UNIVERSITY OF CAPE TOWN

DEPARTMENT OF MATHEMATICS

Certain Relations between Topology and Measure

by

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1969

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Contents

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1. Introduction.

The idea of a weakly outer regular measure arose from the study of two papers :

(i) A note concerning regular measures - R.E.Zink [33] .

(ii) On the sets of regular measures - H.Tsujimoto [30] .

In these papers results are proved using the property that for every measurable set of finite measure there exists an open measurable set, containing it, which is also of finite measure.

Zink uses this condition to extend the outer regularity of μ to the measure ν where

$$\nu(E) = \int_E f \, d\mu ,$$

f being a given non-negative measurable function.

The following problem then presents itself :

Under what conditions, if $E \in \mathbf{S}$, $\mu(E) < \infty$, will there exist an open measurable set U containing E such that $\mu(U) < \infty$.

Although this problem was not solved various other ideas were introduced and results concerning them are proved.

In section 2 the set function $\bar{\mu}$ is defined and shown to be a regular content. The relations between a regular Borel measure μ and the function $\bar{\mu}$ derived from μ are discussed using results from Halmos [7] .

The concept of a topological measure space is introduced in the following section and relations between the various topological measure spaces are studied.

The Micro-Zink and micro-regular properties are formulated in section 4 and are linked with weak outer regularity and the topological measure spaces,

Section 5 concerns the existence of topological measure spaces and in the next section a study is made of the Cartesian products, both finite and infinite, of topological measure spaces.

The last section is a survey of the literature on the relations between measure and topology.

2. Weak Outer Regularity and the Set Function $\bar{\mu}$.

X is a topological space.

We shall denote the class of all compact subsets of X by $\mathbb{C}$, and the class of all those compact subsets of X which are also $\mathcal{G}_\delta$'s by $\mathbb{C}_0$. The σ-ring generated by $\mathbb{C}$ is denoted by $\mathbb{S}$, and the sets of $\mathbb{S}$ are called the Borel sets of X. The σ-ring generated by $\mathbb{C}_0$ is denoted by $\mathbb{S}_0$, and the sets of $\mathbb{S}_0$ are called the Baire sets of X. The class of open Borel sets is denoted by $\mathbb{U}$, and class of open Baire sets by $\mathbb{U}_0$. Throughout this thesis μ will denote a measure on the Borel sets.

<u>2.1 Definition</u>. For any $E \in \mathbb{H}(\mathbb{S})$, we define

$$\bar{\mu}(E) = \inf \{ \mu(G) : G \supset E, G \text{ open Borel} \}.$$

<u>2.2 Proposition</u>. $\bar{\mu}$ is an outer measure on $\mathbb{H}(\mathbb{S})$.

<u>Proof</u>. Obviously $\bar{\mu}$ is non-negative and $\bar{\mu}(\phi) = 0$. Suppose $E, F \in \mathbb{H}(\mathbb{S})$ and $E \subset F$. $\bar{\mu}(E) \leq \mu(U)$ for any open Borel set $U \supset F$, and therefore

$$\bar{\mu}(E) \leq \inf \{ \mu(U) : U \text{ open Borel}, U \supset F \} \leq \bar{\mu}(F).$$

Hence $\bar{\mu}$ is monotone.

Let $\{ E_i \}$ be any sequence of sets in $\mathbb{H}(\mathbb{S})$. If $\bar{\mu}(E_j) = +\infty$, for some j, then $\bar{\mu}$ is clearly countably subadditive. We now suppose that $\bar{\mu}(E_i) < \infty$ $(i = 1, 2, \ldots)$.

Corresponding to each $\varepsilon > 0$ we can find an open Borel set U_i, such that for every i, $E_i \subset U_i$ and $\mu(U_i) \leq \bar{\mu}(E_i) + \frac{\varepsilon}{2^i}$.

Therefore $\bar{\mu}(\bigcup_{i=1}^{\infty} E_i) \leq \mu(\bigcup_{i=1}^{\infty} U_i) \leq \sum_{i=1}^{\infty} \mu(U_i) \leq \sum_{i=1}^{\infty} \bar{\mu}(E_i) + \varepsilon$.

Since this is true for any ε, the countable subadditivity of $\bar{\mu}$ follows. Hence $\bar{\mu}$ is an outer measure on $\mathbf{H}(\mathbf{S})$.

<u>2.3 Remark</u>. It remains an open question as to whether $\bar{\mu}$ is in fact a measure. The difficulty arises from the fact that disjoint Borel sets do not necessarily have disjoint neighbourhoods.

<u>2.4 Definition</u>. A set E in $\mathbf{S}$ is outer regular with respect to the measure μ if

$$\mu(E) = \inf \{ \mu(U) : E \subset U \in \mathbf{U} \} .$$

A set E in $\mathbf{S}$ is inner regular with respect to the measure μ if

$$\mu(E) = \sup \{ \mu(C) : E \supset C \in \mathbf{C} \} .$$

A set E in $\mathbf{S}$ is regular if it is both inner and outer regular. A measure μ is regular if every set E in $\mathbf{S}$ is regular.

<u>2.5 Definition</u>. A set E in $\mathbf{S}$ will be called weakly outer regular with respect to a measure μ, if either $\mu(E) = +\infty$, or $\bar{\mu}(E) < \infty$.

The measure μ is weakly outer regular if every E in $\mathbf{S}$

is weakly outer regular with respect to μ .

<u>2.6 Remark</u>. There do exist measures defined on the Borel sets which are not weakly outer regular. An example is the 1-dimensional Hausdorff measure in $\mathbb{R}^2$ which is finite for any line segment. However, every open set containing a line segment has infinite measure.

<u>2.7 Proposition</u>. Every outer regular measure (and hence every regular measure) is weakly outer regular.

<u>Proof</u>. If $\mu(E) < \infty$ and E is outer regular with respect to μ , then $\bar{\mu}(E) = \mu(E) < \infty$.

<u>2.8 Proposition</u>. Let μ be a weakly outer regular measure. Then, corresponding to each $E \in \mathbb{S}$, there exists an $H \in \mathbb{S} \cap \mathcal{G}_\delta$, such that $E \subset H$ and $\bar{\mu}(E) = \mu(H)$.

<u>Proof</u>. If $\mu(E) = \infty$ then $\mu(G) = \infty$ for each open set $G \supset E$.

Therefore $$\mu(E) = \infty = \mu(G) \geq \bar{\mu}(E) \geq \mu(E) = \infty .$$

Hence $$\bar{\mu}(E) = \mu(G) .$$

If $\mu(E) < \infty$ then by definition $\bar{\mu}(E) < \infty$. Therefore there exists a sequence $\{ G_n \}$ of open sets such that $G_n \supset E$ and

$$\bar{\mu}(E) \leq \mu(G_n) \leq \bar{\mu}(E) + \frac{1}{n} \quad \text{for each } n .$$

Put $H = \bigcap_{n=1}^{\infty} G_n \supset E$.

Thus $H \subset G_n$ and so $\mu(H) \leq \bar{\mu}(E) + \frac{1}{n}$ for each n .

It follows that $\mu(H) \leqslant \bar{\mu}(E)$. Since $\mu(G_n) < \infty$ for each n we also have

$$\mu(H) = \lim_{n \to \infty} \mu(\bigcap_{k=1}^{n} G_k) \geqslant \bar{\mu}(E) .$$

Hence $\mu(H) = \bar{\mu}(E)$.

<u>2.9 Proposition</u>. Let μ be a Borel measure in a locally compact Hausdorff space X . Then the set function $\bar{\mu}$, defined in 2.1 is a regular content.

<u>Proof</u>. $\bar{\mu}$ is certainly defined on $\mathbb{C}$ and clearly $\bar{\mu}(\phi) = 0$, and $\bar{\mu}(C) \geqslant 0$ for any $C \in \mathbb{C}$, since μ is a measure.

We must show that $\bar{\mu}(C) < \infty$ for any $C \in \mathbb{C}$. Being a Borel measure, μ is finite on compact sets. Further, since X is open and $C \subset X$, we know from [7] Theorem 50 D that there exists $C_0 \in \mathbb{C}_0$ and $U_0 \in \mathbb{U}_0$ such that $C \subset U_0 \subset C_0 \subset X$.

Hence $$\bar{\mu}(C) \leqslant \mu(U_0) \leqslant \mu(C_0) < \infty .$$

We have already proved in Proposition 2.2 that $\bar{\mu}$ is monotone and subadditive. So it remains only to show that $\bar{\mu}$ is finitely additive on $\mathbb{C}$. Suppose then that C and D are disjoint compact sets. Since X is locally compact and Hausdorff it follows that there exist two disjoint open sets U and V such that $C \subset U$, $D \subset V$. For any open Borel set W containing $C \cup D$ we therefore have

$$\bar{\mu}(C) + \bar{\mu}(D) \leqslant \mu(W \cap U) + \mu(W \cap V) = \mu(W \cap (U \cup V)) \leqslant \mu(W) .$$

It follows that $\bar{\mu}(C) + \bar{\mu}(D) \leq \bar{\mu}(C \cup D)$. The subadditivity of $\bar{\mu}$ gives the reverse inequality and hence the additivity of $\bar{\mu}$ on $\mathbf{C}$.

The proof that $\bar{\mu}$ is a content is now complete . To show that it is also a regular content we have to show that for each $C \in \mathbb{C}$ we have

$$\bar{\mu}(C) = \inf \{ \bar{\mu}(D) \ : \ C \subset D^0 \subset D \in \mathbb{C} \} .$$

Now from the definition of $\bar{\mu}$ and the finiteness of $\bar{\mu}(C)$, we can find corresponding to each $\varepsilon > 0$ an open Borel set U containing C such that $\mu(U) < \bar{\mu}(C) + \varepsilon$. Using [7] Theorem 50 D again we find $C_0 \in \mathbb{C}_0$ and $U_0 \in \mathbb{U}_0$ such that $C \subset U_0 \subset C_0 \subset U$. Since $C_0^{\,0}$ is the largest open set contained in C_0 we have therefore $C \subset C_0^{\,0} \subset C_0 \in \mathbb{C}$ and $\bar{\mu}(C_0) \leq \mu(U) < \bar{\mu}(C) + \varepsilon$. It follows that

$$\inf \{ \bar{\mu}(D) : C \subset D^0 \subset D \in \mathbb{C} \} < \bar{\mu}(C) + \varepsilon .$$

Since ε is arbitrary , the latter inequality implies that

$$\bar{\mu}(C) = \inf \{ \bar{\mu}(D) : C \subset D^0 \subset D \in \mathbb{C} \} .$$

This completes the proof of Proposition 2.9 .

<u>2.10 Theorem.</u> Suppose μ is a Borel measure on a locally compact , Hausdorff space , and $\bar{\mu}$ is as in definition 2.1 . Then there exists a unique regular Borel measure ν , such that $\nu(C) = \bar{\mu}(C)$ for every $C \in \mathbb{C}$.

<u>Proof.</u> By [7] Theorem 53 E , if $\bar{\mu}^*$ is the outer measure induced by the content $\bar{\mu}$, then $\bar{\mu}^*$ restricted to the Borel

sets is a regular Borel measure . Let $\nu(E) = \bar{\mu}^*(E)$ for every Borel set E . Since $\bar{\mu}$ is a regular content we have , for every $C \in \mathbb{C}$, $\bar{\mu}(C) = \nu(C)$. This follows from [7] Theorem 54 A .

It only remains to show that ν is unique . Suppose that ν_1 , ν_2 are both regular Borel measures such that $\nu_1(C) = \bar{\mu}(C) = \nu_2(C)$ for every C in $\mathbb{C}$. By the inner regularity of ν_1 and ν_2 it follows that , for every $E \in \mathbb{S}$,

$$\begin{aligned} \nu_1(E) &= \sup\{ \nu_1(C) : E \supset C \in \mathbb{C} \} \\ &= \sup\{ \nu_2(C) : E \supset C \in \mathbb{C} \} \\ &= \nu_2(E) . \end{aligned}$$

Hence ν is unique .

<u>2.11 Theorem</u>. Suppose μ is an inner regular Borel measure on a locally compact , Hausdorff space X , and $\bar{\mu}$ is as in definition 2.1 . Then there exists a unique regular Borel measure ν such that $\nu(U) = \mu(U) = \bar{\mu}(U)$ for every U in $\mathbb{U}$.

<u>Proof.</u> Consider the measure ν whose existence is proved in Theorem 2.10 . We need to show that $\nu(U) = \mu(U)$ for every $U \in \mathbb{U}$.

Since μ is inner regular , $\mu(E) = \sup\{ \mu(C) : E \supset C \in \mathbb{C} \}$.

By definition $\mu(U) = \bar{\mu}(U)$

$$= \sup\{ \mu(C) : U \supset C \in \mathbb{C} \}$$

$$\leq \sup\{\bar{\mu}(C) : U \supset C \in \mathbb{C}\}$$
$$\leq \sup\{\nu(C) : U \supset C \in \mathbb{C}\} .$$

Thus $\mu(U) \leq \nu(U)$ since ν is regular .

Since ν inner regular , for each $U \in \mathbb{W}$ and for each $\lambda < \nu(U)$ we can find a $C \in \mathbb{C}$, $C \subset U$ such that $\nu(C) > \lambda$. Hence $\lambda < \bar{\mu}(C) \leq \mu(U)$. Since this is true for each $\lambda < \nu(U)$ we get $\nu(U) \leq \mu(U)$. Therefore $\mu(U) = \nu(U)$ for every U in $\mathbb{W}$.

We must show that ν is unique . Let ν_1 and ν_2 be two regular measures such that $\nu_1(U) = \bar{\mu}(U) = \nu_2(U)$ for every $U \in \mathbb{W}$. Since ν_1 , ν_2 are both outer regular , we get , for every $E \in \mathbb{S}$,

$$\nu_1(E) = \inf\{\nu_1(U) : E \subset U \in \mathbb{W}\}$$
$$= \inf\{\nu_2(U) : E \subset U \in \mathbb{W}\}$$
$$= \nu_2(E) .$$

Hence ν is unique .

2.12 Theorem. Let μ be a regular Borel measure . Then the regular Borel measure ν induced by the regular content $\bar{\mu}$ coincides with μ .

Proof. Since μ is regular it implies that every compact set is outer regular . Hence

$$\mu(C) = \inf\{\mu(U) : C \subset U \in \mathbb{W}\} = \bar{\mu}(C) ,$$

and by Theorem 2.10 $\nu(C) = \mu(C)$ for every C in $\mathbb{C}$. Since ν and μ are both inner regular it follows , as in the proof of Theorem 2.10, that $\nu(E) = \mu(E)$ for every $E \in \mathcal{S}$.

<u>2.13 Proposition.</u> If $\mu_1, \mu_2, \ldots, \mu_n$ are weakly outer regular Borel measures then the measure $\mu = \sum_{i=1}^{n} \mu_i$ is weakly outer regular.

<u>Proof.</u> Consider the case where $\mu = \mu_1 + \mu_2$.
If either $\mu_1(E)$ or $\mu_2(E)$ infinite, the result is true. Hence we may suppose $\mu_1(E)$ and $\mu_2(E)$ both finite.

In this case there exist open Borel sets U, V such that $U \supset E$, $V \supset E$ and $\mu_1(U) < \infty$, $\mu_2(V) < \infty$. Put $W = U \cap V$. Then W is open, $W \supset E$ and

$$\mu(W) = \mu(U \cap V) = \mu_1(U \cap V) + \mu_2(U \cap V)$$
$$\leq \mu_1(U) + \mu_2(V) < \infty .$$

Hence $\mu = \mu_1 + \mu_2$ is weakly outer regular. By mathematical induction we extend the result to any finite sum.

<u>2.14 Remark.</u> From Proposition 2.9 it follows that every compact set in a locally compact Hausdorff space X is weakly outer regular. Since finite sets, and in particular singletons, are compact, the same result holds for them.

<u>2.15 Remark.</u> The class $\mathcal{J}$ of all weakly outer regular sets forms a ring which by remark 2.14 contains all the compact sets.

Proof. Let E , F be sets in $\mathcal{F}$. If either E or F has infinite measure , the result follows .

If both $\mu(E) < \infty$ and $\mu(F) < \infty$ then by the definition of weak outer regularity it follows that $\bar{\mu}(E) < \infty$, $\bar{\mu}(F) < \infty$. By Proposition 2.2 , $\bar{\mu}$ is subadditive and monotone and hence

$$\bar{\mu}(E \cup F) \leq \bar{\mu}(E) + \bar{\mu}(F) < \infty$$

and $\bar{\mu}(E - F) \leq \bar{\mu}(E) < \infty$.

Hence $E \cup F$ and $E - F$ are both weakly outer regular . This implies that $\mathcal{F}$ is a ring .

3. Topological Measure Spaces.

3.1 Definition. A Topological measure space (a TM space) is a measure space $(X, \mathcal{S}, \mu)$, with a topology $\mathfrak{T}$ such that $(X, \mathfrak{T})$ is a topological space.

We denote a TM space by $(X, \mathfrak{T}, \mathcal{S}, \mu)$.

$(X, \mathfrak{T}, \mathcal{S}, \mu)$ is a TM_0 space if every open set is measurable.

$(X, \mathfrak{T}, \mathcal{S}, \mu)$ is a TM_1 space if $\mathfrak{T} \subset \mathcal{S}$ and if corresponding to each x in X there exists an open set V containing x such that $0 < \mu(V) < \infty$.

$(X, \mathfrak{T}, \mathcal{S}, \mu)$ is a TM_2 space if $\mathfrak{T} \subset \mathcal{S}$ and if corresponding to each x in X and each $\varepsilon > 0$ there exists an open set V containing x such that $0 < \mu(V) < \varepsilon$.

3.2 Proposition. Every TM_2 space is a TM_1 space.
Every TM_1 space is a TM_0 space.

3.3 Definition. We define $x \sim y$ to mean either $x = y$ or there exists a measurable open set U such that $x \in U$, $y \in U$ and $\mu(U) = 0$.

3.4 Lemma. $\sim$ is an equivalence relation in X.

Proof. Clearly $\sim$ is reflexive and symmetric.

Suppose $x \sim y$ and $y \sim z$. If either $x = y$ or $y = z$ the result is obvious. If $x \neq y$ and $y \neq z$ then there exist open sets U and V such that $x \in U$, $z \in V$ and $\mu(U) = \mu(V) = 0$. Since $U \cup V$ is an open set of zero

measure containing x and z it follows that $x \sim z$.
Thus $\sim$ is an equivalence relation in X.

<u>3.5 Definition</u>. The equivalence class determined by a point x in X will be denoted by $[x]$.

For any $E \subset X$, we define

$[E] = \{[x] : x \in E\}$, in particular

$[X] = \{[x] : x \in X\}$ = set of all equivalence classes generated in X, by points of X; and

$[\mathfrak{T}] = \{[U] : U \in \mathfrak{T}$ = topology of $X\}$.

<u>3.6 Definition</u>. $\hat{E} = \cup\{[x] : x \in E\}$.

It follows from definitions 3.5 and 3.6 that

(i) $[E] \subset [X]$.

(ii) $E \subset \hat{E} \subset X$.

(iii) $x \sim y \in \hat{E} \Longleftrightarrow x \in \hat{E}$.

<u>3.7 Lemma</u>. If $\{x\} \neq [x]$ then $[x]$ is open.

<u>Proof</u>. Let $\{x\} \neq [x]$. Then there exists an element z in $[x]$ such that $z \neq x$. For any $y \in [x]$ we have $y \sim z \sim x$. Hence there exists an open set U such that $\mu(U) = 0$ and U contains x, y and z.
Hence $y \in U \subset [x]$.

<u>3.8 Lemma</u>. $\hat{E} = E \cup$ open set.

Proof. $$\hat{E} = \bigcup_{x \in E} [x] = \bigcup\{[x] : x \in E \text{ and } [x] = \{x\}\} \cup \bigcup\{[x] : x \in E \text{ and } [x] \neq \{x\}\}.$$

$$= \bigcup\{\{x\} : x \in E\} \cup \bigcup\{[x] : x \in E \text{ and } [x] \neq \{x\}\}.$$

$$= E \cup \text{open set.}$$

3.9 Lemma. If E is open in X then $\hat{E}$ is open in X.

Proof. The proof follows immediately from lemma 3.8 .

3.10 Lemma. $[\mathfrak{T}]$ is a topology in $[X]$.

Proof. (i) We must show that $[\mathfrak{T}]$ is closed under arbitrary unions. We do this by showing

$$\bigcup_{\alpha} [U_{\alpha}] = [\bigcup_{\alpha} U_{\alpha}] \quad \text{for } U_{\alpha} \in \mathfrak{T}.$$

This follows from the following equivalences

$$[x] \in \bigcup_{\alpha} [U_{\alpha}] \Leftrightarrow (\exists\beta)\Big[[x] \in [U_{\beta}]\Big].$$

$$\Leftrightarrow (\exists\beta)(\exists y)[y \in U_{\beta} \text{ and } x \sim y]$$

$$\Leftrightarrow (\exists y)[y \in \bigcup_{\alpha} U_{\alpha} \text{ and } x \sim y]$$

$$\Leftrightarrow [x] \in [\bigcup_{\alpha} U_{\alpha}]$$

(ii) We must still show that $[\mathfrak{T}]$ is closed under finite intersections, and we do this by showing

$\bigcap_{i=1}^{n} [U_i] = [\bigcap_{i=1}^{n} U_i]$ for $U_i \in \mathfrak{T}$. This follows from the following equivalences

$$[x] \in \bigcap_{i=1}^{n} [U_i] \iff (\forall i)\Big[[x] \in [U_i]\Big] .$$

$$\iff (\forall i)(\exists y_i)\Big[y_i \in U_i \text{ and } [x] = [y_i]\Big]$$

$$\iff (\forall i)(\exists y_i)\Big[y_i \in U_i \text{ and } x \in [y_i]\Big]$$

$$\iff (\forall i)\Big[x \in \cup\{[y] : y \in U_i\}\Big]$$

$$\iff (\forall i)\,[x \in \hat{U}_i]$$

$$\iff x \in \bigcap_{i=1}^{n} \hat{U}_i$$

$$\iff [x] \in \Big[\bigcap_{i=1}^{n} \hat{U}_i\Big]$$

Lemma 3.9 implies that $\hat{U}_i \in \mathfrak{T}$, and hence $[\mathfrak{T}]$ is closed under finite intersections.

Hence $[\mathfrak{T}]$ is a topology in $[X]$.

<u>3.11 Remark</u>. The following statements are equivalent.

(i) $x \in [z]$,

(ii) $z \in [x]$,

(iii) $[x] = [z]$.

<u>3.12 Lemma</u>. $[\hat{E}] = [E]$ for every $E \subset X$.

<u>Proof</u>. This follows from the following equivalences.

$$[x] \in [\hat{E}] \iff (\exists y)\left[y \in \hat{E} \text{ and } [x] = [y]\right]$$

$$\iff (\exists y)\left[y \in \hat{E} \text{ and } x \sim y\right]$$

$$\iff (\exists y)(\exists z)\left[z \in E\ ,\ y \in [z]\ ,\ x \sim y\right]$$

$$\iff (\exists z)\left[x \in [z] \text{ and } z \in E\right]$$

$$\iff (\exists z)\left[[x] = [z] \text{ and } z \in E\right]$$

$$\iff [x] \in [E] \ .$$

<u>3.13 Lemma</u>. If $(X, \mathfrak{X}, \mathfrak{S}, \mu)$ is a TM_0 space, then $\hat{F}$ is measurable if F is measurable.

<u>Proof</u>. The proof follows immediately from lemma 3.8 .

<u>3.14 Definition</u>. We define $[\mathfrak{S}] = \left\{[E] \ : \ E \in \mathfrak{S}\right\}$.

<u>3.15 Lemma</u>. If $(X, \mathfrak{X}, \mathfrak{S}, \mu)$ is a TM_0 space, then $[\mathfrak{S}]$ is a σ-ring in $[X]$.

<u>Proof</u>. Consider the sequence $\left\{[E_n]\right\}$ where $E_n \in \mathfrak{S}$ for $n = 1, 2, \ldots$. We have already shown in the proof of lemma 3.10 , that

$$\bigcup_{\alpha} [E_\alpha] = [\bigcup_{\alpha} E_\alpha] \ , \text{ hence}$$

$$\bigcup_{n=1}^{\infty} [E_n] = [\bigcup_{n=1}^{\infty} E_n] \in [\mathfrak{S}] \quad \text{thus}$$

$[\mathfrak{S}]$ is closed under countable unions.

To show that $[\mathbb{S}]$ is closed under differences, we shall show $[E] - [F] = [E - \hat{F}]$. The required result will then follow from lemma 3.13 .

$$
\begin{aligned}
[x] \in [E] - [F] &\Leftrightarrow [x] \in [E] \text{ and } [x] \notin [F] \\
&\Leftrightarrow [x] \in [E] \text{ and } [x] \notin [\hat{F}] \\
&\Leftrightarrow (\exists y)[y \in E \text{ and } x \sim y] \text{ and } \\
&\qquad \sim (\exists z)[z \in \hat{F} \text{ and } x \sim z] \\
&\Leftrightarrow (\exists y)[y \in E \text{ and } x \sim y] \text{ and } \\
&\qquad [x \not\sim \text{ any } z \in \hat{F}] \\
&\Leftrightarrow (\exists y)[y \in E \text{ and } x \sim y \text{ and } y \notin \hat{F}] \\
&\Leftrightarrow (\exists y)[y \in E - \hat{F} \text{ and } x \sim y] \\
&\Leftrightarrow [x] \in [E - \hat{F}] .
\end{aligned}
$$

<u>3.16 Definition</u>. We define $[\mu]$ on $[\mathbb{S}]$ by

$$[\mu]\big([E]\big) = \mu(E) \quad , \text{ where } E \in \mathbb{S} \quad .$$

<u>3.17 Lemma</u>. $[\mu]$ is a measure on $\big([X], [\mathbb{S}]\big)$.

<u>Proof</u>. This follows from the fact that

$$[E] \cap [F] = 0 \Rightarrow E \cap F = 0$$

and that μ is a measure on $(X, \mathbb{S})$.

<u>3.18 Lemma</u>. If $\hat{E} \in \mathbb{S}$ then $\mu(\hat{E}) = \mu(E)$.

<u>Proof</u>. Definition 3.16 and lemma 3.12 gives

$$
\begin{aligned}
\mu(\hat{E}) &= [\mu]\big([\hat{E}]\big) \\
&= [\mu]\big([E]\big) \\
&= \mu(E) .
\end{aligned}
$$

<u>3.19 Definition.</u> $\Big\}\{x\}\Big\{ = x$.

<u>3.20 Lemma.</u> If U is non-empty, open and $\mu(U) = 0$ then $[U]$ has one point, $[\mu]\big([U]\big) = 0$ and $\Big\}[U]\Big\{$ is an isolated point in $[X]$ with respect to the topology $[\mathfrak{T}]$.

<u>Proof.</u> It follows from the definition that $[\mu]\big([U]\big) = 0$.

Since $\mu(U) = 0$ and U is open, all the elements of U are equivalent. Hence $[U]$ has only one point.

By definition, U open implies that $[U]$ open with respect to the topology $[\mathfrak{T}]$. Thus $[U]$ is an open singleton, which implies that $\Big\}[U]\Big\{$ is an isolated point in $[X]$.

<u>3.21 Example.</u> Consider the set $X = [0, 1] \cup \mathbb{N}$. Let $\mathfrak{T}$ be the usual topology of the line, $\mathbb{S}$ the class of Borel sets, and μ the Lebesque measure on $(X, \mathbb{S})$.

In $(X, \mathfrak{T})$, $\{n\}$, for $n > 1$, is a non-empty open set and $\mu(\{n\}) = 0$. In this case,

$$[X] = \Big\{\{x\} : 0 \leqslant x \leqslant 1\Big\} \cup \Big\{\{2, 3, \ldots, n, \ldots\}\Big\} .$$

Let $U = \{2, 3, \ldots, n \ldots\} \subset X$. Then $\mu(U) = 0$ and U is non-empty and open.

Clearly $[U] = \Big\{[2]\Big\}$ has one point.

Also
$$\begin{aligned} [\mu]\big([U]\big) &= [\mu]\big(\{[2]\}\big) \\ &= [\mu]\big([\{2\}]\big) \\ &= \mu(\{2\}) \\ &= 0 . \end{aligned}$$

Finally , $\Big]\,[U]\,\Big[= [2]$ is an isolated point in $[X]$.

In general , let E be any set in $\mathbb{S}$. Then

$[E] = \Big\{\{x\} : x \in E\Big\}$ if $E \subset [0,1]$ and

$[E] = \Big\{\{x\} : x \in E \cap [0,1]\Big\} \cup \Big\{\{2,3,\ldots,n,\ldots\}\Big\}$

if $E \not\subset [0,1]$.

$$[\mu]\Big([E]\Big) = \mu(E) = \mu(E \cap [0,1]) .$$

<u>3.22 Example</u>. Let X be the real line with the discrete topology $\mathfrak{T}$.

$\mathbb{S}$ is the class of Lebesque measurable sets in X , and μ is the Lebesque measure on $\mathbb{S}$.

Clearly all the measurable sets are open . However not all the open sets are measurable , hence $(X, \mathfrak{T}, \mathbb{S}, \mu)$ is not a TM_0 space .

Let x and y be elements in X . Then $\{x , y\}$ is open in X and $\mu\Big(\{x , y\}\Big) = 0$. Hence $x \sim y$. It follows that $[x] = X$ and that $[X] = \{X\}$.

Also $[\mu]\Big([X]\Big) = \mu(X) = \infty$.

<u>3.23 Lemma</u>. If U and V are both non-empty open sets of zero measure , then $[U] = [V] = [U \cup V]$ = a singleton.

<u>Proof</u>. This follows from lemma 3.20 and the fact that $(U \cup V)$ is open and $\mu(U \cup V) = 0$.

<u>3.24 Theorem</u>. Let $(X , \mathfrak{T}, \mathbb{S} , \mu)$ be a TM_0 space . Then

$([X], [\mathfrak{T}], [\mathbb{S}], [\mu])$ is a TM_0 space which has at most one non-empty open set of zero measure . It is the singleton formed by an isolated point in $[X]$.

Proof. Let $[U]$ be a non-empty open set of zero measure . Then $\mu(U) = [\mu]([U]) = 0$. Also U is non-empty and open in X . By lemma 3.20 $[U]$ is a singleton in $[X]$, and $\}[U]\{$ is an isolated point in $[X]$.

If $[V]$ is any other non-empty, open set of zero measure in $[X]$, then

$$\mu(U \cup V) \leqslant \mu(U) + \mu(V) = 0 \text{ , and}$$

therefore $[U] = [V]$.

3.25 Example. Let $X = [0, 1] \cup \{3\} \cup \{3 - \frac{1}{n} : n = 1, 2, \ldots\}$, $\mathfrak{T}$ the inherited topology from the reals, $\mathbb{S}$ the Borel sets in X , and let μ be the Lebesque measure on $(X, \mathbb{S})$.

Clearly $(X, \mathfrak{T}, \mathbb{S}, \mu)$ is a TM_0 space .

$[x] = \{x\}$ for $0 \leqslant x \leqslant 1$.

$[x] = [3]$ for $x = 3 - \frac{1}{n}$, $n = 1, 2, \ldots$.

This follows from the fact that

$$\{3\} \cup \{3 - \frac{1}{n} : n = 1, 2, \ldots\} = X \cap (\tfrac{3}{2}, \tfrac{7}{2}) ,$$

which is open in X and has zero measure.

$[3]$ is an isolated point of $[X]$ since

$$\left[\{3\} \cup \{3 - \frac{1}{n} : n = 1, 2, \ldots\}\right] = \left\{[3]\right\} = \text{open set in } [X].$$

Obviously 3 is not an isolated point of X .
Also $\mathfrak{T}$ is a very special topology, with respect to which X is Hausdorff.

<u>3.26 Example</u>. Let $X = [0, 1] \cup \{\text{rationals in } [2, 3]\}$. Let $\mathfrak{T}$ be the inherited topology from the reals, $\mathbb{S}$ the Borel sets in X , and μ the Lebesque measure on $(X, \mathbb{S})$.

Clearly $(X, \mathfrak{T}, \mathbb{S}, \mu)$ is a TM_0 space.

X has no isolated points, however not all the non-empty open sets have positive measure. This follows from the fact that $\{\text{rationals in } [2, 3]\}$ is a non-empty open set in X of zero measure.

The rationals form a countable set. If we remove the rationals from the space X , in the above example, we are left with a space in which each point has an open set surrounding it, of arbitrary small measure.

This leads us to the following theorem.

<u>3.27 Definition</u>. $(X, \mathfrak{T}, \mathbb{S}, \mu)$ will be called an open-σ-finite space if $X = \bigcup_{n=1}^{\infty} E_n$ where $E_n \in \mathfrak{T}$, $E_n \in \mathbb{S}$ and $\mu(E_n) < \infty$ for each $n = 1, 2, \ldots$.

<u>3.28 Theorem</u>. Let $(X, \mathfrak{T}, \mathbb{S}, \mu)$ be an open-σ-finite TM_0 Hausdorff space. Then there exists a countable set $Y \subset X$, such that, for every x in $X - Y$, and for every $\varepsilon > 0$, there exists an open set V , containing x , such that $\mu(V) < \varepsilon$.

<u>3.29 Lemma</u>. Let $a_1, a_2, \ldots, a_n$ be any finite number of points in X. Then there exist neighbourhoods V_i containing a_i for each $i = 1, 2, \ldots, n$, such that $V_i \cap V_j = \phi$.

<u>Proof of Lemma</u>. Suppose the lemma is true for any set of n points in X.

Consider a set of $n + 1$ points $\{a_1, a_2, \ldots, a_{n+1}\}$ in X. We can by hypothesis choose open sets $U_1, U_2, \ldots, U_n$ such that a_i is contained in U_i and $U_i \cap U_j = \phi$ whenever $1 \leqslant i \neq j \leqslant n$.

We can also, because of the Hausdorff property, choose pairs of open sets $V_1, W_1, V_2, W_2, \ldots, V_n, W_n$ such that $a_i \in V_i$, $a_{n+1} \in W_i$, $V_i \cap W_i = \phi$ for $i = 1, 2, \ldots, n$.

Let $G_i = U_i \cap V_i$ for $i = 1, 2, \ldots, n$, and $G_{n+1} = \bigcap_{i=1}^{n} W_i$.

The sets $G_1, \ldots, G_{n+1}$ are open and a_i is contained in G_i for $i = 1, 2, \ldots, n + 1$.

Moreover, if $1 \leqslant i \neq j \leqslant n$ we have

$$G_i \cap G_j = U_i \cap V_i \cap U_j \cap V_j \subset U_i \cap U_j = \phi ;$$

while if $1 \leqslant i \leqslant n$ we have

$$G_i \cap G_{n+1} = U_i \cap V_i \cap \bigcap_{i=1}^{n} W_i \subset V_i \cap W_i = \phi .$$

Hence the lemma would be true for any set of $n + 1$ points in X. By the Hausdorff property it is true for any set of two points. Hence by induction the lemma is true in general.

Proof of theorem. Case 1 : when $\mu(X) < \infty$.

Let $A_m = \{ y : V \text{ open} , y \in V \Rightarrow \mu(V) > \frac{1}{m} \}$.

Let $a_1 , a_2 , \ldots , a_n$ be n different points of A_m .

Let $V_1 , \ldots , V_n$ be as in lemma 3.29 . Then

$$\begin{aligned} \mu(X) &\geq \mu(V_1 \cup V_2 \cup \ldots \cup V_n) \\ &\geq \sum_{i=1}^{n} \mu(V_i) \\ &> \frac{n}{m} . \end{aligned}$$

Therefore $n < m.\mu(X) < \infty$. Hence A_m is finite for each $m = 1, 2, \ldots$. Let $Y = \bigcup_{m=1}^{\infty} A_m$. Then Y is a countable set .

Suppose x is an element of $X - Y$.

If an $\varepsilon > 0$ were to exist , such that $\mu(V) \geq \varepsilon$ for every open set V containing x , then an integer m could be chosen such that $m > \frac{1}{\varepsilon}$. From this would follow that x is contained in A_m which is false since x is not an element of Y . Hence no such ε can exist . We can therefore conclude that if x is an element of $X - Y$ then , for every $\varepsilon > 0$, there exists an open set V containing x , such that $\mu(V) < \varepsilon$.

Thus case 1 is proved .

Case 2.

Suppose $X = \bigcup_{n=1}^{\infty} X_n$, where $X_n \in \mathfrak{T}$, $\mu(X) < \infty$, for each n.

Consider $(X_n, \mathfrak{T} \cap X_n, \mathbb{S} \cap X_n, \mu)$.

Each X_n, with the inherited topology and measurability is a TM_0 Hausdorff space for which $\mu(X_n) < \infty$. Hence case 1 applies and we know that there exists a countable subset Y_n of X_n such that, if $x \in X_n - Y_n$ then

$(\forall \varepsilon > 0)(\exists V)$ [V open in X_n, $x \in V$, and $\mu(V) < \varepsilon$].

Let $Y = \bigcup_{n=1}^{\infty} Y_n$ = a countable set.

Then $x \in X - Y \Rightarrow x \in X_n$ for some n and $x \notin Y_n$ for any n.

$\Rightarrow x \in X_n - Y_n$ for some n.

$\Rightarrow (\exists n)(\forall \varepsilon > 0)(\exists V)$ [V open in X_n, $x \in V$ and $\mu(V) < \varepsilon$].

Since X_n is open, it follows that openness in X_n implies openness in X. Hence

$x \in X - Y \Rightarrow (\forall \varepsilon > 0)(\exists V)$ [V open in X, $x \in V$, $\mu(V) < \varepsilon$],

and the theorem has been proved.

<u>3.30 Remark.</u> If X is not open-σ-finite, the above proof will fail since we will only have V open in X_n, but not necessarily in X.

<u>3.31 Theorem.</u> Let $(X, \mathfrak{T}, \mathbb{S}, \mu)$ be an open-σ-finite, Hausdorff, TM_0 space such that every non-empty open set has

positive measure. Then there exists a countable subset Y of X such that, for every x in $X - Y$ and every $\varepsilon > 0$ there exists an open set U containing x such that $0 < \mu(U) < \varepsilon$.

Proof. The proof follows directly from Theorem 3.28 .

3.32 Definition. Let $(X, \mathfrak{T}, \mathbb{S}, \mu)$ be a TM_0 space. We denote by X_0 the union of all the open subsets of X of zero measure.

3.33 Lemma. If $x_0 \in X_0$ then $[x_0] = X_0$.

Proof. Suppose that $x_0 \in X_0$. Then we have

$$\begin{aligned} y \in [x_0] &\Leftrightarrow y \sim x_0 \\ &\Leftrightarrow y = x_0 \text{ or } \exists U : U \text{ open}, \\ &\qquad \mu(U) = 0, \ x_0 \in U, \ y \in U. \\ &\Rightarrow y \in X_0 . \end{aligned}$$

Thus $[x_0] \subset X_0$.

Conversely, if $y \in X_0$ then there exists an open set V_0 such that $\mu(V_0) = 0$ and $y \in V_0$. But also there exists an open set U_0 such that $\mu(U_0) = 0$ and $x_0 \in U_0$. Since $U_0 \cup V_0$ is an open set of zero measure which contains both x_0 and y , it follows that $y \sim x_0$.

Thus $X_0 \subset [x_0]$.

3.34 Lemma. If $x \in X - X_0$ then every open set containing x has positive measure.

Proof. If there existed an open set U of zero measure containing x then by definition $U \subset X_0$ and hence $x \in X_0$ which is not so .

3.35 Remarks. The situations illustrated by examples 3.25 and 3.26 are typical of any TM_0 space .

It follows from lemma 3.34 that if $x \in X - X_0$ then $[x] = \{x\}$. For if not and there exists a point y different from x which is equivalent to x , then there exists an open set U of zero measure containing both x and y . Since $U \subset X_0$, it would follow that $x_0 \in X_0$, which is false . Thus we have $X = X_0 \cup (X - X_0)$, where X_0 is either empty or an exact equivalence class $[x_0]$, and the equivalence classes determined by the elements of $X - X_0$ are singletons . Thus either

$$[X] = \Big\{ \{x\} : x \in X \Big\} \quad \text{or}$$

$$[X] = \{X_0\} \cup \Big\{ \{x\} : x \in X - X_0 \Big\} ,$$

where $X_0 = [x_0]$ for some x_0 .

3.36 Remark. Let $(X, \mathfrak{T}, \mathfrak{S}, \mu)$ be a TM_0 space , and $Y \subset X$ where $Y \in \mathfrak{S}$.

Let $\mathfrak{T}_Y = \{ G \cap Y : G \in \mathfrak{T} \} = \mathfrak{T} \cap Y$,

$\mathfrak{S}_Y = \{ E \cap Y : E \in \mathfrak{S} \} = \mathfrak{S} \cap Y$.

Then $(Y, \mathfrak{T}_Y, \mathfrak{S}_Y, \mu)$ is a TM_0 space . Thus a measurable subspace of a TM_0 space is again a TM_0 space .

Moreover , if $(X, \mathfrak{T})$ is Hausdorff then so is $(Y, \mathfrak{T}_Y)$ and if $(X, \mathfrak{T}, \mathfrak{S}, \mu)$ is open-σ-finite then so is $(Y, \mathfrak{T}_Y, \mathfrak{S}_Y, \mu)$.

<u>3.37 Remark</u>. X_0 is open in X and therefore measurable . So is X . Hence $X - X_0$ is a TM_0 space with the inherited structures .

<u>3.38 Definition</u>. A TM_{00} space is a TM_0 space for which the union of all the open sets of zero measure is again a set of zero measure .

<u>3.39 Theorem</u>. Let $(X, \mathfrak{T}, \mathfrak{S}, \mu)$ be an open-σ-finite , Hausdorff TM_{00} space . Then X can be decomposed into three disjoint measurable sets X_0 , X_1 and X_2 such that

(i) X_0 is either empty or an open set of the form $[x_0]$ for a certain $x_0 \in X$, and $\mu(X_0) = 0$,

(ii) X_1 is countable ,

(iii) corresponding to each $x \in X_2$ and each $\varepsilon > 0$ there exists an open set U such that $x \in U$ and $0 < \mu(U) < \varepsilon$.

<u>Proof</u>. We first apply definition 3.32 to construct X_0 . If it is not empty then it has the form $[x_0]$, according to lemma 3.33 . From the definition X_0 is an open subset of X .

It follows next from remark 3.36 that $X - X_0$ is again an open-σ-finite Hausdorff , TM_0 space with respect to the inherited topological and measure structures . Hence we may apply Theorem 3.28 to obtain a countable set $X_1 \subset X - X_0$ such that corresponding to each point x in $(X - X_0) - X_1$ and each $\varepsilon > 0$ a set V containing x can be found which is an open set in the subspace $X - X_0$ and such that $\mu(V) < \varepsilon$. V has the form $U \cap (X - X_0)$, where U is open in X , that is $U \in \mathfrak{T}$. Since $x \in X - X_0$, $x \in U$, we know from Lemma 3.34 that $\mu(U) > 0$.

We have so far assumed only that X is a TM_0 space . If we now bring in the hypothesis that it is in fact a TM_{00} space , we shall know in addition that $\mu(X_0) = 0$. Hence $\mu(V) = \mu(U) - \mu(U \cap X_0) = \mu(U)$. Thus there exists an open set U in the space X such that $x \in U$ and $0 < \mu(U) < \varepsilon$.

<u>3.40 Theorem</u>. Let $(X, \mathfrak{T}, \mathfrak{S}, \mu)$ be an open-σ-finite, Hausdorff, TM_0 space in which every singleton has zero measure. Then the space X can be decomposed into three disjoint measurable sets X_0, X_1, X_2 such that

(i) X_0 is either empty or is an open set of the form $[x_0]$ for a certain x_0 in X ,

(ii) X_1 is countable and has zero measure,

(iii) corresponding to each x in X_2 and each $\varepsilon > 0$ there exists an open set U such that $x \in U$ and $0 < \mu(U) < \varepsilon$.

Proof. We first apply theorem 3.28 to construct a countable subset X_1 of X such that

(i) for every x in $X - X_1$ and every $\varepsilon > 0$ there exists an open set U containing x with $\mu(U) < \varepsilon$,

(ii) for every x in X_1 there exists an $\varepsilon > 0$ such that $\mu(U) > \varepsilon$ for each open set U containing x .

The construction in the proof of theorem 3.28 actually produces an X_1 with these properties, but of course an X_1 with only property (i) could be reduced to one with both properties (i) and (ii) by removal of some of its points if necessary.

Let $$Y = X - X_1 .$$

3.41 Lemma. If U is an open set of zero measure in X then $U \cap X_1 = \phi$.

Proof of lemma. Suppose if possible there were an element x in $U \cap X_1$. Since U is an open set containing x and since $x \in X_1$ it follows from (ii) that $\mu(U) > \varepsilon$ for a certain $\varepsilon > 0$. This would contradict $\mu(U) = 0$. Thus $U \cap X_1 = \phi$.

<u>3.42 Lemma.</u> The following statements are equivalent.

(i) U is an open set in the space X and $\mu(U) = 0$.

(ii) U is an open set in the space Y and $\mu(U) = 0$.

<u>Proof of lemma.</u> If U is an open set in the space X and $\mu(U) = 0$ then by lemma 3.41 we have $U = U \cap Y \in \mathfrak{T}_Y$, so that U is an open set in the space Y.

Conversely, let U be an open set in the space Y which has zero measure. Then there exists an open set V in the space X such that

$$U = V \cap Y = V - (V \cap X_1) .$$

Also $\mu(X_1) = 0$ because X_1 is a countable set.
Hence $\mu(U) = \mu(V) - \mu(V \cap X_1) = \mu(V)$.
Thus V has zero measure.
By lemma 3.41 we know that $V \cap X_1 = \phi$.
Therefore $U = V$ and so is open in the space X.

<u>Proof of theorem.</u> By remark 3.36 $(Y, \mathfrak{T}_Y, \mathfrak{S}_Y, \mu)$ is a TM_0 space.
We now apply definition 3.32 to this space to obtain the set Y_0 which is the union of all the open sets of zero measure in the space Y. By lemma 3.42 $Y_0 = X_0 =$ union of all the open sets of zero measure in the space X. If X_0 is

non-empty then it has the form $[x_0]$ for some x_0 in X, according to lemma 3.33.

Let $X_2 = (X - X_1) - X_0$. Consider any element x in X_2. Since $x \in X - X_1$ we have for every $\varepsilon > 0$ an open set U in the space X such that U contains x and $\mu(U) < \varepsilon$. If $\mu(U)$ were zero then we would have $U \subset X_0$ and so $x \in X_0$ which would contradict $x \in X_2$. Hence we must have $\mu(U) > 0$.

Finally the three sets X_1, $Y - Y_0$, Y_0 are by construction disjoint, and hence $X_0 = Y_0$, X_1, and $X_2 = Y - Y_0$ are also disjoint. This completes the proof of the Theorem.

4. The Micro-Zink Property.

4.1 Definition. A measure μ is micro-regular if, for every sequence $\{E_n\}$ of measurable sets such that $\mu(E_n) \longrightarrow 0$ there exists a sequence $\{U_n\}$ of open measurable sets such that $E_n \subset U_n$ for each n, and $\mu(U_n) \longrightarrow 0$.

4.2 Definition. A measure μ is called micro-Zink if for each contracting sequence $\{E_n\}$ of measurable sets such that $\mu(E_n) \longrightarrow 0$ there exists an open set U such that $\mu(U) < \infty$ and $E_n \subset U$ after some stage.

4.3 Proposition. If μ is a weakly outer regular measure then μ has the micro-Zink property.

Proof. Recall that μ weakly outer regular means that if $\mu(E) < \infty$ then there exists an open set U containing E such that $\mu(U) < \infty$.

Let $\{E_n\}$ be a contracting sequence of sets such that $\mu(E_n) \longrightarrow 0$. Then there certainly exists N such that $\mu(E_N) < \infty$. Accordingly, from the weak outer regularity we know that there exists an open set U such that $U \supset E_N$ and $\mu(U) < \infty$. Thus μ has the micro-Zink property.

4.4 Proposition. Micro-Regularity implies the micro-Zink property.

Proof. We need only observe that since $\mu(U_n) \longrightarrow 0$ it follows that $\mu(U_N) < \infty$ for some N ; U_N will fulfil the rôle of U in definition 4.2 .

<u>4.5 Proposition.</u> Outer regularity implies micro - regularity and the micro - Zink property .

<u>Proof.</u> Obviously outer regularity implies micro - regularity . The rest follows from proposition 4.4 .

<u>4.6 Theorem.</u> Let $(X, \mathfrak{T}, \mathbb{S}, \mu)$ be an open - σ - finite TM_0 space . Then the following statements are equivalent .

(i) μ is micro - Zink .

(ii) μ is weakly outer regular .

<u>Proof.</u> Suppose μ is a micro - Zink measure . Let E be a measurable set and $\mu(E) < \infty$. Since X is open -σ- finite we know that $X = \bigcup_{n=1}^{\infty} X_n$ where X_n is open , $\mu(X_n) < \infty$, $X_n \in \mathbb{S}$ for each n .

Therefore $E = E \cap X = E \cap (\bigcup_{n=1}^{\infty} X_n) = \bigcup_{n=1}^{\infty} (E \cap X_n)$.

Put $E_n = E \cap X_n$ so that $E = \bigcup_{n=1}^{\infty} E_n$.

We now recast $\{E_n\}$ as a disjoint sequence $\{E_n^*\}$ with the same union , as follows :

$$E_1^* = E_1$$

$$E_n^* = E_n - (E_1 \cup E_2 \cup \ldots \cup E_{n-1}) \quad \text{for } n = 2, 3, \ldots .$$

Clearly $E_n^* \subset E_n \subset X_n$, and $E = \bigcup_{n=1}^{\infty} E_n^*$.

Thus $\sum_{n=1}^{\infty} \mu(E_n^*) = \mu(\bigcup_1^{\infty} E_n^*) = \mu(E) < \infty$.

Let $R_n = \bigcup_{i=n+1}^{\infty} E_i^*$.

Then $\{R_n\}$ is a contracting sequence of sets such that $\mu(R_n) = \sum_{i=n+1}^{\infty} \mu(E_i^*) \longrightarrow 0$. Because μ has the micro - Zink property we can therefore find an open set U such that $\mu(U) < \infty$ and $R_n \subset U$ from and after some stage n_0 .

Now $E = \bigcup_{n=1}^{n_0} E_n^* \cup R_{n_0} \subset (\bigcup_{n=1}^{n_0} X_n) \cup U$.

Since the right-hand side is a finite union of open sets, each of finite measure, it is an open set of finite measure containing E .

Hence we have proved that μ is weakly outer regular .

Proposition 4.3 completes the proof of the theorem .

__4.7 Proposition__. Let $(X, \mathfrak{T}, \mathbb{S}, \mu)$ be a σ-finite TM_0 space, and suppose that μ is weakly outer regular . Then $(X, \mathfrak{T}, \mathbb{S}, \mu)$ is open-σ-finite .

__Proof__. We know that $X = \bigcup_{n=1}^{\infty} X_n$, where $X_n \in \mathbb{S}$, $\mu(X_n) < \infty$ for each n . The weak outer regularity of μ gives, for each n, an open set U_n containing X_n such that $\mu(U_n) < \infty$. Clearly $X = \bigcup_{n=1}^{\infty} U_n$ and so X is open-σ-finite.

__4.8 Corollary__. Let $(X, \mathfrak{T}, \mathbb{S}, \mu)$ be a TM_0 open-σ-finite space such that μ is a micro - regular measure . Then μ is weakly outer regular .

<u>Proof</u>. The proof follows directly from Proposition 4.4 and Theorem 4.6 .

<u>4.9 Proposition</u>. Let $(X, \mathfrak{T}, \mathfrak{S}, \mu)$ be an open-σ-finite, TM_0 , Hausdorff space such that μ is a micro - Zink measure. If the non - empty open sets in X have positive measure then $(X, \mathfrak{T}, \mathfrak{S}, \mu)$ is a TM_1 space .

<u>Proof</u>. The Hausdorff property implies that every singleton is measurable and it must have finite measure under the open-σ-finiteness hypothesis . For if $X = \bigcup_{n=1}^{\infty} X_n$, where X_n is open and $\mu(X_n) < \infty$, then $\mu(\{a\}) = \infty$ would imply $\mu(X_n) = \infty$ for some n and so is not possible for any a in X .

By Theorem 4.6 , μ is in fact weakly outer regular and hence there exists an open set U containing a such that $\mu(U) < \infty$.
And $\mu(U) > 0$ since U is non - empty .

<u>4.10 Proposition</u>. Let $(X, \mathfrak{T}, \mathfrak{S}, \mu)$ be a TM_0 space in which singletons are measurable sets of zero measure and non- empty open sets have positive measure . Then micro - regularity of μ implies that $(X, \mathfrak{T}, \mathfrak{S}, \mu)$ is a TM_2 space .

<u>Proof</u>. Let $x \in X$. Since $\mu(\{x\}) = 0$ we may take $E_n = \{x\}$ for each n in Definition 4.1 . Thus we obtain a sequence of open sets $\{U_n\}$ such that $x \in U_n$ for each n ,

and $\mu(U_n) \longrightarrow 0$. Given $\varepsilon > 0$ we can therefore find m such that $\mu(U_m) < \varepsilon$. And $\mu(U_m) > 0$ since U_m contains x and hence is non - empty .

The following theorem, although unconnected with the foregoing propositions, is included here because of its intrinsic interest .

<u>4.11 Theorem</u>. Let $\{\mu_n\}$ be a sequence of micro - regular measures . If $\mu(E) = \sum_{n=1}^{\infty} \mu_n(E)$ for each measurable set and $\mu(X) < \infty$ then the measure μ is micro - regular .

<u>Proof</u>. Let $\{E_m\}$ be a sequence of measurable sets such that $\mu(E_m) \longrightarrow 0$ as $m \longrightarrow \infty$. Therefore $\mu_n(E_m) \longrightarrow 0$ as $m \longrightarrow \infty$ for each n .

Since μ_n is micro - regular, there exists a double sequence $\{U_{n,m}\}$ of open measurable sets such that for each n

(i) $U_{n,m} \supset E_m$ for each m , and

(ii) $\mu_n(U_{n,m}) \longrightarrow 0$ as $m \longrightarrow \infty$.

Let $V_m = U_{1,m} \cap U_{2,m} \cap \dots \cap U_{m,m} \supset E_m$.

If $m > k$ it follows that $\mu_k(V_m) \leq \mu_k(U_{k,m})$.

Thus we have constructed a sequence $\{V_m\}$ of open measurable sets such that $V_m \supset E_m$ for each m, and for each k , $\mu_k(V_m) \longrightarrow 0$ as $m \longrightarrow \infty$.

It remains only to show that $\mu(V_m) \longrightarrow 0$ as $m \longrightarrow \infty$.

Let $\varepsilon > 0$ be given .

Since $\sum_{n=1}^{\infty} \mu_n(X) < \infty$ it follows that there exists N such that $\sum_{N+1}^{\infty} \mu_n(X) < \frac{\varepsilon}{2}$. For each $i = 1, 2, \ldots.$ there exists an m_i such that $\mu_i(V_m) < \frac{\varepsilon}{2N}$ for $m \geq m_i$.

Put $m_o = \max(m_1, m_2, \ldots\ldots, m_N)$.

Then we have

$$\mu(V_m) = \sum_{1}^{N} \mu_n(V_m) + \sum_{N+1}^{\infty} \mu_n(V_m) < \varepsilon$$

provided $m > m_o$.

Thus $\mu(V_m) \longrightarrow 0$, and the proof is complete .

5. Existence of Topological Measure Spaces.

5.1 Theorem. $\mathbb{E}$ a countable class of subsets of X.
$\mathfrak{T}(\mathbb{E})$ the topology generated by $\mathbb{E}$.
$\mathbb{S}(\mathbb{E})$ the σ-algebra generated by $\mathbb{E}$.
Then $\mathfrak{T}(\mathbb{E}) \subset \mathbb{S}(\mathbb{E})$.

Proof. The class $\mathbb{F}$ of all finite intersections of members of $\mathbb{E}$ is contained in $\mathbb{S}(\mathbb{E})$. The class $\mathbb{F}$ is a countable class. Hence any union of members of $\mathbb{F}$ is a countable union of members of $\mathbb{S}(\mathbb{E})$, and so is again a member of $\mathbb{S}(\mathbb{E})$. Therefore
$\mathfrak{T}(\mathbb{E})$ = class of all unions of members of $\mathbb{F} \subset \mathbb{S}(\mathbb{E})$.

5.2 Remark. The result is not necessarily true if $\mathbb{E}$ is not countable.
For let X be any uncountable set and let $\mathbb{E}$ be the class of all singletons in the space X. Then $\mathfrak{T}(\mathbb{E})$ is the discrete topology in X while $\mathbb{S}(\mathbb{E})$ is the class of all countable sets and their complements in X. Hence if E and E' are both uncountable sets then $E \in \mathfrak{T}(\mathbb{E})$ but $E \notin \mathbb{S}(\mathbb{E})$.

5.3 Theorem. Let $\mathbb{E}$ be a countable class of subsets of the space X. Let $\mathfrak{T}$ be the topology and $\mathbb{S}$ be the σ-algebra generated by $\mathbb{E}$ in X. If μ is any measure on $\mathbb{S}$ then $(X, \mathfrak{T}, \mathbb{S}, \mu)$ is a TM_0 space.

Proof. The proof follows at once from Definition 3.1 and Theorem 5.1.

5.4 Theorem. Let $(X, \mathbb{S}, \mu)$ be a totally σ-finite measure space such that $\mu(X) > 0$. Then there exists on X a topology $\mathfrak{T}$ such that $(X, \mathfrak{T}, \mathbb{S}, \mu)$ is an open-σ-finite TM_1 space.

Proof. By hypothesis there exists a sequence $\{E_n\}$ of disjoint sets in $\mathbb{S}$ such that

$$X = \bigcup_1^\infty E_n \quad \text{and} \quad \mu(E_n) < \infty \quad \text{for each} \quad n .$$

By consolidating all those of the E_n which have zero measure with one of the E_n of positive measure we may assume without loss of generality that each set E_n has in fact positive measure.

We define $\mathfrak{T}$ as the topology generated by the countable class $\mathbb{E} = \{E_1, E_2, \ldots\}$ of subsets of X. Since $\mathbb{S} \supset \mathbb{S}(\mathbb{E}) \supset \mathfrak{T}$ by Theorem 5.1, we then have a TM_0 space $(X, \mathfrak{T}, \mathbb{S}, \mu)$. It is in fact a TM_1 space because if $x \in X$ then for some n we have $x \in E_n$ where E_n is open in the topology now in X and $0 < \mu(E_n) < \infty$. The proof is complete.

5.5 Definition. An atom in a measure space $(X, \mathbb{S}, \mu)$ is a measurable set E of positive measure such that for every measurable subset F of E we have either $\mu(F) = 0$ or $\mu(F) = \mu(E)$.

5.6 Definition. A measure space is non-atomic if there are no atoms in it.

5.7 Theorem. If $(X, \mathbb{S}, \mu)$ is a σ-finite non-atomic measure space, if E is a measurable set and if β is a real number such that $0 < \beta < \mu(E)$, then a measurable set F can be found such that $F \subset E$ and $\mu(F) = \beta$.

The proof of this theorem follows from an application of Zorn's lemma. The result is well-known. See Halmos [7] page 174(2) and Menger [21].

5.8 Theorem. Let $(X, \mathbb{S}, \mu)$ be a totally σ-finite non-atomic measure space such that $\mu(X) > 0$. Then corresponding to each positive real number α a sequence $\{E_n\}$ of disjoint measurable sets can be found such that $X = \bigcup_{n=1}^{\infty} E_n$ and $0 < \mu(E_n) < \alpha$ for each $n = 1, 2, \ldots$.

Proof. Suppose firstly that $\mu(X) < \infty$.
If $\mu(X) < \alpha$ we take $E_1 = X$ and let the sequence terminate there. If $\mu(X) \geq \alpha$, let $0 < \beta < \alpha$ and apply Theorem 5.7 to determine a measurable set $E_1 \subset X$ such that $\mu(E_1) = \beta$.
If $\mu(X - E_1) < \alpha$, we put $E_2 = X - E_1$ and terminate the sequence with E_2. Clearly $X = E_1 \cup E_2$, E_1 and E_2 are disjoint and each has positive measure less than α.
If however, $\mu(X - E_1) \geq \alpha$ then apply Theorem 5.7 again to determine a subset E_2 of $X - E_1$ such that $\mu(E_2) = \beta$.

If $\mu(X - E_1 - E_2) < \alpha$ we put $E_3 = X - E_1 - E_2$ and terminate the sequence with E_3. Clearly then $X = E_1 \cup E_2 \cup E_3$, E_1, E_2, E_3 are disjoint and each has positive measure less than α. If $\mu(X - E_1 - E_2) \geqslant \alpha$ we can proceed as before. Either we shall obtain the stated result after constructing in this way a finite number of sets $E_1, E_2, \ldots, E_n$ or we shall obtain an infinite sequence $\{E_n\}$ of disjoint sets such that $X = \bigcup_1^\infty E_n$ and $\mu(E_n) = \beta > 0$ for each n. This would contradict $\mu(X) < \infty$ and so only the first alternative is possible, and the result is proved in this case.

Next suppose that $\mu(X) = \infty$. We express X as $\bigcup_{m=1}^\infty X_m$ where the sets X_m are disjoint and $o < \mu(X_m) < \infty$ for each m. The procedure applied in the previous paragraph to X can be applied to each set X_m to give a finite decomposition $X_m = E_{m,1} \cup E_{m,2} \cup \ldots \cup E_{m,n_m}$ into disjoint measurable sets $E_{m,i}$ such that $o < \mu(E_{m,i}) < \alpha$ for each $i = 1, 2, \ldots, n_m$.

$X = \bigcup_{m=1}^\infty \bigcup_{i=1}^{n_m} E_{m,i}$ is then the desired decomposition of X into disjoint measurable sets each with positive measure less than α.

<u>5.9 Theorem</u>. Let $(X, \mathbb{S}, \mu)$ be a totally σ-finite non-atomic measure space such that $\mu(X) > 0$. Then a topology $\mathfrak{T}$ exists in X such that $(X, \mathfrak{T}, \mathbb{S}, \mu)$ is an open-σ-finite TM_2 space.

Proof. Apply Theorem 5.8 to determine a sequence $\{E_{m,n}\}$ of measurable sets such that for each $m = 1, 2, \ldots$ we have $X = \bigcup_{n=1}^{\infty} E_{m,n}$, $0 < \mu(E_{m,n}) < \frac{1}{m}$ for each $n=1,2,\ldots$ and $E_{m,p} \cap E_{m,q} = \phi$ for $p \neq q$.

Let $\mathfrak{T}$ be the topology in X generated by the countable family of sets

$$\mathbb{E} = \{ E_{m,n} : m = 1,2,\ldots, \; n = 1,2,\ldots \}$$

Since $\mathbb{S} \supset \mathbb{S}(\mathbb{E}) \supset \mathfrak{T}$ by Theorem 5.1, $(X, \mathfrak{T}, \mathbb{S}, \mu)$ is then a TM_0 space.

It is in fact a TM_2 space. For let $\varepsilon > 0$ be given and let $x \in X$. Choose m so large that $\frac{1}{m} < \varepsilon$. Then, since $X = \bigcup_{n=1}^{\infty} E_{m,n}$, we can find n so that $x \in E_{m,n}$. By the construction of $\mathfrak{T}$, $E_{m,n}$ is an open set containing x such that $0 < \mu(E_{m,n}) < \varepsilon$.

The openness of the $E_{m,n}$ and the equation $X = \bigcup_{n=1}^{\infty} E_{m,n}$ also show that $(X, \mathfrak{T}, \mathbb{S}, \mu)$ is open-σ-finite.

5.10 Remark. The topologies constructed in Theorems 5.3, 5.4, 5.9 all have countable bases, hence these topological spaces are in addition separable.

6. The Cartesian Products of Topological Measure Spaces.

Throughout this section we assume that the topological measure spaces are σ - finite . Where a number of topological measure spaces is mentioned in a proposition , X , $\mathfrak{T}$, $\mathbb{S}$, μ will respectively denote the Cartesian products of the spaces, topologies, σ - rings and measures involved .

6.1 Proposition. Let $(X_1, \mathfrak{T}_1, \mathbb{S}_1, \mu_1)$ and $(X_2, \mathfrak{T}_2, \mathbb{S}_2, \mu_2)$ both be TM_0 spaces . Then $(X, \mathfrak{T}, \mathbb{S}, \mu)$ is a topological measure space such that $\mathbb{S}$ contains a base for the product topology $\mathfrak{T}$.

Proof. $\mathbb{S}$ is the σ - ring generated by the measurable rectangles, that is, by sets of the form $A \times B$ where $A \in \mathbb{S}_1$, $B \in \mathbb{S}_2$. Sets of the form $U_1 \times U_2$ where $U_1 \in \mathfrak{T}_1$, $U_2 \in \mathfrak{T}_2$, form a base $\mathbb{B}$ for the product topology $\mathfrak{T}$. Since X_1 , X_2 are both TM_0 spaces $U_1 \in \mathbb{S}_1$ and $U_2 \in \mathbb{S}_2$.
Hence $\mathbb{S} \supset \mathbb{B}$.

6.2 Lemma. If $\mathbb{B}_i$ is a base for the topology $\mathfrak{T}_i$ in the space X_i , for $i = 1,2$, then

$$\mathbb{B}_1 \times \mathbb{B}_2 = \{ U_1 \times U_2 \ : \ U_i \in \mathbb{B}_i \ , \ i = 1,2 \}$$

is a base for the product topology $\mathfrak{T}_1 \times \mathfrak{T}_2$ in $X_1 \times X_2$.
Proof. $W \in \mathfrak{T}_1 \times \mathfrak{T}_2$ means $W = \bigcup_\alpha V_1^\alpha \times V_2^\alpha$ where $V_i^\alpha \in \mathfrak{T}_i$ for each α and for $i = 1,2$.
We also have $V_i^\alpha = \bigcup_{\beta_i} U_i^{\alpha\beta_i}$ where $U_i^{\alpha\beta_i} \in \mathbb{B}_i$ for $i=1,2$ and

for each β_i . So $W = \bigcup_{\alpha} \bigcup_{\beta_1} \bigcup_{\beta_2} U_1^{\alpha\beta_1} \times U_2^{\alpha\beta_2}$ = a union of sets from $\mathbb{B}_1 \times \mathbb{B}_2$.

Hence $\mathbb{B}_1 \times \mathbb{B}_2$ is a base for $\mathfrak{T}_1 \times \mathfrak{T}_2$.

6.3 Proposition. Let $(X_1 , \mathfrak{T}_1 , \mathbb{S}_1 , \mu_1)$ and $(X_2 , \mathfrak{T}_2 , \mathbb{S}_2 , \mu_2)$ both be TM_0 spaces with countable bases . Then $(X , \mathfrak{T} , \mathbb{S} , \mu)$ is also a TM_0 space with a countable base .

Proof. Let $\mathbb{B}_1$ and $\mathbb{B}_2$ be countable bases for $\mathfrak{T}_1$ and $\mathfrak{T}_2$ respectively . Then in the notation of proposition 6.1 and lemma 6.2 we have $\mathbb{S} \supset \mathbf{B} \supset \mathbb{B}_1 \times \mathbb{B}_2$. Since $\mathbb{B}_1 \times \mathbb{B}_2$ is countable and a base for $\mathfrak{T}$, it follows that $\mathfrak{T} \subset \mathbb{S}$.

6.4 Remark. The following example shows that the product of two TM_0 spaces is not necessarily a TM_0 space .

Let X be a set whose cardinal number exceeds that of the continuum . Let $\mathbb{S}$ be the class of all subsets of X .

$D = \{ (x, x) \; ; \; x \in X \}$. Then $D \notin \mathbb{S} \times \mathbb{S}$.

Suppose $D \in \mathbb{S} \times \mathbb{S}$. Then D belongs to the σ- ring generated by the rectangles in $X \times X$. By Theorem 5 D of Halmos [7] there exists a countable class $\mathbb{R}$ of rectangles such that $D \in \mathbb{S}(\mathbb{R})$. Let $\mathbb{E}$ be the countable class of sides of rectangles which belong to $\mathbb{R}$. $\mathbb{S}(\mathbb{E}) \times \mathbb{S}(\mathbb{E})$ is a σ- ring which contains all rectangles in $\mathbb{R}$ since any rectangle in $\mathbb{R}$ is of the form $A \times B$ where both A , B belong to $\mathbb{E}$. Therefore $\mathbb{S}(\mathbb{R}) \subset \mathbb{S}(\mathbb{E}) \times \mathbb{S}(\mathbb{E})$ and so

$D \in \mathbb{S}(\mathbb{E}) \times \mathbb{S}(\mathbb{E})$. Theorem 34 D of Halmos [7] implies that every section of D belongs to $\mathbb{S}(\mathbb{E})$. However the number of sections of D equals the number of elements of X . It follows that the cardinal number of $\mathbb{S}(\mathbb{E})$ also exceeds that of the continuum . This contradicts section 5 (9c) of Halmos [7] .

Let $\mathfrak{T}$ be the discrete topology for X . Then $(X, \mathfrak{T}, \mathbb{S}, \mu)$ is a TM_0 space , However

$D = \bigcup_{x \in X} \{x\} \times \{x\} \in \mathfrak{T} \times \mathfrak{T}$, and so the product space is not a TM_0 space .

<u>6.5 Remark</u>. Let $(X_i, \mathfrak{T}_i, \mathbb{S}_i, \mu_i)$ for $i=1,\ldots,n$, be TM_0 spaces each with a countable base . Then $(X, \mathfrak{T}, \mathbb{S}, \mu)$ is a TM_0 space .

This result can be proved in the same way as lemma 6.2 and proposition 6.3 or we can extend proposition 6.3 to any finite product by mathematical induction .

<u>6.6 Proposition</u>. Let $(X_i, \mathfrak{T}_i, \mathbb{S}_i, \mu_i)$ for $i=1,2,$ be TM_2 spaces each with a countable base .
Then $(X, \mathfrak{T}, \mathbb{S}, \mu)$ is a TM_2 space .

<u>Proof</u>. It follows from proposition 6.3 that $\mathfrak{T} \subset \mathbb{S}$.
Let $(x_1, x_2) \in X_1 \times X_2 = X$.
Let $\varepsilon > 0$ be given . Since each space is TM_2 it implies that there exists $U_1 \in \mathfrak{T}_1$ such that $x_1 \in U_1$ and $0 < \mu_1(U_1) < \sqrt{\varepsilon}$

and there exists $U_2 \in \mathfrak{T}_2$ such that $x_2 \in U_2$ and $o < \mu_2(U_2) < \sqrt{\varepsilon}$. Hence $(x_1, x_2) \in U_1 \times U_2$ where $U_1 \times U_2 \in \mathfrak{T}$ by definition of $\mathfrak{T}$.

Also $\mu(U_1 \times U_2) = \mu_1(U_1).\mu_2(U_2)$ since μ is the unique measure such that $\mu(A \times B) = \mu_1(A).\mu_2(B)$ for $A \in \mathbb{S}_1$, $B \in \mathbb{S}_2$. Thus $o < \mu(U_1 \times U_2) < \varepsilon$.

6.7 Remarks. (i) We do not need the countable base assumption in order to satisfy the second condition for a TM_2 space .

(ii) A similar result can be shown to be true for TM_1 spaces .

6.8 Remark. If $(X_i, \mathfrak{T}_i, \mathbb{S}_i, \mu_i)$ for $i=1,2,\ldots,n$, are TM_2 spaces each with a countable base then $(X, \mathfrak{T}, \mathbb{S}, \mu)$ is a TM_2 space .

This result can be proved as in proposition 6.6 or we can extend the result of proposition 6.6 to a finite product by mathematical induction .

The question now arises as to whether the same results are true for infinite products of topological measure spaces .

6.9 Definitions. Let (X_i, S_i, μ_i) for $i=1,2,\ldots$ be normalized measure spaces .

A rectangle in $\mathop{\mathsf{X}}\limits_{i=1}^{\infty} X_i$ is a set of the form $\mathop{\mathsf{X}}\limits_{i=1}^{\infty} A_i$, where $A_i \subset X_i$ for all i and $A_i = X_i$ for all but a finite number of values of i .

We define a measurable rectangle as a rectangle $\overset{\infty}{\underset{i=1}{\mathbb{X}}} A_i$ for which each A_i is a measurable subset of X_i .

$\mathbb{S} = \overset{\infty}{\underset{i=1}{\mathbb{X}}} \mathbb{S}_i$ is the σ- ring generated by the class of all measurable rectangles .

There exists a unique measure μ on $\mathbb{S}$ such that for every measurable set E of the form $A \times X^{(n)}$ where $X^{(n)} = \overset{\infty}{\underset{n+1}{\mathbb{X}}} X_i$ and $A \in \overset{\infty}{\underset{i=1}{\mathbb{X}}} \mathbb{S}_i$, $\mu(E) = (\mu_1 \times \dots \times \mu_n)(A)$.

If X_i is a class of topological spaces then the class of all sets of the form $\mathbb{X}\{E_i : i \in I\}$, where E_{i_0} is open in X_{i_0} and $E_i = X_i$ for each $i \neq i_0$, is a subbase for the topology of $\overset{\infty}{\underset{i=1}{\mathbb{X}}} X_i$.

<u>6.10 Proposition</u>. Let $(X_i, \mathfrak{T}_i, \mathbb{S}_i, \mu_i)$ for $i=1,2,\ldots$ be normalized TM_0 spaces . Then there exists a base $\mathbb{B}$ for the topology $\mathfrak{T}$, such that $\mathbb{B} \subset \mathbb{S}$.

<u>Proof</u>. Let $\mathbb{B}$ be the base generated by the sub-base whose typical element is $\mathbb{X}\{E_i : i \in I\}$ where E_{i_0} is open in X_{i_0} and $E_i = X_i$ for each $i \neq i_0$. Since each E_i is open and therefore measurable we have $\mathbb{X}\{E_i : i \in I\} \in \mathbb{S}$. Since $\mathbb{S}$ is closed under finite intersections it follows that $\mathbb{B} \subset \mathbb{S}$.

<u>6.11 Proposition</u>. Let $(X_i, \mathfrak{T}_i, \mathbb{S}_i, \mu_i)$ for $i=1,2,\ldots$ be normalized topological measure spaces at least one of which is

a TM_2 space and such that $(X, \mathfrak{T}, \mathfrak{S}, \mu)$ is again a TM_0 space. Then $(X, \mathfrak{T}, \mathfrak{S}, \mu)$ is a TM_2 space.

<u>Proof</u>. Let $\varepsilon > o$ be given.

Let $x \in X$, that is $x = (x_1, \ldots, x_n, \ldots)$ where $x_i \in X_i$ for each $i=1,2,\ldots$.

We may assume without loss of generality that X_1 is a TM_2 space. Hence there exists an open set $U_1 \subset X_1$ such that $x_1 \in U_1$ and $o < \mu_1(U_1) < \varepsilon$. Therefore $U_1 \times X^{(1)}$ is an open set in X containing x and from $\mu(U_1 \times X^{(1)}) = \mu_1(U_1)$ it follows that $o < \mu(U_1 \times X^{(1)}) < \varepsilon$.

<u>6.12 Remark</u>. Let $(X_i, \mathfrak{T}_i, \mathfrak{S}_i, \mu_i)$ for $i=1,2,\ldots$ be normalized topological measure spaces at least one of which is a TM_1 space and such that $(X, \mathfrak{T}, \mathfrak{S}, \mu)$ is again a TM_0 space. Then $(X, \mathfrak{T}, \mathfrak{S}, \mu)$ is a TM_1 space.

The proof is similar to that of proposition 6.11.

Another question now presents itself. What can we say about the product of TM_i and TM_j spaces for $i \neq j$. In general the product of a TM_0 and a TM_1 space will not be a TM_1 space.

Let Y be a TM_1 space. Let X be any space with the indiscrete topology and such that $\mu(X) = \infty$. Then X is a TM_0 space but $X \times Y$ is not a TM_1 space.

<u>6.13 Lemma</u>. Let $(X, \mathfrak{T}, \mathfrak{S}, \mu)$ be an open-σ-finite TM_0 space such that $\mu(X) > 0$. Then $(X, \mathfrak{T}, \mathfrak{S}, \mu)$ is a TM_1 space.

Proof. Since X is open-σ-finite $X = \bigcup_{n=1}^{\infty} U_n$ where U_n is open and $\mu(U_n) < \infty$ for each n.

Let A be the set of positive integers such that $n \in A$ if and only if $\mu(U_n) = 0$. A may be empty. Let $B = \mathbb{N} - A$.

By hypothesis $B \neq \phi$. Let $B = \{b_1, b_2, \ldots\}$.

Let $V_1 = \bigcup_{n \in A} U_n \cup U_{b_1}$

$V_m = U_{b_m}$ for $m > 1$.

Then $X = \bigcup_{m=1}^{\infty} V_m$ where V_m open and $o < \mu(V_m) < \infty$.

Let x be any point in X. Then there exists m such that $x \in V_m$ which is open and such that $o < \mu(V_m) < \infty$.

6.14 Proposition. Let $(X_i, \mathfrak{T}_i, \mathbb{S}_i, \mu_i)$ for $i=1,2$, be open-σ-finite TM_0 spaces each with a countable base and such that $\mu_i(X_i) > 0$.

Then $(X, \mathfrak{T}, \mathbb{S}, \mu)$ is a TM_1 space.

Proof. The proof follows directly from Lemma 6.13 and Remark 6.7.

6.15 Proposition. Let $(X_1, \mathfrak{T}_1, \mathbb{S}_1, \mu_1)$ be a TM_1 space with a countable base. Let $(X_2, \mathfrak{T}_2, \mathbb{S}_2, \mu_2)$ be a TM_2 space with a countable base.

Then $(X, \mathfrak{T}, \mathbb{S}, \mu)$ is a TM_2 space.

Proof. By Proposition 6.3, X has the property that $\mathfrak{T} \subset \mathbb{S}$ and hence X is TM_0.

Let $\varepsilon > o$ be given, and let $(x_1, x_2) \in X_1 \times X_2 = \mathbf{X}$.

Then there exists $U_1 \in \mathfrak{T}_1$ such that $x_1 \in U_1$ and $o < \mu_1(U_1) = c < \infty$, and there exists $U_2 \in \mathfrak{T}_2$ such that $x_2 \in U_2$ and $o < \mu_2(U_2) < \frac{\varepsilon}{c}$.
Therefore $(x_1, x_2) \in U_1 \times U_2$ which is open in X.
Further $o < \mu(U_1 \times U_2) = \mu_1(U_1) \cdot \mu_2(U_2) < \varepsilon$, and so the result follows.

7. A survey of the literature on the relation between measure and topology.

7.1 The existence of an invariant measure on a separable compact group was established by A. Haar in 1933. The separability was required so that the Cantor diagonal process could be used. In 1934 J. von Neumann proved the uniqueness of Haar's measure.

During 1940 André Weil, [31] extended this result to locally compact groups, his existence proof being based on Tychonoff's theorem on the Cartesian products of compact spaces. His proof of the uniqueness was entirely different from that of von Neumann. Weil shows the left-invariant Haar measure to be invariant under the transformation $(x,y) \to (yx,y)$ and $(x,y) \to (y^{-1}x, y)$. Further, if A, B are any measurable subsets of X, of finite measure, $\mu(A \cap xB)$ is a uniformly continuous function of x, belonging to $L^2(X)$, satisfying the formula

$$\int_X \mu(A \cap xB)dx = \mu(A) \cdot \mu(B^{-1}) .$$

From this result it follows that if A is measurable and of positive measure then so is A^{-1}, and further $A^{-1}A$ and AA^{-1} contain an open neighbourhood of the unit element of X. This fact makes it possible to reverse the process and introduce a topology on X, starting from an invariant measure on X . In the first appendix of [31] Weil obtains the following result.

<u>Theorem</u>. Let X be a group without a topology and let μ be a left-invariant measure defined on X such that $f(y^{-1}x)$ is measurable on $X \times X$ whenever $f(x)$ is measurable on X. If there exists, for any measurable subset A of X, of positive, finite measure an $x \in X$ such that $\mu(A \cap xA) < \mu(A)$ then it is possible to introduce a topology into X in such a way that the completion X^* of X, with respect to this topology, is locally compact and that the left-invariant Haar measure on X^* induces the given measure μ on X.

Such a topology is unique and can be obtained by taking the family of all sets of the form $A^{-1}A$ (or AA^{-1}), where A is an arbitrary measurable subset of X of positive measure, as the neighbourhood system of the unit element.

7.2 We will go into more detail using the notation of Halmos [7]. A measurable set E is called outer regular with respect to the measure μ if

$$\mu(E) = \inf\{\mu(U) : E \subset U, \ U \text{ open Borel}\},$$

and inner regular if

$$\mu(E) = \sup\{\mu(C) : E \supset C, \ C \text{ compact}\}.$$

The regularity of all Baire measures is established in [7].

Throughout, the whole space X is assumed to be locally compact and Hausdorff. Starting with a content λ (a non-negative, finite, monotone, finitely additive set function defined on the class of compact sets) a regular Borel measure is induced which, when λ is regular, coincides with it on the class of compact sets. Further, any Baire measure can be uniquely extended to a regular Borel measure. These results lead to the Riesz-Markoff theorem.

<u>Theorem</u>. If Λ is a positive linear functional on $\mathcal{L}$ (the class of real-valued, continuous functions on X, with compact support) then there exists a regular Borel measure μ such that for every $f \in \mathcal{L}$, $\Lambda(f) = \int f \cdot d\mu$.

The representation of a positive linear functional as an integral with respect to a regular Borel measure is unique. A Haar measure is then defined as a Borel measure on a locally compact topological group X, such that $\mu(U) > 0$ for every non-empty open Borel set and $\mu(xE) = \mu(E)$ for any $x \in X$ and E any measurable set. Contents, the theory of regular Borel measures in locally compact Hausdorff spaces and Tychonoff's theorem are used to show the existence of at least one regular Haar measure in every locally compact topological group X.

To prove the uniqueness, Halmos first develops the theory of measurable groups. A measurable group is a σ-finite measure space $(X, \mathbb{S}, \mu)$ such that

(i) μ is not identically zero,

(ii) X is a group,

(iii) $\mathbb{S}$ and μ are invariant under left translation,

(iv) the transformation S of $X \times X$ onto itself, defined by $S(x,y) = (x,xy)$ is measurability preserving.

<u>Theorem</u>. If A is a measurable subset of positive measure, of a measurable group X and $y \in X$, then Ay is a measurable set, of positive measure, and A^{-1} is a measurable set of positive measure. If f is a measurable function, A is a measurable set of positive measure, and for every x in X,

$$g(x) = \frac{f(x^{-1})}{\mu(Ax)}, \text{ then } g \text{ is measurable}$$

provided that $f(x^{-1}) \neq \infty$ and $\mu(Ax) \neq \infty$ for any x.

He then establishes the average theorem.

<u>Theorem</u>. If A and B are measurable sets in a measurable group, $f(x) = \mu(x^{-1} A \cap B)$ then f is a measurable function and $\int f \, d\mu = \mu(A) \cdot \mu(B^{-1})$.

These results, together with certain transformations and

Fubini's theorem, proves the uniqueness, that is, that regular Haar measures differ by a non-zero multiplicative constant.

Halmos carries the study of the relation between measure and topology to greater depth than Weil [31]. If X is a locally compact topological group, μ a regular Haar measure, $\rho(E, F) = \mu(E \Delta F)$, then he proves the following results.

<u>Theorem</u>. (i) If E is a Borel set of finite measure, $f(x) = \rho(xE, E)$ for every $x \in X$, then f is continuous.

(ii) If U is a neighbourhood of e, then there exists a Baire set E, of positive finite measure, and an $\varepsilon > 0$ such that $\{x : \rho(xE, E) < \varepsilon\} \subset U$.

These two results together imply that the class of all sets of the form $\{x : \rho(xE, E) < \varepsilon\}$ is a base at e. Thus it is possible to describe all topological concepts in measure theoretic terms.

We now turn to the development of the Weil topology. The problem is, can a natural topology be introduced in a measurable group such that it becomes a locally compact topological group? To handle this problem one introduces the following classes of measurable sets :

$\mathbb{A} = \{EE^{-1} : E$ a measurable set of positive finite measure$\}$.

$\mathbb{N} =$ class of all sets of the form $\{x : \rho(xE, E) < \varepsilon\}$ for E measurable, of positive finite measure and ε such that $0 < \varepsilon < 2\mu(E)$;

and deduces that every set in $\mathbb{N}$ contains a set of $\mathbb{A}$ and conversely. A measurable group is called separated if, whenever $x \neq e$, there exists a measurable set E of positive finite measure such that $\rho(xE, E) > 0$.

The following very important result introduces the Weil topology.

Theorem. Let X be a separated measurable group. If $\mathbb{N}$ is taken for a base at e then with respect to the induced topology X is a topological group.

The induced topology is called the Weil topology. In order to show that every Haar measure is regular, the theory of quotient groups and invariant σ-rings is required.

Theorem. Let X be a locally compact, σ-compact, topological group. Let μ be a left invariant Baire measure in X, which is not identically zero. If E is any Baire set in X then there exists a compact, invariant Baire subgroup Y of X, such that E is a union of cosets of Y and such that the quotient group $\frac{X}{Y}$ is separable.

From this theorem it follows that every Haar measure in X is completion regular, that is, Haar - Borel and Haar - Baire measures have the same completion. This result is extended to locally compact, but not necessarily σ-compact, topological groups and with the known result that completion regularity implies regularity, the desired result is obtained .

7.3 Let X be a compact Hausdorff space, the closure of whose open sets is open. Further, we assume that the class $\mathcal{B}$ of all open-closed sets, constitute a base for X .
Consider a Jordan measure μ defined on $\mathcal{B}$ such that

(i) $\mu(X) = 1$; $\mu(E) = 0$ if and only if $E = \phi$.

(ii) $\lim_{n\to\infty} \mu(E_n) = \mu((\bigcup_{n=1}^{\infty} E_n)'')$ for any ascending sequence $\{E_n\}$ of sets from $\mathcal{B}$.

Y. Mibu in 1944, [22] , considered the relations between measure and topology in X . He first shows that μ is countably additive on $\mathcal{B}$ and then derives the following results :

(i) G open implies $\mu^*(G) = \mu_*(G)$

(ii) $\mu^*(A) = 0$ is equivalent to A is non-dense, where μ^*, μ_* denote the outer and inner measures respectively.

The main results proved are :-

<u>Theorem</u>. The following conditions are equivalent in X .

(i) A non-dense,

(ii) A is of first category,

(iii) A is of measure zero.

Theorem. The following conditions are equivalent in X .

(i) $\mu^*(A) = \underline{\mu_*(A)}$.

(ii) $(\bar{A})^0 = (A^0)$.

(iii) A is measurable with respect to μ^* .

(iv) There exists an open set G such that $(A \cup G) - (A \cap G)$ is of first category, that is, A has the Baire property.

Theorem. The following are equivalent for a function f defined on X ;

(i) f is a measurable function .

(ii) f is a function having the Baire property.

(iii) There exists a continuous function coinciding with f , except on a set of measure zero.

(iv) The set of points of discontinuity of f is of measure zero.

7.4 Haupt and Pauc, [8] used the following notation :-
μ is a finite valued measure on a Boolean σ-algebra $\mathbb{S}$ of subsets of a topological space X . μ^* is the complete extension of μ defined on $\mathbb{S}^*$.

A measure is called weakly adapted to a topological space if

(i) all the open sets are μ-measurable ,

(ii) the family $\mathcal{G}$ of open sets of X is a family which approximates sets in $\mathbb{S}$ in the sense that to each $M \in \mathbb{S}$ and each $\varepsilon > o$ there corresponds an open set G such that $\mu(G - M) < \varepsilon$.

A set in X is called a null-frontier set, if the measure of its closure equals the measure of its interior .

Proposition. Every finite Borel measure, defined on a topological space, is weakly adapted .

A measure μ is normally adapted to X if

(i) X is compact

(ii) X has a base of open sets whose frontiers are null-sets .

(iii) μ is a regular Borel measure .

Two theorems are stated :

Theorem. If μ is normally adapted to X then every finite covering by open sets has a refinement of disjoint null-frontier sets .

Theorem. If μ_1 and μ_2 are normally adapted to X_1 and X_2 respectively, then the completion of the product measure is normally adapted to $X_1 \times X_2$.

7.5 Various conditions on a Borel measure in a topological space, sufficient to ensure the existence of a decomposition of the space into a set of zero measure and a set of the first category are presented in [19] by Marczewski and Sikorski. They also give a simple proof that the decomposition exists for Hausdorff measures in separable metric spaces.

A cardinal number m is said to have measure zero if every finite measure, defined on the class of all subsets of a set of power m, vanishes identically whenever it vanishes on all finite sets. If a metric space contains a dense subset whose power has measure zero then the decomposition exists for every σ-finite measure that vanishes on all finite sets.

7.6 An attempt at a systematic discussion of the concept of weak convergence of measures is found in Blau [1]. A neighbourhood topology is introduced in the set M of all measures on a given space X, and he discusses the relations between properties of X and the topology of M. His measure is normalized, outer regular and defined on a class of subsets of X, such that all open sets are Caratheodory measurable. The topology in the set M of measures is defined by the subbasic neighbourhoods:

$\{ \varphi : \varphi \in M , \varphi_0(U) < \varphi(U) + a \}$ where $\varphi_0 \in M$, U is open and $a > 0$.

With this topology M is a T_1 space which inherits familar properties when they are ascribed to X. Thus for example M has a countable base when X does. Moreover, if it is assumed that singletons are closed sets, then one can prove :

<u>Theorem</u>. If X is compact, separable and Hausdorff then M is bi-compact .

<u>Theorem</u>. X is compact and metrizable if and only if M has the same properties .

7.7 Katetov [12] obtained results closely related to that of Marczewski and Sikorski . He introduced the concept of reducibility and obtained results applicable also to two-valued measures .

<u>Definition</u>. A closed subset Q of the space X semi-reduces a measure μ if

(i) $\mu(G) > 0$ whenever G is open, $G \in \mathbb{S}$, $G \cap Q \neq \phi$

(ii) $\mu(F) = 0$ whenever F is closed, $F \in \mathbb{S}$, $F \cap Q = \phi$

If in addition $Q \in \mathbb{S}$ and $\mu(X - Q) = 0$ then we say Q reduces μ .

<u>Theorem</u>. Let X be a fully normal space . In order that every (two-valued) Borel or Baire measure in X be semi-reducible, it is necessary and sufficient that every (two-valued) Borel measure in any closed subspace of X , be reducible .

These results of Katetov have been generalized by Ishii [10] to paracompact spaces .

7.8 In [17], Marczewski considers normalized but not necessarily countably additive measures.

A class F, of subsets of X, is called compact if every countable subclass of F that has the finite intersection property, has a non-empty intersection. A class F approximates $\mathbb{S}$ with respect to the measure μ if for every $E \in \mathbb{S}$, $\varepsilon > 0$ there exists $P \in F$ and $D \in \mathbb{S}$ such that $D \subset P \subset E$ and $\mu(E - D) < \varepsilon$. A measure μ is called compact if there exists a compact class F which approximates $\mathbb{S}$. Marczewski develops the basic properties of compact measures and applies them to the theory of measures in product spaces.

<u>Theorem</u>. Every compact measure is countably additive.

From this it follows that every compact measure has a compact, and hence countably additive, extension to the boolean σ-algebra generated by its domain.

<u>Main Theorem</u>. The product of compact measures is compact.

After showing that

<u>Theorem</u>. For a compact, non-atomic, countably additive measure each set of positive measure contains a null set with the power of the continuum,

attention naturally centres on those measures whose null sets are countable. These are called Sierpinski measures.

<u>Theorem</u>. No non-atomic, Sierpinski countably additive measure is compact.

Ryll-Nardzewski continues with these ideas in [26]. A countably additive measure is quasi-compact if to each sequence $\{Q_n\}$ of measurable sets and each $\varepsilon > 0$ there corresponds a measurable set Q_0 such that $\mu(Q_0) > 1 - \varepsilon$ and the sequence $\{Q_0 \cap Q_n\}$ is a compact class. As anticipated by the terminology, each compact, countably additive measure is quasi-compact.

Marczewski and Ryll-Nardzewski together study the compactness of non-direct products of measures in [18']. The direct product of two normalized measures, μ in X, and ν in Y, is λ in $X \times Y$ such that

$$\lambda(A \times B) = \mu(A) \cdot \nu(B) .$$

They weaken this condition and require only

$$\lambda(A \times Y) = \mu(A) \quad \text{and} \quad \lambda(X \times B) = \nu(B) .$$

Such a measure λ is called the product of μ and ν.

<u>Theorem</u>. A product λ of a countably additive measure μ and a compact measure ν is countably additive.

They strengthen the earlier result of Marczewski on the power of null subsets by proving that the conclusion holds even in the absence of countable additivity.
And they finally establish a result complementary to the non-compactness of Sierpinski measures :

<u>Theorem</u>. A purely atomic countably additive measure μ is compact.

7.9 Irregular Borel measures in topological spaces are dealt with in [28] by Swift. If μ fails to be regular in a certain sense at A, we call it irregular in that sense at A.

<u>Theorem</u>. Let X be a topological space and $\{\mu_i\}$ a sequence of Borel measures on X such that $\sum_{i=1}^{\infty} \mu_i(X) < \infty$. Then

(i) the set function μ defined by

$$\mu(A) = \sum_{i=1}^{\infty} \mu_i(A) \quad \text{, for each Borel set } A \text{,}$$

is a Borel measure on X.

(ii) μ will be outer (inner) irregular at a Borel set A if and only if at least one of the μ_i is outer (inner) irregular at A.

In a later paper [29] Swift considers the existence of irregular Borel measures taking at most n values. Such measures on σ-compact Hausdorff spaces have interesting specific properties which enable him to establish the

<u>Theorem</u>. There exists a totally finite, n-valued, irregular Borel measure on a σ-compact Hausdorff

regularity of $\nu = \int f \, d\mu$.

7.11 Let X be a topological space and M the set of all normalized regular outer measures μ satisfying $\mu(U) + \mu(X - U) = 1$ for each open $U \subset X$, and let M be endowed with the weak topology.

Isiwata in [11] considers how compactness of X is carried over into compactness of M.

Theorem. If X is a countably compact T_1 space then M is a compact Hausdorff space.

Mandl in [16] proves two results of a similar kind. If X is a topological space, let P_0 denote the space of Baire probability measures on X and P_1 denote the space of regular probability measures on X, both with the weak topology.

Theorem. (i) If X is countably compact then P_0 is compact.

(ii) If X is countably compact and normal then P_1 is compact.

7.12 In [4], Erohin, generalizes the notion of compactness of measures that was introduced by Marczewski [17]. Let μ be a finitely additive measure defined on an algebra $\mathbb{S}$, of subsets of the space X, and finite on the whole space. $\bar{\mu}$ is an outer Jordan measure induced by μ on X.

Every family, $\mathfrak{T}$, of subsets of X containing ϕ, X and closed under finite intersections and countable unions defines a σ-topology on X. Erohin calls a σ-topology on X regular in the sense of μ if for every A in $\mathbb{S}$

$$\mu(A) = \inf \{ \bar{\mu}(G) : G \in \mathfrak{T}, A \subset G \} .$$

A closed subset Q of X is called almost compact if, for every $\varepsilon > 0$, each countable system of sets from $\mathfrak{T}$ which covers Q, contains a finite subsystem covering $Q - E$, where $\bar{\mu}(E) < \varepsilon$. The measure μ is called almost compact if there exists a σ-topology, regular in the sense of μ, such that X is almost compact. The main result of Erohin is that the measure μ is countably additive if and only if it is almost compact.

7.13 A perfect measure, defined by Sazonov [27], is a measure μ on $\mathbb{S}$ such that given any $\varepsilon > 0$ and any real-valued measurable function f there corresponds to each set $E \subset \mathbb{R}$ for which $f^{-1}(E) \in \mathbb{S}$ an open set $G \supset E$ such that $\mu(f^{-1}(E)) > \mu(f^{-1}(G)) - \varepsilon$.

He first proves that a Borel measure in a topological space X with a countable base is perfect if and only if it is dense, which means that to each $\varepsilon > 0$ corresponds a compact set K in X for which $\mu^*(K) > 1 - \varepsilon$, μ^* being the associated outer measure. He is then able to generalize a theorem due to Gnedenko and Kolmogorov.

<u>Theorem.</u> In order that a measure on a topological space, with a countable base, in which all open sets are measurable be perfect, it is necessary and sufficient that

$$\mu(E) = \sup\{\mu(K) : K \text{ compact}, K \subset E\}.$$

7.14 In [23], Pakshirajan considers a locally compact Hausdorff topological space. He denotes by $\mathfrak{D}$ the class of sets whose intersections with every compact set are Borel sets. $\mathfrak{D}$ is a σ-algebra which cleary contains all open sets in X. Regularity of μ for him means

(i) $\mu(C) < \infty$, for each compact set C, and

(ii) μ is inner regular in the sense of [7].

He proves that to every regular measure μ on $\mathfrak{D}$ there corresponds a unique closed set A_μ, the carrier of μ, with $\mu(X - A_\mu) = 0$ and $\mu(U) > 0$ for every non-null relatively open subset U of A_μ.

7.15 Helmberg establishes the following theorem and corollary in [9].

<u>Theorem.</u> For a Borel measure in a locally compact Hausdorff space which is separable and perfect, there always exist dense σ-bounded open sets of arbitrarily small measure.

<u>Corollary.</u> Under the same assumptions there exists a set of first category whose complement has zero measure.

7.17 Let $(X, \mathbb{S}, \mu)$ be a complete measure space such that $\mu(X) = 1$.

<u>Definition</u>. A sequence $\{K_n\}$ of measurable sets converges to a point x, written $K_n \to x$, if

(i) for each n we have $x \in K_n$ and $\mu(K_n) > 0$;

(ii) $\mu(K_n) \to 0$ as $n \to \infty$.

Let $\mathcal{K}$ be a collection of sequences of measurable sets such that for each x in X there exists at least one sequence $\{K_n\} \in \mathcal{K}$ such that $K_n \to x$.

<u>Definition</u>. The upper outer density of a set E at the point x is

$$D^*(E,x) = \sup\{\limsup_{n \to \infty} \frac{\mu^*(E \cap K_n)}{\mu(K_n)} : \{K_n\} \in \mathcal{K},\ K_n \to x\} .$$

Let $\mathfrak{U}$ be the family of subsets U of X for which $D^*(U', x) = 0$ for each $x \in U$. Martin [20] shows that $\mathfrak{U}$ is a topology on X which he calls the density topology. The density topology and the measure structure on X are related in many satisfying ways, some of which we present below.

<u>Theorem</u>. (i) A set E is of first category if and only if $\mu(E) = 0$;

(ii) Every non-empty open set is of second category ;

(iii) A function is measurable if and only if it is continuous almost everywhere.

<u>Theorem</u>. With respect to the density topology the following statements are equivalent.

(i) Lusin's theorem is valid ;

(ii) μ is a regular Borel measure ;

(iii) Every non-empty open set contains the closure of a non-empty open set ;

(iv) Every null set is contained in open sets of arbitrarily small measure.

7.18 It is well known that every Borel measure in a metric space X is regular provided that X is open-σ-finite (see definition 3.27). Zakon [32] extends this theorem to spaces which have Urysohn's F-property, that is spaces in which every closed set is a G_δ .

<u>Lemma</u>. In every topological measure space $(X, \mathbb{S}, \mu)$, the family R of all regular sets is an algebra. Further, if $\mu(X) < \infty$, then R is a σ-algebra.

<u>Theorem</u>. A Borel measure on an open-σ-finite F-space is regular.

7.19 Rudin [24] showed that if X is a compact Hausdorff space without perfect sets and μ is a regular Borel measure on X such that singletons have zero measure then $\mu(X) = 0$. This result was used by Darst [3] to prove a theorem on the existence of perfect sets of zero measure.

<u>Theorem.</u> Let μ be a non-trivial regular non-atomic Borel measure on a compact Hausdorff space X. Then X contains a non-empty perfect set of zero μ-measure.

7.20 Bledsoe and Morse [2] give two results in topological measure theory that generalize two well known results for metric spaces. Caratheodory has developed a construction that can be applied in metric spaces to produce measures for which the open sets are measurable. The treatment in [2] produces a measure in a regular topological space relative to which the open F_σ sets are measurable. The measure coincides with the Caratheodory measure in the case where the topology is metrizable. Since each open set in a metric space is an F_σ, this provides a generalization of the metric result.

The other result concerns a necessary and sufficient condition for the measurability of open sets. A well-known condition for this in metric spaces is that the measure be additive on any two sets that are a positive distance apart. When the condition is strengthened to require that the additivity hold for two sets whose closures do not intersect, it becomes a necessary and sufficient condition for measurability of open F_σ sets in a normal topological space.

7.21 Let $(X, \mathbb{S}, \mu)$ be a complete, σ-finite measure space with $\mu(X) > 0$. Fillmore [5] records the existence of a topology on X with the property that each measurable real-valued function f on X is equal almost everywhere to a unique continuous real-valued function f^* on X.

N denotes the σ-ideal in $\mathbb{S}$ of sets of measure zero. By a result of Maharam [15] there exists a mapping $\phi : \mathbb{S}/N \to \mathbb{S}$, such that for each p and q in $\mathbb{S}/N$

(i) $\phi(0) = 0$ and $\phi(1) = X$.

(ii) $\phi(p \cap q) = \phi(p) \cap \phi(q)$.

(iii) $\phi(p \cup q) = \phi(p) \cup \phi(q)$.

(iv) $\phi(p) \in p$.

By (i) and (ii) the sets $\phi(p)$ provide a basis for a topology on X and this is the topology in question.

<u>Theorem.</u> Each continuous function on X is measurable.

For each measurable function f on X, there exists a unique continuous function f^* on X, which agrees with f almost everywhere. The mapping $f \to f^*$ preserves algebraic operations where they are defined.

To prove the first part, it is sufficient to show that each open set is measurable. This was done by Maharam in [15]. Those measure spaces for which a mapping ϕ satisfying (i) to (iv) above, exists, have been characterized by Ryan [25].

<u>Definition</u>.. A set is μ-summable if it has finite μ-measure.

<u>Definition</u>. A set is locally μ-null if it meets every μ-summable set in zero measure.

<u>Theorem</u>. A measure space $(X, \mathbf{S}, \mu)$ has the properties (i) to (iv) if and only if $X = \bigcup_{\alpha \in A} Y_\alpha$ where for a certain $\alpha_0 \in A$

(i) Y_{α_0} is locally μ-null and $\mu(Y_{\alpha_0}) = \infty$, or else $Y_{\alpha_0} = \phi$,

(ii) $\{Y_\alpha : \alpha \in A\}$ is a disjoint family of measurable sets, and $0 < \mu(Y_\alpha) < \infty$ for all $\alpha \neq \alpha_0$,

(iii) for each μ-summable set E there exists a countable subset B of A such that

$$E = \bigcup_{\alpha \in B} E \cap Y_\alpha \quad \text{almost everywhere.}$$

7.22 Knowles [13] proves some results concerning the existence of non-atomic measures.

<u>Theorem.</u> If E is an atom with respect to a regular measure ν then there is a point x in E such that $\nu(\{x\}) = \nu(E)$.

<u>Theorem.</u> Suppose X is a compact Hausdorff space. Then

(i) If X is perfect there is a non-atomic Borel measure defined on X ;

(ii) If X contains no perfect subsets there is no atomic Borel measure on X.

<u>Theorem.</u> If X is compact, ν is a non-atomic Borel measure, and E is a Borel set with $\nu(E) > 0$, then

$\nu(E) = \sup\{\nu(P) : P$ is a perfect subset of $E\}$.

Knowles proceeds to generalize the second Theorem. βX denotes the Stone-Čech compactification of X.

<u>Theorem.</u> If X is a G_δ set in βX and contains no isolated point then there is a non-atomic Borel measure on X.

Corollary. If X is a complete, separable metric space with no isolated points then there is a non-atomic Borel measure on X .

7.23 In [14] Knowles studies various types of additivity of Baire and Borel measures in a completely regular Hausdorff space X . He calls X pseudo-compact if every real continuous function on X is bounded . This is equivalent to the property that every non-empty zero set in βX meets X.

Theorem. X is pseudo-compact if and only if every finitely additive measure is σ-additive, that is, there exists no purely finitely additive measure on X .

Knowles calls a measure μ compact-regular if

$$\mu(E) = \sup \{ \mu(C) \ : \ C \text{ compact } \subset E \}$$

for each E of finite measure .

Theorem. A totally finite Borel measure μ is compact-regular if and only if every locally null set is null .

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