\[
\frac{\partial y}{\partial x_j} = \text{derivative of } y \text{ w.r.t. } x_j \text{, while holding all the other } x's \text{ constant.}
\]

What if we cant? ie if \( x_2 \) changes when \( x_3 \) changes?

If we dont have explicit functional forms, we will also not able to solve for our \( y's \) in terms of \( x's \) - we will have interdependence:

\[
\begin{align*}
\text{eg: } & y = C + I_o + G_o \\
& C = C(y, T_o) \\
& \text{To (exog tax)}
\end{align*}
\]

\[
Y = C(Y, T_o) + I_o + G_o
\]

Solving for \( Y \) as an explicit fn is not possible.
Assume $y^*$ exists, under certain conditions.

$$y^* = y^*(I_0, q_0, T_0)$$

& $y^*$ is differentiable.

In some neighbourhood around $y^*$, the identity holds:

$$y^0 = C(y^*, T_0) + I_0 + q_0$$

= Eqn identity.

What is $\frac{\partial y^*}{\partial T_0}$?

$y^*$ is a fn of $T_0$;

$C(y^*, T_0)$ has 2 interdependent arguments.

$T_0$ affects $C$ directly &

$T_0$ affects $C$ indirectly through $y$.

We need total differentiation.
Differentials

\[ \frac{dy}{dx} = f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \]

\[ \frac{\Delta y}{\Delta x} = \frac{dy}{dx} \]

We can say

\[ \frac{\Delta y}{\Delta x} - \frac{dy}{dx} = \delta \text{ where } \delta \to 0 \text{ as } \Delta x \to 0 \]

\[ \Delta y = \frac{dy}{dx} \cdot \Delta x + \delta \Delta x \]

\[ \Delta y = f'(x) \cdot \Delta x + \delta \Delta x \]

\[ f'(x) \Delta x \text{ approximates } \Delta y, \text{ as } \Delta x \to 0. \]
We have \( y = f(x) \).

Slope of \( AD = f'(x) \).

\[
\begin{align*}
\text{CB} &= \Delta y \\
\text{CB} &= \text{CD} + \text{BD} \\
\frac{\text{CB}}{\text{AC}} &= \frac{\Delta y}{\Delta x} \\
\text{CD} &= f'(x_0)\Delta x \quad (\text{Why?} \quad \frac{\text{DC}}{\text{AC}} = \text{slope}) \\
\quad \begin{aligned}
\text{DC} &= \text{AC} \cdot f'(x_0) = \Delta x \cdot f'(x_0)
\end{aligned}
\end{align*}
\]

We know from \( 8.3 \):

\[
\Delta y = f'(x)\Delta x + 8\Delta x
\]

\[
\Delta y = \text{DC} + 8\Delta x
\]

DB = 8\Delta x

As \( \Delta x \to 0 \), B slides towards A.

\[\Delta y \to f'(x)\Delta x\]

\[f'(x)\] becomes a better approx to \( \frac{\Delta y}{\Delta x} \).
relabel AC & CD by \( dx \) & \( dy \) 
\( (CD \text{ in approximation to } CB \text{ as } \Delta x \to 0) \)

\[
\frac{dy}{dx} = \text{slope tangent } AD = f'(x)
\]

\[dy = f'(x)dx\]

- \( dy, \ dx \) = differentials.
- If we know \( dx \), multiply by \( f'(x) \) to get \( dy \).
- We have just shown it is OK to separate \( dy, dx \) previously, only talked about \( \frac{dy}{dx} \) as 1 entity.
- \( dy \) is dependent, \( dx \) independent.
- \[dy = f_n(x, dx) = f'(x)dx\]
- If \( dx = 0 \), \( dy = 0 \).
- \( f'(x) \) converts change \( dx \) into a change \( dy \).
Differentials & Elasticity

DD fn: \( Q = f(P) \)

elasticity \( E_d = \frac{\Delta Q / \Delta P}{\frac{Q}{P}} \)

point elasticity \( E_d = \left( \frac{dQ}{dP} \right) \left( \frac{P}{Q} \right) \)

ie as \( \Delta P \to 0 \)

\[
E_d = \left( \frac{dQ}{dP} \right) \left( \frac{P}{Q} \right)
\]

= marginal fn

average fn

Remember

\( E = 1 \) unit elasticity
\( E < 1 \) inelastic
\( E > 1 \) elastic

P182 eg 1

P182 eg 2: \[ S \text{ fn: } Q = P^2 + 7P \]

Is supply elastic at \( P = 2 \)?

\[ \frac{dQ}{dP} = 2P + 7 \quad Q/P = P + 7 \]

\( E_s = \frac{2P + 7}{P + 7} = \frac{11}{9} \) when \( P = 2 \)

elastic
Using $E = \frac{\text{marginal}}{\text{avg}}$ gives a good graphical way to see pt elasticity.

$y = f(x)$

AB is tangent to $y = f(x)$

Slope AB

$= \frac{\text{marginal fn}}{}$

Slope $OA = \frac{\text{avg fn}}{}$. Why?

At A, $y = x_0A$ & $x = Ox_0$ (labelled distances)

slope of CA = $\frac{x_0A}{Ox_0} = \frac{y}{x}$ (ray from origin)

Average fn = $\frac{y}{x}$

... elasticity at pt A:

if AB is steeper than OA (marg)

then elastic at A,

if AB flatter, inelastic.

Above is it in elastic
See other graphs P183 for comparison.

Also, if $\Theta_m < \Theta a$

$\implies$ marg < avg
$\implies$ inelastic,

if $\Theta_m > \Theta a$, elastic.

Unit Elasticity?

- At a pt where the tangent to f(x) passes through the origin & lies on the ray,
  $\Theta_m = \Theta a$.

- This assumes y is on vertical axis, i.e. for supply, we must have Q on the vertical axis.
Total Differentials

\[ S = S(y, i) \]

- Assume \( S \) is continuous & has continuous partial derivatives.

- For any change in \( y \), \( dy \),

\[ dS = \frac{\partial S}{\partial y} . dy \]

Similarly for \( di \).

\[ dS = \frac{\partial S}{\partial y} . dy + \frac{\partial S}{\partial i} . di \]

If \( i \) doesn't change as \( y \) changes, \( di = 0 \).

\[ dS = \text{total differential} \]

\[ \frac{\partial U}{\partial y} = U(x_1, \ldots, x_n) \]

\[ dU = \sum \frac{\partial U}{\partial x_i} . dx_i \]

\[ U_i = \frac{\partial U}{\partial x_i} \]
\( \sum_{i} u_{i} \, dx_{i} = \text{marginal utility of good } i \text{ consumed} \)

\( du = \text{change in } U \text{ from all possible sources of change} \)

\( S = S(y, i) \text{ now has 2 elasticities} \)

\( U = U(x_{i}, i = 1, 2, \ldots, n) \text{ has } n \)

\[ \begin{align*}
\sum_{i} u_{i} & = \frac{\partial U}{\partial x_{i}} \cdot \frac{x_{i}}{U} \quad i = 1, 2, \ldots, n. \\
\end{align*} \]

\( P \) \( \frac{18b}{x_{2}} \)

\[ u(x_{1}, x_{2}) = x_{1}^{a} x_{2}^{b} \]

\[ du = \left( ax_{1}^{a-1} x_{2}^{b} \right) dx_{1} + \left( bx_{1}^{a} x_{2}^{b-1} \right) dx_{2} \]

\[ \frac{\partial u}{\partial x_{1}} \]

\[ \frac{\partial u}{\partial x_{2}} \]

\[ \sum_{i} u_{i} = \frac{\partial u}{\partial x_{i}} \left/ \frac{U}{x_{i}} \right. \]

\[ = bx_{1}^{a} x_{2}^{b-1} \left/ x_{1}^{a} x_{2}^{b} \right. = b \]
Rules of Differentials

\[ k = \text{constant}, \quad u, v = \text{fns}(x_1, x_2) \]

1. \( dk = 0 \)
2. \( d(u^n) = cn u^{n-1} du \)
3. \( d(u \pm v) = du \pm dv \)
4. \( d(uv) = vdu + u dv \)
5. \( d(u/v) = \frac{v du - u dv}{v^2} \)

\{ \text{no proofs given} \}

Introduce \( w = \text{fns}(x_1, x_2) \)

6. \( d(u \pm v \pm w) = du \pm dv \pm dw \)
7. \( d(uvw) = vw du + uv dw + uwdv \)

\[ \text{ex} \quad y = \frac{x_1 + x_2}{2x_1^2} \]

\[ dy = \frac{\partial y}{\partial x_1} \cdot dx_1 + \frac{\partial y}{\partial x_2} \cdot dx_2 \]

\[ = \left( \frac{-(x_1 + 2x_2)}{2x_1^3} \right) dx_1 + \left( \frac{1}{2x_1^2} \right) dx_2 \]

\# Can prove \#6 & \#7 using \#1-\#5
Total Derivatives

L do not require \( x_1 \) to stay constant when \( x_2 \) changes,
if \( y = f(x_1, x_2) \)

eg \( y = f(x, w) \) & \( x = g(w) \)

\[ y = f(g(w), w) \]

\( w \) affects \( y \) directly, & indirectly through \( x \).

\[
\frac{dy}{dx} = \frac{\partial f}{\partial x} \cdot dx + \frac{\partial f}{\partial w} \cdot dw
\]

\[
\frac{dy}{dw} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dw} + \frac{\partial f}{\partial w}
\]

\[
= f_x \cdot g^1 + f_w \text{ indirect direct effect}
\]
13

\[
\begin{align*}
y & = f(x,w) = 3x - w^2 \\
x & = g(w) = 2w^2 + w + 4
\end{align*}
\]

\[
\frac{dy}{dw} = \frac{dy}{dx} \cdot \frac{dx}{dw} + \frac{dy}{dw}
\]

\[
= 3 \cdot (4w + 1) + (-2w)
\]

\[
= 12w + 3 - 2w = 10w + 3
\]

eg \[ y = f(x_1, x_2, w) \quad x_1 = g(w) \quad x_2 = h(w) \]

\[
\begin{align*}
\frac{dy}{dw} & = f_1 \frac{dx_1}{dw} + f_2 \frac{dx_2}{dw} + f_w \frac{dw}{dw}
\end{align*}
\]

\[
\frac{dy}{dw} = f_1 \frac{dx_1}{dw} + f_2 \frac{dx_2}{dw} + f_w \quad \text{indirect} \quad \text{indirect} \quad \text{direct}
\]

eg \( y = f(x_1, x_2, u, v) \)

\[
\begin{align*}
x_1 & = g(u, v) \\
x_2 & = h(u, v)
\end{align*}
\]
\[ \frac{dy}{du} = \frac{\partial y}{\partial x_1} \frac{dx_1}{du} + \frac{\partial y}{\partial x_2} \frac{dx_2}{du} + \frac{\partial y}{\partial u} \]

\[ + \frac{\partial y}{\partial v} \frac{dv}{du} \]

We hold \( v \) constant to find total derivative of \( y \) w.r.t \( u \)

\[ \frac{dv}{du} = 0 \]

Which derivatives are partials?

- \( \frac{dy}{du} \) = partial total derivative
denoted \( \frac{\partial y}{\partial u} \)

- \( \frac{dx_1}{du}, \frac{dx_2}{du} \) are partials

\[ \frac{\partial y}{\partial u} = \frac{\partial y}{\partial x_1} \frac{dx_1}{du} + \frac{\partial y}{\partial x_2} \frac{dx_2}{du} + \frac{\partial y}{\partial u} \]

Similarly for partial total derivative \( \frac{dy}{dv} \)
Notes p193

1. total derivatives are expressions of the chain rule (fns of fns of fns)
2. Can extend to more than 3 fns
3. total derivatives measure rates of change w.r.t final variables (exog vars)

Implicit functions

\[ y = f(x) = \text{explicit function} \]
\[ y - f(x) = 0 \quad \text{implicit fn, which implies explicit } y = f(x) \]

Generally \( F(y, x) = 0 \)
\( f = \text{explicit fn - 2 arguments} \)
\( F = \text{implicit fn - 1 argument} \)

or \( F(y, x_1, x_2, \ldots, x_n) = 0 \)
with implicit fn \( y = f(x_1, x_2, x_n) \)

We can't always guarantee \( y = f(x) \) is defined.
\(16\)

\[x^2 + y^2 = 0\] is satisfied only at \((0, 0)\), so there is no fn \(y = f(x)\).

\[F(y, x) = x^2 + y^2 - 9 = 0\]

\[y = \pm \sqrt{9 - x^2}\]

\[x^2 + y^2 = 9\]

\[= \text{circle}\]

This is a relation, not a fn - no unique value of \(y\) for each \(x\) value.

So...

Are there known conditions s.t.

\[F(y, x_1, \ldots, x_m) = 0\]

defines \(y = f(x_1, \ldots, x_m)\)?

Around some pt in the domain of \(x\).
Implicit Function Theorem

Given $F$ as above, if

1. $F$ has continuous partial derivatives $F_γ, F_1, \ldots, F_m$ and if
2. at a point $(y_0, x_1, \ldots, x_m)$ satisfying $F(y, x_1, \ldots, x_m) = 0$, $F_γ \neq 0$ then...

there exists a neighbourhood $N$ (in dimensional) in which

$y = f(x_1, \ldots, x_m)$ is defined.

This function satisfies $y_0 = f(x_1, \ldots, x_m)$

Also, in the neighbourhood $N$

$F(y, x_1, \ldots, x_m) = 0$

Also, $f$ is continuous, and has
continuous partial derivatives.
\[ F(y, x) = x^2 + y^2 - 9 = 0 \]

Check

(a) \[ F_y = 2y \quad \text{are continuous} \]
\[ F_x = 2x \]

(b) \[ F_y \neq 0 \quad \text{if } y \neq 0. \]

Around any other pt excluding but \( \text{btwn } (-3, 0) \) \& \((3, 0) \)
there is a neighbourhood where \( y = f(x) \) is defined.

**NOTES**

- Conditions are sufficient, but not necessary.
  - eg if \( F_y = 0 \), may still be a fn \( f \) defined at that pt.
- Even if \( f \) exists, we don't know its specific form, or size of \( N \).
- Still helpful if don't have explicit fn \( f \) & guarantees existence of \( f_1, f_2, \ldots, f_m \).
\[19\]

Hold up! Proof of IF? No.
Given \( F(y, x_1, \ldots, x_m) = 0 \), & we know (using IF theorem) that \( y = f(x_1, \ldots, x_m) \) exists, but we can't solve for it — we can still find \( f_1, f_2, \ldots, f_m \). How?

**FACTS**

1. If \( A = B \) then \( dA = dB \)
2. Differentiate an expression with \( y, x_1, \ldots, x_m \), we get an expr with \( dy, dx_1, \ldots, dx_m \)
3. We can substitute for \( dy \).

   If we can't solve for \( y \):
   
   \[ F(y', x_1, \ldots, x_m) = 0 \]
   
   \[ F_y dy + F_1 dx_1 + F_2 dx_2 + \ldots + F_m dx_m = 0 \]

   \[ y = f(x_1, \ldots, x_m) \]
   
   \[ dy = f_1 dx_1 + \ldots + f_m dx_m \].

Substitute this in for \( dy \) above.
\[
F_y (f_1 dx_1 + f_2 dx_2 + \ldots + f_m dx_m) \\
+ F_1 dx_1 + F_2 dx_2 + \ldots + F_m dx_m = 0
\]
\[
(F_y f_1 + F_1) dx_1 + (F_y f_2 + F_2) dx_2 \\
+ \ldots + (F_y f_m + F_m) dx_m = 0
\]

For this to hold, we need that for each variable \(i\),
\[
F_y f_i + F_i = 0 \quad \text{Why?}
\]
\[
f_i = -\frac{F_i}{F_y}
\]
\[
\frac{\partial y}{\partial x_i} = -\frac{F_i}{F_y}
\]

For \(F(y, x) = 0\)
we get \(\frac{dy}{dx} = -\frac{F_x}{F_y}\).

Now we see why we need \(F_y \neq 0\).
Find \( \frac{\partial y}{\partial x} \) for an IF defined by \( F(y, x, w) = y^3x^2 + w^3 + yxw - 3 = 0 \).

We know \( \frac{\partial y}{\partial x} = -\frac{F_x}{F_y} \).

\[
\frac{\partial y}{\partial x} = -\frac{(2y^3x + yw)}{3y^2x^2 + xw}
\]

Wait Did we check condition of IF hold?

We need
1. \( F_y, F_x, F_w \) to be continuous & to exist
2. At a pt satisfying \( F(y, x, w) = 0 \), \( F_y \neq 0 \)

then \( y = f(x, w) \) is defined around that point & it's okay to use the rule to find

\( \frac{\partial y}{\partial x} \)
for \( y^3 x^2 + w^3 + yxw - 3 = 0 \)

(a) We have
\[
F_y = 3y^2 x^2 + xw \\
F_x = 2y^3 x^2 + yw \\
F_w = 3w^2 + yx
\]

continuous & exist

(b) find a pt satisfying \( F = 0 \)

\[
e g (1, 1, 1) \\
(1)^3 (1)^2 + (1)^3 + (1)(1)(1) - 3 = 0 \\
0 = 0
\]

At \((1,1,1)\) \( F_y = 3(1)^2 (1)^2 + (1)(1) \)
\[
= 4 \neq 0
\]

\( y = f(x, w) \) exists around \((1,1,1)\) & we can talk about \( \frac{dy}{dx} \).

Did we solve for \( y = f(x, w) \)?

No.
eg \( F(Q, K, L) \) implicitly defined production fn \( Q = f(K, L) \)

- We want to know \( MPP_K, MPP_L \)
  (why not just \( MP_K, MP_L \))?
- Assume \( F_Q, F_L, F_K \) exist & are continuous, & for a point which satisfies \( F(Q, K, L) = 0 \), \( F_Q \neq 0 \).

Then \( Q = f(K, L) \) exists, &

\[
\frac{\partial Q}{\partial L} = MPP_L = -\frac{F_L}{F_Q}
\]

& \( \frac{\partial Q}{\partial K} = MPP_K = -\frac{F_K}{F_Q} \)

Also \( \frac{\partial K}{\partial L} = -\frac{F_L}{F_K} \) what is this expr? How we can change \( K \) & \( L \), while keeping \( Q \) constant is slope of the isoquant \( \frac{\partial K}{\partial L} \)

But \( |\frac{\partial K}{\partial L}| = MRTS_{KL} \)
Simultaneous Eqn Case

aka life is never easy.

Given \( n \) functions \( F_1 \) to \( F^n \):
\[
F^1(y_1, y_2, ..., y_n, x_1, ..., x_m) = 0 \\
F^2(y_1, y_2, ..., y_n, x_1, ..., x_m) = 0 \\
F^n(y_1, y_2, ..., y_n, x_1, ..., x_m) = 0
\]
Under what conditions do these define:

\[
y_1 = f^1(x_1, ..., x_m) \\
y_2 = f^2(x_1, ..., x_m) \\
y_n = f^n(x_1, ..., x_m)
\]

**IF Theorem**

Given the set of \( F^1 \) above, if \( F^1 ... F^n \) all have continuous partial derivatives w.r.t. all \( y, x \) and if \( y \) at a point satisfying \( F^i = 0 \), \( |J| \neq 0 \) then \( y_i = f^i(x_1, ... x_m) \) is defined
The point is \((y_{10}, y_{20}, \ldots, y_{m0}), (x_{10}, \ldots, x_{m0})\).

The \(|J|\) is

\[
|J| = \begin{vmatrix}
\frac{\partial F^1}{\partial y_1} & \frac{\partial F^1}{\partial y_2} & \cdots & \frac{\partial F^1}{\partial y_n} \\
\frac{\partial F^2}{\partial y_1} & \frac{\partial F^2}{\partial y_2} & \cdots & \frac{\partial F^2}{\partial y_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial F^n}{\partial y_1} & \frac{\partial F^n}{\partial y_2} & \cdots & \frac{\partial F^n}{\partial y_n}
\end{vmatrix}
\]

Note w.r.t. \(y_i\)'s only.

\(|J| = \begin{vmatrix}
\frac{\partial F^1}{\partial y_1} & \frac{\partial F^1}{\partial y_2} & \cdots & \frac{\partial F^1}{\partial y_n} \\
\frac{\partial F^2}{\partial y_1} & \frac{\partial F^2}{\partial y_2} & \cdots & \frac{\partial F^2}{\partial y_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial F^n}{\partial y_1} & \frac{\partial F^n}{\partial y_2} & \cdots & \frac{\partial F^n}{\partial y_n}
\end{vmatrix}_{n \times n}
\]

We need \(|J| \neq 0\) for the pt satisfying the \(F^i = 0\) above, if so, there exists an \(m\)-dimensional neighbourhood \(N\) of \((x_{10}, x_{20}, \ldots, x_{m0})\) in which \(y_1, \ldots, y_n\) are fns of \(x_1, \ldots, x_m\) as above, & these fns satisfy

\[
y_{10} = f^1(x_{10}, \ldots, x_{m0})
\]

\[
y_{n0} = f^n(x_{10}, \ldots, x_{m0})
\]

\(
\Rightarrow\) the \(F^i\) are identities

\(
\Rightarrow\) \(f^i\) are continuous, have cont partial deriv.
We don't have to solve for the $y_i$ to find $\frac{\partial y_i}{\partial x_j}$ for any $i=1...n$, $j=1...m$.

We can take the total differential of the $F^i = 0$.

$dF^j = 0$ for $j=1,2,...,n$.

We get a set of equations with $dy_1, ... , dy_n; dx_1, ... , dx_n$.

Specifically, for $F^0(y_1, ... , y_n, x_1, ... , x_m) = 0$,

\[
\frac{\partial F^1}{\partial y_1} \ dy_1 + \frac{\partial F^1}{\partial y_2} \ dy_2 + ... + \frac{\partial F^1}{\partial y_n} \ dy_n \\
= - \left( \frac{\partial F^1}{\partial x_1} \ dx_1 + \frac{\partial F^1}{\partial x_2} \ dx_2 + ... + \frac{\partial F^1}{\partial x_m} \ dx_m \right)
\]

\[
\frac{\partial F^2}{\partial y_1} \ dy_1 + \frac{\partial F^2}{\partial y_2} \ dy_2 + ... + \frac{\partial F^2}{\partial y_n} \ dy_n \\
= - \left( \frac{\partial F^2}{\partial x_1} \ dx_1 + ... + \frac{\partial F^2}{\partial x_m} \ dx_m \right)
\]

...
We can write the $dy_i$ from $y_i=f(x_1, x_2, \ldots, x_m)$ as:

\[
\begin{align*}
    dy_1 &= \frac{\partial y_1}{\partial x_1} dx_1 + \frac{\partial y_1}{\partial x_2} dx_2 + \ldots + \frac{\partial y_1}{\partial x_m} dx_m \\
    dy_2 &= \frac{\partial y_2}{\partial x_1} dx_1 + \frac{\partial y_2}{\partial x_2} dx_2 + \ldots + \frac{\partial y_2}{\partial x_m} dx_m \\
    & \vdots \\
    dy_n &= \frac{\partial y_n}{\partial x_1} dx_1 + \frac{\partial y_n}{\partial x_2} dx_2 + \ldots + \frac{\partial y_n}{\partial x_m} dx_m
\end{align*}
\]

We substitute these into the expressions above for $dF^q = 0$.

E.g. for the 1st eqn: Set $dx_1 \neq 0$ other $dx^q = 0$

\[
\begin{align*}
    \frac{\partial F^1}{\partial y_1} \left( \frac{\partial y_1}{\partial x_1} dx_1 + \frac{\partial y_1}{\partial x_2} dx_2 + \ldots + \frac{\partial y_1}{\partial x_m} dx_m \right) \\
    + \frac{\partial F^1}{\partial y_2} dy_2 + \ldots + \frac{\partial F^1}{\partial y_n} = -\left( \frac{\partial F^1}{\partial x_1} dx_1 \right)
\end{align*}
\]

Collect terms.
\[ \frac{\partial F}{\partial y_1} \frac{\partial y_1}{\partial x_1} + \frac{\partial F}{\partial y_2} \frac{\partial y_2}{\partial x_1} + \ldots + \frac{\partial F}{\partial y_n} \frac{\partial y_n}{\partial x_1} \]
\[ = -\frac{\partial F}{\partial x_1} \]

\[ \frac{\partial F^2}{\partial y_1} \frac{\partial y_1}{\partial x_1} + \frac{\partial F^2}{\partial y_2} \frac{\partial y_2}{\partial x_1} + \ldots + \frac{\partial F^2}{\partial y_n} \frac{\partial y_n}{\partial x_1} \]
\[ = -\frac{\partial F^2}{\partial x_1} \]

\[ \vdots \]

\[ \frac{\partial F^n}{\partial y_1} \frac{\partial y_1}{\partial x_1} + \frac{\partial F^n}{\partial y_2} \frac{\partial y_2}{\partial x_1} + \ldots + \frac{\partial F^n}{\partial y_n} \frac{\partial y_n}{\partial x_1} \]
\[ = -\frac{\partial F^n}{\partial x_1} \]

WAIT
Why can we set \( dx_1 \neq 0 \), \( dx_1 = 0 \) if 1?

We want \( \frac{\partial y_1}{\partial x_1}, \frac{\partial y_2}{\partial x_1}, \ldots, \frac{\partial y_n}{\partial x_1} \)

ie derivatives of \( y \)'s w.r.t \( x \) holding other \( x_i \)'s constant

ie \( dx_i \)'s = 0. Why?
If we don't set other $dx_i$'s = 0, we have a huge mess.

Looking at our previous system, it has $\frac{\partial y_i}{\partial x_1}$; these we want $\frac{\partial y_i}{\partial x_1}$ to solve for.

It has $\frac{\partial E_i}{\partial y_i}$'s. At pt satisfying the $F_i$'s = 0, these derivatives are constants.

All in all, we get a linear system:

\[
\begin{bmatrix}
\frac{\partial F_1}{\partial y_1} & \frac{\partial F_1}{\partial y_2} & \cdots & \frac{\partial F_1}{\partial y_n} \\
\frac{\partial F_2}{\partial y_1} & \frac{\partial F_2}{\partial y_2} & \cdots & \frac{\partial F_2}{\partial y_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial F_n}{\partial y_1} & \frac{\partial F_n}{\partial y_2} & \cdots & \frac{\partial F_n}{\partial y_n}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial y_1}{\partial x_1} \\
\frac{\partial y_2}{\partial x_1} \\
\vdots \\
\frac{\partial y_n}{\partial x_1}
\end{bmatrix}
= \begin{bmatrix}
-\frac{\partial F_1}{\partial x_1} \\
-\frac{\partial F_2}{\partial x_1} \\
\vdots \\
-\frac{\partial F_n}{\partial x_1}
\end{bmatrix}
\]

The determinant of this $mx$ is $\det M$, & it is assumed $\neq 0$ if IF conditions are met.
This is a nonhomogeneous system (if RHS $-\frac{\partial F_i}{\partial x_1}$ were all $= 0$, with $|J| \neq 0$), only solution is trivial $\frac{\partial y_i}{\partial x_1} = 0$, which is not very interesting! So there should be a unique nontrivial solution.

Cramer's rule \[ \frac{\partial y_i}{\partial x_1} = \frac{|J_i|}{|J|}, \quad j=1,2,\ldots,n \]

- We can also get the other partial derivatives w.r.t $x_2, \ldots, x_m$ using same procedure.
- The matrix $J$ will not change however, only vectors

\[
\begin{bmatrix}
\frac{\partial F_1}{\partial y_1} & \frac{\partial F_1}{\partial y_2} & \ldots & \frac{\partial F_1}{\partial y_n} \\
\frac{\partial F_2}{\partial y_1} & \frac{\partial F_2}{\partial y_2} & \ldots & \frac{\partial F_2}{\partial y_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial F_n}{\partial y_1} & \frac{\partial F_n}{\partial y_2} & \ldots & \frac{\partial F_n}{\partial y_n}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial y_1}{\partial x_3} \\
\frac{\partial y_2}{\partial x_3} \\
\vdots \\
\frac{\partial y_n}{\partial x_3}
\end{bmatrix}
= \begin{bmatrix}
-\frac{\partial F_1}{\partial x_3} \\
-\frac{\partial F_2}{\partial x_3} \\
\vdots \\
-\frac{\partial F_n}{\partial x_3}
\end{bmatrix}
\]
Remember: $|J|$ has derivatives of $F^o$ only w.r.t the $y_i$.

Constraint: $|J|=0 \implies$ Cramer's rule will yield proper answers.

Eq
\[ xy - w = 0 \]
\[ y - w^3 - 3z = 0 \]
\[ w^3 + z^3 - 2zw = 0 \]
These equations are satisfied at
\[ \left( \frac{1}{4}, 4, 1, 1 \right) \]
Point P

Do the $F^o$'s have continuous derivatives? Yes.

If $|J|=0$ at point $P$, then according to JFT, we can find $\frac{\partial x}{\partial z}$.

\* WAIT

Why $\frac{\partial x}{\partial z}$?

\* Why $\frac{\partial y}{\partial z}$?

\* Why $\frac{\partial w}{\partial z}$?

$z$ = exog, we will find $\frac{\partial y}{\partial z}$ & $\frac{\partial w}{\partial z}$ too.
We take differential (total) of each eqn:

\[
\begin{align*}
(y)\,dx + (x)\,dy + (-1)\,dw + (0)\,dz &= 0 \\
(0)\,dx + (1)\,dy + (-3w^2)\,dw + (-3)\,dz &= 0 \\
(0)\,dx + (0)\,dy + (3w^2)\,dw + \left(\frac{3z^2}{-2w}\right)dz &= 0
\end{align*}
\]

Move \( dz \) terms to RHS:

\[
\begin{bmatrix}
y & x & -1 \\
0 & 1 & -3w^2 \\
0 & 0 & 3w^2 - 2z
\end{bmatrix}
\begin{bmatrix}
dx \\
dy \\
dw
\end{bmatrix}
= \begin{bmatrix}
0 \\
3 \\
\left(\frac{2w}{3z^2}\right)
\end{bmatrix} \,dz
\]

Check, at pt. \( P \), \( |J| \neq 0 \):

\[
|J| = y(3w^2 - 2z) = 3w^2y - 2zy = 4 \text{ at pt } P.
\]

\( \therefore \) IFT holds, \( & y \) are defined around \( P \).
\[
\begin{bmatrix}
1 & 2w \\
\frac{\partial y}{\partial x^2} & \frac{\partial^2 y}{\partial x \partial z} \\
\frac{\partial y}{\partial z^2} & 2w - 3z^2
\end{bmatrix}
\begin{bmatrix}
33 \\
\partial x^2 \\
\partial z
\end{bmatrix}
= \begin{bmatrix}
0 \\
3 \\
2w - 3z^2
\end{bmatrix}
\]

We divided through by \( \partial z \), we get partial derivatives of \( y \):

What is \( \frac{\partial x}{\partial z} \) ?

\[
\begin{vmatrix}
0 & x - 1 \\
3 & 1 - 3w^2 \\
2w - 3z^2 & 0 & 3w^2 - 2z
\end{vmatrix}
= \frac{-1}{4}
\]

at pt \( p \).

Do you know how to get \( \frac{\partial w}{\partial z} \)?

**Example:**

NI model

\[
\begin{align*}
y - C - I_o - G_o &= 0 \\
C - \alpha - B(y - T) &= 0 \\
T - \gamma - 8Y &= 0
\end{align*}
\]

Each LHS is \( \Phi \) \( (Y, C, T, I_o, G_o, x, B, \gamma, S) = 0 \)
To find $\frac{\partial y, c \text{ or } T}{\partial I_0, q_0 \text{ or other } \text{exog}}$ must we find total differentials each time? No.

We know:

$$
\begin{bmatrix}
J \\
\frac{\partial F^i}{\partial y_j}
\end{bmatrix}_{i=1, \ldots, n} = \begin{bmatrix}
\frac{\partial y_1}{\partial x_j} \\
\frac{\partial y_2}{\partial x_j} \\
\vdots \\
\frac{\partial y_n}{\partial x_j}
\end{bmatrix}
= \begin{bmatrix}
-\frac{\partial F^1}{\partial x_j} \\
-\frac{\partial F^2}{\partial x_j} \\
\vdots \\
-\frac{\partial F^n}{\partial x_j}
\end{bmatrix}
$$

You should know why, but don't have to prove it every time. Assuming IFT conditions met, can start from here.
Back to eq

$F^1, F^2, F^3$ have continuous partial derivatives, w.r.t all endg & exog (usually show this) & check $|J| \neq 0$

$$|J| = \begin{vmatrix} \frac{\partial F^1}{\partial y} & \frac{\partial F^1}{\partial c} & \frac{\partial F^1}{\partial t} \\ \frac{\partial F^2}{\partial y} & \frac{\partial F^2}{\partial c} & \frac{\partial F^2}{\partial t} \\ \frac{\partial F^3}{\partial y} & \frac{\partial F^3}{\partial c} & \frac{\partial F^3}{\partial t} \end{vmatrix} = \begin{vmatrix} 1 & -1 & 0 \\ -B & 1 & B \\ -8 & 0 & 1 \end{vmatrix}$$

$$= 1(1) + 1(-B + 8B) = 1 - B + 8B > 0 \quad (1-B > 0)$$

So we can take $Y, C, T$ to be implicit fn's of exog vars at/around a pt which satisfies the original set of eqns $F^i = 0$

$$y^* = f^1(I_0, G_0, x, B, y, s)$$
$$c^* = f^2(\ldots)$$
$$t^* = f^3(\ldots)$$
We can now \( \frac{\partial y}{\partial I_0} \) or \( \frac{\partial y}{\partial g_0} \).

\[
\begin{bmatrix}
    J
\end{bmatrix}
\begin{bmatrix}
    \frac{\partial y}{\partial I_0} \\
    \frac{\partial y}{\partial I_0} \\
    \frac{\partial C}{\partial I_0} \\
    \frac{\partial T}{\partial I_0}
\end{bmatrix}
= \begin{bmatrix}
    1 \\
    0 \\
    0 \\
    0
\end{bmatrix}
\]

\( \text{eg} \quad \frac{\partial T}{\partial I_0} = \frac{|J_b|}{|J|} = \begin{bmatrix} 1 & -1 & 1 \\ -8 & 1 & 0 \\ -8 & 0 & 0 \end{bmatrix} \)

\( \frac{\partial T}{\partial I_0} = \frac{1}{8} \frac{1}{1 - B + BS} \)

Previously, we solved for \( y, C, T \) & then found derivatives. Here used IFT to go straight to derivatives.

CS of General Fn Models

Wait! Did we, will we prove IFT in simultaneous EGN case?
Market Model
\[
\begin{align*}
Q_D &= f_n(\text{Price}, \text{exog } Y_0 - \text{income}) \\
Q_S &= f_n(\text{price alone})
\end{align*}
\]

Generally:
\[
\begin{align*}
Q_D &= Q_S \\
Q_D &= D(P, Y_0) \quad \frac{\partial D}{\partial P} < 0 \quad \frac{\partial D}{\partial Y_0} > 0 \\
Q_S &= S(P) \quad \frac{\partial S}{\partial P} > 0
\end{align*}
\]

- Assume \( D \) & \( S \) have continuous partial derivatives
- Can rewrite
  \[D(P, Y_0) - S(P) = 0\]

Can't solve for \( P^* = f(Y_0) \), but we assume \( P^* \) exists. Can we?

IFT single equation case:
Need \( a) \ F_p, F_{Y_0} \) exist & be continuous
  \( b) \) Need \( F_p \neq 0 \) \( (P = \text{endog var}) \)

\[
F_p = \frac{\partial D}{\partial P} - \frac{\partial S}{\partial P} < 0 \quad \hat{Y}_0 \neq 0
\]

\( \hat{Y}_0 \) can use IFT rule \(
\frac{\partial P}{\partial Y_0} = -\frac{F_{Y_0}}{F_p}
\)
Because IFT holds, we know

1. \( P^* = P^*(y_0) \) is defined around a point B satisfying \( F(P, y_0) = 0 \)
2. \( D(P^*, y_0) - S(P^*) = 0 \) (identity around B)

Rule \( \frac{dP^*}{dy_0} = -\frac{\frac{\partial F}{\partial y_0}}{\frac{\partial F}{\partial P}} = -\frac{\frac{\partial D}{\partial P^*}}{\frac{\partial D}{\partial P^*} - \frac{\partial S}{\partial P^*}} \)

\[\frac{-}{\frac{+}{(-) = -} = +} \]

*NB: Comparative Static derivative \( = \frac{dP^*}{dy_0} \)

\( = \) partial derivatives of the implicit fn/s, evaluated at EQM state

eg for \( y, c, t = fn(s(I_0, g_0, x, b, y_8) \)

\( \frac{\partial y^*}{\partial I_0}, \frac{\partial c^*}{\partial I_0}, \frac{\partial t^*}{\partial I_0} \) are comparative static derivatives.
Can we find \( \frac{dQ^*}{dy_0} \)?

We have \( Q^* = S(P^*) \) in EQM

\& \( P^* = P^*(y_0) \)

\[
\frac{dQ^*}{dy_0} = \frac{dS}{dP^*} \cdot \frac{dP^*}{dy_0} > 0
\]

(+) (+)

We just showed this.

If we knew the functional forms of actual EQM values, we could find the values of the derivative above.

What do results mean? If raise \( y_0 \), this shifts D up,

\( \Rightarrow \) EQM price rises, as does quantity.

Here we've got a general formula.
Can we study $P^*$ and $Q^*$ simultaneously?

From 8.3.2

Let $Q_D = Q_S = Q$

$$F^1(P, Q, Y_0) = D(P, Y_0) - Q = 0$$
$$F^2(P, Q, Y_0) = S(P) - Q = 0$$

What are conditions of IFT in simul. eqn. case?

(a) Continuous partial derivatives assumed. From derivatives of $Q_D$, $Q_S$ funs given.

(b) Need $|J| = 0$ $J$ = mx of partials w.r.t endog vars

$$\begin{vmatrix} \frac{\partial F^1}{\partial P} & \frac{\partial F^1}{\partial Q} \\ \frac{\partial F^2}{\partial P} & \frac{\partial F^2}{\partial Q} \end{vmatrix} = \begin{vmatrix} \frac{\partial D}{\partial P} & -1 \\ \frac{ds}{dp} & -1 \end{vmatrix} = \frac{ds}{dp} - \frac{\partial D}{\partial P} > 0$$

$$(+) - (-)$$
Therefore, if EQM pt. \((P^*, Q^*)\) exists, we can write:

\[ P^* = P^*(\check{Y}_0) \quad Q^* = Q^*(\check{Y}_0) \]

(our implicit functions)

\[ F^1(P^*, Q^*, \check{Y}_0) \equiv 0 \quad \text{identities} \]
\[ F^2(P^*, Q^*, \check{Y}_0) \equiv 0 \]

So, can take total differentials move terms with \(d\check{Y}_0\) to RHS through by \(d\check{Y}_0\) & we will get this:

\[
\begin{bmatrix}
\frac{dP^*}{d\check{Y}_0} \\
\frac{dQ^*}{d\check{Y}_0}
\end{bmatrix} =
\begin{bmatrix}
-\frac{\partial F^1}{\partial \check{Y}_0} \\
-\frac{\partial F^2}{\partial \check{Y}_0}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\frac{dP^*}{d\check{Y}_0} \\
\frac{dQ^*}{d\check{Y}_0}
\end{bmatrix} =
\begin{bmatrix}
-\frac{\partial P^*}{\partial \check{Y}_0} \\
0
\end{bmatrix}
\]

CS derivatives
So use Cramer's Rule to solve:

\[
\frac{dP^*}{dY_0} = \frac{|J_1|}{|J|} = \frac{-\partial D}{\partial Y_0} - 1 \begin{vmatrix} -\partial D \partial Y_0 & -1 \\ 0 & -1 \end{vmatrix} = \frac{\partial D}{\partial Y_0} \frac{dY_0}{|J|}
\]

\[
\frac{dQ^*}{dY_0} = \frac{|J_2|}{|J|} = \frac{\partial D}{\partial p^*} \begin{vmatrix} \partial D \partial Y_0 & -\partial D \partial Y_0 \\ -\partial p^* \partial Y_0 & 0 \end{vmatrix} = \frac{\partial D}{\partial Y_0} \frac{ds}{dp^*} \frac{dY_0}{|J|}
\]

Are these the same as previous answers in single EQN case? Yes.

P209 We could also just take the total derivative wrt \( Y_0 \) of the identities:

\[ D(P^*, Y_0) - S(P^*) = 0 \quad (P^* = P^*(Y_0), \]

\[
\frac{\partial D}{\partial p^*} \frac{dp^*}{dY_0} + \frac{\partial D}{\partial Y_0} - ds \frac{dp^*}{dp^*} \frac{dY_0}{dY_0} = 0
\]

Why? Look back to P192.
We get:
Indirect effect + direct effect = indirect effect of \( y_o \) on \( D \)
Indirect effect of \( y_o \) on \( S \)

* Is this a touch confusing?

* Can do same with simultaneous eqn case to get same answers as before.

* I want you to be able to name the effects as direct & indirect

**National Income Model**

**ISLM model** - want eqm in goods & money mkts
We want \( y^* = y^* (G_o, M^o) \)
\( r^* = r^* (G_o, M^o) \)
How do we know which are endog/exog variables? Context/given.

Example P.211: Closed Model
- Self study, workshop.
- Open Economy

Given:

\[ Y = C(Y^0) + I(r) + G_0 + X(E) - M(Y,E) \]
\[ L(Y, r) = M_0 \]
\[ X(E) - m(Y, E) + k(r, rw) = 0 \]

3 eqns, 3 endog = \( y, r, E \)

exog = \( G_0, M_0, rw \)

\textbf{Given/We know}

- \( X = X(E) \) has \( X'(E) > 0 \)
  \( X = \text{exports, are an increasing fn of the exchange rate} \ E \) (domestic price of foreign currency)

- \( M = M(Y, E) \quad M_y > 0, M_E < 0 \)
- \( M = \text{imports} \)
$E =$ for eg, rand dollar exchange
ie. 8 rand to the dollar
if $E \uparrow$, we import less.

- $K = K(r, rw)$
  - $K =$ net inflow of capital to a country
  - $r =$ domestic int rate (endog)
  - $rw =$ world int rate (exog)

- $Kr > 0 \implies Kw < 0$. Why?

- $BP = \frac{X(E) - M(Y, E)}{K(r, rw)}$
  - $BP =$ current acc + capital acc (purchase of bonds - foreign & domestic)
  - if $E =$ flexible, $BP = 0$
  - (a SS dollars = DD dollars by SA)

Open Economy Eqn
As 3 eqns in box (P44)
Do they make sense?

\[
\begin{align*}
AD &= AS \\
MD &= MS \\
BP &= 0
\end{align*}
\]

V complicated,
be careful.
Can we find $\frac{\partial Y^*, r^*, E^*}{\partial \text{exog}}$? Only if IFT holds?

1. Assume continuous partial derivatives exist? (given this previously with assns about $x'(E), M_Y, M_E, K_r, K_{rw}, C'(y^0), I'(r)$) & we know sum of continuous $f_n = \text{cont} \ f_n$.

$\frac{\partial F^{1,2,3}}{\partial \text{exog, endog}}$ exist & are continuous.

2. We need $|J| \neq 0$

\[
|J| = \begin{vmatrix}
\frac{\partial F^1}{\partial y} & \frac{\partial F^1}{\partial r} & \frac{\partial F^1}{\partial E} \\
\frac{\partial F^2}{\partial y} & \frac{\partial F^2}{\partial r} & \frac{\partial F^2}{\partial E} \\
\frac{\partial F^3}{\partial y} & \frac{\partial F^3}{\partial r} & \frac{\partial F^3}{\partial E}
\end{vmatrix} = \rho \neq 0.47.
\]

BTW. We also know r are given: $C'(y^0)$ btw 0 & 1, $I'(r) < 0$? Why?
\[ |J| = \begin{vmatrix}
I - C'(Y^0)(1-T'(Y)) - I' M_E - X' \\
Ly \\
-\lambda_y \\
\end{vmatrix}
\]

\[ L_r \ 0 \]
\[ K_r \ X' - M_E \]

Apologies, forgot to define

\[ Y^0 = \text{disposable inc} \]
\[ = Y - T \]
\[ T = T(Y) \]
\[ C'(Y^0) \]

is actually meant to be:

\[ \frac{d}{dy} \left[ C(\ Y - T(Y) \) \right] = C'(Y^0)[1 - T'(Y)] \]

by chain rule.

See above.

So is \(|J| \neq 0\).
\[ |J| = (m_E - x') (L_y K_r + L_r M_y) + (x'' - m_E) (L_r (1 - C'(1-T')} + M_y) + I^1 L_y \]

(We used the 3rd col to expand)

(Please don't faint).

Is this \( |J| \neq 0 \)?

We need to tidy up to tell.

Factorise & collect terms

\[ = \left( m_E - x' \right)^2 \left\{ L_y \left( K_r - I^1 \right) + L_r \left( C'(1-T') - 1 \right) \right\} \]

\[ = \left( - - (+) \right)^2 \left\{ (+)(+)-(+) \right\} + (-)(-) \]

\[ = \left( - \right)^2 \left\{ (+) \right\} + (+) \]

\[ < 0 \]

\[ \therefore |J| < 0 \]
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IFT conditions are met

\[ y^* = y^*(q_0, m_o^s, r_w) \]
\[ r^* = r^*(\quad \quad \quad) \]
\[ E^* = E^*(\quad \quad \quad) \]

We can write

\[
\begin{bmatrix}
J & \frac{\partial y^*}{\partial r_w} \\
\frac{\partial r^*}{\partial r_w} & \frac{\partial r^*}{\partial E^*} \\
\frac{\partial E^*}{\partial r_w} & -\frac{\partial F^1}{\partial r_w}
\end{bmatrix} = \begin{bmatrix}
-\frac{\partial F^1}{\partial r_w} \\
-\frac{\partial F^2}{\partial r_w} \\
-\frac{\partial F^3}{\partial r_w}
\end{bmatrix}
\]

We set
\[ d q_0, \quad d m_o^s = 0 \]

(Why w.r.t. \( r_w \)? Question asks for this.)

3rd vector of \( Ax = d \) is
\[
\begin{bmatrix}
0 \\
0 \\
-1
\end{bmatrix}
\]

Now use Cramer's Rule to solve.

(I will spare you this)
So...

Is this insane? Yes
Is it necessary? Yes

Why?

You will encounter general systems like these, must know how to solve for derivatives needed!

Illustration of techniques (determinants, Cramer's Rule, derivatives)

Refresher on ISLM & assumptions of relationships (you're expected to know this)
Assumptions were:

\[ L_y \]
\[ L_r \]
\[ X'(E) \]
\[ m_y \]
\[ m_E \]
\[ K_r \]
\[ K_{rw} \]
\[ T'(Y) \]
\[ C'(YD)(1 - T'(Y)) \]
\[ I'(r) \]

Know signs, (sizes if possible)
In summary, single or simul eqn case

Set up \( F(y_i, x_p) = 0 \)

or \( F'(y_i, x_p) = 0 \)
\( F''(\quad \quad) = 0 \)
\( F^n(\quad \quad) = 0 \)

Check either \( F_{y_j} F_{x_j} \ldots F_{x_m} \) exist & are continuous

or all \( F_{y_1}^1, F_{y_2}^1, F_{x_1}^1, F_{x_m}^1 \)
\( F_{y_1}^n, F_{y_2}^n, F_{x_1}^n, \ldots F_{x_m}^n \) exist & are continua

Then check \( F_y \neq 0 \)

or \( |J| \neq 0 \)

where \( |J| = \frac{\partial F_i}{\partial \text{endog vars}} \)

Then proceed to solve for
\( \partial \text{eqm values of endog vars} \)
\( \partial \text{exog} \) (set d other exog = 0)