CHAPTER 05
How do we know if $A^{-1}$ exists?

**Conditions for Non Singularity**

A must be square = necessary

Necessary = prerequisite

$\[ p \Rightarrow q \]\]

if $p$, then $q$ (if $p$, then had $q$)

$p$ implies $q$

$p$ only if $q$

$a$ = person is father

$b$ = person is male

$a \Rightarrow b$ does $b \Rightarrow a$, not necessarily.

**Sufficient**

$q$ is sufficient for $p$

$\[ p \leq q \]

$p$ = can get to us

$q$ = flies to us

if we see $q$, we def see $p$

but we can see $p$ without $q$, not necessarily.
(2) 

q: both nec & suff

p \iff q.

Is a necessary condition sufficient? Need to be male to be a father (necessary) but is it sufficient?

Is a sufficient condition necessary?

eg: A \iff passing course

A is sufficient for passing but B also sufficient (A is not necessary)

Necessary & sufficient

L best L guarantee

eg: male & your baby is preg with your child (= father)
(3)

Conditions for Nonsingularity (NS)

Squareness = necessary for NS

Sufficient = LI rows or cols.

NS \[=\] Square & LI

\[n \times n \ A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix} = \begin{bmatrix} v_1^1 \\ v_2^1 \\ \vdots \\ v_n^1 \end{bmatrix}\]

where \( v_i^1 = \) row vector 

\( = [a_{i1} \ a_{i2} \ldots a_{in}] \)

\( k_i \) which

\( f_1 \)

\( iv_i^1 = 0 \)

\( 1 \times n \)

eg

\[\begin{bmatrix} 3 & 4 & 5 \\ 0 & 1 & 2 \\ 6 & 8 & 10 \end{bmatrix} \]

\[\begin{bmatrix} v_1^1 \\ v_2^1 \\ v_3^1 \end{bmatrix}\]

\( 2v_1^1 - v_3^1 + 0v_2^1 = 0 \)

or

\( 2v_1^1 - v_3^1 + 0v_2^1 = 0 \)

\( LD = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \)
Can we see LD easily?
No. 2 ways to find it:
1. echelon matrix (rank)
2. determinant

First though, why squareness & LI together?
for n unknowns need n eqns
for unique soln

eg \[
\begin{bmatrix}
10 & 4 \\
5 & 2
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} =
\begin{bmatrix}
d_1 \\
d_2
\end{bmatrix}
\]

L.D

either \(d_1 = 2d_2\) or \(d_1 \neq 2d_2\)
inf solns \(\Rightarrow\) inconsistent

HW - check this out
need square & LD
(5) If rows cos are LI & A is square \[\Rightarrow\] NS 
\[\Rightarrow\] A\(^{-1}\) exists
\[\Rightarrow\] unique Soln

\[Ax = d\]
\[x^* = A^{-1}d\]

Rank

RI only for square?
No. Any m x n
Max no of LI rows for m x n mx = r then rank(A) =

Rank = max no LI rows / cols
\[\leq m\times n\] whichever is smaller

If 2x2, can see it
But otherwise need something else. Need to transform system in OK way which allows to see it.
Use row operations to transform:

1. Swap rows
2. $x$ row by $k \neq 0$
3. Add $k$ times row to another row.

(Why? Same as solving a system.)

These don't alter the rank of a matrix.

**EQ** What is RE form?

\[
\begin{bmatrix}
1 & 0 & -1 \\
0 & 1 & -2 \\
0 & 0 & 1
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

1. Non-zero rows above zero rows.
2. 1st non-zero element = 1.
3. Must be zeroes below 1st non-zero element of a row.
\[ A = \begin{bmatrix}
0 & -11 & -4 \\
2 & 6 & 2 \\
4 & 1 & 0 
\end{bmatrix} \]

\[ A_1 = \begin{bmatrix}
4 & 1 & 0 \\
2 & 6 & 2 \\
0 & -11 & -4 
\end{bmatrix} \]

\[ A_2 = \begin{bmatrix}
1 & \frac{4}{3} & 0 \\
2 & 6 & 2 \\
0 & -11 & -4 
\end{bmatrix} \]

\[ s_2 = \frac{n}{2} \]

\[ A_3 = \begin{bmatrix}
1 & \frac{4}{3} & 0 \\
0 & s_2 & \frac{2}{3} \\
0 & -11 & -4 
\end{bmatrix} \]

\[ A_4 = \begin{bmatrix}
1 & \frac{4}{3} & 0 \\
0 & 1 & \frac{4}{3} \\
0 & -11 & -4 
\end{bmatrix} \]

\[ A_5 = \begin{bmatrix}
1 & \frac{4}{3} & 0 \\
0 & 1 & \frac{4}{3} \\
0 & 0 & \frac{4}{3} \\
0 & 0 & 0
\end{bmatrix} \]

\[ \text{2 LI rows} \]

\[ \text{rank } A = 2 \]
Is there a unique way of getting row echelon form?

Can we do echelon mx transformations for non-square matrices?

For a square mx

to be NS, it must have
n LI rows or cols.

it must have rank n

echelon mx must have
n nonzero rows

eg 3x3 mx with rank(A)=2

A is singular
Determinants to test for NS
Row echelon is one method
\( \det(A) = |A| \) is another.

\( |A| \) = unique scalar.
    = defined only for square matrices.

\( |A| \) where \( A \) is \( 1 \times 1 = a_{11} \)
\( \text{eg } |[3]| = 3 \)

\( |A| \) is not absolute value.

If \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \)
then \( |A| = ad - bc \)

\( \text{eg } \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \text{ = } 4 - 6 = -2 \)

If \( A \) has LI rows, then
\( |A| = 0. \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix} |A| = 16 - 16 = 0 \).
3 order Determinant

\[ A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \]

Expand by any row or column:

1st row =

\[ a_{11} \; \begin{vmatrix} -a_{12} & +a_{13} \end{vmatrix} \]

- How to find signs?
- What goes in the bracket?

Signs:
- If subscripts add to odd number, sign \( < 0 \)
- If even number, sign \( > 0 \)

\[ \begin{align*} a_{13} \; 1 + 3 &= 4 \quad \text{sign} > 0 \\ a_{23} \; 2 + 3 &= 5 \quad \text{sign} < 0 \end{align*} \]
What goes in the 1's?
for a_{ij}, block cut row & col which contain

\[ \begin{bmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23} \\
  a_{31} & a_{32} & a_{33}
\end{bmatrix} \]

eg

\[ a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{12} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{12} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \]

NB
Can expand by any row | col & get same answer

HW - check this

Minor = |mx| associated with each element

eg \[ M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \]
\[ \text{ign} \]
\[ (1)^{i+j} | m_{ij} | \]
\[ i+j \text{ is even.} \]

\[ \frac{\text{eg}}{2} \]

\[ M \]

\[ C_3 \]
A 3x3 matrix has determinant

\[ |A| = a_{11} |M_{11}| - a_{12} |M_{12}| + a_{13} |M_{13}| \]

\[ = a_{11} |C_{11}| + a_{12} |C_{12}| + a_{13} |C_{13}| \]

\[ = \sum_{j=1}^{3} a_{1j} |C_{1j}| \]

or with 2nd row

\[ = \sum_{j=1}^{3} a_{2j} |C_{2j}| \]

or with 3rd col

\[ = \sum_{i=1}^{3} a_{i3} |C_{i3}| \]

HW: Try this with example
\[ \text{for } 4 \times 4 \text{ matrix } B \]

\[ |B| = \sum_{j=1}^{4} b_{ij} |C_{ij}| \text{ 1st row} \]

\[ = \sum_{j=1}^{4} b_{i2} |C_{i2}| \text{ 2nd col} \]

This is called \underline{Laplace Expansion}.

\[ \text{eg } |A| = \begin{vmatrix} 5 & 6 & 1 \\ 2 & 3 & 0 \\ 7 & -3 & 0 \end{vmatrix} \]

3rd col = \[ 1 \begin{vmatrix} 2 & 3 \\ 7 & -3 \end{vmatrix} - 0 \begin{vmatrix} 5 & 6 \\ 7 & -3 \end{vmatrix} + 0 \begin{vmatrix} 5 & 6 \\ 7 & 3 \end{vmatrix} \]

\[ = -6 - 21 \]

\[ = -27 \]

\underline{NB} \text{ Look for cols/rows with zeroes}
\[ |A| = \sum_{j=1}^{n} a_{ij} \cdot C_{ij} \quad \text{ith row} \]
\[ = \sum_{i=1}^{n} a_{ij} \cdot C_{ij} \quad \text{jth col} \]

Properties of Determinants

1. **Swapping rows/cols** does not affect \(|A|\)
   
   \[ \text{ie } |A^T| = |A| \]

2. Swapping any 2 rows/cols changes sign of \(|A|\)

3. \(X\) a row/col by \(k\) yields \(k|A|\)

**HW** - check these 3 properties
4) Adding a multiple of a row/col to another row/col doesn't change $|A|$

5) if 2 rows/cols are multiples of each other, $|A| = 0$

Why? Because getting such an $A$ into row echelon form gives a row/col of zeroes, thus $|A| = 0$

Criterion for a mx to be NS

eg $Ax = d$

Unique Soln iff rows of $A$ are L.I.

ie $A$ is N.S

But if $R_1 = 2R_2$ (for eg) rows are L.D & no unique soln
But if rows are L.D we have a row of zeroes
\[ |A| = 0 \]
for \( Ax = d \) where \( A = n \times n \), 
\[ |A| \neq 0 \iff \text{rows/cols are L.I.} \]
\( \iff \) \( A \) is N.S
\( \iff \) \( A^{-1} \) exists
\( \iff \) a unique soln exists \( x^* = A^{-1}d \).

• This is handy. Why?
• Are values \( x^* \) necessarily admissible values?

\[ \text{RANK (again)} \]
\[ \text{rank } A = \text{max num of L.I. rows} \]
\[ = \text{max order of a non-zero determinant from the rows&cols of a } mx \]

\[ \text{NB why row & col LI?} \]
\[ \text{p96} \]
We know $r(A) \leq \min \{m, n\}$ for $A_{m \times n}$.

For product $AB$:
$$r(AB) \leq \min \{r(A), r(B)\}$$

Why? P98 proof - see next page.

Finding the Inverse

If $A$ is NS, then $A^{-1}$ exists, as does unique soln $x^* = A^{-1}c$.

How to find $A^{-1}$, if $|A| \neq 0$?

Property 6 of $|A|$

The expansion of a determinant by alien cofactors ($c_{ij}$ of wrong row/col) is always zero.
Given \( R(AB) \leq \min \frac{3}{2} r(A), r(B) \),

Given \( m \times A \), with \( r(A) = j \)

& multiply by \( NS \ m \times B \),

then prove \( r(AB) = j \)

2 parts

Consider

Part 1: 3 cases:

1. \( r(A) < r(B) \)
2. \( r(A) = r(B) \)
3. \( r(A) > r(B) \)

Given either 1 or 2, then

\[ r(AB) \leq r(A) = j \]

Given 3

\[ r(AB) \leq r(B) < r(A) = j \]

\[ r(AB) \leq r(A) = j \]
Part 2
Consider \((AB)B^{-1} = A\)

\[ r((AB)B^{-1}) \leq \min \{ r(AB), r(B^{-1}) \} \]

We can say:

\[ r((AB)B^{-1}) \leq r(AB) \]

Simplify

\[ r(A) \leq r(AB) \]

\[ j \leq r(AB) \]

from part 1 & 2 combined

\[ j \leq r(AB) \leq j \]

\[ \text{Part 1 Part 2} \]

\[ r(AB) = j \]

\[ r(AB) = j 's \] roven.
Section 5.4

\[ \frac{1}{2} \begin{vmatrix} 1 & 2 & 0 \\ 3 & 2 & 4 \\ 0 & 1 & 2 \end{vmatrix} \]

= \( a_{11} |C_{21}| + a_{12} |C_{22}| + a_{13} |C_{23}| \)

= 1st row with 2nd row \( |C_{ij}| \)

= \[ 1 \cdot |C_{21}| + 2 \cdot |C_{22}| + 0 \cdot |C_{23}| \]

= \[ -1 \begin{vmatrix} 2 & 0 \\ 1 & 2 \end{vmatrix} + 2 \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} - 0 \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} \]

= \[ -1(4) + 2(2) - 0(1) \]

= \[ 0 \]

\[ \text{eg } [A] = [a_{ij}] \]

We know \( \text{ (1st row elements, 2nd row cofactor: } \sum_{j=1}^{3} a_{1j} |C_{2j}| \text{ (Note mistake in subscripts p99) } \)

= \[ a_{11} |C_{21}| + a_{12} |C_{22}| + a_{13} |C_{23}| \]

= \[ 0 \]

Why? Expand & Check
How else do we know \( \Sigma a_{ij} |C_{ij}| = 0 \)?

This is the same as finding the determinant of this matrix:

\[
|A^*| = \begin{vmatrix}
a_{11} & a_{12} & a_{13} \\
a_{11} & a_{12} & a_{13} \\
a_{31} & a_{32} & a_{33}
\end{vmatrix} \quad \text{(Check)}
\]

But \( |A^*| \) is same as \( |A| \), except 2nd & 1st rows are identical, \( \text{LD} : |A^*| = 0 \)

Expansion by alien cofactors yields zero, for any row/col.

\[
\Sigma a_{ij} |C_{ij'}| = 0 \quad (i \neq i') \text{ row}
\]

\[
\Sigma a_{ij} |C_{ij'}| = 0 \quad (j \neq j') \text{ col}
\]
Matrix Inversion

Assume \( A_{n \times n} \) NS mx.

\[
A = \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{bmatrix}
\]

\( |A| \neq 0 \)

We can form a matrix of cofactors as follows:

\[
C^i = \text{adj } A = \begin{bmatrix}
|C_{11}| & |C_{12}| & \cdots & |C_{1n}| \\
|C_{21}| & |C_{22}| & \cdots & |C_{2n}| \\
\vdots & \vdots & \ddots & \vdots \\
|C_{n1}| & |C_{n2}| & \cdots & |C_{nn}|
\end{bmatrix}
\]

\[
C^i = \text{adj } A = \begin{bmatrix}
|C_{11}| & |C_{21}| & \cdots & |C_{n1}| \\
|C_{12}| & |C_{22}| & \cdots & |C_{n2}| \\
\vdots & \vdots & \ddots & \vdots \\
|C_{1n}| & |C_{2n}| & \cdots & |C_{nn}|
\end{bmatrix}
(\text{adjoint of } A)
\]

\[
AC^i = n \times n \text{ mx}
\]
\[ AC' = \begin{bmatrix}
  a_{11} & a_{12} & \ldots & a_{1n}\\
  a_{21} & a_{22} & \ldots & a_{2n}
\end{bmatrix}
\begin{bmatrix}
  |C_{11}| & |C_{12}| & \ldots & |C_{1n}|
  |C_{21}| & |C_{22}| & \ldots & |C_{2n}|
\end{bmatrix}
\]

\[ = \begin{bmatrix}
  \sum_{j=1}^{m} a_{1j} |C_{1j}|
\end{bmatrix}\]
why do the diagonal element resolve to zero? Because of expansion by a determinant

\[
AC' = \text{adj}(A) = \begin{bmatrix}
|A| & 0 & \cdots & 0 \\
0 & |A| & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & |A|
\end{bmatrix} = |A| \begin{bmatrix} 1 \cdots 0 \\ 0 \cdots 0 \\ \vdots \cdots \vdots \\ 0 \cdots 1 \end{bmatrix} = |A| \cdot I_n
\]

by \( |A| \)

\[
\frac{AC'}{|A|} = I_n
\]

prex by \( A^{-1} \)

\[
\frac{A^{-1}AC'}{|A|} = A^{-1}
\]

\[
A^{-1} = \frac{1}{|A|} C' = \frac{1}{|A|} \text{adj} A
\]
to find $A^{-1}$

1. find $|A|$
2. find $\text{adj} \ A = [\text{cofactors}]^T$
3. find $A^{-1} = \frac{1}{|A|} \cdot \text{adj} \ A$

$$\begin{bmatrix} 4 & 1 & -1 \\ 0 & 3 & 2 \\ 3 & 0 & 7 \end{bmatrix} = A \quad \text{find } A^{-1}$$

$|A| = 99 \neq 0$

$\text{adj} \ A = \begin{bmatrix} 13 & 2 & 0 \\ 0 & 2 & 3 \\ -1 & -1 & 3 \\ -1 & -1 & 3 \\ 11 & 2 & 0 \\ 3 & 2 & 0 \end{bmatrix}^T$

* don't forget signs
* or to take the transpose

$= \begin{bmatrix} 21 & 6 & -9 \\ -7 & 31 & 3 \\ 5 & -8 & 12 \end{bmatrix}$
\[ B^{-1} = \frac{1}{q_1 q_2} \cdot \text{adj} \ A \]

**Cramer's Rule**

\[ Ax = d \quad A = n \times n \]

\[ x^* = A^{-1} d = \frac{1}{|A|} \cdot \text{adj} \ A \cdot d \]

\[
\begin{bmatrix}
  x_1^* \\
  x_2^* \\
  \vdots \\
  x_n^*
\end{bmatrix}
= \frac{1}{|A|} \begin{bmatrix}
  |C_{11}| & \cdots & |C_{1n}| \\
  |C_{21}| & \cdots & |C_{2n}| \\
  \vdots & \ddots & \vdots \\
  |C_{n1}| & \cdots & |C_{nn}|
\end{bmatrix} \begin{bmatrix}
  d_1 \\
  d_2 \\
  \vdots \\
  d_n
\end{bmatrix}
\]

\[
= \frac{1}{|A|} \begin{bmatrix}
  |C_{11}|d_1 + |C_{21}|d_2 + \cdots + |C_{n1}|d_n \\
  |C_{12}|d_1 + |C_{22}|d_2 + \cdots + |C_{n2}|d_n \\
  \vdots \\
  |C_{1n}|d_1 + \cdots + |C_{nn}|d_n
\end{bmatrix}
\]

\[ n \times 1 \]
\[ x^* = \frac{1}{|A|} \begin{bmatrix} \sum_{i=1}^{n} |C_{i1}| d_i \\ \sum_{i=1}^{n} |C_{i2}| d_i \\ \vdots \\ \sum_{i=1}^{n} |C_{in}| d_i \end{bmatrix} \]

\[ n \times 1 \]

\[ x_1^* = \frac{1}{|A|} \sum_{i=1}^{n} |C_{i1}| d_i \]

\[ x_2^* = \frac{1}{|A|} \sum_{i=1}^{n} |C_{i2}| d_i \]

\[ x_n^* = \frac{1}{|A|} \sum_{i=1}^{n} |C_{in}| d_i \]

What are these terms?

\[ x_1^* = \frac{1}{|A|} \cdot |A_1| \]

What is \(|A_1|\)?

\[ |A_1| = \begin{vmatrix} d_1 & a_{12} & a_{13} & \cdots & a_{1n} \\ d_2 & a_{22} & & & \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ d_n & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix} = d_1 |C_{11}| + d_2 |C_{21}| + \cdots + d_n |C_{n1}| = \sum_{i=1}^{n} d_i |C_{i1}| \]
\[ |A_1| = |A| \text{ but with 1st col replaced by } d, \]
\[ |A_1| \text{ calculated by expanding by 1st col.} \]

Similarly,
\[ x_2^* = \frac{1}{|A|} |A_2| \]
\[ |A_2| = |A| \text{ but with col2 replaced by } d. \]

\[ x_j^* = \frac{1}{|A|} |A_j| \]
\[ = \frac{1}{|A|} \begin{vmatrix} a_{11} & a_{12} & d_1 & a_{1n} \\ a_{21} & a_{22} & d_2 & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & d_n & a_{nn} \end{vmatrix} \]

jth col replaced by d

Eq. Solve \[ 5x_1 + 3x_2 = 38 \]
\[ 6x_1 - 2x_2 = 8 \]

\[
\begin{bmatrix} 5 & 3 \\ 6 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 38 \\ 8 \end{bmatrix}
\]
\[ x_1 = \frac{1}{|A|} \cdot |A_1| \]
\[ = \frac{1}{-28} \begin{vmatrix} 30 & 3 \\ 8 & -2 \end{vmatrix} \]
\[ = \frac{1}{-28} \cdot (-84) \]

\[ x_1 = 3 \quad \text{Similarly} \quad x_2 = 5 \]

eg. P105

Homogenous System Equations

if \( d = 0 \)

ie \( Ax = 0 \) ie \( [A][x_0] = [0] \)

= homogenous system

if \( A \) is NS, one solution is \( x^* = 0 \)

Because \( x^* = A^{-1}d = A^{-1}0 = 0 \)

\[ x_j^* = \frac{1}{|A|} \cdot |A_j| \]

but \( A_j \) has a column of zeroes, \( \therefore |A_j| = 0 \)
Only if $|A| = 0 \iff A \in \text{S}$, then $x^* = \frac{|A_j|}{|A|} = \frac{0}{0} = \text{undefined}$

no unique soln

Why infinitely many solns?

Prob - work it out.

\[ Ax = d \]

\[ \begin{array}{c|c|c}
   \hline
   & d \neq 0 & d = 0 \\
   \hline
   |A| \neq 0 & \text{Unique soln } x^* \neq 0 & \text{Unique soln } x^* = 0 \\
   \text{A in NS} & x^* = \frac{1}{|A|} \text{adj} A . d & \text{Inconsistent EQNS} \\
   \hline
   |A| = 0 & \text{Infinite no of solns} & \text{Infinite solns} \& x^* = 0 \\
   \text{A in S} & \text{No soln} & \text{Not possible (Why?)} \\
   \hline
   \text{Inconsistent EQNS} & \end{array} \]


Market & National Income Models

Please work through examples on p107 (market model) & P108 (NI model)

ISLM
2 sectors - goods & monetary

Goods MKT
\[ y = C + I + G \]
\[ C = a + b(1-t)y \]
\[ I = d - e_i \]
\[ G = G_0 \]
\[ y, C, I, i \text{ = endogenous} \]
\[ G \text{ = exog} \]
\[ a, d, e_i, b, t \text{ = parameters} \]

MONEY MKT
\[ Md = Ms \]
\[ Md = kY - li \]
\[ Ms = M_0 \]
\[ M_0 \text{ = exog} \]
\[ k, l \text{ = parameters} \]
\[29\]

Money market

\[ M_0 = kY - e^0 \]

All together

\[ Y - C - I = G_0 \]
\[ b(1-t)Y - C = -a \]
\[ I + e^0 = d \]
\[ kY - e^0 = M_0 \]

In mx format

\[
\begin{bmatrix}
1 & -1 & -1 & 0 \\
0 & b(1-t) & -1 & 0 & 0 \\
0 & 0 & 1 & e \\
0 & 0 & 0 & -1 \\
b(1-t) & 0 & 0 & -e & 0 \\
& & & & 0 & 0 & -1
\end{bmatrix}
\begin{bmatrix}
Y \\
C \\
I \\
e^0 \\
M_0 \\
\end{bmatrix}
= 
\begin{bmatrix}
G_0 \\
-a \\
d \\
M_0 \\
\end{bmatrix}
\]

\[|A| = \text{using 3rd row} \]
\[= 0, |C_{31}| - 0 |C_{32}| + |C_{33}| - e |C_{34}| \]
\[= \begin{vmatrix} 1 & -1 & 0 & 1 & -1 & -1 \\
0 & b(1-t) & -1 & 0 & -e & b(1-t) & -1 & 0 \\
0 & 0 & -e & 0 & 1 \\
k & 0 & -e & 0 & 0 & 0 \\
\end{vmatrix} \]
\[3A\]

\[= -e(-1 + b(1-t)) - e(k(-1))\]

\[= e - be(1-t) + ke\]

\[= e(1 - b(1-t)) + ke\]

How do we know if \(|A| \neq 0\)

Need assns.

Assume \(0 \leq t \leq 1\)  \(t = \text{tax rate}\)

\(0 \leq b \leq 1\)  \(b = \text{mpc}\)

\(k, e > 0\)

\(e > 0\)

\[\therefore |A| > 0.\]

Now, solve for \(C^*\).

\[C^* = \frac{1}{|A|} A_{21}\]

\[= \frac{1}{|A|} \begin{vmatrix}
1 & 9 & -10 \\
-11 & 9 & 0 \\
0 & 1 & e \\
k & M & 0 & e
\end{vmatrix}\]

Expand by 2nd row, or 4th/3rd cols

e tc
Why mx algebra, & not substitution.

- Compact
- Can test for existence of a soln.
- Used in Econometrics from hereon out—very useful.

Leontief Input Output Models

- n industries
- What level of output should each of the n industries produce, to satisfy total demand for each of the n products?
- Interindustry dependence
- Don't want bottlenecks.

Assns

1. Homogeneous product per industry
2. Fixed input ratio per industry
3. Constant returns to scale unrealistic?
The diagram represents an input-output matrix with inputs and outputs. The matrix is structured as follows:

\[
\begin{bmatrix}
I & II & III & N \\
\begin{bmatrix}
a_{i1} & a_{i2} & a_{i3} & \ldots & a_{in} \\
a_{21} & a_{22} & \ldots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \ldots & a_{mn}
\end{bmatrix}
\end{bmatrix}
\]

Each element \( a_{ij} \) represents the amount of good \( i \) required to produce 1 unit of good \( j \).

To produce 1 unit of good \( j \), one needs:
- \( a_{1j} \) of good 1,
- \( a_{2j} \) of good 2,
- \( a_{nj} \) of good \( n \).

- Prices are given.
- \( a_{ij} \) = input coefficient.

What is \( a_{32} \)? What is the sum of the fourth column?

**Open Model**

Final Demand = Input DD (interindustrial) + Consumer, Govt + Foreign DD
All inputs = intermediate inputs + primary inputs (labour etc)

We include an open sector = govt, cons, foreign

\sum_{i=1}^{n} a_{ij} (j=1,2,n) > 1

What if \sum_{i=1}^{n} a_{ij} (j=1,2,n) > 1?

How much do we pay the primary inputs? \[ 1 - \sum_{i=1}^{n} a_{ij} \]

1 - column sum

To meet demand, we need:

\[ x_1 = a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n + d_1 \]

input demand from industry 2 for good 1.

etc etc

\[ x_n = a_{n1} x_1 + a_{n2} x_2 + \ldots + a_{nn} x_n + d_n \]
Move all terms to
\[(1-a_{11})x_1 - a_{12}x_2 - \ldots - a_{1n}x_n = d_1,
- a_{21}x_1 + (1-a_{22})x_2 - a_{2n}x_n = d_2.
\]

\[-a_{nn}x_n - a_n + (1-a_{nn})x_n = d_n\]

\[
\begin{bmatrix}
(1-a_{11}) - a_{12} & \ldots & -a_{1n} \\
-a_{21} & (1-a_{22}) & \ldots & -a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
-a_{nn} & -a_{n2} \ldots & (1-a_{nn})
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{bmatrix}
= \begin{bmatrix}
d_1 \\
d_2 \\
\vdots \\
d_n
\end{bmatrix}
\]

Ignoring 1s, we have \(-A = [E_{ij}]\)
This mx is: \(I - A\) (Check)
\[
(I - A)x = d
\]
\[
x = (I - A)^{-1}d.
\]
I - A = Leontief Matrix.

Example P115

\[ A = \begin{bmatrix} 0.2 & 0.3 & 0.2 \\ 0.4 & 0.1 & 0.2 \\ 0.1 & 0.3 & 0.2 \end{bmatrix} \]

Each col sum < 1 ✓

\[ a_{0j} = \text{amt of primary input} \]

\[ a_{01} = 1 - 0.7 = 0.3 \]

\[ a_{02} = 0.3 \]

\[ a_{03} = 0.4 \]

\((I - A) x = d\)

\[
\begin{bmatrix}
0.8 & -0.3 & -0.2 \\
-0.4 & 0.9 & -0.2 \\
-0.1 & -0.3 & 0.8 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\end{bmatrix}
= 
\begin{bmatrix}
d_1 \\
d \\
d \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
x_1^* \\
x_2^* \\
x_3^* \\
\end{bmatrix} = (I - A)^{-1} = \frac{1}{0.384}
\begin{bmatrix}
0 \\
0.34 \\
0 \\
\end{bmatrix}
\]

H
If \( d = \begin{bmatrix} 10 \\ 5 \\ 6 \end{bmatrix} \) billion dollars,

Show \( x_1^* = 24.84 \)
\( x_2^* = 20.68 \)
\( x_3^* = 18.36 \)

This is done using some amount of primary input. What if the amount is not available?

\[
\sum_{j=1}^{3} a_{0j} x_j^* = 0.3(24.84) + 0.3(20.68) + 0.4(18.36) = 21 \text{ billion}
\]

If we don't have 21 billion worth of primary input, we must revise production downwards.

- Using inverse method, changing \( d \) & then finding \( x^* \) is easy, unlike Cramer's rule.
Non Negative Solutions

If \((I-A)\) is NS, we can find \(x^*\). What if \(x^*\) aren't all non-negative?

This is not going to be the case all the time. When will it be?

Hawkins & Simon

Given (a) \(nxn\) \(mxm\) \(B\), with \(b_{ij} \leq 0\) \((i \neq j)\) is off-diagonal elements \(\leq 0\) and (b) an \(nx1\) vector \(d > 0\), there exists an \(nx1\) vector \(x^* > 0\), s.t. \(Bx^* = d\), iff

\(|B_m| > 0\) \((m = 1, 2, \ldots, n)\)

ie only if leading principal minors of \(B\) are positive.

So (a) \((I-A) = B\), all \(b_{ij} \leq 0\), \(j \neq i\)

(b) \(d > 0\)
What are leading PMs?

PM = principal minor

A kth order PM is obtained by deleting any n-k rows & the same n-k columns & then taking the determinant.

eg \( A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \)

3rd order PM: delete 3-3 rows & cols ie entire matrix

\[ |A_3| = |A| \]

2nd order PM: delete any 3-2 = 1 rows/cols

ie \( |A_2| \) will be 3 2x2 minors

\[ \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}, \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \]

delete 1st row/col 2nd row/col 3rd row/col

1st order PM: delete 3-1 = 2 rows/cols
1st order PMS
\[ \begin{vmatrix} a_{33} \end{vmatrix}, \begin{vmatrix} a_{11} \end{vmatrix}, \begin{vmatrix} a_{22} \end{vmatrix} \]

delete 1st & 2nd rows/cols

Now, what are leading PMS?

\[ \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\
    a_{21} & a_{22} & a_{23} & a_{24} \\
    a_{31} & a_{32} & a_{33} & a_{34} \\
    a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} \]

\[ |A_1| = |a_{11}| \]
\[ |A_2| = \begin{vmatrix} a_{11} & a_{12} \\
    a_{21} & a_{22} \end{vmatrix} \]
\[ \text{etc etc etc} \]

NB: Notation |A_2| is a 2x2 PM
|A_3| is a 3x3 PM
etc

Why principal minors?
Because the diagonal elements of all the minors above are the principal diagonal elements of A.
So a minor $|M_{ij}|$ is the determinant of the matrix obtained by deleting the $i$th row & $j$th column.

A principal minor $|A_k|$ is the det of the $m \times m$ obtained by deleting any $n-k$ rows & cols (same cols as rows).

A leading principal minor $|A_m|$ is obtained by taking the 1st $m$ principal diagonal elements corresponding in $|A|$ & their off-diagonal elements (Remember the pattern).

Back to Hawkins-Simon

L Do we prove it?
L We need the leading PMs of $I-A$ to be positive to yield $x \geq 0$. 
$2 \times 2$ example:

$I - A = \begin{bmatrix} 1 - a_{11} & -a_{12} \\ -a_{21} & 1 - a_{22} \end{bmatrix}$

We need:

(a) $b_{ij} \leq 0$, $i \neq j$

(b) $|B_m| > 0$

leading PMs:

$|B_1| > 0 \implies 1 - a_{11} > 0$

$a_{11} < 1$

$|B_2| > 0 \implies (1 - a_{11})(1 - a_{22}) - a_{12}a_{21} > 0$

$\implies a_{11} + a_{12}a_{21} + (1 - a_{11})a_{22} < 1$

positive

$a_{11} + a_{12}a_{21} < 1$

↑ direct

↑ indirect

uses of (1)
If the conditions are met, we get meaningful results for $x^*$.

Closed Model

If we absorb the exogenous sector into the system as another industry, it becomes a closed model.

- No final DD or primary input
- All goods now intermediate
- $n+1$ industries
- We still assume a fixed input ratio — HHs consume each commodity in fixed prop. to labour they supply.
- Now have a homogenous system $d = 0$
- Will have a soln iff $I-A$ has $\det(I-A) = 0$, because $I-A$ is singular. Why?
- See plot for answer to previous question.
- Why is $\det(I-A) = 0$ in closed system?

$$
\begin{bmatrix}
1-a_{00} & -a_{01} & -a_{02} & -a_{03} \\
-a_{10} & 1-a_{11} & -a_{12} & -a_{13} \\
-a_{20} & -a_{21} & 1-a_{22} & -a_{23} \\
-a_{30} & -a_{31} & -a_{32} & 1-a_{33}
\end{bmatrix}
\begin{bmatrix}
x_0 \\
x_1 \\
x_2 \\
x_3
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
$$

$x_0$ used to be primary input (input requirement for industries 1 through n).

- No final DD, all goods now intermediate.
- Each column sum must equal 1: $a_{0j} + a_{1j} + a_{2j} + a_{3j} = 1$.

or $a_{0j} = 1 - a_{1j} - a_{2j} - a_{3j}$

- In every column of $I-A$, the top row element = 1 - other col element.
- 4 rows L.D
  \[ |I-A| = 0 \]
have nontrivial solutions, infinite number
no I correct answer.

Static Analysis
- ignores how we get to EGM values, & how long it takes to get there (might not be relevant any more if exog vars change)

- EGM state may be unattainable if it is unstable.

Comparative Statics = shifts of EGM state w.r.t exog var changes

Dynamic Analysis:
Attainability/Stability of EGM