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A

Theory Based

Stochastic Investment Model

for

Actuarial Use

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Abstract

This thesis reviews the origins, development and uses of asset-liability modelling, as well as existing largely stochastic investment models, notably those of the Maturity Guarantees Working Party (1980), Wilkie (1986, 1995) and Thomson (1996). A stochastic investment model is developed which describes returns from equities, bonds and cash, as well as inflation and economic growth. The model is consistent with economic theory, adequately fits past data, and is relatively parsimonious compared with other models.

A series of assumptions about the causal relationships between inflation, economic growth and interest rates are made based on standard economic theory. It is noted that consensus does not exist on some of the economic theory. Similarly a series of assumptions on the pricing of assets are made based on financial economic theory on market efficiency, expectations and asset pricing. Notably, it is assumed that financial markets are efficient.

An economic model is described for inflation, economic growth and interest rates based on the set of assumptions. Each variable is modelled such that its value in one period is a function of its value in the previous period, the value of the other economic variables in the current and previous period, and a normally distributed residual. The model is a mixture of a random walk and autoregressive process that has two special cases of a (non-mean-reverting) pure random walk, and a (mean-reverting) pure autoregressive process.

A financial market model is described for bond and equity returns based on the set of assumptions. Expected returns are derived from the expected real interest rate plus a risk premium, where the risk premium is linearly related to the standard deviation of real return. Bond yields are modelled as the sum of expected future short term real interest rates, expected future inflation, and a risk premium. Share prices are modelled as the present value of expected future distributable earnings, discounted at a rate equal to the sum of expected future short term real interest rates, expected future inflation, and a risk premium. The growth in earnings per share is modelled as the sum of inflation, real economic growth and a normal residual, and is also linked to real interest rates. Dividends are modelled as a smoothed function of earnings, with unit-gain from earnings to dividends.

Annual data for a 15 year period is used to parameterise the model for the United States, Britain and South Africa respectively. The modelled volatilities of financial market returns, together with the economic data, are used to fit the economic model. The procedure is similar to the method of moments for statistical estimation. Parameters in the economic model that are not statistically significant or are not consistent with the assumptions are excluded. It was found that neither the random walk nor the autoregressive special case models could adequately explain observed volatility in financial markets, so the general case (mixture model) was adopted for economic variables.

The parameterised models for the three countries studied exhibited a "cascade structure" where all variables are a function of one or two "driving variables", without any circularity/"feedback". The

models for the United States and Britain all have inflation as the driving variable, whereas the South African model has both inflation and economic growth as driving variables.

The model achieves the objectives of consistency with economic theory as well as parsimony (when compared to Wilkie (1995)). With regards to the criterion of producing reasonable output, the model has advantages over existing models. These include that financial market returns simulated by the model are non-normal and exhibit significant leptokurtosis (fat-tails) with higher probabilities of severe down-market returns than are predicted by normal or log-normal distributions. Simulated returns also exhibit the weak and slow mean reversion that is observed in markets, and the simulated yield curve exhibits non-parallel shifts and inversions. However, simulated interest rates (particularly nominal interest rates), and even bond yields can become negative, although the probability of negative nominal interest rates is small in the model, and that of negative bond yields is negligible.

Two areas where a good fit was not achieved were in the models of risk premiums and dividends. It is recommended that alternative approaches for estimating risk premiums be used. The poor fit to dividend data is not regarded as a significant weakness because modelled equity returns are not dependent on dividends.

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1. Introduction

Asset-liability modelling (ALM) is a relatively new field of actuarial science but has become the standard tool of many actuaries, notably those advising defined benefit pension funds on investment strategy and those involved in the financial management of insurance companies. Smith (1996: 1170) defines ALM as modelling "the assets and liabilities of a financial entity, including the interactions between them". ALM has become a standard tool as it not only gives a central projection, as do traditional deterministic actuarial methods, but also information about the possible variability around that projection. A key input to the ALM process is the stochastic investment model, which models the assets, as well as certain variables (such as inflation) that are required to modelling the liabilities.

Despite the widespread use of ALM, no stochastic investment model has been adopted as standard, and there are now several models in existence. This is reflective both of the different approaches used to develop stochastic investment models, and of the fact that the development of stochastic investment models fall into the areas of statistics, economics and financial economics – areas in which actuaries are trained, but are not typically leaders or experts. To date, models have typically been developed using a largely statistical approach i.e. finding a model which best fitted observed data. Unfortunately, this approach often leads to models that are at odds with established economic theory, and even models that incorporate features that do not reflect reality, for example some imply that successful trading strategies exist whereas in practice they do not. The weakness of a purely statistical approach is that when numerous parameters are estimated, as is typically the case when developing a stochastic investment model, statistically significant relationships may be introduced that are spurious. As can be expected, different parameters are found when different data periods or data from different countries is used, but the whole structure of the model can also change.

An alternative approach to developing a stochastic investment model is to base the structure of the model on established economic theory and then to parameterise the model using observed data. Such a model will undoubtedly fit the data used to parameterise it less well than a pure statistical approach, but it addresses the key weakness of a purely statistical approach in that it will not capture spurious, but statistically significant relationships, and has a structure that is stable across different time periods and economies. Importantly, a model that is consistent with economics should produce reasonable output, and make intuitive sense to users. This is beneficial to actuaries interpreting the results from ALM and communicating them to clients.

The objective of this research is to develop a model of inflation, short term interest rates, real economic growth, bond returns and equity returns that is consistent with economic theory, but also adequately fits the data without being overly complex. Ideally the model should be sufficiently flexible to allow adaptation for different applications. Broadly, the approach taken is to specify the structure of the model based on a set of economic and financial economic assumptions, and then parameterise the model on past data. The features of the model will then be evaluated against various criteria.

The research will not consider other explanatory variables, such as money supply and the gold price, which may in practice affect investment returns, and returns from property and inflation-linked bonds, and wages are not modelled. Within equity, sub-sectors of the market will not be considered. While the model is not country specific, it is not a multi-country model and therefore excludes consideration of exchange rates. The model will be parameterised for the US, British and South African economies.

Please note that the research was conducted part-time over several years, so that the data used reflects when particular work was completed and the data available at that time, not any subjective selection of data periods. The thesis has the following structure:

Chapter 2 discusses the origins, developments and uses of ALM. The main published stochastic investment models are then described, particularly those of the Maturity Guarantees Working Party (1980), Wilkie (1986, 1995) and Thomson (1996).

Chapter 3 summarises the approach taken in developing the stochastic investment model and the philosophy behind it. Firstly the evaluation criteria are discussed and how they will be applied. The choice of variables to be included in the model and the available data is then explained. A method of formulating the model is given.

Chapter 4 discusses the economic theory of how inflation, economic growth, and interest rates interact with one another, as well as the efficiency of financial markets. Expectations theory and models for the determination of risk premiums are also discussed. Standard models for the pricing and valuation of bonds and equities are described, including a discussion of the term structure of interest rates, earnings and dividends. Two sets of assumptions are then made, one set for the economic model, and another for the financial market model. These assumptions are used to formulate the full stochastic investment model.

Chapter 5 shows how the assumptions about the functioning of the economy and the financial markets can be described mathematically. The parameters in the mathematical model are then estimated.

Chapter 6 provides a summary description of the model, firstly a diagrammatic representation of the structure of the models, and then through summary statistics. Properties of the model are then more closely examined. In particular, the distribution of returns, including the tail of the distributions is examined. The degree to which mean-reversion of equity returns is incorporated in the model (i.e. to what extent modelled equity risk decays) is then explored. The modelled behaviour of the yield curve is also examined. How the models might be adapted and used is then described, in particular how they might be calibrated to current market conditions. The models are then evaluated under the three criteria of consistency with economic theory, parsimony and reasonableness of output.

Finally, Chapter 7 summarises a number of conclusions and makes recommendations for further research.

Demonstration spreadsheets of each of the three country models are provided in the accompanying CD-ROM, as is the simulation output used to examine the properties of the models.

2. Literature Review

2.1 Asset-Liability Modelling

2.1.1 Origins

Redington (1952) questioned how a portfolio of assets was chosen to match a set of liabilities. He developed the theory of immunisation which shows how a portfolio of fixed interest assets backing fixed liabilities can be immunised against a change in interest rates.

Markowitz (1952) devised what has become known as Modern Portfolio Theory (MPT). This theory aims to find efficient portfolios of assets. An efficient portfolio can be defined as a portfolio which, for a given expected return, has the lowest risk, or for a given risk has the highest expected returns.

These ground breaking papers in actuarial science and financial economics, both dealing with the selection of asset portfolios, led ultimately to what is termed asset-liability modelling (ALM). A comprehensive review of ALM is provided by Ziemba & Mulvey (1998).

2.1.2 Developments

2.1.2.1 *Modern Portfolio Theory*

The original Markowitz (1952) model proposed that an efficient set of portfolios can be found using the means and covariances of returns, which are estimated from past data. The efficient set of portfolios is often referred to as the efficient frontier when plotted in mean-variance (of return) space.

Sharpe (1963) reduced the information and calculation required to find the set of efficient portfolios in the Markowitz model by developing the market model (originally called the diagonal model) which relates the expected return on an asset to the return on a market index. The relationship is defined by the beta of the asset, and this beta is often described as a measure of 'volatility' or 'market sensitivity' or 'systematic risk'. An extension of this model is the Capital Asset Pricing Model (CAPM) developed by Sharpe (1964). The CAPM is derived from a series of assumptions and relates the expected return on an asset to the risk-free return and a market index via its beta.

The market model and CAPM can be described as single-index models. Much research has been conducted into multi-index models. However, Elton & Gruber (1995) argue that while complex models usually describe historical correlation better, they often contain more noise than predictive information, and do not necessarily outperform simpler models. An example of a multi-index model is the Arbitrage Pricing Theory (APT) developed by Ross (1976, 1977).

Given a model of returns, various methods can be used to construct efficient portfolios. A review of these methods is given in Moore (1972). In MPT, an investor would select the efficient portfolio which best suited their risk profile. Utility theory is often applied to the selection of a single efficient portfolio

from an (infinite) set of efficient portfolios. The measure of risk is typically taken to be the variance of return, although a number of other measures exist such as semi-variance (Markowitz (1952)). The process of identifying efficient portfolios is often referred to as optimisation, or mean/variance optimisation in the original Markowitz case. Modern computing power means that the computational problems of finding efficient portfolios have been largely overcome. More complex methods are also now possible, such as dynamic stochastic optimisation as outlined in Dempster et al. (2003), although mean/variance optimisation remains the most widely used method.

2.1.2.2 Actuarial Asset-Liability Modelling

There is no direct reference to the liabilities of investors in MPT. By introducing liabilities as a variable, Wise (1984a, 1984b, 1987) showed how MPT can be applied in actuarial work. Wise (1984a) shows how the variance of surplus can be minimised given a set of liabilities and a choice of assets for which the distribution of returns is known. Wilkie (1985) and Wise (1987) described a technique called PEV optimisation which is used to determine an efficient set of portfolios of assets which maximise return for a given starting value of assets and level of risk. Smith (1996) suggests simpler methods are usually a good approximation to more complex optimisation problems.

Increasingly, the complexity of models has meant that analytic solutions are often impossible to obtain, but by performing a large number of simulations of a model, parameters can be estimated and solutions found. The use of simulations means that it is not necessary to work only with means and variances, but with the full distribution of variables without having to make simplifying assumptions.

ALM theory was first applied to the choice of an efficient asset portfolio for an institutional investor (such as an insurance company or pension fund) with defined liabilities, but is now applied in a number of other areas, which are discussed below. Whereas modelling liabilities has always been a core actuarial skill, complex modelling of assets and the links between assets and liabilities is a more recent addition. The development of stochastic investment modelling is discussed in section 2.2

2.1.3 Uses of Asset-Liability Modelling

2.1.3.1 Asset Allocation

The principle area where ALM has been applied is in determining a portfolio of assets which meets specified criteria. Clark (1992) describes how ALM can form part of the process in determining a long-term investment strategy, and many consulting actuaries use ALM when advising pension funds on asset allocation. Many insurance companies also use the techniques. Thomson (2003) illustrates how ALM, when used in combination with utility theory, can assist in the investment choices of members of defined contribution pension plans.

The strategic asset allocation is typically one of the key determinants of the risk profile of a fund, and usually forms the basis for guidelines given to investment managers, and as a benchmark for monitoring and comparing investment performance.

2.1.3.2 Reserving and Valuations

The Maturity Guarantees Working Party (1980) investigated the reserves required for maturity guarantees given to life insurance policyholders whose policies were backed by volatile investments such as equities. The party concluded that the most suitable method for determining reserves was stochastic modelling. Thomson et al. (1995) also argue that ALM is an appropriate technique when reserving for guarantees and options in life assurance. Examples of these are guaranteed maturity values, guaranteed surrender values, and options to convert lump-sums into annuities at guaranteed rates. These methods also have application in other areas, and if the assets and liabilities of a fund are mismatched in some sense, then ALM can be used to determine an appropriate mismatching reserve. Wilkie et al. (2003) discuss an ALM approach to reserving for guaranteed annuity options. Mehta (1992) shows how ALM using CAPM can be used to find the appraisal value of a life insurance company.

2.1.3.3 Pricing

Just as ALM can be used to reserve for options and guarantees, it can be used to price these, for example, stochastic modelling can be used for pricing options in life insurance. Thomson et al. (1995) describe a stochastic approach to profit-testing new life insurance products which provides information not only about expected profits, but also about the variability of profit i.e. the riskiness of the product. This approach can also be applied in general insurance and other areas. Thomson (2005) shows that when liabilities cannot be fully hedged in the market, and hence a fully hedged asset portfolio is not available, stochastic modelling is required for pricing the liabilities.

2.1.3.4 Other Uses

Smith (1996) argues that the benefit of ALM is in finding optimum capital structure and in capital management. Clark (1992) argues that ALM has more value as a general management tool than just the specific uses described above because ALM aids in the understanding of how a fund operates and what factors affect it, and Kemp (1992) describes ALM as “an excellent tool for helping senior management understand the implications of their decisions”. Haberman et al. (2003) illustrate a stochastic approach, including the use of ALM, to managing defined benefit pension schemes. Dempster et al. (2003) describe how ALM can be used with dynamic techniques to dynamically manage funds with complex liabilities, and argue that the approach has potentially wide applications.

Roff (1992) points out that due to the highly complex nature of with-profit life insurance funds, ALM can be very useful in managing them. The complexity of these types of fund is due in part to the interdependence of investment and bonus strategies, and Wilkie (1987) applies the Black-Scholes option pricing model to the bonus policy of life offices. The liabilities of a general insurance fund tend to be relatively more uncertain than those for a life insurance fund. Consequently, the modelling of liabilities in ALM is given more focus.

Apart from gaining a better understanding of how a fund operates, ALM can be used to identify some of the risks which a fund faces. Ennis & Dorrington (1994) argue that ALM can play an important role

in risk management, particularly in the management of investment risk. Hardy (1993) shows how stochastic simulation provides better information on the risks faced by a life insurance company than traditional sensitivity analysis. By examining the results of an individual simulation in an ALM exercise which resulted in (for example) the insolvency of the fund, the set of circumstances which led to the insolvency can be identified. This is a similar approach to that described in Clarkson (1996).

Thus, ALM can help to identify, understand and quantify the factors affecting:

- the solvency position of an insurance fund,
- the contribution rate and funding level of a pension fund,
- the bonus rates of a with-profit life insurance fund, and
- the profitability of insurance products,

which provides valuable information for the management of life insurance, general insurance and pension funds.

2.2 Stochastic Investment Models

Lee & Wilkie (2000) provide a comprehensive review of stochastic investment models, covering most published models. Arguably, the most influential papers internationally were those of the Maturity Guarantees Working Party (MGWP) (1980), the original Wilkie model as described in Wilkie (1986), and the extended Wilkie model as described in Wilkie (1995). These are discussed in more detail below. In South Africa, the first published model was that of Thomson (1996), which is also discussed in more detail below.

In Australia, Carter (1991) described an autoregressive model of Australian markets. Lang (1998) discusses Australian models, including the Capital Adequacy Sub-Committee, Harris and Cairns (1999) models. The Capital Adequacy Sub-Committee (1992) model is similar to the MGWP model but extended to property, fixed interest stocks and cash. The Regime Switching Vector Autoregressive Model (RSVAM) of Harris (1996) replaced the Harris (1995) model, and models GDP growth, inflation, bond yields and equity prices. This model uses mixture distributions where asset returns can be in two possible states (regimes), one where returns are positive and follow a certain distribution and another where returns are negative and follow a different distribution. In each period there is a transition probability of moving (switching) from one state (regime) to another. The model is further updated in Harris (1999).

Dyson & Exley (1995) describe a cointegrated model based on the rational expectations hypothesis. Two (not necessarily stationary) series are described as cointegrated if some linear combination of them can be represented as a stationary process.

Smith (1996) describes a number of types of stochastic model, the simplest of which is a random walk model where real returns are log normally distributed and follow a random walk. Smith categorises models such as Wilkie and Thomson broadly as autoregressive models which are stationary models

where variables revert to a long-term mean. Clarkson (1996) argues that linear autoregressive models are “highly misleading and potentially dangerous representations of real world financial behaviour” because they understate the risk. Smith describes a Jump Equilibrium Model in which asset prices are modelled as a compound Poisson process with infinitely small jumps occurring with infinite frequency and finite jumps occurring with finite frequency. The Jump Equilibrium Model is applied to UK cash (including the term structure of interest rates), inflation, equities, property, and bonds. Huber (1998) provides for a more detailed description of this model and its properties. Hibbert, Mowbray & Turnbull (2001) produced a model as an alternative to the Wilkie model, which included a regime-switching component for equity returns.

The models of Yakoubov, Teeger & Duval (1999), and Dempster et al. (2003) are closely related to the Wilkie model, whereas Whitten & Thomas (1999) describe a non-linear model.

2.2.1 Maturity Guarantees Working Party (MGWP)

The Maturity Guarantees Working Party (1980) was established to recommend bases of reserving for investment performance guarantees under investment-linked individual life assurance contracts. The Party’s recommended method was that reserves be set at such a level that the probability of ruin was less than a specified value and that these probabilities be obtained through stochastic modelling of assets and liabilities.

The Party considered four alternative direct models for equity returns, as well as a model for dividend and dividend yields that modelled returns indirectly, and selected the latter approach. The Party specified a random-walk model with a non-zero mean for dividends (described as ARIMA(0,1,0) in the methodology of Box & Jenkins (1976)) and an ARIMA(1,0,0) autoregressive process for dividend yields. The Party argued that these models produced reasonable results and fitted both UK and USA data. Finkelstein (1997) shows how the MGWP model can be modified so that residuals follow a general stable distribution, rather than a normal distribution.

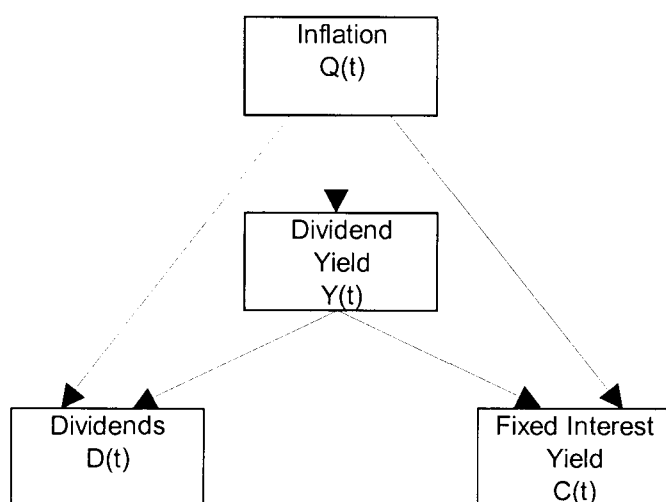
2.2.2 Wilkie Model

2.2.2.1 Original Wilkie Model

Wilkie (1986) aimed to produce a “minimum model that might be used to describe the total investments of a life office or pension fund” which modelled inflation, shares (equity) and fixed interest securities. Wilkie developed this model from the MGWP model for share prices and yields and Wilkie (1981) inflation model.

Share prices are modelled indirectly through dividends and dividend yields, as was the case with MGWP. The model is driven by inflation, so shares and interest rates are dependent variables. Interest rates are represented by a single variable, not as a term structure, so the original model does not contain short-term interest rates. Property and overseas markets were also excluded. The model has a “cascade” structure, as illustrated in Figure 2.2.1.1.

Figure A - Structure of the Original Wilkie Stochastic Investment Model



Specifically, inflation is modelled as a first order autoregressive model or AR(1), where inflation in one year depends on inflation in the previous year. Dividend yields are modelled as a first order autoregressive process and are also dependent on the current level of inflation, higher inflation leads to higher dividend yields. Dividend growth depends on the current level of inflation, the exponentially weighted average of past inflation, and the residual from the dividend yield model. The inclusion of an exponentially weighted average of past inflation means that the model has “unit gain” so that any increase in retail prices will eventually lead to the same increase in dividends. The residual from the dividend yield model is included in the model since it was found that “share prices (and hence yields) to some extent correctly anticipate changes in dividends”. The fixed interest yield is made up of expected future inflation (which is modelled as the exponentially weighted average of past inflation), and a real yield (which is modelled as a third order autoregressive process also dependent on the residual from a dividend yield model). The estimates for the parameters of the Wilkie model were obtained from data over the period 1919 to 1982.

A Working Party was set up under the Financial Management Group (FIMAG) of the Institute of Actuaries in 1989 to review the Wilkie model, and the Party’s finding were reported in Geoghegan et al. (1992). The Party described the model as neither a pure statistical (economically atheoretical) or econometric model (based on economic theory). The Party recommended that only post-1945 data be used in estimating parameters for any model because it was argued that fundamental changes had affected the economic processes generating the data.

It was concluded that no simpler model for inflation could provide a better fit of the data than the Wilkie inflation model, but there was evidence that the residuals are not independent and non-normal. The inflation modelled was described as “too tame” as it did not allow for bursts of high inflation, large, irregular price shocks, and the non-normality of residuals. The Party found that UK inflation had significant auto-regressive conditional heteroscedastic (ARCH) effects (non-constant variances conditional on the past, but constant unconditional variances), and therefore considered the

introduction of an ARCH component to the inflation model. However, because the variance of an ARCH model is constant over the long term, ARCH effects are not required if the model is to be used for long-term projections.

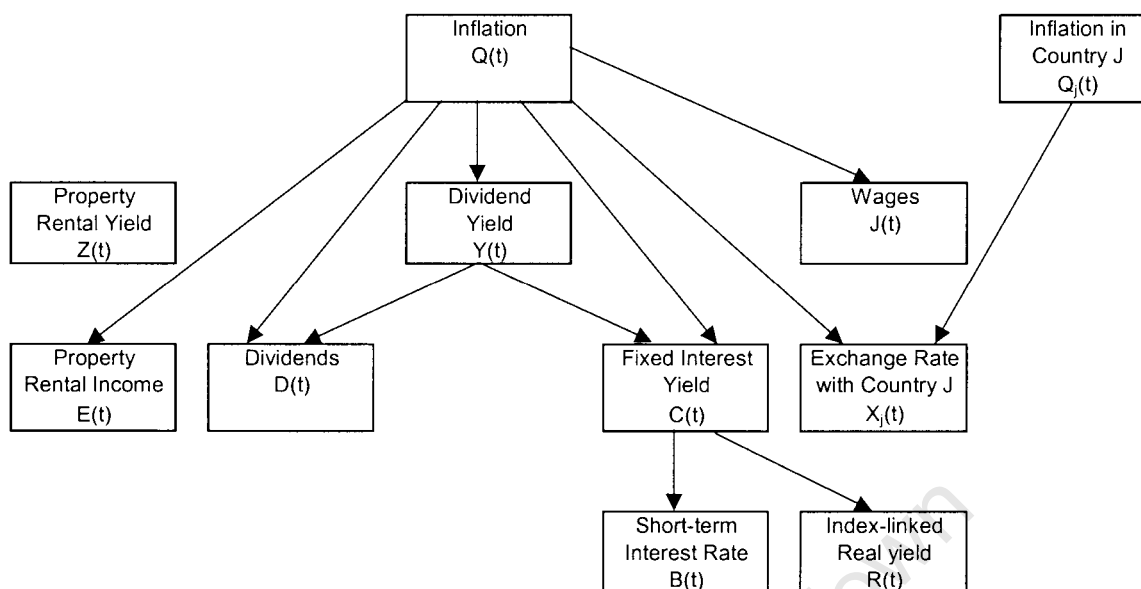
The Party considered using mixture distributions (where two distributions are used to model a single variable) for dealing with random shocks, but concluded that identifying such models was difficult and may again have little effect in long-term projections. The Party also criticised the symmetrical distribution for inflation which has “a significant probability of negative inflation” which “might be unlikely”, and considered using an empirical distribution or alternative non-symmetrical distribution, but again concluded that neither approach would have much effect on the results in long-term projections. However an alternative model for inflation, developed by Clarkson (1991), was presented. Also, in an appendix of the paper by Wilkie, ARCH effects are included in the Wilkie inflation model.

A major concern expressed over the Wilkie model (and any other stochastic model) was whether the parameters were stationary, and in conclusion, the Party neither recommended nor rejected the Wilkie model for use in asset-liability modelling but concluded that more research was required in the area.

2.2.2.2 *Extended Wilkie Model*

Wilkie (1995) published a large amount of research related to the original Wilkie (1986) model. The model was extended to include wages, short-term interest rates, property, index-linked stocks and exchange rates. Figure B illustrates the structure of the extended Wilkie model. Data from a number of countries was examined and alternative models were considered. Wilkie argued that the short-term properties of the Wilkie model are very similar to those of financial economic models (specifically the random walk model), but long-term properties are different.

Figure B - Structure of the Extended Wilkie Stochastic Investment Model



Wilkie showed that UK inflation experience subsequent to the publication of the original model had been adequately explained by the model, although the variability of inflation was lower than predicted. Parameters of the inflation model were estimated for different periods, and it was found that residuals were positively skewed and significantly non-normal. Data from very long periods suggested a higher order autoregressive model may be appropriate, and monthly observations provided evidence of seasonality not seen in annual data. When the model was fitted to data from other countries, the autoregressive parameters were quite similar, although the mean inflation rate often differed substantially between countries. Wilkie also noted that there was some connection between high autoregressive parameters and low standard deviation of residuals, as well as between low average inflation and low variability of inflation. Further investigations into ARCH effects were made, and Wilkie's ARCH model described in Geoghegan et al. (1992) was expanded, and showed a significantly better fit to the data.

Wilkie's tests found that there was no significant cointegration between prices and wages. Wilkie investigated vector autoregressive (VAR) models for wages (models where there is a two-way dependency between the variables, i.e. prices can depend on wages as well as wages on prices) and also fitted the AR(1) inflation model to wages. The proposed model for wages relates wages to inflation.

Wilkie compared his model of dividend yields to a pure random-walk model, and found that as the period between observations is decreased (for example by using monthly or daily data), the estimate for the autoregressive parameter gets closer to one (if the parameter equals one, then dividend yields follow a random walk and are non-stationary). Wilkie argues that over the short term, dividend yields (and hence share prices) may appear to follow a random walk, but do in fact follow an autoregressive process. Wilkie found significant cointegration between dividends and share prices, and therefore argued dividend yields should be modelled as a stationary series. ARCH effects were found, in that

the variance of inflation was higher when dividend yields were higher or lower than usual, but this was not incorporated in the model. When data from other countries is examined, a fairly large range in the estimated parameters was found.

Wilkie argued that the model for share dividends was in some sense efficient, in that share prices react in advance to changes in dividends, and found that the experience subsequent to the parameterisation of the original model was consistent with the model, although dividend growth was in excess of inflation for the period. Wilkie also found that dividends and retail prices were cointegrated, that there was some evidence for ARCH effects in the dividend series. Wilkie argues that the model for share prices is very similar to a random walk in the short term, which is the standard efficient market model for prices, and only differs from it in the long term.

Wilkie argues that the original model for long-term interest rates takes account of the Fisher (1907, 1930) effect where the yield is made up of a real component and a component representing expected future inflation. Wilkie found that experience subsequent to the fitting of the original model for long-term interest did not fit the model well, particularly that the standard deviation had been higher than estimated, and therefore suggested that the standard deviation parameter was increased.

Wilkie described a model for short-term interest rates, where the interest rate is modelled such that the difference between the logs of short term and long term yields is a stationary AR(1) process. While Wilkie acknowledged that a complete model for interest rates requires a yield curve to connect the short-term and long-term rates, it was argued that none of the existing yield curve models satisfactorily combined nominal yields, real yields and inflationary expectations in a single model.

Wilkie argues that property is similar to equity in many ways, and therefore proposes a model for property returns based on the model for share returns, with rental yields akin to dividend yields, and rental income to dividends. A model for index-linked stocks is also described, where the the real yield is modelled as a first order autoregressive process.

Wilkie incorporate exchange rates into the extended model. Wilkie's proposed model is based on purchasing power parity (PPP) where in the long term the exchange rate is solely determined by the purchasing power of the currencies. The exchange rate can depart from PPP in the short term, but since the model is an AR(1) process with mean zero, the exchange rate will tend back to the PPP rate.

2.2.2.3 Reviews of the Wilkie Models

Kitts (1990) notes that residuals from the original Wilkie inflation model are significantly non-normal and non-independent. However more concern is expressed about the estimation of long-term means as Kitts argues that the structure of the process appears to change over time. Clarkson (1991) finds positive skewness of residuals in the inflation model of the Wilkie model, and suggests it can be removed by using a non-linear inflation model.

Thomson (1996) argued that the extended Wilkie model did not adequately describe the South African economy. Clarkson (1996) argues that the non-linearity of UK inflation restricts applicability of the extended Wilkie model, and that the inclusion of ARCH effects is inadequate to fully explain observed

inflation. Smith (1996) argues that the extended Wilkie model “falls between the efficient market of the random walk model and the chaotic model where returns are totally predictable”. As such, Smith argues that the model describes a market which is inefficient (assets are mispriced from time to time) but without arbitrage opportunities, so that, while some strategies may have abnormal expected returns relative to the risk entailed, none of these opportunities are totally riskless.

Huber (1997) aimed to provide a comprehensive review of Wilkie’s extended model. Huber’s review covers the methodology used to derive the model, its theoretical soundness, and its goodness of fit to empirical data (including whether parameters are constant over time).

Huber notes that the Wilkie model was derived from both economic theory and observed data, but that some theory was rejected such as the assumption that markets are efficient. If residuals were found to be autocorrelated then a higher order AR model was fitted. Huber argues that it is “not obvious whether (this approach) leads to an improved model or simply conceals the real problems”. A brief description of Huber’s findings are given below.

Wilkie’s model for long term interest rates incorporates the Fisher (1930) relation, where the yield is made up of a real component and a component representing expected future inflation. Expected future inflation is modelled as the exponentially weighted moving average of past inflation. Huber argues that this contradicts the rational expectations hypothesis which states that investors do not knowingly make systematic ex ante forecasting errors. To be consistent with this hypothesis, then the model for expected future inflation would need to be constrained to be consistent with the price inflation model. Huber also questions the hypothesis that inflation is a stationary process, and cites many authors who argue that it is not stationary.

An implication of the efficient market hypothesis (EMH) is that security returns cannot be forecast. Huber notes that the Wilkie model is not inconsistent with this in the short term as it is virtually the same as a random walk model. Huber notes that some studies have shown that share prices are predictable in the long term, and the Wilkie model is consistent with this too. Additionally, the term in the equity model which allows the price of shares to anticipate future dividends is consistent with the EMH. However, Huber argues that some aspects of the model imply risk-free excess returns can be made by switching between equities and bonds, which is not consistent with the EMH, or supported empirically. Huber also argues that the parameters suggested by Wilkie imply that risk (as measured by standard deviation of returns) and returns for property versus equity, and index-linked bonds versus fixed interest stocks, are inconsistent with portfolio theory (MPT).

In the empirical review, Huber argues that there is not convincing evidence that the parameters in the inflation model are constant over time. Wilkie dealt with the non-normality of residuals by fitting ARCH effects in the inflation model, which Huber notes describe the data better than the original model. Huber found that the residuals in the wage model were not to be independent and identically distributed, but argues that the vector autoregressive model for wages and inflation discussed by Wilkie appear to provide the most promising price and wage inflation models.

Huber questions whether the parameter linking dividend yield with current inflation is constant over time, and argues that without 2 influential values in the data (years 1940 and 1974), the parameter is not significantly different from zero. Huber suggests that the share dividend model is not empirically adequate because the residuals are not independent and identically distributed and the unit gain parameter is not significantly different from zero. Huber also argues that the parameters are not constant over time, and that the unit gain effect may not be appropriate.

Huber finds that the residuals from the long-term interest rate model are correlated with those of the retail price model, the variance of the residuals is not constant over time, and that the parameter relating the yield to the residual from the dividend yield model is not significantly different from zero when a single influential value (year 1974) is excluded from the data. These results lead Huber to conclude that the long-term interest rate model is empirically inadequate.

Huber finds the models for short-term interest rates and index-linked yields to be empirically adequate, but advises caution on the use of the index-linked model given the limited data on which it was derived. Regarding the property model, Huber suggests that the autoregressive parameter in the property yield model may not be constant over time, and, as for the dividend model, questions whether the unit gain parameter is significantly different from zero.

In their review of the inflation model in the extended Wilkie model, Chan & Wang (1998) suggest a modification to overcome the significant positive skewness, non-normality and leptokurtic (fat-tailed) distribution of residuals. Chan & Wang use the Chen & Lui (1993) method to identify significant outliers in the UK inflation data. When these outliers are incorporated into the model, the model has a smaller mean, autoregressive parameter and residual standard deviation, and the non-normality of residuals is eliminated. Chan & Wang also find significant ARCH effects, but introducing these effects into the original Wilkie model does not eliminate the positive skewness and fat-tailed distribution of residuals, and adding ARCH effects to their outlier-adjusted model is found to be insignificant.

2.2.3 Thomson Model

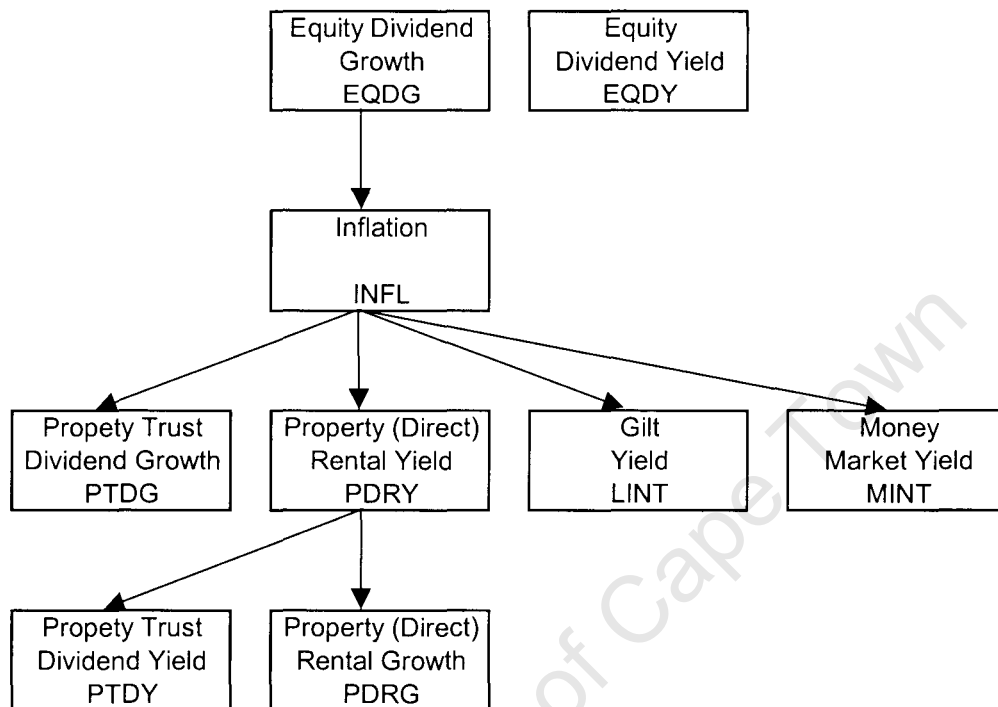
The original Thomson model was published in Thomson (1994), however, all references here are made to the British Actuarial Journal publication, Thomson (1996).

Thomson questions whether the original Wilkie model, which is based on UK data, is applicable to other economies, and found the model did not adequately explain South African data, as had Claasen & Huber (1992) and Claasen (1993). Thomson therefore proposed an alternative model for inflation, long-term interest bearing securities, money market instruments, equities and property. Thomson derives the model using the Box & Jenkins (1976) autoregressive integrated moving average (ARIMA) approach to time series modelling. Thomson did not include exogenous variables in the model, arguing that the effect of such variables is likely to be short-lived and/or insignificant, and that they may add unnecessary complexity to the model.

Thomson takes a four stage approach when determining the structure of the model. First, variables are 'pre-whitened', in other words they are transformed into univariate ARIMA processes, and cross-

correlations between variables and suitable input variables are then determined. Secondly, long-term and any 'unit-gain' relationships are identified. Thirdly, Thomson incorporates relationships previously identified, and finally parameters are estimated and tested for significance. This methodology led to the structure of model illustrated in Figure C.

Figure C - Structure of the Thomson (1996) Stochastic Investment Model



2.2.3.1 Reviews of the Thomson Model

Maitland (1996) reviewed the Thomson model from both a statistical and economic perspective. Using simulations, Maitland estimated the mean, standard deviation and skewness of nominal returns of each variable, together with the correlation between variables. Maitland found that whereas equity returns have historically been higher than property returns (both direct and trusts), the model forecasts the reverse. The forecast standard deviation for nominal equity returns is larger than that for property, and larger than the historic standard deviation. Maitland found the same pattern when real returns are investigated. A number of forecast correlations (of both nominal and real returns) are also found to differ considerably from their historical values, for example the real return for fixed interest securities is zero historically, but forecast to be positive 4%.

Maitland notes the mean-reverting nature of the dividend yield model means that an equity market fall is forecast if the current yield is below the asymptotic mean (as was the case in 1993). It is questioned whether the asymptotic mean is appropriate. Maitland notes that historically, dividend yields have trended downwards, so an appropriate mean may be lower than the historic mean. Maitland presents an alternative ARIMA(0,1,0) model to the AR(1) model for dividend yields.

More generally, Maitland argues that if the true long-run (asymptotic) mean is different from the historic mean, then ARIMA modelling may be inappropriate, and questions the method of testing the significance of the unit-gain functions. Maitland describes an alternative formulation of the model, whereby the parameters can be estimated using maximum likelihood, and some variables are found to be insignificant in formulation. Maitland notes that variables that are trended over time can exhibit spurious correlations, and argues that partial correlations, which eliminate the effect of time, are better suited for model identification.

By using an alternative pre-whitening function, Maitland shows that there is significant (negative) correlation between dividend yield in one year and dividend growth in the following year, and argues that there are both mis-estimations and spurious correlations between returns introduced in the Thomson model. Additionally, Maitland shows that a number of parameters are unstable over time.

Howie (1997) also points out the discrepancies between forecast and historic means in the Thomson model. Howie argues that the modelled property returns, relative to other returns, are inconsistent with standard financial economic models, and makes the case for including wages and economic growth as explanatory variables. Howie also argues that the indices used to parameterise the property model are inappropriate, and that structural changes in the South African economy make historic bond data unreliable.

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3. Approach to Development of a Stochastic Investment Model

3.1 Model Evaluation Criteria

A stochastic investment model is by definition a statistical model. However, it is also a model of economic, or more specifically financial, time series data, and must therefore contain elements from both economics and statistics. The Maturity Guarantees Working Party (1980) model (MGWP) and the Thomson (1996) model are best-fit statistical models where the criteria for choosing the final model was that the model best explained the historical data. By contrast, Smith (1996) set out to construct a model based on financial economic theory. The original and extended Wilkie (1986, 1995) models are somewhat of a compromise between a pure best-fit statistical model and one based on theory because Wilkie's stated aim was to develop a model which was both an adequate representation of past history and based on plausible economic and investment theory.

Geoghegan et al. (1992) describe two types of model: a statistical model and an econometric (i.e. economic) model. Both types of models have both advantages and disadvantages, and in practice most models are compromises between the two approaches. Whether a model is appropriate often depends on what it is being used for. Typically, actuarial applications call for modelling the long term rather than the short term. For this reason Wilkie (1995) argues that a model should adequately model the long-term behaviour of variables, and not necessarily give the best short-term estimates of them. The 'funnel of doubt' or variability over time is more important, Wilkie argues, than estimates of values. However, there is danger in focussing on the long term without worrying about the short term, as famously argued by Keynes (1923): "In the long-run we are all dead. Economists set themselves too easy, too useless a task if in tempestuous seasons they can only tell us that when a storm is long past the ocean is flat again". Ideally, the short term and long term behaviour of models should be consistent.

Practitioners typically use asset-liability modelling more for understanding and managing risk than forecasting the future. Lane (1995) argues that a model which is a poor predictor of the future is not necessarily a poor model. For example a deterministic model (making a single rate assumption about investment return) is a very unreasonable model of the future, but a perfectly acceptable model in many applications. Although Thomson et al. (1995) found that in most practical applications, the choice of model did not make a significant difference to the results, others, such as Smith (1996) and Lang (1998) argue that different models can produce different results. Huber (1997) developed four criteria for the evaluation of a model, which are, in order of importance:

- Consistency with economic theory
- Goodness-of-fit

- Parsimony
- Parameter constancy over time

The first three criteria are discussed in more detail below.

Consistency with economic theory

Centuries of economics research have resulted in an enormous body of work; much of it describing how economic variables behave and interact. When developing a stochastic investment model, such a valuable resource is senselessly overlooked. Indeed, many authors, such as Ennis & Dorrington (1994), Clarkson (1996) and Smith (1996), argue that stochastic investment models need to have some basis in economic theory. Additionally, if a model is based on generally accepted economics, its structure can be explained. For example, in the extended Wilkie (1995) model, dividends depend on inflation, which can be explained in terms of basic economics. By contrast, in the Thomson (1996) model, inflation depends on dividend growth, which is contrary to the economic theory. It is therefore desirable that a model incorporates basic and generally accepted economics.

Goodness-of-fit

If the structure of a model is constrained so as to be consistent with generally accepted economic theory, it is possible (indeed likely) that it will not explain past data as well as a model where the structure is unconstrained. If the structure and parameters of the model are designed to best represent past data, a statistical goodness-of-fit test is likely to indicate a model is superior to a model where the structure is constrained, although not necessarily using out-of-sample data. Even so, a model where the structure is constrained to be consistent with accepted economic theory should adequately explain the data, and parameters should be tested for statistical significance. Additionally, there may be numerous ways of expressing an economic relationship between variables, and statistical tests can be used to determine which structure best explains past data.

Parsimony (Simplicity)

If the aim is to produce a model that is practical and potentially useful, the model needs to be sufficiently easy to use and understand and hence parsimonious. Lane (1995) argues that adding complexity does not necessarily produce a better model and usually only makes the model more difficult to use. Lang (1998) goes further to argue that in many practical applications, solutions to problems can be found using models which are not that complex, provided their limitations are recognised and understood. Wilkie (1986), Clark (1992), Ennis & Dorrington (1994), Thomson et al. (1995) and Booth (1997) also argue for simpler models. A simpler model will almost always be preferable, provided it meets the criteria above, not least because it is easier to communicate to others in practical applications.

Taking the above criteria into account, the steps in constructing a model will be as follows:

1. Describe a simple model based on basic economic theory i.e. express algebraically basic assumptions of how the economy and financial markets operate.

2. Estimate and test parameters.
3. Reject insignificant parameters or parameters whose values contradict the economic theory.

In evaluating the final model, the three criteria used will be consistency with economic theory, parsimony and reasonableness of output. Reasonableness of output is interpreted to mean the modelled variables adequately fit past data and could be understood and explained by a user of the model.

3.2 Variables to be Modelled

The variables originally modelled by Wilkie (1986) are insufficient to properly model the main sectors of the financial markets, notably because the model does not include short term interest rates. At a minimum, the following markets need to be modelled:

- Equity Market – share prices, dividends per share (DPS) and/or earnings per share (EPS)
- Bond (Fixed Interest) Market – benchmark government bond yield
- Money Market – benchmark short term interest rate

Property will be excluded, not because it is an unimportant asset class, but because it has both bond-like and equity-like components in the rental stream and residual value respectively. This means that the theoretical models developed for bonds and equities could be adapted to model property. Additionally, because property is an illiquid asset which is seldom traded, a market price is hardly even known, making property data prone to smoothing and underestimating actual volatility. Exchange rates and non-domestic assets will also be excluded. Foreign asset classes could be modelled in the same way as domestic assets.

Inflation is an obvious explanatory variable, and will be included in the model. Harris (1996) and Howie (1997) argue that economic growth is also an important explanatory variable that should be included, because of its effect on both asset returns (particularly on equity returns) and liabilities. The following explanatory variables will therefore be modelled:

- Rate of Inflation
- Rate of Economic Growth

Wages will be excluded, although it is recognised a model for wages may be required in an ALM exercise where the liability is related to wage growth. For such applications, wage growth could be linked to retail inflation (as in Wilkie (1995)) or to both retail inflation and economic growth.

In summary, the following will be modelled:

- Share prices, dividends per share (DPS) and earnings per share (EPS)
- Government bond yields

- Short term interest rates
- Rate of inflation
- Rate of economic growth

3.3 Choice of Data

3.3.1 Sources and Period of Data

Geoghegan et al. (1992) argue that only post 1945 data should be used in developing a model since they argue that fundamental changes had affected the processes generating the data after this date. If structural changes have taken place, then only the most recent data are relevant.

Data were obtained for three countries, namely the United States, Britain and South Africa. The US and Britain can be classed as developed markets, whereas South Africa can be classed as an emerging market. The relative size of these economies from Eves (1998) is shown in table A.

Table A - GDP in US\$ billion at 31 Dec 1995

Country	GDP in US\$ billion at 31 Dec 1995
United States	7,434
Britain	1,152
South Africa	133

Data for each country for the past 15 years (or the longest period available if less than 15 years) were obtained from I-NET (1999) so that the model could be parameterised separately for each country.

3.3.2 Time Intervals and Transformations

The time interval chosen is restricted by the data, for example GDP is only available quarterly. Most actuarial investigations use annual observations, and there are advantages for using annual data as noted in Thomson (1996) p 772, including:

- Monthly and quarterly seasonality is removed.
- Accounting periods are typically one year, and liability assumptions often are age-related and therefore annual.

Log transformations are also useful since the transformed variables are then additive (when the variables are returns) and asymptotically normal by the Central Limit Theorem. Log transformations are used by Wilkie (1986, 1995), Thomson (1996) and Smith (1996).

The data investigated were therefore annual and log transformed.

3.3.3 Summary of Data

Data obtained from I-NET (1999) is summarised below with full details given in Appendix A. In tables B to E, the means of the untransformed variables equal $\exp[\mu + \sigma^2]$ where μ is the mean of the log transformed variable, and σ is the standard deviation.

Table B - Economic Data: Summary Statistics for the 15 years 30/06/1983 to 30/06/1998

Country	Real Economic Growth			Inflation		
	annual log (1 + real GDP growth)			annual log (1+inflation rate)		
	sample	mean	std dev	sample	mean	std dev
United States	15	0.029	0.016	15	0.033	0.010
Britain	15	0.025	0.020	15	0.044	0.022
South Africa	15	0.015	0.026	15	0.114	0.035

Country	Short Term Interest Rates			Real Short Term Interest Rates		
	annual log (1+money market return)			annual log (1 + real return)		
	sample	mean	std dev	sample	mean	std dev
United States	15	0.061	0.018	15	0.028	0.015
Britain	15	0.093	0.030	15	0.050	0.019
South Africa	15	0.149	0.033	15	0.034	0.041

Table C - Bond Market Data: Summary Statistics for the 15 years 30/06/1983 to 30/06/1998

Country	Term of Bond	Bond Yield			Bond Market Return			Real Bond Market Return		
		log of (1+yield)			annual log (1+return)			annual log (1+ real return)		
		sample	mean	std dev	sample	mean	std dev	sample	mean	std dev
United States	2	5	0.058	0.003	4	0.061	0.005	4	0.037	0.010
	5	5	0.062	0.005	4	0.076	0.027	4	0.052	0.030
	10	6	0.062	0.007	5	0.067	0.077	5	0.043	0.079
	30	16	0.081	0.019	15	0.117	0.146	15	0.084	0.150
Britain	5	2	0.067	0.004	1	0.089	-	1	0.052	-
	10	16	0.088	0.013	15	0.106	0.048	15	0.062	0.061
	30	2	0.062	0.010	1	0.252	-	1	0.215	-
South Africa	2	13	0.138	0.019	12	0.135	0.033	12	0.027	0.047
	5	13	0.144	0.011	12	0.140	0.050	12	0.032	0.057
	10	13	0.151	0.009	12	0.157	0.051	12	0.049	0.057
	30	10	0.151	0.009	9	0.164	0.068	9	0.067	0.069

Table D - Equity Market Dividend and Earnings Data: Summary Statistics for the 15 years 30/06/1983 to 30/06/1998

Country	Index	Yield			Growth			Real Growth		
		log of (1+yield)			annual log (1+growth)			annual log (1+ real growth)		
		sample	mean	std dev	sample	mean	std dev	sample	mean	std dev
United States	Dividends	7	0.024	0.006	6	0.044	0.034	6	0.019	0.034
	Earnings	16	0.059	0.018	15	0.067	0.176	15	0.034	0.179
Britain	Dividends	6	0.036	0.004	5	0.063	0.048	5	0.033	0.051
	Earnings	6	0.053	0.007	5	0.121	0.113	5	0.091	0.113
South Africa	Dividends	16	0.035	0.011	15	0.079	0.064	15	-0.036	0.065
	Earnings	16	0.081	0.020	15	0.100	0.097	15	-0.014	0.104

Table E - Equity Market Data based on FT Actuaries Share Indices : Summary Statistics

Country	Dividend Yield			Dividend Growth			Real Dividend Growth		
	log of (1+yield)			annual log (1+growth)			annual log (1+growth)		
	sample	mean	std dev	sample	mean	std dev	sample	mean	std dev
United States	11	0.027	0.007	11	0.052	0.042	11	0.019	0.039
Britain	11	0.042	0.006	11	0.068	0.069	11	0.025	0.059
South Africa	11	0.030	0.009	11	0.125	0.308	11	0.022	0.310

Country	Equity Return			Real Equity Return		
	annual log (1+return)			annual log (1+ real return)		
	sample	mean	std dev	sample	mean	std dev
United States	10	0.170	0.089	10	0.138	0.093
Britain	10	0.151	0.072	10	0.109	0.080
South Africa	10	0.186	0.213	10	0.084	0.204

3.4 Formulation of Model

3.4.1 Type of Model to be Investigated

A flexible generalised mixture model will be investigated. The Harris (1996) model is an example of a mixture model. The model will consist of a mean-reverting component and a non-mean-reverting component. Special cases of this model will be a vector autoregressive (mean-reverting) model and a random walk model (non-mean-reverting). The modelled volatilities of financial market returns, together with the economic data, are used to fit the economic model. The procedure is similar to the method of moments for statistical estimation.

Specifically, for the financial market model, regression analysis is used to estimate the parameters and the standard deviation of error terms. Similarly, regression is used to estimate risk premiums and the parameters in the dividend and earnings models. The method for estimating the "reversion factor", described later in Section 5, is an iterative one and aims to find a value of the parameters such that the

modelled volatilities are consistent with their historical values, in other words modelled volatilities are broadly equal to their historical values.

The Wilkie (1986, 1995) and Thomson (1996) models are mean-reverting models. Mean-reverting models are typically autoregressive or vector autoregressive (where there is a two-way dependency between the variables). The vector autoregressive component of the model has a maximum order (lag) of one. This restriction is both in the interests of parsimony and because Wilkie (1995) finds that higher orders do not add much to models.

Maitland (1996) criticises mean-reverting (ARIMA) models for being weakly stationary. Other authors have also argued that models should be non-stationary, including Kitts (1990) who observes that the structure of processes appear to change over time, Geoghegan et al. (1992) who argue that stationarity is often the most important limitation of models, and Lane (1995) who argues that it is unreasonable to assume that the underlying relationships do not change with time. The incorporation of a random walk model component will therefore be investigated.

ARCH effects will not be incorporated. Smith (1996) argues that ARCH effects increase short term variance but not the long term variance, and consequently do not have significant impact bearing on optimal decisions. Additionally, Engle (1982) argues that ARCH effects can be interpreted as evidence of misspecification.

3.4.2 Method

A model for inflation, economic growth and short term interest rates will be specified (the economic model) by making assumptions based on economic theory. This model will be parameterised and the parameters tested for significance. A model for bonds and equity will then be specified, based on financial economic theory (the financial market model).

3.5 Practical Issues

Spreadsheets will be produced for the model using Microsoft Excel, as this will allow the widest possible audience to use and test the models. Summary statistics and graphs of distributions will be produced to analyse the model, and following the approach of Smith (1996), will be calculated using simulations for each result. Smith (1996) argues that simulations are required because complex models do not allow analytical computation of results easily.

4. Economic Rationale and Assumptions

4.1 Economic Theory

There are many schools of economic thought, often apparently opposing one another. The theoretical differences between the approaches are typically whether variables affect each other or not. [only barely acceptable – the examiner asked for references not for the statements to be deleted] Typically, competing theories do not contradict each other to the extent that they predict opposite results.

A practical approach is not to pass judgement on any theory up front, but to incorporate any theoretical relationships described in the literature between variables in the initial construction of the model. However, only relationships that are supported by empirical data will ultimately be incorporated into the model

Below is a summary discussion of the theoretical relationships between inflation, economic growth and interest rates.

4.1.1 Inflation

Inflation is an increase in the general level of prices. Bannock (1998) notes three standard theories for the cause of inflation, namely:

1. Demand pull inflation - if there is an increase in aggregate demand for goods and services in the economy, prices will tend to rise.
2. Cost push inflation - if the costs of production, notably wages, are increasing, firms will attempt to increase the prices of goods and services.
3. Increases in money supply - the monetarist view is that money supply growth will lead to higher prices. Real increases in money supply are associated with low or falling real interest rates.

These three reasons are in fact broadly equivalent. An increase in aggregate demand which leads to higher prices may also lead to higher wage demands. Firms can only pass on their higher costs of production through price increases if there is sufficient demand for those goods and services. In addition, price are unlikely to continue to increase (i.e. their being sustained inflation) unless money supply continues to grow to support it.

Bannock (1998) notes that inflationary pressures are often discussed in terms of an output gap. The output gap is defined as actual GDP less potential GDP (defined as the output at full employment). If this is positive it is called an inflationary gap, and if negative a deflationary gap. In essence, inflation is associated with high or excess economic growth, which is why methods employed to reduce inflation also reduce growth. Authorities may respond to higher than desired inflation by reducing government expenditure (which has the effect of reducing aggregate demand) and/or reducing money supply growth (for example by increasing interest rates).

4.1.2 Economic Growth

The aggregate demand in an economy is measured by the gross domestic product (GDP). GDP is made up of consumption, government spending, investment and net exports (exports less imports). Economic growth is measured as a change in GDP, and may be negative. It is meaningful to analyse real changes in GDP i.e. changes in excess of inflation.

Bannock (1998) describes two theoretical components of economic growth:

1. Trend growth - trend growth can be described as the long term, sustainable, non-inflationary growth rate of an economy due to rises in productive capacity. There is some debate as to whether trend growth exists: the neo-classical school argues that adjustment occurs to reach equilibrium, so that balanced growth (where all components of GDP grow at the same rate) or steady-state growth (where variables each grow at constant rate) can occur, whereas the Keynesian view is that the economy does not find a stable growth rate.
2. Business cycle - the business cycle can be described as regular fluctuations in national income or GDP. Several arguments about the causes of the business cycle exist, including the accelerator-multiplier model of economic growth which argues that changes in the level of investment leads to a cyclical pattern of growth. Levels of investment expenditure are stimulated by increases in aggregate demand, technology, high profits, and low interest rates (because investment is often financed by borrowing). Others view the business cycle as a monetary phenomenon where changes in the money supply are argued to play a role in the business cycle. Here, an increase in money supply is thought to lead to an increase in aggregate demand. The alternative view, already mentioned, is that an increase in money supply will only cause an increase in inflation. Again, increases in the real money supply are associated with low or falling real interest rates. Another theory is the real business cycle theory, which argues that real shocks and not fluctuations in aggregate demand cause business cycles. These shocks may be technological, changes in taste, or even wars and weather disturbances.

An economy has some built-in stabilisers which act to dampen the business cycle. For example, when aggregate demand falls, unemployment and welfare benefits paid by the government increase and taxes collected fall. These act to boost aggregate demand. Governments may also employ stabilization policies to control the level of aggregate demand in the economy using both fiscal and monetary policy.

4.1.3 Interest Rates

Interest is the return for lending cash or holding short term deposits, since holding such deposits means delaying consumption. Alternatively, it is the price for borrowing cash. These short term securities are dealt in the money market, which is made up of financial institutions and a central bank or monetary authority. In classical economic theory, aggregate saving equal aggregate investment, and these are brought into line with each other through changes in the interest rate. This is a supply-

demand argument where the price is the interest rate. Long term interest rates, or the yields on fixed interest securities, are dealt with in the next section.

The main theory on interest rates has already been discussed in the sections on inflation and economic growth. An additional point to note is that there is not a single interest rate because financial intermediaries charge for their services by adding to the interest rate charged to borrowers and deducting from the interest rate paid to savers. In addition, risk premiums are added to rates to reflect the differences in the ability of borrowers to repay debts.

4.2 Financial Economic Theory

As is the case in economic theory in general, there are competing theories in financial economics. However, as before, this need not preclude incorporating the broad theories of how markets operate into an investment model. Below is a summary of some of the key ideas in financial economics, which will then be expressed as assumptions and incorporated into the financial market model.

4.2.1 Efficiency

Samuelson (1965) and Fama (1965) developed the efficient market hypothesis (EMH) which proposes that share prices reflect all available information and are thus best estimates of the intrinsic value. Hess & Reinganum (1979) argue that if share prices reflect all relevant and available information, then investors in a certain world cannot use information to earn abnormal or excessive returns. In an uncertain world, an investor cannot expect excessive returns. This does not mean that there are no returns, rather that returns are related to the riskiness of the asset. Since the world is uncertain and prices are thus stochastic, an investor can receive excess returns in an ex post sense, even if the EMH holds.

There have been numerous empirical tests of the EMH. All these tests have the problem that they require an assumption as to how prices reflect information i.e. a function mapping 'information' to prices. This means that any hypothesis test is in fact a joint test that assets fully reflect all available information and that the model of market equilibrium (the function) is correctly specified (Fama (1976)). Additionally, tests also generally assume the rational expectations hypothesis, which is discussed below. Clarkson & Plymen (1988) make the same point that several tests of efficiency are invalidated by an unrealistic model. Marsh & Merton (1986) state that most empirical tests have supported the EMH, or more generally supported stock market rationality. However, there can never be irrefutable proof that markets are efficient.

Assuming efficiency in the field of stochastic investment modelling would mean that all investors knew and believed the model, and used the model to price securities. In terms of the EMH, the information available to investors is the current and past values of variables, and the model itself. If the EMH held, investors would use all that information to price assets. The lack of an efficiency assumption is a major criticism by Huber (1997) of the Wilkie (1986, 1995) models which imply that risk-free excess

returns can be made by switching between equities and bonds. To avoid these apparent anomalies, efficiency will be assumed (in the sense described above).

4.2.2 Expectations

Expectations are described by Bannock (1998) as the views held by economic agents on the future behaviour of relevant economic variables. The simplest model of how expectations are formed is termed adaptive expectations. Here, expectations about a specific variable are formed from the past values and direction of movement of the variable. The assumption of adaptive expectations has been largely rejected as being naïve and ad hoc. The model of rational expectations, where economic agents are assumed not to make systematic errors in forecasting variables, is a more accepted model. This means that forecasts are on average correct, and are only wrong in any instance because of random, non-systematic errors. An additional assumption is that the views of economic agents, as reflected in prices, is based on these rational expectations.

In the context of financial markets, the behaviour of economic agents can be taken to be how investors price securities, and rational expectations is thus closely related to the EMH. A stochastic investment model which assumed rational expectations, would require that investors knew and believed the model, formed expectations from their knowledge of the model, and used those expectations to price securities.

4.2.3 Risk Premiums

Risk can be described generally as the potential extent of deviation of the actual return on an asset from the expected return. Several models exist in financial economics which describe the risk-return relationship in more detail, although the basic premise is that the riskier the asset, the higher its expected return. The main models are briefly discussed below.

The Market Model (originally called the Diagonal Model) was developed by Sharpe (1963), and relates the return (R_j) on a security (j) to the return on market (R_m), where the market is a portfolio of all securities weighted by market capitalisation, as follows:

$$E[R_j] = \alpha_j + \beta_j * R_m.$$

The actual return is equal to $E[R_j] + e_j$, where e_j is an error term which is normally distributed with mean zero. The β_j is often referred to as a measure of volatility, market sensitivity or systematic risk. Fama (1968) showed that the average α_j for all securities that make up the market must be zero and the average β_j must be one.

The Capital Asset Pricing Model (CAPM) was developed by Sharpe (1964), Lintner (1965) and Mossin (1966) from a number of assumptions, notably that investors choose portfolios based on the mean and variance of return, have the same time horizon, can lend and borrow at the riskless rate of return, and have identical estimates of the joint distribution of returns on assets. It was also assumed that there were no restrictions on short sales, no taxes and that no individual investor could affect the price. As

described by Jensen (1979) and Cragg & Malkiel (1982), CAPM relates the expected return on an asset to the market return for each unit of risk and the riskiness of the asset (like the Market Model), but additionally to the rate of return on a risk-free asset (R_f). Although it is derived from a completely different set of assumptions, the equation is similar in form to that of the Market Model:

$$E[R_j] = R_f + \beta_j (E[R_m] - R_f).$$

β_j is systematic risk of security j , and defined as $\text{COV}(R_j, R_m) / \text{VAR}(R_m)$. The term $(E[R_m] - R_f)$ can be described as the expected market return per unit of risk, while the term $(\beta_j (E[R_m] - R_f))$ is the risk premium i.e. the reward investors receive for accepting risk. The risk premium is proportional to the amount of systematic risk of the security. The equation describing the relationship between expected return $E[R_j]$ and β_j is called the Security Market Line.

A variation of CAPM proposed by Black et al. (1972) following the results of a study which showed that the estimated Security Market Line did not correspond to that predicted by CAPM, was to replace the risk free return with the expected return on the zero beta portfolio. The zero beta portfolio is the theoretical portfolio with minimum variance which is uncorrelated to the market return.

An important implication of CAPM is that investors are not rewarded for risks specific to individual shares as these risks can be diversified away. If CAPM holds, investors should hold diversified portfolios. CAPM has been criticised both for the unrealistic nature of its assumptions, and the fact that it is not fully supported by empirical studies. However, the model remains key in understanding financial markets.

The single index models described above are all similar in form, and all relate expected returns linearly to risk. An empirical study by Black et al. (1972) found that there was always a highly linear relationship between risk and return, although this relationship did not always conform to that predicted by CAPM. The models and studies have typically measured risk over a relatively short time horizon, on the assumption that all investors have such a time horizon.

Arbitrage Pricing Theory (APT) was developed by Ross (1976, 1977), and it is described by Cragg & Malkiel (1982) as a multi-factor model. In APT, the return on any asset is a linear function of certain factors (indices), and assets differ from one another in the exposure they have to these factors. CAPM is therefore a special case of such a model. The assumption of no arbitrage means that the risk and returns of assets can be replicated by combinations of other assets. The theory does not specify which factors or indices are appropriate. These could be decided either by theoretical argument or quantitative empirical evidence (provided that neither contradicted the other).

Actuarial studies of risk premiums (Thornton & Wilson (1992), Jones (1993), Wilkie (1995b)) have been largely aimed at setting appropriate valuation or pricing assumptions, rather than testing financial economic models. The studies have been mainly empirical as opposed to theoretical in nature.

Although risk is typically defined as the standard deviation of return, several other measures exist. There are several disadvantages of the standard deviation measure. Clarkson (1996), for example,

argues that risk is not symmetrical and is therefore non-normal, so variance (or standard deviation) is an inappropriate measure of risk.

Another problem is the definition of a risk free rate of return. Real returns are usually more important than nominal returns, and a true risk free real asset usually does not exist. A practical approach is to take the interest rate on short-term treasury bills as the risk free rate i.e. to assume that this has a zero risk premium. Other assets, such as bonds and equities, then have a risk premium over this return, which is related to the standard deviation of expected return. Although treasury bills are not truly risk free real assets, it can be assumed the same relationship broadly holds for real returns.

Assuming a zero risk premium on real money market returns does not mean that they are without volatility. It means that the money market is the minimum risk asset, and the risk premiums of other assets are measured as the excess expected return over the expected return from the money market.

4.2.4 Bonds

Bonds are typically fixed interest security issued by central or local governments, companies, banks or other institutions, although variable rate and index-linked bonds exist. Variable rate bonds are similar in nature to money market instruments, and index-linked bonds are not issued in all markets, so this discussion will focus only on fixed interest bonds.

In Section 6.2, the incorporation of information from the index-linked bond market into the model is discussed. Moreover, the proposed model produces projections of real and nominal interest rates, and can easily be used to model index-linked bonds

Bonds differ from money market instruments in the market where they are traded, and their term to maturity. Bonds can have terms of 30 years or more, although very short term bonds or bonds close to maturity behave much like money market instruments.

McCutcheon & Scott (1986) note that an individual bond can be characterised by the nominal amount of stock held, the coupon rate or rate of interest (which is usually paid half-yearly), the redemption price, and the redemption date. The price of a bond can then be calculated by taking the present value of future cash flows (which are the coupons and redemption value) discounted at an appropriate discount rate. Conversely, if the price is known, an implied yield to redemption can be determined. It is this yield which is quoted in the bond market, rather than the price, and the determinants of those yields are discussed below.

4.2.4.1 The Term Structure of Interest Rates

The term structure of interest rates is the relationship between the yield on a bond and its maturity. This relationship is usually analysed by a yield curve, which plots the yields on bonds against their terms to maturity. Smith (1996) notes that most stochastic models do not distinguish bonds by term, and argues that the full term structure of both real and nominal interest rates should be modelled. Several theories explain the term structure of interest rates, or the "shape" of the yield curve. The main theories are:

1. Expectations of future short term interest rates - this argument assumes that borrowing (or investing) for two years should be equivalent to borrowing (or investing) for one year and rolling over the debt after one year. This theory implies that the yield curve contains implicit forecasts of future short term interest rates.
2. Expectations of future inflation - since real interest rates are nominal rates less inflation, the yield curve will also reflect expectations of future inflation. This is termed the Fisher (1907, 1930) effect, where the yield is composed of 2 parts, one real and one representing expected future inflation. Huber (1997) argues that a model for bonds needs to be consistent with an inflation model, if the assumption of rational expectations is to be satisfied. There is often also assumed to be an inflation risk premium in nominal yields to compensate investors for the risk that inflation is different from that expected.
3. Risk premiums and liquidity preference - liquidity preference can be described as the desire of an investor to hold money and short term assets rather than other forms of wealth such as equities and bonds. A consequence of this is that a premium is required on other non-cash assets. Alternatively, a premium on long term yields over short term yields can be seen as a risk premium representing the uncertainty in both expected future short term interest rates and expected future inflation. Tax, regulation and the liability profile of investors may also influence the demand
4. Another factor which affects the yield curve is the public sector borrowing requirement (PSBR). The PSBR is the supply of government bonds, and is related to government expenditure.

Since government expenditure is a component of GDP, the economic model will indirectly incorporate changes in PSBR, either directly through modelled relationships, or indirectly through error terms.

Chaplin (1998) examines the major mathematical models of term structure in financial economics. Four main models are identified. The first model is the Vasicek (1977) model, which relates the long term bond yield to the money market rate (a one factor model). The Richard (1978) model relates the long term yield to both expected inflation and expected real short term interest rates (a two factor model). The Hull & White (1990) model, described as a modified Vasicek model, where deterministic parameters are replaced with stochastic parameters. The fourth model is a general polynomial model, which is a mathematical model which can be fitted to data and used to describe the yield curve. Hull (2006) divides interest rates models into those described as equilibrium (based on economic theory), such as the Cox, Ingersoll and Ross (1985) model, and those described as no-arbitrage, which are based on the current term structure of the yield curve such as the Ho and Lee (1986) model. A wide variety of interest rate models have been developed in recent times.

4.2.5 Equity

Equities, or shares in companies, distribute a portion of net profits/earnings as dividends. Typically, dividends per share (DPS) are less than earnings per share (EPS), although dividends can be paid out of retained earnings from previous years. Retained earnings in any particular year are defined as

earnings less dividends, and are used to finance companies' activities. The ratio of DPS over EPS is termed the pay-out ratio.

4.2.5.1 Earnings and Dividends

There are several factors that influence the earnings of a single company, but there are three main factors which influence the average earnings of listed companies. These are inflation, real economic growth, and interest rates. These are the three variables modelled in the economic model. The revenue of companies will tend to rise with price inflation, since the price of the goods and services sold will increase with inflation. Revenues will also tend to rise with real economic growth, because more will be sold. The expenses of companies will be linked to price inflation, wage inflation and interest rates. The cost of inputs is linked to price inflation, and wage inflation is linked to both price inflation and economic growth. The expenses of many companies will also be linked to interest rates because companies raise capital from both equity and debt, and the cost of debt financing is interest.

EPS is an accounting measure, and not a market number. It is really only an estimate of earnings. DPS, on the other hand, is cash actually paid to shareholders, so there is no subjectivity over its measurement. Dyson & Exley (1995) note that dividends (and earnings) are broadly linked to inflation and economic growth, and argue that some of the discrepancies between real earnings growth and real economic growth are due to changes in distribution of income (between labour and capital). None of the actuarial models reviewed modelled EPS. Thomson (1996) argues that it is not necessary to model earnings, because the model for dividends can be adapted if pay-out ratios change. However, it is not clear how best to adjust the model, and any modification is likely to be subjective.

Wilkie (1986) models dividends as a function of inflation, where there is unit gain from inflation to dividends. By definition, the unit gain should be from earnings to dividends. There are several reasons why there may be no unit gain from inflation to dividends, including measurement error in the inflation index, a mismatch between the costs and revenues of companies and the chosen inflation index, and any redistribution of income between capital and labour.

The major findings in a study of dividends by Lintner (1956) were that managers have a target pay-out ratio, but prefer to smooth dividends and avoid dividend cuts. This smoothing of dividends was also found by Thomson (1996). In the Marsh and Merton (1986) dividend model, dividends are a function of earnings.

Miller & Modigliani (1958) argued that the dividend policy of a company is irrelevant, particularly in how investors value the share. Although it is earnings rather than dividends that are relevant in the valuation of companies, applications of stochastic models often require forecasts of cashflows, so it is preferable to model both earnings and dividends, so that the actual cashflows from equities to an investor can be modelled.

4.2.5.2 Share Prices

The standard model for equity valuation is the dividend discount model, or more generally a discounted future cashflow model. Here, the value of a share is determined by the present value of its future proceeds (Henning (1984)). In principle this is the same as the valuation of a bond. The difference is that a share has no maturity value, and that future proceeds are dividends, which are uncertain.

Campbell & Shiller (1987) found that the dividend discount model was not an empirically good model for stock prices, although their work was based on the assumption of a fixed discount rate which could be the reason for their findings.

If the Miller & Modigliani (1958) argument holds, then whether or not a dividend is paid does not affect the return to the investor. However, any distributable earnings disclosed could be paid out. A discounted future cashflow model can still hold, provided that the share price is the present value of expected future distributable earnings.

The capital invested in a company is what produces future earnings. The value of that capital is not its accounting measure (whether net asset value, or share capital plus retained earnings), but the value of the earnings it can produce in the future.

4.2.5.3 Distribution of Equity Returns

The standard distribution assumed for equity returns is the normal distribution. Fama (1976) notes that this model was developed, independently, by Bachelier (1900) and Osborne (1959). It assumes that price changes from transaction to transaction are independently and identically distributed, and that the transactions are uniformly distributed across time. If there is a large number of transactions per unit time, then price changes per unit time will be the sum of many independent, identical random variables and thus, by the central limit theorem, approximately normal. An analogous argument for returns is that returns are the product of intermediate returns, just as price changes are the sum of intermediate price changes. If successive returns are independently and identically distributed, log returns will be approximately normal since log returns are the sum of intermediate log returns.

Mandelbrot (1963) questioned the hypothesis of the normality of stock returns. He argued that the limiting distribution of sums of variables is a member of the stable class of distributions of which the normal is only a special case. Stable distributions are invariant under addition i.e. if daily log returns have a stable non-normal distribution then monthly log returns will have a distribution of the same type.

Empirical studies such as that by Fama (1965) found that daily returns are significantly non-normal and distributions which are leptokurtic relative to the normal would be more appropriate, i.e. distributions with higher probabilities of values close to the mean as well as higher probabilities of extreme values. Fama (1965) found that monthly returns were closer to normality, and contradicted Mandelbrot's (1963) hypothesis that daily and monthly returns conform to the same type of stable non-normal distribution. Studies by Officer (1976) and Blattberg & Gonedes (1974) had similar results. Wilkie (1995) also found that monthly residuals were leptokurtic, but that the annual series were not particularly non-normal.

In the model, no explicit assumption will be made for the distribution of equity returns.

4.3 Assumptions

4.3.1 Economic Model

The discussion in section 4.1 can be summarised by the following three assumptions:

1. High inflation causes monetary authorities to increase real interest rates, and governments to reduce expenditure, consequently reducing real economic growth.
2. High real economic growth causes inflation to increase, and monetary authorities to increase real interest rates.
3. High real interest rates cause inflation to decrease, and real economic growth to decrease.

4.3.2 Financial Market Model

The discussion in section 4.2 can be summarised by the following nine assumptions:

1. Financial markets are efficient.
2. Investors have rational expectations and price securities based on those expectations.
3. There is no risk premium on real money market returns (short term interest rates).
4. There is a positive and linear relationship between expected real return and risk, where risk is measured as the standard deviation of real return.
5. The yield on a bond is made up of expected future real short term interest rates, expected future inflation, and a risk premium. Expected future real short term interest rates are assumed to be equal to the corresponding forward rates.
6. Real earnings growth is positively related to real economic growth and negatively to real interest rates.
7. Dividends are a smoothed function of earnings, and there is unit-gain from earnings to dividends.
8. Share prices are determined by the present value of expected future distributable earnings, discounted at a rate equal to expected future short term interest rates plus expected future inflation plus a risk premium.
9. Equity returns are independent of the proportion of distributable earnings that are paid out as dividends.

5. Formulation and Fitting of Model

5.1 Formulation

5.1.1 Economic Model

Defining inflation_t , realgrowth_t , and realinterest_t as the log (1+inflation rate), log (1+real economic growth), and log (1+real money market return) respectively, a mixture model can be formulated with a combination of mean-reverting components and components that are not mean-reverting. These features are incorporated by using a factor, $F_{\text{reversion}}$, where $0 \leq F_{\text{reversion}} \leq 1$. The models for inflation, economic growth and interest rates are mean reverting when $F_{\text{reversion}}$ is 1 (provided all the α , β and γ terms are not equal to 1), and are random walks when $F_{\text{reversion}}$ is zero. When $0 < F_{\text{reversion}} < 1$, the models are not mean reverting, and are generally described as autoregressive processes reverting to a mean, but that the mean itself is a random walk. The models are similar in form, and are expressed algebraically as follows:

$$\mu_{\text{inflation},t} = \mu_{\text{inflation},t-1} + (1 - F_{\text{reversion}}) e_{\text{inf},t}$$

$$\begin{aligned} \text{inflation}_t = & \mu_{\text{inflation},t} + \alpha_{\text{inf}} (\text{inflation}_{t-1} - \mu_{\text{inflation},t-1}) \\ & + \beta_{\text{inf}}^0 (\text{realgrowth}_t - \mu_{\text{realgrowth},t}) + \beta_{\text{inf}}^1 (\text{realgrowth}_{t-1} - \mu_{\text{realgrowth},t-1}) \\ & + \gamma_{\text{inf}}^0 (\text{realinterest}_t - \mu_{\text{realinterest},t}) + \gamma_{\text{inf}}^1 (\text{realinterest}_{t-1} - \mu_{\text{realinterest},t-1}) + F_{\text{reversion}} e_{\text{inf},t} \end{aligned}$$

Where $0 \leq \alpha_{\text{inf}} \leq 1$, $\beta_{\text{inf}}^0, \beta_{\text{inf}}^1 \geq 0$, $\gamma_{\text{inf}}^0, \gamma_{\text{inf}}^1 \leq 0$, and $e_{\text{inf},t} \sim N(0, \sigma_{\text{inf}})$,

$$\mu_{\text{realgrowth},t} = \mu_{\text{realgrowth},t-1} + (1 - F_{\text{reversion}}) e_{\text{gro},t}$$

$$\begin{aligned} \text{realgrowth}_t = & \mu_{\text{realgrowth},t} + \alpha_{\text{gro}} (\text{realgrowth}_{t-1} - \mu_{\text{realgrowth},t-1}) \\ & + \beta_{\text{gro}}^0 (\text{inflation}_t - \mu_{\text{inflation},t}) + \beta_{\text{gro}}^1 (\text{inflation}_{t-1} - \mu_{\text{inflation},t-1}) \\ & + \gamma_{\text{gro}}^0 (\text{realinterest}_t - \mu_{\text{realinterest},t}) + \gamma_{\text{gro}}^1 (\text{realinterest}_{t-1} - \mu_{\text{realinterest},t-1}) \\ & + F_{\text{reversion}} e_{\text{gro},t} \end{aligned}$$

Where $0 \leq \alpha_{\text{gro}} \leq 1$, $\beta_{\text{gro}}^0, \beta_{\text{gro}}^1 \geq 0$, $\gamma_{\text{gro}}^0, \gamma_{\text{gro}}^1 \leq 0$, and $e_{\text{gro},t} \sim N(0, \sigma_{\text{gro}})$, and

$$\mu_{\text{realinterest},t} = \mu_{\text{realinterest},t-1} + (1 - F_{\text{reversion}}) e_{\text{int},t}$$

$$\begin{aligned} \text{realinterest}_t = & \mu_{\text{realinterest},t} + \alpha_{\text{int}} (\text{realinterest}_{t-1} - \mu_{\text{realinterest},t-1}) \\ & + \beta_{\text{int}}^0 (\text{inflation}_t - \mu_{\text{inflation},t}) + \beta_{\text{int}}^1 (\text{inflation}_{t-1} - \mu_{\text{inflation},t-1}) \\ & + \gamma_{\text{int}}^0 (\text{realgrowth}_t - \mu_{\text{realgrowth},t}) + \gamma_{\text{int}}^1 (\text{realgrowth}_{t-1} - \mu_{\text{realgrowth},t-1}) \\ & + F_{\text{reversion}} e_{\text{int},t} \end{aligned}$$

Where $0 \leq \alpha_{int} \leq 1$, $\beta_{int}^0, \beta_{int}^1 \geq 0$, $\gamma_{int}^0, \gamma_{int}^1 \geq 0$, and $e_{int,t} \sim N(0, \sigma_{int})$.

Additionally, in order to ensure no circularity, the following must also hold:

- If $\beta_{inf}^0, \beta_{inf}^1 > 0$, then $\beta_{gro}^0, \beta_{gro}^1 = 0$
- If $\beta_{gro}^0, \beta_{gro}^1 < 0$, then $\beta_{inf}^0, \beta_{inf}^1 = 0$
- If $\gamma_{inf}^0, \gamma_{inf}^1 < 0$, then $\beta_{int}^0, \beta_{int}^1 = 0$
- If $\beta_{int}^0, \beta_{int}^1 > 0$, then $\gamma_{inf}^0, \gamma_{inf}^1 = 0$
- If $\gamma_{gro}^0, \gamma_{gro}^1 < 0$, then $\gamma_{int}^0, \gamma_{int}^1 = 0$
- If $\gamma_{int}^0, \gamma_{int}^1 > 0$, then $\gamma_{gro}^0, \gamma_{gro}^1 = 0$.

In the models above, the μ terms can be described as the “means” or underlying rates. They are independent of other variables and follow a random walk when $F_{reversion} \neq 1$. The actual rates follows an autoregressive process when $F_{reversion} \neq 0$, and revert to the mean, μ , where the α terms are the autoregressive parameters. The β and γ terms incorporate the causal links assumed between variables, and the restrictions on these parameters dictate the nature of this causality i.e. whether the relationship is positive or negative.

5.1.2 Financial Market Model

The financial market model specifies the models for risk premiums, bond yields, equity earnings, equity dividends, and equity prices. These are discussed below.

5.1.2.1 Expected Returns and Risk Premiums

Defining $R_{j,t}$ as the log of $(1 + \text{real return})$ on asset j for period t , and based on the assumptions that there is no risk premium on real money market returns, and that there is a positive and linear relationship between expected real return and the standard deviation of real return, then:

$$E[R_{j,t}] = \text{realinterest}_t + \beta_{\text{return}} [\sigma_j - \sigma_{\text{realinterest}}], \text{ where } \beta_{\text{return}} \geq 0.$$

In this equation, $\sigma_{\text{realinterest}}$ and σ_j are standard deviations of realinterest_t and R_j respectively. If asset j is the treasury bill, then $E[R_{j,t}] = \text{realinterest}_t$, which satisfies the assumption that there is no risk premium on these returns. For any asset more risky than this, such as bonds and equities, σ_j will exceed $\sigma_{\text{realinterest}}$. The risk premium on such an asset is equal to $(\beta_{\text{return}} [\sigma_j - \sigma_{\text{realinterest}}])$. The restriction $(\beta_{\text{return}} \geq 0)$ ensures that the risk premium is positive. [you haven't addressed comment about assumption that market is in equilibrium]

5.1.2.2 Bonds

On the assumption that the yield on a bond is comprised from expected future short term real interest rates, expected future inflation, and a risk premium, then, at time zero, the price of bond with an

outstanding term of m years with an annually paid coupon, c , (that has just been paid), will be as follows:

$$\text{bondprice}_{m,c}(0) = c [v_{m,c}(1) + v_{m,c}(2) + v_{m,c}(3) + \dots + v_{m,c}(m)] + v_{m,c}(m).$$

In this equation, $v_{m,c}(t)$ is the discount factor for the cash flow at time t :

$$v_{m,c}(t) = \exp \left\{ - \sum_{s=1}^t (E[\text{realinterest}_s] + E[\text{inflation}_s] + \text{riskpremium}_{m,c}) \right\}.$$

The risk premium of the bond is $\text{riskpremium}_{m,c}$, and is such that the relationship in section 5.1.2.1 is satisfied. The yield to maturity of the bond can be found by solving for i in the equation $\{\text{bondprice}_{m,c}(0) = c [v^1 + v^2 + v^3 + \dots + v^m] + v^m \}$ where $v = (1+i)^{-1}$. The yield will be a weighted average of the combination of the expected future short term interest rates plus expected future inflation plus and a risk premium, where the weights are the cash flows paid by the bond. Note that a bond with a price $\text{bondprice}_{m,c}(0)$ at time zero will have a price of $\text{bondprice}_{m-1,c}(1)$ at time 1, as the outstanding term of the bond will be one year less.

5.1.2.3 Equity

The three aspects of equities that need to be modelled are earnings, dividends and share prices. For a company whose financial year end was exactly six months ago and pays dividends annually, $\text{shareprice}(0)$ is defined as the share price at time zero. $\text{EPS}(-1)$ and $\text{DPS}(-1)$ are defined as the earnings per share and dividends per share disclosed and paid 6 months ago respectively. In general $\text{EPS}(t)$ relates to the year $(t - \frac{1}{2})$ to $(t + \frac{1}{2})$, so earnings occur on average at time t . Similarly, $\text{DPS}(t)$ relates to the year $(t - \frac{1}{2})$ to $(t + \frac{1}{2})$, and are paid at time $(t + \frac{1}{2})$. This means that $\text{EPS}(-1)$ was earned on average at time (-1) and $\text{DPS}(-1)$ was paid at time $(-\frac{1}{2})$. At time zero, $\text{EPS}(0)$ and $\text{DPS}(0)$ are unknown.

Returns on Shares

The nominal annual return on shares, r , made in period t is the solution to the equation:

$$\text{shareprice}(t-1) (1 + r) = \text{DPS}(t-1) (1 + r)^{1/2} + \text{shareprice}(t).$$

This has the solution:

$$(1 + r)^{1/2} = [\text{DPS}(t-1) + \{ \text{DPS}(t-1)^2 + 4 \text{shareprice}(t-1) \text{shareprice}(t) \}^{1/2}] / [2 \text{shareprice}(t-1)].$$

From this equation, the approximate return is:

$$r \approx [\text{shareprice}(t) + \text{DPS}(t-1)/2] / [\text{shareprice}(t-1) - \text{DPS}(t-1)/2] - 1.$$

It was assumed in Section 4.3 that equity returns are independent of the proportion of distributable earnings that are paid out as dividends. Since the return, r , is independent of the amount of earnings that are distributed as dividends, then $\text{shareprice}(t)$ would have been different if $\text{DPS}(t-1)$ had been

different. Defining $\text{shareprice}'(t)$ as what the share price would need to be in order to obtain the same return, r , if all earnings were distributed, i.e. if $\text{DPS}'(t-1) = \text{EPS}(t-1)$, the equation used to find r can be expanded as follows:

$$\begin{aligned} \text{shareprice}(t-1) (1+r) &= \text{DPS}(t-1) (1+r)^{1/2} + \text{shareprice}(t) \\ &= \text{DPS}'(t-1) (1+r)^{1/2} + \text{shareprice}'(t) \\ &= \text{EPS}(t-1) (1+r)^{1/2} + \text{shareprice}'(t). \end{aligned}$$

This has the result that $\text{shareprice}(t) = \{ \text{EPS}(t-1) - \text{DPS}(t-1) \} (1+r)^{1/2} + \text{shareprice}'(t)$. The interpretation of this result is that any earnings that are not distributed are simply reflected in a higher share price. If all earnings are distributed, $\text{DPS}(t-1) = \text{EPS}(t-1)$, then $\text{shareprice}(t)$ equals $\text{shareprice}'(t)$.

Share Prices

It was assumed in Section 4.3 that share prices are determined by the present value of expected future distributable earnings, and that the discount rate equals expected future short term real interest rates plus expected future inflation plus a risk premium. Under these assumptions:

$$\text{shareprice}(t) = E[\text{EPS}(t)] v_{\text{share}}(t) + E[\text{EPS}(t+1)] v_{\text{share}}(t+1) + E[\text{EPS}(t+2)] v_{\text{share}}(t+2) + \dots$$

Where

$v_{\text{share}}(t+r)$ = discount factor for cash flow at time $(t+r)$,

$$v_{\text{share}}(t+r) = \sum_{s=t+1}^{t+r} \exp \{ - (E[\text{realinterest}_s] + E[\text{inflation}_s] + \text{riskpremium}_{\text{share}}) \},$$

and $\text{riskpremium}_{\text{share}}$ is such as to satisfy the relationship in 5.1.2.1.

Similarly, $\text{shareprice}'(t)$ can be defined as follows:

$\text{shareprice}'(t) = E[\text{EPS}'(t)] v_{\text{share}}'(t) + E[\text{EPS}'(t+1)] v_{\text{share}}'(t+1) + E[\text{EPS}'(t+2)] v_{\text{share}}'(t+2) + \dots$, where $\text{EPS}'(t)$ are the earnings that would have occurred at time t if all the earnings that occurred at time $t-1$ had been distributed. Additionally, these relationships only hold if $E[\text{EPS}(t)]$ and $E[\text{EPS}'(t)]$ are greater than one for at least one t .

If it is assumed that expected return is independent of the proportion of earnings that are distributed as dividends, then $v_{\text{share}}(r) = v_{\text{share}}'(r)$ for all r . This necessarily implies that $E[\text{EPS}(r)] \neq E[\text{EPS}'(r)]$ unless $\text{DPS}(r) = \text{EPS}(r)$. This can be further explained by example. Consider 2 firms, identical in every respect, except for the amount of their most recent earnings that will be distributed as in table F.

Table F - Initial position of identical firms with different dividend policy

	Firm A	Firm B
Shares in Issue	3,000,000	3,000,000
Total Earnings (\$)	45,000,000	45,000,000
Market Capitalisation (\$)	900,000,000	900,000,000
Share Price (\$)	300.00	300.00
EPS (\$)	15.00	15.00
DPS (\$)	15.00	7.50

Assume, for simplicity, that the dividends are due to be paid in one year's time and that all parameters are deterministic, notably that the shares are valued on a risk discount rate of 15%. For firm A, the return is comprised 5% from dividends and 10% from capital, whereas it is 2.5% and 12.5% respectively for firm B, so that after one year, the firms will appear as in table G.

Table G - Position of same firms one year later

	Firm A	Firm B
Shares in Issue	3,000,000	3,000,000
Total Earnings (\$)	49,500,000	50,456,250
Market Capitalisation (\$)	990,000,000	1,009,125,000
Share price (\$)	330.00	336.38
EPS (\$)	16.50	16.82
DPS (\$)	16.50	8.41

The returns achieved over the year by shareholders in each firm are identical, but the firms are no longer identical. Firm B has effectively raised more capital by retaining some earnings. Firm A could have achieved the equivalent result (ignoring costs) by issuing new shares to the value of \$19,125,000. The growth in earnings of the two firms is affected identically by inflation, economic growth and interest rates, but firm B's capital grows by 2.125% more than firm A's. This can be interpreted as the growth in the size of firm B, relative to firm A, due to the capital effectively raised through retained earnings.

In the case of firm B, using the notation of the model, and assuming the actual pay-out ratio was 50%, $shareprice(0)$ is \$300.00 and $EPS(0)$ and $DPS(0)$ are \$15.00 and \$7.50 respectively. $Shareprice(1)$ is \$336.38 and $shareprice'(1)$ is \$330.00. $Shareprice'(1)$ is the price of the share if all earnings had been distributed as dividends, so as to achieve the same return. The variables $shareprice'(t)$ and $EPS'(t)$ are purely notional or dummy variables used in intermediate calculations to identify the change in size effect caused by retained earnings.

Earnings Per Share

It was assumed in Section 4.3 that real earnings growth is related to real economic growth and real interest rates. It is reasonable to exclude the 'change in size' effect described above from the model for earnings growth as this effect does not represent real underlying growth in company earnings, but rather a change in capital through retained prior earnings. If earningsgrowth'_t is defined as $\log [EPS'(t) / EPS(t-1)]$, and earningsgrowth_t as $\log [EPS(t) / EPS(t-1)]$, then earningsgrowth_t is the log of (1+earnings growth), including change in size effect. Recalling that EPS(t) occurs on average at time t, then:

$$\text{earningsgrowth}'_t = \text{inflation}_t + \text{realgrowth}_t + \alpha_{\text{earn}} (\text{realinterest}_t - \mu_{\text{realinterest},t}) + e_{\text{earn},t}$$

where $\alpha_{\text{earn}} \leq 0$, and $e_{\text{earn},t} \sim N(0, \sigma_{\text{earn}})$.

This means that $\log(1+\text{earnings growth})$ in any year (excluding the change in size effect) equals $\log(1+\text{inflation})$ plus $\log(1+\text{real economic growth})$ plus an error term which is distributed normally with mean zero. There is also an effect on earnings from short term interest rates. This is only a short term effect as the unconditional expectation of the term $(\text{realinterest}_{t-1} - \mu_{\text{realinterest},t-1})$ is zero. The condition $(\alpha_{\text{earn}} \leq 0)$ means that this effect is negative i.e. high interest rates reduce earning growth. The 'change in size effect' affects all future earnings proportionately, so that at time zero, $\text{earningsgrowth}'_0 = \text{earningsgrowth}_0$, but for all $t > 0$, $\text{earningsgrowth}'_t \neq \text{earningsgrowth}_t$.

Dividends Per Share

The assumptions in Section 4.3 about dividends were that dividends are a smoothed function of earnings, and that there is unit-gain from earnings to dividends. Defining dividendgrowth_t as \log dividend growth ($\log [DPS(t) / DPS(t-1)]$), and recalling that DPS(t) are the dividends relating to earnings EPS(t), and are paid at time $(t + \frac{1}{2})$, then the Wilkie (1995) dividend model can be adapted by replacing the inflation terms with earnings growth terms, and by removing the mean, dividend yield and error terms as follows:

$$\text{dividendgrowth}_t = \alpha_{\text{div}} \text{smoothearnings}_t + (1 - \alpha_{\text{div}}) \text{earningsgrowth}_t,$$

where $\text{smoothearnings}_t = \beta_{\text{div}} \text{earningsgrowth}_t + (1 - \beta_{\text{div}}) \text{smoothearnings}_{t-1}$, $0 \leq \alpha_{\text{div}} \leq 1$, $0 < \beta_{\text{div}} \leq 1$.

The term smoothearnings_t is the exponentially weighted average of past earnings growth. In this model there are two smoothing terms, α_{div} and β_{div} . The greater the value of α_{div} , the more smoothing that occurs. The greater the value of β_{div} , the less smoothing that occurs. By setting $\beta_{\text{div}} = (1 - \alpha_{\text{div}})$, the model can be simplified so that there is only one smoothing parameter as follows:

$$\text{dividendgrowth}_t = \alpha_{\text{div}} \text{smoothearnings}_t + (1 - \alpha_{\text{div}}) \text{earningsgrowth}_t,$$

where $\text{smoothearnings}_t = \alpha_{\text{div}} \text{smoothearnings}_{t-1} + (1 - \alpha_{\text{div}}) \text{earningsgrowth}_t$, $0 \leq \alpha_{\text{div}} < 1$.

This is the simplest model of dividends if they are a smoothed function of earnings, and there is unit-gain from earnings to dividends i.e. any change in earnings is ultimately reflected as a change in dividends. Note that the pay-out ratio for period t is $DPS(t) / EPS(t)$.

5.1.3 An Additional Constraint on the Economic Model

Since listed companies have limited liability, share prices must by definition be ≥ 0 , it is necessary to impose an additional constraint on the model to ensure that this is the case. This is achieved by imposing a constraint that $\mu_{realinterest,t} = \max (\mu_{realinterest,t-1} + e_{\mu_{realinterest,t}}, \mu_{realgrowth,t} - riskpremium_{share} + C_{\mu_{realinterest}})$. The derivation of this constraint is described below.

At time t , due to the mean-reverting component on the model, there can be found n , such that $E[realinterest_{t+n}] \approx E[realinterest_{t+n+1}]$, $E[inflation_{t+n}] \approx E[inflation_{t+n+1}]$, and $E[earningsgrowth_{t+n}] \approx E[earningsgrowth_{t+n+1}]$, for all values greater than $t+n$. This means that

$$shareprice(t) \approx E[EPS(t)] v_{share}(t) + E[EPS(t+1)] v_{share}(t+1) + \dots + E[EPS(t+n)] v_{share}(t+n) + E[EPS(t+n)] v_{share}(t+n) (1+g) / (i-g),$$

where $g = E[\mu_{inflation,t+n}] + E[\mu_{realgrowth,t+n}]$, and $i = E[\mu_{realinterest,t+n}] + E[\mu_{inflation,t+n}] + riskpremium_{share}$.

At time t , $EPS(t)$ is unknown while $EPS(t-1)$ is known. It is interesting to note that in the model for equities, there is uncertainty about the future, but also about the past. Due to reporting delays, the EPS and DPS are always out of date. In the model, share prices react when actual EPS is different from expected. This means that the model is 'semi-strong' efficient, in that insider trading profits are still possible (if EPS is known before it is published).

This means that $(i - g)$ must be > 0 , therefore using the definitions of i and g above:

$$E[\mu_{realinterest,t}] + E[\mu_{inflation,t}] + riskpremium_{share} - (E[\mu_{inflation,t}] + E[\mu_{realgrowth,t}]) > 0, \text{ which implies that}$$

$$E[\mu_{realinterest,t}] + riskpremium_{share} - E[\mu_{realgrowth,t}] > 0, \text{ which implies that}$$

$$E[\mu_{realinterest,t}] > E[\mu_{realgrowth,t}] - riskpremium_{share}.$$

This constraint can be incorporated into the economic model by modifying the real interest rate model as follows:

$$\mu_{realinterest,t} = \max (\mu_{realinterest,t-1} + e_{\mu_{realinterest,t}}, \mu_{realgrowth,t} - riskpremium_{share} + C_{\mu_{realinterest}}).$$

5.2 Fitting of Model

The parameters in the algebraic model described above will be estimated and tested for significance. Both insignificant parameters and parameters which do not satisfy the constraints imposed by theory will be rejected. At first glance, the economic model may appear to have circularity. However the constraints on parameters will eliminate circularity because the causal links between variables are opposite in sign. For example, in the inflation and growth model, one of β_{inf}^0 , which relates $realgrowth_t$,

to inflation_t, and β_{gro}^0 , which relates inflation_t to realgrowth_t, must be rejected, which will eliminate any circularity.

The parameterisation will use the data described previously and the methods described in Section 3.4.1. The risk-return relationship, and the earnings and dividend models from the financial market model will be fitted first, followed by the economic model and remaining items from the financial market model, as the other variables need to be parameterised before this can be done.

5.2.1 Expected Returns and Risk Premiums

The risk premium model can be parameterised by defining the risk premium over a one year period, $E[R_j - \text{realinterest}]$, as the mean log (1+ real return) on asset j less mean log (1+ real money market return). Using these values for 2, 5, 10 and 30 year bonds, as well as equities for each country, together with σ_j and $\sigma_{\text{realinterest}}$, estimates were made and are shown in Table H. These estimates were obtained by first calculating the historical risk premium for each asset class individually, and estimating the beta parameter from these risk premiums.

Table H – Estimates of Risk Premium Parameters

Country	β_{return}	Significance of T-Statistic
United States	0.620	0.052
Britain	0.380	0.661
South Africa	0.235	0.010

The estimates satisfy the condition that $\beta_{\text{return}} \geq 0$. The estimate for Britain is not significant which is probably because sufficient data to estimate standard deviations was only available for 2 data series, resulting in a sample size that was too small to obtain significant results.

5.2.2 Dividend and Earnings Model

Earnings Per Share

The EPS model can be parameterised by regressing the variable { earningsgrowth'_t - inflation_t - realgrowth_t } on { realinterest_t - $\mu_{\text{realinterest,t}}$ }. For this, $\mu_{\text{realinterest,t}}$ was assumed to be the average log (1+ real interest rate) over the 15 years observed. The variable earningsgrowth'_t is defined as in section 5.1.2.3, and is adjusted for the "increase in size" effect. The initial estimates of parameters are shown in Table I.

Table I – Initial Estimates of EPS Parameters

Country	α_{earn}	Significance of T-Statistic
United States	-1.199	0.423
Britain	0.900	0.351
South Africa	0.449	0.633

The estimates for α_{earn} are not significant for any country, indicating that there is no direct link between growth in earnings per share and real interest rates, and the final estimates are shown in Table J.

Table J – Final Estimates of EPS Parameters

Final Estimates		
Country	α_{earn}	σ_{earn}
United States	0.000	0.054
Britain	0.000	0.030
South Africa	0.000	0.053

Dividends Per Share

The smoothing parameter α_{div} in the DPS model is estimated by simulating 5000 outcomes of earnings growth, and finding the smoothing parameter that produces a standard deviation of dividend growth equal to that observed. The estimated parameters are shown in Table K as follows:

Table K – Estimates of DPS Parameters

Country	Standard Deviation of Earnings Growth	Standard Deviation of Dividend Growth	α_{div}
United States	0.057	0.008	0.931
Britain	0.039	0.015	0.795
South Africa	0.051	0.030	0.655

The estimates satisfy the constraint that $0 \leq \alpha_{\text{div}} \leq 1$. A high value of α_{div} represents a high degree of smoothing.

The mean pay-out ratios over the period observed are shown in Table L.

Table L – Mean Pay-out Ratios

Country	Mean Pay-out Ratio	Sample (Months)
United States	0.513	84
Britain	0.682	71
South Africa	0.422	193

5.2.3 Special Cases

Two special cases of the economic model arise when $F_{\text{reversion}}$ takes the value of one or zero. When $F_{\text{reversion}} = 0$, the economic models are pure random walks. When $F_{\text{reversion}} = 1$, the models are mean reverting. The processes for intermediate cases where $0 < F_{\text{reversion}} < 1$, are a combination of the two processes, but are not mean reverting, as mean reversion only occurs in the special case when $F_{\text{reversion}} = 1$. The auto-regressive parameters (α, β, γ) can be estimated by assuming $F_{\text{reversion}} = 1$. The special cases are described further below.

Random Walk Special Case

The random walk special case occurs when $F_{\text{reversion}} = 0$, and if the initial conditions are such that $\text{inflation}_0 = \mu_{\text{inflation},0}$, $\text{realgrowth}_0 = \mu_{\text{realgrowth},0}$, and $\text{realinterest}_0 = \mu_{\text{realinterest},0}$, then, for all $t > 0$, $\text{inflation}_t = \mu_{\text{inflation},t}$, $\text{realgrowth}_t = \mu_{\text{realgrowth},t}$ and $\text{realinterest}_t = \mu_{\text{realinterest},t}$. This means that the economic models simplify to:

$$\text{inflation}_t = \text{inflation}_{t-1} + e_{\text{inf},t}$$

$$\text{realgrowth}_t = \text{realgrowth}_{t-1} + e_{\text{gro},t}$$

$$\text{realinterest}_t = \text{realinterest}_{t-1} + e_{\text{int},t}$$

Mean Reversion Special Case

If the economic variables revert to a constant mean, it means that $\mu_{\text{inflation},t}$, $\mu_{\text{realgrowth},t}$ and $\mu_{\text{realinterest},t}$ are all constant. This occurs in the special case when $F_{\text{reversion}} = 1$. The constant means can be defined as $\mu_{\text{inflation}}$, $\mu_{\text{realgrowth}}$ and $\mu_{\text{realinterest}}$, and equal $\mu_{\text{inflation},t}$, $\mu_{\text{realgrowth},t}$ and $\mu_{\text{realinterest},t}$ respectively. Estimates of these means are shown in Table M.

Table M – Estimates of Means for Economic Model

Country	$\mu_{\text{inflation}}$	$\mu_{\text{realgrowth}}$	$\mu_{\text{realinterest}}$
United States	0.033	0.029	0.028
Britain	0.044	0.025	0.050
South Africa	0.114	0.015	0.034

The means can then be subtracted from the variables, and the parameters in the economic model estimated by regression. A step-wise regression procedure was followed, where parameters that were not significant at the 5% level or did not satisfy the constraints were eliminated. The regression results are shown in Table N as follows.

Table N – Regression Estimates for Economic Model

Inflation Model						
Country	α_{inf}	β^0_{inf}	β^1_{inf}	γ^0_{inf}	γ^1_{inf}	σ_{inf}
United States	0.680	0	0	0	0	0.008
Britain	0.523	0	0	0	0	0.020
South Africa	0.767	0	0	0	0	0.027

Real Economic Growth Model						
Country	α_{gro}	β^0_{gro}	β^1_{gro}	γ^0_{gro}	γ^1_{gro}	σ_{gro}
United States	0	0	-0.794	0	0	0.011
Britain	0.504	0	-0.559	0	0	0.012
South Africa	0	0	0	0	0	0.022

Real Short Term Interest Rates Model						
Country	α_{int}	β^0_{int}	β^1_{int}	γ^0_{int}	γ^1_{int}	σ_{int}
United States	0.576	0	0	0	0.371	0.008
Britain	0	0	0.610	0	0	0.011
South Africa	0.685	0	0	0	0.705	0.028

5.2.4 General Case

The best-fit model might be one of the special cases, but it is more likely to be an intermediate model between the two extremes when $0 < F_{reversion} < 1$. The mean reversion special case is found when $F_{reversion} = 1$, and the random walk special case is found when $F_{reversion} = 0$. The value of $F_{reversion}$ can be estimated using the observed variances of asset returns, by minimizing a "least squares" function L defined below. This function was chosen so that the modelled volatilities of asset classes broadly

match those observed historically. As stated previously, the approach to fitting the model was in part heuristic, and no better formal method for estimating F was found.

$$L = \sum_{j=1}^n (s_j^{\text{observed}} - s_j^{\text{modelled}})^2$$

where

n = number of asset classes

s_j^{observed} = observed standard deviation of log (1+ real return) of asset class j

s_j^{modelled} = modelled standard deviation of log (1+ real return) of asset class j

Given that data was obtained for 15 years, and that data was not available for the full period for all asset classes, a modified “weighted least squares” function W can be defined as below. Again, the choice of this function is a practical heuristic one with the aim that the modelled volatilities of asset classes broadly match those observed historically.

$$W = \sum_{j=1}^n m_j (s_j^{\text{observed}} - s_j^{\text{modelled}})^2$$

where

n = number of asset classes

m_j = number of years for which asset class j was observed

s_j^{observed} = observed standard deviation of log (1+ real return) of asset class j

s_j^{modelled} = modelled standard deviation of log (1 + real return) over 15 years of asset class j.

The s_j^{modelled} are calculated through simulations of the model as described in Appendix B. It should be noted that the method of weighting to the asset classes results in bond asset classes having a higher weight than equities for the US and South African models. An iterative approach was used to find $F_{\text{reversion}}$ that minimised W. Table O shows the calculation of the values of $F_{\text{reversion}}$ which minimise the W function.

Table O – Calculation of “Reversion Factor”

Country	United States			Britain			South Africa		
n	6			3			6		
W	0.102			0.000			0.077		
F _{reversion}	0.13			0.88			0.83		
Asset category	m _j	s _j ^{observed}	s _j ^{modelled}	m _j	s _j ^{observed}	s _j ^{modelled}	m _j	s _j ^{observed}	s _j ^{modelled}
Money market	15	0.015	0.021	15	0.019	0.017	15	0.041	0.044
2 Year Bonds	4	0.010	0.025	0	-	0.036	12	0.047	0.059
5 Year Bonds	4	0.030	0.044	0	-	0.052	12	0.057	0.093
10 Year Bonds	5	0.079	0.067	15	0.061	0.059	12	0.057	0.110
30 Year Bonds	15	0.150	0.095	0	-	0.067	9	0.069	0.119
Shares	10	0.093	0.167	10	0.080	0.079	10	0.204	0.185

University of Cape Town

6. Description and Evaluation of Final Model

6.1 Description

6.1.1 Structure

The parameterised models for the three countries studied all show a cascade structure without feedback. Figures D, E and F illustrate these structures. The models for the United States and Britain have inflation as the 'driving variable' as does the Wilkie (1986) model. The South African model has both inflation and economic growth as driving variables. The only differences in structure between the country models are the relationships between the economic variables (inflation, economic growth and interest rates) as the relationships between the financial market variables (bond yields, dividends, earnings and share prices) are set by theory. The differences in structure are based on observed statistical relationships in the data, and potentially represent real differences in the way the economies have operated. Examples of these difference may be in the make-up of the economy (e.g. a services versus resource based economy) or how fiscal and monetary policy has been applied and what is has been focused on (e.g. growth stimulation or the control of inflation).

Figure D – Structure of United States Model

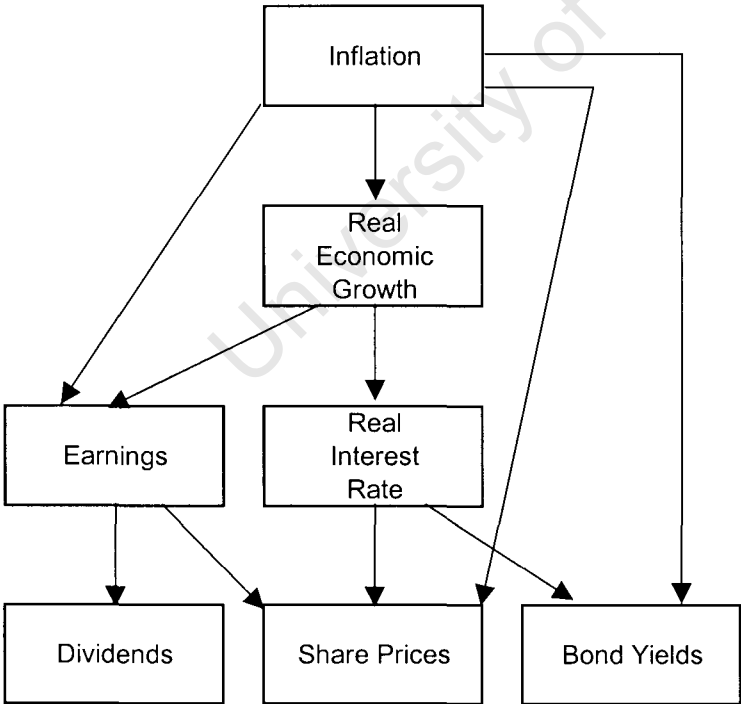


Figure E – Structure of British Model

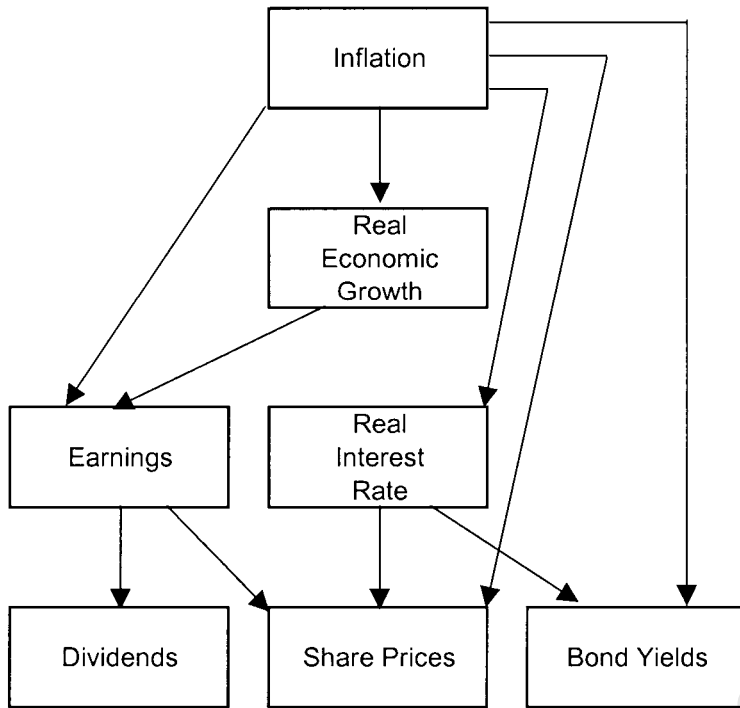
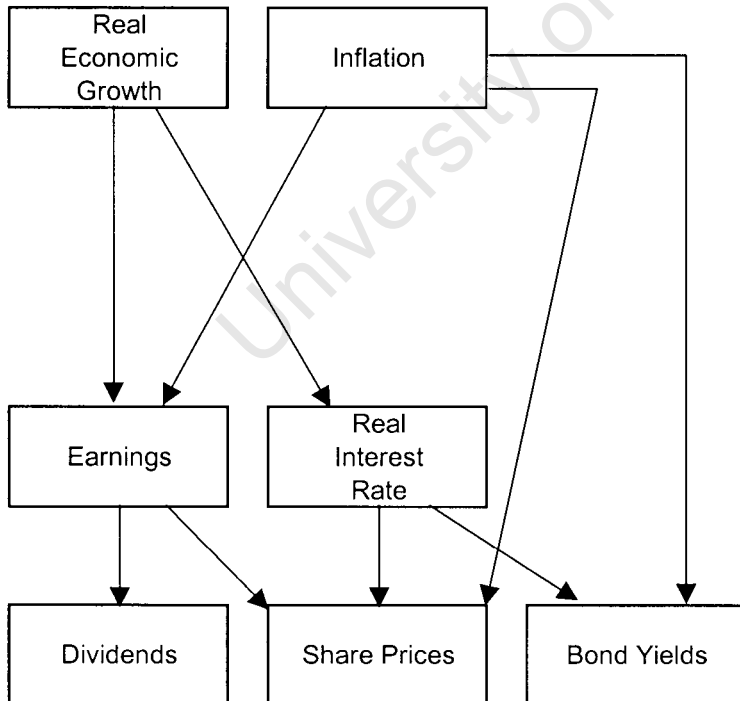


Figure F – Structure of South African Model



The parameterised models are included in demonstration Microsoft Excel spreadsheets on the accompanying CD-ROM. The programs are named “Model (US) demo.xls”, “Model (UK) demo.xls” and “Model (ZA) demo.xls” for the United States, British and South African models respectively. These

provide 25 year stochastic projections and include graphs. In the spreadsheets, the first tab (“Parameters”) lists the parameters, the next 9 tabs show workings, followed by a “Summary” tab and four graphs. A new simulation is performed each time the recalculate button (F9) is pressed.

6.1.2 Summary Statistics

The statistical properties of the models are examined by observing simulated output of the model, and are not analytically determined. This is because the form of the model makes analytical solutions difficult to find. The simulated output used to examine the models is provided in the accompanying CD-ROM. Tables P, Q and R show summary statistics for 667 fifteen year simulated projections for each country model (10,005 individual year simulations in all). These simulations were performed as detailed in Appendix B.

Table P – Summary Statistics of United States Model

Economic Variables						
Summary Statistic	Log (Inflation+1)	Log (Real Economic Growth+)	Log (Real Interest Rate+1)	Nominal Interest Rate	Log (Earnings Growth+1)	Log (Dividend Growth+1)
Mean	0.033	0.028	0.030	0.065	0.113	0.086
Median	0.033	0.028	0.029	0.065	0.113	0.083
Standard Deviation	0.021	0.028	0.020	0.030	0.061	0.018
Kurtosis	0.599	0.752	0.831	0.794	-0.004	0.538
Skewness	-0.015	0.033	0.089	0.022	-0.039	0.539
Minimum	-0.054	-0.085	-0.051	-0.085	-0.098	0.010
Maximum	0.115	0.144	0.122	0.203	0.338	0.167

Financial Market Variables			
Summary Statistic	Long Bond (30 Year) Yield	Earnings Yield	Dividend Yield
Mean	0.116	0.088	0.035
Median	0.115	0.088	0.035
Standard Deviation	0.030	0.030	0.014
Kurtosis	0.901	0.301	2.120
Skewness	0.024	0.064	0.419
Minimum	-0.039	0.010	0.003
Maximum	0.258	0.215	0.159

Asset Returns (Log (1+Real Returns))					
Summary Statistic	Money Market	Short Bonds (5 Year)	Medium Bonds (10 Year)	Long Bonds (30 Year)	Shares
Mean	0.030	0.043	0.057	0.076	0.120
Median	0.029	0.044	0.057	0.076	0.116
Standard Deviation	0.020	0.044	0.067	0.094	0.169
Kurtosis	0.831	0.045	0.019	0.481	3.791
Skewness	0.089	-0.039	-0.034	0.013	0.286
Minimum	-0.051	-0.160	-0.264	-0.430	-1.210
Maximum	0.122	0.219	0.321	0.492	1.209

Correlations							
	Inflation	Real Growth	Real Interest Rate	Short Bond Returns	Medium Bond Returns	Long Bond Returns	Share Returns
Inflation	1	-0.01	-0.05	-0.29	-0.29	-0.28	-0.01
Real Growth	-0.01	1	0.06	0.02	0.01	0.01	0.3
Real Interest Rate	-0.05	0.06	1	0.19	0.03	-0.06	-0.1
Short Bond Returns	-0.29	0.02	0.19	1	0.99	0.96	0.3
Medium Bond Returns	-0.29	0.01	0.03	0.99	1	0.99	0.33
Long Bond Returns	-0.28	0.01	-0.06	0.96	0.99	1	0.33
Share Returns	-0.01	0.3	-0.1	0.3	0.33	0.33	1

Table Q – Summary Statistics of British Model

Economic Variables						
Summary Statistic	Log (1+Inflation)	Log (1+Real Economic Growth)	Log (1 + Real Interest Rate)	Nominal Interest Rate	Log (1+Earnings Growth)	Log (1+Dividend Growth)
Mean	0.043	0.025	0.050	0.098	0.085	0.082
Median	0.043	0.025	0.050	0.097	0.085	0.082
Standard Deviation	0.025	0.022	0.017	0.039	0.039	0.018
Kurtosis	0.075	0.094	0.128	0.081	-0.014	0.021
Skewness	0.006	0.063	0.007	0.087	-0.016	0.006
Minimum	-0.052	-0.049	-0.018	-0.045	-0.056	0.014
Maximum	0.144	0.130	0.120	0.255	0.231	0.149

Financial Market Variables			
Summary Statistic	Long Bond (30 Year) Yield	Earnings Yield	Dividend Yield
Mean	0.119	0.048	0.032
Median	0.119	0.048	0.032
Standard Deviation	0.011	0.006	0.004
Kurtosis	0.324	0.416	0.712
Skewness	0.067	0.169	0.169
Minimum	0.074	0.024	0.015
Maximum	0.162	0.075	0.052

Asset Returns (Log (1+Real Returns))					
Summary Statistic	Money Market	Short Bonds (5 Year)	Medium Bonds (10 Year)	Long Bonds (30 Year)	Shares
Mean	0.050	0.064	0.067	0.070	0.072
Median	0.050	0.064	0.067	0.071	0.072
Standard Deviation	0.017	0.052	0.059	0.066	0.078
Kurtosis	0.128	0.031	0.036	0.050	0.090
Skewness	0.007	-0.018	-0.010	0.002	0.008
Minimum	-0.018	-0.164	-0.193	-0.224	-0.252
Maximum	0.120	0.254	0.285	0.321	0.365

Correlations							
	Inflation	Real Growth	Real Interest Rate	Short Bond Returns	Medium Bond Returns	Long Bond Returns	Share Returns
Inflation	1	-0.41	0.4	-0.72	-0.73	-0.73	-0.44
Real Growth	-0.41	1	-0.47	-0.08	-0.07	-0.07	0.3
Real Interest Rate	0.4	-0.47	1	0.06	0.03	-0.01	-0.16
Short Bond Returns	-0.72	-0.08	0.06	1	1	1	0.62
Medium Bond Returns	-0.73	-0.07	0.03	1	1	1	0.63
Long Bond Returns	-0.73	-0.07	-0.01	1	1	1	0.64
Share Returns	-0.44	0.3	-0.16	0.62	0.63	0.64	1

Table R – Summary Statistics of South African Model

Economic Variables						
Summary Statistic	Log (1+Inflation)	Log (1+ Real Economic Growth)	Log (1 + Real Interest Rate)	Nominal Interest Rate	Log (1 + Earnings Growth)	Log (1 + Dividend Growth)
Mean	0.114	0.015	0.033	0.160	0.158	0.156
Median	0.114	0.015	0.033	0.158	0.158	0.156
Standard Deviation	0.045	0.025	0.045	0.074	0.074	0.052
Kurtosis	0.183	0.098	0.271	0.307	0.092	0.296
Skewness	0.009	0.057	-0.002	0.208	-0.043	-0.027
Minimum	-0.083	-0.076	-0.149	-0.133	-0.113	-0.054
Maximum	0.289	0.110	0.197	0.552	0.439	0.366

Financial Market Variables			
Summary Statistic	Long Bond (30 Year) Yield	Earnings Yield	Dividend Yield
Mean	0.181	0.052	0.021
Median	0.178	0.051	0.021
Standard Deviation	0.038	0.018	0.007
Kurtosis	0.653	0.438	0.544
Skewness	0.383	0.327	0.315
Minimum	0.045	0.008	0.004
Maximum	0.409	0.127	0.058

Asset Returns (Log (1+Real Returns))					
Summary Statistic	Money Market	Short Bonds (5 Year)	Medium Bonds (10 Year)	Long Bonds (30 Year)	Shares
Mean	0.033	0.045	0.050	0.053	0.070
Median	0.033	0.045	0.050	0.053	0.067
Standard Deviation	0.045	0.093	0.111	0.120	0.186
Kurtosis	0.271	-0.006	0.005	0.061	0.703
Skewness	-0.002	-0.007	0.005	0.012	0.112
Minimum	-0.149	-0.309	-0.389	-0.464	-0.972
Maximum	0.197	0.466	0.528	0.545	1.103

Correlations							
	Inflation	Real Growth	Real Interest Rate	Short Bond Returns	Medium Bond Returns	Long Bond Returns	Share Returns
Inflation	1	0	0.01	-0.48	-0.48	-0.48	-0.02
Real Growth	0	1	0.13	0	0	0	0.46
Real Interest Rate	0.01	0.13	1	-0.06	-0.12	-0.15	-0.37
Short Bond Returns	-0.48	0	-0.06	1	1	0.99	0.42
Medium Bond Returns	-0.48	0	-0.12	1	1	1	0.44
Long Bond Returns	-0.48	0	-0.15	0.99	1	1	0.45
Share Returns	-0.02	0.46	-0.37	0.42	0.44	0.45	1

Reversion Factor

The estimated $F_{\text{reversion}}$ factor is much lower for the United States than Britain and South Africa, which means that the current underlying means (such as $\mu_{\text{realinterest},t}$) change more rapidly, so there is a greater probability of them becoming negative over the 15 year period. Since bond yields are linked to expected future interest rates, and these are based more on the current underlying mean ($\mu_{\text{realinterest},t}$) than the current rate (realinterest_t), there is a greater probability of bond yields becoming negative. As a consequence, negative bond yields are generated in some of the simulations of the United States model, whereas this does not occur in the British and South African models. By contrast, the constraint on the economic model (see section 5.1.3) which prevents share prices becoming negative, also ensures share yields (earnings and dividends yields) are always positive.

More generally, the $F_{\text{reversion}}$ factor influences the volatility of asset returns. The data shows that the underlying economic variables in the United States have been stable (e.g. when compared to South Africa) but asset returns have been relatively volatile. A low value of $F_{\text{reversion}}$ was thus required in the model since, ceterus paribus, a lower $F_{\text{reversion}}$ means higher volatility of returns because lower values of F mean less mean-reversion and consequently higher volatility when returns are measured over longer periods.

Central Expectations

Expected earnings growth is greater than the sum of inflation and real economic growth because of the 'change in size' effect described in section 5.1.2 which arises when dividends are less than earnings. Expected dividend growth would be equal to expected earnings growth if the initial value for smoothed earnings growth (smoothearnings_0) were set equal to expected earnings growth. However, in these simulations, the initial value for smoothearnings_0 is set equal to expected inflation plus real economic growth which is why expected dividend growth is lower than expected earnings growth.

Expected returns from various asset classes are a function of the standard deviation of their return and the beta parameter (β_{return}). It is important to note that the beta parameter in this case is not a measure of risk, as it is in CAPM. For instance, the estimated beta for South Africa is lower than that for the United States, but the South African asset returns are more volatile. This is because the modelled underlying economic variables (inflation, growth and interest rates) are more variable than those in the United States.

Volatility and Correlation

The modelled volatility of equity returns is greater than that of bond returns, which in turn is greater than that of cash returns. Modelled longer dated bonds also have more volatile returns than returns on shorter dated bonds, which is also a phenomenon observed in the data. The modelled smoothing of dividends is evident from the lower standard deviation of dividend growth than earnings growth.

The correlations between the economic variables reflect both the structure of the models and the size of the parameters. Whereas there is little correlation between economic variables for both the United States and South African models, there is significant correlation in the British model. The correlations

between asset returns and economic variables are a function of this, but also incorporate built-in theoretical correlations. These include that equities and long bonds are negatively correlated to changes in real interest rates, that equities are positively correlated to economic growth, and that bonds are negatively correlated to inflation.

Returns on shorter bonds will tend to be positively correlated with changes in real interest rates, since the shorter a bond is, the closer it is to being cash. The positive correlation changes to negative correlation as the term on the bond increases. This effect is most pronounced in the United States model. As expected, bond returns are not highly correlated to economic growth as this is not a direct driving variable for bond yields. Short, medium and long bond returns are highly positively correlated to each other in all models, and significantly positively correlated to equity returns.

Equity returns in the United States and South African models are not significantly correlated to inflation, but are negatively correlated in the British model. This is due to the structure of the British economic model where economic growth and inflation are negatively correlated, which is not the case in the other country models.

6.1.3 Distribution of Returns

Figure G shows the distribution of $\log(1 + \text{real return})$ from the simulations of the United States model for the money market, short bonds (5 year), medium bonds (10 year), long bonds (30 year) and equities. The increasing expected return moving from cash to bonds to equity can clearly be seen as the median of the distributions shifts right. This effect is also observed when moving from short through medium to long-dated bonds. The increased volatility associated with higher expected returns is also evident as the spread of the distributions increases. Similar figures for the British and South African models can be produced.

Figure G – Distribution of Simulated Returns (United States)

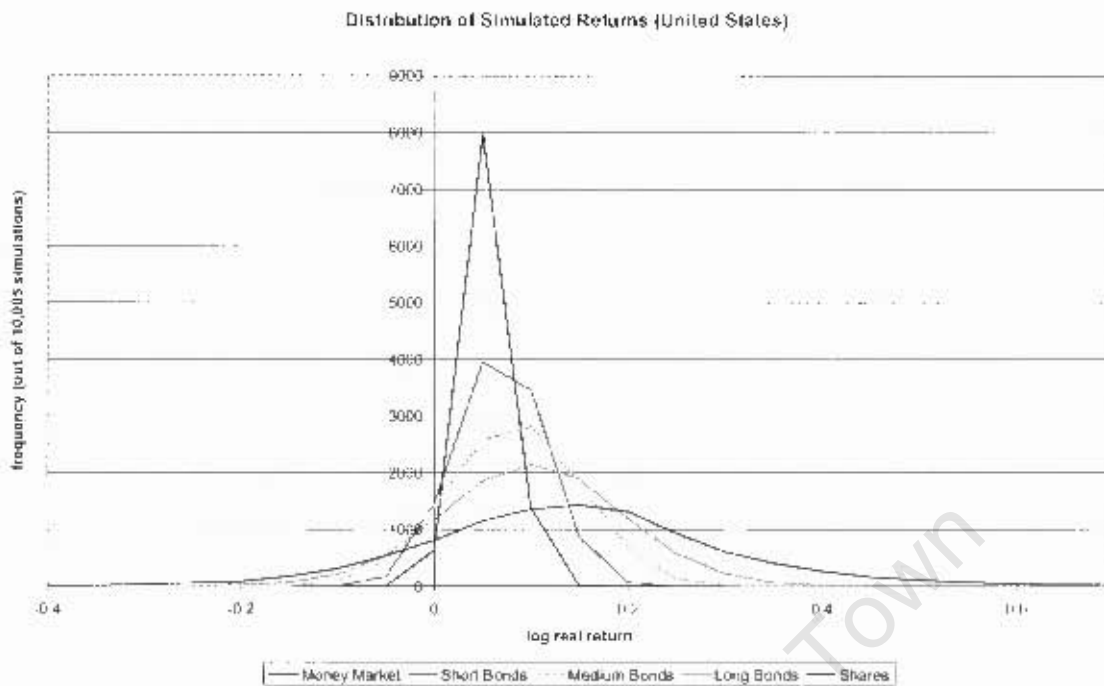


Figure H shows the distribution of simulated equity returns compared to a normal distribution with the same mean and variance. The non-normality of simulated returns is clearly evident. Modelled equity returns can be described as leptokurtic as they show higher probabilities of values close to the mean as well as higher probabilities of extreme values. Simulated bond returns also demonstrate this leptokurtic property, although it is less pronounced

Figure H – Distribution of Simulated Share Returns (United States)

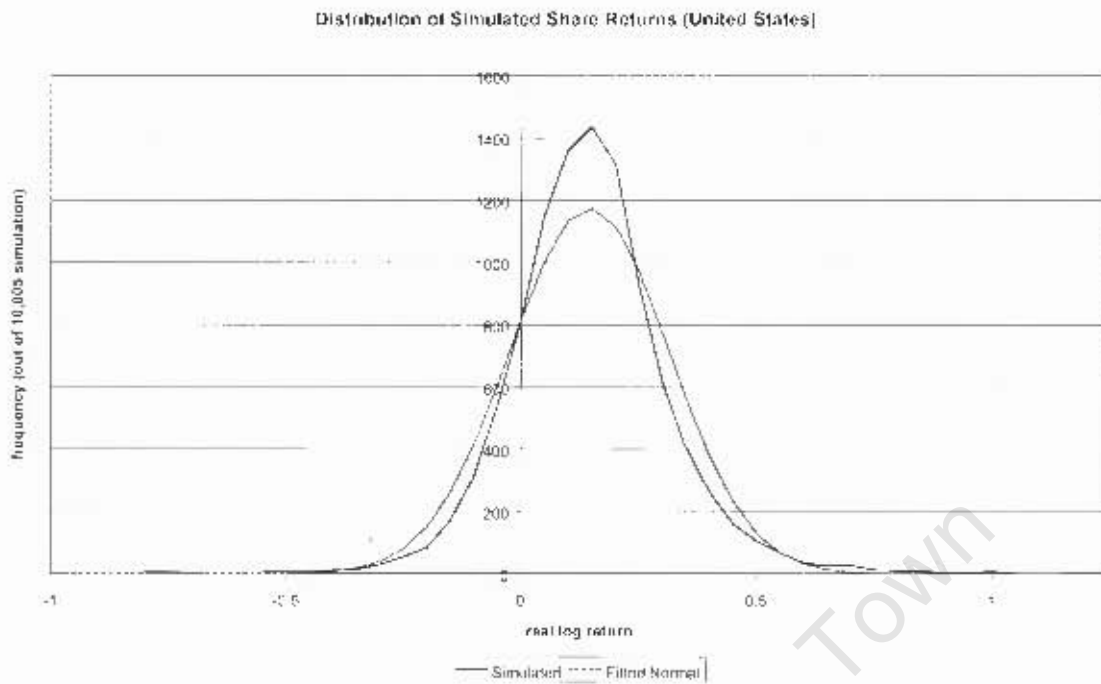
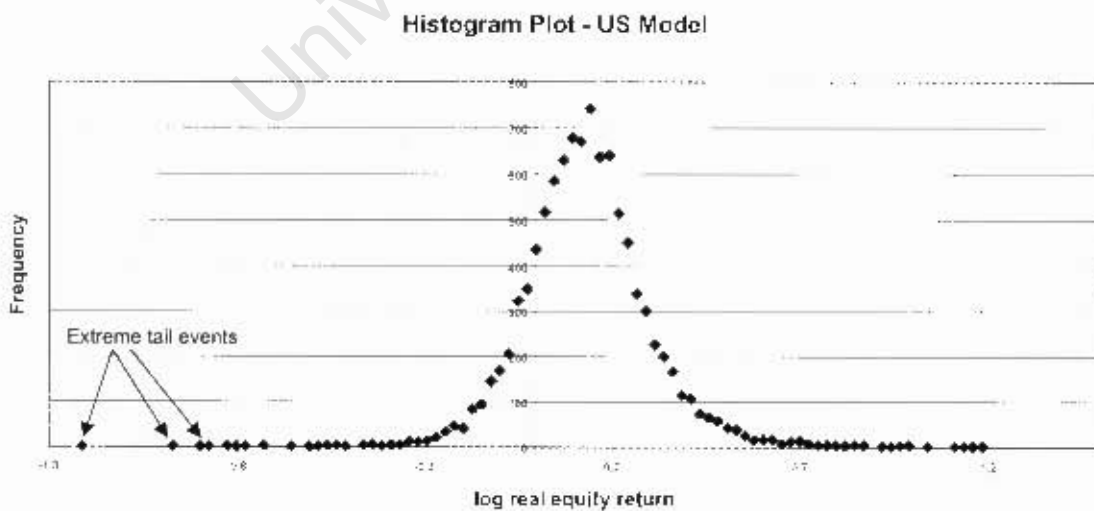


Figure I illustrates the extreme negative tail events which occurred in the 10,005 simulations of the US model. If the returns were normally distributed, a $\log(1 + \text{real return})$ of -1.2 (which corresponds to a nominal return of -69%) or worse would be nearly impossible, whereas here it occurred once in 10,005 simulations. Shiller (1981) suggested that observed volatilities were inconsistent with the assumption of efficient markets. However, here, a model that assumes efficiency produces non-normal returns with extreme negative events.

Figure I – Histogram of Simulated Share Returns (United States)



6.1.3.1 Decay of Equity Risk

In the Wilkie model, equity risk, or the volatility of equity returns, decays over time so for example the annualised volatility of 5 year returns is lower than the annual volatility of 1 year returns. This decay is caused by the mean reversion of dividend yields. This is sometimes referred to as "time diversification". and Howie (1997), based on empirical evidence, argued that equity risk does not decay in this way. From the 10,005 simulations of the US model, the annualised standard deviation of log (1+real equity return) reduces from 0.17 over 1 year to 0.15 over 15 years, which suggests that the model predicts that equity risk marginally decays over time.

The decay of equity risk in models like Wilkie is due to the autoregressive property of variables. This property is inconsistent with the pure form of the efficient market hypothesis. However, recent studies, such as Lo & MacKinlay (1988) and Campbell, Lo & MacKinlay (1997), have found empirical evidence that statistically significant positive autocorrelation exists between asset returns in successive short (less than a year) periods. In other words, the returns are to some extent predictable. They argue that this is not evidence against the efficiency of markets, but reflective of rational factors such as frictions in the trading process. Moreover, whereas this predictability is statistically significant, it is not exploitable when transaction costs are allowed for.

Howie & Davies (2002) find a similar pattern in UK data to the simulated returns shown above, although the decay of equity risk was not shown to be statistically significant. Campbell & Viceira (2002) show that the decay of equity risk and mean-reversion of equity returns are mathematically equivalent. For mean-reversion to occur, equity returns must exhibit negative rather than positive autocorrelation. As can be expected, the simulated equity returns exhibit small negative autocorrelation, which is close to zero over 1 year but -0.1 over 5 years.

In contrast to the positive autocorrelation found in returns over short periods, studies (such as Fama & French (1988) and Poterba & Summers (1988)) have found negative autocorrelation between longer periods and therefore evidence for mean reversion. Given the small sample sizes available for testing long-horizon returns, these studies have tended not to produce statistically significant results (Richardson & Stock (1989) and Richardson (1993)).

As discussed by Howie & Davies (2002) there is some, but limited, evidence for slow and weak mean-reversion in equity markets. By slow, it is meant that the mean-reversion is only observable over long periods so is not exploitable by short-term trading, and by weak it is meant that the effect is small. The modelled equity returns exhibit these two characteristics. By contrast, models such as Wilkie exhibit mean-reversion which is stronger, and imply that the effect can be exploited by trading.

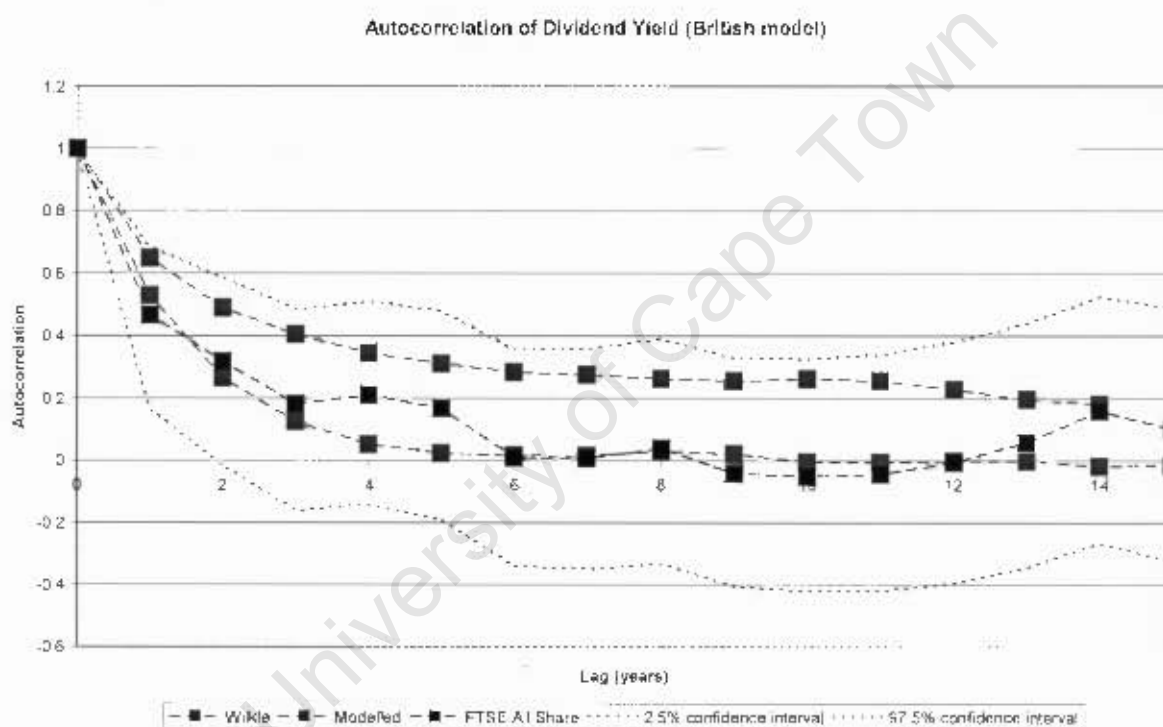
6.1.3.2 Dividend Yields

The mean-reversion of returns in models like Wilkie is because equity prices are modelled on a strongly mean-reverting dividend yield. Here, the modelled dividend yield is not a determinant of equity returns, and the model is constructed so that equity returns are independent of dividends. This means that dividends can be ignored, unless there is a requirement to consider investment income in an

asset-liability study. However, it is worth considering the properties of the modelled dividend yield. These properties do not say anything about the goodness-of-fit of the equity return model, but only about the goodness-of-fit of the dividend model i.e. they are a test of the smoothing function specified in Section 5.1.2.3.

Figure J plots the observed autocorrelations of dividend yields over lags of 1 to 15 years. Five series are shown. The Wilkie (1995) UK model for dividend yields (ignoring the term linking dividend yields to price inflation) is shown for 5,000 simulations, together with the output from 10,005 simulations of the British model. Also shown is the autocorrelation observed between dividend yields of the FTSE All Share Index sampled at the end of each year over the period 31/12/1964 to 31/12/2001, and the 95% confidence interval of these estimates from the FTSE All Share Index.

Figure J – Autocorrelation of Dividend Yield (British Model)



The Wilkie model describes the data well (noting that the data plotted includes data after that used to fit the Wilkie model), but the British dividend model fits less well. Although the pattern is similar to both the data and Wilkie (in that the autocorrelation declines exponentially until a lag of about 5 years and is then relatively constant), the level is too high i.e. the modelled autocorrelation appears too high. However, there are large confidence bounds and the modelled autocorrelation is within the 95% confidence bounds. The fact the Wilkie (1995) fits the dividend data well can be expected given that that model was parameterised directly on the dividend yield data, whereas this model is parameterised indirectly. Specifically, the model for dividends attempts only to fit the observed volatility of dividend yields rather than observed lagged autocorrelations.

These results indicate that the dividend model may be incorrectly specified. It would appear that a model where the lagged autocorrelation tends to zero quicker (an autoregressive process rather than

the exponential smoothing model used for dividends) would be more appropriate. However, the fact that modelled dividend yields do not explain the observed data very well does not detract from the overall model much because the model for equity returns is designed to be independent of dividends. This means that only in applications where returns are required to be separated into income and capital would this aspect of the model be an issue.

6.1.3.3 Yield Curve

The existence of volatility related risk premiums and liquidity preference (see section 4.2.4) results in the 'natural' shape of the yield curve being upward sloping. In the parameterisation of the model (see section 5.2.1) expected long bond returns are set higher than those of shorter bond returns, which means the initial long yields are higher than short yields and thus the yield curve is upward sloping. However, inverted yield curves are often observed in many markets. Section 6.2 below discusses calibration to current market conditions i.e. how the model can be adapted to reflect such economic realities, but it is also worth considering whether the model will generate inverted yield curves in some simulations, given the initial condition of an upward sloping yield curve. Figure K shows 10 simulations from the British model of the difference between short and long yields over 15 years, with the initial conditions of an upward sloping yield curve.

Figure K – 10 Simulations from the British Model

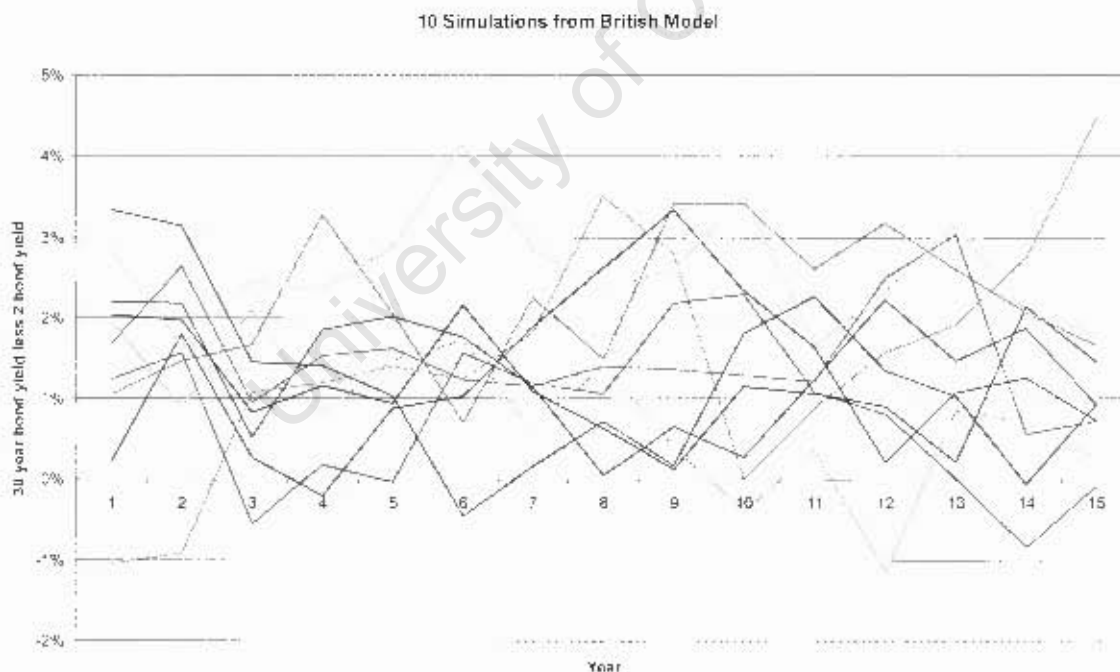


Figure K shows clearly that the model incorporates non-parallel shifts of the yield curve because the simulated changes in short and long yields are not constant. Also, as can be seen, there are occasions when the curve becomes inverted (when the difference between long and short rates is negative). From 667 simulations of the British model, the probability of 30 year bond yields being less than 2 year bond yields was 10.3%. This indicates that the model exhibits a significant probability of the yield curve becoming negative sloping.

At the short end of the yield curve (in extremis, the interest rate on cash) it is very uncommon for nominal yields to be negative. In major markets, only Japan has come close to experiencing negative interest rates, with the official discount rate falling as low as 0.1% (Bank of Japan (2001)). It is deflationary environments when inflation is very low or even negative, which are themselves uncommon, when negative interest rates may occur. By contrast, negative real interest rates are much more common, and often associated with strong monetary policy. Negative yields on long bonds are unheard of in economic history. Table S shows the estimated probability of the model generating negative inflation, nominal interest rates, real interest rates, and long bond yields for each of the 3 countries, based on 10,005 simulations of each model.

Table S – Probability of negative inflation, interest rates and yields

Model	Negative Inflation	Negative Nominal Interest Rate	Negative Real Interest Rate	Negative 30 Year Bond Yield
USA	5.2%	1.7%	6.4%	0.1%
Britain	4.6%	0.4%	0.2%	0.0%
South Africa	0.6%	1.3%	22.4%	0.0%

The South African model shows a lower probability of deflation simply because the average inflation rate is higher. All three country models show low probabilities of negative nominal interest rates and, apart from South Africa, this probability is lower than the probability of negative inflation i.e. even in deflationary environments, interest rates tend to be positive. The modelled standard deviation of South African nominal interest rates is higher than of the USA or Britain, and is also substantially higher than the modelled standard deviation of South African inflation, and hence the probability of negative nominal interest rates is higher than the probability of negative inflation.

Apart from South Africa, the probability of negative real interest rates is relatively low. It is high for South Africa mainly because of the higher volatility of real interest rates. For the British model, the probability is very low, and is lower than the probability of negative nominal rates. This is because the British model has a high mean real interest rate, as well as the lowest volatility of real interest rates. For all three models, the probability of negative long bond yields is negligible or zero.

The discussion above shows how the parameterisation of the model can capture the nuances of how economies and their financial markets differ from each other. Notable is how an emerging market (South Africa) differs from the two developed markets. The parameterisation for other markets, as well as calibration to current market conditions is discussed in the next section.

6.2 Adapting and Using Model

6.2.1 Parameterisation

For any practical application of this model, the parameterisation will need to be considered. This is the same as considering the actuarial assumptions made for an actuarial valuation or premium calculation. When “setting the assumptions” for the model, it is worth considering both the economic history and current market conditions in the market where the model is to be applied.

In general, it is now widely accepted as best practice to deduce as much as possible from current market conditions about the expectations of future economic variables. For example, the current yield curve could be used as an estimator of the expected return on bonds of different durations, as well as the pattern of future interest rates. In markets where inflation-linked bonds are available, the real yield curve is an estimator of the expected real return on inflation-linked bonds of different durations, as well as the pattern of future real interest rates. The difference between the nominal yield curve and the real yield curve is an estimator of inflation over different periods.

In markets without inflation-linked bonds, there are no market estimates of future inflation and real interest rates, but rather an upper limit specified by the nominal yield curve. Here central expectations for inflation and real interest rates can be deduced by the current yield curve and the historic real return generated by bonds (assuming there is a sufficiently long and reliable data series to estimate this), and/or by reference to other markets that do have inflation-linked bonds.

The parameters for the central expectations of the bond, inflation and interest rate models are therefore relatively straightforward to set, given current market conditions. However, other key parameters cannot be deduced from current market variables. These parameters include the central expectation for future economic growth, the equity risk premium (and hence the expected equity return) and the future volatility of all modelled variables.

Some of these parameters can at least be bounded, for example financial theory would argue for a positive equity risk premium, but these types of bounds are not particularly useful in determining parameters. There are a number of multi-country long term studies of financial markets and economies, such as Dimson, Marsh & Staunton (2002), that can assist in setting parameters like the equity risk premium and expected economic growth (and hence dividend and earnings growth). The best estimate of future volatilities are past volatilities.

There are two approaches that can be used to re-parameterise the models:

- (a) Parameterise from first principles. This is the most time-consuming approach and will involve all the steps described in Section 5. This approach will be required if the model is to be applied to a different economy, or if a substantial period of time has elapsed since the last full parameterisation.

- (b) Modify existing parameters. The means and/or volatilities of variables can be modified, whilst keeping the structure of the model unchanged. This approach is most likely to be used when calibrating the model to current market conditions, and is described below.

6.2.2 Calibration to Current Market Conditions

To make the model consistent with current market conditions at any point in time, the parameters will need to be adjusted. This process will be illustrated by means of an example. At 31 December 2002, the conditions set out in Table T prevailed in the UK (from RIMES (2003) and The Economist (2003)).

Table T – Market Conditions at 31 December 2002*

Variable	Value
20 Year Government Bond Yield	4.47%
Over 5 Year Index-Linked Real Yield (Assuming 5% Inflation)	2.02%
Dividend Yield	3.55%
Earnings Yield	5.16%
Retail Price Inflation Over Previous 12 Months	2.9%
Return on Money Market Investments Over Previous 12 Months**	4.0%
Real GDP Growth Over Previous 12 Months	2.2%

* before log transformations

** assuming they earned Base Rate

It is possible to set the initial conditions for inflation_{2002} ($\log(1 + \text{inflation rate})$ in year 2002 = 0.029), realgrowth_{2002} ($\log(1 + \text{real economic growth rate})$ in year 2002 = 0.022), and $\text{realinterest}_{2002}$ ($\log(1 + \text{real short term interest rate})$ in year 2002 or $\log(1 + \text{real money market return}) = 0.011$).

Since the UK has inflation-linked bonds, there is a market-implied estimate for future inflation and hence an estimate for the initial value of $\mu_{\text{inflation},t}$. This is equal to the fixed interest yield less the index-linked real yield (less an inflation risk premium). If for example the inflation risk premium was assumed to be 0.5%, $\mu_{\text{inflation},2003}$ would be set equal to 0.019.

For the expected return on fixed interest bonds to equal the current yield, either the assumed risk premium for asset classes must be fixed and the value of $\mu_{\text{realinterest},t}$ deduced, or $\mu_{\text{realinterest},t}$ must be fixed and the risk premium deduced. The latter of the two is slightly easier: if $\mu_{\text{realinterest},2003}$ is set equal to $\text{realinterest}_{2002}$ (0.011), the $\log(1 + \text{real risk premium})$ (relative to cash) on 20 year bonds equals the current yield less expected future inflation ($\mu_{\text{inflation},2003}$) less expected future real interest rates ($\mu_{\text{realinterest},2003}$), which is 0.014.

Financial theory would argue for a risk premium on shares greater than that on bonds, but how much more is a highly debated topic and beyond the scope of this thesis. Section 5.2.1, based on CAPM,

shows one of the techniques that can be used. It is more important, particularly if the results of the modelling are to be used for mean–variance optimisation, to ensure consistency between expected returns (and hence risk premiums) and modelled volatilities. Smith et al. (2000) provides a useful discussion on setting consistent assumptions.

In order for the initial earnings yield in the model to equal the current earnings yield, either the equity risk premium must be fixed, and $\mu_{\text{realgrowth},t}$ deduced, or $\mu_{\text{realgrowth},t}$ fixed and the equity risk premium deduced. If, for arguments sake, the equity risk premium (relative to bonds) is set at 4%, the log (1 + real risk premium) relative to cash would be 0.053, and $\mu_{\text{realgrowth},2003}$ would need to be 0.008 (found by trial and error) so that the initial earnings yield equalled 5.16%. The initial payout ratio should be set so that the initial dividend yield equalled the current dividend yield.

As can be seen in the example above, judgement is required when adapting and using the model. The conclusion that judgement remains important was supported by Huber & Verrall (1999).

6.2.3 Potential Uses of Model

In Section 2.1.3, the potential uses of asset-liability models were discussed. Given that variables in this model have been modelled on an annual basis, rather than over the shorter term, it lends itself best to use for medium to long term asset allocation decisions, especially for pension funds but also potentially for insurance funds. Uses in risk management, especially over the shorter term such as that conducted by banks, are probably better served by shorter term models. Additionally, given that the model is derived in real-world rather than risk-neutral space, uses in derivative pricing are fairly limited.

6.3 Evaluation of Model

Section 3.1 described three main criteria for evaluating the model, namely consistency with economic theory, parsimony and reasonable output. These are each discussed in turn below.

1. **Consistency with economic theory.** The method of construction ensures that the model is consistent with economic theory, in particular the models for the pricing of bonds and equity, which are based on the well-established theory that prices are the discounted value of (expected) future cashflows. However, although the economic model is consistent with the economic assumptions made, these assumptions are based on economic theory where there is sometimes no consensus or even competing theories. Additionally, certain economic relationships proposed were found not to be statistically significant.
2. **Parsimony (Simplicity).** Parsimony can be measured by the complexity of the model structure, as well as the number of parameters that are required. The structure is relatively straightforward, and compares favourably to other models. One possible exception is the equity model (particularly the earnings per share adjustment), which is more complex than, for example, Wilkie (1995). However, this added complexity can be justified if the goodness-of-fit of equity returns is better. One advantage of the model is that there is no requirement to simulate non-normal variables, even

though the modelled returns are non-normal. In terms of the number of parameters, the British model has 21 parameters (which includes 5 initial values of variables), which compares to 29 (including 6 initial values of variables, but excluding the parameters related to wages, index-linked gilts, property and exchange rates) from the Wilkie (1995) model for the British economy. The US model has the same number of parameters, whereas the South African model has one less. In summary, the model compares favourably to other models on the criteria of parsimony, although some elements, such as the equity pricing model, are still relatively complex.

- 3. Reasonableness of Output.** Given that the model compares favourably on the first two criteria, if the goodness-of-fit is adequate, the model can be regarded as an improvement on existing models. However, no formal goodness-of-fit tests were conducted on the overall model because of the method of estimating parameters. To determine whether the model produces better output, the question must be asked whether the properties of the model better represent the observed properties of financial markets. Firstly, by definition, the model is not meant to be predictive because market efficiency was assumed, and indeed it is argued that calibration to current market conditions (for the means/expectations of variables) is preferable. This means that a comparison of the goodness-of-fit of first moments (means) to past data is not a useful comparison, because the user is meant to change these to reflect current market conditions.

Even though the goodness-of-fit of the economic model is not a criterion for evaluating the model (the goodness-of-fit of the financial model is the criterion), it is worth considering the fit of the economic model. Only statistically significant relationships between variables were included in the economic model, but a number of statistically significant relationships were rejected on theoretical grounds. Consequently the goodness-of-fit of the economic model is likely to be poorer than models where there are no theoretical constraints imposed.

With regards to the financial model, this was fitted to historic volatilities/standard deviations, so fits the data well in this regard. However, a model that fits past standard deviations is easy to achieve e.g. a multi-variate normal model where modelled covariances are set equal to observed past covariances will fit perfectly when measured on the goodness-of-fit to standard deviations. The main advantage of the proposed model is the way it captures features observed in real markets that go beyond a simple standard deviation measure. Most notable is the non-normal distribution of returns, which exhibits significant leptokurtosis (fat-tails) and shows higher probabilities of severe down-market returns than is predicted by a normal or log-normal distributions. The feature of weak and slow mean reversion that is observed in markets is also captured by the proposed model. The modelled yield curve also captures features observed in markets, in particular non-parallel shifts and inversions of the yield curve. In the financial model, one weakness is the fit of dividend yields. However, equity returns are independent from dividends in the model, so do not detract much from the rest of the model, although a better fit would be preferable.

6.3.1 Strengths

The model is theory-based, and consistent with market efficiency (in the semi-strong form). By contrast, several of the main existing models (including Wilkie (1995), Thomson (1996) and Maturity Guarantees Working Party (1980)) are not consistent with market efficiency. The economic and financial economic theory on which the model is based represents basic theory and consequently the structure of the model makes intuitive sense. An example of this intuitive appeal is that the structure implies that equities are riskier than bonds, which in turn are riskier than cash. Also, long bonds are more risky than short bonds (in absolute terms).

The underlying assumption of efficiency means that whatever the initial economic variables, results will be consistent with them and asset returns will be “efficient” in that they will not imply that successful trading strategies exist. This contrasts strongly with models like Wilkie (1995), which do imply that successful trading strategies exist e.g. if dividend yields are below the average, Wilkie implies that equities should be underweighted because the dividend yield should revert to the average and consequently equity returns should be lower than average. The proposed model incorporates the Miller & Modigliani (1958) hypothesis that dividends are irrelevant, and hence dividend yield is irrelevant. It should be noted, however, that overall the model does not conform to all aspects of efficient markets, as is discussed below in Section 6.3.2.

The model is relatively simple and is also quite flexible, which means it can be adapted quite easily e.g. it can be calibrated to current market conditions, or adapted for different economies. It is hoped that the model is not perceived as a “black box”. The method used for estimating risk premium (which does not give a particularly good fit, see below) could easily be replaced with an alternative method.

The model incorporates a full term structure of interest rates/bond yields, and exhibits non-parallel shifts and inversions of the yield curve. The model also includes real economic growth as an explanatory variable, which was not included in the Wilkie (1995) and Thomson (1996) models.

Modelled return distributions are leptokurtic, a feature observed in financial markets. However, the shape of the distribution was not directly assumed, and is generated without having to simulate variables from non-normal distributions. This approach is much easier than trying to generate leptokurtic distributions directly. The return distributions exhibit infrequent extreme negative tail events, with a probability far higher than those predicted by normal or log-normal models, again a feature observed in financial markets. Equity returns also exhibit slow and weak mean-reversion, which is consistent with data and other studies. This slow and weak mean-reversion does not violate the assumption of efficiency because the mean-reversion is so slow and weak that it does not represent a successful trading strategy.

6.3.2 Weaknesses

Whereas theoretical consistency is regarded as a strength of the model, a possible weakness is the theory itself, some of which is debatable and possibly conflicting, especially the economic theory. The theory was discussed and then distilled into a series of assumptions in section 4.3. The fitting process

eliminates any of the economic model assumptions that are not supported by the data, but the financial model assumptions determine the formulation of the financial model and were not tested against the data. Several of the financial model assumptions can and have been challenged by researchers, for example assumptions 1 (market efficiency) and 2 (a linear relationship between expected real return and the standard deviation of real return) have been the subject of a huge body of research, without a consensus view having emerged. If an assumption such as market efficiency does not hold, it is by no means clear what should alternatively be assumed. Therefore, although the assumptions made may not represent reality perfectly and are therefore open to question, it is not clear what other assumptions could have been made that represented reality better.

The method of estimating the parameters uses a blend of classical statistical techniques and heuristic/iterative methods. Method of moments statistical estimation is used where the observed variances (volatilities) in the financial markets have been used to estimate 'the reversion' parameter', $F_{\text{reversion}}$. Although no formal test of goodness-of-fit was conducted, it is reasonable to assume that the model does not fit the data as well as a model that had not been constrained by theoretical assumptions. However, the model produces reasonable output and is indeed superior to some existing models, especially in the features it produces for return distributions.

There are two components of the model which are not borne out by data: the risk/return relationship for determining risk premiums, and dividends. Section 6.2 describes other approaches that can be used for establishing risk premiums, and it is recommended that users of the model consider alternative approaches and exercise judgement when setting assumptions for risk premiums. One aspect that could be included is the covariance between asset classes. Efficient markets allow for the covariance between asset classes in risk premiums, as they represent diversifiable risks, and this is one aspect where the model does not fully conform the efficient market hypothesis. However, the model is sufficiently flexible that risk premiums can be independently specified and this incorporated.

Although the poor fit of the dividend yield model is a weakness in a statistical sense, the dividend model does not affect the equity returns, so this weakness does not detract from the overall model. More generally, the fitting process, which did not allow an overall goodness-of-fit test of the model, and the availability of data (notably on earnings per share) made the statistical evaluation less satisfactory. In particular, the assumption that the reversion factor F was 1 when estimating the autoregressive parameters appears to be invalid for the US economy, but reasonable for Britain and South Africa.

Because of the criterion of parsimony, more complex, possibly better fitting models, were not considered. In particular, with regards the (explanatory) economic variables, these were restricted to 3 (money supply, for example, could also have been considered), lags longer than one year were not considered, and non-normality of residuals was not explored. Despite the goal of parsimony, some aspects of the model are still quite complex, notably the model for share prices, which includes a "dummy" variable.

The model does not include property or inflation-linked bonds. Neither does it model wages or exchange rates, whereas other models such as Wilkie (1995) do. However, the framework developed

means that incorporating property, inflation-linked bonds and wages would be relatively straightforward. Incorporating exchange rates would not be straightforward, but the necessity for exchange rates in the model is unclear. Certainly, having multi-country models linked by exchange rate models would not be parsimonious.

That interest rates (particularly nominal interest rates) and even bond yields can become negative in the model is arguable an unattractive feature of the model. However, the modelled probability of negative nominal interest rates is small, and that of negative bond yields is negligible.

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7. Conclusions

7.1 Conclusions

The proposed model, which is based on economic and financial economic theory and parameterised on past data, has several features that are superior to existing models for certain applications. Most notable is the leptokurtosis and “fat-tails” exhibited in return distributions, a feature of real world financial returns. The applications that would most benefit from these features are those such as reserving and risk analysis. The non-normal return distributions are generated without having to make direct assumptions about the shape of the distribution, or having to simulate any variables from non-normal distributions. Consequently, the model compares favourably on the criterion of parsimony and indeed has fewer parameters than the Wilkie (1995) model, even when the parameters related to wages, index-linked gilts, property and exchange rates are excluded from Wilkie. By design, the model is consistent with the assumptions made at the outset, and consequently with the economic and financial economic theory on which the assumptions are based. The most important of these theoretical bases is that of market efficiency. Although some of this theory is still being debated, much of it is widely accepted, at least in broad terms.

The parameterisation of the model indirectly poses an interesting question: are the variables stationary? Stationarity was not tested statistically, but the parameter estimation, given the assumptions of the model, answers the question whether the observed volatility in the financial markets can be explained by stationary (mean-reverting) economic variables. The estimated parameters for each of the 3 country parameterisations implied the economic variables were not stationary e.g. if inflation and interest rates were stationary, the long bond yield, under the assumptions of the model, would be significantly less volatile than is observed.

7.2 Recommendations for Further Research

The identified weaknesses of the model, as well the many interesting topics touched on that were beyond the scope of the study, present several topics for which further research is recommended. Notably, further research could consider alternative risk-return models (including models that consider risk measures other than standard deviation of return) and alternative dividend yield models that better fit observed autocorrelation.

Parameterisation of the model using longer data series, as well as data for other countries could be considered, as well as extending the model to cover property, inflation-linked bonds and wages. Alternative formulations may be investigated, e.g. incorporating additional constraints that prevent negative nominal interest rates and bond yields, possibly with a yield curve model that can be fully calibrated to the current yield curve. Also, time intervals other than annual could be considered, as well as lags longer than one time period. Non-normal errors could potentially be incorporated, possibly using ARCH effects.

More analysis of the behaviour of economic variables and markets could be undertaken, particularly into the decay of equity risk, both empirically and theoretically, and the stationarity of economic variables in general. The proposed model implies certain insider trading profits, and it would be interesting to measure the extent of such excess returns available.

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8. References

- Bachelier L (1900), *Theorie de la speculation*, Gauthier-Villars, Paris
- Bank of Japan (2001), *Bank of Japan Statistics*, Bank of Japan
- Bannock G (1998), *Dictionary of Economics*, Penguin Books
- Black F, Jensen M & Scholes M (1972), *The Capital Asset Pricing Model: Some Empirical Tests*, *Studies in the Theory of Capital Markets*, Praeger Publishers, New York from Jensen (1979)
- Blattenberg RC & Gonedes NJ (1974), *A Comparison of the Stable and Student Distributions as Statistical Models for Stock Prices*, *Journal of Business*, Vol 47, p 244 - 280
- Booth P (1997), *Knowledge, Wisdom or Understanding*, The Actuary, Staple Inn Actuarial Society
- Box GEP & Jenkins GM (1976), *Time Series Analysis, Forecasting and Control*, Holden-Day, San Francisco
- Cairns AJG (1999), *A Multi-factor Equilibrium Model for the Term Structure and Inflation*, Proceedings of the 9th Annual AFIR Colloquium Tokyo, p 93 - 113
- Capital Adequacy Sub-Committee (1992), *Standard for Determination of Capital Adequacy Reserves*, Transactions of the Institute of Actuaries of Australia, p 643 - 715
- Chan WS & Wang S (1998), *The Wilkie Model for Retail Price Inflation Revisited*, *British Actuarial Journal*, Vol 4, p 637 – 652
- Chaplin GB (1998), *A Review of Term-Structure Models and their Applications*, *British Actuarial Journal*, Vol 4, p 323 - 349
- Chen C & Lui LM (1993), *Joint estimation of model parameters and outlier effects in time series*, *Journal of the American Statistical Association*, Vol 88, p 284 - 297
- Claasen MS & Huber PP (1992), *Integrating Pension Fund Assets and Liabilities: Techniques for the 1990's*, Transactions of the Actuarial Society of South Africa, Vol 9, p 209 - 224
- Claasen MS (1993), *A Systematic Approach to the Financial Management and Risk Control of Employee Benefit Funds*, 3rd AFIR International Colloquium
- Campbell JY, Lo AW & MacKinlay AC (1997), *The Econometrics of Financial Markets*, Princeton University Press
- Campbell JY & Shiller RJ (1987), *Cointegration and Tests of Present Value Models*, *Journal of Political Economy*, Vol 95, p 1062 – 1088
- Campbell JY & Viceira LM (2002), *Strategic Asset Allocation*, Oxford University Press
- Clark G (1992), *Asset & Liability Modelling - The way ahead?*, Staple Inn Actuarial Society

- Clarkson RS (1991), *A Non-Linear Stochastic Model for Inflation*, Transactions of 2nd AFIR International Colloquium, Brighton, Vol 3, p 233 – 253
- Clarskson RS & Plymen J (1988), *Improving the Performance of Equity Portfolios*, Journal of the Institute of Actuaries, Vol 115, p 631 – 675
- Clarkson RS (1996), *Financial Economics - an Investment Actuary's Viewpoint*, British Actuarial Journal, Vol 2, p 809 – 947
- Cox JC, Ingersoll JE and Ross SA (1985), *A Theory of the Term Structure of Interest Rates*, Econometrica, Vol 53, p 385-407
- Cragg JG & Malkiel BG (1982), *Expectations and the Structure of Share Prices*, University of Chicago Press
- Dempster MAH, Germano M, Medova EA & Villaverde, *Global Asset Liability Management*, British Actuarial Journal, Vol 9, p 137 – 195
- Dimson E, Marsh P & Staunton M (2002), *Triumph of the Optimists: 101 Years of Global Investment Returns*, Princeton University Press
- Dyson ACL & Exley CJ (1995), *Pension Fund Asset Valuation and Investment*, British Actuarial Journal, Vol 1, p 471 - 540
- Elton EJ & Gruber MJ (1995), *Modern Portfolio Theory and Investment Analysis*, John Wiley & Sons
- Engle RF (1982), *Autoregressive Conditional Heteroscedasticity with Estimates at the Variance of United Kingdom Inflation*, Ecometrica, Vol 55, p 987 – 1007
- Ennis PW & Dorrington RE (1994), *A Beginners Guide to Asset and Liability Modelling*, Actuarial Society of South Africa
- Eves R (1998), *The Economist World in Figures*, Profile Books, p 22
- Fama EF (1965), *The Behaviour of Stock Market Prices*, *Journal of Business*, Vol 38, p 47-48
- Fama EF (1976), *Foundations of Finance*, Basic Books, New York
- Fama EF (1968), *Risk, Return and Equilibrium: Some Clarifying Comments*, *Journal of Finance*, Vol 23(1), p 29-40
- Fama EF & French K (1988), *Permanent and Temporary Components of Stock Prices*, *Journal of Political Economy*, Vol 96, p 246-273
- Finkelstein GS (1997), *Maturity Guarantees Revisited: allowing for Extreme Stochastic Fluctuations using Stable Distributions*, *British Actuarial Journal*, Vol 3, p 411 - 482
- Fisher I (1907), *The Rate of Interest*, Macmillan Company, New York
- Fisher I (1930), *The Theory of Interest*, Macmillan Company, New York

Geoghegan TJ, Clarkson RS, Feldman KS, Green SJ, Kitts A, Lavecky JP, Ross FJM Smith WJ & Toutounchi A (1992), *Report on the Wilkie Stochastic Investment Model*, Journal of the Institute of Actuaries, Vol 119, p 173

Gonedes N (1973), *Evidence on the Information Content of Accounting Numbers: Accounting Based and Market-Based Estimates of Systematic Risk*, Journal of Financial and Quantitative Analysis, Vol 8, p 407-444

Finkelstein GS (1997), *Maturity Guarantees Revisited: allowing for Extreme Stochastic Fluctuations using Stable Distributions*, British Actuarial Journal, Vol 3, p 411 - 482

Haberman S, Day C, Fogarty D, Khoransanee MZ, McWhirter M, Nash N, Ngwira B, Wright ID & Yakoubov (2003), *A Stochastic Approach to Risk Management and Decision Making in Defined Benefit Pension Schemes*, British Actuarial Journal, Vol 9, p 493 - 586

Hardy MR (1993), *Stochastic Simulation in Life Office Solvency Assessment*, Journal of the Institute of Actuaries, Vol 120, p 131 – 152

Harris GR (1995), *Statistical Data Analysis and Stochastic Asset Model Validation*, International Congress of Actuaries, Brussels

Harris GR (1996), *Market Phases and Cycles? A Regime Switching Approach*, The Institute of Actuaries of Australia Quarterly Journal, Part 2, p 28 – 44

Harris GR (1999), *Markov Chain Monte Carlo Estimation of Regime Switching Vector Autoregressions*, ASTIN Bulletin, Vol 4, p 47 - 79

Hibbert J, Mowbray P & Turnbull CJ (2001), *A stochastic asset model & calibration for long-term financial planning purposes*, 2001 Finance and Investment Conference, Institute of Actuaries

Henning CN (1984), *Financial Markets and the Economy*, 4th Edition, Prentice-Hall

Hess JH & Reinganum MR (1979), *Efficient Capital Markets*, Handbook of Financial Economics, Bicksler JL, Editor, North-Holland Publishing Company

Ho TSY and Lee SB (1986), *Term Structure Movements and Pricing Interest Rate Contingent Claims*, Journal of Finance, Vol 41, p 1011-1029

Howie RJ (1997), *Investment Risk and Asset Modelling - Can Financial Economics Be of Use?*, Transactions of the Actuarial Society of South Africa, Vol 11, p 148 – 173

Howie RJ and Davies H (2002), *Setting Investment Strategy for the Long Term – A closer look at Defined Contribution Investment Strategy*, Finance and Investment Conference, Institute of Actuaries.

Huber PP (1997), *A Review of Wilkie's Stochastic Asset Model*, British Actuarial Journal, Vol 3, p 181 - 210

Huber PP (1998), *A Note on the Jump-Equilibrium Model*, British Actuarial Journal, Vol 4, p 615 – 636

Huber PP & Verrall RJ (1999), *The Need for Theory in Actuarial Economic Models*, British Actuarial Journal, Vol 5, p 377 – 395

Hull JC (2006), *Options, Futures and Other Derivatives*, Prentice Hall

Hull JC & White A (1990), *Pricing Interest-Rate-Derivative Securities*, Review of Financial Studies, Vol 3 p 573-592

I-NET (1999), Deutsche Morgan Grenfell

Jensen MC (1979), *Tests of Capital Market Theory and Implications of the Evidence*, Handbook of Financial Economics, Bicksler J.L., Editor, North-Holland Publishing Company

Jones PD (1993), *The Expected Return on United Kingdom Equities and an Implication for Pension Fund Valuations*, Journal of the Institute of Actuaries, Vol 120, p 253 – 291

Jensen MC (1979), *Tests of Capital Market Theory and Implications of the Evidence*, Handbook of Financial Economics, Bicksler J.L., Editor, North-Holland Publishing Company

Kemp M (1992), *Life Office Investment Strategy*, Staple Inn Actuarial Society

Keynes JM (1923), *A Tract on monetary reform*, Reprinted 1971 in the collected writings of John Maynard Keynes, Vol 4, Macmillan for the Royal Economic Society

Lee PJ & Wilkie AD (2000), *A comparison of stochastic asset models*, Investment Conference, Institute and Faculty of Actuaries

Lo AW and MacKinlay AC (1988), *Stock Market Prices Do Not Follow Random Walks: Evidence from a Simple Specification Test*, Review of Financial Studies, Vol 1, p 41-66

Kitts A (1990), *Comments on a Model of Retail Price Inflation*, Journal of the Institute of Actuaries, Vol 117, p 407 - 412

Lane GB (1995), *A Practical Guide to Stochastic Asset Modelling*, Transactions of the Actuarial Society of South Africa, Vol 10, p 271 - 285

Lang J (1998), *Asset Liability Modelling for Life and General Insurance Companies*, Institute of Actuaries of Australia.

Lintner J (1956), *Distribution of Incomes of Corporation Among Dividends, Retained Earnings and Taxes*, American Economic Review Proceedings, Vol 66 p 97-113

Lintner J (1965), *The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets*, Review of Economics and Statistics, Vol 47, p 13-37

Maitland AJ (1996), *A Review of Thomson's Stochastic Investment Model*, Transactions of the Actuarial Society of South Africa, Vol 11, p 437 – 490

Mandelbrot B (1963), *The Variation of Certain Speculative Prices*, Journal of Business, Vol 36 p 394-419

Markowitz HM (1952), *Portfolio Selection*, Journal of Finance, Vol 7, p 77 - 91

Marsh TA & Merton RC (1986), *Dividend Variability and Variance Bounds Tests for the Rationality of Stock Market Prices*, The American Economic Review, Vol 76 No 3, p 483-498

Maturity Guarantees Working Party (1980), *Report of the Maturity Guarantees Working Party*, Journal of the Institute of Actuaries, Vol 107, p 103 – 214

McCutcheon JJ & Scott WF (1986), *An Introduction to the Mathematics of Finance*, Heinemann Professional Publishing

Mehta SJB (1992), *Allowing for asset, liability and business risk in the valuation of a life office*, Journal of the Institute of Actuaries, Vol 119, p 385 - 455

Miller MH & Modigliani F (1958), *The Cost of Capital, Corporation Finance and the Theory of Investment*, American Economic Review, Vol 48 p 261-297

Moore PG (1972), *Mathematical Models in Portfolio Selection*, Journal of the Institute of Actuaries, Vol 98, p 103 - 130

Mossin J (1966), *Equilibrium in a Capital Asset Market*, Econometrica, Vol 34, p 768-83

Officer RR (1976), *A Time Series Examination of the Market Factor of the New York Stock Exchange*, PhD dissertation, University of Chicago

Osborne MFM (1959), *Brownian Motion in the Stock Market*, Operations Research, Vol 7, p 165-173

Poterba J & Summers (1986), *The Persistence of Volatility and Stock Market Fluctuations*, American Economic Review, Vol 76, p 1142-1151

Redington FM (1952), *Review of the Principles of Life-Office Valuations*, Journal of the Institute of Actuaries, Vol 78, p 287 - 315

Richard S (1978), *An Arbitrage Model of the Tern-Structure of Interest Rates*, Journal of Financial Economics, Vol 6 p 33-57

Richardson M & Stock J (1989), *Drawing Inferences from Statistics Based on Multi-Year Asset Returns*, Journal of Financial Economics, Vol 25, p 323-348

Richardson M (1993), *Temporary Components of Stock Prices: A Skeptic's View*, Journal of Business and Economic Statistics, Vol 11, p 199-207

RIMES (2003), RIMES Technologies Corporation

Ross SA (1976), *The Arbitrage Theory of Capital Asset Pricing*, Journal of Economic Theory, Vol 13, p 341 - 60

Ross SA (1977), *Risk, Return and Arbitrage*, *Risk and Return in Finance*, Cambridge, Massachusetts, Vol 1, p 189 - 222

Roff T (1992), *Asset and Liability Studies on a With Profit Fund*, Staple Inn Actuarial Society

Samuelson PA (1965), *Proof That Properly Anticipated Prices Fluctuate Randomly*, *Industrial Management Review*, Vol 6, p 41-49

Sharpe WF (1963), *A Simplified Model for Portfolio Analysis*, *Management Science*, Vol 9, p 277 - 293

Sharpe WF (1964), *Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk*, *Journal of Finance*, Vol 19, p 425 – 42

Shiller RJ (1981), *The Use of Volatility Measures in Assessing Market Efficiency*, *Journal of Finance*, Vol 36, p 291-304

Smith AD (1996), *How Actuaries can Use Financial Economics*, *British Actuarial Journal*, Vol 2, p 1057 – 1174

Smith AD, van Bezoooyen J, Guthrie KA, Howie RJ, Ludvik PM & Mehta SJB (2000), *Consistent assumptions for multinational asset models*, Finance and Investment Conference, Institute and Faculty of Actuaries

The Economist (2003), *Economic Indicators*, The Economist Newspaper Limited, March 29th 2003, p 112

Thomson CG, Baron ID, Dann M, Hardy MR, Hardwick S, Macdonald AS, Sanders AJ & Wilson D (1995), *Stochastic Modelling for Life Offices*, Joint Actuarial Working Party

Thomson RJ (1994), *A Stochastic Investment Model for Actuarial Use in South Africa*, *Transactions of the Actuarial Society of South Africa*, Vol 9

Thomson RJ (1996), *Stochastic Investment Modelling: the Case of South Africa*, *British Actuarial Journal*, Vol 2, p 765 – 802

Thomson RJ (2003), *The Use of Utility Functions for Investment Channel Choice in Defined Contribution Retirement Funds II: A Proposed System*, *British Actuarial Journal*, Vol 9, p 903 – 958

Thomson RJ (2005), *The pricing of liabilities in an incomplete market using dynamic mean–variance hedging*, *Insurance: Mathematics & Economics*, Vol 36, p 441-455

Thornton PN & Wilson AF (1992), *A Realistic Approach to Pension Funding*, *Journal of the Institute of Actuaries*, Vol 119, p 229 - 312

Vasicek OA (1977), *An Equilibrium Characterization of the Term-Structure of Interest Rates*, *Journal of Financial Economics*, Vol 5 p 177-188

Whitten SP & Thomas RG (1999), *A Non-linear Stochastic Asset Model for Actuarial Use*, *British Actuarial Journal*, Vol 5, p 919 - 953

- Wilkie AD (1981), *Indexing Long Term Financial Contracts*, Journal of the Institute of Actuaries, Vol 108, p 299
- Wilkie AD (1985), *Portfolio Selection in the Presence of Fixed Liabilities: a Comment on 'The Matching of Assets to Liabilities'*, Journal of the Institute of Actuaries, Vol 112, p 229
- Wilkie AD (1986), *A Stochastic Investment Model for Actuarial Use*, Transactions of Faculty of Actuaries, Vol 39, p 341 - 373
- Wilkie AD (1987), *An Option Pricing Approach to Bonus Policy*, Journal of the Institute of Actuaries, Vol 114, p 21 - 77
- Wilkie AD (1995), *More on a Stochastic Asset Model for Actuarial Use*, British Actuarial Journal, Vol 1, p 777 – 963
- Wilkie AD (1995b), *The Risk Premium on Ordinary Shares*, British Actuarial Journal, Vol 1, p 251 - 293
- Wilkie AD (1996), *Notes for a Discussion on the European Single Currency*, Institute and Faculty of Actuaries
- Wilkie AD, Waters HR & Yang SY (2003), *Reserving, Pricing and Hedging for Policies with Guaranteed Annuity Options*, British Actuarial Journal, Vol 9, p 263 - 391
- Wise AJ (1984a), *A Theoretical Analysis of the Matching of Assets to Liabilities*, Journal of the Institute of Actuaries, Vol 111, p 375 - 401
- Wise AJ (1984b), *The Matching of Assets to Liabilities*, Journal of the Institute of Actuaries, Vol 111, p 445 - 485
- Wise AJ (1987), *Matching and Portfolio Selection Parts 1 and 2*, Journal of the Institute of Actuaries, Vol 114, p 113 - 133 and 551 – 568
- Yakoubov YH, Teeger MH & Duval DB (1999), *The TY model: A stochastic investment model for asset and liability management*, Staple Inn Actuarial Society
- Ziemba WT & Mulvey JM (1998), *Worldwide Asset and Liability Modeling*, Cambridge University Press

Appendices

Appendix A – Data

All data is for the 15 years 30/06/1983 to 30/06/1998, or less if data is not available for the full period.

GDP Growth

Data is available for an inflation adjusted GDP index, $GDPR_t$.

The annual log (1 + real economic growth) in year t = $\log [GDPR_t / GDPR_{t-1}]$

= $\log [GDPR_t] - \log [GDPR_{t-1}]$.

The following data series were used:

Gross Domestic Product			
Country	I-NET code	Description	Notes
United States	USAGPR	Total GDP at constant prices	-
Britain	UKGPR	Total GDP at constant prices	-
South Africa	NGDPA	Total GDP at constant prices	-

Inflation

Data is available for a retail price index, CPI_t .

The annual log (1 + inflation rate) in year t = $\log [CPI_t / CPI_{t-1}] = \log [CPI_t] - \log [CPI_{t-1}]$.

The following data series were used:

Inflation			
Country	I-NET code	Description	Notes
United States	USCPI	Consumer Price Index	-
Britain	UKCPI	Consumer Price Index	-
South Africa	ECPI	Consumer Price Index	-

Short-term Interest Rates

Data is available for a short term interest rate, INT_t (which is expressed as an annualised rate).

Monthly data was obtained, and the return in one month is assumed to be MR_t where

$$MR_t = [INT_t + INT_{t-1}] / 24$$

INT_t = annualised interest rate at end of month

INT_{t-1} = annualised interest rate at start of month.

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The annual log (1 + return) in one year is therefore = $\sum_{t=1} \log (MR_t + 1)$

The series for South Africa is a money market index, MMI_n , and not an interest rate, so the annual (1 + log return) in year n simply $\log [MMI_n / MMI_{n-1}]$. The annual log (1 + real return) is the nominal return less the log (1 + inflation rate). The following data series were used:

Interest Rates			
Country	I-NET code	Description	Notes
United States	USTB3M	3 Month Treasury Bill	
	UDDPTN	Call Rate	from 30/06/1994 onwards
Britain	UKODR	Official Discount Rate	
	GPDPTN	Call Rate	from 30/06/1994 onwards
South Africa	GMC1	Money Market Index	Index, not rate

Bonds

Yields

Data is available for a government bond yield, BY_t . This is a yield convertible half yearly, so the annual $\log (1 + \text{yield}) = \log [(1 + BY_t / 2)^2]$.

Returns

Let $C_t = (1 + BY_t / 2)^2 - 1$. Then assume that a bond at time t has term of n years, pays coupons of C_t annually in arrears, has just been issued, has effective annual yield of C_t , and has a price of $P_t = 1$ (as the bond will stand at par). One year later, at time (t+1), the bond will have outstanding term of (n-1) years, pay a coupon of C_t , have an effective annual yield of C_{t+1} , and have a price of P_{t+1} (where $P_{t+1} = C_t a_{n-1} + v^{n-1}$ discounted at rate C_{t+1}).

The nominal return in one year (t) is approximately R_t where

$$R_t = [P_{t+1} + C_t] / P_t - 1 = P_{t+1} + C_t - 1.$$

The log (1 + return) is then $\log (R_t + 1)$, and annual log (1 + real return) is the nominal return less the log (1 + inflation rate). The following data series were used:

Country	I-NET code	Description	Notes
United States	USGB2Y	2 Year Gov Bond Yield	Series starts 31/07/1993
	USGB5Y	5 Year Gov Bond Yield	Series starts 31/07/1993
	USGB10	10 Year Gov Bond Yield	Series starts 30/06/1993
	USGB30	30 Year Gov Bond Yield	-
Britain	UKGB5Y	5 Year Gov Bond Yield	Series starts 31/05/1997
	UKGB10	10 Year Gov Bond Yield	-
	UKGB30	30 Year Gov Bond Yield	Series starts 31/05/1997
South Africa	JAYC02	2 Year Gov Bond Yield	Series starts 31/01/1986
	JAYC05	5 Year Gov Bond Yield	Series starts 31/01/1986
	JAYC10	10 Year Gov Bond Yield	Series starts 31/01/1986
	JAYC30	30 Year Gov Bond Yield	Series starts 31/10/1988

Equities

Dividends and Earnings

Data is available for a share price index P_t , dividend yield DY_t , and earnings yield EY_t , which was obtained annually. A dividend index D_t can be defined as $P_t \cdot DY_t$, and the dividend growth in year n is approximately $[D_{n+1/2} / D_{n-1/2} - 1]$. This is because the dividend index D_t relates to dividends that were paid on average 6 months ago. The $\log(1 + \text{dividend growth})$ is then $\log[D_{n+1/2}] - \log[D_{n-1/2}]$. An earnings index and earnings growth rates can be calculated in the same way. The following data series were used:

<u>Equity Dividend & Earnings Growth</u>			
Country	I-NET code	Description	Notes
United States	FSPI	S & P 500 Composite Index	-
	FSPIDY	Dividend Yield	Series starts 31/01/1992
	FSPIEY	Earnings Yield	-
Britain	FTALL[C]	FTSE All Share Index	Series starts 28/02/1993
	FTALL[Y]	Dividend Yield	Series starts 28/02/1993
	FTALL[X]	Earnings Yield	Series starts 28/02/1993
South Africa	CI01[C]	JSE All Share Index	-
	CI01[Y]	Dividend Yield	-
	CI01[X]	Earnings Yield	-

FT Actuaries Indices

Data is available on I-NET (1999) from 31/10/1987 onwards for a share price index P_t and dividend yield DY_t . The indices are in US\$, and are converted into local currency by multiplying by the relevant exchange rate. Dividend growth can be calculated in the same way as above. Monthly data was obtained, and the return in one month is approximately MR_t where

$$MR_t = [P_t + D_t / 12] / P_{t-1} - 1, \text{ where}$$

P_t = price index in local currency at end of month t

D_t = dividend index in local currency at end of month t.

The annual log (1 + return) in one year is then
$$= \sum_{t=1}^{12} \log (MR_t + 1)$$

The following data series were used:

<u>FT Actuaries Indices</u>			
Country	I-NET code	Description	Notes
United States	FTUSAU[C]	FT Actuaries Index in US\$	-
	FTUSAU[Y]	Dividend Yield	-
Britain	FTUKMU[C]	FT Actuaries Index in US\$	-
	FTUKMU[Y]	Dividend Yield	-
	USDGBP	Dollar Pound exchange rate	-
South Africa	FTSAFU[C]	FT Actuaries Index in US\$	-
	FTSAFU[Y]	Dividend Yield	-
	USDZAR	Dollar Rand exchange rate	-

Appendix B – Simulations of the Stochastic Model

This appendix describes how outcomes of the economic and financial market variables were simulated.

Economic Model

The initial starting values for $\mu_{\text{inflation},0}$, $\mu_{\text{realrowth},0}$, and $\mu_{\text{realinterest},0}$ were set equal to the observed mean over the period. Initial values for the economic variables were set equal to the initial means as follows:

$$\text{inflation}_0 = \mu_{\text{inflation},0}$$

$$\text{realgrowth}_0 = \mu_{\text{realrowth},0}$$

$$\text{realinterest}_0 = \mu_{\text{realinterest},0}$$

These can be described as neutral starting positions, and it means that the unconditional expectation of each variable for all $t \geq 0$ equals the initial value of the variable.

Financial Market Model

The coupons on bonds are set equal to $\exp[\text{inflation}_0 + \text{realinterest}_0 + \text{riskpremium}_{\text{term,coupon}}] - 1$ so that bonds stand at par at time zero. The initial smoothed earnings value, $\text{smoothearnings}_{.1}$, was set equal to $\text{inflation}_0 + \text{realgrowth}_0$, and risk premiums were originally set at zero.

667 simulations of returns over 15 years were produced, which equates to 10,005 individual year simulations. These simulations were used to calculate the standard deviation of returns and re-estimate the risk premiums, using the volatility from the previous simulation and the risk premium model as specified in Section 5.1.2.1. This iterative process was repeated until the standard deviation of returns was calculated to an accuracy of two decimal places.