terms that cannot be added. Actually um, because it is A and B you can't actually add them.

Researcher: [00:01:48] Okay. So you first, I see there. So that's the A's right. These two are the A's okay. So you get that. And then what's this?

Student B1-29: [00:02:02] Five plus five.

Researcher: [00:02:03] Okay. And so you put those two and you added them. And then you got the?

Student B1-29: [00:02:12] Ten

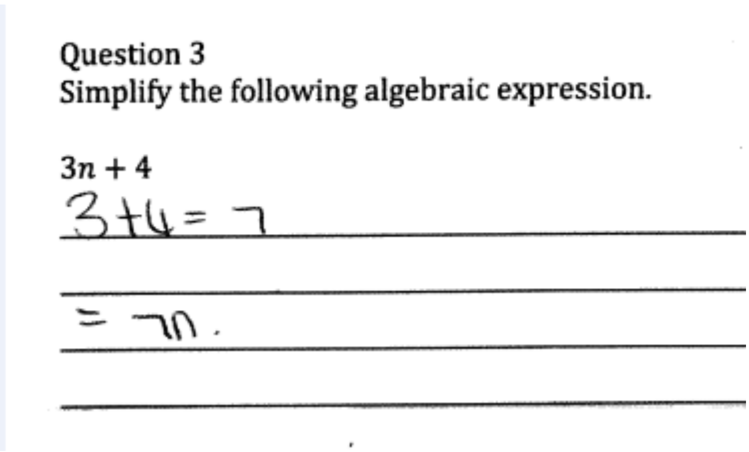
Researcher: [00:02:12] Okay. And then you put the ten there okay. And then you put the A and this and this one?

Student B1-29: [00:02:19] yes

Researcher: [00:02:21] Okay. All right. And then how did you get to the last answer?

Student B1-29: [00:02:25] Because five plus five is ten and then the two A's and then the B just left there.

Researcher: [00:02:31] Okay. So you just you sort of put them in that one answer. Okay. So let quickly ask you something if I. Okay. Let's go to the next one first. Let's see okay. So this one also says simplify. And so there we've got what's our final answer?



Question 3
Simplify the following algebraic expression.

$3n + 4$

$3+4 = 7$

$= 7n.$

Student B1-29: [00:02:49] Seven N.

Researcher: [00:02:50] Okay. So can you tell me about this one quickly?

Student B1-29: [00:02:55] Um the three plus the 3 plus the four is seven and then I just.(inaudible)

Researcher: [00:03:01] Okay, so. If I ask you to explain in your own words, say I gave you this. Okay. Say I gave you. That. And I asked you to tell me what this is? What it means?

Student B1-29: [00:03:21] Um, I don't know. It kind of seems like multiply.

Researcher: [00:03:26] Mhm.

Student B1-29: [00:03:28] But I don't know, you go multiply a seven because if it's unknown. Yeah. So I don't know.

Researcher: [00:03:35] Okay. So if I told you what does it mean in maths to you?.

Student B1-29: [00:03:42] Five.

Researcher: [00:03:43] So what does it actually mean? Like what do you think.

Student B1-29: [00:03:45] Like five times. Like N times five times.

Researcher: [00:03:50] Okay. In times five times. Okay. So is it five N's or is it N fives?

Student B1-29: [00:03:58] I think it's N fives okay.

Researcher: [00:04:01] So is that okay? That's cool. I'm just trying to see because this is your answer for this question. So, also if we go back how do you think this this answer is related to this question? So like what's the relationship?

Student B1-29: [00:04:18] It's.

Researcher: [00:04:21] You don't know

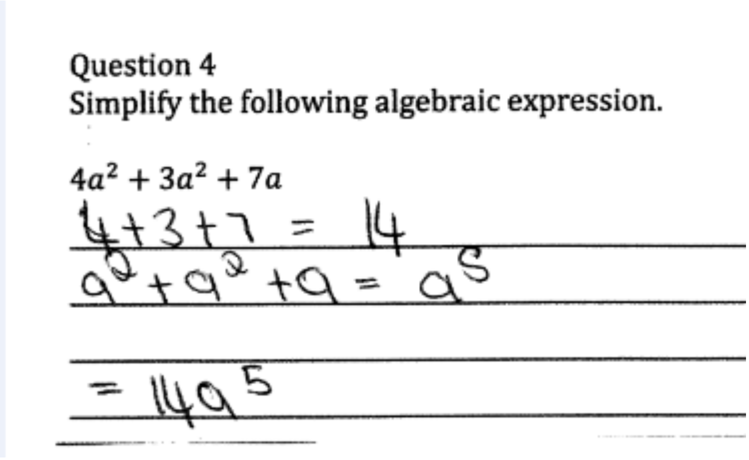
Student B1-29: [00:04:21] I don't understand the question.

Researcher: [00:04:22] Okay. So okay, let's do it with the first one. So six x plus three x. And our answer is nine x squared. So how does this relate to the six x plus the three x? What is that?

Student B1-29: [00:04:38] Six plus three is nine. and the x plus the x is x squared .

Researcher: [00:04:43] Yes. Yeah. Oh, okay. Okay.

Researcher: [00:04:49] So, these are the ones without any here comes the next one, the next one is... What's different now?



Question 4
Simplify the following algebraic expression.

$4a^2 + 3a^2 + 7a$

$4+3+7 = 14$

$a^2+a^2+a = a^5$

$= 14a^5$

Student B1-29: [00:04:59] All of it has the same terms.

Researcher: [00:05:02] Yeah.

Student B1-29: [00:05:02] So you can add it.

Researcher: [00:05:04] Okay. That's good. So in terms of if you say all of it is the same terms, that means that they all.

Student B1-29: [00:05:10] Like the same.

Researcher: [00:05:11] Okay.

Student B1-29: [00:05:12] it has the same variable.

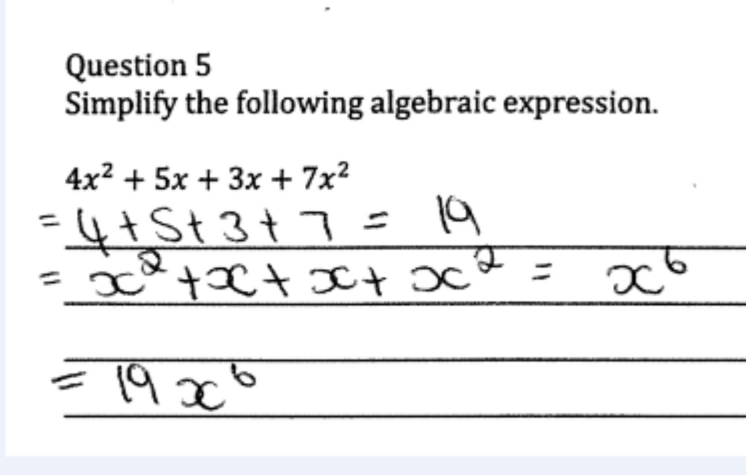
Researcher: [00:05:16] Yes okay okay. Cool. So so you're going to add all of them. And so you got what did you do. Okay. So you first this what's this again.

Student B1-29: [00:05:29] Four a. Four plus three.

Researcher: [00:05:32] Okay. So you first put do it like in two in two parts okay. Um and then this is the from this one and this one in this one from there?. Okay. So then you basically add that and then you get.

Student B1-29: [00:05:49] (Inaudible)

Researcher: [00:05:51] Okay, cool. Right now this one and this one is again similar, you see. So it's sort of like the same thing that you've done. Is it the same or. Yes.



Question 5
Simplify the following algebraic expression.

$4x^2 + 5x + 3x + 7x^2$

$= 4 + 5 + 3 + 7 = 19$

$= x^2 + x + x + x^2 = x^6$

$= 19x^6$

Student B1-29: [00:06:04] Yes it is. It is. It's all like terms.

Researcher: [00:06:07] So they all like terms. Okay. So if you must, if I ask you to give me a definition of like terms, can you remember it from class?

Student B1-29: [00:06:17] Not really. Like I can put it into my own words.

Researcher: [00:06:20] Or even better, if you want, because it's recording, you can write it down there like you can give me an example. Like, just give me an example that you can think of.

Student B1-29: [00:06:34] Um. Maybe four A plus five A plus A. Like thats like terms , you can add it because it has the same variable.

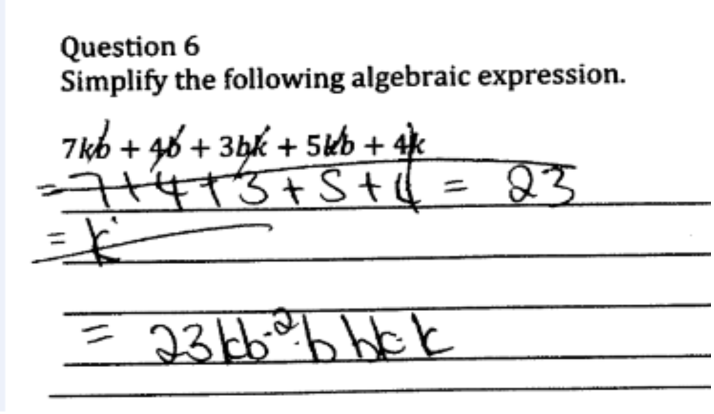
Researcher: [00:06:47] Okay. And then what? What would that answer be.

Student B1-29: [00:06:50] It would be actually ten A. cause there' an imaginary one.

Student B1-29: [00:06:55] Okay okay. Cool. Okay. So this one you got four plus five plus three plus I'm sure that's 19. Okay cool. And then here we get how did you get this six heere?

Student B1-29: [00:07:07] X squared. And x squared is four. And then there's also imaginary ones

Researcher: [00:07:12] So you okay. Cool. And so what I actually what operation are we doing here. There's only one addition. Yes, okay I'm just checking. All right. Now this one is interesting to me because you scratching things I don't know what you scratching.



Question 6

Simplify the following algebraic expression.

$7kb + 4b + 3bk + 5kb + 4k$

$= 7+4+3+5+4 = 23$

$= k$

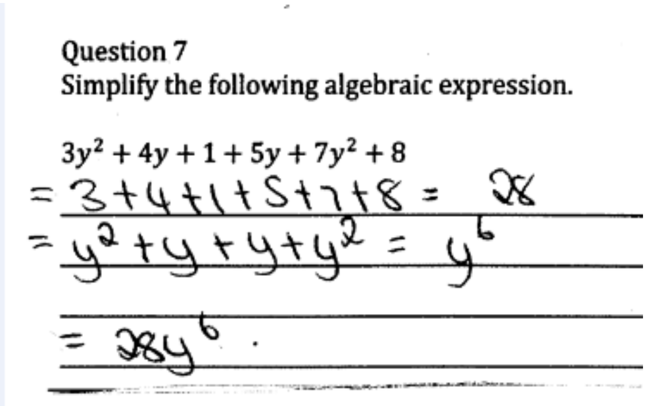
$= 23kb^{2}bbkk$

Student B1-29: [00:07:32] I didn't actually understand this because it was, like, mixed up and. Yeah, so I just wrote the answer.

Researcher: [00:07:40] Okay, that's cool. So you add it again separately, then you get the 23. And then these is the reason why you didn't um, sort of put those or select these or whatever. Um, this one because that's messy. To check the other question. They like this. Oh. Well, you just wanted to leave it like this.

Student B1-29: [00:08:09] Because I didn't. I didn't actually know the answer. I just knew that the numbers is equal to twenty three. All of that added. But the letters I got confused there.

Researcher: [00:08:19] So, this one and this one are again different. Actually this looks similar to the other question. Yeah. It's just that here we have got something different here. Can you tell me what's different between this one and that one with the X? This question?



Question 7

Simplify the following algebraic expression.

$3y^2 + 4y + 1 + 5y + 7y^2 + 8$

$= 3+4+1+5+7+8 = 28$

$= y^2+y+y+y^2 = y^6$

$= 28y^6$.

Student B1-29: [00:08:40] All of. It. Um. If one of these have different variables.

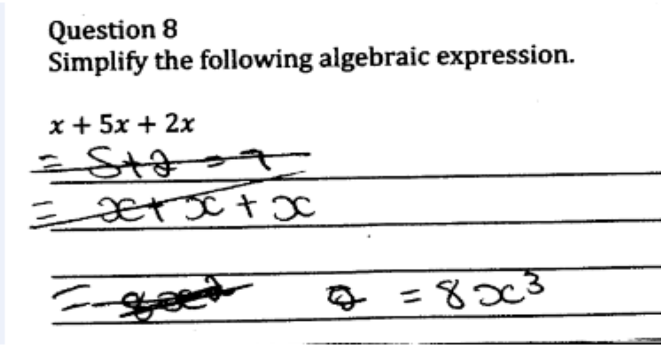
Researcher: [00:08:47] Yeah okay. So okay so what did we do here?

Student B1-29: [00:08:54] I added the numbers again. And then the variables and I got that.

Researcher: [00:08:59] And the six. Okay. And then at the end, what do you do with this and this?

Student B1-29: [00:09:05] Like,I just write it as one answer.

Researcher: [00:09:07] It's one answer. Okay? Okay, cool. Let me just quickly check that I asked everything with this question because... question seven. Um, okay. Yes. That's good. Okay, let's quickly go to the next one. Okay. This one's also interesting. So firstly you got eight. So here you got seven. But then you scratch it after you wrote eight. So can you explain?



Question 8
Simplify the following algebraic expression.

$x + 5x + 2x$

~~$= 5+2 = 7$~~

~~$= x + x + x$~~

~~$= 8x$~~ ~~8~~ $= 8x^3$

Student B1-29: [00:09:37] There's an imaginary one in front of the X.

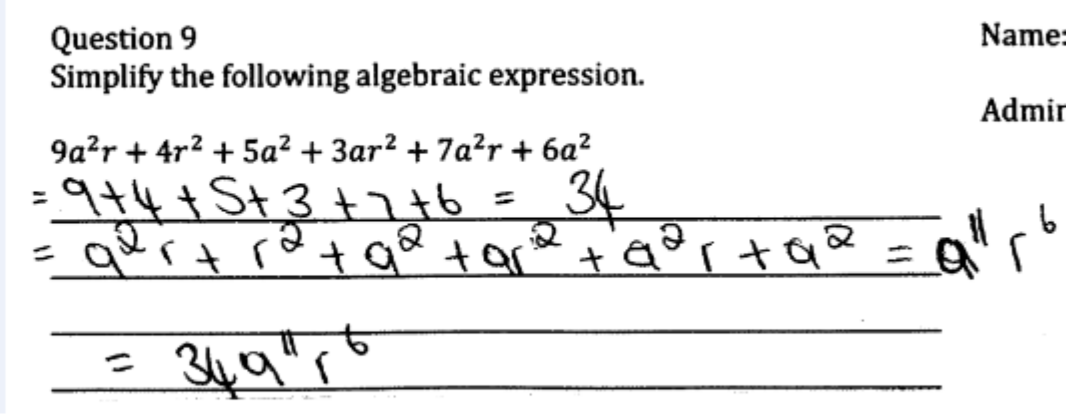
Researcher: [00:09:39] The first x? Okay. So that means okay. And that's actually gives you...

Student B1-29: [00:09:43] eight.

Researcher: [00:09:44] Okay. And then this...

Student B1-29: [00:09:46] x , all of the xs added because there are three xe.

Researcher: [00:09:50] Okay. And so this number there means that there are three xs added together. That number there okay. Let's go to the next one. Okay, this one was a lot, but I see what you've done. You've also, first the numbers and then the letters and then you. So what would you tell me you're doing? If you're doing say give me one word. What do you actually doing when you when you get these answers.



Question 9
Simplify the following algebraic expression.

Name:

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$9a^2r + 4r^2 + 5a^2 + 3ar^2 + 7a^2r + 6a^2$

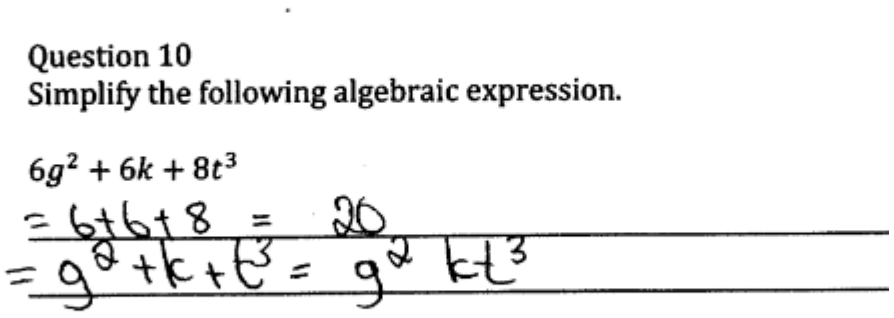
$= 9+4+5+3+7+6 = 34$

$= a^2r + r^2 + a^2 + ar^2 + a^2r + a^2 = a^{11}r^6$

$= 34a^{11}r^6$

Student B1-29: [00:10:19] Adding.

Researcher: [00:10:20] Okay? Okay. This one was very interesting. Can you tell me about this one?



Question 10
Simplify the following algebraic expression.

$6g^2 + 6k + 8t^3$

$= 6+6+8 = 20$

$= g^2 + k + t^3 = g^2\ kt^3$

Student B1-29: [00:10:28] Um, it also has unlike terms because it's different letters.

Researcher: [00:10:33] Mhm.

Student B1-29: [00:10:34] And I just added the numbers and then I just wrote how many cause there's g squared.

Researcher: [00:10:40] Yeah.

Student B1-29: [00:10:41] And then I wrote g, 20 g squared. And then I just put the k and t together and then I wrote three there.

Researcher: [00:10:49] Okay. That's yeah. That's good okay. Now I'm going to ask you some questions from like general stuff. Okay. Um so in general you would tell me like terms, is this the same letter? Okay. Um, now what are unlike terms?

Student B1-29: [00:11:09] unlike terms are like when the letters are different.

Researcher: [00:11:12] When the letters are different. Okay, good. And can you tell me, um, when we simplify in algebra, what does that mean? Again?

Student B1-29: [00:11:21] You just write the answer.

Researcher: [00:11:23] Okay. And the last one is. Yeah, the last one is right. You told me, what's this answer again? You can write it down. You told me. Yeah ten a so you said that was going to be $10a$ and can you tell me this and this. How does it relate. So what does this and this mean. Like if I go from this to this.

Student B1-29: [00:11:51] Add.

Researcher: [00:11:51] Add. Okay and okay that's it. Right. There's one more thing, this is like a little just to see what you would do. I'm just giving this if I ask you to simplify the... This one. What would be your answer?

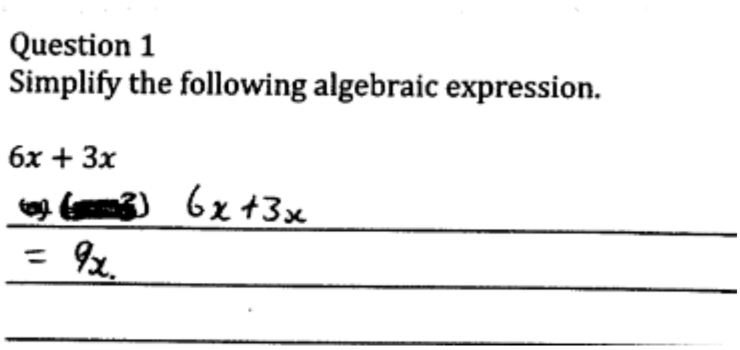
Student B1-29: [00:12:16] Five plus five is ten.

Researcher: [00:12:18] Yeah.

Student B1-29: [00:12:28] I don't know. I think it's x six because it's three plus three and then you just.

Researcher: [00:12:38] Okay. All right. So this is basically, um, like this. I can switch it off now because it's quickly.

8.6 Student B1 − 18



Question 1

Simplify the following algebraic expression.

$6x + 3x$

[illegible] $6x + 3x$

$= 9x.$

Researcher: [00:00:01] So this is your first question here. Okay. Now the question was to simplify. And so you guys said, um, what is it to simplify when you get something like $6x$ plus $3x$, what must you do when you must simplify?

Student B1-18: [00:00:19] Must. I think you must add.

Researcher: [00:00:21] You must add okay. So you add these and then you got your answer of $9x$. Can you tell me how you got that?

Student B1-18: [00:00:29] Um. It's six plus three is equal to nine. So this musn't change, it is equal to x.

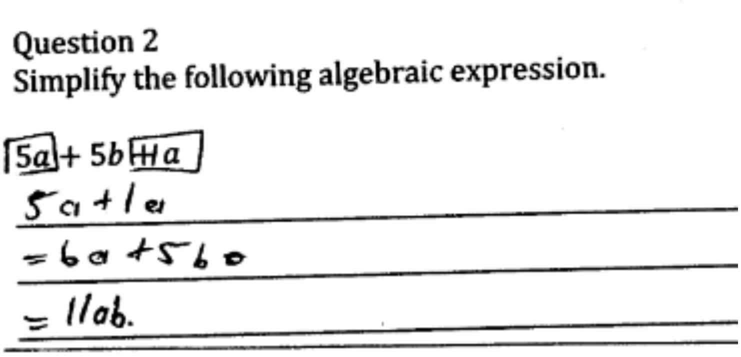
Researcher: [00:00:36] Okay. Okay. So that's how you got the $9x$. And so why I'm musn't this change.

Student B1-18: [00:00:42] Because they are like terms.

Researcher: [00:00:44] Okay. Okay good. So you say something so like terms we're going to discuss as we go on okay. So but for now why are those like those?

Student B1-18: [00:00:57] Because, um. Uh. I think the algebra.

Researcher: [00:01:03] Okay. So the algebra, do you, do you mean the letter like that. Okay. So the letter is the same. All right. So we're going to go to the next one because that's perfect okay. This is your next one. And here you drew boxes around that and you put this. What's that there?



Question 2

Simplify the following algebraic expression.

$\boxed{5a} + 5b \boxed{+4a}$

$5a + 1a$

$= 6a + 5b$

$= 11ab.$

Student B1-18: [00:01:23] It's a,an imaginary one.

Researcher: [00:01:26] Okay, the imaginary one, you put it. Okay, so the boxes, what are the boxes?

Student B1-18: [00:01:32] It simplifies the, it shows the like terms.

Researcher: [00:01:35] Yes. Okay. So you identified the like terms. Now why is this one not with those? Why are they not the same?

Student B1-18: [00:01:41] Because, um, it's different.

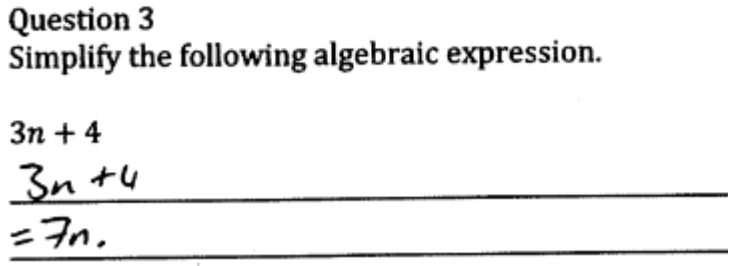
Researcher: [00:01:44] Okay. The letter. Okay. Good. Right. So then you added those and you got this and that's a $5b$ okay. That's nothing what's that? Okay. That's nothing. And then after that we got to this step here. So how did we get there?

Student B1-18: [00:02:04] Um we say 6 plus 5 is equal to 11. Then you put those together and its $11ab$.

Researcher: [00:02:10] Okay. And so when you're doing that, um, you know like the operations in maths, add subtract minus whatever divide okay. Multiply not minus. But can you tell me which operation you're doing from there to there.

Student B1-18: [00:02:29] It's an addition.

Researcher: [00:02:30] From there. Okay. Good. And from here to here is also the addition there okay okay good. So that is now question two. So this one was different from this one because there's another. Another. Yeah another letter okay. This one. Here, we also adding okay so what happened from there to the.



Question 3
Simplify the following algebraic expression.

$3n + 4$

$3n + 4$

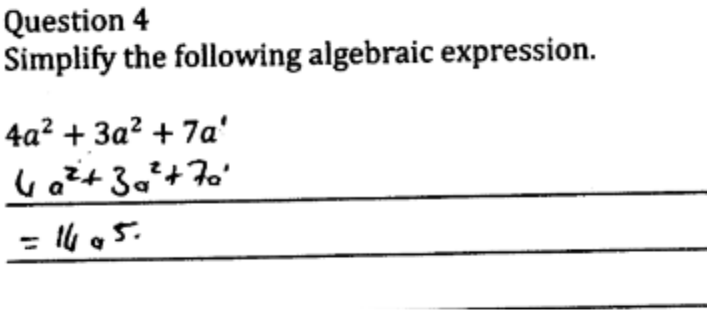
$= 7n.$

Student B1-18: [00:03:04] Four plus three is equal to seven. Then you bring down the n.

Researcher: [00:03:07] Okay. Good. And you put them together like that.

Student B1-18: [00:03:10] Yes.

Researcher: [00:03:10] Okay. So are you telling me. So if I were to ask you if I write this here, you can also write on this page. Okay. Um, can you tell me what this means? If you now break it down? If you break it up. Okay. So they it will equal that. Is there any other way that you can write this. Like can it mean something different or. No. Okay okay. Now let me give you one like this. So if I tell you five uh $5n$, what does that mean? Okay. Now I'm going to give you a trick. Question two n. What is it? Okay, cool. All right, so this is I just want to see how you. Because that's how you understand the seven n like this okay. Okay. Let's go to the next one. Let me take this page away because we wrote now all over this page. Okay. So can you tell me, um, this next one. There's something different here compared to these three here. Can you see? How does this one differ?



Question 4
Simplify the following algebraic expression.

$4a^2 + 3a^2 + 7a'$

$4a^2 + 3a^2 + 7a'$

$= 14a^5.$

Student B1-18: [00:04:39] The exponents aren't the same.

Researcher: [00:04:41] Okay, okay, good. So, that then means what?

Student B1-18: [00:04:47] Instead, you must put an imaginary exponent here, then you say four plus three is equal to seven, and seven plus seven is equal to fourteen. Then you add those. Add the exponents together, then it became A five.

Researcher: [00:05:04] Okay, cool. And at the end you put it.

Student B1-18: [00:05:06] Fourteen A five

Researcher: [00:05:07] Okay. And can you tell me are these like or unlike terms these ones. It's unlike those.

Researcher: [00:05:15] Okay. Which ones are unlike terms.

Student B1-18: [00:05:18] It's a seven.

Researcher: [00:05:19] The seven is unlike with this and these two?

Student B1-18: [00:05:23] Yeah. Like terms.

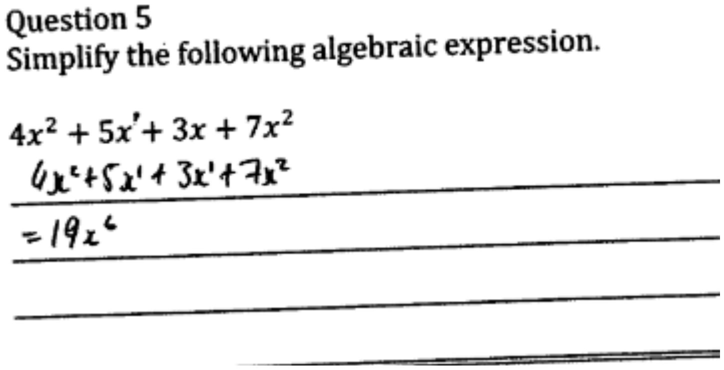
Researcher: [00:05:24] And they like terms. Okay. And so now from class, can you tell me how or even wherever you read or whatever, wherever you learnt it. Um, what are like terms? Just a normal plain language.

Student B1-18: [00:05:38] Um, like terms are things that are the same, like sums like are the same. Like, how can I put it like one one plus one a plus one a is equal to $2a$. Then unlike terms are when the letters on the same.

Researcher: [00:05:58] Okay. Cool. So can if I say write an example of like terms. Here, And unlike terms can you quickly do that. Just like just do like the like terms. Okay. So these are like terms and these are unlike terms. Okay. And if I ask you to simplify this you get 28. And if I ask you to simplify that one.

Student B1-18: [00:06:37] uhm, unlike terms.

Researcher: [00:06:38] Unlike terms. So would you add it or. No, you just leave it like that, okay? Right. Let's look at the next one. What's. This one's very similar to this. Okay, cool. So you can tell me what you did there.



Question 5
Simplify the following algebraic expression.

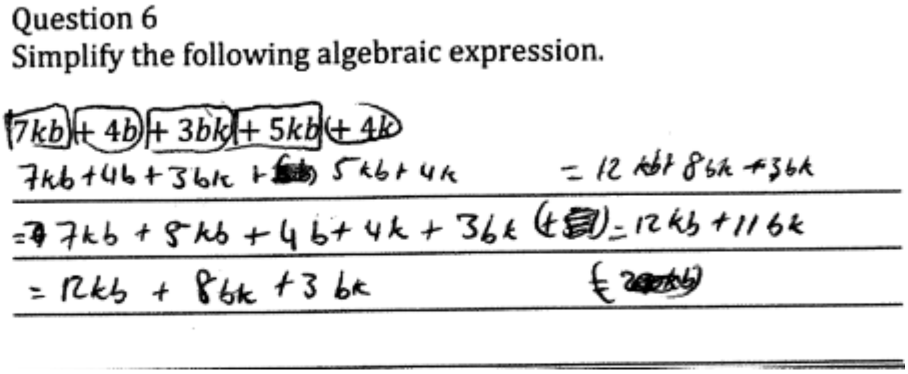
$4x^2 + 5x^1 + 3x + 7x^2$

$4x^2+5x^1+3x^1+7x^2$

$=19x^6$

Student B1-18: [00:07:00] Um. It was supposed to be. So $4x$ two plus $7x$ two is equal to. 11. Then $5x$ one plus $3x$ one is equal to $8x$ one. So you add them both together and you get $19x$ six.

Researcher: [00:07:25] Okay.So you first do it actually in your head you did it separately and then you put it together okay. All right let's look at this. This one was also very interesting because I see, you went here on the side. So I got confused because I didn't know if this is the answer to this. But now you can tell me okay. Oh is that the final answer okay. So firstly tell me what the shapes mean.



Question 6
Simplify the following algebraic expression.

$7kb + 4b + 3bk + 5kb + 4k$

$7kb+4b+3bk+$ [illegible] $5kb+4k$ $= 12kb+8bk+3bk$

$=7kb+5kb+4b+4k+3bk$ $(+$[illegible]$)$ $=12kb+11bk$

$=12kb+8bk+3bk$ $(+$[illegible]$)$

Student B1-18: [00:07:50] The shapes means like it symbolizes the like terms.

Researcher: [00:07:54] Okay. So we have the like terms.

Student B1-18: [00:07:56] One, two, So here. And it's supposed to be kb, but they switched ot around.

Researcher: [00:08:02] Okay, so then where did you so that you rewrote it and then here you and then you added it the, the *kb* because that's the seven and the five that you put next to each other.

Student B1-18: [00:08:17] Yes.

Researcher: [00:08:17] Okay. So that's the $12kb$ you added. And this one here bk just yeah. It's just so you left the b okay. So is there a reason why we just left that one. Oh but we did this. We did this eight come from.

Student B1-18: [00:08:35] It comes from the three.

Researcher: [00:08:38] Is it the three. Yes. And this three bk also. You can't remember this question. Okay, maybe because here the four b and the four k there's the four b and the four k right. Is that the four b and the four. Yeah. And so this one....

Student B1-18: [00:09:00] Comes from. Those two. Okay. And this one. This one is this three. Okay, so I got a final answer of $12kb$ plus $11bk$.

Researcher: [00:09:15] And you, you're going to leave it like that because that's.

Student B1-18: [00:09:18] Unlike terms

Researcher: [00:09:19] Okay. So it's unlikely because the kb and the bk because of the.

Student B1-18: [00:09:26] kb and the bk.

Researcher: [00:09:27] Okay. So if I gave you like ac and ca. Uh would you add terms with those. No. Because they unlike terms okay. And um because that's what you telling me and okay. So how are they different? What makes them so unlike?

Student B1-18: [00:09:55] Because they swapped around.

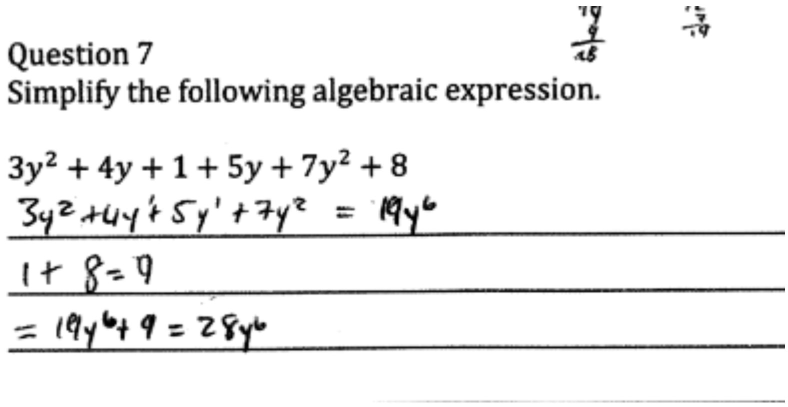
Researcher: [00:09:57] Okay. Because they swapped them. Okay. Cool. And what happened to these two? Here, I just need to make sure that.

Student B1-18: [00:10:06] Um, I added the eight. Then I put the both letters here.

Researcher: [00:10:11] And so why can we add these two. Because that's the $4b$ and the $4a$. Yeah. So, why can we add it?

Student B1-18: [00:10:21] Because they both are single and okay.

Researcher: [00:10:26] Single okay. That is good. This one was a bit long. That was a long one okay. Now this one is similar to the others. But I see you've also got your own method for doing this one. And you broke it up into these ones first. So what are those ones.



$\begin{array}{r} 19 \\ 9 \\ \hline 28 \end{array}$

Question 7

Simplify the following algebraic expression.

$3y^2 + 4y + 1 + 5y + 7y^2 + 8$

$3y^2 + 4y^1 + 5y^1 + 7y^2 = 19y^6$

$1 + 8 = 9$

$= 19y^6 + 9 = 28y^6$

Student B1-18: [00:10:48] Those are, uh like terms.

[00:10:51] [Siren]

Researcher: [00:11:03] Yes. Okay. Yeah. So what does that mean?

Student B1-18: [00:11:07] Um. I added the, the terms here, so I got $19y$ six then. Plus the nine that I got from eight plus one is equal to ten.

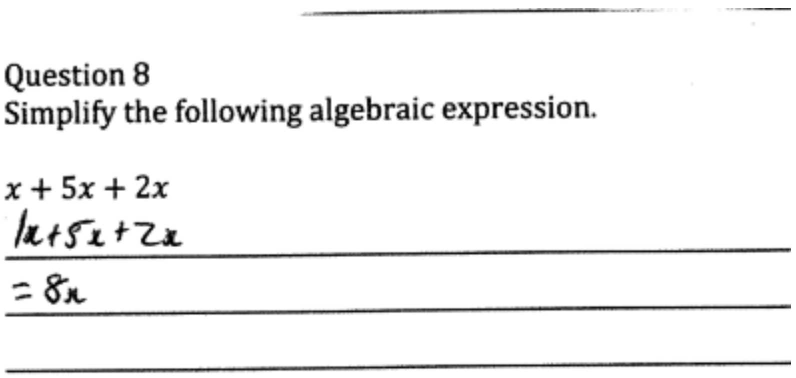
Researcher: [00:11:19] Oh and these here you just added these ones okay. Good. All right. And then you just the nine. And this makes the.

Student B1-18: [00:11:29] 28

Researcher: [00:11:31] Okay. Cool. Okay. We're almost done. Right. So we've got these three questions. So we've got two options. Where must you go next?

Student B1-18: [00:11:42] (Inaudible).

Researcher: [00:11:43] I'm going to go with you to to that class. Um, but I think cause of the noise, we're going to have to just quickly pause and then we're going to continue with the interview. But you can quickly look at this one and think about how you can explain these. And then this one, I think I can see, you know, what you've done. And that last one okay. So just have a look quickly through them. Okay. Right. So what happened there?



Question 8
Simplify the following algebraic expression.

$x + 5x + 2x$

$1x+5x+2x$

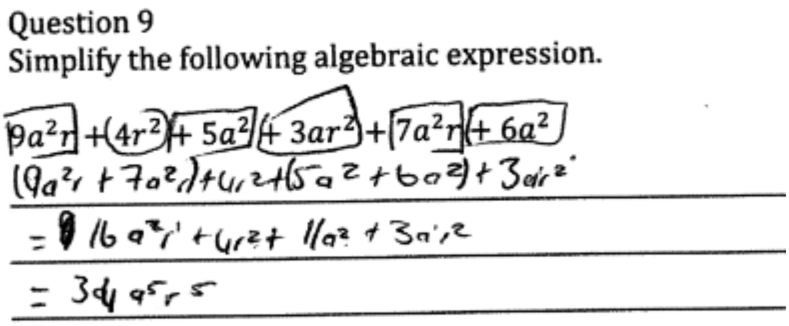
$= 8x$

Student B1-18: [00:12:22] Um, yeah. By the x there was an imaginary one. So one plus. Plus one plus five is equal to six. So we add another two. Is equal to $8x$.

Researcher: [00:12:34] Okay good. And the x is just one. There's just one x there.

Student B1-18: [00:12:39] yes

Researcher: [00:12:40] Okay. Okay. All right. Next, this one you just added. I see you made the like terms with your shapes again. So then you added them and then after at the end you just. What did you do at the end?



Question 9
Simplify the following algebraic expression.

$\boxed{9a^2r} + \boxed{4r^2} \boxed{+ 5a^2} \boxed{+ 3ar^2} + \boxed{7a^2r} \boxed{+ 6a^2}$

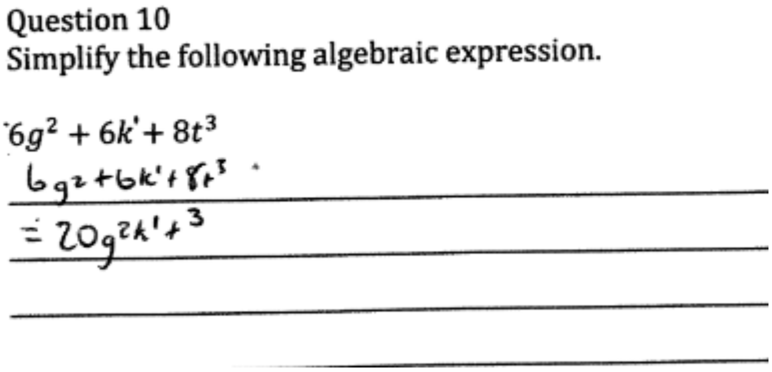
$(9a^2r + 7a^2r) + 4r^2 + (5a^2 + 6a^2) + 3ar^2$

$= 16a^2r^1 + 4r^2 + 11a^2 + 3ar^2$

$= 34a^5r^5$

Student B1-18: [00:12:57] I added them up to get 34a to the five.

Researcher: [00:13:04] Okay. Just like you did with the others. Okay. And this is the last one. What can you tell me?



Question 10
Simplify the following algebraic expression.

$6g^2 + 6k^1 + 8t^3$

$6g^2 + 6k^1 + 8t^3$

$= 20g^2k^1t^3$

Student B1-18: [00:13:16] Here, like. They are all unlike. In order to get down. They all go on (inaudible), I added six plus six is equal to twelve, plus eight is 20. Then I said g two, k one plus t three.

Researcher: [00:13:37] And you put it in one answer like that. Okay. So you say that unlike terms. And then you must basically you first added the number, the number in front.

Student B1-18: [00:13:48] coefficients

Researcher: [00:13:49] Of the coefficients and then you got the last part of it. Okay. Right, Lee, this was very interesting. I just have two general questions you already answered. Simplify. Um, just in your language, you answered what like terms are? Here, you showed me like, terms have the same letters and unlike terms have. Okay. And so. Okay. That's it. Let me just double check that I asked everything. You are very, very good. Okay. And last question. If I gave you let's give you six x. What does that mean.

Student B1-18: [00:14:38] Is there a ... (inaudible).

Researcher: [00:14:42] Like, you can just tell me. Like, how else can you write them? Like we can come from? What does it mean?

Student B1-18: [00:14:49] The number as it symbolises the number. No, just.

Researcher: [00:14:53] Like if I put a six next to a x, what does this mean?

Student B1-18: [00:14:57] Six times x.

Researcher: [00:14:59] Okay. All right. And if I put a seven next to a n, what does that mean?

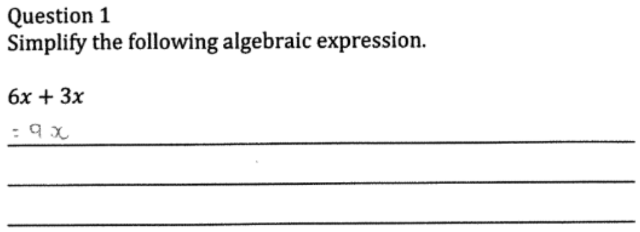
Student B1-18: [00:15:05] (writes 7xn)

Researcher: [00:15:07] Okay. All right. Thank you

8.7 Student A2-09

Student A2-09

[00:00:00] **Researcher:** Okay. So this is your first question here that you have written down, and basically this exercise was about simplifying algebraic expressions right. So if you look at that one there it says, can you read that question?



Question 1

Simplify the following algebraic expression.

$6x + 3x$

$= 9x$

Student A2-09: $6x$ plus $3x$

Researcher: And your answer is

Student A2-09: $9x$

Researcher: So, can you explain how you got there?

[00:00:25] **Student A2-09:** Because it's not any addition sum. I mean a multiplication, which means you don't have to, you don't have to times your constant term, I mean, yes. Your constant term, which means your variables stay the same. They don't change. Then you'll just add your whole numbers, which is, um, 6 plus 3 which is equal to nine. And then you just bring down a variable because it doesn’t change.

[00:00:51] **Researcher:** Okay. You said because it's an addition, the variable doesn’t change.

Student A2-09: Yes

[00:00:57] **Researcher:** Okay. So let's quickly go back to what you said. This nine. How did you get that?

[00:01:09] **Student A2-09:** So, if it was a multiplication sign, it would have been x times x, which is equal to x to the power of 2. But there is no, um, there's no, um, multiplication sign, which means it stays the same and there is a number in front of the variable.

[00:01:33] **Researcher:** Okay. And so the answer, what does this answer mean to you? Like if you just look at $9x$, what does it mean to you?

[00:01:42] **Student A2-09:** Um, there's still more to the sum. We still have to figure out what x is.

[00:01:45] **Researcher:** Okay. And what is x?

[00:01:52] **Student A2-09:** x can be anything. It's a valuable, which means it can be a different number.

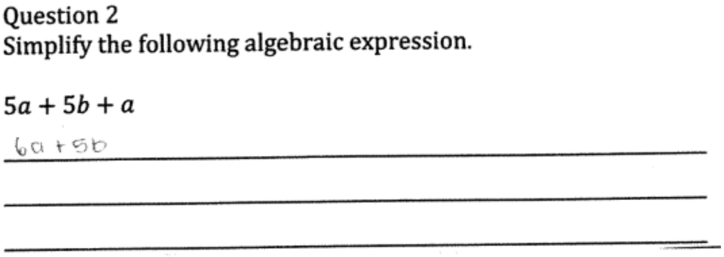
[00:01:58] **Researcher:** Okay. Good. So, whats the reason we can add this like this?

[00:02:10] **Student A2-09:** because it's like, a normal sum.

[00:02:15] **Researcher:** Okay, so once you have added, you put the 9 and the x like this? You've written it next to each other like that. Okay. And so if I were to ask you, what does $9x$ mean?

Student A2-09: 9 times x

[00:02:38] **Researcher:** Okay. I think that's all. That's the first one. This is question two. Can you read your answer to me?



Question 2
Simplify the following algebraic expression.

$5a + 5b + a$

$6a + 5b$

Student A2-09: $6a$ plus $5b$.

[00:02:54] **Researcher:** Okay. So can you explain how you got this?

[00:03:01] **Student A2-09:** So there is $5a$ plus the a, which means that a doesn't have any number in front so it means it is a one in front of it. So then 1 plus 5 is 6 and your variable stays the same.

[00:03:14] **Researcher:** Okay. And so why does your variable stay the same?

[00:03:17] **Student A2-09:** Because there's no multiplication.

Researcher : Good. And then

[00:03:21] **Student A2-09:** So $5b$ stays the same because there's no variable with another b in it.

[00:03:30] **Researcher:** And you decided to stop there.

Student A2-09: Yes.

Researcher: So, in your mind, you can’t simplify this any further. Okay. So why have you decided to stop?

[00:03:42] **Student A2-09:** Because $6a$ and $5b$ are unlike terms.

[00:03:46] **Researcher:** Okay. So now you said something. So what in your own, um, what's your own meaning of unlike terms?

[00:03:54] **Student A2-09:** So unlike, unlike the resolving, the variables are different. So if the variable is a and the variable is b, terms will be different. And if it is a squared and a it's still a different.

[00:04:16] **Researcher:** And there is a reason why we can’t simplify these unlike terms?

[00:04:27] **Student A2-09:** Yes, because, you can do it if it was multiplication. You can say $6a$ times $5b$ which is $30ab$ but there's no multiplication its just addition.

[00:04:42] **Researcher:** Okay, so what's the main reason why we can't really simplify this? The main reason?

[00:04:52] **Student A2-09:** Because they are unlike terms.

[00:04:53] **Researcher:** Okay. And so how are a and b different to you. What do they differ in?

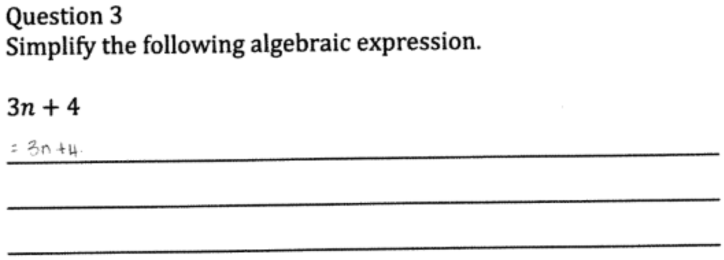
Student A2-09: One’s a and one’s b.

Researcher: Okay. Um, and so if I were to ask you in terms of mathematics, in terms of algebra that you are learning…

[00:05:29] **Student A2-09:** um, a and b can be a different number and variables.

[00:05:35] **Researcher:** We might come back to this one.

[00:05:40] **Researcher:** Excellent. Okay. This one, can you tell me about this one? What do you think?



Question 3
Simplify the following algebraic expression.

$3n + 4$

$= 3n + 4.$

[00:05:45] **Student A2-09:** Because 3n plus 4, there's no variable, which means it all stays the same.

[00:05:56] **Researcher:** So you have left it the same? And so that means that you have decided that you can't simplify this.

Student A2-09: Yes

[00:06:06] **Researcher:** Okay. Um, so… can you tell me what does the 3n mean?

[00:06:12] **Student A2-09:** So, it can be 3 multiplied by n, then it's that number, you add 4.

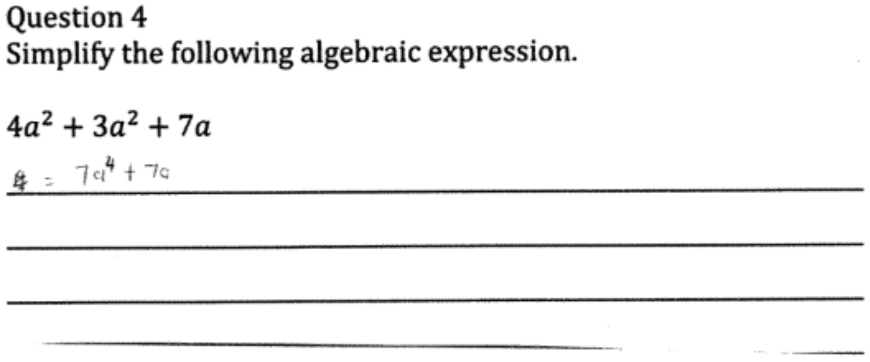
[00:06:29] **Researcher:** okay. And so, your main reason for not simplifying this any further is?

Student A2-09: Because they are unlike terms.

Researcher: And unlike terms can't be simplified because…

[00:06:41] **Student A2-09:** (inaudible)

[00:06:46] **Researcher:** All right. This one is question four. So, can you read your answer to me?



Question 4
Simplify the following algebraic expression.

$4a^2 + 3a^2 + 7a$

$4 = 7a^4 + 7a$

[00:06:57] **Student A2-09:** 7a to the power of 4, plus 7a.

[00:06:58] **Researcher:** Okay, so over here, can you tell me how you thought about this?

[00:07:04] **Student A2-09:** So 4a squared plus 3a squared, I mean sorry, 4 plus 3 is 7. a to the power of two plus a to the power of two is a to the power of four, because you just add the exponents.

[00:07:20] **Researcher:** Okay, so can I ask you to write that, this is a dark pen. Just write like you thought when you were writing this, and you can explain while you do it.

[00:07:41] **Student A2-09:** Okay, so I first said 4 plus 3 which is 7 and then a squared plus a squared is equal to a to the power of 4.

[00:07:55] **Researcher:** And can you tell me how you got the a to the power of 4.

[00:08:04] **Student A2-09:** I just added the two together.

[00:08:11] **Researcher:** Okay. And then what you did was to get this 7a to the power of 4.

Student A2-09: I added the two

Researcher: So, you added the 7 to the a to the power of 4 and when you add, you put it…

Student A2-09: Together.

Researcher: Together. Okay, right. Now, so I quickly wanted to see so what does a to the power of 4 mean to you?

[00:08:38] **Student A2-09:** Like, so if a was 2, then I will say 2 times 2 four times then you will get the answer.

[00:08:45] **Researcher:** Okay. Well, now we need to come back to this again. Let's just look at. Okay. So after that you add your …

Student A2-09: 7a

Researcher: and then you decided the that's the end for this one.

Student A2-09: Yes.

Researcher: Okay. So why, why did you stop at that point?

[00:09:06] **Student A2-09:** Because 7a had no exponents, which means 7a to the power of 4 will, it doesn't have any exponents which means I can't add anything to it.

[00:09:24] **Researcher:** Okay, so this 7a that doesn't have any exponents. So you can't add anything to it? So from this point, you can't simplify further because…

[00:09:33] **Student A2-09:** because $7a$ doesn’t have an exponent.

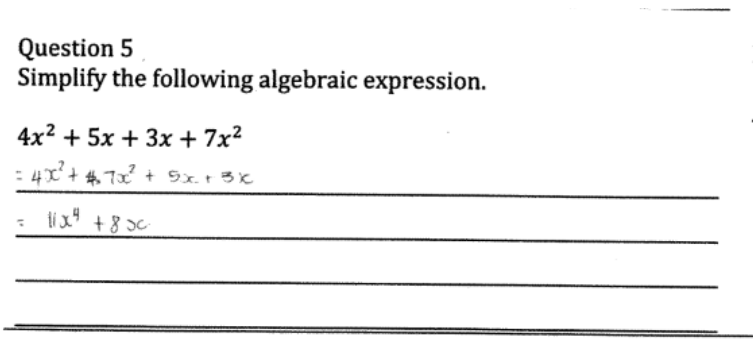
[00:09:37] **Researcher:** Okay. So are you comparing the two here. Are you comparing these two to know when to stop?

Student A2-09: Yes.

[00:09:55] **Researcher:** okay. Anything else you want to say about this one?

Student A2-09: No.

[00:10:05] **Researcher:** This one is question five and your answer was…



Question 5

Simplify the following algebraic expression.

$4x^2 + 5x + 3x + 7x^2$

$= 4x^2 + \cancel{4} 7x^2 + 5x + 3x$

$= 11x^4 + 8x$

[00:10:09] **Student A2-09:** $11x$ to the power of 4 plus $8x$.

[00:10:12] **Researcher:** Okay. So can you explain what you've done here? It looks like you have changed something here.

[00:10:17] **Student A2-09:** So I added all the like terms together, so my terms where $4x$ to the power of 2, plus $5x$ plus $3x$. Which gave me $11x$.

[00:10:33] **Researcher:** Okay, good. And in the second line you took the first line and you…

Student A2-09: I switched it around.

Researcher: So, basically how did you switch it around?

Student A2-09: I put all the like terms together.

[00:10:52] **Researcher:** Okay. Now let's look at the $11x$ to the power of 4. So can you explain how you got the $11x$ to the four?

[00:11:02] **Student A2-09:** Um, so it's the exact same like this one. I just said $4x$ to the power of 2 plus $7x$ to the power of 2 is equal to…

[00:11:26] **Researcher:** Okay. So you said x to the power of 2, to x to the power of 2… and then at the end….

Student A2-09: I added the two together.

[00:11:31] **Researcher:** And when you added the two, which is $11x$ to the power of 4 you got…

[00:11:44] **Student A2-09:** $11x$ to the power of 4.

[00:11:48] **Researcher:** So, you put them together.

Student A2-09: Yes.

[00:12:02] **Researcher:** Alright, so lets look at this point here. So you firstly got the x because…

[00:12:04] **Student A2-09:** x stays the same.

[00:12:09] **Researcher:** And then

Student A2-09: I added the 2 by the 2.

Researcher: Okay. And then you got x to the power of 4, what does that mean to you?

[00:12:18] **Student A2-09:** Any number multiplied four times.

[00:12:28] **Researcher:** Okay, so lets look at this other part the $5x$ plus $3x$ so can you write this and show me how you thought about this?

[00:12:48] **Student A2-09:** So, I said 5 plus 3 is equal to 8 and then I said x plus x is equal to x because there's no exponent.

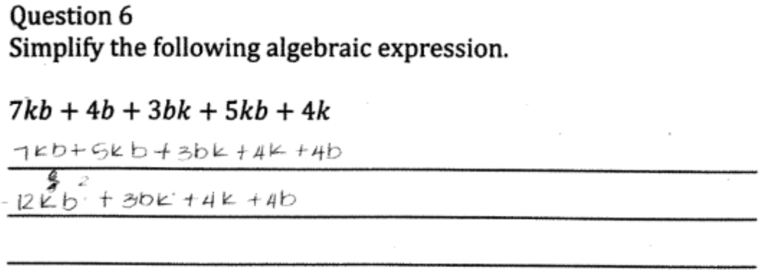
[00:13:08] **Researcher:** So, then there's no exponent, then it stays the same. And once you've got your eight from the 5, plus the 3 and your x from x plus x, you then do what to the 8 and the x…?

[00:13:24] **Student A2-09:** Add them together to get $8x$.

[00:13:27] **Researcher:** So, what does $8x$ mean to you?

[00:13:32] **Student A2-09:** Any number, it's eight multiplied by x.

[00:13:43] **Researcher:** Okay, this is question six. And as you can see it is a bit different. Let's first ask you to read your answer.



Question 6

Simplify the following algebraic expression.

$\boldsymbol{7kb + 4b + 3bk + 5kb + 4k}$

$7kb+5kb+3bk+4k+4b$

$12kb^2 + 3bk + 4k + 4b$

Student A2-09: 12kb squared plus 3bk, plus 4k, plus 4b.

[00:14:00] **Researcher:** Okay. So in the second line. What have you done in the second line?

[00:14:08] **Student A2-09:** I put it in like terms again.

[00:14:08] **Researcher:** Okay, so you've changed the order

Student A2-09: Yes

Researcher: Okay. And so what are the like terms in this question?

[00:14:16] **Student A2-09:** 7kb plus 5kb and then 3bk, 4k and 4b are just unlike terms.

[00:14:26] **Researcher:** Okay. And can you explain how you got from this two, the 7kb and 5kb and what did you get as a result?

[00:14:38] **Student A2-09:** 12kb squared.

[00:14:41] **Researcher:** So can you show me how you've done that part?

[00:14:53] **Student A2-09:** 7kb plus 5kb which is 7 plus 5 is equal to 12 and then kb pluskb is equal to kb squared.

[00:15:07] **Researcher:** Okay. So when you get this, that's the same, like that is the same as the rest. 7 plus the 5 gives you the 12. So the kb plus the kb gave you kb squared. That's what you say. So how did you get that?

[00:15:32] **Student A2-09:** I made a mistake. It's supposed to be kb because there is no exponent.

[00:15:41] **Researcher:** Okay. Alright, we can always look at that again. Plus 3bk, plus 4k, plus 4b. Okay now can you explain why 3bk is not added to anything?

[00:16:01] **Student A2-09:** because most of the numbers they have like terms. So, they do variables like the same. So in order kb that one is in order of bk.

[00:16:13] **Researcher:** okay. So when I say it's not added I actually mean when it's not simplified further. So you've left is as part of this addition here. Yeah. Okay. So you haven't simplified the $3kb\ further$?

[00:16:32] **Student A2-09:** Because the $7kb$ and $5kb$ are in order kb. The rest of the numbers don't have a k or a b or the order is different.

[00:16:51] **Researcher:** okay. So if the order is different they are…

Student A2-09: Unlike terms.

[00:17:02] **Researcher:** So what makes these unlike?

[00:17:07] **Student A2-09:** Because some are missing variables and the order is different.

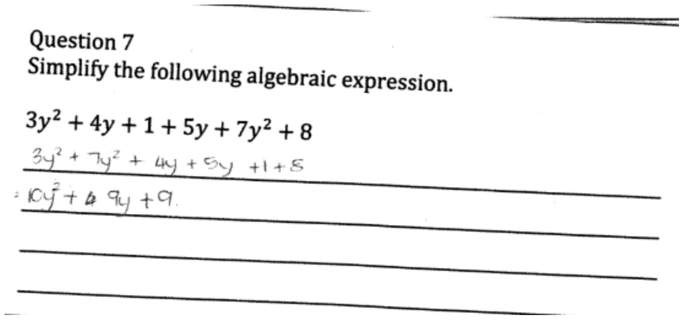
[00:17:12] **Researcher:** The order is different. Okay and then just lastly $4k$ and $4b$ are?

[00:17:29] **Student A2-09:** Unlike terms

Researcher: Because?

Student A2-09: $4k$ doesn't have a b and $4b$ doesn't have a k.

[00:17:37] **Researcher:** Alright, question 7. Question 7 is an addition and here we have a few terms. And I see here, just like the others. Okay, so what have you done in the second line?



Question 7

Simplify the following algebraic expression.

$3y^2 + 4y + 1 + 5y + 7y^2 + 8$

$3y^2 + 7y^2 + 4y + 5y + 1 + 8$

$= 10y^2 + \cancel{4}\ 9y + 9.$

[00:17:56] **Student A2-09:** So, I put all the like terms together.

[00:18:00] **Researcher:** Okay, good. Now, if we look at the first two here, have have you written here?

[00:18:12] **Student A2-09:** $3y$ to the power of 2 and $7y$ to the power of 2.

[00:18:14] **Researcher:** So, have you grouped this as…

[00:18:22] **Student A2-09:** Like terms.

[00:18:22] **Researcher:** And now what was the result of adding these two?

[00:18:27] **Student A2-09:**Uhm I got $10y$ to the power of two

[00:18:31] **Researcher:** Okay so how did you get the 10?

[00:18:32] **Student A2-09:** I said 3 plus 7 which is the 10

[00:18:37] **Researcher:** And the y to the power of 2.

[00:18:39] **Student A2-09:** I said y plus y to the power of 2, is equal to y to the power of 2.

[00:18:48] **Researcher:** Okay, so you've left it the same here, so how did you know to leave this one the same?

[00:18:56] **Student A2-09:** Because I realised all mu other sums where wrong. I did multiplication instead of addition.

[00:19:05] **Researcher:** Okay, and here…?

[00:19:06] **Student A2-09:** I said $4y$ plus $5y$ which is equal to $9y$.

[00:19:17] **Researcher:** Okay and now that one?

Student A2-09: 1 plus 8 is 9.

[00:19:18] **Researcher:** Okay. And you've stopped here? So, what's the reason you stopped here?

[00:19:23] **Student A2-09:** $10y$ to the power of 2 plus $9y$ is two different terms. One's squared and one is y, and then 9 is left on its own because it doesn't have any variables.

[00:19:46] **Researcher:** So, what makes it the $10y$ squared and the $9y$ they are unlike terms because of the …

Student A2-09: Exponents.

[00:20:04] **Researcher:** Okay, and the reason why we can't add these two unlike terms is because …

[00:20:11] **Student A2-09:** Because if you say $10y$ squared you are saying $10y$ times $10y$. $9y$ stays the same because there is no exponent.

[00:20:23] **Researcher:** Alright, okay this one. Can I just quickly go back to this one. So the 9 here, 9 you got from …

[00:20:48] **Student A2-09:** 4 plus 5.

[00:20:53] **Researcher:** And in your mind, can you tell me what y squared means?

[00:20:57] **Student A2-09:** y times y.

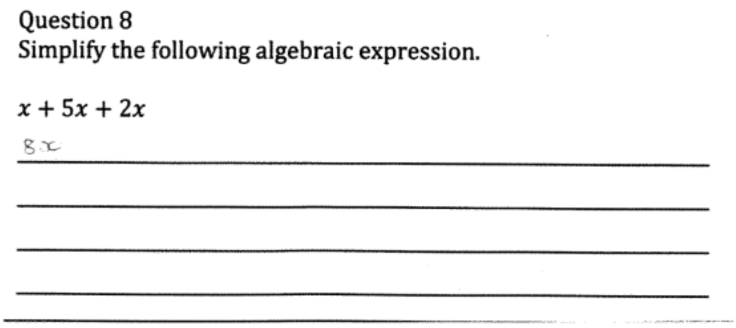
[00:20:58] **Researcher:** And what does $9y$ mean?

[00:21:06] **Student A2-09:** 9 times y

Researcher: And what does $10y$ squared mean?

Student A2-09: $10y$ times $10y$

[00:21:09] **Researcher:** Uh, so let's look at the next question. Asking you to add these terms, and your answer is?



Question 8
Simplify the following algebraic expression.

$x + 5x + 2x$

$8x$

Student A2-09: $8x$.

Researcher: Okay. So can you explain how you got the 8?

[00:21:29] **Student A2-09:** So 5 plus 2 is 7 and there is an invisible 1 in front of the $x$so I add that 1 which makes it $8x$. And the variable stays the same. Nothing for the exponent.

[00:21:48] **Researcher:** So this is 8 and x. So you first added the numbers, that's what you said.

[00:22:01] **Student A2-09:** All the $x's$. Because they are like terms the x stays the same and there's no multiplication.

[00:22:11] **Researcher:** So, the x in front, remind me has a 1…

Student A2-09: A one.

[00:22:28] **Researcher:** So why do you think you can add all of these?

[00:22:30] **Student A2-09:** Because they are like terms, because each term has a variable.

[00:22:38] **Researcher:** Okay. And why can we add like terms like this?

[00:22:46] **Student A2-09:** It's easier than having to write out the whole sum.

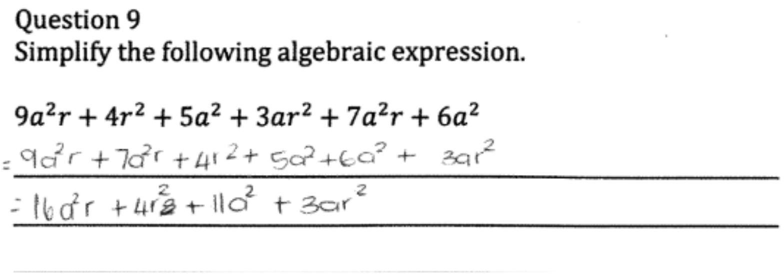
[00:22:56] **Researcher:** And why is this one not x to the power of 3.

Student A2-09: Because it's not multiplication.

[00:23:08] **Researcher:** So if we go back to question 5, or we can go back to question 4. Where we had $4a^2$ plus $3a^2$ gives you $7a^4$. Do you think that is correct?

Student A2-09: No. It's wrong.

[00:23:40] **Researcher:** Right, this question here. Just have a look to see what you have done first.



Question 9
Simplify the following algebraic expression.

$9a^2r + 4r^2 + 5a^2 + 3ar^2 + 7a^2r + 6a^2$

$= 9a^2r + 7a^2r + 4r^2 + 5a^2 + 6a^2 + 3ar^2$

$= 16a^2r + 4r^2 + 11a^2 + 3ar^2$

[00:23:55] **Researcher:** Okay. So what have you done here? In this line and the second line.

[00:24:03] **Student A2-09:** So, I said $9a^2r$ plus $7a^2r$ plus $4r^2$ plus $5a^2$ plus $6a^2$ plus $3ar^2$. So, I put all the like terms together.

[00:24:16] **Researcher:** Okay so you got the 16 from…?

[00:24:19] **Student A2-09:** From 9 plus 7.

[00:24:29] **Researcher:** Can you tell me why didn't you add the $3ar^2$?

[00:24:47] **Student A2-09:** Because the $3ar^2$, the a doesn't have an exponent. Whereas in this one the a has its own exponent.

[00:24:55] **Researcher:** So, what does that a mean actually, a^2 mean?

[00:24:57] **Student A2-09:** So it's 16 times a times r.

[00:25:02] **Researcher:** Okay, so do you want to quickly repeat that?

[00:25:08] **Student A2-09:** 16 times a times a times r.

[00:25:17] **Researcher:** Okay, and this one means.

Student A2-09: 3 times a times r times r

Researcher: Okay. And then, here you see. You've put the $5a$ squared and the s 6 squared next to each other and you've got?

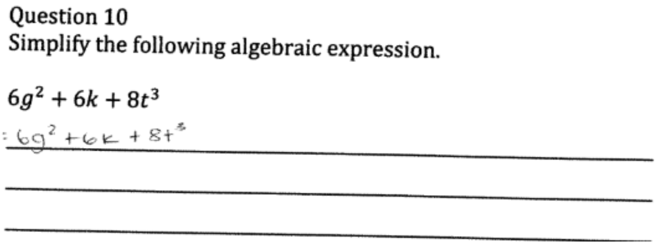
Student A2-09: $11a$ squared

Researcher: And so the 11 comes from…

[00:25:36] **Student A2-09:** The 5 and the 6.

Student A2-09: And the a^2. Because a squared plus Type equation here. A squared. There's no multiplication so it just stays the same.

[00:25:47] **Researcher:** Let's look at this way, this is the last one. And yet you did it, do you just whatever.



Question 10
Simplify the following algebraic expression.

$6g^2 + 6k + 8t^3$

$= 6g^2 + 6k + 8t^3$

[00:25:57] **Student A2-09:** I left it as is.

[00:25:59] **Researcher:** And the reason you left it as is?

Student A2-09: Because there are no like terms

Researcher: And how did you recognise that there are no like terms?

[00:26:02] **Student A2-09:** because g squared, is g times g. Theres nothing with a k and j and k are two different letters which means that they are two different numbers.And then, t is different because t can also be a different number and that number must be multiplied by itself 3 times.

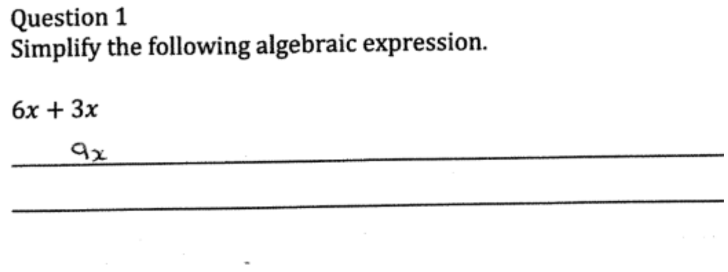
[00:26:29] **Researcher:** So because they all different, you've decided that you just leave it like that? And what's the main, the reason why when they are different, you cannot simplify?

[00:26:42] **Student A2-09:** Because they don't have any like terms.

(Additional questions and content after 10 questions trimmed.)

8.8 Student A1-02

Researcher: [00:00:01] This is Question 1 and the question says you have to simplify the algebraic expression. Can you first tell me, what is an algebraic expression?



Question 1
Simplify the following algebraic expression.

$6x + 3x$

$9x$

Student A1-02: [00:00:10] Uhm, to my knowledge, I think an algebraic expression is like a sum or like with letters and variables.

Researcher: [00:00:24] And lets look at this questions here, can you read it to me?

Student A1-02: [00:00:30] $6x$ plus $3x$.

Researcher: [00:00:32] And your answer is?

Student A1-02: [00:00:34] $9x$

Researcher: [00:00:35] Okay, can you explain how you got to your answer?

Student A1-02: [00:00:38] So, you get 9 because you say 6 plus 3, and you only get one x because its not multiplied, because usually the law is that when you multiply because usually the law is that if the sum was multiplied with the same base then you add the exponents.

Researcher: [00:01:03] Okay, and if you were to write down the question again, and tell me everything you are thinking when you are answering this question?

Student A1-02: [00:01:31] Uhm, so usually I would just go 6 plus 3 that would give me 9. And then then I will say because its not multiply I will just get that one x.

Researcher: [00:01:42] Okay so what have you done with the x?

Student A1-02: [00:01:44] So, basically in my mind it just becomes one. Like it combines together and just becomes one x or we cold have just taken the other one away.

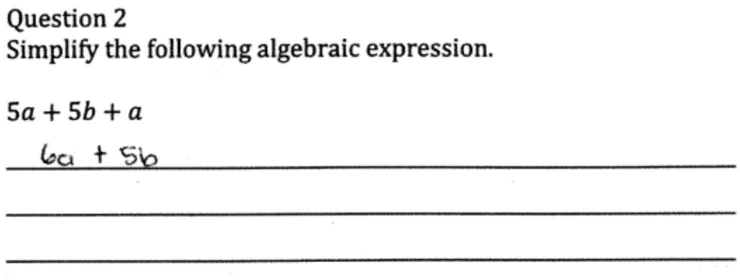
Researcher: [00:02:02] Okay, so you have chosen to represent it like this (referring to $9x$). so how would you describe this? Okay, so just read this again?

Student A1-02: [00:02:14] $9x$

Researcher: [00:02:20] So what does $9x$ mean to you.

Student A1-02: [00:02:22] Just a number with a variable next to it. Like, 9 multiplied by x.

Researcher: [00:02:30] Okay, let us go to the next one. So here we have question two which is $5a$ plus $5b$ plus a, and you answer is?



Question 2

Simplify the following algebraic expression.

$5a + 5b + a$

$6a + 5b$

Student A1-02: [00:02:48] $6a$ plus $5b$.

Researcher: [00:02:58] Now can you explain to me how you have approached this and how you have done it?

Student A1-02: [00:03:00] So first I add the like terms, which is $5a$ and a. Every variable by itself has a one in front of it. So a plus $5a$ is $6a$. And I can't add the other two because they are not like terms.

Researcher: [00:03:20] So you first add the $5a$ and the a and you get the $6a$. And the $5b$ what have you done with that?

Student A1-02: [00:03:32] I just left it as $5b$.

Researcher: [00:03:35] So can you tell me now, you said something about like terms. So what are like terms?

Student A1-02: [00:03:48] Like terms are when the numbers have the same variable and we are able to add it together.

Researcher: [00:04:01] Okay, and you stopped there because …

Student A1-02: [00:04:04] They are not like terms.

Researcher: [00:04:06] And the reason they are not like terms is because

Student A1-02: [00:04:08] They have different variables.

Researcher: [00:04:15] And can you simplify those two? What's the reason why you can't simplify it?

Student A1-02: Because a can't go into b.

Researcher: [00:04:24] And what do you think a and b represent?

Student A1-02: [00:04:27] Just variables to me.

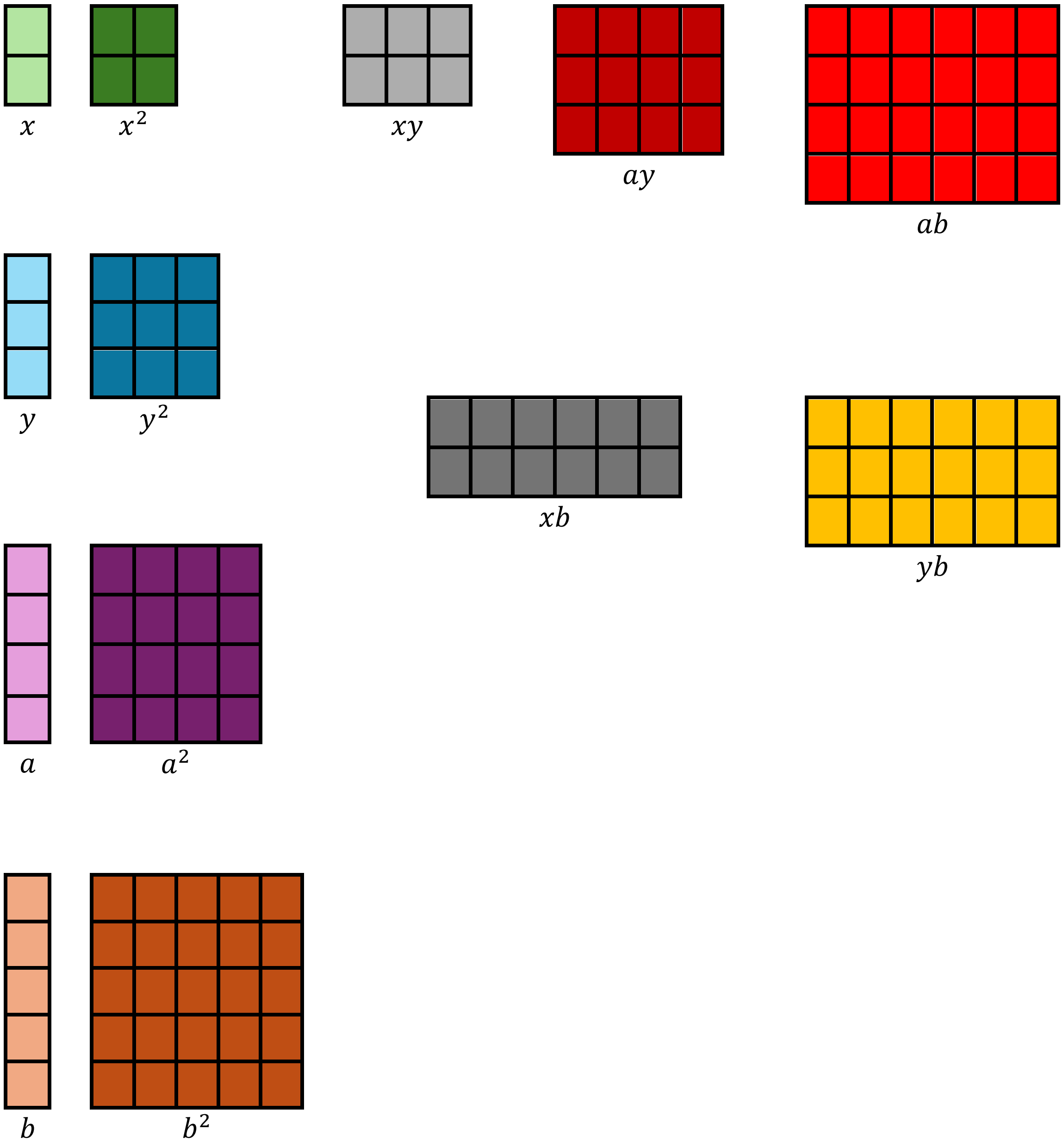
(Due to technical challenges with this video footage, only a part of the transcript is made available.)

9. Area Cards – A teaching strategy

Area Cards

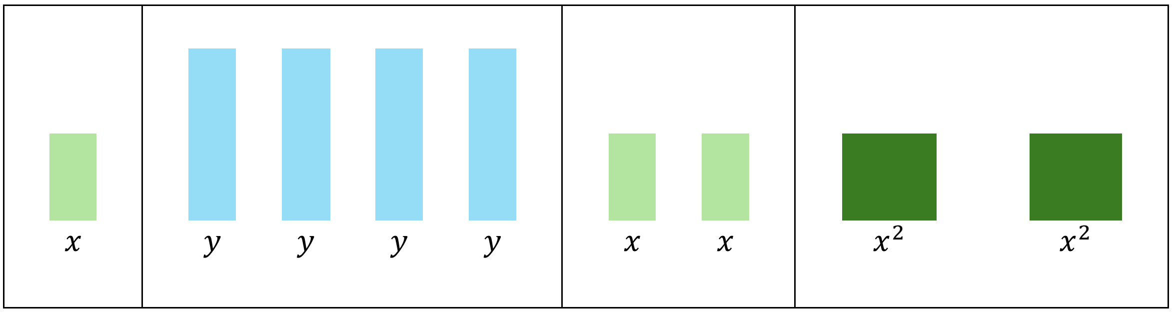
Using a manipulative in order to bridge this cognitive gap. There are limitations to this model because it only allows for the addition of algebraic expressions of degree two and at most two distinct variables per term. However, the hope is that it can serve as a starting point for grade 8 students.

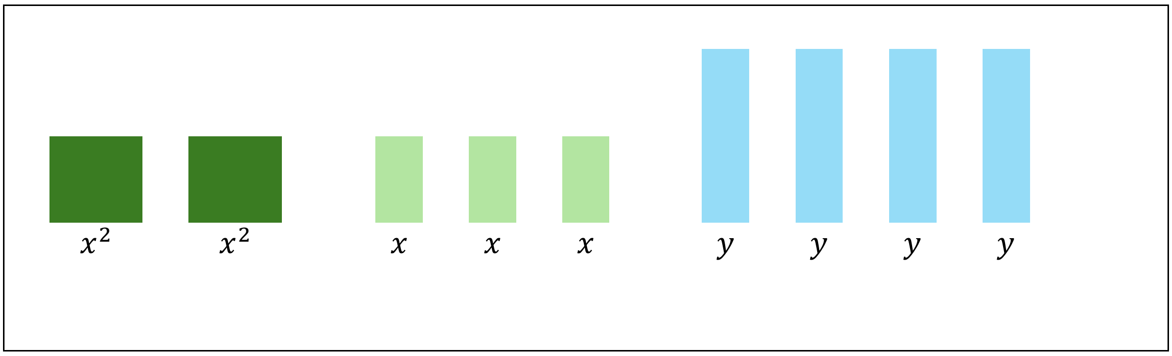
These area cards would be made with centimetre grid paper. I would suggest that you need to have 10 of each card.



How could you use these cards to add like terms?

$x + 4y + 2x + 2x^2 =$





$x + 4y + 2x + 2x^2 = 2x^2 + 3x + 4y$

I would also imagine that you could allow substitution. Unfortunately, this does actually give students the wrong idea about a variable being one fixed value. However, I think it might help students to overcome the error of conjoining terms.