UNIVERSITY OF CAPE TOWN

DEPARTMENT OF MATHEMATICS

STABILITY OF BARRELLED TOPOLOGIES

by

M H BURTON

A thesis prepared under the supervision of
Associate Professor J H Webb in fulfilment
of the requirements of the degree of
Master of Science in Mathematics

PREFACE

In the theory of locally convex topological vector spaces, barrelled topologies have been found to be stable under the formation of products, sums and quotients. We shall in this thesis investigate the stability of barrelled topologies with respect to two further mathematical constructions.

Firstly, we examine the situation with regard to the formation of finite-codimensional and countable-codimensional subspaces. (Of course, barrelled topologies are not stable under the formation of arbitrary subspaces.)

Secondly, we present what is known about the stability of barrelled topologies with respect to enlargements of the dual space - a concept which is defined in the sequel. This aspect of the stability question was tackled in a recent paper by Robertson and Yeomans [11] and was pursued in two subsequent papers by Tweddle and Yeomans [16] and by Robertson, Tweddle and Yeomans [12].

In the next two chapters, we turn our attention to quasibarrelled topologies and we pursue a parallel investigation to that of the first two chapters.

Finally we conduct a similar investigation on σ-barrelled and σ-quasibarrelled spaces. The results 5.2, 5.3, 5.4, 5.5, 5.6, 6.2, 6.3, 6.4 and 6.5 concerning these spaces are original.

I would like to thank my supervisor Associate Professor J H Webb for his invaluable help with this thesis and for the many hours that he has spent on my mathematical education. I would also like to thank Professor K O Househam for encouraging me to continue with post graduate work and for making it possible by offering me a teaching post on his staff. My wife, Suzanne has been a constant source of encouragement and has contributed in no small way to the successful completion of this thesis. Wanda Fouquet did the typing while retaining her cheerful personality, a feat worthy of mention.

CONTENTS

0. INTRODUCTION

It will be assumed that the reader is familiar with the basic theory of topological vector spaces and in particular, the theory of locally convex spaces. The reader is referred to [10], [7] and [28] for background reading. The following conventions and notation will be adopted :

countable : refers always to the infinite case.

w.r.t. : "with respect to" , so we write $f_n \to f$ w.r.t. τ .

$[n] = \{1, 2, 3, \ldots, n\}$.

$\mathbb{N} = \{1, 2, 3, \ldots\}$.

$\mathbb{R} = \{\text{real numbers}\}$.

$\mathbb{C} = \{\text{complex numbers}\}$.

$\mathbb{K}$: denotes $\mathbb{R}$ or $\mathbb{C}$.

dim E : the dimension of the vector space E .

codim F : the codimension of the vector subspace F in E .

$E^* = \mathrm{Hom}(E,\mathbb{K}) = \{\text{linear functionals on } E\}$.

$E' = E(\tau)' = \{f \in E^*: f \text{ is } \tau\text{-continuous}\}$.

$f^{\perp}$: for $f \in E^*$ we let $f^{\perp} = \{x \in E: f(x) = 0\} = \overset{-1}{f}(\{0\})$.

$cl_{\tau}(U)$: the τ-closure of U also denoted $\overline{U}$ if there is no risk of ambiguity.

(g_n) : sequences $(g_n : n \in \mathbb{N})$ will be written as (g_n) for short.

(g_{λ}) : nets $(g_{\lambda} : \lambda \in \Lambda)$ with Λ some directed set will be written as (g_{λ}) for short.

$\tau_1 \vee \tau_2$: the sup topology for τ_1 and τ_2 .

$\bigvee_{i\in I}\tau_i$: the sup topology for $\{\tau_i : i \in I\}$.

A^0_F : the polar of A in $F = \{f \in F : |\langle x,f\rangle| \leq 1$ for all $x \in A\}$.

A^0 : we write A^0 for A^0_F if there is no risk of ambiguity.

$\sigma(E,F)$: the weak topology on E induced by F .

$\mu(E,F)$: the Mackey topology on E induced by F .

$\beta(E,F)$: the strong topology on E induced by F .

ω : the space of all sequences in $\mathbb{K}$. So $\omega = \mathbb{K}^{\mathbb{N}}$.

ℓ_∞ : the space of all bounded sequences.

$\phi = \{(x_n) \in \omega : \{n : x_n \neq 0\}$ is finite$\}$.

$\ell_p = \{(x_n) \in \omega : \sum_{n=1}^{\infty} |x_n|^p < \infty\}$.

$L(E,F)$: for locally convex spaces E and F we let

$$L(E,F) = \{f \in \mathrm{Hom}(E,F) : f \text{ is continuous}\} .$$

The following lemma will be used often and is worth proving here.

0.1 Lemma

If E *is a topological vector space with* U *a neighbourhood of* 0 *and* D *a dense subset of* E *, then* $U \subset \overline{U \cap D}$.

Proof:

Let $x \in U$ and let V be a neighbourhood of 0 . Then we can find a neighbourhood of 0 W such that $x + W \subset U$ and $W \subset V$. Since D is dense in E , $(x+W) \cap D \neq \emptyset$. Let $y \in (x+W) \cap D$. Then $y \in (x+V) \cap (U \cap D)$ and hence $(x+V) \cap (U \cap D) \neq \emptyset$.

□

If E is a vector space with two compatible topologies τ_1 and τ_2 then we say that τ_1 *is linked to* τ_2 iff τ_1 has a local base consisting of τ_2-closed sets. For example, $\beta(E,E')$ is linked to $\sigma(E,E')$. This useful concept is discussed in [28] where the following are proved :

If (x_λ) is a τ_1-Cauchy net and $x_\lambda \to 0$ w.r.t. τ_2 , then $x_\lambda \to 0$ w.r.t. τ_1 . Using this result it is easy to show that if $\tau_2 \subset \tau_1$ and τ_1 is linked to τ_2 , then E is τ_1-complete if E is τ_2-complete.

If $E(\tau)$ is a locally convex space and A is an absolutely convex subset of E , then span $A = \bigcup_{k>0} kA$ is a subspace of E often denoted by E_A . This subspace will be constructed fairly often and we note some facts concerning it here. The *gauge of* A , denoted p_A is defined by :

$$p_A(x) = \inf\{k > 0: x \in kA\} \quad \text{for} \quad x \in E_A .$$

The gauge of A is a seminorm on E_A and we denote the corresponding seminorm topology by τ_A . If A is bounded, then $\tau|_{E_A} \subset \tau_A$ and hence τ_A is also Hausdorff. Consequently p_A is a norm on E_A , in other words, $E_A(\tau_A)$ is a normed space. Furthermore :

$$\{x \in E_A: p_A(x) < 1\} \subset A \subset \{x \in E_A: p_A(x) \leqslant 1\}$$

and so τ_A has $\{kA: k > 0\}$ as a local base. We call A *infracomplete* if $E_A(\tau_A)$ is a Banach space.

0.2 Theorem

If $E(\tau)$ *is a locally convex space and* A *is absolutely convex, closed, bounded and sequentially complete, then* A *is infracomplete.*

Proof:

We have $\tau|_{E_A} \subset \tau_A$ and τ_A is linked to $\tau|_{E_A}$ because A is closed (see [28]). Thus if (x_n) is a τ_A-Cauchy sequence in E_A, then (x_n) is $\tau|_{E_A}$-Cauchy and we can assume that each $x_n \in A$ since (x_n) is τ_A-bounded. Hence there is some $x \in A$ such that $x_n \to x$ w.r.t. $\tau|_{E_A}$. Since τ_A is linked to $\tau|_{E_A}$, it follows that $x_n \to x$ w.r.t. τ_A. □

An infracomplete bounded set is strongly bounded and so, in particular, an absolutely convex compact set is strongly bounded.

A *barrel* in a locally convex space $E(\tau)$ is an absolutely convex, absorbent, closed set and the space is called *barrelled* iff every barrel is a neighbourhood of 0. Dually, $E(\tau)$ is barrelled iff every $\sigma(E',E)$-bounded subset is equicontinuous. Since barrels are just basic $\beta(E,E')$-neighbourhoods of 0 and $\tau \subset \mu(E,E') \subset \beta(E,E')$, $E(\tau)$ is barrelled iff $\tau = \mu(E,E') = \beta(E,E')$. If :

$\mathcal{W} = \{A \subset E' : A$ is $\sigma(E',E)$-bounded$\}$.

$\mathcal{C} = \{A \subset E' : A$ is relatively $\sigma(E',E)$-compact$\}$.

$\mathcal{E} = \{A \subset E' : A$ is equicontinous$\}$.

Then $E(\tau)$ is barrelled iff $\mathcal{W} = \mathcal{C} = \mathcal{E}$.

Because of the foregoing observations, we say that $\mu(E,E')$ *is barrelled* instead of saying that E is a locally convex space and $E(\mu(E,E'))$ is barrelled.

A locally convex space $E(\tau)$ is called *quasi-complete* iff every $\sigma(E,E')$-closed, $\sigma(E,E')$-bounded subset of E is $\sigma(E,E')$-complete (or equivalently $\sigma(E,E')$-compact). Thus E' is quasi-complete iff $\mathcal{W} \subset \mathcal{C}$ and it follows that :

0.3 Lemma

$\mu(E,E')$ *is barrelled iff* E' *is quasi-complete.*

We prove the reverse implication :

Let $U = B^0$ be a barrel with B $\sigma(E',E)$-bounded. If D denotes the closed absolutely convex hull of B, then D is also $\sigma(E',E)$-bounded. Thus D is $\sigma(E',E)$-precompact and since E' is quasi-complete, D is $\sigma(E',E)$-complete. It follows that D is absolutely convex and $\sigma(E',E)$-compact and, since $B \subset D$, $D^0 \subset B^0 = U$ and so U is a $\mu(E,E')$-neighbourhood of 0. □

Pointwise-bounded subsets in the dual of a Banach space are norm-bounded. This is the uniform boundedness principle, otherwise known as the Banach-Steinhaus theorem. It really says that a Banach space is barrelled (because the norm topology on the dual space is the strong topology) and a generalisation of this is :

0.4 Theorem

A locally convex space $E(\tau)$ *is barrelled iff for any locally convex space* F, $A \subset L(E,F)$ *is equicontinuous if* A *is pointwise bounded on* E.

The proof is easy and, since we shall obtain a slight improvement on this in our theorem 2.12, we omit it here.

An important property of barrelled spaces that we shall use is the following :

0.5 Theorem

If E *is barrelled then* E' *is* $\sigma(E',E)$*-sequentially complete.*

Proof:

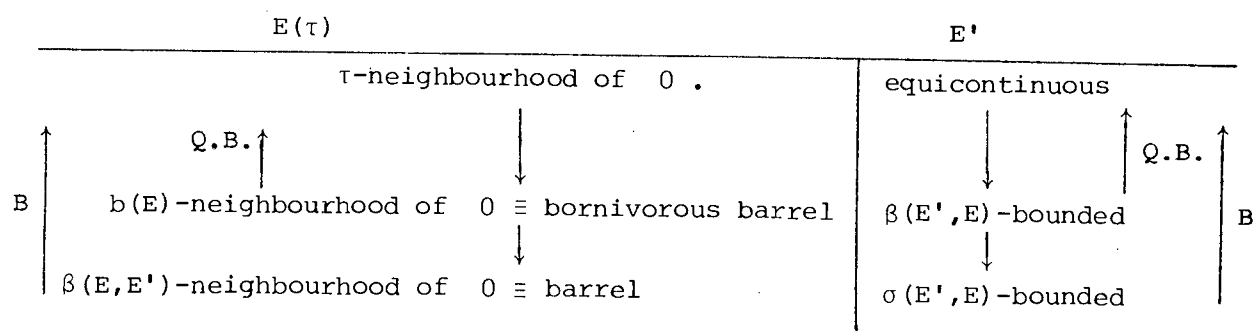
Let (f_n) be a $\sigma(E',E)$-Cauchy sequence in E' and let $F = \{f_n : n \in \mathbb{N}\}$. Then F is $\sigma(E',E)$-bounded and hence F is relatively $\sigma(E',E)$-compact. Let K be $\sigma(E',E)$-compact with $F \subset K$, then F has a cluster point $f \in K$. It follows that $f_n \to f$ w.r.t. $\sigma(E',E)$. □

So if E is a barrelled space with $E' \neq E^*$ then E' is a $\sigma(E^*,E)$-dense, $\sigma(E^*,E)$-sequentially closed subspace of E^*.

A Baire space is barrelled. For if V is a barrel in the Baire space E then $E = \bigcup_{k=1}^{\infty} kV$. Hence some kV has non-empty interior. It is then easy to show that kV, and hence V, contains a neighbourhood of 0. Consequently any Fréchet space or Banach space is barrelled. On the other hand, if ℓ_1 is given the ℓ_1-norm topology then this topology is just $\mu(\ell_1,\ell_\infty)$ because ℓ_1 is then a Banach space and $\ell_1' = \ell_\infty$. Now ϕ is a $\mu(\ell_1,\ell_\infty)$-dense subspace of ℓ_1 but $\phi(\mu(\ell_1,\ell_\infty)|_\phi)$ is not barrelled. To see this consider $A = \{(i\delta_{ij} : j \in \mathbb{N}) : i \in \mathbb{N}\}$ in $\ell_\infty = \phi'$. A is $\sigma(\ell_\infty,\phi)$-bounded but not equicontinuous (because it is not bounded with respect to the ℓ_∞-norm).

We define $E'' = E'(\beta(E',E))'$ and so $\sigma(E',E'') \subset \beta(E',E) \subset \mu(E',E'')$. Consequently the $\sigma(E',E'')$-bounded sets are just the $\beta(E',E)$-bounded sets. Thus the topology $\beta(E'',E')$ has local base $\{B^0 : B \text{ is } \beta(E',E)\text{-bounded}\}$.

We define $b(E) = \beta(E'',E')|_{E'}$ and so $b(E)$ has local base $\{B_E^0 : B \text{ is } \beta(E',E)\text{-bounded}\}$. If τ is the original topology on E then $\tau \subset b(E)$ and we define E to be *quasibarrelled* iff $b(E) \subset \tau$. Dually, E is qusibarrelled iff every $\beta(E',E)$-bounded set is equicontinuous. A subset of $E(\tau)$ is called *bornivorous* if it absorbs τ-bounded sets. Now the bornivorous barrels in E are precisely the basic $b(E)$-neighbourhoods of 0 and so $E(\tau)$ is quasibarrelled iff every bornivorous barrel is a neighbourhood of 0. The following diagram summaries the situation :



Q.B. : $E(\tau)$ is quasibarrelled.

B. : $E(\tau)$ is barrelled.

0.6 Theorem

If $E(\tau)$ *is quasibarrelled then* $\tau = \mu(E,E')$.

Proof:

Let K^0 be a $\mu(E,E')$-neighbourhood of 0 with K absolutely convex, $\sigma(E',E)$-compact. Since K^0 is a barrel we show that K^0 is bornivorous. To this end let B be $\sigma(E,E')$-bounded, then B is $\mu(E,E')$-bounded and hence K^0 absorbs B. □

Because of 0.6 we shall say the $\mu(E,E')$ *is quasibarrelled* to express the fact that E is a locally convex space and $E(\mu(E,E'))$ is quasibarrelled. We have an analogue of 0.5 for quasibarrelled spaces:

0.7 Theorem

If E *is quasibarrelled then* E' *is* $\beta(E',E)$*-sequentially complete.*

Proof:

Let (f_n) be a $\beta(E',E)$-Cauchy sequence, then as in 0.5 we find an $f \in E'$ such that $f_n \to f$ w.r.t. $\sigma(E',E)$. Now $\beta(E',E)$ is linked to $\sigma(E',E)$ and so $f_n \to f$ w.r.t. $\beta(E',E)$. □

The collection of all absolutely convex bornivorous subsets of a locally convex space $E(\tau)$ qualifies as a local base for a locally convex polar topology on E. This topology is called *the associated bornological topology* and is denoted τ^b. It is immediate that $\tau \subset \tau^b \subset \beta(E,E')$ and if $\tau^b = \tau$ we call the space *bornological*. In other words, $E(\tau)$ is bornological iff every absolutely convex bornivorous subset of E is a neighbourhood of 0. We define $E^b = E(\tau^b)'$ and so E^b is a subspace of E^*. We can give an alternative description of E^b :

0.8 Lemma

(1) $\{A \subset E: A$ *is* τ*-bounded*$\} = \{A \subset E: A$ *is* τ^b*-bounded*$\}$.

(2) $\{f \in E^*: f$ *is* τ*-bounded*$\} = \{f \in E^*: f$ *is* τ^b*-bounded*$\}$.

(3) $E^b = \{f \in E^*: f$ *is* τ*-bounded*$\}$.

Proof:

(1) : If A is τ-bounded and U is a τ^b-neighbourhood of 0, then U absorbs A because U is bornivorous and hence A is τ^b-bounded.

(2) : Follows from (1).

(3) : If $f \in E^b$ then f is τ^b-continuous and a foriori, τ^b-bounded. Thus f is τ-bounded.

Conversely if $f \in E^*$ is τ-bounded and U is an absolutely convex neighbourhood of 0 in $\mathbb{K}$, then $\overset{-1}{f}(U)$ is absolutely convex. If B is τ-bounded then $f(B)$ is bounded and hence U absorbs $f(B)$. Thus $\overset{-1}{f}(U)$ absorbs B and this shows that $\overset{-1}{f}(U)$ is bornivorous which means that $\overset{-1}{f}(U)$ is a τ^b-neighbourhood of 0. □

It now follows easily that τ^b is the finest topology on E with the same class of bounded sets as τ and from this it follows that $\tau^b = \mu(E,E^b)$. As a result of this we have :

0.9 Lemma

If $E(\tau)$ *is bornological then* $\tau = \mu(E,E')$.

Again we shall say that $\mu(E,E')$ *is bornological* when we mean that E is a locally convex space and $E(\mu(E,E'))$ is bornological.

If $E(\tau)$ is a locally convex space whose topology is metrizable then $E(\tau)$ is bornological, a fact which is worth proving here :

Let (U_n) be decreasing sequence of neighbourhoods of 0 and let V be absolutely convex and bornivorous. We must show that $U_n \subset V$ for some $n \in \mathbb{N}$. Equivalently we show that $\frac{1}{n} U_n \subset V$ for some $n \in \mathbb{N}$.

Suppose that $\frac{1}{n} U_n \not\subset V$ for any $n \in \mathbb{N}$. Then for each $n \in \mathbb{N}$ can find $x_n \in U_n \setminus nV$. Now $x_n \to 0$ because $U_n \downarrow 0$ and hence $\{x_n : n \in \mathbb{N}\}$ is bounded. Therefore $\{x_n : n \in \mathbb{N}\} \subset mV$ for some $m \in \mathbb{N}$. But this means that $x_m \in mV$ and we have a contradiction. □

From the definitions it is immediate that barrelled spaces and bornological spaces are quasibarrelled. The interesting papers [9] and [14] by Nachbin and Shirota reveal that there is no implication between barrelled and bornological spaces.

1. SUBSPACE TOPOLOGIES OF BARRELLED TOPOLOGIES

We shall first show that subspaces of finite codimension in a barrelled space are barrelled and then show that this result extends to subspaces of countable codimension. In other words we establish that the barrelledness property is stable under the formation of finite and countable codimensional subspaces.

The finite codimensional result was first proved by Dieudonné [2] in 1952, but the proof given here is due to Amemiya [1]. However, the countable codimensional problem resisted solution until 1971, when Saxon and Levin [13] and Valdivia [18] presented their proofs that such subspaces were barrelled. Ten years later Webb [25] presented a simple proof which we give here.

1.1 Lemma

Let $E(\tau)$ *be a barrelled space and* H *a hyperplane in* E . *If* U *is a barrel in* H *then there is a barrel* V *in* E *such that* $V \cap H = U$.

Proof:

Since U is a $\tau|_H$-barrel, U is absolutely convex, absorbent in H and $\tau|_H$-closed. Thus $\overline{U}$ is absolutely convex and τ-closed with $U = \overline{U} \cap H$. There are two possibilities :

(i) If $\overline{U} \subset H$, then $U = \overline{U}$ and so U is τ-closed. Let $E = H \oplus \mathbb{K}x_0$ with $x_0 \in E \setminus H$ and let $B = \{\lambda x_0 : |\lambda| \leqslant 1\}$. Then B is τ-compact and consequently $B + U$ is τ-closed

Furthermore, $B + U$ is absolutely convex and absorbent in E and so we can set $V = B + U$.

(ii) If $\overline{U} \not\subset H$ then there is some $x_0 \in \overline{U} \setminus H$, and hence $E = H \oplus \mathbb{K}x_0$. We show that $\overline{U}$ is absorbent in E . To this end, let $x \in E$ and write $x = h + \lambda x_0$ with $h \in H$ and $\lambda \in \mathbb{K}$. We have U absorbent in H and so $h \in kU \subset k\overline{U}$ for some $k > 0$. Thus $x = h + \lambda x_0 \in k\overline{U} + |\lambda|\overline{U} = (k+|\lambda|)\overline{U}$. Hence $\overline{U}$ is absorbent in E and we can set $V = \overline{U}$. □

1.2 Corollary

A subspace of finite codimension in a barrelled space is barrelled.

Proof:

Let $E(\tau)$ be a barrelled space and let F be a subspace of E . We proceed by induction on the codimension of F .

Firstly, suppose $\operatorname{codim} F = 1$ so that F is a hyperplane in E . If U is a $\tau|_F$-barrel, we can find, by the lemma, a τ-barrel V such that $V \cap F = U$. Since $E(\tau)$ is barrelled, V is a τ-neighbourhood of 0 and thus U is a $\tau|_F$-neighbourhood of 0 .

Now suppose that $\operatorname{codim} F = n$ and that the result holds for all subspaces of codimension less than n . Write $E = F \oplus G \oplus K$ with $\dim G = 1$ and $\dim K = n - 1$. Then $F \oplus G$ is barrelled since it has codimension $n - 1$ and now F is barrelled since it is a hyperplane in $F \oplus G$. □

1.3 Theorem

A subspace of countable codimension in a barrelled space is barrelled.

Proof:

Let $E(\tau)$ be a barrelled space and F a subspace of E of countable codimension. Then there is a subspace G of countable dimension such that $F \oplus G = E$. Let $G = \text{span}\{e_i : i \in \mathbb{N}\}$ and let $F_n = \text{span}\{e_i : i \in [n]\}$. Let $E_1 = F$ and $E_n = F \oplus F_{n-1}$ for $n > 1$. Then for each n, E_n is a hyperplane in E_{n+1}, $E_n \subset E_{n+1}$ and $E = \bigcup_{n=1}^{\infty} E_n$. Now let U be a $\tau|_F$-barrel. We seek a τ-neighbourhood of 0 W such that $U = W \cap F$. By lemma 1.1 we can find, for each $n \in \mathbb{N}$ a barrel U_n in E_n such that :

$$U_1 = U$$

$$U_2 \cap E_1 = U_1$$

and generally $\quad U_n \cap E_{n-1} = U_{n-1} \quad$ for $\quad n > 1$.

Furthermore $\quad U = U_1 \subset U_2 \subset \ldots \subset U_n \subset U_{n+1} \subset \ldots$.

Let $V = \bigcup_{n=1}^{\infty} U_n$ then we have :

V is absolutely convex since if $x,y \in V$ and $|\alpha| + |\beta| \leqslant 1$ then there is some $n \in \mathbb{N}$ such that $x,y \in U_n$. Thus $\alpha x + \beta y \in U_n \subset V$ because U_n is absolutely convex.

V is absorbent in E since if $x \in E$, there is some $n \in \mathbb{N}$ such that $x \in E_n$. Because U_n is a barrel in E_n, U_n absorbs x and so V absorbs x.

Thus $\overline{V}$ is absolutely convex, absorbent and τ-closed, in other words $\overline{V}$ is a τ-barrel and hence a τ-neighbourhood of 0.

We would like to show that $\overline{V} \subset 2V$ since this would imply that $2V$ is a τ-neighbourhood of 0 and hence V would be a τ-neighbourhood of 0 with $V \cap F = U$. To show that $\overline{V} \subset 2V$, we let $z \notin 2V$ and show that $z \notin \overline{V}$. Since $z \in E = \bigcup_{n=1}^{\infty} E_n$, we can find some $m \in \mathbb{N}$ such that $z \in E_m$. Thus $z \in E_n$ and $z \notin 2U_n$ for all $n \geqslant m$. The Hahn-Banach theorem guarantees that for each $n \geqslant m$ there exists $f_n \in E_n'(\tau|_{E_n})'$ with the property that $f_n(z) = 2$ and $|f_n(x)| \leqslant 1$ for all $x \in U_n$. Let g_n be the extension of f_n to E (again guaranteed by the Hahn-Banach theorem), then we have $g_n(z) = 2$ and $|g_n(x)| \leqslant 1$ for all $x \in U_n$ and $n \geqslant m$.

If $A = \{g_n : n \geqslant m\}$ then A is $\sigma(E',E)$-bounded. To see this, let $x \in E$. Then $x \in E_k$ for some $k \in \mathbb{N}$ and so if we define $\ell = \max\{k,m\}$, we have $x \in E_\ell$ with $\ell \geqslant m$. Consequently $|g_n(x)| \leqslant 1$ for all $n \geqslant \ell$. This means that the countable subset $\{g_n : n \geqslant \ell\}$ is $\sigma(E',E)$-bounded and hence A is $\sigma(E',E)$-bounded. Because E is barrelled, $\overline{A}$ is $\sigma(E',E)$-compact and therefore A has a $\sigma(E',E)$-cluster point $g_0 \in E'$.

Let $(g_{n_k} : k \in D)$ be a subnet of the sequence (g_n), with D some directed set, and with $g_{n_k} \to g_0$ w.r.t. $\sigma(E',E)$.
Then $g_{n_k}(x) \to g_0(x)$ for each $x \in E$. In particular, $g_{n_k}(z) \to g_0(z)$.
Now since each $g_{n_k}(z) = 2$, we have $g_0(z) = 2$. Also if $x \in V$, then $x \in U_n$ for some n and we can assume $n \geqslant m$. Thus $g_0(x) = \lim_k g_{n_k}(x)$ and hence $|g_0(x)| \leqslant 1$. So we have $g_0 \in E'$ with $g_0(z) = 2$ and $|g_0(x)| \leqslant 1$ for all $x \in V$.
Let $B = \{\lambda \in \mathbb{K} : |\lambda| < \frac{1}{2}\}$ and $W = g_0^{-1}(B)$.
Then $z + W$ is a τ-neighbourhood of z by the continuity of g_0, $(z + W) \cap V = \emptyset$, and therefore $z \notin \overline{V}$. $\square$

2. ENLARGEMENTS OF BARRELLED TOPOLOGIES

Recall that a barrelled space E carries the Mackey topology $\mu(E,E')$. If M is a subspace of E^* with $M \cap E' = \{0\}$, we shall call $E' + M$ an *enlargement of* E' and we shall call $\mu(E,E'+M)$ an *enlargement of* $\mu(E,E')$. We shall call the dimension of M, the dimension of the enlargement, thus in particular we shall speak of a finite-dimensional enlargement $E' + M$ if M is finite-dimensional, and a countable-dimensional enlargement $E' + M$ if M is countable-dimensional.

We shall in this chapter investigate the stability of the barrelledness property under enlargements of the dual. More precisely, if $\mu(E,E')$ is barrelled, we investigate the barrelledness of $\mu(E,E'+M)$. To anticipate, we shall find that barrelledness is stable under finite-dimensional enlargements but not always under countable-dimensional enlargements.

We first deal with finite-dimensional enlargements and our lemma 2.1 and corollary 2.2 concerning this are from [11] (Theorem 1).

2.1 Lemma

Let E *be a barrelled space and* $E' + M$ *a finite-dimensional enlargement of* E'. *If* B *is* $\sigma(E'+M,E)$*-bounded, then* $B \subset B_1 + B_2$ *where* B_1 *is a* $\sigma(E',E)$*-compact subset of* E' *and* B_2 *is a* $\sigma(M,E)$*-compact subset of* M.

Proof:

The proof is by induction on $\dim M$.

(a) If $\dim M = 1$ let $p: E' + M \to E'$ and $q: E' + M \to M$ be the natural projections.

We first show that $q(B)$ is $\sigma(M,E)$-bounded. To this end let $M = \mathbb{K}f$ where $f \in E^* \setminus E'$ and suppose that $k = \sup \{|\lambda| : \lambda f \in q(B)\}$. If $k = \infty$ then there is a sequence (λ_n) with such that $|\lambda_n| \to \infty$ and $\lambda_n f \in q(B)$. Let $\lambda_n f = q(g_n)$, then (g_n) is a sequence in B . Let $h_n = p(g_n)$ so that $g_n = h_n + \lambda_n f$. Since $|\lambda_n| \to \infty$ we can assume that $\lambda_n \neq 0$ for all n and so :

$$\frac{1}{\lambda_n} g_n = \frac{1}{\lambda_n} h_n + f .$$

We have a sequence (g_n) in a $\sigma(E'+M,E)$-bounded set B and a sequence of scalars $(\frac{1}{\lambda_n})$ with $\frac{1}{\lambda_n} \to 0$. We conclude that

$$\frac{1}{\lambda_n} g_n \to 0 \quad \text{w.r.t.} \quad \sigma(E'+M,E) . \qquad \text{(i)}$$

Now $(-\frac{1}{\lambda_n} h_n)$ is a sequence in E' and from (i) we conclude that

$$-\frac{1}{\lambda_n} h_n \to f \quad \text{w.r.t.} \quad \sigma(E'+M,E)\big|_{E'} = \sigma(E',E) \qquad \text{(ii)}$$

But E is barrelled and so (ii) implies that $f \in E'$ which is a contradiction. Hence $k < \infty$, and this means that $q(B)$ is $\sigma(M,E)$-bounded.

We can now show that $p(B)$ is $\sigma(E',E)$-bounded. For if $(p(g_n))$ is a sequence in $p(B)$, we have

$$g_n = p(g_n) + q(g_n) .$$

Thus $\frac{1}{n} g_n = \frac{1}{n} p(g_n) + \frac{1}{n} q(g_n)$

and so $\frac{1}{n} p(g_n) = \frac{1}{n} g_n - \frac{1}{n} q(g_n)$. (iii)

Now (g_n) is a sequence in the $\sigma(E'+M,E)$-bounded set B and $q(g_n)$ is a sequence in the $\sigma(M,E)$-bounded set $q(B)$. Hence the right-hand side of (iii) converges to 0 (w.r.t. $\sigma(E'+M,E)$) and so then does $\frac{1}{n} p(g_n)$. This shows that $p(B)$ is $\sigma(E',E)$-bounded.

We can now let B_1 be the $\sigma(E',E)$-closure of $p(B)$ and B_2 the $\sigma(M,E)$-closure of $q(B)$.

We note at this stage that $\mu(E,E'+M)$ is barrelled because any $\sigma(E'+M,E)$-bounded set is relatively $\sigma(E'+M,E)$-compact.

(b) If $\dim M = n$ write $M = N \oplus L$ with $\dim N = n - 1$ and $\dim L = 1$ and assume that the lemma holds for all subspaces of dimension less than n . Let $p: E' + M \to E' + N$ and $q: E' + M \to L$ be natural projections. By the induction hypothesis $\mu(E,E'+N)$ is barrelled and so, by the first part, $p(B)$ is $\sigma(E'+N.E)$-bounded and $q(B)$ is $\sigma(L,E)$-bounded. Again by the induction hypothesis $p(B) \subset B_1 + K$ and $q(B) \subset T$ where B_1 is $\sigma(E',E)$-compact, K is $\sigma(N,E)$-compact and T is $\sigma(L,E)$-compact. Thus :

$$B \subset p(B) + q(B) \subset B_1 + K + T .$$

We can now set $B_2 = K + T$ since $K + T$ is $\sigma(M,E)$-compact. □

As an immediate consequence of this lemma we glean :

Barrelledness is stable under finite-dimensional enlargements of the dual space.

More precisely :

2.2 Corollary

If E *is a barrelled space and* E' + M *is a finite-dimensional enlargement of* E' *, then* $\mu(E,E'+M)$ *is barrelled.*

Proof:

Let B be $\sigma(E'+M,E)$-bounded, then from 2.1, B is relatively $\sigma(E'+M,E)$-compact. □

If E' + M is a countable-dimensional enlargement of the dual E' of a barrelled space $E(\tau)$, let $M = \text{span}\{f_i : i \in \mathbb{N}\}$ with $\{f_i : i \in \mathbb{N}\}$ independent and let $M_n = \text{span}\{f_i : i \in [n]\}$. Then $M = \bigcup_{n=1}^{\infty} M_n$, $E' \subset E' + M_1 \subset E' + M_2 \subset \ldots \subset E' + M$ and $E' + M = \bigcup_{n=1}^{\infty} E' + M_n$. Since $E(\tau)$ is barrelled, $\tau = \mu(E,E')$ and from 2.2 we deduce that $\mu(E,E'+M_n)$ is barrelled for each n . What then of the barrelledness of $\mu(E,E'+M)$? We shall first obtain a necessary and sufficient condition for $\mu(E,E'+M)$ to be barrelled. The condition, as might be expected, is one on the $\sigma(E'+M,E)$-bounded sets. We shall then show that $\mu(E,E'+M)$ is not always barrelled; in fact if E has a countable-dimensional bounded set, then we can construct a countable-dimensional enlargement E' + M such that $\mu(E,E'+M)$ is not barrelled. After this we begin the search for countable-dimensional barrelled enlargements and, strangely enough, we shall find that the existence of an uncountable-dimensional bounded set in E allows the construction of a countable-dimensional enlargement E' + M such that $\mu(E,E'+M)$ is barrelled. This means that a barrelled space E has the intriguing property that if it contains a bounded set which is large enough, then countable-dimensional enlargements $E' + M_1$, and $E' + M_2$ can be constructed such that $\mu(E,E'+M_1)$ is barrelled, while $\mu(E,E'+M_2)$ is not!

In order to proceed, we first need a result by De Wilde and Houet [3] on absorbent sequences. If E is a locally convex space, a sequence (X_n) of subsets of E is called an *absorbent sequence* if and only if (i) X_n is absolutely convex for each n.

(ii) $X_n \subset X_{n+1}$ for each n.

(iii) $\bigcup_{n=1}^{\infty} X_n$ is absorbent in E.

2.3 Theorem

If E *is a locally convex space and* (X_n) *is an absorbent sequence of closed sets, then every* $\beta(E,E')$*-bounded subset is absorbed by some* X_n.

Proof:

Suppose that the $\beta(E,E')$-bounded subset A is absorbed by no X_n. Then for each $n \in \mathbb{N}$ we have $A \not\subset nX_n$ and so we can find $x_n \in A \setminus nX_n$. By the Hahn-Banach theorem we can find, for each $n \in \mathbb{N}$, $f_n \in E'$ with the property that $|f_n(x_n)| > n$ and $f_n \in X_n^0$.

Let $F = \{f_n : n \in \mathbb{N}\}$, we claim that F is $\sigma(E',E)$-bounded. To see this, let $x \in E$. We show that $\{|f_n(x)| : n \in J\}$ is bounded for countable $J \subset \mathbb{N}$. Since $\bigcup_{n=1}^{\infty} X_n$ is absorbent, there are positive integers $k(x)$ and $n(x)$ such that $x \in k(x)X_{n(x)}$. Thus if $n \geq n(x)$ we have $X_{n(x)} \subset X_n$ and therefore $x \in k(x)X_n$ for all $n \geq n(x)$. Thus $|f_n(x)| \leq k(x)$ for all $n \geq n(x)$ and so we let $J = \{n \in \mathbb{N} : n \geq n(x)\}$.

It now follows that F^0 is a $\beta(E,E')$-neighbourhood of 0 and consequently F^0 absorbs A. This means that we can find a positive integer k such that $A \subset kF^0$ and hence $|f_n(x)| \leq k$ for all $n \in \mathbb{N}$. In particular, for $n > k$, we have the contradiction :

$$|f_n(x_n)| > n > k . \qquad \square$$

2.4 Corollary

If E *is a barrelled space and* (X_n) *is an absorbent sequence of closed sets, then every* $\sigma(E,E')$*-bounded subset is absorbed by some* X_n.

Proof:

If E is barrelled, we have $\mu(E,E') = \beta(E,E')$ and hence if a subset is $\sigma(E,E')$-bounded, it is $\mu(E,E')$-bounded which in this case means $\beta(E,E')$-bounded. $\square$

Recall that for a locally convex space E, the topology $\mu(E,E')$ has as a local base: $\{K^0 : K$ is absolutely convex, $\sigma(E',E)$-compact$\}$. Thus the search for a condition for $\mu(E,E'+M)$ to be barrelled leads naturally to an investigation of the absolutely convex, $\sigma(E'+M,E)$-compact subsets of $E' + M$. The following lemma and theorem are from [16] (section 3).

2.5 Lemma

Let E *be a barrelled space and* $E' + M$ *a countable-dimensional enlargement of* E'. *Then every absolutely convex,* $\sigma(E'+M,E)$*-compact subset of* $E' + M$ *is contained in a finite dimensional enlargement of* E'.

Proof:

Let $M = \text{span}\{f_i : i \in \mathbb{N}\}$ with $\{f_i : i \in \mathbb{N}\}$ independent, $M_n = \text{span}\{f_i : i \in [n]\}$ and let K be absolutely convex, $\sigma(E'+M,E)$-compact. We claim that there is some $n \in \mathbb{N}$ such that $K \subset E' + M_n$.

Let $H = \text{span}\, K = \bigcup_{n=1}^{\infty} nK$. Then K is absorbent in H. If p is the gauge of K, then p is a norm on H which generates a topology τ on H. The norm topology τ has unit ball K and $H(\tau)$ is a Banach space. Thus $\tau = \beta(H,H')$ and so K is $\beta(H,H')$-bounded. Let $X_n = K \cap (E'+M_n)$. Then $X_n \subset X_{n+1}$ and X_n is absolutely convex for each $n \in \mathbb{N}$. Furthermore, $\bigcup_{n=1}^{\infty} X_n = K \cap \left(\bigcup_{n=1}^{\infty} E' + M_n\right) = K \cap (E'+M) = K$ which is absorbent in H and so (X_n) is an absorbent sequence in H.

We show that each X_n is τ-closed :

Each $\mu(E,E'+M_n)$ is barrelled which implies that each $E' + M_n$ is $\sigma(E'+M_n,E)$-quasi-complete. Since K is $\sigma(E'+M,E)$-closed, X_n is $\sigma(E'+M_n,E)$-closed in $E' + M_n$. Since K is $\sigma(E'+M.E)$-bounded, each X_n is $\sigma(E'+M_n,E)$-bounded. Therefore each X_n is $\sigma(E'+M_n,E)$-compact and hence $\sigma(E'+M,E)$-compact and consequently $\sigma(E'+M,E)$-closed. It now follows that each X_n is τ-closed because $\sigma(E'+M,E)|_H \subset \tau$.

Invoking the DeWilde- Houet theorem we see that K is absorbed by some X_n. Suppose that $K \subset kX_n$, then $K \subset k(E'+M_n) = E' + M_n$.

□

2.6 Theorem

If E *is a barrelled space and* $E' + M$ *is a countable-dimensional enlargement of* E' *, then* $\mu(E,E'+M)$ *is barrelled iff every* $\sigma(E'+M,E)$*-bounded set is contained in a finite dimensional enlargement of* E' .

Proof:

Again let $M = \text{span}\,\{f_i : i \in \mathbb{N}\}$ and $M_n = \text{span}\,\{f_i : i \in [n]\}$ with $\{f_i : i \in \mathbb{N}\}$ independent. Let B be a $\sigma(E'+M,E)$-bounded set and let K be the closed, absolutely convex hull of B . Then K is also $\sigma(E'+M,E)$-bounded and since $\mu(E,E'+M)$ is barrelled, K is $\sigma(E'+M,E)$-compact. By the previous lemma, K is contained in some $E' + M_n$ and hence $B \subset E' + M_n$.

Conversely let K be absolutely convex and $\sigma(E'+M,E)$-bounded. Then by assumption, $K \subset E' + M_n$ for some n . This means that K is $\sigma(E'+M_n,E)$-bounded and, since $\mu(E,E'+M_n)$ is barrelled, K is $\mu(E,E'+M_n)$-equicontinuous. Thus K is $\mu(E,E'+M)$-equicontinuous. □

In [11] (Theorem 2) it is shown that a barrelled space which has a countable-dimensional bounded subset is not stable under countable-dimensional enlargements of its dual.

More precisely :

2.7 Theorem

Let E *be a barrelled space which has a countable-dimensional bounded subset. Then there is a countable-dimensional enlargement* $E' + M$ *such that* $\mu(E,E'+M)$ *is not barrelled.*

Proof:

Let $A = \{e_n : n \in \mathbb{N}\}$ be $\sigma(E,E')$-bounded and independent in E. We seek an independent $B = \{f_n : n \in \mathbb{N}\} \subset E^*$ with the property that: if $M = \text{span } B$, then $M \cap E' = \{0\}$ and B is $\sigma(E'+M,E)$-bounded but not $\beta(E'+M,E)$-bounded (and hence not $\mu(E,E'+M)$-equicontinuous). If A is $\sigma(E,E'+M)$-bounded, then $A^0 \cap (E' + M)$ is a $\beta(E'+M,E)$-neighbourhood of 0, and so we want A^0 not to absorb B. In other words we want : $B \not\subset kA^0$ for all $k > 0$.

Let $H = \{e_\alpha : \alpha \in I\}$ be a Hamel basis for E with $\mathbb{N} \subset I$. Define $g_n : H \to \mathbb{K}$

$$e_n \to n \qquad n \in \mathbb{N}.$$

$$e_\alpha \to 0 \qquad \alpha \notin \mathbb{N}.$$

Extend each g_n by linearity to E and denote the extension by f_n. Thus each $f_n \in E^*$ and so we let $B = \{f_n : n \in \mathbb{N}\}$ and claim that B has the required properties.

To show that B is independent, let $\sum_{n \in J} k_n f_n = 0$ with J a finite subset of $\mathbb{N}$.
Then for $m \in J$: $(\sum_{n\in J} k_n f_n)(e_m) = \sum_{n\in J} k_n f_n(e_m) = k_m f_m(e_m) = mk_m = 0$.
This implies that $k_m = 0$ for all $m \in J$.

B is $\sigma(E^*,E)$-bounded and hence $\sigma(E'+M,E)$-bounded, because if $x \in E$, we can write $x = \sum_{n\in J} k_n e_n + \sum_{\alpha \in K} k_\alpha e_\alpha$ with J a finite subset of $\mathbb{N}$ and K a finite subset of $I \setminus \mathbb{N}$. Let $k = \sum_{n \in J} |k_n| n$. Then for $n \in \mathbb{N}$ we have :

$$|f_n(x)| \leq \sum_{n\in J} |k_n| f_n(e_n) + \sum_{\alpha \in K} |k_\alpha| f_n(e_\alpha) = k + 0 = k .$$

A is $\sigma(E,E'+M)$-bounded since if $f \in E' + M$, we can write $f = g + \sum_{n\in J} k_n f_n$ with $g \in E'$ and J a finite subset of $\mathbb{N}$. Then $|f(e_m)| \leqslant |g(e_m)| + \sum_{n\in J} |k_n|f_n(e_m) \leqslant |g(e_m)| + m|k_m|$. Let $\ell = \max\{m|k_m| : m \in J\}$. Since A is $\sigma(E,E')$-bounded and $g \in E'$, there is some $k > 0$ such that $|g(e_m)| \leqslant k$ for all $m \in \mathbb{N}$. Thus $|f(e_m)| \leqslant k + \ell$ for all $m \in \mathbb{N}$.

Finally, if $B \subset kA^0$ then $|f_n(e_m)| \leqslant k$ for all $m,n \in \mathbb{N}$. Thus in particular $|f_n(e_n)| = n \leqslant k$ for all $n \in \mathbb{N}$ – a contradiction.

□

We note in passing that the existence of an infinite-dimensional bounded subset of E implies that $E' \neq E^*$ since if $E' = E^*$ then every bounded subset is finite-dimensional. Of course, in order to talk about enlargements of E' we need $E' \neq E^*$ a condition that we shall assume from now on.

The next few results : 2.8, 2.9, 2.10 and 2.11 can be found in [12] (theorems 1, 2 and 3).

2.8 Theorem

If E *is barrelled and* $E' + M$ *is a finite or countable dimensional enlargement of* E' *, then* $\mu(E,E'+M)$ *is not complete.*

Proof:

Suppose that $\mu(E,E'+M)$ is complete. Let $\mathcal{H}$ denote the collection of hyperplanes in $E' + M$ and let $\mathcal{K}$ denote the collection of absolutely convex $\sigma(E'+M,E)$-compact subsets of $E' + M$. Then

$\{K^0 : K \in \mathcal{K}\}$ is a local base for $\mu(E,E'+M)$. Thus by Grothendieck's completeness theorem we have :

$\mu(E,E'+M)$ is complete iff $(H \in \mathcal{H},\ K \in \mathcal{K} \Rightarrow H \cap K$ is $\sigma(E'+M,E)$-closed) $\Rightarrow H$ is $\sigma(E'+M,E)$-closed (i)

Let $H \in \mathcal{H}$ with $E' \subset H$. Then since E' is $\sigma(E^*,E)$-dense in E^* , so is H . Thus H is not $\sigma(E^*,E)$-closed and since $H \subset E' + M$, this means that H is not $\sigma(E'+M,E)$-closed.

Hence, by (i), there is some $K \in \mathcal{K}$ such that $H \cap K$ is not $\sigma(E'+M,E)$-closed . (ii)

From 2.5 and 2.1 we deduce that $K \subset A + B$ with A a $\sigma(E',E)$-compact set and B a finite-dimensional compact set. Since $A + B$ is $\sigma(E'+M,E)$-compact and K is $\sigma(E'+M,E)$-closed in $A + B$, $H \cap K$ is $\sigma(E'+M,E)$-closed in $H \cap (A+B)$. Now $H \cap (A+B) = A + (H \cap B)$.

To see this, let $a + b = h \in (A+B) \cap H$, then $b = h - a \in H$ (because $a \in A \subset E' \subset H$) and so $b \in H \cap B$.

Thus $H \cap K$ is $\sigma(E'+M,E)$-closed in $A + (H \cap B) \subset A + B$.

Consequently $H \cap K$ is $\sigma(E'+M,E)$-compact and hence $\sigma(E'+M,E)$-closed, which contradicts (ii). □

2.9 Corollary

If E *is a barrelled space with* $E' \neq E^*$ *, then* E' *has uncountable codimension in* E^* .

Proof:

Write $E^* = E' + M$ and suppose that M is finite or countable-dimensional. Then by 2.8, $\mu(E,E^*) = \mu(E,E'+M)$ is not complete, which is a contradiction. □

Merely re-wording this corollary we obtain :

2.10 Corollary

Let E *be a locally convex space with* $E' \neq E^*$. *If* E' *has finite or countable codimension in* E^* , *then* E *is not barrelled.*

We can now show that any barrelled space is "unstable under countable-dimensional enlargements of its dual". This recovers 2.7 which concerned the special case of barrelled spaces which had countable-dimensional bounded sets. This next theorem, 2.11, includes the case where all bounded sets are finite dimensional. It is worth noting that in 2.7 we did not use the barrelledness of $E(\tau)$ except to assert that $\tau = \mu(E,E')$, whereas the barrelledness of $E(\tau)$ is necessary for 2.11.

2.11 Theorem

If E *is a barrelled space then there is a countable-dimensional enlargement* $E' + M$ *of* E' *such that* $\mu(E,E'+M)$ *is not barrelled.*

Proof:

Let H be a hyperplane in E^* with $E' \subset H$. From 2.10 we infer that $\mu(E,H)$ is not barrelled. Hence there is a $\sigma(H,E)$-bounded subset B such that B is not $\mu(E,H)$-equicontinuous. (i)

If $E' + N \subset H$ is a finite-dimensional enlargement of E', then by 2.2, $\mu(E,E'+N)$ is barrelled. Thus if $B \subset E' + N$, then B would be $\sigma(E'+N,E)$-bounded and hence B would be $\mu(E,E'+N)$-equicontinuous. Consequently B would be $\mu(E,H)$-equicontinuous, contradicting (i).

Thus $B \not\subset E' + N$ for any finite-dimensional enlargement

$$E' + N \subset H . \qquad \text{(ii)}$$

We use (ii) to construct M as follows :

For each $n \in \mathbb{N}$, let $E' + M_n$ be a finite-dimensional enlargement of E' with $E' + M_n \subset H$. Then for each n, $B \not\subset E' + M_n$ and so we can choose $f_n \in B \setminus E' + M_n$. Furthermore we can choose f_n such that $f_n \notin \text{span}\{f_i : i < n\}$. For if not, we would have $B \subset \text{span}\{f_i : i \in [n]\}$ for some n. Writing each $f_i = g_i + h_i$ with $g_i \in E'$ and $h_i \in E^* \setminus E'$, we see that then $B \subset E' + \text{span}\{h_i : i \in [n]\}$ which contradicts (ii). Let $A = \{f_i : i \in \mathbb{N}\}$ and let $M = \text{span}\, A$. Then M has countable dimension and $M \cap E' = \{0\}$ because $A \cap E' = \emptyset$.

Hence we have a countable-dimensional enlargement $E' + M$ of E'. Now since $A \subset B$ and B is $\sigma(H,E)$-bounded, A is $\sigma(E'+M,E)$-bounded. But A is contained in no finite-dimensional enlargement of E' and so by 2.6 we conclude that $\mu(E,E'+M)$ is not barrelled. □

We shall now attempt to gain information about $\mu(E,E'+M)$ in terms of the properties of M^0. Again we shall have $E' + M$ a countable-dimensional enlargement of the dual E' and let $M = \text{span}\{f_i : i \in \mathbb{N}\}$ with $\{f_i : i \in \mathbb{N}\}$ independent and $M_n = \text{span}\{f_i : i \in [n]\}$. Hence

$$M_n^0 = \{x \in E : |f_i(x)| \leq 1 \ \forall\, i \in [n]\} = \{x \in E : f_i(x) = 0 \ \forall\, i \in [n]\} = \bigcap_{i=1}^{n} f_i^{\perp} .$$

Similarly $M^0 = \{x \in E : f_i(x) = 0 \ \forall\, i \in \mathbb{N}\} = \bigcap_{i=1}^{\infty} f_i^{\perp}$.

So we have $E' \subset E' + M_1 \subset \ldots \subset E' + M_n \subset \ldots \subset E' + M$

$$\text{and} \quad E \supset M_1^0 \supset M_2^0 \supset \ldots \supset M_n \supset \ldots \supset M^0 \qquad \text{(i)}$$

Also, $\{f_1, \ldots, f_{n+1}\}$ is independent and hence $\bigcap_{i=1}^{n} f_i^{\perp} \not\subset f_{n+1}^{\perp}$; in other words $M_n^0 \not\subset f_{n+1}^{\perp}$, from which we deduce that $M_{n+1}^0 \neq M_n^0$. This implies that (i) is a strictly decreasing sequence of subspaces of E and since M is countable-dimensional, M^0 is an infinite-codimensional subspace of E .

Recall that one of the characterisations of a barrelled space is that if any collection of continuous linear functions from the space to any other locally convex space is pointwise bounded, then it is equicontinuous (0.4). We now prove a lemma from [11] which improves on this.

2.12 Theorem

Let $E(\tau)$ *and* $F(\tau')$ *be locally convex spaces and* D *a dense, barrelled subspace of* E . *If* $A \subset L(E,F)$ *and* A *is pointwise bounded on* D , *then* A *is* τ*-equicontinuous.*

Proof:

Let V be an absolutely convex closed τ'-neighbourhood of 0 in F. We must show that $\bigcap_{f\in A} \overset{-1}{f}(V) = W$ is a τ-neighbourhood of 0 in E. Since A is pointwise bounded on D we have $\{f(x): f \in A\}$ is τ'-bounded for each $x \in D$. Thus for each $x \in D$ there is some $k(x) > 0$ such that $\{f(x): f \in A\} \subset k(x)V$, and hence $x \in k(x)W$. This means that W absorbs the points of D. (i)

We can now show that $D \cap W$ is a $\tau|_D$-barrel. Since each $f \in A$ is continuous, W is τ-closed and so $D \cap W$ is $\tau|_D$-closed. It is of course absolutely convex and is absorbent on D because of (i).

Since D is $\tau|_D$-barrelled we now have that $D \cap W$ is a $\tau|_D$-neighbourhood of 0. Hence there is a τ-neighbourhood of 0 U, such that $U \cap D = W \cap D$. Since D is dense in E, we invoke 0.1 and claim that $U \subset \overline{U \cap D}$. Thus $U \subset \overline{U} = \overline{U \cap D} = \overline{W \cap D}$. Hence for each $f \in A$ we have :

$$f(U) \subset f(\overline{U\cap D}) \subset \overline{f(U\cap D)} = \overline{f(W\cap D)} \subset \overline{V} = V .$$

So $U \subset \bigcap_{f\in A} \overset{-1}{f}(V) = W$ and hence W is a τ-neighbourhood of 0. □

Some obvious corollaries now follow :

2.13 Corollary

If $E(\tau)$ *is a locally convex space and* D *is a dense barrelled subspace of* E *, then every* $\sigma(E',D)$*-bounded subset is* τ*-equicontinuous.*

Proof:

A set is pointwise bounded on D iff it is $\sigma(E',D)$-bounded.

□

2.14 Corollary

A locally convex space with a dense barrelled subspace is itself barrelled.

Proof:

Let $E(\tau)$ be a locally convex space and D a dense barrelled subspace of E . Let A be $\sigma(E',E)$-bounded. Then A is $\sigma(E',D)$-bounded and hence is τ-equicontinuous. □

If E is a locally convex space then $\mu(E,E^*)$ is barrelled and $\mu(E,E') \subset \mu(E,E^*)$. Thus the collection of all barrelled topologies on E which are stronger than $\mu(E,E')$ is nonempty. Denote this collection by $\{\tau_\alpha : \alpha \in I\}$. The inductive limit topology τ_i is also barrelled and so τ_i is the strongest barrelled topology on E which is weaker than each τ_α . It is also the weakest barrelled topology on E which is stronger than $\mu(E,E')$ (for if τ is a barrelled topology on E stronger than $\mu(E,E')$, then $\tau = \tau_\alpha$ for some α) . The topology τ_i is called the *Kōmura topology* and we shall denote it by $k(E,E')$. (see [8]).

For a subspace G of E^* , the *quasi-completion of* G denoted by $q(G)$ is defined to be the smallest quasi-complete subspace of E^* which contains G . In other words $q(G)$ is the intersection of all quasi-complete subspaces of E^* which contain G .

2.15 Lemma

Let E *be a locally convex space and* G *a subspace of* E* *containing* E' . *Then* $E(k(E,G))' = q(G)$.

Proof:

By definition, $q(G)$ is quasi-complete and hence $\mu(E,q(G))$ is barrelled. It follows that $k(E,G) \subset \mu(E,q(G))$ and then we have :

$$E(k(E,G))' \subset E(\mu(E,q(G)))' = q(G) .$$

On the other hand $k(E,G)$ is barrelled and hence $E(k(E,G))'$ is quasi-complete. Thus $q(G) \subset E(k(E,G))'$. □

We can now sharpen 0.3.

2.16 Corollary

If E *is a locally convex space and* G *is a subspace of* E* *containing* E' , *then* $\mu(E,G)$ *is barrelled iff* $k(E,G) = \mu(E,G)$ *iff* $q(G) = G$.

Proof:

Immediate from 2.15. □

The following theorems 2.17, 2.18 and 2.19 are from [11] (Theorems 3, 4, 5 and 6).

2.17 Theorem

Let E *be a barrelled space and* $E' + M$ *a countable-dimensional enlargement of* E' . *If* M^0 *is dense and barrelled, then* $E' + q(M)$ *is also an enlargement of* E' *and*

$$k(E,E'+M) = \mu(E,E'+q(M)) \ .$$

Proof:

Firstly we must show that $q(M) \cap E' = \{0\}$.
Let M^{00} denote the polar of M^0 in E^* . Then since $\sigma(E^*,E)$ is complete and M^{00} is closed in E^* , M^{00} is $\sigma(E^*,E)$-complete and hence quasi-complete. Thus we have $q(M) \subset M^{00}$ and it suffices to show that $M^{00} \cap E' = \{0\}$. So let $f \in M^{00} \cap E'$, then $f(x) = 0$ for all $x \in M^0$. Then f is continuous and vanishes on the dense subspace M^0 , from which we conclude that f vanishes everywhere on E , in other words $f = 0$.

From 2.16 : $k(E,E'+M) = \mu(E,E'+q(M))$ iff $\mu(E,E'+q(M))$ is barrelled iff $E' + q(M)$ is quasi-complete, so we show that $E' + q(M)$ is quasi-complete. To this end, let $B \subset E' + q(M)$ be $\sigma(E'+q(M),E)$-bounded. Let $p_1: E' + q(M) \to E'$ and $p_2: E' + q(M) \to q(M)$ be the natural projections. For $f \in B$ let $f_1 = p_1(f)$ and $f_2 = p_2(f)$. Then $f_2 \in q(M) \subset M^{00}$ and hence $f_2(x) = 0$ for all $x \in M^0$.
Since B is $\sigma(E'+q(M),E)$-bounded, B is pointwise bounded on E and hence B is pointwise bounded on M^0 . Hence, for $x \in M^0$ we have that :

$$\{f(x): f \in B\} = \{f_1(x) + f_2(x): f \in B\} = \{f_1(x): f \in B\}$$

is bounded.

In other words, $p_1(B)$ is pointwise bounded on M^0. It follows from 2.13 that $p_1(B)$ is equicontinuous and therefore relatively $\sigma(E',E)$-compact. Let $p_1(B) \subset K$ with K an absolutely convex $\sigma(E',E)$-compact set. Now $p_2(B) \subset B - p_1(B) \subset B + K$ and hence $p_2(B)$ is $\sigma(E^*,E)$-bounded. Since $p_2(B) \subset q(M)$ we conclude that $p_2(B)$ is relatively $\sigma(E^*,E)$-compact. Let $p_2(B) \subset L$ with L a $\sigma(E^*,E)$-compact set. Then $B \subset p_1(B) + p_2(B) \subset K + L$ and $K + L$ is $\sigma(E^*,E)$-compact.

□

2.18 Theorem

Let E *be a barrelled space and* $E' + M$ *a countable-dimensional enlargement of* E'.

Let $q(M)$ *denote the quasi-completion of* M.

Let $c(M)$ *denote the completion of* $M(\sigma(M,E))$.

Let M^{00} *denote the polar in* E^* *of* M^0.

If M^0 *has countable codimension in* E, *then* $\mu(E,E'+M)$ *is not barrelled and in this case :* $q(M) = c(M) = M^{00}$.

Proof:

Let $E = M^0 \oplus X$ with X countable dimensional. Then $X(\sigma(X,E^*))$ is topologically isomorphic to E/M^0 with the quotient topology induced by $\sigma(E,E^*)$ (i)

We now assert that $\sigma(M^{00},E) = \sigma(M^{00},E/M^0)$. (ii)

To show (ii) let U be a $\sigma(M^{00},E)$-neighbourhood of 0. Then $U = \{x_1, \ldots, x_n\}^0 = \{f \in M^{00}: |f(x_i)| \leq 1 \ \forall i \in [n]\}$ for some $\{x_1, \ldots, x_n\} \subset E$. Since $x_i \in E = M^0 \oplus X \cong M^0 \oplus E/M^0$ we can write $x_i = x_i^1 + x_i^2$ with $x_i^1 \in M^0$ and $x_i^2 \in E/M^0$.

Then $f(x_i^1) = 0$ for $f \in M^{00}$ and hence

$U = \{f \in M^{00} : |f(x_i^2)| \leqslant 1 \ \forall \ i \in [n]\}$ is a $\sigma(M^{00},E/M^0)$-neighbourhood of 0 . Hence $\sigma(M^{00},E) \subset \sigma(M^{00},E/M^0)$. Conversely, $E/M^0 \cong X$ and X is a subspace of E and so $\sigma(M^{00},E/M^0) \subset \sigma(M^{00},E)$.

Now since E/M^0 is countable-dimensional, $\sigma(M^{00},E/M^0)$ has a countable base. Thus $M^{00}(\sigma(M^{00},E/M^0)) = M^{00}(\sigma(E^*,E)|_{M^{00}})$ is metrisable and hence $M(\sigma(E^*,E)|_M)$ is metrisable since M is a subspace of M^{00}. It follows that $M^{00} = c(M) = q(M)$. We note that since M is countable-dimensional and $c(M)$ is complete, $c(M) \neq M$ by the Baire category theorem.

Suppose that $\mu(E,E'+M)$ were barrelled. Then $q(E'+M) = E' + M$. Hence $E' + M \subset E' + q(M) \subset q(E'+M) = E' + M$ and so $q(M) \subset E' + M$. Write $q(M) = M + q(M) \cap E'$. Then $q(M) = c(M)$ implies that $q(M)$ is $\sigma(E^*,E)$-complete, and hence $\sigma(E^*,E)$-quasi-complete. Since E is barrelled, E' is $\sigma(E^*,E)|_{E'} = \sigma(E',E)$-quasi-complete. Thus we have $q(M) \cap E'$ both quasi-complete and metrisable and so it is $\sigma(E^*,E)$-complete and hence $\sigma(E^*,E)$-closed. Thus we have $q(M) \cap E'$ a closed subspace of the Frechet space $q(M)$ with countable codimension. Since this cannot happen, we have a contradiction. □

From 2.17 and 2.18 we conclude that if M^0 is dense and barrelled and also has countable codimension in E , then $\mu(E,E'+M)$ is not barrelled and $k(E,E'+M) = \mu(E,E'+q(M))$. Thus if it is possible to have M^0 dense, barrelled and of countable codimension in E , we have again proved the existence of a non-barrelled countable-dimensional enlargement (Theorem 2.7).

In fact, the condition of Theorem 2.7 is sufficient to ensure that M^0 has the required properties. More precisely:

2.19 Theorem

If E *is a barrelled space with a countable-dimensional bounded set, then there is a countable-dimensional enlargement* $E' + M$ *of* E' *with the property that* M^0 *has countable codimension in* E *(and is therefore barrelled by 1.3) and* M^0 *is dense in* E .

Proof:

Let $B = \{e_i : i \in \mathbb{N}\}$ be independent and bounded in E . Extend B to a basis $A = \{e_\alpha : \alpha \in I\}$ for E with $\mathbb{N} \subset I$. Let :

$$f_n(e_n) = 1 \quad \text{for} \quad n \in \mathbb{N}$$

$$f_n(e_\alpha) = 0 \quad \text{for} \quad \alpha \neq n . \qquad \text{(i)}$$

and extend each f_n by linearity to E , denoting the extension again by f_n . Thus $f_n \in E^*$ and (i) holds for each f_n . Let $S = \{f_n : n \in \mathbb{N}\}$ and let $J = \{n \in \mathbb{N}: f_n \in E'\}$. We assert :

$$J \text{ is finite.} \qquad \text{(ii)}$$

To prove (ii), let $S_1 = \{nf_n : n \in J\}$. Then $S_1 \subset E'$ and S_1 is $\sigma(E',E)$-bounded. Consequently S_1 is $\beta(E',E)$-bounded (E is barrelled) and hence S_1 is absorbed by B^0 . Thus there is some $k > 0$ such that $S_1 \subset kB^0$ and so we have :

$$|nf_n(e_m)| \leqslant k \quad \text{for all} \quad n \in J \quad \text{and all } m \in \mathbb{N} .$$

Hence J must be finite.

Now let $S_2 = \{f_n : n \in \mathbb{N} \setminus J\} = \{g_n : n \in \mathbb{N}\}$ say, and let $B_2 = \{e_n : n \in \mathbb{N} \setminus J\} = \{x_n : n \in \mathbb{N}\}$ say. Then:

$$g_n(x_n) = 1 \quad \text{for} \quad n \in \mathbb{N}$$

$$g_n(e_\alpha) = 0 \quad \text{for} \quad \alpha \neq n$$

$$g_n \in E^* \setminus E' \; .$$

Let $M_2 = \text{span } S_2$, thus $M_2^0 = \bigcap_{n \in \mathbb{N}} g_n^{\perp}$. Furthermore :

$$E = \text{span } B_2 \oplus M_2^0 \qquad \text{(iii)}$$

To prove (iii), let $x \in E$, then :

$$x = \sum_{\alpha \in I} k_\alpha e_\alpha + \sum_{n \in J} k_n e_n + \sum_{n \in \mathbb{N}} \ell_n x_n \qquad \text{(finite sums)}$$

$$= \quad y_1 \quad + \quad y_2 \quad + \quad y_3 \qquad \text{say.}$$

Then $y_3 \in \text{span } B_2$ and $g_n(y_1 + y_2) = g_n(y_1) + g_n(y_2) = 0 + 0$ (if $g_n = f_m$ then $m \notin J$, so $f_m(e_n) = 0$ for all $n \in J$) . Thus $g_n(y_1 + y_2) = 0$ for all $n \in \mathbb{N}$ implies that $y_1 + y_2 \in \bigcap_{n \in \mathbb{N}} g_n^{\perp} = M_2^0$ and (iii) is proved.

If we set $L = \overline{M_2^0}$ then :

$$\text{codim } L \leqslant \text{codim } M_2^0 = \dim \text{span } B_2 \quad \text{(by (iii))} \qquad \text{(iv)}$$

Let K be an algebraic complement of L contained in $\text{span } B_2$, then $K \oplus L = E$ and we can choose $B_3 \subset B_2$ such that $K = \text{span } B_3$ as follows :

Let $n(1) = \min\{n: x_n \notin L\}$

$n(2) = \min\{n: x_n \notin \text{span}\{L \cup \{x_{n(1)}\}\}\}$

$n(m) = \min\{n: x_n \notin \text{span}\{L \cup \{x_{n(1)}, \ldots, x_{n(m-1)}\}\}\}$.

and let $B_3 = \{x_{n(m)} : m \in \mathbb{N}\}$.

If L were of countable codimension in E , then K would be countable dimensional and hence the bounded subsets of K would be finite-dimensional (K is just ϕ with the weak topology). But $B_3 \subset B$ and B is bounded. Thus B_3 would be bounded and hence $K = \text{span}\, B_3$ would be finite-dimensional. This contradiction, together with (iv) establishes that :

K is finite-dimensional or equivalently,

L is finite-codimensional (v)

If $\dim K = 0$, then $L = \overline{M_2^0} = E$ and we can set $M = M_2$.

If $\dim K = k \in \mathbb{N}$, then $K = \text{span}\{x_{n(1)}, \ldots, x_{n(k)}\}$

Let $M = \text{span}\{g_n : n \in \mathbb{N} \setminus \{n(1), \ldots, n(k)\}\}$, then :

$x_{n(i)} \in M^0$ for $i \in [k]$ (because $g_n(x_m) = 0$ for $m \neq n$)

and $M_2^0 \subset M^0$ (because $M \subset M_2$) .

Thus $L = \overline{M_2^0} \subset \overline{M^0}$. Consequently $L \cup \{x_{n(1)}, \ldots, x_{n(k)}\} \subset \overline{M^0}$.

Thus : $\text{span}\{L \cup \{x_{n(1)}, \ldots, x_{n(k)}\}\} \subset \overline{M^0}$.

In other words : $L \oplus K = E \subset \overline{M^0}$. Hence M^0 is dense in E and by construction M^0 has countable codimension in E .

□

We have seen that there is always a non-barrelled countable-dimensional enlargement of a barrelled space and we now establish some conditions which guarantee the existence of a barrelled countable-dimensional enlargement of a barrelled space. The first theorem regarding this (2.20) is from [16] (Theorem 1).

2.20 Theorem

If a barrelled space E *has a bounded subset* A *with* $\dim(\text{span } A) = c$, *then there is a countable-dimensional enlargement* $E' + M$ *such that* $\mu(E, E'+M)$ *is barrelled.*

Proof:

If B is the absolutely convex hull of A , then B is also bounded and $\text{span } A = \text{span } B$. Let $G = \text{span } B$ then $\dim G = c$. The gauge of B defines a norm $\|\cdot\|$ on G with $\{x: \|x\| < 1\} \subset B \subset \{x: \|x\| \leqslant 1\}$. Let τ be the corresponding norm topology on G , then $\tau = \mu(G,G')$. Since $\dim G = c = \dim \omega$, there is an algebraic isomorphism $T: G \to \omega$. Then $T': \phi \to G^*$ is injective. Let $F = T(\phi)$ then :

$$\text{the bounded subsets of } F \text{ are finite-dimensional} \qquad \text{(i)}$$

If D denotes the unit ball in G' , then D is $\beta(G',G)$-bounded and hence D is $\sigma(G',G)$-bounded. Thus, since $F \cap D \subset D$, $F \cap D$ is a $\sigma(G^*,G)$-bounded subset of F and is therefore finite dimensional by (i). But $F \cap G' = F \cap \text{span } D = \text{span } (F \cap D)$ and so $F \cap G'$ is finite-dimensional.

Let $F = F \cap G' \oplus N$, then N is countable-dimensional.
Let $E = H \oplus G$. For $f \in N$, let $\hat{f}: E \to \mathbb{K}$

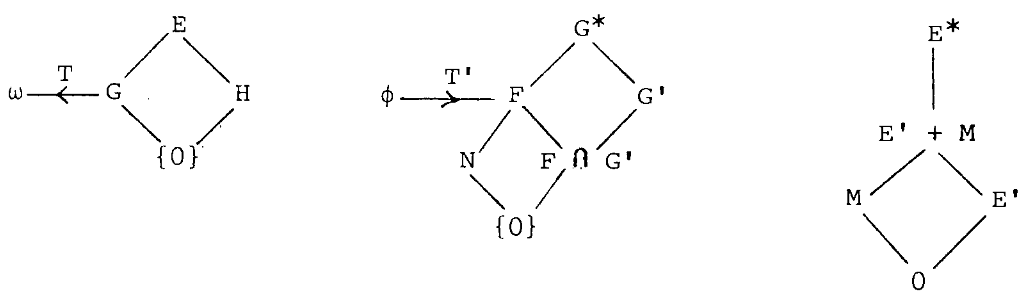
$$x \to f(x) \quad \text{for} \quad x \in G$$

$$x \to 0 \quad \text{for} \quad x \in H .$$

Then $\hat{f} \in E^*$ because $f \in G^*$. Let $M = \{\hat{f}: f \in N\}$, then M is a countable-dimensional subspace of E^* .

To see that $M \cap E' = \{0\}$, let $\hat{f} \in M \cap E'$. Then $\hat{f}$ is $\mu(E,E')$-continuous. Suppose $\hat{f} \neq 0$. Then, since $f \in N$; $f \notin G'$ and hence f is not $\mu(G,G')$-continuous. But $\mu(E,E')|_G \subset \mu(G,G')$ and so $f = \hat{f}|_G$ is $\mu(G,G')$continuous. This contradiction shows that $\hat{f} = 0$.

We have the picture :



Thus we have a countable-dimensional enlargement $E' + M$ and it remains to show that $\mu(E,E'+M)$ is barrelled. This we shall do by employing 2.6, so let X be a $\sigma(E'+M,E)$-bounded subset of $E' + M$. Then each $f \in X$ can be written : $f = g(f) + \hat{h}(f)$ where $g(f) \in E'$ and $h(f) \in N$. Let $Y = \{f|_G: f \in X\}$. Then for $f|_G \in Y$, let $f = g(f) + \hat{h}(f)$ and so :

$$f|_G = (g+\hat{h})|_G = g|_G + h \in G' + N .$$

Hence $Y \subset G' + N$, consequently Y is $\sigma(G'+N,G)$-bounded.

Let $p: G' + N \to G'$, $q: G' + N \to N$ and $r: E' + M \to M$ be natural projections. For $y \in Y$ let $z(y) = \frac{1}{\|p(y)\|} \cdot y$ if $\|p(y)\| \geq 1$

$= y$ otherwise

Let $Z = \{z(y): y \in Y\}$, then we assert that:

$$Z \text{ is } \sigma(G'+N,G)\text{-bounded.} \qquad \text{(ii)}$$

To show this, let $x \in G$. We must show that $\{|\langle x,z(y)\rangle| : y \in Y\}$ is bounded. Now $|\langle x,z(y)\rangle| = |\langle x,\frac{1}{\|p(y)\|} y\rangle| \leq |\langle x,y\rangle|$ if $\|p(y)\| \geq 1$, and so $|\langle x,z(y)\rangle| \leq |\langle x,y\rangle|$ for all $y \in Y$.
Hence $\{|\langle x,z(y)\rangle|: y \in Y\}$ is bounded because $\{|\langle x,y\rangle|: y \in Y\}$ is bounded. For $y \in Y$, let $y = g(y) + h(y)$. Then for $x \in G$ we have:

$$|\langle x,p(z(y))\rangle| = |\langle x,g(y)\rangle| \leq |\langle x,y\rangle| \quad .$$

Thus: $$p(Z) \text{ is } \sigma(G',G)\text{-bounded.} \qquad \text{(iii)}$$

It now follows from (ii) and (iii) that $q(Z)$ is $\sigma(G'+N,G)$-bounded. But $q(Z) \subset N \subset F$ and the bounded subsets of F are finite-dimensional. Hence $q(Z)$ is finite dimensional. Since $\dim q(Z) = \dim q(Y)$, $q(Y)$ is also finite dimensional. Now $q(Y) = \{h(f): f \in X\}$, and $r(X) = \{\hat{h}(f): f \in X\}$. In other words: $r(X) = \{\hat{h}: h \in q(Y)\}$, and hence $r(X)$ is finite-dimensional.

Finally, $X = \{f: f \in X\} = \{g(f) + \hat{h}(f): f \in X\} \subset E' + r(X)$ and we have shown that any $\sigma(E'+M,E)$-bounded set X is contained in a finite-dimensional enlargement $E' + r(X)$ and thus $\mu(E,E'+M)$ is barrelled.

□

This last result has been improved upon in [12] as follows :
For a dual pair $\langle E,F\rangle$, we say that F has *plenty of bounded sequences* iff for every sequence (f_n) in F there is an infinite subsequence (f_{n_k}) and a $\sigma(F,E)$-bounded subset B such that :

$$\{f_{n_k} : k \in \mathbb{N}\} \subset \text{span } B .$$

Equivalently :

F *has plenty of bounded sequence iff for every sequence* (f_n) *in* F *there is a sequence* (λ_n) *in* $\mathbb{K}$ *and a subsequence* (n_k) *with* $\lambda_{n_k} \neq 0$ *for all* k *, such that* $\lambda_n f_n \to 0$ *w.r.t.* $\sigma(F,E)$.

Proof:

Let F have plenty of bounded sequences and let (f_n) be a sequence in F . Let (f_{n_k}) be a subsequence and let B be $\sigma(F,E)$-bounded with $\{f_{n_k} : k \in \mathbb{N}\} \subset \text{span } B$. We can assume that B is absolutely convex and so span $B = \bigcup_{n=1}^{\infty} nB$.

For $f_{n_k} \neq 0$ let $f_{n_k} \in \alpha_{n_k} B$. Let $\lambda_{n_k} = \frac{1}{\alpha_{n_k}} \cdot \frac{1}{n_k}$, then $\lambda_{n_k} f_{n_k} \in \frac{1}{n_k} B$. Thus $\lambda_{n_k} f_{n_k} \to 0$ w.r.t. $\sigma(F,E)$ as $k \to \infty$ (because B is $\sigma(F,E)$-bounded.) So if we set $\lambda_n = 0$ for each $n \notin \{n_k : k \in \mathbb{N}\}$ then :

$$\lambda_n f_n \to 0 \quad \text{w.r.t.} \quad \sigma(F,E) \text{ as } n \to \infty .$$

Conversely given (λ_n) as described, let $B = \{\lambda_{n_k} f_{n_k} : k \in \mathbb{N}\}$. Then $\lambda_{n_k} f_{n_k} \to 0$ w.r.t. $\sigma(F,E)$ as $k \to \infty$ implies that B is $\sigma(F,E)$-bounded. Furthermore $\{f_{n_k} : k \in \mathbb{N}\} \subset \text{span } B$.

□

For example if $\tau \supset \sigma(F,E)$ is a metrisable topology for F, then F has plenty of bounded sequences. To see this, let (U_n) be a decreasing sequence of basic τ-neighbourhood of 0 and let (f_n) be a sequence in F. Then :

For each $n \in \mathbb{N}$, U_n is absorbent so let $f_n \in \lambda_n U_n$.
Let $B = \{\frac{1}{\lambda_n} f_n : n \in \mathbb{N}\}$. Then $U_n \downarrow 0$ and so $\frac{1}{\lambda_n} f_n \to 0$ w.r.t. τ as $n \to \infty$. Thus B is τ-bounded and hence $\sigma(F,E)$-bounded and we have $\{f_n : n \in \mathbb{N}\} \subset \text{span } B$.

For another example let F be the strict inductive limit of a sequence $(F_n(\tau_n))$ of Fréchet spaces and let τ be the inductive limit topology on F. Let $E = F(\tau)'$, then $\langle E,F \rangle$ is a dual pair but F does not have plenty of bounded sequences because if B is $\sigma(F,E)$-bounded then B is τ-bounded. Consequently $B \subset F_n$ for some n and this implies that B is τ_n-bounded. Choose $f_n \in F_n \setminus F_{n-1}$, then we have the sequence (f_n) in F. But if (f_{n_k}) is a subsequence with $\{f_{n_k} : k \in \mathbb{N}\} \subset \text{span } B$ for some $\sigma(F,E)$-bounded B, then :

$$\{f_{n_k} : k \in \mathbb{N}\} \subset F_n \quad \text{for some } n .$$

The following theorem is then proved in [12] (Theorem 4).

Let E *be a barrelled space with* $\dim E \geqslant c$. *If* E *has a subspace* G *with* $\dim G = c$ *and such that* $G' = G(\mu(E,E')|_G)'$ *has plenty of bounded sequences, then there is a countable-dimensional enlargement* $E' + M$ *of* E' *such that* $\mu(E,E'+M)$ *is barrelled.*

The construction of M is similar to the construction in 2.20 and we omit it here. That this result is an improvement of 2.20 is seen as follows :

Let B be a bounded subset of E with $\dim(\text{span } B) = c$. Let $G = \text{span } B = E_B$ (we assume that B is absolutely convex). Let τ_G be the topology induced on G by the gauge of B , then since B is bounded, $\mu(E,E')|_G \subset \tau_G$. Hence

$$G' = G(\mu(E,E')|_G)' \subset G(\tau_G)' = F \quad \text{say.}$$

We must show that G' has plenty of bounded sequences so let (f_n) be a sequence in G' . Then B^0 is absorbent in E' (because B is bounded in E) and so for each $n \in \mathbb{N}$ there is some $\lambda_n \in \mathbb{K}$ such that :

$$f_n \in \lambda_n B^0 \quad \text{so let} \quad f_n = \lambda_n g_n \quad \text{with} \quad g_n \in B^0 .$$

Thus $\{f_n : n \in \mathbb{N}\} \subset \text{span } B^0$ and B^0 is $\sigma(G',G)$-bounded (B is a τ_G-neighbourhood of 0 and hence B^0 is $\sigma(F,G)$-compact. Thus B^0 is $\sigma(F,G)$-bounded and consequently $\sigma(G',G)$-bounded). □

The next theorem (2.21) provides another condition for the existence of a countable-dimensional barrelled enlargement and was proved in [12] (Theorem 5).

2.21 Theorem

If a barrelled space E *has a dense barrelled subspace* F *with* $\text{codim } F \geq c$ *, then there is a countable-dimensional enlargement* $E' + M$ *such that* $\mu(E,E'+M)$ *is barrelled.*

Proof:

Let $F \subset L \subset E$ with codim $L = c$. Then L is dense in E because F is and by 2.14, L is a barrelled subspace of E . Let $E = G \oplus L$, then $\dim G = c = \dim \omega$ and so G is algebraically isomorphic to ω . Endow G with the product topology τ_π and let $H = G(\tau_\pi)'$. For $f \in H$, let $\hat{f}: E \to \mathbb{K}$

$$x \to f(G) \quad \text{for} \quad x \in G$$

$$x \to o \quad \text{for} \quad x \in L .$$

Let $M = \{\hat{f}: f \in H\}$, then M is algebraically isomorphic to ϕ and is therefore a countable-dimensional subspace of E^* . We claim that $M \cap E' = \{0\}$. To see this let $\hat{f} \in M \cap E'$, then $\hat{f}$ is continuous on E and vanishes on the dense subspace L and hence vanishes everywhere on E . In other words $f = 0$. Thus $E' + M$ is a countable-dimensional enlargement of E' .

We next assert that $M^0 = L$. This is because :

$$M^0 = \{x \in E: \hat{f}(x) = 0 \quad \text{for all} \quad f \in H\} .$$

Thus if $x \in E$, write $x = x_1 + x_2$ with $x_1 \in G$ and $x_2 \in L$. Then for $x \in M^0$:

$$\hat{f}(x) = \hat{f}(x_1) + \hat{f}(x_2) = f(x_1) = 0 \quad \text{for all} \quad f \in H .$$

Consequently $x_1 = 0$ and hence $x = x_2 \in L$. Conversely if $x \in L$, then $\hat{f}(x) = 0$ for all $f \in H$ and so $x \in M^0$.

Finally, we show that $\mu(E,E'+M)$ is barrelled :

Let B be $\sigma(E'+M,E)$-bounded.

Let $p: E' + M \to E'$ and $q: E' + M \to M$ be the natural projections.

Then $p(B)$ is $\sigma(E',M^0)$-bounded since if $x \in M^0$ then : $\{|<x,p(f)>| : f \in B\} = \{|<x,f>| : f \in B\}$ (because $q(f) \in M$) which is bounded. In other words, $p(B)$ is pointwise bounded on the dense subspace M^0. Hence, by 2.13, $p(B)$ is $\mu(E,E')$-equicontinuous and hence $p(B)$ is relatively $\sigma(E',E)$-compact. Let $p(B) \subset K_1$ with K_1 $\sigma(E',E)$-compact. Because B and $p(B)$ are bounded, so is $q(B)$. Since $q(B) \subset M$, $q(B)$ is $\sigma(M,E)$-bounded and the bounded subsets of M are finite-dimensional. Hence there is a $\sigma(M,E)$-compact K_2 with $q(B) \subset K_2$. Thus $B \subset p(B) + q(B) \subset K_1 + K_2$ and B is therefore relatively $\sigma(E'+M,E)$-compact. □

We shall call a dense, barrelled subspace F with $\operatorname{codim} F \geqslant c$, a *satisfactory subspace* and we have just seen in Theorem 2.21 that a barrelled space which has a satisfactory subspace has a countable-dimensional barrelled enlargement. We therefore begin the search for satisfactory subspaces of barrelled spaces. We first give an example due to Robertson and Yeomans ([11]), to show that satisfactory subspaces do indeed exist.

2.22 Example

Let $E = \mathbb{R}^{\mathbb{N}}$ *with the product topology, then* E *has a satisfactory subspace.*

Let $S = \sum_{\mathbb{N}} \mathbb{R}$ - the direct sum of a countable number of copies of $\mathbb{R}$. Then $S = \{x = (x_n) \in E: \{n: x_n \neq 0\}$ is finite$\}$. Let τ be the product topology on E, then S is τ-dense in E. Furthermore,

$\dim E = |E| = c$ and S has countable dimension.

Let $B = \{e_\alpha : \alpha \in (0,1)\}$ be a basis for E and let $I = \{1 - 2^{-n} : n \in \mathbb{N}\}$. Then I is a countable subset of $(0,1)$. Let $m(n) = 1 - 2^{-n}$, then $e_{m(n)} \in B$ for each $n \in \mathbb{N}$. Thus if $D = \{e_{m(n)} : n \in \mathbb{N}\}$, D spans a countable-dimensional subspace of E and so D may be assumed to be a basis for S .

Let $L_n = \text{span}\,\{e_\alpha : 2^{-n} \leqslant \alpha < 1\}$, then, since $m(n) \geqslant 2^{-1}$ for all n , $e_{m(n)} \in L_1$ for all n . Thus $D \subset L_1$ and hence $\text{span}\, D = S \subset L_1$. Also, since $(0,1) = \bigcup_{n=1}^{\infty} [2^{-n},1)$ we have $E = \bigcup_{n=1}^{\infty} L_n$ and hence :

$$S \subset L_1 \subset L_2 \subset \ldots \subset L_n \subset \ldots \subset E .$$

Now E is τ-complete and hence E is non-meagre in itself. Thus there is some $k \in \mathbb{N}$ such that L_k is non-meagre in E . Then :

(a) L_k is dense in E because S is dense in E .

(b) $\text{codim}\, L_k = |\{\alpha : 0 < \alpha < 2^{-k}\}| = c$.

(c) L_k is barrelled since :

if V is a barrel in L_k , then $L_k = \bigcup_{n \in \mathbb{N}} nV$.

Since L_k is non-meagre in E , there is some $n \in \mathbb{N}$ and some τ-neighbourhood U such that $U \subset \overline{nV}$ – the τ-closure of nV . Thus $\frac{1}{n} U \subset \overline{V}$ and $\frac{1}{n} U \cap L_k \subset \overline{V} \cap L_k$. Now $\overline{V} \cap L_k$ is the $\tau|_{L_k}$-closure of V and V is $\tau|_{L_k}$-closed. Hence $\frac{1}{n} U \cap L_k \subset V$ and it follows that V is a $\tau|_{L_k}$-neighbourhood. □

The following lemma and corollary (2.23 and 2.24) from [12] (section 3) provide examples of satisfactory subspaces.

2.23 Lemma

Let $E(\tau)$ *be a barrelled space and* E_1 *a barrelled subspace of* E *. Let* G *be an algebraic complement of* E_1 *. If* L_1 *is a satisfactory subspace of* E_1 *, then* $L_1 \oplus G$ *is a satisfactory subspace of* E *.*

Proof:

We must show (i) $L = L_1 \oplus G$ is dense in E

(ii) $\operatorname{codim} L \geqslant c$.

(iii) L is barrelled.

(i) Let $x = x_1 + x_2 \in E = E_1 \oplus G$ with $x_1 \in E_1$ and $x_2 \in G$ and let U be a $\mu(E,E')$-neighbourhood of 0 . Then since L_1 is dense in E_1 and $U \cap E_1$ is a $\mu(E,E')|_{E_1}$-neighbourhood of 0, $(x_1 + U \cap E_1) \cap L_1 \neq \emptyset$. Let $y \in (x_1 + U \cap E_1) \cap L_1$, then $(x + U) \cap L \neq \emptyset$ since :

$y + x_2 \in x_1 + x_2 + U \cap E_1 = x + U \cap E_1 \subset x + U$ and $y + x_2 \in L_1 \oplus G = L$.

(ii) Let $L_1 \oplus H = E_1$, then $\dim H \geqslant c$. Hence

$L \oplus H = L_1 \oplus G \oplus H = E_1 \oplus G = E$ and so $\operatorname{codim} L = \dim H \geqslant c$.

(iii) Let B be a barrel in L , that is a $\tau|_L$-barrel. Then $B \cap L_1$ is a $\tau|_{L_1}$-barrel and hence $B \cap L_1$ is a $\tau|_{L_1}$-neighbourhood of 0 because L_1 is barrelled. Let $B \cap L_1 = U \cap L_1$ where U is a τ-neighbourhood of 0 . Now $L_1 = L_1 \cap E_1$ and so $U \cap L_1 = U \cap E_1 \cap L_1 = V \cap L_1$ where

$V = U \cap E_1$ is a $\tau|_{E_1}$-neighbourhood of 0. Since L_1 is dense in E_1 and V is a $\tau|_{E_1}$-neighbourhood of 0, $V \subset \overline{V \cap L_1} = \overline{B \cap L_1}$ (closure in E) by 0.1. Thus :

$$E_1 = \text{span } V \subset \text{span } \overline{B \cap L_1} \subset \text{span } \overline{B} .$$

Also, since B is a barrel in L and G is a subspace of L, we have :

$$L \subset \text{span } B \quad \text{and hence} \quad G \subset \text{span } B \subset \text{span } \overline{B} .$$

Thus $E_1 \oplus G = E \subset \text{span } \overline{B}$ and so $\overline{B}$ is absorbent in E. Since $\overline{B}$ is also absolutely convex and τ-closed, we conclude that $\overline{B}$ is a τ-barrel and consequently a τ-neighbourhood of 0.

Finally, $B = \overline{B} \cap L$ and hence B is a $\tau|_L$-neighbourhood of 0.

□

As an immediate corollary of 2.23 we obtain :

2.24 Corollary

If E *is the strict inductive limit of a sequence* (E_n) *of barrelled spaces, then if some* E_n *has a satisfactory subspace, so does* E .

We next show that a locally convex space with suitably large dimension can always be written in a convenient form. This lemma is Lemma 3 from [12] and the subsequent corollaries are also from [12].

2.25 Lemma

Let E *be a locally convex space with* $\dim E \geq c$. *Then we can find an increasing sequence* (E_n) *of subspaces of* E *with* $E = \bigcup_{n=1}^{\infty} E_n$ *and* $\operatorname{codim} E_n \geq c$ *for each* n .

Proof:

Let $B = \{e_\alpha : \alpha \in I\}$ be a basis for E with $|I| \geq c$. Write $I = \bigcup_{n=1}^{\infty} I_n$ with $I_1 \subset I_2 \subset \ldots \subset I$ and $|I_n| = |I \setminus I_n| = |I|$.
Let $B_n = \{e_\alpha : \alpha \in I_n\}$ and let $E_n = \operatorname{span} B_n$. Then :

$$\operatorname{codim} E_n = |I \setminus I_n| = |I| \geq c \text{ for each } n \text{ , } E = \bigcup_{n=1}^{\infty} E_n$$

and $E_n \subset E_{n+1}$ for each n because $I_n \subset I_{n+1}$ for each n .

□

This lemma together with a theorem due to Valdivia [18] (Theorem 4) can now be used to establish :

2.26 Corollary

If E *is a barrelled space with* $\dim E \geq c$ *, let* $\hat{E}$ *denote the completion of* E . *If* $\hat{E}$ *is a Baire space then* E *has a dense subspace* F *with* $\operatorname{codim} F \geq c$.

Proof:

Valdivia [18], Theorem 4 states that if E is a barrelled space, $\hat{E}$ is a Baire space and (U_n) is an absorbent sequence of closed sets in E , then there is some n such that $\operatorname{int}(U_n) \neq \emptyset$.
We therefore write $E = \bigcup_{n=1}^{\infty} E_n$ as in 2.25. Then $E = \bigcup_n \bar{E}_n$ and so

for some n, $\text{int}(\bar{E}_n) \neq \emptyset$. Thus there is an absolutely convex neighbourhood of 0 U and some $x \in \bar{E}_n$ such that $x + U \subset \bar{E}_n$. Hence if $u \in U$ then :

$$u = x + u - x \in (x+U) - \bar{E}_n \subset \bar{E}_n - \bar{E}_n \subset \bar{E}_n \quad .$$

Thus $U \subset \bar{E}_n$ and so :

$$E = \text{span } U \subset \text{span } \bar{E}_n = \bar{E}_n \, .$$

Hence E_n is a dense subspace of E.

□

2.27 Corollary

(a) A Baire space E *with* $\dim E \geqslant c$ *has a satisfactory subspace.*

(b) A strict inductive limit of a sequence of Fréchet spaces has a satisfactory subspace.

Proof:

(a) Write $E = \bigcup_{n=1}^{\infty} E_n$ as in 2.26 So $E = \bigcup_n \bar{E}_n$ and there is some $n \in \mathbb{N}$ and some τ-neighbourhood of 0 U such that $U \subset \bar{E}_m$. Thus E_m is dense in E and consequently E_n is dense in E for all $n \geqslant m$. Thus we have E_n non-meagre for some $n \geqslant m$ and hence E_n is barrelled. So E_n is a satisfactory subspace.

(b) If E is the strict inductive limit of a sequence (E_n) of Fréchet spaces and $\dim E \geqslant c$, then each E_n is a Baire space and hence each E_n is also barrelled. For some n we have $\dim E_n \geqslant c$ and so by (a), E_n has a satisfactory subspace. It now follows from 2.24 that E has a satisfactory subspace. □

Further examples of spaces with satisfactory subspaces can be found in the barrelled sequence spaces - see [22] (Lemma D p.360).

The foregoing theory has an interesting application to normed spaces expressed in our theorem 2.28 which is Theorem 7 of [11].

2.28 Theorem

If $E(\tau)$ *is a barrelled normed space then :*

(a) If M *is a finite-dimensional enlargement of* E' *, then* $\mu(E,E'+M)$ *is normable.*

(b) If M *is a countable-dimensional enlargement of* E' *, then* $\mu(E,E'+M)$ *is not normable but is metrisable.*

Proof:

(a) Let $M = \text{span}\{f_1, \ldots, f_n\}$ and let p be the norm on E.

Let τ_i denote the seminorm topology generated by f_i .

Let $B_i = \{x \in E: |f_i(x)| \leqslant 1\}$ and $B_p = \{x \in E: p(x) \leqslant 1\}$.

Define $q = \sup\{p, |f_1|, \ldots, |f_n|\}$ then we assert that q is a norm on E . This is because :

(1) $q(0) = \sup\{p(0), |f_1(0)|, \ldots, |f_n(0)|\} = 0$ and if $q(x) = 0$, then $p(x) = 0$ because $p(x) \leqslant q(x)$ and hence $x = 0$.

(2)
$$\begin{aligned} q(kx) &= \sup\{p(kx), |f_1(kx)|, \ldots, |f_n(kx)|\} \\ &= \sup\{|k|p(x), |k||f_1(x)|, \ldots, |k||f_n(x)|\} \\ &= |k|q(x) . \end{aligned}$$

$$(3)\quad q(x+y) \leqslant \sup\{p(x)+p(y),\ |f_1(x)|+|f_1(y)|,\ \dots,\ |f_n(x)|+|f_n(y)|\}.$$
$$\leqslant \sup\{p(x),\ |f_1(x)|,\ \dots,\ |f_n(x)|\} + \sup\{p(y),\ |f_1(y)|,\ \dots,\ |f_n(y)|\}$$
$$= q(x)+q(y)\ .$$

Let τ_q denote the norm topology on E generated by q . Then :

$\tau_q = \tau \vee \tau_1 \vee \dots \vee \tau_n$ because :

$$B_q = \{x \in E:\ q(x) \leqslant 1\} = B_p \cap B_1 \cap \dots \cap B_n\ .$$

We now show that : $E(\tau_q)' = E' + M$. In other words :

$$E(\tau \vee \tau_1 \vee \dots \vee \tau_n)' = E' + \text{span}\{f_1, \dots, f_n\}\ .$$

We first show that $\tau_q \subset \mu(E,E'+M)$. We have :

$B_q = B_p \cap B_1 \cap \dots \cap B_n = B_p^{00} \cap \{f_1, \dots, f_n\}^0 = \{B_p^0 \cup \{f_1, \dots, f_n\}\}^0 = K^0$ say. Since B_p is a τ-neighbourhood of 0 , B_p^0 is $\sigma(E',E)$-compact and hence, since $B_p^0 \subset E'$, B_p^0 is $\sigma(E'+M,E)$-compact. Thus K is $\sigma(E'+M,E)$-compact. By 2.2, $\mu(E,E'+M)$ is barrelled and so K^{00} is again $\sigma(E'+M,E)$-compact and absolutely convex. (In other words $E' + M$ has the $\sigma(E'+M,E)$-convex compactness property - see [28].) It follows that $B_q = K^0 = K^{000}$ is a $\mu(E,E'+M)$-neighbourhood of 0 .

We now show that $\sigma(E,E'+M) \subset \tau_q$. To this end let :

$$x_\lambda \to 0 \quad \text{w.r.t.} \quad \tau_q\ .$$

Then $x_\lambda \to 0$ w.r.t. $\tau, \tau_1, \dots, \tau_n$. Let $f = g + h \in E' + M$. Then g is τ-continuous and so $g(x_\lambda) \to 0$. If $h = \sum_{i=1}^{n} k_i f_i$, then $f_i(x_\lambda) \to 0$ for each i and so $h(x_\lambda) \to 0$. Thus $f(x_\lambda) \to 0$ for each $f \in E' + M$ and hence $x_\lambda \to 0$ w.r.t. $\sigma(E,E'+M)$.

We have shown that $\sigma(E,E'+M) \subset \tau_q \subset \mu(E,E'+M)$ and it follows that $E(\tau_q)' = E' + M$. Thus $\tau_q = \mu(E,E(\tau_q)') = \mu(E,E'+M)$.

(b) Let $M = \text{span}\{f_i : i \in \mathbb{N}\}$ and let $q_n = \sup\{p,|f_1|,\ldots,|f_n|\}$ for $n \in \mathbb{N}$. From (a) we have:

q_n is a norm on E

$\tau_{q_n} = \tau \vee \tau_1 \vee \ldots \vee \tau_n$

$E(\tau_{q_n})' = E' + M_n$ where $M_n = \text{span}\{f_1,\ldots,f_n\}$.

Thus : $\tau \subset \tau_{q_1} \subset \tau_{q_2} \subset \ldots \subset \tau_{q_n} \subset \ldots$

and : $E' \subset E'+M_1 \subset E'+M_2 \subset \ldots \subset E'+M_n \subset \ldots \subset E'+M$.

Let $\tau_\infty = \tau \vee (\bigvee_{i=1}^{\infty} \tau_i)$. Then τ_∞ has the countable base $\{\frac{1}{m} B_{q_n} : m,n \in \mathbb{N}\}$ and hence τ_∞ is metrisable. We now show that : $E(\tau_\infty)' = E' + M$. In other words :

$$E(\tau \vee \tau_1 \vee \tau_2 \vee \ldots)' = E' + \text{span}\{f_1,f_2,\ldots\} .$$

We first show that $\tau_\infty \subset \mu(E,E'+M)$.
Let U be a basic τ_∞-neighbourhood of 0. Then $U = U_{i_1} \cap \ldots \cap U_{i_n}$ where U_{i_k} is a τ_{i_k}-neighbourhood of 0. Let $m = \max\{i_1,\ldots,i_n\}$, then $\lambda B_{q_m} \subset U$ for some λ. Thus U is a τ_{q_m}-neighbourhood of 0. Since $\tau_{q_m} \subset \mu(E,E'+M_m)$, U is a $\mu(E,E'+M_m)$-neighbourhood of 0 and hence a $\mu(E,E'+M)$-neighbourhood of 0.

To show that $\sigma(E,E'+M) \subset \tau_\infty$ let $x_\lambda \to 0$ w.r.t. τ_∞ and let $f \in E' + M$. Then $f \in E' + M_n$ for some n and so $f(x_\lambda) \to 0$ as in (a). Thus $x_\lambda \to 0$ w.r.t. $\sigma(E,E'+M)$.

We have shown that $\sigma(E,E'+M) \subset \tau_\infty \subset \mu(E,E'+M)$ and so $E(\tau_\infty)' = E' + M$. Thus $\tau_\infty = \mu(E,E(\tau_\infty)') = \mu(E,E'+M)$.

Finally, suppose that $\mu(E,E'+M)$ is normable. We have the norm topology $\beta(E',E)$ on E' and E' is $\beta(E',E)$-complete. In other words $E'(\beta(E',E)$ is a Banach space. Now $E(\mu(E,E'+M))' = E' + M$ and the norm topology on $E' + M$ is $\beta(E'+M,E)$ and $E' + M$ is $\beta(E'+M,E)$-complete. Thus $E' + M$ $(\beta(E'+M,E))$ is also a Banach space and is consequently barrelled. Now E' has countable codimension in $E' + M$ and so by 1.3, E' is $\beta(E'+M,E)|_{E'}$-barrelled.

Now consider the identity map $1 : E'(\beta(E',E)) \to E'(\beta(E'+M,E)|_{E'})$. Since $\beta(E'+M,E)|_{E'} \subset \beta(E',E)$, 1 is continuous and of course linear. If U is an absolutely convex $\beta(E',E)$-neighbourhood of 0 then the $\beta(E'+M,E)|_{E'}$-closure of $1(U)$ is a $\beta(E'+M,E)|_{E'}$-barrel and hence a $\beta(E'+M,E)|_{E'}$-neighbourhood of 0. Hence 1 is almost open and consequently 1 is open because $E'(\beta(E',E))$ is fully complete. Thus $\beta(E',E) \subset \beta(E'+M,E)|_{E'}$ and so we have $\beta(E'+M,E)|_{E'} = \beta(E',E)$.

Thus E' is a complete (hence closed) subspace of $E' + M$. For each $n \in \mathbb{N}$, $E' + M_n$ is closed and we have $E' + M = \bigcup_n (E'+M_n)$. Thus for some n, $E' + M_n$ has nonempty interior. Let U be a neighbourhood of 0 with $U \subset E' + M_n$. Then :

$$\text{span } U = E' + M \subset E' + M_n$$

and we have a contradiction. □

3. SUBSPACE TOPOLOGIES OF QUASIBARRELLED TOPOLOGIES

In chapter 1 we discovered that finite-codimensional and countable-codimensional subspaces of barrelled spaces were again barrelled and we now investigate the quasibarrelled case. For this we need a few preliminary results regarding hyperplanes.

Recall that for a topological vector space, a hyperplane is closed iff it is not dense. We intend to improve on this result. Let $E(\tau)$ be a locally convex space and let :

$$\mathcal{B} = \{B \subset E : B \text{ is absolutely convex, closed and bounded}\}.$$

For $B \in \mathcal{B}$ let $E_B = \text{span } B$ and let τ_B denote the norm topology on E_B generated by the gauge of B. Then $\tau|_{E_B} \subset \tau_B$ because B is bounded. If H is a hyperplane in E then $H \cap E_B$ is a hyperplane in E_B or $H \cap E_B = E_B$ and now :

3.1 Lemma

If $H \cap E_B$ *is* τ_B*-dense in* E_B *then* $H \cap B$ *is* τ_B*-dense in* B.

Proof:

We have :

$$c\ell_{\tau_B}(H \cap B) \subset c\ell_{\tau_B}(B) \subset c\ell_{\tau}(B) = B .$$

Furthermore B is a τ_B-neighbourhood of 0. Hence by 0.1 :

$$B \subset c\ell_{\tau_B}(H \cap E_B \cap B) = c\ell_{\tau_B}(H \cap B) .$$

Thus $B = c\ell_{\tau_B}(H \cap B)$ and so $H \cap B$ is τ_B-dense in B. □

If H is a hyperplane in E we shall say that H is *nearly-closed* iff $B \cap H$ is closed for each $B \in \mathcal{B}$ and we shall say that H is *ultradense* iff for each $B \in \mathcal{B}$ there is some $B_1 \in \mathcal{B}$ such that $B \subset c\ell_\tau(B_1 \cap H)$. We shall write $\overline{X}$ for $c\ell_\tau(X)$ in the sequel.

3.2 Lemma

(a) If H *is closed then* H *is nearly-closed.*

(b) If H *is ultradense then* H *is dense.*

Proof:

(a) Obvious.

(b) Let $x \in E$ then $\{x\}$ is bounded. If X denotes the closed absolutely convex hull of $\{x\}$ then $X \in \mathcal{B}$. Hence there is some $B_1 \in \mathcal{B}$ such that :

$$\{x\} \subset X \subset \overline{B_1 \cap H} \subset \overline{H} .$$

Thus $x \in \overline{H}$ for each $x \in E$ and this means that H is dense in E . □

We now prove a useful theorem due to Valdivia [20] (Lemma 1).

3.3 Theorem

A hyperplane in a locally convex space is nearly-closed iff it is not ultradense.

Proof:

Let $E(\tau)$ be a locally convex space and H a hyperplane in E. Let H be nearly-closed and suppose that H is also ultradense. Let $x \in E \setminus H$, then if X denotes the closed absolutely convex hull of $\{x\}$, we have $x \in X \in \mathcal{B}$. Hence there is some $B_1 \in \mathcal{B}$ such that:

$$x \in X \subset \overline{B_1 \cap H} = B_1 \cap H \subset H .$$

This contradicts $x \notin H$.

For the converse, let H be not nearly-closed. We must show that H is ultradense. By assumption there is some $B \in \mathcal{B}$ such that $B \cap H$ is not closed. Let $P \in \mathcal{B}$, then we seek $Q \in \mathcal{B}$ such that :

$$P \subset \overline{Q \cap H}$$

Let A be the closed absolutely convex hull of $B \cup P$. Then $A \in \mathcal{B}$ and $P \subset A$. If $H \cap E_A = E_A$ then $A \subset H$ and hence $P \subset H$. So we can set $Q = P$. If $H \cap E_A$ is a hyperplane in E_A, two mutally exclusive situations can arise :

(a) $H \cap E_A$ is a τ_A-dense in E_A.

In this case, by 3.1, $H \cap A$ is τ_A-dense in A.

Thus $P \subset A = c\ell_{\tau_A}(H \cap A) \subset c\ell_{\tau}(H \cap A) = \overline{H \cap A}$ and we can set $Q = A$.

(b) $H \cap E_A$ is τ_A-closed in E_A.

Since $B \cap H$ is not closed, there is some $x_0 \in \overline{B \cap H} \setminus B \cap H$.

Then $x_0 \in \overline{B \cap H} \subset \overline{B} = B$ and hence $x_0 \notin H$.

Let $D = \{\lambda x_0 : |\lambda| \leqslant 1\}$, then $x_0 \in \overline{B \cap H} \subset \overline{A \cap H}$. It follows that $D \subset \overline{A \cap H}$ since $A \cap H$ is balanced.

Let $L = \text{span}\{x_0\} = \mathbb{K}x_0$. Then $E = H \oplus L$ and

$E_A = E_A \cap H \oplus L$ (since : $x_0 \in B \subset A \subset E_A$).

Now if $p: E_A \to H \cap E_A$ and $q: E_A \to L$ are the natural projections, these projections are τ_A-continuous because $H \cap E_A$ is τ_A-closed in E_A and $\text{codim}\,(E_A \cap H) = 1$. Hence the projections are bounded. Thus $p(P)$ is τ_A-bounded in $H \cap E_A$ and $q(P)$ is bounded in L. Consequently there are $\alpha, \beta \in \mathbb{K}$ such that :

$$p(P) \subset \alpha(H \cap A) \subset \alpha(\overline{H \cap A})$$

and
$$q(P) \subset \beta D \subset \beta(\overline{H \cap A}) .$$

So we have $P \subset p(P) + q(P) \subset (\alpha + \beta)\,\overline{H \cap A} = \overline{H \cap (\alpha + \beta)A}$ and we can set $Q = (\alpha + \beta)A$. □

3.4 Lemma

Let E *be a locally convex space and* H *a hyperplane in* E . *If* H *is nearly-closed then* H *is sequentially-closed.*

Proof:

Let (x_n) be a sequence in H with $x_n \to x$ for some $x \in E$. Then $A = \{x_n : n \in \mathbb{N}\} \cup \{x\}$ is compact and hence bounded. If B denotes the closed absolutely convex hull of A, then $B \in \mathcal{B}$. Consequently $B \cap H$ is closed. Thus :

$$x \in \overline{A \cap H} \subset \overline{B \cap H} = B \cap H \quad \text{and so} \quad x \in H .$$
□

The following lemma is from [22] (Proposition 1.9).

3.5 Lemma

Let E *be a locally convex space and* H *a hyperplane in* E . *Let* $E = H \oplus L$ *and let* $p: E \to H$ *and* $q: E \to L$ *be the natural projections. If* H *is sequentially-closed then the projections are sequentially-continuous and hence bounded.*

Proof:

Let $L = \text{span}\{x_0\} = \mathbb{K}x_0$ with $x_0 \in E \setminus H$. Let $H = f^{\perp}$ and assume without loss of generality that $f(x_0) = 1$. Now for each $x \in E$ there is some $\lambda(x) \in \mathbb{K}$ and some $h(x) \in H$ such that

$$x = h(x) + \lambda(x).x_0$$

Thus $f(x) = 0 + \lambda(x).1 = \lambda(x)$. Therefore each $x \in E$ can be written as :

$$x = h(x) + f(x).x_0$$

Hence : $p(x) = h(x) = x - f(x).x_0$

and $q(x) = f(x).x_0$.

Let (x_n) be a sequence in E with $x_n \to 0$. Since $H = f^{\perp}$ is sequentially-closed, f is sequentially-continuous and hence :

$$f(x_n) \to 0 .$$

Thus $p(x_n) = x_n - f(x_n).x_0 \to 0 - 0.x_0 = 0$

and $q(x_n) = f(x_n).x_0 \to 0.x_0 = 0$.

This proves that p and q are sequentially-continuous.

If A is bounded in E and $(p(x_n))$ is a sequence in $p(A)$, then (x_n) is a sequence in A and hence $\frac{1}{n} x_n \to 0$. Thus $\frac{1}{n} p(x_n) \to 0$ and we have shown that p is bounded. □

3.6 Corollary

If in 3.5 H *is a sequentially closed finite-codimensional subspace of* E *, then the natural projections* p *and* q *are sequentially continuous.*

Proof:

We proceed by induction on codim H . So let codim $H = n$ and write $E = H \oplus L$ with $\dim L = n$. Now write $L = L_1 \oplus F$ with $\dim L_1 = 1$ and $\dim F = n - 1$. Thus we have :

$$E = H \oplus L = H \oplus F \oplus L_1$$

Now let $r: E \to H \oplus F$ and $s: H \oplus F \to H$ be natural projections. Then $p = s \circ r$ and r is sequentially continuous because $H \oplus F$ is a hyperplane in E . On an induction hypothesis, s is sequentially continuous because H has codimension $n - 1$ in $H \oplus F$. Therefore p (and hence also q) is sequentially continuous. □

With the aid of the foregoing results we can now obtain the analogue of 1.1. This was first proved by Valdivia [17] but the proof presented here is by Webb [24].

3.7 Theorem

Let $E(\tau)$ *be a locally convex space and* H *a hyperplane in* E *. If* V *is a bornivorous barrel in* H *then there is a bornivorous barrel* U *in* E *such that* $U \cap H = V$.

Proof:

By 3.3, there are two situations to consider :

(a) H is ultradense :

We have V absolutely convex, $\tau|_H$-closed and $\tau|_H$-bornivorous. We claim that $\overline{V}$ is a bornivorous barrel in E. Since $V = c\ell_\tau(V)$ is τ-closed and absolutely convex, we need to show that $\overline{V}$ is τ-bornivorous. To this end, let A be τ-bounded and let B be the τ-closed absolutely convex hull of A. Then $B \in \mathcal{B}$ and hence there is some $B_1 \in \mathcal{B}$ such that $A \subset B \subset \overline{B_1 \cap H}$. Now $B_1 \cap H$ is $\tau|_H$-bounded and hence $B_1 \cap H \subset \lambda V$ for some $\lambda \in \mathbb{K}$. Thus $A \subset B \subset \overline{B_1 \cap H} \subset \overline{\lambda V} = \lambda\overline{V}$ and we have shown that A is absorbed by $\overline{V}$.

So we can set $U = \overline{V}$.

(b) H is nearly-closed :

Let $E = H \oplus L$ and let $p: E \to H$, $q: E \to L$ be the natural projections. Then by 3.6, p and q are bounded.

Let $L = \text{span}\,\{x_0\}$ and let $D = \{\lambda x_0 : |\lambda| \leqslant 1\}$.

Then $V + D$ is absolutely convex and we assert that is also τ-bornivorous. To see this, let B be τ-bounded. Then $p(B)$ is $\tau|_H$-bounded and $q(B)$ is bounded in L. Hence there are $\alpha, \beta \in \mathbb{K}$ such that $p(B) \subset \alpha V$ and $q(B) \subset \beta D$. Thus if $k \geqslant \max\,\{|\alpha|, |\beta|\}$, we have :

$$B \subset p(B) + q(B) \subset \alpha V + \beta D \subset k(V + D) .$$

However, $V + D$ is not necessarily closed. But if :

(i) $\overline{V} = c\ell_\tau(V) \subset H$:

Then $V = c\ell_{\tau|_H}(V) = c\ell_\tau(V) \cap H = c\ell_\tau(V)$ and so V is τ-closed. Thus $V + D$ is τ-closed because D is compact and we can set $U = V + D$.

(ii) $\overline{V} \not\subset H$:

Then we can assume that $x_0 \in \overline{V}$ and hence $D \subset \overline{V}$. Consequently $D + V \subset 2\overline{V}$ and hence $\overline{V}$ is τ-bornivorous. Thus we can set $U = \overline{V}$. □

3.8 Corollary

A subspace of finite codimension in a quasibarrelled space is quasibarrelled.

Proof:

This is exactly the same as the proof of 1.2 with "bornivorous barrel" replacing "barrel". □

That there is no analogue of 1.3, namely that a countable-codimensional subspace of a quasibarrelled space is quasibarrelled, was proved by Valdivia [19] when he provided an example of a bornological (hence quasibarrelled) space with a non-quasibarrelled but countable-codimensional subspace.

Countable-codimensional subspaces of quasibarrelled spaces may under certain conditions be quasibarrelled. In this regard see [23].

4. ENLARGEMENTS OF QUASIBARRELLED TOPOLOGIES

A quasibarrelled space E also carries the Mackey topology $\mu(E,E')$ and in this chapter, we shall investigate the stability of the quasibarrelled property under enlargements of the dual.

We shall first consider enlargements within E^b and of course we must therefore assume that $E' \neq E^b$, in other words we need E to be non-bornological. This we shall tacitly assume whenever we talk of enlargements within E^b. We shall find that in this case, the results obtained in chapter 2 on barrelled spaces have analogues for quasibarrelled spaces. In particular, we shall see that the quasibarrelled property is stable under finite-dimensional enlargements within E^b, but not necessarily under countable-dimensional enlargements. We shall establish a necessary and sufficient condition for an enlargement to retain the quasibarrelled property and it will be no surprise to find that this condition is one on the strongly-bounded subsets of the enlargement. (Compare 4.4 with 2.6).

We then show that the quasibarrelled property is stable under arbitrary finite-dimensional enlargements.

After this we show that countable-dimensional enlargements may or may not be quasibarrelled.

The results in this chapter are all due to Webb [27] except for Theorems 4.5 and 4.10 which are due to Tweddle and Yeomans and were extracted from private correspondence.

4.1 Theorem

Let E *be a quasibarrelled space and let* $E' + M \subset E^b$ *be a finite-dimensional enlargement of* E' . *Then* $\mu(E,E'+M)$ *is quasibarrelled.*

Proof:

We first observe that :

$$\beta(E',E) = \beta(E'+M,E)\big|_{E'} . \qquad \text{(i)}$$

To see this let B be $\sigma(E,E')$-bounded and let $f \in E' + M$. Then since $E' + M \subset E^b$, f is bounded and so $f(B)$ is bounded. Thus B is $\sigma(E,E'+M)$-bounded and hence $\beta(E',E) \subset \beta(E'+M,E)\big|_{E'}$. We always have $\beta(E'+M,E)\big|_{E'} \subset \beta(E',E)$ and so (i) follows.

Now let A be $\beta(E'+M,E)$-bounded. Since E is quasibarrelled, E' is $\beta(E',E)$-sequentially complete and hence E' is $\beta(E'+M,E)$-sequentially closed in $E' + M$. Thus by 3.6 the natural projections $p: E' + M \to E'$ and $q: E' + M \to M$ are sequentially continuous and hence bounded. Therefore $p(A)$ is $\beta(E',E)$-bounded (by (i)). Consequently, because E is quasibarrelled, $p(A)$ is relatively $\sigma(E',E)$-compact. Since $q(A)$ is a bounded subset of the finite-dimensional space M , $q(A)$ is relatively $\sigma(M,E)$-compact. So we have :

$A \subset p(A) + q(A)$ with $p(A)$ and $q(A)$ relatively $\sigma(E'+M,E)$-compact. Thus A is relatively $\sigma(E'+M,E)$-compact and so $\mu(E,E'+M)$ is quasibarrelled. □

From the proof of 4.1 we extract a corollary analogous to 2.1.

4.2 Corollary

If E *is a quasibarrelled space and* $E' + M \subset E^b$ *is a finite-dimensional enlargement of* E' *then if* A *is* $\beta(E'+M,E)$*-bounded,* $A \subset B + D$ *where* B *is a* $\sigma(E',E)$*-compact set and* D *is a compact subset of* M *.*

4.3 Theorem

Let E *be a quasibarrelled space and let* $E' + M \subset E^b$ *be a countable-dimensional enlargement of* E' *. Then every absolutely convex* $\sigma(E'+M,E)$*-compact subset of* $E' + M$ *is contained in a finite-dimensional enlargement of* E' *.*

Proof:

Let $M = \text{span}\,\{f_i : i \in \mathbb{N}\}$ and $M_n = \text{span}\,\{f_i : i \in [n]\}$ so that: $M = \bigcup_{n=1}^{\infty} M_n$ and $E' \subset E' + M_1 \subset \ldots \subset E' + M_n \subset \ldots \subset E' + M$ as before.

Let K be absolutely convex and $\sigma(E'+M,E)$-compact.

Let $K_n = K \cap (E'+M_n)$ then :

K_n is $\sigma(E'+M_n,E)$-closed in $E' + M_n$ because K is $\sigma(E'+M,E)$-closed.

K_n is $\beta(E'+M_n,E)$-bounded because K is $\beta(E'+M,E)$-bounded and $\beta(E'+M_n,E) = \beta(E'+M,E)|_{E'+M_n}$ as in 4.1.

It follows that K_n is $\sigma(E'+M_n,E)$-compact since $\mu(E,E'+M_n)$ is quasibarrelled. Consequently K_n is $\sigma(E'+M,E)$-compact.

Let $E_K = \text{span } K = \bigcup_{n \in \mathbb{N}} nK$ and let τ_K denote the norm topology induced on E_K by the gauge of K. Then $E_K(\tau_K)$ is a Banach space and has unit ball K. Furthermore $\sigma(E'+M,E)|_{E_K} \subset \tau_K$ because K is $\sigma(E'+M,E)$-compact and hence $\sigma(E'+M,E)$-bounded. Thus each K_n is τ_K-closed and we have (K_n) a τ_K-closed absorbent sequence in E_K. Since K is the unit ball of E_K, K is $\beta(E_K,E_K')$-bounded and consequently, by 2.3 K is absorbed by some K_n. Suppose $K \subset \lambda K_n$ for some $\lambda \in \mathbb{K}$ then :

$$K \subset \lambda K_n \subset \lambda(E'+M_n) = E' + M_n .$$

□

4.4 Theorem

Let E *be a quasibarrelled space and let* $E' + M \subset E^b$ *countable-dimensional enlargement of* E'. *Then* $\mu(E,E'+M)$ *is quasibarrelled iff every* $\beta(E'+M,E)$*-bounded subset of* $E' + M$ *is contained in a finite-dimensional enlargement of* E'.

Proof:

Write $M = \bigcup_{n=1}^{\infty} M_n$ as in 4.3, and let $\mu(E,E'+M)$ be quasibarrelled.

Let K be $\beta(E'+M,E)$-bounded, then K is $\mu(E,E'+M)$-equicontinuous. Hence there is an absolutely convex $\sigma(E'+M,E)$-compact set A with $K \subset A$. Thus by 4.3 there is some $n \in \mathbb{N}$ such that :

$$K \subset A \subset E' + M_n .$$

Conversely, let A be $\beta(E'+M,E)$-bounded. Then, by assumption, $A \subset E' + M_n$ for some n. Hence A is $\beta(E'+M_n,E)$-bounded and, since $\mu(E,E'+M_n)$ is quasibarrelled, A is $\mu(E,E'+M_n)$-equicontinuous. Thus A is $\mu(E,E'+M)$-equicontinuous.

□

We now consider enlargements $E' + M$ of E' where $E' + M$ is not necessarily a subspace of E^b. Before proceeding we state some facts concerning bornological spaces due to Dieudonné [2].

Let E be a locally convex space and let $E^b + M$ be a finite-dimensional enlargement of E^b. Then $\mu(E,E^b+M)$ is bornological. (Compare this with 2.2). Furthermore, if A is $\beta(E^b+M,E)$-bounded subset of $E^b + M$, then $A \subset B + D$ where B is absolutely convex and $\sigma(E^b,E)$-compact, and D is an absolutely convex compact subset of M. (Compare this with 2.1.)

We can now improve 4.1 :

4.5 Theorem

If $E(\tau)$ *is a quasibarrelled space and* $E' + M$ *is a finite-dimensional enlargement of* E', *then* $\mu(E,E'+M)$ *is quasibarrelled.*

Proof:

The proof is by induction on $\dim M$ and so we start by letting $\dim M = 1$. Thus $M = \text{span}\,\{g\}$ with $g \in E^* \setminus E'$. If $g \in E^b$, then $E' + M \subset E^b$ and we invoke 4.1. We therefore assume that $g \notin E^b$.

Now $\tau^b = \mu(E,E^b)$ and so $\mu(E,E^b)$ is bornological. Hence (by [2]) $\mu(E,E^b+M)$ is bornological which implies that $\mu(E,E^b+M)$ is quasibarrelled.

We assert that: $\beta(E'+M,E) = \beta(E^b+M,E)|_{E'+M}$ (i)

To prove (i), it is clearly sufficient to show that for $A \subset E$:

A is $\sigma(E,E'+M)$-bounded iff A is $\sigma(E,E^b+M)$-bounded. (ii)

Since $E' + M \subset E^b + M$, the reverse implication is clear, so let A be $\sigma(E,E'+M)$-bounded and let $f \in E^b + M$. Then $f = f_1 + f_2$ with $f_1 \in E^b$ and $f_2 \in M$. Thus $f_1(A)$ is bounded and since $M \subset E' + M$ and A is $\sigma(E,E'+M)$-bounded, $f_2(A)$ is bounded. Consequently $f(A)$ is bounded. This establishes (ii) which in turn establishes (i).

Now let A be a $\beta(E'+M,E)$-bounded subset of $E' + M$. Then A is $\beta(E^b+M,E)$-bounded (by (i)) and hence (by [2]) $A \subset B + D$ with B an absolutely convex, $\sigma(E^b,E)$-compact subset of E^b and D an absolutely convex, compact subset of M. Since $g \notin E^b$, $M \cap E^b = \{0\}$ and so :

$$A \subset (E'+M) \cap (B+D) = B \cap E' + D \qquad \text{(iii)}$$

To see (iii), let $f + h = b + d \in (E'+M) \cap (B+D)$ with $f \in E'$, $h \in M$, $b \in B$, $d \in D$. Then $f - b = d - h \in E^b \cap M = \{0\}$. Therefore $f \in E' \cap B$ and $h = d \in D$ and (iii) follows.

Now B is $\beta(E^b,E)$-bounded and hence $B \cap E'$ is $\beta(E',E)$-bounded, absolutely convex and $\sigma(E',E)$-closed. Consequently $B \cap E'$ is also $\sigma(E',E)$-compact (because E is quasibarrelled) and we conclude that $B \cap E' + D$ is absolutely convex and $\sigma(E'+M,E)$-compact. Hence A is $\mu(E,E'+M)$-equicontinuous.

If now $\dim M = n$, let $M = \text{span}\{f_1, \ldots, f_n\}$, with $\{f_1, \ldots, f_n\}$ independent (mod E') in $E^* \setminus E'$. Let $N = \text{span}\{f_1, \ldots, f_{n-1}\}$, then on an induction hypothesis, $\mu(E,E'+N)$ is quasibarrelled. Since $M = N + \text{span}\{f_n\}$, $\mu(E,E'+M)$ is quasibarrelled by the first part. □

We now investigate countable-dimensional enlargements and we start with a theorem analogous to 2.8.

4.6 Theorem

If E *is a quasibarrelled space and* E' + M *is a finite or countable dimensional enlargement of* E' *, then* $\mu(E,E'+M)$ *is not complete.*

Proof:

We mimic the proof of 2.8, the only change occurring directly after (ii) in that proof. In this case, since K is $\sigma(E'+M,E)$-compact, it is therefore $\beta(E'+M,E)$-bounded (and absolutely convex) and we appeal to 4.2 in order to assert that $K \subset A + B$ with A a $\sigma(E',E)$-compact set and B a compact subset of M .

□

4.7 Corollary

Let $E(\tau)$ *be a quasibarrelled space. If* E *is* τ^b*-complete then* E' *has uncountable codimension in* E^b .

Proof:

Let $E^b = E' \oplus M$ and let E be τ^b-complete. Now $\tau^b = \mu(E,E^b) = \mu(E,E'\oplus M)$ and so it follows from 4.6 that M has uncountable dimension. □

This last result which is reminiscent of 2.9 enables us to show that not all countable-dimensional enlargements retain the quasibarrelled property. Compare this with 2.11.

4.8 Theorem

Let $E(\tau)$ *be a quasibarrelled, non-bornological space. Then if* E *is* τ^b*-complete, there is a countable-dimensional enlargement* $E' + M \subset E^b$ *such that* $\mu(E,E'+M)$ *is not quasibarrelled.*

Proof:

From 4.7 we know that E' has uncountable codimension in E^b and so we have $E^b = E' \oplus L$ with $\dim L \geqslant c$. We can therefore find an infinite subset $\{\ell_i : i \in \mathbb{N}\} \subset L$ which is independent mod E'. Let $M_n = \text{span}\{\ell_i : i \in [n]\}$, then we have :

$$E' \subset E' + M_1 \subset \ldots \subset E' + M_n \subset \ldots \subset E^b .$$

If H is a hyperplane in E^b with $H \supset E'$ then, since H has finite codimension in E^b, we deduce from 4.7 that $\mu(E,H)$ is not quasibarrelled. Consequently, we can find some $B \subset H$ with the property that :

B is $\beta(H,E)$-bounded but not $\mu(E,H)$-equicontinuous (i)

We now claim that :

$$B \not\subset E' + M_n \quad \text{for any } n . \tag{ii}$$

For if $B \subset E' + M_n$ for some n , then B would be $\beta(E'+M_n,E)$-bounded $(\beta(H,E)|_{E'+M_n} = \beta(E'+M_n,E))$. Hence by 4.4 B would be $\mu(E,E'+M_n)$-equicontinuous and consequently B would be $\mu(E,H)$-equicontinuous. This contradiction establishes (ii).

We now use (ii) to construct M just as we did in 2.11 - namely, we find an infinite-dimensional subset $A \subset B$ with $A \cap E' = \phi$ and let $M = \text{span } A$.

Thus we have a countable-dimensionsal enlargement $E' + M$ of E' and now, by 4.4 $\mu(E,E'+M)$ is not quasibarrelled because A is $\beta(E'+M,E)$-bounded but A is not contained in any finite-dimensional enlargement of E' . □

We next prove the direct analogy of 2.14.

4.9 Lemma

If $E(\tau)$ *is a locally convex space and* D *is a dense quasibarrelled subspace of* E *, then* E *is quasibarrelled.*

Proof:

Let U be a bornivorous barrel in E . Then $U \cap D$ is a bornivorous barrel in D . (If B is a $\tau|_D$-bounded subset of D then B is also τ-bounded. Thus B is absorbed by U and hence by $U \cap D$, since $B \subset D$.) It follows that $U \cap D$ is a

$\tau|_D$-neighbourhood of 0 , so let $U \cap D = V \cap D$, with V a τ-neighbourhood of 0 . By 0.1 :

$$V \subset \overline{V \cap D} = \overline{U \cap D} \subset \overline{U} = U .$$

Hence U is a τ-neighbourhood of 0 . □

Armed with 4.9 we can now prove that countable-dimensional quasibarrelled enlargements do exist. The statement of this next theorem is just the statement of 2.21 with "quasibarrel" replacing "barrel" throughout.

4.10 Theorem

If a quasibarrelled space has a dense quasibarrelled subspace K *with* $\operatorname{codim} K \geq c$ *, then there is a countable-dimensional enlargement* $E' + M$ *such that* $\mu(E,E'+M)$ *is quasibarrelled.*

Proof:

Let L be a subspace of E with $\operatorname{codim} L = c$ and $K \subset L$. Then L is dense in E because K is and by 4.9, L is also quasibarrelled.

Let G be an algebraic complement of L in E . Then $E = G \oplus L$. Since $\dim G = c = \dim \omega$, G is algebraically isomorphic to ω . Equip ω with the product topology τ_π and let $H = \omega(\tau_\pi)'$. Now each $f \in H$ can be identified with an element of G^* so let :

$$\hat{f}(x) = f(x) \quad \text{for} \quad x \in G$$

$$\hat{f}(x) = 0 \quad \text{for} \quad x \in L .$$

Now define $M = \{\hat{f} : f \in H\}$, then $M \subset E^*$ and :

$$\dim M = \dim H = \dim \phi \quad ,$$

thus M has countable dimension. Furthermore $M \cap E' = \{0\}$, since if $\hat{f} \in M \cap E'$ then $\hat{f}$ is τ-continuous and vanishes on the dense subspace L .

Thus $E' + M$ is indeed a countable-dimensional enlargement of E' and it remains to show that $\mu(E,E'+M)$ is quasibarrelled. To this end, let B be $\beta(E'+M,E)$-bounded, let $p: E' + M \to E'$ and $q: E' + M \to M$ be the natural projections. Because L is dense in E we have $L' = E'$ and now we observe that :

If $A \subset L$ is $\sigma(L,L')$-bounded, then A is $\sigma(E,E'+M)$-bounded. (i)

Let $f = g + \hat{h} \in E' + M$ with $g \in E'$ and $\hat{h} \in M$. Then $f(A) = \{g(x) + \hat{h}(x): x \in A\} = \{g(x) + 0: x \in A\} = g(A)$ which is bounded and (i) is established.

We now show that $p(B)$ is $\beta(L',L)$-bounded (ii)

If $A^0_{E'}$ is a $\beta(L',L)$-neighbourhood of 0 with A $\sigma(L,L')$-bounded, then A is $\sigma(E,E'+M)$-bounded and hence $A^0_{E'+M}$ is a $\beta(E'+M,E)$-neighbourhood of 0 . Hence $A^0_{E'+M}$ absorbs B and so there is some $k > 0$ such that $B \subset k\, A^0_{E'+M}$. Hence $p(B) \subset kA^0_{E'}$ because if $p(g+\hat{h}) = g \in p(B)$ and $x \in A$ then :

$$|g(x)| = |(g+\hat{h})(x)| \leq k \ .$$

This proves (ii).

Now from (ii) and the fact that L is quasibarrelled we have that $p(B)$ is $\mu(L,L')$-equicontinuous and therefore $p(B)$ is $\mu(E,E')$-equicontinuous and hence $p(B)$ is relatively $\sigma(E',E)$-compact. It follows from the fact that B is $\sigma(E'+M,E)$-bounded and $p(B)$ is $\sigma(E',E)$-bounded, that $q(B)$ is $\sigma(M,E)$-bounded. In other words $q(B)$ is $\sigma(\phi,\omega)$-bounded and so $q(B)$ is finite dimensional. Hence $q(B)$ is relatively $\sigma(M,E)$-compact and it follows that B is relatively $\sigma(E'+M,E)$-compact. □

5. σ-BARRELLED SPACES

If E is a locally convex space, we call a sequence (f_n) in E' *a* $\sigma(E',E)$-*bounded sequence* iff $\{f_n : n \in \mathbb{N}\}$ is a $\sigma(E',E)$-bounded set. A subset U of E is called a *σ-barrel* iff there is a $\sigma(E',E)$-bounded sequence (f_n) in E' with $U = \{f_n : n \in \mathbb{N}\}^0$. $E(\tau)$ (or just τ) is then called *σ-barrelled* iff every σ-barrel is a neighbourhood of 0 . Equivalently, E is σ-barrelled iff every $\sigma(E',E)$-bounded sequence is equicontinuous. These spaces, also known as *ω-barrelled spaces*, were introduced by De Wilde and Houet [3] and have been studied by Levin, Saxon, Husain, Khaleelulla, Webb and others. A compendium of results can be found in [5] where it is shown that a finite or countable codimensional subspace of a σ-barrelled space is σ-barrelled. We shall concern ourselves here with the stability of a σ-barrelled space under enlargements of its dual. Since a σ-barrelled space does not necessarily have the Mackey topology (see [5]: chap. VI, example 2), we shall have to impose this restriction in order to proceed. Thus the object under investigation is a σ-barrelled space $E(\tau)$ for which $\tau = \mu(E,E')$. In other words, E is a σ-barrelled Mackey space.

If $E' + M$ is an enlargement of E' , we ask the question : is $\mu(E,E'+M)$ again σ-barrelled?

Before attempting to answer we prove the elementary :

5.1 Theorem

If E *is* σ*-barrelled then* E' *is* $\sigma(E',E)$*-sequentially complete.*

Proof:

Let (f_n) be a $\sigma(E',E)$-Cauchy sequence in E' and let $F = \{f_n : n \in \mathbb{N}\}$. Then since (f_n) is also $\sigma(E^*,E)$-Cauchy and E is $\sigma(E^*,E)$-complete, there is some $f \in E^*$ such that $f_n \to f$ w.r.t. $\sigma(E^*,E)$. Now F is equicontinuous and consequently so is F^{00} . Thus $f \in E'$ because $f \in F^{00}$ and $F^{00} \subset E'$. So we have $f_n \to f$ w.r.t. $\sigma(E',E)$. □

As we did in chapter 2 we first consider finite-dimensional enlargements :

5.2 Theorem

Let E *be a* σ*-barrelled Mackey space. If* $E' + M$ *is a finite-dimensional enlargement of* E' *then* $\mu(E,E'+M)$ *is* σ*-barrelled.*

Proof:

Firstly suppose that $\dim M = 1$ so that $M = \text{span}\{g\}$ with $g \in E^* \setminus E'$. Let (f_n) be a $\sigma(E'+M,E)$-bounded sequence in $E' + M$ and let $F = \{f_n : n \in \mathbb{N}\}$. Write $f_n = g_n + \lambda_n g$ with $g_n \in E'$ and $\lambda_n \in \mathbb{K}$. Now E' is $\sigma(E',E)$-sequentially complete and this is all that we need to establish that $\{\lambda_n : n \in \mathbb{N}\}$ is a bounded subset of $\mathbb{K}$ as in 2.1. Consequently $G = \{g_n : n \in \mathbb{N}\}$ is $\sigma(E',E)$-bounded and hence G is $\mu(E,E')$-equicontinuous. Let $F_1 = G^{00}$. Then F_1 is absolutely convex and $\sigma(E',E)$-compact. Since $\{\lambda_n g: n \in \mathbb{N}\}$ is a bounded subset of the 1-dimensional space M, we have $\{\lambda_n g: n \in \mathbb{N}\} \subset F_2$ for some absolutely convex $\sigma(M,E)$-compact set F_2. Thus we have :

$$F \subset F_1 + F_2 = K$$

with K an absolutely convex, $\sigma(E'+M,E)$-compact set. Therefore $K^0 \subset F^0 = \bigcap_{n=1}^{\infty} \bar{f}_n^{1}(B)$ where $B = \{\lambda \in \mathbb{K}: |\lambda| \leqslant 1\}$. Thus, since K^0 is a $\mu(E,E'+M)$-neighbourhood of 0, F is $\mu(E,E'+M)$-equicontinuous.

A simple inductive argument as in 2.1 establishes the general result. □

We now move on to countable-dimensional enlargements and with one eye on 2.5 we state :

5.3 Theorem

Let E *be a* σ*-barrelled Mackey space and let* $E' + M$ *be a countable-dimensional enlargement of* E'. *Then every absolutely convex,* $\sigma(E'+M,E)$*-compact subset of* $E' + M$ *is contained in a finite-dimensional enlargement of* E'.

Proof:

Let $M = \text{span}\{f_i : i \in \mathbb{N}\}$ with $\{f_i : i \in \mathbb{N}\}$ independent mod E', $M_n = \text{span}\{f_i : i \in [n]\}$ and let K be absolutely convex, $\sigma(E'+M,E)$-compact. Let $E_K = \text{span}\, K$ and let the gauge of K induce the norm topology τ_K on E_K. For $n \in \mathbb{N}$ let $X_n = K \cap (E'+M_n)$. Then X_n is absolutely convex. Also :

$$\bigcup_{n=1}^{\infty} X_n = K \cap \left(\bigcup_{n=1}^{\infty} E' + M_n \right) = K$$

and so (X_n) is an absorbent sequence in E_K. We show that X_n is τ_K-closed.

Since $\mu(E,E'+M_n)$ is σ-barrelled (5.2), $E' + M_n$ is $\sigma(E'+M_n,E)$-sequentially complete (5.1). Thus X_n is $\sigma(E'+M_n,E)$-sequentially complete because X_n is $\sigma(E'+M_n,E)$-closed in $E' + M_n$. Therefore X_n is $\sigma(E'+M,E)$-sequentially complete. Now τ_K has the countable local base : $\{nK : n \in \mathbb{N}\}$ consisting of $\sigma(E'+M,E)$-closed sets. Thus τ_K is linked to $\sigma(E'+M,E)$ (see introduction) and so X_n is τ_K-sequentially complete. Hence X_n is τ_K-complete and consequently X_n is τ_K-closed.

It now follows from 2.3 that K is absorbed by some X_n and a fortiori, by $E' + M_n$. Thus $K \subset E' + M_n$. □

The analogue of 2.6 now follows quickly.

5.4 Theorem

Let E *be a* σ*-barrelled Mackey space and let* $E' + M$ *be a countable-dimensional enlargement of* E' . *Then* $\mu(E,E'+M)$ *is* σ*-barrelled iff every* $\sigma(E'+M,E)$*-bounded sequence is contained in a finite-dimensional enlargement of* E' .

Proof:

Suppose $\mu(E,E'+M)$ is σ-barrelled. Let $M = \bigcup_{n=1}^{\infty} M_n$ with M and M_n defined as in 5.3. Let (f_n) be a $\sigma(E'+M,E)$-bounded sequence in $E' + M$. Then $F = \{f_n : n \in \mathbb{N}\}$ is $\mu(E,E'+M)$-equicontinuous and hence there is an absolutely convex, $\sigma(E'+M,E)$-compact set K with $K^0 \subset F^0$. Thus :

$$F \subset F^{00} \subset K^{00} = K$$

and, since $K \subset E' + M_n$ for some n (by 5.3), we have $F \subset E' + M_n$.

Conversely, let (f_n) be a $\sigma(E'+M,E)$-bounded sequence in $E' + M$. Then by assumption $F = \{f_n : n \in \mathbb{N}\} \subset E' + M_n$ for some n . Thus F is $\sigma(E'+M_n,E)$-bounded and therefore (by 5.2) F is $\mu(E,E'+M_n)$-equicontinuous. Hence F is $\mu(E,E'+M)$-equicontinuous. □

We now show that countable-dimensional enlargements of σ-barrelled Mackey topologies may or may not be σ-barrelled.

5.5 Theorem

If E *is a Mackey space which has a countable-dimensional bounded subset then there is a countable-dimensional enlargement* E' + M *such that* $\mu(E,E'+M)$ *is not* σ*-barrelled.*

Proof:

In the proof of 2.7 we use only the fact that E is a Mackey space and the countable-dimensional bounded subset to produce a countable-dimensional enlargement E' + M and a $\sigma(E'+M,E)$-bounded sequence (f_n) which is not $\mu(E,E'+M)$-equicontinuous. Thus $\mu(E,E'+M)$ is not σ-barrelled. □

5.6 Theorem

If a σ*-barrelled Mackey space* E *has a bounded subset* A *with* dim (span A) = c , *then there is a countable-dimensional enlargement* E' + M *such that* $\mu(E,E'+M)$ *is* σ*-barrelled.*

Proof:

In the proof of 2.20 we use only the fact that E is a Mackey space and the subset A to produce a countable-dimensional enlargement E' + M . If now (f_n) is a $\sigma(E'+M,E)$-bounded sequence in E' + M , let $X = \{f_n : n \in \mathbb{N}\}$. Then, as in 2.20 X is contained in a finite-dimensional enlargement E' + r(X) . Thus by 5.4, $\mu(E,E'+M)$ is σ-barrelled. □

6. σ-QUASIBARRELLED SPACES

A locally convex space E is called *σ-quasibarrelled* iff every $\beta(E',E)$-bounded sequence is equicontinuous. The previous chapter prompts the question: if E is a σ-quasibarrelled Mackey space and $E' + M$ is an enlargement of E', is $\mu(E,E'+M)$ σ-quasibarrelled?

We need the basic result:

6.1 Theorem

If E *is a σ-quasibarrelled space then* E' *is* $\beta(E',E)$-*sequentially complete.*

Proof:

Let (f_n) be a $\beta(E',E)$-Cauchy sequence in E'. Then $F = \{f_n : n \in \mathbb{N}\}$ is $\beta(E',E)$-bounded and hence F is equicontinuous. Now (f_n) is also $\sigma(E',E)$-Cauchy and as in 5.1 we have:

$$f_n \to f \quad \text{w.r.t.} \quad \sigma(E',E) \quad \text{for some} \quad f \in E' .$$

Since $\beta(E',E)$ is linked to $\sigma(E',E)$ we then have :

$$f_n \to f \quad \text{w.r.t.} \quad \beta(E',E) .$$

□

We first consider finite-dimensional enlargements within E^b and in this case we recall from 4.1 that :

$$\text{if} \quad E' + M \subset E^b \quad \text{then} \quad \beta(E',E) = \beta(E'+M,E)\big|_{E'} .$$

6.2 Theorem

If E *is a* σ*-quasibarrelled Mackey space and* $E' + M \subset E^b$ *is a finite-dimensional enlargement of* E' *then* $\mu(E,E'+M)$ *is* σ*-quasibarrelled.*

Proof:

Let (f_n) be a $\beta(E'+M,E)$-bounded sequence in $E' + M$ and let $p: E' + M \to E'$ and $q: E' + M \to M$ be natural projections. Since E' is $\beta(E',E)$-sequentially complete, E' is $\beta(E'+M,E)$-sequentially complete. Hence E' is $\beta(E'+M,E)$-sequentially closed and so p is sequentally continuous. Thus p (and hence q) is bounded.

If $F = \{f_n : n \in \mathbb{N}\}$ then $p(F)$ is a $\beta(E',E)$-bounded sequence in E' and hence $p(F)$ is $\mu(E,E')$-equicontinuous. Thus $p(F)$ is $\mu(E,E'+M)$-equicontinuous so let $p(F) \subset K_1$ with K_1 absolutely convex and $\sigma(E'+M,E)$-compact.

Since $q(F)$ is a bounded sequence in the finite-dimensional space M , $q(F) \subset K_2$ with K_2 some absolutely convex, $\sigma(E'+M,E)$-compact subset of M .

So we have :

$$F \subset p(F) + q(F) \subset K_1 + K_2 = K$$

where K is absolutely convex and $\sigma(E'+M,E)$-compact. Thus F is $\mu(E,E'+M)$-equicontinuous.

□

We now consider arbitrary finite-dimensional enlargements of E', that is, enlargements which are not necessarily in E^b. We recall from 4.5 that :

$$\beta(E'+M,E) = \beta(E^b+M,E)\big|_{E'+M} .$$

6.3 Theorem

If E *is a* σ*-quasibarrelled Mackey space and* $E'+M$ *is a finite-dimensional enlargement of* E' *then* $\mu(E,E'+M)$ *is* σ*-quasibarrelled.*

Proof:

Let $\dim M = 1$ so that $M = \mathrm{span}\,\{g\}$ with $g \in E^* \setminus E'$. If $g \in E^b$ then we invoke 6.2, so we assume that $g \notin E^b$.

Now $\mu(E,E^b+M)$ is bornological (Dieudonné [2]) and hence $\mu(E,E^b+M)$ is quasibarrelled. Thus $\mu(E,E^b+M)$ is σ-quasibarrelled.

Let (f_n) be a $\beta(E'+M,E)$-bounded sequence in $E'+M$. Then (f_n) is $\beta(E^b+M,E)$-bounded and so $F = \{f_n : n \in \mathbb{N}\}$ is $\mu(E,E^b+M)$-equicontinuous. We can therefore find an absolutely convex, $\sigma(E^b+M,E)$-compact set K such that $F \subset K$.

Let $K \subset B + D$ with B an absolutely convex, $\sigma(E^b,E)$-compact subset of E^b and D an absolutely convex, $\sigma(M,E)$-compact subset of M (Dieudonné [2]). Then :

$$F \subset (B+D) \cap (E'+M) = (B \cap E') + D$$

as in 4.5.

Write $f_n = g_n + h_n$ with $g_n \in B \cap E'$ and $h_n \in D$ and let $G = \{g_n : n \in \mathbb{N}\}$. Because B is $\beta(E^b,E)$-bounded, $B \cap E'$ is $\beta(E',E)$-bounded. Consequently G is $\beta(E',E)$-bounded and hence G is $\mu(E,E')$-equicontinuous (because E is σ-quasibarrelled). Thus G is $\mu(E,E'+M)$-equicontinuous and so there is some absolutely convex $\sigma(E'+M,E)$-compact set L such that $G \subset L$. We then have :

$$F \subset G + D \subset L + D .$$

Since $L + D$ is absolutely convex and $\sigma(E'+M,E)$-compact, F is $\mu(E,E'+M)$-equicontinuous.

Induction on dim M establishes the general result. □

We now turn to countable-dimensional enlargements of E' and here we restrict ourselves to enlargements with E^b .

6.4 Theorem

Let E *be a* σ*-quasibarrelled Mackey space and let* $E' + M \subset E^b$ *be a countable-dimensional enlargement of* E' . *Then every absolutely convex* $\sigma(E'+M,E)$*-compact set is contained in a finite-dimensional enlargement of* E' .

Proof:

As before we write $M = \bigcup_{n=1}^{\infty} M_n$ with $\dim M_n = n$. Let K be absolutely convex and $\sigma(E'+M,E)$-compact. Then K is $\sigma(E'+M,E)$-closed and hence K is $\beta(E'+M,E)$-closed.

For $n \in \mathbb{N}$ let $K_n = K \cap (E'+M)$. Then:

By 6.3, $\mu(E,E'+M_n)$ is σ-quasibarrelled. By 6.1, $E' + M_n$ is $\beta(E'+M_n,E)$-sequentially complete and hence $E' + M_n$ is $\beta(E'+M,E)$-sequentially complete because $\beta(E'+M,E)|_{E'+M_n} = \beta(E'+M_n,E)$. Now K_n is $\beta(E'+M_n,E)$-closed in $E' + M_n$ and it follows that K_n is $\beta(E'+M,E)$-sequentially complete.

Let E_K = span K and let τ_K be the norm topology induced on E_K by the gauge of K . Then since $\{nK: n \in \mathbb{N}\}$ is a local base for τ_K consisting of $\beta(E'+M,E)$-closed sets, τ_K is linked to $\beta(E'+M,E)|_{E_K}$. Thus K_n is τ_K-sequentially complete and this means that K_n is τ_K-complete. Consequently K_n is τ_K-closed and we have (K_n) an absorbent sequence of τ_K-closed sets. The result follows as before.

□

From this we obtain:

6.5 Theorem

Let E *be a* σ*-quasibarrelled Mackey space and let* $E' + M \subset E^b$ *be a countable-dimensional enlargement of* E' . *Then* $\mu(E,E'+M)$ *is* σ*-quasibarrelled iff every* $\beta(E'+M,E)$*-bounded sequence is contained in a finite-dimensional enlargement of* E' .

Proof:

Same as 4.4 with bounded sequences replacing bounded sets.

□

7. BIBLIOGRAPHY

[1] Amemiya, I and Y Kómura, "Über nicht-vollstandige Montelraüme", Math. Ann. 177 (1968), 273-277.

[2] Dieudonné J, "Sur les propriétés de permanence de certains espaces vectoriels topologiques", Ann. Polon. Math. 25 (1952), 50-55.

[3] De Wilde, M and C Houet, "On increasing sequences of absolutely convex sets in locally convex spaces", Math. Ann. 192 (1971), 257-261.

[4] De Wilde, and J Schmets, "Locally convex topologies strictly finer than a given topology and preserving barrelledness or similar properties", Bull. Soc. Roy. Sci. Liège 41 (1972), 268-271.

[5] Husain, T and S M Khaleelulla, "Barrelledness in Topological and Ordered Vector Spaces", Springer-Verlag, LNM 692, 1978.

[6] Kelley, J L and I Namioka, "Linear Topological Spaces", Springer-Verlag, GTM 36, 1963.

[7] Köthe, G, "Topological Vector Spaces I", Springer-Verlag, GMW 159, 1969.

[8] Kömura, Y, "On linear topological spaces", Kum. J. Sci. Ser. A5 (1962), 148-157.

[9] Nachbin, L, "Topological vector spaces of continuous functions", Proc. Nat. Acad. Sci. USA 40 (1954), 471-474.

[10] Robertson, A P and W J Robertson, "Topological Vector Spaces", Cambridge U.P., 1973.

[11] Robertson, W J and F E Yeomans, "On the stability of barrelled topologies I", Bull. Austral. Math. Soc. 20 (1979), 385-395.

[12] Robertson, W J, I Tweddle and F E Yeomans, "On the stability of barrelled topologies III", Bull. Austral. Math. Soc. 22 (1980), 99-112.

[13] Saxon, S and M Levin, "Every countable-codimensional subspace of a barrelled space is barrelled", Proc. Amer. Math. Soc. 29 (1971), 91-96.

[14] Shirota, T, "On locally convex vector spaces of continuous functions", Proc. Japan Acad. 30 (1954), 294-298.

[15] Tsirulnikov, B, "Sur la conservation des propriétés de tonnelage dans les espaces localement convexes", C.R. Acad. Sc. Paris Comptes Rendus A 288 (1979), 875-876.

[16] Tweddle, I and F E Yeomans, "On the stability of barrelled topologies II", Glasgow Math. J. 21 (1980), 91-95.

[17] Valdivia, M, "A hereditary property in locally convex spaces", Ann. Inst. Fourier, Grenoble 21, 2 (1971), 1-2.

[18] Valdivia, M, "Absolutely convex sets in barrelled spaces", Ann. Inst. Fourier, Grenoble 21, 2 (1971), 3-13.

[19] Valdivia, M, "Some examples on quasi-barrelled spaces", Ann. Inst. Fourier, Grenoble 21, 2 (1972), 21-26.

[20] Valdivia, M, "On subspaces of countable codimension of a locally convex space", J. Reine Angew. Math. 256 (1972), 185-189.

[21] Valdivia, M, "On nonbornological barrelled spaces", Ann. Inst. Fourier 22 (1972), 27-30.

[22] Webb, J H, "Sequential convergence in locally convex spaces", Proc. Camb. Phil. Soc. 64 (1968), 341-364.

[23] Webb, J H, "Subspaces of bornological and quasi-barrelled spaces", J. London Math. Soc. (2), 7 (1973), 157-158.

[24] Webb, J H, "Finite-codimensional subspaces of countably quasi-barrelled spaces", J. London Math. Soc. (2), 8 (1974), 630-632.

[25] Webb, J H, "Subspaces of barrelled spaces", Quaestiones Mathematicae 4 (1981), 323-324.

[26] Webb, J H, "Countable-codimensional subspaces of locally convex spaces", Proc. Edinburgh Math. Soc. 18 (series II), part 3, (1973), 167-172.

[27] Webb, J H, "Enlargements of quasi-barrelled topologies", Research Report, Dept. Mathematics, University of Cape Town, RR 005, Oct. 1981.

[28] Wilansky, A, "Modern Methods in Topological Vector Spaces", McGraw-Hill, 1978.