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DEPARTMENT OF MATHEMATICAL STATISTICS

The distribution of the complex rectangular co-ordinates and its applications

by

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A C K N O W L E D G E M E N T S

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§1. INTRODUCTION AND SUMMARY.

N.R. Goodman (3) and Wooding (11) have discussed some aspects of the complex multivariate normal distribution. Goodman also discussed the analogues of the Wishart distribution and of multiple and partial correlations. It can be noted that for every distributional result of multivariate real Normal statistical analysis obtained in closed (explicit) form, the counterpart result in the multivariate complex Normal statistical analysis is also obtainable in closed (explicit) form.

Let U_i (p components) be independently distributed according to $N(0,\Sigma)$ $(i = 1,2,...k)$. The Wishart matrix is defined by

$$S = \sum_{r=1}^{k} U_r U_r'$$

Furthermore, let T be a lower triangular square root of S. The elements t_{ij} of T are called the rectangular co-ordinates (Mahalanobis, Bose and Roy (7), Roy (8)). T is also called the square root of a matrix. The distribution of the complex rectangular co-ordinates will be derived here.

Goodman derived the complex Wishart distribution with the aid of characteristic functions and Fourier transforms. From this the distribution of the complex rectangular co-ordinates were derived as an application of the complex Wishart distribution.

In the present paper we give a direct and simplified method of deriving the distribution of the complex rectangular co-ordinates. From this distribution the complex Wishart distribution will be derived as an application.

Some properties of the distribution of the complex rectangular coordinates will be given. In addition some applications and in particular the application of the distribution in the derivation of the distribution of the "complex" generalized variance will also be given.

§2. NOTATION AND PRELIMINARIES;

For a complex number Z, $\bar{Z}$ will denote the conjugate. A matrix M of elements will be denoted by (m_{jk}), the determinant of a square matrix by $|M|$, the transpose by M' and the trace of a square matrix by $\text{tr } M$.

A p-variate complex Normal random variable $\xi' = (Z_1, Z_2, \ldots, Z_p)$ is a p-tuple of complex Normal random variables such that the vector of real and imaginary parts $\eta' = (X_1 Y_1, X_2, Y_2, \ldots, X_p, Y_p)$ is 2p-variate Normally distributed.

The probability density function of a p-variate complex Normal random variable $\xi' = (Z_1, Z_2, \ldots, Z_p)$ is

$$P(\xi) = \frac{1}{\pi^p |\Sigma|} e^{-\bar{\xi}'\Sigma^{-1}\xi}$$

assuming that each of the random variables X_i and Y_i $(i = 1, 2, \ldots, p)$ have mean zero, where $Z_j = X_j + iY_j$ $(j = 1, 2, \ldots, p)$ and $\Sigma = E\xi\bar{\xi}'$. For full discussion and definitions the reader is referred to Goodman (3).

In what follows some results of complex matrix algebra are needed, and they will be listed in this section, without any proof, in the form of lemmas. The material summarized here can be found, for example, in Macduffee (6).

Lemma 2.1

Every Hermitian positive definite (semidefinite) matrix H, is uniquely expressible as $H = B\bar{B}'$, where B is Hermitian positive definite (semidefinite).

Lemma 2.2

Let the square matrix A be such that

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

If A_{11} is non-singular then

$$|A| = |A_{11}| \cdot |A_{22} - A_{21}A_{11}^{-1}A_{12}|$$

Lemma 2.3

Any Hermitian matrix H, may be expressed in the form $H = P\bar{P}'$, where P is lower triangular.

Lemma 2.4

For every Hermitian positive definite matrix H, there exists a complex non-singular matrix B, such that $BH\bar{B}' = I$.

§3. THE DISTRIBUTION OF THE COMPLEX RECTANGULAR CO-ORDINATES.

3.1. Derivation of the distribution.

Let ξ_i $(i = 1,2,\ldots,k)$ be independently distributed according to $CN(0,\Sigma)$ with $\xi_i' = (Z_{1i}, Z_{2i}, \ldots, Z_{pi})$.

Let

$$(3.1) \qquad S = \sum_{i=1}^{k} \xi_i \bar{\xi}_i'$$

$$= \sum_{i=1}^{k} \begin{bmatrix} Z_{1i} \\ Z_{2i} \\ \cdot \\ \cdot \\ \cdot \\ Z_{pi} \end{bmatrix} \left(\bar{Z}_{1i}, \bar{Z}_{2i}, \ldots, \bar{Z}_{pi}\right)$$

$$= \begin{bmatrix} \sum_{i=1}^{k} Z_{1i}\bar{Z}_{1i} & \sum_{i=1}^{k} Z_{1i}\bar{Z}_{2i} & \ldots & \sum_{i=1}^{k} Z_{1i}\bar{Z}_{pi} \\ \sum_{i=1}^{k} Z_{2i}\bar{Z}_{1i} & & & \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ \sum_{i=1}^{k} Z_{pi}\bar{Z}_{1i} & & \ldots\ldots & \sum_{i=1}^{k} Z_{pi}\bar{Z}_{pi} \end{bmatrix}$$

Also let

$$(3.2) \qquad T = \begin{bmatrix} t_{11} & 0 & \ldots & 0 \\ t_{21} & t_{22} & & \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & & \cdot \\ t_{p1} & t_{p2} & \ldots & t_{pp} \end{bmatrix} \qquad t_{jk} = t_{jkR} + it_{jkI}$$

and consider the equation

(3.3) $S = T\bar{T}'$

$$= \begin{bmatrix} t_{11}\bar{t}_{11} & t_{11}\bar{t}_{21} & \cdots & t_{11}\bar{t}_{p1} \\ t_{21}\bar{t}_{11} & t_{21}\bar{t}_{21}+t_{22}\bar{t}_{22} & & \\ \cdot & & \cdot & \\ \cdot & & \cdot & \\ \cdot & & \cdot & \\ t_{p1}\bar{t}_{11} & & & t_{p1}\bar{t}_{p1}+\cdots+t_{pp}\bar{t}_{pp} \end{bmatrix}$$

It has been shown that, with the additional restriction that the t_{jj} are real and positive $(j = 1,2,\ldots,p)$, there exists a unique T of the form (3.2), satisfying (3.3) (Goodman (3)). The elements t_{ij} $i \geq j$, $i,j = 1,2,\ldots,p$ with t_{jj} real and positive $j = 1,2,\ldots,p$ are called the complex rectangular co-ordinates. Their joint distribution will now be derived.

We partition

$$T = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}$$

into $(p-1)\times p$ and $1\times p$ submatrices T_1 and T_2. Thus

$$S = T\bar{T}'$$
$$= \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} \left(\bar{T}_1', \bar{T}_2' \right)$$
$$= \begin{bmatrix} T_1\bar{T}_1' & T_1\bar{T}_2' \\ T_2\bar{T}' & T_2\bar{T}_2' \end{bmatrix}$$
$$= \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}$$

It is clear that $S_{21} = \bar{S}_{12}'$.

$$(3.4) \quad |S| = |T\bar{T}'|$$
$$= |T|\cdot|\bar{T}'|$$
$$= (t_{11}t_{22} \ldots t_{pp})(t_{11}t_{22} \ldots t_{pp})$$
$$= t_{11}^2\, t_{22}^2 \ldots t_{pp}^2 \quad .$$

Also

$$(3.5) \quad |S_{11}| = |T_1\, \bar{T}_1'|$$
$$= |T_1|\cdot|\bar{T}_1'|$$
$$= (t_{11}t_{22} \ldots t_{p-1,p-1})(t_{11}t_{22} \ldots t_{p-1,p-1})$$
$$= t_{11}^2\, t_{22}^2 \ldots t_{p-1,p-1}^2 \quad .$$

Let $S^{-1} = (S^{ij}) = (S_{ij})^{-1}$

Therefore

$$(3.6) \quad S^{pp} = \frac{|S_{11}|}{|S|}$$
$$= \frac{1}{t_{pp}^2} \quad \text{by (3.4) and (3.5)} .$$

For S_{11} nonsingular we have by Lemma 2.2

$$|S| = |S_{11}|\cdot|S_{22} - S_{21}S_{11}^{-1}S_{12}| \quad .$$

Clearly $S_{22\cdot1} = S_{22} - S_{21}S_{11}^{-1}S_{12}$ is a scalar and

$$(3.7) \quad S_{22\cdot1} = \frac{|S|}{|S_{11}|} = \frac{1}{S^{pp}} = t_{pp}^2 \quad \text{by (3.6)} .$$

Thus to find the distribution of t_{pp}^2 we need only find the distribution of $S_{22\cdot1}$ where $S = \sum_{i=1}^{k} \xi_i\bar{\xi}_i'$ and $\xi_i \sim CN(0,\Sigma)$.

Partition Σ in a manner corresponding to the partitioning of S i.e.

(3.8) $$\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$$

where Σ_{11}, Σ_{12} and Σ_{22} are $(p-1)\times(p-1)$, $(p-1)\times 1$ and 1×1 respectively.

Let $\Sigma^{-1} = (\sigma^{ij}) = (\sigma_{ij})^{-1}$.

Theorem 3.1

t_{pp}^2 is distributed according to $\frac{1}{2\sigma^{pp}} \chi^2_{2k-2p+2}$.

Proof

Troskie (10) has stated and proved the following theorem.

Let $Y : P\times M$ be distributed according to $CN(\Gamma W, \Phi)$ where $\Gamma : P\times r$ and $W : r\times M$ are complex matrices and Φ is Hermitian positive definite. Let $G = Y\bar{W}'H^{-1}$ where $H = W\bar{W}'$ and is nonsingular. Then $Y\bar{Y}' - GH\bar{G}'$ is distributed as $U\bar{U}'$ where $U : P\times(m-r)$ is distributed according to $CN(0,\Phi)$.

From this theorem it is easy to deduce that the distribution of $S_{22\cdot 1}$ is that of $\sum_{\alpha=1}^{k-(p-1)} U_\alpha \bar{U}_\alpha'$ (1) where $U_\alpha (\alpha=1,2,\ldots,k-p+1)$ are independently distributed according to $CN(0,\Sigma_{22\cdot 1})$.

But, $\Sigma_{22\cdot 1} = \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}$ is a scalar and equals $\frac{|\Sigma|}{|\Sigma_{22}|} = \frac{1}{\sigma^{pp}}$ by Lemma 2.2 .

In view of (1) we assume from now on that $k > p - 1$ i.e. $k \geq p$.

Thus, since $t_{pp}^2 = S_{22\cdot 1}$, by (3.7), t_{pp}^2 is distributed as $\sum_{\alpha=1}^{k-(p-1)} U_\alpha \bar{U}'_\alpha$ where the U_α are independent, Univariate complex normal random variables with means 0, and variances $\frac{1}{\sigma^{pp}}$

i.e. $U_\alpha = U_{\alpha R} + i\, U_{\alpha I} \sim CN(0, \frac{1}{\alpha^{pp}})$ $(\alpha = 1,2,\ldots,k-p+1)$

$E[U_\alpha] = E[U_{\alpha R}] + i\, E[U_{\alpha I}] = 0$

Therefore $E[U_{\alpha R}] = E[U_{\alpha I}] = 0$

Also $\text{Var}\,[U_\alpha] = E[U_\alpha \bar{U}_\alpha]$

$$= E[U_{\alpha R}^2] + E[U_{\alpha I}^2] = \frac{1}{\sigma^{pp}}$$

Thus $E[U_{\alpha R}^2] = \text{Var}[U_{\alpha R}] = \frac{1}{2\sigma^{pp}}$

and $E[U_{\alpha I}^2] = \text{Var}\,[U_{\alpha I}] = \frac{1}{2\sigma^{pp}}$

Thus we have $U_{\alpha R} \sim N(0, \frac{1}{2\sigma^{pp}}$ $\alpha = 1,2,\ldots,k-p+1$

$U_{\alpha I} \sim N(0, \frac{1}{2\sigma^{pp}}$

Thus $\sqrt{2\sigma^{pp}}\; U_{\alpha R} \sim N(0,1)$

$\sqrt{2\sigma^{pp}}\; U_{\alpha I} \sim N(0,1)$

Therefore $(\sqrt{2\sigma^{pp}}\; U_{\alpha R})^2 + (\sqrt{2\sigma^{pp}}\; U_{\alpha I})^2 \quad \chi_2^2$

i.e. $2\sigma^{pp}\,(U_{\alpha R}^2 + U_{\alpha I}^2) \sim \chi_2^2$

Thus $2\sigma^{pp} \sum_{\alpha=1}^{k-(p-1)} (U_{\alpha R}^2 + U_{\alpha I}^2) \sim \chi_{2(k-p-1)}^2$

since all the U_α are independent.

i.e. $2\sigma^{pp} \sum_{\alpha=1}^{k-(p-1)} U_\alpha \bar{U}'_\alpha \sim \chi_{2k-2p+2}^2$

Thus $t_{pp}^2 \sim \frac{1}{2\sigma^{pp}}\; \chi_{2k-2p+2}^2$

We consider the special case that $\Sigma = I$. From the relations (3.1) and (3.3)

$$\sum_{i=1}^{k} \xi_i \overline{\xi}_i' = S = T\overline{T}'$$

we see that

$$\sum_{i=1}^{k} Z_{pi} \overline{Z}_{pi} = S_{22} = T_2 \overline{T}_2'$$

$$= [t_{p1}, t_{p2}, \ldots, t_{pp}] \begin{bmatrix} \overline{t}_{p1} \\ \overline{t}_{p2} \\ \cdot \\ \cdot \\ \cdot \\ \overline{t}_{pp} \end{bmatrix}$$

$$= |t_{p1}|^2 + |t_{p2}|^2 + \ldots + |t_{p,p-1}|^2 + t_{pp}^2$$

i.e. We have that

$$(3.9) \quad 2 \sum_{i=1}^{k} Z_{pi} \overline{Z}_{pi} = 2|t_{p,1}|^2 + 2|t_{p,2}|^2 + 2|t_{p,p-1}|^2 \; 2t_{pp}^2$$

Since $\Sigma = I$ it follows from Theorem 3.1 that

$$(3.10) \quad 2t_{pp}^2 \sim \chi^2_{2k-2p+2} = \chi^2_{2k-2(p-1)}$$

and that

$$Z_{pi} = Z_{piR} + i\, Z_{piI} \sim CN(0,1)$$

independent $i = 1, 2, \ldots, k$.

Therefore $E[Z_{pi}] = E[Z_{piR}] + i\, E[Z_{piI}] = 0$

Thus $E[Z_{piR}] = E[Z_{piI}] = 0$.

Also $\text{Var}\,[Z_{pi}] = E[Z_{pi} \overline{Z}_{pi}]$

$$= E[Z^2_{piR}] + E[Z^2_{piI}] = 1 \; .$$

Thus $E[Z^2_{piR}] = \text{Var}\,[Z_{piR}] = \frac{1}{2}$.

$$E[Z^2_{piI}] = \text{Var}\,[Z_{piI}] = \tfrac{1}{2} \; .$$

i.e. $Z_{piR} \sim N(0,\frac{1}{2})$

$$i = 1,2,\ldots,k$$

$Z_{piI} \sim N(0,\frac{1}{2})$

Thus $\sqrt{2}\, Z_{piR} \sim N(0,1)$, $\sqrt{2}\, Z_{piI} \sim N(0,1)$

Therefore $2(Z^2_{piR} + Z^2_{piI}) \sim \chi^2_2 \qquad i = 1,2,\ldots,k$

Because the Z_{pi} $(i = 1,2,\ldots,k)$ are independent

we have

$$2 \sum_{i=1}^{k} (Z^2_{piR} + Z^2_{piI}) \sim \chi^2_{2k}$$

i.e.

$$(3.11) \quad 2 \sum_{i=1}^{k} Z_{pi}\bar{Z}_{pi} \sim \chi^2_{2k} .$$

By (3.10) and (3.11) and applying Cochrane's Theorem to (3.9) we conclude that the quantities $2|t_{pi}|^2$ $i = 1,2,\ldots,p-1$, are all independent, each being distributed according to χ^2_2.

Thus $\forall\ i = 1,2,\ldots,p-1$,

$t_{piR} \sim N(0,\frac{1}{2})$, $t_{piI} \sim N(0,\frac{1}{2})$

and are independent.

Let $S_{p-1,p-1}$ denote the $(p-1)$th diagonal element of S. Again by (3.1) and (3.3) we have that

$$\sum_{i=1}^{k} Z_{p-1,i}\bar{Z}_{p-1,i} = S_{p-1,p-1} = t_{p-1,1}\bar{t}_{p-1,1} + t_{p-1,2}\bar{t}_{p-1,2} + \ldots + t^2_{p-1,p-1}$$

i.e.

$$(3.12) \quad 2 \sum_{i=1}^{k} Z_{p-1,i}\bar{Z}_{p-1,i} = 2|t_{p-1,1}|^2 + 2|t_{p-1,2}|^2 + \ldots + 2t^2_{p-1,p-1} .$$

<u>Lemma 3.1</u>

In the case that $\Sigma = I$

$$2t_{ii}^2 \sim \chi^2_{2k-2(i-1)} \qquad 1 \leq i \leq p$$

Proof

Delete the last i elements $1 \leq i \leq p$,

$Z_{i+1,r}\ Z_{i+2,r} \ldots Z_{p,r}$ of $\xi_r (r = 1,2,\ldots,k)$ and the last i rows and columns of S and T. Then applying Theorem 3.1 to t_{ii} we have that $2t_{ii}^2$ is distributed as $\chi^2_{2k-2i+2}$

i.e. $2t_{ii}^2 \sim \chi^2_{2k-2(i-1)}$.

From Lemma 3.1, we have

(3.13) $2t^2_{p-1,p-1} \sim \chi^2_{2k-2(p-2)}$.

As before, since $Z_{p-1.i} \sim CN(0,1)$ $(i = 1,2,\ldots,k)$ we conclude that

(3.14) $2 \sum_{i=1}^{k} Z_{p-1,i}\bar{Z}_{p-1,i} \sim \chi^2_{2k}$

Thus by (3.13) and (3.14) and by applying Cochrane's Theorem to (3.12) we again conclude that the quantities $2|t_{p-1,i}|^2$ $(i = 1,2,\ldots,p-2)$ are all independent, each being distributed according to χ^2_2 .

Thus $\forall\ i = 1,2,\ldots,p-2$,

$t_{p-1,iR} \sim N(0,\frac{1}{2})$ $\quad t_{p-1,iI} \sim N(0,\frac{1}{2})$,

and are independent.

Repeating the above procedure we conclude that

(3.15) $t_{ijR} \sim N(0,\frac{1}{2})$

$$(i > j \quad i\ j = 1,2,\ldots,p),$$

$t_{ijR} \sim N(0,\frac{1}{2})$

and that

(3.16) $2t_{ii}^2 \sim \chi^2_{2k-2(i-1)}$ $(i = 1,2,\ldots,k)$

It will now be shown that all the t_{ij} are independent. (We know only that they are independent along the rows).

$$\sum_{i=1}^{k} \xi_i \bar{\xi}_i' = S = T\bar{T}'$$

therefore $\mathrm{tr} \sum_{i=1}^{k} \xi_i \bar{\xi}_i' = \mathrm{tr}\, T\bar{T}'$

Now $\mathrm{tr} \sum_{i=1}^{k} \xi_i \bar{\xi}_i' = \mathrm{tr} \sum_{i=1}^{k} \bar{\xi}_i' \xi_i$

$= \sum_{i=1}^{k} \bar{\xi}_i' \xi_i$ since $\Sigma\, \bar{\xi}_i' \xi_i$ is a scalar.

$$= \sum_{i=1}^{k} [\bar{Z}_{1i}, \bar{Z}_{2i} \ldots \bar{Z}_{pi}] \begin{bmatrix} Z_{1i} \\ Z_{2i} \\ \cdot \\ \cdot \\ \cdot \\ Z_{pi} \end{bmatrix}$$

$$= \sum_{i=1}^{k} [|Z_{1i}|^2 + |Z_{2i}|^2 + \ldots + |Z_{pi}|^2]$$

Therefore

(3.17) $2\, \mathrm{tr}\, T\bar{T}' = 2 \sum_{i=1}^{k} |Z_{1i}|^2 + 2 \sum_{i=1}^{k} |Z_{2i}|^2 + \ldots + 2 \sum_{i=1}^{k} |Z_{pi}|^2$

Since $\Sigma = I$, the terms Z_{ji} $(j = 1,2,\ldots,p,\ i = 1,2,\ldots,k)$ are independent, each distributed according to $CN(0,1)$. Thus each sum on the right hand side of (3.17) is distributed independently as χ^2_{2k}.

Therefore

$$(3.18) \quad \chi^2_{2kp} = \chi^2_{2k} + \chi^2_{2k} + \ldots + \chi^2_{2k}$$

$$= 2 \text{ tr } T\bar{T}'$$

$$= 2t^2_{11} + 2 \left(|t_{21}|^2 + t^2_{22}\right) + \ldots$$

$$+ \ldots + 2 \left(|t_{p1}|^2 + |t_{p2}|^2 + \ldots + t^2_{pp}\right) .$$

$$= 2S_{11} + 2S_{22} + \ldots + 2S_{pp}$$

where S_{ii} denotes the ith diagonal element of S $(i = 1,2,\ldots,p)$.

We have that

$2S_{11} = 2t^2_{11} \sim \chi^2_{2k}$ by (3.16)

$2S_{22} = 2|t_{21}|^2 + 2t^2_{22}$.

Now $2|t_{21}|^2 \sim \chi^2_2$ by (3.15) and

$2t^2_{22} \sim \chi^2_{2k-2}$ by (3.16)

and t_{21} and t_{22} are independent.

Thus $2S_{22} \sim \chi^2_{2k}$.

Similarly,

$2S_{pp} = 2|t_{p1}|^2 + 2|t_{p2}|^2 + \ldots + 2t^2_{pp}$

Now $2|t_{pi}|^2 \sim \chi^2_2$ $i = 1,2,\ldots,p-1$, by (3.15)

$2t^2_{pp} \sim \chi^2_{2k-2p+2}$ by (3.16)

and $t_{p1}, t_{p2}, \ldots t_{pp}$ are all independent.

Thus $2S_{pp} \sim \chi^2_{2k}$.

Hence, applying Cochrane's Theorem in (3.15), we conclude that all the t_{ij} are independent.

Theorem 3.2

The joint probability density function of the t_{ij} when $\Sigma = I$ is

(3.19) $$Ae^{-\text{tr}\, T\bar{T}'} \, 2^p t_{11}^{2k-1} \, t_{22}^{2k-3} \, \ldots \, t_{pp}^{2k-(2p-1)}$$

where

(3.20) $$A = \left[\Pi^{\frac{1}{2}p(p-1)} \prod_{i=1}^{p} \Gamma(k-i+1) \right]^{-1}$$

(For the density to exist we must have $k \geq p$).

<u>Proof</u>

We have established that for $\Sigma = I$, $2t_{ii}^2$ has the χ^2 distribution with $2k-2i+2$ degrees of freedom $(i = 1,2,\ldots,p)$. Thus the density of $2t_{ii}^2$ is

(3.21) $$\frac{1}{2^{\frac{1}{2}(2k-2i+2)}\Gamma\left[\frac{1}{2}(2k-2i+2)\right]} (2t_{ii}^2)^{\frac{1}{2}(2k-2i+2)-1} \, e^{-\frac{1}{2}(2t_{ii}^2)}$$

$$= \frac{1}{2^{k-i+1}\Gamma(k-i+1)} (2t_{ii}^2)^{k-i} \, e^{-t_{ii}^2} \qquad 0 < 2t_{ii}^2 < \infty$$

Since $\dfrac{d2t_{ii}^2}{dt_{ii}} = 2^2 t_{ii}$, the density of t_{ii} is

(3.22) $$f(t_{ii}) = \frac{2\, t_{ii}^{2k-2i+1}}{\Gamma(k-i+1)} \, e^{-t_{ii}^2} \qquad 0 < t_{ii} < \infty$$

Also for $\Sigma = I$, we have established that t_{ijR} and t_{ijI} have normal distributions with means 0, and variances $\frac{1}{2}$ $(i>j, ij = 1,2,\ldots,p)$. Thus the density of t_{ijR} is

$$g(t_{ijR}) = \frac{1}{\sqrt{2\Pi(\frac{1}{2})}} \, e^{-\frac{1}{2(\frac{1}{2})} t_{ijR}^2} \qquad -\infty < t_{ijR} < \infty$$

$$= \frac{1}{\sqrt{\Pi}} \, e^{-t_{ijR}^2} \qquad -\infty < t_{ijR} < \infty$$

Since all the t_{ijR}, t_{ijI}, and t_{ii} are independent their joint density is

$$(3.23)\quad \prod_{i=1}^{p} f(t_{ii}) \prod_{i>j} g(t_{ijR}) \prod_{i>j} g(t_{ijI})$$

$$= \prod_{i=1}^{p} \frac{2}{\Gamma(k-i+1)} t_{ii}^{2k-2i+1} e^{-t_{ii}^2} \prod_{i>j} \frac{1}{\sqrt{\Pi}} e^{-t_{ijR}^2} \prod_{i>j} \frac{1}{\sqrt{\Pi}} e^{-t_{ijI}^2}$$

Now the number of elements in T is P^2. The number of non-diagonal elements in T is P^2-P. Therefore the number of elements in the lower triangle of T is $\frac{1}{2}(P^2-P)$. Thus there are $\frac{1}{2}(P^2-P)$ elements of the form t_{ijR} and $\frac{1}{2}(P^2-P)$ elements t_{ijI}.

Thus (3.23) reduces to

$$(3.24)\quad \frac{2^p}{\Pi^{\frac{1}{2}(p^2-p)}} \left[\prod_{i=1}^{p} \frac{t_{ii}^{2k-2i+1} e^{-t_{ii}^2}}{\Gamma(k-i+1)}\right] e^{-\sum_{i>j}(t_{ijR}^2 + t_{ijI}^2)}$$

$$= 2^p \left[\Pi^{\frac{1}{2}(p^2-p)} \prod_{i=1}^{p} \Gamma(k-i+1)\right]^{-1} \left[\prod_{i=1}^{p} t_{ii}^{2k-2i+1} e^{-t_{ii}^2}\right] e^{-\sum_{i>j}(t_{ijR}^2 + t_{ijI}^2)}$$

But $\sum_{i>j} (t_{ijR}^2 + t_{ijI}^2) + \sum_{i=1}^{p} t_{ii}^2$

$$= t_{11}^2 + t_{21R}^2 + t_{21I}^2 + t_{22}^2 + \ldots$$

$$+ \ldots + t_{p1R}^2 + t_{p1I}^2 + t_{p2R}^2 + t_{p2I}^2 + \ldots + t_{pp}^2$$

$$= t_{11}^2 + (|t_{21}|^2 + t_{22}^2) + \ldots + (|t_{p1}|^2 + |t_{p2}|^2 + \ldots + t_{pp}^2)$$

$= \operatorname{tr} T\bar{T}'$ by (3.18)

Thus from (3.24) the joint density of t_{ij} is

$$2^p A \prod_{i=1}^{p} t_{ii}^{2k-2i+1} e^{-\mathrm{tr}\, T\bar{T}'}$$

$$= 2^p A\, e^{-\mathrm{tr}\, T\bar{T}'}\, t_{11}^{2k-1}\, t_{22}^{2k-3} \cdots t_{pp}^{2k-(2p-1)}$$

Lemma 3.2 †

Let $T = AG$ where T, A and G are lower triangular complex matrices of order P, with real diagonal elements. The Jacobian of the transformation $T = AG$ is

$$J(T \to G) = \prod_{j=1}^{p} a_{jj}^{2j-1}$$

Proof

Since A and G are lower triangular matrices

$$t_{ij} = \sum_k a_{ik}\, g_{kj}$$

with $a_{ik} = 0$ for $i < k$, and $g_{kj} = 0$ for $k < j$. Writing the Jacobian in the following form

$$J = \frac{\partial(t_{11}, t_{21R}, t_{21I}, \ldots, t_{p1R}, t_{p1I}, t_{22}, \ldots, t_{pp})}{\partial(g_{11}, g_{21R}, g_{21I}, \ldots, g_{11R}, g_{11I}, g_{22}, \ldots, g_{pp})}$$

one immediately sees that the Jacobian matrix is triangular with diagonal elements being given by

$$\frac{\partial t_{jj}}{\partial g_{jj}} = a_{jj} \qquad \frac{\partial t_{jkR}}{\partial g_{jkR}} = a_{jj}$$

$$\frac{\partial t_{jkI}}{\partial g_{jkI}} = a_{jj} \qquad j > k,\ j,k = 1,2,\ldots,p$$

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†Khatri gave this lemma in (5), Page 100. Unfortunately his result is incorrect, so I give my own proof.

Thus

$$J(T \to G) = a_{11}\, a_{22}^{3}\, a_{33}^{5} \cdots a_{pp}^{2p-1}$$

$$= \prod_{j=1}^{p} a_{jj}^{2j-1}$$

We note that (3.19) is the probability density function of the joint distribution of the rectangular co-ordinates, in the special case that $\Sigma = I$. To derive the joint distribution of the rectangular co-ordinates for any Σ, we proceed as follows.

Let T^* be a lower triangular complex random matrix with diagonal elements real and positive having the density (3.19).

We have that

$$S^* = T^* \bar{T}^{*\prime}$$

$$= \sum_{i=1}^{k} \xi_i^* \bar{\xi}_i^{*\prime} \quad \text{where}$$

ξ_i^* $(i = 1,2,\ldots,k)$ are independently distributed according to $CN(0,I)$.

Make the transformation

(3.25) $T = CT^*$

where C is a lower triangular oomplex matrix with diagonal elements real and positive (1), and

(3.26) $C\bar{C}' = \Sigma$ (See Lemma 2.3)

Let $S = T\bar{T}'$

We will have that

---

(1) Diagonal elements real and positive implies that C is non-singular.

$$S = CT^* (\overline{CT}^*)'$$
$$= CT^* \overline{T}^{*\prime} \overline{C}'$$
$$= CS^* \overline{C}'$$
$$= C \sum_{i=1}^{k} \xi_i^* \overline{\xi}_i^{*\prime} \overline{C}'$$
$$= \sum_{i=1}^{k} (C\xi_i^*) (\overline{C} \overline{\xi}_i^*)'$$
$$= \sum_{i=1}^{k} \xi_i \overline{\xi}_i' \quad \text{where} \quad \xi_i = C\xi_i^*$$

and ξ_i $(i = 1,2,\ldots,k)$ is independently distributed according to $N(0,CI\overline{C}')$ (Giri (2))

$$= N(0,C\overline{C}') = N(0,\Sigma) \quad \text{by (3.26).}$$

The Jacobian of the transformation $T = CT^*$ is by Lemma 3.2

$$J(T \to T^*) = \prod_{j=1}^{p} c_{jj}^{2j-1}$$

Thus

$$(3.27) \quad J(T^* \to T) = \frac{1}{\prod_{j=1}^{p} c_{jj}^{2j-1}}$$

We note that

$$T = CT^{*} = \begin{bmatrix} c_{11} & 0 & \cdots & 0 \\ c_{21} & c_{22} & \cdots & 0 \\ \vdots & & \ddots & \\ c_{p1} & c_{p2} & \cdots & c_{pp} \end{bmatrix} \begin{bmatrix} t^{*}_{11} & 0 & & 0 \\ t^{*}_{21} & t^{*}_{22} & & 0 \\ \vdots & & \ddots & \\ t^{*}_{p1} & t^{*}_{p2} & & t^{*}_{pp} \end{bmatrix}$$

$$= \begin{bmatrix} c_{11}\, t^{*}_{11} & 0 & \cdots & 0 \\ & c_{22}\, t^{*}_{22} & \cdots & 0 \\ & & \ddots & \vdots \\ & & & c_{pp}\, t^{*}_{pp} \end{bmatrix}$$

Thus if t_{jj} $j = 1,2,\ldots,p$ denote the diagonal elements of T, then $t_{11} = c_{11}t^{*}_{11}$, $t_{22} = c_{22}t^{*}_{22},\ldots,t_{pp} = c_{pp}t^{*}_{pp}$.

Thus $t^{*}_{11} = {}^{t_{11}}/_{c_{11}}$ $t^{*}_{22} = {}^{t_{22}}/_{c_{22}},\ldots,t^{*}_{pp} = {}^{t_{pp}}/_{c_{pp}}$

Also $\operatorname{tr} T^{*}\bar{T}^{*\prime}$

$$= \operatorname{tr} (C^{-1}T)(\overline{C^{-1}T})'$$

$$= \operatorname{tr} (C^{-1}T)(\bar{C}^{-1}\bar{T})'$$

$$= \operatorname{tr} C^{-1}T\bar{T}'\bar{C}'^{-1}$$

$$= \operatorname{tr} \bar{C}'^{-1}C^{-1}T\bar{T}'$$

$$= \operatorname{tr} (C\bar{C}')^{-1}T\bar{T}'$$

$$= \operatorname{tr} \Sigma^{-1}\, T\bar{T}'$$

$$= \operatorname{tr} \bar{T}'\Sigma^{-1}T \;.$$

Thus the density of T is

$$2^p\, Ae^{-\mathrm{tr}\, \bar{T}'\Sigma^{-1}T}\left[\frac{t_{11}}{c_{11}}\right]^{2k-1}\left[\frac{t_{22}}{c_{22}}\right]^{2k-3}\cdots\left[\frac{t_{pp}}{c_{pp}}\right]^{2k-(2p-1)} \times J(T^{*}\to T)$$

$$= 2^p\, Ae^{-\mathrm{tr}\, \bar{T}'\Sigma^{-1}T}\left[\frac{t_{11}}{c_{11}}\right]^{2k-1}\left[\frac{t_{22}}{c_{22}}\right]^{2k-3}\cdots\left[\frac{t_{pp}}{c_{pp}}\right]^{2k-(2p-1)} \frac{1}{c_{11}^{1}c_{22}^{3}\cdots c_{pp}^{2p-1}}$$

$$= 2^p\, Ae^{-\mathrm{tr}\, \bar{T}'\Sigma^{-1}T}\ \frac{t_{11}^{2k-1}t_{22}^{2k-3}\cdots t_{pp}^{2k-(2p-1)}}{c_{11}^{2k}\,c_{22}^{2k}\cdots c_{pp}^{2k}}$$

But $\Sigma = C\bar{C}'$.

Therefore

$$|\Sigma| = |C\bar{C}'| = |C|\cdot|\bar{C}'|$$
$$= (c_{11}c_{22}\cdots c_{pp})(c_{11}c_{22}\cdots c_{pp})$$
$$= c_{11}^2c_{22}^2\cdots c_{pp}^2 .$$

Thus the density of T is

$$(3.28)\quad 2^p\, A|\Sigma|^{-k}\, e^{-\mathrm{tr}\, \bar{T}'\Sigma^{-1}T}\, t_{11}^{2k-1}\, t_{22}^{2k-3}\cdots t_{pp}^{2k-(2p-1)} \qquad k \geq p.$$

This is the density function of the complex rectangular co-ordinates for any Σ. We have thus established the following theorem.

<u>Theorem 3.3</u>

Let $S = \sum_{i=1}^{k} \xi_i\bar{\xi}_i'$ where the ξ_i $(i = 1,2,\ldots,k)$ are independently distributed according to $CN(0,\Sigma)$. Let $S = T\bar{T}'$ where T is a lower triangular complex random matrix with diagonal elements real and positive. Let $T = (t_{ij})$ where $t_{ij} = t_{ijR} + t_{ijI}$. Then the probability density function of the joint distribution of $t_{11}, t_{21R}, t_{21I},\ldots$ $\ldots t_{p1R}, t_{p1I}, t_{22},\ldots,t_{pp}$ is (3.28) where A is given by (3.20) i.e.

$$P(T) = \frac{2^p|\Sigma|^{-k} e^{-\mathrm{tr}\,\bar{T}'\Sigma^{-1}T} t_{11}^{2k-1} t_{22}^{2k-3} \cdots t_{pp}^{2k-(2p-1)}}{\pi^{\frac{1}{2}p(p-1)} \prod_{i=1}^{p} \Gamma(k-i+1)}$$

The density function is defined over the domain D_T of lower triangular complex matrices with real diagonal elements.

Notation: If T has the above density we shall say that T is distributed according to $R(\Sigma,k)$.

3.2. Some properties of the distribution of the complex rectangular co-ordinates.

Theorem 3.4

Let T be distributed according to $R(\Sigma,k)$. Let $Q = CT$ where C is a lower triangular complex matrix with diagonal elements real and positive. Then Q is distributed according to $R(\Phi,k)$ where $\Phi = C\Sigma\bar{C}'$.

Proof.

The density of Q is obtained from the density of T, $R(\Sigma,k)$ by replacing T by

(3.29) $T = C^{-1}Q$

and multiplying by the Jacobian of the transformation which by Lemma 3.2 is

$$J(T \to Q) = \frac{1}{\prod_{j=1}^{p} c_{jj}^{2j-1}}$$

The transformation (3.29) carries

$\operatorname{tr} \bar{T}'\Sigma^{-1}T$ into

(3.30) $$\operatorname{tr} \bar{T}'\Sigma^{-1}T = \operatorname{tr} \overline{(C^{-1}Q)}' \Sigma^{-1} (C^{-1}Q)$$
$$= \operatorname{tr} \bar{Q}' \bar{C}^{-1}{}' \Sigma^{-1} C^{-1} Q$$
$$= \operatorname{tr} \bar{Q}' (C\Sigma\bar{C}')^{-1}Q$$
$$= \operatorname{tr} \bar{Q}' \Phi^{-1} Q \quad \text{where} \quad \Phi = C\Sigma\bar{C}' .$$

Thus the density of $Q = (Q_{ij})$ is

$$2^p A|\Sigma|^{-k} e^{-\operatorname{tr} \bar{Q}'\Phi^{-1}Q} \left[\frac{Q_{11}}{c_{11}}\right]^{2k-1} \left[\frac{Q_{22}}{c_{22}}\right]^{2k-3} \cdots \left[\frac{Q_{pp}}{c_{pp}}\right]^{2k-(2p-1)}$$
$$\times \frac{1}{c_{11}^{1} c_{22}^{3} \cdots c_{pp}^{2p-1}}$$

$$= 2^p A|\Sigma|^{-k} e^{-\operatorname{tr} \bar{Q}'\Phi^{-1}Q} \frac{Q_{11}^{2k-1} Q_{22}^{2k-3} \cdots Q_{pp}^{2k-(2p-1)}}{c_{11}^{2k} c_{22}^{2k} \cdots c_{pp}^{2k}}$$

But $|\Phi| = |C\Sigma\bar{C}'| = |C|\circ|\Sigma|\circ|\bar{C}'|$

(3.31) $$= c_{11}^2 c_{22}^2 \cdots c_{pp}^2 |\Sigma| .$$

Therefore Q will have the density

$$2^p A|\Phi|^{-k} e^{-\operatorname{tr} \bar{Q}'\Phi^{-1}Q} Q_{11}^{2k-1} Q_{22}^{2k-3} \cdots Q_{pp}^{2k-(2p-1)}$$

i.e. Q is distributed according to $R(\Phi, k)$. This proves the theorem.

Marginal distributions

If T is distributed according to $R(\Sigma,k)$ the marginal distribution of any arbitrary set of elements of T may be obtained (although the expression may be awkward). However the marginal distribution of some sets of elements can be found easily. We have the following theorem.

Theorem 3.5

Let $T(p\times p)$ be distributed according to $R(\Sigma,k)$ $(k \geq p)$.

We partition

$$T = \begin{bmatrix} T^{(1)} \\ T^{(2)} \end{bmatrix} \tag{3.32}$$

such that $T^{(1)}$ and $T^{(2)}$ are $q\times p$ and $(p-q)\times p$ submatrices respectively.

Let

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} \tag{3.33}$$

be partitioned in a manner corresponding to the partitioning of T. (i.e. Σ_{11} and Σ_{22} are $q\times q$ and $(p-q)\times(p-q)$ submatrices. Then $T^{(1)}$ is distributed according to $R(\Sigma_{11},k)$.

Proof.

Let ξ_i $(i=1,2,\ldots,k)$ be independently distributed according to $CN(0,\Sigma)$. Let $S = \sum_{i=1}^{k} \xi_i\bar{\xi}_i' = T\bar{T}'$ where T is a lower triangular complex random matrix with diagonal elements real and positive. Then T is distributed according to $R(\Sigma,k)$. (Theorem 3.3).

Partition ξ_i into subvectors of q and $p-q$ components.

(3.34) $$\xi_i = \begin{bmatrix} \xi_i^{(1)} \\ \xi_i^{(2)} \end{bmatrix}$$

Then the $\xi_i^{(1)}$ $(i = 1,2,\ldots,k)$ are independent, each with the distribution $CN(0,\Sigma_{11})$. (Giri (2)).

$$S = \sum_{i=1}^{k} \xi_i \bar{\xi}_i' = \sum_{i=1}^{k} \begin{bmatrix} \xi_i^{(1)} \\ \xi_i^{(2)} \end{bmatrix} \begin{bmatrix} \bar{\xi}_i^{(1)'}, & \bar{\xi}_i^{(2)'} \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{i=1}^{k} \xi_i^{(1)} \bar{\xi}_i^{(1)'} & \sum_{i=1}^{k} \xi_i^{(1)} \bar{\xi}_i^{(2)'} \\ \sum_{i=1}^{k} \bar{\xi}_i^{(2)} \bar{\xi}_i^{(1)'} & \sum_{i=1}^{k} \bar{\xi}_i^{(2)} \bar{\xi}_i^{(2)'} \end{bmatrix}$$

$$= \begin{bmatrix} S_{(1,1)} & S_{(1,2)} \\ S_{(2,1)} & S_{(2,2,)} \end{bmatrix}$$

But $S = T\bar{T}' = \begin{bmatrix} T^{(1)} \bar{T}^{(1)'} & T^{(1)} \bar{T}^{(2)'} \\ T^{(2)} \bar{T}^{(1)'} & T^{(2)} \bar{T}^{(2)'} \end{bmatrix}$

Thus

$$\sum_{i=1}^{k} \xi_i^{(1)} \bar{\xi}_i^{(1)'} = S_{(1,1)} = T^{(1)} \bar{T}^{(1)'} .$$

Applying Theorem 3.3, we see that $T^{(1)}$ is distributed according to $R(\Sigma_{11},k)$.

§4. Applications.

From the distribution of the complex rectangular co-ordinates the complex Wishart distribution can be derived.

Let T be distributed according to $R(\Sigma, k)$ and let $S = T\bar{T}'$. To obtain the distribution of S we need the following lemma.

Lemma 4.1

Let T be a lower triangular complex matrix of order p, with diagonal elements real and positive. The Jacobian of the transformation $S = T\bar{T}'$ is

(4.1) $J(S \to T) = 2^p \, t_{11}^{2p-1} \, t_{22}^{2p-3} \cdots t_{pp}$.

Proof.

Since T is a lower triangular complex matrix

$$s_{ij} = \sum_r t_{ir} \bar{t}_{jr}$$

with $t_{ir} = 0$ for $i < r$. Writing the Jacobian in the following form

$$J = \frac{\partial(s_{11}, s_{21R} s_{21I}, \ldots, s_{p1R} s_{p1I}, s_{22}, \ldots, s_{pp})}{\partial(t_{11}, t_{21R}, t_{21I}, \ldots, t_{p1R} t_{p1I}, t_{22}, \ldots, t_{pp})}$$

one immediately sees that the Jacobian matrix is triangular with diagonal elements being given by

$$\frac{\partial s_{jj}}{\partial t_{jj}} = 2t_{jj} \qquad \frac{\partial s_{jkR}}{\partial t_{jRk}} = t_{kk}$$

$$\frac{\partial s_{jkI}}{\partial t_{jkI}} = t_{kk} \qquad j > k, \quad j, k = 1, 2, \ldots, p.$$

Thus $J(S \to T) = 2^p t_{11}^{2p-1} \, t_{22}^{2p-3} \cdots t_{pp}$

The density of S is obtained from the density of T, $R(\Sigma,k)$ by replacing $T\bar{T}'$ by S, and multiplying by the Jacobian of the transformation which by Lemma 4.1 is

$$J(T \to S) = \frac{1}{2^p t_{11}^{2p-1} t_{22}^{2p-3} \ldots t_{pp}}$$

The transformation $S = T\bar{T}'$ carries $\operatorname{tr} \bar{T}'\Sigma^{-1}T$ into

$$(4.2) \quad \operatorname{tr} \bar{T}'\Sigma^{-1}T = \operatorname{tr} \Sigma^{-1}T\bar{T}' = \operatorname{tr} \Sigma^{-1}S \, .$$

In addition $t_{11}^{2k-1}\, t_{22}^{2k-3} \ldots t_{pp}^{2k-(2p-1)}\, J(T \to S)$

$$= t_{11}^{2k-1}\, t_{22}^{2k-3} \ldots t_{pp}^{2k-(2p-1)} \frac{1}{2^p t_{11}^{2p-1} t_{22}^{2p-3} \ldots t_{pp}}$$

$$= \frac{1}{2^p}\, t_{11}^{2k-2p}\, t_{22}^{2k-2p} \ldots t_{pp}^{2k-2p}$$

$$= \frac{1}{2^p}\, |S|^{k-p} \quad \text{since}$$

$$|S| = |T\bar{T}'| = t_{11}^2\, t_{22}^2 \ldots t_{pp}^2 \, .$$

Thus the density of S is

$$(4.3) \quad 2^p A|\Sigma|^{-k} e^{-\operatorname{tr} \Sigma^{-1}S}\, \frac{|S|^{k-p}}{2^p}$$

$$= \frac{|S|^{k-p} e^{-\operatorname{tr} \Sigma^{-1}S}}{\Pi^{\frac{1}{2}p(p-1)}|\Sigma|^k \prod_{i=1}^{p} \Gamma(k-i+1)}$$

This is the density of the complex Wishart distribution (Goodman (3), Khatri (5), Srivastava (9)). The density is defined over the domain D_S where S is Hermitian positive semi-definite.

Many important properties of the complex Wishart distribution can be derived from the distribution of the complex rectangular co-ordinates. For example the characteristic function of S may be found as follows. Let

$$(4.4) \qquad \Theta = (\theta_{jk})$$

where $\theta_{kj} = \bar{\theta}_{jk}$ and $\theta_{jk} = \theta_{jk} = \theta_{jkR} + i\,\theta_{jkI}$, $j,k = 1,2,\ldots,p$. We have that

$$(4.5) \quad \Psi_\Theta(S) = E\left[e^{i \operatorname{tr} S\Theta}\right]$$

$$= E\left[e^{i \operatorname{tr} T\bar{T}'\Theta}\right]$$

$$= E\left[e^{i \operatorname{tr} \bar{T}'\Theta T}\right]$$

$$= \int\cdots\int 2^p A|\Sigma|^{-k}\, e^{-\operatorname{tr} \bar{T}'\Sigma^{-1}T}$$

$$t_{11}^{2k-1}\, t_{22}^{2k-3} \ldots t_{pp}^{2k-(2p-1)}\, e^{i \operatorname{tr} \bar{T}'\Theta T} dT$$

$$= |\Sigma|^{-k} \int\cdots\int 2^p\, A\, e^{-\operatorname{tr} \bar{T}'(\Sigma^{-1}-i\Theta)T} t_{11}^{2k-1} t_{22}^{2k-3} \ldots t_{pp}^{2k-(2p-1)} dt$$

$$= |\Sigma|^{-k}\, |\Sigma^{-1} - i\Theta|^{-k} \int\cdots\int\; 2^p\, A\, |\Sigma^{-1} - i\Theta|^{k}$$

$$e^{-\operatorname{tr} \bar{T}'(\Sigma^{-1}-i\Theta)T}\, t_{11}^{2k-1}\, t_{22}^{2k-3} \ldots t_{pp}^{2k-(2p-1)} dt$$

$$= |\Sigma|^{-k}\, |\Sigma^{-1}-i\Theta|^{-k}$$

which is the characteristic function of the complex Wishart distribution with parameters Σ and k.

Lemma 4.2

Let B be distributed according to $CW(\Sigma,k)$. Let C be a lower triangular complex matrix, with diagonal elements real and positive, and let $A = C B \bar{C}'$. Then A is distributed according to $CW(\Phi,k)$ where $\Phi = C\Sigma\bar{C}'$.

Proof.

Since B is Hermitian, we can by Lemma 2.3, let $B = T\bar{T}'$, where T is a lower triangular complex random matrix, diagonal elements real and positive. Since B is distributed according to $CW(\Sigma,k)$, T is distributed according to $R(\Sigma,k)$. Thus

$$(4.6) \qquad A = CB\bar{C}' = CT\bar{T}'\bar{C}'$$

$$= CT(\bar{C}\,\bar{T})'$$

$$= Q\bar{Q}' \qquad \text{where}$$

$Q = CT$ is distributed according to $R(\Phi,k)$ where $\Phi = C\Sigma\bar{C}'$ (By Theorem 3.4). Thus A is distributed as $CW(\Phi,k)$, as required.

Lemma 4.3

Let $S(p\times p)$ be distributed according to $CW(\Sigma,k)$. Let S and Σ be partitioned into q and $(p-q)$ rows and columns

$$(4.7) \qquad S = \begin{bmatrix} S_{(1,1)} & S_{(1,2)} \\ S_{(2,1)} & S_{(2,2)} \end{bmatrix} \qquad \Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$$

Then $S_{(1,1)}$ is distributed according to $CW(\Sigma_{11},k)$.

Proof.

Let $S = T\bar{T}'$, where T is a lower triangular complex matrix as before. T is distributed according to $R(\Sigma,k)$. Partition T in a manner corresponding to the partitioning of S and Σ.

i.e.
$$T = \begin{bmatrix} T^{(1)} \\ T^{(2)} \end{bmatrix} \begin{matrix} q\times p \\ (p-q)\times p \end{matrix} .$$

Then by Theorem (3.5), $T^{(1)}$ is distributed according to $R(\Sigma_{11},k)$. Since $S_{(1,1)} = T^{(1)}\bar{T}^{(1)'}$ and $T^{(1)}$ has the distribution $R(\Sigma_{11},k)$, it follows that $S_{(1,1)}$ is distributed as $CW(\Sigma_{11},k)$ as required.

The distribution of the complex rectangular co-ordinates is also useful in deriving the distributions of functions of the elements of a complex Wishart distributed matrix. Two such functions are the sample multiple coherence.

$$(4.8)\quad \hat{R}^2_{p\cdot p-1,p-2\ldots1} = 1 - (S_{pp}S^{pp})^{-1}$$
$$= 1 - \left[\sum_{j=1}^{p-1} |T_{pj}|^2\right]\left[\sum_{j=1}^{p} |T_{pj}|^2\right]^{-1}$$

and the sample conditional coherence.

$$(4.9)\quad \hat{R}^2_{p,p-1|p-2\ldots1} = |S^{p-1,p}|^2 \left[S^{p-1,p-1}\ S^{p,p}\right]^{-1}$$
$$= |T_{p,p-1}|^2 \left[|T_{p,p-1}|^2 + T^2_{pp}\right]^{-1}$$

For a full discussion of these, the reader is referred to Goodman (3).

The distribution of the complex rectangular co-ordinates may be utilized to obtain the distribution of the "complex" generalized variance as follows

Let U_α (p components) $\alpha = 1,2,\ldots,k$, be distributed according to $N(0,\Sigma)$. The sample generalized variance is defined by (Anderson (1));

$$|S| = \left|\frac{1}{k}\sum_{j=1}^{k} U_\alpha U_\alpha'\right| \quad .$$

In an anolagous way, the complex Generalized variance is defined. Let Z_α (p components) $\alpha = 1,2,\ldots,k$, be distributed according to $CN(0,\Sigma)$. The complex generalized variance is defined by (Goodman (4)).

$$(4.10) \qquad |S| = \left|\frac{1}{k}\sum_{j=1}^{k} Z_\alpha \bar{Z}_\alpha'\right|$$

The distribution of $|S|$ will be derived. It is the same as the distribution of

$$\frac{1}{k^p}\,|A|$$

where

$$(4.11) \qquad A = \sum_{\alpha=1}^{k} Z_\alpha \bar{Z}_\alpha' \; .$$

Let

$$(4.12) \qquad W_\alpha = CZ_\alpha$$

where C is a complex matrice chosen in such a way that

$$C\Sigma\bar{C}' = I. \qquad \text{(See Lemma 2.4).}$$

Then W_α $(\alpha = 1,2,\ldots,k)$ will be independently distributed according to $CN(0,C\Sigma\bar{C}') = CN(0,I)$. Giri (2)

Let

$$(4.13) \qquad B = \sum_{\alpha=1}^{k} W_\alpha \overline{W}'_\alpha$$

$$= \sum_{\alpha=1}^{k} (CZ_\alpha)(\overline{CZ_\alpha})'$$

$$= \sum_{\alpha=1}^{k} CZ_\alpha \overline{Z}'_\alpha \overline{C}'$$

$$= C \sum_{\alpha=1}^{k} Z_\alpha \overline{Z}'_\alpha \overline{C}'$$

$$= CA\overline{C}'$$

Thus

$$(4.14) \qquad |B| = |CA\overline{C}'| = |C| \cdot |A| \cdot |\overline{C}'|$$

$$= \frac{|A|}{|\Sigma|} \qquad .$$

(4.14) holds, since $C\Sigma\overline{C}' = I$. Therefore

$$|C\Sigma\overline{C}'| = |C| \cdot |\Sigma| \cdot |\overline{C}'| = 1$$

Thus $$|C| \cdot |\overline{C}'| = \frac{1}{|\Sigma|}$$

Since W_α $(\alpha = 1,2,\ldots,k)$ are independently distributed according to $CN(0,I)$ it follows that $B = \sum_{\alpha=1}^{k} W_\alpha \overline{W}'_\alpha$ is distributed according to $CW(I,k)$.

Let $B = T\overline{T}'$ where T is a lower triangular complex random matrix with diagonal elements real and positive. Clearly T is distributed according to $R(I,k)$

$$|B| = |T\overline{T}'|$$

$$= t_{11}^2 \; t_{22}^2 \ldots t_{pp}^2$$

where all the t_{ii} $(i = 1,2,\ldots,k)$ are independent.

Thus to find the distribution of $|B|$, we need only find the distribution of the right hand side when $\Sigma = I$. By Lemma 3.1, we have that

$$2t_{ii}^2 \sim \chi^2_{2k-2i+2} \qquad 1 \le i \le p .$$

Thus $|B|$ will be distributed as

(4.15) $$\frac{1}{2^p} \chi^2_{2k} \chi^2_{2k-2} \cdots \chi^2_{2k-2p+2} .$$

Since $|B| = \frac{|A|}{|\Sigma|}$ by (4.14) we have that $|A|$ is distributed as

$$\frac{|\Sigma|}{2^p} \chi^2_{2k} \chi^2_{2k-2} \cdots \chi^2_{2k-2p+2}$$

Thus, since $|S| = \frac{1}{k^p} |A|$ we have that $|S|$ is distributed as

$$\frac{|\Sigma|}{(2k)^p} \chi^2_{2k} \chi^2_{2k-2} \cdots \chi^2_{2k-2p+2}$$

i.e. $\frac{(2k)^p}{|\Sigma|} |S|$ is distributed as

$$\chi^2_{2k} \chi^2_{2k-2} \cdots \chi^2_{2k-2p+2}$$

We thus have the following theorem

Theorem 4.1

The distribution of the generalized variance $|S|$ of a set $Z_1, Z_2, \ldots, Z_k$ from $CN(0,\Sigma)$ is the same as the distribution of $\frac{|\Sigma|}{(2k)^p}$ times the product of p independent factors, the distribution of the ith factor being the χ^2 distribution with $2k-2i+2$ degrees of freedom.

This result is in agreement with Goodman (4).

Since the hth movement of a χ^2 variable with m degrees of freedom is $2^h \frac{\Gamma(\frac{1}{2}m+h)}{\Gamma(\frac{1}{2}m)}$ and the moment of a product of independent variables is the product of the moments of the variables, the hth moment of $|S|$ is

$$(4.16) \quad E[|S|^h] = \frac{|\Sigma|^h}{(2k)^{ph}} 2^h \prod_{i=1}^{p} \frac{\Gamma[\frac{1}{2}(2k-2i+2)+h]}{\Gamma[\frac{1}{2}(2k-2i+2)]}$$

$$= \frac{2^h|\Sigma|^h}{(2k)^{ph}} \prod_{i=1}^{p} \frac{\Gamma(k-i+1+h)}{\Gamma(k-i+1)}$$

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