Smooth Particle Interpolation (SPI) is an interpolation scheme based on the Smooth Particle Hydrodynamics (SPH) technique of solving PDE’s.

Consider a function \( f(x) \). From the definition of the \( \delta \)-function,

\[
f(\vec{x}) = \int_{-\infty}^{+\infty} f(x') \delta(\vec{x} - x') \, dx'.
\]

The smoothed approximation of SPH involves replacing the \( \delta \)-function by a kernel function \( W(\vec{x} - x'; h) \), with \( h \) a parameter known as the smoothing length. The kernel function satisfies the following criteria:

- \( W(\vec{x} - x'; h) \geq 0; \)
- \( \int_{-\infty}^{+\infty} W(\vec{x} - x'; h) \, dx' = 1 \) (normalisation);
- \( W(\vec{x} - x'; h) \) has compact support (i.e. \( \exists \, \delta \) such that \( W(\vec{x} - x'; h) = 0 \) when \( |\vec{x} - x'| > \delta \));
- \( W(\vec{x} - x'; h) \) approaches \( \delta(\vec{x} - x') \) as \( h \to 0; \)
- \( W(\vec{x} - x'; h) \) should be an even function of \( \vec{x} - x' \) to ensure reasonable accuracy.

This results in:

\[
f(\vec{x}) \approx \int_{-\infty}^{+\infty} f(x') W(\vec{x} - x'; h) \, dx' \equiv \langle f(\vec{x}) \rangle.
\]

The second approximation involves evaluating the integral in the smoothed approximation \( \langle f(\vec{x}) \rangle \) using only the function values at the \( N \) points \( x_1, x_2, \ldots x_N \):

\[
f(\vec{x}) \approx \sum_{i=1}^{N} \Delta x_i f(x_i) W(\vec{x} - x_i; h) \equiv [f(\vec{x})].
\]

In SPI terminology, particles are said to be positioned at \( x_1, x_2, \ldots x_N \) and the \( \Delta x_i \) are then particle spacings. Given \( f(x) \) evaluated at the \( N \) points \( x_1, x_2, \ldots x_N \), the particle approximation allows the estimation of the function at some point \( \vec{x} \) (i.e. interpolation).

A number of SPI kernel functions exist, although the most common is the Gaussian kernel:

\[
W(\vec{x} - x'; h) = \frac{1}{h\sqrt{\pi}} \exp\left(-\left(\frac{\vec{x} - x'}{h}\right)^2\right).
\]

Strictly speaking, the Gaussian kernel does not have compact support. However, it does tend to zero quickly enough for most applications.
A very useful feature of SPI is that it can be used to approximate derivatives as well as functions. Consider the smoothed approximation of the first derivative \( f'(x) \):

\[
\langle f'(\bar{x}) \rangle \equiv \int_{-\infty}^{+\infty} f'(x') W(\bar{x} - x'; h) \, dx'.
\]

We can use integration by parts to shift the derivative to the kernel function:

\[
\langle f'(\bar{x}) \rangle = \left. f(x') W(\bar{x} - x'; h) \right|_0^{\infty} - \int_{-\infty}^{+\infty} f(x') \frac{dW(\bar{x} - x'; h)}{dx'} \, dx',
\]

where the compact support of the kernel function has been used to cancel the boundary term. Finally, using the particle approximation, we obtain an estimate for the first derivative:

\[
f'(\bar{x}) \approx -\sum_{i=1}^N \Delta x_i f(x_i) W'(\bar{x} - x_i; h) \equiv [f'(\bar{x})].
\]

A similar procedure can be applied to higher derivatives.

Questions:

1. Show that SPI is not strictly speaking an interpolation scheme by considering the particle approximation \([f(x_i)]\).

2. Show that normalisation of the kernel function is required in order to ensure agreement between the smoothed approximation \(\langle f(\bar{x}) \rangle\) and \(f(\bar{x})\) for a constant function \(f(x) = C\).

3. Show that \(\langle f(\bar{x}) \rangle = f(\bar{x}) + \frac{h^2}{2} f''(\bar{x}) + O(h^4),\) for the Gaussian kernel. (Hint: Taylor expand \(f(x')\) about \(\bar{x}\), multiply by \(W(\bar{x} - x'; h)\) and integrate.)

4. Derive the particle approximation to the second derivative \([f''(\bar{x})]\).

5. Use the associated EJS simulation to investigate the use of SPI in approximating the function \(f(x) = 3x^4 - x^3 + 2x + 3\), defined over the interval \([-10, +10]\).

   (a) Confirm that SPI is not truly an interpolation scheme.

   (b) Account for the behaviour of the SPI approximation to \(f(x)\) with a parameter choice \(h = 0.1\) and \(N = 10\).

   (c) **Focus on the central region** \(x \in [-2, 2]\).
       Investigate the effect that variation of \(h\) and \(N\) has on the SPI approximation to \(f(x)\) over this interval. Obviously there is considerable interplay between the parameters \(h\) and \(N\). Can you suggest a combination of the two that serves as a more sensible single parameter to keep track of?

   (d) **Consider the entire interval** \([-10, 10]\).
       What do you notice near the interval boundaries? Does adjusting the number of SPI particles and/or the smoothing length \(h\) remedy the situation?
i. Can you suggest a reason for this failure? (Hint: consider the criteria that the kernel function is assumed to meet.)

ii. Propose a correction to alleviate the problem near the boundaries.

(e) Consider the SPI approximations to the first and second derivative of $f(x)$.

i. Do the derivative approximations suffer from similar weaknesses at the boundaries?

ii. Investigate the effect that variation of $h$ and $N$ has on the SPI approximations to the derivatives.

(f) Is the $h$-dependence of the error derived in Question 3 confirmed by the simulation? Discuss.