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A Covariant Approach to Gravitational Lensing

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Abstract

The main focus of this thesis is to study the properties of null geodesics in general relativistic models. This thesis is divided into two parts. In the first part, we introduce the (1+3)-covariant approach which will be used in our study of null geodesics and their applications to gravitational lensing. The dynamics of the null congruence can be better understood through the propagation and constraint equations in the direction of the congruence. Thus, we derive these equations after describing the geometry of a ray. We also derive a general form of the null geodesic deviation equation (NGDE) which can be used in any given space-time. Various applications of this equation are studied, including its role in determining area-distance relations in an Friedmann-Lemaître-Robertson-Walker (FLRW) cosmological model. We also use the NGDE in deriving a covariant form of the angle of deflection, showing its versatile applications in gravitational lensing theory.

In the second part of this thesis, we apply the (1+1+2)-covariant approach to gravitational lensing in spherically symmetric space-times. A covariant form of the deflection angle is found in this general setting. The validity of the deflection angle is tested using a Schwarzschild space-time. We conclude by deriving the null geodesic deviation equation in a Schwarzschild space-time.
Contents

Acknowledgements ........................................... i
Abstract ...................................................... ii
List of tables .............................................. v
List of figures ............................................... vi
Conventions and abbreviations .............................. vii

1 Introduction ................................................ 1
1.1 Outline of thesis .......................................... 1

I A Covariant Approach to Observations in Cosmology 4

2 The Covariant Approach To Cosmology ............... 5
  2.1 Introduction ........................................... 5
  2.2 Kinematics of Cosmological models ................ 5
    2.2.1 Kinematical quantities and their properties .... 5
    2.2.2 Choosing an observer ............................ 8
  2.3 Matter Description .................................... 9
    2.3.1 The Energy Momentum Tensor .................... 9
    2.3.2 Energy-momentum Conservation ................. 9
    2.3.3 Equations of State ................................ 10
  2.4 Dynamics of Cosmological models .................. 12
    2.4.1 The Ricci Identities .............................. 12
    2.4.2 The Bianchi Identities ........................... 14
    2.4.3 The Basic Propagation Equations ............... 14
    2.4.4 Constraint Equations ............................. 15
    2.4.5 The vorticity-free case : (\( \omega = 0 \)) ......... 16
  2.5 FLRW cosmologies ..................................... 18
    2.5.1 Basic Description of FLRW universes .......... 18
3 Null Geodesics and the Screen-space

3.1 Introduction .............................................. 24
3.2 The Screen-space Approach .............................. 24
3.3 The Ricci Identities for $k^a$ ............................ 28
  3.3.1 The Propagation Equations ......................... 28
  3.3.2 The Null Raychaudhuri Equation ...................... 28
  3.3.3 The Shear Propagation Equation ..................... 29
  3.3.4 The Vorticity Propagation Equation .................. 33
  3.3.5 The Energy Propagation Equation .................... 34
  3.3.6 Constraint Equation ................................. 34

3.4 Null Geodesic Deviation .................................. 35
  3.4.1 The Geodesic Deviation Equation .................... 35
  3.4.2 The General Null Geodesic Deviation Equation ..... 37
  3.4.3 The Null Geodesic Deviation Equation ............... 40

4 The Null Geodesic Deviation Equation in FLRW Universes 41
  4.1 Introduction .............................................. 41
  4.2 A solution in terms of $v$ .............................. 42
  4.3 A solution in terms of redshift ....................... 43
    4.3.1 A matter dominated Universe ....................... 44
    4.3.2 A mixture of dust and radiation .................... 45
    4.3.3 Matter and a Cosmological Constant ............... 47
  4.4 Distance-Redshift Relation in Inhomogeneous Cosmologies 49
    4.4.1 Magnitude-redshift $(m - z)$ Plots for Clumpy universes 53

5 A Covariant Form of the Angle of Deflection ............. 56
  5.1 Introduction .............................................. 56
  5.2 The Screen-space Projected Null Geodesic Deviation Equation 56
  5.3 Application of the screen-space projected NGDE to the perturbed FLRW dust model ................................. 58
    5.3.1 A quasi-Newtonian frame in a perturbed FLRW dust model ................................. 58
  5.3.2 The matter-comoving (Lagrangian) frame ............. 61
  5.3.3 Screen-space Projected NGDE in the Lagrangian Frame 62
  5.4 $(1 + 3)$-Covariant form of the Deflection Angle .......... 64
    5.4.1 Comparison with the Standard form of the Deflection Angle ................................. 70
5.4.2 A more general form of the deflection angle 71

II A (1+1+2)-Covariant Approach to Gravitational Lensing 73

6 (1+1+2)-Approach to Gravitational Lensing in Spherically Symmetric Space-times 74
6.1 Introduction 74
6.2 The (1+1+2)-Covariant sheet Approach 75
6.2.1 Motivation for using (1+1+2)-approach 75
6.2.2 Formalism of (1+1+2)-Covariant sheet Approach 76
6.3 Gravitational Lensing in Spherically Symmetric Space-times 80
6.3.1 Kinematical Quantities 80
6.3.2 Lensing Geometry 83
6.3.3 General Form of the Deflection Angle 83
6.3.4 General Propagation Equations along $k^a$ 85

7 Application to Lensing in Schwarzschild space-times 87
7.1 Introduction 87
7.2 Solutions for $A$, $E$ and $\phi$ 87
7.3 Solutions for the lensing variables 89
7.4 Form of the Scalar Deflection Angle $\alpha$ 91
7.5 Standard Form of the Deflection Angle in a Schwarzschild space-time 92
7.5.1 Null Geodesic Deviation Equation 93

8 Summaries and Conclusion 97
8.1 Summary of Part I 97
8.2 Summary of Part II 100
8.3 Conclusion and Future Research 102

A History of the Distance-Redshift Relation neglecting Gravitational lensing effects 104
List of Figures

4.1 Plot of the magnitude of null geodesic deviation vector η(z) according to differential equation (4.33) .................................................. 46
4.2 Plot of the magnitude of the null geodesic deviation vector η(z) according to differential equation (4.41) .................................................. 48
4.3 Hubble curves generated using equation (4.50) for various values of Ω_m, Ω_A and ν. As can be seen, the type of Hubble curve generated depends on how the matter is distributed in a chosen universe model .................................................. 52
4.4 m – z curves for z = 0.83 and Ω_A = 0.1 .................................................. 53
4.5 m – z curves for z = 0.5 and Ω_A = 0.1 .................................................. 54
4.6 m – z curves for z = 0.83 and Ω_m = 0.2 .................................................. 54
5.1 Illustration of the lens geometry showing the deviation vectors and angles used in the derivation of the covariant form of the deflection angle .................................................. 66
6.1 The lensing geometry of a photon in a spherically symmetric space-time .................................................. 84
6.2 Geometry of the deflection angle .................................................. 85
Conventions and abbreviations

Sign conventions

Signature: \([-, +, +, +]\).

Riemann tensor:
\[ V^a_{\ ;bc} - V^a_{\ ;cb} = R^a_{\ dcb} V^d, \]

where \(\ ;\) denotes covariant differentiation with respect to the metric tensor.

Ricci tensor:
\[ R_{ab} = R^c_{\ aeb}. \]

Units:
\[ c = 8\pi G = \kappa = 1. \]

Latin and Greek indices assume the values 0, 1, 2, 3 and 1, 2, 3 respectively.

Sign conventions follow those of Ellis (1971) and Ellis & van Elst (1998) [13, 14].

For a tensor \( T^{a\ldots b}_{c\ldots d\ldots e\ldots f} \) we have:

symmetrization:
\[ T^{a\ldots b}_{c\ldots (d\ldots e)\ldots f}, \]
anti-symmetrization:
\[ T^{a\ldots b}_{c\ldots [d\ldots e]\ldots f}. \]

The Newtonian operators of div and curl can be generalized by defining the covariant spatial divergence [44, 33]:
\[ \text{div} V = D^a V_a, \quad (\text{div} T)_a = D^b T_{ab}, \]

and curl:
\[ \text{curl} V_a = \varepsilon_{abc} D^b V^c, \quad \text{curl} T_{ab} = \varepsilon_{cde} (a D^e T_{bd}). \]

The completely anti-symmetric pseudotensor \( \eta^{abcd} \) is defined by
\[ \eta^{0123} = (-g)^{-\frac{1}{2}}, \]

where \(g\) is the determinant of the metric \(g_{ab}\).

The projected permutation tensor \( \varepsilon_{abc} \) is defined by:
\[ \varepsilon_{abc} = \eta_{abcd} u^d. \]

Abbreviations

FLRW  \quad \text{Friedmann-Lemaitre-Robertson-Walker.}
LHS  \quad \text{left hand side.}
RHS  \quad \text{right hand side.}
SPSTF  \quad \text{screen-space projected symmetric trace-free.}
OFE  \quad \text{Optical Focussing Equation.}
Chapter 1

Introduction

Null geodesics describe the path that light rays follow in a space-time. Thus, the properties and dynamics of null geodesics in any given space-time should be fully understood in order to interpret and understand the information that they convey to an observer. An extensive part of this thesis is devoted to investigating the dynamics of a null congruence in a general space-time. This involves setting up the geometry for these geodesics so that propagation and constraint equations in the direction of the ray can be derived in this general framework. The null geodesic deviation equation plays a significant role in studying the behavior of the null congruence, so that for a chosen space-time we are able to simulate a propagating ray bundle. There are various applications of the behavior of null geodesics to observational cosmology. In particular, in this thesis we will apply our results to the area of gravitational lensing, which involves the deflection of light rays in curved space-times.

1.1 Outline of thesis

This thesis has been divided into two parts: PART I and PART II. PART I (chapters 2 to 5) applies the covariant approach, first presented by Ehlers [12], Ellis [13, 14] and Hawking [26], to null congruences. A detailed examination of a covariant approach to null geodesics and their geometry was first presented by Sachs [40]. Using the covariant approach, it is possible to study the dynamics of null geodesics in a completely general framework. Chapter 2 begins with an outline of the covariant approach. A review of FLRW cosmologies [16] is also provided. We are particularly interested in the observational relations found in the FLRW models. These relations are
used extensively in chapter 4.

In chapter 3, we look at the geometry of a null congruence using the (1+3) covariant approach. We define the screen-space as the two-dimensional plane orthogonal to the null geodesic tangent vector. The propagation equations and one of the constraint equations along the congruence are derived. The propagation equations show how the various screen-space defined kinematical quantities, the expansion $\Theta$, shear $\sigma_{ab}$ and vorticity $\omega_{ab}$, change along the congruence. Finally, we derive the general form of the null geodesic deviation equation (NGDE) following the work done in [17]. This equation can be interpreted as the various forces acting on a ray bundle in a given universe model.

Chapter 4 begins with the investigation of the NGDE in an FLRW cosmology. Following [17], the NGDE can be written in terms of redshift and the density parameters once a suitable universe model has been specified. Numerical and analytic solutions to the NGDE are obtained and plotted. In the case of flat space-times, one finds that the null geodesic deviation vector can be directly related to the area-distance. Thus, we are able to relate our results found from the NGDE to observations in standard FLRW models. We complete this chapter by looking at solutions to the distance-redshift relation in perturbed FLRW models. This distance is commonly known as the Dyer-Roeder distance and is discussed in [8], [9] and [30]. This analysis shows the importance of taking into account the distribution of matter in the form of inhomogeneities when determining quantities such as the Hubble parameter on small angular scales.

In chapter 5 we consider another application of the NGDE. In particular, the NGDE is applied to gravitational lensing theory. A full discussion of standard lensing results can be seen in [41]. Using the quasi-Newtonian approach discussed in [21, 20], we derived a covariant form of the angle of deflection due to some fluctuation in the gravitational potential.

PART II applies the (1+1+2)-covariant approach, formulated by [4], to gravitational lensing in spherically symmetric space-times. Chapter 7 gives a brief review of the 1+1+2 approach before restricting the kinematical variables to locally rotationally symmetric (LRS) space-times. We then find general expressions for the deflection angle and propagation equations for the 'lensing variables' along a null ray, valid in any spherically symmetric
space-time.

Finally, in chapter 8 we apply the general expressions for the deflection angle and propagation equations to a Schwarzschild space-time recovering the standard form of the angle given in Weinberg [47]. We complete this chapter by deriving the 1+1+2 form of the NGDE in the Schwarzschild space-time, leaving its applications to future research.
Part I

A Covariant Approach to Observations in Cosmology
Chapter 2

The Covariant Approach To Cosmology

2.1 Introduction

This chapter serves as a review of the covariant approach to cosmology which was first presented in the classic paper by Ehlers [12], as well as by Ellis [13, 14] and Hawking [26]. We will adopt the notation introduced by Maartens in [33]. The material in this chapter is drawn mostly from [14, 6].

2.2 Kinematics of Cosmological models

2.2.1 Kinematical quantities and their properties

In cosmology, the average motion of matter at each space-time event defines a unique 4-velocity vector

\[ u^a = \frac{dx^a}{d\tau}, \quad u^a u_a = -1, \tag{2.1} \]

where \( \tau \) is the proper time measured along the fundamental worldlines. The vector field \( u^a \) is tangent to the worldline of the fundamental observer so that \( \tau \) represents the affine parameter along it. Given \( u^a \), the projection tensor

\[ h_{ab} = g_{ab} + u_a u_b, \tag{2.2} \]

projects into the local rest spaces of comoving observers and satisfies the conditions:

\[ h^a c h^c_b = h^a_b, \quad h^a_a = 3, \quad h_{ab} u^b = 0. \tag{2.3} \]
The volume element for the rest-spaces is given by

\[ \varepsilon_{\alpha\beta\gamma} = \eta_{\alpha\beta\gamma\delta} u^\delta \Rightarrow \varepsilon_{\alpha\beta\gamma} = \varepsilon_{[\alpha\beta\gamma]}, \quad \varepsilon_{\alpha\beta\gamma\delta} u^\delta = 0, \quad (2.4) \]

where \( \eta_{\alpha\beta\gamma\delta} \) is the 4-dimensional volume element or the spacetime permutator. The volume element (2.4) satisfies the following identities [33]:

\[ \varepsilon_{\alpha\beta\gamma\delta} \varepsilon_{\epsilon\delta\gamma\delta} = 3! h^a_d h^b_e h^c_f, \quad (2.5) \]
\[ \varepsilon_{\alpha\beta\gamma\delta} \varepsilon_{\epsilon\delta\gamma\delta} = 2 h_f [b h_{\epsilon g} c], \quad (2.6) \]
\[ \varepsilon_{\alpha\beta\gamma\delta} \varepsilon_{\epsilon\delta\alpha\beta} = 2 h_{\epsilon g} c, \quad (2.7) \]
\[ \varepsilon_{\alpha\beta\gamma\delta} \varepsilon_{\epsilon\delta\alpha\beta} = 3! . \quad (2.8) \]

There are two derivatives defined: the covariant time derivative (denoted by a dot) for any tensor, \( T^{a..b..c..d} \), along the fundamental worldlines is defined by:

\[ \dot{T}^{a..b..c..d} = u^c \nabla_c T^{a..b..c..d}, \quad (2.9) \]

and a fully orthogonally projected covariant derivative, \( D \) [14]

\[ D^c T^{a..b..c..d} = h^a_f h^b_e h^c_g h^d_h \nabla_c T^f..g..p..q, \quad (2.10) \]

with total projection on all free indices\(^1\). Angle brackets denote the orthogonal projection of vectors and the projected symmetric trace-free (PSTF) part of tensors [33]:

\[ V^{(a)} = h^a_b V^b, \quad T^{(ab)} = \left[h^{(a}_c h^{b)}_d - \frac{1}{3} h^{ab}_c h_{cd} \right] T^{cd}. \quad (2.11) \]

By analogy to the Newtonian divergence and curl operators, it is possible to define covariant forms of these operators for curved space-times (see [44, 33]):

\[ \text{div } V = D^a V_a, \quad \left(\text{div } T\right)_a = D^b T_{ab} \]
\[ \text{curl } V_a = \varepsilon_{abc} D^b V^c, \quad \text{curl } T_{ab} = \varepsilon_{cd(a} D^c T_{b)} d. \quad (2.12) \]

We may split the covariant derivative of \( u_a \) into its irreducible parts [45, 14]:

\[ \nabla_a u_b = -u_a \dot{u}_b + D_a u_b = -u_a \ddot{u}_b + \frac{1}{3} \Theta h_{ab} + \sigma_{ab} + \omega_{ab}. \quad (2.13) \]

The vector \( \dot{u}_b = u^a \nabla_a u_b \) is the acceleration vector; it represents the degree to which matter moves under non-gravitational forces, for example pressure.

\(^1\)If \( u^a \) has non-zero vorticity, \( D \) is not a proper 3-dimensional covariant derivative.
gradients. It vanishes for matter in free fall (i.e. moving only under gravity plus inertia). The trace $\Theta = D^a u_a$ is called the expansion scalar (volume expansion) and represents the rate of expansion of the fluid. It can be used to define a representative length scale $\mathcal{S}$ along the observers’ worldline which in turn (in a FLRW universe model) defines the Hubble parameter as follows:

$$\frac{\dot{\mathcal{S}}}{\mathcal{S}} = \frac{1}{3} \Theta = H.$$  \hfill (2.14)

The shear tensor, $\sigma_{ab}$, is the symmetric trace-free part of the spatial change in the 4-velocity defined as:

$$\sigma_{ab} = D_{\langle a} u_{b\rangle},$$ \hfill (2.15)

where

$$\sigma_{ab} = \sigma_{(ab)} , \quad \sigma_{ab} u^b = 0 , \quad \sigma^a_a = 0 .$$ \hfill (2.16)

The shear tensor describes the rate of distortion of the matter flow. The shear magnitude is

$$\sigma^2 = \frac{1}{2} \sigma_{ab} \sigma^{ab} \geq 0 \Rightarrow \sigma^2 = 0 \Leftrightarrow \sigma_{ab} = 0 .$$ \hfill (2.17)

The anti-symmetric vorticity tensor, $\omega_{ab}$, describes the rotation of matter relative to a non-rotating frame. It is defined as:

$$\omega_{ab} = D_{[a} u_{b]} ,$$ \hfill (2.18)

with

$$\omega_{ab} = \omega_{[ab]} , \quad \omega_{ab} u^b = 0 .$$ \hfill (2.19)

Its magnitude is given by:

$$\omega^2 = \frac{1}{2} \omega_{ab} \omega^{ab} = \omega^a \omega_a \geq 0 , \quad \Rightarrow \omega^2 = 0 \Leftrightarrow \omega^a = 0 \Leftrightarrow \omega_{ab} = 0 ,$$ \hfill (2.20)

where $\omega^a$ is the vorticity vector. The vorticity vector, has the following properties:

$$\omega^a = \frac{1}{2} \varepsilon^{abc} \omega_{bc} \Rightarrow \omega_{ab} = \varepsilon_{abc} \omega^c ,$$

$$\omega_{a} u^a = 0 ,$$

$$\omega = 0 \Leftrightarrow \omega^a = 0 \Leftrightarrow \omega_{ab} = 0 .$$ \hfill (2.21)

We see that the covariant derivative of $u^a$ given in (2.13) is completely determined by the kinematical quantities introduced above.
2.2.2 Choosing an observer

We can think of two families of worldlines representing the 4-velocities of sets of observers $\mathcal{O}_u$ and $\mathcal{O}_e$. These worldlines, $u^a$ and $e^a$ respectively, are unit future directed timelike smooth vector fields, such that $u^a u_a = e^a e_a = -1$. We can define the following projection tensors into a subspace $H_p$ of the tangent space $T_p$ at each point $p$ of the spacetime:

$$h_{ab} = g_{ab} + u_a u_b, \quad \bar{h}_{ab} = g_{ab} + e_a e_b.$$  \hspace{1cm} (2.22)

These define the spatial part of the instantaneous rest spaces of the observers $\mathcal{O}_u$ and $\mathcal{O}_e$, respectively. The projection tensor $h_{ab}$ (or $\bar{h}_{ab}$) is the metric in the surface $H_p$ if the vector field $u^a$ (or $e^a$) is hypersurface orthogonal.\footnote{A detailed discussion is given in section 2.4.5 which describes the case of irrotational flows.}

The relation between $u^a$ and $e^a$ can be characterized by the hyperbolic tilt angle $\beta$ [32] such that

$$u^a e_a = - \cosh \beta, \quad \beta \geq 0.$$  \hspace{1cm} (2.23)

The direction $\bar{c}^a$ of the motion of $\mathcal{O}_u$ in the instantaneous rest space of $\mathcal{O}_e$ is the projection of $u^a$ into this space. Similarly, the direction $-c^a$ of the motion of $\mathcal{O}_e$ is the projection of $e^a$ into the instantaneous rest space of $\mathcal{O}_u$. This gives

$$\bar{h}^a u^b = \sinh \beta \bar{c}^a = \bar{V}^a \Rightarrow \bar{V}^a e_a = \bar{c}_a e^a = 0, \quad \bar{c}_a e^a = 1,$$  \hspace{1cm} (2.24)

$$h^a u^b = - \sinh \beta c^a = V^a \Rightarrow V^a u_a = c_a u^a = 0, \quad c_a c^a = 1.$$  \hspace{1cm} (2.25)

where either of these directions can be used to specify the direction of tilt. The tilt angle $\beta$ is related to the relativistic Lorentz contraction factor $\gamma$ by

$$\gamma \equiv \cosh \beta = (1 - v^2)^{-\frac{1}{2}}, \quad v = \tanh \beta$$  \hspace{1cm} (2.26)

so that $\beta \approx v \ll 1$ corresponds to a non-relativistic relative velocity $v$ between $\mathcal{O}_u$ and $\mathcal{O}_e$. In this case

$$d^a \equiv u^a - e^a \simeq \beta \bar{e}^a \equiv \bar{V}^a \simeq \beta e^a \equiv -V^a, \quad \bar{h}_{ab} \simeq h_{ab} + 2 u_a (V_b).$$  \hspace{1cm} (2.27)

We can refer to the change between two arbitrary frames $u^a$ and $e^a$ with a small relative velocity as a change of first order in $\beta$. 
2.3 Matter Description

2.3.1 The Energy Momentum Tensor

The relativistic energy, momentum and stresses associated with a matter field can be described by the energy-momentum tensor (EMT) $T_{ab}$. The EMT can be decomposed relative to $u^a$ by splitting it up into parts parallel and orthogonal to $u^a$ as follows\textsuperscript{3}:

$$T_{ab} = \mu u_a u_b + q_a u_b + u_a q_b + p h_{ab} + \pi_{ab},$$  \hspace{1cm} (2.28)

$$\Rightarrow T = T^a_a = 3p - \mu ,$$  \hspace{1cm} (2.29)

with

$$q_a u^a = 0, \quad \pi_{ab} = \pi_{(ab)}, \quad \pi^a_a = 0, \quad \pi_{ab} u^a = 0.$$  \hspace{1cm} (2.30)

We define the following quantities:

- $\mu = T_{ab} u^a u^b$, is the 	extit{relativistic energy density} relative to $u^a$;
- $p = \frac{1}{3} h^a_b T_{bc} u^c$, is the 	extit{isotropic pressure};
- $q^a = -h^a_b T_{bc} u^c$, is the 	extit{relativistic momentum density} (also the energy flux) to $u^a$;
- $\pi_{ab} = h^a_c h^b_d T_{cd} - \frac{1}{3} h_{ab} (h_{cd} T^{cd})$, is the trace-free 	extit{anisotropic pressure} (stress)\textsuperscript{4}.

We also note that the EMT $T_{ab}$ is Lorentz invariant but the quantities $\mu, p, q^a$ and $\pi_{ab}$ are not.

2.3.2 Energy-momentum Conservation

The condition for the conservation of energy and momentum is:

$$\nabla_b T^{ab} = 0 .$$  \hspace{1cm} (2.31)

Substituting for the general form of the EMT (2.28) into (2.31), gives

$$\dot{\mu} + (\mu + p) \Theta + \pi^{ab} \sigma_{ab} + q^{a} \dot{u}_a + D_a q^a = 0,$$  \hspace{1cm} (2.32)

\textsuperscript{3}This is the general form of the EMT used to describe imperfect fluids.

\textsuperscript{4}Note that in [16] a relativistic stress tensor is defined which includes the isotropic and anisotropic pressures as measured by $u^a$, i.e. $\Pi_{ab} = \pi_{ab} + p h_{ab}$. 

9
for the component parallel to the 4-velocity $u^a$. This is the energy conservation equation and determines the rate of change of relativistic energy $\mu$ along the fundamental world lines. The component of (2.31) orthogonal to $u^a$ gives the momentum conservation equation

$$(\mu + p) \dot{u}_a + D_a p + D_b \pi_{ab} + \dot{q}_a + \left( \omega^b_a + \sigma^b_a + \frac{4}{3} \Theta h^b_a \right) q_b = 0,$$  

(2.33)

determining the acceleration caused by various pressure contributions. The conservation equations (2.32) and (2.33) can also be derived from the twice contracted Bianchi identities (2.61). Note that $(\mu + p)$ is defined as the inertial mass density [16].

### 2.3.3 Equations of State

The physics of the matter is given by an equation of state which relates the quantities $\mu$, $p$, $q_a$, $\pi_{ab}$ and possibly other thermodynamic variables such as the temperature and entropy. There are many possible descriptions of the matter and radiation in the universe which should obey at least one of the four constraints:

(i) We usually require the energy condition

$$\mu > 0,$$  

(2.34)

to hold for any matter distribution.

(ii) The inertial mass density of matter is positive. From (2.32), this condition can be restated as:

$$\mu + p > 0.$$  

(2.35)

This condition states that matter will tend to move in the direction of a pressure gradient applied to it. By equation (2.33), we see that this condition also implies that when matter expands, its density decreases rather than increases.

(iii) The gravitational mass density [16] of matter is positive which can be shown to be equivalent to

$$\mu + 3p > 0.$$  

(2.36)

(iv) The speed of sound must be less than the speed of light (i.e. no signal
can be sent faster than light.). If this condition is not obeyed, we can have violations of special relativistic causality such as, a signal can be sent faster by sound than by light. For example, a barotropic fluid with equation of state \( p = p(\mu) \), has the speed of sound given by \( v_s^2 = (dp/d\mu) \). In this case, one has the condition

\[ 0 \leq \frac{dp}{d\mu} \leq 1 \quad (2.37) \]

Similar restrictions can be imposed on non-barotropic fluids.

We expect matter present in our universe today to obey these conditions. However, when dealing with quantum matter, violations of certain conditions are expected. We can thus conclude that these conditions can be applied when looking at a classical description of matter, but may not be valid when a quantum description is applicable.

We may restrict the EMT given in (2.28) to describe a \textit{perfect fluid} which will be used later. Perfect fluids are characterized by the conditions:

\[ q^a = \pi_{ab} = 0 \iff T_{ab} = \mu u_a u_b + p h_{ab} \quad (2.38) \]

Usually the additional assumption of \( p = 0 \) is made resulting in the simplest case: \textit{pressure-free matter} or 'dust'. Otherwise we need to specify an equation of state in which \( p \) can be determined from the energy density \( \mu \) and possibly other thermodynamical variables. Whatever these relations may be, we usually require that at least one of the conditions stated above, (2.34)-(2.37), should hold. Perfect fluids should be sufficiently good for weak lensing, as these effects occur post decoupling when \( u_a \) and \( \pi_{ab} \) are at most second order. Under the perfect fluid assumption the energy conservation equation (2.32) becomes:

\[ \dot{\mu} + (\mu + p) \Theta = 0, \quad (2.39) \]

and the momentum conservation equation (2.33) reduces to the form \( 5 \):

\[ D_a p + (\mu + p) u_a = 0. \quad (2.40) \]

\(^5\)This equation becomes \( u_a = 0 \) in the case of dust.
2.4 Dynamics of Cosmological models

In this section, we look at the general dynamical relations that hold in any cosmological model, without initially restricting the equation state. The metric $g_{ab}$ satisfies the *Einstein Field Equations* (EFE)\(^6\):

$$R_{ab} = T_{ab} - \frac{1}{2} T g_{ab} + \Lambda g_{ab},$$  \hspace{1cm} (2.41)

$$\Rightarrow R = \mu - 3p + 4\Lambda,$$  \hspace{1cm} (2.42)

$R_{ab} = R_{ac}^{\ c} \_{\ bc}$ is the *Ricci tensor* and $R = R_{\ a}^{\ a}$ is the *Ricci scalar*. $T_{ab}$ is the energy-momentum tensor of the matter present and the cosmological constant, $\Lambda$, has been included so that results remain as general as possible. The EFE's are a set of ten non-linear partial differential equations for the components of the metric which should all be satisfied in order for a solution to exist.

Substituting for the general form of the EMT given by equation (2.28) and its trace $T$ (2.29), into the EFE (2.41) gives:

$$R_{ab} = (\mu + p) u_a u_b + \frac{1}{2} (\mu - p + 2\Lambda) g_{ab} + 2\eta_{(a} u_{b)} + \pi_{ab},$$  \hspace{1cm} (2.43)

which can be written as

$$R_{ab} = R_{ab}^P + R_{ab}^I,$$  \hspace{1cm} (2.44)

where

$$R_{ab}^P = (\mu + p) u_a u_b + \frac{1}{2} (\mu - p + 2\Lambda) g_{ab},$$  \hspace{1cm} (2.45)

is the *perfect fluid* contribution to the Ricci Tensor and

$$R_{ab}^I = 2\eta_{(a} u_{b)} + \pi_{ab},$$  \hspace{1cm} (2.46)

is the extra contribution arising when we deviate from the non-equilibrium situation.

2.4.1 The Ricci Identities

Any given vector field must obey the *Ricci identity* given by:

$$\nabla_c \nabla_c u^a - \nabla_d \nabla_c u^a = R_{abcd} u^a,$$  \hspace{1cm} (2.47)

\(^6\)Remembering that we have employed geometrized units characterized by $c = 1 = 8\pi G/c^2$. 

12
where $u^a$ is the preferred velocity field and $R_{abcd}$ is the Riemann curvature tensor. The Riemann tensor has the following symmetry properties:

$$R_{abcd} = R_{[ab][cd]} = R_{cdab}, \quad R_{a[bcd]} = 0,$$

which give 20 independent components. The Riemann tensor can be algebraically separated into its trace and trace free parts, the Ricci tensor $R_{ab}$ and the Weyl tensor $C_{abcd}$, respectively. Both have 10 independent components. The Ricci tensor is determined locally at each point by the matter tensor, through the EFE (2.41), and the Weyl tensor is defined by:

$$C_{abcd} = R_{abcd} - \frac{1}{2} \left( R_{ac}g_{bd} - R_{ad}g_{bc} + R_{bd}g_{ac} - R_{bc}g_{ad} \right) + \frac{1}{6} R g_{abcd},$$

(2.49)

where $g_{abcd} \equiv g_{acbd} - g_{adcb}$. As implied by this definition, the Weyl tensor satisfies all of the symmetries of the Riemann tensor (2.48) and is trace-free:

$$C^a_{\ bcd} = 0.$$

(2.50)

Substituting for the Ricci Tensor (2.41) and the Ricci scalar (2.42) into (2.49), allows the curvature tensor to be written as

$$R_{abcd} = C_{abcd} + R_{abcd}^P + R_{abcd}^I,$$

(2.51)

where

$$R_{abcd}^P = \frac{1}{2} \left( \mu + p \right) \left( u_a \nabla c g_{bd} - u_b \nabla d g_{ac} + u_b \nabla d g_{ac} - u_b \nabla c g_{bd} \right) + \frac{1}{3} \left( \mu + \Lambda \right) g_{abcd}$$

$$= 2 \left( \mu + p \right) u_{[a} g_{d][b} u_{c]} + \frac{1}{3} \left( \mu + \Lambda \right) g_{abcd},$$

(2.52)

and

$$R_{abcd}^I = R_{[a[c} g_{d][b} + R_{[a[c} g_{d][a].$$

(2.53)

The Weyl tensor can be split relative to $u^a$ into `electric' and `magnetic' Weyl curvature parts\(^7\) according to:

$$E_{ab} = C_{abcd} u^c u^d \Rightarrow E^a_{\ a} = 0, \ E_{a b} = E_{(a b)}, \ E_{a b} u^b = 0,$$

$$H_{ab} = \frac{1}{2} \varepsilon_{ade} C_{b c d} u^c \Rightarrow H^a_{\ a} = 0, \ H_{a b} = H_{(a b)}, \ H_{a b} u^b = 0.$$  

(2.54)

Thus, we are able to write:

$$C_{abcd} = C_{abcd}^{E} + C_{abcd}^{H},$$

(2.55)

\(^7\)These are defined in analogy with the way the electromagnetic field tensor $F_{ab}$ can be split into electric and magnetic parts.
where
\[
C^E_{\ abcd} = (g_{abpq}g_{cdrs} - \eta_{abpq}\eta_{cdrs})u^pu^rE^{qs},
\]
(2.56)
\[
C^H_{\ abcd} = - (\eta_{abcd}g_{cdrs} + g_{abpq}\eta_{cdrs})u^pu^rH^{qs}.
\]
(2.57)

The curvature tensor (2.51) may now be written as \(^8\)
\[
R_{abcd} = R^P_{\ abcd} + R^I_{\ abcd} + C^E_{\ abcd} + C^H_{\ abcd}.
\]
(2.58)

2.4.2 The Bianchi Identities

The Riemann tensor always satisfies the Bianchi identities:
\[
\nabla_{[a}R_{bc]de} = 0 \Rightarrow \nabla_aR_{bcde} + \nabla_cR_{abde} + \nabla_bR_{cade} = 0.
\]
(2.59)

This identity is equivalent to the field equations (2.41), as there are the same number of components in each set. These can be contracted once to give
\[
\nabla^aR_{abcd} = \nabla_cR_{bd} - \nabla_dR_{bc}.
\]
(2.60)

Contracting again gives the twice contracted Bianchi identities:
\[
\nabla^aR_{ac} = \frac{1}{2}\nabla_cR.
\]
(2.61)

By projecting (2.61) parallel and perpendicular to \(u^a\), one obtains the conservation equations (2.32) and (2.33), respectively.

2.4.3 The Basic Propagation Equations

The evolution equations for the kinematical quantities defined in section 2.2.1 arise from the Ricci identity (2.47) for \(u^a\). We begin by first contracting (2.47) with the timelike vector \(u^a\) and allowing for the various substitutions of the first covariant derivative of \(u^a\) (2.13) and the EFE (2.41). We are then able to separate the result into a trace, symmetric trace-free and skew symmetric parts, giving the basic propagation equations shown below.

(A) The Raychaudhuri Equation

The evolution equation for the expansion \(\Theta\) is
\[
\dot{\Theta} + \frac{1}{3}\Theta^2 + 2(\sigma^2 - \omega^2) - \Lambda + \frac{1}{2}(\mu + 3p) - \Lambda = 0,
\]
(2.62)

\(^8\) This form of the curvature tensor will be useful in later calculations.
where \( A = \nabla^a \dot{u}_a \). Using the definition of the expansion as the relative change in scale factor (2.14), the equation above can be rewritten in the form

\[
3 \frac{\dot{S}}{S} = 2 (\sigma^2 - \omega^2) + A - \mu + 3p + \Lambda. \tag{2.63}
\]

This shows how the scale factor \( S \) is directly determined at each spacetime point by the matter density at that point, and leads to the identification of \( \mu + 3p \) as the active gravitational mass. It also reveals the repulsive effect of a positive cosmological constant \( \Lambda \), the tendency of vorticity to hold matter apart and contribution of the shear to contraction. The acceleration, which represents spatial pressure gradients, affects the average distance of world-lines through its divergence.

(B) The shear propagation equation

This is the symmetric trace-free part of the equation obtained by projecting on the indices \( c \) and \( a \) of the Ricci identity (2.47):

\[
\sigma^{(ab)} - D^{(a} \dot{u}^{b)} + \frac{2}{3} \Theta \sigma^{ab} - \dot{\sigma}^{(a} x^{b)} + \sigma^{(a} \sigma^{b)c} + \omega^{(a} \omega^{b)} + (E^{ab} - \frac{1}{2} \pi^{ab}) = 0, \tag{2.64}
\]

showing how the anisotropic pressure \( \pi^{ab} \) and the tidal gravitational field \( E^{ab} \) directly induce a shear distortion in the fluid flow.

(C) The vorticity propagation equation

This is derived by taking the antisymmetric part of the equation obtained from the Ricci identity (2.47) by projecting on the indices \( c \) and \( a \).

\[
\dot{\omega}^{(a)} - \frac{1}{2} (\text{curl } \dot{u})^a + \frac{2}{3} \Theta \omega^a - \sigma^a \omega^b = 0. \tag{2.65}
\]

2.4.4 Constraint Equations

These, like the propagation equations, are derived from the Ricci identity (2.47). The constraint equations are those components of the identity which are perpendicular to \( u^a \) on the index \( b \), and do not involve time derivatives of kinematic quantities.
(A) Shear constraint

This equation is obtained by contracting on the indices $c$ and $a$ of the Ricci
identity (2.47):

\[ (\text{div } \sigma)^a - \frac{2}{3} D^a \Theta + (\text{curl } \omega)^a + 2 \varepsilon^{abc} \dot{u}_b \omega_c + q^a = 0. \]  

(2.66)

This equation shows how the energy flux vector $q^a$ relates to the spatial inhomogeneity of the expansion, as well as to the spatial gradients of vorticity and shear.

(B) Vorticity constraint

From (2.47) it follows that the identity $R_{a[bcd]} u^a = 0$ implies $D[c D_d u_b] = 0$.

Multiplying by $\varepsilon^{bcd}$, we can write this identity in the form:

\[ (\text{div } \omega) - \omega^a \dot{u}_a = 0. \]  

(2.67)

(C) H constraint

The third constraint equation is also obtained from the Ricci identity. In this case the equation (2.47) is multiplied by $\varepsilon^{cfd}$ and symmetrizing on $a$ and $d$ gives:

\[ H^{ab} + 2 \dot{u}^{(a} \omega^{b)} + D^{(a} \omega^{b)} - \text{curl } \sigma^{(ab)} = 0. \]  

(2.68)

This shows that the magnetic part of the Weyl tensor is related to the curl of the shear and to the velocity distortion.

2.4.5 The vorticity-free case : $(\omega = 0)$

The case of non-zero vorticity, has two important characterizations:

(1) A local inertial frame rotates relative to a defined rest frame;
(2) The 4-velocity $u^a$ is not a gradient.

In detail, we have\(^9\)

\[ \omega = 0 \iff u_{[b} u_{c;d]} = 0 \iff u_{[b} u_{c;d]} = 0 \]

\[ \iff u^a = -f t_a \text{ for some local functions } f(x^a), \ t(x^a); \]  

(2.69)

\(^9\)Note that the semi-colon in these expressions is just the covariant derivative $\nabla$. 

16
that is, \( u^a \) is proportional to a gradient in the case of vanishing vorticity. Geometrically, this condition states that \( u^a \) is orthogonal to the surfaces \( t = \text{constant} \). These instantaneous rest spaces, orthogonal to \( u^a \), mesh together to form a spacelike hypersurface (3-surface) in space-time with \( t = \text{constant} \) and are spanned by the metric \( h_{ab} \) given in (2.2) \(^{10}\). The surfaces are unique with respect to the vector field \( u^a \). The function \( t \) can be thought of as a cosmological time coordinate defined by the fluid flow. If \( \dot{u}^a = 0 \) it can be normalized to measure proper time along the matter world lines. We then have the condition:

\[
\begin{align*}
\omega = 0 = \dot{u}^a & \iff u_{cd} = 0 \iff u_{cd} = 0 \\
& \iff u_a = -t_a, \text{ for some local function } t(x^a). 
\end{align*}
\]  

(2.70)

So that if in addition the acceleration vanishes, we can set \( f = 1 \) in (2.69). However, when the acceleration does not vanish for \( \omega = 0 \), it is possible to normalize the cosmic time to measure proper time along one world line but then, even though it synchronizes instantaneous events on the different world lines, it will not measure proper time along other world lines.

We can define an intrinsic curvature for these 3-spaces using the Ricci identity (2.47) in the surfaces. We then have:

\[
2D_cD_dX_a = (3)R_{abcd}X^b, 
\]

(2.71)

for any vector field \( X^a \) in the 3-spaces with \( X^a u_a = 0 \). Using this identity in the hypersurfaces, and using (2.41), (2.28) and (5.13), we obtain the 3-Ricci tensor \((3)R_{ab} = h^{cd}(3)R_{abcd} \) of the 3-spaces:

\[
(3)R_{ab} = -\sigma_{(ab)} - \Theta \sigma_{ab} + D_{(a} \dot{u}_{b)} + \dot{u}_{(a} \dot{u}_{b)} + \pi_{ab} + \frac{1}{3}h_{ab} \left[ 2\mu - \frac{2}{3}\Theta^2 + 2\sigma^2 + 2\Lambda \right]. 
\]

(2.72)

Using (2.72), we can obtain the 3-Ricci scalar, \((3)R = h^{ab}(3)R_{ab} \) of the 3-spaces:

\[
(3)R = 2\mu - \frac{2}{3}\Theta^2 + 2\sigma^2 + 2\Lambda, 
\]

(2.73)

which is a generalized Friedmann equation, showing how the matter content of the space-time determines the 3-space curvature. Equations (2.72) and (2.73) fully determine the 3-curvature tensor \((3)R_{abcd} \) of the orthogonal hypersurfaces.

\(^{10}\)In general, these surface elements do not mesh together to form a surface in space-time.
2.5 FLRW cosmologies

In this section, we give a basic description of the FLRW universe models, looking in particular at their dynamics and observational relations. These relations will be used quite extensively in chapter 4.

2.5.1 Basic Description of FLRW universes

The simplest solutions to the Einstein field equations (2.41) for an isotropic, homogeneous geometry are known as Friedmann-Lemaître Robertson Walker (FLRW) cosmological models with the metric given by [37]:

\[ ds^2 = dt^2 - S^2(t) \left[ \frac{dr^2}{1 - Kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right], \]  \hspace{1cm} (2.74)

where \( r, \theta \) and \( \phi \) are the usual spherical coordinates, \( K = +1, 0, \) or \(-1\) and \( S(t) \) is a scaling factor.

The FLRW universe model can be characterized by a perfect fluid matter tensor and the condition that local isotropy holds everywhere:

\[ \sigma_{ab} = \omega_{ab} = \dot{u}^a = 0 \iff E_{ab} = H_{ab} = 0 \iff \nabla_a p = 0 . \]  \hspace{1cm} (2.75)

The first condition states that the kinematical quantities are locally isotropic. This is true if and only if both the electric and magnetic tensors vanish, so that the universe model is conformally flat. The vanishing of all spatial gradients in the 3-surfaces allows for spatially homogeneous spatial sections.

The dynamics of this universe model is governed by the Friedmann equation which takes the form

\[ 3\dot{S}^2 = \mu - \Lambda = -\frac{3K}{S^2} , \]  \hspace{1cm} (2.76)

where \( K \) is constant. The solution of the Friedmann equation (2.76) depends on what form is assumed for the matter\(^{11}\).

FLRW models can be fully characterized by specifying the current rate of expansion of the universe, as well as the current mass density \( \mu_0 \). This is

\(^{11}\) Usually it is taken to be a perfect fluid with equation of state \( p = p(\mu) \), or as a superposition of fluids and sometimes as a scalar field given by the potential \( V(\phi) \).
usually expressed as a dimensionless ratio of density to critical density $\rho_c$, defined as the density parameter

$$\Omega_0 = \frac{\rho_0}{\rho_c} = \frac{\rho_0}{3H_0^2}.$$

(2.77)

If the current mass density is greater than the critical density, then $\Omega > 1$ and there is enough mass in the universe for its gravitational influence to eventually cause the universe to stop expanding, and to re-collapse in on itself [2]. This is known as a closed model. If $\rho_0 < \rho_c$, then $\Omega < 1$ and the universe will expand eternally as an open model. In between these two alternatives, if $\rho_0 = \rho_c$, the density parameter $\Omega$ is exactly equal to one, and the universe will still expand forever, but its expansion velocity will tend asymptotically towards zero. This is known as the Einstein-de Sitter model, and is said to have a flat geometry. The amount of matter in the universe also affects the curvature of space, which determines the value of $K$ in equation (2.74). Open, flat, and closed models have $K = -1, 0$ and $+1$ respectively, corresponding to the spatial hypersurfaces having negative, zero and positive curvature. We also note that the effective density parameters are defined as [16]

$$\Omega_m = \frac{\rho_m}{3H_0^2},$$

(2.78)

$$\Omega_r = \frac{\rho_r}{3H_0^2},$$

(2.79)

$$\Omega_\Lambda = \frac{\Lambda}{3H_0^2},$$

(2.80)

for the matter, radiation and cosmological constant, respectively.

The density of the universe can also be quantified through the use of the cosmic deceleration parameter $q_0$ defined by:

$$q_0 = -\frac{1}{H_0^2} \frac{\ddot{a}(t)}{\dot{a}(t)} |_{t_0}.$$

(2.81)

This specifies, in a dimensionless form, the rate at which the expansion of the universe is currently decelerating.
2.5.2 Observational Relations

Redshift

Astronomical observations are based on radiation travelling on null geodesics towards an observer, forming the past null cone. We note that the spatial homogeneity of these models allows us to choose the origin of coordinates of any light ray of interest, while the isotropic nature expresses the equivalence of all light rays regardless of the direction they are travelling in. Thus, in the case of an FLRW universe, we may consider only radial null rays as these are completely generic. These light rays then travel in the metric (2.74) such that \( ds^2 = 0 = d\theta^2 = d\phi^2 \), resulting in

\[
\frac{dt^2}{S^2(t)} = dr^2, \tag{2.82}
\]

where \( K = 0 \) for flat space-times. We then have the following relation:

\[
r = \int_E^O dr = \int_E^{t_O} \frac{dt}{S(t)} = \int_{S_E}^{S_O} \frac{dS}{S(t)S(t)}, \tag{2.83}
\]

where the radiation has been emitted at a point \( E \) and received at \( O \). Once a suitable matter description has been chosen, \( \dot{S} \) can be found from the Friedmann equation (2.76).

Now consider two successive light pulses sent from \( E \) to \( O \), each remaining at the same comoving coordinate position. If the pulses of radiation are emitted by \( E \) at times \( t_E \) so that \( t_E' = t_E + \Delta t_E \), and received by \( O \) at times \( t_O' = t_O + \Delta t_O \). Then the observed time dilation for all events at \( E \) as seen by \( O \) is \( \Delta t_O/\Delta t_E \), and we can relate the period \( \Delta t \) to the wavelength of light \( \lambda \) by setting \( \lambda = c\Delta t \), with the observed cosmological redshift defined by \( z_c \equiv \Delta \lambda/\lambda \). It then follows that in an FLRW model we have

\[
(1 + z_c) = \frac{\lambda_0}{\lambda_E} = \frac{\Delta t_O}{\Delta t_E} = \frac{S(t_O)}{S(t_E)}, \tag{2.84}
\]

using (2.83). From the relation (2.84), we can see that the cosmological redshift is a direct measurement of the expansion of the universe between the time the light was emitted and when it was received. In general, there could be local gravitational and Doppler contributions at the observer \( z_O \) and at the emitter \( z_E \), each redshift corresponding to an observed time dilation so that the overall redshift \( z \) for \( E \) as measured by \( O \) is given by

\[
(1 + z) = (1 + z_O)(1 + z_c)(1 + z_E). \tag{2.85}
\]
No direct observation by the observer can distinguish between the cosmological and local effects at the source/observer. These have to be inferred indirectly.

**Distance Measurements**

In order to determine one of the most fundamental cosmological parameters, the current mass density of the universe $\mu_0$, it is necessary to measure volumes of space and hence the distances of any sources which are observed. These distances are not directly observable, and the extragalactic distance scale is built largely on secondary distance indicators, which need to be well calibrated.

One of the most important cosmological results, the *Hubble law*, was originally obtained without good absolute distance calibration [2]. The law states that the redshift $z$ of a galaxy is proportional to its distance $D$.

$$D = \frac{z}{H_0}.$$  \hfill (2.86)

The *Hubble constant*, $H_0$, is usually written in the form

$$H_0 = 100h\text{ km s}^{-1}\text{Mpc}^{-1},$$  \hfill (2.87)

where the dimensionless constant $h$ has a true value somewhere between 0.64 and 0.8 [25].

**Area and Luminosity Distance**

In cosmology, or in any astrophysical situation, the notion of distance is meaningless unless one defines it by relating it to some observable quantity. Thus, there is no unique definition of distance. Two distance measurements which are frequently used in cosmology are the angular-diameter and luminosity distances. In this section we look at how these distances are defined and their relation to each other.

*Apparent size* is one of the fundamental issues we confront in observational cosmology. Consider a past-directed set of null geodesics, converging to an observer at time $t_0$ so that they form a solid angle of $d\Omega = \sin \theta d\theta d\phi$. From the FLRW metric (2.74), the corresponding null rays will be described by
constant values of $\theta$ and $\phi$, and will encompass an area $dA = S^2(t_E) f^2(r) d\Omega$ at time $t_E$ which is orthogonal to the light rays. Here, $r$ will be given by (2.83). Thus, we may use the standard area relation in defining the observer area distance $r_A(z)$ given by

$$r_A^2 := \frac{dA}{d\Omega}(\nu_E) \Rightarrow r_A^2 = S^2(t_E) f^2(r) . \quad (2.88)$$

The isotropic nature of these models allows one to relate this area distance to the observed angle $\alpha$ corresponding to a linear length scale $l$ orthogonal to the rays:

$$l = r_0 \alpha . \quad (2.89)$$

To obtain specific formula for the area distance $r_0$, one must assume a specific equation of state together with the Friedmann equation (2.76), or use the geodesic deviation equation [17].

First, we want to define the luminosity distance $D_l$. In flat space, the observed flux $f_0$ of a source of intrinsic luminosity $L_s$ at a distance $D$ is given by

$$f_0 = \frac{L_s}{4\pi D^2} . \quad (2.90)$$

Conversely, if one knows the source luminosity $L_s$ and the observer's flux $f_0$, one can infer the distance to the source. Hence, one can define the luminosity distance as:

$$D_l^2 := \frac{L_s}{4\pi f_0} . \quad (2.91)$$

This is often called the uncorrected luminosity distance in the literature in contrast to the corrected one, which includes the correction due to the redshift of photon frequencies.

The corrected luminosity distance can be defined by

$$D_C^2 := \frac{dA_l}{d\omega_l}(0) , \quad (2.92)$$

where $d\omega_l$ is the infinitesimal solid angle at source $E$ subtended by the area element $dA_l(0)$ at $O$. The relation between $D_l$ and $D_C$ is [43]

$$D_l = (1 + z)D_C . \quad (2.93)$$
It is known that the luminosity and area distance are not mutually independent but are essentially the same, apart from a redshift factor. This is known as the reciprocity theorem and it states that

\[ D_C = (1 + z)r_A \quad \Leftrightarrow \quad D_l = (1 + z)^2 r_A . \quad (2.94) \]

**Number Counts**

For a given solid angle \( d\Omega \) in the distance range \( (r, r + dr) \), the corresponding volume will be \( dV = S^3(t_E) r_A^2 dr d\Omega \). If we observe sources in this given solid angle \( d\Omega \), with source density \( n(t_E) \) and probability of detection \( p \), the number of sources observed will be

\[ dN = p \, n(t_E) dV = d \left[ \frac{n(t_E)}{(1 + z^3)} \right] S^3(t_0) f^2(r) dr d\Omega , \quad (2.95) \]

where \( r \) is given by (2.83). This is the basic number count relation, where \( dr \) can be expressed in terms of observable quantities such as \( dz \). If source numbers are conserved in a FLRW model so that

\[ n(t_E) = n(t_0)(1 + z)^3 , \quad (2.96) \]

then the quantity in the brackets of equation (2.95) will be constant.

It is worth noting that the FLRW number count predictions agree with observations only if source numbers and/or luminosity evolution is allowed (see [19] for a more detailed discussion).
Chapter 3

Null Geodesics and the Screen-space

3.1 Introduction

In this chapter, we introduce a congruence of null geodesics whose behavior in a particular space-time can be deduced from a set of propagation and constraint equations along the ray bundle. We also see how the various kinematical and dynamical quantities affect the dynamics of the null congruence. A general form of the geodesic deviation equation for null rays is also derived. This equation forms the basis of the exploration of the geometry of the standard Friedmann-Lemaître-Robertson-Walker (FLRW) models of relativistic cosmology discussed in section 4 below (also see [17]).

3.2 The Screen-space Approach

Null geodesics will be represented by the family of curves, $x^a(\nu)$, where $\nu$ is the affine parameter along each of these curves. For this family of curves,

$$k^a = \frac{dx^a}{d\nu}(\nu),$$

(3.1)

is the tangent vector to a null geodesic. Null geodesics have the following properties involving the null tangent vector $k^a$:

$$k^a k_a \equiv 0, \quad \frac{\delta k^a}{\delta \nu} = k^b \nabla_b k_a = 0 \Rightarrow k^b \nabla_a k_b = 0,$$

(3.2)
where we define the derivative along the ray as, $\frac{\delta}{\delta \nu} = k^a \nabla_a$. The null tangent vector $k^a$ can be split into a timelike and spacelike component

$$k^a = E(u^a + n^a), \quad n^a n_a = 1, \quad n^a u_a = 0,$$

where $E \equiv -u_a k^a$ and $n^a$ is the (unit) spatial direction of the light propagation vector. The scalar quantity $E$ can be interpreted as the energy associated with the ray. The propagation equation for $E$ along the ray can be easily determined by differentiating equation (3.3) and substituting for (2.13) giving:

$$\frac{\delta E}{\delta \nu} = -E^2 \left[ \frac{1}{3} \Theta + \sigma_{ab} n^a n^b + \dot{u}_a n^a \right],$$

which tells us the isotropic redshifting of photons due to expansion, and the anisotropic redshifting due to the shear and acceleration.

The \textit{screen-space projection tensor} may be defined as:

$$\tilde{h}_{ab} \equiv g_{ab} + 2k_{(a}l_{b)}, \quad \tilde{h}^a_\alpha = 2, \quad \tilde{h}_{ac} \tilde{h}^c_b = \tilde{h}_{ab}, \quad \tilde{h}_{ab} k^b = 0,$$

which projects vectors and tensors into a two-dimensional screen-space orthogonal to the light propagation vector $k^a$. The null vector, $l^a$, obeys the following properties:

$$l^a l_a = 0, \quad k^a l_a = -1,$$

together with

$$\frac{\delta l^a}{\delta \nu} = k^b \nabla_b l^a = 0;$$

The latter condition implies that $\tilde{h}_{ab}$ is constant along a ray:

$$\frac{\delta}{\delta \nu} \tilde{h}_{ab} = 0,$$

$$\frac{\delta^2}{\delta \nu^2} \tilde{h}_{ab} = 0.$$

We shall require a special case later where $l^a$ takes the form

$$l^a \equiv \frac{1}{2E} (u^a - n^a),$$

allowing the screen-space projection tensor (3.5) to be written as

$$\tilde{h}_{ab} = g_{ab} + u_a u_b - n_a n_b,$$
provided that $u^a$ and $n^a$ satisfy the conditions:

$$
\frac{\delta}{\delta \nu} u^a = - \frac{1}{E} \frac{\delta E}{\delta \nu} n^a,
$$

(3.12)

$$
\frac{\delta}{\delta \nu} n^a = - \frac{1}{E} \frac{\delta E}{\delta \nu} u^a,
$$

(3.13)

as may be seen by substituting equation (3.10) into (3.6). We would like to know in which situations these conditions will be satisfied. Substituting for $\frac{\delta E}{\delta \nu}$ from (3.4) into (3.12) gives

$$
\frac{\delta}{\delta \nu} u^a = E \left( \frac{1}{3} \Theta + \sigma_{bc} n^b n^c + \dot{u}_b n^b \right) n^a.
$$

(3.14)

However, we also have

$$
\frac{\delta}{\delta \nu} u^a = k^b \nabla_b u^a = E (u^b + n^b) \nabla_b u^a
$$

$$
= E \left[ \dot{u}^a + \frac{1}{3} \Theta n^a + \sigma_{ab} n_b + \omega_{ab} n_b \right],
$$

(3.15)

so that equations (3.14) and (3.15) are equivalent if and only if $\dot{u}_a = \sigma_{ab} = \omega_{ab} = 0$ which is true for FLRW cosmologies. It therefore follows that (3.7) together with (3.8) imposes a condition which is too restrictive for the analysis of null geodesics in general cosmological models - which is our aim in this chapter. We therefore drop the first requirement for now but bring it back in chapter 5 when we consider the screen-space projected null geodesic deviation equation in an almost FLRW model. In this case $\tilde{h}_{ab}$ acts only on first order quantities and so we only require it to FLRW order.

Any vector or tensorial object lying completely in the screen-space will be denoted by

$$
\mathcal{V}^a = \tilde{h}^a_b \mathcal{V}^b , \quad \mathcal{T}^{a..c}_{b..d} = \tilde{h}^a_e \tilde{h}^c_f \tilde{h}^b_g \tilde{h}^d_h T^{e..g..f..h} ,
$$

(3.16)

where we projected on all free indices of the particular object. Derivatives in the screen-space will be defined by

$$
\tilde{D}_a T_{b..c} = \tilde{h}^d_a \tilde{h}^e_b \tilde{h}^f_c \nabla_d T_{e..f}.
$$

(3.17)

A Screen-space Projected Symmetric and Trace-free (SPSTF) object will be denoted by

$$
\tilde{A}_{(ab)} = \tilde{h}^c_{(a} \tilde{h}^d_{b)} A_{cd} - \frac{1}{2} \tilde{A}^c_{(a} \tilde{h}^d_{b)}
$$

(3.18)
The 2-volume element of the screen-space is defined as

$$\tilde{S}_{ab} \equiv \eta_{cdab}u^c n^d = \tilde{S}_{[ab]} ,$$

(3.19)

with $\eta_{abcd}$ being the space-time permutator. Using (3.19), we can derive the identity

$$\tilde{S}_{ab}\tilde{S}^{cd} = 2\tilde{h}_{[a} \tilde{c} \tilde{h}_{b]} ,$$

(3.20)

which will be used in later calculations. $\tilde{S}_{ab}$ is invariant under parallel transport in the direction of $k^a$, so that

$$\tilde{D}_a\tilde{S}_{bc} = 0 = \frac{\delta}{\delta V} \tilde{S}_{bc} .$$

(3.21)

Note, we also have

$$\tilde{D}_a \tilde{h}_{ab} = 0 .$$

(3.22)

The covariant derivative of the null vector $k^a$ can be decomposed into

$$\nabla_b k_a = \frac{1}{2} \tilde{h}_{ab} \tilde{\Theta} + \tilde{\sigma}_{ab} + \tilde{\omega}_{ab} + \tilde{X}_a k_b + k_a \tilde{Y}_b + \lambda k_a k_b ,$$

(3.23)

where

$$\tilde{X}_a = \frac{1}{E} n^d \nabla_d k_a ,$$

$$\tilde{Y}_b = \frac{1}{E} n^c \nabla_b k_c ,$$

$$\lambda = -\frac{1}{E^2} n^c n^d \nabla_d k_c .$$

(3.24)

We also have $\tilde{X}_a k^a = \tilde{Y}_a k^a = 0$ and $\tilde{\sigma}^a a = 0$. Equation (3.23) may be rewritten as

$$\nabla_b k_a = \frac{1}{2} \tilde{h}_{ab} \tilde{\Theta} + \tilde{\sigma}_{ab} + \tilde{S}_{ab} \tilde{\omega} + \tilde{V}_a k_b + \tilde{W}_a k_b + \lambda k_a k_b ,$$

(3.25)

where $\tilde{V}_a = \tilde{X}_a + \tilde{Y}_a$ and $\tilde{W}_a = \tilde{X}_a - \tilde{Y}_a$.

For later convenience we define an additional vector $p^a$ as:

$$p^a \equiv u^a + n^a \quad \Rightarrow \quad k^a = Ep^a .$$

(3.26)

1Note that the vorticity tensor is just $\omega_{ab} = \tilde{S}_{ab} \tilde{\omega}$.
3.3 The Ricci Identities for $k^a$

The Ricci identity for the light propagation vector $k^a$ is

$$\nabla_c \nabla_d k_a - \nabla_d \nabla_c k_a = R_{abcd} k^b.$$  

(3.27)

3.3.1 The Propagation Equations

Contracting the Ricci Identity (3.27) with $k^d$ and using the fact that $k^a$ is geodesic, we obtain

$$R_{abcd} k^b k^d + \frac{\delta}{\delta \nu} (\nabla_c k_a) + \nabla_a k^d \nabla_d k_a = 0.$$  

(3.28)

We will see that each of the propagation equations along $k^a$ will be derived by taking the trace, SPSTF and antisymmetric part of this equation\(^2\).

3.3.2 The Null Raychaudhuri Equation

We begin by taking the trace of equation (3.28). This is done by contracting on $a$ and $c$ resulting in

$$R_{ab} k^a k^b + \frac{d}{d\nu} (\nabla_a k^a) + \nabla_a k^b \nabla_b k^a = 0.$$  

(3.29)

Now consider each term separately. We can evaluate the first term of this expression by substituting for the Ricci Tensor from the EFE (2.41). Then

$$R_{ab} k^a k^b = \left( T_{ab} - \frac{1}{2} T^{c}_{\phantom{c}cd} g_{ab} + \Lambda g_{ab} \right) k^a k^b$$

$$= T_{ab} k^a k^b$$

$$= E^2 \left( \mu + p - 2q_a n^a + \pi_{ab} n^a n^b \right).$$  

(3.30)

Using equation (3.23) we can substitute for the covariant derivative of the null tangent vector $k^a$ into the two remaining terms in (3.29). The following results are obtained:

$$\frac{\delta}{\delta \nu} (\nabla_a k^a) = \frac{\delta}{\delta \nu} \tilde{\Theta},$$  

(3.31)

$$\left( \nabla_a k^b \right) \left( \nabla_b k^a \right) = \frac{1}{2} \tilde{\Theta}^2 + 2\tilde{\sigma}^2 - 2\bar{\omega}^2.$$

(3.32)

---

\(^2\text{This is similar to the derivation of the evolution equations from the Ricci identity for } u^a \text{ described in chapter 2.}\)
Equations (3.30-3.32) can be substituted back into (3.29) giving the Null Raychaudhuri Equation or Expansion Propagation Equation:

\[
\frac{\delta \tilde{\Theta}}{\delta \nu} = -E^2 \left( \mu + p - 2q_a n^a + \pi_{ab} n^a n^b \right) - 2 \left( \bar{\sigma}^2 - \bar{\omega}^2 \right) - \frac{1}{2} \bar{\Theta}^2. \tag{3.33}
\]

By writing \( \frac{1}{\sqrt{A}} \frac{\delta \sqrt{A}}{\delta \nu} = \tilde{\Theta} \), where \( A \) represents the screen-space area, we can transform the Null Raychaudhuri Equation (3.33) into

\[
\frac{\delta^2 \sqrt{A}}{\delta \nu^2} = - \left[ \frac{1}{2} E^2 \left( \mu + p - 2q_a n^a + \pi_{ab} n^a n^b \right) + (\bar{\sigma}^2 - \bar{\omega}^2) \right] \sqrt{A}, \tag{3.34}
\]

which is just the Optical Focussing Equation \([42]\). In the case of a perfect fluid, (3.34) reduces to

\[
\frac{\delta^2 \sqrt{A}}{\delta \nu^2} = - \left[ \frac{1}{2} E^2 (\mu + p) + (\bar{\sigma}^2 - \bar{\omega}^2) \right] \sqrt{A}. \tag{3.35}
\]

### 3.3.3 The Shear Propagation Equation

We would like to extract the screen-space projected symmetric trace-free (SPSTF) part of the derivative, of the shear down the null geodesics. This can be done by taking the SPSTF part of equation (3.28) giving

\[
\tilde{h}_{(e a \tilde{h}_f)} \left( R_{abck} k^b k^d + \frac{\delta}{\delta \nu} (\nabla_c k_a) + \nabla_c k^d \nabla_d k_a \right) = 0. \tag{3.36}
\]

Now consider each term in equation (3.36) separately. Starting with the final term, we can immediately substitute for the covariant derivative of the null tangent vector from equation (3.25):

\[
\tilde{h}_{(e a \tilde{h}_f)} \nabla_c k^d \nabla_d k_a = \tilde{\sigma}_{(e f)} d + \left( \frac{1}{4} \tilde{\Theta}^2 - 2 \bar{\omega}^2 \right) \tilde{h}_{(e f)} + \tilde{\Theta} (\tilde{\sigma}_{(e f)} + \tilde{\omega}_{(e f)}) = \tilde{\sigma}_{e f} \tilde{\Theta}. \tag{3.37}
\]

The first term on the RHS of equation (3.37) vanishes due to the fact that the shear is SPSTF. The second term is zero because the metric has no trace-free part. In the final term, we may drop the \( \tilde{\omega}_{ab} \) because of its antisymmetric nature. In the final result the angular brackets on the shear can be neglected since the shear itself is SPSTF. To evaluate the second
term in (3.36), we again start by substituting for equation (3.25) and look
at the SPSTF part of this expression. We obtain

$$\frac{\delta}{\delta \nu} \left( \nabla_b k_f \right) = \frac{\delta \sigma_{(cf)}}{\delta \nu},$$

(3.38)

where we have used the result in equation (3.9).

The first term in equation (3.36) involves taking the SPSTF part of the
curvature tensor. Substituting for $R_{abcd}$ from equation (2.58) we are left with

$$\tilde{h}_{(e} \tilde{h}_{f)}^c R_{abcd} k^b k^d = \tilde{h}_{(e} \tilde{h}_{f)}^c \left( C_{abcd}^E + C_{abcd}^H \right) k^b k^d;$$

(3.39)

where the contributions from the perfect and imperfect fluid curvature terms
vanish, leaving only the trace-free part of the Riemann curvature tensor.

The electric and magnetic parts of the Weyl tensor need to be evaluated separately as described below.

**(A) The Electric Weyl Term**

We begin by substituting for the null vector $k^a$ from (3.3) giving

$$C_{abcd}^E k^b k^d = C_{abcd}^E E^2 \left( u^b u^d + 2u^{(b} n^{d)} + n^b n^d \right).$$

(3.40)

We immediately note that

$$C_{abcd}^E u^b u^d = E_{ac},$$

(3.41)

so that only the remaining two terms in (3.40) need to be calculated. Now
consider the second term in equation (3.40):

$$2C_{abcd}^E u^{(b} n^{d)} = 2(g_{abpq} g_{cdrs} - \eta_{abpq} \eta_{cdrs}) u^p u^r u^{(b} n^{d)} E^{qs}$$

$$= -(g_{aq} g_{bp} g_{cr} g_{ds} u^b n^d + g_{aq} g_{bp} g_{cs} g_{dr} u^d n^b) u^p u^r E^{qs}$$

$$= 2u_{(a} E_{c)} u^b n^d,$$

(3.42)

where we have substituted for the definition of the electric Weyl tensor
$C_{abcd}^E$ from equation (2.56) in the first step of the calculation.

---

3The Weyl tensor is the trace-free part of the curvature tensor as seen in section (2.4.1).
Next we need to evaluate the final term in (3.40):
\[
G_{abcd}^{\nu_{ab}} n^d = (g_{abpq}g_{cdrs} - \eta_{abpq}\eta_{cdrs}) u^p u^r n_b n_d E^{qs} \\
= (g_{abpq}g_{cdrs} n_b n_d u^p u^r - S_{aq} S_{cs}) E^{qs} \\
= u_a u_c E_{bc} n^b n^d - 2\tilde{h}_{ca} \tilde{h}_{cq} E^{qs} \\
= \tilde{E}_{ac} + E_{bd} n^b n^d (\tilde{h}_{ac} + u_a u_c). 
\]
(3.43)

In the second step we have used the property of the of the 2-volume element described in equation (3.20). The last step uses the fact that \( E^a_a = 0 \) implying that \( \tilde{E}^b_b = -E_{ab} n^a n^b \).

All the results obtained, equations (3.41)-(3.43), can be substituted back into (3.40) giving
\[
G_{abcd}^{\nu_{ab}} n^d = E^2 \left[ E_{ac} + \tilde{E}_{ac} + 2u_{(a} E_{c)b} n^b + E_{bd} n^b (\tilde{h}_{ac} + u_a u_c) \right] \\
= E^2 [2\tilde{E}_{ac} + E_{bd} (2u_{(a} g_{c)b} n^b + 2n_{(a} g_{c)b} n^b - n_a n_b n_c n^d + u_a n^b u_c n^d + \tilde{h}_{ac} n^b n^d)] \\
= E^2 \left( 2\tilde{E}_{ac} + E_{bd} n^b n^d\tilde{h}_{ac} \right) + 2k_{(a} E_{c)b} k^b - E_{bd} k^b k^d (n_a n_c - u_a u_c) \\
= 2E^2 \tilde{E}_{(ac)} + 2k_{(a} E_{c)b} k^b - E_{bd} k^b k^d (n_a n_c - u_a u_c) , 
\]
(3.44)

where we have used \( k^a E_{ab} = E n^a E_{ab} \).

Finally, we are now in position to calculate the term containing the electric part of the Weyl tensor in equation (3.39) giving
\[
\tilde{h}_{(e} h_{f)} \gamma_{abcd}^{\nu_{ab}} n^d = \tilde{h}_{(e} h_{f)} \gamma_{abcd}^{\nu_{ab}} [2E^2 \tilde{E}_{(ac)} + 2k_{(a} E_{c)b} k^b \\
- E_{bd} k^b k^d (n_a n_c - u_a u_c)] \\
= 2E^2 \tilde{E}_{(ef)} .
\]
(3.45)

(B) The Magnetic Weyl Term

We proceed in the same way as done for the electric part of the Weyl term above, starting by substituting for the null vector tangent \( k^a \) from (3.3) giving
\[
C_{abcd}^{\nu_{ab}} n^d = C_{abcd}^{H} E^2 \left( u^b u^d + 2u^{(b} n^d) + n^b n^d \right) . 
\]
(3.46)

\footnote{This is due to the fluid 4-velocity \( u^a \) being orthogonal to both \( E_{ab} \) and \( H_{ab} \). Analogously, we also have \( k^a H_{ab} = E n^a H_{ab} \).}
From the definition of the magnetic part of the Weyl tensor given in equation (2.57), we can evaluate each term in this expression separately.

From the skew symmetry of $\eta_{abcd}$, we immediately notice that

$$C_{abcd}^{H}u^{b}u^{d} = -(\eta_{abpq}g_{cdrs} + g_{abpq}\eta_{cdrs})u^{p}u^{r}H^{qs}u^{b}u^{d} = 0.$$  (3.47)

We now calculate the remaining two terms in (3.46):

$$2C_{abcd}^{H}u^{b}n^{d} = -(\eta_{abpq}g_{cdrs}u^{p}u^{r}n^{b} + g_{abpq}\eta_{cdrs}u^{p}u^{r}n^{d})H^{qs}$$

$$= (g_{cdrs}u^{d}u^{r}\tilde{S}_{cs} + g_{abpq}u^{p}u^{b}\tilde{S}_{cs})H^{qs}$$

$$= 2\tilde{S}_{b(a}u_{c)d}H^{bd},$$  (3.48)

$$C_{abcd}^{H}n^{b}n^{d} = -(\eta_{abpq}g_{cdrs} + g_{abpq}\eta_{cdrs})u^{p}n^{b}n^{d}H^{qs}$$

$$= (\tilde{S}_{aq}g_{cdrs}u^{d}n^{a} + \tilde{S}_{cs}g_{abpq}u^{b}n^{d})H^{qs}$$

$$= 2\tilde{S}_{b(a}u_{c)d}H^{bd}n_{d}.$$  (3.49)

The results obtained in (3.47 - 3.49) can be substituted back into equation (3.46) giving

$$C_{abcd}^{H}k^{b}k^{d} = 2E^{2}\left(\tilde{S}_{b(a}h_{c)d} + \tilde{S}_{b(a}u_{c)d}n_{d}\right)H^{bd}$$

$$= E^{2}\left[\tilde{S}_{bc}(h_{cd} + u_{c}n_{d}) + \tilde{S}_{bc}(h_{ad} + u_{a}n_{d})\right].$$  (3.50)

Now we notice that

$$(h_{ab} + u_{a}n_{b})H^{bc} = (g_{ab} + u_{a}u_{b} + u_{a}n_{b})H^{bc} = (\tilde{h}_{ab} + p_{a}n_{b})H^{bc} = (\tilde{h}_{ab} + p_{a}p_{b})H^{bc}.$$  (3.51)

where the vector $p^{a}$ is defined in (3.26). Using this result (3.51) in equation (3.50) gives

$$C_{abcd}^{H}k^{b}k^{d} = E^{2}\left[\tilde{S}_{ba}(\tilde{h}_{cd} + p_{c}p_{d}) + \tilde{S}_{bc}(\tilde{h}_{ad} + p_{a}p_{d})\right]H^{bd}$$

$$= 2\left(E^{2}\tilde{S}_{b(a}h_{c)d} + \tilde{S}_{b(a}k_{c)d}\right)H^{bd},$$  (3.52)

where we have used $H_{ab}u^{b} = 0$ and $k^{a} = Ep^{a}$.

We can now use this result (3.52) in the evaluation of equation (3.39):

$$\tilde{h}_{(c}^{a}\tilde{h}_{f)}^{c}C_{abcd}^{H}k^{b}k^{d} = \tilde{h}_{(c}^{a}\tilde{h}_{f)}^{c}(2E^{2}\tilde{S}_{b(a}h_{c)d} + 2\tilde{S}_{b(a}k_{c)d})H^{bd}$$

$$= 2E^{2}\tilde{h}_{b}^{b}(\tilde{S}_{f})_{b}.$$  (3.53)
Substituting for (3.37 - 3.53) into equation (3.36) we get the Shear Propagation Equation:

\[ \frac{\delta \tilde{\sigma}_{(ab)}}{\delta \nu} = -\tilde{\sigma}_{ab} \tilde{\Theta} - 2E^2 \left( \tilde{E}_{(ab)} + \tilde{H}_c^{(a} \tilde{S}^{b)c} \right) \]  \hspace{1cm} (3.54)

### 3.3.4 The Vorticity Propagation Equation

We now need the projected anti-symmetric part of equation (3.28). This can be obtained by multiplying through by the antisymmetric tensor \( \tilde{S}^{ca} \):

\[ \tilde{S}^{ca} \left( R_{abcd} k^b k^d + \frac{\delta}{\delta \nu} (\nabla c k_a) + \nabla a k^d \nabla d k_a \right) = 0 \]  \hspace{1cm} (3.55)

We now consider each term in equation (3.55) separately. Starting with the final term in this expression, we can immediately substitute for the covariant derivative of the null tangent vector from (3.25) giving

\[ \tilde{S}^{ca} \nabla c k^d \nabla d k_a = \tilde{S}^{ca} \left( \tilde{\sigma}^d_{ca} \tilde{\sigma}_{ad} + \tilde{\Theta} \tilde{\sigma}_{ac} + \frac{1}{4} \tilde{\Theta} \tilde{h}_{ac} + \tilde{\Theta} \tilde{\omega}_{ac} - 2 \tilde{h}_{ac} \tilde{\omega}^2 \right) \]

\[ = \tilde{S}_{ca} \tilde{\Theta} \tilde{\omega}_{ac} \]

\[ = -2 \tilde{\Theta} \tilde{\omega} \]  \hspace{1cm} (3.56)

Most of the terms evaluate to zero because they are symmetric in a and c, whereas \( S^{ca} \) is antisymmetric. We have also used the fact that the vorticity tensor can be written as \( \tilde{\omega}_{ac} = \tilde{S}_{ac} \tilde{\omega} \).

We now evaluate the second term in equation (3.55):

\[ \tilde{S}^{ca} \frac{\delta}{\delta \nu} (\nabla c k_a) = \tilde{S}^{ca} \frac{\delta}{\delta \nu} (\tilde{\omega}_{ac}) = \tilde{S}^{ca} \frac{\delta}{\delta \nu} (\tilde{S}_{ac} \tilde{\omega}) = \tilde{S}^{ca} \tilde{S}_{ac} \frac{d \tilde{\omega}}{d \nu} = -2 \frac{d \tilde{\omega}}{d \nu} , \] \hspace{1cm} (3.57)

where we have used the fact that \( \frac{\delta}{\delta \nu} \tilde{S}_{ab} = 0 \) (from (3.21)) and the antisymmetric property (3.19) in obtaining the final expression. Finally, we have

\[ \tilde{S}^{ca} R_{abcd} k^b k^d = 0 \]  \hspace{1cm} (3.58)

Substituting the results obtained in (3.56-3.58) into equation (3.55) we obtain the Vorticity Propagation Equation:

\[ \frac{\delta \tilde{\omega}}{\delta \nu} = -\tilde{\Theta} \tilde{\omega} \]  \hspace{1cm} (3.59)
3.3.5 The Energy Propagation Equation

For completeness we also include the energy propagation equation:\footnote{We note that $\sigma_{ab}$ and $\Theta$ in the energy propagation equation refer to the matter flow, not the flow along the null geodesics.}

$$\frac{\delta E}{\delta \nu} = -E^2 \left[ \frac{1}{3} \Theta + \sigma_{ab} n^a n^b + \dot{u}_a n^a \right]. \quad (3.60)$$

We see that this equation is independent of the cosmological constant $\Lambda$. This equation is the basic redshift equation used in FLRW models and in the Sachs-Wolfe effect (CBR anisotropy) calculations [7].

3.3.6 Constraint Equation

In this section we will derive the constraint equation that will be useful in the analysis done later in this thesis. We start with the Ricci identity (3.27) and multiply through by $\tilde{h}^{ce} \tilde{h}^{ad}$ to obtain the required result. So we need to evaluate the following:

$$\tilde{h}^{ce} \tilde{h}^{ad} (\nabla_c \nabla_d k_e - \nabla_d \nabla_c k_e) = \tilde{h}^{ce} R_{bced} k^b \tilde{h}^{ad}. \quad (3.61)$$

We now consider each term in (3.61). Starting with the first term we obtain:

$$\tilde{h}^{ce} \tilde{h}^{ad} \nabla_c \nabla_d k_e = \tilde{D}^e (\nabla_a k_e)$$

$$= \tilde{D}^e \tilde{\sigma}_{ae} + \frac{1}{2} \tilde{D}_a \tilde{\Theta} + \tilde{\sigma}_{ae} \tilde{D}^e \tilde{\omega} + \frac{1}{2} \tilde{\sigma}_{ae} \left( \tilde{V}^e - \tilde{W}^e \right)$$

$$+ \frac{1}{4} \tilde{\Theta} \left( 3 \tilde{V}_a - \tilde{W}_a \right), \quad (3.62)$$

where we have substituted for the covariant derivative of the null tangent vector $k^a$ from (3.25) in the first line. Next, we deal with the second term in equation (3.61), where once again we are able to substitute for (3.25) giving

$$\tilde{h}^{ce} \tilde{h}^{ad} \nabla_d \nabla_c k_e = \tilde{D}_a \tilde{h}_b \nabla_d (\nabla c k_e)$$

$$= \tilde{D}_a \tilde{\Theta} + \tilde{V}_e \nabla_a k^e + k^e \nabla_a \tilde{V}_c$$

$$= \tilde{D}_a \tilde{\Theta} + \tilde{V}_e \left( \tilde{\sigma}_{a e} + \frac{1}{2} \tilde{h}^{e a} \tilde{\Theta} \right). \quad (3.63)$$

Here we have used the fact that

$$\nabla_d \tilde{\sigma}_{ec} \tilde{h}^{ed} \tilde{h}_a = \tilde{D}_a \tilde{\sigma}_{e c} - \tilde{\sigma}_{e c} \nabla_d \tilde{h}_{ec} \tilde{h}^{d a} = 0. \quad (3.64)$$
Now we need to evaluate the term containing the curvature tensor on the RHS of (3.61). Substituting for the general form of the curvature tensor (2.58) results in

\[ \tilde{h}^{ce} R^{P}_{\text{ebcd}} k^{b} \tilde{h}^{ad} = \tilde{h}^{ce} [ R^{P}_{\text{ebcd}} + R^{I}_{\text{ebcd}} + C^{E}_{\text{ebcd}} + C^{H}_{\text{ebcd}} ] k^{b} \tilde{h}^{ad}. \] (3.65)

Each term in (3.65) can be found using the definitions of the various components of the curvature tensor given in equations (2.52), (2.53), (2.56) and (2.57) respectively:

\[ \tilde{h}^{ce} R^{P}_{\text{ebcd}} k^{b} \tilde{h}^{ad} = \frac{1}{3} (\mu + \Lambda) g_{\text{ebcd}} k^{b} \tilde{h}^{ad} \tilde{h}^{ce} = 0, \] (3.66)

\[ \tilde{h}^{ce} R^{I}_{\text{ebcd}} k^{b} \tilde{h}^{ad} = \frac{1}{2} R^{I}_{\text{bc}} k^{b} \tilde{h}^{ac} \]
\[ = \frac{1}{2} E(\tilde{h}^{ab} \pi_{bc} n^{c} - \tilde{q}^{a}), \] (3.67)

\[ \tilde{h}^{ce} C^{E}_{\text{ebcd}} k^{b} \tilde{h}^{ad} = (u^{e} u^{e} - n^{e} n^{e}) g_{\text{ebpg}} g_{\text{cdqs}} u^{p} u^{s} E^{qs} k^{b} \tilde{h}^{ad} \]
\[ = \tilde{\xi}^{ac} E_{bc} k^{c}, \] (3.68)

\[ \tilde{h}^{ce} C^{H}_{\text{ebcd}} k^{b} \tilde{h}^{ad} = n^{c} n^{e} g_{\text{ebpg}} g_{\text{cdrs}} u^{p} u^{r} H^{qs} k^{b} \tilde{h}^{ad} \]
\[ = \tilde{\xi}^{ac} H_{bc} k^{c}. \] (3.69)

Combining all the results obtained (3.61-3.69), we obtain the Constraint Equation:

\[ \tilde{D}_{\theta} \tilde{\xi}^{ab} = \frac{1}{2} \tilde{D}^{a} \tilde{\Theta} - \tilde{S}^{ab} \tilde{D}_{\theta} \tilde{\omega} + \frac{1}{2} (\tilde{\xi}^{ab} - \frac{1}{2} \tilde{h}^{ab} \Theta)(\tilde{V}_{b} + \tilde{W}_{b}) \]
\[ + (\tilde{h}^{ab} E_{bc} + \tilde{S}^{ab} H_{bc}) k^{c} + \frac{1}{2} E(\tilde{h}^{ab} \pi_{bc} n^{c} - \tilde{q}^{a}), \] (3.70)

showing the dynamics of \( \tilde{\xi}_{ab}, \tilde{\Theta} \) and \( \tilde{\omega} \) relative to each other in the screen-space.

### 3.4 Null Geodesic Deviation

#### 3.4.1 The Geodesic Deviation Equation

Consider the one-parameter family of geodesics, \( x^{a}(\nu, \omega) \), where \( \omega \) labels each geodesic and \( \nu \) is an affine parameter along each of them. The general
form of the Geodesic Deviation Equation (GDE) [16] is:

\[
\frac{\delta^2 \eta^a}{\delta\nu^2} = -R^a{}_{bcd} V^b \eta^c V^d ,
\]

(3.71)

where \( V^a = dx^a/d\nu \) represents a geodesic vector field and \( \eta^a := dx^a/d\omega \) is the geodesic deviation vector field. The geodesic deviation vector \( \eta^a \) represents the displacement between neighboring geodesics in the congruence. This deviation vector commutes with the vector \( V^a \):

\[
\frac{\delta \eta^a}{\delta\nu} = \eta^b \nabla_b V^a .
\]

(3.72)

As \( V^a \) is geodesic, it can easily be shown that (see [15])

\[
V^b \nabla_b (\eta^a V_a) = \frac{1}{2} \eta^b \nabla_b (V^a V_a) .
\]

(3.73)

Thus, if the geodesics are such that their magnitude is everywhere the same, then \( \eta^a V_a \) will be constant along each geodesic i.e.

\[
\frac{\delta (\eta_a V^a)}{\delta\nu} = 0 \Leftrightarrow \left( \eta_a V^a \right) = \text{const} .
\]

(3.74)

We can restrict the deviation vector even further. When \( V^a \) is not parallel to \( u^a \), the vector \( \eta^a \) lies in the two dimensional screen-space of \( u^a \) if both \( (\eta_a V^a) = 0 \) and \( (\eta^a u_a) = 0 \). It is possible to choose these conditions initially. We now check whether the condition \( (\eta^a u_a) = 0 \) will be maintained along the integral curves of any geodesic vector field \( V^a \). Using the equation for the covariant derivative of \( u^a \) (2.13), one finds

\[
\frac{\delta}{\delta\nu} (\eta_a u^a) = 2\omega_{ab} \eta^a \xi^b + E \eta_a \dot{u}^a - \nabla_b E \eta^b - \dot{u}_a \eta^a (u_b \eta^b) ,
\]

(3.75)

which reduces to

\[
\frac{\delta}{\delta\nu} (\eta_a u^a) = \eta^b \nabla_b (u_a \eta^a) ,
\]

(3.76)

when \( \omega_{ab} = \dot{u}_a = 0 \). The LHS of equation (3.76) is zero if \( \eta^b \nabla_b (u_a \eta^a) = 0 \). We can propagate condition (3.76) along the integral curves of \( u^a \) to confirm its preservation. We then find

\[
\left[ \frac{\delta}{\delta\nu} (\eta_a u^a) - \eta^b \nabla_b (u_a \eta^a) \right] = 0 ,
\]

(3.77)
provided that $\nabla_b u_\alpha$ is symmetric in its indices which is true when $\omega_{ab} = \delta_{ab} = 0$. In deriving (3.77), we have also used the following relation given in [17]:

$$u_\alpha \left[ u^c (\nabla_c V^b)(\nabla_b \eta^a) + v^b u^c \nabla_c \nabla_b \eta^a - u^c (\nabla_c \eta^b)(\nabla_b V^a) - \eta^b u^c \nabla_c \nabla_b V^a \right] = 0.$$  \hspace{1cm} (3.78)

Thus, the consistent solution to the above equations is

$$(\eta_a u^a) = 0 \text{ and } (\eta_a V^a)$$  \hspace{1cm} (3.79)

which follows iff the space-time is both irrotational and geodesic.

We are interested in null geodesics, so that we allow $V^a = k^a$. Since $k^a k_a = 0$ everywhere, equation (3.73) tells us that $\eta^a k_a$ is constant along the null geodesics. The Null Geodesic Deviation Equation (NGDE) is:

$$\frac{\delta^2 \eta^a}{\delta \nu^2} = -R_{abcd} k^b \eta^c k^d.$$  \hspace{1cm} (3.80)

We choose $\eta^a$ initially orthogonal to $k^a$ (i.e. $\eta^a k_a = 0$) so that through (3.74), one ensures that they remain orthogonal everywhere. It may be convenient to resolve $\eta^a$ as follows:

$$\eta^a = \tilde{\eta}^a + \eta_k k^a,$$  \hspace{1cm} (3.81)

where $\tilde{\eta}^a$ is a unit vector in the screen space, so that $\tilde{\eta}^a = \eta \tilde{\eta}^a$. We also note that $\eta^a \eta_a = \tilde{\eta}^a \tilde{\eta}_a = 1^a$.

### 3.4.2 The General Null Geodesic Deviation Equation

We would like to evaluate the RHS of the NGDE (3.80) explicitly. Substituting from equation (2.58) yields

$$\begin{align*}
-R_{abcd} k^b \eta^c k^d &= -R^p_{abcd} k^b \eta^c k^d - R^l_{abcd} k^b \eta^c k^d - C^E_{abcd} k^b \eta^c k^d \\
&= -C^H_{abcd} k^b \eta^c k^d 
\end{align*}$$  \hspace{1cm} (3.82)

Each term in (3.82) can now be calculated separately.

(A) The Perfect Fluid Curvature Term
The calculation of this term can be simplified by initially neglecting the contraction with $\eta^c$. Substituting for the perfect fluid curvature tensor (2.52) and the definition of the null vector (3.3) gives

$$-R^P_{abcd}k^b k^d = -E^2 \left[ 2(\mu + p)u_{[a}g_{d[b}u_{c]} + \frac{1}{3}(\mu + \Lambda)g_{abcd} \right] (u^b u^d + 2u^{(b}n^{d)} + n^b n^d) \, . \quad (3.83)$$

So we make the following calculations:

$$-R^P_{abcd}u^b u^d = -2(\mu + p)u_{[a}g_{d[b}u_{c]}u^b u^d - \frac{1}{3}(\mu + \Lambda)g_{abcd}u^b u^d$$
$$= -\frac{1}{6}(\mu + 3p - 2\Lambda)(g_{ac} + u_a u_c) \, , \quad (3.84)$$

$$-R^P_{abcd}u^{(b}n^{d)} = -2(\mu + p)u_{[a}g_{d[b}u_{c]}u^{(b}n^{d)} - \frac{1}{3}(\mu + \Lambda)g_{abcd}u^{(b}n^{d)}$$
$$= -\frac{1}{6}(\mu + 3p - 2\Lambda)u_{a}n_{c} \, , \quad (3.85)$$

$$-R^P_{abcd}n^b n^d = -2(\mu + p)u_{[a}g_{d[b}u_{c]}n^b n^d - \frac{1}{3}(\mu + \Lambda)g_{abcd}n^b n^d$$
$$= -\frac{1}{2}(\mu + p)u_{a}u_{c} - \frac{1}{3}(\mu + \Lambda)(g_{ac} - n_a n_c) \, , \quad (3.86)$$

Substituting equations (3.84-3.86) into (3.83) gives

$$-R^P_{abcd}k^b k^d = -\frac{1}{2}E(\mu + p)(Eg_{ac} + 2u_{(a}k_{c)}) + \frac{1}{3}(\mu + \Lambda)k_{a}k_{c} \, . \quad (3.87)$$

Finally, contracting with $\eta^c$ we obtain

$$-R^P_{abcd}k^b \eta^c k^d = -\frac{1}{2}E(\mu + p)[E\eta_{a} + (u_{(a}\eta^{c)})k_{c}] \, . \quad (3.88)$$

(B) The Imperfect Perfect Fluid Curvature Term

We first consider

$$-R^I_{abcd}k^b k^d = R^I_{abcd}(u^b u^d + 2u^{(b}n^{d)} + n^b n^d) \, , \quad (3.89)$$
where we have substituted for the null vector given in (3.3). Calculating each term and using the definition of the imperfect Riemann curvature tensor gives (2.53) one finds

\begin{align*}
-R_{abcd}^{I}u^{b}u^{d} &= \frac{1}{2} \pi_{ac}, \quad (3.90) \\
-R_{abcd}^{I}u^{b}n^{d} &= -\frac{1}{2} \left[ -(q_{d}n^{d})(u_{a}u_{c} + g_{ac}) + n_{a}q_{c} - u_{c}(\pi_{ad}n^{d}) \right] \quad (3.91) \\
-R_{abcd}^{I}u^{d}n^{b} &= -\frac{1}{2} \left[ -(q_{d}n^{d})(u_{a}u_{c} + g_{ac}) + q_{a}n_{c} - u_{a}(\pi_{cd}n^{d}) \right] \quad (3.92) \\
-R_{abcd}^{I}n^{b}n^{d} &= -\frac{1}{2} \left[ -(q_{d}n^{d})(u_{a}n_{c}) + 2u_{a}(q_{c}) + (\pi_{bd}n^{d}n^{d})g_{ac} \right. \\
&\quad - (\pi_{bc}n^{b})n_{a} - (\pi_{ad}n^{d})n_{c} + \pi_{ac} \right]. \quad (3.93)
\end{align*}

Combining all the results obtained in equations (3.90-3.93) gives

\begin{align*}
-R_{abcd}^{I}k^{b}k^{d} &= -\frac{1}{2} E[(q_{c} - \pi_{bc}n^{b} - (q_{d}n^{d})u_{c})k_{a} + (q_{a} - \pi_{ad}n^{d} \\
&\quad -(q_{d}n^{d})u_{a})k_{c} + E(\pi_{bd}n^{b}n^{d} - 2q_{d}n^{d})g_{ac}]. \quad (3.94)
\end{align*}

Contracting (3.94) with $\eta^{c}$ yields

\begin{align*}
-R_{abcd}^{I}k^{b}\eta^{c}k^{d} &= -\frac{1}{2} E[(q_{c} - \pi_{bc}n^{b} - (q_{d}n^{d})u_{c})\eta^{c}k_{a} \\
&\quad + E(\pi_{bd}n^{b}n^{d} - 2q_{d}n^{d})\eta_{a}] , \quad (3.95)
\end{align*}

where we have used the fact that $\eta^{c}k_{a} = 0$ along the null ray.

\textbf{(C) The Electric Weyl Curvature Term}

If we choose to leave out the contraction with $\eta^{c}$ in equation (3.82), the result (3.44) obtained before, can be used directly. We can immediately contract equation (3.44) with $\eta^{c}$ giving

\begin{align*}
-C_{abcd}^{E}k^{b}\eta^{c}k^{d} &= -E^{2} \left[ 2E_{ac}\eta^{c} + E_{bd}n^{b}n^{d}(\eta_{a} + 2u_{c}\eta^{c}u_{a}) \\
&\quad + E_{bc}n^{b}\eta^{c}(u_{a} - n_{a}) + (u_{c} - n_{c})\eta^{c}E_{ab}n^{b} \right]. \quad (3.96)
\end{align*}

\textbf{(D) The Magnetic Weyl Curvature Term}
We now use the result obtained in equation (3.52). So we are able to contract (3.52) with \( \eta^c \) to recover the \( C^H_{abcd} k^b \eta^c k^d \) term:

\[
-C^H_{abcd} k^b \eta^c k^d = -2E^2 \left[ \bar{S}_{d(a} H_{b)c} \eta^c + \bar{S}_{d(a} u_{c)} \eta^c H_b \eta^c H_b \eta^c \right]. \tag{3.97}
\]

### 3.4.3 The Null Geodesic Deviation Equation

We are now in a position to write down the desired *null geodesic deviation equation*. This is done by substituting (3.88), (3.95) (3.96) and (3.97) into equation (3.82) leading to the general result:

\[
\frac{\delta^2 \eta_a}{\delta \nu^2} = \frac{1}{2} E^2 \left[ \mu + p + \pi b \eta^b \eta^a - 2q_b \eta^b \right] \eta_a \\
-\frac{1}{2} E \left[ (\mu + p) u_b \eta^c + (q_b - \pi c \eta^b - q_d \eta^d u_a) \eta^c \right] k_a \\
- E^2 \left[ 2E_{ab} \eta^c + E_{bd} \eta^b \eta^d (\eta_a + 2u_c \eta^c u_a) + E_{bd} \eta^b \eta^c (u_a - n_a) \\
+(u_c - n_c) \eta^c E_{ab} \eta^b + 2\bar{S}_{d(a} H_{b)c} \eta^c + 2\bar{S}_{d(a} u_{c)} \eta^c H_b \eta^c \right]. \tag{3.98}
\]

We note that the cosmological constant \( \Lambda \) does not appear in (3.98). The quantities entering the equation through dropping the prefect fluid assumption only contribute parallelly to the deviation vector by acting like additional energy density terms. They also contribute in the \( k^a \) direction.

When dealing with perfect fluids, equation (3.98) reduces to the form:

\[
\frac{\delta^2 \eta_a}{\delta \nu^2} = \frac{1}{2} E^2 (\mu + p)(\eta_a + u_c \eta^c k_a) \\
- E^2 \left[ 2E_{ab} \eta^c + E_{bd} \eta^b \eta^d (\eta_a + 2u_c \eta^c u_a) + E_{bd} \eta^b \eta^c (u_a - n_a) \\
+(u_c - n_c) \eta^c E_{ab} \eta^b + 2\bar{S}_{d(a} H_{b)c} \eta^c + 2\bar{S}_{d(a} u_{c)} \eta^c H_b \eta^c \right]. \tag{3.99}
\]
Chapter 4

The Null Geodesic Deviation Equation in FLRW Universes

4.1 Introduction

In this chapter, we are interested in obtaining solutions to the NGDE (3.98) in an FLRW cosmology. In flat space-times we find that these solutions can be directly related to the area-distance $r_A$, discussed at the end of chapter 2. Thus, given a particular equation of state, we are able to determine solutions for $r_A$ using the NGDE. These results will give us a better understanding of how the distribution of matter in an FLRW universe model affect the propagation of null geodesics within it.

FLRW universes are conformally flat so that the Weyl curvature tensor vanishes everywhere (i.e. $C_{abcd} = 0$ in this case). We also have

$$\bar{u}_a = \omega_{ab} = 0 \Rightarrow u^a \eta_a = 0 \Leftrightarrow \eta^a \eta_a = 0 ,$$

which was shown in chapter 3. Thus, the Weyl parts in the NGDE (3.98) vanish, leaving us with

$$\frac{\delta^2 \eta_a}{\delta \nu^2} = -R^P_{\ abcd}k^b \eta^c k^d = -\frac{1}{2} E^2 (\mu + p) \eta_a ,$$

where we have used condition (4.1).

From the energy propagation equation (3.60), we have

$$\frac{\delta E}{\delta \nu} = -\frac{1}{3} E^2 \Theta .$$

41
Furthermore, we also find

\[ \frac{\delta u^a}{\delta \nu} = \frac{1}{3} \Theta E n^a. \]  

(4.4)

The null geodesic deviation vector in the screen-space is

\[ \bar{\eta}^a = \eta e^a, \quad e_a e^a = 1 \Rightarrow e_a u^a = e_a n^a = 0, \]  

(4.5)

where \( e^a \) is a unit vector in the screen. Using a parallely propagated and aligned basis allows for \( \frac{d e^a}{d \nu} = 0 \). Substituting for (4.5) into (4.2) gives

\[ \frac{d^2 \eta}{d \nu^2} = -\frac{1}{2} E^2 (\mu + p) \eta, \]  

(4.6)

which now involves the spatial component \( \eta \). It is this second order differential equation which we wish to obtain solutions for. In this chapter, we obtain explicit solutions to (4.6) in terms of the affine parameter \( \nu \) and redshift \( z \) \(^1\).

We will also like to have the relation between the affine parameter \( \nu \) and cosmic time \( t \). As we are dealing with FLRW universes, we know that spatial gradients of all functions must vanish, for otherwise we would lose homogeneity and be able to choose preferred spatial directions (i.e. the universe would be anisotropic as well). So we must have

\[ D_a f = 0, \]  

(4.7)

and the spatial part of the differential of the null affine parameter, for a suitable congruence of null geodesics, vanishes leaving the following equation relating it to the cosmic time:

\[ dt = (-u_a k^a) d \nu = E d \nu. \]  

(4.8)

### 4.2 A solution in terms of \( \nu \)

Firstly, we would like to obtain a solution for equation (4.6) explicit in the affine parameter \( \nu \). Here we will only look at the dust case. So we need to

\(^1\)As we shall see later, the solutions in terms of the redshift will give useful information regarding the behavior of these null geodesics in FLRW universes.
solve
\[ \frac{d^2 \eta}{dv^2} = -\frac{1}{2} (k_\alpha u^\alpha)^2 \mu \eta . \] (4.9)

We can set \((k_\alpha u^\alpha)|v_0 = 1\) by rescaling the affine parameter so that \(k_\alpha u^\alpha = (S_0/S)\). Because we are dealing with dust, we also have \(\mu = \mu_0 S_0^3/S^3\). So (4.6) becomes
\[ \frac{d^2 \eta}{dv^2} + \frac{1}{2} \frac{\mu_0 S_0^5}{S^5} \eta = 0 . \] (4.10)

We are interested in the case when \(K = 0\) (i.e. flat space-times), so solving the Friedmann Equation (2.76) for this particular case yields
\[ S^{\frac{1}{2}} dS = \left( \frac{1}{3} \mu_0 S_0^3 \right)^{\frac{1}{2}} dt = \left( \frac{1}{3} \mu_0 S_0^3 \right)^{\frac{1}{2}} dv \frac{dt}{dv} . \] (4.11)

Using the fact that \(dt = -(u^\alpha k_\alpha) dv \Rightarrow \frac{dt}{dv} = -\frac{S_0}{S}\) this reduces (4.11) to
\[ S^{\frac{1}{2}} dS = -\left( \frac{1}{3} \mu_0 S_0^5 \right)^{\frac{1}{2}} dv , \] (4.12)

which can be integrated to give
\[ S = \left( \frac{25}{12} \mu_0 S_0^5 \right)^{\frac{1}{2}} v^{\frac{3}{2}} . \] (4.13)

We then substitute for \(S\) into equation (4.10) resulting in the following Euler/Cauchy equation:
\[ \nu^2 \frac{d^2 \eta}{dv^2} + \frac{6}{25} = 0 , \] (4.14)

which can be solved easily to give a solution in terms of \(v\)
\[ \eta = av^{\frac{3}{2}} + bv^{\frac{3}{2}} . \] (4.15)

4.3 A solution in terms of redshift

If we allow for a change of variables to redshift \(z\), we find a solution that is true for open, closed or flat spatial sections. Thus, our solution will be independent of the type of universe model used.

We can change variables using the following relation for FLRW:
\[ \frac{dz}{dv} = E_0 H (1 + z)^2 , \] (4.16)
where the affine parameter can be rescaled such that $E_0 = -1$ and

$$\frac{dt}{d\nu} = E = \frac{1}{S(t)} H_0^{-1} = (1 + z) H_0^{-1}.$$ (4.17)

We now look at the solutions of (4.6) by considering the various forms matter takes on in a chosen universe model. We start with dust.

4.3.1 A matter dominated Universe

We would like a solution for the pure dust case:

$$\rho = 0, \quad \mu = \mu_0 (1 + z)^3.$$ (4.18)

Using this form of the energy density $\mu$ in the Friedmann Equation (2.76) gives

$$H^2 = \frac{1}{3} \mu + H_0^2 (1 - \Omega_0)(1 + z)^2,$$ (4.19)

where $\Omega_0$ is the present-day density parameter and is defined as

$$\Omega_0 \equiv \frac{\mu_0}{3H_0^2}.$$ (4.20)

We can substitute for $H$ in terms of $\mu_0$ into (4.16) using equations (4.19) and (4.20), resulting in

$$\frac{dz}{d\nu} = H_0 (1 + z)^3 \sqrt{1 + \Omega_0 z}.$$ (4.21)

We are interested in calculating $\frac{d^2\eta}{d\nu^2}$ which is given by the expression

$$\frac{d^2\eta}{d\nu^2} = \left(\frac{dz}{d\nu}\right)^2 \frac{d^2\eta}{dz^2} + \frac{d^2\nu}{dz^2} \frac{d\eta}{dz}.$$ (4.22)

We can now substitute directly for $\frac{dz}{d\nu}$ and $\frac{d^2\nu}{dz^2}$ from (4.21) into (4.22) giving

$$\frac{d^2\eta}{d\nu^2} = H_0^2 (1 + z)^5 \left[(1 + z)(1 + \Omega_0 z) \frac{d^2\eta}{dz^2} + (3 + \frac{\Omega_0}{2} + \frac{7}{2} \Omega_0 z) \frac{d\eta}{dz}\right].$$ (4.23)

Finally, on substituting for the energy density (4.18) into equation (4.6) we obtain

$$\frac{d^2\eta}{d\nu^2} = -\frac{1}{2} \mu_0 (1 + z)^5 \eta,$$ (4.24)
which can be equated with the RHS of (4.23) giving

\[(1 + \Omega_0 z)(1 + z) \frac{d^2 \eta}{dz^2} + \left(3 + \frac{7}{2} \Omega_0 z\right) \frac{d\eta}{dz} + \frac{3}{2} \Omega_0 \eta = 0. \tag{4.25}\]

It is worth noting that the angle that an observer subtends is given by, \(\theta = d\eta(z)/d\omega|_{z=0}\), where \(\omega\) labels the neighboring geodesics. Then, we have \(\frac{d\theta}{dz} = 0\), so that \(\frac{d\eta}{dz} = \theta \frac{dr_A(z)}{dz}\), which can be directly substituted into equation (4.25), giving

\[(1 + \Omega_0 z)(1 + z) \frac{d^2 r_A}{dz^2} + \left(3 + \frac{7}{2} \Omega_0 z\right) \frac{dr_A}{dz} + \frac{3}{2} \Omega_0 r_A = 0, \tag{4.26}\]

where all the \(\theta\)'s have cancelled in this equation. Thus, by solving for \(\eta\) we are simultaneously solving for the observer area distance \(r_A\). The solution to equation (4.26) was found to be

\[r_A = \frac{2}{H_0 \Omega_0^2 (1 + z)^2} \left[\Omega_0 z - (2 - \Omega_0)(\sqrt{1 + \Omega_0 z} - 1)\right], \tag{4.27}\]

giving the observer area distance as a function of redshift \(z\) in units of the present-day Hubble parameter \(H_0\).

### 4.3.2 A mixture of dust and radiation

Here we assume that the matter and the radiation do not interact with each other [17]. So each will contribute separately to the total density parameter, \(\Omega = \Omega_m + \Omega_r\), where \(\Omega_m\) is the density parameter of the matter and \(\Omega_r\) the density parameter of the radiation. So the contribution of the matter and radiation to the energy density is

\[\mu = \mu_m + \mu_r = 3H_0^2 (1 + z)^3 (\Omega_m_0 + \Omega_r_0 (1 + z)). \tag{4.28}\]

Using the Friedmann Equation (2.76) and this equation for the energy density we obtain

\[H^2 = H_0^2 (1 + z)^2 [1 + (2\Omega_0 + \Omega_m_0 + \Omega_r_0 z)]z. \tag{4.29}\]

As for the dust case, we can substitute for \(H\) into (4.16) using equation (4.29) resulting in

\[\frac{dz}{d\omega} = H_0 (1 + z)^3 [1 + (2\Omega_0 + \Omega_m_0 + \Omega_r_0 z)]^{\frac{1}{2}}. \tag{4.30}\]
Figure 4.1: Plot of the magnitude of null geodesic deviation vector $\eta(z)$ according to differential equation (4.33).

We can then substitute (4.30) into the RHS of (4.22) leaving us with

$$ \frac{d^2\eta}{dz^2} = H_0^2(1 + z)^5 \left[ (1 + z)(1 + (2\Omega_0 + \Omega_m + \Omega_0 z) z) \frac{d^2\eta}{dz^2} + \left( 3 + \Omega_0 + \frac{1}{2} \Omega_m + 8\Omega_0 z + \frac{7}{2} \Omega_m z + 4\Omega_0 z^2 \right) \frac{d\eta}{dz} \right] \quad (4.31) $$

Also, from equation (4.6) we have

$$ \frac{d^2\eta}{dz^2} = -\frac{1}{2} (1 + z)^5 H_0^2 \left[ 3\Omega_0 + 4\Omega_0(1 + z) \right] \eta, \quad (4.32) $$

where we have used the energy density in (4.28) and the fact that only the radiation will contribute to the pressure term (i.e. $p = \frac{4}{3} \mu_r = H_0^2 \Omega_0 (1 + z)^4$). Equating equations (4.31) and (4.32) we obtain:

$$ (1 + z) \left[ 1 + (2\Omega_0 + \Omega_0 + \Omega_0 z) z \right] \frac{d^2\eta}{dz^2} + \left[ 3 + \Omega_0 + \frac{1}{2} \Omega_m + 8\Omega_0 z + \frac{7}{2} \Omega_m z + 4\Omega_0 z^2 \right] \frac{d\eta}{dz} + \frac{1}{2} \left[ 3\Omega_0 + 4\Omega_0(1 + z) \right] \eta = 0. \quad (4.33) $$

Figure 4.1 shows the solutions we obtain by numerical integration of equation (4.33). A general solution to (4.33) could also be found.
\[ \eta(z) = \frac{1}{(1+z)^2} \left[ C_1 (2 - \Omega_m 0 - 2 \Omega_r 0 + \Omega_m 0 z) + C_2 (1 + \Omega_m 0 z + \Omega_r 0 z (2 + z))^\frac{1}{2} \right]. \] (4.34)

where \(C_1\) and \(C_2\) are integration constants. We can relate this expression to the observer area distance \(r_A(z)\). Using \(d/d\omega = E_0^{-1}(1+z)^{-1}d/d\nu = H(1+z)d/dz\), where \(\omega\) is the affine parameter labelling neighboring geodesics, and choosing the integration constants in equation (4.34) such that \(\eta(z = 0) = 0\), we obtain from the definition of the area distance [17]

\[ r_A := \sqrt{\frac{dA_0(z)}{d\Omega_0}} = \frac{\eta(z)|_{z'=z'}}{d\eta(z)/d\omega}|_{z=0}, \] (4.35)

\[ r_A(z) = H_0^{-1} \left[ 2\Omega_m 0 - (2 - \Omega_m 0 - 2 \Omega_r 0)(\Omega_m 0 + 2 \Omega_r 0) \right] \frac{1}{(1+z)^2} \\\n\left[ (2 - \Omega_m 0 - 2 \Omega_r 0 + \Omega_m 0 z) - \left( 2 - \Omega_m 0 - 2 \Omega_r 0 (1 + \Omega_m 0 z + \Omega_r 0 z (2 + z))^\frac{1}{2} \right) \right], \] (4.36)

which is the observer area distance in a universe containing both matter and radiation.

We now proceed in a similar way as for the matter dominated case, by substituting for \(\frac{d\theta}{dz} = \theta \frac{d}{dz} r_A\) into equation (4.33) giving

\[ (1 + z) \left[ 1 + (2\Omega_r 0 + \Omega_m 0 + \Omega_r 0 z) z \right] \frac{d^2 r_A}{dz^2} + \left[ 3 + \Omega_r 0 + \frac{1}{2} \Omega_m 0 + 8 \Omega_r 0 z \right] \left[ \frac{7}{2} \Omega_m 0 z + 4 \Omega_r 0 z^2 \right] \frac{dr_A}{dz} + \frac{1}{2} \left[ 3 \Omega_m 0 + 4 \Omega_r 0 (1 + z) \right] r_A = 0. \] (4.37)

Thus, the numerical curves that we have obtained in Figure 4.1 are just solutions for the observer area distance \(r_A(z)\).

### 4.3.3 Matter and a Cosmological Constant

In this case, the total density parameter is given by \(\Omega = \Omega_m + \Omega_\Lambda\), where \(\Omega_\Lambda\) is the density parameter due to the presence of a cosmological constant. The energy density contribution will then take the form

\[ \mu = \mu_m + \mu_\Lambda = 3H_0^2 \Omega_m 0 (1 + z)^3 + \Lambda. \] (4.38)
Once again we use the Friedmann Equation (2.76) and the energy density (4.38) to obtain

\[ H^2 = H_0^2 (1 + z)^2 (1 - \Omega_{\Lambda 0} + \Omega_{m0} z) + \frac{1}{3} \Lambda , \quad (4.39) \]

from which

\[ \frac{dz}{d\nu} = \left[ H_0^2 (1 + z)^2 (1 - \Omega_{\Lambda 0} + \Omega_{m0} z) + \frac{1}{3} \Lambda \right] \frac{1}{2} (1 + z)^2 , \quad (4.40) \]

by direct substitution of (4.39) into equation (4.6). We can use this expression in (4.22) resulting in

\[ (1 + z) \left[ (1 + z)^2 (1 - \Omega_{\Lambda 0} + \Omega_{m0} z) + \Omega_{\Lambda 0} \right] \frac{d^2 \eta}{dz^2} + [3 (1 + z)^2 (1 - \Omega_{\Lambda 0} + \Omega_{m0} z) + \frac{1}{2} (1 + z)^3 \Omega_{m0} + 2 \Omega_{\Lambda 0} \frac{d\eta}{dz} + \frac{3}{2} (1 + z)^2 \Omega_{m0} \eta = 0 . \quad (4.41) \]

Solutions to this equation can be seen in Figure 4.2. As before, we can write this equation in terms of the observer area distance \( r_A \):

\[ (1 + z) \left[ (1 + z)^2 (1 - \Omega_{\Lambda 0} + \Omega_{m0} z) + \Omega_{\Lambda 0} \right] \frac{d^2 r_A}{dz^2} + [3 (1 + z)^2 (1 - \Omega_{\Lambda 0} + \Omega_{m0} z) + \frac{1}{2} (1 + z)^3 \Omega_{m0} + 2 \Omega_{\Lambda 0} \frac{dr_A}{dz} + \frac{3}{2} (1 + z)^2 \Omega_{m0} r_A = 0 . \quad (4.42) \]
The solutions obtained in Figure 4.2 is just the area distance as a function of redshift. We start with a universe that is matter dominated (i.e. $\Omega_m = 1$ and $\Omega_A = 0$). The cosmological parameter $\Omega_A$ is then increased with the corresponding decrease in the matter density. As the cosmological constant becomes the more dominant component contributing to the total density of the universe, the value of $\eta(z)$ tends to increase. We can relate the observer area distance $r_A$ to the flux of light received from the source using the relation

$$F = \frac{L(t_E)}{(4\pi) (1 + z)^4 r_A^2},$$

where $L(t_E)$ is the luminosity of the object at emission. At a given redshift we find that the area distance becomes larger as $\Omega_{\Lambda_0}$ is increased. Thus, using relation (4.43) and assuming that the luminosity of the object is known, we see that an increase in $r_A$ results in the observed object appearing fainter in a cosmological constant dominated universe model.

### 4.4 Distance-Redshift Relation in Inhomogeneous Cosmologies

It has been customary to compare observations of distant galaxies with predictions made by strictly homogeneous models. When looking on the scale of galaxies and clusters of galaxies, the universe appears to be quite inhomogeneous, so that it is of great interest to obtain relations valid for locally inhomogeneous models which are homogeneous in the mean.

Matter which has been homogeneously spread inside an observing beam of light (and thus in the universe model being used), gravitationally focuses the beam much differently than does an equal-mass clump of externally lensing matter (see [8] and [9]). We introduce a new parameter $\nu$, where $0 \leq \nu \leq 2$, to account for this gravity-light effect. Thus, the fraction $\frac{\rho_l}{\rho_0} = \frac{\nu(\nu+1)}{6} < 1$ of the present-day mass density of the universe $\rho_0$, represents the fraction of the average mass density which is inhomogeneously distributed in observing beams (i.e. $\nu$ is the fraction of matter in the form 'clumps' and is thus called the clumpiness parameter), where $\rho_l$ is the mass density of these clumps. $\nu = 0$ corresponds to the FLRW case when all the matter in the universe

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2Note that this parameter $\nu$ should not be confused with the affine parameter along the null geodesics, which is also represented by $\nu$ elsewhere in this thesis.
is smoothed out, resulting in a completely homogeneous and isotropic universe. Conversely, \( \nu = 2 \) corresponds to the case when all matter is stored in the form of mass clumps \(^3\). As we are dealing with an \( \Omega = 1 \) universe, the fraction of matter homogeneously spread in the beam will just be given by

\[
\frac{\rho_H}{\rho_0} = \left| 1 - \frac{\rho_I}{\rho_0} \right| = \left| 1 - \frac{\nu(\nu - 1)}{6} \right| = \frac{(3 + \nu)(2 - \nu)}{6},
\]

(4.44)

where \( \rho_H \) is the mass density of the beam. We are interested in bundles of light rays not passing through these 'clumps', so that we are able to replace the energy density of the beam \( \mu \) by \( \frac{(3 + \nu)(2 - \nu)}{6} \mu \). Essentially, we are assuming that the cross-section of these beams are quite narrow so that they pass 'clumps' without being gravitational lensed. In this limiting case, we may neglect the effects of shear and vorticity along the beam.

For pressure-free FLRW models, the Optical Focusing Equation (3.35) reduces to

\[
\frac{\delta^2}{\delta \nu^2} \sqrt{A} = -\frac{1}{2} E^2 \left[ \frac{(3 + \nu)(2 - \nu)}{6} \mu \right] \sqrt{A}
\]

(4.45)

where we have neglected gravitational lensing effects, and we have substituted for the corrected energy density of the beam given. Using the form of the dust energy density equation (4.18), and the definition of \( E \) given in (4.17), equation (4.45) can be evaluated further

\[
\frac{\delta^2}{\delta \nu^2} \sqrt{A} = -\frac{1}{2} (1 + z)^2 \frac{1}{H_0^2} \left[ \frac{(3 + \nu)(2 - \nu)}{6} \right] (1 + z)^3 \mu_0 \sqrt{A}
\]

\[
= -\frac{1}{3H_0^2} (1 + z)^5 \frac{(3 + \nu)(2 - \nu)}{4} \mu_0 \sqrt{A}
\]

\[
= -\frac{\mu_0}{\mu_\sigma} (1 + z)^5 \frac{(3 + \nu)(2 - \nu)}{4} \sqrt{A}
\]

\[
= -\Omega_0 (1 + z)^5 \frac{(3 + \nu)(2 - \nu)}{4} \sqrt{A}
\]

(4.46)

\(^3\)These 'clumps' can take the form of galaxies or clusters of galaxies.
where the critical density\(^4\) is defined as \(\mu_{cr} \equiv 3H_0^2\) and the present-day density parameter is just \(\Omega_0 = \frac{\rho_0}{\mu_{cr}}\). We now introduce the observer area-distance \(r_A\) by requiring that \(r_A = \sqrt{A}\) so that (4.46) becomes

\[
\frac{\delta^2}{\delta \nu^2} r_A + \frac{(3 + \nu)(2 - \nu)}{4} \Omega_0 (1 + z)^3 r_A = 0 .
\] (4.47)

See appendix B for a short history of the distance-redshift relation\(^5\). Equation (4.16) can be used to convert (4.47) to a differential equation with respect to redshift \(z\) where we have chosen \(E_0 = H_0^{-1}\). As for the pure dust case, we obtain the conversion relation:

\[
\frac{dz}{d\nu} = (1 + z)^3 \sqrt{1 + \Omega_0 z} ,
\] (4.48)

giving rise to the equation:

\[
(1 + z)(1 + \Omega_0 z) \frac{d^2 r_A}{dz^2} + \left( 3 + \frac{\Omega_0}{2} + \frac{7}{2} \Omega_0 z \right) \frac{dr_A}{dz} + \frac{(3 + \nu)(2 - \nu)}{4} \Omega_0 r_A = 0 .
\] (4.49)

We see that this equation is similar to the one obtained in the matter-dominated case (4.25), where the correction factor \(\frac{(3+\nu)(2-\nu)}{4}\) in the final term needs to be present when taking into account the effects of inhomogeneities in our model.

This equation can be generalized to the case where both matter and a cosmological constant are present in our universe model. Once again we see that the equation looks similar to that obtained in (4.42), with a correction factor accounting for inhomogeneities in the final term:

\[
(1 + z) \left[ (1 + z)^2(1 - \Omega_{\Lambda 0} + \Omega_{m0} z) + \Omega_{\Lambda 0} \right] \frac{d^2 r_A}{dz^2} + \left[ 3(1 + z)^2(1 - \Omega_{\Lambda 0} + \Omega_{m0} z) + \frac{1}{2}(1 + z)^3 \Omega_{m0} + 2\Omega_{\Lambda 0} \right] \frac{dr_A}{dz} + \frac{(3 + \nu)(2 - \nu)}{4} (1 + z)^2 \Omega_{m0} r_A = 0 .
\] (4.50)

\(^4\)For FLRW we usually have \(\mu_{cr} = \frac{3\pi G}{8\pi G} \), but we are using geometrised units so that \(8\pi G = 1\).

\(^5\)In this discussion we have chosen to use the luminosity distance \(D_l\) rather than area distance \(r_A\) in the distance-redshift relations.
Figure 4.3: Hubble curves generated using equation (4.50) for various values of $\Omega_m$, $\Omega_\Lambda$ and $\nu$. As can be seen, the type of Hubble curve generated depends on how the matter is distributed in a chosen universe model.

This second order differential equation can be expressed in terms of the luminosity distance $D_L$ using relation (2.94) from which the Hubble curves given in figure 4.3 can be obtained. To generate these Hubble curves we plot $z/D_L = H_0$ against redshift $z$, illustrating the constant behavior of these curves at small $z$. We expect the $\Omega_m = 0.3$, $\Omega_\Lambda = 0.7$ and $\nu = 0$ curve to correspond to current observations (remembering that $\nu = 0$ is just the standard FLRW case). The extreme cases of a matter or $\Lambda$-dominated homogeneous universe model has also been plotted. One sees that the $\Lambda$-dominated model experiences the least amount of deviation from the expected curve. We have also included the "empty beam" case in which all the matter take the form of clumps. Here, large deviation from the expected curve is observed. It is clear that one needs to take into account the distribution of matter in a universe model in determining quantities such as the Hubble parameter given the density parameters [30].
4.4.1 Magnitude-redshift ($m - z$) Plots for Clumpy universes

In this section, several $m - z$ plots are given to illustrate the effects that density clumps can have on this relation and consequently in the determination of the parameters $\Omega_m$ and $\Omega_A$ (see [30]). The magnitude $m$ is defined as $m = 5 \log_{10} D_l$ [38] and is just a measure of the luminosity distance $D_l$. In the diagrams, we have plotted multiple $m - z$ curves for various values of all three parameters, $\Omega_m$, $\Omega_A$ and $\nu$ to illustrate the effects of the clumpy parameter $\nu$.

In figures 4.4 and 4.5 the sensitivity of observed magnitudes to variations of $\Omega_{m0}$ is illustrated by fixing $z$ and $\Omega_A = 0.1$. If, for example, the distance modulus of a source such as SN 1997ap at $z = 0.83$ were precisely known then a determination of $\Omega_{m0}$ could be made, assuming that $\Omega_A$ were known somehow. It can be clearly seen that the determined value of $\Omega_{m0}$ depends on the clumping parameter $\nu$. From these plots we find that the $\Omega_{m0}$ will be about 95 percent larger for the $\nu = 2$ completely clumpy universe than it will be for a $\nu = 0$ completely smooth FLRW universe. Equivalently, $\Omega_{m0}$ could be underestimated by as much as 50 percent if the FLRW model is used. The maximum underestimate is reduced to 33 percent at the smaller redshift of $z = 0.5$. These conclusions are not sensitive to the value of $\Omega_A$.

The the case of fixed $\Omega_m$ is shown in figure 4.6. Slightly different con-
Figure 4.5: $m - z$ curves for $z = 0.5$ and $\Omega_{\Lambda} = 0.1$

Figure 4.6: $m - z$ curves for $z = 0.83$ and $\Omega_m = 0.2$
clusions follow from this figure. The discrepancy in the determined value of \( \Omega_{A0} \) is approximately 0.07 for \( \nu = 2 \) compared to \( \nu = 0 \) and is not sensitive to the distance modulus. This discrepancy is halved at smaller redshifts.
Chapter 5

A Covariant Form of the Angle of Deflection

5.1 Introduction

In this chapter, we will derive the (1+3)-covariant form of the deflection angle in the weak field limit [41, 43]. The use of a quasi-Newtonian frame allows us to obtain the deflection angle in a form similar to that in the standard lensing literature. A detailed discussion of the quasi-Newtonian frame can be found in [21] and [20]. A full review of the standard lensing literature is presented in [41], from both an observational and theoretical point of view.

5.2 The Screen-space Projected Null Geodesic Deviation Equation

We are particularly interested in the form of the NGDE equation given by (3.98), in the two-dimensional screen-space orthogonal to the null tangent vector \( k^a \). Projecting the NGDE (3.98) into the screen yields:

\[
\frac{\delta^2 \eta_a}{\delta u^2} = -\frac{1}{2} E^2 \left[ \mu + p + n_a n^d n^d - 2 q_b n^b + 2 E_{bd} n^b n^d \right] \eta_a - E^2 h_a \left[ 2 E_{ec} + 2 E_{bd} n^d u_c u_e + 2 E_{bc} n^b u_c 
+ (u_c - n_c) E_{eb} n^b + 2 \tilde{S}_{d(c)H_c} n^d + 2 \tilde{S}_{d(e)H_e} n^b \right] \eta^c , \tag{5.1}
\]
where we have used the property of the projection tensor $\tilde{h}_{ab}$ given in (3.9).

In the special case of vanishing acceleration and vorticity, we find (see (3.79))

$$u_a \eta^a = 0 \Leftrightarrow \eta_a \eta^a = 0 ,$$

(5.2)

so that if these conditions are imposed on the NGDE (5.1) it reduces to the form:

$$\frac{\delta^2 \eta_a}{\delta \nu^2} = -\frac{1}{2} E^2 \left[ \mu + p + \pi_{b\alpha n^b n^d} - 2 q_b n^b \right] \eta_a$$

$$- E^2 \tilde{h}_{ab} \left[ 2 E_{ac} + 2 \tilde{S}_{(c} H_{c')} \right] \eta^c .$$

(5.3)

Using the form of the projection tensor $\tilde{h}_{ab}$ given in (3.11), equation (5.3) may be rewritten in terms of the screen-space quantities $\bar{E}_{ab}$ and $\tilde{H}_{ab}$ giving

$$\frac{\delta^2 \eta^a}{\delta \nu^2} = \left[ -\frac{1}{2} E^2 \left( \mu + p + \pi_{b\alpha n^b n^c} - 2 q_b n^b \right) \tilde{h}_{ab}^{(a \tilde{S}^b)} \right] \eta_b$$

$$- 2 E^2 \left( \bar{E}^{(c \tilde{S}^b)} + \tilde{H}^{(c \tilde{S}^b)} \right) \eta_b .$$

(5.4)

In this way,

$$\frac{\delta^2 \eta^a}{\delta \nu^2} = T^{ab} \eta_b ,$$

(5.5)

where

$$T^{ab} = -\frac{1}{2} E^2 \left( \mu + p + \pi_{b\alpha n^b n^c} - 2 q_b n^b \right) \tilde{h}_{ab}^{(a \tilde{S}^b)} \tilde{h}_{ab} - 2 E^2 \left( \bar{E}^{(a \tilde{S}^b)} + \tilde{H}^{(a \tilde{S}^b)} \right)$$

(5.6)

Relation (5.6) defines the $(2 \times 2)$ Optical Tidal Matrix of gravitational lensing theory. This tensor takes on this name because it shows how the Ricci and conformal curvatures (which are present in the 2D screen-space) govern the evolution of infinitesimal light beams. This matrix which represents the source of beam deformation is investigated in [42]. $T^{ab}$ can be decomposed as follows:

$$T_{ab} \equiv (T_1)_{ab} + (T_2)_{ab} ,$$

(5.7)

with

$$(T_1)_{ab} \equiv -\frac{1}{2} E \left( \mu + p + \pi_{b\alpha n^b n^c} - 2 q_b n^b \right) \tilde{h}_{ab} ,$$

(5.8)

$^1$Note that the projection tensor (3.11) will be used throughout this chapter.
and

\( (T_2)_{ab} \equiv -2E^2 \left( \tilde{E}_{(ab)} + \tilde{H}^c_{(a} \tilde{S}_{b)c} \right) \). \quad (5.9)

In order to see to what extent the quantities \((T_1)_{ab}\) and \((T_2)_{ab}\) affect the deflection of the beam, write

\( (T_1)_{ab} \eta^b = -\frac{1}{2} E \left( \mu + p + \pi^c_{bc} n^b n^c - 2q_b n^b \right) \eta_a \), \quad (5.10)

and

\( (T_2)_{ab} \eta^b = -2E^2 \left( \tilde{E}_{(ab)} + \tilde{H}^c_{(a} \tilde{S}_{b)c} \right) \eta^b \). \quad (5.11)

Equation (5.10) shows how the Ricci curvature affects the light bundle. We see that \((T_1)_{ab}\) brings about changes in the cross-sectional area of the bundle. \((T_2)_{ab}\) contains the Weyl curvature parts acting in the screen-space. \(\tilde{E}_{ab}\) and \(\tilde{H}_{ab}\) cause shearing effects and rotation of the ray bundle. It will be shown that the covariant form of the deflection angle can be determined from the form of the NGDE given in (5.4).

### 5.3 Application of the screen-space projected NGDE to the perturbed FLRW dust model

We now choose a universe model in which we may set up the screen-space projected NGDE (5.1) and from which other useful results will be obtained. These results will be helpful in deriving the covariant deflection angle.

#### 5.3.1 A quasi-Newtonian frame in a perturbed FLRW dust model

We choose to use a quasi-Newtonian frame for our analysis, as this choice of frame only admits scalar modes, simplifying the analysis considerably [21]. A \((1 + 3)\)-covariant analogue of Bardeen’s ‘longitudinal’ or quasi-Newtonian gauge \(^2\) [1] was developed in [21]. The quasi-Newtonian gauge is based on time slices which have zero-shear normals. The space-time manifolds of the relativistic quasi-Newtonian cosmologies, which we will denote by \((M, g, u)\) are subject to the Einstein Field Equations (2.41), containing a congruence of worldlines that is both irrotational and shear-free\(^3\). The congruence \(u\) is

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\(^2\)A gauge well adapted for the study of the evolution of matter perturbations and uses a metric-based approach.

\(^3\)These conditions are analogous to the gauge fixing conditions of the quasi-Newtonian gauge.
called a *Newtonian-like timelike congruence*.  

The quasi-Newtonian frame with 4-velocity $u^a$ is irrotational  

$$\omega_{ab}(u) = 0 , \quad (5.12)$$

and shear-free  

$$\sigma_{ab}(u) = 0 . \quad (5.13)$$

The vanishing of vorticity (5.12) reflects that all vector modes will be absent, while the vanishing of both the vorticity and shear (5.13) results in the magnetic Weyl curvature field being zero, so that tensor modes are absent from the preferred frame. Thus, the *peculiar velocity* [20] and matter density perturbations will be described by the remaining scalar modes. It follows that the magnetic Weyl curvature field of $(M, g, u)$ is zero:  

$$H_{ab}(u) = 0 , \quad (5.14)$$

reflecting the absence of gravitational radiation in a quasi-Newtonian cosmology [18]. For a given vector $W_a$ and trace-free tensor $P_{ab}$ orthogonal to $u^a$, we find:  

$$W_a = D_a W , \quad P_{ab} = D_a (D_b P) , \quad (5.15)$$

where $W$ and $P$ are scalars. The vector and tensor modes are automatically zero to linear order ($\omega_a = H_{ab} = 0$). We may then write  

$$A_a \equiv \dot{u}_a = D_a \Phi \quad \text{and} \quad u_a = D_a \left( \frac{2 \Theta}{3 \mu} \right) , \quad (5.16)$$

where $u_a$ is the peculiar velocity and $\Phi$ is the *peculiar gravitational potential* [20]. Note that we have chosen the matter in the perturbed FLRW model to be dust so that $p = 0$. However, to linear order in $v^a$, we find  

$$\pi_{ab} = 0 \quad \text{and} \quad q_a \neq 0 \Rightarrow A_a \neq 0 . \quad (5.17)$$

Imposing the quasi-Newtonian conditions on the linearized shear propagation equation for dust$^4$:  

$$\dot{\sigma}_{ab} = D_{(a} A_{b)} - 2H \sigma_{ab} - E_{ab} , \quad (5.18)$$

$^4$Here, we have included the scalar, vector and tensor modes in the evolution equation for $\sigma_{ab}$ as can be seen in [20].

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59
gives

\[ E_{ab} = D_{(a} D_{b)} \Phi , \]  

implying that the electric Weyl curvature field is a purely tidal field determined by the peculiar gravitational potential \( \Phi \). This result is in direct correspondence to the Newtonian situation [19].

The relativistic equations in the (1+3)-covariant approach do not directly provide an analogue of the Newtonian Poisson equation for the peculiar gravitational potential. Rather one finds that the equations governing the peculiar gravitational potential are the integrability conditions arising from the condition (5.13), turning the shear propagation equation (5.18) into a new constraint:

\[ \epsilon_{ab} \equiv E_{ab} - D_{(a} A_{b)} = 0 . \]  

The time evolution \( \dot{\epsilon}_{ab} \) and the spatial divergence \( D_a \epsilon_{ab} \) of this constraint equation give us the dynamical equations for the peculiar gravitational field:

\[ \dot{\Phi} = -\frac{1}{3} \Theta , \]  

and

\[ D_a (D^2 \Phi) = \frac{1}{2} D_a \mu - \frac{K}{S^2} D_a \Phi - H D_a \Theta , \]  

where \( H \) is the Hubble parameter, \( D^2 \) is the covariant Laplace operator for a constant curvature space, \( \frac{K}{S^2} \), with \( K = 0, \pm 1 \) and \( S \) is the background scale factor. Substituting for the expansion \( \Theta \) from equation (5.21) into (5.22) results in

\[ D_a \left[ D^2 \Phi - 3H \dot{\Phi} + \frac{3}{S^2} K \Phi - \frac{1}{2} \mu \right] = 0 , \]  

so that when integrated we are left with

\[ D^2 \Phi - 3H^2 (f - \Omega_K) \Phi = \frac{1}{2} \mu + C(t) , \]  

which is the (1+3)-covariant relativistic generalization of the Newtonian Poisson equation \( \nabla^2 \Phi = \frac{1}{2} \mu \). As discussed in [20], equation (5.24) can be written as

\[ D^2 \Phi - 3H^2 (f - \Omega_K) \Phi = \frac{1}{2} \mu + C(t) , \]  

where \( f(t) \) is defined as the position-independent dimensionless growth rate and \( \Omega_K \) is the mass density parameter given by \( \Omega_K = 1 - \Omega_m - \Omega_\Lambda \). As done
in [20], we may use the gauge freedom in $\Phi$, $\Phi \to \Phi + \beta(t)$, to remove the background part $\mu$ of the energy density on the right hand side of equation (5.25). This gauge freedom allows the integration constant, $C(t)$, to be absorbed into the time-dependant function $\beta(t)$, so that the Poisson-type equation can now be written as

$$D^2\Phi - 3H^2(f - \Omega_K)\Phi = \frac{1}{2} \delta \mu,$$  \hspace{1cm} (5.26)

We find a solution to (5.26) can be obtained using the Green function in a constant curvature space for the inhomogeneous Helmholtz equation. This Green function is given by [20]:

$$\Phi(t, r) = -\frac{1}{8\pi} \int \frac{\delta \mu(t, r')}{|r - r'|} e^{-H(t)\sqrt{3f(t)}|r - r'|} d^3r',$$  \hspace{1cm} (5.27)

in the case of a spatially flat background ($\Omega_K = 0$).

### 5.3.2 The matter-comoving (Lagrangian) frame

The dust matter flow of our cosmological model is moving with 4-velocity $\hat{u}$ ($\hat{u}^a \hat{u}_a = -1$) where

$$\hat{u}^a = u^a + v^a, \quad u^a v_a = 0$$  \hspace{1cm} (5.28)

and $v^a v_a \ll 1$. The quasi-Newtonian and the matter-comoving frames are in non-relativistic relative motion. Then, to linear order in $v^a$ we have the material quantities\(^5\):

$$\mu(\hat{u}) = \mu(u),$$  \hspace{1cm} (5.29)

$$p(\hat{u}) = 0,$$  \hspace{1cm} (5.30)

$$q^a(\hat{u}) = q^a - \mu v^a = 0,$$  \hspace{1cm} (5.31)

$$\pi_{ab}(\hat{u}) = 0,$$  \hspace{1cm} (5.32)

and

$$\hat{\Theta} = \Theta + D_a v^a,$$  \hspace{1cm} (5.33)

$$\hat{u}^a = \hat{u}^a + \dot{v}^a + \frac{1}{3} \Theta v^a = 0,$$  \hspace{1cm} (5.34)

$$\hat{\omega}^a = \omega^a - \frac{1}{2} (\text{curl } v)^a = -\frac{1}{2} (\text{curl } v)^a = 0,$$  \hspace{1cm} (5.35)

$$\hat{\sigma}_{ab} = \sigma_{ab} + D_a v_b = D_a v_b,$$  \hspace{1cm} (5.36)

\(^5\)It should be noted that the Lorentzian boost in equation (5.28) preserves the pressure and the energy density to linear order in $v_a$ i.e. $\hat{p} = p$ and $\hat{\mu} = \mu$.  

61
are the *kinematical variables*. The electric and magnetic Weyl curvature fields are frame-invariant to linear order in $\sigma^a$ so that

$$E_{ab}(\bar{u}) = E_{ab}(u) , \quad H_{ab}(\bar{u}) = H_{ab}(u) = 0 . \quad (5.37)$$

In general, the null geodesic deviation vector $\eta^a$ is defined as a space-time vector having both timelike and spatial components. However, in chapter 3 it was shown that

$$\text{if} \quad \dot{\bar{u}}_a = \bar{\omega}_{ab} = 0 \quad \Rightarrow \quad \bar{u}_a \eta^a = 0 , \quad (5.38)$$

so that $\eta^a$ is a completely spatial vector. Equations (5.34) and (5.35) imply that the matter-comoving frame is both irrotational and geodesic, satisfying the conditions in (5.38) and thus allowing for $\bar{u}_a \eta^a = 0$.

### 5.3.3 Screen-space Projected NGDE in the Lagrangian Frame

Imposing the restrictions (5.29)-(5.37) on the screen-space projected NGDE (5.4) results in

$$\frac{\delta^2 \eta^a}{\delta \nu^2} = -E^2 \left( \frac{1}{2} \bar{u} \bar{h}^{ab} + 2 \bar{\varepsilon}^{(ab)} \right) \eta_b , \quad (5.39)$$

where the anisotropies $\bar{q}_a$ and $\bar{\pi}_{ab}$ fall away as they are zero to linear order in $\nu^a$. It should be noted that the energy density and electric part of the Weyl tensor are frame invariant, i.e. $\mu = \bar{\mu}$ and $E_{ab} = \bar{E}_{ab}$, allowing us to write the NGDE (5.39) in terms of quantities defined in the quasi-Newtonian frame. Taking the SPSTF part of equation (3.52) gives

$$\bar{h}_{(e}^{a}(\bar{t}) f) c \left( 2E^2 \bar{\varepsilon}_{b(a} k_{c)d} + 2 \bar{\varepsilon}_{b(a} k_{c) k_d} \right) H^{bd} = 2E^2 \bar{H}^{b}_{(e} \bar{\varepsilon}_{f)b} , \quad (5.40)$$

so that if $H^{ab}$ vanishes in the preferred frame $u$, it follows from this expression that $\bar{H}^{ab} = 0$ (i.e. in the screen-space we only experience the effects of a tidal force). We then have

$$\bar{H}^{ab} = H^{ab}(\bar{u}) = 0 . \quad (5.41)$$

We would like to write equation (5.39) in terms of the peculiar gravitational potential $\Phi$. This form of the NGDE (i.e. the NGDE in terms of $\Phi$) will
be needed later when finding the deflection angle. The SPSTF part of the electric Weyl curvature field may be written as \( \tilde{E}_{(ab)} \) as

\[
\tilde{E}_{(ab)} \equiv \tilde{h}^c_{(a} \tilde{h}^d_{b)} E_{cd} - \frac{1}{2} \tilde{h}_{ab} \tilde{h}^{cd} \tilde{E}_{cd}
\]

\[= \tilde{E}_{ab} - \frac{1}{2} \tilde{h}_{ab} \tilde{h}^{cd} \tilde{E}_{cd} \]  

(5.42)

Projecting equation (5.19) into the 2D screen-space and using (5.42) gives

\[
\tilde{E}_{ab} = \tilde{D}_{(a} \tilde{D}_{b)} \Phi - \frac{1}{2} \tilde{h}_{ab} (D^2 \Phi - D^c D_c \Phi) .
\]

(5.43)

Now, the Poisson-type equation (5.26) reduces to the form

\[D^2 \Phi = \frac{1}{2} \delta \mu , \]  

(5.44)

when considering a flat universe model (i.e. \( K = 0 \)) on small scales\(^6\) in which case the expansion, \( \Theta \), can be neglected. It can be shown, that the 3-Laplacian(\( D^2 \)) is related to the Laplacian in the screen (\( \tilde{D}_c \tilde{D}^c \)) through the expression:

\[
\tilde{D}_c \tilde{D}^c \Phi = D^2 \Phi - 4 \ddot{\Phi} - \frac{1}{3} \Theta \dot{\Phi} + \dot{u}_c \nabla^c \Phi ,
\]

(5.45)

so that when using a stationary potential, i.e. \( \ddot{\Phi} = 0 \), together with \( \dot{u}_c = 0 \), equation (5.45) reduces to

\[D^2 \Phi = \tilde{D}_c \tilde{D}^c \Phi = \frac{1}{2} \delta \mu .
\]

(5.46)

In this case we find:

\[\tilde{E}_{ab} = \tilde{D}_{(a} \tilde{D}_{b)} \Phi = \tilde{E}_{(ab)}
\]

(5.47)

so that the projected NGDE (5.39) becomes\(^7\):

\[\frac{\delta^2 \eta^a}{\delta \nu^2} = -E^a \left[ 2 \tilde{D}^a \tilde{D}^b \Phi + (\tilde{D}_c \tilde{D}^c \Phi) \tilde{h}^{ab} \right] \eta_b , \]

(5.48)

\(^6\)Here, we are looking at scales well below the Hubble radius.

\(^7\)In equation (5.39), we have substituted for \( \frac{1}{2} \mu \), in terms of the potential \( \Phi \) given in (5.46). It should be noted that in equation (5.39), we are dealing with the perturbation of the energy density \( \mu \), as it is this perturbation that causes the ray bundle to be gravitationally lensed.
so that the (2 × 2) optical tidal matrix now has the form

\[ T_{ab} = -E^2 \left[ 2\tilde{D}_a\tilde{D}_b\Phi + \left( \tilde{D}_c\tilde{D}^c\Phi \right) \tilde{h}_{ab} \right]. \tag{5.49} \]

We expect to recover the standard form of the lens equation from the first term of equation (5.48). This term represents lensing due to a massive object such as a galaxy or cluster of galaxies. The second term represents lensing due to the presence of matter along the path of the ray bundle in our universe model. This is what is known as the cumulative lensing effect [27].

### 5.4 (1 + 3)-Covariant form of the Deflection Angle

In deriving the (1 + 3)-covariant form of the deflection angle, we will use an approach similar to [43] when deriving the cosmological lens equation. The unit vector, \( n^a \), has been introduced before and lies in the 3-dimensional rest-space formed by the timelike vector \( u^a \):

\[ n^a = \frac{1}{E} \left( k^a - E u^a \right), \quad E \equiv -k_au^a, \tag{5.50} \]

where \( E \) is the energy (or frequency) of the light ray measured in the cosmic rest frame \( u^a \). From the above definition of \( n^a \), the proper distance \( dy \) that rays travel during an infinitesimal affine distance \( d\nu \) in the rest frame \( u^a \) is be given by:

\[ \left( \frac{dy}{d\nu} \right)^2 = E^2 n^a n_a = E^2, \tag{5.51} \]

so that

\[ dy = E |d\nu|. \tag{5.52} \]

Our aim is to integrate the NGDE along a past-directed null geodesic \( \gamma(\nu) \) from an observer \( O \) to the source \( S \) through the lens plane. The initial conditions are

\[ \eta^a \big|_0 = 0 \quad \text{and} \quad \left( \frac{\delta \eta^a}{\delta \nu} \right) \big|_0 = E_o \left( \frac{\delta \eta^a}{\delta y} \right) \big|_0 = E_o d\phi^a, \tag{5.53} \]
where \( d\vec{\theta}^a \) is an infinitesimal angular vector. For the unperturbed (undeflected) ray

\[
\left( \frac{\delta \eta^a}{\delta \nu} \right)_{t_-} = E_t \left( \frac{d\eta^a}{dy} \right)_{t_-} = E_t d\vec{\theta}^a, \tag{5.54}
\]

while

\[
\left( \frac{\delta \eta^a}{\delta \nu} \right)_{t_+} = E_t \left( \frac{d\eta^a}{dy} \right)_{t_+} = E_t d\vec{\theta}^a, \tag{5.55}
\]

for the deflected ray, where the subscripts \( l_-, l_+ \) and \( l_+ \) denote the quantities evaluated at, just before and just after the lens plane respectively. We have assumed that the photon energy \( E \) is conserved during this deflection (i.e. the photon energy remains \( E \) throughout the deflection process).

The lensing geometry is illustrated in figure 5.1. \( d\vec{\theta}^a \) represents the angular vector that \( \eta^a \) (i.e. the geodesic deviation vector at the lens plane) subtends at the observer \( O \), while \( d\vec{\theta}^a \) is the angular vector that \( \eta^a \), (i.e. the geodesic deviation vector at the source) subtends at the lens plane once deflection of the ray has taken place.

We also assume a geometrically thin lens and small deflection of light rays. Motivated by these assumptions, the ray path can be approximated by combining its incoming and outgoing asymptotes. An infinitesimal deflection 'angle', \( d\vec{\alpha}^a \), will be defined as the difference between the initial and final ray direction

\[
\left( \frac{\delta \eta^a}{\delta \nu} \right)_{l_-} - \left( \frac{\delta \eta^a}{\delta \nu} \right)_{l_+} = E_t \left( d\vec{\theta}^a - d\vec{\theta}^a \right) = E_t d\vec{\alpha}^a, \tag{5.56}
\]

at the lens plane.

We may write

\[
\left( \frac{\delta \eta^a}{\delta \nu} \right)_{l_+} - \left( \frac{\delta \eta^a}{\delta \nu} \right)_{l_-} = \lim_{\epsilon \to 0} \int_{\nu_1 - \epsilon}^{\nu_1 + \epsilon} d\nu \left( \frac{\delta^2 \eta^a}{\delta \nu^2} \right), \tag{5.57}
\]

\footnote{In our analysis, the term "angle" refers to a vectorial quantity possessing both magnitude and direction. Naturally, we allow this vector to change in the direction of the ray.}

\footnote{As the tilde on \( \alpha^a \) suggests, the deflection angle lies completely in the screen-space. For small angles, \( \vec{\alpha}^a \) will be described as angular vectors in the observer tangent plane. The 2D screen-space is just the lens plane at \( \nu = \nu_1 \).}
so that the NGDE can be substituted from equation (5.48) giving the following integral with respect to proper distance $\eta$ (see (5.52)):

$$
\left( \frac{\delta \eta^a}{\delta \nu} \right)_{l+} - \left( \frac{\delta \eta^a}{\delta \nu} \right)_{l-} = \int_{-\infty}^{\infty} dy \frac{E_i}{E_l} \left[ -E_i^2 (2\tilde{D}^a \tilde{D}_b \Phi + (\tilde{D}_c \tilde{D}^c \Phi) \tilde{h}^{a}_{b} ) \right] \eta^b_l
$$

$$
= -2 \int_{-\infty}^{\infty} dy \frac{E_i}{E_l} \left( \tilde{D}^a \tilde{D}_b \Phi \right) \eta^b_l
$$

$$
+ \int_{-\infty}^{\infty} dy \frac{E_i}{E_l} \left( \tilde{D}_c \tilde{D}^c \Phi \right) \eta^a_l .
$$

(5.58)

At the lens plane, we made the following approximations:

$$
E = E_l ,
$$

$$
\eta^a = \eta^a_l ,
$$

(5.59)

so that quantities such as the photon energy and the null geodesic deviation vector, are evaluated at the lens plane as the subscript $l$ suggests. The proper
distance \( y \) is measured in the direction of the ray so that the derivatives defined in the screen-space \( \tilde{D}^a \) can be taken out of the integral, i.e.

\[
\left( \frac{\delta \eta^a}{\delta \nu} \right)_{\nu^+} - \left( \frac{\delta \eta^a}{\delta \nu} \right)_{\nu^-} = -2E_i \tilde{D}^a \tilde{D}_b \left[ \int_{-\infty}^{\infty} dy \Phi \right] \eta^b_i + E_i \dot{\tilde{D}}_a \tilde{D}^c \left[ \int_{-\infty}^{\infty} dy \Phi \right] \eta^a_i .
\]

(5.60)

Finally, we can write (5.60) as:

\[
\left( \frac{\delta \eta^a}{\delta \nu} \right)_{\nu^+} - \left( \frac{\delta \eta^a}{\delta \nu} \right)_{\nu^-} = -2E_i \left( \tilde{D}^a \tilde{D}_b \Psi \right) \eta^b_i + E_i \left( \dot{\tilde{D}}_a \tilde{D}^c \Psi \right) \eta^a_i ,
\]

(5.61)

where \( \Psi = \int_{-\infty}^{\infty} dy \Phi \).

We will use a potential of the form (5.27) which can be written as:

\[
\Phi(r^a) = -\frac{G}{c^2} \int d^3 r' \frac{\delta \mu(r^{a'})}{|r^a - r^{a'}|},
\]

(5.62)

when working in the \((1+3)\)-covariant framework and the Hubble parameter \( \tilde{H} = 0 \) due to the vanishing of the expansion. It should be noted that we will re-introduce the units at this stage as this will allow us to directly compare the covariant form of the deflection angle with the standard result. Using (5.62), we may evaluate the following integral:

\[
\int_{-\infty}^{\infty} dy \Phi(r^a) = -\frac{G}{c^2} \int_{-\infty}^{\infty} dy \int d^3 r' \frac{\delta \mu(r^{a'})}{|r^a - r^{a'}|} \\
= -\frac{G}{c^2} \int_{-\infty}^{\infty} dy \int d^2 z' dy' \frac{\delta \rho(z', y')c^2}{\sqrt{|\bar{z}^a - \bar{z}^{a'}|^2 + |y - y'|^2}} \\
= -\frac{G}{c^2} \int d^2 z' \tilde{\Sigma}(\bar{z}^{a'}) \int_{-\infty}^{\infty} dy \frac{dy}{\sqrt{|\bar{z}^a - \bar{z}^{a'}|^2 + |y - y'|^2}} \\
= 2G \frac{\int d^2 z' \tilde{\Sigma}(\bar{z}^{a'}) \ln |\bar{z}^a - \bar{z}^{a'}|}{r_c} \\
= \Psi(\bar{z}^a),
\]

(5.63)

where comoving coordinates are chosen such that\(^{10}\) \( r^a = (\bar{z}^a, y) \) with \( y \) being along the direction of the light path and \( \bar{z}^a \) being the two dimensional comoving coordinates on the lens plane. We also have \( r_c \) being an arbitrary

\(^{10}\)We have chosen coordinates \( \bar{z}^a \), in the lens plane, to be Eulerian.
cutoff length. The integral in the final expression in (5.63) extends over the whole lens plane. The surface mass density is defined by

\[ \tilde{\Sigma}(\tilde{x}^a) \equiv \int d\tilde{y}^i \delta\rho(\tilde{r}^a), \]  

(5.64)

where \( \delta\rho(\tilde{r}^a) \) is the perturbation in the mass density. It should be noted that in expression (5.63), we first perform the integration of \( \delta\rho \) with respect to \( y' \), allowing the surface mass density, \( \Sigma \), to be dependent on \( \tilde{x}^a \). In (5.63), the integration with respect to \( y \) can be pulled through the 2-dimensional screen integration \( d^2\tilde{x}' \), as we are integrating in the direction of the null ray with the screen remaining orthogonal to the null tangent vector. When doing the integration with respect to \( y \), \( \tilde{x}^a \) is treated like a constant giving the \( \ln \)-function observed in (5.63).

Using equations (5.56) and (5.61), one finds an expression for the infinitesimal deflection angle:

\[ d\tilde{\alpha}^a = 2 \left( \tilde{D}^a \tilde{D}_b \psi \right) \eta^b - \left( \tilde{D}_c \tilde{D}^c \psi \right) \eta_a \]  

(5.65)

From the definition of the angular diameter distance, the null geodesic deviation vector evaluated at the lens plane will be given by the expression

\[ \tilde{\eta}^a = r_A(z_l) d\tilde{\theta}^\alpha, \]  

(5.66)

where \( r_A(z_l) \) is the angular-diameter distance from observer O to the lens at redshift \( z_l \), and \( d\tilde{\theta}^\alpha \) is the infinitesimal angular vector that \( \eta^a \) subtends at O.

We may now substitute (5.66) into (5.65) resulting in

\[ d\tilde{\alpha}^a = 2 \left( \tilde{D}^a \tilde{D}_b \psi \right) r_A(z_l) d\tilde{\theta}^b - \left( \tilde{D}_c \tilde{D}^c \psi \right) r_A(z_l) d\tilde{\theta}^a. \]  

(5.67)

The Eulerian coordinates, \( \tilde{x}^a \) can be re-scaled through the relation:

\[ \tilde{\theta}^a = \frac{1}{r_A(z_l)} \tilde{x}^a, \]  

(5.68)

where \( \tilde{\theta}^a \) represents the angular coordinates seen by an observer. We would like to rewrite the function \( \psi \) in terms of these angular coordinates. This results in,

\[ \psi = \frac{2G}{c^2} \int d^2\tilde{\theta}^a \tilde{\Sigma}(\tilde{x}^a) \ln \left( \frac{\tilde{\theta}^a - \tilde{\theta}^a}{\theta_c} \right), \]  

(5.69)

68
where \( \theta_c \equiv \frac{r}{r_A(z_l)} \). Using the fact that the derivatives \( \tilde{D}^a \) commute in the lens plane, equation (5.67) becomes

\[
d\tilde{\alpha}^a = 2r_A(z_l)d\tilde{\theta}^b \left( \tilde{D}_b \tilde{D}^a \Psi \right) - r_A(z_l)d\tilde{\theta}^a \left( \tilde{D}_c \tilde{D}^c \Psi \right)
= 2r_A(z_l)d\tilde{\theta}^b \frac{\partial}{\partial \tilde{\theta}^b} \left[ \frac{\partial \Psi}{\partial \tilde{\theta}^a} \right] - r_A(z_l)d\tilde{\theta}^a \frac{\partial^2 \Psi}{\partial \tilde{\theta}^c \partial \tilde{\theta}^c}
= d \left[ 2r_A(z_l) \frac{\partial}{\partial \tilde{\theta}^a} \Psi \right] - r_A(z_l)d\tilde{\theta}^a \frac{\partial^2 \Psi}{\partial \tilde{\theta}^c \partial \tilde{\theta}^c},
\] (5.70)

where the differential operator \( d \) is defined as \( d = d\tilde{\theta}^a \frac{\partial}{\partial \tilde{\theta}^a} \). Substituting for \( \Psi \) into equation (5.70) yields

\[
d\tilde{\alpha}^a = d \left[ \frac{4G}{c^2} r_A(z_l) \int d^2 \tilde{\theta}^b \tilde{\Sigma}(\tilde{\theta}^b) \frac{\tilde{\theta}^a - \tilde{\theta}^a}{|\tilde{\theta}^d - \tilde{\theta}^d|^2} \right]
- r_A(z_l)d\tilde{\theta}^a \frac{\partial^2}{\partial \tilde{\theta}^c \partial \tilde{\theta}^c} \left( \frac{2G}{c^2} \int d^2 \tilde{\theta}^b \tilde{\Sigma}(\tilde{\theta}^b) \ln \left| \frac{\tilde{\theta}^d - \tilde{\theta}^d}{\theta_c} \right| \right),
\] (5.71)

which is the infinitesimal deflection a ray undergoes in the presence of some deflecting object. The first term in equation (5.71) represents the contribution to lensing due to the presence of some deflecting mass, and is not a cumulative effect (i.e. statistical lensing effects due to many deflecting bodies). We expect the second term of (5.71) to be a direct result of cumulative lensing effects. We would like to compare our result to that found in the standard lensing literature in which lensing due to large structures are considered. Thus, we are only interested in the first term in equation (5.71). We find:

\[
d\tilde{\alpha}^a = d \left[ \frac{4G}{c^2} r_A(z_l) \int d^2 \tilde{\theta}^b \tilde{\Sigma}(\tilde{\theta}^b) \frac{\tilde{\theta}^a - \tilde{\theta}^a}{|\tilde{\theta}^d - \tilde{\theta}^d|^2} \right],
\] (5.72)

which is just the differential form of \( \tilde{\alpha}^a \), i.e.

\[
\tilde{\alpha}^a(\tilde{\theta}^a) = \frac{4G}{c^2} r_A(z_l) \int d^2 \tilde{\theta}^b \tilde{\Sigma}(\tilde{\theta}^b) \frac{\tilde{\theta}^a - \tilde{\theta}^a}{|\tilde{\theta}^d - \tilde{\theta}^d|^2},
\] (5.73)

giving the (1+3)-covariant form of the deflection angle.
5.4.1 Comparison with the Standard form of the Deflection Angle

We would like to compare the covariant form of the deflection angle (5.73) to that obtained in the standard lensing literature (see [41], [43]). In the conventional approach of deriving the lens equation, one may use the thin-lens approximation introduced earlier. Then the deflection angle, $\tilde{\alpha}$, for a lens of point mass $M$ at position $\tilde{\theta}'$ is

$$
\tilde{\alpha}(\tilde{\theta}) = \frac{4GM}{c^2} \frac{\tilde{\theta} - \tilde{\theta}'}{|\tilde{\theta} - \tilde{\theta}'|^2} D_l ;
$$

(5.74)

where $\tilde{\theta}$ describes the position of the light ray in the lens plane and $D_l$ is the distance between the lens and the observer. When looking at the deflection angle of several point masses, the various contributions to $\tilde{\alpha}$ can just be added. Decomposing a general matter distribution into small components of mass $M_i$ (so that we have a system of point mass¹), allows the deflection angle to be written as:

$$
\tilde{\alpha}(\tilde{\theta}) = \sum_i \frac{4GM_i}{c^2} \frac{\tilde{\theta} - \tilde{\theta}_i}{|\tilde{\theta} - \tilde{\theta}_i|^2} D_l ,
$$

(5.75)

where $\tilde{\theta}_i$ is the position of a mass element $M_i$ in the lens plane. When looking at a continuous mass distribution of lens, we may generalize (5.75) by taking the continuum limit and replace the sum by an integral. If we define $dM = \tilde{\Sigma}(\tilde{\theta}) d^2\tilde{\theta}$ we find

$$
\tilde{\alpha}(\tilde{\theta}) = \frac{4G}{c^2} \int d^2\tilde{\theta} \tilde{\Sigma}(\tilde{\theta}') \frac{\tilde{\theta} - \tilde{\theta}'}{|\tilde{\theta} - \tilde{\theta}'|^2} D_l ,
$$

(5.76)

where $d^2\tilde{\theta}$ is the surface element on the lens and $\tilde{\Sigma}(\tilde{\theta})$ is the surface mass density at position $\tilde{\theta}$ which results if the volume mass distribution of the lens is projected onto the lens plane. This is the standard form of the lens equation, which can only be used once the geometries with and without the lens object have been well-defined¹¹. If a perturbed FLRW universe is used, we may identify the distance measure, $D_l$, with the angular-diameter distance $r_A(z_l)$ measured from the observer (at $z = 0$) to the lens at redshift $z_l$. On doing this, one notices a remarkable similarity between the standard form of the deflection angle, and that obtained in (5.73).

¹¹Only then will the distance, $D_l$, and surface mass density, $\tilde{\Sigma}(\tilde{\theta})$, have a unambiguous meaning as discussed in [43].
5.4.2 A more general form of the deflection angle

In deriving the deflection angle (5.71), several plausible but rather idealized assumptions were made. We would now like to see how far we can relax these assumptions without changing much of the result and its implications. As stated before, equation (5.73) describes lensing due to a massive object such as a galaxy or cluster of galaxies. We would like to understand the effect that inhomogeneities, other than this lensing object, have on gravitational lensing of a light beam.

Consider a perturbed FLRW universe model with some parameter\textsuperscript{12} \( \beta \) measuring the fraction of matter in 'clumpy' form or the degree of inhomogeneity in the model. As in the Dyer-Roeder approximation (see chapter 4), we assume that the light beam is very narrow so that it passes these matter inhomogeneities without the effect of being gravitationally lensed. To this inhomogeneous model, we add a deflector (represented by a galaxy, cluster of galaxies, etc.), so that light rays in the vicinity of this lens may be deflected. Again we assume that deflection angle of the light rays will be small. As in the previous section, the NGDE will be integrated along a past-directed null geodesic \( \gamma(\nu) \) from an observer \( O \) to a source \( S \).

The linearity of the NGDE

\[
\frac{\delta^2}{\delta \nu^2} \eta^a(\nu) = -(R^a_{cde} k^c k^e) \eta^d, \tag{5.77}
\]

implies that the value of a solution at the end point \( S \) depends linearly on the initial values at \( O \):

\[
\frac{\delta^2 \eta^a}{\delta \nu^2}(\nu) = \frac{1}{E_0} A^a_b(\nu) \left( \frac{\delta \eta^b}{\delta \nu} \right)_0 = A^a_b(\nu) d\delta^b, \tag{5.78}
\]

where the linear map, \( A^a_b(\nu) \), maps vectors at the observer \( O \) into vectors at \( \nu \).\textsuperscript{13} Substituting this relation into the NGDE (5.77) results in

\[
\frac{\delta^2}{\delta \nu^2} (A^a_b(\nu)) = -(R^a_{cde} k^c k^e) A^d_b, \tag{5.79}
\]

\textsuperscript{12}Note that in chapter 4, this parameter was labelled \( \nu \).

\textsuperscript{13}This mapping is formally known as the 'Jacobi map'.

71
with initial conditions \( A^a_b(0) = 0 \) and \( \frac{\delta}{\delta \psi^a} A^a_b(0) = \delta^a_b E_0 \). At the lens plane, the geodesic deviation vector \( \eta^a \) becomes \(^{14}\)

\[
\eta^a = A^a_b(\nu) d\theta^b .
\]

Substituting for (5.80) into equation for the deflection angle, \( d\alpha^a \), given in (5.65) results in

\[
d\alpha^a = 2 \left( \tilde{D}^a \tilde{D}_b \psi \right) A^b_c d\delta^c - \left( \tilde{D}_c \tilde{D}^c \psi \right) A^a_c d\delta^c .
\]

(5.81)

We may write the screen-space derivative as \( \tilde{D}^a = \frac{\partial}{\partial \theta^a} \), and using the fact that these derivatives commute, (5.81) becomes

\[
d\alpha^a = \frac{\partial}{\partial \theta^b} \left( \frac{\partial}{\partial \theta^a} \psi \right) A^b_c d\delta^c - \frac{\partial}{\partial \theta^c} \left( \frac{\partial}{\partial \theta^a} \psi \right) A^a_c d\delta^c ,
\]

(5.82)

so that substituting for \( \psi \) into this expression results in

\[
d\alpha^a = \left[ \frac{\partial}{\partial \theta^b} \left( \frac{2G}{c^2} \int d^2 \hat{v} \hat{L} \left( \hat{r}^b \right) \frac{\hat{r}^a - \hat{r}^a'}{\hat{r}^d - \hat{r}^d'} \right) A^b_c 
- \frac{\partial}{\partial \theta^c} \frac{\partial}{\partial \theta^a} \left( \frac{2G}{c^2} \int d^2 \hat{v} \hat{L} \left( \hat{r}^b \right) \ln \left| \frac{\hat{r}^d - \hat{r}^d'}{\hat{r}^c} \right| \right) A^a_c \right] d\delta^c .
\]

(5.83)

This is the differential form of the deflection angle in the general situation. The fact that our solution contains an unspecified linear mapping, \( A^a_b \), makes (5.83) a more general form of the deflection angle compared to that obtained in (5.71). The deflection angle (5.71) makes use of the area-distance \( r_A \) as a distance measure in a perturbed FLRW cosmology. Equation (5.83) can be used in any space-time once a distance relation/measure has been established through the mapping \( A^a_b \).

\(^{14}\)Note that this form of the null geodesic deviation vector is more general than that used in the definition (5.66). This is due to the fact that we have not specified the form of the mapping \( A^a_b \) in defining \( \eta^a \), making our results more general and versatile.

72
Part II

A (1+1+2)-Covariant Approach to Gravitational Lensing
Chapter 6

(1+1+2)-Approach to Gravitational Lensing in Spherically Symmetric Space-times

6.1 Introduction

In this chapter, we describe the 1+1+2 covariant approach which can be applied to gravitational lensing of a light ray in any given spherically symmetric space-time. A more detailed discussion of this approach can be seen in [4], from which most of the material presented in section 7.2 has been drawn. We begin with a motivation for using the 1+1+2 approach rather than the conventional 1+3 covariant method used successfully in part A of this thesis. We also write the null tangent vector $k^a$, introduced in chapter 3, in terms of its 1+1+2 components. Allowing for this decomposition gives one more insight of the lensing geometry for a light ray travelling in a spherically symmetric space-time. From this geometry, we can directly obtain the general form of the deflection angle $\alpha$, for spherically symmetric lensing situations. We conclude this chapter by deriving the propagation equations along $k^a$ for the 'lensing variables'. These equations are derived in a local rotationally symmetric (LRS) space-time. The propagation equations for the lensing variables are used in chapter 8, where we investigate lensing effects in a Schwarzschild space-time.
6.2 The (1+1+2)-Covariant sheet Approach

6.2.1 Motivation for using (1+1+2)-approach

The (1+3)-covariant approach described in Part A of this thesis has proven to be a very useful technique in cosmology, not only in studying null congruences and gravitational lensing, but in many other areas of cosmology. This approach has also been useful in developing a good understanding of many aspects in relativistic cosmology, whether working with the full nonlinear GR equations or by the application of the gauge-invariant, covariant perturbation formalism [22]. These methods can be applied to the formation and evolution of density perturbations [23] in the universe and to cosmic microwave background physics [36, 5], amongst other things. The strength of this approach lies in the fact that all the essential information describing a particular system is captured in a set of (1+3)-covariant variables which are defined relative to a preferred time-like congruence, allowing these variables to have an immediate physical and geometrical significance. These variables then satisfy a set of evolution and constraint equations which are derived from the Einstein Field Equations, the Bianchi and Ricci identities, forming a closed system of equations for a chosen equation of state describing the matter.

The extension of gauge-invariant, covariant perturbations in an astrophysical setting was explored in [4], using a (1+1+2) approach. In particular, it could be applied to linear perturbations of a Schwarzschild model. The advantage of this formalism to the (1+3) approach lies in the fact that in general astrophysical situations (such as the interior of compact objects), significant results can be obtained quite simply. Allowing for a (1+3) decomposition of variables relative to a timelike observer results in complicated equations and constraints from which the physics cannot be easily recovered.

In addition to the splitting with respect to a timelike vector $u^a$, the (1+1+2)-approach relies on the further splitting of the space-time using a spacelike vector $e^a$. The Bianchi and Ricci identities can then be split using both $u^a$ and $e^a$ giving rise to a set of coupled first order differential equations and constraints. Differential operators are along the two vector fields resulting in a set of evolution and propagation (along $e^a$) equations (see [4] for the full set of equations). We also denote a differential operator orthogonal to the spacelike vector $e^a$ (and thus lying in the two-dimensional sheet) by $\delta_a$. 

75
6.2.2 Formalism of (1+1+2)-Covariant sheet Approach

In the (1+3)-covariant approach, a 4-velocity field, \( u^a = dx^a/d\tau \), was introduced which represents the average 4-velocity of a set of observers in the space-time. The projection tensor, \( h_{ab} = g_{ab} + u_a u_b \), can be used to project any vector or tensor orthogonal to \( u^a \). Thus, \( h_{ab} \) allows for the decomposition of any 4-vector into a component parallel to \( u^a \) and a component lying in the 3-space orthogonal to \( u^a \). This (1+3)-formalism is based on choosing a timelike vector field \( u^a \) from which a 3-space, whose projection tensor is given by \( h_{ab} \), can be clearly identified.

Another vector field can be introduced from which we are able to perform an additional split of the (1+3) equations found in Part A. Introducing a unit spatial vector \( e^a \) which lies orthogonal to the \( u^a \) so that \( u^a e_a = 0 \) and \( e^a e_a = 1 \). The projection tensor

\[
N^a_b \equiv h^a_b - e^a e^b = g^a_b + u_a u_b - e^a e^b , \quad N^a_a = 2 ,
\]

(6.1)

projects vectors orthogonal to \( e^a \) and the timelike vector \( u^a \), i.e. \( e^a N_{ab} = 0 = u^a N_{ab} \). Thus, \( N_{ab} \) projects an object onto 2-surfaces which we call the 'sheet'. In using the (1+3)-covariant approach in the study of null congruences (see chapter 2), we introduced the spatial vector, \( n^a \), which was chosen such that \( n^a n_a = 1 \) and \( u^a n_a = 0 \) (i.e. \( n^a \) is a unit spatial vector). The tensor, \( \tilde{h}^a_b \equiv h^a_b - n^a n_b \), could then be defined as a projection tensor, which projects a vector or tensor into the 2-dimensional screen-space, orthogonal to the null tangent vector \( k^a \). In the analysis done below, we have not allowed for the spatial vectors \( n^a \) and \( e^a \) to coincide. However, if we choose \( n^a \) to coincide with \( e^a \), it immediately follows that \( \tilde{h}^a_b = N^a_b \), so that the screen and sheet represent the same 2-dimensional surface.

Any 3-vector, \( \psi^a \), can be irreducibly split into a component along \( e^a \) and a sheet component \( \Psi^a \), orthogonal to \( e^a \) i.e.

\[
\psi^a = \psi^a + \Psi^a , \quad \Psi \equiv \psi^a e_a \quad \text{and} \quad \Psi^a \equiv N^a_b \psi_b .
\]

(6.2)

A similar decomposition can be done for a PSTF tensor, \( \psi_{ab} \), which can be split into scalar, vector and tensor part:

\[
\psi_{ab} = \psi_{(ab)} = \psi (e_a e_b - \frac{1}{2} N_{ab}) + 2 \psi_{(a} e_{b)} + \Psi_{ab} ,
\]

(6.3)
where
\[
\Psi \equiv e^a e^b \psi_{ab} = -N^{ab} \psi_{ab} , \\
\Psi_a \equiv N_a^b e^c \psi_{bc} , \\
\Psi_{ab} \equiv \psi_{\{ab\}} \equiv \left( N_c^c N_{(ab)}^d - \frac{1}{2} N_{ab} N^{cd} \right) \psi_{cd} ,
\]
(6.4)
and the curly brackets denote the PSTF part of a tensor with respect to $e^a$. We also have
\[
h_{(ab)} = 0 , \quad N_{(ab)} = -\varepsilon_{(a} e_{b)} = N_{ab} - \frac{2}{3} h_{ab} .
\]
(6.5)
We now define the alternating Levi-Civita 2-tensor as:
\[
\varepsilon_{ab} \equiv \varepsilon_{abc} e^c = \eta_{abcd} e^c u^d ,
\]
(6.6)
where $\varepsilon_{abc}$ is the 3-space permutation symbol and $\eta_{abcd}$ is just the space-time permutator. With the definition of $\varepsilon_{ab}$ given by equation (6.6) above, we also have the following relations \footnote{Note that for any 2-vector $\Psi^a$, we may use $\varepsilon_{ab}$ to form a vector that will lie orthogonal to $\Psi^a$ but of the same length.}:
\[
\varepsilon_{abc} e^b = 0 = \varepsilon_{(ab)} ,
\]
(6.7)
\[
\varepsilon_{abc} = e_a \varepsilon_{bc} - e_b \varepsilon_{ac} + e_c \varepsilon_{ab} ,
\]
(6.8)
\[
\varepsilon_{ab} \varepsilon^{cd} = N_a^c N_b^d - N_a^d N_b^c ,
\]
(6.9)
\[
\varepsilon_a^c \varepsilon_{bc} = N_{ab} , \quad \varepsilon^{ab} \varepsilon_{ab} = 2 .
\]
(6.10)
From these definitions it follows that any object in the 1+1+2 setting can be split into scalars, 2-vectors in the sheet, and PSTF 2-tensors (also defined in the sheet.) These are the three objects that remain after the splitting has occurred. In the 1+1+2 formalism we are introduced to two new derivatives:
\[
\hat{\psi}_{a..b} \varepsilon^{c..d} \equiv e^f \nabla_f \psi_{a..b} \varepsilon^{c..d} , \\
\delta_f \psi_{a..b} \varepsilon^{c..d} \equiv N_a^f \ldots N_b^g N_h^c \ldots N_t^d N_j^f \nabla_j \psi_{f..g}^{i..j} .
\]
(6.11)
The hat-derivative represents the derivative along the $e^a$ vector-field. The $\delta$-derivative is the projected derivative onto the sheet\footnote{We note that one needs to project on every free index when calculating the $\delta$-derivative.}.
We are now able to decompose the covariant derivative of $e^a$ orthogonal to $u^a$ giving

$$D_a e_b = e_a a_b + \frac{1}{2} \phi N_{ab} + \xi e_{ab} + \zeta_{ab} ,$$  \hspace{1cm} (6.12)

where

$$a_a = e^a D e_a = \delta_a ,$$  \hspace{1cm} (6.13)

$$\phi = \delta_a e^a ,$$  \hspace{1cm} (6.14)

$$\xi = \frac{1}{2} e^{ab} \delta_a e_b ,$$  \hspace{1cm} (6.15)

$$\zeta_{ab} = \delta_{(a e_b)} .$$  \hspace{1cm} (6.16)

We see that when travelling along $e^a$, $\phi$ represents the expansion of the sheet, $\zeta_{ab}$ is the shear of $e^a$ (i.e. the distortion of the sheet) and $a^a$ its acceleration. We can also interpret $\xi$ as the vorticity associated with $e^a$ so that it is a representation of the 'twisting' or rotation of the sheet.

We now split the (1+3) kinematical variables and Weyl tensors into the irreducible 1+1+2 set \{$\Theta, A, \Omega, \Sigma, \varepsilon, H, A^a, \Sigma^a, \varepsilon^a, H^a, \Sigma_{ab}, \varepsilon_{ab}, H_{ab}$\} using equations (6.2) and (6.3) giving:

$$\dot{u}^a = A e^a + A^a ,$$  \hspace{1cm} (6.17)

$$\omega^a = \Omega e^a + \Omega^a ,$$  \hspace{1cm} (6.18)

$$\sigma_{ab} = \Sigma \left( e_a e_b - \frac{1}{2} N_{ab} \right) + 2 \Sigma (a e_b) + \Sigma_{ab} ,$$  \hspace{1cm} (6.19)

$$E_{ab} = \varepsilon \left( e_a e_b - \frac{1}{2} N_{ab} \right) + 2 \varepsilon (a e_b) + \varepsilon_{ab} ,$$  \hspace{1cm} (6.20)

$$H_{ab} = H \left( e_a e_b - \frac{1}{2} N_{ab} \right) + 2 H (a e_b) + H_{ab} .$$  \hspace{1cm} (6.21)

We have now obtained the 1+1+2 form of the kinematical and Weyl variables corresponding to the irreducible parts of $\nabla_a e_b$ (6.22) given below.

The full covariant derivative of $e^a$ is:

$$\nabla_a e_b = -Au_a u_b - u_a \alpha_b + \left( \Sigma + \frac{1}{3} \Theta \right) e_a u_b + [\Sigma_a - \varepsilon_{ac} \Omega^c] u_b$$

$$+ e_a a_b + \frac{1}{2} \phi N_{ab} + \xi e_{ab} + \zeta_{ab} ,$$  \hspace{1cm} (6.22)
and

\[ \nabla_a u_b = -u_a (A e_b + A_b) + e_a e_b \left( \frac{1}{3} \Theta + \Sigma \right) + e_a (\Sigma_b + e_{bc} \Omega^c) + (\Sigma_a - \varepsilon_{ac} \Omega^c) e_b + N_{ab} \left( \frac{1}{3} \Theta - \frac{1}{2} \Sigma \right) + \Omega e_{ab} + \Sigma_{ab} \, , \] (6.23)

for the decomposition of the (1+3)-covariant derivative of \( u^a \) (2.13). These derivatives will be used in later calculations. The derivative of \( e^a \) in the direction of \( u^a \) is given by

\[ \dot{e}_a = A u_a + \alpha_a \, , \quad \text{where} \quad A = e^a \dot{u}_a \, , \] (6.24)

and \( \alpha_a \) is the component lying in the sheet. The new variables \( a_a, \phi, \xi, \zeta_{ab} \) and \( \alpha_a \) are fundamental objects of the space-time and their dynamics give us information about the space-time geometry. Essentially, they are treated on the same footing as the kinematical variables of \( u^a \) in the 1 + 3 approach.

The covariant derivative of a scalar \( \Psi \) is

\[ D_a \Psi = \dot{\Psi} e_a + \delta_a \Psi \, . \] (6.25)

While for any vector \( \Psi^a \) orthogonal to both \( u^a \) and \( e^a \) (i.e. \( \Psi^a \) lies in the sheet), the various parts of its spatial derivative may be decomposed as follows\(^3\):

\[ D_a \Psi_b = -e_a e_b \Psi_c e^c + e_a \dot{\Psi}_b - e_b \left[ \frac{1}{2} \phi \Psi_a + (\xi \varepsilon_{ac} + \zeta_{ac}) \Psi^c \right] + \delta_a \Psi_b \, . \] (6.26)

Similarly, for a tensor \( \Psi_{ab} \) (where \( \Psi_{ab} = \Psi_{(ab)} \)):

\[ D_a \Psi_{bc} = -2e_a e_b \Psi_{(d)} e^d + e_a \dot{\Psi}_{bc} - 2e_b \left[ \frac{1}{2} \phi \Psi_{(c)} e^d + \Psi_c (\xi \varepsilon_{ad} + \zeta_{ad}) \right] + \delta_a \Psi_{bc} \, . \] (6.27)

We include the following relations which may be useful:

\[ \dot{N}_{ab} = 2u_{(a} \dot{u}_{b)} - 2e_{(a} \varepsilon_{b)} = 2u_{(a} A_{b)} - 2e_{(a} \alpha_{b)} \, , \]

\[ \dot{N}_{ab} = -2e_{(a} \alpha_{b)} \, , \]

\[ \delta_{c} N_{ab} = 0 \, , \] (6.28)

\(^3\)Note that a bar on a particular index indicates that the vector or tensor lies in the sheet.
while
\[
\begin{align*}
\dot{\varepsilon}_{ab} &= -2u_{[a}e_{b]c}A^c + 2e_{[a}e_{b]c}a^c, \\
\varepsilon_{ab} &= 2e_{[a}e_{b]c}a^c, \\
\delta_c\varepsilon_{ab} &= 0 .
\end{align*}
\] (6.29)

We may also split the anisotropic fluid variables \(q^a\) and \(\pi_{ab}\) giving:
\[
\begin{align*}
q^a &= Qe^a + Q^a , \\
\pi_{ab} &= \Pi \left( e_a e_b - \frac{1}{2} N_{ab} \right) + 2\Pi (e_a e_b) + \Pi_{ab} .
\end{align*}
\] (6.30, 6.31)

6.3 Gravitational Lensing in Spherically Symmetric Space-times

This section begins with a presentation of the various kinematical relations valid in the general local rotationally symmetric (LRS) space-time. A detailed discussion of the covariant approach to LRS perfect fluid space-times can be seen in [22]. However, in deriving the lensing geometry in section 7.3.2, we restrict ourselves to the case of spherically symmetric space-times. We also derive the propagation equations for the lensing variables along \(k^a\) in the general LRS space-time. These equations will be needed in chapter 8 where we derive the deflection angle in the Schwarzschild space-time.

6.3.1 Kinematical Quantities

As discussed in section 7.2, the (1+1+2)-approach requires a further splitting of space-time variables, in addition of the split with respect to the time-like vector \(u^a\). This extra splitting was done relative to the vector \(e^a\). In the case of LRS space-times, \(e^a\) is just a vector pointing along the axis of symmetry and is thus called a 'radial' vector. LRS space-times are defined to be isotropic allowing for the vanishing of all \((1+1+2)\) vectors and tensors so that there are no preferred directions in the sheet. The full covariant derivative of the radial vector \(e^a\) (6.22) and 4-velocity \(u^a\) (6.23) reduce to the forms:
\[
\nabla_a e_b = -A u_a u_b + \left( \Sigma + \frac{1}{3} \Theta \right) e_a u_b + \frac{1}{2} \phi N_{ab} + \xi e_{ab} ,
\] (6.32)
and
\[ \nabla_a u_b = -\mathcal{A} u_a e_b + e_a e_b \left( \frac{1}{3} \Theta + \Sigma \right) + N_{ab} \left( \frac{1}{3} \Theta - \frac{1}{2} \Sigma \right) + \Omega e_{ab}, \quad (6.33) \]
respectively. We also find
\[ \dot{e}_a = \mathcal{A} u_a, \quad (6.34) \]
is the evolution of \( e^a \).

The kinematical quantities and Weyl tensors (6.17) - (6.31) reduce to the form:
\begin{align*}
\dot{\omega}^a &= \mathcal{A} e^a, \\
\omega^a &= \Omega e^a, \\
\sigma_{ab} &= \Sigma (e_a e_b - \frac{1}{2} N_{ab}), \\
q^a &= Q e^a, \\
\pi_{ab} &= \Pi \left( e_a e_b - \frac{1}{2} N_{ab} \right), \\
E_{ab} &= \mathcal{E} (e_a e_b - \frac{1}{2} N_{ab}), \\
H_{ab} &= \mathcal{H} (e_a e_b - \frac{1}{2} N_{ab}),
\end{align*}
in an LRS space-time where we have not made the perfect fluid assumption (i.e. \( Q \neq \Pi \neq 0 \)).

We now list the set of 1+1+2 equations resulting in any LRS (and thus any spherically symmetric) space-time:\footnote{These equations were obtained through private communication with Chris Clarkson.}
\begin{align*}
\phi &= \left( \frac{2}{3} \Theta - \Sigma \right) \left( \mathcal{A} - \frac{1}{2} \phi \right) + 2 \xi \Omega + Q, \\
\dot{\xi} &= \left( \frac{1}{2} \Sigma - \frac{1}{3} \Theta \right) \xi + \left( \mathcal{A} - \frac{1}{2} \phi \right) \Omega + \frac{1}{2} \mathcal{H}, \\
\dot{\Omega} &= + \mathcal{A} \xi + \Omega \left( \Sigma - \frac{2}{3} \Theta \right), \\
\dot{\mathcal{A}} - \frac{1}{3} \mathcal{A} - \Sigma &= - \mathcal{A}^2 + \left( \frac{1}{3} \Theta + \Sigma \right)^2 + \frac{1}{6} (\mu + 3p - 2\Lambda) + \mathcal{E}
\end{align*}
\begin{equation}
\frac{-1}{2}\Pi ,
\end{equation}
\begin{equation}
\dot{A} - \dot{\Theta} = -(\dot{u} + \phi)A + \frac{1}{3}\Theta^2 + \frac{3}{2}\Sigma^2 - 2\Omega^2 + \frac{1}{2}(\mu + 3p)
\end{equation}
\begin{equation}
-\Lambda ,
\end{equation}
\begin{equation}
\dot{\Sigma} - \frac{2}{3}\dot{A} = \frac{1}{3}(2\dot{A} - \phi)\dot{u} - \left(\frac{2}{3}\Theta + \frac{1}{2}\Sigma\right)\Sigma - \frac{2}{3}\Omega^2 - \varepsilon
\end{equation}
\begin{equation}
+ \frac{1}{2}\Pi ,
\end{equation}
\begin{equation}
\dot{\mu} + \dot{Q} = -\Theta(\mu + p) - (\phi + 2\dot{A})Q - \frac{3}{2}\Sigma,\Pi ,
\end{equation}
\begin{equation}
\dot{Q} + \dot{\rho} + \dot{\Pi} = -\left(\frac{3}{2}\phi + A\right)\Pi - \left(\frac{4}{3}\Theta + \Sigma\right)Q - (\mu + p)A ,
\end{equation}
\begin{equation}
\dot{\epsilon} + \frac{1}{2}\dot{\Pi} + \frac{1}{3}\dot{Q} = + \left(\frac{3}{2}\Sigma - \Theta\right)\epsilon - \frac{1}{2}\left(\frac{1}{3}\Theta + \frac{1}{2}\Sigma\right)\Pi
\end{equation}
\begin{equation}
+ \frac{1}{3}\left(\frac{1}{2}\phi - 2A\right)Q + 3\xi\mathcal{H} - \frac{1}{2}(\mu + p)\Sigma ,
\end{equation}
\begin{equation}
\dot{\mathcal{H}} = -3\xi\epsilon + \left(\Theta + \frac{3}{2}\Sigma\right)\mathcal{H} + \Omega Q + \frac{3}{2}\xi\Pi ,
\end{equation}
\begin{equation}
\dot{\phi} = -\frac{1}{2}\phi^2 + 2\xi^2 + \left(\frac{1}{3}\Theta + \Sigma\right)\left(\frac{2}{3}\Theta - \Sigma\right) - \frac{2}{3}(\mu + \Lambda)
\end{equation}
\begin{equation}
- \frac{1}{2}\Pi - \varepsilon ,
\end{equation}
\begin{equation}
\dot{\xi} = -\phi\xi + \left(\frac{1}{3}\Theta + \Sigma\right)\Omega ,
\end{equation}
\begin{equation}
\dot{\Sigma} - \frac{2}{3}\dot{\Theta} = -\frac{3}{2}\phi\Sigma - 2\xi\Omega - Q ,
\end{equation}
\begin{equation}
\dot{\Omega} = +(A - \phi)\Omega ,
\end{equation}
\begin{equation}
\dot{\epsilon} - \frac{1}{3}\dot{\mu} + \frac{1}{2}\dot{\Pi} = -\frac{3}{2}\phi\left(\epsilon + \frac{1}{2}\Pi\right) + \frac{1}{2}\Sigma Q + 3\Omega\mathcal{H} ,
\end{equation}
\begin{equation}
\dot{\mathcal{H}} = -\frac{3}{2}\phi\mathcal{H} - \left(3\epsilon + \mu + p - \frac{1}{2}\Pi\right)\Omega - Q\xi .
\end{equation}
\begin{equation}
0 = (2\mathcal{A} - \phi)\Omega - 3\xi\Sigma + \mathcal{H}
\end{equation}
82
6.3.2 Lensing Geometry

In chapter 3, we introduced the spatial vector, \( n^a \), which is a 3-vector in the direction of light propagation. Allowing for a further split of this vector with respect to \( e^a \) gives:

\[
n^a = \kappa e^a + \kappa^a .
\]  
(6.60)

Thus, \( \kappa \) is the magnitude of the radial component and \( \kappa^a \) is the component lying in the 2-dimensional sheet. We are then able to write the null tangent vector \( k^a \) (3.3) as

\[
k^a = E(u^a + \kappa e^a + \kappa^a ) .
\]  
(6.61)

Using this form of the null vector, we can determine the lensing geometry of a photon experiencing a deflection about the center of symmetry\(^5\).

6.3.3 General Form of the Deflection Angle

Figure 6.1 traces the path of a photon in the presence of a strong gravitational field such as a black hole, with \( k^a \) lying tangent to the null geodesic. We can also see the directions of the various vector components of the null vector. The two-dimensional sheet is represented as a sphere about a central point. It should be noted that the deflection seen in figure 6.1 has been greatly exaggerated.

When considering infinitesimal small angles of deflection \( d\alpha \), the lensing geometry will be given by figure 6.2. The angle \( d\alpha \) substands an infinitesimal small displacement of the photon’s path, given by \( |E\kappa^a|\). \( d\alpha \) can be obtained from figure 6.2 giving:

\[
d\alpha = \frac{1}{r} |E\kappa^a| \, d\nu ,
\]  
(6.62)

which may be integrated giving the total deflection angle as

\[
\alpha = \int_{\nu_1}^{\nu_2} \frac{1}{r} |E\kappa^a| \, d\nu - \alpha_0 ,
\]  
(6.63)

where \( \alpha_0 \) is the integration constant. This is a completely general form of the scalar deflection angle which can be applied to any spherically symmetric space-time. The deflection angle \( \alpha \) (6.63) may be written as

\[
\alpha = \int_{\nu_1}^{\nu_2} \frac{1}{r} |E| \sqrt{\kappa^a \kappa_a} \, d\nu - \alpha_0 ,
\]  
(6.64)

\(^5\)In the case of a Schwarzschild space-time explored in the next chapter, this deflection will is due to a strong gravitational field. Thus, we cannot use any of the assumptions made in chapter 5 as these are only valid in the weak field limit.
where the magnitude of $\kappa^a$ (i.e. $|\kappa^a|$) is just $\sqrt{\kappa^a \kappa_a}$. Substituting for $k^a$ (6.61) into the null condition, $k_a k^a = 0$, gives a restriction on $\kappa^a$:

$$\kappa_a k^a = 1 - \kappa^2,$$

(6.65)
Figure 6.2: Geometry of the deflection angle.

so that the general deflection angle takes the form:

$$\alpha = \int_{\nu_1}^{\nu_2} \frac{1}{r} |E| \sqrt{1 - \kappa^2} d\nu - \alpha_0 .$$  \hspace{1cm} (6.66)

Thus, if we know the relations $E(\nu)$, $\kappa(\nu)$ and $r(\nu)$, for a given spherically symmetric space-time, it is possible to find an explicit form of the deflection angle. The differential equations for these variables are derived in the next section.

6.3.4 General Propagation Equations along $k^a$

We begin this section by finding propagation equations for the lensing variables $E$ and $\kappa$ in the direction of the null ray. Each ray is parameterized by the affine parameter, $\nu$, so that the geodesic condition can be written as:

$$k^b \nabla_b k^a = \frac{\delta k^a}{\delta \nu} = (k^a)' = 0 ,$$  \hspace{1cm} (6.67)

where the prime derivative (') denotes change along the ray (i.e. with respect to the affine parameter $\nu$). The geodesic condition (6.67) can be used to
derive propagation equations for $E$ and $\kappa$. Substituting for the null vector $k^a$ (6.61) into (6.67) gives

$$
\left[ E' + E^2 \kappa A + E^2 \kappa^2 \left( \Sigma + \frac{1}{3} \Theta \right) \right] u_a + \left[ E' \kappa + E^2 A + E^2 \kappa \left( \frac{1}{3} \Theta + \Sigma \right) + E \kappa' \right] e_a

+ \left[ E' + \frac{1}{2} E^2 \phi + E^2 \left( \frac{1}{3} \Theta - \frac{1}{2} \Sigma \right) \right] \kappa_a + E^2 (\Omega + \xi) \varepsilon_{ab} \kappa^b + E \kappa_a' = 0 ,
$$

(6.68)

where we have also used

$$
u_a = E A e_a + E \kappa \left( \frac{1}{3} \Theta + \Sigma \right) e_a + E \left( \frac{1}{3} \Theta - \frac{1}{2} \Sigma \right) \kappa_a$$

$$
+ E \Omega \varepsilon_{ab} \kappa^b ,
$$

(6.69)

$$
e'_a = E A u_a + E \kappa \left( \Sigma + \frac{1}{3} \Theta \right) u_a + \frac{1}{2} E \phi \kappa_a + E \xi \varepsilon_{ab} \kappa^b ,
$$

(6.70)

which are obtained from (6.23) and (6.22), respectively. When (6.68) is projected along the timelike direction ($u^a$) and radial direction ($e^a$), we obtain the general equations of $E'$ and $\kappa'$, respectively. These are:

$$
E' = -E^2 \kappa A - 2 E^2 \kappa^2 \Sigma - E^2 \left( \frac{1}{3} \Theta - \Sigma \right) ,
$$

(6.71)

$$
\kappa' = -E \left( 1 - \kappa^2 \right) \left( A - \frac{1}{2} \phi + 2 \kappa \Sigma \right) ,
$$

(6.72)

which can be used in a general LRS space-time.

The LRS class II space-time [22], admits spherically symmetric solutions. This space-time is free of rotation allowing for the vanishing of the variables $\mathcal{H}$, $\Omega$ and $\xi$. If in addition we describe the matter to be a perfect fluid so that $Q = \Pi = 0$, then we can determine the propagation of some of the scalars, $\{\mu, p, \Theta, \Sigma, A, \mathcal{E}, \phi\}$, along $k^a$ using

$$
X' = E \left( X + \kappa X \right) ,
$$

(6.73)

where $X$ can be found through the addition and subtraction of the relevant 1+1+2 equations (6.42)-(6.59).

\footnote{Note that $\delta_a X = 0$ in a spherically symmetric space-time.}

\footnote{We also note that in using a spherically symmetric space-time, these equations will reduce to simpler expressions as $\mathcal{H} = \Omega = \xi = 0$ and $Q = \Pi = 0$.}
Chapter 7

Application to Lensing in Schwarzschild space-times

7.1 Introduction

Shortly after the advent of the theory of general relativity, Schwarzschild obtained a static spherically symmetric asymptotically flat vacuum solution to the Einstein field equations. Later, it was found that these solutions have an event horizon when maximally extended, implying that this solution represents the gravitational field of a spherically symmetric black hole [28]. Schwarzschild gravitational lensing in a weak gravitational field (i.e. where the deflection angle is small) is well known [41]. An extension of Schwarzschild lensing in the strong field limit is investigated in [24] and [3].

We have general forms for the deflection angle and the propagation equations of the lensing variables along $k^a$. These will now be applied to the Schwarzschild space-time in which we hope to recover the standard lensing results found in [47]. We finish off this chapter with the derivation of a 1+1+2 form of the NGDE in the Schwarzschild space-time.

7.2 Solutions for $A$, $E$ and $\phi$

The Schwarzschild space-time is characterized as being static, spherically symmetric and vacuum. This covariant characterization of the space-time allows for the quantities $A$, $E$ and $\phi$ to be non-zero, with all the other
covariant quantities vanishing\(^1\). The remaining propagation equations are [4]:

\[
\begin{align*}
\phi &= -\frac{1}{2} \phi^2 - \mathcal{E}, \\
\dot{\mathcal{E}} &= -\frac{3}{2} \phi \mathcal{E},
\end{align*}
\]  
(7.1)  
(7.2)

together with an extra equation

\[\mathcal{E} + \mathcal{A}\phi = 0.\]  
(7.3)

These are a closed set of equations from which we are able to find solutions for the variables \(\mathcal{A}, \phi\) and \(\mathcal{E}\). The coupled differential equations seen in (7.1) and (7.2) can be solved explicitly when the hat-derivative is associated with an affine parameter, say \(\rho\), so that \(\dot{\rho} = d/d\rho\). The parametric solutions are:

\[
\begin{align*}
\mathcal{E} &= -\frac{1}{(2m)^2} \text{sech}^6 x = -\frac{2m}{r^3}, \\
\phi &= \frac{1}{m} \text{sinh} x \text{sech}^3 x = \frac{2}{r} \sqrt{1 - \frac{2m}{r}}, \\
\mathcal{A} &= \frac{1}{4m} \text{cosech} x \text{sech}^3 x = \frac{m}{r^2} \left(1 - \frac{2m}{r}\right)^{-1/2},
\end{align*}
\]  
(7.4)  
(7.5)  
(7.6)

where solutions can be written in terms of either parameter \(x\) or \(r\), and are related to the affine parameter through the relation

\[\rho = 2m (x + \text{sinh} x \text{cosh} x).\]  
(7.7)

\(r\) is the usual Schwarzschild radial coordinate related to parameter \(x\) by

\[r = 2m \text{cosh}^2 x.\]  
(7.8)

Solutions (7.4) - (7.6) are a one-parameter family of solutions, parameterized by the Schwarzschild mass \(m\). Note that the Schwarzschild solution is given for \(2m < r < \infty\) for \(0 < x < \infty\).

\(^1\)This covariant characterization of the 1+1+2 variables in a Schwarzschild space-time is also discussed in [4].
7.3 Solutions for the lensing variables

As stated in the previous section, the only non-zero 1+1+2 variables in a Schwarzschild spacetime are the scalars \( \{ A, \mathcal{E}, \phi \} \) (and their derivatives \( \{ \dot{A}, \dot{\mathcal{E}}, \dot{\phi} \} \)). Thus, the general propagation equations of \( E \) (6.71) and \( \kappa \) (6.72), in the direction of the ray, reduce to the form:

\[
E' = -E^2 A \kappa ,
\]

\[
\kappa' = E(1 - \kappa^2)(\frac{1}{2} \phi - A) .
\]

Equation (6.65) gives the propagation equation for \( \kappa^a \) when differentiated with respect to \( \nu \):

\[
\kappa^{a'} = -E \kappa \left( \frac{1}{2} \phi - A \right) \kappa^a - \frac{1}{2} E \phi \left( 1 - \kappa^2 \right) e^a .
\]

In order to find solutions to these differential equations, we also include the propagation equations for \( A, \mathcal{E} \) and \( \phi \). These are obtained by substituting for the 1+1+2 form of \( k^a \) (6.61) into the definition of the prime derivative of a particular variable. We then find:

\[
A' = k^a \nabla_a A = -E \kappa (\phi + A) A ,
\]

\[
\phi' = k^a \nabla_a \phi = E \kappa (A - \frac{1}{2} \phi) \phi ,
\]

\[
\mathcal{E}' = k^a \nabla_a \mathcal{E} = -\frac{3}{2} E \kappa \phi \mathcal{E} .
\]

The differential equations (7.12)-(7.14) can be used in solving for \( E \) and \( \kappa \) as will be described below.

Dividing equation (7.13) by \( \phi \) gives:

\[
\frac{\phi'}{\phi} = E \kappa A - \frac{1}{2} E \kappa \phi .
\]

We can then use the differential equations for \( E \) (7.9) and \( \mathcal{E} \) (7.14) to rewrite (7.15) as:

\[
\frac{E'}{E} = -\frac{\phi'}{\phi} + \frac{1}{3} \frac{\mathcal{E}'}{\mathcal{E}} ,
\]

89
which can be solved as a total differential equation, i.e.

\[
(ln E)' = \left(\ln \phi - \ln E^{1/3}\right)' = \left(\ln \frac{\phi}{E^{1/3}}\right)'
\]

\[\Rightarrow E = \frac{\phi}{C_1 E^{1/3}}, \quad (7.17)\]

with \(C_1\) being an integration constant with respect to affine parameter \(\nu\). Similarly, we are able to solve for \(\kappa\) starting with (7.13) and using the differential for \(\kappa\) (7.10). We find:

\[
\frac{\kappa \kappa'}{1 - \kappa^2} = -\frac{\phi'}{\phi}, \quad (7.18)
\]

which can be written in total differential form as

\[
\left[\frac{1}{2} \ln (1 - \kappa^2)\right]' = (\ln \phi)' \quad (7.19)
\]

This equation can be integrated giving

\[
\kappa = \left(1 - \frac{\phi^2}{C_2}\right)^{1/2}, \quad (7.20)
\]

where \(C_2\) is a constant of integration with respect to \(\nu\).

We can now rewrite the solutions for \(E\) and \(\kappa\) in terms of the radial coordinate \(r\) and Schwarzschild mass \(m\). In order to do this, the integration constants \(C_1\) and \(C_2\) need to be determined. \(C_1\) can be obtained by taking the limit of equation (7.17) as \(r\) tends to \(\infty\), and given the solutions of \(E\) and \(\phi\) from equations (7.4) and (7.5), respectively. We find:

\[
C_1 = -\frac{(2m)^{1/3}}{2E_\infty}, \quad (7.21)
\]

where we denote the energy at a large radial distance as, \(E_\infty\).

At the point of closest approach\(^2\), \(r = r_0\), we must have \(\kappa = 0\) so that we are able to determine \(C_2\) from equation (7.20) above. We obtain:

\[
C_2 = \sqrt{\frac{4(r_0^3 - 2m)}{r_0^3}}, \quad (7.22)
\]

\(^2r_0\) is just the closest point that the ray would reach in the vicinity of the lensing object.
where $r_0$ is the distance of closest approach. Substituting for the constants of integration (7.21) and (7.22), into (7.17) and (7.20) respectively, gives:

$$E = E_\infty \left(1 - \frac{2m}{r}\right)^{-\frac{1}{2}} ,$$  \hspace{1cm} (7.23)

$$\kappa = \pm \left[1 - \frac{r_0^3}{r^2(r_0 - 2m)} \right]^{\frac{1}{2}} ,$$  \hspace{1cm} (7.24)

Equations (7.23) and (7.24), together with the solutions for $\{A, E, \phi\}$, form a complete set of solutions from which the lensing geometry in a Schwarzschild space-time can be determined. This is described in the next section.

### 7.4 Form of the Scalar Deflection Angle $\alpha$

We would like to rewrite the scalar deflection angle (6.66) in terms of the Schwarzschild mass $m$ and radial coordinate $r$ so that it may be compared to that given in the standard lensing literature [41]. Substituting for the solutions for $E$ (7.23) and $\kappa$ (7.24) into (6.66) gives

$$\alpha = \int_{\nu_1}^{\nu_2} \frac{E_\infty}{r} \left(\frac{r_0^3}{r^2(r_0 - 2m)}\right)^{1/2} d\nu - \alpha_0$$

$$= \int_{\nu_1}^{\nu_2} \frac{J}{r^{\frac{3}{2}}} d\nu - \alpha_0 ,$$  \hspace{1cm} (7.25)

where $J$ is the impact parameter defined as

$$J = r_0 \left(1 - \frac{2m}{r_0}\right)^{-1/2} .$$  \hspace{1cm} (7.26)

We need a transformation relation between the affine parameter $d\nu$ and radial distance $dr$. Differentiating the solution for $E$ (7.4) with respect to $\nu$ gives:

$$E' = \frac{6m}{r^4} r' ,$$  \hspace{1cm} (7.27)

and substituting for $\kappa$ (7.24), $\phi$ (7.5) and $E$ (7.4) into the differential equation for $E$ (7.14) results in:

$$E' = E \frac{6m}{r^4} \left(1 - \frac{2m}{r}\right)^{1/2} \left[1 - \frac{r_0^3}{r^2(r_0 - 2m)} \left(1 - \frac{2m}{r}\right)\right]^{1/2} .$$  \hspace{1cm} (7.28)
The transformation relation can be obtained in equating equations (7.27) and (7.28) giving
\[dr = E_\infty \left[1 - \frac{r_0^3}{r^2 (r_0 - 2m) \left(1 - \frac{2m}{r}\right)}\right]^{1/2} d\nu ,\]
\[= E_\infty \left[1 - \frac{J^2}{r^2} \left(1 - \frac{2m}{r}\right)\right]^{1/2} d\nu .\] (7.29)

Using this transformation in equation (7.25), gives the form of the scalar deflection angle in terms of the impact parameter \(J\) and radial coordinate \(r\) as:
\[\alpha = 2 \int_{r_0}^{\infty} \frac{J}{r^2} \left[1 - \frac{J^2}{r^2} \left(1 - \frac{2m}{r}\right)\right]^{-1/2} dr - \alpha_0 ,\] (7.30)
which is the exact expression obtained in the lensing literature [47] where a metric based approach is used (and if we take \(\alpha_0 = \pi\)). We will provide an outline of these results in the next section.

### 7.5 Standard Form of the Deflection Angle in a Schwarzschild space-time

The line element of the Schwarzschild spacetime is given by \(^3\)
\[ds^2 = \left(1 - \frac{2m}{r}\right) dt^2 - \left(1 - \frac{2m}{r}\right) dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2 ,\] (7.31)

Consider the motion of a photon in this static isotropic gravitational field. Since the field is isotropic, we may choose the orbit of the photon to lie in an equatorial plane such that
\[\theta = \frac{\pi}{2} ,\] (7.32)
where the path of the photon will then be given by \(\phi\). An extensive derivation of the scalar deflection angle can be found in chapters 8.4 and 8.5 in Weinberg [47]. The form of the deflection angle \(\alpha\) for a photon with closest distance of approach \(r_0\) is given by
\[\alpha(r_0) = 2 \int_{r_0}^{\infty} \frac{J^2}{r^2} \left[1 - \frac{J^2}{r^2} \left(1 - \frac{2m}{r}\right)\right]^{-1/2} dr - \pi ,\] (7.33)

\(^3\)This solution to the EFE was found by K. Schwarzschild in 1916.
with impact parameter $J$ defined in (7.26). This is the result we obtain for the angle of deflection when restricting the general form of the angle (6.63) to a Schwarzschild space-time. We see that this form of the deflection angle is identical to (7.30).

It’s also worth noting that a timelike hypersurface in a spacetime is defined as a photon sphere if the Einstein bending angle of a light ray with closest distance of approach $r_0$ becomes unboundedly large. For the Schwarzschild metric, $r_0 = 3m$ is the only photon sphere and thus the deflection angle $\alpha$ is finite for $r_0 > 3m$.

### 7.5.1 Null Geodesic Deviation Equation

The null geodesic deviation vector, $\eta^a$, is a vector connecting the adjacent rays in the ray bundle. As before, we allow the null vector $k^a$ to be initially orthogonal to the deviation vector (i.e. $\eta^a k_a = 0$). By choosing this initial condition, one allows $k^a$ to always remain orthogonal to the deviation vector $\eta^a$. In general, $\eta^a$ is a 4-vector, so that the extra decomposition with respect to radial vector $e^a$ gives

$$\eta^a = \lambda u^a + \chi e^a + \chi'^a, \quad (7.34)$$

where $\chi^a$ is the component lying in the sheet.

Differentiating the form of $\eta^a$ found in (7.34) twice with respect to the affine parameter $\nu$, gives the LHS of the NGDE:

$$\frac{\delta^2 \eta_a}{\delta \nu^2} = \left[ \lambda'' + E^2 A^2 \lambda + 2E A \chi' + 2E^2 \kappa A^2 \chi - E^2 \kappa \phi \chi \right] u^a$$

$$+ \left[ 2E A \chi' - E^2 \kappa A^2 \lambda + E^2 \kappa \phi \phi + \chi'' + E^2 A^2 \chi - \frac{1}{4} E^2 \phi^2 (1 - \kappa^2) \chi \right] e^a$$

$$+ \left[ E \phi \chi' + \frac{1}{2} E^2 \kappa A \chi \phi - \frac{1}{2} E^2 \kappa \phi \phi \chi + \frac{1}{2} E^2 A \phi \lambda \right] \kappa^a + \chi'^{a \nu}, \quad (7.35)$$

which is in terms of the $(1+1+2)$ variables and their derivatives. In deriving (7.35), we used the general propagation equations for $u^a$ (6.69) and $e^a$ (6.70) which reduce to the following form in a Schwarzschild space-time:

$$u_a' = E \phi e_a, \quad (7.36)$$

$$e_a' = E \alpha u_a + \frac{1}{2} E \phi \kappa a. \quad (7.37)$$
The RHS of the NGDE can be obtained using the general form of the curvature tensor (2.51), in which we have included both Ricci and Weyl curvature contributions. As stated before, the Schwarzschild space-time being considered is vacuum so that the Ricci curvature parts in (2.51) vanish. It is also irrotational and has vanishing magnetic part of the Weyl tensor. Thus, in a Schwarzschild space-time the NGDE (3.80) reduces to the form

$$\frac{\delta^2 \eta^a}{\delta \nu^2} = -C^E_{abcd} k^b \eta^c k^d,$$  \hspace{1cm} (7.38)

where $C^E_{abcd}$ is the electric part of the Weyl curvature tensor. Note that the LHS of the NGDE (7.38) is given by equation (7.35).

Evaluating the RHS of equation (7.38) in terms of the 1+1+2 variables gives the NGDE as:

$$\frac{\delta^2 \eta_a}{\delta \nu^2} = -\frac{3}{2} E^2 E \left[ \kappa (\chi - \kappa \lambda) u_a + (\chi - \kappa \lambda) e_a + (\lambda - \kappa \chi) \kappa_a - (1 - \kappa^2) \chi_a \right],$$  \hspace{1cm} (7.39)

where we have used the identity\(^4\)

$$\chi_a \kappa^a = \lambda - \kappa \chi.$$  \hspace{1cm} (7.40)

From the spherical symmetric geometry, it can be inferred that

$$\chi^a \propto \kappa^a \Rightarrow \chi^a = C \kappa^a,$$  \hspace{1cm} (7.41)

where $C$ is some proportionality factor. Contracting this solution for $\chi^a$ with $\kappa^a$ gives

$$\chi^a \kappa_a = C \kappa^a \kappa_a = C \left(1 - \kappa^2\right)$$  \hspace{1cm} (7.42)

We may use equations (7.40) and (7.42) to solve for $C$. One finds:

$$C = \left(1 - \kappa^2\right)^{-1} (\lambda - \kappa \chi),$$  \hspace{1cm} (7.43)

giving the solution of $\chi^a$ as

$$\chi^a = \left(1 - \kappa^2\right)^{-1} (\lambda - \kappa \chi) \kappa^a.$$  \hspace{1cm} (7.44)

\(^4\)This follows directly from the initial condition, $\eta^a \kappa_a = 0$, where we have substituted for the 1+1+2 form of $\eta^a$ and $k^a$. 

94
Substituting this solution of $\chi^a$ into the NGDE (7.39) results in

$$\frac{\delta^2 \eta_a}{\delta \nu^2} = -\frac{3}{2} \epsilon E^2 (\chi - \kappa \lambda) (\kappa u_a + e_a) . \tag{7.45}$$

Thus, the contributions to the NGDE are in the timelike and radial directions. The NGDE can be described as a force acting on a null congruence. In a Schwarzschild space-time, the force causing spatial separation of null geodesics acts in the radial direction.

We would like to obtain solutions for $\chi$ and $\lambda$ in order to study the behavior of the null congruence. Using both the LHS (7.35) and RHS (7.39) of the NGDE, we can find second order differential equations for $\lambda$ and $\chi$, when projecting along the timelike $u^a$ and radial $e^a$ directions, respectively. Thus, projection along $u^a$ gives:

$$\lambda'' + E^2 A \left( A + \frac{1}{2} \phi (3 \kappa^2 - 1) \right) \lambda' + 2 E A (\chi' - E \kappa (\phi + A) \chi) = 0 , \tag{7.46}$$

while the projection along $e^a$ results in

$$\chi'' + E \kappa \phi \chi' + E (2A - \phi) (\lambda' - E \kappa A \lambda)$$

$$+ E^2 \chi \left[ \frac{1}{2} \phi (1 - \kappa^2) \left( \frac{1}{2} \phi - A \right) - 2A \left( \phi - \frac{1}{2} A \right) \right] = 0 . \tag{7.47}$$

Equations (7.46) and (7.47) can be integrated to give the first order differential equations:

$$\lambda' = -2 E A \chi + E \kappa A \lambda + 2 E c_1 , \tag{7.48}$$

$$\chi' = -\frac{1}{2} \frac{E}{\kappa} (\kappa^2 \phi + 2A - \phi) \chi - \frac{c_1 \left( -c_2 \sqrt{A (1 - \kappa^2)} - 2 \phi E^{3/2} \right)}{\kappa \sqrt{E \phi}} . \tag{7.49}$$

where $c_1$ and $c_2$ are integration constants with respect to $\nu$. Finally by defining a new variable,

$$\Psi \equiv \chi - \kappa \lambda , \tag{7.50}$$

allows the NGDE (7.45) to be written as

$$\frac{\delta^2 \eta_a}{\delta \nu^2} = -\frac{3}{2} \epsilon E^2 \Psi (\kappa u_a + e_a) , \tag{7.51}$$
where we also found the first order DE of $\Psi$ to be:

$$\Psi' + \frac{1}{2} \frac{E}{\kappa} ( -4\kappa^2 A + \kappa^2 \phi + 2A - \phi ) \Psi + \frac{c_1(\kappa^2 - 1)}{\sqrt{E\kappa\phi}} \left( c_2 \sqrt{A} + 2\phi E^{3/2} \right) = 0.$$  

(7.52)
Chapter 8

Summaries and Conclusion

8.1 Summary of Part I

Part I began with an overview of the (1+3)-covariant approach to cosmological models and introduced the various kinematical and dynamical quantities. We also looked at how matter can be described in a cosmological model and decided to use the fluid description throughout this thesis. The case of non-zero vorticity was also explored showing what complications can arise in these models.

In chapter 3, the null congruence $x^a(\nu)$ was introduced with affine parameter $\nu$ along each of these curves. Tangent to each null geodesic we have the null tangent vector $k^a$ which lies orthogonal to a two-dimensional screen-space. The most general form of the screen-space projection tensor is given in (3.5), from which all dynamical quantities in the screen could be defined. As for the timelike vector $u^a$, one can determine the propagation and constraint equations from the Ricci Identities for $k^a$:

\[
\begin{align*}
\frac{d\Theta}{d\nu} &= -E^2 \left( \mu + p - 2\eta_{a}n^a + \pi_{ab}n^a n^b \right) - 2 \left( \tilde{\sigma}^2 - \tilde{\omega}^2 \right) - \frac{1}{2} \tilde{\Theta}^2 , \\
\frac{\delta \tilde{\sigma}_{(ab)}}{\delta \nu} &= -\tilde{\sigma}_{ab} \tilde{\Theta} - 2E^2 \left( \tilde{E}_{(ab)} + \tilde{H}^c_{(a} \tilde{S}_{b)c} \right) , \\
\frac{d\tilde{\omega}}{d\nu} &= -\tilde{\omega} \tilde{\Theta} , \\
\frac{dE}{d\nu} &= -E^2 \left( \frac{1}{3} \tilde{\Theta} + \sigma_{ab}n^a n^b + \dot{u}_a n^a \right) ,
\end{align*}
\]

97
$$\tilde{D}_b \tilde{\sigma}^{ab} = \frac{1}{2} D^a \tilde{\Theta} - \tilde{S}^{ab} \tilde{D}_b \tilde{\omega} + \frac{1}{2} \left( \tilde{\sigma}^{ab} - \frac{1}{2} \tilde{\Pi}^{ab} \Theta \right) \left( V_b + \tilde{W}_b \right)$$

$$+ \left( \tilde{\Pi}^{ab} E_{bc} + \tilde{S}^{ab} H_{bc} \right) k^c + \frac{1}{2} E \left( \tilde{\Pi}^{ab} \pi_{bc} n^c - \tilde{q}^a \right).$$

These equations show how the screen-space expansion (\( \tilde{\Theta} \)), shear (\( \tilde{\sigma}_{ab} \)) vorticity (\( \tilde{\omega}_{ab} \)) and energy \( E \) propagate along the null geodesic.

The general form of the geodesic deviation equation can be restricted to the case of null geodesics. An implicit form of the null geodesic deviation equation could be found by substituting for the Riemann and Weyl curvature tensors resulting in:

$$\frac{\delta^2 \eta_a}{\delta \nu^2} = \frac{-1}{2} E^2 \left[ \mu + p + \pi_{bc} n^b n^d - 2q_b n^b \right] \eta_a$$

$$- \frac{1}{2} E \left[ (\mu + p) u_c \eta^c + (\mu - \pi_{bc} n^b - q_d n^d u_c) \eta^c \right] k_a$$

$$- E^2 \left[ 2E_{ac} \eta^c + E_{bd} n^b n^d (\eta_a + 2u_d \eta^d u_a) + E_{bc} n^b \eta^c (u_a - u_b) \right]$$

$$(u_c - n_c) \eta^c E_{ab} n^b + 2 \tilde{S}_{d(a} \eta^d n^b \right) + 2 \tilde{S}_{d(a} u_{c)} \eta^d H_b d^b,$$

where \( \eta^a \) is the null geodesic deviation vector connecting adjacent rays. Again it is worth mentioning that we have not dropped the cosmological constant, \( \Lambda \), rather it yields no contribution to the NGDE\(^1\). By choosing a suitable cosmological model, we were able to use the NGDE quite extensively in the chapters that followed.

Chapter 4 deals mainly with finding solutions to the NGDE in an FLRW universe model by assuming a particular equation of state. These solutions can be directly related to the area distance \( r_A \), given that \( \eta = \theta r_A \) in flat FLRW cosmologies, \( \theta \) corresponding to angle between adjacent rays and \( \eta \) is the magnitude of the null geodesic deviation vector. Solutions to the NGDE could be easily determined in terms of the affine parameter \( \nu \). For various universe models we were also able to derive second order differential equations for the area distance \( r_A \) in terms of the redshift \( z \). These results are consistent with area-distance results found in the literature. It was found that in a universe containing both a matter (\( \Omega_m \)) and cosmological (\( \Omega_\Lambda \))

\(^1\)This is a very peculiar result as we expect \( \Lambda \) to affect light propagation in some way because of the dominant role it plays in the density of our universe models.
parameter, increasing the cosmological parameter (with a corresponding decrease in the matter density), results in a significant demagnification of a particular source.

We also found solutions for the distance-redshift relation in perturbed FLRW models. Here, one may account for any inhomogeneities in the model by introducing a "clumpiness" parameter $\nu$, which is just a measure of the fraction of matter outside an observing beam. In this section we have chosen to neglect the shearing effects arising from gravitational lensing of the beam in the presence of these clumps. The Optical Focussing Equation (OFE) reduces to the form

$$\frac{\delta^2}{\delta t^2} \sqrt{A} = -\Omega_0 (1 + z)^{5} \left( \frac{3 + \nu}{4} \right) \sqrt{A},$$

where $A$ is the cross-sectional area of the beam. In terms of the area-distance $r_A$ and redshift $z$, the OFE becomes

$$(1 + z) \left[ (1 + z)^2 (1 - \Omega_{A0} + \Omega_{m0} z) + \Omega_{\Lambda0} \right] \frac{d^2 r_A}{dz^2} + \left[ 3(1 + z)^2 (1 - \Omega_{A0} + \Omega_{m0} z) + \frac{1}{2} (1 + z)^3 \Omega_{m0} + 2\Omega_{\Lambda0} \right] \frac{dr_A}{dz} + \frac{(3 + \nu)(2 - \nu)}{4} (1 + z)^2 \Omega_{m0} r_A = 0,$$

from which the Hubble and magnitude-redshift curves were generated. The importance of taking into account the distribution of matter in a universe model can be clearly seen from these diagrams.

In chapter 5, our aim was to obtain a covariant form of the deflection angle. We used the screen-space projected NGDE in a perturbed FLRW dust model. A quasi-Newtonian frame was found suitable in obtaining a form of the screen-space projected NGDE which can be integrated to give the deflection angle. In deriving the angle of deflection, various assumptions were made which are valid in the weak gravitational field limit (i.e. for small deflections of light rays). A thin lens approximation allows one to define the deflection angle as the difference between the incoming and outgoing ray directions, giving the covariant form of the deflection angle:
\[ d\tilde{\alpha}^{a} = d \left[ \frac{4G}{c^2} r_{A}(z_{1}) \int d^2 \tilde{\theta}' \tilde{\Sigma}(\tilde{\theta}') \frac{\tilde{a}^{a} - \tilde{a}^{a'}}{|\tilde{a}^{d} - \tilde{a}^{d'}|^2} \right] - r_{A}(z_{1}) d\tilde{\theta}^{a} \frac{\partial}{\partial \tilde{\theta}_{c} \partial \tilde{\theta}^{c}} \left( \frac{2G}{c^2} \int d^2 \tilde{\theta}' \tilde{\Sigma}(\tilde{\theta}') \ln \frac{|\tilde{a}^{d} - \tilde{a}^{d'}|}{\theta_{c}} \right), \]

where the first term represents the contribution to lensing due to the presence of some deflecting mass while the second term is a cumulative lensing effect. We also found a more general form of the deflection angle to be:

\[ d\tilde{\alpha}^{a} = \left[ \frac{\partial}{\partial \tilde{\theta}_{b}} \left( \frac{2G}{c^2} \int d^2 \tilde{\theta}' \tilde{\Sigma}(\tilde{\theta}') \frac{\tilde{a}^{a} - \tilde{a}^{a'}}{|\tilde{a}^{d} - \tilde{a}^{d'}|^2} \right) A_{c}^{b} \right] d\tilde{\theta}^{c} - \frac{\partial}{\partial \tilde{\theta}_{c} \partial \tilde{\theta}^{c}} \left( \frac{2G}{c^2} \int d^2 \tilde{\theta}' \tilde{\Sigma}(\tilde{\theta}') \ln \frac{|\tilde{a}^{d} - \tilde{a}^{d'}|}{\theta_{c}} \right) A_{c}^{a} d\tilde{\theta}^{c}. \]

where we have not specified the form of the linear mapping \( A_{b}^{a} \), making this form of the deflection angle more general.

### 8.2 Summary of Part II

Part II begins with a brief overview of the 1+1+2 approach from which all the kinematical variables and important relations are defined. The 1+1+2 approach requires the further splitting of (1+3) vectors relative to the space-like vector \( e^{a} \). The general relations are then restricted to LRS space-times so that \( e^{a} \) is a vector pointing along the axis of symmetry. The vanishing of all 1+1+2 vectors and tensors in these space-times allows the kinematical and Weyl quantities to be written in terms of the scalars \( \{\Theta, A, \Omega, \Sigma, E, H\} \).

A 1+1+2 decomposition of the null tangent vector \( k^{a} \) gives

\[ k^{a} = E(u^{a} + \kappa e^{a} + \kappa^{a}), \]

so that \( k^{a} \) has a component along \( u^{a} \), the radial direction \( e^{a} \) and in the two-dimensional sheet \( \kappa^{a} \). The lensing geometry, figures 7.1 and 7.2, can be inferred from this form of \( k^{a} \) and can be used in any spherically symmetric space-time. Figure 7.2 implies that the deflection angle must take the form

\[ d\alpha = \frac{1}{r} |E \kappa^{a}| d\nu, \]

100
which can be written as
\[ \alpha = \int_{\nu_1}^{\nu_2} \frac{1}{r} |E| \sqrt{1 - \kappa^2} d\nu - \alpha_0 , \]

using \( \kappa_0, \kappa = 1 - \kappa^2 \) in spherically symmetric space-times. Thus, in order to find an explicit form of the deflection angle, one has to find relations for \( E(\nu) \) and \( \kappa(\nu) \). These variables were named the 'lensing variables' whose propagation equations, along the ray, could be obtained from the geodesic condition (6.67) giving
\[ E' = - E^2 \kappa A - 2E^2 \kappa^2 \Sigma - E^2 \left( \frac{1}{3} \Theta - \Sigma \right) , \]
\[ \kappa' = -E \left( 1 - \kappa^2 \right) \left( A - \frac{1}{2} \phi + 2\kappa \Sigma \right) , \]
in any LRS space-time. Thus, once the space-time geometry is known - giving \( A, \phi, \Sigma \) and \( \Theta \), we can solve these equations for \( E \) and \( \kappa \).

The covariant characterization of the Schwarzschild space-time allows for the scalars \( A, \phi \) and \( E \) to be non-zero, with all other quantities vanishing. In chapter 8, solutions to these variables were found together with those for the lensing variables \( E \) and \( \kappa \), in the special case of a Schwarzschild space-time. Solutions for \( E \) and \( \kappa \) are given in terms of the radial coordinate \( r \) and the Schwarzschild mass \( m \);
\[ E = E_\infty \left( 1 - \frac{2m}{r} \right)^{-\frac{1}{2}} , \]
\[ \kappa = \pm \left[ 1 - \frac{r_0^3}{r^2(r_0 - 2m)} \left( 1 - \frac{2m}{r} \right) \right]^{\frac{1}{2}} . \]
These solutions together with the transformation relation between the affine parameter \( d\nu \) and radial coordinate \( dr \)
\[ dr = E_\infty \left[ 1 - \frac{J^2}{r^2} \left( 1 - \frac{2m}{r} \right) \right]^{1/2} d\nu . \]
could be substituted into the general expression for the angle of deflection resulting in
\[ \alpha = 2 \int_{r_0}^{\infty} \frac{J}{r^2} \left[ 1 - \frac{J^2}{r^2} \left( 1 - \frac{2m}{r} \right) \right]^{-1/2} dr - \alpha_0 . \]
As discussed before, this corresponds to the form of the deflection angle found in the standard lensing literature. The final section, was devoted to deriving the form of the null geodesic deviation vector in a Schwarzschild space-time. By allowing for a 1+1+2 decomposition of the deviation vector $\eta^a$,

$$\eta^a = \lambda u^a + \chi e^a + \chi^a,$$

we found that the NGDE to take the form

$$\frac{\delta^2 \eta_a}{\delta u^2} = -\frac{3}{2} \mathcal{E} E^2 \Psi (\kappa u_a + e_a),$$

where a new variable $\Psi = \chi - \kappa \lambda$ was defined, and was shown to satisfy a first order differential equation.

### 8.3 Conclusion and Future Research

The covariant form of the deflection angle (5.71) obtained in chapter 5 was derived using a quasi-Newtonian approximation and the assumption that the gravitational field is weak. This covariant form of the deflection angle can be applied to any space-time and always maintain its physical meaning. The approach used in deriving the covariant deflection angle has been used successfully throughout the history of gravitational lens theory. However, in Part II we extended these lensing results to the strong field regime. In chapter 7 we tried to maintain a general setting by working in an LRS or the special case of spherically symmetric space-times. This allows us to use these results in any spherically symmetric space-time, once it has been specified. Apart from finding a general form of the deflection angle, one should be able to determine other observational results in this general framework, which can be tested by specifying the geometry and matter of a particular model.

The significance of the NGDE lies in the fact that it describes the change in the shape of a ray bundle due to the presence of various elements in a chosen universe model. For example, equation (7.52) can be integrated for $\Psi$ so that solutions to the NGDE (7.51) may be found. These solutions will give us a better understanding of how null geodesics, and thus the entire null congruence, behaves in a Schwarzschild space-time. This is just one of many applications of the NGDE.

102
In deriving the form of the deflection angle (7.30), we only considered the path of a photon confined to an equatorial plane. This restriction allowed us to determine the scalar deflection angle $\alpha$. The NGDE (as seen in chapter 5), can be also used in the study of more general lensing situations, such as the propagation of a null congruence or ray bundle in the Schwarzschild geometry. Thus, we could determine a more general form of the deflection angle once its relationship to the NGDE (7.51) is known. However, it is crucial that a relation between the NGDE and the deflection angle be determined in the strong field limit, as their relationship is not the same as those obtained in the perturbed FLRW case.

The general form of the deflection angle can be subjected to more testing by choosing the form of the gravitational potential but maintaining the spherical symmetry. In particular, solutions to the lensing variables can be found using, for example, a Vilenkin monopole and Ellis wormhole, used by Perlick [39] from which we may directly compare results. Another application of our results could be to lensing of gravitational waves in which magnetic part of the Weyl tensor does not vanish. These results can be closely related to observational data.
Appendix A

History of the Distance-Redshift Relation neglecting Gravitational lensing effects

In the paper by Kantowski [30], analytic results obtained for the distance-redshift relations in FRW models were extended to FLRW (and thus including the cosmological constant \( \Lambda \)). A Swiss cheese model was used to derive a general form of the Optical Focussing Equation (OFE). In this particular model, as a light beam from a distant source propagates through the universe, the "cheese" of the model produces the same focusing effect as does the transparent material appearing within the beam. The holes in the cheese with their condensed central masses reproduce the optical effects of the remaining Friedmann matter that has been condensed into clumps, e.g. galaxies. The OFE then takes the form

\[
\frac{\sqrt{A}''}{\sqrt{A}} + \frac{\langle \epsilon^2 \rangle}{A^2} = -\frac{3}{2} \frac{\rho_D}{\rho_0} \Omega_m (1 + z)^5 ,
\]

(A.1)

where \( ' \) denotes differentiation with respect to the affine parameter \( \nu \) along the null ray. In (A.1), \( \langle \epsilon^2 \rangle \) is the average of the square of the wavefront's shear over it's area \( (\sigma/A)^2 \), also given as

\[
\langle \epsilon^2 \rangle = \frac{15}{2} \frac{\rho_l}{\rho_0} B_0 \Omega_m \int \frac{A^2 (1 + z)^6}{z'} dz .
\]

(A.2)
\( \rho_D \) (\( D \) is for dust) is the average mass density of all transparent material in the light beam and \( \rho_I \) (\( I \) is for inhomogeneous) is the average mass density of all mass found in clumps excluded from the beam. Note that \( B_0 \) is the dimensionless gravitational lensing parameter [30].

The average shear term in (A.1) comes from the Weyl curvature tensor of inhomogeneous material exterior to the beam. The term containing the fraction, \( \rho_D/\rho_0 \), comes from the Ricci tensor of transparent material within the beam. It has been found by Dyer and Roeder [10] that the inclusion of the cosmological constant \( \Lambda \) (i.e. using FLRW rather than FRW) only modifies the functional relationship between redshift and affine parameter.

The earliest analytic solutions to the OFE (A.1), in which lensing has been neglected (i.e. \( \langle \epsilon^2 \rangle = 0 \)) and \( \Lambda = 0 \) (FRW models), were written down before the equation were formulated by Kantowski [31]. The standard homogeneous FRW solution was given by Mattig [34]:

\[
D_I(\Omega_m = \Omega_0, \Omega_\Lambda = 0, \nu = 0; z) = \frac{2}{H_0 \Omega_0^2} [\Omega_0 z + (\Omega_0 - 2)(\sqrt{1 + \Omega_0 z} - 1)],
\]

(A.3)

for the case of \( \rho_I = 0 \) or \( \nu = 0 \). The \( \Omega_0 = 1 \) and general \( \nu \) was given by Dashevskii and Slysh [11]:

\[
D_I(\Omega_m = 1, \Omega_\Lambda = 0, \nu; z) = \frac{1}{H_0} \frac{1}{(\nu + \frac{1}{2})} \left[ (1 + z)^{\frac{2}{\nu} + 1} - (1 + z)^{-\frac{2}{\nu} + \frac{1}{2}} \right].
\]

(A.4)

The \( \nu = 2 \) or \( \rho_D = 0 \) solution is due to Dyer and Roeder [8]:

\[
D_I = \frac{1}{H_0} \frac{\Omega_0(1 + z)^2}{4(1 - \Omega_0)^{3/2}} \left[ \frac{3\Omega_0}{2(1 - \Omega_0)} \ln \left( \frac{1 + \sqrt{1 - \Omega_0}}{1 - \sqrt{1 - \Omega_0}} \right) \left( \frac{\sqrt{1 + \Omega_0 z} - \sqrt{1 - \Omega_0}}{\sqrt{1 + \Omega_0 z} + \sqrt{1 - \Omega_0}} \right) \right]
\]

\[
+ \frac{3}{\sqrt{1 - \Omega_0}} \left( \frac{\sqrt{1 + \Omega_0 z} - 1}{1 + z} \right) + \frac{2\sqrt{1 - \Omega_0}}{\Omega_0} \left( 1 - \frac{\sqrt{1 + \Omega_0 z}}{(1 + z)^2} \right),
\]

(A.5)

where this is the case of extreme clumping effects on the standard Mattig result above\(^1\). The \( \nu = 1 \) or \( \rho_D = 2\rho_I \) solution was given by Dyer and Roeder [9]:

\[
D_I(\Omega_m, z) = \frac{4}{3\Omega_0^2} \left[ \frac{3}{2} \Omega_0 - 1 + \frac{1}{2} \Omega_0 z \sqrt{1 + \Omega_0 z} - \frac{3}{2} \Omega_0 - 1 \right]
\]

(A.6)

\(^1\)Note that for this solution no matter converges the light beam. Only external lensing material gravitationally focuses the light beam.
In [30], analytic solutions to equation (A.1) are found in any Swiss cheese model where \( \langle e^2 \rangle \) is neglected and with the inclusion of the cosmological constant \( \Lambda \). These solutions are given in terms of Heun functions and used to illustrate the significance of inhomogeneities in the determination of the cosmological parameters. It can also be seen that the value of these parameters inferred from a given set of observations depend on the fractional amount of matter in inhomogeneities and can differ significantly from those obtained using the standard magnitude-redshift \( (m - z) \) result for the pure dust FLRW models. Kantowski's paper [29], gives the distance-redshift relations in terms of elliptic integrals rather than the less familiar Heun functions used before. Here, three different mass distributions are considered\(^2\) in a pressure-free FLRW cosmology. These new expressions for the distance-redshift were found to significantly reduce computer evaluation times.

\(^2\)The three chosen values of \( \nu \) singled out because their distance-redshift relations can be given in terms of incomplete elliptic integrals.
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