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CONTRIBUTIONS
TO
MODERN PORTFOLIO THEORY

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_The race is not to the swift_  
_Or the battle to the strong._  
_Nor does food come to the wise_  
_Or wealth to the brilliant_  
_Or favour to the learned;_  
_But time and chance happen to them all._

Ecclesiastes 9:11
# Table of Contents

**ACKNOWLEDGEMENTS** ............................................................................................................. 1

Chapter 1: Introduction and Literature Review ......................................................... 1 - 1 to 24

Chapter 2: Analysis of Tracking Error............................................................................. 2 - 1 to 17

Chapter 3: Empirical Demonstration of Mean Variance and Tracking Error

Variance Optimisation ........................................................................................................... 3 - 1 to 20

Chapter 4: Index Funds: Serial Correlation and TEV Bias............................................. 4 - 1 to 12

Chapter 5: Leverage and the Separation Theorem under Constrained

Leverage ................................................................................................................................. 5 - 1 to 10

Chapter 6: An Empirical Examination of Constrained Leverage

Conditions ............................................................................................................................... 6 - 1 to 17

Chapter 7: Constrained Leverage in a TEV Framework ............................................... 7 - 1 to 18

Chapter 8: Conclusions .......................................................................................................... 8 - 1 to 6

**REFERENCES** ................................................................................................................... iii to vii

**APPENDICES** ................................................................................................................... viii to xxv
# INTRODUCTION & LITERATURE REVIEW

## 1.1 INTRODUCTION

## 1.2 THE HISTORY OF MODERN PORTFOLIO THEORY

1.2.1 Markowitz ........................................... 3
1.2.2 Tobin .............................................. 4
1.2.3 Sharpe ............................................ 5

## 1.3 FROM MARKOWITZ TO TRACKING ERROR

1.3.1 Reasons for the move ......................... 7
1.3.1.1 Manager Evaluation and Fees .................. 7
1.3.1.2 Criticisms of Mean/Variance Optimisation .... 9
1.3.1.3 Intuitive advantages of the Tracking Error Variance (TEV) Model .......................... 10
1.3.2 Disadvantages and Criticisms of the TEV approach ...................... 12
1.3.2.1 Intuitive Disadvantages ..................... 13
1.3.2.2 Choice of Benchmark ....................... 13
1.3.2.3 Serial Correlation ............................ 15
1.3.2.4 Continuity and Robustness of Performance Measures ............... 16
1.3.2.5 TEV Optimisation as an Error Maximiser .......................... 16

## 1.4 IMPORTANT RESEARCH IN TRACKING ERROR OPTIMISATION

1.4.1 Roll (1992) ......................................... 17
1.4.2 Franks (1992) ..................................... 19
1.4.3 Chow (1995) ...................................... 19
1.4.4 Tierney and Bailey (1995) ..................... 22
1.4.5 Leibowitz et al (1993) ......................... 23

## 1.5 STRUCTURE OF THE THESIS

23
1.1 INTRODUCTION

Fund managers and investors are confronted with the problem of selecting a single investment portfolio from a large number of possible combinations of available assets. In South Africa the set of possible portfolios has become even larger with the gradual relaxing of the constraints on foreign investment from 1995 to the present day, thereby expanding the investment universe for South African investors. Moreover, portfolio selection in South Africa is being transformed increasingly from being the exclusive domain of high net worth individuals, trustees and their investment managers to being the domain and responsibility of the man on the street.

The Unit Trust industry started in South Africa in 1965 and gave the lower net worth individual a vehicle with which to invest in a diverse investment portfolio. This industry has proved very popular and has expanded from only 8 funds in 1980 to 338 funds and 136 billion rands under management in November 2000. Moreover the past two years, 1999 and 2000, has seen a change in the pension fund industry from defined benefit (DB) to defined contribution (DC) pension funds, transferring more of the risk and the responsibility of portfolio selection onto pension fund members. With increasing demand for fund management and investment advice by pension fund members and individual investors alike, the financial services industry in South Africa has also expanded. The consequent competition for assets of all descriptions have led, one hopes, to a more efficient market in equity, fixed income and derivative products. Thus modern portfolio theory has come a long way and will have to go further in meeting the demand to assist investors in their decision making.

1.2 THE HISTORY OF MODERN PORTFOLIO THEORY

In the past, portfolio managers might have been content to construct an investment portfolio comprised simply of securities that promised high returns without

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1 Mutual Funds
considering their portfolio's exposure to risk. The thinking behind this one-dimensional view was that it was more profitable to select top-performing stocks than to select a good combination of assets. We now know that the opposite is in fact true. The co-movements of asset returns, or rather their absence, can significantly reduce the amount of risk or uncertainty that a portfolio bears. Traditionally, uncertainty was seen to be relatively unimportant because "smart people are supposed to know what they are doing."\(^2\) Now we know that this statement is flawed for three reasons. Firstly, the more "smart" people there are, the more difficult it is for any of them to outsmart the others (market efficiency). Secondly, the only decision regarding portfolio selection that can be made in advance is the level of risk an individual investor is prepared to bear. Lastly, not even "smart" people can predict the future. Risk simply must be managed.

1.2.1 Markowitz

The 1990 Nobel prize winner, Harry Markowitz, wrote a now classic 1952 article, "Portfolio Selection". In this article he proposed that investors should no longer merely select portfolios which maximise expected returns but should also consider minimising the volatility (or variance) of these returns. He proposed what he called the E-V maxim: selecting portfolios that offer the highest expected return for a given level of variance.

Markowitz also showed that, although a portfolio's expected return is simply the weighted average of the expected returns of the component securities, calculating the variance of a portfolio's returns is not as simple. He recognised the concept of co-movement among returns for different securities and showed that the variance of any portfolio is equal to the product of portfolio weights (vector \(\mathbf{W}^\top\)) transposed, the variance-covariance matrix of the returns of all the component securities (\(\Sigma\)) and the portfolio weights (vector \(\mathbf{W}\)) again.

\[ \sigma_{\text{portfolio}}^2 = \mathbf{W}^\top \Sigma \mathbf{W} \]

\(^2\) Bernstein (1991)
Thus investors became able (with the assumption of a fixed variance-covariance matrix and some computational expense) to measure any portfolio's risk in terms of the variance, or more commonly the standard deviation, of a portfolio's returns.

Furthermore, Markowitz offered investors two key insights into the notion of the risk of a portfolio. Firstly (for non-negative weights), unless all the assets in a portfolio are perfectly correlated with each other, a portfolio's standard deviation will always be less than the weighted average of the standard deviations of the component securities. And so, by diversifying their investments across several assets, investors may reduce their portfolio's risk without necessarily sacrificing expected return. Secondly, Markowitz showed that, unless a portfolio is comprised of two sub-portfolios that are perfectly inversely correlated with each other, it is impossible to eliminate portfolio risk entirely through diversification.

In accord with the "E-V maxim", Markowitz considered all portfolios that maximise expected return for a given level of variance to be efficient and termed a continuum of such portfolios in mean/standard deviation space, an "efficient frontier". Markowitz suggested that investors should restrict their choice of investment portfolio to this efficient set but did not explicitly consider the role of a zero-variance or risk-free asset.

1.2.2 Tobin

The 1981 Nobel Prize winner, James Tobin, extended the work of Markowitz in Tobin (1958) and showed that the portfolio selection process was twofold. Firstly an investor should construct and select from an efficient set of portfolios (as recommended by Markowitz (1952)). Secondly, an investor should decide to what extent he/she would combine this portfolio with a riskless asset i.e. borrow or lend at a fixed interest rate. This twofold process arises from Tobin’s Separation Theorem.
1.2.3 Sharpe

William Sharpe (1964) extended the insights of Markowitz and Tobin with his theory of market equilibrium under conditions of risk, for which he shared the Nobel prize with Markowitz in 1990. He showed that, for a fixed variance-covariance matrix and a specific risk-free asset with a fixed rate of return, there is a unique portfolio which, when combined with a riskless asset, dominates all other combinations of efficient portfolios with this same riskless asset. That is, borrowing or lending any proportion of funds at a risk-free rate of interest and investing borrowed or remaining funds in this one particular portfolio will offer higher expected return per unit of risk than any other available portfolio under the assumption of limitless borrowing or lending.

Figure 1 shows a concave, positively sloped efficient frontier in the two-dimensional mean/standard deviation space. The riskless asset \( R_f \) is marked at the risk-free rate of interest along the vertical axis since it has, by definition, no risk and therefore a standard deviation of zero. The straight line from the risk-free rate which is tangent to the concave efficient frontier represents the efficient frontier when limitless borrowing or lending is allowed at the risk-free rate. The tangent point (where the straight line touches the concave efficient frontier) is marked as portfolio M. This unique portfolio discussed above provides the optimal combination of risky assets. The part of the straight line to the left of portfolio M represents a partial investment in portfolio M with remaining funds invested in the risk-free asset. The straight line to the right of M represents borrowing supplementary funds at the risk-free rate \( R_f \), and investing all funds in portfolio M. All combinations of portfolio M and the risk-free rate will always offer the highest expected return for a given level of risk and thus the most efficient set of portfolios.
Sharpe showed that, in equilibrium, all rational, risk-averse investors prefer to invest along the straight line from the risk-free rate through portfolio M. The assumptions he made were that a) investors have the same expectation with regard to expected returns, variances and correlations and b) that there is one interest rate at which investors are free to borrow or lend limitless amounts. Furthermore, he showed that, if all possible risky assets are included in the feasible set from which the efficient frontier is calculated, then portfolio M is the market portfolio and represents the maximum possible diversification available in a portfolio.

Within this framework, Sharpe demonstrated that the risk of an asset could be divided into two component parts: market/unsystematic risk and unique/systematic risk. Market risk is a result of changes in the position of the market portfolio, as its name implies, and cannot be diversified away. Unique risk, however, is a result of non-market related factors which are unique to each asset and can be diversified away.

Sharpe also showed that the expected excess return (that is, the return in excess of the risk-free rate) of an asset is linearly related to the market’s expected excess return. The slope of this linear relationship is beta (β). The beta of any asset can be estimated by performing a linear regression of the asset’s excess returns against the excess returns of the market. The residual or unexplained variation from this regression quantifies the unique risk of the asset while the beta coefficient quantifies
the market risk of the asset. This decomposition of risk in turn implies that, in an efficient market, investors are only rewarded for holding market risk and are not rewarded for any undiversified unique risk in their portfolio. Thus the Capital Asset Pricing Model (CAPM) arose and, along with Tobin's separation theorem and Markowitz's EV Maxim, became the foundation for modern portfolio theory.

1.3 FROM MARKOWITZ TO TRACKING ERROR

1.3.1 Reasons for the move

Markowitz's E-V maxim recommended that investors optimise their expected return while simultaneously minimising the uncertainty of these expectations (i.e. their risk). He proposed MV optimisation which finds the set of portfolios with the maximum mean return for every level of variance and vice versa. The MV model therefore assumes that the only two factors that should influence any investor's decision are return and risk. However, these factors do not necessarily represent investors' (or their managers') goals and decision criteria accurately or comprehensively. It is argued that both investment managers and their sponsors (the fund's trustees) are sometimes more concerned with their relative performance than their mean and variance of performance.

1.3.1.1 Manager Evaluation and Fees

Portfolio managers, in particular, do not necessarily have the same objective as a "Markowitz investor". The performance of modern fund managers is often evaluated in terms of return over-and-above a given benchmark since, it is argued, it is the skill to deliver these "abnormal" returns for which professional asset managers are employed. In the past, it may have been acceptable to pay a professional asset manager to prudently invest and administer one's assets. However, with investment advice and trading information becoming increasingly accessible and the investment
profession becoming increasingly well regulated, some would argue that the sole fee an investor needs to pay is for a broker to execute trades on his/her behalf.

Historically unit trust funds in particular offered the man-on-the-street a more diverse portfolio than he or she could achieve alone. This ability to diversify perhaps justified the management fees to remunerate the service of co-ordinating pooled funds and investing in a broad range of stocks or bonds according to the managers’ mandate. However, with the advent of listed index tracking instruments such as the SATRIX\(^3\) and the range of “passive” unit trust funds which track individual stock market indices (for smaller fees), diversification may no longer be worth the management fee.

Thus modern investors may reason that management fees are a fair exchange only for the expected benefit of achieving returns on their investment which exceed those which they can achieve themselves or which can be achieved by investing in a passively managed fund. This reasoning explains the trend to distinguish between “passively” managed funds such as benchmark or index trackers which simply mimic their benchmark and “active” funds which attempt to exceed the returns on their given benchmark by means of specialist stock selection and/or market timing.

If active management is the goal then techniques are required to both evaluate and select successful active managers. The periods over which fund performance (i.e. manager performance) is evaluated are relatively short, and so it becomes difficult to determine whether the fund being evaluated has achieved a return higher than that of the benchmark. This difficulty has led managers and investors to focus on the interim volatility of the difference between their fund’s returns and the benchmark’s returns. Gadkari Spindel and Thum (1988) describe the tracking error variance (TEV) model that minimises the variance of portfolios’ returns relative to the benchmark’s returns rather than minimising the absolute variance of portfolios’ returns. The tracking error (TE) or active return of a portfolio refers to the return of

\(^3\) The SATRIX is a listed instrument on the Johannesburg Stock Exchange which tracks the performance of the ALSI 40 – see www.jse.co.za
the portfolio in excess of its benchmark and its tracking error variance (TEV) or active risk consequently refers to the variance of the tracking errors over time. The efficient set under the TEV model is thus the set of portfolios with the maximum return for every level of tracking error variance.

### 1.3.1.2 Criticisms of Mean/Variance Optimisation

A second possible reason for the popularity of TEV optimisation and modern emphasis on active performance measurement stems from criticisms of the practical application of the traditional Markowitz model – mean/variance (MV) optimisation. Typically the problem with the Markowitz MV model has been that it often yields optimal portfolios which do not appeal to investors, chiefly because of their lack of diversification. Moreover the optimal portfolio selected by this traditional model often differs substantially from the benchmark in its composition and, without adequate diversification, often under-performs this benchmark. Fisher and Statman (1997) argue that investors care about more than expected returns and variance: they are also concerned with the “palatability” of their portfolio. One of the reasons that mean-variance optimal portfolios are dismissed is their lack of intuitive appeal, chiefly because of their lack of diversification and their dissimilarity to the benchmark. These authors suggest that investors’ tastes be taken into account rather than using a straightforward prescriptive model such as the Markowitz model and refer to the descriptive Shefrin-Statman model as an alternative. Behavioural models such as the Shefrin-Statman model are one of the methods of getting around the use of MV optimisers thereby avoiding the use of additional constraints on the weights of individual assets, or the number of counters to achieve a more “realistic” efficient set.

One reason for MV optimisation’s failure in practice is that the selection of MV optimal portfolios tends to be particularly sensitive to errors in the estimation of expected return\(^4\), which can be substantial in the field of portfolio selection. In fact,

\(^4\) Practitioners often replace mean returns in a MV optimisation problem with some other estimation of expected returns for example using fundamental analysis, Bayesian estimation methods etc. Errors in these estimates can only be detected in hindsight by comparing estimated returns with actual returns.
MV optimisation tends to maximise errors in expected return estimation by favouring assets for which the expected returns have been overestimated and the risk underestimated. As the number of assets increases, the estimation errors accumulate rather than diversify. Thus, not only is the expected return of the ex ante optimised portfolio overestimated and its risk underestimated, but also the composition of the ex ante "optimised" portfolio is not necessarily optimal ex post. Kritzman (1992) suggests two simple methods to scale or shrink portfolio estimates to a more realistic value in order to take account of upwardly biased returns but admits that these techniques are not free of flaws and ultimately the problem remains the same: a prospective MV optimiser often seems to be an error maximiser!

1.3.1.3 Intuitive advantages of the Tracking Error Variance (TEV) Model

Practical issues aside, there are some intuitive reasons for the current trend to measuring return and risk relative to a benchmark rather than in absolute terms. Franks (1992) mentions several advantages for this shift from the traditional notion of risk i.e. variance, to one of tracking error variance (TEV). Firstly, the TEV model defines return and risk in a more meaningful way to the client: the client’s goal is to beat the benchmark with minimum likelihood of under-performing this same benchmark.

Furthermore, the TEV model applies to any asset universe and any benchmark, whatever its components. This flexibility contrasts with the notion of a unique global optimal portfolio in the Markowitz MV framework. The TEV model is thus client-specific in that it allows for a personal choice of benchmark and, as a result, a choice of optimal portfolio that differs from other investors with the same feasible set but with a different benchmark.

A further advantage of a TEV model is that its benchmark portfolio always lies along the efficient frontier (where tracking error and tracking error variance are zero) rather than, as is usually the case in the MV model, somewhere Southeast of the efficient set in MV space. Thus a TEV model orientates the portfolio selection
Chapter 1: Introduction and Literature Review

problem around a particular benchmark which, in this relative framework, is effectively the risk-free asset.

Speidell, Miller and Ullman (1989) relate tolerance toward Tracking Error Variance to the style of a portfolio. Since the benchmark is typically a market index, the authors reason that the more TEV a fund manager is prepared to take on, the more the fund moves toward a particular style and away from a typical core/index portfolio. Thus relative risk may be used as an indicator for the intensity of the style that a manager adopts.

However, Divecha (1989) emphasises the importance of customising the benchmark or normal portfolio so that it encompasses the style, strategy and limitations of the manager and fully represents the contract the manager has with the fund sponsor. Thus, Divecha reasons, the use of a benchmark divides the responsibility (and guilt) between the active fund manager and the sponsor. The sponsor takes responsibility for the risk and return of the benchmark portfolio and the manager takes responsibility for value added over-and-above the benchmark.

Clarke, Krase and Statman (1994) point out that the tracking error framework is not intuitively grounded in the Markowitz framework. While the MV model constructs portfolios based on an aversion to risk, the TEV model bases its choice on aversion to regret. In fact, Clarke et al (1994) posit that, since the tracking error framework has as its starting point a chosen benchmark, the investor already knows what their desired “optimal” portfolio is – the benchmark itself. The chosen optimal/benchmark portfolio a) reflects the investor’s attitude to risk and b) becomes a benchmark against which performance can be measured. Thus the benchmark portfolio exposes an investor to risk and any decisions that lead to investments which differ from the benchmark, expose the investor to regret. Therefore, gains and losses that come with holding the benchmark portfolio are considered an “act of God” commensurate with the investment’s risk but losses that come as a result of deviating from the benchmark are considered an “act of man” and thus incur regret. It follows that investors who are extremely averse to regret will invest in a passive, benchmark portfolio.
Clarke et al (1994) describe the expected utility of a regret-averse investor (presumably assuming that the investor's risk-aversion has been satisfied by the choice of benchmark) as follows:

\[ E[U] = E[R_{portfolio} - R_{benchmark}] - \psi \cdot \text{Var}[R_{portfolio} - R_{benchmark}] \]

where

- \( E[R_{portfolio}] \) is the expected return of the portfolio,
- \( E[R_{benchmark}] \) is the expected return of the benchmark,
- \( \text{Var}[R_{portfolio} - R_{benchmark}] \) is the variance of the tracking errors of the portfolio,

and \( \psi \) is the regret-aversion of the investor.

This expected utility function takes the same form as the objective function of the traditional Markowitz optimisation, thereby allowing investors to select portfolios along the TEV efficient frontier according to their regret-aversion.

Roll (1992) suggests a further possible practical advantage of using expected TE rather than expected returns is that this may reduce the noise in estimation. The MV model is particularly susceptible to errors in estimating expected returns and these errors are usually substantial. However, Roll suggests it may be easier and more accurate for portfolio managers to estimate expected returns relative to a benchmark than absolute returns thus reducing errors in estimation.

### 1.3.2 Disadvantages and Criticisms of the TEV approach

Benchmarks and relative risk management may solve some of the problems facing managers and their sponsors but the TEV framework brings with it some disadvantages and compromises.
1.3.2.1 Intuitive Disadvantages

One could argue that models which focus on TEV are designed to look after the fund managers’ jobs rather than the investor’s interests! This argument may apply particularly, for example, if the benchmark delivers negative performance. Portfolio managers who have designed funds such that they deliver the same negative performance will be rewarded regardless of the investors’ loss. However, investors who have carefully managed the risk of their portfolios and ignore TEV, are unlikely to forego profit in this same situation. Thus while TEV optimisation enables fund managers to contextualise their portfolio selection task, it can compromise the management and optimisation of the investor’s risk of loss (i.e. absolute risk).

The position that the two models (MV and TEV) take on cash is also worth noting. Cash (borrowing or lending) is desirable in the MV framework since cash is a riskless asset. However, in a TEV framework, cash is undesirable since it has tremendous tracking error volatility\(^5\). So while the MV model might encourage an investor to manage the risk of a portfolio of risky assets with an allocation to cash, the TEV model will be averse to cash (and any other asset which causes the portfolio to depart from the benchmark). For a traditional (absolute) variance-averse investor, aversion to a certain reward (i.e. the risk-free asset) may be a hard pill to swallow!

1.3.2.2 Choice of Benchmark

Many of the criticisms of the TEV approach concern the choice of the benchmark. An ill-chosen benchmark can, once again, compromise the investor/sponsor in favour of the fund manager. If the benchmark given to fund managers is not critical for the fund as a whole, the fund may avoid strategies which incur benchmark risk but which could benefit the fund’s position as a whole. For example when a peer aggregate benchmark is used for segments of a pension fund when in fact the most critical

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\(^5\) Refer to Chapter 2.
objective for the fund as a whole is to meet its liabilities, the fund may avoid benchmark risk at the expense of its liabilities.

Hence it is critical not to mis-specify the benchmark. Benchmarks are typically set by the fund manager in collaboration with the fund sponsor. Left to their own devices, some fund managers may be tempted to set a "slow rabbit" benchmark which is easier to beat. However, irrespective of the discomfort managers may feel by having their performance evaluated relative to a benchmark, it is in their best interest in the long run to make sure that this benchmark is a fair representation of their investment strategy. The ideal benchmark or "normal portfolio" should be customised to cover every aspect of a manager's contract with the fund sponsor: style, cash holdings, liabilities, restrictions on investment etc. However, as benchmarks become more precise and customized, it becomes increasingly difficult for managers to add value. Thus the benchmark can become "more of a stick than a carrot"!\(^6\)

Bailey and Tierney (1993) show that a "slow rabbit" benchmark does not benefit the manager and can lead to client terminations. They reason that while a skilful manager can enhance benchmark-relative returns constructing a "slow rabbit" benchmark, this choice of benchmark will in all likelihood make the volatility of the benchmark-relative returns appear greater. Therefore, if the manager's performance is measured using a risk-adjusted measure, the inferior benchmark is unlikely to make risk-adjusted performance appear better than it is. Moreover, the apparent increased volatility of excess of benchmark returns may generate an interim under-performance of the benchmark large enough to result in client terminations.

Bailey (1992) lists the qualities of a good benchmark:

- The benchmark should have a high proportion of the assets in common with the managed portfolio.
- Since the benchmark represents the passive alternative, the turnover required to track the benchmark should be low.

\(^6\) Divecha (1989)
Chapter 1: Introduction and Literature Review

- The active weights of the managed portfolio (i.e. the weight of each asset in the portfolio less the weight of the same asset in the benchmark) should be mostly positive. That is, if a manager finds an asset unattractive, one would expect the portfolio to exclude the asset altogether. Market capitalization benchmarks can be inappropriate because they assign weights to single securities which the prudent manager, even if he/she finds the asset attractive, cannot match and will thus take a negative active weight in the asset.

- The benchmark should have weighted positions in all its assets which are large enough to be realistically investable given the funds total assets and the liquidity of the securities.

- The fund’s TEV against the benchmark should be lower than the fund’s TEV against a broader market index.

- The correlation of the benchmark’s excess of market returns and the fund’s excess of market returns should be significantly positive.

- The correlation between the benchmark’s excess of market returns and the fund’s excess of benchmark returns should be close to zero.

- The style exposure of the benchmark and the fund, as defined by a multi-factor risk model such as the BARRA system, should be similar.

Two of these criteria and their consequences are discussed in further detail in Tierney and Bailey (1995) – refer to section 1.4.4 of this chapter.

1.3.2.3 Serial Correlation

Some managers may be elected to avoid "adding value" by being mandated to track a given benchmark exactly (i.e. passive fund management) for which benchmark-relative performance monitoring is particularly useful. Passive funds must be constantly re-balanced to bring their constituent parts in line with the index that they track. However, practical constraints such as liquidity differences among stocks and short-term market inefficiency can inhibit a fund manager’s ability to perfectly track an index. Short-term monitoring of TEV enables investors and managers to monitor the success with which the tracker fund is mimicking its benchmark. However, Pope
and Yadav (1994) caution that an index fund, which is overweight in either relatively less or relatively more liquid stocks, is expected to exhibit negative serial correlation in its excess-of-benchmark returns. Consequently, estimates of TEV are expected to overstate the true TEV, particularly when based on high-frequency returns. The subject of serial correlation and potential remedies is pursued in further detail in Chapter 4.

1.3.2.4 Continuity and Robustness of Performance Measures

Although TEV may be a more intuitively pleasing alternative to variance when it comes to performance measurement, Green (1986) shows that when a benchmark is not mean-variance efficient, the relative rankings of funds in terms of their performance relative to the benchmark is particularly sensitive to the choice of benchmark. Thus while a benchmark provides fund managers with a clear and implicit mandate, a relative framework may not be the most appropriate context in which to evaluate the performance of competing fund managers.

1.3.2.5 TEV Optimisation as an Error Maximiser

Given the criticism of mean/variance optimisers as error maximisers, it is important to note that a TEV optimiser, while producing more diverse portfolios, does not solve the error maximisation problem. As with the traditional mean variance optimisation, the contribution to risk management of the TEV optimiser depends on the quality of the return and covariance forecasts.

Even with improved covariance estimators, Rohweder (1998) questions the use of TEV optimisation to control risk. Rohweder suggests the portfolio segmentation (PS) model as an alternative to avoid the consequences of incorrect variance/covariance estimates. The PS model reduces the portfolio selection problem to dividing assets between two portfolios, a passive portfolio (representing the benchmark) and a well-diversified active portfolio which implements stock selection ideas without adjusting for tracking error variance. This model neglects the potential
Chapter 1: Introduction and Literature Review

diversification benefits of exploiting the covariance between assets in favour of avoiding the potential loss of inaccurate covariance estimates. Portfolio segmentation has various organisational advantages such as a lower turnover rate and consequently lower costs than those usually associated with TEV optimisation. Rohweder suggests that the fund management task be divided between an active and a passive manager thus reducing management fees since active management fees will be paid on fewer assets. Rohweder shows empirically that portfolios composed in this way have been superior (ex post) to TEV optimal portfolios.

1.4 IMPORTANT RESEARCH IN TRACKING ERROR OPTIMISATION

Benchmark-relative portfolio selection and performance monitoring has been a hot topic for the last 10 years or so, and there is much literature available to both academics and practitioners on the subject. Probably the best known authors on the subject of active portfolio management are Grinold and Kahn who published a second edition of their book, “Active Portfolio Management” in 2000. However, a handful of academic articles have made major contributions to the development of a benchmark-relative portfolio selection framework. These contributions are discussed in detail in this section.

1.4.1 Roll (1992)

Roll (1992) provides a comprehensive theoretical discussion of the use of TEV optimisation as opposed to MV optimisation. The portfolio selection framework within which Roll (1992) works allows for unrestricted short sales and focuses on self-financing adjustments to existing portfolios rather than the selection of an efficient long-only portfolio from scratch. Thus minimising TEV for a given TE without constraints on short sales allows an explicit solution. Roll views the consequent TEV efficient frontier and optimal portfolio from this model in the MV framework. His model is as follows:
Minimise $(\Delta x)^{\prime}\Sigma(\Delta x)$ subject to
\[
\begin{align*}
(\Delta x)^{\prime}1 &= 0 \\
(\Delta x)^{\prime}R &= G
\end{align*}
\]
where $\Delta x$ is a self-financing change in portfolio weights from the benchmark weights.

$R$ is the vector of expected returns.

and $G$ is the target expected TE

This model has an explicit solution:
\[
\Delta x = \frac{G}{R_1 - R_0}(x_1 - x_0)
\]
where "0" is the global minimum variance portfolio.

and "1" is the portfolio which lies on the efficient frontier on a ray from the MV origin through "0".

Under these conditions, Roll (1992) makes five major assertions. Firstly, by design, MV optimisation will never produce portfolios which are superior in a TEV sense to those produced by TEV optimisation and vice versa, even under perfect return assumptions. Moreover, realistically any pre-selected benchmark is unlikely to be MV efficient. Roll shows that TEV efficient portfolios in a MV context are inefficient by the same amount of variance for every level of expected return as the benchmark. In other words the TEV efficient frontier is equidistant horizontally in MV space from the MV efficient frontier and the distance between the two frontiers is the amount of variance by which the benchmark is inefficient. This assertion highlights the importance of benchmark choice and its influence in the departure of TEV optimisation from the traditional MV optimisation.

However, Roll also proves that, under identical return and risk assumptions, two TEV-optimising managers with the same TE target but different portfolios and different benchmarks will conduct identical trades. Thus self-financing adjustments to any portfolio under TEV optimisation are identical, irrespective of the original portfolio and the chosen benchmark. This result implies a restricted role for the benchmark in active fund management.
However, if the fund sponsor elects to constrain the beta of the managed portfolio, the optimised, self-financing adjustments to the portfolio for managers with different benchmarks are no longer identical. In fact, Roll shows that beta-constrained TEV efficient frontiers can dominate unconstrained TEV efficient frontiers in MV space and the practise of constraining betas in a TEV optimisation is particularly efficient in a MV sense when the benchmark is far from the MV efficient frontier. Moreover, Roll proves that it is best to constrain betas to a value less than one in order for the resultant TEV efficient frontier to produce MV dominating portfolios.

1.4.2 Franks (1992)

Franks (1992) uses TEV optimisation with a short-run target return strategy. Using simulation Franks showed actual returns approached various pre-specified target excess-of-benchmark returns with reliability over fifty years and with low risk of under-performing the benchmark. These results are stable over a wide range of look-back periods. Thus both intuitively and practically, TEV optimisation appears to work well.

1.4.3 Chow (1995)

Most portfolio selection models concern themselves with maximising return and minimising a particular form of risk: absolute volatility, variance of returns relative to a benchmark, downside variation or downside tracking error variation. But Chow argues that an investor’s preferences aren’t always as simple as a linear trade-off between one kind of risk and return. An investor may want to minimise his/her absolute risk but may still be uncomfortable with large deviations from the benchmark portfolio. Chow confirms that non-Markowitz optimisation (i.e. TEV optimisation) yields portfolio selections that are closer to the portfolios investors actually hold. However, simply substituting relative risk for absolute risk in an optimisation model may be too extreme for many investors. Thus Chow (1995) suggests a combination of the Markowitz and TEV optimisation procedures: an objective function for
optimisation which maximises expected return and minimises both variance (absolute risk) and TEV (relative risk) subject to an aversion to each risk. Chow asserts that this “benchmark-sensitive” objective function generates portfolios that more closely resemble the portfolios investors actually choose than objective functions that do not take tracking error into account.

The mean-variance/tracking error (MVTE) optimisation problem as proposed by Chow is stated as follows:

Maximise \( W^T R - \lambda W^T \Sigma W - \psi W^T \Sigma^* W \)

subject to \( W_i \geq 0 \) and \( \sum W_i = 1 \)

where \( R \) is the vector of returns of assets indexed by \( i \)
\( W \) is the vector of weights of each asset in the portfolio
\( \Sigma \) is the variance/covariance matrix of returns
\( \Sigma^* \) is the variance/covariance matrix of tracking error (relative returns)
\( \lambda \) is the inverse of risk (variance) tolerance
\( \psi \) is the inverse of tracking error variance tolerance

and \( w_i \) is the weight of the asset \( i \) in the portfolio

Chow’s MVTE model requires an investor to stipulate tolerances for volatility and tracking error volatility separately, presuming an orthogonality (linear tradeoffs) between expected return, variance and tracking error variance. The MVTE model generates a three-dimensional surface of efficient portfolios. This surface has as its three boundaries i) the efficient frontier of the MV model, ii) the efficient frontier of the TEV model and iii) the straight line from the minimum risk portfolio to the benchmark (minimum TEV) portfolio. Thus the efficient set of the MVTE model contains the whole set of M/V and M/TEV efficient portfolios and all convex combinations of these two sets. Chow refers to Black and Litterman (1992) and concludes that the MVTE model is a special case of the class of objective functions with two risk benchmarks. In the case of the MVTE model, one benchmark is the risk-free investment (cash) - which accounts for the MV part of the objective function.
- and the other is a benchmark portfolio (such as the market index) – which accounts for the MTE part of the benchmark function.

However, it is important to note that, while Chow touches on the TEV model’s aversion for cash, he excludes the riskless asset from his analysis. This exclusion has important implications for the TEV vs variance debate since the asset with zero variance has a high level of TEV. Naturally the converse is also true since the benchmark has no TEV but usually has significant variance. See also Section 1.3.1.3.

Chow’s empirical results show that the MVTE model selects a middle-of-the-road portfolio as opposed to the two extremes chosen by MV and TEV models with the same tolerance toward risk and TEV respectively. Chow mentions the estimation error problem but asserts that, while the effect of estimation errors may force investors to reject optimal portfolios generated by the MV model, optimal portfolios generated by the MVTE model “approximate the actual portfolios investors hold”. Chow rejects attempts to amend estimated return inputs in favour of a MVTE model which allows for subsequent adjustments to the parameters of risk and TEV tolerance instead.

Chow also criticises incorporating constraints to the weightings held in each asset, or the number of assets held in total, within the traditional MV optimisation. This constrained strategy has been used in practise in an effort to make MV portfolios more meaningful. Chow argues that this kind of constrained MV optimisation implies a discontinuous objective function on the part of the investor while MVTE allows for a smoother trade-off between TEV, variance and return. Moreover, Chow argues that quantifying the constraints on asset weights is less intuitive than setting the relative importance of return, TEV and absolute risk in the MVTE model.
1.4.4 Tierney and Bailey (1995)

Tierney and Bailey (1995) give some guidance in the choice of benchmarks by examining two fundamental properties of a valid benchmark with two consequent implications:

Firstly, over the long-term, the covariance of the benchmark's return and the active/abnormal return of the portfolio should be zero. Consequently, the portfolio’s beta relative to its benchmark should be one.

\[ \text{Cov}(\text{Benchmark, Active Component}) = 0 \Rightarrow \beta(\text{Portfolio, Benchmark}) = 1 \]

Secondly, over the long-term, the covariance of the market proxy and the active/abnormal return of the portfolio should be zero. Consequently, the portfolio’s beta relative to its market proxy should be identical to the benchmark’s beta relative to the market proxy.

\[ \text{Cov}(\text{Active Component, Market Proxy}) = 0 \Rightarrow \beta(\text{Portfolio, Market Proxy}) = \beta(\text{Benchmark, Market Proxy}) \]

Thus two tests for a valid benchmark are:

- The beta of the portfolio against the benchmark must not be significantly different from one and
- The beta (market risk) of the portfolio and the benchmark should not differ significantly.

The latter test also enables the fund sponsor to set an appropriate cash position for the benchmark. The idea being that the benchmark represents the style of the fund and any departure of the benchmark from the market proxy is a "style mismatch".

Another implication is that the returns relative to the market proxy cannot be less variable than the returns relative to a valid benchmark, and thus any risk-adjusted measure of performance relative to a valid benchmark will always be larger (more exact) than a risk-adjusted performance measure relative to the market proxy. Thus a test of the manager's active skills are more accurately measured using a valid benchmark.
1.4.5 Leibowitz et al (1993)

Leibowitz et al (1993) deal with three aspects of the portfolio optimisation problem: return, relative return (or TE) and surplus control. The authors demonstrate surplus control (using a shortfall constraint) as it applies to asset allocation relative to a benchmark. A shortfall constraint limits the probability of over- or under-performing the benchmark by a particular amount (egg-shaped region). Using equity/bond/cash portfolios and benchmarks, the authors demonstrate how one may find a feasible region and manage a portfolio within that region. "Efficient frontiers" then are made up of those portfolios which eliminate interest rate risk by matching the duration of the asset portfolio to that of the benchmark, while simultaneously falling within the shortfall constraint region.

The authors conclude that the investor should not simply opt for the maximum allowable equity and expected return portfolio since a slightly less optimal position allows a broader range of duration choices. This approach to the problem, they say, suggests a far more flexible solution compared to the optimisation "black box".

1.5 Structure of the Thesis

The thesis begins by exploring the theoretical framework of relative risk and return (TEV and TE), particularly under CAPM assumptions. Chapter 2, specifically examines the Capital Market Line (CML), cash and the benchmark itself in an asset allocation framework that replaces variance with TEV. Chapter 3 compares empirically the difference between MV and TEV optimisation using CAPM and historical risk and return estimation.

Chapter 4 of the thesis concerns performance monitoring under the benchmark-relative paradigm. In particular, this chapter presents evidence of serial correlation in passive or index tracking fund performance. The extent of the influence of serial dependence on risk estimation is discussed along with a correction procedure to reduce bias in risk estimation.
Chapters 5 through 7 are concerned with leverage in both the traditional MV framework and the TEV framework with particular emphasis on constrained investments in the risk-free rate or the benchmark. Chapter 5 contains a discussion of the effect of constrained leverage (investing or borrowing at a risk-free rate) on MV utility optimisation and Chapter 6 strengthens the discussion with an empirical study based on leverage constraints found in practice. Finally Chapter 7 discusses the constrained use of the risk-free rate and the benchmark in a benchmark-relative framework of portfolio selection. The thesis ends with Conclusions and References.
Chapter 2: An Analysis of Tracking Error

AN ANALYSIS OF TRACKING ERROR

2.1 INTRODUCTION ................................................................. 2

2.2 AN ANALYSIS OF TRACKING ERROR .................................. 3
  2.2.1 Decomposing Tracking Error Variance (TEV) ...................... 3
    2.2.1.1 Definitions .................................................. 3
    2.2.1.2 Decomposition ............................................. 3
  2.2.2 Tracking Error in an Asset Allocation Framework .................. 6
    2.2.2.1 TE/TEV Space ............................................. 9
      2.2.2.1.1 The relationship between consensus TE and TEV ........... 10
      2.2.2.1.2 The risk-free asset .................................. 13
    2.2.2.2 TE/TE Standard Deviation Space ........................... 14
      2.2.2.2.1 The Capital Market Line ................................ 15

2.3 CONCLUSIONS ............................................................... 17
2.1 INTRODUCTION

The Mean-Variance (MV) approach to portfolio selection has yielded insights that are both sensible and intuitive and consistent with rational investor behaviour. The work of Markowitz, Tobin and Sharpe conveyed a deeper appreciation of risk and the benefits of diversification and led neatly into a theory of asset pricing i.e. the Capital Asset Pricing Model (CAPM). Recognising that total risk (i.e. variance) can be decomposed into two interpretable terms was central to the development of the CAPM. This development not only equipped practitioners and academics with a quantitative approach for measuring different kinds of risk but also made clear conceptual sense.

However, with the current emphasis on tracking error (TE)$^1$ (see for example Franks (1992), Chow (1995), Roll (1992) and Clarke et al (1994)), the conceptual foundations of Capital Market Theory may have been shaken. An important consideration regarding the replacement of variance with tracking error variance (TEV), is whether the neat interpretations and decompositions of risk are lost! This chapter begins with an investigation into the implications of replacing variance with tracking error variance with a particular focus on interpretable risk decomposition. This decomposition is pursued into a form that relates tracking error to a portfolio’s beta.

The chapter continues by bringing TEV into an asset allocation framework. To enable this analysis, the second decision parameter, expected return (or relative return) is introduced. This analysis takes a consensus view on the return parameter by using the CAPM as the return-generating model. Thus the chapter ends with a graphical summary by presenting a Mean$^2$ TE-TEV framework under the assumption

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$^1$ In this paper the terminology used is similar to that defined by Roll(1992)$^1$. In particular, tracking error is taken to mean the return relative to a benchmark portfolio, whereas tracking error variance is taken to mean the variance of these relative returns.

$^2$ In this chapter, reference is made to expected (consensus) returns and thus "Mean" refers to an expectation rather than a historical average.
of consensus (CAPM) returns³. This last section is particularly concerned with the interpretations of risk within this two-dimensional TE-TEV framework.

2.2 **AN ANALYSIS OF TRACKING ERROR**

2.2.1 Decomposing Tracking Error Variance (TEV)

This section investigates the links between Tracking Error Variance (TEV), the relative measure of risk, and the traditional variance measure. In particular, one can decompose relative risk (TEV) in much the same way as one can decompose the traditional variance measure of risk i.e. into Market Risk and Unique Risk.

2.2.1.1 Definitions

**Tracking error** or relative return (TE) is the difference between the return on a portfolio \( R_p \) and the return on a particular benchmark portfolio \( R_b \).

\[
TE_p = R_p - R_b
\]

**Tracking error variance** (TEV) is the variance of this TE time series (following Roll (1992)).

\[
TEV = \text{Var}[R_p - R_b] = \frac{\sum_{t=1}^{n}[(R_{pt} - R_{bt}) - (R_p - R_b)]^2}{n-1}
\]

where \( n \) is the number of observations in the time series.

2.2.1.2 Decomposition

The above expression (2.1) can be written in terms of its expectation⁴:

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³ Refer to F Black & R Litterman (Financial Analysts Journal 1992 September-October) to see how consensus returns may be modified by views.
\[ TEV_p = \sigma_p^2 + \sigma_b^2 - 2\sigma_{pb} \]  

It is thus evident that TEV can be decomposed into three components: the variance of the portfolio \( \sigma_p^2 \), the variance of the benchmark return \( \sigma_b^2 \) and the covariance between the portfolio return and the benchmark return \( \sigma_{pb} \). This form\(^5\) immediately establishes the link between TEV and the traditional portfolio variance measure \( \sigma_p^2 \).

But a portfolio’s \( \beta \) is a function of the covariance component found in equation (2.2). In particular, the well-known Ordinary Least Squares (OLS) market\(^6\) model expected value is:

\[ \beta_{pb} = \frac{\sigma_{pb}}{\sigma_b^2} \]  

(2.3)

where \( \beta_{pb} \) is the benchmark beta i.e. the beta of the portfolio against the benchmark.

\( \sigma_b^2 \) is the variance of the benchmark portfolio

and \( \sigma_{pb} \) is the covariance of the portfolio and the benchmark portfolio.

Thus rearranging the terms in (2.3) and substituting the covariance term into (2.2), TEV can be further decomposed as follows:

\[ TEV_p = \sigma_p^2 + \sigma_b^2 (1 - 2\beta_{pb}) \]  

(2.4)

\(^4\) Equation (2.1) can be expanded as follows:

\[
TEV = \frac{\sum_{t=1}^{n} [(R_{pt} - R_{mt}) - (\bar{R}_p - \bar{R}_m)]^2}{n-1}
\]

\[= \frac{\sum_{t=1}^{n} (R_{pt} - \bar{R}_p)^2}{n-1} + \frac{\sum_{t=1}^{n} (R_{mt} - \bar{R}_m)^2}{n-1} - 2 \frac{\sum_{t=1}^{n} (R_{pt} - \bar{R}_p)(R_{mt} - \bar{R}_m)}{n-1} \]

The expression in Equation (2.2) follows when the above expression is written in terms of its expected value.

\(^5\) Franks (1992) arrives at the same decomposition of tracking error variance.

\(^6\) The market model traditionally uses the returns of the market portfolio as the independent series when estimating \( \beta \) however, it is not uncommon to use a benchmark instead of the elusive market portfolio. The results of the ordinary least squares regression model still hold.
Furthermore, it is well known (see Sharpe (1964)) that the market model decomposes the variance (i.e. total risk) of portfolio returns into market/benchmark risk and unique risk. That is:

\[ \sigma_p^2 = \beta_{pb}^2 \sigma_b^2 + \sigma_e^2 \]  

(2.5)

where \( \beta_{pb}^2 \sigma_b^2 \) is the benchmark risk of the portfolio and \( \sigma_e^2 \) is the unique risk of the portfolio (or the residual variance)

And so, by replacing the expression for \( \sigma_p^2 \) in (2.4) tracking error variance can be rewritten as follows:\footnote{TEV\textsubscript{p} = \beta_p^2 \sigma_m^2 + \sigma_e^2 + \sigma_m^2 (1 - 2 \beta_p) 
= \sigma_e^2 (\beta_p^2 - 2 \beta_p + 1) + \sigma_e^2}

\[ TEV_p = \sigma_b^2 (\beta_{pb} - 1)^2 + \sigma_e^2 \]  

(2.6)

From equation (2.6) it is evident that, like the traditional variance measure of risk, tracking error variance also decomposes into two neat risk components. The second term of equation (2.6), \( \sigma_e^2 \) has the same interpretation as the unique risk component of the traditional variance decomposition. In fact, if the benchmark portfolio is the market portfolio, the value of \( \sigma_e^2 \) will be exactly the same as the unique risk component of traditional variance. As with variance, this unique risk component of tracking error variance penalises a manager for taking on any \textbf{unique/diversifiable risk}, that is, the risk of holding a portfolio of stocks which differs from that of the benchmark. Thus the lessons of diversification are just as important in the TEV framework as they are in the traditional variance framework.

The second component of tracking error variance (the first term of equation (2.6) i.e. \( \sigma_m^2 (\beta_{pb} - 1)^2 \) penalises a manager for shifting the portfolio’s exposure to the benchmark away from that of the benchmark itself (i.e. shifting the portfolio’s benchmark beta away from one). This component could be called “\textbf{relative}”
**benchmark risk.** In the traditional variance framework, a manager incurs more market risk with increasing exposure to the market (i.e. as the portfolio’s beta increases away from $\beta=0$ or the so-called “risk-free” position). In the TEV framework, a manager incurs more benchmark risk as the portfolio’s benchmark beta departs from that of the benchmark itself (i.e. $\beta=1$), irrespective of whether the portfolio’s $\beta$ is greater than or smaller than one. In other words, the squared term in parenthesis in (2.6) ensures that portfolios with benchmark betas which depart from unity, whether above or below, systematically incur the same additional risk in a tracking error variance framework.

This decomposition of Tracking Error Variance into such an intuitive and interpretable form is very encouraging for the use of TEV in place of variance. The fact that all of the well-entrenched notions of variance decomposition (such as the benefits of diversification and the control of benchmark) are still upheld for TEV, suggests that the use of TEV in the place of variance (and a benchmark portfolio instead of the market portfolio) does not necessarily conflict with Capital Market Theory.

### 2.2.2 Tracking Error in an Asset Allocation Framework

In the tradition of portfolio selection within a risk-reward framework, this chapter considers TEV within such a framework. To this end, the second decision parameter, that of expected return or, in this case, expected tracking error (TE) is introduced. A consensus view on this parameter is taken by using the CAPM as the return-generating model.

\[
E[R_p] - R_f = \beta_m (E[R_m] - R_f)
\]

where $R_m$ is the market portfolio

$R_f$ is the risk-free rate

Hence equation (2.6) follows. Grinold and Kahn have also derived this equation in their book “Active Portfolio Management”.
and $\beta_{pm}$ is the portfolio's beta against the market.

Thus if the CAPM is the return-generating model, then the expected TE, of any portfolio is a function of the market premium:

$$E[TE_p] = E[R_p - R_b]$$
$$= [(E[R_p] - R_f) - (E[R_b] - R_f)]$$
$$= \beta_{pm} (E[R_m] - R_f) - \beta_{bm} (E[R_b] - R_f)$$

(2.7)

However, the expected return of the benchmark is also a function of the market premium:

$$(E[R_b] - R_f) = \beta_{bm} (E[R_m] - R_f)$$
$$\Leftrightarrow E[R_m] - R_f = \frac{(E[R_b] - R_f)}{\beta_{bm}}$$

Thus, replacing the market premium in expression (2.7) with the benchmark premium, the expected tracking error of any portfolio can be expressed as a function of the benchmark premium:

$$E[TE_p] = \frac{\beta_{pm} (E[R_b] - R_f) - \beta_{bm} (E[R_b] - R_f)}{\beta_{bm}}$$
$$= \left(\frac{\beta_{pm}}{\beta_{bm}} - 1\right) (E[R_b] - R_f)$$

(2.8)

That is, a portfolio's expected tracking error is the product of the portfolio's beta against the benchmark relative to the benchmark's beta against the market less one, and the benchmark premium. It follows that any portfolios with the same $\beta$'s and the same benchmarks will have the same expected TE's (and the same expected returns). However, portfolios with the same $\beta$'s are expected to have different expected TE's if their benchmarks differ with respect to their benchmarks' betas against the market.

If the benchmark's beta against the market is one (or in fact, if the benchmark is the market), the relationship between TE and the benchmark-market premium is simpler:

$$E[TE_p] = (\beta_{pm} - 1)(E[R_b] - R_f) = (\beta_{pm} - 1)(E[R_m] - R_f)$$

(2.9)

Equation (2.9) shows that the expected TE will be positive for any portfolio with
Chapter 2: An Analysis of Tracking Error

\[ \beta \text{ greater than one and negative for any portfolio with } \beta \text{ less than one. This relationship is intuitive since, under the CAPM, a portfolio must have a } \beta \text{ greater than one in order to expect the portfolio to earn more than the market. If a portfolio earns more than the market, its tracking error relative to the market will be positive. The converse is true for portfolios with } \beta \text{ less than one.} \]

Figure 2.1 presents a graphical summary of this relationship in a framework that is analogous to risk-reward (MV) representations. Firstly, in a CAPM world, it is known that all portfolios with identical \( \beta \)'s will have identical expected returns and therefore identical expected TE’s (relative to the same benchmark). And so, portfolios with identical \( \beta \)'s plot along horizontal lines in TE-TEV space as illustrated in Figure 2.1.

**Figure 2.1: Beta Rays in Consensus Tracking Error (TE)/ Tracking Error Variance (TEV) Space**

Figure 2.1 also illustrates the relationship between the benchmark premium and a portfolio’s TE and \( \beta \)-value. From equation (2.8) it is clear that expected TE is given by the departure from \( \beta = 1 \) relative to the benchmark’s beta, times the market premium. For example, any portfolio with a \( \beta \) of 1.5 will have an expected TE (against a benchmark with a beta of one) of half the market premium. In the same way, given the vertical position of any portfolio in TE-TEV space and the
benchmark's beta against the market, one would be able to determine the portfolio's beta or vice versa.

Notice that the risk-free asset \( (R_f \text{ or cash}) \) has a fixed position in this framework. Since cash has zero market risk (i.e. \( \beta = 0 \)) its TE is minus the benchmark premium. Thus, against a benchmark with a beta of one against the market portfolio, the expected return of the risk-free rate is minus the market/benchmark premium. The issue of the risk-free asset in the TE-TEV framework will be discussed in more detail later on in this chapter.

### 2.2.2.1 TE/TEV Space

**Returning to the risk (TEV) parameter:** recall from equation (2.6) that, although portfolios with identical \( \beta \)'s will have the same expected TE, these portfolios may have different values of TEV (tracking error variance). The difference in TEV for portfolios with identical \( \beta \)'s (in a consensus/CAPM environment) depends only on the amount of unique risk \( (\sigma^2) \) each portfolio has. Therefore, it is evident from equation (2.6) that, amongst portfolios with the same reward (TE) and therefore the same \( \beta \) value, the portfolio with the minimum risk (TEV) will have the least unique risk. This line of inference is, of course, also true in the MV framework. Thus the portfolios that offer the best TE/TEV trade-offs are identical to the portfolios which offer the best risk/reward trade-off in the traditional Markowitz asset allocation framework since, under the assumption of consensus returns, unique risk is not priced.

Ultimately, according to Tobin's separation theorem, the most efficient risk-reward opportunities are provided by the Capital Market Line which comprises all combinations of the risk-free asset with the market portfolio. The following section examines the TE/TEV framework to determine whether the same relationship exists in a benchmark-relative setting.
2.2.2.1.1 The relationship between consensus TE and TEV

For the remainder of this analysis, it is assumed that the benchmark’s beta relative to
the market is one and that the benchmark has no unique risk i.e. the benchmark is the
market portfolio.

The relationship between expected Tracking Error and Tracking Error Variance along
the curve which bounds all investment opportunities can be established by combining
(2.6) and (2.8) to define TE and TEV implicitly:

\[ E[TE_p] = (\beta_{pm} - 1)E[R_m - R_f] \]

\[ TEV_p = \sigma_m^2(\beta_{pm} - 1)^2 + \sigma_e^2 \]

\[ \Leftrightarrow (\beta_{pm} - 1)^2 = \frac{E[TE_p]}{E[R_m - R_f]} = \frac{TEV_p - \sigma_e^2}{\sigma_m^2} \]

Thus expected TE and TEV can be defined implicitly:

\[ F(TEV_p, E[TE_p]) = \frac{E[TE_p]}{E[R_m - R_f]} - \frac{TEV_p - \sigma_e^2}{\sigma_m^2} = 0 \]

Since different portfolios have different amounts of unique risk ( \( \sigma_e^2 \geq 0 \) ), one can
also write the implicit function of TE and TEV as follows to describe a family of
curves in TE-TEV space, each with a different amount of unique risk:

\[ \frac{E[TE_p]}{E[R_m - R_f]} - \frac{TEV_p}{\sigma_m^2} = -\frac{\sigma_e^2}{\sigma_m^2} \]  

(2.10)

The gradient of the implicit function above is as follows\(^8\):

\[ \frac{dF(TEV_p, E[TE_p])}{d(TEV_p)} = \frac{-1}{\sigma_m^2} = \frac{E[R_m - R_f]}{E[TE_p]} \]

\[ \frac{dF(TEV_p, E[TE_p])}{d(E[TE_p])} = \frac{-1}{2E[TE_p]} = \frac{2E[TE_p]}{2E[TE_p] \sigma_m^2} \]

Substituting the consensus/CAPM expression for expected tracking error in the
denominator, the expression in (2.10) follows.

\(^8\) The gradient of the function in (2.9) is calculated as follows (provided, of course,
that the expected TE of the portfolio and the market premium is non-zero):

\[ \frac{d(E[TE_p])}{d(TEV_p)} = \frac{-1}{\sigma_m^2} = \frac{E[R_m - R_f]}{E[TE_p]} \]

\[ \frac{d(E[TE_p])}{d(E[TE_p])} = \frac{-1}{2E[TE_p]} = \frac{2E[TE_p]}{2E[TE_p] \sigma_m^2} \]
\[
\frac{d[E[TE_{p}]]}{d(TEV_{p})} = \frac{1}{(\beta_{pm} - 1)} \frac{E[R_{m} - R_{f}]}{2\sigma_{m}^{2}}
\] 

(2.11)

And so, a continuum of portfolios which are CAPM-compliant and have the same unique risk will form a frontier with a slope described by equation (2.10). Naturally the most favourable frontier in TE/TEV space will be the frontier which describes a continuum of portfolios with zero unique risk (since these portfolios will have the least TEV). This zero-unique risk frontier will constitute the most risk-reward efficient investment opportunities and this frontier is referred to henceforth as the “boundary frontier”. Frontiers describing increasing amounts of unique risk will from concentric hyperbolas to the right of the boundary frontier in TE-TEV space.

The opportunities along any of these frontiers (each with their own fixed amount of unique risk) are determined by 3 factors: the largest-β portfolio in the feasible set with a particular amount of unique risk, the benchmark/market premium and the variance of the market/benchmark. Therefore the higher the benchmark/market premium and the lower the variance (risk) of this same benchmark/market, the steeper these frontiers will be. The steeper the frontier, the more opportunities there are in a higher (more favourable) position in TE-TEV space. This is intuitive since a market with low risk (in the traditional variance sense) and a high expected excess return is always more attractive to investors. (This result has an analogy in MV space - see Appendix A)

The gradient of each frontier also depends on the β-value of each of the portfolios with the same amount of unique risk in the opportunity set. As β approaches one (the benchmark), the gradient of the frontier (equation (2.11)) approaches infinity (i.e. the gradient becomes vertical). Conversely, as β becomes much larger than one, the slope of the frontier approaches zero (i.e. the gradient becomes horizontal). Since the gradient of the frontiers (2.11) are all symmetrical about \( \beta=1 \), their slopes flatten as \( \beta \) becomes smaller than one. The factors influencing the shape of these frontiers will have the same effect on each of the frontiers describing portfolios of a particular fixed amount of unique risk (\( \sigma_{u}^{2} \)). Thus the shape and gradient of each of these frontiers
will be identical.

The boundary frontier that traces the positions of all possible portfolios with zero unique risk is plotted with a bold line in TE-TEV space in Figure 2. Figure 2 illustrates how TEV relates to $\beta$ and unique risk. Portfolio $P$, for example has a $\beta$ of 1.5 but it does not lie on the boundary frontier. Portfolio $P$ lies on a frontier directly to the right, in a more risky position. As a consequence of the decomposition of TEV, the distance between $P$ and the boundary curve stems only from the amount of unique risk $P$ has. Thus the decomposition of TEV for portfolio $P$ can be restated in terms of the distances shown in Figure 2, as follows:

$$TEV_P = CP = CA + AP$$

**Figure 2.2: Unique Risk in Consensus Tracking Error (TE) / Tracking Error Variance (TEV) Space**

Therefore the distance $AP$ represents the unique risk of portfolio $P$. In this way one can illustrate the unique risk of any portfolio that lies off the boundary frontier in Figure 2 by measuring the horizontal distance between the portfolio and the boundary frontier of portfolios with no unique risk. In fact, the entire frontier on which portfolio $P$ lies should be the same distance ($AP$) from the boundary frontier.

The "relative" market risk, of portfolio $P$ can be established by measuring the
horizontal distance $CA$ in Figure 2. In general this distance is given by $\sigma^2_m (\beta_p - 1)^2$.

In the absence of fully diversified (zero unique risk) portfolios (particularly in higher market risk regions) the efficient set of portfolios (portfolios which offer the best TE/TEV trade-off) will depart from the boundary frontier into less favourable TE/TEV space. Essentially, the boundary frontier will end and the next frontier, which has slightly more unique risk, will become the most efficient frontier as high-beta portfolios with little or no unique risk cease to be available.

However assuming limitless borrowing and lending at the risk free rate, it is easy to show that investors can extend their opportunities along the boundary frontier indefinitely in exactly the same way as they can shift up and down the CML in the traditional Markowitz framework (see Appendix B). Section 2.2.2.1.2 and 2.2.2.2 elaborate further on the issue of the risk-free asset and the CML in the benchmark-relative framework.

2.2.2.1.2 The risk-free asset

Notice the position of the risk-free asset in the TE/TEV framework. Since the risk-free asset has no unique risk, it lies on the boundary sketched in Figure 2. Furthermore, since this asset has a $\beta$ of zero, its TEV is simply the market variance ($\sigma_m^2$). When the boundary of available opportunities is viewed in TE-TEV space, it becomes clear that the asset which is considered "risk-free" in a mean-variance context, in fact carries a high TEV risk and a negative expected TE and as such lies in a very unfavourable position relative to other opportunities. Thus, even without the assumption of a consensus/CAPM environment, the MV risk free asset is in fact very risky in the benchmark-relative sense.

In the Markowitz Mean-Variance framework, the inclusion of a "risk-free" asset in the efficient set is familiar. When risk is expressed in terms of variance, "riskless" means zero variance and the riskless asset is therefore any asset that has a constant
return. However, when risk is expressed in terms of TEV, the only “riskless” asset is the asset that never strays from the benchmark, namely the benchmark itself.

In the Mean-Variance context it is well known that the risk-free asset combines linearly with any portfolio in the feasible set and can be held long or short (borrow or lend at a fixed interest rate). In the TE-TEV context, the benchmark simply replaces the role of the risk-free asset.

It is easy to show then (see Appendix D) that the benchmark combines linearly with any other asset in TE-TE Standard Deviation Space. Furthermore, combining the benchmark with any asset/portfolio with no unique risk (such as the risk-free asset) will form a portfolio which lies along the boundary frontier in TE/TEV (or TE/TE standard deviation space (see Appendix E)).

2.2.2.2 TE/TE Standard Deviation Space

Since the Markowitz efficient frontier is usually shown in Mean-Standard Deviation Space, this section considers the TE-TEV frontiers in TE-TE Standard Deviation space as well. (Once again the analogous development in Mean-Standard Deviation space is included in the Appendix C).

Given the relationship between TEV and expected TE in the implicit function (2.10), the gradient of the boundary curve in TE/TE standard deviation space is as follows:

\[ \frac{d}{d\sqrt{\text{TEV}_p}} \left( E]\text{TEV}_p \right) \frac{2\sqrt{\text{TEV}_p}}{\sigma^2} \frac{E[\text{R}_m - R_f]^2 \sqrt{\text{TEV}_p}}{E[R_m - R_f]} \frac{E[\text{TEV}_p]}{E[R_m - R_f]^2} \]

Substituting the decomposition of TEV (2.5) in the numerator and the consensus expression for expected TE in the denominator, the expression for the slope may be written as follows:

\[ 2 - 14 \]
\[
\frac{d \langle E[TE_p] \rangle}{d \langle \sqrt{TEV_p} \rangle} = \pm \frac{E[R_m - R_f]}{\sigma^2_m (\beta_p - 1)} \left[ \frac{\sigma^2_m (\beta_p - 1)^2 + \sigma^2_i}{(\beta_p - 1)} \right]
\]  
(2.12)

The simple relationship between beta and the slope of the frontiers formed by groups of portfolios with the same amount of unique risk no longer applies in the TE/TE Standard Deviation framework.

### 2.2.2.2.1 The Capital Market Line

However, if one considers only portfolios with no unique risk (i.e. \( \sigma^2_e = 0 \)), the expression in (2.12) simplifies to a familiar\(^{10}\) constant:

\[
\frac{d \langle E[TE_p] \rangle}{d \langle \sqrt{TEV_p} \rangle} = \pm \frac{E[R_m - R_f]}{\sigma^2_m (\beta_p - 1)} \left[ \frac{\sigma^2_m (\beta_p - 1)^2}{(\beta_p - 1)} \right] = \pm \frac{E[R_m - R_f]}{\sigma^2_m}
\]  
(2.12)

Thus the boundary frontier in TE-TE Standard Deviation space which traces portfolios with no unique risk (i.e. \( \sigma^2_e = 0 \)), consists of two linear segments (constant slope) whose slopes (2.11) are determined only by the market premium and variance. The slope of this curve is independent of the individual portfolios in the feasible set and therefore constant on either side of \( \beta = 1 \) (the benchmark): the gradient is positive when \( \beta > 1 \) and negative when \( \beta < 1 \). Also note that, as before, the boundaries of the opportunity set widen with increased market/benchmark premium and decreased benchmark variance (risk).

In fact, this boundary frontier is the Capital Market Line. Figure 2.3 shows the Capital Market Line which forms the boundary frontier of opportunities in TE-TE Standard Deviation space. The CML represents all portfolios that have no unique risk and has the same components as the boundary frontier represented in Figure 2.

---

\(^{10}\) The market premium divided by the market risk is the slope of the Capital Market Line.
The distinction between the Capital Market Line in a Mean-Standard Deviation framework and a TE-Standard Deviation framework is this: portfolios along the upper line ($\beta > 1$) will be more attractive to benchmark-sensitive investors than portfolios along the lower line ($\beta < 1$) since they offer a higher expected TE for the same level of TEV. For this reason, the lower line is dominated in the TE-TEV sense and therefore does not form part of the TE-TEV efficient frontier. In other words, a combination of any fixed amount held short (borrowed) in the risk-free asset with the benchmark will incur the same TEV as the same fixed amount held long (lent) with the benchmark. However, the expected TE for the former is positive and the expected TE for the latter is negative and thus the portfolio with a short position on the risk-free asset will always dominate a portfolio with a long position. Effectively then, the CML is cut in half in a benchmark-relative asset allocation framework.

Once again, in the absence of fully diversified portfolios, particularly at higher levels of market risk ($\beta$), this linear boundary will begin to curve downwards (SouthEast) in TE/TE Standard Deviation space. However, investors can “lever up” in TE/TE Standard Deviation space using the benchmark in the same way as they can when shifting an investment along the CML in MV space using the risk free asset (see
Appendix E).

2.3 CONCLUSIONS

The objective of this chapter was to establish whether the lessons of Capital Market Theory are still upheld when variance is replaced with TEV in an asset allocation framework. The first section demonstrated that TEV decomposes into two neat interpretable forms much like the traditional market model decomposition of variance. One term was unique risk and the other was termed relative market risk. Hence, if managers are penalised for bearing TEV, they can identify the above mentioned two component sources of risk.

The second section demonstrated how these risk components bounded the scope of opportunities in a consensus (CAPM) environment. It was shown that for every level of unique risk, the three determinants of the slope of the frontier of investments are the market/benchmark premium, the market/benchmark variance and the highest beta asset with that level of unique risk available in the opportunity set. These results are also consistent with the Markowitz framework and the Capital Market Line theory.

Furthermore, it was noted that the (MV) risk-free asset has significant TEV risk (even without the assumption of consensus returns) and that the benchmark portfolio takes its place as the riskless asset in the TEV framework. However, since neither the benchmark nor the traditional risk-free asset have unique risk, they still combine linearly in a TE-TE Standard Deviation framework and, as such, may be used to shift along the Capital Market Line assuming limitless borrowing or lending is allowed. In other words, Tobin’s Separation Theorem applies in the benchmark-relative framework.
Chapter 3: An Empirical Demonstration of Mean Variance and Tracking Error Variance Optimisation

**AN EMPIRICAL DEMONSTRATION OF MEAN VARIANCE & TRACKING ERROR VARIANCE OPTIMISATION**

3.1 INTRODUCTION........................................................................................................... 2

3.2 DATA AND ANALYSIS.................................................................................................. 3
  3.2.1 The Data.................................................................................................................. 3
  3.2.2 CAPM Estimation.................................................................................................... 3
  3.2.3 Generating the MV Efficient Frontier........................................................................ 4

3.3 MV AND TEV OPTIMISATION USING A COMPOSITE INDEX AS A
BENCHMARK/MARKET PROXY..................................................................................... 5
  3.3.1 MV and TEV Optimisation under CAPM Risk and Return Behaviour................. 6
  3.3.2 MV and TEV Optimisation under CAPM Return Behaviour.................................. 9
  3.3.3 MV and TEV Optimisation Given Historical Return and Risk Estimation........... 13

3.4 MV AND TEV OPTIMISATION USING A MV EFFICIENT
BENCHMARK/MARKET PROXY..................................................................................... 15
  3.4.1 MV and TEV Optimisation under CAPM Return Behaviour............................... 16
  3.4.2 MV and TEV Optimisation Given Historical Return and Risk Estimation........... 18

3.5 CONCLUSIONS........................................................................................................... 20
AN EMPIRICAL DEMONSTRATION OF MEAN VARIANCE & TRACKING ERROR VARIANCE OPTIMISATION

3.1 INTRODUCTION ........................................................................................................... 2

3.2 DATA AND ANALYSIS ................................................................................................. 3
   3.2.1 The Data ............................................................................................................. 3
   3.2.2 Return Estimation .............................................................................................. 3
   3.2.3 Generating the MV Efficient Frontier ................................................................. 4

3.3 MV AND TEV OPTIMISATION USING A COMPOSITE INDEX AS A BENCHMARK/MARKET PROXY .................................................................................. 5
   3.3.1 MV and TEV Optimisation in the absence of unique risk ................................... 6
   3.3.2 MV and TEV Optimisation under CAPM Return Behaviour ........................... 9
   3.3.3 MV and TEV Optimisation Given Historical Return and Risk Estimation ....... 13

3.4 MV AND TEV OPTIMISATION USING A MV EFFICIENT BENCHMARK/MARKET PROXY .......................................................................................... 15
   3.4.1 MV and TEV Optimisation under CAPM Return Behaviour ....................... 16
   3.4.2 MV and TEV Optimisation Given Historical Return and Risk Estimation ....... 18

3.5 CONCLUSIONS ............................................................................................................. 20
3.1 **INTRODUCTION**

This chapter follows the analysis of TE and TEV in Chapter 2 and is concerned with the use of TEV in risk-reward asset allocation framework. The analysis in this chapter seeks to compare the use of MV optimisation and TEV optimisation under various return estimation techniques and benchmarks.

The first section discusses the data that was used in this chapter as well as the use of the CAPM and the optimisation procedure for both MV and TEV optimisation. The second section compares the results of MV and TEV optimisation under three return estimation techniques: a) both return and risk are estimated directly from the CAPM, b) only return is estimated using the CAPM and c) return and risk are estimated directly from historical performance. The benchmark for this analysis is the Allshare index which represents a traded market proxy.

The final section compares the same optimisation under the same return estimation techniques but with a different benchmark. Since the Allshare index of the previous section turned out to be MV inefficient, the ex-post efficient frontier under historical return estimates was used as a benchmark (and as a market proxy in the CAPM) in this final section.

The analysis in this chapter illustrates the similarity and differences between the two methods of asset allocation, MV and TEV optimisation. If the benchmark is MV efficient (an unlikely event), the MV and TEV efficient frontiers overlap from the highest return portfolio to the benchmark itself. However, the extent to which the benchmark is MV inefficient, is the extent to which the MV and TEV efficient frontiers will separate in risk-reward space, the MV frontier dominating in MV space and the TEV frontier dominating in TEV space. The TEV efficient frontier never extends beyond the benchmark to portfolios of lower return than the benchmark, irrespective of their MV efficiency since the benchmark is, by definition, the "riskless" certainty equivalent in the TEV sense. Moreover, the risk-free rate which is the point of lowest risk (or certainty equivalent) in the MV efficient frontier, is
undesirable in the TEV framework and is seldom if ever part of any portfolios along the TEV efficient frontier.

3.2 DATA AND ANALYSIS

3.2.1 The Data

To represent the investment environment available to South African investors, the universe of assets considered here consists of 53 Johannesburg Stock Exchange (JSE) Actuaries Indices, one of which is the All Share Index (JOHMKT) which is used as a benchmark in section 3.3. The overnight interest rate at the time of this study was 10.10% NACQ\(^1\) (about 0.84% per month) and this rate is used as a proxy for the risk-free rate in this study.

3.2.2 Return Estimation

In part of the analysis that follows, returns consistent with the CAPM were generated to correspond with the theory of Chapter 2. Thus, (expected) returns for each of these indices in the sample set were calculated as follows:

\[
R_i = R_f + \hat{\beta}_i (R_{\text{Benchmark}} - R_f)
\]

where \(R_i\) is the estimated (expected) return of each index,

\(R_f\) is the risk-free rate (in this case, the current overnight interest rate),

\(R_{\text{Benchmark}}\) is the historical return\(^2\) on the benchmark (either JOHMKT or another specified benchmark) each month from January 1999 to August 2000,

---

\(^1\) Nominal Annual Compounded Quarterly (October 2000).

\(^2\) The analysis could also have used an expected benchmark return to achieve the same demonstration of the results in Chapter 2. However, to more easily compare the results in sections 3.3.1, 3.3.2 and 3.4.1 with those in sections 3.3.3 and 3.4.2 which use historical return estimation, the historical average return for the benchmark was used.
and $\hat{\beta}_i$ is the best fit ordinary least squares\(^3\) estimate of each index's beta based on the monthly returns of the benchmark and those of the index in question from January 1996 to December 1998.

"Out of period" returns were used to estimate betas to be consistent with the CAPM as a forward-looking, expectations model. Thus the beta of each asset relative to the JOHMKT index is estimated in one period (January 1996 to December 1998) and used to estimate returns in the following period (January 1999 to August 2000). However, instead of using expected (forward-looking) returns on the benchmark against which to estimate expected returns on each index, the actual average return on the benchmark from January 1999 to August 2000 was used.

### 3.2.3 Generating the MV Efficient Frontier

Mean-Variance (MV) optimisation and Mean-TEV (MTEV) optimisation are used in this chapter and those that follow. The formula for traditional mean-variance optimisation is:

$$\begin{align*}
\text{Minimise} & \quad -W^T R + \lambda W^T \Sigma W \\
\text{subject to} & \quad W^T 1 = 1 \\
& \quad 0 \leq W \leq 1
\end{align*}$$

Solve for $W$

where $W$ is the vector of the proportional investment in each asset,

$R$ is the vector of returns for each asset,

$\lambda$ is the risk aversion which ranges from zero (risk-indifferent) to infinity (highly risk-averse)

and $\Sigma$ is the variance/covariance matrix of returns.

The MV optimisation formula can also be described by the following algorithm:

Select a portfolio for each level of risk aversion such that the portfolio return is maximised and the portfolio risk is minimised subject to the following constraints:

\[^3\] $R_i = \hat{\alpha}_i + \hat{\beta}_i R_{\text{benchmark}} + e_i$ where $\hat{\alpha}_i$ and $\hat{\beta}_i$ are best fit regression coefficients and $e_i$ is the residual return.
the sum of proportional investments must equal 100%,
• there are no short sales i.e. the proportional investment in any asset must be at least zero,
• there is no leverage i.e. the proportional investment in any asset must be less than one.

The formula for optimisation of returns and minimising TEV rather than variance is identical in logical structure. In the case of MTEV optimisation, a variance/covariance matrix of relative returns/TE’s \((R_i - R_{\text{benchmark}})\) is used instead of a variance/covariance of absolute returns \((R_i)\). Thus the formula for MTEV optimisation is as follows:

\[
\text{Minimise} \quad -W' \Sigma W + \psi W' \Sigma^* W \quad \text{for } \psi = 0\ldots\infty
\]

\[
\text{subject to} \quad \begin{cases} W' 1 = 1 \\ 0 \leq W \leq 1 \end{cases}
\]

Solve for \(W\)

where \(W\) is the vector of the proportional investment in each asset,
\(R\) is the vector of returns for each asset\(^4\),
\(\psi\) is the TEV aversion which ranges from zero (TEV-indifferent) to infinity (highly TEV-averse),
and \(\Sigma^*\) is the variance/covariance matrix of tracking errors.

3.3 MV AND TEV OPTIMISATION USING A COMPOSITE INDEX AS A BENCHMARK/MARKET PROXY

In this section, the JOHMKT (JSE Allshare Index) is used as a market proxy and benchmark. It is common practice to use a published index such as the JSE Allshare

\(^4\) It would be consistent with the previous chapter to maximise tracking error (TE) rather than returns while minimising TEV, however the use of either (TE) or returns will produce the same resulting optimal portfolios. It is the choice of risk measurement i.e. TEV or variance that distinguishes MTEV optimisation from MV optimisation.
as a benchmark since it’s composition and performance is published regularly and it is exempt from the control of any one fund manager. A summary of the historical data used in this section is provided in Appendix F along with a summary of the benchmark-relative statistics.

3.3.1 MV and TEV Optimisation in the absence of unique risk

The analysis begins by imposing a simple linear relationship between the expected return of the benchmark and the expected return for each asset. Thus, the risk of each asset is purely market related and no asset in the set has any unique risk. Thus, for every asset, \( i \):

\[
R_i = R_f + \beta_i (R_{Benchmark} - R_f)
\]

and consequently:

\[
Var[R_i] = \beta_i^2 Var[R_{Benchmark}]
\]

The feasible set of assets is plotted in risk-return space in Figure 3.1: the series determined by the linear relationship above (in pink) is shown on the same axes as the historical risk and return for each asset (in blue). The JOHMKT index, which formed the market proxy in this section, is plotted as an enlarged pink circle as a reference point.

Notice that the assets in the feasible set plot in a straight line emanating from the market proxy if the simple linear relationship described above is true. With the exception of the four assets which had negative betas\(^5\) over the measurement period, all the assets are risk-reward efficient. Historically however, some of these assets (seven to be exact) had a higher risk-reward\(^6\) ratio than the JOHMKT (refer to Appendix F), demonstrating the shortcomings of the CAPM when the market proxy is not mean-variance efficient.

---

\(^5\) It is unusual to find negative betas on any stock exchange and one does not expect these to persist over time. Negative betas indicate that the correlation of the stock with the market/benchmark is less than one.

\(^6\) Risk-reward without taking the risk-free rate into account.
Figure 3.1: Historical Return and Risk vs Systematic Return and Risk

It is no surprise then that the efficient frontier, when every asset’s returns are linearly related to the benchmark and have no unique risk, includes every asset except those which have negative betas. Under these conditions there is no benefit to diversification and all assets combine in a straight line, the Capital Market Line. Figure 3.2 plots the MV efficient frontier under these conditions, clearly showing that the efficient set extends from the highest beta portfolio through the market portfolio (JOHMKT) and ends at the lowest non-negative beta portfolio, the risk-free asset which has a beta of zero.
The TEV efficient frontier under the same conditions corresponds almost exactly with the MV efficient frontier. Figure 3.3 depicts both the MV and the TEV efficient frontier in return-TE standard deviation space. Notice that the two frontiers overlap from the highest beta portfolio to the market proxy (which has a beta of one). However, for assets with betas less than one (i.e. from JOMKT towards the risk-free asset), CAPM expectations would predict returns less than the benchmark/market and thus all assets with betas less than one, although MV efficient, are excluded from the TEV efficient frontier.

Notice too the extreme position of the risk-free asset. The risk-free asset, as shown in Chapter 2, has the greatest departure from the benchmark in respect of both its risk and return (with the exception of the four assets which had negative betas over the period). The risk-free asset will thus seldom, if ever, form part of the portfolios along the TEV efficient frontier.
The conclusions in this section hold true irrespective of the market proxy used provided the benchmark is the market proxy. In the absence of unique risk, all the individual assets in the feasible set lie along the efficient frontier. Moreover the MV and TEV efficient frontiers overlap from the asset with the highest TE to the benchmark portfolio, from which the MV frontier extends to the portfolio comprising only the risk-free asset. This latter section of the MV frontier is inefficient under the TEV criterion.

### 3.3.2 MV and TEV Optimisation under CAPM Return Behaviour

The analysis continues by generating expected returns using the same relationship as described in section 3.2.2 but all risk statistics are now calculated from historical monthly returns over the period January 1999 to August 2000. Thus

\[ R_x = R_f + \beta_x (R_{\text{Benchmark}} - R_f) \]

but the risk is estimated using the variance of the historical monthly returns over the period January 1999 to August 2000.
Figure 3.4 summarises the data used in this section. The horizontal axis describes the historical risk while the vertical axis describes the return, either historical (the blue series) or CAPM-predicted (the pink series). The benchmark index (JOHMKT) is highlighted as a reference point. Notice that the historical and the CAPM-predicted return for the JOHMKT are identical in contrast to the other indices, many of which have very different historical and CAPM returns.

**Figure 3.4: CAPM vs Historical Returns (Historical Risk)**

MV optimisation was performed on the set of assets shown in Figure 3.4 (as described in section 3.2.3) using the historical Variance-Covariance matrix and CAPM-predicted returns. The resultant efficient frontier is shown in Figure 3.5 along with the Capital Market Line (CML), the risk-free rate and the tangent portfolio. The fine blue line represents the risky assets-only efficient frontier (excluding the risk-free asset) while the bold blue line represents the efficient frontier of all investments that

\[ \text{Figure 3.5 have been truncated to best display the efficient frontier. Thus some of the assets which are visible in Figure 3.4 are not displayed in Figure 3.5 although these assets still form part of the analysis.} \]
are held long\(^8\) (including the risk-free asset). For comparison, the MTEV optimisation efficient frontier is included in Figure 3.5. This frontier is discussed in further detail later on in this section.

**Figure 3.5: The Mean-Variance Efficient Frontier with CAPM-generated returns**

Once again, the efficient frontier extends from the asset with the highest return to the risk-free asset. Having introduced unique risk into the asset allocation environment, the efficient frontier is no longer a straight line and the benefits of diversification among risky-assets is evident. The individual asset portfolios are no longer part of the efficient frontier (except for the asset with the highest return, JOHPLAT). Instead, the efficient frontier is comprised of a more diverse set of investments. Under these conditions, the separation theorem becomes part of the portfolio selection process. The tangent portfolio (highlighted in red) offers the highest risk-reward efficiency and the rational, risk-averse investor must select a position along the CML.

---

\(^8\) To hold an asset long means that the weight of the asset in the portfolio is positive i.e. there is no borrowing.
which satisfies their particular risk-aversion by borrowing or lending at the risk-free rate and investing in the tangent portfolio.

Notice that the benchmark portfolio (JOHMKT) in this case is not MV efficient i.e. JOHMKT does not form part of the MV efficient frontier but it is MTEV efficient. This contrast illustrates the difference between the MV and the MTEV efficient frontier unlike the overlap that was observed in Figure 3.3. Figure 3.6 illustrates this difference by displaying the MTEV efficient frontier in Mean-TE standard deviation space alongside its MV counterpart.

Figure 3.6: The TEV Efficient Frontier with CAPM-generated returns

Once again, the TEV efficient portfolio extends from the single asset with the highest return to the benchmark portfolio. The curve of the frontier demonstrates the benefits of diversification exist in a TEV framework as well as the traditional MV framework. Since the benchmark portfolio is not MV efficient in this case, it is not surprising that the MV tangent portfolio is not efficient in the TEV sense. In fact, there are even some individual assets in this feasible set which dominate the MV frontier in TEV space. Thus the TEV efficient frontier dominates in TEV space and the MV efficient

3 - 12
frontier dominates in MV space. The extent to which these frontiers differ is the extent to which the benchmark portfolio is MV inefficient (i.e. the amount of unique risk the benchmark portfolio has compared to the MV tangent portfolio).

From this analysis and the discussions in Chapter 2, note that the risk-free asset is always as far horizontally from the benchmark portfolio in TEV space as the benchmark is risky; the risk-free asset has TEV equal to the benchmark’s variance by definition. Thus confirming the TEV inefficiency of the risk-free asset and the CML.

3.3.3 MV and TEV Optimisation Given Historical Return and Risk Estimation

In this section CAPM conditions are dropped altogether and only historical averages are used to estimate return, variance and TEV. Figure 3.7 depicts the MV efficient frontier based on these historical inputs in MV space. Once again the fine blue line represents the efficient frontier of risky assets while the bold blue line represents the efficient frontier of all assets held long (including the risk-free asset). The tangent portfolio is marked in red and has a different composition to the portfolio that was tangent in section 3.3.2. The MTEV efficient frontier under the same conditions is included for completeness. Note that the benchmark portfolio (JOHMKT) is still MV inefficient and, using historical estimation of return, this benchmark is not only more risky than the tangent portfolio but also less profitable.
Figure 3.7: MV Efficient Frontier Using Historical Risk and Return

Figure 3.8 shows the TEV efficient frontier for the same set of assets in TEV space alongside the MV efficient frontier, with and without the risk-free asset. As expected from previous discussions and analysis, the TEV efficient frontier dominates in TEV space by definition. The MV efficient frontier without the risk-free asset in turn dominates the efficient frontier with the risk-free asset held long in TEV space as a consequence of the risk-free asset’s significant contribution to its TEV inefficiency.

It is clear from the analysis in this section that the risk-free asset and the choice of benchmark are important factors in the difference between MV and TEV efficiency. Whether historical or CAPM estimates of risk and return are used, the risk-free rate is exceptionally “risky” in a TEV sense and the MV efficient frontier (and CML) are dominated in a TEV framework as a consequence. The choice of benchmark and more particularly, the MV inefficiency of the benchmark, determines the disparity between the MV and TEV efficient frontier in higher return areas (i.e., from the portfolio comprising only the asset with the highest return to the benchmark portfolio).
3.4 MV AND TEV OPTIMISATION USING A MV EFFICIENT BENCHMARK/MARKET PROXY

It is clear from the preceding analysis that the MV inefficiency of the benchmark is a significant factor in the separation of the MV and TEV efficient frontiers. The following section addresses the effect of using an ex-post MV efficient tangent portfolio as a benchmark and market proxy. In this section, the ex-post tangent portfolio based on historical estimates (section 3.3.3) is used as a market proxy and benchmark. The summary statistics relative to the ex-post tangent portfolio (henceforth the "market proxy") of all the indices used in this section of the analysis are presented in Appendix G. Figure 3.9 summarises graphically the data used in this section of the analysis and is analogous to Figure 3.4 except for the choice of market proxy. The horizontal axis represents historical risk and the vertical axis represents returns, either CAPM-generated (the pink series) or historical (the blue series). The
former is used to generate efficient frontiers in section 3.4.1 and the latter in section 3.4.2.

**Figure 3.9: CAPM vs Historical Returns (Historical Risk) in MV Space**

![Image](https://example.com/image.png)

**3.4.1 MV and TEV Optimisation under CAPM Return Behaviour**

The market proxy used in this section is the MV tangent portfolio under historical return estimates (found in section 3.3.3). This market proxy is therefore not necessarily efficient under CAPM conditions.

Figure 3.10 illustrates the MV and TEV efficient frontiers and the CML in return-standard deviation space and Figure 3.11 illustrates the same frontiers in TEV space. Notice that, as a consequence of the market proxy not being MV efficient under CAPM conditions, the tangent portfolio in this case differs from the market proxy portfolio. However the improved efficiency of the market proxy compared to the JOHMKT (used as a benchmark in section 3.3) brings the MV and TEV efficient frontier closer together in risk-reward space than the two frontiers shown in Figure 3.6 of section 3.3.2.
Also note that the TEV and the MV efficient frontiers still begin with the asset with the highest return and end with the market proxy and risk-free rate respectively. Regardless of the return estimation techniques and the choice of benchmark, the risk-free rate and the benchmark itself will remain “riskless” in the MV and TEV framework respectively. Moreover, the element of the MV efficient frontier which lies between a return similar to the benchmark and the risk-free rate bends back in TEV space into a less efficient area (more TEV and less return) – refer to Figure 3.11.
Figure 3.11: MV and TEV Efficient Frontiers in TEV Space

3.4.2 MV and TEV Optimisation Given Historical Return and Risk Estimation

Since the market proxy and benchmark used in this section is the MV tangent portfolio under historical return estimates, this benchmark will form part of the MV and the TEV efficient frontier. Figure 3.12 and Figure 3.13 show both efficient frontiers in MV and TEV space and confirm that, if the benchmark is MV efficient, the MV and TEV efficient frontiers will overlap from the asset with the highest return to the benchmark, irrespective of the return estimation technique. From the benchmark onwards, the MV efficient frontier is dominated by the TEV efficient frontier in TEV space and ends with the risk-free rate.
Chapter 3: An Empirical Demonstration of Mean Variance and Tracking Error Variance Optimisation

Figure 3.12: MV and TEV Efficient Frontiers in MV Space

Figure 3.13: MV and TEV Efficient Frontiers in TEV Space
3.5 CONCLUSIONS

The analysis in this chapter illustrates the difference between minimising absolute (variance) or relative (TEV) risk in the portfolio selection process. It is unlikely that a MV efficient benchmark will be selected en route to selecting a TEV optimal portfolio since benchmark selection is usually determined by a fund mandate and is determined a priori. However, the analysis in this chapter shows that the MV and TEV efficient frontier overlap from the highest return portfolio to the benchmark itself if the benchmark is MV efficient. Under a more realistic benchmark choice, the extent to which the benchmark is MV inefficient is the extent to which the MV and TEV efficient frontiers will separate in risk-reward space. Thus the selection of a benchmark is an important factor for the manager who is concerned with TEV optimisation. Comparison with MV optimisation may be a prudent way for such a manager to keep absolute risk in check.

The analysis in this chapter also highlights two theoretical discussions from the previous chapter. Firstly that diversification reduces both absolute (MV) and relative (TEV) risk. The benefits of diversification can be clearly observed when historical risks (market and unique risk) are used in the optimisation process. Secondly, the "riskless" or certainty equivalent asset for the TEV framework is the benchmark and the "riskless" asset for the MV framework is the risk-free rate. Thus the TEV efficient frontier never extends beyond the benchmark to portfolios of lower return than the benchmark, irrespective of their MV efficiency. Moreover, the risk-free rate is highly undesirable in the TEV framework and seldom, if ever, forms part of any portfolios along the TEV efficient frontier. Thus managers who are constrained to invest in the risk-free rate (or a similarly low risk asset) should apply TEV models with caution. This point will be discussed further in Chapter 7.
Chapter 4: Index Funds: Serial Correlation and TEV Bias

INDEX FUNDS: SERIAL CORRELATION AND TEV BIAS

4.1 INTRODUCTION ................................................................. 2
4.2 THE RATIONALE BEHIND TEV BIAS .................................... 3
4.3 ANALYSIS OF SA TRACKER FUNDS .................................... 4
  4.3.1 The Data ................................................................. 4
  4.3.2 Test for Negative Serial Correlation ............................... 5
  4.3.3 TEV bias ............................................................... 7
4.4 ADJUSTED TEV MEASUREMENT ....................................... 9
  4.4.1 Lo and MacKinlay Adjustment ................................... 9
  4.4.2 Implementing the Lo-MacKinlay Adjustment .................. 9
4.5 CONCLUSIONS ............................................................... 12
4.1 INTRODUCTION

Benchmark-relative performance monitoring is particularly useful in passive fund management or index tracking. Index funds must be constantly re-balanced to bring their constituent parts in line with the index that they track. However, practical constraints such as liquidity differences among stocks and short-term market inefficiency can inhibit a fund manager’s ability to perfectly track an index. Short-term monitoring of TEV enables investors and managers to monitor the success with which tracker funds mimic their benchmark.

Pope and Yadav (1994) show that an index fund that is overweight in either relatively less or relatively more liquid stocks, is expected to exhibit negative serial correlation in its excess-of-benchmark returns (TE’s). Consequently, the usual estimate of TEV will be upwardly biased. That is:

\[
\frac{\sum_{t=1}^{T} (R_{\text{fund}} - R_{\text{benchmark}}) - \left(\frac{R_{\text{fund}} - R_{\text{benchmark}}}{T-1}\right)^2}{\text{Var}\left[R_{\text{fund}} - R_{\text{benchmark}}\right]} > \text{Var}\left[R_{\text{fund}} - R_{\text{benchmark}}\right]
\]

The bias is expected to be greater when estimated over higher frequency data.

There are consequent implications for the performance monitoring of funds, particularly on the shorter term where the bias is expected to be the greatest. Firstly, funds that are monitored on a shorter term than their investment horizon (which is typically the case) will appear to have greater TEV than is actually the case. This estimation error may consequently lead to greater “churn” or turnover than necessary in the fund’s assets thus incurring greater and unnecessary transaction costs. Secondly, when comparing the success of index funds across various indices/benchmarks (and mandates), the funds which track indices that are composed of more liquid stocks may appear to be more successful (i.e. have lower TEV) than funds for which the estimate of TEV is biased to a greater degree.

The first section in this chapter explores the rationale behind the expectation of biased
TEV estimation and serial correlation. The second section applies this rationale to the South African Mutual Fund environment and index tracking funds in particular. In this section, evidence for negative serial correlation is sought and the extent of the bias in TEV estimation for high frequency data is explored. The final section of the analysis in this chapter describes a possible correction for the TEV bias, the Lo-MacKinlay adjustment, and measures the success of this adjustment on the estimation of TEV for local tracker funds.

4.2 **The Rationale Behind TEV Bias**

Pope and Yadav (1994) show that an index fund that is overweight in either relatively less or relatively more liquid stocks, is expected to exhibit negative serial correlation in its excess-of-benchmark returns (TE’s). This expectation is based on the argument that less liquid stocks are less price efficient than more liquid stocks i.e. the adjustment of the price of less liquid stocks to new information is slower than the adjustment of the price of more liquid stocks to the same information. Thus the cross-covariance (across lagged time periods) between the returns on stocks with different relative liquidity is expected to be positive, i.e.

\[
\begin{align*}
\text{Cov}[R_{(less\text{ liquid})}, R_{(more\text{ liquid})}] &> 0 \\
\text{Cov}[R_{(more\text{ liquid})}, R_{(less\text{ liquid})}] &> 0
\end{align*}
\]

(1)

where \( R_{(less\text{ liquid})} \) and \( R_{(more\text{ liquid})} \) are the returns on less liquid and more liquid stocks respectively at time \( t \) or a lagged period, \( t-k \).

The covariance term in (1) is expected to be increasingly positive, as the frequency of returns increases i.e. the cross-covariance of weekly return data is expected to be greater than the cross-covariance of monthly return data.

The serial covariance of the TE of two stocks of different liquidity in a portfolio takes the following form:

\[
(w_{less\text{ liquid}} - x_{less\text{ liquid}})(w_{more\text{ liquid}} - x_{more\text{ liquid}}) \text{Cov}[R_{(less\text{ liquid})}, R_{(more\text{ liquid})}] > 0
\]

(2)
where \( w \) is the weight of the share in the tracker portfolio and \( x \) is the weight of the share in the index/benchmark.

If a tracker fund is overweight in relatively less liquid stocks \( (w_{\text{less liquid}} > x_{\text{less liquid}}) \) and consequently underweight in relatively more liquid stocks \( (w_{\text{more liquid}} < x_{\text{more liquid}}) \) then the first two terms in parentheses in (2) will be of opposite sign. The same is true if the tracker fund is underweight in relatively less liquid stocks. Furthermore, the covariance terms for the returns are expected to be positive (refer to (1)). Therefore, the serial covariance of the TE between stocks of different liquidity in a portfolio (the term in (2)) is expected to be negative if the tracker fund is overweight in relatively less or relatively more liquid stocks. The greater the difference in the weights of less (or more) liquid stocks between the fund and the benchmark/index and the higher the frequency of the returns, the more negative the covariance between the TE of stocks of different liquidity is expected to be. Conversely, for stocks of similar liquidity, the covariance term in (1) is expected to be close to zero and of random sign and consequently the covariance term in (2) for stocks of similar liquidity is expected to be small.

Consequently, when the tracker portfolio is overweight in less liquid stocks, estimates of TEV are expected to overstate the true TEV, particularly when TEV is estimated using high-frequency returns.

### 4.3 Analysis of SA Tracker Funds

#### 4.3.1 The Data

The South African Unit Trust industry currently has 12 equity index tracker funds listed: seven JSE All Share index tracker funds, three JSE ALSI-40 index tracker funds and two JSE Financial and Industrial (FINDI) index tracker funds. The monthly and weekly returns of the funds which track each of these equity indices were
calculated\textsuperscript{1} from February 1996 to October 2000. No income was taken into account on the indices or the tracker funds and all returns are thus capital growth only. The returns on the funds were calculated from offer prices only, thus excluding the funds’ front-end fees from the calculation as well as all administrative fees. Transaction costs are however implicit in the returns.

4.3.2 Test for Negative Serial Correlation

Both weekly and monthly excess-of-benchmark returns (TE) were tested for negative serial correlation over time periods of various lengths. The time periods were specifically chosen to include each fund as it was launched and each series was lagged by one, two, three and four week (a month) periods. The null and alternate hypotheses are as follows:

\[ H_0 : \rho(j) = 0 \]
\[ H_1 : \rho(j) < 0 \]

\( j = 1, 2, 3, 4 \text{ weeks or 1 month} \)

There was very little evidence of negative serial correlation in the funds’ monthly TE’s over any of the periods tested, however there was significant evidence of serial correlation in the weekly TE’s of the same funds, particularly over the longer test periods. Table 4.1 summarises the results of the first order serial correlation tests on the weekly returns. The figures highlighted in pink, yellow and grey are significant at better than the 1%, 5% and 10% level respectively. A similar table of results for the monthly returns is included in Appendix H (refer to Table III) and the third and fourth order results for tests of serial correlation on weekly TE’s are summarised in Table IV and Table V of the same Appendix.

\textsuperscript{1} Source: Micropal
Table 4.1: Weekly Serial Correlation (lag = 1 week) of each Fund’s TE against its Benchmark for periods ending October 2000

<table>
<thead>
<tr>
<th>Fund Type</th>
<th>243</th>
<th>225</th>
<th>194</th>
<th>163</th>
<th>150</th>
<th>141</th>
<th>124</th>
<th>48</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>23/02/56</td>
<td>28/06/95</td>
<td>31/01/97</td>
<td>29/08/97</td>
<td>28/11/97</td>
<td>30/01/98</td>
<td>29/05/98</td>
<td>25/11/99</td>
<td>31/03/00</td>
</tr>
<tr>
<td>AllShareTrackers</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Fedvex Index</td>
<td>-0.23</td>
<td>-0.38</td>
<td>-0.37</td>
<td>-0.38</td>
<td>-0.39</td>
<td>-0.39</td>
<td>-0.23</td>
<td>-0.17</td>
<td>-0.35</td>
</tr>
<tr>
<td>Investec Index R</td>
<td>-0.23</td>
<td>-0.25</td>
<td>-0.25</td>
<td>-0.25</td>
<td>-0.25</td>
<td>0.10</td>
<td>-0.03</td>
<td>-0.05</td>
<td>-0.00</td>
</tr>
<tr>
<td>Standard BK Index</td>
<td>-0.24</td>
<td>-0.26</td>
<td>-0.25</td>
<td>-0.25</td>
<td>-0.25</td>
<td>-0.25</td>
<td>0.22</td>
<td>0.23</td>
<td>0.19</td>
</tr>
<tr>
<td>Grysphor Imp SA Tracker</td>
<td>-0.42</td>
<td>-0.43</td>
<td>-0.43</td>
<td>-0.43</td>
<td>-0.43</td>
<td>-0.43</td>
<td>-0.25</td>
<td>-0.35</td>
<td>-0.27</td>
</tr>
<tr>
<td>NIB LT Wealth Creator</td>
<td>-0.23</td>
<td>-0.25</td>
<td>-0.25</td>
<td>-0.25</td>
<td>-0.25</td>
<td>-0.25</td>
<td>0.10</td>
<td>0.22</td>
<td>0.08</td>
</tr>
<tr>
<td>NIB Quant Core Equity</td>
<td>-0.12</td>
<td>-0.12</td>
<td>-0.12</td>
<td>-0.12</td>
<td>-0.12</td>
<td>-0.12</td>
<td>-0.12</td>
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<tr>
<td>Sunam index</td>
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<td></td>
<td></td>
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<tr>
<td>ALSI-40 Trackers</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>Coronation ALSI 40</td>
<td>-0.22</td>
<td>-0.17</td>
<td>-0.21</td>
<td>-0.32</td>
<td>-0.39</td>
<td>-0.39</td>
<td>-0.20</td>
<td>-0.04</td>
<td>-0.01</td>
</tr>
<tr>
<td>Tracker</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Libary ALSI 40 C</td>
<td>-0.33</td>
<td>-0.33</td>
<td>-0.33</td>
<td>-0.33</td>
<td>-0.33</td>
<td>-0.33</td>
<td>-0.22</td>
<td>-0.20</td>
<td>-0.04</td>
</tr>
<tr>
<td>RMB Top 40 Index</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>FINDI Trackers</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>ABSA Find &amp; Industrial Index</td>
<td>-0.43</td>
<td>-0.44</td>
<td>-0.44</td>
<td>-0.44</td>
<td>-0.44</td>
<td>-0.44</td>
<td>-0.20</td>
<td>-0.04</td>
<td>-0.04</td>
</tr>
<tr>
<td>Brent FINDI</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
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<td></td>
<td></td>
</tr>
</tbody>
</table>

The JSE All Share trackers have the longest history among SA tracker funds which allows for tests of more statistical power. The funds which track this index show significant serial correlation over the long to medium term (five to three years). The ALSI-40 is a smaller (fewer shares) and more liquid index: ALSI-40 futures and options are traded regularly on the short and long term and hedging and speculative trading render the forty composite stocks among the most liquid on the JSE. However, there is still significant evidence of serial correlation on ALSI-40 trackers. The FINDI index, by contrast, is fairly illiquid: the FINDI is no longer listed on the futures exchange and the demand for some of its composite stocks is low. The results in Table 4.1 confirm the argument set out in section 4.2 that funds which track an index comprising relatively less liquid stocks are expected to be negatively serially correlated in the short term. Even on the small amount of data available, there is highly significant evidence of negative serial correlation on the FINDI trackers (P value <0.01).
4.3.3 TEV bias

Given the serial correlation that exists for most of these tracker funds over one, two and three weeks (refer to the Appendix H) and in the light of Pope and Yadav (1994), one would expect to find that the annualised estimates of TEV using weekly returns are larger than the annualised estimates of TEV using monthly returns.

Figure 4.1 shows the extent to which the annualised TE standard deviation for index trackers are overestimated using weekly data compared to monthly data for each of the sample time periods listed in Table 4.1. The bars in Figure 4.1 depict the ratio of weekly return-based estimates to the monthly return-based estimates of TE standard deviation. Thus, if the estimation of TE standard deviation based on higher frequency data is biased upwards, the ratios depicted in Figure 4.1 will be larger than one.

It is clear from Figure 4.1 that the annualised TE standard deviation calculated on weekly data has almost always been greater than the annualised TE standard deviation calculated on monthly data over the same period. Only five out of the 69 estimates of TE standard deviation based on weekly returns were smaller than their corresponding estimates of TE standard deviation based on monthly returns. These five estimates represented only three funds and were calculated among the shorter time periods (with fewer observations). Furthermore, as expected and reasoned by Pope and Yadav (1994), funds which track indices comprised of stocks of a wider range of liquidity exhibit greater bias. Accordingly it is clear from the results in Figure 4.1 that the bias in high frequency TEV estimates is smaller for the ALSI-40 trackers on the whole than for the other tracker funds. By contrast the FINDI, which contains some fairly illiquid stocks, exhibits a bias in the TE standard deviation measured on weekly data of between 6 and 16 times for the two funds tracking this index.
The bias in benchmark-relative risk measurement has important consequences, particularly for index funds. Firstly, if benchmark tracking ability is compared across funds with different benchmarks with different liquidity characteristics such as the ALSI-40 and the FINDI, the funds which track benchmarks comprised of stocks with more diverse liquidity characteristics will be severely prejudiced. These funds will appear to be far less successful in their tracking ability than they actually are and will be rated poorly among competing tracking funds. Secondly, tracker funds as a whole are intended to offer much less benchmark-relative risk than active funds. The bias in TEV estimation can thus be damaging to the passive fund industry as a whole by making passive funds appear far more “risky” or actively managed than they actually are.
4.4 **ADJUSTED TEV MEASUREMENT**

4.4.1 *Lo and MacKinlay Adjustment*

Lo and MacKinlay (1988) show that the ratio of the variance calculated over low frequency data to the variance calculated over high frequency data are related as follows:

\[
\frac{TEV_n}{TEV_1} = 1 + \frac{2(n-1)}{n} \rho(1) + \frac{2(n-2)}{n} \rho(2) + \ldots + \frac{2}{n} \rho(n-1)
\]

(1)

where \(TEV_n\) and \(TEV_1\) are the variance over \(n\) period (e.g. monthly) and single period (e.g. weekly) intervals respectively, and \(\rho(j)\) is the \(j^{th}\) order serial correlation coefficient for the single period TE series.

Naturally, if the single period series is not serially correlated, all the correlation terms in (1) are zero and the ratio of variances becomes one.

4.4.2 *Implementing the Lo-MacKinlay Adjustment*

The formulation of the variance ratios in (1) enables us to correct for the bias in the variance of a high frequency series (such as weekly TE’s) if the low frequency series (such as monthly TE’s) is not also serially correlated. The analysis in section 4.3.2 showed that there was little evidence of serial correlation in the monthly TE series. Thus if we were to estimate annualised TEV using a weekly TE series, we could adjust for serial correlation as follows (assume 4 weeks in a month):

\[
TEV = 52.\overline{TEV}_{\text{weekly}} \left(1 + \frac{2(4-1)}{4} \rho(1) + \frac{2(4-2)}{4} \rho(2) + \frac{2(4-3)}{4} + \frac{2}{4} \rho(3)\right)
\]

(2)

\[
= 52.\overline{TEV}_{\text{weekly}} \left(1 + \frac{3}{2} \rho(1) + \rho(2) + \frac{1}{2} \rho(3)\right)
\]
We expect the term in parenthesis in (2) to be less than one on account of negative serial correlation. Thus the adjusted weekly estimate of annualised TE standard deviation should be smaller than the unadjusted estimate which assumes no serial correlation.

Figure 4.2 depicts the ratio of the adjusted to the unadjusted weekly estimates of TE standard deviation. All but six of the shorter term estimates are reduced by the correction described in section 4.4.1 i.e. the ratio of adjusted to unadjusted estimate is less than one.

**Figure 4.2: Ratio of Adjusted Weekly Estimate of TE Std Dev to the Unadjusted Weekly Estimate**

The true test of the success of the adjustment of weekly TEV estimates is to compare the adjusted estimates to unbiased, unadjusted estimates of TEV. If we can safely assume that there is no serial correlation in the monthly series of TE's, it is sufficient to compare the adjusted weekly estimate to the standard monthly estimate of annualised TE standard deviation. Figure 4.3 depicts the ratio of Lo-MacKinlay-
adjusted weekly to monthly estimates of annualised TE standard deviation.

Figure 4.3: Ratio of Adjusted Weekly Estimates to Monthly Estimates of Annualised TE Std Dev

![Ratio of Weekly Adjusted to Monthly Annualised TE Std Dev Ratio](image)

When compared with Figure 4.1, Figure 4.3 illustrates the success of the Lo-MacKinlay adjustment in bringing high frequency (in this case, weekly) estimates of TE standard deviation in line with lower frequency (in this case, monthly), unbiased\(^2\) estimates of TE standard deviation. In particular, the bias of the most significantly serially correlated funds, namely the FINDI trackers, has been reduced by almost half.

\(^2\) There is some evidence of serial correlation in the monthly TE's of some of these funds (refer to section 4.3.2). These funds may still have biased monthly estimates of TE standard deviation and these estimates should be adjusted in turn to bring them in line with lower frequency estimates (e.g. quarterly).
4.5 CONCLUSIONS

In this chapter the evidence of serial correlation and its consequent impact on TEV estimation was examined on locally (South African) listed index tracking mutual funds. The analysis shows clear evidence of negative serial correlation of weekly TE’s of tracker funds across three indices (and mandates) and consequently upwardly biased estimates of TEV when estimated over weekly data. There seems to be little evidence of serial correlation of monthly TE’s and thus estimates of TEV based on monthly data seem to be relatively accurate in the SA environment.

The analysis also shows that the serial correlation and consequent bias of TEV estimation is most pronounced on indices comprised of stocks with varying liquidity characteristics such as the JSE FINDI Index. Therefore comparison of index tracking deviation across indices and mandates should be applied with caution.

TE measurements should be examined for negative serial correlation before estimates of TEV are made. If serial correlation is detected, estimates of TEV should either be made from lower frequency return measurements, if they are available, or an adjustment technique such as the Lo-MacKinlay adjustment should be used to obtain unbiased estimates.
Chapter 5: The Separation Theorem under Constrained Leverage Conditions

THE SEPARATION THEOREM UNDER CONSTRAINED LEVERAGE CONDITIONS

5.1 INTRODUCTION ..............................................................................................................................2
5.2 THE GEOMETRY OF FIXED CONSTRAINED LEVERAGE .................................................................4
5.3 UTILITY OPTIMISATION UNDER DIFFERENT LEVERAGE CONSTRAINTS .........................5
5.4 CONCLUSIONS .............................................................................................................................9
5.1 Introduction

The major problem facing investors has always been the maximisation of wealth in a world of uncertainty. Two main conceptual frameworks have been developed to deal with the problem; the state-preference framework developed by Arrow (1951) and later by Debreu (1959), and the mean-variance or parameter-preference framework developed by Markowitz (1952).

Tobin (1958) extended this theory by showing that the investment decision could be separated into two phases: firstly, the selection of a unique optimum combination of risky assets and secondly, the allocation of funds between the unique optimum combination of risky assets and a single riskless asset. Under conditions of market equilibrium, Roll (1977) pointed out that the former choice would be the unique “market portfolio” (or tangent portfolio) and consequently investors need only concern themselves with the latter stage of the investment decision (namely establishing the desirable combination with the riskless asset).

The focus in this chapter is on the restrictions of the latter separation case, where investors find themselves in a constrained leverage position. That is they are unable to “lever up” or “down” along the Capital Market Line (CML). In this restricted situation, the second phase of the investment decision is fixed and the decision reverts to finding an optimal combination of risky assets which no longer turns out to be the “market portfolio” or tangency portfolio in the usual sense.

Several extensions of Markowitz theory consider a variety of restrictions imposed on investors. Some of these have concerned themselves with restrictions on leverage. For example, Brennan (1971) considers a model where risk-free borrowing and lending are allowed, but at different interest rates while Black (1972) regards a model constraining an investor to hold the risk-free asset long. Likewise Fama (1976) discusses a model where risk-free borrowing and lending are allowed with a restriction on the amount of borrowing. More recently Michaud (1993) cites the constrained borrowing imposed by long-short equity strategies. Still other extensions
of Markowitz theory involve restrictions on portfolio weights. For example, Frost and Savarino (1988) consider a model which constrains the portfolio weights whilst Best and Grauer (1992) discuss the circumstances necessary for minimum variance portfolios to have all positive weights. Similarly, Green and Hollifield (1992) find the necessary conditions for mean/variance efficient portfolios to have small weights in each of their assets (i.e. to be well diversified). And more recently Chang et al (1998) consider models which constrain the number of assets in a portfolio as well as the weights of these assets.

Little has however emerged in the literature which directly considers the impact on the choice (and the risk) of the optimal mean/variance portfolio when investors are constrained to lend or borrow a set proportion of their funds\(^1\).

This note highlights the impact of restricted leverage on optimal decision making in the usual mean-variance framework. Mutual Funds in South Africa, for example, are restricted to a minimum of 5% liquid assets (and will thus be constrained to a fixed lending position). Another example of restricted leverage is when a homeowner is constrained to hold a housing bond (and will thus be constrained to a fixed borrowing position). Section 5.2 briefly discusses the geometry of restricted leverage in mean variance space. The following section (5.3) gives a general proof for the intuitive notion that the higher the proportion of funds an investor is constrained to borrow, the lower the risk of this investor's optimal portfolio of risky assets should be and vice versa.

---

\(^1\) The same result can be derived by using efficient set mathematics (see Best and Grauer (1991) and Roll (1977)). I thank an anonymous referee of the note, "A Note on Portfolio Selection with Restrictions on Leverage" by DJ Bradfield and H Raubenheimer for pointing this out.

5 - 3
5.2 **The Geometry of Fixed Constrained Leverage**

Figure 5.1: Efficient Frontiers with Fixed Borrowing and Lending

Figure 5.1 illustrates the geometrical consequences of the restricted leverage cases. This figure depicts three efficient frontiers with a selection of portfolios, A, B, C, M and D. The efficient frontier labelled ABCMD represents the typical unconstrained efficient frontier constructed from an opportunity set of available investments. $R_f$ represents the return on a risk-free asset. Consequently portfolio M represents the portfolio tangent to the CML, henceforth referred to as the tangency portfolio. Investors who have unrestricted borrowing and lending opportunities are able to select their positions along the CML, consistent with their particular risk aversion, by borrowing or lending at the risk-free rate ($R_f$) and investing the available funds in the tangency portfolio of risky assets (M).

Figure 5.1 shows that investors who borrow $X_b$ at the risk-free rate and invest all funds in the tangency portfolio (M), would be located at $M_0$ (on the CML).
respectively. In the same way we can find the location in risk return space of the portfolios A, B, C and D when a proportion $X_b$ is borrowed at the fixed rate, $R_f$. A continuous spanning $A_b$ to $D_b$, i.e. $A_b B_b C_b M_b D_b$ can be constructed (as shown in Figure 5.1). This curve represents the new constrained efficient frontier from which investors who are constrained to borrow a fixed proportion $X_b$ must make their choice. Similarly, investors who invest $X_i$ in the risk-free asset (i.e. lend), will be constrained to choose from the efficient frontier spanning $A_i B_i C_i M_i D_i$.

On either of these constrained efficient frontiers, the final portfolios chosen will depend on the investor’s personal indifference curves. The ensuing discussion incorporates indifference curves into the choice framework.

5.3 Utility Optimisation Under Different Leverage Constraints

This section considers the change in the composition of the optimal portfolio choice when the constrained proportion of the risk free asset changes. It is shown that, as the restricted proportion of funds borrowed at the risk-free rate increases, the optimal portfolio choice tends to become less risky and vice versa.

The expected utility of a risk-averse investor can be defined as follows:

$$U(\sigma, \mu) = v(\mu) - w(\sigma)$$

(5.1)

where $v(\mu)$ is concave and $w(\sigma)$ is convex reflecting a rational investor’s attitude to return ($\mu$) and risk ($\sigma$) respectively\(^2\). This form of the expected utility is a generalisation of some common models\(^3\).

\(^2\) The fact that this expected utility function is additive relies on the assumption that the two attributes with which we are concerned, namely risk ($\sigma$) and return ($\mu$), satisfy the corresponding trade-offs condition. See Keeney and Raiffa (1976 (page 91)).

\(^3\) (See also Kallberg and Ziemba (1983) for a comparison of utility functions in the portfolio selection context.)
Since $w(\sigma)$ is convex, $w''(\sigma) > 0$ by definition and so $w'(\sigma)$ is increasing. Consequently, for any $\sigma_1$ and $\sigma_2$ such that $\sigma_1 < \sigma_2$, it follows that $w'(\sigma_1) < w'(\sigma_2)$.

Similarly, since $v(\mu)$ is concave, $v''(\mu) < 0$ and $v'(\mu)$ is decreasing. Consequently, for any $\mu_1$ and $\mu_2$ such that $\mu_2 > \mu_1$, it follows that $v'(\mu_1) > v'(\mu_2)$.

Since $U$ is an implicit function of $\sigma$ and $\mu$, the gradient of the relationship between $\mu$ and $\sigma$ along the iso-utility line is calculated as follows:

$$\frac{d\mu_{\text{utility}}}{d\sigma_{\text{utility}}} = -\frac{\partial U}{\partial \sigma} \frac{\partial \mu}{\partial U} = \frac{-w'(\sigma)}{v'(\mu)}$$

Thus the slopes of the iso-utility lines at $(\sigma_1;\mu_1)$ and $(\sigma_2;\mu_2)$ (where $\sigma_1 < \sigma_2$ and $\mu_1 < \mu_2$) respectively, are:

$$\frac{d\mu_{\text{utility}}}{d\sigma_{\text{utility}}} \bigg|_{\sigma_1;\mu_1} = \frac{w'(\sigma_1)}{v'(\mu_1)}$$

and

$$\frac{d\mu_{\text{utility}}}{d\sigma_{\text{utility}}} \bigg|_{\sigma_2;\mu_2} = \frac{w'(\sigma_2)}{v'(\mu_2)}$$

But we have already shown that $w'(\sigma_1) < w'(\sigma_2)$ and $v'(\mu_1) > v'(\mu_2)$, therefore:

$$\frac{w'(\sigma_1)}{v'(\mu_1)} < \frac{w'(\sigma_2)}{v'(\mu_2)}$$

and consequently:

$$-\frac{\partial U}{\partial \sigma} \frac{\partial \mu}{\partial U} \bigg|_{\sigma_1;\mu_1} < -\frac{\partial U}{\partial \sigma} \frac{\partial \mu}{\partial U} \bigg|_{\sigma_2;\mu_2}$$

Which leads to:

$$\frac{d\mu_{\text{utility}}}{d\sigma_{\text{utility}}} \bigg|_{\sigma_1;\mu_1} < \frac{d\mu_{\text{utility}}}{d\sigma_{\text{utility}}} \bigg|_{\sigma_2;\mu_2} \quad (5.2)$$

In other words, the slope of the iso-utility line at $(\sigma_2;\mu_2)$ is steeper than the slope of the iso-utility line at $(\sigma_1;\mu_1)$.

The expected utility of a risk-averse investor is represented by indifference curves $U_1$, $U_2$ and $U_3$ in Figure 5.2.
Let efficient frontiers 1 and 2 be two leverage constrained efficient frontiers (as shown in Figure 5.2) with frontier 2 levered to a greater extent than frontier 1. In particular, for any point \((\sigma_1, \mu_1)\) on efficient frontier 1, there is a corresponding point \((\sigma_2, \mu_2)\) on efficient frontier 2 with coordinates \((K\sigma_1, K\mu_1 + (1-K)R_f)\) where \(K > 1\) is the factor by which frontier 2 has been leveraged relative to frontier 1. Consequently for any two corresponding points on frontier 1 and 2, \(\sigma_1 < \sigma_2\) and \(\mu_2 > \mu_1\). It follows (from the result in (2)) that the slope of the iso-utility line at any point on frontier 2 is steeper than the slope of the iso-utility line at a corresponding point on frontier 1.

Hence the following propositions will be proved:

**Proposition 1 (increasing leverage):** If \(B_1\) is the optimal expected utility portfolio on Frontier 1, then the optimal portfolio on Frontier 2 is \(A_2\) where \(\sigma_{A_2} < \sigma_{B_1}\) and \(\mu_{A_2} < \mu_{B_1}\).
Chapter 5: The Separation Theorem under Constrained Leverage Conditions

**Proposition 2 (decreasing leverage):** If $A_2$ is the optimal expected utility portfolio on Frontier 2, then the optimal portfolio on Frontier 1 is $B_1$ where $\sigma_{B_1} > \sigma_{A_1}$ and $\mu_{B_1} > \mu_{A_1}$.

To prove Proposition 1, let us assume that $B_1$ is the utility-maximizing portfolio along frontier 1. This means that the iso-utility curve is tangent to the efficient frontier at $B_1$ and hence the slope of the iso-utility function is the same as the slope of the efficient frontier at $B_1$. But since efficient frontier 2 is simply a linear projection of efficient frontier 1, the slope of the efficient frontier at $B_1$ is the same as the slope of efficient frontier 2 at $B_2$ (see Appendix I). Therefore:

$$\frac{d\mu_{Utility}}{d\sigma_{Utility}} \bigg|_{\sigma_{B_1}; \mu_{B_1}} = \frac{d\mu_{Eff frontier}}{d\sigma_{Eff frontier}} \bigg|_{\sigma_{B_1}; \mu_{B_1}} = \frac{d\mu_{Eff frontier}}{d\sigma_{Eff frontier}} \bigg|_{\sigma_{B_1}; \mu_{B_1}}$$

But we know that the slope of the iso-utility line at $B_2$ is steeper than the slope of the iso-utility line at $B_1$. Thus the slope of the iso-utility line at $B_2$ is also steeper than the slope of the efficient frontier at $B_2$, i.e.

$$\frac{d\mu_{Utility}}{d\sigma_{Utility}} \bigg|_{\sigma_{B_2}; \mu_{B_2}} = \frac{\partial U}{\partial \mu} \bigg|_{\sigma_{B_2}; \mu_{B_2}} > \frac{d\mu_{Eff frontier}}{d\sigma_{Eff frontier}} \bigg|_{\sigma_{B_2}; \mu_{B_2}}$$

Rearranging the terms:

$$\frac{d\mu_{Eff frontier}}{d\sigma_{Eff frontier}} \bigg|_{\sigma_{B_1}; \mu_{B_1}} + \frac{\partial U}{\partial \sigma} \bigg|_{\sigma_{B_2}; \mu_{B_2}} < 0$$

Multiplying through by $-\frac{\partial U}{\partial \mu} \bigg(d\sigma_{Eff frontier} \big|_{\sigma_{B_1}; \mu_{B_1}}\big)$, we obtain the following result:

$$\frac{\partial U}{\partial \mu} \bigg(-d\mu_{Eff frontier} \bigg|_{\sigma_{B_1}; \mu_{B_2}}\big) + \frac{\partial U}{\partial \sigma} \bigg(-d\sigma_{Eff frontier} \bigg|_{\sigma_{B_2}; \mu_{B_2}}\big) > 0$$

The left-hand side of this inequality is the inner product of two paths: a) the normal to the expected utility function and b) a path along the efficient frontier in the direction of negative return ($\mu$) and negative risk ($\sigma$). The fact that this product is positive means that the expected utility increases as one moves down efficient frontier 2 from $B_2$ in the direction of decreasing return and decreasing risk (southwest in risk return space). This implies that, although portfolio B is the utility optimal portfolio of risky
assets along frontier 1 for our hypothetical investor, when the same investor is confined to invest along frontier 2, there exist portfolios of higher utility than B in a lower risk ($\sigma$) position.

Similarly, Proposition 2 can be proved for the case of decreasing leverage by assuming that $A_2$ is the utility-maximizing portfolio along frontier 2.

So finally it has been proven that the higher an investor’s constrained borrowing, the lower the risk ($\sigma$) of this investor’s optimal portfolio (proposition 1) and conversely, the higher an investor’s constrained lending, the higher the risk ($\sigma$) of this investor’s optimal portfolio (proposition 2). Conversely, as the restricted proportion of funds invested at the risk-free rate increases, the optimal portfolio choice tends to become more risky.

5.4 CONCLUSIONS

In this chapter some consequences of restricted leverage on the allocation of risky assets were highlighted. Although the consequences are intuitive, this chapter aims at giving theoretical justification to these intuitions. In particular it is shown that, if an investor is constrained to borrow a certain amount, as the constrained amount of borrowing increases, the rational, risk-averse investor’s optimal portfolio of risky assets will become less risky. Conversely, if an investor is constrained to invest in the risk-free asset, the greater his/her constrained investment in the risk-free asset, the more risky their optimal portfolio of risky assets will become. Thus it is important to note that, in the context of constrained leverage, the tangent portfolio is not necessarily optimal for all investors.
Chapter 6: An Empirical Examination of Constrained Leverage Conditions

AN EMPIRICAL EXAMINATION OF CONSTRAINED LEVERAGE CONDITIONS

6.1 INTRODUCTION ............................................................................................................. 2

6.2 DATA AND ANALYSIS ............................................................................................. 3
   6.2.1 Data..................................................................................................................... 3
   6.2.2 Generating the Efficient Frontier ...................................................................... 3
   6.2.3 Utility Optimisation ......................................................................................... 5

6.3 CONSTRAINED INVESTMENT IN THE RISK-FREE ASSET – THE CASE OF SA UNIT TRUSTS ................................................................................................................. 6
   6.3.1 5% Fixed Constraint in the Risk-Free Rate ....................................................... 7
   6.3.2 5% Minimum Constraint in Liquid Assets ...................................................... 10

6.4 CONSTRAINED BORROWING – THE CASE OF EMPLOYEE SHARE SCHEMES ................................................................................................................................. 12
   6.4.1 Fixed Borrowing Constraints ........................................................................... 12
   6.4.2 Fixed Borrowing Constraints with Constrained Investment in a Particular Share 14

6.5 CONCLUSIONS ......................................................................................................... 17
6.1 INTRODUCTION

The previous chapter showed that, if a rational, risk-averse investor is constrained to borrow a certain amount at the risk-free rate, the more this investor is constrained to borrow, the more their remaining investments should tilt towards a portfolio of less risky assets. Conversely, if an investor is constrained to invest a certain amount at the risk-free rate, the greater the proportion that this investor is constrained to invest in the risk-free rate, the more their remaining investment should tilt towards a more risky portfolio of assets.

This chapter explores two particular cases where constrained leverage exists and examines the optimal investment strategy in each. The first is the case of unit trust investments in South Africa: all unit trusts are constrained by law to hold at least 5% in liquid assets at all times. Section 0 investigates the consequence of this constraint first by examining the effects of a fixed 5% invested in the risk-free rate (section 6.3.1) on an investor who would otherwise invest only in equity. The analysis extends to consider a minimum 5% invested in the risk-free rate (section 6.3.2). The second case of constrained leverage considered in this chapter is that of an employee in an employee share scheme who is given a loan with which to buy shares in the company i.e. constrained borrowing with a minimum investment in one share (section 6.4).

The analysis shows that, when constrained to lend or borrow, the assumptions underlying Tobin's separation theorem are violated (as discussed in Chapter 8). Thus the constrained investor, who simply chooses a utility maximising portfolio from an efficient frontier of risky assets and adds in the constrained amount of the risk-free asset, will hold a sub-optimal portfolio. The constrained investor should select portfolios from an appropriately constrained efficient frontier in order to maximise his/her utility on wealth.
6.2 DATA AND ANALYSIS

6.2.1 Data

The analysis in this chapter uses an unconstrained ex-post efficient frontier as a reference point. To represent the investment environment available to South African investors, the universe of assets considered here consists of 54 JSE Actuaries Indices and the Morgan Stanley Capital International (MSCI) World Index. A summary of the historical data used in this study is provided in Appendix J.

In all the scenarios considered in this study, the investor is constrained to hold a maximum of 15% in the MSCI index in accordance with the Pension Fund Act\(^1\) and the prevailing legislation regarding Asset Swap Capacity\(^2\). The overnight interest rate at the time of this study was 10.10% NACQ\(^3\) (0.84% per month) and this rate is used as a proxy for the risk-free rate in this study.

6.2.2 Generating the Efficient Frontier

The standard mean-variance optimisation was performed on the universe of investments mentioned above:

*Maximise portfolio return and minimise portfolio risk subject to the following constraints:*

- no short sales i.e. the proportional investment in any asset must be at least zero
- the sum of proportional investments must equal 100%

---

\(^1\) Pension Fund Act 24 of 1956, Regulation 28 section c states, “a registered fund may invest only in an asset…Provided that the total fair value of investments in assets in territories outside the Republic …shall not exceed 15% of the total fair value of the total assets of the fund.”

\(^2\) Asset Managers are permitted to engage in an asset swap of up to 15% of their assets under management in order to invest offshore.

\(^3\) Nominal Annual Compounded Quarterly (October 2000).
no more than 15% may be invested in the MSCI

The resultant efficient frontier is sketched in Figure 6.1 along with the individual indices, the capital market line (CML) and the tangent portfolio.

According to Tobin's separation theorem, the rational investor who is free to borrow or lend at the risk-free rate, would have only one investment decision: the extent of their borrowing or lending. The choice of portfolio for this rational, unconstrained investor would simply be the tangent portfolio since it is, by definition, the most risk-return efficient portfolio available. The investor would then select their optimal borrowing or lending position by maximising their utility on wealth along the CML.

Figure 6.1: Mean/Variance Efficient Frontier

The axes displayed in the Figures are fixed to best illustrate the efficient frontier(s). Thus some of the assets in the investment universe may be hidden although all assets listed in the Appendix form part of the analysis throughout this chapter.
6.2.3 Utility Optimisation

For the purpose of this discussion, let us assume that the hypothetical investors discussed here can be represented by an exponential utility function on wealth\(^5\). That is, their utility function on wealth, \(u(w)\) can be described as follows:

\[
u(w) = 1 - e^{-cw} \quad c \geq 0 \]

where \(c\) is the coefficient describing risk aversion.

The exponential utility function is characterised by having a constant relative risk aversion\(^6\), that is:

\[
\frac{u'(w)}{u''(w)} = \frac{-c^2 e^{-cw}}{ce^{-cw}} = c
\]

The indifference or iso-utility curves of this utility function have the following form\(^7\):

\[
u(R, \sigma) = cR - \frac{1}{2}c^2 \sigma^2
\]

Table 6.1 summarises the optimal unconstrained choices made by investors with different relative risk aversions. These choices are plotted in Figure 6.2. In the case of this particular investment environment with these given parameters, the hypothetical investor with a relative risk aversion of 3.4 would prefer to hold the tangent portfolio without borrowing or lending. In the same case, investors with greater relative risk aversion than 3.4 will choose to combine the risk-free asset with the tangent portfolio and investors with a relative risk aversion greater than 135 would prefer to hold only the risk-free asset. Conversely, investors with small relative aversion to risk would prefer to borrow additional funds with which to invest in the tangent portfolio thereby gearing up their performance and their risk.

\(^5\) See Kallberg and Ziemba for alternative forms of utility functions in the finance context.

\(^6\) Refer to Arrow (1971) and Pratt (1964).

\(^7\) See Keeney and Raiffa (1976) (page 186).
Table 6.1: Utility-Maximising Cash Holdings for Unconstrained Investors

<table>
<thead>
<tr>
<th>Risk Aversion (c)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>3.4</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>10</th>
<th>15</th>
<th>30</th>
<th>45</th>
<th>&gt;135</th>
</tr>
</thead>
<tbody>
<tr>
<td>Risk-Free Investment (%)</td>
<td>-12</td>
<td>-48</td>
<td>-14</td>
<td>-20</td>
<td>30</td>
<td>41</td>
<td>51</td>
<td>58</td>
<td>65</td>
<td>80</td>
<td>90</td>
<td>95</td>
<td>100</td>
<td></td>
</tr>
<tr>
<td>Risk (%) (f)</td>
<td>15.9</td>
<td>10.5</td>
<td>8.1</td>
<td>7.1</td>
<td>5.7</td>
<td>5.0</td>
<td>4.0</td>
<td>3.4</td>
<td>3.0</td>
<td>2.5</td>
<td>1.4</td>
<td>0.7</td>
<td>0.4</td>
<td>-</td>
</tr>
<tr>
<td>Return (%) (g)</td>
<td>4.6</td>
<td>3.3</td>
<td>2.8</td>
<td>2.5</td>
<td>2.2</td>
<td>2.0</td>
<td>1.8</td>
<td>1.7</td>
<td>1.5</td>
<td>1.4</td>
<td>1.2</td>
<td>1.0</td>
<td>0.9</td>
<td>0.8</td>
</tr>
</tbody>
</table>

Figure 6.2: Utility-Maximising Investments for Unconstrained Investors of Different Relative Risk-Aversion

6.3 CONSTRAINED INVESTMENT IN THE RISK-FREE ASSET – THE CASE OF SA UNIT TRUSTS

The Separation Theorem’s assumption of limitless borrowing or lending seldom applies to institutional investors. Many publicly listed funds are not permitted to gear up their returns and risks (i.e. to borrow) and have constraints placed on their

\[ A \text{ negative proportion under "Risk-Free Investment" means that additional funds are borrowed at the risk-free rate.} \]
investment in "risk-free" instruments. For example, the Unit Trusts Control Act No 54 of 1981 constrains all unit trust funds to hold at least 5% of the value of their portfolio in liquid assets\(^9\). The spirit of this restriction is to have at least 5% of the fund in assets which can be liquidated quickly and easily without risk to their price/return in order to provide for fund redemptions\(^10\).

### 6.3.1 5% Fixed Constraint in the Risk-Free Rate

First, consider a constraint which prescribes exactly 5% in the risk-free rate, no more and no less. The efficient frontier of investments under this constraint is a linear shrinkage of the unlevered efficient frontier towards the risk-free asset in mean-standard deviation space, as discussed in section 6.2.2. Every portfolio along the unlevered efficient frontier is projected south west in risk/return space as a consequence of investing 5% of each portfolio invested in the risk-free rate.

The risk-return co-ordinates of any and each constrained portfolio (including the efficient set) can be calculated as a function of each unlevered portfolio as follows:

\[ R_{\text{constrained}} = XR_f + (1 - X)R_{\text{unlevered}} \]  
\[ \sigma_{\text{constrained}} = \sqrt{\text{Var}\left[ XR_f + (1 - X)R_{\text{unlevered}}\right]} = (1 - X)\sigma_{\text{unlevered}} \]

where \(X\) is the proportion (in this case 5%) invested in the risk-free asset, \(R_f\)

\[ R_{\text{constrained}}, R_{\text{unlevered}} \] are the returns

\[ \sigma_{\text{constrained}}, \sigma_{\text{unlevered}} \] are the risks of the constrained and the unlevered portfolios.

\(^9\) Section 6 number 2 of the Unit Trusts Control Act No. 54 of 1981 states, “Every management company, other than a management company in property shares, shall include in every unit portfolio liquid assets with an aggregate market value of not less than five per cent of the aggregate market value of all the assets comprised in the unit portfolio.”

\(^10\) For the purpose of this study, assume that the overnight Johannesburg Interbank (JIBAR) rate is an adequate proxy for a “risk-free” liquid investment.
Consequently, the underlying composition and relative weighting of the risky assets in the portfolios along the constrained efficient frontier are the same as in the unlevered efficient frontier: the same portfolios that are risk-return efficient without a risk-free investment are efficient with a constrained investment in the risk-free rate. Figure 6.3 demonstrates the efficient frontier with 5% constrained lending alongside the unlevered efficient frontier and the CML (as in Figure 6.1).

**Figure 6.3: Mean/Variance Efficient Frontier with 5% Fixed Constraint on Lending**

Although the same portfolios make up the efficient set irrespective of the borrowing/lending constraint, the optimal (utility-maximising) portfolio will differ under different leverage conditions if the constraint is binding on the investor. Thus, as shown in Chapter 5, if an investor is constrained to invest in the risk-free rate when he/she would otherwise have invested only in risky assets, the more risky his/her optimal choice of portfolio is.

Table 6.2 demonstrates the difference between the optimal portfolio along the 5% levered efficient frontier and the optimal portfolio along the unlevered efficient
frontier for investors with a range of relative risk aversions (as in Table 6.1). The first section of Table 6.2 comprises three descriptive statistics for the optimal portfolios of investors who are constrained to hold 5% in the risk-free asset. The second section of Table 6.2 describes the risky ("unlevered") component of the same optimal portfolio\(^{11}\) and the third section contains three descriptive statistics for the optimal portfolio for investors who are constrained to hold only risky assets.

### Table 6.2: Utility-Maximising Portfolios for Constrained Investors

<table>
<thead>
<tr>
<th>Risk Aversion (c)</th>
<th>Constrained: 5% in Risk-Free Investment</th>
<th>Risky Component of the Portfolio</th>
<th>Constrained 0% in Risk-Free Investment</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Risk – Free Investment</td>
<td>Return (% p.a.)</td>
<td>Risk – Free Investment</td>
</tr>
<tr>
<td>1</td>
<td>5%</td>
<td>7.33</td>
<td>2.51</td>
</tr>
<tr>
<td>2</td>
<td>5%</td>
<td>6.84</td>
<td>2.34</td>
</tr>
<tr>
<td>3</td>
<td>5%</td>
<td>6.69</td>
<td>2.42</td>
</tr>
<tr>
<td>3.4</td>
<td>5%</td>
<td>6.64</td>
<td>2.42</td>
</tr>
<tr>
<td>4</td>
<td>5%</td>
<td>6.59</td>
<td>2.41</td>
</tr>
<tr>
<td>5</td>
<td>5%</td>
<td>6.58</td>
<td>2.41</td>
</tr>
<tr>
<td>6</td>
<td>5%</td>
<td>6.23</td>
<td>2.28</td>
</tr>
<tr>
<td>7</td>
<td>5%</td>
<td>6.23</td>
<td>2.28</td>
</tr>
<tr>
<td>8</td>
<td>5%</td>
<td>5.38</td>
<td>1.92</td>
</tr>
<tr>
<td>10</td>
<td>5%</td>
<td>4.73</td>
<td>1.61</td>
</tr>
<tr>
<td>15</td>
<td>5%</td>
<td>4.02</td>
<td>1.23</td>
</tr>
<tr>
<td>30</td>
<td>5%</td>
<td>3.13</td>
<td>0.54</td>
</tr>
<tr>
<td>45</td>
<td>5%</td>
<td>2.94</td>
<td>0.38</td>
</tr>
<tr>
<td>&gt;135</td>
<td>5%</td>
<td>2.79</td>
<td>0.11</td>
</tr>
</tbody>
</table>

Naturally, the "constrained 5%" portfolios are less risky than the "constrained 0%" portfolios as a consequence of their greater investment in the risk-free asset. However, when only their risky assets are compared ("unlevered"), the risk of the optimal 5% constrained portfolio of risky assets is greater than or equal to the risk of the optimal 0% constrained portfolio of risky assets. Thus, rational risk-averse investors who are bound by a constrained investment in the risk-free asset will compensate by investing in a riskier portfolio of risky assets.

A selection of the investors and their corresponding portfolio choices are represented in Figure 6.4. Note that the 5% constrained optimal portfolios and their unlevered

---

\(^{11}\) Refer to equations (6.1) and (6.2).
equivalents (the portfolios marked in small yellow dots) have identical compositions aside from the 5% investment in the risk-free asset. However, the optimal portfolios along the unlevered (no borrowing or lending) efficient frontier (marked in red dots) are different in composition and less risky (lower down) than the risky component of the 5% constrained lending optimal portfolios.

Figure 6.4: Utility Maximising Portfolios with 5% Fixed Constraint on Lending

6.3.2 5% Minimum Constraint in Liquid Assets

Legislative or regulatory constraints on investments tend to take the form of a minimum or maximum constraint rather than a fixed proportion, such as Regulation 28 of the Pension Funds Act. As mentioned earlier in this section, the Unit Trusts Control Act constrains all unit trust funds to hold at least 5% of the value of their portfolio in liquid assets. That is, the fund may hold between 100% and 5% in the risk-free asset. The efficient frontier under this constraint is a combination of the CML (5% to 100% lending) and the 5% constrained efficient frontier (refer to Figure 6.10).
6.3). The purple line in Figure 6.5 represents the resultant frontier that allows for no less than 5% in the risk-free asset.

Figure 6.5: Mean/Variance Efficient Frontier with No Less than 5% Invested in the Risk-Free Asset

![Graph showing the Mean/Variance Efficient Frontier with constraints.

Thus the more risk-averse investors under the constraint of minimum lending, will select portfolios which fall along the CML, with increasing proportions held in the risk-free asset the greater their risk aversion. These investors are not bound by the maximum constraint on lending. They choose the same portfolio that they would have chosen if they were unconstrained, that is the tangent portfolio. The least risk-averse investors will select portfolios along the 5% constrained-lending efficient frontier, with increasingly risky portfolios of risky assets the smaller their risk-aversion. These investors are bound by the maximum constraint on lending and will choose riskier portfolios than they would have chosen if they were not constrained to hold the risk-free asset (as proved in Chapter 5 and shown in section 6.3.1). These investors will expect less reward per unit of risk as a consequence of constraints which affect their choice, unlike the more risk-averse investors who are unaffected by the constraint and...
who thus achieve maximum expected reward per unit of risk by investing along the CML.

6.4 CONSTRAINED BORROWING – THE CASE OF EMPLOYEE SHARE SCHEMES

Many listed companies create loyalty and performance incentives for their employees by means of an employee share scheme. Employees are made an offer of a specific number of shares (determined by management) and enabled to buy these shares by means of a loan from the company. Employees are thus liable to the company for the repayment of this debt and able to participate financially in the growth of the company. The debt usually incurs interest expense and the full amount is payable on sale of these shares. Trading in these shares is often restricted for a specified time period. Thus an employee share scheme can be considered to be a constrained borrowing situation: the employee, on accepting the offer, is constrained to hold a certain amount of his/her portfolio in debt which funds their shares in the company. Not only is the employee constrained to borrow a certain amount but he/she is also constrained to invest this amount in the company's shares.

In Chapter 5 it was shown that an investor who is constrained to borrow should select a less risky portfolio of assets in which to invest their funds, in order to optimise their utility on wealth. This section considers the additional constraint of holding a constrained amount in one share.

6.4.1 Fixed Borrowing Constraints

To understand the effects of constrained borrowing on the efficient set and the optimal choice, consider investors who are constrained to borrow 25%, 50%, 75% and 100% at the risk-free rate relative to their own funds (i.e. they invest 125%, 150%, 175% and 200% of their own funds). The efficient frontiers for such investors are sketched in Figure 6.6. Each frontier in Figure 6.6 has a marked optimal portfolio for
a hypothetical investor with relative risk aversion (c) of 4\(^2\). Table 6.3 reports on the risk and reward of the optimal portfolios under the four different borrowing constraints alongside the risk of their risky component i.e. without the risk-free rate. Notice that the risk of the optimal portfolio (without the risk-free rate) decreases with increasing borrowing constraints, as proven in Chapter 5. Thus rational investors compensate for binding constraints on borrowing by decreasing the risk of their portfolio of risky assets.

Figure 6.6: Mean/Variance Efficient Frontiers with 25%, 50%, 75% and 100% Borrowed at the Risk-Free Rate

Refer to Appendix K for the compositions of these optimal portfolios.
Table 6.3: Risk of Optimal Portfolios under Borrowing Constraints for Relative Risk Aversion c = 4

<table>
<thead>
<tr>
<th>Constrained Borrowing</th>
<th>Optimal Portfolio</th>
<th>Risky Component$^{13}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Standard Deviation</td>
<td>Return</td>
</tr>
<tr>
<td>25%</td>
<td>8.80%</td>
<td>2.88%</td>
</tr>
<tr>
<td>50%</td>
<td>10.08%</td>
<td>3.04%</td>
</tr>
<tr>
<td>75%</td>
<td>10.93%</td>
<td>3.09%</td>
</tr>
<tr>
<td>100%</td>
<td>11.65%</td>
<td>2.87%</td>
</tr>
</tbody>
</table>

6.4.2 Fixed Borrowing Constraints with Constrained Investment in a Particular Share

Section 6.4.1 confirmed that, the greater the amount of debt rational investors are constrained to carry, the less risky their choice of portfolio is. However, when the same investors are additionally constrained to invest the borrowed amount in a particular share (as with the employee share scheme) their optimum portfolio choices may become more risky.

Figure 6.7 depicts the case of investors who are constrained to borrow 25%, 50%, 75% and 100% at the risk-free rate and invest these additional funds in the Financial Services (JOHFIN Index$^{14}$). Notice that the efficient frontiers in Figure 6.7 are less risk-reward efficient than the efficient frontiers in Figure 6.6 where no stock-specific constraints were considered. The optimal portfolio choices for an investor with a relative risk aversion (c) of 4 are marked on the frontiers in Figure 6.7$^{13}$.

Figure 6.7 shows once again that, with increasing constraints and restrictions, the efficient set offers decreasing return per unit of risk. The risky component of the optimal portfolios for each of the constraints discussed here are much the same and, in fact, increase slightly with increasing debt. Although one expects the risk of optimal

---

$^{13}$ Refer to equations (1) and (2) of section 6.3.1.

$^{14}$ The JOHFIN Index is used here as a proxy for an individual investment in the financial services.

$^{12}$ Refer to Appendix L for the compositions of these optimal portfolios.
portfolios to decrease with increasing constrained debt, the corresponding increasing constrained investment in JOHFIN, a high risk share (standard deviation = 11%), has the opposite effect.

However, the investor only exercises his/her choice on the element of the portfolio that is unconstrained i.e. the selection of stocks excluding JOHFIN. The risk of this "additional" selection of stocks is recorded in Table 6.4 alongside the risk and return of each optimal portfolio. The additional risk measures the risk of the portfolio without the risk-free rate and the constrained share, JOHFIN. The results in Table 6.4 show that the additional risk of the optimal portfolios for this investor increases with increasing constrained borrowing and investment in JOHFIN to compensate for the lower risk-reward trade-offs that result from these constraints.

Thus, when increased borrowing incurs increased constrained investment in a particular asset, the rational investor will not necessarily choose portfolios with less risk. However, the principle of constrained choice remains the same: when constrained, an investor should incorporate the constraints when determining the efficient set and select from the appropriately constrained set.
Figure 6.7: Mean/Variance Efficient Frontiers with 25%, 50%, 75% and 100% Borrowed at the Risk-Free Rate and Invested in JOHFINS

Table 6.4: Risk of Optimal Portfolios under Borrowing and Stock-Specific Constraints for Relative Risk Aversion $c = 4$

<table>
<thead>
<tr>
<th>Constrained Borrowing and Investment in JOHFINS</th>
<th>Optimal Portfolio</th>
<th>Additional Risk</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Standard Deviation</td>
<td>Return</td>
</tr>
<tr>
<td>0%</td>
<td>6.92%</td>
<td>2.49%</td>
</tr>
<tr>
<td>25%</td>
<td>8.64%</td>
<td>2.60%</td>
</tr>
<tr>
<td>50%</td>
<td>10.40%</td>
<td>2.63%</td>
</tr>
<tr>
<td>75%</td>
<td>12.23%</td>
<td>2.60%</td>
</tr>
<tr>
<td>100%</td>
<td>14.25%</td>
<td>2.58%</td>
</tr>
</tbody>
</table>
6.5 CONCLUSIONS

It is often assumed that, on account of Tobin's Separation Theorem, to first choose an optimal equity portfolio (without considering borrowing or lending) and then to simply add in the constrained amount of liquid assets will result in an optimal portfolio under fixed lending constraints. However, when borrowing or lending is constrained, the assumption of limitless borrowing and lending that underpins the Separation Theorem is violated. Thus it is clear from the evidence here and from discussions in Chapter 5 that selecting the optimal risky asset portfolio and diluting the investment with the constrained proportion of the risk-free asset results in a less risky and sub-optimal portfolio selection. An investor who is constrained with regards to their investment in the risk-free asset should make their selection from an appropriately constrained efficient frontier.

The converse is true when an investor is constrained to borrow a fixed proportion. However, when the investor is also constrained to invest the borrowed funds in a particular share, their selection of additional assets (those assets which are not constrained) may well be more risky than their unconstrained selection.

Overall, the implication is that investors who are constrained to borrow, lend or invest in a particular asset should incorporate their constraints into the efficient set and make a utility maximising selection from the appropriately constrained set. Adding constrained investments in after selecting a utility maximising portfolio results in a sub-optimal selection.
Chapter 7: Constrained Leverage in a TEV Framework

CONstrained Leverage in a TEV FrameWork

7.1 INTRODUCTION .............................................................................................................. 2

7.2 CONSTRAINED LENDING AND BORROWING IN A TEV FRAMEWORK ................................................................. 3
  7.2.1 Data and Analysis........................................................................................................ 3
  7.2.2 Constrained Investment in the Risk-Free Asset within SA Unit Trusts .......... 5
  7.2.3 Leverage Using the Risk-Free Asset........................................................................ 7

7.3 "LEVERAGE" IN A TE-TEV FRAMEWORK .................................................................... 9
  7.3.1 Leverage Using the Benchmark ............................................................................... 9
  7.3.2 TE-TEV Utility Optimisation Under Different Passive Management Constraints .............................................. 12
  7.3.3 Empirical Example of Constrained Investment in the Benchmark .................. 16

7.4 CONCLUSIONS ............................................................................................................. 18
7.1 INTRODUCTION

Chapters 5 and 6 explored the separation theorem and the effect of constrained leverage in a traditional MV framework. In the MV framework, the risk-free asset (i.e. cash or equivalent fixed interest instrument) has no risk (variance) and zero correlation with any risky asset and thus combines linearly with any of these risky assets in MV space. However, as shown in Chapter 2, the risk-free asset is very risky in a TEV sense since its returns are divergent from a benchmark of risky-assets. Thus the only truly "risk-free" asset in this benchmark-relative framework of portfolio selection is the benchmark itself.

This paradigm shift to a benchmark-relative or active framework has important implications for the active manager. Active managers cannot gear their fund linearly along the same active risk – active return trade-off by investing in or borrowing at the risk-free rate. Instead, fund managers who seek to gear their fund in TE/TEV space but maintain the same relative return (TE) – relative risk (TEV) trade-off must borrow/lend the benchmark.

This chapter begins by empirically demonstrating the implications of MV leverage constraints (constrained investment in the risk-free asset) on an active manager. The second section shows that the TEV or "active" analogy of constrained leverage in a MV framework (as presented in Chapter 5) is constrained passive management. This section contains analysis and proofs of the intuitive result that managers who are subject to greater constrained passive management will choose to incur more active risk in the remaining fund than they would if they were unconstrained. Finally, the chapter concludes with an empirical demonstration of passive management constraints.
7.2 Constrained Lending and Borrowing in a TEV Framework

7.2.1 Data and Analysis

The data and optimisation methods in this section are identical to that of Chapter 3 and a summary of the data is listed in Table VII of Appendix M. The resultant efficient frontier is sketched in Figure 7.1 along with the individual indices, the benchmark (JOHMKT) and the risk-free rate (the overnight interest rate).

**Figure 7.1: TE/TEV Efficient Frontier**

As in Chapter 6, let us assume that the hypothetical investors discussed here can be represented by an exponential utility function on wealth or, in this case, relative wealth. That is, their utility function on wealth, \( u(w) \) can be described as follows:

\[
u(w) = 1 - e^{-dr} \quad d \geq 0
\]
where \( d \) is the coefficient describing aversion to benchmark-relative risk (TEV). Consequently (as explained in Chapter 6) the indifference or iso-utility curves of this utility function have the following form:

\[
U(T_E, \sqrt{TEV}) = dT_E - \frac{1}{2}d^2TEV
\]

Table 7.1 summarises the optimal unconstrained choices made by investors with different aversions to TEV and these choices are plotted in Figure 7.2. In the case of this particular investment environment with these given parameters, only an investor with an extremely high aversion to TEV \((d = 2000)\) would elect to hold the benchmark thereby adopting a passive strategy. The smaller the investor's TEV aversion, the more active their optimal strategy and the greater their investment's distance from the benchmark in TE/TE standard deviation space. Notice once again in Figure 7.2, the unfavourable position of the risk-free asset (in this case, the overnight rate).

### Table 7.1: Utility-Maximising Cash Holdings for Unconstrained Investors

<table>
<thead>
<tr>
<th>Risk Aversion ((d))</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>10</th>
<th>20</th>
<th>25</th>
<th>30</th>
<th>40</th>
<th>50</th>
<th>100</th>
<th>150</th>
<th>250</th>
<th>1000</th>
<th>2000</th>
</tr>
</thead>
<tbody>
<tr>
<td>TE Std Dev (% p.a.)</td>
<td>9.00</td>
<td>6.64</td>
<td>5.19</td>
<td>4.52</td>
<td>3.97</td>
<td>3.43</td>
<td>2.72</td>
<td>2.45</td>
<td>2.19</td>
<td>1.97</td>
<td>1.74</td>
<td>1.08</td>
<td>0.82</td>
<td>0.68</td>
<td>0.36</td>
<td>0.06</td>
</tr>
<tr>
<td>Std Dev (% p.m.)</td>
<td>12.66</td>
<td>11.04</td>
<td>8.60</td>
<td>7.45</td>
<td>7.01</td>
<td>6.78</td>
<td>6.66</td>
<td>6.52</td>
<td>6.46</td>
<td>6.36</td>
<td>6.23</td>
<td>6.10</td>
<td>5.98</td>
<td>5.81</td>
<td>5.62</td>
<td>5.40</td>
</tr>
<tr>
<td>TV (% p.m.)</td>
<td>4.23</td>
<td>4.01</td>
<td>3.80</td>
<td>3.69</td>
<td>3.58</td>
<td>3.42</td>
<td>3.09</td>
<td>2.93</td>
<td>2.77</td>
<td>2.62</td>
<td>2.42</td>
<td>1.77</td>
<td>1.47</td>
<td>1.29</td>
<td>0.55</td>
<td>0.06</td>
</tr>
</tbody>
</table>
Chapter 7: Constrained Leverage in a TEV Framework

Figure 7.2: Utility-Maximising Investments for Unconstrained Investors of Different TEV-Aversion

![Diagram showing TE/TEV Efficient Frontier with various points and labels indicating JSE Indices, Overnight Rates, Unconstrained Efficient Frontier, Benchmark, and Utility Optimal.]

7.2.2 Constrained Investment in the Risk-Free Asset within SA Unit Trusts

As mentioned in Chapter 6, the Unit Trusts Control Act constrains all unit trust funds to hold at least 5% of the value of their portfolio in liquid assets. This restriction is at odds with the active manager who typically (albeit unfairly) has a fully invested benchmark and would ideally invest the full value of the fund in assets of the same class as that benchmark. Figure 7.3 demonstrates the disadvantages associated with a constrained investment in the risk-free rate in an active management framework.
The TE/TEV efficient frontiers move south-east with increasing constrained investment in the risk-free asset as a consequence of the high TEV and the negative TE of the risk-free asset. Thus the segment of the TE/TEV efficient frontier that represents low TEV-aversion will become less risky with increasing investment in the risk-free rate but the segment representing high TEV aversion will become more risky. Notice that, under constrained investment in the risk-free rate, there is no passive alternative and the benchmark itself cannot be held or mimicked with 100% of the funds. Therefore, the constrained efficient frontier stops short of the "riskless" position in this framework, namely the benchmark. The more risky (in the MV sense) the benchmark and the greater its excess returns (over-and-above the risk-free rate), the more dramatic the effect of a constrained investment in the risk-free rate will be in a TE/TEV paradigm.
Table 7.2 comprises the co-ordinates of the utility optimal portfolios with and without a constrained investment in the risk-free rate. In contrast to the effects of such a constraint in a MV framework (refer to Chapters 5 and 6), the efficient frontiers under a constrained investment in cash are not linear projections of each other and the risks of the selected portfolios are not uniformly affected in one direction. That is, a constrained investment in cash does not necessarily mean that an active manager should take on greater TEV. In this particular example, managers with high aversion to TEV (\(d = 2000\)) would have chosen a fund with more TEV under a constrained investment in the risk-free rate since the risk-free option of investing in the benchmark is not available under these constraints. By contrast, less TEV-averse managers would have selected a fund with less TEV under a constrained investment in the risk-free rate simply because of the shape and the shrinkage of the efficient frontier towards the risk-free rate. Ultimately the risk-free rate in a TE/TEV framework behaves much like any other badly-performing risky asset and contributes negatively to the TE of a fund. The following section explores the relationship between the risk-free rate and other assets in the active management framework.

7.2.3 Leverage Using the Risk-Free Asset

It is well known that the risk-free rate combines linearly with any other risky asset in MV space. That is, if a fixed proportion, \(X\), is invested in the risk-free rate, \(R_f\) and the remainder of the fund, \((1-X)\), is invested in any risky asset, the return and risk of
Chapter 7: Constrained Leverage in a TEV Framework

the fund as a whole is a linear combination of the risk and return of the two component assets:

\[ R_p = XR_f + (1-X)R_{risky-asset} \]
\[ \sigma_p = (1-X)\sigma_{risky-asset} \]

where \( R_p, R_{risky-asset} \) are the returns of the whole fund and the risky asset respectively and \( \sigma_p, \sigma_{risky-asset} \) are the standard deviations (risks) of the fund and the risky asset respectively.

These relationships hold true whether funds are borrowed at \((X<0)\) or invested in \((X>0)\) the risk-free rate. Thus, in a MV framework, the risk-return co-ordinates of a fund/asset that is levered by borrowing or lending at the risk-free rate are simply a linear extension along a line connecting the risk-free rate with the non-levered fund/asset.

\[ (\sigma_{risky-asset}, R_{risky-asset}) \underset{\text{levered}}{\rightarrow} ((1-X)\sigma_{risky-asset}, XR_f + (1-X)R_{risky-asset}) \]

Consequently, as shown in Chapter 5, every element of a non-levered efficient frontier in MV space can be projected up or down along linear rays from the risk-free asset to form efficient frontiers representing fixed borrowing or lending constraints (refer to Figure 5.1 of Chapter 5).

The same simple relationship between the risk-free rate and any other asset does not apply in a TEV framework. If a fixed proportion, \( X \), is invested in the risk-free rate, \( R_f \), and the remainder of the fund, \((1-X)\), is invested in any risky asset, the TE and TEV of the fund are as follows:

\[ TE_p = XR_f + (1-X)R_{risky-asset} - R_{benchmark} \]
\[ = TE_{risky-asset} - X(R_{risky-asset} - R_f) \]
Chapter 7: Constrained Leverage in a TEV Framework

\[ TEV_p = \text{Var}[TE_{\text{risky-asset}} - X(R_{\text{risky-asset}} - R_f)] \]
\[ = TEV_{\text{risky-asset}} - 2X \text{Cov}[R_{\text{risky-asset}} - R_{\text{benchmark}}, R_{\text{risky-asset}} - R_f] + X^2 \sigma_{\text{risky-asset}}^2 \]
\[ = TEV_{\text{risky-asset}} - 2X \text{Cov}[R_{\text{risky-asset}} - R_{\text{benchmark}}, R_{\text{risky-asset}}] + X^2 \sigma_{\text{risky-asset}}^2 \]
\[ = TEV_{\text{risky-asset}} + X(X - 2) \sigma_{\text{risky-asset}}^2 + 2X \sigma_{\text{risky-asset,benchmark}} \]

where \( R_{\text{benchmark}} \) is the return of the benchmark,

\( TE_p, TEV_p \) are the TE and TEV of the fund as a whole and

\( \sigma_{\text{risky-asset,benchmark}} \) is the covariance between the returns of the risky-asset and the returns of the benchmark.

Drawing on the decomposition of TEV in Chapter 2:

\[ TEV_{\text{risky-asset}} = \sigma_{\text{risky-asset}}^2 + \sigma_{\text{benchmark}}^2 - 2\sigma_{\text{risky-asset,benchmark}} \]
\[ \Leftrightarrow 2\sigma_{\text{risky-asset,benchmark}} = TEV_{\text{risky-asset}} + \sigma_{\text{risky-asset}}^2 + \sigma_{\text{benchmark}}^2 \]

(7.2)

And replacing the covariance term in (7.1):

\[ TEV_p = TEV_{\text{risky-asset}} + X(X - 2) \sigma_{\text{risky-asset}}^2 + X(\sigma_{\text{risky-asset}}^2 + \sigma_{\text{benchmark}}^2 - TEV_{\text{risky-asset}}) \]
\[ = (1 - X)TEV_{\text{risky-asset}} - X(1 - X) \sigma_{\text{risky-asset}}^2 + X \sigma_{\text{benchmark}}^2 \]

(7.3)

Thus the TEV of a combination of the risk-free rate and any risky asset is not a simple linear combination of the original TEV of the non-levered risky asset and the amount of leverage \( X \). The reason for this effect is that the risk-free rate is not without risk in the TEV environment and it thus combines with any other risky asset non-linearly in a similar way to the combination of any two risky assets in either MV or TEV space.

7.3 "LEVERAGE" IN A TE-TEV FRAMEWORK

7.3.1 Leverage Using the Benchmark

The true "risk-free" asset in an active management sense is the benchmark itself. Thus if a proportion, \( Y \), is invested in the benchmark and the remainder is invested in
any risky asset in the TEV framework, the combination is linear in active reward (TE) – active risk (TE standard deviation) space.

\[
TE_p = YR_{\text{benchmark}} + (1 - Y)R_{\text{risky-asset}} - R_{\text{benchmark}} \\
= (1 - Y)(R_{\text{risky-asset}} - R_{\text{benchmark}}) \\
= (1 - Y)TE_{\text{risky-asset}}
\]

and

\[
\sqrt{TEV_p} = \sqrt{\text{Var}[(1 - Y)TE_{\text{risky-asset}}]} \\
= |1 - Y|\sqrt{TEV_{\text{risky-asset}}}
\]

Thus, in an active management framework, a fund/asset can be levered along a line connecting the benchmark (the origin) with the non-levered fund/asset by borrowing \((Y < 0)\) or lending \((Y > 0)\) the benchmark to any extent. That is:

\[
(\sqrt{TEV_{\text{risky-asset}}, TE_{\text{risky-asset}}})_{\text{levered}} \rightarrow [(1 - Y)|\sqrt{TEV_{\text{risky-asset}}}, (1 - Y)TE_{\text{risky-asset}}]
\]

Of course the risk-free asset also combines in this way with the benchmark:

\[
(\sqrt{TEV_{\text{risk-free}}, TE_{\text{risk-free}}}) = (\sigma_{\text{benchmark}} - (R_{\text{benchmark}} - R_f)) \\
\text{levered} \rightarrow [(1 - Y)|\sigma_{\text{benchmark}}, (Y - 1)(R_{\text{benchmark}} - R_f)]
\]

Thus if, for a particular portfolio, less than 100% of funds are invested in the benchmark (i.e. \(0 < Y \leq 1\)) and the remainder is invested in the risk-free rate, the portfolio will have a negative TE and increasing TEV with increasing investment in the risk-free rate. However, if some funds are borrowed at the risk-free rate and all funds are invested in the benchmark (i.e. \(Y > 1\)), the fund will be geared up in TE-TE standard deviation space in the same linear way as in MV space.

Investors are seldom able to hold assets short and borrowing is usually only available at the risk-free rate. However, one can still gear up in TE-TE standard deviation space without holding any assets short by splitting the fund between the benchmark and any other asset. Figure 7.4 depicts the gearing possibilities in TE-TE standard deviation space.
A, B, C and D are hypothetical assets in the investment universe. Figure 7.4 shows that these assets combine linearly with the benchmark. If the benchmark is the tangent portfolio, the two straight lines between the origin (benchmark) and the risk-free rate and the origin and the “Long Benchmark & Short Risk-Free Rate” represent the CML and will offer the highest TE/TEV ratio and consequently will have the steepest slopes in this framework. However, even if the benchmark is not the tangent portfolio, it still combines linearly with the risk-free rate and all other risky assets although its combination with the risk-free rate may not necessarily provide the greatest TE/TEV ratio in the investment universe. Figure 7.4 shows that the combination of the benchmark with a poorly performing hypothetical asset D can provide greater gearing than the risk-free rate. If any asset which underperforms the benchmark may be held short (i.e. borrowed), the resultant set of combinations of the benchmark and this asset will form a positively sloping component as shown in Figure 7.4. Conversely, holding this same asset long (lending) will incur negative TE and large TEV. The downward sloping component in Figure 7.4 makes this clear.
Lastly, the curved lines between assets A, B and C demonstrate the non-linear relationship between risky assets and the risk-free asset in an active management framework whereas, in the traditional MV framework, these curved lines would be straight and positively sloped.

7.3.2 TE-TEV Utility Optimisation Under Different Passive Management Constraints

The developments in Chapter 5 have an analogy when it comes to constrained leverage in an active management framework. That is, the greater the extent of constrained lending (borrowing), the greater (lower) the risk of the remaining unconstrained utility-optimal portfolio. However, in this active framework, leverage is attained via the benchmark rather than the risk-free rate. It would thus be more accurate to refer to leverage in a TE/TEV framework as constrained passive management. To constrain fund managers to invest in the benchmark itself is to constrain a proportion of funds under their management to be managed passively.

Therefore, just as an investor should select a higher risk portfolio when bound by a constraint to invest a greater proportion in the risk-free rate, a manager who is bound by a constraint to manage a greater proportion of their fund passively should take more active risk on the remainder of the fund. This presumes that the active manager’s utility is a function of TE and TEV rather than the utility function of return and variance which formed the basis of the discussion in Chapter 5.

The expected utility of a risk-averse benchmark-sensitive manager can be defined as follows:

\[ U(\sqrt{TEV}, TE) = \nu(TE) - w(\sqrt{TEV}) \]

where \( \nu(TE) \) is concave and \( w(\sqrt{TEV}) \) is convex reflecting a rational manager’s attitude to relative return and relative risk respectively\(^1\).

\(^1\) The fact that this expected utility function is additive relies, once again, on the assumption that the two attributes with which we are concerned satisfy the corresponding trade-offs condition. See Keeney and Raiffa (1976 (page 91)). See 7 - 12
Chapter 7: Constrained Leverage in a TEV Framework

The proof that the greater the portion of the fund which is managed passively, the greater the active risk of the active portion of the fund should be, follows the same reasoning as the discussions in Chapter 5.

Since \( w(\sqrt{TEV}) \) is convex, \( w'(\sqrt{TEV}) > 0 \) by definition and so \( w'(\sqrt{TEV}) \) is increasing. Similarly, since \( v(TE) \) is concave, \( v'(TE) < 0 \) and \( v'(TE) \) is decreasing.

Since \( U \) is an implicit function of \( \sqrt{TEV} \) and \( TE \), the gradient of the relationship between \( TE \) and \( \sqrt{TEV} \) along the iso-utility line is as follows:

\[
\frac{dTE_{Utility}}{d\sqrt{TEV}_{Utility}} = \frac{w'(\sqrt{TEV})}{v'(TE)}
\]

Thus the slopes of the iso-utility lines are steeper at \( (\sqrt{TEV_2}, TE_2) \) than \( (\sqrt{TEV_1}, TE_1) \) if \( \sqrt{TEV_1} < \sqrt{TEV_2} \) and \( TE_1 < TE_2 \) since:

\[
\frac{w'(\sqrt{TEV_1})}{v'(TE_1)} < \frac{w'(\sqrt{TEV_2})}{v'(TE_2)}
\]

Therefore:

\[
\frac{dTE_{Utility}}{d\sqrt{TEV}_{Utility}} \bigg|_{\sqrt{TEV_1}, TE_1} < \frac{dTE_{Utility}}{d\sqrt{TEV}_{Utility}} \bigg|_{\sqrt{TEV_2}, TE_2}
\]  (7.4)

The expected utility of a risk-averse investor is represented by indifference curves \( U_1 \), \( U_2 \) and \( U_3 \) in Figure 7.5.

also Clarke, Krase and Statman (1994) for an example of the use of such a utility function.
Figure 7.5: Constrained Efficient Frontiers and Utility Functions

Let efficient frontiers 1 and 2 be two passive management-constrained efficient frontiers (as shown in Figure 7.5) with frontier 2 constrained to hold a smaller proportion, \( Y_2 \), in the benchmark than the proportion, \( Y_1 \), held by frontier 1. In particular, for any point \((\sqrt{TEV}_1, TE_1)\) on efficient frontier 1, there is a corresponding point \((K\sqrt{TEV}_2, KTE_2)\) on efficient frontier 2 with coordinates \((K\sqrt{TEV}_1, KTE_1)\) where \(K>1\) is the factor by which frontier 1 has been constrained relative to frontier 2. Consequently for any two corresponding points on frontier 1 and 2, \(\sqrt{TEV}_1 < \sqrt{TEV}_2\) and \(TE_1 < TE_2\). It follows (from the result in (7.4)) that the slope of the iso-utility line at any point on frontier 2 is steeper than the slope of the iso-utility line at a corresponding point on frontier 1.

Let us assume that \(A_2\) is the utility-maximizing portfolio along frontier 2. This means that the iso-utility curve is tangent to the efficient frontier at \(A_2\) and hence the slope of the iso-utility function is the same as the slope of the efficient frontier at \(A_2\).
Chapter 7: Constrained Leverage in a TEV Framework

But the slope of the efficient frontier at $A_2$ is also the same as the slope of efficient frontier 1 at $A_1$ (refer to Appendix N). Therefore:

$$\frac{d\mu_{\text{Utility}}}{d\sigma_{\text{Utility}}} = \frac{d\mu_{\text{Eff frontier}}}{d\sigma_{\text{Eff frontier}}} = \frac{d\mu_{\text{Eff frontier}}}{d\sigma_{\text{Eff frontier}}}$$

We also know that the slope of the iso-utility line at $A_2$ is steeper than the slope of the iso-utility line at $A_1$. Thus the slope of the efficient frontier at $A_1$ is also steeper than the slope of the iso-utility line at $A_1$:

$$\frac{d\mu_{\text{Eff frontier}}}{d\sigma_{\text{Eff frontier}}} > \frac{d\mu_{\text{Utility}}}{d\sigma_{\text{Utility}}} = \frac{\partial U}{\partial \sigma} \frac{\partial \mu}{\partial \sigma}$$

Rearranging the terms:

$$\frac{d\mu_{\text{Eff frontier}}}{d\sigma_{\text{Eff frontier}}} + \frac{\partial U}{\partial \mu} \frac{\partial \mu}{\partial \sigma} > 0$$

Multiplying through by $\frac{\partial U}{\partial \mu} (d\sigma_{\text{Eff frontier}})$ we obtain the following result:

$$\frac{\partial U}{\partial \mu} (d\mu_{\text{Eff frontier}}) + \frac{\partial U}{d\sigma} (d\sigma_{\text{Eff frontier}}) > 0$$

The left-hand side of this inequality is the inner product of two paths: a) the normal to the expected utility function and b) a path along the efficient frontier in the direction of positive return ($\mu$) and positive risk ($\sigma$). The fact that this product is positive means that the expected utility increases as one moves up efficient frontier 1 from $A_1$ in the direction of increasing return and increasing risk (north-east in TE/TE standard deviation space). This increase implies that, although portfolio A is the utility optimal portfolio of actively managed assets along frontier 2 for our hypothetical manager, when the same manager is confined to invest along frontier 1, there exist portfolios of higher utility than A in a higher active risk (TEV) position, such as portfolio $B_I$.

---

$^2$ $K = (1 - Y_2) / (1 - Y_1)$ where $0 < Y_1 < Y_2 < 1$ and thus $K > 1$
To conclude, the higher a manager’s constrained investment in the benchmark, the higher the risk ($\sqrt{TEV}$) of this manager’s optimal remaining portfolio is and conversely, the lower a manager’s constrained investment in the benchmark, the lower the risk ($\sqrt{TEV}$) of this manager’s optimal remaining portfolio is. Thus, intuitively, the utility-optimal remedy to increased constrained passive management of a fund is to increase the risk of the active component of the same fund.

### 7.3.3 Empirical Example of Constrained Investment in the Benchmark

This section illustrates the effect of constrained passive management on the active manager. It is not uncommon for fund managers to be instructed to hold a certain proportion of assets under management in passive portfolios which emulate the benchmark and to focus their active management skills on the remainder of the fund. Managers who have as a benchmark a published market index will usually be able to invest a proportion of their fund in securities which mimic the benchmark exactly such as exchange traded funds (ETF’s) or futures and options. For more complex benchmarks, an index tracker will have to be designed.

The previous section concluded that the greater the proportion of a fund which is constrained to be invested in a passive style, the more active risk the remainder of the fund should take. Figure 7.6 shows an example of these constraints to manage 80%, 60%, 40% and 20% of a fund passively by constraining the investment in the benchmark to be at least 20%, 40%, 60% and 80% respectively. Table 7.3 lists the co-ordinates of the optimal unconstrained portfolios, the optimal constrained portfolios and the active segment of the constrained portfolios for various levels of TEV aversion and a 20% minimum constrained investment in the benchmark.
It is clear from the chart that greater constrained investment in the benchmark, and thus greater investment in an asset without risk, will decrease the overall risk of the portfolios chosen by managers. However, the active component of these portfolios (that part of the portfolio which is not invested in the benchmark) will have greater TEV than the same manager’s unconstrained portfolio. The only exception to this pattern is a manager who would under unconstrained conditions invest the same
constrained amount passively and who is thus effectively unbound by the constraint, such as, in this example, the manager with risk aversion $d=2000$.

### 7.4 Conclusions

This chapter extends the findings of Chapters 5 and 6 into the TE/TEV paradigm. The analysis here shows that active managers cannot gear their fund linearly along the same active risk -- active return trade-off by investing in or borrowing at the risk-free rate. To the contrary, constrained investment in the risk-free rate will force the fund into less efficient TE/TEV space. Instead, fund managers who seek to gear their fund in TE/TEV space but maintain the same relative return (TE) -- relative risk (TEV) trade-off must borrow/lend the benchmark.

While borrowing the benchmark (i.e. selling the benchmark short) is a practice which is usually beyond regulatory and legislative restrictions, it is possible and common to constrain a long investment in the benchmark. This strategy translates into a constraint on active management by constraining the manager to invest a portion of the fund passively. The analysis in this chapter shows that the greater the constrained investment in the benchmark, when this constraint is binding, results in the active segment of the fund optimising the manager's utility at a higher active risk level.
CONCLUSIONS

8.1 BENCHMARK-RELATIVE PORTFOLIO SELECTION ..................2
8.2 BENCHMARK-RELATIVE PORTFOLIO MONITORING .................3
8.3 CONSTRAINED LEVERAGE AND THE SEPARATION THEOREM ......4
The objective of this thesis was to explore modern portfolio theory in the context of the additional constraints that are often made in practice. The thesis comprises three major sections. The first section (Chapters 2 and 3) explores the introduction of a benchmark into the modern portfolio selection framework against which both risk and return are measured. The second section (Chapter 4) analyses the potential pitfalls of benchmark-relative performance monitoring with particular reference to benchmark-relative risk measurement. The third and final section (Chapters 5 through 7) considers the impact of constrained investment in the risk-free rate and the consequences for the separation theorem and the benchmark-relative portfolio selection framework.

8.1 BENCHMARK-RELATIVE PORTFOLIO SELECTION

The objective of this section as a whole was to establish whether the lessons of Capital Market Theory are still upheld when variance is replaced with TEV in an asset allocation framework. Chapter 2 demonstrated that TEV decomposes into two neat interpretable forms much like the traditional market model decomposition of variance: unique risk and relative market risk. Hence, managers can attribute TEV to the two component sources of risk.

Chapter 2 also demonstrated how these risk components bound the scope of opportunities in a consensus (CAPM) environment. It was shown that for every level of unique risk, the three determinants of the slope of the frontier of investments are the market/benchmark premium, the market/benchmark variance and the highest beta asset with that level of unique risk available in the opportunity set. These results are also consistent with the Markowitz framework and the Capital Market Line theory.

Furthermore, it was noted that the (MV) risk-free asset has significant TEV and that the benchmark portfolio takes its place as the riskless asset in the TEV framework. The role of the risk-free rate in the MV and the TEV framework was explored further
in Chapters 5 through 7.

Chapter 3 illustrated the difference between minimising absolute (variance) or relative (TEV) risk in the portfolio selection process. The analysis in this chapter showed that the MV and TEV efficient frontier overlap from the highest return portfolio to the benchmark itself if the benchmark is MV efficient. Under a more realistic benchmark choice, the extent to which the benchmark is MV inefficient is the extent to which the MV and TEV efficient frontiers will separate in risk-reward space. Thus the selection of a benchmark is an important factor for the manager who is concerned with TEV optimisation. Comparison with MV optimisation may be a prudent way for such a manager to keep absolute risk in check.

The results in Chapter 3 confirmed two theoretical discussions from the previous chapter. Firstly that diversification reduces both absolute (MV) and relative (TEV) risk. Secondly, that the “riskless” or certainty equivalent asset for the TEV framework is the benchmark and that the “riskless” asset for the MV framework is the risk-free rate. Thus the TEV efficient frontier never extends beyond the benchmark to portfolios of lower return than the benchmark, irrespective of their MV efficiency. Moreover, the risk-free rate is highly undesirable in the TEV framework and seldom, if ever, forms part of any portfolios along the TEV efficient frontier.

8.2 BENCHMARK-RELATIVE PORTFOLIO MONITORING

In Chapter 4 the evidence of serial correlation and its consequent impact on TEV estimation was examined on locally (South African) listed index tracking mutual funds. The analysis showed clear evidence of negative serial correlation of weekly TE’s of tracker funds across three indices (and mandates) and consequently upwardly biased estimates of TEV when estimated from weekly data. There seemed to be little evidence of serial correlation of monthly TE’s and thus estimates of TEV based on monthly data can be considered relatively accurate in the SA environment.
Chapter 8: Conclusions

The analysis in this chapter also showed that the serial correlation and consequent bias of TEV estimation is most pronounced on indices comprised of stocks with varying liquidity characteristics such as the JSE FINDI Index. Therefore comparison of index tracking deviation across indices and mandates should be applied with caution.

TE measurements should be examined for negative serial correlation before estimates of TEV are made. If serial correlation is detected, estimates of TEV should either be made from lower frequency return measurements, if they are available, or an adjustment technique such as the Lo-MacKinlay adjustment should be used to obtain unbiased estimates.

8.3 CONSTRAINED LEVERAGE AND THE SEPARATION THEOREM

Chapter 5 showed that, if an investor is constrained to borrow a certain amount, as the constrained amount of borrowing increases, the rational, risk-averse investor's optimal portfolio of risky assets will become less risky. Conversely, if an investor is constrained to invest in the risk-free asset, the greater his/her constrained investment in the risk-free asset, the more risky their optimal portfolio of risky assets will become. Thus it is important to note that, in the context of constrained leverage, the tangent portfolio is not necessarily optimal for all investors.

It is often assumed that, on account of Tobin's Separation Theorem, to first choose an optimal equity portfolio (without considering borrowing or lending) and then to simply add in the constrained amount of liquid assets will result in an optimal portfolio under fixed lending constraints. However, when borrowing or lending is constrained, the assumption of limitless borrowing and lending that underpins the Separation Theorem is violated. The evidence in Chapters 5 and 6 makes it clear that selecting the optimal risky asset portfolio and diluting the investment with the constrained proportion of the risk-free asset results in a less risky and sub-optimal portfolio selection. An investor who is constrained with regards to their investment
in the risk-free asset should make their selection from an appropriately constrained efficient frontier.

The converse is true when an investor is constrained to borrow a fixed proportion. However Chapter 6 showed that, when the investor is additionally constrained to invest the borrowed funds in a particular share, their selection of additional assets (those assets which are not constrained) may well be more risky than their unconstrained selection, depending on the risk and return of the share in which they are constrained to invest.

Overall, the implication of these two chapters was that investors who are constrained to borrow, lend or invest in a particular asset should incorporate their constraints into the efficient set and make a utility maximising selection from the appropriately constrained set. Adding constrained investments after selecting a utility maximising portfolio results in a sub-optimal selection.

Finally, Chapter 7 extended the findings of Chapters 5 and 6 into the TE/TEV paradigm. The analysis in this chapter shows that active managers cannot gear their fund linearly along the same active risk – active return trade-off by investing in or borrowing at the risk-free rate. To the contrary, constrained investment in the risk-free rate will force the fund into less efficient TE/TEV space. Instead, fund managers who seek to gear their fund in TE/TEV space but maintain the same relative return (TE) – relative risk (TEV) trade-off must borrow/lend the benchmark.

While borrowing the benchmark (i.e. selling the benchmark short) is a practice which is usually beyond regulatory and legislative restrictions, it is possible and common to constrain a long investment in the benchmark. This translates into a constraint on active management by constraining the manager to invest a portion of the fund passively. The analysis in this chapter showed that the greater the constrained investment in the benchmark, provided this constraint is binding, the higher the active risk level in the active segment of the fund which optimises the manager's utility.
Chapter 8: Conclusions

Thus the thesis aims to provide assistance to portfolio managers, investors and analysts who are constrained by a benchmark-relative portfolio selection and monitoring framework as well as those who are constrained by mandatory restrictions on their investment in the so-called "risk-free" asset.
REFERENCES


References


APPENDICES

APPENDIX A:  THE RELATIONSHIP BETWEEN EXPECTED RETURN AND VARIANCE ........................................... IX

APPENDIX B:  THE SEPARATION THEOREM IN TE/TEV SPACE ........ X

APPENDIX C:  THE BOUNDARY FRONTIER IN MEAN/STANDARD DEVIATION SPACE ........................................... XI

APPENDIX D:  PROOF THAT THE BENCHMARK COMBINES LINEARLY WITH ANY ASSET IN TE-TE STANDARD DEVIATION SPACE ....................... XII

APPENDIX E:  PROOF THAT THE COMBINATION OF THE BENCHMARK WITH ANY FULLY DIVERSIFIED PORTFOLIO/ASSET WILL LIE ALONG THE BOUNDARY CURVE IN TE-TE STANDARD DEVIATION SPACE .................................................. XIII

APPENDIX F:  DATA SET FOR SECTION 3.3 OF CHAPTER 3 ............. XIV

APPENDIX G:  DATA SET FOR SECTION 3.4 OF CHAPTER 3 .......... XVI

APPENDIX H:  RESULTS OF ANALYSIS IN CHAPTER 4 .............. XVIII

APPENDIX I:  PROOF THAT THE SLOPES OF FRONTIERS 1 AND 2 AT CORRESPONDING POINTS ARE EQUAL ................................. XIX

APPENDIX J:  DATA SET FOR CHAPTER 6 ................................ XX

APPENDIX K:  PORTFOLIO COMPOSITIONS FOR SECTION 6.4.1 OF CHAPTER 6 XXII

APPENDIX L:  PORTFOLIO COMPOSITIONS FOR SECTION 6.4.2 OF CHAPTER 6 XXII

APPENDIX M:  DATA SET FOR CHAPTER 7 ................................ XXIII

APPENDIX N:  PROOF OF EQUAL SLOPES OF EFFICIENT FRONTIERS AT CORRESPONDING POINTS .............................. XXV
APPENDIX A: THE RELATIONSHIP BETWEEN EXPECTED RETURN AND VARIANCE

The relationship between expected return and variance can be established by combining the CAPM and the market model decomposition of risk as follows:

\[
E[R_p] = R_f + \beta_p E[R_m - R_f] \quad \Leftrightarrow \quad \beta_p^2 = \frac{E^2[R_p - R_f]}{E^2[R_m - R_f]}, \quad \sigma_p^2 = \sigma_m^2 \beta_p^2 + \sigma_e^2
\]

(i)

Thus expected return can be defined implicitly as a function of variance:

\[
F(\sigma_p^2, E[R_p]) = \frac{E^2[R_p - R_f]}{E^2[R_m - R_f]} - \frac{\sigma_p^2 - \sigma_e^2}{\sigma_m^2} = 0
\]

(ii)

The gradient of the implicit function in (ii) is as follows:

\[
\frac{dF(\sigma_p^2, E[R_p])}{d(\sigma_p^2)} = -\frac{1}{\sigma_m^2} = -\frac{2E[R_p - R_f]}{E^2[R_m - R_f]} = \frac{E^2[R_m - R_f]}{2E[R_p - R_f] \sigma_m^2}
\]

(iii)

Substituting the consensus/CAPM expression for expected return in the denominator of (iii), the expression describing the gradient simplifies as follows:

\[
\frac{d(E[R_p])}{d(\sigma_p^2)} = \frac{E^2[R_m - R_f]}{2\sigma_m^2 \beta_p E[R_m - R_f]} = \frac{E[R_m - R_f]}{2\sigma_m^2 \beta_p}
\]

(iv)

It is evident from (iv) that any frontier of investment opportunities with a particular level of unique risk is determined by three factors: the largest-\(\beta\) portfolio with that particular level of unique risk, the market/benchmark premium and the variance of the benchmark. One can conclude that the higher the benchmark premium and the lower the variance (risk) of this benchmark, the steeper the frontier will be. A steeper boundary to the frontier translates into greater choice and more opportunities in a higher (more favourable) position in risk/return space. This inference is intuitive since a market with low risk (in the traditional variance sense) and a high expected excess return is always more attractive to investors.
APPENDIX B: THE SEPARATION THEOREM IN TE/TEV SPACE

By borrowing or lending at the risk free rate and investing the remaining or additional funds in the benchmark, an investor can shift his/her portfolio along the boundary curve in TE/TEV space.

Consider a two-asset portfolio with a proportion, \( X_{RF} \), invested in the risk free asset and the remaining/additional funds, \( X_m \), invested in the benchmark. These two proportions must add to 1 although, if limitless borrowing or lending is allowed, either asset may be held long or short.

Since the beta of the benchmark is one and the beta of the risk-free rate is zero, the beta value of this two-asset portfolio is the same as the proportion of the funds invested in the benchmark.

\[
\beta_p = X_m \beta_m + X_{RF} \beta_{RF} = X_m \cdot 1 + X_{RF} \cdot 0 = X_m
\]

If borrowed funds are invested in the benchmark portfolio (i.e. \( X_{RF} < 0 \) and \( X_m > 1 \)), the portfolio’s beta will be greater than one and if funds are lent (\( X_{RF} > 0 \) and \( X_m < 1 \)), this two-asset portfolio’s beta will be smaller than one.

Since neither of these two assets have any unique risk, the two asset portfolio will also have no unique risk. Thus the two-asset portfolio offers the best risk (TEV) reward (TE) trade-off for any value of beta and any proportion (\( X_{RF} \) or \( X_m \)). And so investors may place themselves anywhere on the boundary frontier in TE/TEV space by investing in the benchmark portfolio and lending or borrowing funds at the risk-free rate.
APPENDIX C: THE BOUNDARY FRONTIER IN MEAN/STANDARD DEVIATION SPACE

Given the relationship between variance and expected return (implicit function (ii)), the gradient of the boundary curve in return/standard deviation space is as follows:

\[ \frac{d\left( E[R_p] \right)}{d\left( \sigma_p \right)} = \frac{dF(\sigma_p, E[R_p])}{d(\sigma_p)} = -\frac{2\sigma_p}{\sigma_m^2} \frac{E\left[R_m - R_f\right]}{E\left[R_p - R_f\right]} \frac{\sigma_p}{\sigma_m^2} = \frac{E^2\left[R_m - R_f\right]}{E\left[R_p - R_f\right]^2 \sigma_m^2} \]  

(vi)

Substituting the consensus/CAPM expression for expected return \( \beta_p E\left[R_m - R_f\right] \) in the denominator and the market model decomposition of variance in the numerator, (vi) can be rewritten as follows:

\[ \frac{d\left( E[R_p] \right)}{d\left( \sigma_p \right)} = \frac{E\left[R_m - R_f\right]^4 \sqrt{\beta^2 \sigma_m^2 + \sigma_e^2}}{\sigma_m^2 E\left[R_m - R_f\right]\beta_p} = \frac{E\left[R_m - R_f\right] \sqrt{\beta^2 \sigma_m^2 + \sigma_e^2}}{\beta_p} \]  

(vii)

Considering only portfolios with no unique risk (i.e. \( \sigma_e^2 = 0 \)), the expression in (vii) can be further simplified as follows:

\[ \frac{d\left( E[R_p] \right)}{d\left( \sigma_p \right)} = \frac{\beta_p \sigma_m E\left[R_m - R_f\right]}{\beta_p \sigma_m^2} = \frac{E\left[R_m - R_f\right]}{\sigma_m} \]  

(viii)

Thus the boundary frontier in Mean/Standard Deviation space which traces portfolios with no unique risk (i.e. \( \sigma_e^2 = 0 \)), is the Capital Market Line (CML). The slope of the CML is determined by the market premium and variance and is constant since it is independent of the individual portfolios in the feasible set.
APPENDIX D: PROOF THAT THE BENCHMARK COMBINES LINEARLY WITH ANY ASSET IN TE-TE STANDARD DEVIATION SPACE

Clearly the expected relative return $E[TE_p]$ of the benchmark is zero (irrespective of the consensus views). The variance/uncertainty of the benchmark’s TE is zero as is the correlation of the benchmark’s TE with any variable time series of TE’s. Consequently, any combination of the benchmark and any other risky asset will be linear in TE/TE Standard Deviation space. The proof follows.

Consider two assets, one without TEV (i.e. the benchmark) and one with TEV. Let $X_m$ be the (non-negative) proportion invested in the benchmark and let $X_i = 1 - X_m$ be the proportion invested in asset $i$.

The TEV of this two-asset portfolio is as follows:

$$TEV_p = \text{Var}(R_p - R_m)$$  \hfill (ix)

Substituting for the portfolio’s return in (ix) gives the following expression for the TEV of this portfolio of two assets:

$$TEV_p = \text{Var}[X_mR_m + (1 - X_m)R_i - R_m]$$ \hfill (x)

Simplifying (x) further:

$$TEV_p = \text{Var}[(1 - X_m)(R_i - R_m)]$$

$$= (1 - X_m)^2 \text{Var}(R_i - R_m)$$

$$= (1 - X_m)^2 TEV_i$$ \hfill (xi)

And so the TE Standard Deviation of this two-asset portfolio is as follows:

$$\sqrt{TEV_p} = (1 - X_m)\sqrt{TEV_i}$$ \hfill (xii)

The expected TE of a portfolio consisting of these two assets, since the benchmark has zero expected TE, is as follows:

xii
\[ E[TE_p] = (1 - X_m)E[TE_i] \]  

(xiii)

And so (xii) and (xiii) combine to express expected TE in terms of Tracking Error standard deviation:

\[ E[TE_p] = \frac{E[TE_i]}{\sqrt{TEV_i}} \frac{1}{\sqrt{TEV_p}} \]  

(xiv)

Thus the expected TE of a portfolio which combines the benchmark with any other asset is linearly related to the TE standard deviation of the same portfolio i.e. the gradient of this line is independent of the proportion invested in either asset. Consequently, the combination of any asset with the benchmark portfolio is linear in TE/TE standard deviation space.

**APPENDIX E:** PROOF THAT THE COMBINATION OF THE BENCHMARK WITH ANY FULLY DIVERSIFIED PORTFOLIO/ASSET WILL LIE ALONG THE BOUNDARY CURVE IN TE/TE STANDARD DEVIATION SPACE.

Appendix D shows that the benchmark combines linearly with any other asset with the following slope:

\[ \frac{E[TE_i]}{\sqrt{TEV_i}} \]  

(xv)

However, both the TE and the TEV of an asset are linked (under consensus assumptions) to the opportunities offered by the market and the asset/portfolio’s beta. Thus the slope of all possible combinations of the benchmark and any other asset is as follows:

\[ \frac{E[TE_i]}{\sqrt{TEV_i}} = \frac{(\beta - 1)E[R_m - R_f]}{\pm \sqrt{(\beta - 1)^2 \sigma_m^2 + \sigma_i^2}} \]  

(xvi)

If asset \( i \) had no unique risk, (xvi) would simplify as follows:

\[ \frac{E[TE_i]}{\sqrt{TEV_i}} = \pm \frac{(\beta - 1)E[R_m - R_f]}{(\beta - 1)\sigma_m} = \pm \frac{E[R_m - R_f]}{\sigma_m} \]  

(xvii)
The expression in (xvi) is a constant and thus the combination of the benchmark with an asset/portfolio with no unique risk, such as the risk-free rate, is linear in $TE - TE$ Standard Deviation space. Furthermore this slope is the same as the slope of the boundary frontier or the CML (see (2.12) in 2.2.2.2.1 of Chapter 2).

**APPENDIX F: DATA SET FOR SECTION 3.3 OF CHAPTER 3**

The historical month-end, last-traded prices of various indices on the Johannesburg Stock Exchange (JSE) were downloaded from Bloomberg for the dates January 1996 to August 2000. The monthly returns of these indices from January 1996 to December 1998 were used to calculate betas\(^1\) for each index against the Allshare (JOHMKT) index. These betas were then used to estimate monthly returns according to the CAPM for the period of January 1999 to August 2000, as described in section 3.2.2.

The table below summarises the returns (historical and CAPM-estimated) and variances of the indices over the period of January 1999 to August 2000 which were used to generate the efficient frontiers in this chapter. The first column identifies each index by its Bloomberg "Ticker" code and the second and third column report the average of the monthly CAPM-estimated and historical returns respectively. The fourth column reports the standard deviation of the historical monthly returns. The fifth column is comprised of the rank of the ratio of historical return to standard deviation, from largest (rank of one) to smallest (rank of 55) i.e. an asset with a rank of 1 means that this asset earned the highest return per unit of risk over the period. The sixth column reports the beta values estimated over a prior period (January 1996 to December 1998) using the JOHMKT as the independent series. The last two columns report the average monthly tracking error (return of the index less the return of JOHMKT) and the standard deviation of the monthly tracking error.

\(^1\) Ordinary least squares regression coefficient: $y = \alpha + \beta x$ where $x$ is the JOHMKT return series and $y$ is the return series on each of the indices listed here.
Table 1: Summary Statistics of the Empirical Investment Universe used in Section 3.3 of Chapter 3 (Jan '99 – Aug '2000)

<table>
<thead>
<tr>
<th>Index</th>
<th>CAPM Estimated Return p.a.</th>
<th>Average Return p.m.</th>
<th>Standard Deviation p.m.</th>
<th>Ranked Return/Std Dev Ratio</th>
<th>Beta against JOHMTK</th>
<th>Average Tracking Error p.m.</th>
<th>Tracking Error Std Dev p.m.</th>
</tr>
</thead>
<tbody>
<tr>
<td>JOHMTK</td>
<td>2.42%</td>
<td>2.42%</td>
<td>5.94%</td>
<td>8</td>
<td>1.00</td>
<td>0%</td>
<td>0%</td>
</tr>
<tr>
<td>JOHPLAT</td>
<td>3.71%</td>
<td>5.80%</td>
<td>8.82%</td>
<td></td>
<td>1.81</td>
<td>4.37%</td>
<td>7.78%</td>
</tr>
<tr>
<td>JOHJNTO</td>
<td>2.62%</td>
<td>4.11%</td>
<td>7.23%</td>
<td></td>
<td>1.12</td>
<td>1.69%</td>
<td>5.38%</td>
</tr>
<tr>
<td>JOHINVTR</td>
<td>2.29%</td>
<td>5.24%</td>
<td>9.85%</td>
<td></td>
<td>0.92</td>
<td>2.81%</td>
<td>8.78%</td>
</tr>
<tr>
<td>JOHJAM</td>
<td>3.33%</td>
<td>5.42%</td>
<td>11.28%</td>
<td></td>
<td>1.57</td>
<td>3.06%</td>
<td>8.26%</td>
</tr>
<tr>
<td>JOHMIN</td>
<td>2.86%</td>
<td>4.45%</td>
<td>9.44%</td>
<td></td>
<td>1.27</td>
<td>2.02%</td>
<td>6.14%</td>
</tr>
<tr>
<td>JOHRES</td>
<td>2.73%</td>
<td>4.48%</td>
<td>9.74%</td>
<td></td>
<td>1.19</td>
<td>2.66%</td>
<td>6.39%</td>
</tr>
<tr>
<td>JOHLOAN</td>
<td>1.58%</td>
<td>2.13%</td>
<td>5.12%</td>
<td></td>
<td>0.47</td>
<td>-0.29%</td>
<td>6.76%</td>
</tr>
<tr>
<td>JOHMMIGUS</td>
<td>3.10%</td>
<td>4.89%</td>
<td>12.08%</td>
<td></td>
<td>1.42</td>
<td>2.47%</td>
<td>8.74%</td>
</tr>
<tr>
<td>JOHCHIEF</td>
<td>2.13%</td>
<td>4.98%</td>
<td>12.38%</td>
<td></td>
<td>1.19</td>
<td>2.56%</td>
<td>9.15%</td>
</tr>
<tr>
<td>JOHINDI</td>
<td>2.38%</td>
<td>2.61%</td>
<td>6.69%</td>
<td></td>
<td>1.97</td>
<td>0.19%</td>
<td>4.10%</td>
</tr>
<tr>
<td>JOHJNMN</td>
<td>2.25%</td>
<td>4.87%</td>
<td>12.95%</td>
<td></td>
<td>0.89</td>
<td>2.45%</td>
<td>9.82%</td>
</tr>
<tr>
<td>JOHJH1</td>
<td>1.84%</td>
<td>1.69%</td>
<td>4.38%</td>
<td></td>
<td>0.63</td>
<td>-0.83%</td>
<td>6.16%</td>
</tr>
<tr>
<td>JOHIPPOR</td>
<td>2.31%</td>
<td>5.35%</td>
<td>18.34%</td>
<td></td>
<td>0.87</td>
<td>3.92%</td>
<td>15.90%</td>
</tr>
<tr>
<td>JOHJNDST</td>
<td>2.26%</td>
<td>2.09%</td>
<td>6.18%</td>
<td></td>
<td>0.90</td>
<td>-0.33%</td>
<td>3.65%</td>
</tr>
<tr>
<td>JHJFIND</td>
<td>2.31%</td>
<td>2.12%</td>
<td>6.38%</td>
<td></td>
<td>0.93</td>
<td>-0.31%</td>
<td>3.33%</td>
</tr>
<tr>
<td>JHJFTH</td>
<td>2.75%</td>
<td>3.10%</td>
<td>9.66%</td>
<td></td>
<td>1.21</td>
<td>0.73%</td>
<td>7.61%</td>
</tr>
<tr>
<td>JHJPROPT</td>
<td>1.69%</td>
<td>1.65%</td>
<td>5.23%</td>
<td></td>
<td>0.54</td>
<td>-0.78%</td>
<td>3.77%</td>
</tr>
<tr>
<td>JHJRFMA</td>
<td>1.59%</td>
<td>1.53%</td>
<td>5.21%</td>
<td></td>
<td>0.48</td>
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2 Arithmetic average.
Appendices

APPENDIX G: DATA SET FOR SECTION 3.4 OF CHAPTER 3

Table II: Summary Statistics of the Empirical Investment Universe used in Section 3.4 of Chapter 3 (Jan '99 – Aug '00)

<table>
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<tr>
<th>Index</th>
<th>CAPM Estimated Return p.m.</th>
<th>Average Standard Deviation p.m.</th>
<th>Ranked Return/Std Dev Ratio</th>
<th>Beta against JOHMKT</th>
<th>Average Tracking Error Std Dev p.m.</th>
<th>Tracking Error Std Dev p.m.</th>
</tr>
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<td>0.97</td>
<td>-2.55%</td>
<td>5.47%</td>
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<td>-3.15%</td>
<td>17.64%</td>
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<td>-3.22%</td>
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Ordinary least squares regression coefficient: \( y = \alpha + \beta x \) where \( x \) is the market proxy return series and \( y \) is the return series on each of the indices listed here.

xvi
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<th>CAPM Estimated Return p.m.</th>
<th>Average Return p.m.</th>
<th>Standard Deviation p.m.</th>
<th>Ranked Return/Std Dev Ratio</th>
<th>Beta against JOHMKT</th>
<th>Average Tracking Error p.m.</th>
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## Appendix H: Results of Analysis in Chapter 4

### Table III: Monthly Serial Correlation of each Fund’s TE against its Benchmark for periods ending October 2000

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### Table IV: Weekly Serial Correlation (Lag = 2 weeks) of each Fund’s TE against its Benchmark for periods ending October 2000

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<th>31/01/97</th>
<th>28/08/97</th>
<th>28/11/97</th>
<th>30/01/98</th>
<th>29/05/98</th>
<th>29/11/98</th>
<th>31/03/00</th>
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<tr>
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<td>0.00</td>
<td>0.00</td>
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Table V: Weekly Serial Correlation (Lag = 3 weeks) of each Fund’s TE against its Benchmark for periods ending October 2000

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<tbody>
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<td>31/01/97</td>
<td>28/02/97</td>
<td>22/11/97</td>
<td>30/01/98</td>
<td>25/05/98</td>
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<td>-0.07</td>
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<td>-0.03</td>
<td>-0.02</td>
<td>-0.02</td>
<td>-0.07</td>
<td>-0.14</td>
<td>-0.18</td>
</tr>
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<td>0.01</td>
<td>0.01</td>
<td>0.02</td>
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<tr>
<td><strong>ALSI-40 Trackers</strong></td>
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<td>Corporation ALSI 40 Tracker</td>
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<td>Liberty ALSI 40 C</td>
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<tr>
<td>ABFA Fine  &amp; Industrial Index</td>
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<td>0.28</td>
<td>0.11</td>
<td>-0.00</td>
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<tr>
<td>Brail FINDI</td>
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<td>0.28</td>
<td>0.28</td>
<td>0.12</td>
<td>-0.02</td>
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</tbody>
</table>

**APPENDIX I: PROOF THAT THE SLOPES OF FRONTIERS 1 AND 2 AT CORRESPONDING POINTS ARE EQUAL**

It is shown here that the slopes of the efficient frontiers 1 and 2 (in Figure 5.2 of Chapter 5) at any two corresponding points, (σ₁,μ₁) and (σ₂,μ₂), are equal.

Let \( E_f(\sigma) \) and \( E_2(\sigma) \) represent frontiers 1 and 2 respectively. Since frontier 2 is levered relative to frontier 1 by a factor \( K \), for two corresponding points \((σ₁,μ₁)\) and \((σ₂,μ₂)\):

\[
E_2(σ_2) = (1 - K)R_f + KE_1(σ_1)
\]

Since \( σ_2 = Kσ_1 \)

\[
E_2(σ_2) = (1 - K)R_f + KE_1 \left( \frac{σ_2}{K} \right)
\]

It follows that

\[
\frac{dE_2}{dσ_2} = K \frac{dE_1}{dσ_1} \left( \frac{1}{K} \right) = \frac{dE_1}{dσ_1}
\]
Thus the slopes of an efficient frontier and its leveraged counterpart at any two corresponding points are the same.

**APPENDIX J: DATA SET FOR CHAPTER 6**

The historical month-end, last-traded prices of various indices on the Johannesburg Stock Exchange (JSE) as well as the Morgan Stanley Capital International (MSCI) World Index were downloaded from Bloomberg for the dates January 1996 to August 2000. The table below summarises the returns and risks of the indices over this period which were used to generate the efficient frontiers in this chapter.

The first column identifies each index by its Bloomberg “Ticker” code, the second column contains the geometric average of the monthly returns as a percentage and the third column contains the Standard Deviation (also of the monthly returns expressed as a percentage). The last column contains the rank of the ratio of return to standard, from largest (rank of one) to smallest (rank of 55) i.e. a rank of 1 means that the index earned the highest return per unit of risk.

**Table VI: Summary Statistics of the Empirical Investment Universe**

<table>
<thead>
<tr>
<th>Index</th>
<th>Effective Monthly Return</th>
<th>Standard Deviation (p.m.)</th>
<th>Ranked Return/Sd Dev Ratio</th>
</tr>
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<td>11.4%</td>
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</tr>
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<tr>
<td>JOHINVTR Index</td>
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<td>9.4%</td>
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<tr>
<td>JOHDCM Index</td>
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<td>7</td>
</tr>
<tr>
<td>JOHFLS Index</td>
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<tr>
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<td>Standard Deviation (p.m.)</td>
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APPENDIX K: PORTFOLIO COMPOSITIONS FOR SECTION 6.4.1 OF CHAPTER 6

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APPENDIX L: PORTFOLIO COMPOSITIONS FOR SECTION 6.4.2 OF CHAPTER 6

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APPENDIX M: DATA SET FOR CHAPTER 7

The historical month-end, last-traded prices of various indices on the Johannesburg Stock Exchange (JSE) were downloaded from Bloomberg for the last two years (August 1998 to August 2000). The table below summarises the active and passive returns and risks of these indices over the period which were used to generate the efficient frontiers in this chapter.

The first three columns comprise each index’s Bloomberg “Ticker” code, the arithmetic average of the monthly returns, and the standard deviation of these monthly returns. The risk-reward ratio is in the fourth column. The following three columns comprise the average monthly return less that of the benchmark (JOHMKT) i.e. the TI of each index, the standard deviation of these TE’s and the ratio of TE to TE standard deviation from largest (rank of one) to smallest (rank of 54).

Table VII: Summary Statistics of the Empirical Investment Universe

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<th>Index</th>
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<th>Std Dev % p.m.</th>
<th>Ranked Std Dev % p.m.</th>
<th>Average TE (% p.m.)</th>
<th>Std Dev % p.m.</th>
<th>Ranked TE to TE Std Dev Ratio</th>
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