Quantitative Methods for Economics

Tutorial 4

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TUTORIAL 4

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ECO3021S

PART 1

1. The system of equations below describes the market for chocolates, where G is the price of substitute goods (e.g. jellytots), and N is the price of inputs.

$$\begin{aligned} Q_d^{choc} &= \alpha - \theta P + \gamma G \\ Q_s^{choc} &= \beta + \lambda P - \delta N \text{ where } 0 < \beta, \delta, \theta, \lambda < 1 \\ Q_d^{choc} &= Q_s^{choc} \end{aligned}$$

   (a) Use Cramer's Rule to solve for the equilibrium price and quantity of chocolates.

   (b) Show that an increase in price of inputs will reduce equilibrium quantity and increase equilibrium price. Be sure to explain the economic reasoning in your argument.

   (c) Show that an increase in the price of substitutes will lead to an increase in equilibrium price and quantity of chocolates. Be sure to explain the economic reasoning in your argument.

2. Does the system of equations below exhibit functional dependence?

$$\begin{aligned} y_1 &= 5x_1^2 + 2x_2 \\ y_2 &= 3x_1^4 + 2x_1^2(x_2 + 1) + x_2(x_2 + 6) + 4 \end{aligned}$$

3. Is $y = x^4 + 2x - 1$ a monotonic function?

4. Find the total differentials of the following:

   (a) $y = 3x_1 + 5x_1x_2 - 2x_2^2$

   (b) $y = \dfrac{2x}{x + z^2}$

5. An individual faces the following utility curve:

$$U = U(a,b) = 3a + 2ab + b^2$$

   (a) Find dU.
   (b) Find $\varepsilon_{U.a}$ and $\varepsilon_{U.b}$.

6. Find the total derivative of the expression below:
   $y = f(m,n) = 2m - 3n^2$ where $m = h(n) = 2n^2 + n + 1$

PART 2

7. A firm's production function is given by:

$$Q = 20K^{0.5}L^{0.5}$$

   (a) Is the firm experiencing increasing returns to scale? Explain.
   (b) Derive an expression for the marginal product of labour, and the marginal product of capital.
   (c) Show that the marginal product of labour, and the marginal product of capital, both decrease as one moves along the isoquant. (*Hint:* Find the second derivative, and evaluate the sign)

8. Find the total derivatives of the expressions below:

   (a) $y = f(a,b) = 4a^2 + 12b$ where $b = g(a) = 3a^2 - 5a - 3$
   (b) $S = f(Y,i)$ where $i = g(Y)$

9. Imagine you get utility from consumption as well as having money. Money can have a direct effect on your level of utility (the pleasure of knowing you're rich?) plus an indirect effect (the more money you have, the more food you can buy and therefore eat!). Formally, this relation might be expressed as $U = U(c(y), y)$ where c refers to consumption and y refers to income. Derive an expression for the marginal utility of income. What would you expect the sign of $\frac{dU}{dy}$ to be? Explain your reasoning.

TUTORIAL 4 SOLUTIONS

ECO3021S

PART 1

1. The system of equations below describes the market for chocolates, where G is the price of substitute goods (e.g. jellytots), and N is the price of inputs.

$$\begin{aligned} Q_d^{choc} &= \alpha - \theta P + \gamma G \\ Q_s^{choc} &= \beta + \lambda P - \delta N \text{ where } 0 < \beta, \delta, \theta, \lambda < 1 \\ Q_d^{choc} &= Q_s^{choc} \end{aligned}$$

(a) Use Cramer's Rule to solve for the equilibrium price and quantity of chocolates.

By now, solving for the reduced from using Cramer's Rule should be second nature.

Step 1: Rewrite the system of equations so that all the endogenous variables are on one side of the equals sign,and the exogenous variables are on the other.

$$\begin{aligned} Q_d + \theta P + 0Q_s &= \alpha + \gamma G \\ 0Q_d - \lambda P + Q_s &= \beta - \delta N \\ Q_d + 0P - Q_s &= 0 \end{aligned}$$

Step 2: Transform this into matrix notation:

$$\begin{bmatrix} 1 & \theta & 0 \\ 0 & -\lambda & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} Q_d \\ P \\ Q_s \end{bmatrix} = \begin{bmatrix} \alpha + \gamma G \\ \beta - \delta N \\ 0 \end{bmatrix}$$

Step 3: Find the determinant, $|\mathbf{A}|$ to ensure that you can solve this system of equations:

$$\begin{aligned} |\mathbf{A}| &= 1\begin{vmatrix} -\lambda & 1 \\ 0 & -1 \end{vmatrix} + 1\begin{vmatrix} \theta & 0 \\ -\lambda & 1 \end{vmatrix} \text{ expanding by column 1} \\ &= \lambda + \theta \end{aligned}$$

As long as $\lambda \neq -\theta, |\mathbf{A}| \neq 0$.

Step 4: To solve for Q_d, use Cramer's Rule, which entails substituting the constant values into the first column of the coefficient matrix $\mathbf{A}$.

$$\begin{aligned}
|\mathbf{A}_1| &= \begin{vmatrix} \alpha+\gamma G & \theta & 0 \\ \beta-\delta N & -\lambda & 1 \\ 0 & 0 & -1 \end{vmatrix} \\
&= -1 \begin{vmatrix} \alpha+\gamma G & \theta \\ \beta-\delta N & -\lambda \end{vmatrix} \\
&= -1\left[-\lambda\left(\alpha+\gamma G\right)-\theta\left(\beta-\delta N\right)\right] \\
&= \lambda\left(\alpha+\gamma G\right)+\theta\left(\beta-\delta N\right)
\end{aligned}$$

Therefore

$$\begin{aligned}
Q_d = Q_s = \overline{Q} &= \frac{|\mathbf{A}_1|}{|\mathbf{A}|} \\
&= \frac{\lambda\left(\alpha+\gamma G\right)+\theta\left(\beta-\delta N\right)}{\lambda+\theta}
\end{aligned}$$

As long as $\beta > \delta N, \overline{Q} > 0$.

Step 5: To solve for equilibrium P, use Cramer's Rule, which entails substituting the constant values into the second column of the coefficient matrix $\mathbf{A}$.

$$\begin{aligned}
|\mathbf{A}_2| &= \begin{vmatrix} 1 & \alpha+\gamma G & 0 \\ 0 & \beta-\delta N & 1 \\ 1 & 0 & -1 \end{vmatrix} \\
&= 1 \begin{vmatrix} \beta-\delta N & 1 \\ 0 & -1 \end{vmatrix} + 1 \begin{vmatrix} \alpha+\gamma G & 0 \\ \beta-\delta N & 1 \end{vmatrix} \\
&= -\left(\beta-\delta N\right)+\left(\alpha+\gamma G\right)
\end{aligned}$$

Therefore

$$\begin{aligned}
\overline{P} &= \frac{|\mathbf{A}_2|}{|\mathbf{A}|} \\
&= \frac{-\left(\beta-\delta N\right)+\left(\alpha+\gamma G\right)}{\lambda+\theta}
\end{aligned}$$

If $\beta > \delta N$,and $\left(\alpha+\gamma G\right) > \left(\beta-\delta N\right), \overline{P} > 0$.

(b) Show that an increase in price of inputs will reduce equilibrium quantity and increase equilibrium price. Be sure to explain the economic reasoning in your argument.

$$\frac{\partial \bar{Q}}{\partial N} = \frac{-\delta\theta}{(\theta + \lambda)} < 0$$

Therefore, if the price of inputs increases then equilibrium quantity will fall.

$$\frac{\partial \bar{P}}{\partial N} = \frac{\delta}{\theta + \lambda} > 0$$

Thus, if the price of inputs increases, equilibrium price will increase too. (This happens because higher input prices makes production more expensive, which causes the supply curve to shift to the left).

(c) Show that an increase in the price of substitutes will lead to an increase in equilibrium price and quantity of chocolates. Be sure to explain the economic reasoning in your argument.

$$\begin{aligned}\frac{\partial \bar{Q}}{\partial G} &= \frac{\lambda\gamma(\lambda + \theta)}{(\lambda + \theta)^2} \\ &= \frac{\lambda\gamma}{\lambda + \theta} > 0\end{aligned}$$

Therefore, as the price of substitutes increases, our good becomes relatively cheaper, and thus more of it is bought (i.e. the demand curve shifts out to the right) causing an increase in equilibrium quantity.

$$\begin{aligned}\frac{\partial \bar{P}}{\partial G} &= \frac{\gamma(\lambda + \theta)}{(\lambda + \theta)^2} \\ &= \frac{\gamma}{\lambda + \theta} > 0\end{aligned}$$

Because the demand for our product increases when substitute goods become relatively more expensive, this causes an outward shift in the demand curve, resulting in rising prices.

2. Does the system of equations below exhibit functional dependence?

$$\begin{aligned} y_1 &= 5x_1^2 + 2x_2 \\ y_2 &= 3x_1^4 + 2x_1^2(x_2+1) + x_2(x_2+6) + 4 \end{aligned}$$

So we need to find the determinant of the Jacobian matrix

$$\begin{aligned} |J| &= \begin{vmatrix} \partial y_1/\partial x_1 & \partial y_1/\partial x_2 \\ \partial y_2/\partial x_1 & \partial y_2/\partial x_2 \end{vmatrix} \\ &= \begin{vmatrix} 10x_1 & 2 \\ 12x_1^3 + 4x_1(x_2+1) & 2x_1^2 + 2x_2 + 6 \end{vmatrix} \\ &= 10x_1(2x_1^2 + 2x_2 + 6) - 2(12x_1^3 + 4x_1(x_2+1)) \\ &= 20x_1^3 + 20x_1x_2 + 60x_1 - 24x_1^3 - 8x_1x_2 - 8x_1 \\ &= x_1(-4x_1^2 + 12x_2 + 52) \end{aligned}$$

As long as the above does not work out to zero, then you know that the determinant isn't zero. Thus there is no functional dependence in the system of equations.

3. Is $y = x^4 + 2x - 1$ a monotonic function?

$$\frac{dy}{dx} = 4x^3 + 2$$

Now consider the following two possibilities:

$$\begin{aligned} \text{If } x &= 1 \text{ then } \frac{dy}{dx} = 4(1)^3 + 2 = 6 \\ \text{If } x &= -1 \text{ then } \frac{dy}{dx} = 4(-1)^3 + 2 = -2 \end{aligned}$$

Therefore, because the sign of the derivative changes depending on the value of x, this function is not monotonic.

4. Find the total differentials of the following:

(a)

$$\begin{aligned} y &= 3x_1 + 5x_1x_2 - 2x_2^2 \\ dy &= (3 + 5x_2)dx_1 + (5x_1 - 4x_2)dx_2 \end{aligned}$$

(b)

$$\begin{aligned} y &= \frac{2x}{x+z^2} \\ dy &= \left[\frac{2(x+z)^2 - 1(2x)}{(x+Z^2)^2}\right] dx + \left[\frac{0 - 2z(2z)}{(x+z^2)^2}\right] dz \\ &= \frac{2z^2}{(x+z^2)^2} dx - \frac{4xz}{(x+z^2)^2} dz \end{aligned}$$

5. An individual faces the following utility curve:

$$U = U(a, b) = 3a + 2ab + b^2$$

(a) Find dU.

$$dU = (3 + 2b)da + (2a + 2b)db$$

(b) Find $\varepsilon_{U.a}$ and $\varepsilon_{U.b}$.

$$\begin{aligned} \varepsilon_{U.a} &= \frac{\partial U/\partial a}{U/a} = \frac{(3+2b)}{\frac{3a+2ab+b^2}{b}} = \frac{3a+2ab}{3a+2ab+b^2} \\ \varepsilon_{U.b} &= \frac{\partial U/\partial b}{U/b} = \frac{2a+2b}{\frac{3a+2ab+b^2}{b}} = \frac{2ab+2b^2}{3a+2ab+b^2} \end{aligned}$$

6. Find the total derivative of the expression below:

$$\begin{aligned} y &= f(m, n) = 2m - 3n^2 \text{ where } m = h(n) = 2n^2 + n + 1 \\ \frac{dy}{dn} &= \frac{\partial y}{\partial m} \times \frac{dm}{dn} + \frac{\partial y}{\partial n} \times \frac{dn}{dn} \\ &= 2(4n+1) - 6n \\ &= 2n + 2 \end{aligned}$$

PART 2

7. A firm's production function is given by:

$$Q = 20K^{0.5}L^{0.5}$$

(a) Is the firm experiencing increasing returns to scale? Explain.

Firstly, note that the production function Q is a function of K and L. i.e. $Q = f(K, L)$

Now, suppose we increase (or decrease) both factors by a scalar k (> 1). Then we observe the following:

$$\begin{aligned} f(kK, kL) &= 20(kK)^{0.5}(kL)^{0.5} \\ &= 20k^{0.5}K^{0.5}k^{0.5}L^{0.5} \\ &= k^{0.5+0.5}20K^{0.5}L^{0.5} \\ &= k20K^{0.5}L^{0.5} \\ &= kf(K, L) \\ &= kQ \end{aligned}$$

i.e. if we increase both factors by a scalar k, Q increases by k which indicates that we have constant returns to scale.

Note: If $f(kK, kL) = k^r f(K, L)$, then:

- The production function will display increasing returns to scale if and only if $r > 1$.
- The production function will display constant returns to scale if and only if $r = 1$.
- The production function will display decreasing returns to scale if and only if $r < 1$.

The shortcut for Cobb-Douglas production functions, as you probably already know, is to consider the indices of K and L. Here we see that they sum up to 1 $(0.5 + 0.5)$ hence we have constant returns to scale and not increasing returns to scale. (If the sum of the indices is greater than 1, then we have increasing returns to scale. If the sum of the indices is less than 1, then we have decreasing returns to scale)

(b) Derive an expression for the marginal product of labour, and the marginal product of capital.

$$\begin{aligned} MP_L &= \frac{\partial Q}{\partial L} = 10K^{0.5}L^{-0.5} = \frac{10\sqrt{K}}{\sqrt{L}} \quad \text{which is positive (as it should be!)} \\ MP_K &= \frac{\partial Q}{\partial K} = 10K^{-0.5}L^{0.5} = \frac{10\sqrt{L}}{\sqrt{K}} \quad \text{which is positive (as it should be!)} \end{aligned}$$

(c) Show that the marginal product of labour, and the marginal product of capital, both decrease as one moves along the isoquant. (*Hint:* Find the second derivative, and evaluate the sign)

$$\begin{aligned} \frac{\partial}{\partial L}(MP_L) &= Q_{LL} = -5K^{0.5}L^{-1.5} = \frac{-5\sqrt{K}}{\sqrt{L^3}} < 0 \quad \text{because } K > 0 \text{ and } L > 0. \\ \frac{\partial}{\partial K}(MP_K) &= Q_{KK} = -5K^{-1.5}L^{0.5} = \frac{-5\sqrt{L}}{\sqrt{K^3}} < 0 \quad \text{because } K > 0 \text{ and } L > 0. \end{aligned}$$

8. Find the total derivatives of the expressions below:

(a)

$$\begin{aligned} y &= f(a,b) = 4a^2 + 12b \text{ where } b = g(a) = 3a^2 - 5a - 3 \\ \frac{dy}{da} &= \frac{\partial y}{\partial a} \times \frac{da}{da} + \frac{\partial y}{\partial b} \times \frac{db}{da} \\ &= 8a + 12(6a - 5) \\ &= 80a - 60 \end{aligned}$$

(b)

$$\begin{aligned} S &= f(Y, i) \text{ where } i = g(Y) \\ \frac{dS}{dY} &= \frac{\partial S}{\partial Y} \times \frac{dY}{dY} + \frac{\partial S}{\partial i} \times \frac{di}{dY} \end{aligned}$$

9. Imagine you get utility from consumption as well as having money. Money can have a direct effect on your level of utility (the pleasure of knowing you're rich?) plus an indirect effect (the more money you have, the more food you can buy and therefore eat!). Formally, this relation might be expressed as $U = U(c(y), y)$ where c refers to consumption and y refers to income. Derive an expression for the marginal utility of income. What would you expect the sign of $\frac{dU}{dy}$ to be? Explain your reasoning.

$$
\begin{aligned}
dU &= \frac{\partial U}{\partial c}dc + \frac{\partial U}{\partial y}dy \\
\frac{dU}{dy} &= \frac{\partial U}{\partial c}.\frac{dc}{dy} + \frac{\partial U}{\partial y}.\frac{dy}{dy} \\
&= \frac{\partial U}{\partial c}.c'(y) + \frac{\partial U}{\partial y}
\end{aligned}
$$

$\frac{\partial U}{\partial y} > 0$ since $\frac{\partial U}{\partial y}$ =marginal utility of money.

So sign of $\frac{dU}{dy}$ depends on the sign of the first term.

For large quantities of food, the marginal change in utility as food consumption increases might be negative. But if some other good can be consumed once the marginal utility of consumption reaches zero, then $c'(y)$ might also be negative, in which case $\frac{dU}{dy}$ is always positive.