UNIVERSITY OF CAPE TOWN

MASTER'S THESIS

Two-loop corrections to the rho-propagator in the Kroll-Lee-Zumino Theory

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Declaration of authorship

I, Mawande Lushozi, declare that this thesis titled, 'Two-loop Corrections in the Kroll-Lee-Zumino Theory' and the work presented in it are my own. I confirm that:

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- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
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ABSTRACT

The Abelian Kroll-Lee-Zumino theory is a renormalizable quantum field theory of charged pions and neutral rho-mesons. It provides the quantum field theory platform for the vector meson dominance model, allowing for a systematic computation of quantum corrections in perturbation theory (the ρ-π-π coupling is small enough for perturbation theory to make sense). At one-loop level, the theory makes a prediction for the electromagnetic form factor of the pion in good agreement with experiment, but there is room for improvement. This makes the theory an attractive prospect for filling the gap between chiral perturbation theory (which is valid for low-energy interactions) and perturbative QCD (which is valid at high energies). In this thesis we perform a two-loop calculation of the rho-propagator. This quantity is closely related to the pion's electromagnetic form factor, and will allow its calculation at this order.

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CONTENTS

1. INTRODUCTION

1.1 Vector Meson Dominance and the KLZ Model

The electromagnetic form factor of the pion, $F_\pi(p^2)$, is a quantity that is closely related to the distribution of electric charge in a pion. One obtains F_π by calculating the diagram shown in figure 1.1(a). Before the advent of Quantum Chromodynamics (QCD), the most popular model for the interaction at the γ-π-π vertex used the notion of vector meson dominance (VMD). VMD, proposed by Sakurai, postulates that the photon turns into a neutral rho-meson which then interacts strongly with the pions shown in figure 1.1(b) [1, 2, 3]. This model manages to predict $F_\pi(p^2)$ within about 20 percent of experiment. The trouble with VMD, in its original formulation, is that it is a tree level model; it does not allow a systematic computation of corrections to this result.

The problem is remedied by the renormalizable quantum field theory of charged pions and a neutral rho-meson, proposed by Kroll, Lee and Zumino (KLZ). In their 1967 paper, Kroll,Lee and Zumino showed that VMD could be formulated as of a renormalizable quantum field theory [4] with the following lagrangian.

$$\mathscr{L} = |D_\mu \phi|^2 - m_\pi^2 \phi^* \phi - \frac{1}{4}\rho^{\mu\nu}\rho_{\mu\nu} + \frac{1}{2}m_\rho^2 \rho^\mu \rho_\mu , \tag{1.1}$$

where m_ρ and m_π are the masses of the pion and the rho-meson respectively. $g_{\rho\pi\pi} \approx 5$ is the coupling of the rho-mesons to pions. We also have the field strength tensor

$$\rho^{\mu\nu} = \partial^\mu \rho^\nu - \partial^\nu \rho^\mu$$

and the covariant derivative

$$D_\mu = \partial_\mu - i g_{\rho\pi\pi} \rho_\mu .$$

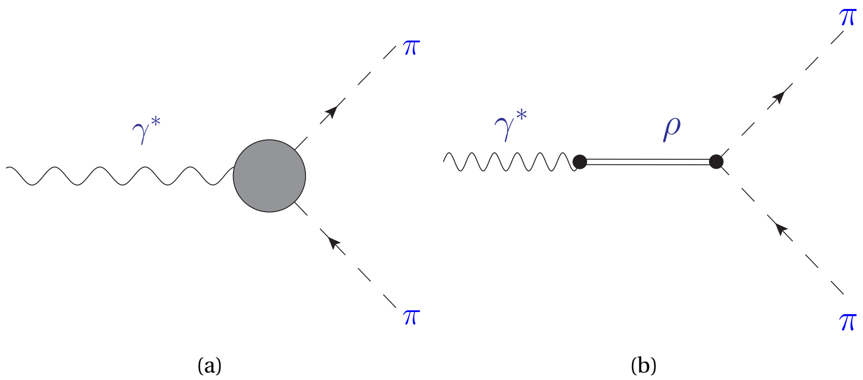


Fig. 1.1: Interaction of a virtual photon with pions

This lagrangian is renormalizable even though there is an explicit ρ^0 mass term. Kroll, Lee and Zumino, in their paper [4], show that this is follows from coupling of the rho-field to the following conserved current:

$$j_\mu = \phi^* \partial \phi - \phi \partial \phi^* .$$

Renormalizability of KLZ has the advantage that no free parameters appear at any order in perturbation theory, and this gives the theory great predictive power. Also, in dimensional regularization, the expansion parameter turns out to be $g_{\rho\pi\pi}/4\pi \approx 0.4$ which is less than unity and therefore allows for a sensible perturbative expansion. These properties make KLZ an ideal platform for computing corrections to VMD.

It is very surprising that higher order corrections in the KLZ theory were not carried out for a long time after its conception. In fact it was not until 1991, in a paper by Gale and Kapusta [5], that one-loop corrections in the KLZ theory appeared in the literature. The reason might be that the ideas of KLZ were pursued at about the same time as the development of QCD. In the 1970's, when QCD was shown to be the theory of strong interactions, the pursuit of ideas such as the KLZ theory seemed futile and so they were forgotten. It turns out that these ideas were abandoned too soon since QCD turned out to be difficult to solve. At low energies the strong coupling constant of QCD becomes so large that low-energy phenomena are inaccessible to perturbation theory in QCD. This problem is (partially) solved by chiral perturbation theory (ChPT), which is an effective field theory of QCD at low energy with a lagrangian expressed through pion degrees of freedom. ChPT describes experimental data reasonably well for energies below the rho-meson peak, but there's no reason to expect it to hold beyond this point since the theory does not include rho-meson degrees of freedom. The abelian theory of Kroll, Lee and Zumino (KLZ) - a quantum field theory of neutral rho-mesons and pions - takes over (partially) from ChPT at energies near the rho-meson pole.

In [6], KLZ was used compute corrections to this result; a brief summary of their methods and results is given below. It is shown that the electromagnetic form factor of the pion is given by

$$F_\pi(p^2) = \frac{m_\rho^2 - \Pi(0)}{m_\rho^2 - p^2 + \Pi(p^2)} + \frac{m_\rho^2}{m_\rho^2 - p^2}\left[\tilde{G}_z(p^2) - \tilde{G}_z(0)\right], \tag{1.2}$$

where $\Pi(p^2)$ is the self energy of the rho-meson (figure 1.2) and $\tilde{G}_z(q^2)$ is related to the ρ-π-π vertex correction shown in figure 1.3. [1]

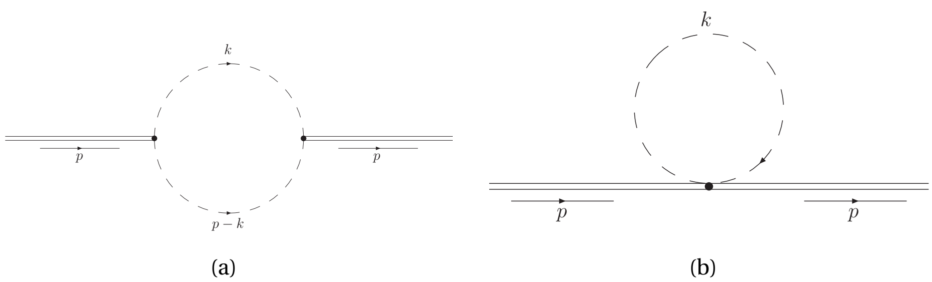


Fig. 1.2: Rho Self Energy Diagrams

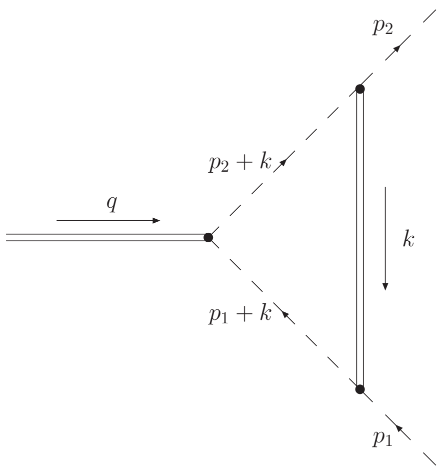


Fig. 1.3: 1-Loop correction to ρ-π-π vertex

[1] The precise meaning of these Feynman diagrams will be made clearer in chapter 2, where the KLZ Feynman rules are introduced.

In the space-like region ($p^2 < 0$), the one-loop correction to the form factor gives good agreement with experiment. The theory fits the data with a chi-squared-per-degree-of-freedom of $\chi_F^2 \approx 1.1$ [6], a vast improvement from naïve VMD which gives $\chi_F^2 \approx 5.0$.

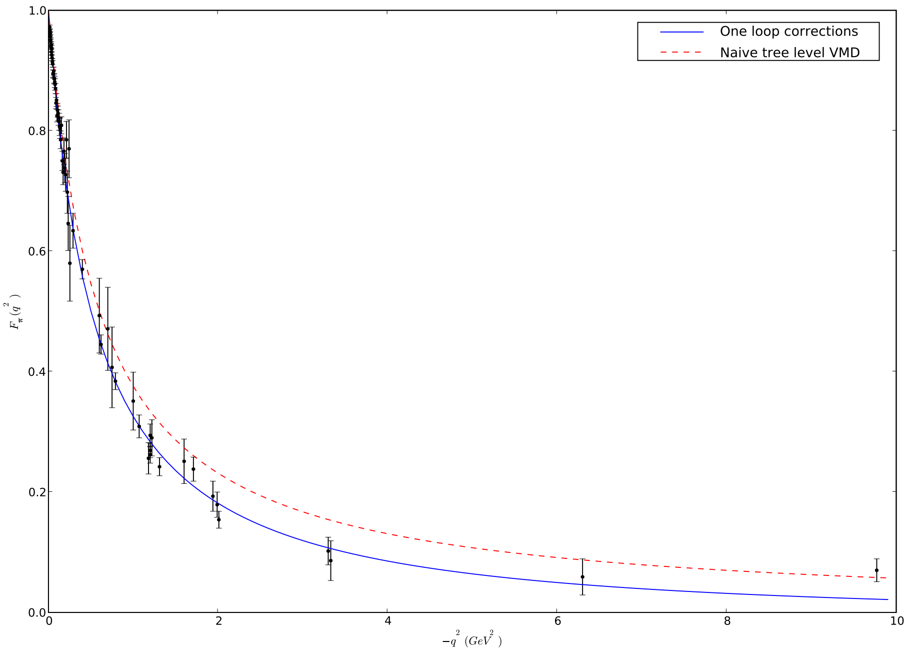


Fig. 1.4: Space-like Form Factor at Zero Temperature [7]

In the time-like region, KLZ one-loop corrections give a momentum dependent rho-width Γ_ρ which is close to the Gounaris-Sakurai width near the peak [6]. The Gounaris-Sakurai width is empirical so it is remarkable that the parameter-free KLZ is in close agreement with this result. However, agreement with experiment is not as good as in the case of the form factor in the space-like region [7, 8].

In this thesis we take steps towards an improvement of this result with two-loop corrections. The result can be improved by calculating two-loop corrections to $\Pi(p^2)$. Such a calculation involves two steps, viz.: regularization of Feynman integrals and renormalization. The thesis will focus on regularization. [2] In future, the two-loop results obtained here will be used to determine the electromagnetic form factor of the pion, by taking the imaginary part of the neutral rho-meson's self-energy.

[2] Given the two-year time constraint to complete this thesis, the renormalization procedure has been left out. It will be performed later (in the PhD thesis) as part of the two-loop calculation of the pion form factor

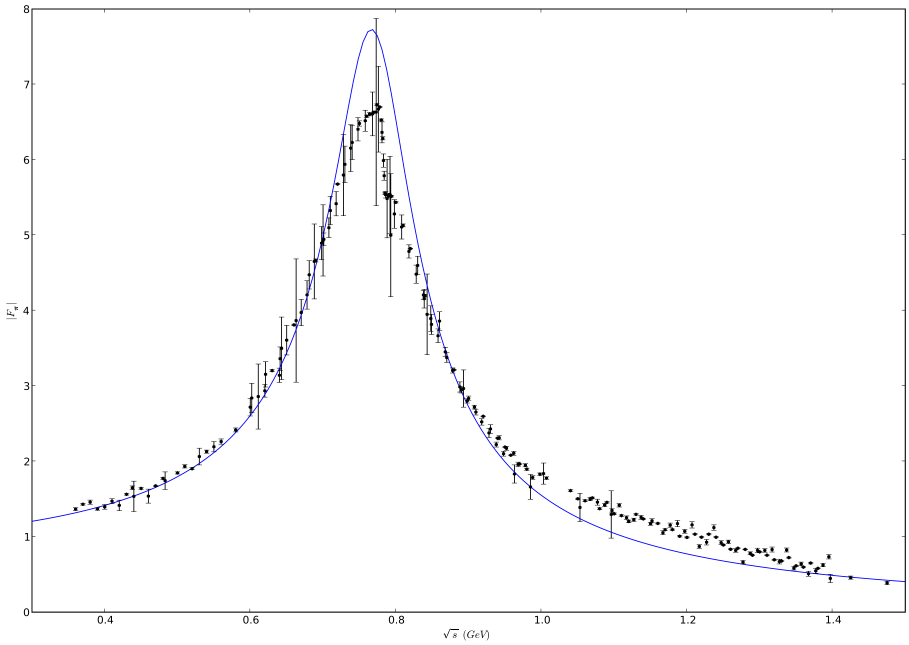


Fig. 1.5: Time-like Form Factor at Zero Temperature [7]

The Abelian KLZ theory has also been used to calculate the scalar radius of the pion at one-loop order which is related to $\bar{l}_4$, the low energy constant of chiral perturbation theory [9], thereby providing an independent determination of $\bar{l}_4$ from other known methods [10, 11]. The extension of KLZ to a non-abelian SU(2) theory will allow the inclusion of charged rho-mesons, and the theory can be used to compute π-π scattering lengths [12]. Should the predictions of this extended theory turn out to be as good as those of the abelian theory, it would undoubtedly cement the role of the KLZ as the bridge between chiral perturbation theory and perturbative QCD.

1.2 Outline

In chapter 2 we review properties of dimensionally regularized Feynman integrals. We then evaluate the one-loop self energy in preparation for two-loop renormalization. In chapter 3 we calculate two-loop corrections to the ρ^0 self-energy followed by conclusions. Some of the detailed calculations required in chapter 3 are given in the appendix.

2. ONE-LOOP SELF ENERGY

When calculating multi-loop Feynman diagrams it is very likely to obtain divergent integrals. A regularization parameter is introduced in order to make the integrals finite and to expose the source of the divergence. In dimensional regularization we regulate the divergence by extending the space-time dimension away from $n = 4$ to $n = 4 - 2\epsilon$ where the integrals are convergent. This extension can only make sense if integration in $n \in \mathbb{C}$ dimensions is a well defined procedure. Such a definition is given in [13] such that the usual properties of integration hold:

$$\int d^n k \, [a f(k) + b g(k)] = a \int d^n k f(k) + b \int d^n k g(k) \tag{2.1}$$

$$\frac{\partial}{\partial p} \int d^n k h(k,p) = \int d^n k \frac{\partial h(k,p)}{\partial p}. \tag{2.2}$$

A change of variables may be done as in usual integration. The following interesting properties also hold:

- Integration by parts

$$\int d^n k \frac{\partial}{\partial k_\mu} p_\mu f(k) = 0 \tag{2.3}$$

$$\int d^n k \frac{\partial}{\partial k_\mu} k_\mu f(k) = 0. \tag{2.4}$$

- Veltman's formula

$$\int d^n k \left(k^2\right)^\alpha = 0 \qquad \forall \alpha \in \mathscr{R}. \tag{2.5}$$

- The metric tensor in n dimensions has the following property

$$g_{\mu\nu} g^{\mu\nu} = n. \tag{2.6}$$

Similarly, we have

$$\frac{\partial k^{\mu}}{\partial k^{\mu}} = n\,. \tag{2.7}$$

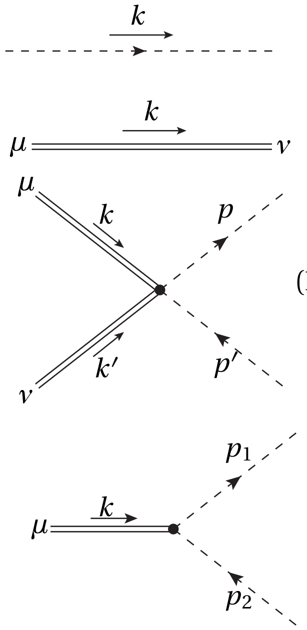
The KLZ lagrangian must have a mass dimension of $-n$ in order to keep the action dimensionless. This means the ρ and pion fields each have mass dimension $\frac{n-2}{2}$. In order to keep the coupling constant dimensionless, we introduce a constant μ with mass dimension equal to unity. This gives us the following lagrangian in n-dimensional space-time[1]

$$\begin{aligned}\mathscr{L} = {}& \partial_{\mu}\phi^{*}\partial^{\mu}\phi - m_{\pi}^{2}\phi^{*}\phi - \frac{1}{4}\rho^{\mu\nu}\rho_{\mu\nu} + m_{\rho}^{2}\rho^{\mu}\rho_{\mu} \\ & + i\,g_{\rho\pi\pi}\,(\mu)^{\frac{4-n}{2}}\,\rho^{\mu}\phi^{*}\overleftrightarrow{\partial_{\mu}}\phi + g_{\rho\pi\pi}^{2}\,(\mu)^{4-n}\,\phi^{*}\phi\rho^{\mu}\rho_{\mu}\,.\end{aligned} \tag{2.8}$$

We will suppress the scale through the replacement

$$g_{\rho\pi\pi}\mu^{\frac{4-n}{2}} = g\,. \tag{2.9}$$

With this lagrangian at hand, the following Feynman rules are obtained[2]:



$$\Delta(k) = \frac{i}{k^{2} - m_{\pi}^{2} + i\varepsilon}$$

$$D_{\mu\nu}(k) = \frac{-i\,g_{\mu\nu}}{k^{2} - m_{\rho}^{2} + i\varepsilon}$$

$$(\Gamma^{(0)}_{\rho\rho\pi\pi})_{\mu\nu} = 2i\,g^{2}g_{\mu\nu}$$

$$(\Gamma^{(0)}_{\rho\pi\pi})^{\mu} = i\,g(p_{1} + p_{2})^{\mu}$$

Fig. 2.1: KLZ feynman rules. The dashed lines represent pions and the double lines represent ρ^{0} mesons.

[1] The quantities ϕ,ρ^{μ},m_{ρ},m_{π} and $g_{\rho\pi\pi}$ are unrenormalized.

[2] The $i\varepsilon$ term in the propagator will is often suppressed to simplify calculations.

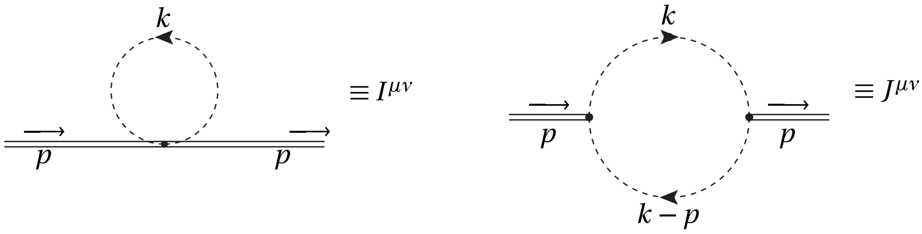


Fig. 2.2: Diagrams that contribute to the one-loop vacuum polarization

The self-energy $\Pi^{\mu\nu}(p^2)$ of the ρ^0 is the sum of all one-particle-irreducible diagrams with two external ρ^0 lines both carrying momentum p. Lorentz covariance implies that the self energy must have the following form:

$$\Pi^{\mu\nu}(p^2) = \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right)\Pi_T(p^2) + \frac{p^\mu p^\nu}{p^2}\Pi_L(p^2) \tag{2.10}$$

In the KLZ theory the Ward-Takahashi identities imply the vacuum polarization is transverse [14]. So we expect the vacuum polarization to be of the following form:

$$\Pi^{\mu\nu}(p^2) = \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right)\Pi(p^2). \tag{2.11}$$

This is expected to hold to all orders of perturbation theory if we use dimensional regularization [15]. We now calculate the one-loop contribution to $\Pi^{\mu\nu}(p^2)$ as this will be necessary in two-loop renormalization. The diagrams in figure 2.2 contribute to the vacuum polarization at one-loop order. The Feynman rules give the following:

$$I^{\mu\nu}(p^2) = -2g^2 \int \frac{d^n k}{(2\pi)^n} \frac{g_{\mu\nu}}{k^2 - m_\pi^2} \tag{2.12}$$

$$J^{\mu\nu}(p^2) = g^2 \int \frac{d^n k}{(2\pi)^n} \frac{(2k-p)^\mu (2k-p)^\nu}{(k^2 - m_\pi^2)[(p-k)^2 - m_\pi^2]}, \tag{2.13}$$

where $I^{\mu\nu}$ and $J^{\mu\nu}$ are the tadpole and bubble diagrams of figure 2.2 respectively. Summing these together, we get the following contribution to the vacuum polarization at one-loop order

$$\Pi_1^{\mu\nu}(p^2) = \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right)\Pi_{1T}(p^2) + \frac{p^\mu p^\nu}{p^2}\Pi_{1L}(p^2)$$
$$= I^{\mu\nu} + J^{\mu\nu}\,. \tag{2.14}$$

We deliberately assume a longitudinal part $\Pi_{1L}(p^2)$ so that, in the end, we may check the correctness of our computations by verifying that it vanishes. We get the longitudinal part of $\Pi_1^{\mu\nu}(p^2)$ by using the longitudinal projector as follows:

$$\frac{p^\mu p^\nu}{p^2}\Pi_{\mu\nu 1}(p^2) = \Pi_{1L}(p^2)$$
$$= \frac{p^\mu p^\nu}{p^2} I_{\mu\nu} + \frac{p^\mu p^\nu}{p^2} J_{\mu\nu}\,. \tag{2.15}$$

Using the longitudinal projector on 2.12 and 2.13 we obtain the following:

$$\frac{p^\mu p^\nu}{p^2} I_{\mu\nu} = -2g^2 \int \frac{d^n k}{(2\pi)^n} \frac{1}{k^2 - m_\pi^2} \tag{2.16}$$

and

$$\frac{p^\mu p^\nu}{p^2} J_{\mu\nu} = g^2 \frac{1}{p^2} \int \frac{d^n k}{(2\pi)^n} \frac{p\cdot(2k-p)p\cdot(2k-p)}{(k^2-m_\pi^2)[(p-k)^2-m_\pi^2]}$$
$$= g^2 \frac{1}{p^2} \int \frac{d^n k}{(2\pi)^n} \frac{\left[(k^2-m_\pi^2)-((p-k)^2-m_\pi^2)\right]^2}{(k^2-m_\pi^2)[(p-k)^2-m_\pi^2]}$$
$$= g^2 \frac{1}{p^2}\left\{\int \frac{d^n k}{(2\pi)^n} \frac{(k^2-m_\pi^2)}{([(p-k)^2-m_\pi^2]} + \int \frac{d^n k}{(2\pi)^n} \frac{(p-k)^2-m_\pi^2}{(k^2-m_\pi^2)} - 2\int \frac{d^n k}{(2\pi)^n} 1\right\}.$$

The third integral on the right above vanishes by (2.5), and we make the variable change $k \rightarrow k+p$ on the first integral yielding

$$\frac{p^\mu p^\nu}{p^2} J_{\mu\nu} = g^2 \frac{1}{p^2} \int \frac{d^n k}{(2\pi)^n} \frac{(k+p)^2 - m_\pi^2 + (k-p)^2 - m_\pi^2}{(k^2-m_\pi^2)}$$
$$= g^2 \frac{1}{p^2}\left\{2\int \frac{d^n k}{(2\pi)^n} 1 + \int \frac{d^n k}{(2\pi)^n} \frac{2p^2}{(k^2-m_\pi^2)}\right\}$$
$$= 2g^2 \int \frac{d^n k}{(2\pi)^n} \frac{1}{k^2-m_\pi^2}\,. \tag{2.17}$$

In going from the second line to the third, the first integral on the right vanishes by (2.5). Now using (2.16) and (2.17) we see that the longitudinal part of the polarization indeed vanishes

$$\begin{aligned}\Pi_{1L}(p^2) &= \frac{p^\mu p^\nu}{p^2} I_{\mu\nu} + \frac{p^\mu p^\nu}{p^2} J_{\mu\nu} \\ &= -2g^2 \int \frac{d^n k}{(2\pi)^n} \frac{1}{k^2 - m_\pi^2} + 2g^2 \int \frac{d^n k}{(2\pi)^n} \frac{1}{k^2 - m_\pi^2} \\ &= 0 .\end{aligned}$$

Now that we have verified that the longitudinal part of $\Pi_1^{\mu\nu}(p^2)$ vanishes, we may find its transverse part by projecting with $g_{\mu\nu}$. We have

$$g_{\mu\nu} I^{\mu\nu} = -2ng^2 \int \frac{d^n k}{(2\pi)^n} \frac{1}{k^2 - m_\pi^2} \tag{2.18}$$

and

$$\begin{aligned}g_{\mu\nu} J^{\mu\nu} &= g^2 \int \frac{d^n k}{(2\pi)^n} \frac{(2k-p)^2}{(k^2 - m_\pi^2)[(p-k)^2 - m_\pi^2]} \\ &= g^2 \left\{ 4 \int \frac{d^n k}{(2\pi)^n} \frac{1}{k^2 - m_\pi^2} + (4m_\pi^2 - p^2) \int \frac{d^n k}{(2\pi)^n} \frac{1}{(k^2 - m_\pi^2)[(p-k)^2 - m_\pi^2]} \right\} .\end{aligned} \tag{2.19}$$

So we get the following for the transverse part:

$$\Pi_{1T}(p^2) = \frac{g^2}{n-1} \left\{ 2(2-n) \int \frac{d^n k}{(2\pi)^n} \frac{1}{k^2 - m_\pi^2} + (4m_\pi^2 - p^2) \int \frac{d^n k}{(2\pi)^n} \frac{1}{(k^2 - m_\pi^2)[(p-k)^2 - m_\pi^2]} \right\} ,$$

which may be written as

$$\Pi_{1T}(p^2) = \frac{i g_{\rho\pi\pi}^2}{(4\pi)^2} \frac{1}{n-1} \left\{ 2(2-n) A_0(m_\pi^2) + (4m_\pi^2 - p^2) B_0(p^2, m_\pi^2) \right\} , \tag{2.20}$$

where we have used (2.9) together with the definitions

$$A_0(m^2) = \int \frac{d^n k}{(2\pi\mu)^{n-4} i\pi^2} \frac{1}{k^2 - m^2} \tag{2.21}$$

$$B_0(p^2, m^2) = \int \frac{d^n k}{(2\pi\mu)^{n-4} i\pi^2} \frac{1}{(k^2 - m^2)[(k+p)^2 - m^2]} \,. \tag{2.22}$$

The above integrals are given in [16] but they have also been included in A.3 for completeness. Substituting the integrals A_0 and B_0 into (2.20) we get the following result:

$$\begin{aligned}
\Pi_{1T}(p^2) = \frac{i g_{\rho\pi\pi}^2}{(4\pi)^2} \Bigg[& -\frac{p^2}{3\epsilon} + \frac{1}{3}\left(\left(4m_\pi^2 - p^2\right)\left(\frac{m_\pi^2 (r_1 - r_2)(\log(r_1) - \log(r_2))}{2p^2} - L_{m_\pi} + 2 \right) + 4m_\pi^2 L_{m_\pi} \right) \\
& - \frac{2p^2}{9} + \epsilon \Bigg\{ \left(\left(-\frac{1}{6} L_{m_\pi}^2 + \frac{8L_{m_\pi}}{9} - \frac{\pi^2}{36} - \frac{52}{27} \right) p^2 - \frac{8}{3} L_{m_\pi} m_\pi^2 + \frac{m_\pi^2}{6} (r_1 - r_2) L_{m_\pi} \left(\log\left(\frac{r_1}{r_2}\right) \right) \right. \\
& - \frac{1}{9} (\log(r_1) - \log(r_2))(r_1 - r_2) m_\pi^2 - \frac{1}{6} \left(-\mathrm{Li}_2\left(\frac{1-r_2}{r_1 - r_2}\right) - \mathrm{Li}_2\left(-\frac{(1-r_1) r_2}{r_1 - r_2}\right) \right. \\
& + \mathrm{Li}_2\left(\frac{1-r_1}{r_2 - r_1}\right) + \mathrm{Li}_2\left(-\frac{r_1(1-r_2)}{r_2 - r_1}\right) + 2\log(r_1) - 2\log(r_2) \\
& \left. - \log\left(\frac{1-r_2}{r_1 - r_2}\right) \log\left(-\frac{(1-r_1) r_2}{r_1 - r_2}\right) + \log\left(\frac{1-r_1}{r_2 - r_1}\right) \log\left(-\frac{r_1(1-r_2)}{r_2 - r_1}\right) \right) (r_1 - r_2) m_\pi^2 \\
& + \frac{64 m_\pi^2}{9} + \frac{1}{p^2} \left(-\frac{2}{3} L_{m_\pi} (\log(r_1) - \log(r_2))(r_1 - r_2) m_\pi^4 + \frac{4}{9} (\log(r_1) - \log(r_2))(r_1 - r_2) m_\pi^4 \right. \\
& + \frac{2}{3} \left(-\mathrm{Li}_2\left(\frac{1-r_2}{r_1 - r_2}\right) - \mathrm{Li}_2\left(\frac{(r_1 - 1) r_2}{r_1 - r_2}\right) + \mathrm{Li}_2\left(\frac{1-r_1}{r_2 - r_1}\right) + \mathrm{Li}_2\left(\frac{r_1(r_2 - 1)}{r_2 - r_1}\right) + 2\log\left(\frac{r_1}{r_2}\right) \right. \\
& \left. \left. \left. - \log\left(\frac{1-r_2}{r_1 - r_2}\right) \log\left(\frac{(r_1 - 1) r_2}{r_1 - r_2}\right) + \log\left(\frac{1-r_1}{r_2 - r_1}\right) \log\left(\frac{r_1(r_2 - 1)}{r_2 - r_1}\right) \right) (r_1 - r_2) m_\pi^4 \right) \Bigg\} \Bigg] \\
& + O(\epsilon^2),
\end{aligned} \tag{2.23}$$

where

$$L_{m_\pi} = \gamma + \ln\left(\frac{m_\pi^2}{4\pi\mu^2}\right) \tag{2.24}$$

with $\gamma \equiv$ Euler's constant. r_1 and r_2 are the roots of the following quadratic equation

$$m_\pi^2 \left(r + \frac{1}{r} \right) = 2m_\pi^2 - p^2 \,. \tag{2.25}$$

We keep the order-ϵ term because it is necessary for performing renormalization to two-loop order.

3. TWO-LOOP SELF ENERGY

The main goal of this thesis is the computation of the two-loop self-energy of the rho-meson in the KLZ field theory. This means we have to evaluate the Feynman diagrams appearing in figure 3.1. Each diagram comes with an associated symmetry factor; the table below shows the symmetry factor of each diagram.

Diagram	Symmetry weight
ξ	$\frac{1}{2}$
ζ	1
Ω	1
W	1
X	4
Z	2
A	1

These weights can be obtained by considering the number of diagrams with the same topology. Alternatively one can look at the combinatorics of Wick contractions in the perturbative expansion of the two-point function to obtain these factors. The rho-self energy is obtained by summing the diagrams with the associated weights in the following way

$$\Pi_2^{\mu\nu} = \frac{1}{2}\xi^{\mu\nu} + \zeta^{\mu\nu} + \Omega^{\mu\nu} + W^{\mu\nu} + 4X^{\mu\nu} + 2Z^{\mu\nu} + A^{\mu\nu}\,. \tag{3.1}$$

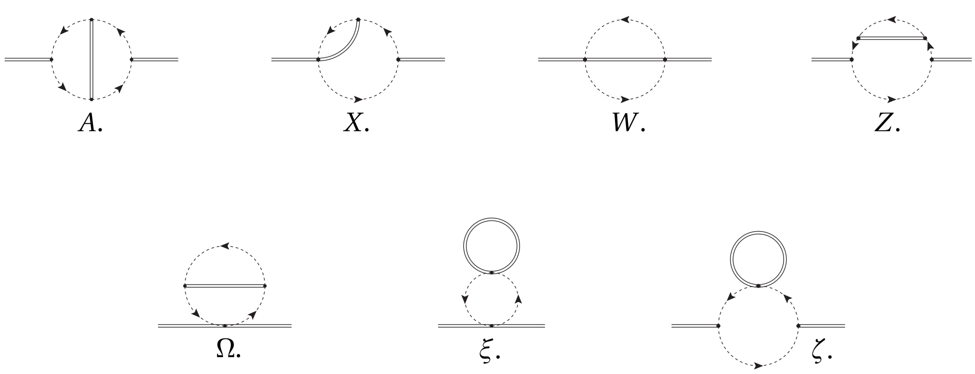


Fig. 3.1: Two-Loop 1PI diagrams for the rho propagator

Since the rho-meson is a vector particle, the integrals above carry tensor indices. A direct evaluation of such integrals is not a trivial task, so the best thing one could do is consult a reference for these. Unfortunately, there does not seem to be a repository available anywhere where one can simply look-up index-carrying integrals. However, there are a lot of results in the literature for scalar two-loop integrals, so it would be wise to reduce the problem to that of finding known scalar integrals. In this chapter we try to carry out such a reduction program in the most efficient way.

First we calculate the longitudinal part of the self-energy. This means finding the longitudinal part of each diagram and then summing all contributions with appropriate symmetry weights. This weighted sum will give a vanishing result, implying that the rho meson's self-energy is transverse at two-loop order. This means that, at the two-loop level, the self-energy of the rho-meson is of the form

$$\Pi_2^{\mu\nu}(p^2) = \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right) \Pi_2(p^2). \tag{3.2}$$

$$\tag{3.3}$$

It is easy to see that the scalar factor above is given by

$$\Pi_2(p^2) = \frac{1}{n-1} g_{\mu\nu} \Pi_2^{\mu\nu}(p^2). \tag{3.4}$$

In essence, determining the self-energy is reduced to the task of contracting each of the integrals in figure 3.1 with $g_{\mu\nu}$. These contractions will yield sums of two-loop scalar integrals which one would hope are easier to handle. It turns out that all two-loop self-energy diagrams can be reduced to a class of so-called T-integrals [17]. The T-integrals have been well studied and there are known methods to evaluate them either analytically or numerically [18, 19, 16].

3.1 Notation

Before we give a definition of the T-integrals it is helpful to introduce some notation. We will write:

$$\begin{aligned} &k_1 = k \qquad && k_2 = k + p \qquad && k_3 = q - k \\ &k_4 = q \qquad && k_5 = q + p\,, && \end{aligned} \tag{3.5}$$

where k and q are the loop momenta in our diagrams and p is the external momentum. These definitions are consistent with [18, 19, 16, 17] from which we will be taking various integration results and relations. With these definitions at hand we proceed to define the T-integrals as follows:

$$\tilde{T}_{i_1 i_2 \dots i_r}(p^2; m_{j_1}^2, m_{j_2}^2, \dots, m_{j_r}^2) = \int \frac{d^n k}{(2\pi)^n} \frac{d^n q}{(2\pi)^n} \frac{1}{(k_{i_1}^2 - m_{j_1}^2)(k_{i_2}^2 - m_{j_2}^2)\dots(k_{i_r}^2 - m_{j_r}^2)} \tag{3.6}$$

The indices are used to indicate the momenta that appear in the T-integral. Here are a few examples to clarify the notation:

$$\begin{aligned}
\tilde{T}_{123}(p^2; m_a^2, m_b^2, m_c^2) &= \int \frac{d^n k}{(2\pi)^n} \frac{d^n q}{(2\pi)^n} \frac{1}{(k_1^2 - m_a^2)(k_2^2 - m_b^2)(k_3^2 - m_c^2)} \\
&= \int \frac{d^n k}{(2\pi)^n} \frac{d^n q}{(2\pi)^n} \frac{1}{(k^2 - m_a^2)\left[(k+p)^2 - m_b^2\right]\left[(q-k)^2 - m_c^2\right]} \\
\tilde{T}_{1123}(p^2; m_a^2, m_b^2, m_c^2, m_d^2) &= \int \frac{d^n k}{(2\pi)^n} \frac{d^n q}{(2\pi)^n} \frac{1}{(k_1^2 - m_a^2)(k_1^2 - m_b^2)(k_2^2 - m_c^2)(k_3^2 - m_d^2)}
\end{aligned}$$

We will often shorten the notation by omitting the masses and external momentum unless confusion may arise. For example, we will write:

$$\tilde{T}_{12345} = \int \frac{d^n k}{(2\pi)^n} \frac{d^n q}{(2\pi)^n} \frac{1}{(k_1^2 - m_1^2)(k_2^2 - m_2^2)(k_3^2 - m_3^2)(k_4^2 - m_4^2)(k_5^2 - m_5^2)}$$

Massless propagators will be denoted by primed indices, for example:

$$\tilde{T}_{1'234'5} = \int \frac{d^n k}{(2\pi)^n} \frac{d^n q}{(2\pi)^n} \frac{1}{k_1^2 (k_2^2 - m_2^2)(k_3^2 - m_3^2) k_4^2 (k_5^2 - m_5^2)}$$

In the reduction of Feynman integrals, we will come across Y-integrals. That is, integrals of the form:

$$Y_{i_1 i_2 \dots i_r}^{l_1 l_2 \dots l_s}(p^2; m_{j_1}^2, m_{j_2}^2, \dots, m_{j_r}^2) = \int \frac{d^n k}{(2\pi)^n} \frac{d^n q}{(2\pi)^n} \frac{k_{l_1}^2 k_{l_1}^2 \dots k_{l_s}^2}{(k_{i_1}^2 - m_{j_1}^2)(k_{i_2}^2 - m_{j_2}^2)\dots(k_{i_r}^2 - m_{j_r}^2)}$$

We will again omit the masses in this notation, for example:

$$Y_{234}^1 = \int \frac{d^n k}{(2\pi)^n} \frac{d^n q}{(2\pi)^n} \frac{k_1^2}{(k_2^2 - m_2^2)(k_3^2 - m_3^2)(k_4^2 - m_4^2)}$$

Note that the numerator can't be cancelled against the denominator by completing the square. However, the Y-integrals can, as will be shown later, be reduced to T-integrals. So the T-integrals are all we need for two-loop corrections to the propagator.

In order to adapt this notation to the KLZ model, we have to talk about masses of rho-mesons and pions. In the sequel it will be understood that:

$$m_1 = m_2 = m_4 = m_5 = m_\pi \quad \text{and} \quad m_3 = m_\rho \,. \tag{3.7}$$

For example:

$$\tilde{T}_{234} = \int \frac{d^n k}{(2\pi)^n} \frac{d^n q}{(2\pi)^n} \frac{1}{\left(k_2^2 - m_\pi^2\right)\left(k_3^2 - m_\rho^2\right)\left(k_4^2 - m_\pi^2\right)}$$

It can also be seen that the T and Y-integrals are invariant under the following permutations of indices [17]:

$$(1 \longleftrightarrow 2)(4 \longleftrightarrow 5)\,, \qquad (2 \longleftrightarrow 4)(1 \longleftrightarrow 5)\,, \qquad (1 \longleftrightarrow 4)(2 \longleftrightarrow 5) \tag{3.8}$$

Lastly, the following notation will help make for less cumbersome algebra:

$$\begin{aligned} D_1 &= \left(k_1^2 - m_\pi^2\right) & D_2 &= \left(k_2^2 - m_\pi^2\right) & D_3 &= \left(k_3^2 - m_\rho^2\right) \\ D_4 &= \left(k_4^2 - m_\pi^2\right) & D_5 &= \left(k_5^2 - m_\pi^2\right). \end{aligned} \tag{3.9}$$

With this notation in hand, we proceed to calculate the rho-self energy. Note that in what follows we are only interested in two things :

1. Writing the longitudinal parts of each diagram in terms of T and Y-integrals.

2. Contracting each diagram with $g_{\mu\nu}$ and expressing the result in terms of T and Y-integrals.

This means we will be contracting each diagram with the tensors $\frac{p^\mu p^\nu}{p^2}$ and $g_{\mu\nu}$. Only once these contractions have been done for all diagrams will we demonstrate the transversal nature of the self-energy and, subsequently, compute the regularized two-loop self energy of the rho-meson.

3.2 Reduction

Of the seven diagrams that appear in figure 3.1, two (i.e. diagrams ξ and ζ) are reducible to one-loop integrals. We begin by calculating diagram ξ, for which the Feynman rules yield:

$$\begin{aligned}
\xi^{\mu\nu} &= -4ig^4 n \int \frac{d^n k}{(2\pi)^n} \frac{d^n q}{(2\pi)^n} \frac{g^{\mu\nu}}{\left(k_1^2 - m_\pi^2\right)^2 \left(k_3^2 - m_\rho^2\right)} \\
&= -4ig^4 n \tilde{T}_{113} g^{\mu\nu} \\
\implies g_{\mu\nu}\xi^{\mu\nu} &= -4ig^4 n^2 \tilde{T}_{113} \qquad (3.10) \\
\frac{p^\mu p^\nu}{p^2}\xi_{\mu\nu} &= -4ig^4 n \tilde{T}_{113}\,. \qquad (3.11)
\end{aligned}$$

For the second one-loop-reducible diagram, ζ, the Feynman rules give the following integral:

$$\zeta^{\mu\nu} = 2ig^4 n \int \frac{d^n k}{(2\pi)^n} \frac{d^n q}{(2\pi)^n} \frac{(k_1+k_2)^\mu (k_1+k_2)^\nu}{\left(k_1^2 - m_\pi^2\right)^2 \left(k_2^2 - m_\pi^2\right)\left(k_3^2 - m_\rho^2\right)}$$

We project using the tensors $g^{\mu\nu}$ and $\frac{p^\mu p^\nu}{p^2}$:

$$\frac{p^\mu p^\nu}{p^2}\zeta_{\mu\nu} = 2ig^4 n \int \frac{d^n k}{(2\pi)^n} \frac{d^n q}{(2\pi)^n} \frac{\left[p\cdot(k_1+k_2)\right]^2}{p^2\left(k_1^2 - m_\pi^2\right)^2 \left(k_2^2 - m_\pi^2\right)\left(k_3^2 - m_\rho^2\right)} \qquad (3.12)$$

Focusing on the numerator above we see the following:

$$\begin{aligned}
\left[p\cdot(k_1+k_2)\right]^2 &= \left[p\cdot(2k+p)\right]^2 \\
&= (k_1^2 - k_2^2)^2 \\
&= (D_1 - D_2)^2 \\
&= D_1^2 - 2D_1 D_2 + D_2^2 \qquad (3.13)
\end{aligned}$$

Putting this into equation (3.12) gives:

$$\begin{aligned}
\frac{p^\mu p^\nu}{p^2}\zeta_{\mu\nu} &= 2ig^4 n \int \frac{d^n k}{(2\pi)^n}\frac{d^n q}{(2\pi)^n}\frac{D_1^2 - 2D_1 D_2 + D_2^2}{p^2 D_1^2 D_2 D_3} \\
&= 2ig^4 n \left[\frac{1}{p^2}\tilde{T}_{23} - \frac{2}{p^2}\tilde{T}_{13} + \int \frac{d^n k}{(2\pi)^n}\frac{d^n q}{(2\pi)^n}\frac{D_2}{p^2 D_1^2 D_3}\right] \\
&= 2ig^4 n \left[-\frac{1}{p^2}\tilde{T}_{13} + \int \frac{d^n k}{(2\pi)^n}\frac{d^n q}{(2\pi)^n}\frac{\left(k_2^2 - m_\pi^2\right)}{p^2 D_1^2 D_3}\right] \\
&= 2ig^4 n \left[-\frac{1}{p^2}\tilde{T}_{13} + \int \frac{d^n k}{(2\pi)^n}\frac{d^n q}{(2\pi)^n}\frac{D_1 + 2p\cdot k + p^2}{p^2 D_1^2 D_3}\right] \\
&= 2ig^4 n \left[-\frac{1}{p^2}\tilde{T}_{13} + \frac{1}{p^2}\tilde{T}_{13} + \int \frac{d^n k}{(2\pi)^n}\frac{d^n q}{(2\pi)^n}\frac{2p\cdot k}{p^2 D_1^2 D_3} + \tilde{T}_{113}\right] \\
&= 2ig^4 n \tilde{T}_{113}
\end{aligned} \tag{3.14}$$

In the third line we used a variable shift to give $\tilde{T}_{13} = \tilde{T}_{23}$ and in the second last line the integrand that appears is odd and therefore the integral vanishes.

We also have:

$$\begin{aligned}
g_{\mu\nu}\zeta^{\mu\nu} &= 2ig^4 n \int \frac{d^n k}{(2\pi)^n}\frac{d^n q}{(2\pi)^n}\frac{(k_1+k_2)^2}{\left(k_1^2 - m_\pi^2\right)^2\left(k_2^2 - m_\pi^2\right)\left(k_3^2 - m_\rho^2\right)} \\
&= 2ig^4 n \int \frac{d^n k}{(2\pi)^n}\frac{d^n q}{(2\pi)^n}\frac{2k_1^2 + 2k_2^2 - p^2}{\left(k_1^2 - m_\pi^2\right)^2\left(k_2^2 - m_\pi^2\right)\left(k_3^2 - m_\rho^2\right)} \\
&= 2ig^4 n \int \frac{d^n k}{(2\pi)^n}\frac{d^n q}{(2\pi)^n}\frac{2D_1^2 + 2D_2^2 + 4m_\pi^2 - p^2}{\left(k_1^2 - m_\pi^2\right)^2\left(k_2^2 - m_\pi^2\right)\left(k_3^2 - m_\rho^2\right)} \\
&= 2ig^4 n \left[2\tilde{T}_{123} + 2\tilde{T}_{113} + (4m_\pi^2 - p^2)\tilde{T}_{1123}\right]
\end{aligned} \tag{3.15}$$

The above expression can be reduced using integration by parts relations as follows:

$$\int \frac{d^n k}{(2\pi)^n}\frac{d^n q}{(2\pi)^n}\frac{\partial}{\partial k_\mu}\frac{k_\mu}{\left(k_1^2 - m_\pi^2\right)\left(k_2^2 - m_\pi^2\right)\left(k_3^2 - m_\rho^2\right)} = 0 \tag{3.16}$$

The above relation is a well known property of dimensionally regularised integrals. Carrying out the differentiation of the integrand yields:

$$
\begin{aligned}
0 &= \int \frac{d^n k}{(2\pi)^n} \frac{d^n q}{(2\pi)^n} \left\{ \frac{n}{D_1 D_2 D_3} - \frac{2k_1^2}{D_1^2 D_2 D_3} - \frac{2k_1 \cdot k_2}{D_1^2 D_2 D_3} \right\} \\
\implies 0 &= (n-3)\tilde{T}_{123} - (4m_\pi^2 - p^2)\tilde{T}_{1123} - \tilde{T}_{113} \\
\therefore (4m_\pi^2 - p^2)\tilde{T}_{1123} &= (n-3)\tilde{T}_{123} - \tilde{T}_{113}. \qquad (3.17)
\end{aligned}
$$

Putting (3.17) into (3.15) yields:

$$
g_{\mu\nu}\zeta^{\mu\nu} = 2ig^4 n \left[(n-1)\tilde{T}_{123} + \tilde{T}_{113} \right]. \qquad (3.18)
$$

The remaining diagrams are pure two loop and we will evaluate them in what follows. The sunset diagram, W, is easily seen to reduce to the following:

$$
\begin{aligned}
W^{\mu\nu} &= -4ig^4 \int \frac{d^n k}{(2\pi)^n} \frac{d^n q}{(2\pi)^n} \frac{g^{\mu\nu}}{\left(k_2^2 - m_\pi^2\right)\left(k_3^2 - m_\rho^2\right)\left(k_4^2 - m_\pi^2\right)} \\
&= -4ig^4 \tilde{T}_{234} g^{\mu\nu} \\
\implies \frac{p^\mu p^\nu}{p^2} W_{\mu\nu} &= -4ig^4 \tilde{T}_{234}. \qquad (3.19)
\end{aligned}
$$

Contracting with the metric tensor we get

$$
g_{\mu\nu} W^{\mu\nu} = -4ig^4 n \tilde{T}_{234} \qquad (3.20)
$$

Evaluating diagram Ω, we get the following :

$$
\Omega^{\mu\nu} = 2ig^4 g^{\mu\nu} \int \frac{d^n k}{(2\pi)^n} \frac{d^n q}{(2\pi)^n} \frac{(k_1 + k_4)^2}{\left(k_1^2 - m_\pi^2\right)^2 \left(k_3^2 - m_\rho^2\right)\left(k_4^2 - m_\pi^2\right)}. \qquad (3.21)
$$

The numerator can be written as

$$
\begin{aligned}
(k_1 + k_4)^2 &= k_1^2 + k_4^2 + k_1^2 - k_3^2 + k_4^2 \\
&= 2D_1 - D_3 + 2D_4 + 4m_\pi^2 - m_\rho^2.
\end{aligned}
$$

So that (3.21) becomes:

$$
\begin{aligned}
\Omega^{\mu\nu} &= 2ig^4 g^{\mu\nu} \int \frac{d^n k}{(2\pi)^n} \frac{d^n q}{(2\pi)^n} \frac{2D_1 - D_3 + 2D_4 + 4m_\pi^2 - m_\rho^2}{D_1^2 D_3 D_4} \\
&= 2ig^4 g^{\mu\nu} \left[2\tilde{T}_{134} - \tilde{T}_{114} + 2\tilde{T}_{113} + (4m_\pi^2 - m_\rho^2)\tilde{T}_{1134} \right]. \qquad (3.22)
\end{aligned}
$$

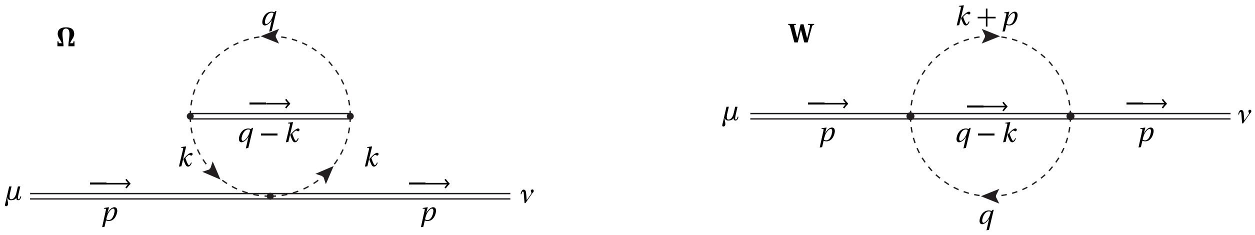


Fig. 3.2: Diagrams Ω and W

So the $\frac{p^\mu p^\nu}{p^2}$ and $g_{\mu\nu}$ projections are

$$\frac{p^\mu p^\nu}{p^2}\Omega_{\mu\nu} = 2ig^4\left[2\tilde{T}_{134} - \tilde{T}_{114} + 2\tilde{T}_{113} + (4m_\pi^2 - m_\rho^2)\tilde{T}_{1134}\right]. \tag{3.23}$$

$$g_{\mu\nu}\Omega^{\mu\nu} = 2ig^4 n\left[2\tilde{T}_{134} - \tilde{T}_{114} + 2\tilde{T}_{113} + (4m_\pi^2 - m_\rho^2)\tilde{T}_{1134}\right] \tag{3.24}$$

The same reduction procedure as above can be carried out for diagrams Z, A and X yielding:

$$\frac{p^\mu p^\nu}{p^2}X_{\mu\nu} = \frac{p^\mu p^\nu}{p^2}\left(2ig^4\int\frac{d^n k}{(2\pi)^n}\frac{d^n q}{(2\pi)^n}\frac{(k_1+k_4)_\mu(k_1+k_2)_\nu}{\left(k_1^2-m_\pi^2\right)\left(k_2^2-m_\pi^2\right)\left(k_3^2-m_\rho^2\right)\left(k_4^2-m_\pi^2\right)}\right)$$
$$= 2ig^4\tilde{T}_{234} \tag{3.25}$$

$$g_{\mu\nu}X^{\mu\nu} = ig^4\left[5\tilde{T}_{234} + \tilde{T}_{134} - 2\tilde{T}_{124} + \tilde{T}_{123} + Y^1_{2345} + (7m_\pi^2 - 2m_\rho^2 - 2p^2)\tilde{T}_{1234}\right], \tag{3.26}$$

$$\frac{p^\mu p^\nu}{p^2}A_{\mu\nu} = \frac{p^\mu p^\nu}{p^2}\left(-ig^4\int\frac{d^n k}{(2\pi)^n}\frac{d^n q}{(2\pi)^n}\frac{(k_1+k_2)_\mu(k_4+k_5)_\nu(k_1+k_4)\cdot(k_2+k_5)}{\left(k_1^2-m_\pi^2\right)\left(k_2^2-m_\pi^2\right)\left(k_3^2-m_\rho^2\right)\left(k_4^2-m_\pi^2\right)\left(k_5^2-m_\pi^2\right)}\right)$$
$$= ig^4\left[-\frac{4}{p^2}\int\frac{d^n k}{(2\pi)^n}\frac{d^n q}{(2\pi)^n}\frac{2p\cdot k_2}{D_2D_3D_4} + \frac{2\left(4m_\pi^2-m_\rho^2\right)}{p^2}\left(\tilde{T}_{234}-\tilde{T}_{134}\right)\right] \tag{3.27}$$

$$g_{\mu\nu}A^{\mu\nu} = ig^4\left[-4\tilde{T}_{234} - 4\tilde{T}_{134} - 4\tilde{T}_{123} + 8\tilde{T}_{124} - 4Y^1_{2345} - 4(7m_\pi^2 - 3m_\rho^2 - 3p^2)\tilde{T}_{1234}\right.$$
$$\left. + (8m_\pi^2 - 2m_\rho^2 - 4p^2)\tilde{T}_{1245} - \left(4m_\pi^2 - 2m_\rho^2 - p^2\right)\left(4m_\pi^2 - m_\rho^2 - 2p^2\right)\tilde{T}_{12345}\right]. \tag{3.28}$$

We also have

$$\frac{p^\mu p^\nu}{p^2} Z_{\mu\nu} = \frac{p^\mu p^\nu}{p^2} \left(-i g^4 \int \frac{d^n k}{(2\pi)^n} \frac{d^n q}{(2\pi)^n} \frac{(k_1+k_2)^\mu (k_1+k_2)^\nu (k_1+k_4)^2}{\left(k_1^2 - m_\pi^2\right)^2 \left(k_2^2 - m_\pi^2\right) \left(k_3^2 - m_\rho^2\right) \left(k_4^2 - m_\pi^2\right)} \right)$$
$$= i g^4 \left[\frac{2}{p^2} \int \frac{d^n k}{(2\pi)^n} \frac{d^n q}{(2\pi)^n} \frac{2p \cdot k_2}{D_2 D_3 D_4} - \frac{(4m_\pi^2 - m_\rho^2)}{p^2} \left(\tilde{T}_{234} - \tilde{T}_{134} \right) - 2\tilde{T}_{234} - 2\tilde{T}_{134} \right.$$
$$\left. - 2\tilde{T}_{113} + \tilde{T}_{114} - (4m_\pi^2 - m_\rho^2) \tilde{T}_{1134} \right] \tag{3.29}$$
$$g_{\mu\nu} Z^{\mu\nu} = i g^4 \left[-4\tilde{T}_{234} - 2(8m_\pi^2 - m_\rho^2 - p^2)\tilde{T}_{1234} - 4\tilde{T}_{134} - 2\left((n-1)\tilde{T}_{123} + \tilde{T}_{113}\right) \right.$$
$$\left. + \left((n-1)\tilde{T}_{124} + \tilde{T}_{114}\right) - 2(4m_\pi^2 - m_\rho^2)\tilde{T}_{1134} - (4m_\pi^2 - p^2)(4m_\pi^2 - m_\rho^2)\tilde{T}_{11234} \right]. \tag{3.30}$$

The algebra required to get the above results is lengthy and so the full calculation has been relegated to the appendix. Now that the reduction of each diagram is complete, we are in a position to verify that $\Pi_2^{\mu\nu}(p^2)$ is transverse. From equation 3.1 we have:

$$\frac{p^\mu p^\nu}{p^2} \Pi_{2\mu\nu}(p^2) = \frac{1}{2} \frac{p^\mu p^\nu}{p^2} \xi_{\mu\nu} + \frac{p^\mu p^\nu}{p^2} \zeta_{\mu\nu} + \frac{p^\mu p^\nu}{p^2} \Omega_{\mu\nu} + \frac{p^\mu p^\nu}{p^2} W_{\mu\nu}$$
$$+ 4\frac{p^\mu p^\nu}{p^2} X_{\mu\nu} + 2\frac{p^\mu p^\nu}{p^2} Z_{\mu\nu} + \frac{p^\mu p^\nu}{p^2} A_{\mu\nu} \tag{3.31}$$

This gives a vanishing result. This follows immediately from making substitutions of equations (3.11),(3.14),(3.23),(3.19),(3.25),(3.27) and (3.29) into the right hand side of the first line above. Thus we see that $\Pi_2^{\mu\nu}(p^2)$ is indeed transverse. As it was discussed above, object of interest becomes the scalar quantity $\Pi_2(p^2)$ in equation (3.4). Using (3.1) we get the following:

$$\Pi_2(p^2) = \frac{1}{n-1} \left\{ \frac{1}{2} g_{\mu\nu} \xi^{\mu\nu} + g_{\mu\nu} \zeta^{\mu\nu} + g_{\mu\nu} \Omega^{\mu\nu} \right.$$
$$\left. + g_{\mu\nu} W^{\mu\nu} + 4 g_{\mu\nu} X^{\mu\nu} + 2 g_{\mu\nu} Z^{\mu\nu} + g_{\mu\nu} A^{\mu\nu} \right\}. \tag{3.32}$$

Substituting (3.10),(3.18),(3.24),(3.20),(3.26),(3.28) and (3.30) into the equation above gives:

$$\Pi_2(p^2) = \frac{ig^4}{n-1}\{2(n-2)\left[2\tilde{T}_{134} - 2\tilde{T}_{234} + (4m_\pi^2 - m_\rho^2)\tilde{T}_{1134} + (n-1)\left(\tilde{T}_{123} - \tilde{T}_{113}\right)\right]$$
$$+ 2(n-1)\left[\tilde{T}_{124} - \tilde{T}_{114}\right] - 8(4m_\pi^2 - m_\rho^2 - p^2)\tilde{T}_{1234} + 2(4m_\pi^2 - m_\rho^2 - p^2)\tilde{T}_{1245}$$
$$- 2(4m_\pi^2 - p^2)(4m_\pi^2 - m_\rho^2)\tilde{T}_{11234} - (4m_\pi^2 - 2m_\rho^2 - p^2)(4m_\pi^2 - m_\rho^2 - 2p^2)\tilde{T}_{12345}\} \tag{3.33}$$

With this result the problem has been reduced to finding scalar integrals. In the next section we evaluate the scalar integrals that appear in the above equation.

3.3 Evaluation of T-Integrals

We will be using results on T-integrals from [16, 19], and so we must make our notation consistent with that of the those references. The T-integrals in references [16, 19] are defined using the integration measure $\mathscr{D}^n k = \dfrac{d^n k}{(2\pi\mu)^{n-4} i\pi^2}$ instead of the measure we have been using so far, $\dfrac{d^n k}{(2\pi)^n}$.We write down, explicitly, the scale μ using (2.9) and then we relate the above-mentioned measures as follows

$$\begin{aligned} g^2 \frac{d^n k}{(2\pi)^n} &\to g_{\rho\pi\pi}^2 \mu^{4-n} \frac{d^n k}{(2\pi)^n} \\ &= \frac{i g_{\rho\pi\pi}^2}{(4\pi)^2} \frac{d^n k}{(2\pi\mu)^{n-4} i\pi^2} \\ &= \frac{i g_{\rho\pi\pi}^2}{(4\pi)^2} \mathscr{D}^n k \end{aligned}$$

There are two factors of integration measure in our two-loop T-integrals, therefore the following replacement may be made

$$\begin{aligned} g^4 \tilde{T}_{abc} &\to \left(\frac{i g_{\rho\pi\pi}^2}{(4\pi)^2}\right)^2 T_{abc} \\ &= -\left(\frac{g_{\rho\pi\pi}}{4\pi}\right)^4 T_{abc}. \end{aligned}$$

This allows us to write (3.33) in the following way:

$$\Pi_2(p^2) = -\left(\frac{g_{\rho\pi\pi}}{4\pi}\right)^4 \frac{i}{n-1}\{2(n-2)\left[2T_{134} - 2T_{234} + (4m_\pi^2 - m_\rho^2)T_{1134} + (n-1)(T_{123} - T_{113})\right]$$
$$+ 2(n-1)[T_{124} - T_{114}] - 8(4m_\pi^2 - m_\rho^2 - p^2)T_{1234} + 2(4m_\pi^2 - m_\rho^2 - p^2)T_{1245}$$
$$- 2(4m_\pi^2 - p^2)(4m_\pi^2 - m_\rho^2)T_{11234} - (4m_\pi^2 - 2m_\rho^2 - p^2)(4m_\pi^2 - m_\rho^2 - 2p^2)T_{12345}\}. \tag{3.34}$$

The integrals $T_{123}, T_{113}, T_{124}, T_{114}$ and T_{1245} that appear above may be written as products of one loop integrals. For example:

$$T_{123} = \int \frac{d^n k}{i\pi^2 (2\pi\mu)^{n-4}} \frac{d^n q}{i\pi^2 (2\pi\mu)^{n-4}} \frac{1}{\left(k_1^2 - m_\pi^2\right)\left(k_2^2 - m_\pi^2\right)\left(k_3^2 - m_\rho^2\right)}$$
$$= \int \frac{d^n k}{(2\pi\mu)^{n-4} i\pi^2} \frac{1}{(k^2 - m_\pi^2)[(k+p)^2 - m_\pi^2]} \int \frac{d^n q}{(2\pi\mu)^{n-4} i\pi^2} \frac{1}{(q-k)^2 - m_\rho^2} \qquad q \to q + k$$
$$= \int \frac{d^n k}{(2\pi\mu)^{n-4} i\pi^2} \frac{1}{(k^2 - m_\pi^2)[(k+p)^2 - m_\pi^2]} \int \frac{d^n q}{(2\pi\mu)^{n-4} i\pi^2} \frac{1}{q^2 - m_\rho^2}$$
$$= B_0(p^2, m_\pi^2) A_0(m_\rho^2),$$

where

$$A_0(m^2) = \int \frac{d^n k}{(2\pi\mu)^{n-4} i\pi^2} \frac{1}{k^2 - m^2} \tag{3.35}$$
$$B_0(p^2, m^2) = \int \frac{d^n k}{(2\pi\mu)^{n-4} i\pi^2} \frac{1}{(k^2 - m^2)[(k+p)^2 - m^2]}. \tag{3.36}$$

Order ϵ expressions for integrals A_0 and B_0 can be found in [16, 20] and they have also been written down in the appendix.Similarly, we have

$$T_{124} = B_0(p^2, m_\pi^2) A_0(m_\pi^2) \tag{3.37}$$
$$T_{1245} = B_0^2(p^2, m_\pi^2). \tag{3.38}$$

Consider the following integration by parts relation:

$$0 = \int \frac{d^n k}{(2\pi\mu)^{n-4} i\pi^2} \frac{\partial}{\partial k^\mu} \frac{k^\mu}{k^2 - m^2}$$
$$\Longrightarrow \int \frac{d^n k}{(2\pi\mu)^{n-4} i\pi^2} \frac{1}{(k^2 - m^2)^2} = \frac{n-2}{2m^2} A_0(m^2). \tag{3.39}$$

This means we may write

$$
\begin{aligned}
T_{113} &= \int \frac{d^n k}{i\pi^2 \,(2\pi\mu)^{n-4}} \frac{d^n q}{i\pi^2 \,(2\pi\mu)^{n-4}} \frac{1}{\left(k_1^2 - m_\pi^2\right)^2 \left(k_3^2 - m_\rho^2\right)^2} \\
&= \int \frac{d^n k}{(2\pi\mu)^{n-4} i\pi^2} \frac{1}{\left(k^2 - m_\pi^2\right)^2} \int \frac{d^n q}{(2\pi\mu)^{n-4} i\pi^2} \frac{1}{q^2 - m_\rho^2} \\
&= \frac{n-2}{2m_\pi^2} A_0(m_\pi^2) A_0(m_\rho^2).
\end{aligned} \tag{3.40}
$$

Similarly,

$$
T_{114} = \frac{n-2}{2m_\pi^2} A_0^2(m_\pi^2). \tag{3.41}
$$

Analytical expressions for T_{134} and T_{1134} are well-known [21] but we have put them in the appendix for ease of reference. The remainder of the scalar integrals that appear in equation 3.34 (i.e. T_{234}, T_{1234}, T_{11234} and T_{12345}) have been treated in [19, 22, 16]. We will shortly give an illustration of the methods in [19, 22] used to evaluate these. Integrals T_{112345} and T_{12345} are finite so they may be evaluated numerically.On the other hand T_{234} and T_{1234} are evaluated using the semi-numerical approach of [19]. We treat the case of T_{234} as a way of illustrating the methods of [18, 22].

3.4 A demonstration of the semi-numerical method

The example given in this section can be found in [19]. We include it again here, with a lot of detail fleshed out, for completeness. We consider the scalar sunset diagram with arbitrary masses $T_{234}(p^2; m_2^2, m_3^2, m_4^2)$. The idea is to write down the integral as a sum of two pieces; one that is known analytically, and the other to be evaluated numerically. That is $T = T_A + T_N$. Where T_A has a known analytical representation and T_N is finite, and therefore has no dependence on ϵ from dimensional regularisation. It is useful to note the following:

$$
\begin{aligned}
\frac{1}{k_i^2 - m_i^2} &= \frac{k_i^2}{k_i^2(k_i^2 - m_i^2)} \\
&= \frac{k_i^2 - m_i^2 + m_i^2}{k_i^2(k_i^2 - m_i^2)} \\
&= \frac{1}{k_i^2} + \frac{m_i^2}{k_i^2(k_i^2 - m_i^2)}.
\end{aligned}
$$

This formula may be applied to the integrand in $T_{234}(p^2; m_2^2, m_3^2, m_4^2)$ in the following way

$$
\begin{aligned}
T_{234}(p^2; m_2^2, m_3^2, m_4^2) &= \int \mathscr{D}^n k \mathscr{D}^n q \frac{1}{(k_2^2 - m_1^2)} \left[\left(\frac{1}{k_3^2} + \frac{m_3^2}{k_3^2(k_3^2 - m_3^2)} \right) \left(\frac{1}{k_4^2} + \frac{m_4^2}{k_4^2(k_4^2 - m_4^2)} \right) \right] \\
&= \int \mathscr{D}^n k \mathscr{D}^n q \frac{1}{(k_2^2 - m_1^2)} \left[\frac{1}{k_3^2 k_4^2} + \frac{m_4^2}{k_3^2 k_4^2 (k_4^2 - m_4^2)} + \frac{m_3^2}{k_3^2 k_4^2 (k_3^2 - m_3^2)} + \right. \\
&\quad \left. \frac{m_3^2 m_4^2}{k_3^2 k_4^2 (k_3^2 - m_3^2)(k_4^2 - m_4^2)} \right] \\
&= \int \mathscr{D}^n k \mathscr{D}^n q \frac{1}{(k_2^2 - m_1^2)} \left[\frac{1}{k_3^2 k_4^2} + \frac{m_4^2 - k_4^2 + k_4^2}{k_3^2 k_4^2 (k_4^2 - m_4^2)} + \frac{m_3^2 - k_3^2 + k_3^2}{k_3^2 k_4^2 (k_3^2 - m_3^2)} \right. \\
&\quad \left. + \frac{m_3^2 m_4^2}{k_3^2 k_4^2 (k_3^2 - m_3^2)(k_4^2 - m_4^2)} \right] \\
&= \int \mathscr{D}^n k \mathscr{D}^n q \frac{1}{(k_2^2 - m_1^2)} \left[-\frac{1}{k_3^2 k_4^2} + \frac{1}{k_3^2 (k_4^2 - m_4^2)} + \frac{1}{k_4^2 (k_3^2 - m_3^2)} \right. \\
&\quad \left. + \frac{m_3^2 m_4^2}{k_3^2 k_4^2 (k_3^2 - m_3^2)(k_4^2 - m_4^2)} \right] \\
&= -T_{234}(p^2; m_2^2, 0, 0) + T_{234}(p^2; m_2^2, 0, m_4^2) + T_{234}(p^2; m_2^2, m_3^2, 0) \\
&\quad + T_{23344}(p^2; m_2^2, m_3^2, 0, m_4^2, 0)
\end{aligned}
\tag{3.42}
$$

The first three terms in the expression above are known analytically [16, 19]. The fourth term, by power counting, is convergent in four dimensions and will be treated numerically. The analytical and numerical parts of the sunset diagram are respectively:

$$
T_{234A} = -T_{234}(p^2; m_2^2, 0, 0) + T_{234}(p^2; m_2^2, 0, m_4^2) + T_{234}(p^2; m_2^2, m_3^2, 0) \tag{3.43}
$$

$$
T_{234N} = T_{23344}(p^2; m_2^2, m_3^2, 0, m_4^2, 0) \tag{3.44}
$$

The convergent integral, $T_{23344}(p^2; m_2^2, m_3^2, 0, m_4^2, 0)$ can be given a two-dimensional integral representation[19]. The analytical terms $T_{234}(p^2; m_2^2, 0, m_4^2) + T_{234}(p^2; m_2^2, m_3^2, 0)$ are given in the appendix of [19], and $T_{234}(p^2; m_2^2, 0, 0)$ may be found in [16].

The numerical part of T_{234}

It turns out the numerical piece can be expressed as follows[19]:

$$
T_{234N} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx dy \, -4p^2 \log \left(\frac{(w_2 + w_3 + w_4)(w_2 + \tilde{w}_3 + \tilde{w}_4)}{(w_2 + \tilde{w}_3 + w_4)(w_2 + w_3 + \tilde{w}_4)} \right), \tag{3.45}
$$

where

$$
\begin{aligned}
w_1 &= \left[x^2 - \frac{m_1^2}{p^2} + i\varepsilon\right]^{\frac{1}{2}}, & w_2 &= \left[(x+1)^2 - \frac{m_2^2}{p^2} + i\varepsilon\right]^{\frac{1}{2}}, \\
w_3 &= \left[(x+y)^2 - \frac{m_3^2}{p^2} + i\varepsilon\right]^{\frac{1}{2}}, & w_4 &= \left[y^2 - \frac{m_4^2}{p^2} + i\varepsilon\right]^{\frac{1}{2}}, \\
w_5 &= \left[(y-1)^2 - \frac{m_5^2}{p^2} + i\varepsilon\right]^{\frac{1}{2}}, & \tilde{w}_3 &= \left[(x+y)^2 + i\varepsilon\right]^{\frac{1}{2}}, \\
\tilde{w}_4 &= \left[y^2 + i\varepsilon\right]^{\frac{1}{2}},
\end{aligned} \tag{3.46}
$$

with ε infinitesimal. In order to facilitate numerical integration, we map the above integration region to the a unit square in the first quadrant of the x-y plane. This is done in two steps; the first is to express T_{234N} as an integral over the first quadrant using the simple relation that follows

$$
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\,dy\,F(x,y) = \int_{0}^{\infty}\int_{0}^{\infty} dx\,dy\,\left[F(x,y) + F(-x,y) + F(x,-y) + F(-x,-y)\right]
$$

Then one may change the integration region to a unit square through the map $x(u,v) = \frac{u}{1-u}$, $y(u,v) = \frac{v}{1-v}$. It follows that the numerical part of the sunset diagram is expressible as:

$$
\begin{aligned}
T_{234N} = \int_0^1\int_0^1 du\,dv \frac{-4p^2}{(1-u)^2(1-v)^2} &\{M[x(u,v), y(u,v)] + M[-x(u,v), y(u,v)] \\
&+ M[x(u,v), -y(u,v)] + M[-x(u,v), -y(u,v)]\}
\end{aligned}
$$

$$
M[x,y] = \log\left(\frac{(w_2 + w_3 + w_4)(w_2 + \tilde{w}_3 + \tilde{w}_4)}{(w_2 + \tilde{w}_3 + w_4)(w_2 + w_3 + \tilde{w}_4)}\right) \tag{3.47}
$$

After fixing the values of the parameters m_2, m_3 and m_4 a suitable numerical integration algorithm can be used to determine this T_{234N} as a function of p^2 with arbitrary precision. In order to make a comparison with [19], the masses are given the values $m_2 = 3, m_3 = 5, m_4 = 7$ a table is made on which are shown values of p^2 against values of $\frac{1}{p^2}\text{Im}(T_{234N})$. The numerical integration was done on Mathematica using the package Suave from the CUBA library and the values are compared to those of [19]. There is good agreement with [19] for values of p^2 not close to 15, where convergence of the algorithm was slow.

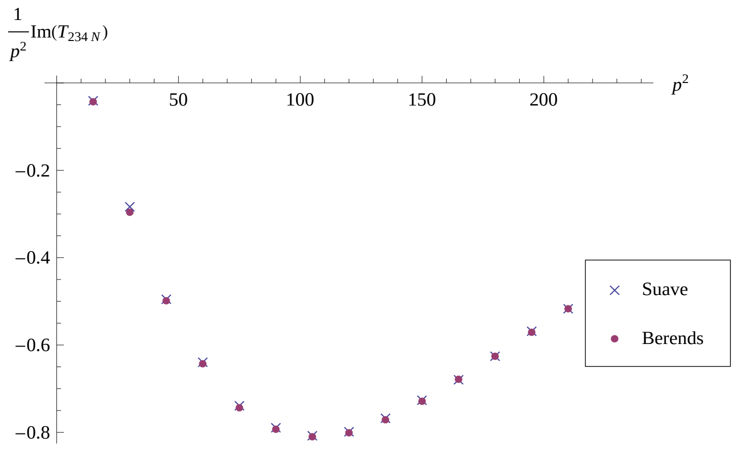


Fig. 3.3: Graph of $\frac{1}{p^2}\mathrm{Im}(T_{234N})$ vs p^2. Results of the Suave integration are compared to Berends and Tausk [19]

Tab. 3.1: $\frac{1}{p^2}\mathrm{Im}T_{234N}$ at different values of p^2. A = Suave integration result, B = Berends and Tausk results [19], C = relative difference between A and B in percentage terms

p^2	A	B	C
15	−0.037674	−0.0417649	9.8
30	−0.280545	−0.2942085	4.6
45	−0.491801	−0.4967213	1
60	−0.636653	−0.6411420	0.7
75	−0.736474	−0.7418943	0.73
90	−0.786393	−0.7915522	0.65
105	−0.804537	−0.8086538	0.51
120	−0.796005	−0.7991393	0.39
135	−0.76496	−0.7691024	0.54
150	−0.723223	−0.7264257	0.44
165	−0.676853	−0.6766883	0.02
180	−0.622943	−0.6235912	0.10
195	−0.565457	−0.5692483	0.67
210	−0.514348	−0.5151574	0.16
225	−0.461229	−0.4619533	0.16
240	−0.413092	−0.4138414	0.18

This concludes the example of the semi-numerical method of [19]. We will use this method to evaluate $\Pi_2(p^2)$.

3.5 Evaluating the analytical and numerical parts of $\Pi_2(p^2)$

$\Pi_2(p^2)$ will be separated into an analytical part and a numerical part using T_{234} and T_{1234}. Consider the following:

$$
\begin{aligned}
T_{234} &= T_{234}(p^2; m_\pi^2, m_\rho^2, m_\pi^2) \\
&= \int \mathscr{D}^n k \mathscr{D}^n q \frac{1}{\left[(k+p)^2 - m_\pi^2\right]\left[(q-k)^2 - m_\rho^2\right](q^2 - m_\pi^2)} \\
&= \int \mathscr{D}^n k \mathscr{D}^n q \frac{1}{\left[(k+p)^2 - m_\rho^2\right]\left[(q-k)^2 - m_\pi^2\right](q^2 - m_\pi^2)} \qquad (k \to -k+q-p) \\
&= T_{234}(p^2; m_\rho^2, m_\pi^2, m_\pi^2).
\end{aligned}
$$

We may then use equations 3.43 and 3.44 with $m_2 = m_\rho$ and $m_3 = m_4 = m_\pi$ to separate T_{234} into a numerical and an analytical piece as follows

$$T_{234A} = 2T_{234'}(p^2; m_\rho^2, m_\pi^2, 0) - T_{23'4'}(p^2; m_\rho^2, 0, 0) \tag{3.48}$$

$$T_{234N} = m_\pi^4 T_{233'44'}(p^2; m_\rho^2, m_\pi^2, 0, m_\pi^2, 0). \tag{3.49}$$

In a similar fashion we can decompose T_{1234} as $T_{1234A} + T_{1234N}$, where

$$T_{1234A} = T_{123'4'}(p^2; m_\pi^2, m_\pi^2, 0, 0) \tag{3.50}$$

$$
\begin{aligned}
T_{1234N} &= m_\rho^2 T_{1233'4'}(p^2; m_\pi^2, m_\pi^2, m_\rho^2, 0, 0) + m_\pi^2 T_{123'44'}(m_\pi^2, m_\pi^2, 0, m_\pi^2, 0) \\
&\quad + m_\pi^2 m_\rho^2 T_{1233'44'}(p^2; m_\pi^2, m_\pi^2, m_\rho^2, 0, m_\pi^2, 0)\,.
\end{aligned} \tag{3.51}
$$

This allows for $\Pi_2(p^2)$ to be separated into an analytical and a numerical piece as follows

$$
\begin{aligned}
\Pi_2(p^2) &= \Pi_A(p^2) + \Pi_N(p^2) \\
\Pi_A(p^2) &= -\left(\frac{g}{4\pi}\right)^4 \frac{i}{n-1}\Big[2(n-2)\Big(2T_{134} - 2T_{234A} + (4m_\pi^2 - m_\rho^2)T_{1134} + (n-1)(T_{123} - T_{113})\Big) \\
&\quad +2(n-1)(T_{124} - T_{114}) + 2\left(4m_\pi^2 - m_\rho^2 - p^2\right)T_{1245} - 8\left(4m_\pi^2 - m_\rho^2 - p^2\right)T_{1234A}\Big]
\end{aligned}
\tag{3.52}
$$

$$
\begin{aligned}
\Pi_N(p^2) &= -\left(\frac{g}{4\pi}\right)^4 \frac{i}{n-1}\Big[-4(n-2)T_{234N} - 8\left(4m_\pi^2 - m_\rho^2 - p^2\right)T_{1234N} \\
&\quad -2\left(4m_\pi^2 - p^2\right)\left(4m_\pi^2 - m_\rho^2\right)T_{11234} - \left(4m_\pi^2 - 2m_\rho^2 - p^2\right)\left(4m_\pi^2 - m_\rho^2 - 2p^2\right)T_{12345}\Big] .
\end{aligned}
\tag{3.53}
$$

3.5.1 The analytical part of $\Pi_2(p^2)$

One gets the following result:

$$
\Pi_A(p^2) = \frac{i g^4}{(4\pi)^4}\left\{T^{(-2)}\frac{1}{\epsilon^2} + T^{(-1)}\frac{1}{\epsilon} + T^{(0)} + O(\epsilon)\right\} ,
$$

where

$$
T^{(-2)} = -\frac{2}{3}(p^2 - 3m_\pi^2 - 6m_\rho^2) \tag{3.54}
$$

$$
\begin{aligned}
T^{(-1)} &= \frac{1}{9p^2}\{-46p^4 + 6p^2 m_\pi^2(21 - (r_1 - r_2)(3\ln r_1 + \ln r_2)) + 3m_\pi^4(r_1 - r_2)(21\ln r_1 + 11\ln r_2) \\
&\quad - 12m_\rho^2\left(3p^2\left(3 + \mathrm{L}_{m_\rho}\right) + m_\pi^2(r_1 - r_2)(3\ln r_1 - \ln r_2)\right) + 12p^2\mathrm{L}_{m_\pi}\left(p^2 - 3m_\pi^2 - 3m_\rho^2\right)\},
\end{aligned}
\tag{3.55}
$$

and the finite piece is given by the following lengthy expression

$$
\begin{aligned}
T^{(0)} &= \frac{1}{27}\Big\{-8(p^2 - 3m_\pi^2 - m_\rho^2) + \frac{6}{p^2}\Big(m_\pi^4(r_1 - r_2)(21\ln r_1 - \ln r_2) + 4p^2\left(p^2 - 3m_\pi^2 - 3m_\rho^2\right) \\
&\quad - 2p^2\left(7p^2 + \left(22 + 6\mathrm{L}_{m_\rho}\right)m_\rho^2\right) - 2m_\pi^2\Big(p^2(-19 + 3(r_1 - r_2)\ln r_1 + (r_1 - r_2)\ln r_2) \\
&\quad + 2m_\rho^2(r_1 - r_2)(\ln r_1 - \ln r_2)\Big)\Big) \\
&\quad + 18\{p^2 + 4m_\pi^2 - 8m_\pi^2\mathrm{L}_{m_\pi}\frac{m_\pi^2}{p^2}\left(2p^2 + 4p^2\mathrm{L}_{m_\pi} + m_\pi^2(r_1 - r_2)(\ln r_1 - \ln r_2)\right) \\
&\quad - \frac{m_\pi^2}{4}\left(-12 - 13\pi^2 + 36\mathrm{L}_{m_\pi} - 42\mathrm{L}_{m_\pi}^2\right) + \frac{1}{6}\left(2 - \mathrm{L}_{m_\pi} + \frac{m_\pi^2}{2p^2}(r_1 - r_2)(\ln r_1 - \ln r_2)\right) \\
&\quad \times\left(2m_\pi^2(1 - \mathrm{L}_{m_\pi}) + m_\rho^2(1 - \mathrm{L}_{m_\rho})\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2}\Bigg[8+\frac{\pi^2}{6}+\mathrm{L}_{m_\pi}^2+\frac{m_\pi^2}{p^2}(r_1-r_2)\left(\mathrm{Li}_2\left(\frac{r_1-1}{r_1-r_2}\right)-\mathrm{Li}_2\left(\frac{(r_1-1)r_2}{r_1-r_2}\right)-\mathrm{Li}_2\left(\frac{r_2-1}{r_2-r_1}\right)\right.\\
&+\mathrm{Li}_2\left(\frac{(r_2-1)r_1}{r_2-r_1}\right)+2(\ln r_1-\ln r_2)+\ln\left(\frac{r_1-1}{r_1-r_2}\right)\ln\left(\frac{(r_1-1)r_2}{r_1-r_2}\right)-\ln\left(\frac{r_2-1}{r_2-r_1}\right)\\
&\left.\times\ln\left(\frac{(r_2-1)r_1}{r_2-r_1}\right)\right)+2\mathrm{L}_{m_\pi}\left(-2-\frac{(r_1-r_2)(\ln r_1-\ln r_2)m_\pi^2}{2p^2}\right)\Bigg]\left(2p^2-11m_\pi^2-4m_\rho^2\right)\\
&+3m_\rho^2+2m_\rho^2\mathrm{L}_{m_\pi}+\frac{5m_\rho^2}{p^2}\left(2p^2+2p^2\mathrm{L}_{m_\pi}+2p^2\mathrm{L}_{m_\rho}+m_\pi^2(r_1-r_2)(\ln r_1-\ln r_2)\right)\\
&-\frac{m_\rho^2}{2}\left(12+\pi^2-12\mathrm{L}_{m_\pi}+6\mathrm{L}_{m_\rho}^2-4(6+\pi^2+3\mathrm{L}_{m_\pi}^2+6(-1+\mathrm{L}_{m_\rho})\mathrm{L}_{m_\pi}-6\mathrm{L}_{m_\rho}+3\mathrm{L}_{m_\rho}^2)\right)\\
&-\left(-p^2+4m_\pi^2-m_\rho^2\right)\left(-2+\mathrm{L}_{m_\pi}-\frac{(r_1-r_2)(\ln r_1-\ln r_2)}{2p^2}\right)^2\\
&-2\Bigg(\mathrm{F}(m_\pi^2,m_\rho^2,m_\pi^2)+14m_\pi^2+\frac{\pi^2}{3}m_\pi^2-12m_\pi^2\mathrm{L}_{m_\pi}+4m_\pi^2\mathrm{L}_{m_\pi}7m_\rho^2+\frac{\pi^2}{6}m_\rho^2-6m_\rho^2\mathrm{L}_{m_\rho}\\
&+2m_\rho^2\mathrm{L}_{m_\rho}^2-m_\rho^2\ln^2\left(\frac{m_\rho^2}{m_\pi^2}\right)+\left(4m_\pi^2-m_\rho^2\right)\left[-\mathrm{L}_{m_\pi}+\mathrm{L}_{m_\pi}^2+\frac{1}{12}\left(6+\pi^2\frac{6\mathrm{F}(m_\pi^2,m_\rho^2,m_\pi^2)}{\lambda_k(m_\pi^2,m_\rho^2,m_\pi^2)}\right)\right]\\
&\frac{1}{4}\Bigg[-13p^2+4p^2\mathrm{L}_{m_\rho}+2m_\rho^2(12+\pi^2)+8m_\rho^2\mathrm{Li}_2\left(\frac{p^2}{m_\rho^2}\right)+8m_\rho^2(-3+\mathrm{L}_{m_\rho})\mathrm{L}_{m_\rho}\\
&+\frac{4}{p^2}(p^4-m_\rho^4)\ln\left(1-\frac{p^2}{m_\rho^2}\right)\\
&-4\Bigg\{p^2\left(-\frac{13}{2}+\ln\left(-\frac{m_\pi^2}{p^2}\right)+\ln\left(-\frac{m_\rho^2}{p^2}\right)\right)+2p^2\mathrm{L}_p+4m_\pi^2\left(-3+\mathrm{L}_{m_\pi}\right)\mathrm{L}_{m_\pi}\\
&+4m_\rho^2(-3+\mathrm{L}_{m_\rho})\mathrm{L}_{m_\rho}+\frac{4m_\rho^2}{p^2}(p^2-m_\pi^2)\left(\mathrm{Li}_2\left(1-\frac{1}{\tilde r_1}\right)+\mathrm{Li}_2\left(1-\frac{1}{\tilde r_2}\right)-\mathrm{Li}_2\left(1-\frac{m_\pi^2}{m_\rho^2}\right)\right)\\
&+\frac{m_\pi^2}{p^2}\left(p^2-m_\rho^2\right)\left(\mathrm{Li}_2(1-\tilde r_1)+\mathrm{Li}_2(1-\tilde r_2)-\mathrm{Li}_2\left(1-\frac{m_\rho^2}{m_\pi^2}\right)\right)\\
&2(m_\pi^2-m_\rho^2)\left(-\mathrm{Li}_2\left(1-\frac{m_\pi^2}{m_\rho^2}\right)+\mathrm{Li}_2\left(1-\frac{m_\rho^2}{m_\pi^2}\right)\right)+\frac{1}{3}\left(m_\pi^2+m_\rho^2\right)\left(36+\pi^2-3\ln^2\left(\frac{m_\rho^2}{m_\pi^2}\right)\right)\\
&\frac{m_\pi^2}{p^2}\left(p^2+m_\pi^2+m_\rho^2\right)(\tilde r_1-\tilde r_2)(\ln\tilde r_2-\ln\tilde r_1)-\frac{m_\pi^4-m_\rho^4}{p^2}\ln\left(\frac{m_\rho^2}{m_\pi^2}\right)\Bigg\}\Bigg]\Bigg)\\
&+(-p^2+4m_\pi^2-m_\rho^2)\Bigg(38+\pi^2+2\mathrm{Li}_2\left(\frac{p^2}{m_\pi^2}\right)+4\mathrm{L}_{m_\pi}^2+4\left(-1+\frac{m_\pi^2}{p^2}\right)\ln\left(1-\frac{p^2}{m_\pi^2}\right)\\
&-4\mathrm{L}_{m_\pi}\left(5+\frac{2m_\pi^2(r_1-r_2)\ln r_1}{p^2}\right)+\frac{2m_\pi^2}{p^2}(r_1-r_2)\\
&\times\Bigg[-\mathrm{Li}_2(1-r_1)-2\mathrm{Li}_2\left(\frac{1}{1+r_1}\right)+\mathrm{Li}_2\left(r_1-r_1^2\right)+\mathrm{Li}_2(1-r_2)+2\mathrm{Li}_2\left(\frac{1}{1+r_2}\right)\\
&-\mathrm{Li}_2\left(r_2-r_2^2\right)+8\ln r_1-\ln^2(1+r_1)+\ln^2(1+r_2)+\eta\left(1-\frac{p^2}{m_\pi^2},r_1\right)\\
&-\ln((1-r_2)r_2)\eta\left(1-\frac{p^2}{m_\pi^2},r_2\right)\Bigg]\Bigg)\Bigg\}\Bigg\}.
\end{aligned}
\tag{3.56}
$$

where

$$\eta(a,b) = \ln(ab) - (\ln(a) + \ln(b))$$
$$= 2\pi i\,\{\theta(-\mathrm{Im}a)\theta(-\mathrm{Im}b)\theta(\mathrm{Im}(ab)) - \theta(\mathrm{Im}a)\theta(\mathrm{Im}b)\theta(\mathrm{Im}(-ab))\}$$

$$L_{m_i} = \gamma_{\mathrm{E}} + \ln\left(\frac{m_i^2}{4\pi\mu^2}\right)$$
$$L_p = \gamma_{\mathrm{E}} + \ln\left(\frac{-p^2}{4\pi\mu^2}\right)$$

and γ_{E} is Euler's constant. We also have r_1 and r_2 representing the roots of the following quadratic equation

$$m_\pi^2\left(r + \frac{1}{r}\right) = 2m_\pi^2 - p^2\,.$$

$\tilde{r}_1$ and $\tilde{r}_2$ are the roots of the equation

$$m_\rho^2 r + \frac{m_\pi^2}{r} = m_\pi^2 + m_\rho^2 - p^2$$

$$\lambda_k(a,b,c) = (a-b-c)^2 - 4bc\,.$$

$$F(m_\pi^2, m_\rho^2, m_\pi^2) = m_\rho^2\,\lambda^2\left(\frac{m_\pi^2}{m_\rho^2}\right)\Phi\left(\frac{m_\pi^2}{m_\rho^2}\right),$$

where

$$\Phi(x) = \frac{1}{\lambda(x)}\left\{\frac{\pi^2}{3} - 2\mathrm{Li}_2\left(\frac{-\lambda(x) - x + 2}{2}\right) - 2\mathrm{Li}_2\left(\frac{x - \lambda(x)}{2}\right)\right.$$
$$\left. + 2\ln\left(\frac{-\lambda(x) - x + 2}{2}\right)\ln\left(\frac{x - \lambda(x)}{2}\right)\right\}$$

and

$$\lambda(x) = \sqrt{x^2 - 4x}\,.$$

We have also used the following notation for the dilogarithm

$$\mathrm{Li}_2(z) = -\int_0^1 dt\,\frac{\ln(1-zt)}{t}.$$

$T^{(0)}$ may also be represented graphically as shown in figures 3.4 and 3.5. In the graphical representation, we arbitrarily set the scale μ equal to 1 GeV.

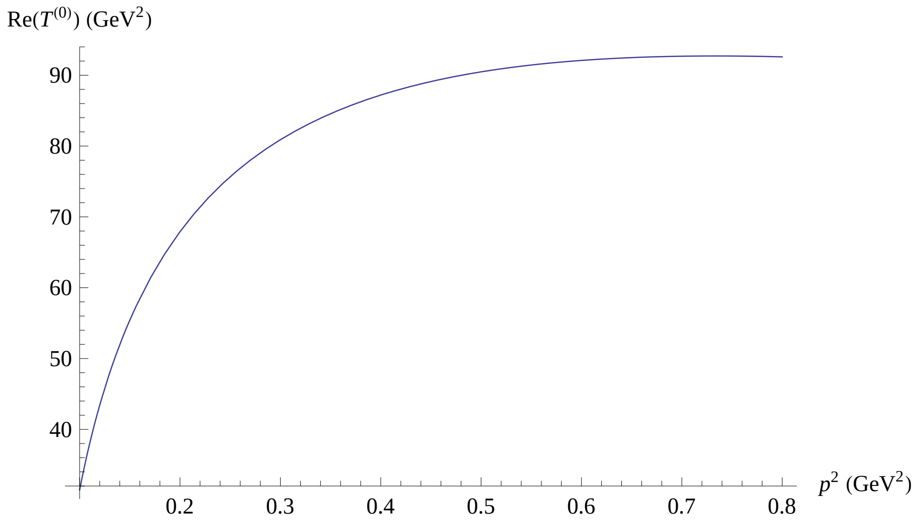


Fig. 3.4: The real part of $T^{(0)}$ as a function of p^2

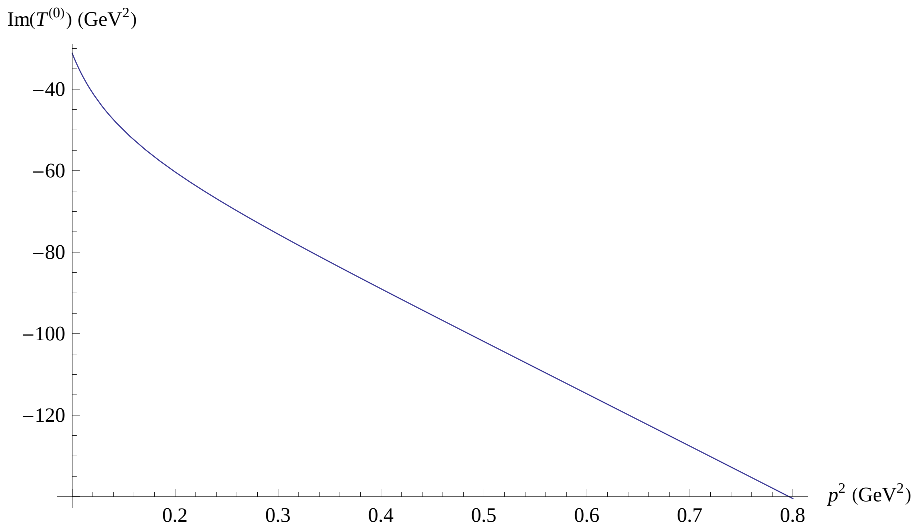


Fig. 3.5: The imaginary part of $T^{(0)}$ as a function of p^2

3.5.2 The numerical part of $\Pi_2(p^2)$

Given below are double integral representations of the T-integrals appearing in $\Pi_N(p^2)$.

$$T_{234N} = -4p^2 \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \log\left(\frac{(w_2+w_3+w_4)(w_2+\tilde{w}_3+\tilde{w}_4)}{(w_2+\tilde{w}_3+w_4)(w_2+w_3+\tilde{w}_4)}\right),$$

$$T_{1234N} = 4\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \frac{1}{w_1^2-w_2^2} \log\left(\frac{(w_1+w_3+w_4)(w_2+\tilde{w}_3+\tilde{w}_4)}{(w_2+w_3+w_4)(w_1+\tilde{w}_3+\tilde{w}_4)}\right), \tag{3.57}$$

$$\begin{aligned} T_{11234} &= \int \frac{d^4k}{i\pi^2}\frac{d^4q}{i\pi^2} \frac{1}{\left(k_1^2-m_\pi^2\right)^2\left(k_2^2-m_\pi^2\right)\left(k_3^2-m_\rho^2\right)\left(k_4^2-m_\pi^2\right)} \\ &= \frac{4}{p^2}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \frac{1}{\left(w_1^2-w_2^2\right)^2}\left\{\log\left(\frac{(w_1+w_3+w_4)(w_2+\tilde{w}_3+\tilde{w}_4)}{(w_2+w_3+w_4)(w_1+\tilde{w}_3+\tilde{w}_4)}\right)\right. \\ &\left. -\frac{(w_1^2-w_2^2)(\tilde{w}_3+\tilde{w}_4-w_3-w_4)}{2w_1(w_1+w_3+w_4)(w_1+\tilde{w}_3+\tilde{w}_4)}\right\} \end{aligned} \tag{3.58}$$

$$T_{12345} = -\frac{4}{p^2}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \frac{1}{w_1^2-w_2^2}\frac{1}{w_4^2-w_5^2} \log\left(\frac{(w_1+w_3+w_4)(w_2+w_3+w_5)}{(w_2+w_3+w_4)(w_1+w_3+w_5)}\right). \tag{3.59}$$

These integral representations are derived in [22, 19]. The w's above are defined in equation 3.46 with $m_1 = m_2 = m_4 = m_5 = m_\pi = 0.13957$ GeV^2 and $m_3 = m_\rho = 0.77549$ GeV^2 for all the above integrals except T_{234N}, for which the values m_2 and m_3 are swapped. These integrals are treated numerically in the same manner as was shown in the earlier example. We give plots of the results for $0.1 \le p^2 \le 1$ GeV^2.

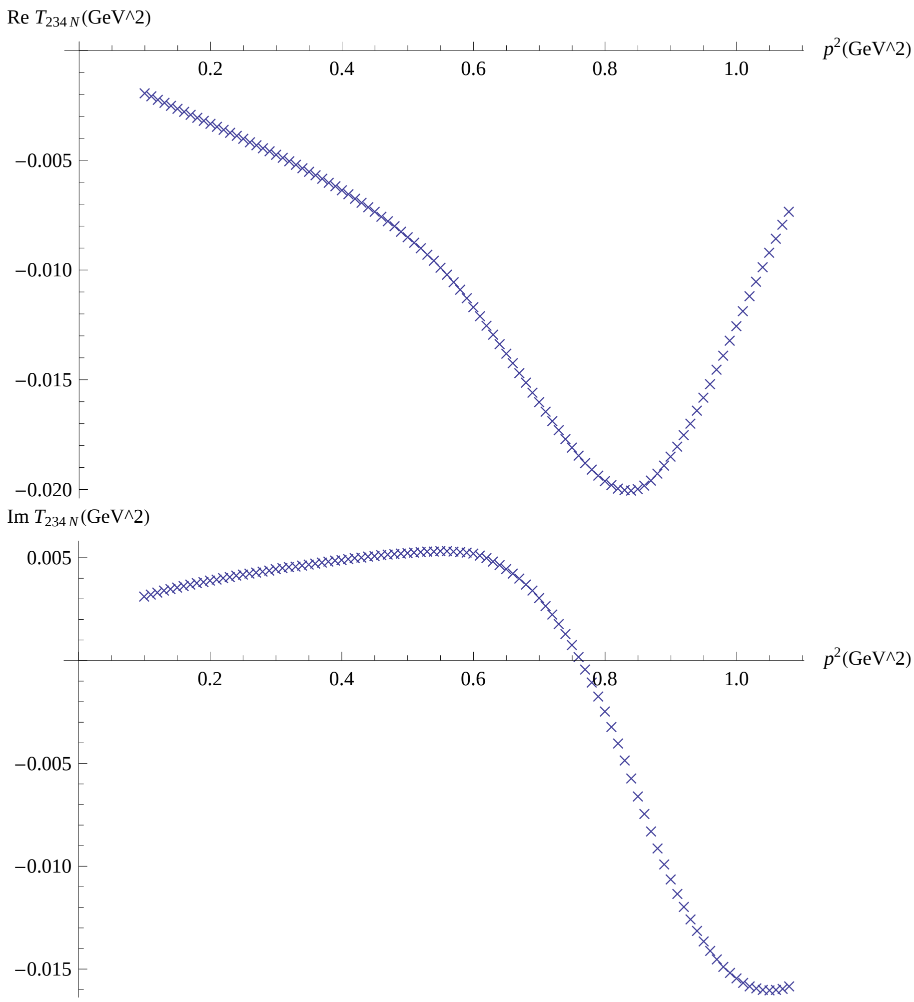


Fig. 3.6: The real and imaginary parts of T_{234N} as functions of p^2

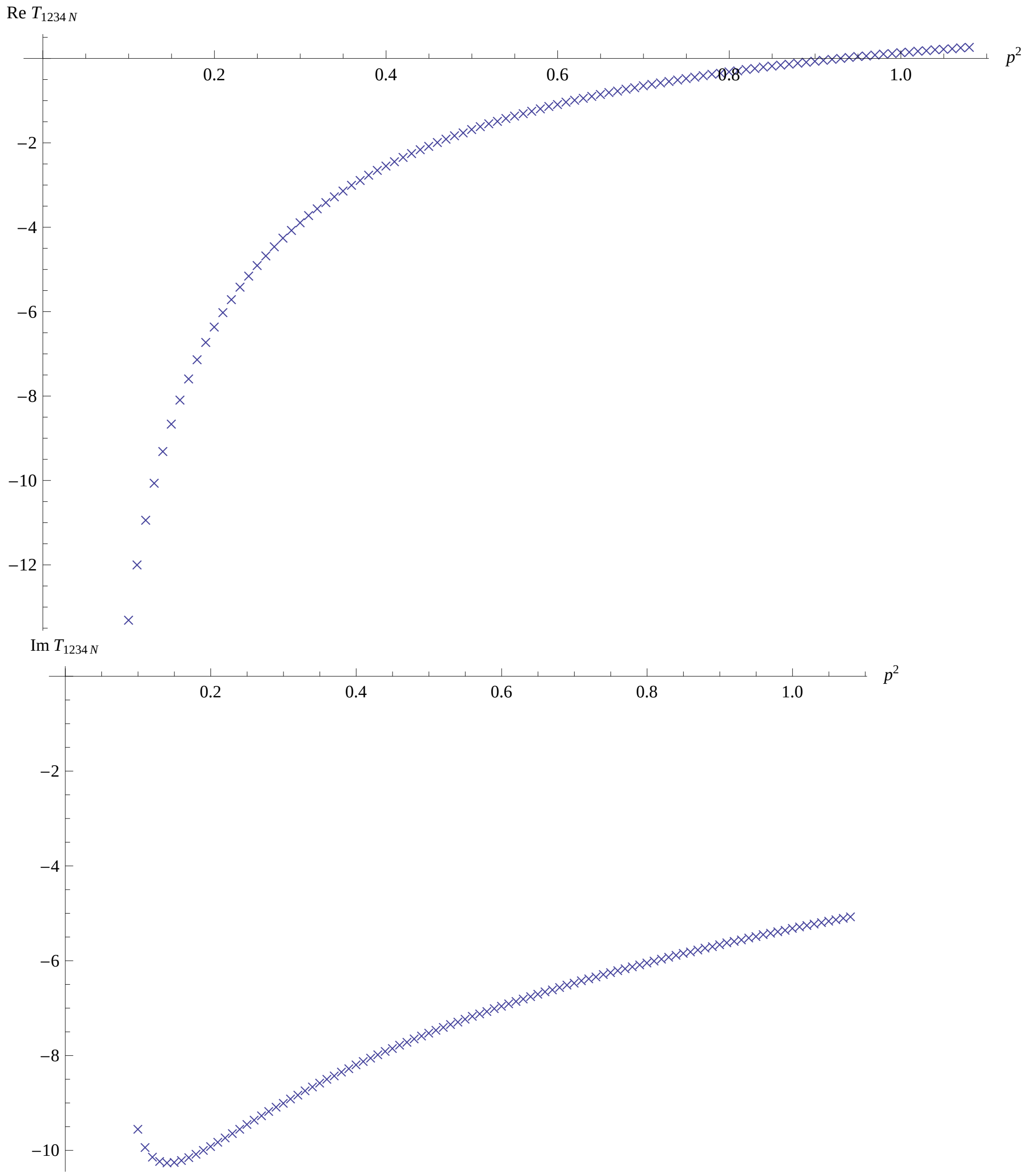


Fig. 3.7: The real and imaginary parts of T_{1234N} as functions of p^2

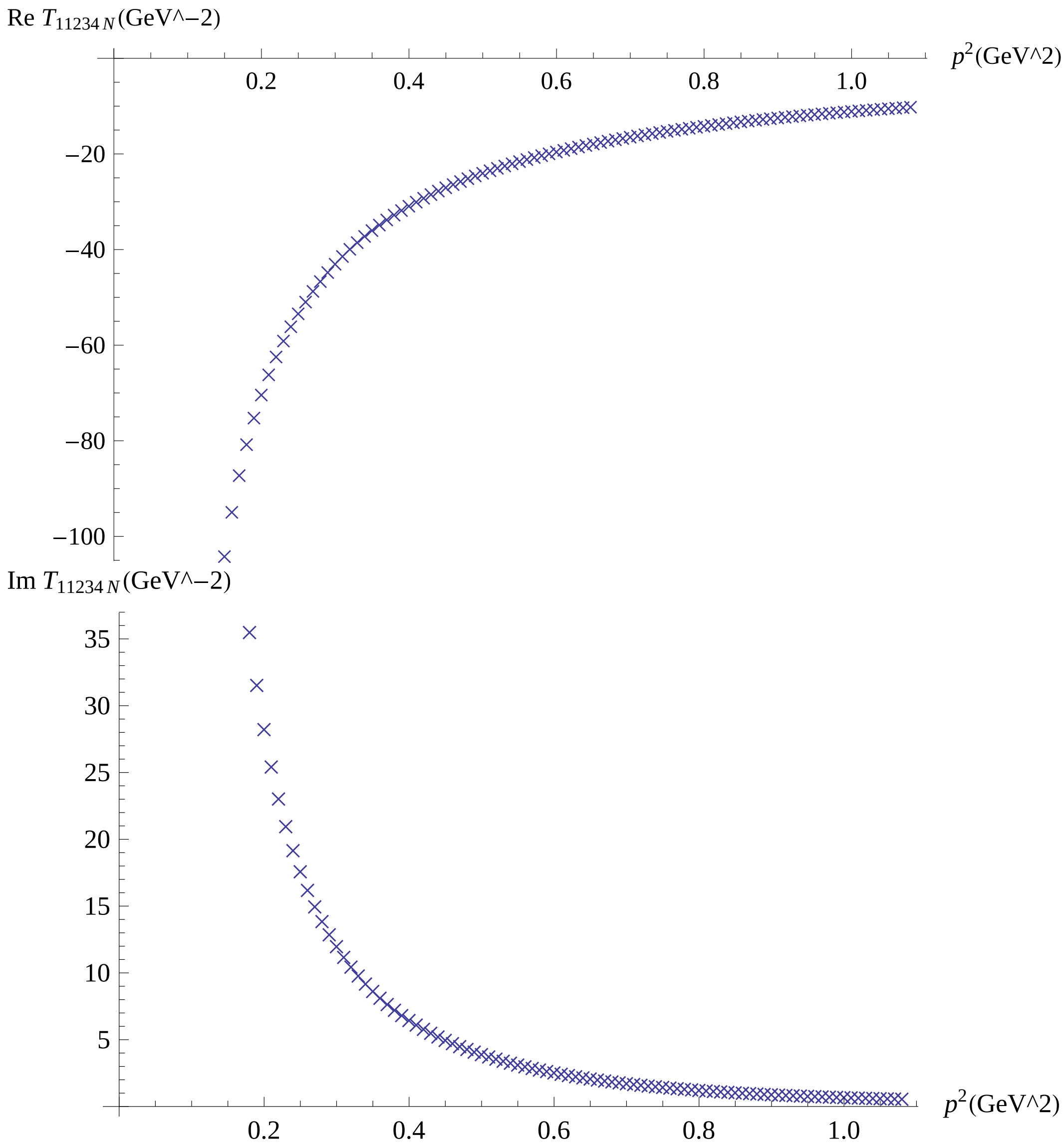


Fig. 3.8: The real and imaginary parts of T_{11234} as functions of p^2

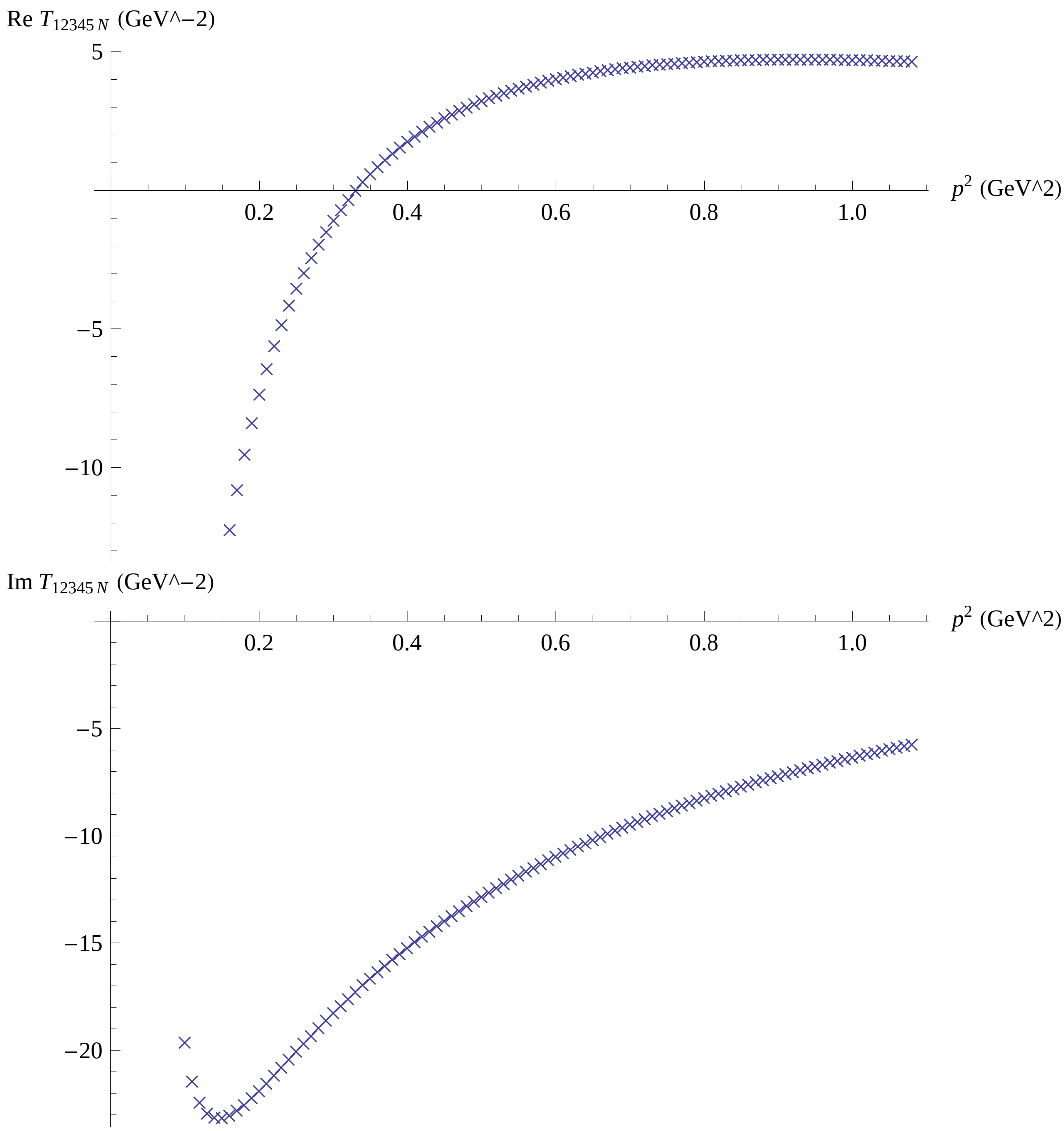


Fig. 3.9: The real and imaginary parts of T_{12345} as functions of p^2

We collect all the above numerical results to get $\Pi_N(p^2)$. We begin by writing $\Pi_N(p^2)$ as follows

$$\Pi_N(p^2) = i\left(\frac{g}{4\pi}\right)^4 F_N(p^2), \tag{3.60}$$

where (3.61)

$$F_N(p^2) = \frac{1}{3}\Big[8T_{234N} + 8(4m_\pi^2 - m_\rho^2 - p^2)T_{1234N} + 2(4m_\pi^2 - m_\rho^2)(4m_\pi^2 - p^2)T_{11234} + (4m_\pi^2 - 2m_\rho^2 - p^2)(4m_\pi^2 - m_\rho^2 - 2p^2)T_{12345}\Big]. \tag{3.62}$$

$F_N(p^2)$ was evaluated numerically and the results are collected in table 3.2 below.

p^2 (GeV^2)	ReF_N (GeV^2)	ImF_N (GeV^2)
0.1	1.35534	3.33582
0.12	0.571648	1.74946
0.14	0.320112	0.586499
0.16	0.292828	-0.330386
0.2	0.553265	-1.71824
0.22	0.769936	-2.26056
0.24	1.02142	-2.72816
0.26	1.29836	-3.13316
0.3	1.9055	-3.78858
0.34	2.55804	-4.27518
0.36	2.89493	-4.46488
0.4	3.5817	-4.75271
0.42	3.92873	-4.85486
0.46	4.62581	-4.98472
0.5	5.32153	-5.02537
0.52	5.66736	-5.01522
0.54	6.01119	-4.98632
0.56	6.35265	-4.93963
0.6	7.02647	-4.79566
0.62	7.35834	-4.70049
0.64	7.68656	-4.59065
0.66	8.01119	-4.46689
0.7	8.64805	-4.17992
0.74	9.26794	-3.84405
0.76	9.57144	-3.65918
0.8	10.1644	-3.25742
0.82	10.4542	-3.04149
0.84	10.7395	-2.81594
0.86	11.0206	-2.58095
0.9	11.5685	-2.08281
0.92	11.836	-1.81968
0.94	12.0977	-1.5482
0.96	12.3544	-1.26835
0.98	12.6055	-0.980132
1.	12.8506	-0.683888
1.02	13.0897	-0.380037
1.04	13.323	-0.0681988
1.06	13.5499	0.249931
1.08	13.7704	0.575117

Tab. 3.2: The numerical part of the self energy at values of p^2 in the range [0.1GeV^2 – 1.08GeV^2]

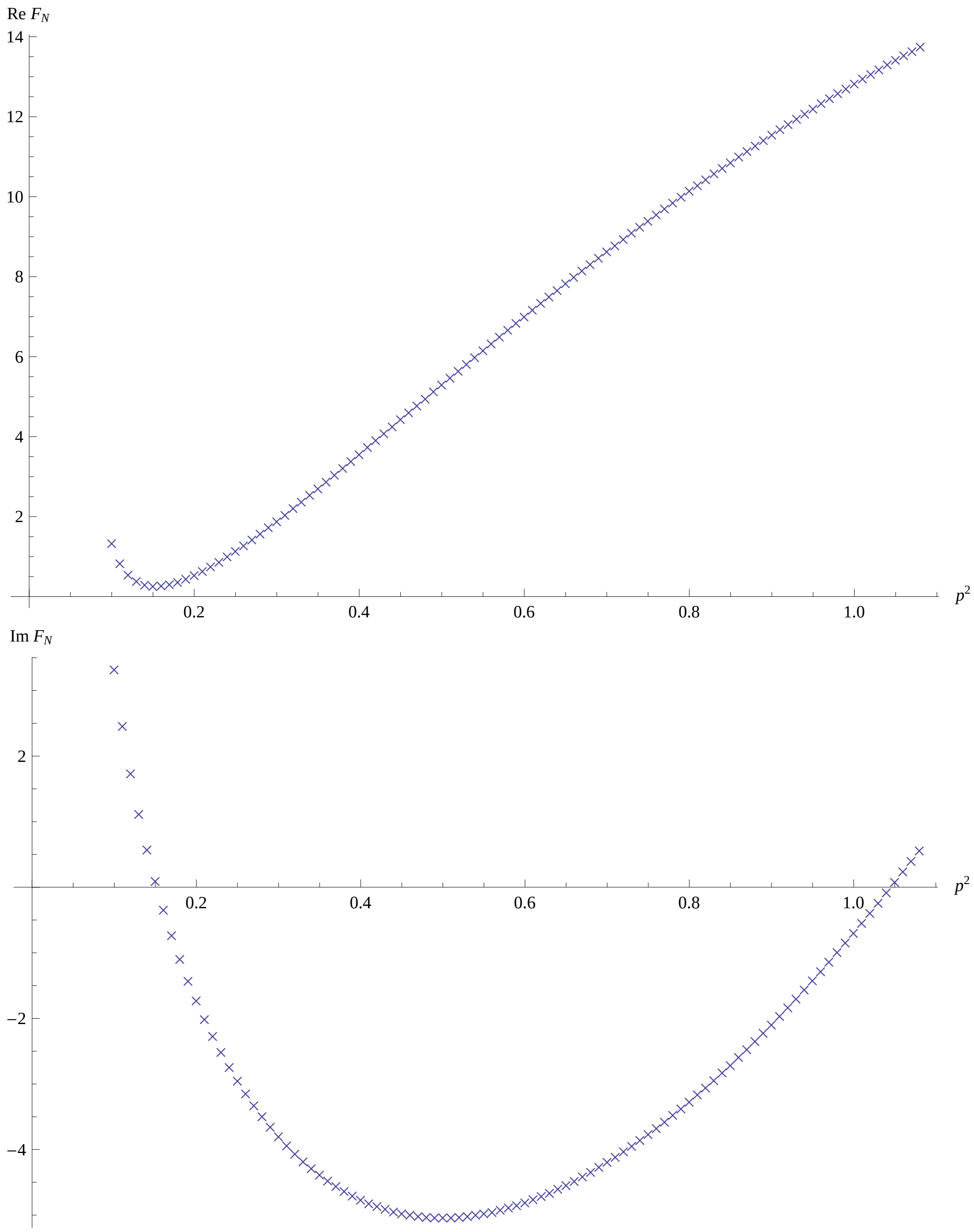


Fig. 3.10: The real and imaginary parts of F_N as functions of p^2

4. CONCLUSION

We set out on a program to calculate two-loop corrections to the pion form factor using the KLZ theory. This involves regularization and renormalization of the ρ^0 self-energy at the two-loop level. In this thesis, using dimensional regularization, we calculated the $1/\epsilon^2$, $1/\epsilon$ terms of the self-energy. We also calculated the constant term using analytical and numerical techniques thereby completing the regularization of the ρ^0 self-energy. The exact final result is summarised by the following

$$\Pi_2(p^2) = \frac{ig^4}{(4\pi)^4}\left\{T^{(-2)}\frac{1}{\epsilon^2} + T^{(-1)}\frac{1}{\epsilon} + T^{(0)} + F_N(p^2) + O(\epsilon)\right\},$$

where $T^{(-2)}$ and $T^{(-1)}$ are given by (3.54) and (3.55) respectively. The terms $T^{(0)}$ and $F_N(p^2)$ are represented graphically in figure 3.4 and figure 3.10 respectively. It was not possible to complete the two-loop renormalization due to the time constraints on an MSc project. Renormalization, followed by comparison to experiment, will be carried out as part of my PhD thesis.

APPENDIX

A.

A.1 Reduction of two-loop integrals

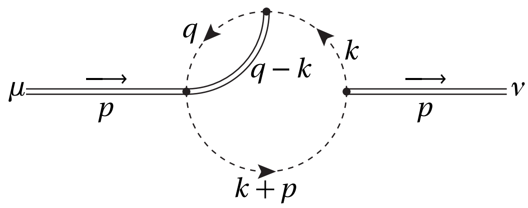


Fig. A.1: $X^{\mu\nu}$

Given below are detailed calculations needed to get to the results given in equations (3.25),(3.26),(3.27),(3.28),(3.29) and (3.30) of the third chapter. We start with the reduction of the integral represented by figure A.1

$$X^{\mu\nu} = 2ig^4 \int \frac{d^n k}{(2\pi)^n} \frac{d^n q}{(2\pi)^n} \frac{(k_1+k_4)^\mu (k_1+k_2)^\nu}{\left(k_1^2 - m_\pi^2\right)\left(k_2^2 - m_\pi^2\right)\left(k_3^2 - m_\rho^2\right)\left(k_4^2 - m_\pi^2\right)}. \tag{A.1}$$

We have to project using the operators $g_{\mu\nu}$ and $\frac{p^\mu p^\nu}{p^2}$. In order compute these projections, it is useful to note the following:

$$\begin{aligned}(k_1+k_4)\cdot(k_1+k_2) &= \frac{1}{2}\Big\{3\left(k_1^2-m_\pi^2\right)+\left(k_2^2-m_\pi^2\right)-2\left(k_3^2-m_\rho^2\right)+\left(k_4^2-m_\pi^2\right)\\ &\quad +k_5^2+5m_\pi^2-2m_\rho^2-2p^2\Big\}\\ &= \frac{1}{2}\Big\{3D_1+D_2-2D_3+D_4+k_5^2+5m_\pi^2-2m_\rho^2-2p^2\Big\}\end{aligned} \tag{A.2}$$

$$\begin{aligned}p\cdot(k_1+k_4)\,p\cdot(k_1+k_2) &= \frac{1}{2}\Big\{k_2^2\left(k_2^2-m_\pi^2\right)+k_1^2\left(k_1^2-m_\pi^2\right)+k_5^2\left(k_2^2-m_\pi^2\right)\\ &\quad -k_5^2\left(k_1^2-m_\pi^2\right)-2\left(k_1^2-m_\pi^2\right)\left(k_2^2-m_\pi^2\right)-\left(k_2^2-m_\pi^2\right)\left(k_4^2-m_\pi^2\right)\\ &\quad -2(m_\pi^2+p^2)\left(k_2^2-m_\pi^2\right)+2p^2\left(k_1^2-m_\pi^2\right)\Big\}\\ &= \frac{1}{2}\Big\{k_2^2D_2+k_1^2D_1+k_5^2D_2-k_5^2D_1-2D_1D_2\\ &\quad -D_2D_4-2(m_\pi^2+p^2)D_2+2p^2D_1\Big\}\end{aligned} \tag{A.3}$$

Using (A.2) and then cancelling terms in the numerator against the denominator we get

$$\begin{aligned}g_{\mu\nu}X^{\mu\nu} &= 2ig^4\int\frac{d^nk}{(2\pi)^n}\frac{d^nq}{(2\pi)^n}\frac{(k_1+k_4)\cdot(k_1+k_2)}{D_1D_2D_3D_4}\\ &= ig^4\left[3\tilde{T}_{234}+\tilde{T}_{134}-2\tilde{T}_{124}+\tilde{T}_{123}+\tilde{Y}^5_{1234}+(5m_\pi^2-2m_\rho^2-2p^2)\tilde{T}_{1234}\right].\end{aligned} \tag{A.4}$$

We can shuffle the indices of $\tilde{Y}^5_{1234}$ above by (3.8), giving

$$= ig^4\left[3\tilde{T}_{234}+\tilde{T}_{134}-2\tilde{T}_{124}+\tilde{T}_{123}+\tilde{Y}^1_{2345}+(5m_\pi^2-2m_\rho^2-2p^2)\tilde{T}_{1234}\right]. \tag{A.5}$$

Similarly with the help of (A.3) we get

$$\begin{aligned}\frac{p^\mu p^\nu}{p^2}X_{\mu\nu} &= ig^4\Big[\frac{1}{p^2}\tilde{Y}^2_{134}+\frac{1}{p^2}\tilde{Y}^1_{234}+\frac{1}{p^2}\tilde{Y}^5_{134}-\frac{1}{p^2}\tilde{Y}^5_{234}-\frac{2}{p^2}\tilde{T}_{34}-\frac{1}{p^2}\tilde{T}_{13}\\ &\quad +\frac{1}{p^2}\tilde{T}_{23}-\frac{2}{p^2}(m_\pi^2+p^2)\tilde{T}_{134}+2\tilde{T}_{234}\Big] &(A.6)\\ &= ig^4\Big[\frac{1}{p^2}\tilde{Y}^2_{134}+\frac{1}{p^2}\tilde{Y}^1_{234}+\frac{1}{p^2}\tilde{Y}^2_{134}-\frac{1}{p^2}\tilde{Y}^1_{234}-\frac{2}{p^2}\tilde{T}_{34}-\frac{1}{p^2}\tilde{T}_{13}\\ &\quad +\frac{1}{p^2}\tilde{T}_{13}-\frac{2}{p^2}(m_\pi^2+p^2)\tilde{T}_{134}+2\tilde{T}_{234}\Big] &(A.7)\\ &= ig^4\left[\frac{2}{p^2}\left\{\tilde{Y}^2_{134}-\tilde{T}_{34}-(m_\pi^2+p^2)\tilde{T}_{134}\right\}+2\tilde{T}_{234}\right] &(A.8)\\ &= \tilde{T}_{234}. &(A.9)\end{aligned}$$

In going from (A.6) to (A.7) we used the symmetry of the Y and T-integrals under the permutations of indices (3.8). In the last step follows from the equality

$$\tilde{Y}^2_{134} = \tilde{T}_{34} + (m_\pi^2 + p^2)\tilde{T}_{134} \tag{A.10}$$

which can be proved in the following way

$$\begin{aligned}
\tilde{Y}^2_{134} &= \int \frac{d^n k}{(2\pi)^n}\frac{d^n q}{(2\pi)^n}\frac{k_2^2}{D_1 D_3 D_4} \\
&= \int \frac{d^n k}{(2\pi)^n}\frac{d^n q}{(2\pi)^n}\frac{(k+p)^2}{D_1 D_3 D_4} \\
&= \int \frac{d^n k}{(2\pi)^n}\frac{d^n q}{(2\pi)^n}\frac{k^2 - m_\pi^2 + m_\pi^2 + p^2 + 2k\cdot p}{D_1 D_3 D_4}
\end{aligned}$$

$$\begin{aligned}
&= \int \frac{d^n k}{(2\pi)^n}\frac{d^n q}{(2\pi)^n}\frac{D_1}{D_1 D_3 D_4} + \int \frac{d^n k}{(2\pi)^n}\frac{d^n q}{(2\pi)^n}\frac{m_\pi^2 + p^2}{D_1 D_3 D_4} + \int \frac{d^n k}{(2\pi)^n}\frac{d^n q}{(2\pi)^n}\frac{2k\cdot p}{D_1 D_3 D_4} \\
&= \tilde{T}_{34} + (m_\pi^2 + p^2)\tilde{T}_{134} + \int \frac{d^n k}{(2\pi)^n}\frac{d^n q}{(2\pi)^n}\frac{2k\cdot p}{D_1 D_3 D_4}.
\end{aligned}$$

The last integral above vanishes. This is because the inner integral (i.e the integral with respect to q) is an odd function of k.

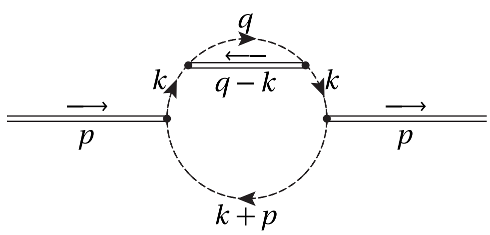


Fig. A.2: $Z^{\mu\nu}$

The following is the amplitude represented by the diagram in figure A.2:

$$Z^{\mu\nu} = \int \frac{d^n k}{(2\pi)^n}\frac{d^n q}{(2\pi)^n}\frac{(k_1+k_2)^\mu (k_1+k_2)^\nu (k_1+k_4)^2}{\left(k_1^2 - m_\pi^2\right)^2 \left(k_2^2 - m_\pi^2\right)\left(k_3^2 - m_\rho^2\right)\left(k_4^2 - m_\pi^2\right)} \tag{A.11}$$

We now project with $\dfrac{p^\mu p^\nu}{p^2}$ as follows

$$\frac{p^\mu p^\nu}{p^2} Z_{\mu\nu} = -i g^4 \int \frac{d^n k}{(2\pi)^n} \frac{d^n q}{(2\pi)^n} \frac{\left[p \cdot (k_1 + k_2)\right]^2 (k_1 + k_4)^2}{p^2 \left(k_1^2 - m_\pi^2\right)^2 \left(k_2^2 - m_\pi^2\right) \left(k_3^2 - m_\rho^2\right) \left(k_4^2 - m_\pi^2\right)} \tag{A.12}$$

We focus on the numerator of the integrand above. To avoid unwieldy algebra, we do the computation on small pieces of the numerator at a time.

$$\begin{aligned}
(k_1 + k_4)^2 &= k_1^2 + k_4^2 + 2k_1 \cdot k_4 \\
&= k_1^2 + k_4^2 + k_1^2 - k_3^2 + k_4^2 \\
&= 2k_1^2 + 2k_4^2 - k_3^2 \\
&= 2\left(k_1^2 - m_\pi^2\right) + 2\left(k_4^2 - m_\pi^2\right) - \left(k_3^2 - m_\rho^2\right) + 4m_\pi^2 - m_\rho^2 \\
&= 2D_1 + 2D_4 - D_3 + 4m_\pi^2 - m_\rho^2 ,
\end{aligned} \tag{A.13}$$

$$\begin{aligned}
\left[p \cdot (k_1 + k_2)\right]^2 &= \left[p \cdot (k + k + p)\right]^2 \\
&= \left[2k \cdot p + p^2\right]^2 \\
&= \left[k_2^2 - k_1^2\right]^2 \\
&= \left[\left(k_2^2 - m_\pi^2\right) - \left(k_1^2 - m_\pi^2\right)\right]^2 \\
&= \left(k_1^2 - m_\pi^2\right)^2 + \left(k_2^2 - m_\pi^2\right)^2 - 2\left(k_1^2 - m_\pi^2\right)\left(k_2^2 - m_\pi^2\right) \\
&= \left(k_1^2 - m_\pi^2\right)^2 + k_2^2\left(k_2^2 - m_\pi^2\right) - m_\pi^2\left(k_2^2 - m_\pi^2\right) - 2\left(k_1^2 - m_\pi^2\right)\left(k_2^2 - m_\pi^2\right) \\
&= D_1^2 + k_2^2 D_2 - m_\pi^2 D_2 - 2D_1 D_2
\end{aligned} \tag{A.14}$$

Multiplying (A.13) and (A.14) together we get:

$$\begin{aligned}
\left[p \cdot (k_1 + k_2)\right]^2 (k_1 + k_4)^2 = {} & 2k_2^2 D_1 D_2 + 2k_2^2 D_2 D_4 - k_2^2 D_2 D_3 + (4m_\pi^2 - m_\rho^2) k_2^2 D_2 \\
& + (-10m_\pi^2 + 2m_\rho^2) D_1 D_2 - 2m_\pi^2 D_2 D_4 + m_\pi^2 D_2 D_3 \\
& - m_\pi^2 (4m_\pi^2 - m_\rho^2) D_2 - 4D_1^2 D_2 - 4D_1 D_2 D_4 + 2D_1 D_2 D_3 \\
& + 2k_1^2 D_1^2 + 2D_1^2 D_4 - D_1^2 D_3 + (2m_\pi^2 - m_\rho^2) D_1^2
\end{aligned} \tag{A.15}$$

We now substitute (A.15) into the numerator in (A.12), and after cancelling terms against the denominator we get

$$\frac{p^\mu p^\nu}{p^2} Z_{\mu\nu} = -i g^4 \frac{1}{p^2} \Big[2\tilde{Y}^2_{134} + 2\tilde{Y}^2_{113} - \tilde{Y}^2_{114} + (4m_\pi^2 - m_\rho^2)\tilde{Y}^2_{1134} + (-10m_\pi^2 + 2m_\rho^2)\tilde{T}_{134}$$
$$- 2m_\pi^2 \tilde{T}_{113} + m_\pi^2 \tilde{T}_{114} - m_\pi^2(4m_\pi^2 - m_\rho^2)\tilde{T}_{1134} - 4\tilde{T}_{34} - 4\tilde{T}_{13} + 2\tilde{T}_{14}$$
$$+ 2\tilde{Y}^1_{234} + 2\tilde{T}_{23} - \tilde{T}_{24} + (2m_\pi^2 - m_\rho^2)\tilde{T}_{234} \Big] .$$

We reduce some of the Y-integrals above in a way similar to (A.10). We also use the invariance of the integrals under permutations (3.8) yielding

$$\frac{p^\mu p^\nu}{p^2} Z_{\mu\nu} = -i g^4 \frac{1}{p^2} \Big[2(\tilde{T}_{34} + (p^2 + m_\pi^2)\tilde{T}_{134}) + 2(\tilde{T}_{13} + (p^2 + m_\pi^2)\tilde{T}_{113}) - (\tilde{T}_{14} + (p^2 + m_\pi^2)\tilde{T}_{114})$$
$$(4m_\pi^2 - m_\rho^2)(\tilde{T}_{134} + (p^2 + m_\pi^2)\tilde{T}_{1134}) + (-10m_\pi^2 + 2m_\rho^2)\tilde{T}_{134} - 2m_\pi^2 \tilde{T}_{113} + m_\pi^2 \tilde{T}_{114}$$
$$- m_\pi^2(4m_\pi^2 - m_\rho^2)\tilde{T}_{1134} - 4\tilde{T}_{13} - 4\tilde{T}_{13} + 2\tilde{T}_{14} + 2\tilde{Y}^1_{234} + 2\tilde{T}_{13} - \tilde{T}_{14} + (2m_\pi^2 - m_\rho^2)\tilde{T}_{234} \Big]$$
$$= -i g^4 \left[-\frac{2}{p^2}\tilde{T}_{13} + \frac{-4m_\pi^2 + m_\rho^2 + 2p^2}{p^2}\tilde{T}_{134} + 2\tilde{T}_{113} - \tilde{T}_{114} + (4m_\pi^2 - m_\rho^2)\tilde{T}_{1134} \right.$$
$$\left. + \frac{2}{p^2}\tilde{Y}^1_{234} + \frac{2m_\pi^2 - m_\rho^2}{p^2}\tilde{T}_{234} \right] . \tag{A.16}$$

We now project with $g^{\mu\nu}$ and find

$$g^{\mu\nu} Z_{\mu\nu} = -i g^4 \int \frac{d^n k}{(2\pi)^n} \frac{d^n q}{(2\pi)^n} \frac{(k_1 + k_2)^2 (k_1 + k_4)^2}{\left(k_1^2 - m_\pi^2\right)^2 \left(k_2^2 - m_\pi^2\right) \left(k_3^2 - m_\rho^2\right) \left(k_4^2 - m_\pi^2\right)} .$$

The numerator above can be rewritten as follows

$$(k_1 + k_2)^2 (k_1 + k_4)^2 = (2k_1^2 + 2k_2^2 - p^2)(2k_1^2 + 2k_4^2 - k_3^2)$$
$$= (2D_1 + 2D_2 + 4m_\pi^2 - p^2)(2D_1 + 2D_4 - D_3 + 4m_\pi^2 - m_\rho^2)$$
$$= 4D_1^2 + 4D_1 D_4 - 2D_1 D_3 + 2(8m_\pi^2 - m_\rho^2 - p^2)D_1 + 4D_1 D_2$$
$$+ 4D_2 D_4 - 2D_2 D_3 + 2(4m_\pi^2 - m_\rho^2)D_2 + 2(4m_\pi^2 - p^2)D_4$$
$$- (4m_\pi^2 - p^2)D_3 + (4m_\pi^2 - p^2)(4m_\pi^2 - m_\rho^2).$$

We can then use this form to cancel terms against the denominator, yielding

$$
\begin{aligned}
g^{\mu\nu} Z_{\mu\nu} = -i g^4 \Big[& 4\tilde{T}_{234} + 4\tilde{T}_{123} - 2\tilde{T}_{124} + 2(8m_\pi^2 - m_\rho^2 - p^2)\tilde{T}_{1234} + 4\tilde{T}_{134} \\
& + 4\tilde{T}_{113} - 2\tilde{T}_{114} + 2(4m_\pi^2 - m_\rho^2)\tilde{T}_{1134} + 2(4m_\pi^2 - p^2)\tilde{T}_{1123} \\
& - (4m_\pi^2 - p^2)\tilde{T}_{1124} + (4m_\pi^2 - p^2)(4m_\pi^2 - m_\rho^2)\tilde{T}_{11234} \Big] .
\end{aligned}
$$

The terms $\tilde{T}_{1123}$ and $\tilde{T}_{1123}$ can be reduced using the integration-by-parts relation (3.17) and this allows us to rewrite the above expression as follows

$$
\begin{aligned}
g_{\mu\nu} Z^{\mu\nu} = i g^4 \Big[& -4\tilde{T}_{234} - 2(8m_\pi^2 - m_\rho^2 - p^2)\tilde{T}_{1234} - 4\tilde{T}_{134} - 2\big((n-1)\tilde{T}_{123} + \tilde{T}_{113}\big) \\
& + \big((n-1)\tilde{T}_{124} + \tilde{T}_{114}\big) - 2(4m_\pi^2 - m_\rho^2)\tilde{T}_{1134} - (4m_\pi^2 - p^2)(4m_\pi^2 - m_\rho^2)\tilde{T}_{11234} \Big]
\end{aligned}
$$

We reduce the diagram in figure A.3. Projecting with $\dfrac{p^\mu p^\nu}{p^2}$ we get

$$
\frac{p^\mu p^\nu}{p^2} A_{\mu\nu} = -\frac{i g^4}{p^2} \int \frac{d^n k}{(2\pi)^n} \frac{d^n q}{(2\pi)^n} \frac{[p\cdot(k_1+k_2) p\cdot(k_4+k_5)](k_1+k_4)\cdot(k_2+k_5)}{\left(k_1^2 - m_\pi^2\right)\left(k_2^2 - m_\pi^2\right)\left(k_3^2 - m_\rho^2\right)\left(k_4^2 - m_\pi^2\right)\left(k_5^2 - m_\pi^2\right)} .
$$

Focusing on the numerator and multiplying it out gives

$$
\begin{aligned}
[p\cdot(k_1+k_2) p\cdot(k_4+k_5)](k_1+k_4)\cdot(k_2+k_5) = {} & D_1 D_2 D_5 + k_2^2 D_2 D_5 - 2D_2 D_3 D_5 + D_2 D_4 D_5 \\
& + k_5^2 D_2 D_5 + (2m_\pi^2 - 2m_\rho^2 - p^2) D_2 D_5 - D_1 D_2 D_4 \\
& - k_2^2 D_2 D_4 + 2D_2 D_3 D_4 - k_4^2 D_2 D_4 - D_2 D_4 D_5 \\
& - (2m_\pi^2 - 2m_\rho^2) D_2 D_4 - k_1^2 D_1 D_5 - D_1 D_2 D_5 + \\
& - D_1 D_4 D_5 - k_5^2 D_1 D_5 - (2m_\pi^2 - 2m_\rho^2 - p^2) D_1 D_5 \\
& + k_1^2 D_1 D_4 + D_1 D_2 D_4 - 2D_1 D_3 D_4 + k_4^2 D_1 D_4 \\
& 2D_1 D_3 D_5 + D_1 D_4 D_5 + (2m_\pi^2 - 2m_\rho^2 - p^2) D_1 D_4 .
\end{aligned}
$$

Substituting this back into the numerator we get

$$
\begin{aligned}
\frac{p^\mu p^\nu}{p^2} A_{\mu\nu} &= -\frac{i g^4}{p^2} \left[4\tilde{Y}^2_{134} - 4\tilde{Y}^1_{234} + 2(2m_\pi^2 - 2m_\rho^2 - p^2)\tilde{T}_{134} - 2(2m_\pi^2 - 2m_\rho^2 - p^2)\tilde{T}_{234} \right] \\
&= -\frac{i g^4}{p^2} \left[4\tilde{T}_{13} - 4\tilde{Y}^1_{234} + (8m_\pi^2 - 4m_\rho^2 + 2p^2)\tilde{T}_{134} - 2(2m_\pi^2 - 2m_\rho^2 - p^2)\tilde{T}_{234} \right] .
\end{aligned}
$$

Next we will contract $A_{\mu\nu}$ with $g^{\mu\nu}$. The following identity is helpful for reducing this contraction

$$
\begin{aligned}
(k_1+k_2)\cdot(k_4+k_5)(k_1+k_4)\cdot(k_2+k_5) = {} & k_1^2 D_1 + k_2^2 D_2 + 2k_3^2 D_3 + k_4^2 D_4 + k_5^2 D_5 \\
& + (-m_\pi^2 + c_2 + c_1) D_1 + (-m_\pi^2 + c_2 + c_1) D_2 \\
& + (-2m_\rho^2 - 2c_2 - c_1) D_3 + (-m_\pi^2 + c_2 + c_1) D_4 \\
& + (-m_\pi^2 + c_2 + c_1) D_5 + 2D_1 D_2 - 3D_1 D_3 + 2D_1 D_4 \\
& + 2D_1 D_5 - 3D_2 D_3 + 2D_2 D_4 + 2D_2 D_5 - 3D_3 D_4 \\
& - 3D_3 D_5 + 2D_4 D_5 + c_1 c_2 \,, \qquad \text{(A.17)}
\end{aligned}
$$

where

$$
\begin{aligned}
c_1 &= 4m_\pi^2 - 2m_\rho^2 - p^2 \\
c_2 &= 4m_\pi^2 - m_\rho^2 - 2p^2 \,.
\end{aligned}
$$

Using (A.17) we get

$$
\begin{aligned}
g_{\mu\nu} A^{\mu\nu} &= -i g^4 \int \frac{d^n k}{(2\pi)^n} \frac{d^n q}{(2\pi)^n} \frac{(k_1+k_2)\cdot(k_4+k_5)(k_1+k_4)\cdot(k_2+k_5)}{\left(k_1^2 - m_\pi^2\right)\left(k_2^2 - m_\pi^2\right)\left(k_3^2 - m_\rho^2\right)\left(k_4^2 - m_\pi^2\right)\left(k_5^2 - m_\pi^2\right)} \\
&= i g^4 \Big[-4\tilde{T}_{234} - 4\tilde{T}_{134} - 4\tilde{T}_{123} + 8\tilde{T}_{124} - 4Y^1_{2345} - 4(7m_\pi^2 - 3m_\rho^2 - 3p^2)\tilde{T}_{1234} \\
&\quad + (8m_\pi^2 - 2m_\rho^2 - 4p^2)\tilde{T}_{1245} - \left(4m_\pi^2 - 2m_\rho^2 - p^2\right)\left(4m_\pi^2 - m_\rho^2 - 2p^2\right)\tilde{T}_{12345} \Big] \,.
\end{aligned}
$$

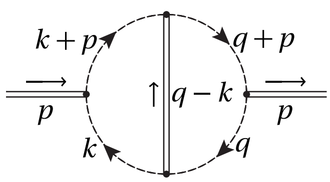


Fig. A.3: Master Diagram

A.2 Numerics

The following are tables of the scalar integrals that were done numerically.

p^2(GeV^2)	ReT_{234N} (GeV^2)	ImT_{234N} (GeV^2)
0.1	-0.00187221	0.00318803
0.14	-0.00245499	0.00354155
0.18	-0.00300638	0.00382778
0.22	-0.00355044	0.00407539
0.26	-0.00410375	0.00429831
0.3	-0.0046799	0.00450386
0.34	-0.00529195	0.0046957
0.38	-0.00595391	0.0048749
0.42	-0.00668233	0.00504007
0.46	-0.00749824	0.00518676
0.5	-0.00843022	0.00530545
0.54	-0.00952035	0.00537657
0.58	-0.010838	0.00535387
0.62	-0.0124655	0.0050404
0.66	-0.0141836	0.00430464
0.7	-0.0159589	0.00311058
0.74	-0.0176433	0.00135266
0.78	-0.0190347	-0.00100759
0.82	-0.0198859	-0.00396489
0.86	-0.019758	-0.00740054
0.9	-0.018439	-0.0105762
0.94	-0.0163499	-0.0130818
0.98	-0.0138283	-0.0148162
1.02	-0.0111372	-0.0157662
1.06	-0.00850382	-0.0159543

Tab. A.1: The real and imaginary parts of T_{234N}

p^2 (GeV^2)	ReT_{1234N}	ImT_{1234N}
0.1	-13.2771	-9.52907
0.14	-9.2883	-10.2388
0.18	-7.11177	-10.0576
0.22	-5.68025	-9.71197
0.26	-4.64958	-9.34013
0.3	-3.86481	-8.97932
0.34	-3.24385	-8.63994
0.38	-2.73851	-8.32393
0.42	-2.31836	-8.03029
0.46	-1.96313	-7.75717
0.5	-1.65873	-7.50253
0.54	-1.39498	-7.26441
0.58	-1.1644	-7.04108
0.62	-0.961277	-6.83097
0.66	-0.781218	-6.63275
0.7	-0.620758	-6.44523
0.74	-0.477141	-6.26739
0.78	-0.348133	-6.09832
0.82	-0.231921	-5.93726
0.86	-0.127007	-5.7835
0.9	-0.0321549	-5.63645
0.94	0.053665	-5.49556
0.98	0.131296	-5.36036
1.02	0.201427	-5.23041
1.06	0.264588	-5.10534

Tab. A.2: The real and imaginary parts of T_{1234N}

p^2 (GeV^2)	Re T_{11234} (GeV^{-2})	Im T_{11234} (GeV^{-2})
0.1	-222.132	173.315
0.14	-115.338	63.6668
0.18	-80.4392	35.616
0.22	-62.1037	23.1519
0.26	-50.6163	16.3142
0.3	-42.6913	12.0994
0.34	-36.8759	9.29929
0.38	-32.4199	7.33873
0.42	-28.8939	5.91156
0.46	-26.0333	4.8413
0.5	-23.6662	4.01938
0.54	-21.6753	3.37599
0.58	-19.978	2.86446
0.62	-18.5144	2.45236
0.66	-17.2398	2.11671
0.7	-16.1204	1.84084
0.74	-15.1299	1.61231
0.78	-14.2478	1.42173
0.82	-13.4577	1.26196
0.86	-12.7463	1.12737
0.9	-12.1029	1.01359
0.94	-11.5186	0.917091
0.98	-10.9859	0.835134
1.02	-10.499	0.765357
1.06	-10.0526	0.705946

Tab. A.3: The real and imaginary parts of T_{11234}

p^2(GeV^2)	ReT_{12345} (GeV^{-2})	ImT_{12345}(GeV^{-2})
0.1	-26.7717	-19.5749
0.14	-15.6834	-23.0696
0.18	-9.47554	-22.4731
0.22	-5.56087	-21.104
0.26	-2.90959	-19.6218
0.3	-1.02689	-18.2047
0.34	0.354593	-16.9002
0.38	1.39253	-15.7148
0.42	2.18559	-14.6422
0.46	2.79908	-13.6707
0.5	3.27748	-12.7895
0.54	3.65224	-11.9877
0.58	3.9462	-11.256
0.62	4.17623	-10.5861
0.66	4.35507	-9.97094
0.7	4.4925	-9.40441
0.74	4.59615	-8.88114
0.78	4.67192	-8.39664
0.82	4.72457	-7.94683
0.86	4.75785	-7.52831
0.9	4.77466	-7.13804
0.94	4.77795	-6.77325
0.98	4.76928	-6.43187
1.02	4.74999	-6.11159
1.06	4.7216	-5.8106

Tab. A.4: The real and imaginary parts of T_{12345}

A.3 Scalar integrals

In this section we give all the scalar integrals that are required to evaluate the one-loop and two-loop self energies that appear in the main text. The integrals defined in equations (2.21) and (2.22) are given by the following expressions to order ϵ

$$A_0(m^2) = \frac{m^2}{\epsilon} + m^2(1 - L_m) + \epsilon\left\{\frac{m^2}{2}\zeta(2) + \frac{m^2}{2}L_m^2 - m^2 L_m + m^2\right\}$$

$$\begin{aligned} B_0(p^2, m^2) = \frac{1}{\epsilon} - &\left\{L_m - 2 - \frac{1}{2}\frac{m^2(r_1 - r_2)}{p^2}(\ln(r_1) - \ln(r_2))\right\} + \frac{\epsilon}{2}\left\{\zeta(2) + 8 + L_m^2\right. \\ &+ 2L_m\left(-2 - -\frac{1}{2}\frac{m^2(r_1 - r_2)}{p^2}(\ln(r_1) - \ln(r_2))\right) + \frac{m^2(r_1 - r_2)}{p^2}\left[2\ln(r_1) - 2\ln(r_2)\right. \\ &+ \ln\left(\frac{1 - r_1}{r_2 - r_1}\right)\ln\left(\frac{-r_1(1 - r_2)}{r_2 - r_1}\right) - \ln\left(\frac{1 - r_2}{r_1 - r_2}\right)\ln\left(\frac{-r_2(1 - r_1)}{r_1 - r_2}\right) \\ &\left.\left. + \mathrm{Li}_2\left(\frac{-r_1(1 - r_2)}{r_2 - r_1}\right) - \mathrm{Li}_2\left(\frac{-r_2(1 - r_1)}{r_1 - r_2}\right) - \mathrm{Li}_2\left(\frac{1 - r_2}{r_1 - r_2}\right) + \mathrm{Li}_2\left(\frac{1 - r_1}{r_2 - r_1}\right)\right]\right\}. \end{aligned} \tag{A.18}$$

The following are the analytical parts of the T-integrals which are reduced using the semi-numerical method of [19]:

$$
\begin{aligned}
T_{234A} &= \frac{1}{\epsilon^2}\left(\frac{m_\rho^2}{2}+m_\pi^2\right)+\frac{1}{4\epsilon}\left(-4m_\rho^2 L_{m_\rho}-8m_\pi^2 L_{m_\pi}+6m_\rho^2+12m_\pi^2-p^2\right)\\
&+2\left(\left(1-\frac{m_\pi^2}{p^2}\right)m_\rho^2\left(-\mathrm{Li}_2\left(\frac{m_\rho^2-m_\pi^2}{m_\rho^2}\right)+\mathrm{Li}_2\left(-\frac{1-\tilde{r}_1\left(p^2\right)}{\tilde{r}_1\left(p^2\right)}\right)+\mathrm{Li}_2\left(-\frac{1-\tilde{r}_2\left(p^2\right)}{\tilde{r}_2\left(p^2\right)}\right)\right)\right.\\
&+m_\pi^2\left(1-\frac{m_\rho^2}{p^2}\right)\left(-\mathrm{Li}_2\left(\frac{m_\pi^2-m_\rho^2}{m_\pi^2}\right)+\mathrm{Li}_2\left(1-\tilde{r}_1\left(p^2\right)\right)+\mathrm{Li}_2\left(1-\tilde{r}_2\left(p^2\right)\right)\right)\\
&+\frac{1}{2}\left(m_\rho^2-m_\pi^2\right)\left(\mathrm{Li}_2\left(\frac{m_\rho^2-m_\pi^2}{m_\rho^2}\right)-\mathrm{Li}_2\left(\frac{m_\pi^2-m_\rho^2}{m_\pi^2}\right)\right)+m_\rho^2\left(L_{m_\rho}^2-3L_{m_\rho}\right)\\
&+m_\pi^2\left(L_{m_\pi}^2-3L_{m_\pi}\right)+\frac{1}{2}p^2\mathrm{lp}\left(p^2\right)+\frac{1}{4}p^2\left(\ln\left(-\frac{m_\rho^2}{p^2}\right)+\ln\left(-\frac{m_\pi^2}{p^2}\right)-\frac{13}{2}\right)\\
&+\frac{m_\pi^2}{4p^2}\left(m_\rho^2+m_\pi^2+p^2\right)\left(\tilde{r}_1\left(p^2\right)-\tilde{r}_2\left(p^2\right)\right)\left(\ln\left(\tilde{r}_2\left(p^2\right)\right)-\ln\left(\tilde{r}_1\left(p^2\right)\right)\right)\\
&\left.+\frac{1}{4}p^2\left(\frac{m_\rho^4}{p^4}-\frac{m_\pi^4}{p^4}\right)\ln\left(\frac{m_\rho^2}{m_\pi^2}\right)+\left(m_\rho^2+m_\pi^2\right)\left(-\frac{1}{4}\ln^2\left(\frac{m_\rho^2}{m_\pi^2}\right)+\frac{\pi^2}{12}+3\right)\right)\\
&-m_\rho^2\mathrm{Li}_2\left(\frac{p^2}{m_\rho^2}\right)-\frac{1}{2}p^2 L_{m_\rho}-m_\rho^2\left(L_{m_\rho}^2-3L_{m_\rho}\right)+\frac{m_\rho^2}{2}\left(\frac{m_\rho^2}{p^2}-\frac{p^2}{m_\rho^2}\right)\ln\left(1-\frac{p^2}{m_\rho^2}\right)\\
&-\left(3+\frac{\pi^2}{4}\right)m_\rho^2+\frac{13p^2}{8},
\end{aligned}
$$

$$
\begin{aligned}
T_{1234A} &= \frac{1}{2\epsilon^2}+\frac{1}{\epsilon}\left(-\frac{m_\pi^2 r_2\ln(r_1)}{p^2}+\frac{m_\pi^2 r_1\ln(r_1)}{p^2}-L_{m_\pi}+\frac{5}{2}\right)\\
&+\frac{1}{2p^2}\left(m_\pi^2(r_1-r_2)\left(-\mathrm{Li}_2(1-r_1)+\mathrm{Li}_2((1-r_1)r_1)-2\mathrm{Li}_2\left(\frac{1}{r_1+1}\right)\right.\right.\\
&+\mathrm{Li}_2(1-r_2)-\mathrm{Li}_2((1-r_2)r_2)+2\mathrm{Li}_2\left(\frac{1}{r_2+1}\right)+\eta\left(1-\frac{p^2}{m_\pi^2},r_1\right)\\
&\left.\left.-\ln^2(r_1+1)+8\ln(r_1)-\eta\left(1-\frac{p^2}{m_\pi^2},r_2\right)\ln((1-r_2)r_2)+\ln^2(r_2+1)\right)\right)\\
&-L_{m_\pi}\left(\frac{2m_\pi^2(r_1-r_2)\ln(r_1)}{p^2}+5\right)+\frac{1}{2}\mathrm{Li}_2\left(\frac{p^2}{m_\pi^2}\right)+\left(\frac{m_\pi^2}{p^2}-1\right)\ln\left(1-\frac{p^2}{m_\pi^2}\right)\\
&+L_{m_\pi}^2+\frac{\pi^2}{4}+\frac{19}{2},
\end{aligned}
$$

where

$$
\begin{aligned}
\eta(a,b) &= \ln(ab)-(\ln(a)+\ln(b))\\
&= 2\pi i\left\{\theta(-\mathrm{Im}a)\theta(-\mathrm{Im}b)\theta(\mathrm{Im}(ab))-\theta(\mathrm{Im}a)\theta(\mathrm{Im}b)\theta(\mathrm{Im}(-ab))\right\}.
\end{aligned}
$$

$$T_{1134}(p^2, m_\pi^2, m_\pi^2, m_\rho^2, m_\pi^2) = \frac{1}{2\epsilon^2} + \frac{1}{\epsilon}\left(\frac{1}{2} - L_{m_\pi}\right) - \frac{m_\rho^2 F\left(m_\pi^2, m_\pi^2, m_\rho^2\right)}{2\lambda_k\left(m_\pi^2, m_\rho^2, m_\pi^2\right)} + L_{m_\pi}^2 - L_{m_\pi} + \frac{1}{2} + \frac{\pi^2}{12}$$

$$T_{134}(p^2; m_\pi^2, m_\rho^2, m_\pi^2) = \frac{1}{2\epsilon^2}(2m_\pi^2 + m_\rho^2) + \frac{1}{\epsilon}\left\{\frac{3}{2}(2m_\pi^2 + m_\rho^2) - 2m_\pi^2 L_{m_\pi} - m_\rho^2 L_{m_\rho}\right\} + \left(\frac{7}{2} + \frac{1}{2}\zeta(2)\right)\left(2m_\pi^2 + m_\rho^2\right) + 2m_\pi^2\left(L_{m_\pi}^2 - 3L_{m_\pi}\right) + m_\rho^2\left(L_{m_\rho}^2 - 3L_{m_\rho}\right) - \frac{1}{2}\ln^2\left(\frac{m_\pi^2}{m_\rho^2}\right) + F\left(m_\pi^2, m_\rho^2, m_\pi^2\right)$$

We have used the following notation:

$$L_{m_i} = \gamma_{\mathrm{E}} + \ln\left(\frac{m_i^2}{4\pi\mu^2}\right) \tag{A.19}$$

and r_1 and r_2 are the roots of the following quadratic equation

$$m_\pi^2\left(r + \frac{1}{r}\right) = 2m_\pi^2 - p^2. \tag{A.20}$$

$\tilde{r}_1$ and $\tilde{r}_2$ are the roots of the equation

$$m_\rho^2 r + \frac{m_\pi^2}{r} = m_\pi^2 + m_\rho^2 - p^2 \tag{A.21}$$

$$\lambda_k(a,b,c) = (a-b-c)^2 - 4bc \tag{A.22}$$

$$F(m_\pi^2, m_\rho^2, m_\pi^2) = m_\rho^2 \lambda^2\left(\frac{m_\pi^2}{m_\rho^2}\right)\Phi\left(\frac{m_\pi^2}{m_\rho^2}\right), \tag{A.23}$$

where

$$\Phi(x) = \frac{1}{\lambda(x)} \left\{ \frac{\pi^2}{3} - 2\mathrm{Li}_2\left(\frac{-\lambda(x) - x + 2}{2}\right) - 2\mathrm{Li}_2\left(\frac{x - \lambda(x)}{2}\right) \right.$$
$$\left. + 2\ln\left(\frac{-\lambda(x) - x + 2}{2}\right) \ln\left(\frac{x - \lambda(x)}{2}\right) \right\}$$

and

$$\lambda(x) = \sqrt{x^2 - 4x}\,.$$

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