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Construction of quasi-metrics determined by orders

by

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Abstract

The second half of the last century has seen a growing interest in the area of quasi-pseudometric spaces and related asymmetric structures. Problems arising from theoretical computer science, applied physics and many more areas can easily be expressed in that setting. In the asymmetric framework, many investigations on general topology have been done in order to extend known results of the classical theory. It is the aim of this thesis to dig more into this theory by investigating the existence of quasi-pseudometrics that *produce* a given partially ordered metric space. We show that whenever one has an ordered metric space, it is often possible to describe both the metric and the order using quasi-pseudometrics. We establish respectively topological and algebraic conditions for the existence of such quasi-pseudometrics. We deduct some specific conditions when the ordered metric space $(X, m, \leq)$ is assumed to have some topological properties such as compactness. When a *producing* quasi-pseudometric exists, the question of uniqueness is also studied. A characterization of *produced* spaces is given. More precisely, given an ordered metric space $(X, m, \leq)$, we investigate the conditions of existence of a quasi-pseudometric d such that $d \vee d^{-1} =: d^s = m$ and $\leq_d = \leq$ where d^{-1} is the *conjugate* of d and $\leq_d$ the *specialization order* of d. We explicitly give such existence conditions, namely the *interval condition* when the partial order $\leq$ of the ordered metric space $(X, m, \leq)$ is assumed to be linear. When the order $\leq$ is an arbitrary order, we show that under certain conditions we can construct such a quasi-pseudometric. In our investigation, we also explore the above existence

question when the ordered metric space $(X, m, \leq)$ is moreover endowed with a group structure and when the underlining poset $(X, \leq)$ of the ordered metric space is a lattice. Some interesting and crucial examples and counterexamples, to illustrate the results, are also provided.

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Dedication

To God Almighty, to Jesus-Christ His Son, my king and saviour, my all and especially my inspiration, for His continuous presence and His faithfulness. I dedicate this thesis to my parents Prudence A. Gaba and Gilberte Adjagba epse Gaba, to my brother Nathan O. Gaba, and to my sisters Marie-Bénédicte B. Gaba and Charlène O. Gaba, and to my beloved friend Clémence O. Akadiri, for their patience, understanding and care during my doctoral studies.

Declaration

I, Yaé O. K. Ulrich Gaba, hereby declare that the work on which this thesis is based is my original work (except where acknowledgements indicate otherwise) and that neither the whole work nor any part of it has been, is being, or is to be submitted for another degree in this or any other university. I authorise the University to reproduce for the purpose of research either the whole or any portion of the contents in any manner whatsoever.

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Contents

Introduction

One of the fundamental concepts of modern mathematics is the notion of a metric space: a set together with a metric which separates points (i.e. the metric between two points is 0 if and only if they are identical), is symmetric and satisfies the triangle inequality. The theory of metric spaces has been well studied over the years and actually provides the foundation of many branches of pure mathematics such as geometry, analysis and topology as well as more applied areas. A more general and natural idea about metric is that of a distance (which is a function that satisfies the triangle inequality and does not assume symmetry), but again in the literature, there is generally no distinction between a metric and a distance. This is due to the fact that, although distance functions appear frequently in numerous fields, they have not been studied thoroughly. Moreover in many practical applications, it is often easier to make use of metrics and this is often achieved by "symmetrising" other distance functions. In location theory for instance (see [42]), practical distances (e.g. in terms of time or cost) are rarely symmetric but are still measured using metrics. In fact, the early studies of distance functions were developed in an abstract topological framework and could not be of great utility.

A quasi-pseudometric is a distance function which satisfies the triangle inequality but is not necessarily symmetric. There are two versions of the separation axiom:

- either for a distance between two points to be 0 they must be the same (as in the case of a metric),
- or, it is allowed that one distance between two different points be 0 but not both.

In all cases the distance between two identical points has to be 0. Hence, for any pair of points in a quasi-pseudometric space there are two distances which need not be the same. Quasi-pseudometrics were first introduced in the 1930s (see Wilson [41]) and are a subject of intensive research in the context of topology (see Gaba [14, 16, 20] and Künzi [26]) and theoretical computer science (see Plastria [36]). Recent work in the area can also be read in [13, 15, 17, 32, 33]. Although many of the results from the theory of metric spaces can easily be applied in the quasi-pseudometric setting, there are some concepts which are unique to the quasi-pseudometrics, and among those we have "duality". In the theory of quasi-pseudometric spaces, "duality" is probably the most important concept, as it is unique and specific to this type of space. Duality has no metric counterpart. Every quasi-pseudometric has its *conjugate (dual)* quasi-pseudometric which is obtained by reversing the order of each pair of points before computing the distance. The existence of two quasi-pseudometrics, the original one and its conjugate, leads to other dual structures depending on which quasi-pseudometric is used: balls, neighbourhoods etc. We distinguish them by calling the structures obtained from the original quasi-pseudometric the *left (or forward) structures* while the structures obtained from the *conjugate* quasi-pseudometric are referred to as the *right (or backward) structures*. The *join* or *symmetrisation* of the left and right structures produce a corresponding metric structure. Another important concept unique to quasi-pseudometrics is that of an associated partial order[1].

[1] We nevertheless mention here that a metric space can always be considered as a partially

Every quasi-pseudometric can be associated with a partial order and every partial order can be shown to arise from a quasi-pseudometric (Proposition 0.2.1). Hence, quasi-pseudometrics not only generalise metrics, but they also generalise partial orders. This fact has been important for applications in theoretical computer science and will also be one of the main axes in our present investigation.

More formally,

Definition -1.0.1. Let X be a nonempty set. A function $m : X \times X \to \mathbb{R}$ is called a **metric** on X if for all $x, y, z \in X$:

i) $m(x, y) \geq 0$ (non-negativity);

ii) $m(x, y) = 0$ if and only if $x = y$ (identity of indiscernibles, or coincidence axiom);

iii) $m(x, y) = m(y, x)$ (symmetry axiom);

iv) $m(x, z) \leq m(x, y) + m(y, z)$ (subadditivity / triangle inequality axiom).

These conditions reflect intuitive notions about distance, or more precisely, metric. For example, that the distance between distinct points is positive and the distance from x to y is the same as the distance from y to x. It also appeared in practice that the axiom iii) is not a requirement and it seems more natural to define a closely related structure by omitting symmetry. We are therefore led to the structure of quasi-pseudometric (which we shall define more precisely later). A quasi-pseudometric is a generalized metric space in which the distance between two distinct points can be zero.

A metric on a set X induces on it a topology and as we know not all topologies can be generated by a metric. But a topological space whose topology can be

ordered set ordered with 'equality'.

described by a metric is said to be *metrizable*. The problem of finding such a metric is called the *metrization problem* and a crucial result in its resolution is that of Urysohn and Tychonoff that establishes that every second-countable regular space (T_3-space) is metrizable.

A quasi-pseudometric space is a pair (X, d) such that X is a (nonempty) set and d is a quasi-pseudometric on X. A quasi-pseudometric d on a set X induces a topology on it. A topological space $(X, \mathcal{T})$ is called *quasi-pseudometrizable* if there is a quasi-pseudometric d on X compatible with $\mathcal{T}$, i.e. the topology $\mathcal{T}$ can be described by d. The problem of finding such quasi-pseudometric is referred to as the *quasi-pseudometrization problem*.

When A.W.Wilson (1931) introduced quasi-pseudometric spaces, it was for purely topological reasons. However, several generalized distance functions were already used by E. W. Chittenden (1927), V.W. Niemytzki (1927) and others, in their study of the metrization problem. In a paper published in 1963, J.C.Kelly began a systematized study of quasi-pseudometrizable spaces from a bitopological point of view. Thus the problem of characterizing both the topological spaces and the bitopological spaces that admit a compatible quasi-pseudometric appears in a natural way. In this direction several contributions have been obtained (S. Salbany (1972), R. Fox (unpublished), R. D. Kopperman (1993), etc.) but the cited problem is still open since researchers are unhappy with the known results.

If we enrich the structure of a topological space by endowing it with a preorder, we make the above problem more interesting and this is our main focus in this thesis. The problem of *quasi-pseudometrization* of a given preordered topological space has been studied by P. Fletcher and W. F. Lindgren [12], E. Minguzzi [30], L. Nachbin [31] and many others. In particular Minguzzi generalizes the result of Urysohn, for topological spaces endowed with a preorder $\leq$ and proved that every second-countable completely regular preordered space

$(E, \mathcal{J}, \leq)$ is quasi-pseudometrizable. The term *quasi-pseudometrizable*[2], for a preordered topological space $(E, \mathcal{J}, \leq)$ refers to the existence of a quasi-pseudometric p on E for which the pseudometric $p \vee p^{-1}$ induces $\mathcal{J}$ and the graph of $\leq$ is exactly the set $\{(x, y) : p(x, y) = 0\}$. When the preorder $\leq$ is a partial order, known results characterize ordered topological spaces as being order homeomorphic to subspaces of the ordered Hilbert cube (see [30]). Related to the problem of quasi-pseudometrizability of an ordered topological space is that of quasi-pseudometrizability of a bitopological space. Indeed, it is possible to associate to every ordered topological space a bitopological space by taking the topology $\mathcal{T}^\sharp$ made of all upper sets, and the topology $\mathcal{T}^\flat$ made of all lower sets. Bitopological spaces were introduced by Kelly [21] and subsequently investigated in [28, 35].

In this work, we look in a slightly modified direction by addressing the problem of *quasi-pseudometrization* of an ordered metric space by imposing that the metric associated to the desired quasi-pseudometric be equal to the initial metric.

The main aim of this thesis is to investigate quasi-pseudometrizability in the setting of ordered metric spaces (and by doing so developing a little further the theory of quasi-pseudometric spaces). Our questions of interest here are the following:

Problem -1.0.1. Given a partially ordered metric space $(X, m, \leq)$, when does there exist a T_0-quasi-metric d on X such that whenever $x, y \in X$

$$d^s(x, y) := \max\{d(x, y), d(y, x)\} = m(x, y)$$

and

$$d(x, y) = 0 \iff x \leq y \ ?$$

Problem -1.0.2. We shall see that, when the order $\leq$ on $(X, m, \leq)$ is linear, and that such a T_0-quasi-metric d exists, therefore it is unique. Hence the question of uniqueness arises in general.

[2]It is important to state that there are many different formulations of the pseudometrization problem and that it is therefore useful to always specify the one we are dealing with.

Even though we shall not address it in this thesis, an interesting and natural question is to investigate the reverse of Problem -1.0.2, namely, if for a given partially ordered metric space $(X, m, \leq)$, there exists a unique T_0-quasi-metric d satisfying the conditions of Problem -1.0.1, can we prove that the order $\leq$ on the partially ordered metric space $(X, m, \leq)$ is linear?

In the coming chapter, we shall make clear what we understand by T_0-quasi-metric d. To motivate these questions, let us look at the following example.

Example -1.0.1. We consider the real line $\mathbb{R}$ with its standard metric defined by $m(x, y) = |x - y|$ whenever $x, y \in \mathbb{R}$. We define the function u by

$$u(x, y) = \begin{cases} 0 & , \text{ if } x \leq y, \\ x - y & , \text{ if } x > y, \end{cases}$$

whenever $x, y \in \mathbb{R}$. We show that u verifies the following properties:

1. $u(x, x) = 0$ whenever $x \in \mathbb{R}$,
2. $u(x, z) \leq u(x, y) + u(y, z)$ whenever $x, y, z \in \mathbb{R}$,
3. $u(x, y) = 0 = u(y, x) \Longrightarrow x = y$ whenever $x, y \in \mathbb{R}$ (T_0-condition),
4. $\leq_u = \leq$ i.e. $x \leq_u y \Longleftrightarrow x \leq y$ (where $\leq$ is the usual order on $\mathbb{R}$ and $\leq_u$ is the order on $\mathbb{R}$ defined by $x \leq_u y \Longleftrightarrow u(x, y) = 0$) whenever $x, y \in \mathbb{R}$,
5. $u^s(x, y) := \max\{u(x, y), u(y, x)\} = m(x, y)$, whenever $x, y \in \mathbb{R}$.

Indeed:

1. $u(x, x) = 0$ since $x \leq x$ whenever $x \in \mathbb{R}$.
2. We show that u satisfies the triangle inequality, i.e.

$$u(x, z) \leq u(x, y) + u(y, z) \quad \text{whenever } x, y, z \in \mathbb{R}.$$

• If $x \leq z$, so $u(x,z) = 0$ and the triangle inequality is satisfied.

• If $z < x$, so $u(x,z) = x - z$. In this sub-case, if $x = y$ or $z = y$, the triangle inequality is satisfied. The most interesting situation is where $x \neq y$ and $z \neq y$. Hence, we are led to the following inequalities :

$$y < z < x, \text{ or } z < y < x, \text{ or } z < x < y.$$

a) If $y < z < x$, then $u(x,y) = x - y$ and $u(y,z) = 0$ and since $x - z < x - y$, the triangle inequality is satisfied.

b) If $z < y < x$, then $u(x,y) = x - y$ and $u(y,z) = y - z$ and since $x - z = x - y + y - z$, the triangle inequality is satisfied.

c) If $z < x < y$, then $u(x,y) = 0$ and $u(y,z) = y - z$ and since $x - z < y - z$, the triangle inequality is satisfied.

3. u trivially satisfies the so-called T_0-condition. Indeed for any $x, y \in \mathbb{R}$ $u(x,y) = 0 \Longrightarrow x \leq y$ and $u(y,x) = 0 \Longrightarrow y \leq x$. Combining these two inequalities, we get that $x = y$.

4. Furthermore, it is also clear that for any $x, y \in \mathbb{R}$,

$$x \leq_u y \Longleftrightarrow u(x,y) = 0 \Longleftrightarrow x \leq y.$$

5. It is trivial that

$$u^s(x,y) = \max\{x - y, 0, y - x\} = |x - y|$$

whenever $x, y \in \mathbb{R}$.

So for $(\mathbb{R}, m, \leq)$, we would have found a T_0-quasi-metric d, namely u, answering the question asked in Problem -1.0.1. Moreover, in this particular case, u is unique.

Below, in the last section of this introduction, we give an outline of this dissertation.

Overview of the Chapters

In Chapter 0, we give a brief overview of some basic concepts from the theory of quasi-pseudometric and ultra-quasi-pseudometric spaces including a few words on asymmetric normed spaces. Some interesting examples of T_0-quasi-metric spaces (Example 0.1.4 and Example 0.1.5), T_0-ultra-quasi-metric spaces (Example 0.1.5) and asymmetric norms (Example 0.1.6) are presented. We also give a brief introduction to the theory of order and quasi-uniformities as well as some motivating examples (see Example 0.3.2 and Example 0.3.3).

In Chapter 1, we actually start our investigation and begin the study of our problem as stated in Problem -1.0.1. The chapter is divided in two main sections. In the first one, where the main result is Proposition 1.1.2, we give necessary and sufficient conditions of unique existence of a T_0-quasi-metric d *producing* a given linearly order metric space $(X, m, \leq)$. In the second section, we use a purely order theoretical approach to introduce what we called the *pseudo-dimension* of a partially ordered metric space and prove that for some specific values of the pseudo-dimension, the problem we suggested can be solved. The main result is formulated in Theorem 1.2.2. For each section, we provide very crucial examples, namely Examples 1.1.2 and 1.1.3 on the one hand and Examples 1.2.2 and 1.2.3 on the other hand, to illustrate our results.

In Chapter 2, the longest of this manuscript, we provide a general framework for the study of the suggested problem. More precisely, given a metric space (X, m), we say that a T_0-quasi-metric d defined on X is *m-splitting* provided that $d^s := d \vee d^{-1} = m$. In this chapter, we present the results in six sections. After some introductory definitions and preliminary results that we provide in the first section, we show in the second section, how, via Proposition 0.2.1, Szpilrajn's theorem naturally extends to the case of quasi-pseudometrics. In the third section, we show (see Proposition 2.3.1) that below each m-splitting T_0-quasi-metric there is a minimal m-splitting T_0-quasi-metric (called minimally m-splitting). Moreover a partial order $\leq$ on a metric space (X, m) that is the specialization order of an m-splitting T_0-quasi-metric will be called *m-produced*. Hence, in the fourth section, we focus on m-produced partial orders that are maximal among the m-produced partial orders (see Theorem 2.4.1) and for this purpose, we make use of the very powerful quasi-pseudometric $D_{(m,\leq)}$ (Cf. Definition 2.4.2). We show that a minimally m-splitting T_0-quasi-metric need not have a maximally m-produced partial order (see Example 2.6.1), but each maximally m-produced partial order is the specialization order of a minimally m-splitting T_0-quasi-metric (see Proposition 2.4.1). In the fifth section, we explain how the problem of quasi-pseudometrization of a partially ordered metric space $(X, m, \leq)$ can often be related to that of its "linear extensions" $(X, m, \preceq)$ when X is of finite cardinality. We close the chapter, in the sixth section by giving an example (Example 2.6.1) showing that a partial order that is m-produced on a metric space (X, m) need not be the intersection of a family of maximally m-produced partial orders on X.

Chapter 3 presents two new different approaches to the problem. In the first section we describe the first approach which is based on the concept of convexity. We prove that our problem can only be solved if the ordered metric space $(X, m, \leq)$ satisfies a convexity condition (see Proposition 3.1.1 and Lemma 3.1.2). The conditions are necessary and are shown to be equivalent (see Lemma 3.1.1) to an "interval type condition". The second section elaborates on the second approach based on basic but fundamental topological concepts. In particular we show in Proposition 3.2.6, that any partially T_2-ordered compact metric space is determined by a quasi-uniformity. This result is similar to a previous one already obtained by Nachbin [31]. We conclude the chapter with a related result, Proposition 3.2.7, due to Minguzzi ([30, Theorem 2.5] but to which we provide a more elegant proof.

We discuss in Chapter 4 the suggested problem in the context of order theory. We present the results in four major sections. In the first one, we redefine the problem at hand. Our first main result, Lemma 4.2.2 which appears in the second section, gives sufficient conditions to derive the main result of Section 4.3, Lemma 4.3.1 (which establishes existence results). One of the key implications is the possibility to express an arbitrary metric m as the supremum of minimally m-splitting T_0-quasi-metrics (see Corollary 4.3.1) and to give a characterization of maximally m-produced orders (see Corollary 4.3.2). Again the introduction of the function $d_{ab,l}$ (Cf. Definition 4.2.1) allows us in the fourth section, to describe an iterative process via Corollary 4.4.1 which is specific to finite sets.

Throughout Chapter 2, the m-splitting-theory did not give us an explicit simple algorithm to compute $D_{(m,\leq)}$. We therefore provide in Chapter 5 more or less detailed computations of $D_{(m,\leq)}$. Two sections are presented: the first one where an explicit form is given for $D_{(m,\leq)}$ when the underlining ordered metric space $(X, m, \leq)$ is a lattice (see Theorem 5.1.1) and the second one when the underlining ordered metric space $(X, m, \leq)$ is a group $(X, *)$ (see Theorem 5.2.1). As we shall see, to guarantee the existence of a producing $D_{(m,\leq)}$, we shall impose some closedness assumptions on upper and downward sets generated by singletons as well as the convexity of the closed balls.

The Chapter 6 describes the construction of a natural embedding between a produced ordered metric space $(X, m, \leq)$ and the set of bounded real-valued functions defined on X. The formulated result is to be read in Proposition 6.0.2.

In Chapter 7, the last one of this thesis, we conclude our investigations by summarizing the main results of the dissertation. Furthermore we make mention of some open problems which can constitute some topics for further research. One of the most interesting ones being that of *amalgamation*. Another natural direction also being the study of our problem in a non-archimedean setting.

Some of the results presented in this thesis have been discussed in

1. Y.U. Gaba and H.-P.A. Künzi; *Splitting metrics by T_0-quasi-metrics*, Topology and its Applications 193 (2015), pp. 84–96.
2. Y.U. Gaba and H.-P.A. Künzi; *Partially ordered metric spaces produced by T_0-quasi-metrics*, under review.

Although we did not mention it anywhere in the thesis, the formulation of many of our problems, as well as the ideas used in their study can be expressed in the setting of Category Theory. The author is aware of that fact and intends to do further studies in that direction.

Chapter 0

Terminologies and basic notions

In this chapter we recall some definitions, notations and results from the literature that we shall use in the later chapters of the thesis. We use [19, 40] as the references for topological concepts. We give a brief summary about some facts in ultra-quasi-pseudometric spaces (see [11]). In the aim to ease the reading of the manuscript and to make this thesis as self-contained as possible, we also give a little bit of background on order theory and recall as well the definition of some algebraic structures and that of a quasi-uniformity; the material is mainly taken from [12, 37]. Our main structures being quasi-pseudometrics and orders, we shall therefore, for any new concept we introduce, connect it with these two fundamental notions. We also provide some elementary examples, very useful though.

0.1 Quasi-pseudometric structures

The present section consists mostly of the review of the literature and basic concepts illustrated by examples. We introduce the concept of a quasi-pseudometric space and related notions. Intuitively a quasi-pseudometric can be thought of as an "asymmetric metric". For the purpose of our investigations, we shall adopt here a definition of quasi-pseudometric which has the advantage of naturally inducing a partial order. Thus, a notion of a quasi-pseudometric that generalises both distances and partial orders is presented.

0.1.1 Basic definitions

Definition 0.1.1. Let X be a nonempty set. A function $d : X \times X \to [0, \infty)$ is called a **quasi-pseudometric** on X if:

i) $d(x, x) = 0$ whenever $x \in X$,

ii) $d(x, z) \leq d(x, y) + d(y, z)$ whenever $x, y, z \in X$.

Moreover, if

iii) $d(x, y) = 0 = d(y, x) \Longrightarrow x = y$, then d is said to be a T_0**-quasi-metric** or a **di-metric**. The latter condition is referred as the T_0-condition.

For a nonempty set X and a quasi-pseudometric (resp. a T_0-quasi-metric) d, the pair (X, d) will be called quasi-pseudometric space (resp. T_0-quasi-metric space or di-metric space).

A quasi-pseudometric d on a set X which is such that:

iv) $d(x, y) = d(y, x)$ whenever $x, y \in X$ is called a **pseudo-metric** on X.

v) $d(x, y) = 0 \Longrightarrow x = y$ whenever $x \in X$ is called a **quasi-metric** on X.

Finally, a T_0-quasi-metric which satisfies iv) is a **metric**.

If we allow the function d to take values in $[0, \infty]$ (i.e. in particular ∞), then we shall speak of an *extended quasi-pseudometric* and with the value ∞, the triangle inequality is interpreted in the obvious way.

Duality is a very important phenomenon associated with asymmetric structures. Some of its topological aspects are investigated thoroughly in the paper by Kopperman [24]. As we mentioned in the introduction, in the case of quasi-pseudometrics, duality is expressed by the presence of two structures, left and right. The symmetrisation (or 'join') of these two structures corresponds to a metric structure.

Definition 0.1.2. Let (X, d) be a quasi-pseudometric space, the **conjugate**[1] (or ***dual***) of d is the function denoted d^{-1} and defined, whenever $x, y \in X$ by

$$d^{-1}(x, y) = d(y, x).$$

Remark 0.1.1. It is easy to see that the function d^s defined, whenever $x, y \in X$, by

$$d^s(x, y) = \max\{d(x, y), d(y, x)\}, \quad (\text{ i.e. } d^s := d \vee d^{-1})^{2},$$

defines a metric on X whenever d is a T_0-quasi-metric. Indeed,

- $d^s(x, x) = 0$ and $d^s(x, y) = 0 \Longrightarrow x = y \quad \forall\ x, y \in X$.
- $d^s(x, y) = \max\{d(x, y), d(y, x)\} = d^s(y, x) \quad \forall\ x, y \in X$.
- Given $x, y, z \in X$, we have that

$$d(x, z) \leq d(x, y) + d(y, z) \leq d^s(x, y) + d^s(y, z).$$

[1] The conjugate quasi-pseudometric is also denoted d^t or $\bar{d}$, specially in the case where X is a vector space, see Definition 0.1.5.

[2] This construction (of d^s) is actually a very common way of symmetrizing a quasi-pseudometric.

and

$$d(z,x) \leq d(z,y) + d(y,x) \leq d^s(y,z) + d^s(x,y).$$

Hence

$$d^s(x,z) \leq d^s(x,y) + d^s(y,z) \quad \forall x,y,z \in X.$$

We often call d^s the ***associated pseudometric*** to the quasi-pseudometric d and we observe that the associated pseudometric d^s is the smallest metric majorising the T_0-quasi-metric d.

Another often used symmetrisation of a quasi-pseudometric is the 'sum' pseudometric d^+, defined, whenever $x,y \in X$, by

$$d^+(x,y) = d(x,y) + d(y,x).$$

It is also an easy exercise to check that the function d^+ defines a metric on X whenever d is a T_0-quasi-metric on X. Indeed whenever $x,y,z \in X$

- $d^+(x,x) = 0$ and $d^+(x,y) = 0 \Longrightarrow x = y$,
- $d(x,z) \leq d(x,y) + d(y,z)$ and $d(z,x) \leq d(z,y) + d(y,x)$ imply that

  $$d(x,z) + d(z,x) \leq d(x,y) + d(y,z) + d(z,y) + d(y,x)$$

  i.e.

  $$d^+(x,z) \leq d^+(x,y) + d^+(y,z).$$

Definition 0.1.3. A T_0-quasi-metric space (X,d) is **bicomplete** if and only if the metric space (X,d^s) is complete [3].

[3] By referring to Definiton 1.0.5, the natural observation will be that if the partially ordered metric space $(X,m,\leq)$ is complete and produced by a T_0-quasi-metric d, then the T_0-quasi-metric space (X,d) is bicomplete.

Definition 0.1.4. An **ultra-quasi-pseudometric** d on a nonempty set X is a quasi-pseudometric d which satisfies the strong form of the triangle inequality, namely

$$d(x,z) \leq \max\{d(x,y), d(y,z)\} \text{ whenever } x,y,z \in X.$$

Ultra-quasi-pseudometrics are also called **non-archimedean** quasi-pseudometrics. An ultra-quasi-pseudometric d will be called aT_0**-ultra-quasi-metric** if it satisfies the T_0-condition.

0.1.2 Examples

We now present some well-known examples of quasi-pseudometric spaces.

Example 0.1.1. Let X be a set and let $d : X \times X \to [0, \infty)$ be the map defined by

$$d(x,y) = \begin{cases} 0 & \text{, if } x = y, \\ 1 & \text{, if } x \neq y, \end{cases}$$

whenever $x, y \in X$. It can easily be checked that d is a metric. It is called the ***discrete metric***.

Next, we give some examples of quasi-pseudometrics on $\mathbb{R}$.

Example 0.1.2.

$$d(x,y) = \begin{cases} \min\{1, y-x\} & \text{, if } x \leq y, \\ 1 & \text{, otherwise.} \end{cases}$$

In this case d induces a T_1-topology $\mathcal{T}$ on $\mathbb{R}$ whose base consists of all left balls centred at $x \in \mathbb{R}$. The left balls are of the form $B_d(x,\epsilon) = [x, x+\epsilon)$ where $x \in \mathbb{R}$ and $0 < \epsilon < 1$ (note that for any $x \in \mathbb{R}$ and $\epsilon \geq 1$, $B_d(x,\epsilon) = \mathbb{R}$). The topological space $(\mathbb{R}, \mathcal{T})$ is called the ***Sorgenfrey line***, and is a well-known object in topology and a source of many counter-examples. The associated metric d^s is the discrete metric.

Any unbounded quasi-metric can be converted to a bounded quasi-metric while preserving the topology in the following way.

Example 0.1.3. Let (X, d) be an extended quasi-pseudometric. Then the map $\rho : X \times X \to [0, \infty)$ defined by

$$\rho(x, y) = \min\{1, d(x, y)\},$$

whenever $x, y \in X$, is a quasi-pseudometric such that the topologies generated by the two quasi-pseudometrics coincide. The proof of quasi-pseudometric axioms is trivial and the fact that the topologies coincide follows from the fact that all open balls of radius not greater than 1 coincide.

Example 0.1.4. Given two nonnegative real numbers a and b we shall write $a \dot{-} b$ for $\max\{a - b, 0\}$. In a more lattice theoretic notation, we shall also denote it by $(a - b) \vee 0$. It should be noted that $u(x, y) = x \dot{-} y$ with $x, y \in [0, \infty)$ defines the standard T_0-quasi-metric on $[0, \infty)$. Thus $([0, \infty), u)$ is a T_0-quasi-metric space.

Example 0.1.5. Let $X = [0, \infty)$ be equipped with $n(x, y) = x$ if $x > y$ and $n(x, y) = 0$ if $x \leq y$. It is easy to check that (X, n) is a T_0-ultra-quasi-metric space. Let us verify that the strong triangle inequality

$$n(x, z) \leq \max\{n(x, y), n(y, z)\},$$

whenever $x, y, z \in X$ holds. The case that $n(x, y) = x$ is trivial, since then $n(x, z) \leq n(x, y)$. Similarly the case $n(x, y) = 0$ and $n(y, z) = y$ are obvious, since then $x \leq y$ and $n(x, z) \leq n(y, z)$. In the remaining case that $n(x, y) = 0 = n(y, z)$, we see by transitivity of $\leq$ that $x \leq z$, and thus $n(x, z) = 0$. It is also obvious that n satisfies the T_0-condition.

0.1.3 Asymmetric normed spaces

Important examples of quasi-metrics are provided by asymmetric norms, the asymmetric versions of norms. The research area of quasi-normed spaces has seen a significant development in recent years both in theory and in applications (see [7–9, 34]).

Definition 0.1.5. (compare [7, Section 1.1]) Let X be a nonempty real vector space and $p : X \to [0, \infty)$ be a mapping from X into the set $[0, \infty)$ of nonnegative reals. Then p is called an **asymmetric norm** on X if whenever $x, y \in X$ and $\alpha \geq 0$, we have:

i) $p(x) = p(-x) = 0 \Rightarrow x = 0$,

ii) $p(\alpha x) = \alpha p(x)$,

iii) $p(x + y) \leq p(x) + p(y)$.

For a nonempty real vector space X and an asymmetric norm p, we call the pair (X, p) an asymmetric normed space.

Sometimes p will be allowed to take value ∞, in which case we shall call it an *extended asymmetric norm.* We define the conjugate (see Definition 0.1.2) p^t of p as $p^t(x) = p(-x)$, whenever $x \in X$. It is not difficult to see that $p^s = \max\{p, p^t\}$ is a norm on X. To denote an asymmetric norm, the symbol $\|.|$ is often used (see for instance [25, Ch. IX, Section 5]).

The following are examples of an asymmetric norms on $\mathbb{R}$.

Example 0.1.6. (compare [7, Example 1.2]) Define the map $u : \mathbb{R} \to [0, \infty)$ by $\alpha \mapsto u(\alpha) = \alpha^+ := \max\{\alpha, 0\}$. Then it is not hard to see that u is an asymmetric norm on $\mathbb{R}$. The conjugate u^t of u is defined by $u^t(\alpha) = \alpha^- := \max\{-\alpha, 0\}$ and $u^s(\alpha) = \max\{u(\alpha), u^t(\alpha)\} = |\alpha|$ is a norm on $\mathbb{R}$.

Example 0.1.7. (*The general quasi-metric "segment I_{ab}"*) (see [1, Remark 2]) Let $X = [0, 1]$. Choose $a, b \in [0, \infty)$ such that $a + b \neq 0$. Set $d_{ab}(x, y) = (x - y)a$ if $x > y$ and $d_{ab}(x, y) = (y - x)b$ if $y \geq x$. Then $([0, 1], d_{ab})$ is a T_0-quasi-metric space as it is readily checked, by considering the various cases for the underlying asymmetric norm n_{ab} on $\mathbb{R}$ defined by $n_{ab}(x) = xa$ if $x > 0$ and $n_{ab}(x) = -xb$ if $x \leq 0$.

In the next section, we discuss a bit of order theory. Again the material presented is quite standard.

0.2 Ordered sets

We now proceed to indicate the ideas relative to ordered sets which we shall have the opportunity to use. Our main references for this section are [31, 37]. For a nonempty set X, whenever for any two elements of X, there is defined a concept of "greater than or equal", we say that this concept defines a preorder. More formally, we have

Definition 0.2.1. Let X be a nonempty set. A binary relation $\leq \subseteq X \times X$ on a set X is called a **preorder** if it is:

i) reflexive, that is, $x \leq x$, whenever $x \in X$,

ii) transitive, that is, if $x \leq y$ and $y \leq z$, then $x \leq z$ whenever $x, y, z \in X$.

A preordered set is a set equipped with a preorder, that is, a set on which a preorder is given.

Moreover, if the preorder $\leq$ is

iii) antisymmetric, that is, if $x \leq y$ and $y \leq x$, then $x = y$, whenever $x, y \in X$, it is called a **partial order**.

A partially ordered set or poset is a set equipped with a partial order, that is, a set on which a partial order is given.

Remark 0.2.1. Consider a set X and define $x \leq y$ by $x = y$. In this fashion, we obtain an order relation on X which we call discrete order[4].

Definition 0.2.2. A partial order will be called a **linear order** (or **total order**) if it is

iv) decisive, that is, $x \leq y$ or $y \leq x$, whenever $x, y \in X$.

A totally ordered set or a chain is a set equipped with a total order, that is, a set on which a total order is given.

Although our main focus in this thesis is to use partial orders, we shall nevertheless present the theory using preorders which represent a more general class. Moreover, in this manuscript, the set (or any subset) of real numbers will always be considered as equipped with its well-known *natural total order*, unless otherwise stated.

Definition 0.2.3. Given a set X, the graph of a preorder $\leq$ on X, that we shall denote as $G(\leq)$ is the subset of the square X^2 formed by the points (x, y) such that $x \leq y$, hence

$$G(\leq) = \{(x, y) : x \leq y\}.$$

[4]Logically speaking, there is no distinction between the "discrete order" and the "relation of equality". However it is of a great advantage to look at the equality relation as an order relation, because in doing so, many of the results relative to non-ordered sets can then be considered as particular cases of results on ordered sets.

Using this notation, the "reflexive" and "transitive" properties of a preorder $\leq$ on X can be respectively expressed as

$$\Delta_X := \{(x, x) : x \in X\} \subseteq G(\leq) \text{ and } G(\leq) \circ G(\leq) \subseteq G(\leq).$$

Here the symbol "$\circ$" denotes the usual composition of binary relations.

In a preordered set, we write $x \geq y$ to say that $y \leq x$. There is a natural duality between the relations $\leq$, $\geq$ which often relieves one from duplicative formulation of a definition or theorem. Once a definition or theorem is formulated for one relation, the formulation for the dual relation follows automatically.

Definition 0.2.4. Let $(X, \leq)$ be a partially ordered set, and $Y \subseteq X$ be an arbitrary subset.

i) An element $u \in X$ is said to be an **upper bound** of Y if $y \leq u$ for each $y \in Y$.

ii) An upper bound u of Y is said to be its **least upper bound**, or **join**, or **supremum**, if $u \leq v$ for each upper bound v of Y.

iii) Dually, $l \in X$ is said to be a **lower bound** of Y if $l \leq y$ for each $y \in Y$. A lower bound l of Y is said to be its **greatest lower bound**, or **meet**, or **infimum**, if $k \leq l$ for each lower bound k of Y.

Definition 0.2.5. Let $(X, \leq)$ be a partially ordered set.

i) $(X, \leq)$ is called a **join-semilattice** if each two-element subset $\{a, b\} \subseteq X$ has a join (i.e. least upper bound) denoted $a \vee b$.

ii) Dually $(X, \leq)$ is called a **meet-semilattice** if each two-element subset $\{a, b\} \subseteq X$ has a meet (i.e. greatest lower bound) denoted $a \wedge b$.

iii) Finally a partially ordered set $(X, \leq)$ is called a **lattice** if it is both a join-semilattice and a meet-semilattice.

It is important at this point to show the interdependence between quasi-pseudometrics and partial orders.

Definition 0.2.6. Let (X, d) be a quasi-pseudometric space. We associate to the quasi-pseudometric d the binary relation denoted $\leq_d$ and defined by

$$x \leq_d y \Longleftrightarrow d(x, y) = 0.$$

It is easy to see that $\leq_d$ is a partial order, we call it **the specialization order of** d. Hence one can associate a partial order to every quasi-pseudometric. The converse is also true. In fact we establish the following result.

Proposition 0.2.1. *Let X be a nonempty set. The collection of all two-valued T_0-quasi-metrics on X can be identified with the set of partial orders on X.*

Proof. Let $(X, \leq)$ be a partially ordered set. Then $d_{\leq} : X \times X \to \{0, 1\}$ defined by for $x, y \in X$, $d_{\leq}(x, y) = 0$ if and only if $x \leq y$, yields a T_0-quasi-metric on X. Obviously the value 1 is highly non-canonical, and in some contexts it is better to work with extended quasi-pseudometrics using ∞ instead of 1 [5]. Note here the crucial fact that the specialization order $\leq_{(d_{\leq})}$ of $d_{\leq}$ is equal to $\leq$. Furthermore, given any T_0-quasi-metric $d : X \times X \to \{0, 1\}$ and $\leq_d$ its specialization order, we have that $d_{(\leq_d)} = d$. Hence the T_0-quasi-metric $d : X \times X \to \{0, 1\}$ can be identified with the partial orders on X. □

We now give the following example, as it illustrates very well the above proposition.

Example 0.2.1. For any set X and any partial order $\leq$ on X, the T_0-quasi-metric $d_{\leq}$ is such that $(d_{\leq})^s = d_=$ and $\leq_{(d_{\leq})} = \leq$, where we note that $d_=$ is the standard discrete metric on X.

[5]Later on, we shall work with this slightly modified definition (see Definition 0.2.7).

We conclude this section by the following very crucial definition:

Definition 0.2.7. Let X be a set and $\leq$ be a partial order on X. The map $d_{\leq}$ defined on $X \times X$ defined

$$d_{\leq}(x,y) = \begin{cases} 0 & , \text{ if } x \leq y, \\ \infty & , \text{ otherwise,} \end{cases}$$

is an extended quasi-pseudometric on X.

0.3 Topologies and quasi-uniformities

Here again, we present a very basic material on the topic. There is substantial amount of publications about topological and uniform structures related to quasi-pseudometric spaces. The major review by Künzi [26] contained useful information that we used in this section.

Let (X, d) be a quasi-pseudometric space. Then for each $x \in X$ and $\epsilon > 0$, the set

$$B_d(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\}$$

denotes the *open ϵ-ball centered at x, of radius ε with respect to d*. It should be noted that the collection

$$\{B_d(x, \epsilon) : x \in X, \epsilon > 0\}$$

yields a base for the topology $\tau(d)$ induced by d on X. In a similar manner, for each $x \in X$ and $\epsilon \geq 0$, we define

$$C_d(x, \epsilon) = \{y \in X : d(x, y) \leq \epsilon\},$$

known as the *closed ϵ-ball centered at x, of radius ϵ with respect to d.*

Hence, for any $A \in X$, we shall respectively denote by $int_{\tau(d)}A$ and $cl_{\tau(d)}A$, the interior and the closure of the set A with respect to the topology $\tau(d)$.

Of course the collection

$$\{B_{d^{-1}}(x, \epsilon) : x \in X, \epsilon > 0\}$$

also yields a base for the topology $\tau(d^{-1})$ induced by d^{-1} on X. The set $C_d(x, \epsilon)$ is $\tau(d^{-1})$-closed, but not $\tau(d)$-closed in general.

The balls with respect to d are often called *forward balls* and the topology $\tau(d)$ is called *forward topology*, while the balls with respect to d^{-1} are often called *backward balls* and the topology $\tau(d^{-1})$ is called *backward topology*.

We shall say that a subset $E \subseteq X$ is ***join-closed*** if it is $\tau(d^s)$-closed, i.e. closed with respect to the topology generated by d^s. The topology $\tau(d^s)$ is finer than the topologies $\tau(d)$ and $\tau(d^{-1})$.

Remark 0.3.1. Note that for any quasi-pseudometric space (X, d),

$$B_{d^s}(x, \epsilon) = B_{d^{-1}}(x, \epsilon) \cap B_d(x, \epsilon)$$

and hence a base of the metric topology $\tau(d^s)$ consists exactly of intersections of forward and backward open balls of the same radius, centred at any point. Therefore, $\tau(d^s)$ is the supremum of $\tau(d)$ and $\tau(d^{-1})$:

$$\tau(d^s) = \tau(d) \vee \tau(d^{-1}).$$

Hence, one can naturally associate the bitopological space $(X, \tau(d), \tau(d^{-1}))$ to a quasi-metric space (X, d). The relationships between quasi-pseudometric and bitopological spaces are well researched in [26].

Definition 0.3.1. A topological space (X, τ) is **quasi-pseudometrisable** if there exists a quasi-pseudometric d such that $\tau = \tau(d)$.

The question of which topologies are quasi-pseudometrisable (i.e. can be induced from a quasi-pseudometric) has been long open. Some characterisations have been given by Kopperman [22, 23] in terms of bitopological spaces and by Vitolo [39] in terms of hyperspaces of metric spaces.

It is well known that concepts like total boundedness, uniform continuity (very crucial in metric spaces) cannot be formulated in arbitrary topological spaces, due to the fact that there is no obvious way to compare the sizes of the neighbourhoods of distinct points [6]. In trying to fix this problem, we are naturally led to the simple but very useful concept of a quasi-uniform space.

Definition 0.3.2. Let X be a nonempty set. A **filter** $\mathscr{F}$ on X is a collection of subsets of X which satisfies the following properties:

i) $X \in \mathscr{F}$,

ii) if $V \in \mathscr{F}$ and $V \subseteq W \subseteq X$, then $W \in \mathscr{F}$,

iii) if $U, V \in \mathscr{F}$, then $U \cap V \in \mathscr{F}$,

iv) $\emptyset \notin \mathscr{F}$.

Example 0.3.1. Let X be a topological space and consider a point $x \in X$. Recall that we define a set V to be a neighbourhood of x if there exists an open set U such that $x \in U \subseteq V$. Let $\mathcal{N}_x$ be the set of all neighbourhoods of x. Then, $\mathcal{N}_x$ is a filter.

Definition 0.3.3. A family $\mathscr{B}$ of subsets of X will be called a **filter base** if it satisfies the following properties:

[6] (Compare [27])Of course, one natural way to overcome this is to choose for a given topological space X a large ordered index I set and for each point $x \in X$ an indexed neighbourhood base $\{N_i(x), i \in I\}$ such that if $i_1, i_2 \in I$ such that $i_1 \leq i_2$, then $N_{i_1}(x) \subseteq N_{i_2}(x)$. In fact, it turned out that it was necessary to enrich the structure of I in order to define a "triangle inequality" like property for this family of neighbourhoods.

i) $\emptyset \notin \mathscr{B}$,

ii) $\mathscr{B} \neq \emptyset$ and for any $B_1, B_2 \in \mathscr{B}$, there exists $B_3 \in \mathscr{B}$ such that $B_3 \subseteq B_1 \cap B_2$.

Remark 0.3.2.

- Since any arbitrary intersection of filters is again a filter, for any subset A of a set X, we shall define the **filter generated** by A as the intersection of all filters containing A.

- For a given filter base $\mathscr{B}_* \in \mathscr{P}(X)$ on a set X, define

$$\mathscr{B} = \{F \subseteq X : E \subseteq F \text{ for some } E \in \mathscr{B}_*\},$$

then $\mathscr{B}$ is the filter generated by $\mathscr{B}_*$.

- Conversely, for a given filter $\mathscr{F} \subseteq \mathscr{P}(X)$ on a set X, we say that a filter base $\mathscr{B}$ is a basis for the filter $\mathscr{F}$ if

$$\text{for any } F \in \mathscr{F}, \text{ there exists } B \in \mathscr{F} : B \subseteq F.$$

Definition 0.3.4. A **quasi-uniformity** $\mathcal{U}$ on a set X is a filter on $X \times X$ such that

i) each member U of $\mathcal{U}$ contains the diagonal $\Delta_X = \{(x, x) : x \in X\}$ of X,

ii) for each member U of $\mathcal{U}$ there exists a $V \in \mathcal{U}$ such that $V^2 \subseteq U$, where $V^2 := V \circ V = \{(x, z) \in X \times X : \exists y \in X : (x, y) \in V \text{ and } (y, z) \in V\}$.

The members U of $\mathcal{U}$ are called **entourages** of $\mathcal{U}$ and the pair $(X,\mathcal{U})$ is called a **quasi-uniform space**.

To any quasi-uniformity $\mathcal{U}$ corresponds a dual quasi-uniformity $\mathcal{U}^{-1}$ defined as

$$\mathcal{U}^{-1} = \{U : U^{-1} \in \mathcal{U}\}$$

where

$$U^{-1} = \{(x, y) : (y, x) \in U\}.$$

A quasi-uniformity is a **uniformity** if $V \in \mathcal{U}$ implies $V^{-1} \in \mathcal{U}$.

Example 0.3.2. Let T be a reflexive and transitive relation, that is, a preorder on a set X. Then the filter on $X \times X$ generated by the base $\{T\}$ is a quasi-uniformity on X.

Hence for any partially ordered set $(X, \leq)$, we shall denote by $\mathcal{U}_{\leq}$ the quasi-uniformity generated by the base $\{\leq\}$ on X.

Each quasi-uniformity $\mathcal{U}$ on a set X induces a topology $\tau(\mathcal{U})$ as follows: For each $x \in X$ and $U \in \mathcal{U}$, set $U(x) = \{y \in X : (x, y) \in U\}$. A subset $G \subseteq X$ belongs to $\tau(\mathcal{U})$ if and only if for each $x \in G$, there exists $U \in \mathcal{U}$ such that $U(x) \subseteq G$.

The neighbourhood filter $\tau(\mathcal{U})(x)$, at $x \in X$ with respect to the topology $\tau(\mathcal{U})$ is given by

$$\tau(\mathcal{U})(x) = \{U(x) : U \in \mathcal{U}\}.$$

Proposition 0.3.1. *(Compare [12, Proposition 1.9]) Let $(X, \mathcal{U})$ be a quasi-uniform space. Then $\tau(\mathcal{U})$ is a T_0-topology if and only if $\bigcap \mathcal{U}$ is a partial order. Furthermore, $\tau(\mathcal{U})$ is a T_1-topology if and only if $\bigcap \mathcal{U}$ is equal to the diagonal of X.*

Remark 0.3.3. (Compare [27, page 242]) Given a quasi-pseudometric d on a set X, $\tau(\mathcal{U}_d)$ (see Example 0.3.3) is the standard quasi-pseudometric topology $\tau(d)$ on X having all open balls $B_d(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$ ($x \in X$ and $\varepsilon > 0$) as a base. Observe that $\tau(d)$ is a T_0-topology if and only if for all $x, y \in X$, $d(x, y) = 0 = d(y, x)$ implies that $x = y$. Similarly, $\tau(d)$ is a T_1-topology if and only if for all $x, y \in X$, $d(x, y) = 0$ implies that $x = y$.

Given a quasi-uniformity $\mathcal{U}$, the collection $\{V \cap W^{-1}, V, W \in \mathcal{U}\}$ yields a base for the supremum uniformity $\mathcal{U}^*$ (that we shall denote by $\mathcal{U}^s$ for obvious reasons), the coarsest uniformity containing $\mathcal{U}$. The symmetric topology of the quasi-uniformity $\mathcal{U}$ is $\tau(\mathcal{U}^s)$. Moreover the intersection $\bigcap \mathcal{U}$ is the graph of a preorder on X (see [31]). Thus given a quasi-uniformity $\mathcal{U}$ on X, one naturally obtains a topological preordered space $(X, \tau(\mathcal{U}^s), \bigcap \mathcal{U})$ (a topological space equipped with a preorder).

Example 0.3.3. (Compare [27]) Let d be a quasi-pseudometric on a set X. The filter on $X \times X$ generated by the base $\{U_\epsilon : \epsilon > 0\}$ where

$$U_\epsilon = \{(x, y) \in X \times X : d(x, y) < \epsilon\},$$

is a quasi-uniformity called **quasi-pseudometric quasi-uniformity** and denoted $\mathcal{U}_d$. It is the quasi-uniformity induced by d on X. Indeed, just observe that for each $\epsilon > 0$, $U^2_{\epsilon/2} \subseteq U_\epsilon$. The dual (conjugate) quasi-uniformity $\mathcal{U}_d^{-1}$ is generated by the entourages

$$U_\epsilon^{-1} = \{(x, y) \in X \times X : d(y, x) < \epsilon\}$$

and the symmetrisation $\mathcal{U}_{d^s} = \mathcal{U}_d \vee \mathcal{U}_d^{-1}$ produces a uniformity. It is easy to see that for any T_0-quasi-pseudometric d, the uniformity $\mathcal{U}^s$ is equivalent to the uniformity generated by the associated metric d^s.

This example concludes our introductory chapter.

Chapter 1

Linearity conditions

In this chapter, we are interested in investigating the conditions of existence of a T_0-quasi-metric d *producing* an ordered metric space $(X, m, \leq)$ in the case where the order on the ordered metric space is linear. We give and prove, as a necessary condition, what we called *the interval condition* in Lemma 1.1.1. We then establish in Proposition 1.1.2 that *the interval condition* is actually sufficient. Using the idea of Dushnik-Miller [10], we introduce the concept of *pseudo-dimension* and show that it is indeed an appropriate way of measuring how the ordered metric space $(X, m, \leq)$ can be produced if its "linear extension", in a sense that we shall specify, is produced. We begin with the following definition

Definition 1.0.5. Let $(X, m, \leq)$ be a partially ordered metric space. We say that a T_0-quasi-metric d **produces** $(X, m, \leq)$ if

$$\leq_d = \leq \quad \text{and} \quad d^s = m.$$

When such a T_0-quasi-metric d exists, we shall say that the ordered metric space $(X, m, \leq)$ is **produced** by d or that d is a **producing** T_0-quasi-metric for the ordered metric space $(X, m, \leq)$.

The lemmas below are immediate consequences of Definition 1.0.5.

Lemma 1.0.1. *Let $(X, m, \leq)$ be a partially ordered metric space and $Y \subseteq X$ a subset of X. If $(X, m, \leq)$ is produced, then $(Y, m_Y, \leq_Y)$ is produced. Here m_Y is the induced metric on Y by m and $\leq_Y$ is the induced order on Y by $\leq$.*

Lemma 1.0.2. *Let $[(X_i, m_i, \leq_i)]_{i\in I}$ be a family of uniformly bounded partially ordered metric spaces. If $(X_i, m_i, \leq_i)$ is produced by some d_i for any $i \in I$, then $(X, m, \leq)$ is produced by $d = \sup\limits_{i\in I} d_i$. Here*

$$X = \prod_{i\in I} X_i, \quad m = \sup\{m_i : i \in I\},$$

and for any $x = \{x_i\}, y = \{y_i\} \in X$,

$$x \leq y \Longleftrightarrow x_i \leq_i y_i \quad \text{for any } i \in I.$$

1.1 The linear case

We start with the following very important characterization which answers the question of uniqueness.

Proposition 1.1.1. *Let $(X, m, \leq)$ be a linearly ordered metric space. If there exists a T_0-quasi-metric d which produces $(X, m, \leq)$, then d is unique.*

Proof. The map u defined by

$$u(x, y) = \begin{cases} 0 & , \text{ if } x \leq y, \\ m(x, y) & , \text{ otherwise,} \end{cases}$$

is the only choice for d since $x \leq y \Longrightarrow d(x, y) = 0$ and $x > y$ certainly implies that $d^s(x, y) = \max\{d(x, y), d(y, x)\} = m(x, y)$. Since for $x > y$, $d(y, x) = 0$, it follows that $m(x, y) = \max\{d(x, y), 0\} = d(x, y)$, because $m(x, y) > 0$. □

Remark 1.1.1. The above proof establishes that, under existence conditions, the candidate for the producing T_0-quasi-metric d is known but does not guarantee that the candidate is actually a T_0-quasi-metric. Indeed, as the following example shows, the map u does not always satisfy the triangle inequality in general.

Example 1.1.1. We consider $X = \{0, 1, 2\}$ equipped with the natural linear order $\leq$ such that $0 < 1 < 2$. We define the metric $m : X \times X \to [0, \infty)$ by

- $m(i, i) = 0$ for all $i \in X$,
- $m(0, 1) = m(1, 0) = 1 = m(0, 2) = m(2, 0)$,
- $m(1, 2) = m(2, 1) = 2$.

On the other hand, using the map u (compare Example -1.0.1) defined above, we have $d(2, 1) = m(2, 1) = 2, d(2, 0) = m(2, 0) = 1$ and $d(0, 1) = 0$. Hence, d does not satisfy the triangle inequality, since

$$2 = d(2, 1) > d(2, 0) + d(0, 1) = 1 + 0 = 1.$$

In the following, we show how we derive necessary and sufficient conditions for the existence of a producing T_0-quasi-metric d, in other words, for the candidate map u defined above, to satisfy the triangle inequality.

If such a T_0-quasi-metric d exists, then for any $x, y, z \in X$, we must have

$$d(x, z) \leq d(x, y) + d(y, z).$$

Let $(X, m, \leq)$ be a linearly ordered metric space. Assume that there exists a T_0-quasi-metric d that produces $(X, m, \leq)$.

Given the linear order $\leq$, and $x, y, z \in X$, without loss of generality, we can assume that x, y and z are pairwise distinct (i.e. $x \neq y \neq z, x \neq z$.). Therefore we have six (06) ways of arranging x, y, z with respect to this linear order.

1. <u>Case 1</u>, $z < x < y$ which gives us under the triangular inequality that

$$d(x,z) = m(x,z) \leq 0 + m(y,z) = d(x,y) + d(y,z).$$

2. <u>Case 2</u>, $y < z < x$ which gives us under the triangular inequality that

$$d(x,z) = m(x,z) \leq m(x,y) + 0.$$

3. <u>Case 3</u>, $x < y < z$ which obviously gives us the triangular inequality since

$$d(x,z) = 0 \leq 0 + 0 = d(x,y) + d(y,z).$$

4. <u>Case 4</u>, $x < z < y$ which obviously gives us the triangular inequality since

$$d(x,z) = 0.$$

5. <u>Case 5</u>, $y < x < z$ which obviously gives us the triangular inequality since

$$d(x,z) = 0.$$

6. <u>Case 6</u>, $z < y < x$ which gives us under the triangular inequality that

$$d(x,z) = m(x,z) \leq m(x,y) + m(y,z) = d(x,y) + d(y,z).$$

Among all these six (06) cases, only Case 1 and Case 2 will be of interest for us since the four (04) other are trivial and do not bring any additional information.

Hence, necessary conditions for the existence of a T_0-quasi-metric d are: for all $x, y, z \in X$,

$$(\mathcal{L}) \quad \begin{cases} z < x < y \Longrightarrow m(x,z) \leq m(y,z) & (\mathcal{L}_1), \\ y < z < x \Longrightarrow m(x,z) \leq m(x,y) & (\mathcal{L}_2). \end{cases}$$

When we rewrite conditions $(\mathcal{L})$ in a more natural way, we obtain that for all $x, y, z \in X$,

$$\boxed{(\mathcal{L}_1) \Longleftrightarrow (x < y < z \Longrightarrow m(y,x) \leq m(z,x))}.$$

$$\boxed{(\mathcal{L}_2) \Longleftrightarrow (x < y < z \Longrightarrow m(z,y) \leq m(z,x))}.$$

Remark 1.1.2. It is worth mentioning that the conditions $(\mathcal{L}_1)$ and $(\mathcal{L}_2)$ are not equivalent as shown in the example below. We basically repeat the previous Example 1.1.1. We set $x = 0, y = 1, z = 2$.

So

$$m(y,x) = m(1,0) = 1 \leq 1 = m(z,x) = m(2,0),$$

i.e. the condition $(\mathcal{L}_1)$ is satisfied

but

$$m(z,y) = m(2,1) = 2 > 1 = m(z,x) = m(2,0),$$

i.e. the condition $(\mathcal{L}_2)$ is not satisfied.

We are now in a position to state the main results of this section. The first one is the so-called "interval condition", named so because of the connection this result establishes between our theory and that of *Robinsonian dissimilarities* (see e.g. [6, p. 523]).

Lemma 1.1.1. *(Interval condition) Let $(X, m, \leq)$ be a partially ordered metric space. If there exists a T_0-quasi-metric d which produces $(X, m, \leq)$, then for any $x, y, z \in X$ such that $x \leq y \leq z$, we have that*

$$\max\{m(x,y), m(y,z)\} \leq m(x,z).$$

Proof. Suppose that a T_0-quasi-metric d on X that produces the totally ordered metric space $(X, m, \leq)$, exists. Consider any $x, y, z \in X$ such that $x \leq y \leq z$. We necessarily have $d(x,y) = 0$ since $x \leq y$ and $d(y,x) = d^s(x,y) = m(x,y)$ (see proof of Proposition 1.1.1). Therefore $y \geq x$ implies that $d(y,x) = m(x,y)$. This similarly goes for the pairs (z,x) and (z,y).

Thus

$$m(y,z) = d(z,y) \leq d(z,x) + d(x,y) = m(x,z) + 0 = m(x,z).$$

Furthermore, $m(x,y) = d(y,x) \leq d(y,z)+d(z,x) = 0+m(x,z) = m(x,z)$. Hence the given condition is satisfied. □

We give the following characterization that will be useful in the proof of Proposition 1.1.2, but also in the rest of the manuscript, specially for computational purposes.

Corollary 1.1.1. *Let $(X, m, \leq)$ be a linearly ordered metric space. If there is a T_0-quasi-metric d that produces $(X, m, \leq)$ then it is necessarily equal to the function $r = \min\{m, d_{\leq}\}$ where $d_{\leq}$ is as defined in Definition 0.2.7.*

Proof. The result follows from Proposition 1.1.1. Since the conditions

(1) $d^s = m$ and,

(2) for any $x, y \in X$, $x \leq y$ if and only if $d(x,y) = 0$,

obviously do not leave any freedom to define d differently from r provided that $\leq$ is a linear order. Indeed for $x, y \in X$ with $x \leq y$[1], we have

$$d(x,y) = 0 = \min\{m(x,y), d_{\leq}(x,y) = 0\},$$

and

$$d(y,x) = m(y,x) = \min\{m(y,x), d_{\leq}(y,x) = \infty\}.$$

□

[1]This is always possible and does not introduce any loss of generality since the order is total.

The interval condition is crucial as our next result shows. It gives a characterization of linearly ordered metric spaces that are produced by a T_0-quasi-metric. The following is the main result of this section.

Proposition 1.1.2. *Suppose that $\leq$ is a linear order on a metric space (X, m). Then there exists a T_0-quasi-metric d on X that produces the totally ordered metric space $(X, m, \leq)$ if and only if m satisfies the following condition: For any $x, y, z \in X$ we have that $x \leq y \leq z$ implies that $m(y, z) \leq m(x, z)$ and $m(x, y) \leq m(x, z)$.*

Proof. One implication follows directly from Lemma 1.1.1. In order to prove the converse, suppose that a linearly ordered metric space $(X, m, \leq)$ is given such that m satisfies the interval condition (with respect to $\leq$).

We want to show that $d := r = \min\{m, d_{\leq}\}$ is a T_0-quasi-metric that produces $(X, m, \leq)$, where the natural extended T_0-quasi-metric $d_{\leq}$ on X is defined by $d_{\leq}(x, y) = 0$ whenever $x \leq y$ and $d_{\leq}(x, y) = \infty$ otherwise.

It is obvious, by our definition, that d satisfies conditions

(1) $d^s = m$ and,

(2) for any $x, y \in X$, $x \leq y$ if and only if $d(x, y) = 0$.

So we only have to check that d satisfies the triangle inequality.

Let $x, y, z \in X$. If $x \leq z$, then $d(x, z) = 0$ and therefore the triangle inequality $d(x, z) \leq d(x, y) + d(y, z)$ is clearly satisfied.

So by linearity of $\leq$ we can assume that $z < x$ in the following three (3) cases.

- For the first case assume that $z \leq y \leq x$. Then by the triangle inequality of m, $d(x, z) = m(x, z) \leq m(x, y) + m(y, z) \leq d(x, y) + d(y, z)$. Hence the triangle inequality for d holds in this case.

- Next, we consider the second case that $y \leq z \leq x$. Then by the interval assumption satisfied by m,

$$d(x,z) = m(z,x) \leq m(y,x) + 0 = d(x,y) + d(y,z).$$

Again, the triangle inequality for d is satisfied in the case under consideration.

- Finally we consider the third case that $z \leq x \leq y$. Hence again by the interval condition satisfied by m we have

$$d(x,z) = m(z,x) \leq m(z,y) = 0 + m(y,z) = d(x,y) + d(y,z).$$

Therefore we conclude that d satisfies the triangle inequality. □

The above proposition summarizes the results of our investigation in this first section. We close the section by giving two examples. The first example is a direct application of the above proposition.

Example 1.1.2. Let $X = \{a, b, c\}$ be equipped with the linear order $\leq$ such that $a < b < c$ and the metric m such that $m(a,b) = 2$, $m(b,c) = 3$, $m(a,c) = 4$ and $m(x,x) = 0$ whenever $x \in X$. The interval condition is indeed satisfied since

$$3 = \max\{m(a,b), m(b,c)\} \leq m(a,c) = 4.$$

We then know that the T_0-quasi-metric d given by $d(a,b) = d(b,c) = d(a,c) = 0$, $d(b,a) = 2, d(c,b) = 3, d(c,a) = 4$ and $d(x,x) = 0$ whenever $x \in X$, produces $(X, m, \leq)$.

The second example elaborates on the fact that the interval condition is a necessary one.

Example 1.1.3. Let $X = \{a, b, c\}$ be equipped with the linear order $\leq$ such that $a < b < c$ and the metric m such that $m(a, b) = 2$, $m(b, c) = 2$, $m(a, c) = 1$ and $m(x, x) = 0$ whenever $x \in X$. Then any producing T_0-quasi-metric d on X would satisfy

$$2 = m(b, a) = d(b, a) \leq d(b, c) + d(c, a) = 0 + m(c, a) = 1$$

which is a contradiction. So $(X, m, \leq)$ is not produced by any T_0-quasi-metric on X. Observe also that since $m(a, b) = 2 > 1 = m(a, c)$, the interval condition is not satisfied and this can be inferred from the above contradiction.

In the next section, we define a notion of "dimension", namely the "pseudo-dimension", appropriate in the class of partially ordered metric spaces.

1.2 The pseudo-dimension of a partially ordered metric space

We give here a brief motivation for our next result. Most of the material is taken from [10] and [37, Ch. VII]

Definition 1.2.1. Let S be a nonempty set on which we define two partial orders $\preceq$ and a linear order $\leq$. If every ordered pair in $G(\preceq)$ occurs in $G(\leq)$, then $\leq$ will be called a **linear extension** of $\preceq$.

Definition 1.2.2. Let S be a nonempty set and $\mathcal{L}$ be any collection of linear orders, each defined on S. We define a partial order $\preceq$ on S as follows. For any $x, y \in S$, $x \preceq y$ if and only if $x\ L\ y$ for any $L \in \mathcal{L}$. The partial order $\preceq$ so obtained will be said to be **realized** by the linear orders of $\mathcal{L}$ and the elements of $\mathcal{L}$ are called the **realizers** of $\preceq$.

Some of the main results in this theory are:

Lemma 1.2.1. *Every partial order $\preceq$ possesses a linear extension $\leq$. Moreover, if a and b are any two non-comparable elements of $\preceq$, there exist an extension $\leq_1$ in which $a \leq_1 b$ and an extension $\leq_2$ in which $b \leq_2 a$.*

Theorem 1.2.1. *If $\preceq$ is a partial order on a set S, then there exists a collection of linear orders which realize $\preceq$.*

Definition 1.2.3. By **dimension** or **Dushnik-Miller dimension** of a partial order $\preceq$ defined on a nonempty set S, we mean the smallest cardinal number $\mathbf{m}$ such that $\preceq$ is realized by $\mathbf{m}$ linear orders on S. We denote it by $\mathbf{dim}(S, \leq)$.

We then see that any partially ordered metric space $(X, m, \leq)$ can be order-embedded into $(X, m, \preceq)$ where $\preceq$ is a linear extension of $\leq$. Hence, we shall refer to $(X, m, \preceq)$ as being a linear extension of $(X, m, \leq)$ in the sense that $\preceq$ is a linear extension of $\leq$.

We introduce a concept of dimension, appropriate for the class of partially ordered metric spaces. For this let us consider a partially ordered metric space $(X, m, \leq)$ such that $\mathbf{dim}(X, \leq) = k < \infty$. We denote by $\mathcal{R}_k(X, \leq)$ the collection of elements $\mathcal{A}$ such that

$$\mathcal{A} \in \mathcal{R}_k(X, \leq) \Longleftrightarrow \mathcal{A} \text{ is a set of } k \text{ realizers of } \leq .$$

Since $\mathbf{dim}(X, \leq) = k$, then $\mathcal{R}_k(X, \leq) \neq \emptyset$. So $\mathcal{A}$ is made of k linear extensions of $\leq$. We call the **index** $I_{\mathcal{A}}$ of $\mathcal{A}$ the cardinal of the set

$$\{i : \preceq_i \in \mathcal{A} \text{ and } (X, m, \preceq_i) \text{ is produced }\},$$

i.e.

$$I_{\mathcal{A}} = \#\{i : \preceq_i \ \in \mathcal{A} \text{ and } (X, m, \preceq_i) \text{ is produced}\}.$$

Definition 1.2.4. Let $(X, m, \leq)$ be a partially ordered metric space such that $\mathbf{dim}(X, \leq) = k < \infty$. By **pseudo-dimension** of $(X, m, \leq)$, and we denote $\mathbf{pdim}(X, m, \leq)$, we mean

$$\mathbf{pdim}(X, m, \leq) := \sup_{\mathcal{A} \in \mathcal{R}_k(X, \leq)} I_{\mathcal{A}}.$$

Remark 1.2.1. It follows from the definition that for any produced linearly ordered metric space $(X, m, \mathcal{R})$, the pseudo-dimension is 1 (the same as the dimension of $(X, \mathcal{R})$).

For a partially ordered metric space $(X, m, \leq)$, the Dushnik-Miller dimension $\mathbf{dim}(X, \leq)$ of $(X, \leq)$ provides an upper bound for $\mathbf{pdim}(X, m, \leq)$. Moreover, if $(X, \tau, \leq)$ is a preordered topological space, the idea of pseudo-dimension can be extended to $(X, \tau, \leq)$.

We provide an example showing the difference between the Dushnik-Miller dimension[2] of an ordered metric space $(X, m, \leq)$ and its pseudo-dimension.

Example 1.2.1. Let $X = \{a, b, c\}$ be equipped with the partial order $\leq$ such that $a \leq c, b \leq c$ and the metric m such that $m(a, b) = 3$, $m(b, c) = 2$, $m(a, c) = 5$, and $m(x, x) = 0$ whenever $x \in X$. Since $(X, \leq)$ is not a chain, it has dimension at least 2 and we are done if we can find a set of two realizers. The realizers are given as follows:

- the first linear $\leq_1$ order is such that

$$a \leq_1 b \leq_1 c,$$

and

- the second linear $\leq_2$ order is such that

$$b \leq_2 a \leq_2 c.$$

[2]Of course the the Dushnik-Miller dimension of $(X, m, \leq)$ refers to the Dushnik-Miller dimension of $(X, \leq)$, i.e. $\mathbf{dim}(X, m, \leq) = \mathbf{dim}(X, \leq)$.

We then conclude that $\mathbf{dim}(X, \leq) = 2$.

If we consider $(X, m, \leq_1)$: Since the interval condition

$$3 = \max\{m(a,b), m(b,c)\} \leq m(a,c) = 5,$$

is satisfied, then $(X, m, \leq_1)$ is produced by a T_0-quasi-metric.

Considering now $(X, m, \leq_2)$: Since

$$3 = \max\{m(b,a), m(a,c)\} > m(b,c) = 2,$$

the interval condition is not satisfied, then $(X, m, \leq_2)$ is not produced by any T_0-quasi-metric.

Hence $\mathbf{pdim}(X, m, \leq) = 1$. Of course, since $\{\leq_1, \leq_2\}$ is the only set of two realizers, its index is automatically the pseudo-dimension.

The next example we present now is less trivial, as we have more sets of realizers.

Example 1.2.2. Let $X = \{a, b, c\}$ be equipped with the partial order $\leq$ such that $a \leq b$ and the metric m such that $m(a,b) = 2$, $m(b,c) = 3$, $m(a,c) = 4$ and $m(x,x) = 0$ whenever $x \in X$. Since $(X, \leq)$ is not a chain, it has dimension at least 2 and we are done if we can find a set of two realizers. Since we aim at computing the pseudo-dimension of $(X, m, \leq)$, we shall give the complete list of sets of two realizers.

The realizers are given as follows:

- the first linear order is

$$a \leq_1 b \leq_1 c,$$

- the second linear order is

$$a \leq_2 c \leq_2 b,$$

and

- the third linear order is

$$c \leq_3 a \leq_3 b.$$

It is therefore easy to see that $\mathbf{dim}(X, \leq) = 2$. The sets of two realizers are

$$\mathcal{A}_{1,2} = \{\leq_1, \leq_2\}, \quad \mathcal{A}_{1,3} = \{\leq_1, \leq_3\}, \quad \mathcal{A}_{2,3} = \{\leq_2, \leq_3\},$$

and one can convince oneself that the respective indices are

$$I_{\mathcal{A}_{1,2}} = 1, \quad I_{\mathcal{A}_{1,3}} = 1, \quad I_{\mathcal{A}_{2,3}} = 0.$$

Hence $\mathbf{pdim}(X, m, \leq) = 1$.

The introduction of the pseudo-dimension aims at investigating a possible correlation between a partially ordered metric space $(X, m, \leq)$ and its linear extensions $(X, m, \preceq)$ (in the sense that $\preceq$ is a linear extension of $\leq$) in the "producing" problem. Namely, if $(X, m, \leq)$ (resp. $(X, m, \preceq)$) is produced, what can we say about $(X, m, \preceq)$ (resp. $(X, m, \leq)$)? In this direction, Lemma 1.0.1 gives a partial answer.

The following provides an example of a partially ordered metric space which is produced but admits a linear extension which is not produced. This motivates the theorem we give just after, illustrating the importance of the pseudo-dimension.

Example 1.2.3. (Compare Example 1.2.1) Let $X = \{a, b, c\}$ be equipped with the partial order $\leq$ such that $a \leq b, a \leq c$ and the metric m such that $m(a, b) = 2$, $m(b, c) = 2$, $m(a, c) = 3$ and $m(x, x) = 0$ whenever $x \in X$. The T_0-quasi-metric d given by $d(a, b) = 0 = d(a, c), d(b, a) = 2 = d(b, c) = d(c, b), d(c, a) = 3$ and $d(x, x) = 0$ whenever $x \in X$ produces $(X, m, \leq)$. The order $a \leq_1 c \leq_1 b$ is a linear extension of $\leq$ but since

$$3 = \max\{m(a, c), m(c, b)\} > m(a, b) = 2,$$

$(X, m, \leq_1)$ is not produced.

Theorem 1.2.2. *The partially ordered metric space* $(X, m, \leq)$ *is produced if* $\mathbf{pdim}(X, m, \leq) = \mathbf{dim}(X, \leq) < \infty$.

Proof. Since $\mathbf{pdim}(X, m, \leq) = \mathbf{dim}(X, \leq) = k < \infty$, there exist k linear orders $\leq_1, \cdots, \leq_k$ that realize $\leq$ and k T_0-quasi-metrics $d_1, \cdots, d_k$ such that the spaces $(X, m, \leq_i), i = 1, \cdots, k$ are produced respectively by $d_i, i = 1, \cdots, k$. The T_0-quasi-metric $D = \max\{d_i, i = 1, \cdots, k\}$ produces $(X, m, \leq)$. Indeed, whenever $x, y \in X$, we trivially have

$$D^s(x, y) = m(x, y)$$

and

$$\begin{aligned} D(x, y) = 0 &\Longleftrightarrow d_i(x, y) = 0 \text{ for any } i = 1, \cdots, k. \\ &\Longleftrightarrow x \leq_i y \text{ for any } i = 1, \cdots, k. \\ &\Longleftrightarrow x \leq y. \end{aligned}$$

□

In the next chapter, we shall prove a result that establishes a connection between a partially ordered metric space $(X, m, \leq)$ and its linear extensions $(X, m, \preceq)$. The results we obtained in Section 2.5 of the next chapter (e.g. Corollary 2.5.2) give a clear characterization of partially ordered metrics space which admit produced linear extensions.

Chapter 2

m-splitting theory

In this chapter, we present a general theory to tackle Problem -1.0.1. We are now interested in the case where the order on the metric space X is not necessarily linear. We give an adequate framework for our study, namely the *m-splitting* theory, and give the main results obtained. We also prove that these results actually generalize some results obtained in the previous chapter. The introduction of the idea of *m-splitting* quasi-pseudometrics also aims at extending the well-known theorem by Szpilrajn. In this line, it is important to note that the specialization order of a minimally m-splitting T_0-quasi-metric need not be linear (see Example 2.6.1). In the subcollection of all two-valued T_0-quasi-metrics on a set (which can be identified with the set of partial orders) the existence of a minimally m-splitting T_0-quasi-metric below a given T_0-quasi-metric is indeed equivalent to the well-known result due to Szpilrajn [5, 38] which states that each partial order can be extended to a linear order.

2.1 Preliminary definitions and results

Definition 2.1.1. Let (X, m) be a metric space.

i) We shall say that a quasi-pseudometric d on X is m-**splitting** provided that $d^s = m$.

ii) If (X, m) is equipped with a partial order $\leq$, then we shall say that the partially ordered metric space $(X, m, \leq)$ is **produced** by a quasi-pseudometric d on X provided that

    1) d is m-splitting and

    2) $\leq$ is the specialization order of d.

iii) A partial order $\leq$ on (X, m) is called m-**produced** by a quasi-pseudometric d on X if d produces $(X, m, \leq)$[1].

Example 2.1.1. Each T_0-quasi-metric d on a set X produces the ordered metric space $(X, d^s, \leq_d)$.

Remark 2.1.1. From the definition, we see that if d is an m-splitting quasi-pseudometric then it is a T_0-quasi-metric and whenever d is an m-splitting quasi-pseudometric, so is its conjugate d^{-1}. These facts look trivial but they cannot be omitted.

As we mentioned in the introduction, we shall be interested in the existence of minimal elements in the class of m-splitting quasi-pseudometrics. We shall also look at the existence of maximal specialization orders that can be obtained from the collection of m-splitting T_0-quasi-metrics [2]. Hence, we introduce the following definitions:

[1]Hence in particular d is m-splitting.

[2]Observe that the terminology of m-splitting T_0-quasi-metrics is superfluous. Indeed m-splitting quasi-pseudometrics are T_0-quasi-metrics.

Definition 2.1.2. Let (X, m) be a metric space.

i) A quasi-pseudometric d on X with $d^s = m$ is called **minimally m-splitting** if whenever q is a quasi-pseudometric on X that is m-splitting and $q \leq d$, then $q = d$.

ii) An m-produced partial order $\leq$ is called **maximally m-produced** if there is no partial order $\preceq$ on X such that $\leq \subset \preceq$ (note the strict inclusion!) and $(X, m, \preceq)$ is produced by a quasi-pseudometric on X.

Remark 2.1.2. Note that the conjugate d^{-1} of a minimally m-splitting T_0-quasi-metric d on X is minimally m-splitting too. Observe also that the dual order of a partial order that is m-produced by a T_0-quasi-metric d is m-produced by the conjugate T_0-quasi-metric d^{-1}. Hence the dual order of a maximally m-produced partial order is maximally m-produced too.

Next we illustrate the latter definition with the simple example that we have already discussed in the introduction.

Example 2.1.2. (Compare Example -1.0.1) Let the set $\mathbb{R}$ of the reals be equipped with the usual metric $m(x, y) = |x - y|$ whenever $x, y \in \mathbb{R}$ and the usual linear order $\leq$. Then $(\mathbb{R}, m, \leq)$ is produced by the standard T_0-quasi-metric u defined by $u(x, y) = 0$ whenever $x \leq y$ and $u(x, y) = \max\{x - y, 0\}$ otherwise. The T_0-quasi-metric u is minimally m-splitting and the partial order $\leq$ is maximally m-produced.

2.2 Connection with Szpilrajn's theorem

The connection with Szpilrajn's theorem is obtained via the two-valued T_0-quasi-metrics which we presented in Proposition 0.2.1 and for which we also provided Example 0.2.1.

Observe there that given two partial orders $\leq_1$ and $\leq_2$ on X we have that

$$\leq_1 \subseteq \leq_2 \quad \text{if and only if} \quad d_{\leq_1} \geq d_{\leq_2}.$$

Hence Szpilrajn's Theorem can obviously be reformulated in the following form:

In the collection of all T_0-quasi-metrics on a set X mapping into $\{0,1\}$ below each ($d_=$-splitting) T_0-quasi-metric d there is a minimally $d_=$-splitting T_0-quasi-metric. Such a minimally $d_=$-splitting T_0-quasi-metric is characterized by the property that it has a linear specialization order.

In the coming sections, we present the main results we have obtained so far in the m-splitting Theory.

2.3 Existence of minimally m-splitting T_0-quasi-metrics

In this section we shall now consider arbitrary T_0-quasi-metrics on a nonempty set X.

Definition 2.3.1. Let X be a nonempty set. We denote by $\mathcal{QP}(X)$ the set of all quasi-pseudometrics on X . If m is a metric on X, we shall denote by $[\mathcal{QP}(X)]_m$, the set of all m-splitting quasi-pseudometrics on (X,m) i.e.

$$[\mathcal{QP}(X)]_m = \{d \in \mathcal{QP}(X) : d^s = m\}.$$

We put on $\mathcal{QP}(X)$ the natural order $\leq_{[\mathcal{QP}(X)]}$, i.e. for $d, d' \in \mathcal{QP}(X)$

$$d \leq_{[\mathcal{QP}(X)]} d' \Longleftrightarrow d \leq d' \Longleftrightarrow d(x,y) \leq d'(x,y) \text{ whenever } x, y \in X.$$

The collection $\mathcal{QP}(X)$ along with this order shall be denoted by $\Big(\mathcal{QP}(X), \leq_{[\mathcal{QP}(X)]}\Big)$.

We can then state our first lemma for which the proof is straightforward.

Lemma 2.3.1. *Let $(X, m, \leq)$ be a partially ordered metric space and let d_1, d_2, d_3 be T_0-quasi-metrics on X such that $d_1 \leq d_2 \leq d_3$. If d_1 and d_3 produce $(X, m, \leq)$, then d_2 also produces $(X, m, \leq)$.*

Proof. On the set of T_0-quasi-metrics on X, taking the supremum metric is a monotone operation while taking the specialization order is an anti-tone operation. Since $\leq_{d_3}=\leq_{d_1}=\leq$ and $d_1^s = d_3^s = m$, the assertion follows. More precisely,

$$d_1 \leq d_2 \leq d_3 \Longrightarrow G(\leq) = G(\leq_{d_3}) \subseteq G(\leq_{d_2}) \subseteq G(\leq_{d_1}) = G(\leq),$$

and

$$m = d_1^s \leq d_2^s \leq d_3^s = m.$$

□

The second result, less trivial, goes as follows:

Proposition 2.3.1. *Let (X, m) be a metric space and q be an m-splitting T_0-quasi-metric on X. Then there is a minimally m-splitting T_0-quasi-metric s on X such that $s \leq q$.*

Proof. Let $(d_i)_{i \in I}$ be a nonempty chain of quasi-pseudometrics below q on a metric space (X, m) such that $d_i^s = m$ whenever $i \in I$.

It is obvious that $\bigwedge\limits_{i \in I} d_i = \inf\limits_{i \in I} d_i$ is well defined since $(d_i)_{i \in I}$ is bounded from below.

1. We first prove that $\bigwedge_{i\in I} d_i$ is a quasi-pseudometric on X.

   - Since $d_i(x,x)=0$ for any $x\in X$, then $\left[\bigwedge_{i\in I} d_i\right](x,x)=0$ for any $x\in X$.
   - For any $x,y,z\in X$ and any $\varepsilon>0$, there exist $j,k\in I$ such that

$$d_j(x,z)\le\left[\bigwedge_{i\in I} d_i\right](x,z)+\frac{\varepsilon}{2}$$

   and

$$d_k(z,y)\le\left[\bigwedge_{i\in I} d_i\right](z,y)+\frac{\varepsilon}{2}$$

   Since $(d_i)_{i\in I}$ is a chain, we can assume that $d_j\le d_k$.

   Hence

$$\left[\bigwedge_{i\in I} d_i\right](x,y)\le d_j(x,y)\le d_j(x,z)+d_j(z,y)\le d_j(x,z)+d_k(z,y)$$
$$\le\left[\bigwedge_{i\in I} d_i\right](x,z)+\left[\bigwedge_{i\in I} d_i\right](z,y)+\varepsilon.$$

   Hence, for any $x,y,z\in X$ and for any $\varepsilon>0$

$$\left[\bigwedge_{i\in I} d_i\right](x,y)\le\left[\bigwedge_{i\in I} d_i\right](x,z)+\left[\bigwedge_{i\in I} d_i\right](z,y)+\varepsilon,$$

   therefore for any $x,y,z\in X$

$$\left[\bigwedge_{i\in I} d_i\right](x,y)\le\left[\bigwedge_{i\in I} d_i\right](x,z)+\left[\bigwedge_{i\in I} d_i\right](z,y).$$

   <u>Conclusion:</u> $\bigwedge_{i\in I} d_i$ is a quasi-pseudometric, i.e. $\bigwedge_{i\in I} d_i\in\mathcal{QP}(X)$.

2. We prove that $\bigwedge_{i\in I} d_i$ is m-splitting, i.e $\left[\bigwedge_{i\in I} d_i\right]^s = m$.

   For any $x, y \in X$, by definition

$$\left[\bigwedge_{i\in I} d_i\right]^s (x,y) := \left[\bigwedge_{i\in I} d_i\right](x,y) \bigvee \left[\bigwedge_{i\in I} d_i\right](y,x).$$

   For any $x, y \in X$, it is clear on the one hand that

$$\left[\bigwedge_{i\in I} d_i\right]^s (x,y) = \left[\bigwedge_{i\in I} d_i\right](x,y) \bigvee \left[\bigwedge_{i\in I} d_i\right](y,x) \leq m(x,y),$$

   since

$$\left[\bigwedge_{i\in I} d_i\right](x,y) \leq d_i(x,y) \leq m(x,y) \text{ whenever } i \in I.$$

   On the other hand, for any $\varepsilon > 0$, there exist $k, l \in I$ such that

$$d_k(x,y) \leq \left[\bigwedge_{i\in I} d_i\right](x,y) + \frac{\varepsilon}{2}$$

   and

$$d_l(y,x) \leq \left[\bigwedge_{i\in I} d_i\right](y,x) + \frac{\varepsilon}{2}.$$

   Since $(d_i)_{i\in I}$ is a chain, we can assume that $d_l \leq d_k$, hence we have

$$d_l(x,y) \leq \left[\bigwedge_{i\in I} d_i\right](x,y) + \frac{\varepsilon}{2}$$

   and

$$d_l(y,x) \leq \left[\bigwedge_{i\in I} d_i\right](y,x) + \frac{\varepsilon}{2}$$

   which entail that, for any $\varepsilon > 0$

$$m(x,y) = d_l^s(x,y) \leq \left[\bigwedge_{i\in I} d_i\right](x,y) \bigvee \left[\bigwedge_{i\in I} d_i\right](y,x) + \varepsilon.$$

Hence for any $x, y \in X$,

$$m(x,y) \leq \left[\bigwedge_{i\in I} d_i\right](x,y) \bigvee \left[\bigwedge_{i\in I} d_i\right](y,x) = \left[\bigwedge_{i\in I} d_i\right]^s (x,y).$$

<u>Conclusion:</u>

$$\left[\bigwedge_{i\in I} d_i\right]^s = m.$$

Hence $\bigwedge_{i\in I} d_i = m$ is m-splitting, that is $\bigwedge_{i\in I} d_i \in [\mathcal{QP}(X)]_m$.

So the chain $(d_i)_{i\in I}$ of m-splitting quasi-metrics has a lower bound $\bigwedge_{i\in I} d_i$ which is m-splitting. By Zorn's Lemma there is a minimal T_0-quasi-metric t on (X, m) such that $t \leq q$ and $t^s = m$, i.e. there is a minimally m-splitting T_0-quasi-metric on (X, m). □

The above result is no longer true if instead of "m-splitting" we consider "producing". In other words, the statement that below each T_0-quasi-metric q producing $(X, m, \leq)$ there is a minimal T_0-quasi-metric s on X producing $(X, m, \leq)$ does not hold. In particular we observe that the infimum of a chain of T_0-quasi-metrics $(d_i)_{i\in I}$ producing a partially ordered metric space $(X, m, \leq)$ need not produce $(X, m, \leq)$. The problem is that although $\bigwedge_{i\in I} d_i$ is an m-splitting T_0-quasi-metric on X (according to Proposition 2.3.1) its specialization order is possibly larger than $\leq$. We illustrate our argument with the following example.

Example 2.3.1. The T_0-quasi-metric on $X = \{0, 1, 2\}$ given by

$$\mathbf{D} = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 2 \\ 3 & 2 & 0 \end{pmatrix},$$

produces the ordered metric space $(X, m, \leq)$ where the metric m is defined by the matrix

$$\mathbf{M} = \begin{pmatrix} 0 & 2 & 3 \\ 2 & 0 & 2 \\ 3 & 2 & 0 \end{pmatrix}, \text{that is, } m(i,j) = m_{i,j},$$

and $\leq = \{(1,2),(1,3)\} \cup \Delta_X$ on X where Δ_X denotes the diagonal of X.

The family of T_0-quasi-metrics given by the matrices

$$\mathbf{D}_\varepsilon = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & \varepsilon \\ 3 & 2 & 0 \end{pmatrix},$$

with $0 < \varepsilon < 2$ also produce $(X, m, \leq)$, while the T_0-quasi-metric on X given by

$$\mathbf{D}' = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 3 & 2 & 0 \end{pmatrix} = \inf_{\varepsilon > 0} \mathbf{D}_\varepsilon,$$

produces a topological ordered space with a larger specialization order.

Another example, on a 4-point set is given in Example 2.6.2.

2.4 Existence of maximally m-produced order

The next question of interest is the following. Given a metric space (X, m), are there maximal orders $\preceq$ on X for which $(X, m, \preceq)$ is produced? The following facts are trivial.

Lemma 2.4.1. *Let $(\leq_i)_{i \in I}$ be a chain of partial orders on a nonempty set X. Then $\mathcal{R} := \bigcap_{i \in I} \leq_i$ is a partial order and a lower bound for the chain $(\leq_i)_{i \in I}$.*

Lemma 2.4.2. *Let $(d_i)_{i\in I}$ be a chain of quasi-pseudometrics on a nonempty set X. Then*

$$d_k \ \leq \ d_j \Longrightarrow \leq_j \subseteq \leq_k \ ,$$

where $\leq_i := \leq_{d_i}$ whenever $i \in I$.

Proof. Let $x, y \in X$ such that $x \leq_j y$. Then $d_j(x, y) = 0$ and this implies that $d_k(x, y) = 0$, since $d_k \leq d_j$, hence $x \leq_k y$. □

In general the reverse implication in Lemma 2.4.2 does not hold but the result of Theorem 2.4.1 guarantees an equivalence. In the coming example, we prove that the reverse implication does not hold in general.

Example 2.4.1. Let $X = \{a, b, c\}$ and define on X the following two quasi-pseudometrics.

$$\mathbf{d_1} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix} \qquad \mathbf{d_2} = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 2 \\ 1 & 0 & 0 \end{pmatrix}.$$

It is clear that $G(\leq_{d_1}) \subseteq G(\leq_{d_2})$ but d_1 and d_2 are not even comparable.

The following, although quite trivial, needs to be stated but the proof will be omitted.

Corollary 2.4.1. *Let $(X, m, \leq)$ be a partially ordered metric space. If $(d_i)_{i\in I}$ where I is an index set, is a chain of m-producing quasi-pseudometrics, then*

$$\leq \ \subseteq \ \leq_{[\bigwedge\limits_{i\in I}(d_i)]} \ .$$

Let $(X, m, \leq)$ be a partially ordered metric space. Assume that $(X, m, \leq)$ is produced and denote by $\mathcal{P}([X, m, \leq])$ the collection of all quasi-pseudometrics that produces $(X, m, \leq)$.

Now we set $r := \min\{d_{\leq}, m\}$[3] (see Corollary 1.1.1). We know that in general r is not a quasi-pseudometric, since it does not satisfy the triangle inequality and to correct this deficiency, we introduce the idea of "path".

Definition 2.4.1. Let x, y be two arbitrary elements of a partially ordered metric space $(X, m, \leq)$. A **path** from x to y is a finite sequence of points of X starting at x and ending at y, it will be denoted by P_{xy}. Hence P_{xy} can be written as $P_{xy} = (x_0, \cdots, x_n)$ where $x_0 := x$ and $x_n := y$.

For any $x, y \in X$, we denote by $\mathcal{P}([x, y])$ the collection of all paths starting at x and ending at y. For a given path $P_{xy} \in \mathcal{P}([x, y])$, we define the **length** $l(P_{xy})$ of P_{xy} by

$$l(P_{xy}) = \sum_{i=0}^{n-1} r(x_i, x_{i+1}).$$

We now introduce the map $D_{(m,\leq)}$[4] that we shall sometimes denote $D_{\leq}$ (when there is no confusion).

Definition 2.4.2. Let $(X, m, \leq)$ be a partially ordered metric space. For any $(x, y) \in X \times X$, we set

$$D_{(m,\leq)}(x, y) = \inf\{l(P_{xy}), P_{xy} \in \mathcal{P}([x, y])\}.$$

Remark 2.4.1. It is well known and obvious that $D_{\leq}$ is a *quasi-pseudometric* on X. Furthermore we see (by applying the triangle inequality of q) that if q is

[3] The map $d_{\leq}$ refers to the one in Definition 0.2.7. If here we consider instead a partially ordered T_0-quasi-metric space $(X, d, \leq)$ then r will be given by $r := \min\{d_{\leq}, d\}$.

[4] Hence for an arbitrary T_0-quasi-metric space $(X, d, \leq)$, we shall write $D_{(d,\leq)}$ to avoid confusion.

a quasi-pseudometric on X such that $q \leq m$ and $q \leq d_{\leq}$, then $q \leq D_{\leq}$. Thus we can write $D_{\leq} = m \wedge d_{\leq}$ since $D_{\leq}$ is the largest quasi-pseudometric $\leq m$ on X with specialization preorder extending $\leq$. In general $D_{\leq}$ will not produce $(X, m, \leq)$ (compare for instance Example 1.1.3). However in the case that there exists a T_0-quasi-metric d that produces the space $(X, m, \leq)$, then $D_{\leq}$ produces $(X, m, \leq)$ and $D_{\leq}$ is the largest quasi-pseudometric on X with this property as we prove in the next lemma.

Lemma 2.4.3. *Let $(X, m, \leq)$ be a partially ordered metric space and let $D_{\leq}$ be as constructed above. If $\mathcal{P}([X, m, \leq]) \neq \emptyset$, then $D_{\leq} \in \mathcal{P}([X, m, \leq])$ and it is the largest quasi-pseudometric below m having this property.*

Proof. Suppose $\mathcal{P}([X, m, \leq]) \neq \emptyset$. We claim that for any $d \in \mathcal{P}([X, m, \leq])$,

$$d(x, y) \leq D_{\leq}(x, y) \text{ for any } x, y \in X.$$

Indeed, let $d \in \mathcal{P}([X, m, \leq])$

- if $x \leq y$, then $d(x, y) = 0 \leq r(x, y) = 0$,
- if $x \not\leq y$, then $d(x, y) \leq r(x, y) = m(x, y)$,

hence $d \leq r$ and therefore $d \leq D_{\leq}$.

Next we shall prove that $D_{\leq} \in \mathcal{P}([X, m, \leq])$, i.e. that $D_{\leq}$ produces $(X, m, \leq)$.

Indeed, for any $d \in \mathcal{P}([X, m, \leq])$, since $d \leq D_{\leq}$ therefore $m = d^s \leq [D_{\leq}]^s \leq m$, i.e. $D_{\leq}$ is m-splitting.

Again, since $d \leq D_{\leq}$ for any $d \in \mathcal{P}([X, m, \leq])$, we have that

$$\left[\leq_{(D_{\leq})}\right] \subseteq [\leq] = [\leq_d] \, .$$

We also know by construction that $[\leq] = [\leq_d] \subseteq \left[\leq_{(D_{\leq})}\right]$, since for any $x, y \in X$,

$$x \leq_d y \Longleftrightarrow d(x,y) = 0 \Longrightarrow D_{\leq}(x,y) = 0 \Longleftrightarrow x \leq_{D_{\leq}} y.$$

Hence $\left[\leq_{(D_{\leq})}\right] = [\leq] = [\leq_d]$ and $D_{\leq} \in \mathcal{P}([X, m, \leq])$. □

Corollary 2.4.2. *Let $(X, m, \leq)$ be a partially ordered metric space. Then it is produced by a T_0-quasi-metric if and only if $m \leq [D_{\leq}]^s$ and $\left[\leq_{D_{\leq}}\right] \subseteq [\leq]$.*

Remark 2.4.2. Note that the second part of this condition is superfluous if $\leq$ is a linear order on X, since $x, y \in X$ with $x \not\leq y$ then implies that $y \leq x$ and thus $D_{\leq}(y,x) = 0$, and $D_{\leq}(x,y) > 0$ by the first part of the condition. The second part of the condition is also superfluous in the case that X is finite. We shall elaborate more on this observation in the proof of the Lemma 2.5.1.

Now comes the main result of this section.

Theorem 2.4.1. *Let $(X, m, \leq)$ be a partially ordered metric space that is produced by a T_0-quasi-metric on X. Then there is a maximally m-produced partial order $\preceq$ on X such that $\leq \subseteq \preceq$.*

Proof. In the proof of the theorem, we shall make use of the important so-called **zero's argument** which description is given below.

The zero's argument:

For an arbitrary partially ordered metric space $(X, m, \leq)$, the expression of $D_{\leq}$ can be rewritten. Indeed for any $x, y \in X$, and a given $P_{xy} \in \mathcal{P}([x, y])$, if in the path, there are two consecutive terms x_i and x_{i+1} such that $x_i \leq x_{i+1}$, then $r(x_i, x_{i+1}) = 0$ and we can reduce $l(P_{xy})$ to an expression containing only the non-zero terms.

We now start the proof of the theorem. Let

$$\mathcal{M} = \{\preceq : (X, \preceq) \text{ is a poset, } (X, m, \preceq) \text{ is produced and } \leq \subseteq \preceq\}$$

be ordered by the set-theoretic inclusion. We need to show that $\mathcal{M}$ has maximal elements. Let $\mathcal{K} = \{\leq_i : i \in I\} \subseteq \mathcal{M}$ be a nonempty chain in $\mathcal{M}$. Then

$$\mathcal{U} = \bigcup \{\leq : \leq \in \mathcal{K}\}$$

is a partial order on X, since $\mathcal{K}$ is a chain.

For each $(X, m, \leq_i), \leq_i \in \mathcal{K}$ consider the largest producing T_0-quasi-metric

$$D_i := D_{(m, \leq_i)} \text{ (see Lemma 2.4.3)} .$$

Claim 1:

$$\leq_k \subseteq \leq_l \iff D_l \leq D_k \text{ whenever } k, l \in I.$$

In view of the Lemma 2.4.2, it is enough to prove that

$$\leq_k \subseteq \leq_l \Longrightarrow D_l \leq D_k \text{ whenever } k, l \in I.$$

By the zero's argument,

$$D_l(x,y) = \inf_{\mathcal{P}([x,y])} \sum_{x_i \not\leq_l x_{i+1}} r(x_i, x_{i+1}) \quad \text{whenever } x, y \in X.$$

Since $\leq_k \subseteq \leq_l$, (we can always assume that the inclusion is strict), there exist $a, b \in X$ such that $a \leq_l b$ but $a \not\leq_k b$. Hence the term $r(a,b) \neq 0$ for D_k becomes $r(a,b) = 0$ for D_l, and doing so, reducing the value of the infimum. Therefore $D_l \leq D_k$.

Claim 2: $D_{\mathcal{U}}$ produces $\left(X, m, \leq_{[\bigwedge_{i\in I} D_i]}\right)$.

According to Claim 1, $(D_i)_{i\in I}$ is a chain of m-splitting quasi-pseudometrics, hence $\bigwedge_{i\in I} D_i$ is an m-splitting quasi-pseudometric (see Proposition 2.3.1). Therefore $\left(X, m, \leq_{[\bigwedge_{i\in I} D_i]}\right)$ is produced by $\mathscr{D}$ where $\mathscr{D} := \bigwedge_{i\in I} D_i$. Furthermore, if $x, y \in X$ are such that $x \,\mathcal{U}\, y$, then there exists $i \in I$ such that $x \leq_i y$ which entails that $D_i(x,y) = 0$ and hence $\left[\bigwedge_{i\in I} D_i\right](x,y) = 0$ which in return gives that

$$\mathcal{U} \subseteq \leq_{\mathscr{D}}.$$

Hence we have found an upper bound $\leq_{[\bigwedge_{i\in I} D_i]}$ of the chain $\mathcal{K}$ in $\mathcal{M}$. Then by Zorn's Lemma we conclude that there exists a maximal element in $\mathcal{M}$.

Moreover since $D_{\leq_{\mathscr{D}}}$ is the largest (finest) quasi-pseudometric producing $\left(X, m, \leq_{[\bigwedge_{i\in I} D_i]}\right)$, then $\mathscr{D} \leq D_{\leq_{\mathscr{D}}}$.

We also know by the zero's argument that $D_{\leq_{\mathscr{D}}} \leq D_{\mathcal{U}}$ (since $\mathcal{U} \subseteq \leq_{\mathscr{D}}$).

Again, from Claim 1, $D_{\mathcal{U}} \leq D_i$ for any $i \in I$ and hence $D_{\mathcal{U}} \leq \mathscr{D}$.

Therefore

$$D_{\mathcal{U}} \leq \mathscr{D} \leq D_{\leq_{\mathscr{D}}} \leq D_{\mathcal{U}}.$$

In conclusion, $D_{\mathcal{U}} = D_{\leq_{\mathscr{D}}}$ and $D_{\mathcal{U}}$ produces $\left(X, m, \leq_{[\bigwedge_{i\in I} D_i]}\right)$. ☐

We end this section by showing an interesting connection between minimally m-splitting T_0-quasi-metrics and maximally m-produced partial orders.

Proposition 2.4.1. *Let (X, m) be a metric space. Each partial order $\leq$ on X that is maximally m-produced by a T_0-quasi-metric is the specialization order of a minimally m-splitting T_0-quasi-metric n on X.*

Proof. Suppose that e is a T_0-quasi-metric on X that produces $(X, m, \leq)$ and that $\leq$ is maximally m-produced. By Proposition 2.3.1 there exists a minimally m-splitting T_0-quasi-metric n on X such that $n \leq e$. Then $\leq_e = \leq \subseteq \leq_n$ and thus $\leq_n = \leq_e$, since $\leq$ is maximally m-produced. Hence the assertion is satisfied. ☐

2.5 Produced linear extensions of $(X, m, \leq)$: case of finite sets

The ultimate attempt of this section is to transfer the resolution of the quasi-pseudometrization problem of a partially ordered metric space $(X, m, \leq)$ to that of its linear extensions $(X, m \preceq)$. We present some more general results though.

We begin with the following interesting lemma.

Lemma 2.5.1. *Let $(X, m, \leq)$ be a finite ordered metric space. Then the specialization order of $D_{(m,\leq)}$ is $\leq$.*

Proof. The order $\leq$ is obviously contained in the specialization order of $D_{(m,\leq)}$. Assume now that for $x, y \in X$ with $x \neq y$, we have $D_{(m,\leq)}(x, y) = 0$. For an efficient computation of $D_{(m,\leq)}$, note that it is surely enough to consider paths from x to y that do not repeat any points. Therefore, since X is finite, we will indeed only have a finite number of paths to take into account. It follows that there exists a path from x to y such that $\sum_{i=0}^{k} r(x_i, x_{i+1}) = 0$ where for any $i, r(x_i, x_{i+1}) = m(x_i, x_{i+1}) = 0$ or $x_i \leq x_{i+1}$. But this would imply by transitivity that $x \leq y$. □

Corollary 2.5.1. *Let $(X, m, \leq)$ be a finite ordered metric space and $\leq_1, \leq_2$ be two partial orders on X such that $\leq_1 \subseteq \leq_2$. Then $(X, m, \leq_1)$ is produced if $(X, m, \leq_2)$ is produced.*

Proof. It is enough to show that the T_0-quasi-metric $D_{(m,\leq_1)}$ on X is m-splitting. From Theorem 2.4.1, it is clear that

$$D_{(m,\leq_2)} \leq D_{(m,\leq_1)} \text{ and } m \leq D^s_{(m,\leq_2)} \leq D^s_{(m,\leq_1)} \leq m.$$

Thus $D^s_{(m,\leq_1)} = m$. By Lemma 2.5.1, the order $\leq_1$ and the specialization order of $D_{(m,\leq_1)}$ are equal. □

The preceding result does no longer hold for infinite metric spaces, as the following example demonstrates.

Example 2.5.1. (Compare Example 1.2.3) Let $X = \left\{\frac{1}{n} : n \in \mathbb{Z} \setminus \{0\}\right\}$ be equipped with the usual metric on $\mathbb{R}$. We define on X the partial order $\leq$ by

$$G(\leq) = \Delta_X \cup \left\{\left(-\frac{1}{n+1}, 1\right) : n \in \mathbb{N}\right\} \cup \left\{\left(-1, \frac{1}{n+1}\right) : n \in \mathbb{N}\right\}.$$

Note that

$$\begin{aligned} D_{(m,\leq)}(-1,1) &\leq D_{(m,\leq)}\left(-1,\frac{1}{n+1}\right)+D_{(m,\leq)}\left(\frac{1}{n+1},-\frac{1}{n+1}\right)+D_{(m,\leq)}\left(-\frac{1}{n+1},1\right)\\ &\leq 0+\frac{2}{n+1}+0=\frac{2}{n+1}, \end{aligned}$$

whenever $n \in \mathbb{N}$. So $D_{(m,\leq)}(-1,1)=0$ but we do not have $-1 \leq 1$. Therefore $(X,m,\leq)$ is not produced by a T_0-quasi-metric on X.

On the other hand, $(\mathbb{R},m,\preceq)$ is produced, where $\preceq$ is the usual linear order on $\mathbb{R}$ which extends $\leq$ since $(\mathbb{R},m,\preceq)$ is a subspace of the produced space $(\mathbb{R},m,\preceq)$ (see Example -1.0.1 in the introduction).

The following corollary is immediate from Corollary 2.5.1.

Corollary 2.5.2. *Let $(X,m,\leq)$ be a finite ordered metric space and $\preceq$ be a linear extension of $\leq$. Then $(X,m,\leq)$ is produced if $(X,m,\preceq)$ is produced.*

It is worth mentioning for the above Corollary 2.5.2, the converse does not hold, i.e. a produced finite ordered metric space $(X,m,\leq)$ can admit a linear extension $(X,m,\preceq)$ which is not produced. This is illustrated in Example 2.6.1.

We conclude this section with a more general result, similar to Lemma 1.0.2.

Lemma 2.5.2. *Let I be a nonempty set. Suppose that for each $i \in I$, the ordered metric space $(X,m_i,\leq_i)$ is produced by a T_0-quasi-metric d_i. Moreover, assume that the family $(d_i)_{i\in I}$ is uniformly bounded. Then the ordered metric space*

$$\left(X,\bigvee_{i\in I} m_i=\sup_{i\in I}\, m_i,\bigcap_{i\in I}\leq_i\right)$$

is produced by the T_0-quasi-metric $\bigvee_{i\in I} D_{(m_i,\leq_i)}$.

Although the above lemma is important, we shall not provide any proof to it, but instead we shall prove the following corollary, as it fits more in the scope of our problem.

Corollary 2.5.3. *Let I be a nonempty set. Suppose that for each $i \in I$, the ordered metric space $(X, m, \leq_i)$ is produced. Then the ordered metric space $(X, m, \bigcap_{i\in I} \leq_i)$ is produced by the T_0-quasi-metric $\bigvee_{i\in I} D_{(m,\leq_i)}$.*

Proof. For each $i \in I$ we have that $D_{(m,\leq_i)}$ produces $(X, m, \leq_i)$, in particular $D^s_{(m,\leq_i)} = m$ for each $i \in I$, so $\left(\bigvee_{i\in I} D_{(m,\leq_i)}\right)^s = m$. Moreover, note that $\left(\bigvee_{i\in I} D_{(m,\leq_i)}\right)(x, y) = 0$ if and only if $D_{(m,\leq_i)}(x, y) = 0$ whenever $i \in I$. Furthermore, since $(X, m, \leq_i)$ is produced by $D_{(m,\leq_i)}$, we have $\leq_{D_{(m,\leq_i)}} = \leq_i$ whenever $i \in I$. Therefore the specialization order of $\bigvee_{i\in I} D_{(m,\leq_i)}$ is $\bigcap_{i\in I} \leq_i$ and we are done.

Note that it then also follows that $D_{\left(m, \bigcap_{i\in I} \leq_i\right)}$ produces $\left(X, m, \bigcap_{i\in I} \leq_i\right)$. □

Example 2.5.2. Let I be a nonempty set. We equip $[0, 1]^I$ with the metric $m(f, g) = \sup_{x\in I} |f(x) - g(x)|$ whenever $f, g \in [0, 1]^I$ and the partial order $f \preceq g$ if $f(x) \leq g(x)$ whenever $x \in I$. Then $([0, 1]^I, m, \preceq)$ is produced by the T_0-quasi-metric $d(f, g) = \sup_{x\in I} u(f(x), g(x))$ whenever $f, g \in [0, 1]^I$, [5] where u refers to the map defined in Examples -1.0.1 and 0.1.4.

The following section presents simple examples illustrating some of the results discussed in the previous sections.

[5] Chapter 6 will show the importance of a variant of this example in the theory of produced T_0-quasi-metrics.

2.6 Concrete examples

We shall give two examples, one on a three-point set and another one on a four-points set. These examples will allow us to see in practice how the m-splitting theory works.

Example 2.6.1. (Cf. Example 2.3.1) We are given the following metric m on the three points set $X = \{a, b, c\}$.

$$\mathbf{m} = \begin{pmatrix} 0 & 2 & 3 \\ 2 & 0 & 2 \\ 3 & 2 & 0 \end{pmatrix}$$

We define on (X, m) the partial order $\leq$ by $G(\leq) = \{(a, b), (a, c)\} \cup \Delta_X$ where Δ_X denotes the diagonal of X. The quasi-pseudometric d given by

$$\mathbf{d} = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 2 \\ 3 & 2 & 0 \end{pmatrix}$$

produces is the largest T_0-quasi-metric producing $(X, m, \leq)$ while the quasi-pseudometric d' given by

$$\mathbf{d}' = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 2 \\ 3 & 1 & 0 \end{pmatrix}$$

is minimally m-splitting. Indeed, for the quasi-pseudometric d' to be m-splitting, the nonzero terms of the first and third columns cannot be modified and modifying the entry 1 at the bottom row (by any smaller value) implies the loss of the triangle inequality.

$$\begin{pmatrix} 0 & 0 & 0 \\ \boxed{2} & 0 & \boxed{2} \\ \boxed{3} & \boxed{1} & 0 \end{pmatrix}$$

We shall prove that $d = D_{\leq}$. We then compute $r = \min\{d_{\leq}, m\}$ and we have

$$r(a,b) = \min\{d_{\leq}(a,b), m(a,b)\} = 0; \qquad r(a,c) = \min\{d_{\leq}(a,c), m(a,c)\} = 0,$$
$$r(b,a) = \min\{d_{\leq}(b,a), m(b,a)\} = 2; \qquad r(c,a) = \min\{d_{\leq}(c,a), m(c,a)\} = 3,$$
$$r(b,c) = \min\{d_{\leq}(b,c), m(b,c)\} = 2; \qquad r(c,b) = \min\{d_{\leq}(c,b), m(c,b)\} = 2.$$

Hence

$$\mathbf{r} = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 2 \\ 3 & 2 & 0 \end{pmatrix}.$$

The following is devoted to the computation of $D_{\leq}$. We summarize in an array all the possible paths and compute their lengths.

Startpoint x	Endpoint y	Paths	Lengths	$D_{\leq}(x,y)$
a	b	$a \to b$	0	
a	b	$a \to c \to b$	2	0
a	c	$a \to c$	0	
a	c	$a \to b \to c$	2	0
b	a	$b \to a$	2	
b	a	$b \to c \to a$	5	2
b	c	$b \to c$	2	
b	c	$b \to a \to c$	2	2
c	a	$c \to a$	3	
c	a	$c \to b \to a$	4	3
c	b	$c \to b$	2	
c	b	$c \to a \to b$	3	2

Hence

$$\mathbf{D}_{\leq} = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 2 \\ 3 & 2 & 0 \end{pmatrix} = \mathbf{d}$$

is the largest T_0-quasi-metric producing on X $(X, m, \leq)$.

Now we enlarge the order on X by adding the pair (b, c). We therefore have a new order that we denote $\preceq$ which is such that $G(\preceq) = G(\leq) \cup \{(b, c)\}$. The quasi-pseudometric d_1 given by

$$\mathbf{d_1} = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 3 & 2 & 0 \end{pmatrix}$$

produces $(X, m, \preceq)$ and is minimally m-splitting.

We now check that $d_1 = D_{\preceq}$. We first compute $r' = \min\{d_{\preceq}, m\}$ and we obtain

$$\mathbf{r}' = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 3 & 2 & 0 \end{pmatrix}.$$

We then compute explicitly $D_{\preceq}$. So

$$\mathbf{D}_{\preceq} = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 3 & 2 & 0 \end{pmatrix} = \mathbf{d_1}$$

is the largest T_0-quasi-metric on X producing $(X, m, \preceq)$. The order $\preceq$ is maximally m-produced, since it is linear.

Remark 2.6.1.

- Since $\leq \subseteq \preceq$, then we have $D_{\preceq} \leq D_{\leq}$ as predicted by Theorem 2.4.1. It is therefore worth mentioning that although d' produces $(X, m, \leq)$, since $d' < D_{\leq}$ (actually $d' \neq D_{\leq}$), it cannot be compared with d_1 which produces $(X, m, \preceq)$.
- Observe also that $d = d' \vee d_1$.
- Although d' is minimally m-splitting, the corresponding specialization order $\leq_{d'}$ is not maximally m-produced.

If instead, we consider the order $\sqsubseteq = \{(a, b), (a, c), (c, b)\} \cup \Delta_X$ on (X, m), the T_0-quasi-metric $D_{\sqsubseteq}$ is given by the matrix

$$\mathbf{R} = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 2 \\ 2 & 0 & 0 \end{pmatrix}$$

and as such is not m-splitting.

Thus we see that the m-produced partial order $\leq$ on X cannot be written as the intersection of maximally m-produced partial orders on X, since $\leq$ is not maximally m-produced and of the two possible extensions of $\leq$, namely $\preceq$ and the $\sqsubseteq$, only the first one is m-produced. The order $\sqsubseteq$ also provides an example explaining why the converse of Corollary 2.5.2 does not hold.

Note that d' is an example of a minimally m-splitting T_0-quasi-metric that does not have a linear specialization order. Indeed its specialization order is the specialization order of $D_{\leq}$ that equals $\leq$ which shows that d' is also an example of a minimally m-splitting T_0-quasi-metric on X whose specialization order $\leq$ is not maximally m-produced.

While the T_0-quasi-metric $D_{\leq}$ can be written as the supremum of minimally m-splitting T_0-quasi-metrics below it, namely as the supremum of d' and $D_{\preceq}$, the T_0-quasi-metric $t = \frac{d'+D_{\leq}}{2}$ which produces $(X, m, \leq)$ according to Lemma 2.3.1 and is given by the matrix

$$\mathbf{t} = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 2 \\ 3 & 1.5 & 0 \end{pmatrix}$$

cannot be written as the supremum of minimally m-splitting T_0-quasi-metrics, since d' is the only minimally m-splitting T_0-quasi-metric on X below t.

We conclude this section with the following example which illustrates that in Proposition 2.3.1, we cannot replace m-splitting by m-producing, at least in a general setting. In other words, there need not exist a "minimally producing" T_0-quasi-metric below a given producing quasi-pseudometric. The example shows that the infimum of a chain of T_0-quasi-metrics need not be producing (compare page 40).

Example 2.6.2. On the four-point set $X = \{a, b, c, f\}$, we are given the following metric m

$$\mathbf{m} = \begin{pmatrix} 0 & 2 & 3 & 3 \\ 2 & 0 & 2 & 3 \\ 3 & 2 & 0 & 2 \\ 3 & 3 & 2 & 0 \end{pmatrix}.$$

We have on (X, m) with the partial order $\leq = \{(a, b), (a, c)\} \cup \Delta_X$ where Δ_X denotes the diagonal of X. The chain of quasi-pseudometrics d_α given by

$$\mathbf{d}_\alpha = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 2 & 0 & 2 & 3 \\ 3 & 1 & 0 & 2 \\ 3 & 1 & \alpha & 0 \end{pmatrix},$$

with $\alpha \in (0, 2)$ is a chain of m-producing quasi-pseudometrics.

Observe that

$$\mathbf{d_0} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 2 & 0 & 2 & 3 \\ 3 & 1 & 0 & 2 \\ 3 & 1 & 0 & 0 \end{pmatrix}$$

is a minimally m-splitting quasi-pseudometric but is not producing.

To check:

$$\mathbf{D}_\leq = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 2 & 0 & 2 & 3 \\ 3 & 2 & 0 & 2 \\ 3 & 2 & 2 & 0 \end{pmatrix},$$

we first compute $r_1 = \min\{d_\leq, m\}$ and we obtain

$$\mathbf{r_1} = \begin{pmatrix} 0 & 0 & 0 & 3 \\ 2 & 0 & 2 & 3 \\ 3 & 2 & 0 & 2 \\ 3 & 3 & 2 & 0 \end{pmatrix}.$$

Now, after identifying the different paths, we can easily derive, using r_1, the values of $D_{\leq}$.

Now we enlarge the order on X by adding the pair (a, f). We therefore have a new order that we denote $\preceq_1$ which is such that $G(\preceq_1) = G(\leq) \cup \{(a, f)\}$. It is easy to check that

$$\mathbf{D}_{\preceq_1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 2 & 0 & 2 & 3 \\ 3 & 2 & 0 & 2 \\ 3 & 2 & 2 & 0 \end{pmatrix}.$$

By defining the consecutive orders $\preceq_2,\ \preceq_3,\ \preceq_4$ such that

$$G(\preceq_2) = G(\preceq_1) \cup \{(b, c)\},\ G(\preceq_3) = G(\preceq_2) \cup \{(b, f)\},\ G(\preceq_4) = G(\preceq_3) \cup \{(c, f)\},$$

we respectively have

$$D_{\preceq_2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 3 \\ 3 & 2 & 0 & 2 \\ 3 & 2 & 2 & 0 \end{pmatrix}, \quad D_{\preceq_3} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 3 & 2 & 0 & 2 \\ 3 & 2 & 2 & 0 \end{pmatrix}, \quad D_{\preceq_4} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 3 & 2 & 2 & 0 \end{pmatrix},$$

and logically, we get

$$D_{\preceq_4} \ \leq \ D_{\preceq_3} \ \leq \ D_{\preceq_2} \ \leq \ D_{\preceq_1} \leq \ D_{\leq}.$$

We conclude this chapter by a few words on the **consistency** of the m-splitting Theory. So given a partially ordered metric space $(X, m, \leq)$, as an illustration we give another proof of Lemma 1.1.1 by showing how the condition $m \leq D^s_{(m,\leq)}$ (that is, that $D_{(m,\leq)}$ is m-splitting) implies the interval condition (compare Corollary 2.4.2): For convenience set $D := D_{(m,\leq)}$ and $r = \min\{m, d_{\leq}\}$.

Suppose that $x, y, z \in X$ and $x \leq y \leq z$. Then

$$\begin{aligned} m(x,y) &\leq \max\{D(y,x), D(x,y)\} \\ &\leq \max\{r(y,z) + r(z,x), r(x,x) + r(x,y)\} \\ &\leq \max\{m(x,z), m(x,x)\}, \end{aligned}$$

since $r(y,z) = 0 = r(x,y)$. Hence $m(x,y) \leq m(x,z)$.

Similarly

$$\begin{aligned} m(y,z) &\leq \max\{D(y,z), D(z,y)\} \\ &\leq \max\{r(y,z) + r(z,z), r(z,x) + r(x,y)\} \\ &\leq \max\{m(z,z), m(z,x)\} = m(z,x). \end{aligned}$$

Thus $\max\{m(x,y), m(y,z)\} \leq m(x,z)$. Hence the interval condition is satisfied in $(X, m, \leq)$.

Chapter 3

Convexity and topological conditions

We recall here some notions of order theory that we shall exclusively use in this chapter. We change our approach to the problem and consider an approach based on the convexity of the closed balls generated by the quasi-pseudometric d producing m and $\leq$. The background material can be read in details in [31, 37].

3.1 Convexity conditions

Definition 3.1.1. Let $(X, \leq)$ be a partially ordered set.

i) An **increasing set** or **upset** (also called an **upward set** or an **upper set**) is a subset U with the property that,

$$\text{if } x \text{ is in } U, y \text{ is in } X, \text{ and } x \leq y, \text{ then } y \text{ is in } U.$$

ii) We have the dual notion, namely **decreasing set** or **downset** (also called **downward set** or **lower set**) which is a subset L with the property that,

$$\text{if } x \text{ is in } L, y \text{ is in } X, \text{ and } y \leq x, \text{ then } y \text{ is in } L.$$

iii) An **order-convex set** is the intersection of an increasing set and a decreasing set.

Example 3.1.1. Let d be a quasi-pseudometric on a nonempty set X. We consider the partially ordered quasi-pseudometric space $(X, d \leq_d)$. Then for any $x \in X, \varepsilon \geq 0$, $C_d(x, \varepsilon)$ is an increasing set and $C_{d^{-1}}(x, \varepsilon)$ is a decreasing set.

Proof. Indeed, let $y \in C_d(x, \varepsilon)$ and $z \in X$ such that $y \leq z$. By the triangle inequality, we have $d(x, z) \leq d(x, y) + d(y, z)$ and since $d(y, z) = 0$, we conclude that $d(x, z) \leq \varepsilon$, i.e. $z \in C_d(x, \varepsilon)$.

Similarly, if $y \in C_{d^{-1}}(x, \varepsilon)$ and z is a point of X such that $z \leq y$, then, by $d^{-1}(x, z) \leq d^{-1}(x, y) + d^{-1}(y, z)$, we conclude that $d^{-1}(x, z) \leq \varepsilon$ since $d^{-1}(y, z) = d(z, y) = 0$. □

Definition 3.1.2. [37] Given an arbitrary subset Q of partially ordered set $(X, \leq)$ and $x_0 \in X$, we define

$$\downarrow Q := \{y \in X : \exists x \in Q \text{ and } y \leq x\}, \text{ (read "down } Q\text{")}$$

and

$$\uparrow Q := \{y \in X : \exists x \in Q \text{ and } y \geq x\} \text{ (read "up } Q\text{")}.$$

For simplicity we shall write $\downarrow x_0$ for $\downarrow \{x_0\}$, and $\uparrow x_0$ for $\uparrow \{x_0\}$, hence

$$\downarrow x_0 := \{y \in X : y \leq x_0\} = \downarrow \{x_0\} \text{ and } \uparrow x_0 := \{y \in X | y \geq x_0\} = \uparrow \{x_0\}.$$

Remark 3.1.1. It is easy to check that $\downarrow Q$ is the smallest downset containing Q and that Q is a downset if and only if $Q = \downarrow Q$. Dually for $\uparrow Q$.

If a subset A of a partially ordered set X is order-convex, then it has the form $A = U \cap L$ where U is an upset and L a downset. Hence, by definition, we have $\uparrow A \subseteq U$ and $\downarrow A \subseteq L$, i.e. $A \subseteq (\uparrow A) \cap (\downarrow A) \subseteq (U \cap L) = A$. Therefore, A is order-convex if and only if $A = \ (\uparrow A) \cap (\downarrow A)$.

Example 3.1.2. Consider $\mathbb{R}^2$ endowed with the supremum distance m_∞, i.e. for $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$, we set $m_\infty(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$. Moreover, we endow $\mathbb{R}^2$ with the product order, i.e. $x \leq y \iff x_i \leq y_i$ for $i = 1, 2$. For any $x \in \mathbb{R}^2$ and $\varepsilon \geq 0$, we know that the closed ball $C_{m_\infty}(x, \varepsilon)$ is the square which vertices are $A_1 = (x_1 - \varepsilon, x_2 - \varepsilon), A_2 = (x_1 + \varepsilon, x_2 - \varepsilon)$,
$A_3 = (x_1 - \varepsilon, x_2 + \varepsilon)$ and $A_4 = (x_1 + \varepsilon, x_2 + \varepsilon)$.

$C_{m_\infty}(x, \varepsilon)$ is neither an upset nor a downset. Indeed, just consider the points $a = (x_1 - 2\varepsilon, x_2 - \varepsilon)$ and $b = (x_1 + 2\varepsilon, x_2 + \varepsilon)$. For any $y \in C_{m_\infty}(x, \varepsilon)$, $a \leq y \leq b$ but neither a nor b belongs to $C_{m_\infty}(x, \varepsilon)$.

Claim:

$$\uparrow C_{m_\infty}(x, \varepsilon) = [x_1 - \varepsilon, \infty) \times [x_2 - \varepsilon, \infty).$$

Proof. Set $E = [x_1 - \varepsilon, \infty) \times [x_2 - \varepsilon, \infty)$. Indeed, for any $y = (y_1, y_2) \in E$, $y_1 \geq x_1 - \varepsilon$ and $y_2 \geq x_2 - \varepsilon$, i.e. there exists $x^0 = (x_1 - \varepsilon, x_2 - \varepsilon) \in C_{m_\infty}(x, \varepsilon)$ such that $x^0 \leq y$. Hence $\boxed{E \subseteq \uparrow C_{m_\infty}(x, \varepsilon)}$. Conversely, if $y \in \uparrow C_{m_\infty}(x, \varepsilon)$, then there exists $x^0 \in C_{m_\infty}(x, \varepsilon)$ such that $x^0 \leq y$. Since $x^0 \in C_{m_\infty}(x, \varepsilon)$ therefore $x_1^0 \geq x_1 - \varepsilon$ and $x_2^0 \geq x_2 - \varepsilon$. Hence, $y \in E$ since $y = (y_1, y_2) \geq x_0 = (x_1^0, x_2^0) \geq (x_1 - \varepsilon, x_2 - \varepsilon)$. This entails that $\boxed{\uparrow C_{m_\infty}(x, \varepsilon) \subseteq E}$. So

$$\uparrow C_{m_\infty}(x, \varepsilon) = [x_1 - \varepsilon, \infty) \times [x_2 - \varepsilon, \infty).$$

Similarly, we prove that

$$\downarrow C_{m_\infty}(x, \varepsilon) = (-\infty, x_1 + \varepsilon] \times (-\infty, x_2 + \varepsilon].$$

□

Proposition 3.1.1. *Let $(X, m, \leq)$ be a partially ordered metric space. If there exists a T_0-quasi-metric d which produces $(X, m, \leq)$, then each closed ε-ball $C_m(x, \varepsilon)$ is order-convex, in the sense that*

$$C_m(x, \varepsilon) = C_d(x, \varepsilon) \cap C_{d^{-1}}(x, \varepsilon).$$

Proof. Indeed, since the quasi-pseudometric d produces $(X, m, \leq)$ then, $d^s = m$. Hence

$$\begin{aligned} y \in C_m(x, \varepsilon) &\Longleftrightarrow \max\{d(x, y), d(y, x)\} \leq \varepsilon \\ &\Longleftrightarrow d(x, y) \leq \varepsilon \text{ and } d^{-1}(x, y) \leq \varepsilon \\ &\Longleftrightarrow y \in C_d(x, \varepsilon) \cap C_{d^{-1}}(x, \varepsilon). \end{aligned}$$

□

Proposition 3.1.1 remains true if we consider instead the open ε-ball $B_m(x, \varepsilon)$.

Example 3.1.3. According to Example 3.1.2, for any $x \in \mathbb{R}^2$ and, $\varepsilon \geq 0$,

$$\uparrow C_{m_\infty}(x, \varepsilon) = \{(b_1, b_2) \in \mathbb{R}^2 : b_1 \geq x_1 - \varepsilon, b_2 \geq x_2 - \varepsilon\}$$

and

$$\downarrow C_{m_\infty}(x, \varepsilon) = \{(a_1, a_2) \in \mathbb{R}^2 : a_1 \leq x_1 + \varepsilon, a_2 \leq x_2 + \varepsilon\}.$$

It is very easy to check that $\boxed{C_{m_\infty}(x, \varepsilon) = (\uparrow C_{m_\infty}(x, \varepsilon)) \cap (\downarrow C_{m_\infty}(x, \varepsilon))}$, i.e. the closed ε-balls of $(\mathbb{R}^2, m_\infty, \leq)$ are order-convex.

Remark 3.1.2. This condition is very important in our research for an existing quasi-pseudometric that produces a partially ordered metric space $(X, m, \leq)$. It is a necessary condition as stated in Proposition 3.1.1 and when it fails (as we shall see in the next example), we can conclude without any doubt that the partially ordered metric space $(X, m, \leq)$ cannot be produced.

Example 3.1.4. Consider $\mathbb{R}^2$ endowed with the Euclidean distance m_e, i.e. for $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$, we set $m_e(x, y) = [(x_1 - y_1)^2 + (x_2 - y_2)^2]^{\frac{1}{2}}$, and with the usual coordinatewise product order inherited form $\mathbb{R}$. It is easy to show that each $C_{m_e}(x, \varepsilon)$ is neither an upset nor a downset. In the sequel, we prove that $C_{m_e}(x, \varepsilon)$ is not an order-convex set and for that, it is enough to prove that

$$C_{m_e}(x, \varepsilon) \subsetneq (\uparrow C_{m_e}(x, \varepsilon)) \cap (\downarrow C_{m_e}(x, \varepsilon)).$$

By a simple computation, we see that the point

$$a = (x_1 - \varepsilon, x_2 + \varepsilon) \in (\uparrow C_{m_e}(x, \varepsilon)) \cap (\downarrow C_{m_e}(x, \varepsilon))$$

and

$$a \notin C_{m_e}(x, \varepsilon).$$

Indeed $m_e(x, a) = \sqrt{2}\varepsilon > \varepsilon$ and $(x_1 - \varepsilon, x_2) \leq a \leq (x_1, x_2 + \varepsilon)$ which entails that $a \in (\uparrow C_{m_e}(x, \varepsilon)) \cap (\downarrow C_{m_e}(x, \varepsilon))$ since $(x_1 - \varepsilon, x_2), (x_1, x_2 + \varepsilon) \in C_{m_e}(x, \varepsilon)$.

Hence, by Proposition 3.1.1 there is no T_0-quasi-metric that produces the Euclidean metric m_e and the usual coordinatewise product order $\leq$ on the plane $\mathbb{R}^2$.

However, we note that the interval condition in this example is satisfied, because for any $x, z \in \mathbb{R}^2$ with $x \leq z$ and any point y belonging to the order-interval

$$[x, z] = \{y \in \mathbb{R}^2 : x \leq y \leq z\}$$

spanned by x and z, we get that

$$\max\{m_e(x, y), m_e(y, z)\} \leq m_e(x, z),$$

since for any $a, b \in \mathbb{R}^2$ with $a \leq b$ the Euclidean diameter denoted diam satisfies $\text{diam}[a, b] = m_e(a, b)$. We therefore conclude interval condition is fulfilled, since $[x, y] \cup [y, z] \subseteq [x, z]$.

Using an analogous argument, we also prove that in $\mathbb{R}^2$ endowed with the Euclidean metric (or 1-distance m_1) defined by

$$m_1(x, y) = |x_1 - y_1| + |x_2 - y_2| \text{ for } x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$$

and with the product order, the closed ε-balls are not order-convex.

In the general case, via the m-splitting theory, the interval condition easily translates to Corollary 2.4.2 and in the last part on this section, we wish to illustrate, with the help of this corollary, that the existence of a producing T_0-quasi-metric for a partially ordered metric space implies order-convexity of all its closed metric balls of points.

The next results of this section can be seen as extensions of Lemma 1.1.1 (the interval condition).

Lemma 3.1.1. *Let $(X, m, \leq)$ be an ordered metric space. The following two conditions are equivalent:*

(a) For any $c \in X$ and $\epsilon > 0$ we have that the closed ε-ball $C_m(c, \epsilon)$ is order-convex.

(b) For any $c, x, y, z \in X$ such that $x \leq y \leq z$ we have

$$m(c, y) \leq \max\{m(c, x), m(c, z)\}. \tag{*}$$

Proof. $(b) \implies (a)$: Let $c, x, y, z \in X$ and $\epsilon > 0$ be such that $x \leq y \leq z$ and $x, z \in C_m(c, \epsilon)$. By assumption (b) it follows that $m(c, y) \leq \epsilon$ and thus $C_m(c, \epsilon)$ is order-convex.

$(a) \implies (b)$: Suppose that (b) is not satisfied. Then there are $c, x, y, z \in X$ and $\epsilon > 0$ such that $x \leq y \leq z$ and $m(c, y) > \epsilon > \{m(c, x), m(c, z)\}$. Consequently $C_m(c, \epsilon)$ is not order-convex. Therefore we conclude that (a) implies (b). □

Lemma 3.1.2. *Let $(X, m, \leq)$ be a partially ordered metric space that is produced by a T_0-quasi-metric. Then $(X, m, \leq)$ satisfies condition (*) and thus for any $c \in X$ and $\epsilon > 0$ we have that the closed ε-ball $C_m(c, \epsilon)$ is order-convex.*

Proof. Consider $D_{(m,\leq)}$ and, as before, define, for any $x, y \in X$, the function r by

$$r(x,y) = \begin{cases} 0 & , \text{ if } x \leq y, \\ m(x,y) & , \text{ if } x \not\leq y. \end{cases}$$

By Corollary 2.4.2 we have $m \leq D^s$, i.e., in particular, using the definition of $D_{(m,\leq)}$, we have for any $c, x, z \in X$ and $y \in [x, z]$

$$\begin{aligned} m(c,y) &\leq \max\{D(y,c), D(c,y)\} \\ &\leq \max\{r(y,z) + r(z,c), r(c,x) + r(x,y)\} \\ &\leq \max\{m(c,z), m(c,x)\} \end{aligned}$$

since $r(y, z) = 0 = r(x, y)$. It follows that for any $\epsilon > 0$ and $c \in X$ we have that $C_m(c, \epsilon)$ is order-convex by Lemma 3.1.1 above. ☐

Remark 3.1.3. Let $(X, m, \leq)$ be an ordered metric space. By Lemma 3.1.1, the order-convexity of closed ε-balls implies the fact that for any $x, y, z \in X$ such that $x \leq y \leq z$,

$$m(x,y) \leq \max\{m(x,z), m(x,x)\}$$

and

$$m(z,y) \leq \max\{m(z,z), m(z,x)\}.$$

Thus, it implies the condition that for any points $x, y, z \in X$, with $x \leq y \leq z$ we have that $m(x, y) \leq m(x, z)$ and $m(y, z) \leq m(x, z)$, that is, the interval condition.

In concluding this section, we finally note — as expected — that the interval condition in a linearly ordered metric space $(X, m, \leq)$ implies condition (*): Indeed let $c, x, y, z \in X$ be such that $x \leq y \leq z$. By the interval condition, if $c \leq y$, then $m(z,c) \geq m(y,c)$, and if $y < c$, then $m(x,c) \geq m(y,c)$. Thus in either case $m(c,y) \leq \max\{m(c,x), m(c,z)\}$.

3.2 Topological conditions

A partially ordered metric space is a particular case of a partially ordered topological space (see Definition 3.2.1), it is therefore expected that some topological properties induced by the metric reflect on the conditions for the existence of a producing T_0-quasi-metric. In the study of dynamical systems [3], general relativity [29] and computer science [18], there is an extensive use of topological preordered spaces and in most of these applications, it is crucial to establish if such topological spaces are quasi-uniformizable. More precisely, given a preordered topological space $(X, \mathcal{P}, \leq)$, when can we find a quasi-uniformity $\mathcal{U}$ such that $\mathcal{P} = \tau(\mathcal{U}^s)$ and $G(\leq) = \bigcap \mathcal{U}$?

Whenever a quasi-pseudometrizable space is equipped with a specialization order, it is a T_2-preordered space, thus the original order on our initial space must be closed in order to have any chance to come from a quasi-uniformity and when that is not the case, some current work shows that it is convenient to study some new closed preorder related to the original preorder.

Definition 3.2.1. A **topological preordered space** $(X, \tau, \leq)$ is a topological space (X, τ) equipped with a preorder $\leq$.

Remark 3.2.1. Some authors, in the definition of a topological preordered space (see [12]), require $G(\leq) := \{(x,y) \in X \times X : x \leq y\}$ to be a closed subset of $X \times X$; but as we will see next, it comes as a consequence when the topological preordered space is "determined" by a quasi-uniformity.

Definition 3.2.2. [27] A topological preordered space $(X, \tau, \leq)$ is said to be **determined by a quasi-uniformity** $\mathcal{U}$ on X if there exists a quasi-uniformity $\mathcal{U}$ on X such that $\tau(\mathcal{U}^s) = \tau$ and $\bigcap \mathcal{U} = \leq$. One also says that the quasi-uniformity $\mathcal{U}$ **determines** the topological preordered space $(X, \tau, \leq)$. The topological preordered spaces determined by a quasi-uniformity are called the **completely regularly ordered spaces** (or **uniform ordered spaces**).

Proposition 3.2.1. *If the partially ordered metric space $(X, m, \leq)$ is produced by a T_0-quasi-metric d on X, then the preorder $\leq$ is closed in $(X^2, \tau(m)^2)$*[1].

Proof. The convergence is with respect to the topology $\tau(m)$ generated by m on X. Consider $G(\leq) := \{(x, y) \in X \times X : x \leq y\}$ and let $(x, y) \in cl(G(\leq))$ where $cl(G(\leq))$ is the closure of $G(\leq)$ in $(X^2, \tau(m)^2)$.

$$
\begin{aligned}
(x, y) \in cl(G(\leq)) &\Longleftrightarrow \text{ there exists } (x_n, y_n) \in G(\leq) \text{ such that } (x_n, y_n) \longrightarrow (x, y),\\
&\Longleftrightarrow x_n \longrightarrow x, y_n \longrightarrow y \text{ and } x_n \leq y_n \ \forall\ n,\\
&\Longrightarrow m(x, x_n) \longrightarrow 0, m(y_n, y) \longrightarrow 0, \text{ and } x_n \leq y_n \ \forall\ n,\\
&\Longrightarrow d(x, x_n) \longrightarrow 0, d(y_n, y) \longrightarrow 0 \text{ and } d(x_n, y_n) = 0 \ \forall\ n,\\
&\Longrightarrow d(x, y) = 0 \text{ since } d(x, y) \leq d(x, x_n) + d(x_n, y_n) + d(y_n, y),\\
&\Longrightarrow x \leq y \text{ i.e. } (x, y) \in G(\leq).
\end{aligned}
$$

□

More generally,

Definition 3.2.3. A topological ordered space $(X, \tau, \leq)$ is called T_2**-ordered** if the order $\leq$ is closed in $(X^2, \tau \times \tau)$[2].

In view of what we discussed above, given an ordered metric space $(X, m, \leq)$, it seems clear that the existence of a T_0-quasi-pseudometric d which produces m

[1] We use the supremum metric on the Cartesian product.

[2] We so refer to the product topology on X^2.

and $\leq$ is demanding. Indeed, the condition $d^s = m$ is very strong in the sense that if $d^s = m$, then $\tau(d^s) = \tau(m)$, i.e. the topologies generated by d^s and m are the same. Hence, in the particular setting, where we only require the equality between topologies, we obtain a weaker formulation of our problem and another necessary condition.

Lemma 3.2.1. *Let $(X, m, \leq)$ be partially ordered metric space. If there exists a T_0-quasi-metric d that produces $(X, m, \leq)$, then $\mathcal{U}_d$ determines $(X, m, \leq)$.*

Proof. Indeed, if there exists a T_0-quasi-metric d that produces $(X, m, \leq)$, then $d^s = m$ and $\leq_d = \leq$ which entail that $\tau(d^s) = \tau(m)$ and $\bigcap \mathcal{U}_d = \leq_d = \leq$. □

Proposition 3.2.2. *Let $(X, m, \leq)$ be a partially ordered metric space. If there exists a T_0-quasi-metric d that produces $(X, m, \leq)$, then $(X, m, \leq_d)$ is T_2-ordered.*

Proof. This is obvious, due to the the fact that if there exists a T_0-quasi-metric d that produces $(X, m, \leq)$, then the quasi-uniformity $\mathcal{U}_d$ determines $(X, m, \leq)$ which entails that the order $\leq$ is $\tau(m)^2$-closed in $X \times X$, according to Proposition 3.2.1, i.e. $(X, m, \leq_d)$ is T_2-ordered. □

In summary, Lemma 3.2.1 and Proposition 3.2.2 state that if a partially ordered metric space is produced, then it is completely regularly T_2-ordered.

Example 3.2.1. Let $(X, m, \leq)$ be a finite metric space and let $\mathcal{D}$ denote the discrete uniformity on X. Obviously there is $\epsilon > 0$ such that $B_\epsilon = \{(x, y) \in X \times X : m(x, y) < \epsilon\} = \Delta_X$ where $\Delta_X = \{(x, x) : x \in X\}$ denotes the diagonal of X. Hence $\mathcal{U}_\leq$ (Cf Example 0.3.2) is a quasi-uniformity such that $(\mathcal{U}_\leq)^s = \mathcal{U}_m = \mathcal{D}$ and $\bigcap \mathcal{U}_\leq = \leq$. In general there is no T_0-quasi-metric d on X such that $\leq_d = \leq$ and $d^s = m$ as the 3 point Example 1.1.3 shows.

In connection with our study, some interesting results can be found in [27] but we cite here just two of them which will be useful.

Proposition 3.2.3. *[27, Lemma 2.2.2] Let $(H_n)_{n\in\mathbb{N}}$ be a sequence of binary reflexive relations on a set X such that for each $n \in \mathbb{N}$, $H_{n+1} \circ H_{n+1} \circ H_{n+1} \subseteq H_n$. Then there is a quasi-pseudometric d on X such that*

$$H_{n+1} \subseteq \{(x,y) \in X \times X : d(x,y) < 2^{-n}\} \subseteq H_n$$

whenever $n \in \mathbb{N}$. If each H_n is a symmetric relation, then d can be constructed to be a pseudometric.

Proposition 3.2.4. *[27, Corollary 2.2.3] A (quasi-)uniformity $\mathcal{U}$ on a set X with a countable base is (quasi-)pseudometrizable, i.e. there is a (quasi-)pseudometric d on X such that $\mathcal{U}_d = \mathcal{U}$.*

The next result, due to Nachbin, is taken from Fletcher [12].

Proposition 3.2.5. *[12, Theorem 1.20] Let $(X, \mathcal{T})$ be a compact Hausdorff space and let G be a closed partial order on X. There is exactly one quasi-uniformity $\mathcal{U}$ on X such that $\bigcap \mathcal{U} = G$ and $\mathcal{T} = \mathcal{T}(\mathcal{U}^s)$.*

This result can be interpreted in the metric case by saying that if $(X, m, \leq)$ is a T_2-ordered compact metric space, then there is a T_0-quasi-uniformity $\mathcal{U}$ on X with a countable base that determines $(X, \tau(m), \leq)$. We now state the main result of this section, a version of [12, Theorem 1.20] in connection with our work.

Proposition 3.2.6. *Let $(X, m, \leq)$ be a partially T_2-ordered compact metric space. There is one quasi-metrizable quasi-uniformity $\mathcal{U}$ on X with a countable base such that $\bigcap\, \mathcal{U} = \leq$ and $\mathcal{U}^s = \mathcal{U}_m$.*

Proof. Let $\mathcal{U} = \text{fil}\{U_n \circ \leq \circ\, U_n : n \in \mathbb{N}\}$ where

$$U_n = \{(x,y) \in X \times X : m(x,y) < 2^{-n}\} \text{ for each } n \in \mathbb{N}.$$

Given $m \in \mathbb{N}$, suppose that $\{(U_n \circ \leq \circ\, U_n)^2 \setminus (U_m \circ \leq \circ\, U_m) \neq \emptyset, n \in \mathbb{N}\}$ is a filter base on $(X \times X) \setminus (U_m \circ \leq \circ\, U_m)$.

Since $U_m \circ \leq \circ\, U_m$ is open in $X \times X$, so $(X \times X) \setminus (U_m \circ \leq \circ\, U_m)$ is compact in $(X^2, \tau(m)^2)$. Hence there must be a cluster point (x, y) with

$$(x, y) \in \bigcap_{n \in \mathbb{N}} \overline{A_n} \cap [(X \times X) \setminus (U_m \circ \leq \circ\, U_m)].$$

where

$$A_n = (U_n \circ \leq \circ\, U_n)^2 \setminus (U_m \circ \leq \circ\, U_m), \qquad n \in \mathbb{N}.$$

By setting $B_n = (U_n \circ \leq \circ\, U_n)^2,\ n \in \mathbb{N}$, it is clear that $A_n \subseteq B_n$, hence

$$(x, y) \in \bigcap_{n \in \mathbb{N}} \overline{B_n} \cap [(X \times X) \setminus (U_m \circ \leq \circ\, U_m)].$$

Now, for each n, $(x, y) \in \overline{(U_n \circ \leq \circ\, U_n)^2}$, there exist a_n, b_n such that $(x, a_n) \in U_n, (b_n, y) \in U_n, (a_n, b_n) \in (U_n \circ \leq \circ\, U_n)^2$, i.e. there exist $c_n, d_n, d'_n, e_n, e'_n$ such that

$$(a_n, c_n) \in U_n, \quad c_n \leq c'_n, \quad (c'_n, d'_n) \in U_n$$

and

$$(d'_n, d_n) \in U_n, \quad d_n \leq e_n, \quad (e_n, b_n) \in U_n.$$

Since X is compact, there is a subsequence $(c'_{n(k)})$ of (c'_n) such that $c'_{n(k)} \to a$ for some $a \in X$ and by $(c'_n, d'_n) \in U_n$ we have that $d_{n(k)} \to a$. By construction, we know that $c_n \to x$ and $e_n \to y$. Since $(X, m, \leq)$ is T_2-ordered, a short straightforward computation gives that $x \leq a \leq y$ which implies that $x \leq y$, so $(x, y) \in U_m \circ \leq \circ\, U_m$ — a contradiction.

<u>Conclusion</u> For any $m \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that

$$(U_n \circ \leq \circ\, U_n)^2 \subseteq U_m \circ \leq \circ\, U_m.$$

Next, we check that $\bigcap_{n\in\mathbb{N}}(U_n\circ\leq\circ U_n)=\leq$.

Indeed, let $(x,y)\in U_n\circ\leq\circ U_n$ for all $n\in\mathbb{N}$. For each $n\in\mathbb{N}$, there exist $a_n,a'_n\in X$ such that

$$(x,a_n)\in U_n,\quad a_n\leq a'_n,\quad (a'_n,y)\in U_n.$$

Since $a_n\to x$ and $a'_n\to y$, then $x\leq y$, by T_2-ordered assumption.

The reverse inclusion is obvious. So $\mathcal{U}$ is a quasi-uniformity such that $\cap\,\mathcal{U}=\leq$.

We just have to prove that $\mathcal{U}_m\subseteq\mathcal{U}^s$ since the inclusion $\mathcal{U}^s\subseteq\mathcal{U}_m$ is obvious. In this aim, we show that, given any $k\in\mathbb{N}$, there is $n\in\mathbb{N}$ such that

$$(U_n\circ\leq\circ U_n)\cap(U_n\circ\geq\circ U_n)\subseteq U_k.$$

Otherwise, set $\mathcal{F}=\{[(U_n\circ\leq\circ U_n)\cap(U_n\circ\geq\circ U_n)]\setminus U_k:n\in\mathbb{N}\}$ which is a filter base on the compact set $(X\times X)\setminus U_k$.

Thus, there exists $(x,y)\in\bigcap_{F\in\mathcal{F}}\overline{F}$ such that $(x,y)\notin U_m$.

Therefore

$$(x,y)\in\overline{(U_n\circ\leq\circ U_n)\cap(U_n\circ\geq\circ U_n)}\quad\forall n\in\mathbb{N}.$$

Hence for each n, there exist a_n,b_n such that $(x,a_n)\in U_n,\ (b_n,y)\in U_n,\ (a_n,b_n)\in(U_n\circ\leq\circ U_n)\cap(U_n\circ\geq\circ U_n)$. So, for each $n\in\mathbb{N}$, there exist c_n,d_n,d'_n,e_n,e'_n such that

$$(a_n,d_n)\in U_n,\ d_n\leq d'_n,\ (d'_n,b_n)\in U_n,\ (a_n,e_n)\in U_n,\ e_n\geq e'_n,\ (e'_n,b_n)\in U_n.$$

We have

$$d_n\to x,\ e'_n\to y,\ e_n\to x,\ d'_n\to y,$$

i.e. $x\geq y,\ x\leq y$, thus $x=y$. Contradiction, since $(x,y)\notin U_k$, so $\mathcal{U}_m=\mathcal{U}^s$. In particular $\tau(\mathcal{U}_m)=\tau(\mathcal{U}^s)$ and $\mathcal{U}$ determines $(X,\tau(\mathcal{U}_m),\leq)$, where $\mathcal{U}$ has a countable base. $\square$

The next proposition follows from a result due to Minguzzi ([30, Theorem 2.5] (in the preordered setting). We include a proof of Minguzzi's result that seems slightly more direct than the original one.

Proposition 3.2.7. *Let $(X, \tau, \leq)$ be a completely regularly T_2-ordered space such that the topology τ is second countable. Then there is a T_0-quasi-uniformity on X with a countable base that determines $(X, \tau, \leq)$.*

Proof. Let $\mathcal{B} = \{B_n : n \in \mathbb{N}\}$ be a countable base of τ and let $\mathcal{C}$ be a quasi-uniformity determining $(X, \tau, \leq)$.

Then for all $(n, m) \in \mathbb{N} \times \mathbb{N}$ such that there is $C \in \mathcal{C}$ with $B_n \subseteq C^s(B_n) \subseteq B_m$ choose such a C and denote it by C_{nm}. Furthermore for each $(n, m) \in \mathbb{N} \times \mathbb{N}$ such that there is $C \in \mathcal{C}$ with $C \cap (B_n \times B_m) = \emptyset$ choose such a C and call it $C'_{n,m}$.

Let $\mathcal{D}$ be the set of the entourages C_{nm} and $C'_{n,m}$ of $\mathcal{C}$ chosen. For each $D \in \mathcal{D}$ choose a sequence $(D_n)_{n\in\mathbb{N}}$ of entourages in $\mathcal{C}$ such that $D_1 \subseteq D$ and for each $n \in \mathbb{N}$, $D^2_{n+1} \subseteq D_n$. Then by a standard argument, $\{D_n : n \in \mathbb{N}, D \in \mathcal{D}\}$ is a subbase for a quasi-uniformity $\mathcal{H}$ on X having a countable base. Of course, $\mathcal{H} \subseteq \mathcal{C}$.

We want to show that $\mathcal{H}$ determines $(X, \tau, \leq)$. Of course $\leq = \bigcap \mathcal{C} \subseteq \bigcap \mathcal{H}$ and $\tau(\mathcal{H}^s) \subseteq \tau(\mathcal{C}^s) = \tau$ are obvious, since $\mathcal{H} \subseteq \mathcal{C}$.

Let $x, y \in X$ with $x \not\leq y$ be given. Since $\bigcap \mathcal{C} = \leq$, there is $C \in \mathcal{C}$ such that $(x, y) \notin C^3$. Then $(C(x) \times C^{-1}(y)) \cap C = \emptyset$. Furthermore $\tau = \tau(\mathcal{C}^s)$ and thus there exist $n, m \in \mathbb{N}$ such that $x \in B_n \subseteq C(x)$ and $y \in B_m \subseteq C^{-1}(y)$. Thus C'_{nm} exists and $(x, y) \notin C'_{nm}$. Furthermore $\leq \subseteq \bigcap \mathcal{D}$ and hence the intersection of all $C'_{nm} \in \mathcal{D}$ is equal to $\leq$. It follows that $\bigcap \mathcal{H} = \leq$.

Let $x \in X$ and $G \in \tau$ be such that $x \in G$. Thus there are $m, n \in \mathbb{N}$ and $C \in \mathcal{C}$ such that $x \in B_m \subseteq G$, $(C^s)^2(x) \subseteq B_m$ and $x \in B_n \subseteq C^s(x)$. It follows that $C^s(B_n) \subseteq (C^s)^2(x) \subseteq B_m$. Consequently C_{nm} must exist. Furthermore, by the definition of C_{nm} and since $x \in B_n$, $C^s_{nm}(x) \subseteq C^s_{nm}(B_n) \subseteq B_m \subseteq G$. We conclude that $\tau(\mathcal{H}^s) = \tau$. Hence we are done. $\square$

Chapter 4

Order theoretic approach

This chapter elaborates on an order theoretic approach in the resolution of the quasi-pseudometrization problem as defined in our context. We propose here a solution (on finite sets) using ideas from Discrete Mathematics. The approach builds on a construction, step by step, of an appropriate producing T_0-quasi-metric. We start by providing a global framework in Definition 4.2.1, from which we deduct our main result, Lemma 4.2.2. The direct implications are straightforward and can be read in Lemma 4.3.1 (for the metric case) and Corollary 4.4.1 (for applications to Discrete Mathematics).

4.1 Considering a special case of the problem

It appears from the previous chapters that it is not trivial to formulate conditions for the existence of a quasi-pseudometric that produces a given partially ordered metric space. Hence, in the following section, we tackle the problem by weakening the assumptions on the initial metric space. More precisely, we formulate our new problem as follows:

Problem 4.1.1. Let $(X, m, \leq)$[1] be a partially ordered *finite* metric space. For two non-comparable points $a, b \in X$, we consider partially ordered metric space $(X, m, \leq_{ab})$ where the order $\leq_{ab}$ is the partial order generated by $\leq \cup\{(a, b)\}$. We are looking for conditions for the existence of a quasi-pseudometric that produces $(X, m, \leq_{ab})$.

4.2 First results

We begin with the following lemma which is of great importance.

Lemma 4.2.1. *Let $(X, m, \leq)$ be a partially ordered metric space. Then for any pair $a, b \in X$ such that a and b are not comparable, the partial order $\leq_{ab}$ generated by $\leq \cup\{(a, b)\}$ is given by*

$$\leq_{ab} \; = \; \leq \cup[(\downarrow a) \times (\uparrow b)].$$

Proof. Let $\{P_i : i \in I\}$ denote the collection of all partial orders containing $\leq \cup\{(a, b)\}$. Then we obviously have that $\leq \cup\{(a, b)\} \subseteq \bigcap_{i \in I} P_i$. It is also trivial that

$$(\leq \cup\{(a, b)\}) \; \subseteq \; (\leq \cup[(\downarrow a) \times (\uparrow b)]) \; \subseteq \; \bigcap_{i \in I} P_i.$$

We conclude the proof by showing that $\preceq := \leq \cup[(\downarrow a) \times (\uparrow b)]$ is a partial order.

Clearly, $\preceq$ is reflexive. Now, investigating for transitivity and given $x, y, z \in X$, we analyse the four following cases:

- <u>Case 1:</u>

  $x \leq y$ and $y \leq z$ imply that $x \leq z$, i.e. $x \preceq z$.

[1] We shall generally start with "=" as our partial order, and in that case, non-comparable elements are just elements which are distinct (see Section 0.2).

- Case 2:

  $x \leq y$ and $(y, z) \in [(\downarrow a) \times (\uparrow b)]$ give that $x \leq y \leq a$ and $b \leq z$ which in return imply that $x \preceq z$.

- Case 3:

  $(x, y) \in [(\downarrow a) \times (\uparrow b)]$ and $y \leq z$ give that $x \leq a$ and $b \leq y \leq z$ which in return imply that $x \preceq z$.

- Case 4:

  $(x, y) \in [(\downarrow a) \times (\uparrow b)]$ and $(y, z) \in [(\downarrow a) \times (\uparrow b)]$ which give that $x \in (\downarrow a)$ and $z \in (\uparrow b)$ which in return imply that $x \preceq z$.

Hence

$$x \preceq y \text{ and } y \preceq z \Longrightarrow x \preceq z,$$

and $\preceq$ is a partial order and $\preceq = \leq_{ab}$. □

The following idea can be understood as a kind of method of splitting T_0-quasi-metrics that are not necessarily metrics. In this context let us refer to another method of 'splitting' distance functions in the literature [2].

Definition 4.2.1. Let (X, d) be a T_0-quasi-metric space and let $a, b \in X$ be incomparable with respect to the specialization order $\leq_d$ of d. Furthermore let $\ell \in [0, d(a, b))$. Set

$$d_{ab,\ell}(x, y) = \min\{d(x, a) + \ell + d(b, y), d(x, y)\} \text{ whenever } x, y \in X.$$

Instead of $d_{ab,0}$ we shall more briefly write d_{ab}.

We then have the following crucial lemma which is the key in the step by step construction mentioned at the very beginning of this chapter.

Lemma 4.2.2. *Let (X, d) be a T_0-quasi-metric space and let $a, b \in X$ be incomparable with respect to the specialization order $\leq_d$ of d, and $d_{ab,\ell}$ as defined above for $\ell \in [0, d(a,b))$. Then $d_{ab,\ell}$ is a T_0-quasi-metric on X such that $d_{ab,\ell} < d$. Furthermore $d_{ab} \vee d_{ba} = d$. Moreover $d_{ab,\ell}$ is the largest T_0-quasi-metric $q \leq d$ on X such that $q(a,b) = \ell$.*

Proof. For any $x \in X$, it is obvious that

$$d_{ab,\ell}(x,x) = \min\{d(x,a) + \ell + d(b,x), d(x,x)\} = \min\{d(x,a) + \ell + d(b,x), 0\} = 0.$$

One easily checks that the map $d_{ab,\ell}$ satisfies the triangle inequality, for the four possible cases of the right-hand side, for any $a, b \in X$ and $\ell \in [0, d(a,b))$. More precisely, for any a, b which are incomparable with respect to the specialization order $\leq_d$ of d, and $x, y, z \in X$ and $\ell \in [0, d(a,b))$, let us prove that

$$d_{ab,\ell}(x,y) \leq d_{ab,\ell}(x,z) + d_{ab,\ell}(z,y). \tag{4.1}$$

Let us set

$$t_1 = d(x,a) + \ell + d(b,y), \qquad t_2 = d(x,y),$$

$$u_1 = d(x,a) + \ell + d(b,z), \qquad u_2 = d(x,z),$$

$$v_1 = d(z,a) + \ell + d(b,y), \qquad v_2 = d(z,y).$$

Since $d_{ab,\ell}(x,y) \in \{t_1, t_2\}$, $d_{ab,\ell}(x,z) \in \{u_1, u_2\}$ and $d_{ab,\ell}(z,y) \in \{v_1, v_2\}$, proving Inequality (4.1) reduces to proving that

$$\min\{t_1, t_2\} \leq u_i + v_j, \quad i, j, k = 1, 2$$

for any $a, b, x, y, z \in X$ and $\ell \in [0, d(a,b))$.

- For $i = j = 1$, i.e. $d_{ab,\ell}(x,z) = u_1$ and $d_{ab,\ell}(z,y) = v_1$. Just observe that

$$d(x,a) + \ell + d(b,y) \leq d(x,a) + \ell + d(b,z) + d(z,a) + \ell + d(b,y),$$

hence

$$\boxed{t_1 \leq u_1 + v_1}.$$

- For $i = 1, j = 2$, i.e. $d_{ab,\ell}(x,z) = u_1$ and $d_{ab,\ell}(z,y) = v_2$. Just observe that

$$d(x,a) + \ell + d(b,y) \leq d(x,a) + \ell + d(b,z) + d(z,y),$$

hence

$$\boxed{t_1 \leq u_1 + v_2}.$$

- For $i = 2, j = 1$, i.e. $d_{ab,\ell}(x,z) = u_2$ and $d_{ab,\ell}(z,y) = v_1$. Just observe that

$$d(x,a) + \ell + d(b,y) \leq d(x,z) + d(z,a) + \ell + d(b,y),$$

hence

$$\boxed{t_1 \leq u_2 + v_1}.$$

- For $i = 2, j = 2$, i.e. $d_{ab,\ell}(x,z) = u_2$ and $d_{ab,\ell}(z,y) = v_2$. Just observe that

$$d(x,y) \leq d(x,z) + d(z,y),$$

hence

$$\boxed{t_2 \leq u_2 + v_2}.$$

In order to conclude that $d_{ab,\ell}$ is a T_0-quasi-metric, we just need to verify that d_{ab} fulfils the T_0-condition, since $d_{ab} \leq d_{ab,\ell}$. But this fact is established by the following argument.

For $x, y \in X$ suppose that $d_{ab}(x, y) = 0$ and $d_{ab}(y, x) = 0$. We want to show that $x = y$. We shall see that only the first case is of interest, as the three others lead to contradictions.

- Case 1: $d(x, y) = 0$, together with $d(y, x) = 0$ imply that $x = y$, as stated, since d is a T_0-quasi-metric.
- Case 2: Since

  $$d(b, a) \leq d(b, x) + d(x, a) \leq d(y, a) + d(b, x) + d(x, a) + d(b, y),$$

  $d(x, a) + d(b, y) = 0$ and $d(y, a) + d(b, x) = 0$, then $d(b, a) = 0$ and this is a contradiction, since a, b are incomparable with respect to $\leq_d$. Hence this case is impossible under our assumptions.
- Case 3: $d(x, y) = 0$ and $d(y, a) + d(b, x) = 0$ imply that $d(b, a) = 0$ which is a contradiction.
- Case 4: $d(y, x) = 0$ and $d(x, a) + d(b, y) = 0$ imply that $d(b, a) = 0$ which is a contradiction.

Since $d_{ab,\ell}(a, b) = \ell < d(a, b)$ and $d_{ab,\ell} \leq d$, we have $\boxed{d_{ab,\ell} < d}$.

Next, we establish that $d_{ba} \vee d_{ba} = d$. Let $x, y \in X$ and suppose by contradiction that $(d_{ab} \vee d_{ba})(x, y) < d(x, y)$. Then

$$d(x, a) + d(b, y) < d(x, y) \text{ and } d(x, b) + d(a, y) < d(x, y).$$

Thus

$$d(x, y) + d(x, y) \leq d(x, a) + d(a, y) + d(x, b) + d(b, y) < d(x, y) + d(x, y)$$

which is a contradiction. Hence we indeed have $\boxed{d_{ab} \vee d_{ba} = d}$.

As already observed earlier, $d_{ab,\ell}(a,b) = \ell$. Moreover, for any quasi-pseudometric q on X such that $q(a,b) = \ell$ and $q \leq d$, we have that, for any $x, y \in X$,

$$q(x,y) \leq q(x,a) + q(a,b) + q(b,y) \leq d(x,a) + \ell + d(b,y)$$

and therefore by definition of $d_{ab,\ell}$ we have $q \leq d_{ab,\ell}$.

Hence the T_0-quasi-metric $d_{ab,\ell}$ can be described as the supremum (or maximum) of all the T_0-quasi-metrics q on X with the properties that $q(a,b) = \ell$ and $q \leq d$. □

The following observations point out some connections with investigations conducted in Chapter 2.

Remark 4.2.1. Let (X,d) be a T_0-quasi-metric space and $a, b \in X$ be incomparable with respect to the specialization order of d.

(a) Then

$$d_{ab} = D_{(d, \Delta \cup \{(a,b)\})}.$$

The inclusion $d_{ab} \geq D_{(d,\Delta\cup\{(a,b)\})}$ comes from the definitions of the two distance functions. For $x, y \in X$, to evaluate $D_{(d,\Delta\cup\{(a,b)\})}(x,y)$, with the paths from x to y, we just need to consider only the paths that pass the directed edge ab at most once and (by possibly repeating the start point x and endpoint y) we can assume that we start and end with an edge the length of which is measured by d. Then using the triangle inequality of d, we see that $D_{(d,\Delta\cup\{(a,b)\})} \geq d_{ab}$. So our equality is proved.

(b) Observe that the specialization order of d_{ab} is equal to

$$\leq_d \cup \{(x,y) \in X \times X : x \leq_d a \text{ and } b \leq_d y\}$$

(Cf Lemma 4.2.1).

Example 4.2.1. Using the notation introduced above, note that in Example 2.6.1 we showed that $(D_{(m,\leq)})_{23} = D_{(m,\preceq)}$ is m-splitting, while $(D_{(m,\leq)})_{32} = D_{(m,\sqsubseteq)}$ is not m-splitting.

Example 4.2.2. Let (X, m) be a metric space. Then for any $(a, b) \in X \times X$ with $a \neq b$ we have that the T_0-quasi-metric $m_{ab} = D_{(m,\Delta\cup\{(a,b)\})}$ produces the partially ordered metric space $(X, m, \Delta \cup \{(a, b)\})$.

Proof. Since m is a metric, we have $m_{ab} = (m_{ba})^{-1}$, and thus

$$m = m_{ab} \vee (m_{ab})^{-1} =^{2} (m_{ab})^s$$

by Lemma 4.2.2. Obviously the specialization order of m_{ab} is $\Delta \cup \{(a, b)\}$.

□

The three results, presented in the next section, elaborate on the implications of Lemma 4.2.2 in the metric case (i.e. when the T_0-quasi-metric considered in Definition 4.2.1 is actually a metric).

4.3 The metric case

In light of what is done in the previous sections, we have the following.

Lemma 4.3.1. *Let (X, m)[3] be a metric space. Then for any pair $a, b \in X$ such that $a \neq b$[4]. Then $(X, m, \leq_{ab})$ and $(X, m, \leq_{ba})$ are both produced.*

[2] This equality is actually a special case of the more general equality $(d_{ab})^{-1} = \left(d^{-1}\right)_{ba}$ used in the proof of Lemma 4.3.2.

[3] We can also write $(X, m, =)$ since $=$ can be considered as a partial order, as stated in Section 0.2 of the preliminaries.

[4] This just means that a and b are not comparable.

Lemma 4.3.2. *Let a, b be elements of a produced partially ordered metric space $(X, m, \leq)$,such that a and b are $\leq$-incomparable. Then $(X, m, \leq_{ab})$ is produced by d_{ab} if and only if $(X, m, \preceq)$, where*

$$\preceq := \leq_{d^{-1}} \cup \{(x, y) \in X \times X : x \leq_{d^{-1}} a \text{ and } b \leq_{d^{-1}} y\},$$

is produced by $(d_{ba})^{-1}$.

Proof. It is enough to observe that since

$$(d_{ab})^{-1} = \left(d^{-1}\right)_{ba},$$

therefore d_{ab} is m-splitting if and only if $\left(d^{-1}\right)_{ba}$ is m-splitting. Moreover

$$c \leq_{ab} e \iff d_{ab}(c, e) = 0 = \left(d^{-1}\right)_{ba}(c, e) \iff e \leq_{ba} c,$$

for any $c, e \in X$, and the result follows by the part (b) of Remark 4.2.1. □

Corollary 4.3.1. *Let (X, m) be a metric space. Then $m = \bigvee \mathcal{D}$ where $\mathcal{D}$ is a collection of minimally m-splitting T_0-quasi-metrics on X.*

Proof. By Example 4.2.2 and Proposition 2.3.1 for each $(a, b) \in X \times X$ with $a \neq b$ there is a minimally m-splitting T_0-quasi-metric d on X such that $d \leq m_{ba}$. Then $d(b, a) = 0$, since $m_{ba}(b, a) = 0$ and thus $d(a, b) = m(a, b)$, since d is m-splitting. Therefore the statement holds. □

Remark 4.3.1. Given a metric space (X, m), by an argument similar to the one just presented, we see that we can write

$$m = \bigvee \{D_{(m, \leq_{ab})} : a, b \in X, a \neq b\}$$

where each $\leq_{ab}$ is a maximally m-produced partial order on X containing (a, b) which exists according to Theorem 2.4.1 and Example 4.2.2. Recall that the T_0-quasi-metric $D_{(m, \leq_{ab})}$ produces the partially ordered metric space $(X, m, \leq_{ab})$ by Remark 2.4.1.

Corollary 4.3.2. *Let (X,m) be a metric space and let $\leq$ be an m-produced partial order on X. Then $\leq$ is maximally m-produced if and only if for each $\leq$-incomparable pair $(a,b) \in X \times X$ we have that the T_0-quasi-metric $(D_{(m,\leq)})_{ab}$ is not m-splitting.*

Proof. Suppose that $\leq$ is maximally m-produced on X. Consider $a, b \in X$ such that a and b are not comparable with respect to $\leq$. Then the T_0-quasi-metric $D_{(m,\leq)}$ produces the partially ordered metric space $(X, m, \leq)$ (see Remark 2.4.1). Moreover by Lemma 4.2.2 the specialization order of the T_0-quasi-metric $(D_{(m,\leq)})_{ab}$ is strictly larger than $\leq$, because it contains (a,b). Hence $(D_{(m,\leq)})_{ab}$ cannot be m-splitting by our assumption on maximality.

Conversely, assume that $\leq$ is not maximally m-produced. Then there is a partial order $\preceq$ that is m-produced by $D_{(m,\preceq)}$ on X (see Remark 2.4.1), but for which we have $\leq \subset \preceq$. There exist $a, b \in X$ that are incomparable with respect to $\leq$, but $a \preceq b$. As stated earlier, since $\leq$ is m-produced, $D_{(m,\leq)}$ produces the partially ordered metric space $(X, m, \leq)$ (see Remark 2.4.1). Hence by applying Lemma 4.2.2, we obtain $D_{(m,\leq)} \geq (D_{(m,\leq)})_{ab} \geq D_{(m,\preceq)}$, where the last inequality follows from the definition of the operations D and $\cdot_{ab}$ and the fact that $(\leq \cup \{(a,b)\}) \subseteq \preceq$. Hence $(D_{(m,\leq)})_{ab}$ is m-splitting, since $D_{(m,\leq)}$ and $D_{(m,\preceq)}$ are m-splitting, too. □

Remark 4.3.2. Let (X,m) be a metric space and let d be a T_0-quasi-metric on X that is m-splitting, but not minimally m-splitting. Consequently there is an m-splitting T_0-quasi-metric $e \leq d$ on X such that for some $a, b \in X$ we have that $e(a,b) < d(a,b)$.

Also note that

$$m(a,b) \geq d(a,b) > e(a,b) \geq 0, \text{ and so } 0 < m(a,b) = e(b,a) \leq d(b,a),$$

i.e. $a, b \in X$ are not comparable with respect to the specialization order of d.

Let $t = e(a,b)$. Since e is m-splitting and since by Lemma 4.2.2 $e \leq d_{ab,t} < d$, we see that $d_{ab,t}$ is m-splitting, too. Such a case is illustrated for instance with the matrices $\mathbf{d}$ and $\mathbf{d}'$ of Example 2.6.1.

We conclude that a T_0-quasi-metric d on a metric space (X, m) is minimally m-splitting if and only if for any $a, b \in X$ that are $\leq_d$-incomparable and any $\ell \in [0, d(a, b))$ we have that $d_{ab,\ell}$ is not m-splitting. We can then formulate the following characterization.

Proposition 4.3.1. *Let (X, m) be a metric space. Then a T_0-quasi-metric d on X is minimally m-splitting (that is, is minimal among the m-splitting T_0-quasi-metrics) if and only if it is minimally producing the ordered metric space $(X, m, \leq_d)$.*

Proof. Let d be a minimally m-splitting T_0-quasi-metric on X and e a T_0-quasi-metric on X such that $e < d$. Then e cannot be m-splitting and thus will not produce $(X, m, \leq_d)$. Hence d is minimally producing the ordered metric space $(X, m, \leq_d)$.

For the converse suppose that d is a T_0-quasi-metric on X that is minimally producing $(X, m, \leq_d)$. Assume that there are an m-splitting T_0-quasi-metric e on X with $a, b \in X$ such that $e(a, b) < d(a, b)$. Set $\ell = \frac{d(a,b)+e(a,b)}{2}$. Then $a, b \in X$ are $\leq_d$-incomparable (see Remark 4.3.2) and the T_0-quasi-metric $f = d_{ab,\ell}$ is m-splitting since $e \leq d_{ab,\ell} \leq d$. But $\leq_f = \leq_d$, since $\ell > 0$. Thus f produces $(X, m, \leq_d)$. Since $f(a, b) < d(a, b)$ we see that d is not minimally producing $(X, m, \leq_d)$ —a contradiction. Thus we conclude that d is minimally m-splitting on X. □

The following observation may explain why the theory of two-valued[5] T_0-quasi-metrics is much simpler than that of arbitrary T_0-quasi-metrics.

[5] See Section 2.2.

Remark 4.3.3. (Compare Lemma 4.2.2) Let (X, d) be a T_0-quasi-metric space such that $d(X \times X) \subseteq \{0, 1\}$ and let $a, b \in X$ be incomparable with respect to $\leq_d$. Since d_{ab} is a T_0-quasi-metric, obviously $d^s_{ab} = d^s = d_=$. Similarly $d^s_{ba} = d^s = d_=$. Then both d_{ab} and d_{ba} are d^s-splitting. But this is not the case in general.

We observe that for general m-splitting T_0-quasi-metrics d on a set X the situation is much more complicated. Indeed Example 4.2.1 provides us with a case where exactly one of the two T_0-quasi-metrics d_{ab} and d_{ba} under consideration remains m-splitting on X, while Example 4.4.1 presents a case where neither of the two T_0-quasi-metrics remains m-splitting on X.

4.4 Implication for finite sets

In this section, we elaborate on a few applications (on finite sets), of the m-splitting theory using the order theoretic approach.

Corollary 4.4.1. *Let X be a finite set and e a T_0-quasi-metric on X. Then e can be written as the maximum of finitely many T_0-quasi-metrics of the form $d = D_{(e,\leq)}$ where $\leq$ is a linear order on X and the intersection of the set of the used linear orders $\leq$ is equal to the specialization order of e.*

Proof. We iterate the procedure suggested by Lemma 4.2.2. The proof is by induction, starting with $\mathcal{D} = \{e\}$. First observe that $D_{(e,\leq_e)} = e$.

Our induction hypothesis (H) goes as follows: Suppose that at a certain stage we have $e = \bigvee \mathcal{D}$, for a finite nonempty family $\mathcal{D}$ of T_0-quasi-metrics d on X of the form $d = D_{(e,\leq_d)}$ such that the intersection of the specialization orders $\leq_d$ is equal to the specialization order of e.

The hypothesis is satisfied at the start of the induction, i.e. for the base case.

If the specialization order of some $d \in \mathcal{D}$ is not yet linear, then there are $a, b \in X$ such that a, b are incomparable with respect to $\leq_d$. Then set $d = d_{ab} \vee d_{ba}$ with the use of Lemma 4.2.2. Since $d \leq e$, we have $d_{ab} \leq D_{(e, \leq_{d_{ab}})} \leq D_{(e, \leq_d)} = d \leq e$. Moreover note that d_{ab} and $D_{(e, \leq_{d_{ab}})}$ have the same specialization order. In particular note that the equality $d = d_{ab} \vee d_{ba}$ implies that

$$\leq_d = \leq_{d_{ab}} \cap \leq_{d_{ba}} .$$

So we can replace $d \in \mathcal{D}$ by both $D_{(e, \leq_{d_{ab}})}$ and $D_{(e, \leq_{d_{ba}})}$. Clearly in this way the representation of $e = \bigvee \mathcal{D}$ remains true.

Furthermore, since for instance $D_{\left(e, \leq_{d_{ba}}\right)} = D_{\left(e, \leq_{D_{\left(e, \leq_{d_{ba}}\right)}}\right)}$, we see that the two newly introduced T_0-quasi-metrics in $\mathcal{D}$ are of the correct form, so that the induction hypothesis (H) remains satisfied.

Since X is finite, the procedure will stop after finitely many steps, as soon as all $d \in \mathcal{D}$ used in the representation of e have a linear specialization order on X. Hence the assertion is established. □

In general, the T_0-quasi-metrics d used in the previous representation of e will not be e^s-splitting. Of course, starting with a given finite metric space (X, m), we may want to stop splitting an obtained m-splitting T_0-quasi-metric d as soon as no pair of $\leq_d$-incomparable elements a, b in X can be found such that both d_{ab} and d_{ba} are m-splitting (compare Examples 4.2.2 and 4.4.1).

Example 4.4.1. Define a metric m on the four element set $X = \{a, b, c, d\}$ as follows: (We assume that $a < b < c < d$ is the given linear order on the index set.) As before, $m_{ij} = m(i, j)$ where

$$\mathbf{M} = \begin{pmatrix} 0 & 1 & 2 & 2 \\ 1 & 0 & 1 & 3 \\ 2 & 1 & 0 & 2 \\ 2 & 3 & 2 & 0 \end{pmatrix}.$$

One readily checks that the triangle inequality for m is satisfied, since the case $3 \leq 1 + 1$ cannot occur.

The distance matrix of the T_0-quasi-metric m_{bc} is as follows

$$\mathbf{D} = \begin{pmatrix} 0 & 1 & 1 & 2 \\ 1 & 0 & 0 & 2 \\ 2 & 1 & 0 & 2 \\ 2 & 3 & 2 & 0 \end{pmatrix}.$$

The obtained T_0-quasi-metric m_{bc} is m-splitting, as expected, see Example 4.2.2.

Case 1: We compute the distance matrix for the T_0-quasi-metric $(m_{bc})_{ab}$ as follows:

$$\mathbf{L} = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 1 & 0 & 0 & 2 \\ 2 & 1 & 0 & 2 \\ 2 & 2 & 2 & 0 \end{pmatrix}$$

which means that the T_0-quasi-metric $(m_{bc})_{ab}$ is not m-splitting.

Case 2: Similarly, we compute the distance matrix for the T_0-quasi-metric $(m_{bc})_{ba}$ as follows:

$$\mathbf{P} = \begin{pmatrix} 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 2 \\ 1 & 1 & 0 & 2 \\ 2 & 3 & 2 & 0 \end{pmatrix}$$

which shows that the T_0-quasi-metric $(m_{bc})_{ba}$ is not m-splitting.

As expected in light of Lemma 4.2.2, we have that $m_{bc} = (m_{bc})_{ab} \vee (m_{bc})_{ba}$.

Let v be the T_0-quasi-metric $\leq m_{bc}$ given by the following matrix:

$$\mathbf{V} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 3 & 2 & 0 \end{pmatrix}.$$

One indeed checks that v satisfies the triangle inequality and is m-splitting.

We see that v is minimally m-splitting, since neither $v(a,b)$ nor $v(b,a)$ can be made smaller without losing m-splittability, since we have $3 = v(d,b) \leq v(d,a) + v(a,b) = 2 + v(a,b)$ and $2 = v(c,a) \leq v(c,b) + v(b,a) = 1 + v(b,a)$.

4.5 The quasi-uniformity $\mathcal{U}_{ab}$

In this short section, we now discuss the connection between the map d_{ab}[6] in theory of produced ordered metric spaces and the theory of ordered metric spaces determined by a quasi-uniformity. Note that the result presented in Lemma 4.2.2 is related to the construction of the quasi-uniformity $\mathcal{U}_{ab}$ which we describe below in Proposition 4.5.1.

[6]The map $d_{ab,\ell}$ for $\ell = 0$

Proposition 4.5.1. *Let* $(X,\mathcal{U})$ *be a* T_0*-quasi-uniform space endowed with the partial order* $\leq = \bigcap \mathcal{U}$[7]*. Suppose that* $a, b \in X$ *be such that* $(b,a) \notin \bigcap \mathcal{U}$ *and set* $U_{ab} = U \cup [U^{-1}(a) \times U(b)]$ *whenever* $U \in \mathcal{U}$*. Then* $\mathcal{S} := \{U_{ab} : U \in \mathcal{U}\}$ *is a base for the* T_0*-quasi-uniformity* $\mathcal{U}_{ab}$ *on* X *such that*

$$\mathcal{U}^s = (\mathcal{U}_{ab})^s \text{ and } \left(\bigcap \mathcal{U}\right) \cup (\downarrow a \times \uparrow b) = \bigcap \mathcal{U}_{ab}.$$ [8]

Proof. It is straightforward to check that $\mathcal{S}$ is a base of a quasi-uniformity on X. By assumption there is $U \in \mathcal{U}$ such that $(b,a) \notin U^3$. It is then readily verified that $U_{ab} \cap (U_{ab})^{-1} = U \cap U^{-1}$. Hence $(\mathcal{U}_{ab})^s = \mathcal{U}^s$. Furthermore we obtain that $\bigcap_{U \in \mathcal{U}} [U^{-1}(a) \times U(b)] \cap \bigcap \mathcal{U} = \emptyset$ and $\bigcap \mathcal{U}_{ab} = (\bigcap \mathcal{U}) \cup (\downarrow a \times \uparrow b)$ which is a partial order. □

Corollary 4.5.1. *Let* $(X, \tau, \leq)$ *be a completely regularly* T_2*-ordered space such that* $a, b \in X$ *are incomparable with respect to the partial order* $\leq$ *. Then the space* $(X, \tau, \leq \cup [\downarrow a \times \uparrow b])$ *is also completely regularly* T_2*-ordered.*

Proof. Suppose that the quasi-uniformity $\mathcal{U}$ determines the topological ordered space $(X, \tau, \leq)$. The topological ordered space $(X, \tau, \leq \cup [\downarrow a \times \uparrow b])$ is obviously determined by the quasi-uniformity $\mathcal{U}_{ab}$ introduced in Proposition 4.5.1. □

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[7] A quasi-uniformity is a T_0-*quasi-uniformity* if $\bigcap \mathcal{U}$ is a partial order.

[8] Note that $\bigcap \mathcal{U} = \bigcap \mathcal{U}_{ab}$ if $a \leq b$.

Chapter 5

Algebraic conditions

It appears that the function $D_{(m,\leq)}$ (Cf Definition 2.4.2) is crucial in our study, but is not easy to manipulate. In practice, the computation of the function $D_{(m,\leq)}$ for an arbitrary partially ordered metric space $(X, m, \leq)$ is not trivial. So we use characterisations which simplify it. The present chapter gives specific calculations of $D_{(m,\leq)}$ in some special cases, namely when the partially ordered metric space $(X, m, \leq)$ is a join (resp. meet) semi-lattice, a lattice and finally a group. The results formulated both for the lattice case and the group case are very close but are structurally distinct, as our examples will demonstrate.

5.1 The lattice $(X, m, \leq)$

This section elaborates on the computation of the function $D_{(m,\leq)}$ for an arbitrary partially ordered metric space $(X, m, \leq)$ when the partial order $\leq$ provides a structure of (semi-)lattice. We give the result for the meet semi-lattice and infer the result for the join semi-lattice (by duality). In both cases, we just arrive at more refined characterizations of $D_{(m,\leq)}$. Finally, to provide a tentative solution to the problem, we consider a lattice plus some additional conditions. We start with the following definitions.

Definition 5.1.1. Let $(X, m, \leq)$ be a partially ordered metric space. Then we shall call $(X, m, \leq)$:

i) An m**-meet semi-lattice** provided that $(X, \leq)$ is a meet semi-lattice such that

$$m(a \wedge x, a \wedge y) \leq m(x, y)$$

whenever $x, y, a \in X$.

ii) An m**-join semi-lattice** provided that $(X, \leq)$ is a join semi-lattice such that

$$m(a \vee x, a \vee y) \leq m(x, y)$$

whenever $x, y, a \in X$.

iii) An m**-lattice**[1] if it is both an *m-meet semi-lattice* and an *m-join semi-lattice*.

We then give the following characterisations.

Lemma 5.1.1. *Let $(X, m, \leq)$ be a metric space equipped with a lattice order $\leq$. Then X is an m-lattice if and only if for each $a, b, x, y \in X$ we have that $m(a \vee x, b \vee y) \leq m(a, b) + m(x, y)$ and $m(a \wedge x, b \wedge y) \leq m(a, b) + m(x, y)$.*

Proof. Suppose that $(X, m, \leq)$ is an m-lattice. Then for any $a, b, x, y \in X$ we have that $m(a \vee x, b \vee y) \leq m(a \vee x, b \vee x) + m(b \vee x, b \vee y) \leq m(a, b) + m(x, y)$ and

[1]We observe that a metric lattice in the sense of Birkhoff [4] satisfies the stronger condition that

$$m(a \wedge x, a \wedge y) + m(a \vee x, a \vee y) \leq m(x, y)$$

whenever $x, y, a \in X$.

analgously $m(a \wedge x, b \wedge y) \leq m(a \wedge x, b \wedge x) + m(b \wedge x, b \wedge y) \leq m(a,b) + m(x,y)$. Hence the two stated conditions are satisfied.

The converse follows by putting $a = b$ in the latter two inequalities and using that $m(a,b) = 0$. $\square$

Corollary 5.1.1. *Let $(X, m, \leq)$ be an m-lattice. Then the lattice operations $\vee$ and $\wedge$ are uniformly continuous on the product $(X \times X, m \times m)$. (Here the metric $m \times m$ is defined by $(m \times m)((x_1, x_2), (y_1, y_2)) = m(x_1, y_1) \vee m(x_2, y_2)$ whenever $(x_1, x_2), (y_1, y_2) \in X \times X$.*

Let $(X, m, \leq)$ be a partially ordered metric space. In the rest of the section, we shall denote by D_1 and D_2 respectively, the maps

$$D_1(x,y) = \inf\{m(x,y') : y' \in X \text{ and } y' \leq y\} \quad \text{for any } x, y \in X,$$

and

$$D_2(x,y) := \inf\{m(x,y') : y' \in X \text{ and } y' \geq y\} \quad \text{for any } x, y \in X.$$

Proposition 5.1.1. *Let $(X, m, \leq)$ be an m-*meet semi-lattice.

Then

$$D_{(m,\leq)} = D_1.$$

Proof. <u>Claim:</u> D_1 is a quasi-pseudometric on X. Moreover, $D_{(m,\leq)} \leq D_1 \leq m$.

- Since, by reflexivity $y \leq y$ for any $y \in X$, then $\boxed{D_1(x,y) \leq m(x,y)}$.
- $0 \leq D_1(x,x) \leq m(x,x) = 0$, hence $D_1(x,x) = 0$, for arbitrary $x \in X$.
- Consider $y' \in X$ such that $y' \leq y$. Then

$$D_{(m,\leq)}(x,y) \leq r(x,y') + r(y',y) \leq m(x,y') + 0 = m(x,y').$$

Thus $D_{(m,\leq)}(x,y) \leq D_1(x,y)$ and therefore $D_{(m,\leq)} \leq D_1$.

- We prove that D_1 satisfies the triangle inequality. Let $x, y, z \in X$. We show that $D_1(x,z) \leq D_1(x,y) + D_1(y,z)$.

  Let $\varepsilon > 0$, then there exits $y'_\varepsilon \in X$ with $y'_\varepsilon \leq y$ and $z'_\varepsilon \in X$ with $z'_\varepsilon \leq z$ such that

  $$m(x, y'_\varepsilon) \leq D_1(x,y) + \frac{\varepsilon}{2} \text{ and } m(y, z'_\varepsilon) \leq D_1(y,z) + \frac{\varepsilon}{2}.$$

  Since $y'_\varepsilon \wedge z'_\varepsilon \leq z'_\varepsilon \leq z$, then we have that $D_1(x,z) \leq m(x, y'_\varepsilon \wedge z'_\varepsilon)$.

  Furthermore since m is a metric, so

  $$\begin{aligned} D_1(x,z) \leq m(x, y'_\varepsilon \wedge z'_\varepsilon) &\leq m(x, y'_\varepsilon) + m(y'_\varepsilon, y'_\varepsilon \wedge z'_\varepsilon) \\ &\leq m(x, y'_\varepsilon) + m(y'_\varepsilon \wedge y, y'_\varepsilon \wedge z'_\varepsilon) \quad \text{since } y'_\varepsilon \wedge y = y'_\varepsilon \\ &\leq m(x, y'_\varepsilon) + m(y, z'_\varepsilon) \quad \text{by hypothesis on } m \\ &\leq D_1(x,y) + \frac{\varepsilon}{2} + D_1(y,z) + \frac{\varepsilon}{2}. \end{aligned}$$

  Hence, for any $\varepsilon > 0$ and $x, y, z \in X$,

  $$D_1(x,z) \leq D_1(x,y) + D_1(y,z) + \varepsilon,$$

  i.e.

  $$D_1(x,z) \leq D_1(x,y) + D_1(y,z).$$

Therefore, D_1 is a quasi-pseudometric on X such that $D_{(m,\leq)} \leq D_1 \leq m$. Observe also that if $x, y \in X$ with $x \leq y$, we have that $D_1(x,y) \leq m(x,x) = 0$. So in view of Remark 2.4.1, since $D_{m,\leq}$ is the largest quasi-pseudometric $\leq m$ on X with specialization preorder extending $\leq$, we conclude that

$$D_{(m,\leq)} = D_1.$$

□

We also have the following similar result, for which the proof follows from an analogous argument as Proposition 5.1.1.

Proposition 5.1.2. *Let* $(X, m, \leq)$ *be an* m-join semi-lattice

Then

$$D_{(m,\leq)} = (D_2)^{-1},$$

where $(D_2)^{-1}$ *denotes the conjugate of* D_2.

Example 5.1.1. In the sequel, we prove that indeed $D_1^s \neq m$ (in general) by just providing a counter-example.

We consider $X = \{0, 1, 2\}$ with the natural order $0 \leq 1 \leq 2$ (of course all chains are lattices) and we define the metric $m : X \times X \to [0, \infty)$ by

$$\mathbf{m} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}.$$

Then $(X, m, \leq)$ is a meet semi-lattice. It is easy to check the condition

$$m(a \wedge x, a \wedge y) \leq m(x, y) \qquad \text{for any } a, x, y \in X,$$

is satisfied and that $D_1(1, 2) \leq m(1, 0)$ and $D_1(2, 1) \leq m(2, 0)$ but we have $D_1^s(1, 2) = 1 < 2 = m(1, 2)$. This completes the proof.

The next two lemmas correct slightly a possible deficiency of the previous results in the sense that, with an additional condition, D_1 and D_2 will become T_0-quasi-metrics with the right specialization order.

Lemma 5.1.2. *Let* $(X, m, \leq)$ *be an* m-meet semi-lattice*.*

Then the map D_1

is a T_0*-quasi-metric on* X *such that* $\leq_{D_1} = \leq$ *if* $\downarrow x$ *is* $\tau(m)$*-closed for any* $x \in X$*.*

Proof. Indeed, as we showed earlier, for $x, y \in X$, we have $D_1(x, y) = 0$ if $x \leq y$, since $m(x, x) = 0$. Assume now that $D_1(x, y) = 0$, then there exists a sequence $(y'_n)_{n \geq 1}$ in X with $y'_n \leq y$ for each $n \geq 1$ and $m(x, y'_n) \longrightarrow 0$ in $(X, \tau(m))$. Hence $y'_n \longrightarrow x$ and $x \leq y$. In conclusion

$$D_1(x, y) = 0 \Longleftrightarrow x \leq y.$$

The T_0-condition is just a straightforward from the above equivalence. □

Similarly

Lemma 5.1.3. *Let* $(X, m, \leq)$ *be an* m-join semi-lattice*. Then the map* D_2 *is a* T_0*-quasi-metric on* X *such that* $\leq_{(D_2)^{-1}} = \leq$ *if* $\uparrow x$ *is* $\tau(m)$*-closed for any* $x \in X$*.*

Remark 5.1.1. Observe that if $(X, m, \leq)$ is an *m-meet semi-lattice* then $\uparrow x$ is $\tau(m)$-closed for any $x \in X$.

Indeed for any $x \in X$ and for any sequence $(a_n) \subseteq X$ such that $x \leq a_n$ and $m(y, a_n) \to 0$ for some $y \in X$, we have $m(y \wedge x, x) = m(y \wedge x, x \wedge a_n) \leq m(y, a_n)$. Since $m(y, a_n) \to 0$ we get $m(y \wedge x, x) = 0$, that is $y \wedge x = x$ and hence $y \in \uparrow x$.

In an analogous way, if $(X, m, \leq)$ is an *m-join semi-lattice* then $\downarrow x$ is $\tau(m)$-closed for any $x \in X$.

We then formulate the following existence result for a producing T_0-quasi-metric. We basically combine both results from Lemma 5.1.2 and Lemma 5.1.3 into a single sufficient condition for a partially ordered metric space to be produced.

Theorem 5.1.1. *Let $(X, m, \leq)$ be an m-*lattice*. We consider the maps D_1 and D_2 respectively defined by*

$$D_1(x, y) = \inf\{m(x, y') : y' \in X \text{ and } y' \leq y\}, \quad \text{for any } x, y \in X,$$

$$D_2(x, y) = \inf\{m(x, y') : y' \in X \text{ and } y' \geq y\}. \quad \text{for any } x, y \in X.$$

Then $D_1 = (D_2)^{-1}$ produces $(X, m, \leq)$ if the m-closed balls are order convex [2].

Proof. Let $c \in X$ and $\epsilon > 0$. By the previous Propositions 5.1.1 and 5.1.2 we have that $D_{(m,\leq)} = D_1$ and $(D_{(m,\leq)})^{-1} = D_2$.

Obviously $D_1 \vee D_2 \leq m$. It is therefore enough to show that $D_1 \vee D_2 = m$ in order to conclude that $D_{(m,\leq)}$ is m-splitting.

Indeed suppose that for $x, y \in X$ there is $\delta > 0$ such that

$$D_1(x, y) \vee D_2(x, y) + \delta < m(x, y).$$

Then by definition of D_1 and D_2 there are $y_1 \in X$ such that $y_1 \leq y$ and $y_2 \in X$ such that $y_2 \geq y$ with $d(x, y_1) < m(x, y) - \delta$ and $d(x, y_2) < m(x, y) - \delta$. Since the closed ball $C_m(x, m(x, y) - \delta)$ is order-convex and $y_1, y_2 \in C_m(x, m(x, y) - \delta)$, we have that $y \in C_m(x, m(x, y) - \delta)$ —a contradiction. We conclude that

$$m = \max\{D_1, D_2\} = \max\left\{D_{(m,\leq)}, (D_{(m,\leq)})^{-1}\right\}.$$

But the specialization order of $D_{(m,\leq)} = D_1 = (D_2)^{-1}$ is equal to $\leq$ by Lemmas 5.1.2 and 5.1.3. □

We give this interesting result, it connects the interval condition (Lemma 1.1.1) with the m-lattice condition (Theorem 5.1.1).

[2] See Definition 3.1.1 and Example 3.1.2.

Lemma 5.1.4. *A linearly ordered metric space $(X, m, \leq)$ is an m-lattice if and only if it satisfies the interval condition.*

Proof. Suppose that a linearly ordered metric space $(X, m, \leq)$ satisfies the interval condition. We are going to show that it is an m-lattice. Let $x, y, a \in X$. We consider the following three cases:

(1) a belongs to the interval spanned by x and y,

(2) a is below that interval,

(3) a is above that interval.

Indeed in the two cases (2) and (3) the m-lattice condition

$$\max\{m(a \vee x, a \vee y), m(a \wedge x, a \wedge y)\} \leq m(x, y)$$

requires that $m(a, a) \leq m(x, y)$ and $m(x, y) \leq m(x, y)$, while in case (1) it requires the interval condition to hold. Therefore $(X, m, \leq)$ is indeed an m-lattice.

On the other hand, given an m-lattice and $x, y, z \in X$ such that $x \leq y \leq z$ we have that $m(y \wedge x, y \wedge z) = m(x, y) \leq m(x, z)$ and $m(y \vee x, y \vee z) = m(y, z) \leq m(x, z)$. Therefore the interval condition is satisfied. $\square$

Example 5.1.2. Let m and $\leq$ be the usual metric and order on $\mathbb{R}$. Then $(\mathbb{R}, m, \leq)$ is a linearly ordered metric space that satisfies all conditions of Proposition 5.1.1. Note that it follows readily from this result that for any nonempty set I the set B_I of bounded real-valued functions on I equipped with the usual supremum metric s and the coordinatewise order is an m-lattice that satisfies all conditions of Proposition 5.1.1. In particular, for any $(a_i)_{i \in I}, (x_i)_{i \in I}, (y_i)_{i \in I} \in B_I$ we have that $\sup_{i \in I} |a_i \vee x_i - a_i \vee y_i| \leq \sup_{i \in I} |x_i - y_i|$ where the latter inequality also remains true if we replace $\vee$ by $\wedge$. This space will be discussed in some detail in Chapter 6.

Example 5.1.3. Example 3.1.4 shows that an m-lattice need not have order-convex closed metric balls. Here, we prove that this example is indeed an m-lattice. For any $(a_1, a_2), (x_1, x_2)$ and (y_1, y_2) in $\mathbb{R}^2$,

$$\sqrt{\sum_{i=1}^{2}(a_i \vee x_i - a_i \vee y_i)^2} \leq \sqrt{\sum_{i=1}^{2}(x_i - y_i)^2},$$

and

$$\sqrt{\sum_{i=1}^{2}(a_i \wedge x_i - a_i \wedge y_i)^2} \leq \sqrt{\sum_{i=1}^{2}(x_i - y_i)^2}.$$

Hence the m-lattice condition is satisfied.

Our next example shows that a metric equipped with a lattice order $\leq$ that is produced by a T_0-quasi-metric need not be an m-lattice. The example will make use of the following proposition.

Proposition 5.1.3. *Let $(X, m, \leq)$ be an m-lattice and $a, b \in X$. Then we have that $m(a \vee b, a) = m(b, a \wedge b)$ and $m(a \vee b, b) = m(a, a \wedge b)$.*

Proof. We see that

$$m(a \vee b, b) \geq m(a \wedge (a \vee b), a \wedge b) = m(a, a \wedge b) \geq m(b \vee a, b \vee (a \wedge b)) = m(b \vee a, b).$$

Therefore $m(a \vee b, b) = m(a, a \wedge b)$. The first statement follows by interchanging a and b and the commutativity of the lattice operations $\vee$ and $\wedge$. $\square$

Example 5.1.4. Let $X = \{1, 2, 3, 4\}$ and define the T_0-quasi-metric t on X by

$$\mathbf{T} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 3 & 2 & 0 & 0 \\ 3 & 2 & 1 & 0 \end{pmatrix},$$

where $T_{ij} := t(i,j)$ whenever $i \in X$. We endow X with the order $\leq$ given as $1 \leq 2,\ 1 \leq 3,\ 1 \leq 4,\ 2 \leq 4,\ 3 \leq 4$ and $x \leq x$ whenever $x \in X$. One easily checks that the lattice order $\leq$ is the specialization order of t but the ordered metric space $(X, t^s, \leq)$ is not an m-lattice by Proposition 5.1.3, since

$$2 = t^s(2,4) \neq t^s(1,3) = 3.$$

5.2 The group $(X, m, \leq, *)$

This section elaborates on the computation of the function $D_{(m,\leq)}$ for an arbitrary partially ordered metric space $(X, m, \leq)$ when the space X is moreover endowed with a group structure satisfying some conditions. Like in the lattice case, we give the results for two kinds of 'half conditions' (using duality). In both cases, we arrive at more refined characterizations of $D_{(m,\leq)}$ and to provide a solution to the problem, we finally consider more elaborated conditions.

Let Let $(X, m, \leq, *)$ be an ordered metric space equipped with a group operation $*$. In the rest of the section, we shall denote by G_1 and G_2 respectively, the maps

$$G_1(a,b) = \inf\{m(a,b') : b' \in X \text{ and } b' \leq b\} \quad \text{whenever } a, b \in X,$$

and

$$G_2(a,b) = \inf\{m(a,b') : b' \in X \text{ and } b' \geq b\} \quad \text{whenever } a, b \in X.$$

Proposition 5.2.1. *Let $(X, m, \leq, *)$ be an ordered metric space equipped with a group operation $*$*[3] *such that the two following conditions are satisfied:*

$$(1) \qquad m(a,b) \leq m(xa, xb) \qquad \text{for any } a,b,x \in X,$$

$$(2) \qquad a \leq b \Longleftrightarrow ac \leq bc \quad \text{for any } a,b,c \in X,$$

whenever $a, b \in X$. Then

$$D_{(m,\leq)} = G_1.$$

Proof. The proof here follows an analogous argument as in the proof of Proposition 5.1.1. We shall just prove here that the map G_1 satisfies the triangle inequality (as it is the only novelty). Let $a, b, c \in X$ and let $\varepsilon > 0$. Then there exists $b'_\varepsilon \in X$ with $b'_\varepsilon \leq b$ such that

$$m(a, b'_\varepsilon) \leq G_1(a,b) + \frac{\varepsilon}{2}.$$

Similarly, there exists $c'_\varepsilon \in X$ with $c'_\varepsilon \leq c$ such that

$$m(b, c'_\varepsilon) \leq G_1(b,c) + \frac{\varepsilon}{2}.$$

Since $b'_\varepsilon \leq b$, we have that $b'_\varepsilon b^{-1} c'_\varepsilon \leq b b^{-1} c'_\varepsilon = c'_\varepsilon \leq c$. Hence

$$\begin{aligned} G_1(a,c) \leq m(a, b'_\varepsilon b^{-1} c'_\varepsilon) &\leq m(a, b'_\varepsilon) + m(b'_\varepsilon, b'_\varepsilon b^{-1} c'_\varepsilon) \\ &\leq m(a, b'_\varepsilon) + m(b, c'_\varepsilon) \quad \text{multiplying by } b(b'_\varepsilon)^{-1} \\ &\leq G_1(a,b) + G_1(b,c) + \varepsilon. \end{aligned}$$

Hence, for any $\varepsilon > 0$ and $a, b, c \in X$,

$$G_1(a,c) \leq G_1(a,b) + G_1(b,c) + \varepsilon,$$

i.e.

$$G_1(a,c) \leq G_1(a,b) + G_1(b,c).$$

□

[3] We write ab for $a * b$. We shall use this notation for simplification, even though the group operation will be denoted $*$.

Proposition 5.2.2. *Let $(X, m, \leq, *)$ be an ordered metric space equipped with a group operation $*$ such that the two following conditions are satisfied:*

$$(1) \qquad m(a,b) \leq m(xa, xb) \qquad \text{for any } a, b, x \in X,$$
$$(2) \qquad a \leq b \Longleftrightarrow ac \leq bc \quad \text{for any } a, b, c \in X.$$

Then

$$D_{(m,\leq)} = (G_2)^{-1}.$$

Like in the lattice case, we present here two variants of the above results which have the advantage of giving G_1 and G_2 as T_0-quasi-metrics with the right specialization order.

Lemma 5.2.1. *Let $(X, m, \leq, *)$ be an ordered metric space equipped with a group operation $*$ such that the two following conditions are satisfied:*

$$(1) \qquad m(a,b) \leq m(xa, xb) \qquad \text{for any } a, b, x \in X,$$
$$(2) \qquad a \leq b \Longleftrightarrow ac \leq bc \quad \text{for any } a, b, c \in X.$$

Then the map G_1 is a T_0-quasi-metric on X such that $\leq_{G_1} = \leq$ if $\downarrow x$ is $\tau(m)$-closed for any $x \in X$.

Similarly

Lemma 5.2.2. *Let $(X, m, \leq, *)$ be an ordered metric space equipped with a group operation $*$ such that the two following conditions are satisfied:*

$$(1) \qquad m(a,b) \leq m(xa, xb) \qquad \text{for any } a, b, x \in X,$$
$$(2) \qquad a \leq b \Longleftrightarrow ac \leq bc \quad \text{for any } a, b, c \in X.$$

Then the map G_2 is a T_0-quasi-metric on X such that $\leq_{(G_2)^{-1}} = \leq$ if $\uparrow x$ is $\tau(m)$-closed for any $x \in X$.

Our sufficient condition for producing then reads:

Theorem 5.2.1. *Let $(X, m, \leq, *)$ be an ordered metric space equipped with a group operation $*$ such that the two following conditions are satisfied:*

$$(1) \qquad m(a,b) \leq m(xa, xb) \qquad \text{for any } a,b,x \in X, \text{ and}$$

$$(2) \qquad a \leq b \Longleftrightarrow ac \leq bc \quad \text{for any } a,b,c \in X.$$

We consider the maps G_1 and G_2 respectively defined by

$$G_1(a,b) = \inf\{m(a,b') : b' \in X \text{ and } b' \leq b\}, \qquad \text{whenever } a,b \in X$$

$$G_2(a,b) = \inf\{m(a,b') : b' \in X \text{ and } b' \geq b\}, \qquad \text{whenever } a,b \in X.$$

Then $G_1 = (G_2)^{-1}$ produces $(X, m, \leq)$ if

i) the m-closed balls are order convex [4]*,*

ii) $\downarrow x$ and $\uparrow x$ are $\tau(m)$-closed whenever $x \in X$[5]*.*

Remark 5.2.1. All the results obtained in this section can be reformulated if we substitute the conditions (1) and (2) by

$$(3) \qquad m(a,b) \leq m(ax, bx) \qquad \text{for any } a,b,x \in X,$$

$$(4) \qquad a \leq b \Longleftrightarrow ca \leq cb \quad \text{for any } a,b,c \in X,$$

since it is nowhere required that the group $(X, *)$ be abelian.

Remark 5.2.2. Note that if $*$ is commutative, then left invariance of m under $*$ is equivalent to $m(a * x, b * y) \leq m(a,b) + m(x,y)$ whenever $a,b,x,y \in X$ which also implies that $*$ is uniformly continuous on the product $(X \times X, m \times m)$ (for proofs compare Lemma 5.1.1 and Corollary 5.1.1).

The additive group structure on $(\mathbb{R}, u^s, \leq)$ (Cf. Example -1.0.1) with its usual order $\leq$ satisfies the two stated invariance conditions under $*$. Similarly, the conditions are fulfilled for the ordered metric space $(B_X, s, \preceq)$ (Cf. Chapter 6) with the usual addition of real-valued functions as group operation.

[4] See Definition 3.1.1 and Example 3.1.2.

[5] Such spaces are called T_1-ordered.

We end this chapter with the following interesting result. According to [4] a group $(X, *)$ equipped with a partial order $\leq$ is called a *po-group* if for each $x, y \in X$ with $x \leq y$ we have that $a * x * b \leq a * y * b$ whenever $a, b \in X$. Obviously the latter condition is a two-sided version of our condition (2) introduced above and is equivalent to it for a commutative $*$.

Corollary 5.2.1. *Let $(X, *, \leq)$ be a po-group with identity element e and equipped with a left-invariant metric m with respect to $*$ (that is, $m(x, y) = m(a*x, a*y)$, whenever $x, y, a \in X$) . If $\uparrow e$ is $\tau(m)$-closed and for each $\varepsilon > 0$ the closed ball $C_m(e, \varepsilon)$ is order-convex, then the map G_1 as defined above is a left-invariant T_0-quasi-metric that produces $(X, m, \leq)$.*

Proof. Let us first prove that all closed balls of points of X are order-convex: Let $\varepsilon > 0$ and $x, a, b, c \in X$ be such that $a \leq b \leq c$ and $a, c \in C_m(x, \varepsilon)$. Then $m(a, x) \leq \varepsilon$ and $m(c, x) \leq \varepsilon$. Therefore

$$m(x^{-1}a, e) \leq \varepsilon,\ m(x^{-1}c, e) \leq \varepsilon,\ x^{-1}a \leq x^{-1}b \leq x^{-1}c,$$

and thus, by our assumption on order convexity at e, we have that $m(x^{-1}b, e) \leq \varepsilon$ and therefore $m(b, x) \leq \varepsilon$. Hence we are done.

We next prove the statement that $\uparrow e$ is $\tau(m)$-closed implies that $\uparrow x$ and $\downarrow x$ are $\tau(m)$-closed whenever $x \in X$. Let $x \in X$ and $a \in cl_{\tau(m)}(\uparrow x)$[6]. Then there is a sequence $(x_n)_{n \in \mathbb{N}}$ such that $m(a, x_n) \to 0$ and $x_n \geq x$ whenever $n \in \mathbb{N}$. Thus $x^{-1}x_n \geq e$ and $m(x^{-1}a, x^{-1}x_n) \to 0$. By our assumption it follows that $x^{-1}a \in \uparrow e$. Therefore $a \geq x$ and $a \in \uparrow e$ which shows that $\uparrow x$ is $\tau(m)$-closed whenever $x \in X$.

Fix now $x \in X$ and suppose that $a \in cl_{\tau(m)}(\downarrow x)$. Then there is a sequence $(b_n)_{n \in \mathbb{N}}$ in X such that $m(a, b_n) \to 0$, and $b_n \leq x$ whenever $n \in \mathbb{N}$. By left invariance of m under $*$ and condition (2) we have that $m((xb_n^{-1})a, x) \to 0$ and $a \leq x(b_n^{-1}a)$

[6] The closure of the upset with respect to the metric topology

whenever $n \in \mathbb{N}$. Since $\uparrow a$ is $\tau(m)$-closed, we conclude that $a \leq x$. Thus $\downarrow x$ is $\tau(m)$-closed.

Using the results of Proposition 5.2.1, it is enough to show that the T_0-quasi-metric $G = G_1$ is left-invariant:

Let $a, x, y \in X$, then

$$\begin{aligned} G_1(ax, ay) &= \inf\{m(ax, b') : b' \in X \text{ and } b' \leq ay\} \\ &= \inf\{m(x, a^{-1}b') : b' \in X \text{ and } a^{-1}b' \leq y\} \geq G_1(x, y). \end{aligned}$$

Therefore

$$G_1(x, y) = G_1(a^{-1}(ax), a^{-1}(ay)) \geq G_1(ax, ay) \geq G_1(x, y)$$

whenever $x, y \in X$.

Consequently G_1 is left-invariant on X. Since $G_1 = G_2^{-1}$ and the conjugate of a left-invariant quasi-pseudometric is left-invariant under $*$, we directly deduce that G_2 is left-invariant with respect to $*$, too. $\square$

The next example illustrates previous result.

Example 5.2.1. Consider the group B of all increasing bijections on the usually ordered real unit interval $[0, 1]$ with the composition $\circ$ of maps as group operation. The elements of B are partially ordered by the property that for $f, g \in B$ we set $f \leq g$ provided that $f(x) \leq g(x)$ whenever $x \in [0, 1]$. Then $(B, \circ, \leq)$ is a po-group. Furthermore a left-invariant metric m on B can be defined by setting $m(f, g) = \sup_{x \in [0,1]} |g^{-1}(x) - f^{-1}(x)|$ whenever $f, g \in B$. For the identity map id of the group we have that $\uparrow (id)$ is $\tau(m)$-closed: Indeed if $(g_n)_{n \in \mathbb{N}}$ is a sequence in B such that $x \leq g_n(x)$ whenever $x \in [0, 1]$ and $f \in B$ is such that $m(g_n, f) \to 0$, then $g_n^{-1}(x) \leq x, f^{-1}(x) \leq x$ and $x \leq f(x)$ whenever $x \in [0, 1]$. Furthermore $C_m(id, \varepsilon)$ is order-convex whenever $\varepsilon > 0$. The latter follows from the fact that closed intervals are order convex in $[0,1]$ and that by the po-property for $f, g \in B$ with $f \leq g$ we have that $g^{-1} = f^{-1} \circ f \circ g^{-1} \leq f^{-1} \circ g \circ g^{-1} = f^{-1}$.

Chapter 6

Embedding

In this small chapter, we aim, at characterizing partially ordered metric spaces that are produced by a T_0-quasi-metric.

As stated in the introduction, known results characterize ordered topological spaces as spaces which are order homeomorphic to subspaces of the ordered Hilbert cube. We give here a different type of embedding.

Let X be a nonempty set, and let us denote by B_X the set of all bounded real-valued $f : X \to \mathbb{R}$ functions defined on X, i.e.

$$f \in B_X \Longleftrightarrow \exists\, M(f) > 0 : \ |f(x)| < M(f) \ \text{ whenever } x \in X.$$

We endow B_X with the following T_0-quasi-metric that we call b:

$$\begin{aligned} b \ : \ B_X \times B_X \ &\to \ [0, \infty) \\ (f, g) \ &\mapsto \ b(f, g) \ = \ \max\left\{\sup_{x \in X}(f(x) - g(x)), 0\right\}. \end{aligned}$$

By symmetrizing the T_0-quasi-metric b, we naturally obtain on B_X the metric s

defined by

$$s(f,g) = \sup_{x \in X} |f(x) - g(x)| \quad \text{whenever } f, g \in B_X.$$

We can then write $b^s = s$.

Also we define on B_X the natural partial order $\preceq$ given by

$$f \preceq g \Longleftrightarrow f(x) \leq g(x) \text{ whenever } x \in X.$$

Note that if we denote by $\leq_b$ the specialization order of b, we have

$$f \preceq g \Longleftrightarrow f(x) \leq g(x) \text{ whenever } x \in X \Longleftrightarrow f \leq_b g,$$

hence

$$\leq_b = \preceq .$$

Remark 6.0.3. We observe that with an arbitrary nonempty set X, we can construct an ordering on B_X.

Let us now assume that (X, d) is a T_0-quasi-metric space.

Proposition 6.0.3. *Let (X, d) be a nonempty T_0-quasi-metric space. Then there exists an isometric embedding from (X, d) to (B_X, b).*

Proof. Let $a_0 \in X$ be a fixed element. Let

$$\begin{array}{rcccc} F & : & (X,d) & \to & (B_X, b) \\ & & a & \mapsto & F(a) =: f_a \end{array}$$

with

$$f_a(x) = d(a,x) - d(a_0,x) \text{ whenever } x \in X.$$

For any $a \in X$, it is clear that $f_a \in B_X$ since $\sup_{x \in X} |f_a(x)| \leq d^s(a, a_0)$.

For any $a, e \in X$, we have that $f_a(x) - f_e(x) = d(a, x) - d(e, x)$, evaluating the later expression at $x = e$, we can write:

$$\begin{aligned} d(a,e) &= f_a(e) - f_e(e) \\ &\leq \max\left\{\sup_{x\in X}(f_a(x) - f_e(x)), 0\right\} \\ &\leq \max\left\{\sup_{x\in X}(d(a,x) - d(e,x)), 0\right\} \\ &\leq d(a,e), \end{aligned}$$

i.e.

$$b(f_a, f_e) = d(a, e).$$

Moreover, we know that f is injective, since d is a T_0-quasi-metric.

We draw the conclusion that

$$F : (X, d) \hookrightarrow (B_X, b)$$

is an isometric injection. □

Observe that using F, we can define on the T_0-quasi-metric space (X, d) a partial order $\preccurlyeq_X$ by setting for $a, c \in X$,

$$a \preccurlyeq_X c \Longleftrightarrow d(a, x) \leq d(c, x) \text{ for each } x \in X \Longleftrightarrow f_a \leq_b f_c.$$

In other words, the partial order $\preccurlyeq_X$ on X is the restriction of $\leq_b$ on B_X via F.

Theorem 6.0.2. *A (nonempty) partially ordered metric space $(X, m, \leq_X)$ is produced by a T_0-quasi-metric if and only if it is isometric to a subspace of the space of bounded real-valued functions on the set X equipped with the supremum metric $s(f, g) = \sup_{x\in X}|f(x) - g(x)|$ whenever $f, g \in B_X$. and the partial order $\preceq$ given by $f \preceq g \Longleftrightarrow f(x) \leq g(x)$ whenever $x \in X$.*

Proof. Just observe that

$$\begin{aligned} x \leq_X y &\iff d(x,y) = 0 \\ &\iff b(f_x, f_y) = 0 \\ &\iff f_x \leq_b f_y \\ &\iff d(x,a) \leq d(y,a) \text{ for each } a \in X \\ &\iff x \preccurlyeq_X y. \end{aligned}$$

□

Remark 6.0.4. If the order of the ordered metric space $(X, m, \leq_X)$ is not produced, such an embedding does not exist, since any subspace of $(B_X, b^s, \leq_b)$ will carry an order that is produced by the restriction of F.

Chapter 7

Conclusion and open problems

In this last chapter, we draw the conclusions of our investigations and formulate some open problems found throughout the work that can constitute topics of further research.

The dissertation established some new results for the *quasi-pseudometrization of a partially ordered metric space*: namely, given a partially ordered metric space $(X, m, \leq)$, we found several conditions under which there exists a T_0-quasi-metric d for which the symmetrization d^s is m and the specialization order $\leq_d$ of d is $\leq$.

Below we give, in a first section, a summary of the work which we presented in each chapter of the dissertation and then suggest, in a second section important areas of future research which are related to the above stated problem.

7.1 Summary of the achieved work

In Chapter 0, we fixed the terminologies while presenting some preliminaries from the theory of quasi-pseudometric spaces, quasi-uniformities and orders that should ease the reading of this work. The notations were standard and the chapter was assumed to provide the reader with the basic notions needed for a good understanding of the dissertation.

In Chapter 1, we presented some results for the case that the partial order $\leq$ on the partially ordered metric space $(X, m, \leq)$ is total. The main result in this chapter was Proposition 1.1.2 which establishes that when $(X, m, \leq)$ is a linearly ordered metric space, the interval condition is necessary and sufficient for $(X, m, \leq)$ to be produced. The introduction of the idea of *pseudo-dimension* provided with a tool to tackle the Problem -1.0.1 by considering the so-called linear extensions. In that section, the principal result, formulated in Theorem 1.2.2 states that when the *pseudo-dimension* and the "classical" dimension are equal, then the partially ordered metric space $(X, m, \leq)$ is produced. A couple of examples were given to illustrate the results.

In Chapter 2, we developed a general framework for the solution of the suggested problem, namely the m-splitting theory. Our main results, Proposition 2.3.1 (which shows the existence of minimally m-splitting T_0-quasi-metrics) and Theorem 2.4.1 (which establishes the existence of maximally m-produced partial orders) made use of the concept of m-splitting quasi-metrics. We proved that the class of m-splitting T_0-quasi-metrics admits minimal elements and also that the original metric m can be written as the supremum of such minimally m-splitting T_0-quasi-metrics. Proposition 2.4.1 states that a maximally m-produced order is the specialization order of a minimally m-splitting T_0-quasi-metric.

In Chapter 3, we introduced the concept of order-convexity in solving our problem. We gave necessary conditions for the existence of a producing quasi-metric. The construction is very similar to the one already given by Nachbin [31]. The key results are Proposition 3.1.1 (which describes the order-convexity of the closed balls as a necessary condition for the partially ordered metric space $(X, m, \leq)$ to be produced), Proposition 3.2.6 (where we formulated the unique existence of a quasi-pseudometrizable quasi-uniformity with a countable base under compactness in the metric case) and Proposition 3.2.7 which is a variant of Proposition 3.2.6 under the second countability assumption.

In Chapter 4, we discussed the Problem -1.0.1 in the context of order theory. The introduction of the function $d_{ab,l}$ (Cf. Definition 4.2.1) leads to key implications among which the possibility to express an arbitrary metric m as the supremum of m-splitting T_0-quasi-metrics (See Corollary 4.3.1) and to give a characterization of maximally m-produced orders (see Corollary 4.3.2). The main result was Lemma 4.2.2 which states that for a given quasi-metric space (X, d), $d_{ab,\ell}$ is the largest T_0-quasi-metric $q \leq d$ on X such that $q(a, b) = \ell$.

In Chapter 5, we concentrated on some algebraic structures, that of a group and that of a lattice. We gave more or less detailed computations of $D_{(m,\leq)}$ initially defined in Chapter 2. More precisely, we gave explicit forms of $D_{(m,\leq)}$ respectively when we rewrote the conditions of existence in this case. The results were formulated in Theorems 5.1.1 (which give a detailed computation of $D_{(m,\leq)}$ when $(X, m, \leq)$ is a lattice) and 5.2.1 (which give a detailed computation of $D_{(m,\leq)}$ when $(X, m, \leq, *)$ is a group).

In the short Chapter 6, we finally characterized the partially ordered metric spaces that are produced by a T_0-quasi-metric with the help of an embedding theorem. The main result, in Theorem 6.0.2 described this characterization.

The next section outlines some interesting problems that can be considered for further research on the quasi-pseudometrization problem.

7.2 On amalgamation, extension and inverse problems

Problem 7.2.1. Let $(X, m, \leq)$ be a partially ordered metric space. Suppose that there exists a proper subset $E \subsetneq X$ such that $(E, m, \leq)$ is produced by a quasi-pseudometric d. For a given point $a \in X \setminus E$, we want to investigate the conditions under which $(E \cup \{a\}, m, \leq)$ is produced. This leads us to the following question: Let $(X, m, \leq)$ be a partially ordered metric space. Suppose that there exists a proper subset $E \subsetneq X$ such that $(E, m, \leq)$ is produced by a quasi-pseudometric d. For two given distinct points $a, b \in X \setminus E$, if the subspaces $(E \cup \{a\}, m, \leq)$ and $(E \cup \{b\}, m, \leq)$ are produced, what can we infer about $(E \cup \{a, b\}, m, \leq)$?

Problem 7.2.2. For many results in topology, the notion of *density* permits the generalization to the whole space of certain properties which are true only on a dense subset. Let $(X, m, \leq)$ be a partially ordered metric space. Suppose that there exists a proper subset $E \subsetneq X$ such that $(E, m, \leq)$ is produced by a quasi-pseudometric d. What topological or order property can we assume that E possesses in the aim to conclude that $(X, m, \leq)$ is produced? The type of property we are looking for is very closed to the density as a topological concept but also density as an order theoretic concept.

Problem 7.2.3. The reverse of Problem -1.0.2, namely if for a given partially ordered metric space $(X, m, \leq)$, there exists a unique T_0-quasi-metric d satisfying the conditions of Problem -1.0.1, can we prove that the order $\leq$ on the partially ordered metric space $(X, m, \leq)$ is linear?

Bibliography

[1] C. Agyingi, P. Haihambo, and H.-P. Künzi, *Tight extensions of T_0-quasi-metric spaces*, Ontos-Verlag: Logic, Computations, Hierarchies, (2014), pp. 9–22.

[2] H.-J. Bandelt and A. Dress, *A canonical decomposition theory for metrics on a finite set*, Advances in Mathematics, 92 (1992), pp. 47–105.

[3] N. P. Bathia and G. P. Szegö, *Dynamical systems: stability theory and applications*, Springer-Verlag, Berlin, 1967.

[4] G. Birkhoff, *Lattice theory*, American Mathematical Society, 25 (1967). AMS Colloquium Publications, Providence, Rhode Island.

[5] G. Bosi and G. Herden, *On a possible continuous analogue of the Szpilrajn theorem and its strengthening by Dushnik and Miller*, Order, 23 (2006), pp. 271–296.

[6] V. Chepoi and M. Seston, *Seriation in the presence of errors: A factor 16 approximation algorithm for l_∞-fitting Robinson structures to distances*, Algorithmica, 59 (2011), pp. 521–568.

[7] S. Cobzaş, *Functional analysis in asymmetric normed spaces*, Springer Science & Business Media, 2012.

[8] J. Conradie, *Asymmetric norms, cones and partial orders*, Topology and its Applications, 193 (2015), pp. 100–115.

[9] J. Conradie and M. Mabula, *Completeness, precompactness and compactness in finite-dimensional asymmetrically normed lattices*, Topology and its Applications, 160 (2013), pp. 2012–2024.

[10] B. Dushnik and E. W. Miller, *Partially ordered sets*, American Journal of Mathematics, 63 (1941), pp. 600–610.

[11] P. Fletcher and W. F. Lindgren, *Transitive quasi-uniformities*, Journal of Mathematical Analysis and Applications, 39 (1972), pp. 397–405.

[12] ——, *Quasi-uniform spaces; Lectures notes in pure and applied mathematics; v. 77*, Marcel Dekker, Inc., New York, 1982.

[13] Y. U. Gaba, *New results in the startpoint theory for quasipseudometric spaces*, Journal of Operators, 2014 (2014).

[14] ——, *An order theoretic approach in fixed point theory*, Mathematical Sciences, 8 (2014), pp. 87–93.

[15] ——, *Startpoints and-contractions in quasi-pseudometric spaces*, Journal of Mathematics, 2014 (2014).

[16] ——, *(C_α, C_β)-admissible functions in quasi-pseudometric type spaces*, Sohag Journal of Mathematics, 2 (2015), pp. 51–55.

[17] ——, *Advances in startpoint theory for quasi-pseudometric spaces*, Bulletin of the Allahabad Mathematical Society, (in press).

[18] G. Gierz, *Continuous lattices and domains*, no. 93, Cambridge University Press, 2003.

[19] I. James, *Topological and Uniform Spaces*, Springer Science & Business Media, 2012.

[20] E. F. Kazeem, C. A. Agyingi, and Y. U. Gaba, *On quasi-pseudometric type spaces*, Chinese Journal of Mathematics, 2014 (2014).

[21] J. C. Kelly, *Bitopological spaces*, Proceedings of the London Mathematical Society, 3 (1963), pp. 71–89.

[22] R. Kopperman, *All topologies come from generalized metrics*, American Mathematical Monthly, 95 (1988), pp. 89–97.

[23] ——, *Which topologies are quasimetrizable?*, Topology and its Applications, 52 (1993), pp. 99–107.

[24] ——, *Asymmetry and duality in topology*, Topology and its Applications, 66 (1995), pp. 1–39.

[25] M. Krein and A. Nudelman, *Markov's problem of moments and extremum problems*, 1973. In Russian, English translation: American Mathematical Society, Providence, R.I. 1977.

[26] H.-P. A. Künzi, *Nonsymmetric distances and their associated topologies: About the origins of basic ideas in the area of asymmetric topology*, in Handbook of the history of general topology, vol. 3, Springer, 2001, pp. 853–968.

[27] ——, *An introduction to quasi-uniform spaces*, Beyond Topology, 486 (2009), pp. 239–304.

[28] E. P. Lane, *Bitopological spaces and quasi-uniform spaces*, Proceedings of the London Mathematical Society, 3 (1967), pp. 241–256.

[29] E. Minguzzi, *Time functions as utilities*, Communications in Mathematical Physics, 298 (2010), pp. 855–868.

[30] ——, *Quasi-pseudo-metrization of topological preordered spaces*, Topology and its Applications, 159 (2012), pp. 2888–2898.

[31] L. Nachbin, *Topology and order*, Von Nostrand, Princeton, NJ, 1976.

[32] K. W. odarczyk and R. Plebaniak, *Asymmetric structures, discontinuous contractions and iterative approximation of fixed and periodic points*, Fixed Point Theory and Applications, 2013 (2013), pp. 1–15.

[33] O. O. Otafudu, *Quasi-metric tree in T_0-quasi-metric spaces*, Topology and its Applications, 160 (2013), pp. 1794–1801.

[34] ——, *A fixed point theorem in non-archimedean asymmetric normed linear spaces*, Fixed Point Theory, 16 (2015), pp. 175–184.

[35] C. W. Patty, *Bitopological spaces*, Duke Mathematical Journal, 34 (1967), pp. 387–391.

[36] F. Plastria, *Asymmetric distances, semidirected networks and majority in fermat–weber problems*, Annals of Operations Research, 167 (2009), pp. 121–155.

[37] B. S. W. Schröder, *Ordered sets: an Introduction*, Boston: Birkhäuser, 2003.

[38] E. Szpilrajn, *Sur l'extension de l'ordre partiel*, Fundamenta mathematicae, 1 (1930), pp. 386–389.

[39] P. Vitolo, *A representation theorem for quasi-metric spaces*, Topology and its Applications, 65 (1995), pp. 101–104.

[40] S. Willard, *General topology*, Courier Corporation, 2004.

[41] W. A. Wilson, *On quasi-metric spaces*, American Journal of Mathematics, (1931), pp. 675–684.

[42] C. Witzgall and A. J. Goldman, *Optimal location of a central facility: mathematical models and concepts*, US Department of Commerce, National Bureau of Standards, 1964.