A contribution to the magnetic field of a shaded pole motor

by

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submitted to the University of Cape Town in partial fulfilment of the requirements for the degree of Master of Science in Engineering.

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I, EGN VÔSS, submit this thesis in partial fulfilment of the requirements for the degree of Master of Science in Electrical Engineering.

I claim that this is my original work and that it has not been submitted in this or similar form for a degree at any other university.

SIGNED: .................. (EGON VOSS)

AT: CAPE TOWN

Beginning with the developed version of a shaded pole motor a contour of the area concerned is defined which follows the boundary between rotor iron and rotor conducting medium; the pole axis; the salient pole; the boundary between stator iron and airgap and the axis of the pole gap, figure 1. In this case the analysis is restricted to a region which is given in y-direction as the distance between rotor- and stator iron and over half a pole pitch in x-direction. Due to the shading ring it is not possible to define boundary conditions for the contour parts along the pole- and the pole-gap axis. Following the two axis theory of any machine the given problem is resolved into d- and q-components which guarantees such boundary conditions along the whole contour defined above whereby the shading ring itself will contribute to the d- and q-component of the analysis whilst the main winding constitutes a d-component only. Since the problem is symmetrical with respect to the pole axis, the full pole pitch is considered automatically.
As the problem is assumed to be plane in z-direction the solutions will be functions of x and y only. Since the conducting part cannot be described using the scalar-potential the analysis is done with the vector-potential which is appropriate to describe the whole area enclosed by the defined contour.

From Maxwell's equations general solutions of the vector-potential are derived whereby Laplace's equation is solved for the airgap and the skin-equation for the conducting rotor layer. Furthermore from figures 2 to 4 it is obvious that the solutions have to have zeros at the boundaries and for this reason the separation constants are introduced in such a way that harmonic solutions are obtained in x-direction, i.e. along and parallel to the rotor surface.

Imposing these boundary conditions on to the general solutions of the vectorpotentials specifies the functional behaviour and determines the constant of separation, leaving only the constants of the functions unknown. Due to the geometry; the shading ring and the rotor-airgap boundary it is not possible to describe the whole area with only one function which is therefore subdivided into 4 sections.
The result of this are three interfaces at which the potentials must be equal for all x-values. The conditions to be satisfied at these boundaries are those of the normal and tangential components of the magnetic field strength which are derivatives of their respective vectorpotentials.

Application of these conditions results in a set of simultaneous matrix-equations whose solution yields defining equations for these constants giving the complete analytical description of the potential functions. The entire procedure is done for both d- and q-components and following the superposition principle the resultant solution is found by combining the individual parts respectively.

From the point of the "Theory of Potentials" the analysis is completed and the practical evaluation can be followed up. For this the real-part of the potentials is formed enabling field-patterns to be drawn as a function of time whereby lines of constant real parts define these magnetic field lines. Since the general definition of a potential is that of a field-describing Quantity whose derivatives give the fieldstrengths the magnetic field strength can be found from the vectorpotential. In this work the flux into the rotor is considered as well as its distribution along the rotor surface so that only the y-component is looked at.
For the arbitrarily chosen time $t=0$ the flux into the rotor is shown as a function of the airgap and of the thickness of the conducting rotor layer for the two interesting materials aluminium and copper. The time dependancy of the magnetic field strength is shown for one set of dimensional parameters to emphasise the influence of the shading ring. The work concludes with a discussion of the results.
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# Nomenclature

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<thead>
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<th>Symbol</th>
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<tr>
<td>$E$</td>
<td>Electric field strength</td>
<td>(V/m)</td>
</tr>
<tr>
<td>$H$</td>
<td>Magnetic field strength</td>
<td>(A/m)</td>
</tr>
<tr>
<td>$B$</td>
<td>Magnetic field density</td>
<td>(Vs/m²)</td>
</tr>
<tr>
<td>$G$</td>
<td>Conduction current density</td>
<td>(A/m²)</td>
</tr>
<tr>
<td>$A$</td>
<td>Magnetic vector potential</td>
<td>(Vs/m)</td>
</tr>
<tr>
<td>$V_m$</td>
<td>Magnetic scalar potential</td>
<td>(A)</td>
</tr>
<tr>
<td>$\psi$</td>
<td>Magnetic flux</td>
<td>(Vs)</td>
</tr>
<tr>
<td>$z$</td>
<td>Impedance</td>
<td>(V/A)</td>
</tr>
<tr>
<td>$i$</td>
<td>Conduction current</td>
<td>(A)</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>Conductivity</td>
<td>(A/Vm)</td>
</tr>
<tr>
<td>$\mu$</td>
<td>Permeability</td>
<td>(Vs/Am)</td>
</tr>
<tr>
<td>$\omega$</td>
<td>Radian frequency</td>
<td>(rad/s)</td>
</tr>
<tr>
<td>$T$</td>
<td>Time</td>
<td>(s)</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>Skin constant</td>
<td>(1/m)</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>Complex wave constant</td>
<td>(1/m)</td>
</tr>
<tr>
<td>$\delta$</td>
<td>Dirac-delta function</td>
<td></td>
</tr>
<tr>
<td>$dr$</td>
<td>Elementary distance</td>
<td>(m)</td>
</tr>
<tr>
<td>$e$</td>
<td>Half the pole width</td>
<td>(m)</td>
</tr>
<tr>
<td>$a$</td>
<td>Half the pole pitch</td>
<td>(m)</td>
</tr>
<tr>
<td>$r$</td>
<td>Thickness of rotor conducting layer</td>
<td>(m)</td>
</tr>
<tr>
<td>$b$</td>
<td>Airgap between rotor and stator</td>
<td>(m)</td>
</tr>
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</table>
NOMENCLATURE cont.

- h - distance between shading ring and rotor layer (m)
- s - distance between shading ring and pole face (m)
- b - airgap between rotor and stator (m)
- k - sheet current density (A/m)
- 1; 2; 3; 4 - indices for subsections
- q; d; e - indices for Q; D; E - components respectively
2.0 THE PROBLEM

To be found is the instantaneous magnetic flux distribution at the rotor surface of a shaded pole motor of the salient-pole type as well as the fieldlines in the conducting rotor part as a function of time.

A cartesian coordinate system \((x, y, z)\) is located in such a way that the positive \(y\)-axis coincides with the field pole axis and the positive \(x\)-axis with the rotor surface. The shading ring whose purpose is to delay the flux through it compared with the flux through the unshaded portion is in practically built machines solidly embedded in the pole iron. The analysis of such an arrangement is in general possible but leads to analytical problems. The ring having a width of \(\Delta x = 2g\) is therefore placed with its centre at \(x = d\) and at a height of \(y = h\) separated from the pole face by a distance, \(a\).
During the evaluation this distance is made infinitely small so that the ring is practically attached to the pole face giving a close approach to practical machines, figure 1.

Figure 1. Developed stator and rotor bore with x-y-axis

For symmetry reasons only the part of the right hand side of the north-pole including shading ring and conducting rotor medium is analysed.
From figure 1 it is obvious that the field components for the contour at \(x = 0\) and \(x = a\) are not defined. This is due to the shading ring for without it the boundary conditions for the \(x\)- and \(y\)-components of the magnetic field strength would be \(H_x(0,y)=0\) and \(H_y(a,y)=0\) constituting \(d\)- and \(q\)-axis. The shading ring remaining in the position as shown yields definite boundary conditions at \(x = 0\) and \(x = a\) only when split up into two components. The two axis model which holds for any machine has therefore a \(d\)-component from the shading ring and the main-field-winding whilst the \(q\)-component is determined by the shading ring alone.

For the rotor layer the following operations show that the induced Eddy currents have the direction of the vectorpotential. 

\[
\text{curl } \mathbf{E} = -\frac{\partial}{\partial t} \mathbf{B} = -\frac{\partial}{\partial t} \text{curl } \mathbf{A} = \text{curl} \left( -\frac{\partial}{\partial t} \mathbf{A} \right)
\]

\[
\Rightarrow \mathbf{E} = -\frac{\partial}{\partial t} \mathbf{A} = -j \omega \mathbf{A}
\]

assuming harmonic functions.

With \(\mathbf{G} = \omega \mathbf{E}\) it follows that \(\mathbf{G} = -j \omega \mathbf{A}\) so that the induced Eddy currents have a \(z\)-component only.

Because they have to obey Lenz's law their magnetic field will act against the main- and the shading ring field.
For this reason the rotor layer is included in all separately analysed parts, see figures 2 to 4.
The two d-components could be dealt with together but the separation has the advantage of being easier to check than their combination as the results to be expected are known in general. If only the stator-excitation is considered the magnetic field components are completely defined along the contour shown in figure 2.

Figure 2. Defined boundary conditions of B-part.
As mentioned previously, two parts must be analyzed when the shading ring is considered in order to obtain definite boundary conditions at $x = 0$ and $x = a$. For this the asymmetrical problem is resolved into two symmetrical ones with shading ring current of $\frac{I_k}{2}$ whose superposition will then represent the magnetic field caused by the shading ring alone.

Figures 3 and 4 show that the ring current adds up to $I_k$ for $0 \leq x \leq a$ and cancels out for $-a \leq x \leq 0$.

---

**Figure 3. Defined boundary conditions of D-part.**

---

**Figure 4. Defined boundary condition of Q-part.**
2.1 Magnet field components derived from the vector potential

From Maxwell's equations we obtain

\[
\text{div } \mathbf{B} = 0 \quad \text{and} \quad \text{curl } \mathbf{H} = 0 \quad \text{for airgap}
\]

\[
\text{curl } \mathbf{H} = \frac{\mu_0}{\alpha} \mathbf{A} \quad \text{for rotor layer}
\]

Where \( \mathbf{B} \) is the magnetic field density in \( \text{Vs/m}^2 \); \( \mathbf{H} \) the magnetic field strength in \( \text{A/m} \) and \( \mathbf{A} \) the density of the conduction current in \( \text{A/m}^2 \).

From \( \text{curl } \mathbf{H} = 0 = \text{curl } (-\text{grad } V_m) \) one can represent the magnetic field using the scalar potential \( V_m \) as \( \mathbf{H} = -\text{grad } V_m \). From \( \text{div } \mathbf{B} = 0 = \text{div } (\text{curl } \mathbf{A}) \) the magnetic field can be written using the vector potential \( \mathbf{A} \) as \( \mathbf{B} = \mu_0 \mathbf{H} = \text{curl } \mathbf{A} \Rightarrow \mathbf{H} = \frac{1}{\mu_0} \text{curl } \mathbf{A} \).

Whilst \( \text{curl } \mathbf{H} = 0 \) holds for the current-free volume only (i.e. for the airgap) the equation \( \text{div } \mathbf{B} = 0 \) is of general nature (i.e. holds for the rotor layer as well) and for this reason the fields will be described with the vector potential only.
From Faraday's law one obtains \[ \mathbf{E} = -\mathbf{j} \omega \mathbf{A} \quad (I) \] and from Oersted's law (for the case of negligible displacement current) we have \[ \text{curl } \mathbf{A} = \mathbf{G} \] which results with \[ \mathbf{B} = \mathbf{\mu} \mathbf{A} \] and \[ \mathbf{G} = \omega \mathbf{E} \] in \[ \text{curl } \mathbf{A} = \text{curl} \left( \frac{1}{\mathbf{\mu}} \mathbf{B} \right) = \omega \mathbf{E} \Rightarrow \text{curl } \mathbf{B} = \omega \mathbf{E} \quad (II). \]

After substituting \[ \mathbf{B} = \text{curl } \mathbf{A} \] and (I) into (II) one obtains \[ \text{curl } \frac{1}{\mathbf{\mu}} \mathbf{B} = \omega \mathbf{E} = -j \omega \mathbf{E} = -j \omega \mathbf{\omega} \mathbf{\mu} \mathbf{A} = \text{curl} (\text{curl } \mathbf{A}) \]

Using the vector identity \[ \text{curl} (\text{curl } \mathbf{A}) = \text{grad} (\text{div } \mathbf{A}) - \Delta \mathbf{A} \]
for Cartesian coordinates one obtains with the divergence-free vector potential \[ \text{curl} (\text{curl } \mathbf{A}) = -\Delta \mathbf{A} = -j \omega \mathbf{E} \mathbf{\mu} \mathbf{A} \]
from which the skin equation

\[ \Delta \mathbf{A} = \lambda^2 \mathbf{A} \quad (1) \]

for the conducting medium follows.

\[ \lambda = j \omega \mathbf{\mu} \] is called the skin constant which contains \( \omega, \mathbf{\mu} \) as parameters. The curl of \( \mathbf{A} \) is determined by \( \mathbf{B} \) but no equation defines the divergence which means conditions can be imposed arbitrarily. The gauge condition used here to define the divergence of \( \mathbf{A} \) is that of Buchhols namely

\[ \text{div } \mathbf{A} = 0 \]

For the air gap (\( \varepsilon = 0 \)) the skin equation reduces to

\[ \Delta \mathbf{A} = 0 \quad (2) \]
Due to the fact that the problem is plane in z-direction the field describing vector potential has only a z-component which is a function of the coordinates $x$ and $y$.

With

$$\text{curl } \vec{A} = \begin{vmatrix} e_x & e_y & e_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & 0 \end{vmatrix} = e_x \frac{\partial}{\partial y} A_z - e_y \frac{\partial}{\partial x} A_z$$

and

$$\vec{H} = e_x H_x + e_y H_y$$

it follows from $\vec{H} = \frac{\mu}{2 \pi} \text{curl } \vec{A}$ that

$$H_x = \frac{\mu}{2 \pi} \frac{\partial}{\partial y} A_z \quad (\text{III})$$

and

$$H_y = -\frac{\mu}{2 \pi} \frac{\partial}{\partial x} A_z \quad (\text{IV})$$

To be determined are however the vectorpotentials first which have to satisfy equations (1) and (2) respectively whose general solutions are

$$A_z(x,y) = (A_0 + B_0 x)(C_0 + D_0 y) + \sum_{k=1}^\infty (A_k \cos[kx] + B_k \sin[kx])(C_k \sinh[ky] + D_k \cosh[ky])$$

for the airgap $\Delta \vec{A} = 0$ and

$$A_z(x,y) = (A_0 + B_0 x)(C_0 \sinh[ky] + D_0 \cosh[ky]) + \sum_{k=1}^\infty (A_k \cos[kx] + B_k \sin[kx])(C_k \sinh[ky] + D_k \cosh[ky])$$

for the rotor layer $\Delta \vec{A} = dx \vec{A}$

To satisfy the boundary conditions at $x = 0$ and $x = a$ the separation has been done in such a way that harmonic functions are obtained in $x$. 
Neither the boundary conditions of the contour nor the conditions at the interfaces of the subsections within the contour are imposed on to (3) and (4) for which reason the eigenvalues $k$; the complex constant $y = \sqrt{k^2 + d^2}$ and all constants $A; B; C$ and $D$ are still unknown.

From (3) and (4) which satisfy (1) and (2) respectively special solutions are found by applying boundary conditions along the contour, figures 2 to 4.

This specifies the eigenvalues $k$ and with the known skinconstant $d$ the complex wave constant $y$ is determined. The remaining complex constants are found from the conditions at the interfacing subsections.
2.2 Boundary conditions; vectorpotentials and constants

Since the general solutions of the equations (1) and (2) are of the form

\[ A_z(x,y) = \sum_{k=0}^{\infty} X_k(x) Y_k(y) \]

they can conveniently be written down using matrices which will be done for the Q; D and E-part respectively. Using an upper T to denote the transpose; an upper (-1) to denote the inverse; the numbers 1, 2, 3 and 4 to indicate the subsections and \( \gamma_n = n \cdot \pi \)

\[ q_n = (2n-1) \frac{\pi}{2} \]

\( \beta_n = n \cdot \pi \)

the conditions can be written in the following way.

2.2.1 Analysis of Q-part

Region 4 - rotor layer \( 0 \leq x \leq a; \; -r \leq y \leq 0 \)

\[ \frac{\partial}{\partial x} A_4(0,y) = 0 \quad \frac{\partial}{\partial y} A_4(x,0) = 0 \quad \frac{\partial}{\partial y} A_4(a,y) = 0 \quad (5) \]

Region 1 - rotor surface to shading ring \( 0 \leq x \leq a; \; 0 \leq y \leq h \)

\[ \frac{\partial}{\partial x} A_4(y) = 0 \quad \frac{\partial}{\partial y} A_4(a,y) = 0 \quad (6) \]

Region 2 - shading ring to pole face \( 0 \leq x \leq a; \; h \leq y \leq h+s \)

\[ \frac{\partial}{\partial x} A_4(y) = 0 \quad \frac{\partial}{\partial y} A_4(a,y) = 0 \quad (7) \]
Region 3 - pole face to stator iron $e \leq x \leq a; \ h+s \leq y \leq b$

$$
\frac{\partial}{\partial x} A_3(x, y) = 0 \quad \frac{\partial}{\partial y} A_3(x, b) = 0 \quad \frac{\partial}{\partial y} A_3(a, y) = 0 \quad (8)
$$

Inner boundary conditions at interfacing subsections:

boundary at regions 4 and 1 $0 \leq x \leq a; \ y = 0$

$$
A_4(x, 0) = A_1(x, 0) \quad (9) \quad \frac{\partial}{\partial y} A_4(x, 0) = \frac{\partial}{\partial y} A_1(x, 0) \quad (10)
$$

boundary at regions 1 and 2 $0 \leq x \leq a; \ y = h$

$$
A_1(x, h) = A_2(x, h) \quad (11)
$$

$$
\frac{\mu}{\partial x} \frac{\partial}{\partial y} A_1(x, h) - \frac{\mu}{\partial x} \frac{\partial}{\partial y} A_2(x, h) = k_s(x) \quad (12)
$$

where $k_s(x)$ is the sheet current density of the shading ring whose current has been referred to the length $a$ of the whole boundary.

Boundary at regions 2 and 3 $e \leq x \leq a; \ y = h+s$

$$
A_2(x, h+s) = A_3(x, h+s) \quad \text{for} \quad e \leq x \leq a \quad (13)
$$

$$
0 \quad \text{for} \quad 0 \leq x \leq e \quad (14)
$$

$$
\frac{\partial}{\partial y} A_2(x, h+s) = 0 \quad \text{for} \quad 0 \leq x \leq e
$$

$$
\frac{\partial}{\partial y} A_3(x, h+s) \quad e \leq x \leq a
$$
With the following definitions of the row vectors \( \cos^T(Q \frac{x}{a}) \), \( \sin^T(Q \frac{a-x}{a}) \)

full matrices \((m \times n)\) SOT; S0 (see appendix)

column-vectors \( D_4; C_1; D_1; C_2; D_2; D_3 \)

and all hyperbolicus matrices as diagonal ones the vectorpotentials for the respective regions become:

\[
A_1(xy) = \mu_0 \alpha \cos^T(Q \frac{x}{a}) \cosh(T \frac{d+y}{a}) \cosh^{-1}(T \frac{d-x}{a}) D_4
\]

from equations (4) + (5) \hspace{10cm} (15)

\[
A_2(xy) = \mu_0 \alpha \cos^T(Q \frac{x}{a}) \left[ \sinh(Q \frac{b+y}{a}) \cosh^{-1}(Q \frac{b}{a}) C_1 + \cosh(Q \frac{b-y}{a}) \cosh^{-1}(Q \frac{b}{a}) D_4 \right]
\]

from equations (3) + (6) \hspace{10cm} (16)

\[
A_3(xy) = \mu_0 \alpha \cos^T(Q \frac{x}{a}) \left[ \sinh(Q \frac{b+s-y}{a}) \cosh^{-1}(Q \frac{s}{a}) C_2 + \cosh(Q \frac{b-s+y}{a}) \cosh^{-1}(Q \frac{s}{a}) D_2 \right]
\]

from equations (3) + (7) \hspace{10cm} (17)

\[
A_4(xy) = \mu_0 \alpha \left( \frac{a-x}{a} \right) \sin^T(Q \frac{a-x}{a}) \cosh(Q \frac{b-y}{a}) \cosh^{-1}(Q \frac{b-y}{a} D_3)
\]

from equations (3) + (8) \hspace{10cm} (18)

All potential functions have been multiplied by the permeability \( \mu_0 \) of air to obtain the correct unit for the vectorpotential. In addition to this the factor \( k_o \) representing the sheet current density of the mainwinding for rated conditions has been added into the potentials so that a ratio of \( k_s / k_o \) could be defined to avoid the specification of these densities in terms of their individual values.
The vectorpotentials (15; 16; 17; 18) which are functions of \( x \) and \( y \) yield together with the boundary conditions (9; 10; 11; 12; 13; 14) functional relations between the complex constants which are functions of the eigenvalues

\[
\tanh \left( Q \frac{h}{a} \right) C_1 + D_1 = D_4
\]

from equations (9) + (15, 16)

\[
T \tanh \left( T \frac{\alpha}{a} \right) D_4 = (-Q) \left[ C_1 + \tanh \left( Q \frac{\alpha}{a} \right) D_4 \right]
\]

from equations (10) + (15, 16)

\[
cosh^{-1} \left( Q \frac{h}{a} \right) D_4 = \tanh \left( Q \frac{\alpha}{a} \right) C_2 + D_2
\]

from equations (11) + (16, 17)

\[
\tanh \left( Q \frac{\alpha}{a} \right) D_2 + C_2 = \cosh^{-1} \left( Q \frac{h}{a} \right) C_1 = 2k_0 Q^{-1} \left[ \cos \left( Q \frac{d_2}{a} \right) - \cos \left( Q \frac{d_1}{a} \right) \right]
\]

from equations (12) + (16, 17)

where \( k_0 \) is the rated sheet current density of the main field winding whose current has been referred to the whole length \( b-h-s \) of the pole. With the sheet current density \( k_s \), the ratio \( k_s/k_0 = k_{so} \) is a sheet current ratio (see equation 22).
\frac{2a}{(a-e)^2} \mathbf{S} \mathbf{C} \mathbf{T} \cosh^{-1}(Q \frac{b}{a}) \mathbf{D}_2 - \mathbf{D}_3 \quad (23)

from equations (13) + (17,18)

\begin{align*}
Q \cosh^{-1}(Q \frac{b}{a}) \mathbf{C}_2 = & \left( \frac{2a}{(a-e)^2} \right) \mathbf{S} \mathbf{C} \mathbf{T} \cdot \mathbf{Q} \cdot \tanh(\frac{b-h-s}{a-e}) \mathbf{D}_3 \\
\end{align*}

from equations (14) + (17,18)

Equations (19)......(24) constitute a system of 6 simultaneous matrix equations of order n whose first constant \(D_2\) is found as the solution of

\[ X_q \cdot D_2 = Y_q \quad (V) \]

where the number of equations (6) coincides with the number of constants to be found namely \(D_2; D_3; C_1; C_2; D_1\) and \(D_4\). The matrices in (V) are defined as follows:

- \(X_q = \tanh(Q \frac{s}{a}) - N_2 + K_2 - N_2 \cdot \tanh(Q \frac{s}{a}) K_2\) [full matrix]
- \(N_2 = \cosh^{-1}(Q \frac{b}{a}) M_1^{-1} L_1 \cdot \cosh(Q \frac{b}{a}) = M_1^{-1} L_1\) [diagonal matrix]
- \(M_1 = T \cdot \tanh(T \frac{b}{a}) \cdot \tanh(Q \frac{b}{a}) + Q\) [diagonal matrix (25)]
- \(L_1 = (-Q) \cdot \tanh(Q \frac{b}{a}) - T \cdot \tanh(T \frac{b}{a})\) [diagonal matrix (26)]
- \(K_2 = \cosh(Q \frac{b}{a}) G^{-1} \mathbf{S} \mathbf{C} \mathbf{T} \cdot \tanh(Q \frac{b-h-s}{a-e}) \frac{4}{(a-e)^2} \cdot \mathbf{S} \mathbf{T} \cdot \cosh^{-1}(Q \frac{s}{a})\) [full matrix (27)]
- \(Y_q = 2k \cdot s \cdot Q^{-1} \left[ \cos(Q \frac{d}{a}) - \cos(Q \frac{d-e}{a}) \right]\) [column-vector (28)]

These matrices determine the constant column-vector \(D_2\).
The remaining constants are given by:

\[ D_3 = \frac{2 \pi}{\sigma (a - e)^2} \mathcal{G} \mathcal{T} \cosh^{-1}(Q \frac{e}{c}) D_2 \] from equation (23)

\[ C_2 = K_2 \times D_2 \] from equations (24) + (27)

\[ D_1 = \cosh(Q \frac{b}{c}) \left[ \tanh(Q \frac{b}{c}) C_2 + D_2 \right] \] from equation (21)

\[ C_1 = M_1 \times L_1 \times D_1 \] from equations (25) + (26)

\[ D_0 = \tanh(Q \frac{b}{c}) C_1 + D_1 \] from equation (19)

With these constants the vectorpotentials (15)....(18)
and the magnetic field strengths derived from them are
defined for the \( q \)-component of the shading ring, figure 4.

A special word must be said about equation (12).
According
to theory is the difference between the tangential components
of the magnetic field strength given by the sheet current
density \( k_s(x) \) in the boundary of regions 1 and 2 at
\( y = h \). \( k_s(x) \) has only a \( x \)-component and values only at
\( x = d - g \) and \( x = d + g \).
Since the analysis does not assume any dimensions for the shading ring the Dirac - delta function is most appropriate

\[ k_s(x) = k_s \left\{ (x-(d-g)) - (x-(d+g)) \right\} \]

From (12) and (16) and (17) we obtain in the first instance

\[
\cos^2(Q \frac{x}{a}) \left[ \cosh^{-1}(Q \frac{x}{a}) \right] C_t - \tanh(Q \frac{x}{a}) D_t = \frac{1}{k_s} k_s(x)
\]

and after orthogonal expansion with the eigen-functions

\[
\cos(Q \frac{x}{a})
\]

between the limits \( x = 0 \) and \( x = a \)

\[
\int \frac{2}{a} \cos \left( \frac{Q x}{a} \right) \cos \left( \frac{Q x}{a} \right) dx = \frac{2}{a} \int \frac{1}{k_s(x)} k_s(x) dx
\]

Since the integral on the left hand side of this equation yields the \((n \times m)\) unity matrix emphasis is placed on the right hand side. Integrating the sheet current density gives the current in the shading ring which has values only at the places \( x = d-g \) and \( x = d+g \).

\[
\int \frac{2}{a} \cos \left( \frac{Q x}{a} \right) dx = \frac{2}{a} \int \cos \left( \frac{Q (d-g)}{a} \right) - \cos \left( \frac{Q (d+g)}{a} \right)
\]

with \((Ia/a) = k_s\) and \((ka/ke) = k_s\)

the right hand side of the equation above becomes

\[
2k_s \frac{1}{a} \left[ \cos \left( \frac{Q (d-g)}{a} \right) - \cos \left( \frac{Q (d+g)}{a} \right) \right]
\]

which equals the defined column-vector \( Y_q \) (28).
At the shading ring Faraday's law must be satisfied from which the current can be calculated as follows if $e$ denotes the induced electro motive force and $\psi$ the flux penetrating through the ring and $Z$ the impedance of the ring

$$e = -\frac{d}{dt} \psi = i \cdot Z \Rightarrow i = -\frac{1}{Z} \frac{d}{dt} \psi$$

Assuming harmonic variation in time the most general description of the flux is given by

$$\psi = \psi_0 \exp(j\omega t + \phi)$$

from which one obtains

$$i = -\frac{1}{Z} \frac{d}{dt} \psi_0 \exp(j\omega t + \phi) = -\frac{1}{Z} j\omega \psi_0 \exp(j\omega t + \phi) = -\frac{1}{Z} j\omega \psi$$

Literature search however gave no results as to what the impedance can be taken so that this approach could not be followed.

Another problem occurs when looking at the flux linked with the ring:

$$\psi = \int \int_{\Gamma} \vec{B} \cdot d\vec{F} = \int \int_{\Gamma} \text{curl} \vec{A} \cdot d\vec{F} = \oint \vec{A} \cdot d\vec{s}$$

This vectorpotential however, is due to the main field current and the effect of the induced eddy currents in the rotor layer. In the stage of analysis this vector-potential is not known as this work is concerned with finding exactly this resultant potential. For these reasons the ring current was considered as an additional excitation having the sheet current density $k_a(x)$ as given previously.
As explained before the shading ring is creating an asymmetrical problem which is resolved into two symmetrical ones with d and q components according to the two axis theory. Similarly to the Q-part analysis one obtains the following for the D-part referring to the conditions given in figure 3.

2.2.2 Analysis of D-part

Region 4 - rotor layer \(0 \leq x \leq a; -r \leq y \leq 0\)

\[
\frac{\partial}{\partial y} A_4(x,y) = 0 \quad \frac{\partial}{\partial y} A_4(x,r) = 0 \quad \frac{\partial}{\partial x} A_4(a,y) = 0 \tag{29}
\]

Region 1 - rotor surface to shading ring \(0 \leq x \leq a; 0 \leq y \leq h\)

\[
\frac{\partial}{\partial y} A_1(x,y) = 0 \quad \frac{\partial}{\partial x} A_1(a,y) = 0 \tag{30}
\]

Region 2 - shading ring to pole face \(0 \leq x \leq a; h \leq y \leq h+s\)

\[
\frac{\partial}{\partial y} A_2(x,y) = 0 \quad \frac{\partial}{\partial x} A_2(a,y) = 0 \tag{31}
\]

Region 3 - pole face to stator iron \(e \leq x \leq a; h+s \leq y \leq b\)

\[
\frac{\partial}{\partial x} A_3(x,y) = 0 \quad \frac{\partial}{\partial y} A_3(x,b) = 0 \quad \frac{\partial}{\partial x} A_3(a,y) = 0 \tag{32}
\]
Inner boundary conditions at interfacing subsections:

boundary at regions 4 and 1 \( 0 \leq x \leq a; \ y = 0 \)

\[
A_4(x,0) = A_4(x,0) \quad (33) \quad \frac{\partial}{\partial y} A_4(x,0) = \frac{\partial}{\partial y} A_4(x,0) \quad (34)
\]

boundary at regions 1 and 2 \( 0 \leq x \leq a; \ y = h \)

\[
A_1(x,h) = A_2(x,h) \quad (35) \quad \frac{1}{\mu} \frac{\partial}{\partial y} A_1(x,h) - \frac{1}{\mu} \frac{\partial}{\partial y} A_2(x,h) = k_5(x) \quad (36)
\]

boundary at regions 2 and 3 \( e \leq x \leq a; \ y = h+s \)

\[
A_2(x,h+s) = A_3(x,h+s) \quad \text{for} \ e \leq x \leq a \quad (37)
\]

\[
\frac{\partial}{\partial y} A_2(x,h+s) = \begin{cases} 
0 & \text{for} \ e \leq x \leq e \\
\frac{\partial}{\partial y} A_3(x,h+s) & \text{for} \ e \leq x \leq a
\end{cases} \quad (38)
\]

The equations (33).....(38) are of course the same as equations (9).....(14) since the only difference between the \( d \) and \( q \) - components of the shading ring are the conditions along the contour at \( x = 0 \) and \( x = a \), figures 3 and 4.
With the definitions of the row vectors 
\[ \sin^T \left( \frac{Q^x}{a} \right) \] \[ \cos^T \left( \frac{p^a-x}{a-e} \right) \]
full matrices \((nxn)\) \(V(x); V(y)\) (see appendix)
column vectors \(F_4; E_1; F_1; E_2; F_2; F_3\)
complex constant \(Q^3\)

and all hyperbolicus matrices as diagonal matrices the vectorpotentials for the respective regions are:

\[ A_4(x,y) = \mu_0 \kappa_0 \frac{a}{2} \sin^T \left( \frac{Q^x}{a} \right) \cosh^T \left( \frac{1-\mu^a+y}{a} \right) \cosh^T (\eta^2 \eta_3) F_4 \] \[ \text{from equations (4) and (29)} \] \[ A_6(x,y) = \mu_0 \kappa_0 \frac{a}{2} \sin^T \left( \frac{Q^x}{a} \right) \left[ \sinh \left( Q^b \frac{b+y}{a} \right) \cosh \left( Q^s \frac{s}{a} \right) F_e + \cosh \left( Q^b \frac{b+y}{a} \right) \cosh \left( Q^s \frac{s}{a} \right) F_2 \right] \] \[ \text{from equations (3) and (30)} \] \[ A_6(x,y) = \mu_0 \kappa_0 \frac{a}{2} \sin^T \left( \frac{Q^x}{a} \right) \left[ \sinh \left( Q^b \frac{b+y}{a} \right) \cosh \left( Q^s \frac{s}{a} \right) F_e + \cosh \left( Q^b \frac{b+y}{a} \right) \cosh \left( Q^s \frac{s}{a} \right) F_2 \right] \] \[ \text{from equations (3) and (31)} \] \[ A_6(x,y) = \mu_0 \kappa_0 \frac{a}{2} \left[ Q_3 + \cos \left( \frac{p^a-x}{a-e} \right) \cosh \left( \frac{p^b+y}{a-e} \right) \cosh \left( \frac{p^b-s}{a-e} \right) F_3 \right] \] \[ \text{from equations (3) and (32)} \]

Again the factors \(\mu_0\) and \(\kappa_0\) have been added for reasons given before.
With the potentials \((39; 40; 41; 42)\) and the boundary conditions \((33; 34; 35; 36; 37; 38)\) the relations between the still unknown constants are

\[
\tanh\left(\frac{Q}{a}\right) E_1 + F_1 = F_4
\]

from equations \((33)\) and \((39, 40)\)

\[
\prod \tanh\left(\frac{h}{a}\right) F_4 = (-Q)[E_1 + \tanh\left(\frac{Q}{a}\right) F_1]
\]

from equations \((34)\) and \((39, 40)\)

\[
\cosh^{-1}\left(\frac{h}{a}\right) F_4 = \tanh\left(\frac{Q}{a}\right) E_2 + F_2
\]

from equations \((35)\) and \((40, 41)\)

\[
\tanh\left(\frac{Q}{a}\right) F_2 + E_2 - \cosh^{-1}\left(\frac{h}{a}\right) E_1 = 2k_0 \frac{Q^{-1}}{\sinh\left(\frac{Q}{a}\right)} - \sinh\left(\frac{Q}{a}\right)
\]

from equations \((36)\) and \((40, 41)\)

\[
\frac{2a}{(a-e)^2} V_0 \frac{tanh^2}{\left(\frac{Q}{a}\right)} F_2 = F_3
\]

from equations \((37)\) and \((41, 42)\)

\[
Q \cosh^{-1}\left(\frac{Q}{a}\right) E_2 = \left(\frac{2}{a}\right) V_0 \prod \tanh\left(P \frac{b - h - s}{a - e}\right) F_3
\]

from equations \((38)\) and \((41, 42)\)
The first constant $F_2$ is found from the equations (43).....(48) which constitute the matrix equation

$$X_d * F_2 = Y_d \quad (VI)$$

Equation (49) is not needed to solve the system (VI) as it is the defining equation for the additive constant $Q_3$ no other constant depends on. The matrices in (VI) are defined as follows:

\[
X_d = \tanh(Q \frac{e}{a}) - N_2 + K_2 - N_2 \tanh(Q \frac{e}{a}) K_2
\]

full matrix

\[
K_2 = \cosh(Q \frac{e}{a}) \sinh(P \frac{b+h}{a}) \frac{4}{(a-b)} \sqrt{\cosh^{-1}(Q \frac{e}{a})}
\]

full matrix

\[
Y_d = 2kse Q^{-1} \left[ \sin\left( Q \frac{d-e}{a} \right) - \sin\left( Q \frac{d+e}{a} \right) \right]
\]

column vector

and all the other matrices $N_2; M_1; L_1$ as in the $Q$-part.
The remaining constants are then given as:

\[ F_3 = \frac{\lambda a}{(a-e)^2} VOT \cosh^{-1}(Q \frac{e}{a}) F_2 \]  from equation (47)

\[ E_2 = K_2 F_2 \]  from equations (48) and (27)

\[ F_1 = \cosh(Q \frac{a}{2}) \left[ \tanh(Q \frac{e}{a}) E_2 + F_2 \right] \]  from equation (45)

\[ E_1 = M_1^{-1} L_1 F_1 \]  from equations (25) and (26)

\[ F_4 = \tanh(Q \frac{a}{2}) E_1 + F_1 \]  from equation (43)

and the additive constant

\[ Q_3 = \left( \frac{a}{a-e} \right)^2 \cos^{-1}(Q \frac{e}{a}) Q \cosh^{-1}(Q \frac{e}{a}) F_2 \]  from equation (49)

These constants and the pertaining vectorpotentials (39) ... (42)

define the conditions for the D-part of the shading ring completely.
2.2.3 Analysis of E-part

Although not necessary the E-part is subdivided into the same subsections (1; 2; 3 and 4) as the Q- and D-parts to obtain solutions of similar mathematical description. It also provides the same number of constants as for the D-part and hence exactly the same pattern can be followed in solving for them. As the E-part is like the D-part of the shading ring essentially a d-component problem it has the same boundary conditions along the contour (figures 2 and 3) and differs only in the constants. In especially the E-part has no shading ring but the stator excitation which introduces an additional constant into the potential function of the third region. For these reasons the vector potentials can be written as follows:

\[ A_1(x,y) = \mu_0 k_0 \sin \left[ \frac{Q}{x} \right] \cosh \left( \frac{r}{a} \right) \cosh \left( \frac{r}{e} \right) H_4 \]  
\[ A_2(x,y) = \mu_0 k_0 \sin \left[ \frac{Q}{x} \right] \sinh \left( \frac{Q}{2} \right) \cosh \left( \frac{Q}{2} \right) G_1 + \cosh \left( \frac{Q}{2} \right) \cosh \left( \frac{Q}{2} \right) H_4 \]  
\[ A_3(x,y) = \mu_0 k_0 \sin \left[ \frac{Q}{x} \right] \sinh \left( \frac{Q}{2} \right) \cosh \left( \frac{Q}{2} \right) G_2 + \cosh \left( \frac{Q}{2} \right) \cosh \left( \frac{Q}{2} \right) H_4 \]  
\[ A_4(x,y) = \mu_0 k_0 \sin \left[ \frac{Q}{x} \right] \sinh \left( \frac{Q}{2} \right) \cosh \left( \frac{Q}{2} \right) G_3 + \cosh \left( \frac{Q}{2} \right) \cosh \left( \frac{Q}{2} \right) H_4 \]  

Where \[ P_2(x,y) = \frac{A}{\mu_0 k_0} \left[ \frac{(a-x)^2}{a-e} - \frac{(b-y)^2}{a-e} \right] \] guarantees the boundary condition \[ \frac{1}{\mu_0} \frac{\partial}{\partial x} A_3(x,y) = k e \] in region 3.
The boundary conditions inside the contour are as for D- and Q-part except for the missing shading ring at \( y = h \). Here equation (36) becomes with \( k_s(x) = 0 \)

\[
\frac{1}{\mu} \frac{\partial}{\partial y} A_1(x, h) = \frac{1}{\mu} \frac{\partial}{\partial y} A_2(x, h)
\]

With the column vector \( B_0 \) and the additional constant term \( C_0 \) which counts for the main field winding (see appendix) the relations between the constants are:

\[
I_t = \tanh \left( \frac{Q}{\alpha} x \right) G_1 + H_1
\]

\[
tanh \left( \frac{Q}{\alpha} x \right) H_1 = -Q \left[ G_1 + \tanh \left( \frac{Q}{\alpha} x \right) H_1 \right]
\]

for the boundary at regions 4 and 1

\[
cosh^{-1} \left( \frac{Q}{\alpha} x \right) H_1 = \tanh \left( \frac{Q}{\alpha} x \right) G_2 + H_2
\]

\[
tanh \left( \frac{Q}{\alpha} x \right) H_2 + G_2 - \cosh^{-1} \left( \frac{Q}{\alpha} x \right) G_1 = 0
\]

for the boundary at regions 1 and 2
\[
\frac{2a}{(a-e)^2} \text{VOT} \cosh^{-1}(Q \frac{a}{a-e})H_2 = \left( \frac{2}{a-e} \right)B0H_3 \\
Q \cosh^{-1}(Q \frac{a}{a-e})G_2 \left[ \frac{-2(b-h-s)}{(a-e)} \right]Q^{-1} \cos(Q \frac{a}{a-e}) + \left( \frac{2}{a-e} \right)\text{VOT} \tanh(P \frac{b-h-s}{a-e})B0H_3 \\
\cos^2(Q \frac{a}{a-e})Q^{-1} \cosh^{-1}(Q \frac{a}{a-e})H_2 = C0 + G_3 \left( \frac{a-e}{a} \right)^2
\]
for the boundary at regions 2 and 3.

Equations (51f.)... (59) constitute the set of simultaneous matrix equations written as \(X_0 \cdot H_2 = Y_0\) (VII) from which \(H_2\) is found as first constant.

Except for the column-vector \(Ye = N_2 \tanh(Q \frac{a}{a-e})K_4 - K_1\) with

\[
K_4 = -\cosh(Q \frac{a}{a-e})Q^{-1} \left[ \frac{-2(b-h-s)}{(a-e)} \right]Q^{-1} \cos(Q \frac{a}{a-e}) + \left( \frac{4}{a(q-a-e)} \right)\text{VOT} \tanh(P \frac{b-h-s}{a-e})B0
\]
all matrices are the same as in the D-part. Again equation (60) serves solely for the determination of the constant \(G_3\) (see also page 24) and is not needed for the solution of the system (VII).

The remaining constants are found from:

\[
H_3 = \frac{2a}{(a-e)^2} \text{VOT} \cosh^{-1}(Q \frac{a}{a-e})H_2 - \left( \frac{2}{a-e} \right)B0 \\
G_2 = K_2H_2 + K_1 \\
H_1 = \cosh(Q \frac{a}{a-e}) \tanh(Q \frac{a}{a-e})G_2 + H_2
\]
$G_1 = M_1 \ast L_1 \ast H_1$ \hfill (64)

$H_+ = \tanh \left( \frac{Q}{a} \right) G_1 + H_1 \hfill (65)$

and the additive term

$G_3 = \left( \frac{a}{a - e} \right)^2 \left[ \cos \left( \frac{Q}{2} \right) \cosh^{-1} \left( \frac{Q}{a} \right) H_2 - C \right]$\hfill follows

from (60). Potentials and field strength derived from them are now completely defined for the E-part.
3.0 Practical Evaluation

Of prime interest in this work was the magnetic field strength at the rotor surface as a function of thickness of rotor layer and airgap. Five particular points were looked at, i.e. the centre of the pole (x 1) at \( x = 0\) mm
left side of shading ring (x 2) at \( x = 6\) mm
centre of shading ring (x 3) at \( x = 7.75\) mm
right side of shading ring (x 4) at \( x = 9.5\) mm
centre between adjacent poles (x 5) at \( x = 10\) mm
all with reference to the coordinate system in figure 2.

Kimberley has measured airgap fluxes published in his work from 1949 and built a model of the shaded pole motor with an airgap of 3.3 mm in order to house the exploring coils.

Vaske showed in his dissertation the currents in the main winding and shading ring as function of the speed (see literature reference). Both researchers gave flux density distributions as a function of time. From these works the values of 3.3 mm for the airgap and 3.8 \( \exp(-120) \) for the ratio kso were chosen to provide a connection between measurements and analysis in this work, which was done for the time \( T=0 \).
The distance between two adjacent poles equals 1mm to approach the machines used by Kimberley and Vaske as both investigated in motors having no salient poles (the shading rings were embedded in the stator iron).

While Kimberley had the exploring coils cemented to the stator iron Vaske placed them on the rotor surface. Both measured therefore different airgap fluxes with the characteristic peaks appearing close to the stator fading away when moving towards the rotor. This work shows both limiting cases including intermediate cases for $0^\circ \leq t \leq 150^\circ$ in steps of $30^\circ$ ($180^\circ$ is the mirror-image of $0^\circ$). Furthermore the magnetic fields are shown inside the rotor layer for different times.

According to praxis the investigation was done for the material aluminium in the rotor. The theoretical difference between copper and aluminium is shown by the listing of tables giving the calculated values.
To implement this the following steps have to be done:

1. Solving the matrix-equations

   (V) \( X_q \cdot D_2 = Y_q \)
   (VI) \( X_d \cdot F_2 = Y_d \)
   (VII) \( X_e \cdot H_2 = Y_e \)

   For the Q; D and E-part respectively which are generally of order \( n \). Since the underlying principle is the series character of the solutions of the skin-(1) and Laplace-equation (2) they consist of infinite many terms and had to be truncated to a value investigated on pages 48 and 49.

   The field strength as a function of \( n \) is shown in figure 15.

2. Determining all remaining constants of the individual parts.

3. Calculating the vector potentials for all four regions.

   For each pair of \( x,y \)-coordinates they are the sum of \( n \) - terms.
4. Finding the time dependency of the potentials by taking the real part of the complex-values.

5. Determining the field strength from the vector potential by forming the derivatives (IV) which yields the \( y \)-component only representing the flux into the rotor.

After forming the real part of the complex field strength

\[
H_y(x,y,t) = \text{Re} \left[ (H_\theta + jH_d) \cdot \exp(j\omega t) \right] = H_\cos \omega t - H_\sin \omega t
\]

one obtains the \( y \)-component which is the superposition of the respective components of the \( D, Q \) and \( R \)-part.

The following theoretical possibilities exist:

\[
\begin{align*}
H_y(x,y,t) &= H_\theta(x,y,t) + H_d(x,y,t) + H_q(x,y,t) \quad (66) \\
H_y(x,y,t) &= H_\theta(x,y,t) - H_d(x,y,t) + H_q(x,y,t) \quad (67) \\
H_y(x,y,t) &= H_\theta(x,y,t) + H_d(x,y,t) - H_q(x,y,t) \quad (68) \\
H_y(x,y,t) &= H_\theta(x,y,t) - H_d(x,y,t) - H_q(x,y,t) \quad (69)
\end{align*}
\]

However, imposing the conditions that

a) the shading ring field must lag the main field in time and

b) the \( d \)- and \( q \)-components must add up to zero for negative values of \( x \),

the only superposition which reflects the physics is (69).
Writing this as $H_y(x,y,t) = H_0(x,y,t) - \{H_d(x,y,t) + H_q(x,y,t)\}$ one realises that the components produced by the induced currents act against the main field which is true as they have to obey Lenz's law. Equation (66) does not fulfil condition a) and equations (67) and (68) not condition b). Only (69) satisfies both.

The field strength can now be evaluated along the rotor surface as a function of time and any other dimensional parameter.

The following train of thoughts shows how the magnetic field is found from the vectorpotential.

The differential equation of the magnetic field lines is $d^t \times \hat{m}(r) = \mathbf{0}$ from which follows due to the vectorpotential having a z-component only that

$$\mathbf{0} = d^t \times \hat{m} = d^t \times \text{curl} \hat{A} = d^t \times \text{curl}(e_z A)$$

and furthermore from vectoranalysis with

$$d^t \text{ grad } \hat{A} = dA \quad \text{and} \quad e_z d^t = 0$$

$$\mathbf{0} = -d^t \times (e_z \times \text{grad } \hat{A}) = -e_z (d^t \text{ grad } \hat{A}) + (e_z d^t) \text{grad } \hat{A} \quad \Rightarrow \quad 0 = -e_z dA$$

For fields which are independent of the third coordinate the magnetic field lines are given for constant potentials as indicated by $dA = 0$, meaning $A = A(x,y)$ = constant.
The following pages show:

a) field strength as a function of the thickness of rotor layer

\[ HY = f(R) \quad 0 \leq R \leq 3\text{mm} \]

pages 36 - 37

b) field strength as a function of airgap

\[ HY = f(h) \quad 0 \leq h \leq 3.3\text{mm} \]

page 40

c) field strength as a function of time

\[ HY = f(T) \quad 0^\circ \leq T \leq 150^\circ \]

pages 42 - 46

d) field strength as a function of the number of terms added in the solutions

\[ HY = f(N) \quad 0 \leq N \leq 50 \]

page 48

e) magnetic fields in the rotor aluminium as a function of time

\[ A = f(T) \quad 0^\circ \leq T \leq 150^\circ \]

page 50

The discussion is given on the pages following immediately after the individual graphs.
Figure 5

Figure 6
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</table>

Figure 7
Figures 5 and 6 show the y-component of the field strength as a function of the thickness of the rotor layer. As the graphs do not indicate any significant difference between the materials copper and aluminium due to the small difference in conductivity their actual values are given in the tables to the left. Each line represents a certain point at the rotor surface as described on page 30. For the case of no material present \( R = 0 \) the skin-equation (1) for region (4) reduces to Laplace's equation (2) for \( \mathcal{E} = 0 \). Here for both figures 5 and 6 the lines have to have the same values which holds true for all points investigated. With increasing thickness less flux penetrates into the rotor meaning the influence of the induced Eddy currents increases.

This stands in good accordance with practical experience as lamin-ation is used to reduce these Eddy currents in machines and trans-formers. Laminating material however, means reduction of the area available. No lamination would increase the influence of the Eddy currents the effect of this is evident from figures quoted. Underneath the left side of the shading ring \( x_2 \) the change is obviously less dominant as for all other points.

Figure 7 again shows the same dependancy for aluminium only but with an increased thickness to \( R = 4\text{mm} \) with the actual values listed to the left.
Here as well is a decrease of flux into the rotor observable with increasing $R$. Furthermore, the values for copper are in general higher indicating that copper allows more flux into the rotor than aluminium. Because copper has a higher conductivity this is well in accordance with the theory.
Figures 8 and 9 show the y-component of the field strength as a function of the airgap evaluated at the rotor surface for the same points as in figures 5, 6 and 7. The increase of the airgap shows an effect which is not as clear from the change in rotor layer thickness. For both materials it can be seen that for small airgaps flux is coming out of the rotor underneath the centre of the shading ring while there is flux penetrating into the rotor underneath the right side of the shading ring.

Both graphs however, show a decrease of flux with increasing airgap. Since for an infinite airgap no flux would penetrate or leave the rotor the values are ever decreasing.
Figures 12
HY = f(x) at \( y = 3.0 \text{cm} \)

Figures 13
As explained on pages 30 and 31 Kimberley and Vaske have shown extreme cases of the flux distribution due to their different location of exploring coils.

Pages 42 to 46 show this distribution for a constant airgap of 3.3mm at different distances from the rotor surface and as a function of time.

The position of the shading ring is at 0.006m (left) and 0.0095m (right). The time lag of the field in the shaded portion against the unshaded area is clearly visible. At the shading ring \((y = 3.3\text{mm})\) the distribution shows the characteristic peaks with no field strength in the centre of the shading ring conductors. The Transition from the distributions measured by Vaske to those measured by Kimberley is obvious from the previous graphs.
HY = F(N) at shading-ring height

![Graph showing HY = F(N) at shading-ring height with data points labeled as diamonds, circles, and triangles.](image-url)
As mentioned on page 32 the solutions are in general of order \( n \) and it has therefore to be investigated from what value for \( n \) onwards no significant fluctuations occur anymore.

For the five points mentioned on page 30 the field strengths were calculated for \( 15 \leq n \leq 50 \).

This is shown on page 48 supported by a list of values each symbol representing a different point namely:

- \( \square \) - \( x_1 \) centre of pole
- \( \bigcirc \) - \( x_2 \) shading ring left
- \( \triangle \) - \( x_3 \) centre of shading ring
- \( \diamond \) - \( x_4 \) shading ring right
- \( \ast \) - \( x_5 \) centre of pole-pitch

The graphs show that the greatest fluctuations occur up to \( n = 25 \) and that \( n \geq 30 \) is sufficient for stable solutions.

In this thesis \( n = 50 \) was used for the field line determination and \( n = 30 \) for all other investigations.
Figures 16 show the magnetic field lines in the rotor aluminium as derived from the vectorpotential. The numbers represent the constant value of the vectorpotential for different x-y-coordinates describing magnetic field lines. Equal values have been chosen in order to see the direction of field travel as a function of time. Since the field lines limit the area of equal flux their distance is a measure of flux density and for this reason equidistances of 0.25 V are chosen. The drawing shows the conducting rotor layer underneath the poles according to figure 1 penetrated by the total field which is the superposition of individual Q; D and E-parts. The shading ring is situated in the right part of each pole section. T = 0 shows the field due to the main winding while T = 90 indicates that of shading ring and Eddy currents in the rotor. Comparing these two it can be seen that the main field is more equally distributed which must be attributed to the fact that the main winding covers the full pole and the shading ring only a part of it.

According to the analysis in this work the current in the main winding initially decreases with time from T = 0 onwards. The currents induced in the shading ring will therefore build up a field trying to eliminate this effect and so act in the same direction as the main field.
The right part of the left pole is penetrated by field lines from top to bottom defining the direction of all field lines for the following times shown. Taking the O-Vs field line it can be seen that a movement to the right takes place which makes the poles north or south as a function of time. This also indicates a main field travel from the unshaded to the shaded portion which coincides with practical experience. Since for a divergence free vector potential in Cartesian coordinates the current density $\mathbf{j}$ fulfills the equation

$$\Delta \mathbf{A} = -\mu \mathbf{j}$$

the figure also indicates the highest current density at places with highest derivate of $A$ (for instance 2.5 V/s for $T = 0$).

These regions also form the boundary for up- or downwards directed field lines and furthermore show a low field density. This is particularly obvious for $T = 90$ indicating that the disturbing field (shading ring and rotor layer) weaker is than the main field (see $T = 0$). Due to this the main field will although delayed always be the dominant part.
SUMMARY

An analytical method was used to analyse a shaded pole motor with respect to what was partly measured in previous works. It was shown that the model coincides with practical experience and current theory.

The model allows to investigate into the effects of all possible parameter variations which can effectively be shown by drawing field patterns.

In order to calculate the forces generated in the rotor layer the equation \( \mathbf{F} = \int (\mathbf{G} \times \mathbf{B}) \, dv \) must be solved where \( \mathbf{G} \) is found from values of constant field strength and \( \mathbf{B} \) is the resultant field without the main winding. This can be found from this work as the total field was already split up in its individual parts.

The theoretical results were justified by reference to respective literature.
1. Hannakam L.
   Einfuehrung in die Feldtheorie

2. Kimberley E.E.
   The field fluxes of the shaded pole motor
   AIEE - Transactions 1949 Volume 68

3. Vaske P.
   Beitrag zur Theorie des Spaltpolmoters
   Archiv fuer Elektrotechnik 1962
   Heft 1
   (PhD - Dissertation)

4. Kraus and Garver
   Electromagnetics
   McGrawhill second edition 1973
6.0 APPENDIX A

(Summary of the necessary integrals)

\[ VOT(i, n) = \int_{\frac{a}{e}}^{a} \cos(\frac{\rho}{a-x}) \sin(\frac{q}{a-x}) \ dx = (-1)^i \cos(\frac{q}{a}) \frac{\left(\frac{q}{a}\right)}{\left(\frac{q}{a}\right)^2 - \left(\frac{\rho}{a-e}\right)^2} \]

\[ VO(i, n) = \int_{\frac{a}{e}}^{a} \sin(\frac{q}{a}) \cos(\frac{q}{a}) \ dx = (-1)^n \cos(\frac{q}{a}) \frac{\left(\frac{q}{a}\right)}{\left(\frac{q}{a}\right)^2 - \left(\frac{\rho}{a-e}\right)^2} \]

\[ SOT(i, n) = \int_{\frac{a}{e}}^{a} \sin(\frac{q}{a}) \cos(\frac{q}{a}) \ dx = (-1)^i \sin(\frac{q}{a}) \frac{\left(\frac{q}{a}\right)}{\left(\frac{q}{a}\right)^2 - \left(\frac{\rho}{a-e}\right)^2} \]

\[ SO(i, n) = \int_{\frac{a}{e}}^{a} \cos(\frac{q}{a}) \sin(\frac{q}{a}) \ dx = (-1)^n \sin(\frac{q}{a}) \frac{\left(\frac{q}{a}\right)}{\left(\frac{q}{a}\right)^2 - \left(\frac{\rho}{a-e}\right)^2} \]
\[ B_0(c, t) = \int_{e}^{a} \cos(\rho; \frac{a-x}{a-e}) P_3(x, h + s) \, dx = \left(\frac{a-e}{\rho^2}\right)(-1)^i \]

\[ D_0 = \int_{e}^{a} P_3(x, h + s) \, dx = \left(\frac{a-e}{6}\right) - \frac{(b-h-s)^2}{2(a-e)} \]

\[ C_0 = \left(\frac{a-e}{a^2}\right) D_0 \]

\[ \int_{e}^{a} \sin(q_n \frac{x}{a}) \, dx = \left(\frac{a}{q_n}\right)\cos(q_n \frac{e}{a}) \]

\[ \int_{e}^{a} \cos(\rho; \frac{a-x}{a-e}) \, dx = 0 \]

with

\[ P_3(x, h + s) = \frac{4}{\pi^2} \left[ \frac{(a-x)^2}{a-e} - \frac{(b-h-s)^2}{a-e} \right] \]

and

\[ q_n = (2n-1)\frac{\pi}{2} \quad j \quad q_i = (2i-1)\frac{\pi}{2} \quad j \quad \rho_n = n \pi \quad j \quad \rho_i = i \pi \]