# A risk-budgeting framework for the combination of factor equity portfolios

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# Declaration

I declare that this dissertation is my own, unaided work. It is being submitted for the Degree of Master of Philosophy in the University of the Cape Town. It has not been submitted before for any degree or examination in any other University.


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# Abstract

This dissertation examines a risk-budgeting approach to the construction of factor equity portfolios, proposed by de Carvalho *et al.* (2014). The approach begins with the construction of active-weighted portfolios with exposure to factors that historically have been linked to excess returns in the market. These factor portfolios are then combined using a risk-budgeting approach. Implied stock-level returns are then estimated using this combined active allocation, and a further optimisation allows for the incorporation of specific investor constraints. The framework constitutes a risk-based approach to portfolio construction in the sense that no direct estimation of expected stock returns is required, but is dependent on a robust estimation of the covariance structure of stock returns. The framework is first evaluated in the context of a simulation study. This section provided confirmation for the risk model estimation methodology used, as well as insight into the intricacies of the framework, in an environment where the underlying structure of data was known. The framework is useful for investors who wish to combine a set of active portfolios, by controlling the allocation of risk, and understanding the exposure of the final portfolio to each of the factor portfolio components. Based on the findings of the simulation study and a back-test of the framework on JSE data, it was found that at the risk-budgeting juncture, the level of prior information imposed (with regard to the performance of factor portfolios) has a significant impact on the performance of final portfolios. In addition, the application of investor constraints, such as long-only and absolute weight limits, ultimately hinder the investor's ability to retain the views taken on in the factor portfolio components. Furthermore, due to significant discrepancies in ex-ante and ex-post tracking error risk measurement, the use of alternative, or adjusted, risk measures is recommended.

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## Chapter 1

# Introduction

This research was undertaken from the point of view of a quantitative portfolio manager of equities. In the equities market, it has become increasingly apparent that there are opportunities for investors to consistently generate excess return by taking an active view on the market. Portfolios constructed based on certain stock characteristics have been shown to outperform market capitalisation weighted indices on average. The reported presence of rewarded factors in the market mean that active portfolio managers aim for the capture of excess returns relating to these factors - in as transparent and controlled a manner as possible. The increased availability of *smart beta* exchange-traded funds on the market, which offer purported exposure to these rewarded factors at a low cost, point to this investor demand. Risk-based approaches to equity investment have also gained a lot of traction, both in literature and industry. Papers such as those by de Carvalho *et al.* (2012), Amenc *et al.* (2014), Roncalli and Weisang (2015) and Homescu (2014) are examples of investigations into the nature and the success of these risk-based approaches. These approaches to portfolio construction tend to base asset-allocation decisions on the riskiness of components rather than on their return profiles. While there are empirical studies showing superior performance of these types of strategies, there is still no theory to confirm that risk-based approaches are guaranteed to outperform the market (Lee, 2011).

de Carvalho *et al.* (2014) propose an approach for the construction of portfolios in a constrained investor environment, which aims for the simultaneous capture of multiple factors shown to have historical success. It is a risk-based approach to portfolio construction in the sense that no direct estimation of expected stock returns is required, yet is dependent on the estimation of the covariance structure of returns. This proposed framework was implemented and evaluated in the context of a simulation study, followed by a back-test using data from the Johannesburg Stock Exchange (JSE).

The next section in this introductory chapter reviews relevant literature in order to contextualize this research. Chapter 2 provides an outline of the framework to be implemented. Thereafter, Chapter 3 contains the methodology and results of the simulation study, which was carried out in order to gain insight into the intricacies of the framework. Finally, Chapter 4 presents the back-test of the performance of the framework on data from the JSE, and Chapter 5 completes the paper with some concluding and summarising remarks.

## 1.1 Literature Review

### 1.1.1 Asset Pricing Theory in the context of equity portfolio management

In order to fully understand the idea of a risk-budgeting approach to portfolio construction, it is helpful to begin with some of the basics of Asset Pricing Theory.

The Capital Asset Pricing Model (CAPM) emerged in the 1960's from numerous independent authors on the back of the earlier work of Markowitz on modern portfolio theory (Markowitz, 1952). The CAPM states that the expected return of an individual asset can be linearly related to expected excess market returns through a beta estimate, which is a measure of the sensitivity of the asset's expected returns to the expected market returns. This basic model has been shown to be empirically flawed, yet still has its place due to its simplicity. Research into asset pricing models was extended by Ross (1976), among others. Arbitrage Pricing Theory (APT), introduced by Ross (1976), proposed that the expected return of an asset can be modelled as a linear function of a number of factors, where the exposure to factors is different for differently-priced assets.

In a practical sense, the CAPM can be considered to be a single-factor APT model, with the single factor in the CAPM being the value of a market index, and the exposure to this factor is represented by the beta estimate. The common theme in modern APT models is then that financial securities earn their premium through their exposure to certain factors. The question as to what selection of factors best explain individual asset returns does not have a clear answer. It is, however, increasingly clear that the CAPM is limited in its explanation of asset returns; certain factors are more consistently linked with returns in excess of market returns.

It is common to speak of stocks or portfolios that generate excess returns as having positive alpha. This stems largely from the Jensen's work in 1968 where *Jensen's Alpha* was used to measure the performance of mutual fund managers (Jensen, 1968). Situations when stocks or portfolios generate consistent alpha can be associated with violations of the underlying assumptions of the CAPM, as explained by Fama and French (2004), for example.

In portfolio management, investors can typically employ a passive or an active strategy. A passive strategy involves investing purely in a market-capitalisation-weighted index and tracking the performance of the market. In order to generate alpha an investor would need to take on an active investment view, by over- or under-weighting equities relative to a market index, which were expected to perform better or worse respectively. In what follows, we review some of the equity-related factors (also referred to in the literature as risk premia) that at stock- and portfolio-level have been shown to be linked with positive alpha.

### 1.1.2 Factors linked with excess returns

In the market for equities, there have been specific factors that have been shown to explain or generate systematic alpha. Examples of these factors, by numerous authors, are size, value, momentum and low-risk, to name a few. Fama and French (1993) developed the well-known "HML" and "SMB" factors which captured the value and size factors respectively. The value factor was represented by a portfolio which was long value stocks and short growth stocks, while the size factor was long stocks with smaller market-capitalisations and short larger market-cap stocks. Banz (1981) had also found that smaller cap stocks were expected to perform better. Stocks showing good performance in the past year have been shown to have continued good performance in the short-term, as shown by Novy-Marx (2012) and Fama and French (2012), for example. This is the momentum effect. Most empirical work on the existence of such factors is fairly outdated, and also not a lot has been done in a South African context. It is important to note that studies showing that these rewarded factors exist are seldom conclusive; results may often be market- or period-specific.

For the sake of this study it is important to note the research into the existence of these risk premia in the context of the JSE. Van Rensburg (2001), Strugnell *et al.* (2011) and Muller and Ward (2013) are examples of examinations on the subject in the South African market. Van Rensburg (2001) linked earnings/price ratios (i.e. value), market-cap (i.e. size) and returns over the past 12 months (i.e. momentum) to future expected returns. Strugnell *et al.* (2011) confirm the size and value effects, although tentatively observe a withering away of the size effect over time. Muller and Ward (2013) made a comprehensive study of the commonly targeted factors of size, value and momentum as well as a number of accounting-type performance factors of companies such as earnings yield and dividend yield measures. While many factors were shown to be linked with excess returns in isolation, the performance of portfolios which combined these factors was also investigated. For example, it was found that a portfolio of stocks that included a combination of momentum, high return-on-capital (ROC), cashflow-to-price and earnings yield (a way to distinguish between value and growth stocks) outperformed the market index by around 14 % per annum from a period from 1986 to 2011.

### 1.1.3 Implementing strategies to capture factors

Assume that an observable factor that has been said to help capture excess returns has been isolated. A question then arises as to how one would form portfolios that incorporate this factor. The "HML" and "SMB" portfolio construction methods by Fama and French (1993) were previously mentioned. Fundamental indices, which are readily available in developed markets, can also provide exposure to certain factors. Fundamental indices are portfolios that are weighted not on market-capitalisation but in proportion to stocks' economic fundamentals, such as book-to-market ratios; such an example would be aimed at capturing the value factor. Although investing in an index usually suggests a passive strategy, Blitz and Swinkels (2008) noted that these fundamental indices are in fact active strategies in disguise. This type of approach to investing, which aims to use alternative methods for index construction, falls under what is often termed a *smart beta* approach. Smart beta approaches aim to address the criticisms of market-capitalisation-weighted indices (which are only mean-variance efficient if the CAPM holds true). In this context, relevant criticisms of market-cap indices are that they are under-exposed to certain rewarded factors, and also poorly diversified due to being highly concentrated in a subset of large-cap stocks. Amenc *et al.* (2014) argue that the strategies employed in constructing the fundamental indices often gain the desired factor exposure but not in an optimal fashion. They propose a two-step process to constructing factor portfolios. Firstly, a subset of equities are chosen based on their exposure to a certain factor. Thereafter, weights are assigned using five different diversification strategies, all of which achieve a different objective. The final factor portfolio weights are an average of the weights obtained from the five strategies.

Other less intricate quantitative strategies exist for capturing performance relating to economic fundamentals and other factors. For example, strategies such as those employed by Muller and Ward (2013), involve actively ranking stocks according to factors, and constructing equal-weighted portfolios out of stocks from one end of the ranking scale, depending on whether high- or low-ranked stocks are expected to perform. There is also scope for long-short strategies, as described by Jacobs *et al.* (1999), for example. de Carvalho *et al.* (2014) employ a simple example of long-short factor portfolio construction in their paper. It is noted at this point that the focus of this research will be on *combining* factor portfolios in an optimal fashion - not necessarily in finding alpha-capture strategies that themselves are the most optimal.

### 1.1.4 Multiple-factor and risk-based strategies

It is a straightforward concept that if one had a number of strategies on hand, found to be successful in extracting excess returns, it might be beneficial to consider a simultaneous use of these strategies. However, a simple aggregation of a number of multiple strategies into one portfolio may in fact cancel out their individual positive effects, due to an interaction effect between strategies. For example, a strategy aiming to capture momentum premiums may not be simply combined with a strategy aiming to capture value premiums, because these strategies have been shown to be negatively correlated and can detract from one another's performance (Blitz, 2011). Hence, more proactive quantitative techniques of combining strategies have a place in portfolio management.

Studies on the implementation of such techniques are present in the literature. In an investigation into factor investing, Clarke *et al.* (2015) find that investing in a combination of factor portfolios can lead to worse results than those of more traditional securities portfolios. Important considerations for improving the efficiency of multi-factor investments include the correlation structure between factor portfolios, as well as the need to take care to change factor-weightings over time in accordance with investor views of predicted factor performance in the market.

The factor portfolio indices constructed as part of the research of Amenc *et al.* (2014) (as explained in 1.1.3) were further used as building blocks in a "multi-smart-beta" framework. The allocations to a group of factor-tilted portfolios were chosen such that each of the portfolios' contributions to tracking error risk (relative to a cap-weighted benchmark) were equal. This is an example of an equal-risk or risk-parity approach to risk allocation.

Roncalli and Weisang (2015) performed an investigation into the diversification of risk in a portfolio, by decomposing a portfolio's total risk into risk factor contributions. One of the applications of their methodology involves assigning a risk budget to the above-mentioned Fama and French equity factors. The view of Homescu (2014) is that the key to achieving consistent above-average performance in equity markets lies with effective management of risks. As such, a thorough examination was carried out as to what is required to construct a risk-robust portfolio that yields market-superior performance. The study was practical in nature and involved addressing the successes and pitfalls of a range of existing portfolio construction strategies, with particular focus given to factor investing, risk-parity and smart beta techniques.

de Carvalho *et al.* (2012) considered five risk-based portfolio construction techniques, which do not rely explicitly on stock return forecasts. It was shown that the risk and excess returns over a market-cap index of the portfolios formed from these techniques can be explained by a latent exposure to certain factors. It is suggested in this paper that research should be extended to exploring ways to be more directly exposed to factors and subsequently combining these strategies. Their research was indeed extended, and the framework they implemented in their subsequent paper will be used as a primary basis for this research.

## Chapter 2

# Outline of Framework

This section provides an outline of the portfolio-construction framework to be implemented (de Carvalho *et al.*, 2014). A short, high-level description is provided here before each step of the framework is presented in slightly more detail.

First, factor portfolios are constructed (or pre-specified), in stock-allocation form, and are essentially the building blocks for the final portfolio. These are then combined into an optimal unconstrained portfolio, where the factor portfolios' contributions to the total risk of the portfolio can be assigned based on their information ratios and correlations. This is the risk-budgeting facet of the framework. Thereafter, a risk model (an estimate of the covariance structure of stock returns) is used to imply excess returns, such that the unconstrained stock weights are mean-variance optimal. Intuitively, this is the reverse of the usual mean-variance optimisation process where a covariance matrix and a set of returns estimates are used to yield optimal stock weights. Finally, these implied stock returns, which should incorporate the views taken on in the factor portfolios to some extent, can then be used as part of a final optimisation where relevant constraints may be applied.

## 2.1 Factor Portfolios

A portfolio can be tilted towards factors that are expected to generate positive alpha or tilted away from those expected to generate negative alpha, or both. Factors that have been linked to excess returns were selected for use based on their performance in a South African context. Factors such as size, momentum, value, earnings yield and dividend yield were front-runners for consideration. It is necessary for the implementation of the framework that factor portfolios are represented in active weights (i.e. the deviation in weighting from a chosen market index). Weights were chosen via optimisations with chosen ex-ante tracking error targets for portfolios. Portfolios should be rebalanced at regular intervals (either monthly or quarterly), and at each re-balancing point the entire framework implemented.

## 2.2 Risk-budgeting allocation to strategies

The risk budget allocation is the point at which an investor's view on strategy-weighting should be incorporated. Risk-adjusted return (information ratios) forecasts for factor portfolios, as well as forecasts for the correlation structure between these portfolios are necessary for certain risk-budgeting approaches. It is noted that a historical approach to factor portfolio forecasts - used in this paper and also by de Carvalho *et al.* (2014) - can lead to inconsistent views of what the risk budget allocation should be.

The optimal risk budget, $RB$, can be given by the equation below, where $\eta$ is a risk-aversion metric that is adjusted in order to set the overall ex-ante tracking error risk. $\Theta$ is the estimated correlation matrix of factors, and $IR$ is the vector of information ratios:

$$RB = \frac{1}{\eta}\Theta^{-1}IR \tag{2.1}$$

This equation can be adjusted for chosen levels of prior information (i.e. when there is no knowledge of correlations and/or when there is no differentiation between information ratios between strategies). This will be discussed in more detail in later chapters.

## 2.3 Portfolio Construction

### 2.3.1 The aggregation of strategies into an optimal unconstrained target allocation

The vector of unconstrained active stock weights, $P_A$ can be built from a matrix containing the stock weights for each of the strategies, $P_S$, and a weighting vector $w$. The vector $w$ is determined by the ex-ante tracking errors, $\sigma_i$, of the factor portfolios and the corresponding risk budget vector, $RB$. Specifically, we have the following:

$$P_A = P_S \cdot w \tag{2.2}$$

where

$$w_i = \sigma_i^{-1} \cdot RB_i \tag{2.3}$$

### 2.3.2 The risk model

For the estimation of the variance-covariance structure of asset returns, a linear factor model is considered. Principal Component Analysis is used to set up the risk model. The variance-covariance matrix of factor returns, the choice of the number of factors, and the loadings on these factors can be obtained from the spectral decomposition of a sample time series of weekly stock returns. The methodology of Plerou *et al.* (2002) will be referenced here. The resulting risk model is represented by $\Sigma$, where $\Lambda$ is the variance-covariance matrix of factor returns (simply a diagonal matrix of eigenvalues), $\Phi$ is the exposure to factors and $\Delta$ are the stock-specific risks. Note that the factors here will not be representative of those chosen to create strategy portfolios, but are rather a set of orthogonalized factors with little or no economic interpretation. The model is as follows:

$$\Sigma = \Phi' \Lambda \Phi + \Delta \tag{2.4}$$

### 2.3.3 Implied active stock returns

The vector of implied stock returns, $R_I$, is defined as the set of stock returns that make the unconstrained active portfolio, $P_A$, mean-variance efficient. These returns incorporate the risk-budgeting allocation to each of the factor portfolios. $R_I$ can be estimated from $\Sigma$ and $P_A$ as follows:

$$R_I = \lambda \Sigma \cdot P_A \tag{2.5}$$

where $\lambda = IR_{P_A} \cdot \frac{T}{\sigma_{P_A}}$ and $IR_{P_A}$ and $\sigma_{P_A}$ refer to the information ratio and the ex-ante risk of the unconstrained portfolio respectively, and where $T = 12$, $52$ or $260$ depending on whether we are working with monthly, weekly or daily returns.

As is mentioned above, this set of implied returns is by definition the solution to the unconstrained optimisation, and is represented below:

$$\gamma^* = \frac{1}{\eta} \Sigma^{-1} R_I \tag{2.6}$$

for a specified level risk aversion $\eta$ and where $\gamma^* = (\frac{\lambda}{\eta}) P_A$ in the unconstrained case.

### 2.3.4 Handling Constraints

It is now possible to run a final mean-variance optimisation using the implied excess returns (which incorporate the views of the different strategies) and the estimated risk model, with desired constraints added.
With $k$ linear constraints, the optimisation can be expressed as follows:

$$\begin{aligned} &\gamma^* = \text{argmin} \quad \frac{\eta}{2}\gamma' \Sigma \gamma - \gamma' R_I \\ &\text{subject to} \\ &v_i' \gamma \geq u_i \qquad\qquad \forall 1 \leq i \leq k \end{aligned} \tag{2.7}$$

where $\gamma^*$ is the constrained solution which is in the form of active portfolio weights and $v_i$ and $u_i$ are coefficients and constants (respectively) that are needed to formulate constraints. All other variables are as defined before.
Using the fact that the unconstrained solution is $\gamma^* = (\frac{\lambda}{\eta})P_A$, it is also possible to express the optimisation as follows:

$$\begin{aligned}
&\gamma^* = \text{argmin} \quad (\gamma - \frac{\lambda}{\eta}P_A)'\Sigma(\gamma - \frac{\lambda}{\eta}P_A) \\
&\text{subject to} \\
&v_i'\gamma \geq u_i \qquad\qquad\qquad\qquad \forall 1 \leq i \leq k
\end{aligned} \tag{2.8}$$

This has the interpretation of minimizing the tracking error risk between the unconstrained portfolio of active weights relative to the constrained portfolio, at the same level of risk aversion.

This framework is meant to allow the user to move from a set of factor portfolios to a single portfolio of stocks that maintains the views taken in the factor portfolios. It also provides the facility to gauge and monitor the impact of constraints on the retention of these views. Initially, this can be done by observing the differences in active weights between the constrained and unconstrained portfolios, while it will also be shown to be possible to compare the factor exposures in the constrained solutions with the target exposures (the assigned risk budget). In the original implementation of this framework, de Carvalho *et al.* (2014) show that risk budgets are preserved to a reasonable extent, provided constraints are not too tight. It is also shown that when too little or too much ex-ante risk is taken on, there is a greater impact on the expected return of the portfolio (i.e. the discrepancy between the the expected performance of the unconstrained and constrained portfolios becomes larger). The aim of this research will be to set up, experiment with, and evaluate the usefulness of this framework.

# Chapter 3

# Simulation study to test framework

## 3.1 Outline

It is an obvious notion that conclusions made in many studies in the field of finance, particularly studies in asset-management, are dependent on the dataset used. A portfolio construction technique, such as that which is described above, may be tested over a certain period in a certain market. The study may yield promising results, and the portfolio construction technique deemed 'successful' based on certain performance measures. The fragility of such conclusions should be noted. The uncertainty is due to the fact that financial time series data holds little or no repetition in observations, and an asset-management strategy that is shown to have performed well over a certain period can very seldom, if ever, be guaranteed to work indefinitely. Well-reasoned data management and selection is therefore crucial to the proper evaluation of portfolio construction techniques.

On a related note, the simulation of stock price data is a common tool in mathematical finance. Data that emulates real-life data (to some extent) can be created; the benefit is that for real-life data an underlying statistical distribution is often unknown, whereas for simulated data one has the ability to control the underlying structure of the data. This characteristic of simulated data is used in what follows.

It is therefore proposed that before attempting to test the framework with JSE data, a rigid simulation study is carried out, which will allow some insights into the usefulness of the methodology explained above. For this, the simulation of stock price paths is required as well as the simulation of factors associated with these stocks, in order for factor portfolios to be constructed.

### 3.1.1 Simulating Stock Paths

Stock price paths will follow the dynamics of Geometric Brownian Motion (GBM). While GBM supplies useful characteristics such as non-negativity of stock prices and the fact that the size of returns are independent of the actual stock price, there are established drawbacks to modelling stock prices with GBM. Notably, the stylized facts of non-constant volatility in stock prices over time and stock-price jumps (Cont, 2001) are not accounted for by GBM dynamics. However, the greater simplicity and control over the correlation structure of returns afforded by this approach justify its use. A component of the drift will also be simulated with Arithmetic Brownian Motion (ABM) to emulate a random deviation from market movement or an 'alpha' for each stock. This can be expressed as follows:

$$d\vec{S}_t = D[\vec{S}_t]\big((\vec{\alpha}_t + \mu_0)dt + \sigma d\vec{W}_t^1\big) , \tag{3.1}$$

$$d\vec{\alpha}_t = \vec{a}dt + bd\vec{W}_t^2 . \tag{3.2}$$

In the above, $\vec{S}_t$ is a vector of stock prices at time $t$, $\mu_0$ is some benchmark drift and $\sigma$ is the volatility matrix of the process. $\vec{W}_t^1$ and $\vec{W}_t^2$ represent vectors of standard Brownian Motion processes (the sources of noise which drive the process). The vector $\vec{a}$ and the matrix $b$ represent the drifts and the volatility matrix of the $\alpha$ processes respectively. We can impose a correlation structure on $\vec{W}_t^1$ and $\vec{W}_t^2$ in order to control the structure of $\sigma$ and $b$. The notation $D[.]$ represents a diagonal matrix of the enclosed vector. With a simple application of Ito's Lemma, it is easy to see that the following is true in the one-dimensional case:

$$(\alpha_t)_i \sim N\Bigg(\alpha_0 + a_i t\ ,\ b_{ii}^2 t\Bigg) , \tag{3.3}$$

$$\ln(S_t)_i \sim N\Bigg(\ln(S_0)_i + \Big(\mu_0 + \alpha_0 + a_i t - \frac{1}{2}\sigma_{ii}^2\Big)t\ ,\ b_{ii}^2 t^3 + \sigma_{ii}^2 t\Bigg) . \tag{3.4}$$

Then extending to a multi-dimensional case and observing the distribution of log-returns as opposed to log-prices, the following is true:

$$\ln\left(\frac{\vec{S}_t}{\vec{S}_0}\right) \sim N\left(\left(\mu_0 + \alpha_0 + \begin{bmatrix} a_1 \\ . \\ a_n \end{bmatrix} t - \frac{1}{2}\begin{bmatrix} \sigma_{11}^2 \\ . \\ \sigma_{nn}^2 \end{bmatrix}\right)t\ ,\ b^2 t^3 \cdot Z_2 + \sigma^2 t \cdot Z_1\right) . \tag{3.5}$$

In the above expression for the covariance structure, $Z_1$ and $Z_2$ represent the covariance structures of the independent standard normal random variables which are used to form the processes $\vec{W}_t^1$ and $\vec{W}_t^2$. In the expression above, $Z_1$ and $Z_2$ are in actual fact identity matrices, while the covariance structure between returns is captured in $\sigma^2$ and $b^2$. The reason for expressing the covariance structure as above is to highlight how return correlations are simulated. In the equities market, the returns on different stocks are not independent of one another, and in the simulation process it is in $Z_1$ and $Z_2$ where a correlation structure between these returns can be incorporated. Through a Cholesky decomposition of a specified correlation matrix, we generate correlated random normal variables and transfer this structure to $\sigma^2$ and $b^2$. The fact that stock price returns are multi-variate normal in the simulated system is vital, as is it is simple to transform stock returns distributions to portfolio returns distributions when given a vector of portfolio weights (be it active weights or actual weights).

For example, consider the situation where we have stock returns, $\vec{R}$, and two portfolios with weights $\vec{w_1}$ and $\vec{w_2}$. Then if:

$$\vec{R} \sim N\left(\vec{\mu}\,,\,\Sigma\right)\;, \tag{3.6}$$

We have:

$$\vec{w_1}^t\vec{R} \sim N\left(\vec{w_1}^t\vec{\mu}\,,\,\vec{w_1}^t\Sigma\vec{w_1}\right)\;, \tag{3.7}$$

$$\vec{w_2}^t\vec{R} \sim N\left(\vec{w_2}^t\vec{\mu}\,,\,\vec{w_2}^t\Sigma\vec{w_2}\right)\;, \tag{3.8}$$

and

$$\mathbb{COV}[\vec{w_1}^t\vec{R}\,,\,\vec{w_2}^t\vec{R}] = \begin{bmatrix} \vec{w_1}^t\Sigma\vec{w_1} & \vec{w_1}^t\Sigma\vec{w_2} \\ \vec{w_2}^t\Sigma\vec{w_1} & \vec{w_2}^t\Sigma\vec{w_2} \end{bmatrix}\;. \tag{3.9}$$

### 3.1.2 The Risk Model

A crucial component of the framework is the risk model. As explained above, in reality it is necessary to estimate this risk model from weekly asset returns. This risk model is a representation of the covariance structure of these asset returns. In the simulation framework, we have a perfect understanding of the covariance of returns for any given length of time (as evident from equation 3.5). Hence, whenever the risk model is required in this simulation study, the theoretical covariance matrix of price-returns can be used.

Seeing that the theoretical covariance of returns structure is known, this simulation study also provides a useful framework for testing the methodology of setting up the risk model as used in Plerou *et al.* (2002). A slightly more detailed outline of the risk model estimation methodology is provided here, followed by an explanation of how this methodology was tested.

A principal components analysis based on two years of weekly stock returns is performed. Plerou *et al.* (2002) showed that the eigenvalues, $\lambda$, of a $T$ X $N$ matrix of returns (where $T$ is the number of observations and $N$ the number of stocks) have an asymptotic cap of $\lambda_{max} = \sigma^2(1+N/T+2(N/T)^{\frac{1}{2}})$, where $\sigma^2$ is the variance of the eigenvalues. Therefore, a natural discard point arises and only the factor exposures associated with the eigenvalues larger than $\lambda_{max}$ are considered.

In order to test the methodology, 1000 sets of two-years worth of stock price paths with weekly observations are simulated. All sets of stock price paths are simulated with an identical, but randomly-generated correlation structure. For each set of stock price paths, the risk model is estimated from the associated returns data. A Monte-Carlo type average of the 1000 estimated risk models is then compared to the theoretical covariance matrix. It was found that the difference of all entries in this matrix are of order 10e-06 or smaller. The same result was found for other randomly generated correlations structures, as well as some extreme cases (correlation structures designed to have high and low absolute values). This small difference between the average of the risk models estimated from simulated stock price returns and the theoretical covariance matrix, regardless of correlation structure, lends merit to the methodology used going forward.

### 3.1.3 Simulating Factor Portfolios

Simulating factors such as Size, Value, or Momentum for a set of stocks seems a daunting task. However, the framework above is designed to combine any such portfolios that are designed to capture above-market returns. In fact, it is designed to combine any portfolios with active views. Therefore, the strength of the framework depends on its ability to preserve risk budget allocation to these portfolios, while yielding returns that somewhat reflect the performance of these individual factor portfolios. The actual views taken on in these factor portfolios are not vital and it is not imperative that these views are expected to yield positive alpha. Therefore, for the sake of this simulation study, random factor portfolios are sufficient.

Two manners of constructing factor portfolios will be considered, both of which depend only so much on the value of factors as to how they are ranked in context of the other stocks. Therefore, we can simulate random rankings for the stocks (simply a random reordering of the original ordering of the stocks) and use these to construct the portfolios. Method (1) of constructing these factor portfolios is as implemented in an example implementation of the framework in de Carvalho *et al.* (2014). The upper 30% of stocks according to factor-ranking are constrained to have equal, positive active weights and the bottom 30% of stocks are constrained to have equal, negative weights, while the mid-ranked stocks are given zero active-weighting (i.e. the same weights as the stocks have in an index). Moreover, the absolute value of all non-zero weights are the same and are chosen such that the portfolios meet a chosen target ex-ante tracking error. Method (2) is a simplified version of a construction technique explained in an article by Amenc *et al.* (2014). Here, the lower 50% of stocks according to factor-ranking are given zero absolute weighting in the portfolio, while the upper 50% of stocks are long relative to the market index. In this second case, weights are chosen such that the ex-ante tracking error falls in a certain range.

Given that factor portfolios are specified in active-weight form, it is necessary at this point to mention what is used as an index in this simulation study. We assume a universe of stocks with arbitrary initial prices, where there is only a single stock of each type available on the market. A fictitious market-cap index is therefore easily calculable for each point in time.

In reality, a steady rebalancing of the portfolios would be required due to a change in the characteristics of stocks over time and hence a reordering of factors. With the random factor portfolios used in this simulation study, one could simply choose completely new random portfolios at each juncture, but this would not provide a good reflection of what might happen in reality - a full reshuffle of factor ordering for the stocks is not realistic. Therefore, at the outset fully random factors are simulated. Then at each rebalance point, a small random noise element is added to each of the stocks' respective factors. This enables a better reflection of a rebalancing process; the reordering of stocks is not extreme. For example, a stock that was ranked highest in terms of one of the random factors at a point of rebalancing will not jump to being ranked much lower at the next rebalancing point. In a real scenario, it may be that a stock, is ranked highest in terms of a momentum factor at one point. It is implausible that at the next rebalancing point its momentum ranking will be much different.

The first step in the framework that these simulated factor portfolios are used is as part of the risk budget allocation process, an explanation of which follows below.

### 3.1.4 Risk budget allocation

The risk-budgeting allocation was carried out with three different approaches: an equal risk budget, a maximum diversification approach and a mean-variance approach. The equal risk budget approach assumes no prior information, and therefore indirectly assumes that all factor portfolios are equally correlated with equal expected risk-adjusted returns. The maximum diversification approach involves taking a view on the expected correlation structure between the factor portfolios, while the mean-variance approach, in addition to this, requires a view on the portfolios' expected information ratios (the ratio of excess returns above the index to tracking error with the index). The latter two approaches therefore require an estimation period, whereas the first does not.

This estimation process is done by setting up factor portfolios, and rebalancing them monthly over a simulation period of two years. Using the results from subsection *3.1.1*, for each set of factor portfolios we can calculate a theoretical correlation matrix and a set of information ratios (using active portfolio weights, expected log-returns and the covariance matrix). The average of these correlation matrices serve as an estimate for $\Theta$, while the average of the information ratios over the two year period provide an estimate for $IR$. The parameters $\Theta$ and $IR$ are as represented in the risk-budgeting section of the framework.

### 3.1.5 Combined Portfolio Construction

#### Unconstrained portfolio

Given that factor portfolios have been set up and a risk budget has been allocated, the combined portfolio construction section follows as explained in the framework above. The ex-ante tracking errors of each factor portfolios are known at each rebalance point and are used in equations 2.3 and 2.2 to find the active weights of the unconstrained active portfolio. Note that a risk aversion parameter is chosen such that the ex-ante tracking error of this unconstrained portfolio meets a target level.

Thereafter, the risk model can be used to find the implied excess returns according to equation 2.5, which can be used as an input into a further optimisation which incorporates constraints.

#### Incorporating constraints

The final optimisations now use equations 2.7 and 2.8 to form combined portfolios. They ideally should retain the views taken in the individual factor portfolios as best as possible while maintaining the desired level of tracking error.

For this pilot simulation study, we observe three kinds of constrained portfolios:

(1) A long only portfolio that constrains absolute weights of individual stocks to 10% (labelled 'LOLim10');

(2) a portfolio with the same constraints as the first, but with an additional liquidity constraint that does not allow investment in stocks with the lowest

weight in the market-cap index (bottom 10% bracket) (labelled 'LOLimLiq');

(3) and a portfolio that constrains the absolute weights of the individual stocks to lie between -5% and 5% (labelled 'Lim55').

## 3.2 Simulation Study Results

For both the sets of results from the simulation study, the same set of simulated data was used. Table 3.1 shows the parameters used for simulating the stock price data. Parameters $a$, $b$, $\mu_0$ and $\sigma_{ii}$ were chosen to reasonably reflect stock price data, through observation of past prices.

**Tab. 3.1:** Parameters for simulation of stock prices and factors

| Parameter | Explanation | Value |
|---|---|---|
| $\alpha_0$ | initial value of $\alpha_t$ process | 0 |
| $a_i$ | drift of $\alpha_t$ process | $\sim U(-0.1, 0.3)$ |
| $b_{ii}$ | volatility of $\alpha_t$ process | $\sim U(0.05, 0.25)$ |
| $(\alpha_t)_i$ | random drift component of $S_t$ process, representing stock 'alpha' | $\sim N(a_i t, b_{ii}^2 t)$ |
| $\mu_0$ | constant drift component of $S_t$ process, representing benchmark drift | 0.15 |
| $\sigma_{ii}$ | diagonal of $\sigma$ - volatility of stock price process | $\sim U(0.2, 0.6)$ |
| $\rho_{ij}$ | off-diagonals of correlation structure | $\sim U(-1, 1)$ |
| $\sigma_{ij}$ | off-diagonals of $\sigma$ - covariances of stock price process | $\rho_{ij}\sigma_{ii}\sigma_{jj}$ |
| $S_{0_i}$ | initial stock prices | $\sim U(5, 55)$ |
| $F_0$ | initial random factors | $\sim N(0, 1)$ |
| $F_i$ | random factor noise at rebalance points | $\sim U(-0.05, 0.05)$ |

As mentioned previously, a risk model is estimated from numerous sets of two years of weekly returns, but the discrepancy between an average of these estimates and the theoretical covariance matrix is miniscule. Monthly stock prices and factors are then simulated so that factor portfolios can be rebalanced and the framework implemented at each rebalance point. A universe of 50 stocks is used.

### 3.2.1 Basic random factor portfolio results

The first result presented shows that the framework is being implemented correctly at each iteration, for the simplest case of an equal risk budget. Tables A.2 and A.3 in Appendix A show the weights of the optimized portfolios and the factor portfolios and their respective tracking errors. We display a table for each manner of constructing factor portfolios.

Tables A.2 and A.3 demonstrate in a very basic sense that the framework is being implemented correctly. The factor portfolios for method (1) are set up such that non-zero active weights have equal absolute value (with a long- or short-tilt), whereas those created using method (2) have either zero absolute weight, or positive active weights. The ex-ante tracking error targets are met for both methods - for method (1) the target is 5% and for method (2) the target region is 2.25-3.25% (note that tracking errors are annualized). The portfolio weights in the unconstrained portfolio reflect the equal-risk-weighted view of the factor portfolios, and attain the target tracking error of 5%. The constrained portfolio weights can be seen to adhere to constraints, and these constraints have an impact on the tracking errors. The more constrained the portfolio, the lower the tracking error. Logically, the constraints pose a trade-off between preserving the views of the factor portfolios and tracking the benchmark. Being the most constrained, the long-only portfolios with additional constraints on liquidity have the lowest tracking errors. The results are consistent for both the long-short method and the long-only factor portfolio construction methods. The impact of the various constraints can be seen visually in Figures 3.1 and 3.2. Many stock weights are seen to be tight on the lower and upper bounds that have been set, suggesting that constraints are impactful in this example. The portfolios that allow stocks to be shorted up to a -5% weighting are the closest fit to the unconstrained portfolios. Although the tracking errors of the combined portfolios are a good rudimentary measure for how well the views of the factor portfolios are preserved, the results in the following section provide further insight into this matter.

### 3.2.2 Extended random factor portfolio results

In this section, further analysis is made on the simulated data. The first two years of the simulated data is used as an estimation period for the correlation structure and the information ratios of the factor portfolios. These estimates are shown in Table 3.2.

The test period of the simulated data is then used to obtain realised returns of the factor portfolios and optimised portfolios for each rebalance period. A regression analysis is then used to extract information on how well the variation in the returns of the combined portfolios can be explained by the returns of the factor portfolios. If the realised excess returns of the factor portfolios are regressed against the excess returns of the combined portfolios, the coefficients of the multiple linear regression model provide estimations of the realised exposure of the

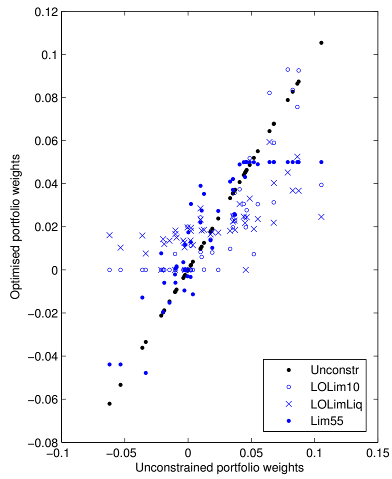


**Fig. 3.1:** Visual impact of constraints on portfolio weights - method (1)

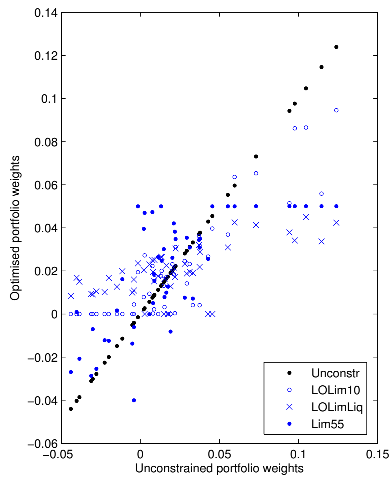


**Fig. 3.2:** Visual impact of constraints on portfolio weights - method (2)

**Tab. 3.2:** Estimations based on factor portfolios for risk-budgeting

| Method (1) | | | | |
|---|---|---|---|---|
| $IR$ | $\Theta$ | | | |
| 0.040 | 1.000 | 0.182 | 0.046 | 0.364 |
| 0.117 | 0.182 | 1.000 | 0.521 | 0.246 |
| -0.003 | 0.046 | 0.521 | 1.000 | 0.243 |
| 0.398 | 0.364 | 0.246 | 0.243 | 1.000 |
| Method (2) | | | | |
| $IR$ | $\Theta$ | | | |
| 0.267 | 1.000 | 0.367 | 0.527 | 0.548 |
| 0.487 | 0.376 | 1.000 | 0.539 | 0.541 |
| 0.613 | 0.527 | 0.539 | 1.000 | 0.427 |
| 0.845 | 0.548 | 0.541 | 0.427 | 1.000 |

combined portfolio to the factor portfolios. These can be compared with the risk budget allocation.

Table 3.3 and Table A.1 in Appendix A correspond to methods (1) and (2) for factor portfolio construction respectively and contain regression statistics for the 3 different risk-budgeting approaches. The first thing to note is that the unconstrained combined portfolios have $R^2$ statistics of very close to 1, which is expected given that they are expected to retain the views of the factor portfolios. The $R^2$ statistics for all regressions are fairly high (i.e. larger than 0.7), with the exception of those corresponding to the 'LOLimLiq' portfolios. This means that a large proportion of the variance in returns in most of these combined portfolios is explained by the variation in the returns of the factor portfolios. The low $R^2$ statistics of the 'LOLimLiq' portfolios resonate with the lower tracking errors found in the previous section. The average ex-ante tracking errors and realised information ratios are also displayed. The impact of the constraints is again evident here; the information ratios for the more constrained portfolios are lower, in general. Although factor portfolios are randomly constructed, and hence information ratios can not be expected to be positive, this trend of lower information ratios for the constrained portfolios is present, and investigated in the following section.

The realised exposures to the factor portfolio returns also yield information. The less constrained the portfolios, the closer these exposures are to the average risk budget over the period. The exposures are somewhat proportionate to the risk budget, even in the constrained cases. The weighting of the exposures is different, as expected, for the separate risk-budgeting approaches. The equal risk budget approach is clear in Tables 3.3 and A.1 - note that for method (2), the factor portfolios do not have exactly the same tracking errors and hence the exposure is slightly different for each portfolio despite the total contribution to risk being identical. The

**Tab. 3.3:** Regression analysis for different risk budget allocations - method (1) factor portfolios

| Equal Risk Budget | | | | | |
|---|---|---|---|---|---|
| | RB | Unconstr. | LOLim10 | LOLimLiq | Lim55 |
| | 0.000 | 0.000 | -0.008 | -0.006 | 0.007 |
| | 0.452 | 0.438 | 0.265 | 0.179 | 0.492 |
| | 0.452 | 0.462 | 0.456 | 0.197 | 0.344 |
| | 0.452 | 0.451 | 0.265 | 0.151 | 0.332 |
| | 0.452 | 0.466 | 0.261 | 0.119 | 0.325 |
| $R^2$ | - | 0.999 | 0.944 | 0.700 | 0.935 |
| Tracking Error | - | 5.00% | 3.93% | 2.35% | 4.37% |
| Information Ratio | - | 0.24 | 0.00 | -0.09 | 0.36 |
| Maximum Diversification approach | | | | | |
| | RB | Unconstr. | LOLim10 | LOLimLiq | Lim55 |
| | 0.000 | 0.000 | -0.009 | -0.006 | 0.007 |
| | 0.584 | 0.569 | 0.309 | 0.205 | 0.613 |
| | 0.344 | 0.355 | 0.390 | 0.161 | 0.260 |
| | 0.511 | 0.506 | 0.285 | 0.149 | 0.412 |
| | 0.391 | 0.405 | 0.285 | 0.126 | 0.274 |
| $R^2$ | - | 0.999 | 0.949 | 0.724 | 0.947 |
| Tracking Error | - | 5.00% | 3.80% | 2.27% | 4.40% |
| Information Ratio | - | 0.21 | -0.03 | -0.10 | 0.35 |
| Mean-variance approach | | | | | |
| | RB | Uconstr. | LOLim10 | LOLimLiq | Lim55 |
| | 0.000 | 0.000 | -0.008 | -0.006 | -0.005 |
| | -0.298 | -0.299 | -0.169 | -0.079 | -0.270 |
| | 0.248 | 0.251 | 0.112 | 0.067 | 0.225 |
| | -0.359 | -0.364 | -0.355 | -0.213 | -0.284 |
| | 0.972 | 0.966 | 0.461 | 0.223 | 0.786 |
| $R^2$ | - | 1.000 | 0.697 | 0.416 | 0.899 |
| Tracking Error | - | 5.00% | 3.62% | 2.01% | 4.10% |
| Information Ratio | - | 0.37 | 0.03 | -0.07 | 0.24 |

maximum diversification approach gives less exposure to the factor portfolios that are more correlated with the others. The mean-variance approach rewards low correlations, but also high information ratios; the negative weightings align with the negative information ratio estimates in Table 3.2.

Negative information ratio estimates would be a worry for an actual portfolio manager's factor portfolio, but given that these portfolios are randomly constructed, this is a coherent result. It in fact shows that if a manager 'disagrees' with a view taken in a portfolio, this can be incorporated in the framework. However, in the following chapter, when a back-test of the framework is made on actual JSE data, factor portfolios that are expected to generate alpha should not exhibit negative information ratios.

### 3.2.3 Non-random factor portfolios

The following section specifically examines the impact of constraints on portfolios that *are* expected to outperform the market. In the previous section of the simulation study, factor portfolios were set up that represented a random active view, and hence were not expected to attain excess return over and above the fictitious market index. However, given that we have control over the distribution of the returns of each asset, we can construct portfolios that are expected to outperform the market over a period - this is representative of factor portfolios that are known to generate *alpha* in reality.

Three factors are isolated that are directly related to the size of returns in this simulation study:

- The randomly generated drift vector, with components $a_i$, of the $\alpha_t$ process;
- the randomly generated volatilities of the stock price process, $\sigma_{ii}$;
- and the expected log-returns for each stock, which are distributed as expressed in equation 3.5.

Method (1) of factor portfolio construction is used to favour stocks with high $a_i$, low $\sigma_{ii}$, and high expected log returns in the three respective factor portfolios. The correlation structure and the information ratios of these factor portfolios for the simulation period are shown in Table 3.4. Note that all factor portfolios have high information ratios. The volatility and expected log-returns factor portfolios are highly correlated, as can be expected once again through observation of equation 3.5.

**Tab. 3.4:** Non-random factor portfolio risk-budgeting inputs

| Factor | $IR$ | $\Theta$ | | |
|---|---|---|---|---|
| Drift | 0.43 | 1.000 | 0.367 | 0.460 |
| Volatility | 1.47 | 0.367 | 1.000 | 0.944 |
| Expected log-returns | 1.52 | 0.460 | 0.944 | 1.000 |

The three factor portfolios are then used as components in the combination framework. To demonstrate the impact of constraints, the information ratios of three combined portfolios will be considered - an unconstrained portfolio, and two 'LOLim10'-style long-only portfolios. The first is set to have the same level of risk-aversion to the unconstrained portfolio and the second is set to have the same tracking error. This target tracking error for both the factor portfolios and the unconstrained portfolios is set to be equal. It was found that the level at which this target tracking error is set has little or no impact on the information ratios of the unconstrained portfolios. There is nonetheless an impact on the information ratios for the constrained portfolios. If the target tracking error is set too high or too low, this eats into the information ratio of the constrained portfolios. This is consistent with the findings of de Carvalho *et al.* (2014). As such, a target tracking error of 3% was chosen for the results displayed here. The three combined portfolios are constructed using each of the three different risk-budgeting approaches used in the previous section.

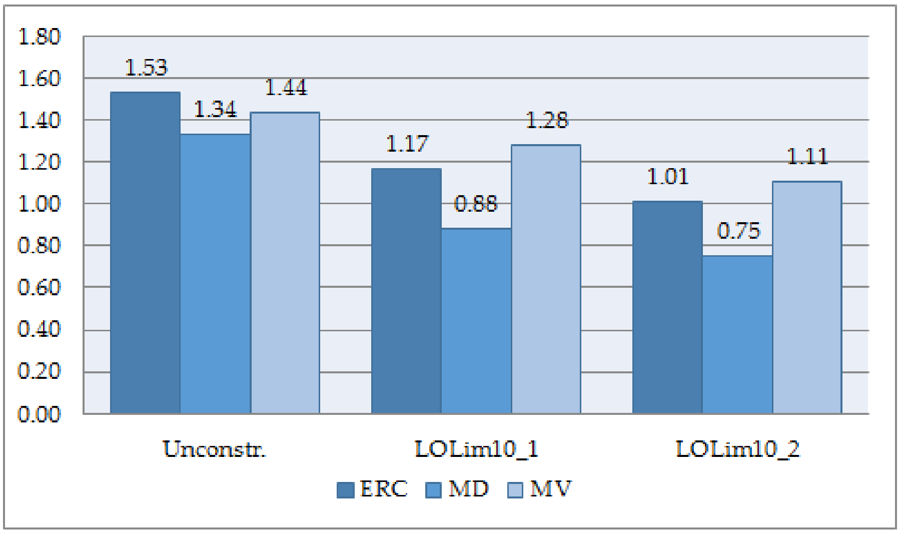


**Fig. 3.3:** Impact of constraints on information ratios

To directly observe the impact of constraints on information ratios, the correct comparison is between the unconstrained portfolios and the portfolios constrained to have the same tracking error (labelled 'LOLim10_2'). For all risk-budgeting approaches, a significant drop in the information ratios is apparent when constraints are applied. The smallest drop is evident in the case of the mean-variance risk-budgeting approach - factor portfolios with higher information ratios are given higher weighting, and hence this weighted combination yields greater excess returns. The lower level of performance for the portfolios constructed using the maximum-diversification approach can be explained by the fact that portfolios with the highest correlation with the other portfolios are down-weighted, and that in this case the down-weighted expected log-returns factor portfolio is (naturally) the best-performing component.

In this simulated example, where information ratios are unrealistically high, the information ratios of the constrained portfolios are still very healthy. However, when component portfolios are not known to have positive information ratios with surety, the drop in information ratios when constraints are applied could be very damaging.

## Chapter 4

# Back-testing of framework on JSE data

## 4.1 Data Description and Factor Portfolio Performance

Data pertaining to 293 equities on the JSE, dating from 2003-2015 was used for back-testing the framework in a South African context. The All-Share index (ALSI) was the chosen benchmark, relative to which portfolios were built. Six factors were considered for factor portfolio construction: value (for which price-to-book ratios were used); size (market-cap); momentum (average of monthly returns over the past 12 months); low-risk (2-year volatility of weekly asset returns); earnings yield (12-month earnings yield as percentage of stock price); and dividend yield (12-month dividend yield as percentage of stock price). Factor portfolios were set up according to method (1) (see section 3.1.3) and were rebalanced at the start of each month. Method (2) was considered, but the factor portfolios formed using this method yielded inferior performance compared to those constructed using method (1). It must be noted that at each rebalancing, only equities with non-zero weighting in the chosen benchmark index and for which factor information was available were considered. As a result, the factor portfolios typically consist of about 150 stocks.

In Table A.4 of A, the performance of these unconstrained factor portfolios are presented for the period from January 2005 - December 2014. The leverage of the portfolios was set such that their ex-ante tracking errors were 3%. This target risk level was chosen carefully in the knowledge that too high or too low a target level can be detrimental once constraints are applied. The information displayed in Table A.4 is also used in the risk-budgeting process. It is evident that the factor portfolios are positively and strongly correlated in general with all correlations falling in the range of 0.60-0.94. It is also evident that this correlation structure is similar whether measured across the entire 10-year period or either of the 5-year periods. The information ratios (measured as the ratio of average realised excess returns to ex-post tracking error) are less consistent across the two periods of estimation. Four of the six information ratios are positive measured across the entire 10-year period, while the value and size factors perform poorly. The poor performance of the size factor is in alignment with the findings of Strugnell *et al.* (2011), and this factor will therefore be discarded as a component of the combined portfolios. With the exception of the value factor, all factor portfolios perform better in the period from January 2010 - December 2014, when the average annualised monthly benchmark return was just 8.9% (annualised) as evident from Table 4.1. The benchmark performance during this period was markedly better in the five years prior to this. Note that the period in the lead-up to the global financial crisis of 2008-2009 was a period associated with high market returns and therefore strategies aiming to outperform the market would have been less likely to achieve that goal. A visual representation of the performance of the factor portfolios is provided in Figure 4.1.

**Tab. 4.1:** Benchmark Performance

| Period | Jan '05 - Dec '14 | Jan '05 - Dec '09 | Jan '10 - Dec '14 |
|---|---|---|---|
| Average Benchmark Return | 22.0% | 25.9% | 8.9% |

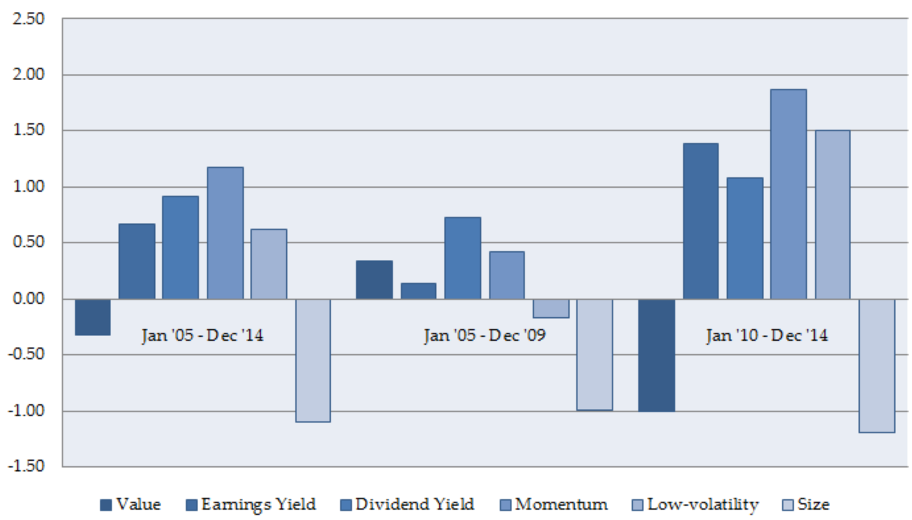


**Fig. 4.1:** Information Ratios of Factor Portfolios

## 4.2 Combined Portfolio Performance

This section assesses the performance of the portfolios formed by implementation of the framework, using factor portfolios described above as constituents. In the analysis of results, the following are reported: average return, ex-post volatility, average excess returns, ex-post tracking error risk, and realised information ratios. For a short review of the discrepancy between ex-ante and ex-post risk measures see Steiner (2013). Multiple regression analyses results, similar to those displayed previously, were used to assess risk budget adherence.

de Carvalho *et al.* (2014) present only in-sample results; the dataset used for the estimation of parameters required for risk budgeting is the same dataset used for performance-assessment of the framework. For this study, some in-sample results from the period January 2010 - December 2014 are presented in Tables 4.2 and 4.3. The information from Table A.4 pertaining to the period in question are used as inputs for the risk-budgeting process.

**Tab. 4.2:** Performance by Risk-budgeting Approach - In-sample

| Mean-Variance Approach | | | |
|---|---|---|---|
| | Unconstr. | LOLim_1 | LOLim_2 |
| Average Return (%) | 60.58 | 19.19 | 18.74 |
| Volatility (%) | 18.55 | 11.86 | 12.49 |
| Average Excess Return (%) | 45.89 | 4.50 | 4.04 |
| Tracking Error Risk (%) | 20.24 | 7.35 | 8.80 |
| Information Ratio | 2.27 | 0.61 | 0.46 |
| Maximum Diversification Approach | | | |
| | Unconstr. | LOLim_1 | LOLim_2 |
| Average Return (%) | 23.18 | 13.42 | 13.97 |
| Volatility (%) | 17.76 | 11.72 | 11.91 |
| Excess Return (%) | 8.49 | -1.28 | -0.72 |
| Tracking Error Risk (%) | 10.69 | 6.34 | 8.85 |
| Information Ratio | 0.79 | -0.20 | -0.08 |
| Equal-risk Approach | | | |
| | Unconstr. | LOLim_1 | LOLim_2 |
| Average Return (%) | 38.22 | 18.07 | 18.63 |
| Volatility (%) | 12.99 | 9.98 | 9.90 |
| Excess Return (%) | 23.53 | 3.38 | 3.93 |
| Tracking Error Risk (%) | 10.00 | 6.92 | 9.43 |
| Information Ratio | 2.35 | 0.49 | 0.42 |

Table 4.2 reveals that the performance of constrained portfolios is markedly lower than the respective unconstrained portfolios. Most drastically, in the mean-variance case, the information ratio drops from 2.27 for the unconstrained portfolio to 0.46 for the LOLim_2 portfolio. With the highest information ratios, the mean-variance approach performs the best for this period. This is expected given that risk-budget allocation was done with in-sample estimates. Another notable result is that the ex-post tracking error risk measures are markedly higher than the ex-ante

targets. This issue will be addressed later in this section. The poorer performance of the maximum diversification approach is explicable through observation of Table 4.3. A large positive weighting is budgeted to the value factor portfolio, despite its negative estimated information ratio for the period. This positive weighting to a poor-performing component detracts from the performance of the other factor portfolio components. Table 4.3 provides evidence of which factor exposures are most affected by constraints, and the likely reasons for decreased performance. For example, in the mean-variance risk-budgeting case, the large negative exposure to the value portfolio is not preserved, while the hefty positive exposure to the earnings yield portfolio is also not retained.

**Tab. 4.3:** Risk-budget adherence - In-sample

| Mean-Variance Approach | | | | |
|---|---|---|---|---|
| | RB | Unconstr. | LOLim_1 | LOLim_2 |
| Intercept | - | 0.00 | -0.06 | -0.09 |
| Value | -1.33 | -1.33 | *0.01* | *-0.07* |
| EY | 1.00 | 1.00 | *0.16* | *0.30* |
| DY | -0.20 | -0.20 | -0.17 | -0.18 |
| Momentum | 0.64 | 0.64 | 0.45 | 0.53 |
| Low-Volatility | -0.07 | -0.07 | -0.03 | -0.14 |
| R squared | - | 1.00 | 0.50 | 0.46 |
| Maximum Diversification Approach | | | | |
| | RB | Unconstr. | LOLim_1 | LOLim_2 |
| Intercept | - | -0.01 | -0.12 | -0.17 |
| Value | *0.98* | *1.05* | 0.36 | 0.40 |
| EY | -0.35 | -0.34 | -0.04 | 0.04 |
| DY | 1.04 | 0.86 | 0.34 | 0.46 |
| Momentum | 0.61 | 0.77 | 0.42 | 0.50 |
| Low-Volatility | -0.19 | -0.19 | 0.01 | 0.06 |
| R squared | - | 0.93 | 0.64 | 0.60 |
| Equal Risk Approach | | | | |
| | RB | Unconstr. | LOLim_1 | LOLim_2 |
| Intercept | - | -0.01 | -0.07 | -0.10 |
| Value | 0.40 | 0.36 | 0.19 | 0.22 |
| EY | 0.40 | 0.42 | *0.12* | *0.13* |
| DY | 0.40 | 0.46 | *0.10* | *0.14* |
| Momentum | 0.40 | 0.39 | 0.16 | 0.20 |
| Low-Volatility | 0.40 | 0.37 | 0.31 | 0.45 |
| R squared | - | 0.97 | 0.50 | 0.52 |

An out-of-sample application of the framework was also implemented from the period of January 2007 - December 2014. Risk-budgeting estimates for the current year were based on a two year-period prior to the year in question, and applied at each rebalancing point throughout the year. This estimation window was rolled forward at the end of each year. A set of results are displayed in Tables 4.4 and 4.5.

**Tab. 4.4:** Performance by Risk-budgeting Approach - Out-of-sample

| Mean-Variance Approach | | | |
|---|---|---|---|
| | Unconstr. | LOLim_1 | LOLim_2 |
| Average Return (%) | 17.26 | 11.62 | 9.82 |
| Volatility (%) | 21.83 | 19.50 | 20.86 |
| Excess Return (%) | 4.02 | -1.62 | -3.42 |
| Tracking Error Risk (%) | 11.44 | 8.87 | 10.71 |
| Information Ratio | 0.35 | -0.18 | -0.32 |
| Maximum Diversification Approach | | | |
| | Unconstr. | LOLim_1 | LOLim_2 |
| Average Return (%) | 27.18 | 14.72 | 15.04 |
| Volatility (%) | 15.34 | 12.16 | 12.82 |
| Excess Return (%) | 13.94 | 1.48 | 1.80 |
| Tracking Error Risk (%) | 11.14 | 8.11 | 12.06 |
| Information Ratio | 1.25 | 0.18 | 0.15 |
| Equal-risk Approach | | | |
| | Unconstr. | LOLim_1 | LOLim_2 |
| Average Return (%) | 31.56 | 15.71 | 16.07 |
| Volatility (%) | 14.35 | 11.75 | 11.79 |
| Excess Return (%) | 18.32 | 2.47 | 2.83 |
| Tracking Error Risk (%) | 13.26 | 9.28 | 11.92 |
| Information Ratio | 1.38 | 0.27 | 0.24 |

Contrary to the in-sample results, the mean-variance approach to risk-budgeting yields the poorest performing portfolios. This points to the danger of incorporating forecasts of expected returns in portfolio construction techniques - be it at a stock- or portfolio-level. Although still positive, the information ratios of the constrained maximum diversification and equal-risk portfolios are not strong. This weakness can be attributed to positive realised exposures to the value factor once again, as well as the non-exposure to the strong momentum factor portfolio in the equal-risk case.

Once more, ex-post measures of risk are excessively higher than the ex-ante targets. At first glance, these numbers are a significant worry. However, literature on the subject suggests that there is some explanation. The results of Satchell and Hwang (2001) suggest that realised measurements of tracking error are necessarily higher than planned measures due to the stochastic nature of portfolio weights. A study by Lawton-Browne (2001) made similar findings, citing volatility clustering and autocorrelation in returns (i.e. a version of the momentum effect) as possible explanations for the downward bias in ex-ante measures. A calculation that incor-

**Tab. 4.5:** Risk-budget adherence - Out-of-sample

| | | | | |
|---|---|---|---|---|
| Mean-Variance Approach | | | | |
| Intercept | - | 0.00 | -0.01 | -0.04 |
| Value | -0.03 | -0.09 | -0.22 | -0.27 |
| EY | 0.39 | 0.32 | 0.21 | 0.25 |
| DY | 0.59 | -0.11 | -0.18 | -0.14 |
| Momentum | 0.00 | 0.47 | 0.09 | 0.17 |
| Low-Volatility | -0.60 | -0.46 | -0.27 | -0.33 |
| R squared | - | 0.61 | 0.42 | 0.43 |
| Maximum Diversification Approach | | | | |
| | RB | Unconstr. | LOLim_1 | LOLim_2 |
| Intercept | - | -0.01 | -0.06 | -0.09 |
| Value | *0.78* | *0.81* | 0.21 | 0.28 |
| EY | -0.20 | -0.13 | 0.05 | 0.07 |
| DY | 0.89 | 0.82 | 0.26 | 0.37 |
| Momentum | 0.61 | 0.65 | 0.15 | 0.20 |
| Low-Volatility | -0.03 | -0.01 | 0.18 | 0.29 |
| R squared | - | 0.95 | 0.55 | 0.56 |
| Equal Risk Approach | | | | |
| | RB | Unconstr. | LOLim_1 | LOLim_2 |
| Intercept | - | -0.01 | -0.05 | -0.07 |
| Value | 0.39 | 0.34 | 0.09 | 0.10 |
| EY | 0.39 | 0.42 | 0.22 | 0.21 |
| DY | 0.39 | 0.44 | 0.12 | 0.18 |
| Momentum | 0.39 | 0.41 | *0.02* | *0.02* |
| Low-Volatility | 0.39 | 0.30 | 0.29 | 0.40 |
| R squared | - | 0.99 | 0.68 | 0.65 |

porates the aforementioned stochastic nature of portfolio weights is recommended. Scowcroft and Sefton (2001) too questioned the suitability of tracking error as a risk measure. In their study, ex-post tracking errors were often found to be in the region of double those that were predicted, while Lawton-Browne (2001) also suggested that the discrepancy could lie in this region. These studies were carried out in developed markets and realised tracking errors in emerging markets (such as the South African market) have been known to be greater than those in developed markets (Johnson *et al.*, 2013). Given this fact, the ex-post tracking error levels found in this study seem less infeasible than one might think. Furthermore, it is postulated that these realised tracking errors would be reduced given a larger sample of excess returns.

However, it is not good risk-management practice for the realised measures of risk to be so distanced from predicted outcomes. It would be worth considering a new - or at least adjusted - measure of active portfolio risk in the implementation of this portfolio construction framework.

# Chapter 5

# Conclusions

de Carvalho *et al.* (2014) proposed a risk-based approach for the construction of portfolios, which aims to capture multiple historically successful factors in the face of investor constraints. This framework was implemented and evaluated in the context of a simulation study, followed by a back-test using data from the JSE. Throughout, relative portfolio performance and risk measures are used (i.e. with reference to a market benchmark as opposed to in absolute terms).

As a first step, factor portfolios aiming to capture excess returns were set up by tilting portfolios toward (away from) stocks with favourable (unfavourable) characteristics. Although portfolio weights were constrained in the sense that portfolios are constrained to have a target level ex-ante tracking error, they were unconstrained in the sense that no upper or lower bound on absolute portfolio weights was enforced. Constraints were incorporated following the construction of a combined portfolio. It is noted that the framework is suitable for the combination of any group of portfolios that take an active view on the market. A combined portfolio is formed from the component portfolios using a risk-budgeting approach. The total tracking error risk of the combined portfolio is set, and allocations of this total risk are made to each of the factor portfolios - the investor can choose to take on varying levels of prior information about factor portfolio performance and the relationship between them at this risk-budgeting juncture. Constrained portfolios were then formed using the returns implied by the unconstrained combined allocation.

The simulation study provided useful insight into the operation of the framework, and provided intuition moving into the back-test of the framework on JSE data. Stock prices were simulated in such a way that the distribution of log-returns was multi-variate normal, with controllable correlation structures being imposed. Known theoretical covariance structures were compared to risk models estimated from sets of simulated stock prices. This provided confirmation for the risk model estimation methodology that was used in that the discrepancy between a Monte-Carlo type estimate of the estimated risk models and the theoretical covariance structures was negligible.

The simulation study also provided insight into the impact of constraints on the combined portfolios, when compared with the unconstrained portfolios. The more constrained the optimal combined portfolio, the lower the ex-ante tracking errors; as such it is more difficult to adhere to the risk views taken on the factor portfolios the more constraints the investor faces. It is also seen that the information ratios are lower for the more constrained portfolios. This was observed using random factor portfolios as well as factor portfolios based on simulation parameters that were known to generate excess returns. This suggests that in a highly constrained environment, where factor portfolio components are not known to generate positive excess return with surety, the above framework should be used with caution.

Both the simulation study and the back-test on real data revealed that the risk-budgeting approach has a marked effect on the performance of the final portfolios. The mean-variance approach requires a forecast on the information ratios of the factor portfolios - it was seen from out-of-sample implementations of the framework that inaccurate forecasts can be highly detrimental to portfolio performance. An equal-risk approach is the safest approach, which incorporates an implicit view of equal information ratios and correlations between factor portfolios. Multiple regression analyses provided useful insights into the realised exposures of the combined portfolios to the factor portfolio components; it is possible to see where exposure is lost in the face of constraints, and these lost exposures can often explain reductions in portfolio performance.

The ex-post measures of risk were found to be a lot larger than the ex-ante measures for budgeting risk. This finding is possibly explained by some literature on the subject, which cite this discrepancy as a common and explicable occurrence. It is however recommended that the use of alternative risk measures be investigated - or the downward bias of ex-ante tracking error measurements be reconciled. Such differences in risk realisations and risk forecasts are indeed not conducive to accurate portfolio or investor performance measurement.

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# Appendix A

# Appendix

## A.1 Simulation Study Results

**Tab. A.1:** Regression analysis for different risk-budget allocations - method (2) factor portfolios

| Equal Risk Budget | | | | | |
|---|---|---|---|---|---|
| | RB | Unconstr. | LOLim10 | LOLimLiq | Lim55 |
| | 0.000 | 0.000 | -0.001 | -0.004 | -0.024 |
| | 0.643 | 0.704 | 0.390 | -0.239 | 0.262 |
| | 0.750 | 0.733 | 0.492 | 0.383 | 0.588 |
| | 0.768 | 0.753 | 0.545 | 0.220 | 0.841 |
| | 0.687 | 0.675 | 0.614 | 0.546 | 0.868 |
| $R^2$ | - | 0.999 | 0.954 | 0.189 | 0.850 |
| Tracking Error | - | 5.00% | 3.64% | 2.51% | 4.48% |
| Information Ratio | - | 0.86 | 0.86 | 0.57 | 0.45 |
| Maximum Diversification approach | | | | | |
| | RB | Unconstr. | LOLim10 | LOLimLiq | Lim55 |
| | 0.000 | 0.000 | 0.000 | -0.028 | -0.022 |
| | 0.695 | 0.759 | 0.391 | -0.390 | 0.297 |
| | 0.805 | 0.786 | 0.518 | 0.279 | 0.644 |
| | 0.723 | 0.712 | 0.519 | 0.654 | 0.801 |
| | 0.617 | 0.603 | 0.577 | 1.351 | 0.799 |
| $R^2$ | - | 0.999 | 0.945 | 0.510 | 0.848 |
| Tracking Error | - | 5.00% | 3.63% | 2.52% | 4.48% |
| Information Ratio | - | 0.83 | 0.86 | 0.58 | 0.45 |
| Mean-variance approach | | | | | |
| | RB | Unconstr. | LOLim10 | LOLimLiq | Lim55 |
| | 0.000 | 0.000 | -0.018 | -0.014 | -0.018 |
| | -0.853 | -0.828 | -0.321 | -0.197 | -0.633 |
| | -0.275 | -0.211 | -0.107 | 0.072 | -0.389 |
| | 1.115 | 1.152 | 0.654 | 0.289 | 0.919 |
| | 1.786 | 1.704 | 0.963 | 0.565 | 1.558 |
| $R^2$ | - | 0.989 | 0.814 | 0.721 | 0.890 |
| Tracking Error | - | 5.00% | 3.22% | 1.93% | 4.09% |
| Information Ratio | - | 1.05 | 0.41 | 0.21 | 0.63 |

**Tab. A.2:** Table showing portfolios using factor portfolio method (1)

| | Absolute Weights | | | | | Active Weights | | | |
|---|---|---|---|---|---|---|---|---|---|
| Portfolios: | Benchmark | Unconstrained | LOLim10 | LOLimLiq | Lim55 | Rand1 | Rand2 | Rand3 | Rand4 |
| Stocks: | 1.35% | 4.08% | 3.74% | 3.06% | 4.90% | 4.46% | -4.26% | 0.00% | 5.58% |
| | 0.39% | -0.23% | 0.00% | 0.00% | 1.16% | 0.00% | 4.26% | 0.00% | -5.58% |
| | 2.21% | 0.20% | 0.00% | 1.36% | -0.33% | 0.00% | -4.26% | 0.00% | 0.00% |
| | ... | ... | ... | ... | ... | ... | ... | ... | ... |
| | ... | ... | ... | ... | ... | ... | ... | ... | ... |
| | 2.72% | 6.79% | 5.90% | 4.03% | 5.00% | 0.00% | 4.26% | 4.34% | 0.00% |
| | 1.13% | 7.88% | 9.30% | 4.52% | 5.00% | 4.46% | 4.26% | 0.00% | 5.58% |
| | 3.45% | -3.34% | 0.00% | 0.76% | -4.78% | -4.46% | 0.00% | -4.34% | -5.58% |
| Sum of Weights | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| Tracking error | - | 5.00% | 4.11% | 2.28% | 4.38% | 5.00% | 5.00% | 5.00% | 5.00% |

**Tab. A.3:** Table showing portfolios using factor portfolio method (2)

| | Absolute Weights | | | | | Active Weights | | | |
|---|---|---|---|---|---|---|---|---|---|
| Portfolios: | Benchmark | Unconstrained | LOLim10 | LOLimLiq | Lim55 | Rand1 | Rand2 | Rand3 | Rand4 |
| Stocks: | 1.35% | 3.12% | 2.32% | 1.96% | 3.09% | 2.22% | -1.35% | 1.51% | 0.17% |
| | 0.39% | 1.48% | 1.31% | 0.00% | 3.01% | 0.50% | 1.82% | -0.39% | -0.39% |
| | 2.21% | -0.41% | 0.00% | 1.09% | -4.00% | -2.21% | -2.21% | 2.72% | -2.21% |
| | ... | ... | ... | ... | ... | ... | ... | ... | ... |
| | ... | ... | ... | ... | ... | ... | ... | ... | ... |
| | 2.72% | 1.35% | 2.01% | 2.64% | 2.47% | -2.72% | 0.00% | 0.51% | 0.00% |
| | 1.13% | 5.96% | 6.36% | 4.25% | 5.00% | 4.17% | 0.00% | 2.63% | 0.00% |
| | 3.45% | -3.11% | 0.00% | 0.94% | -2.87% | -3.45% | 1.43% | -3.45% | -3.45% |
| Sum of Weights | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| Tracking Error | - | 5.00% | 3.71% | 2.09% | 4.47% | 3.08% | 2.69% | 2.54% | 2.67% |

## A.2 JSE Data Results

**Tab. A.4:** Estimations based on factor portfolios for risk-budgeting

| January 2005 - December 2014 | Average Benchmark Return: 22.0% | | | | | | |
|---|---|---|---|---|---|---|---|
| Factor | $IR$ | $\Theta$ | | | | | |
| Value | -0.32 | 1.00 | 0.90 | 0.63 | 0.84 | 0.85 | 0.91 |
| Earnings Yield | 0.66 | 0.90 | 1.00 | 0.74 | 0.87 | 0.90 | 0.89 |
| Dividend Yield | 0.91 | 0.63 | 0.74 | 1.00 | 0.67 | 0.74 | 0.67 |
| Momentum | 1.17 | 0.84 | 0.87 | 0.67 | 1.00 | 0.91 | 0.88 |
| Low-Volatility | 0.62 | 0.85 | 0.90 | 0.74 | 0.91 | 1.00 | 0.84 |
| Size | -1.10 | 0.91 | 0.89 | 0.67 | 0.88 | 0.84 | 1.00 |
| January 2005 - December 2009 | Average Benchmark Return: 25.9% | | | | | | |
| Factor | $IR$ | $\Theta$ | | | | | |
| Value | 0.34 | 1.00 | 0.92 | 0.66 | 0.88 | 0.91 | 0.91 |
| Earnings Yield | 0.14 | 0.92 | 1.00 | 0.75 | 0.90 | 0.94 | 0.90 |
| Dividend Yield | 0.73 | 0.66 | 0.75 | 1.00 | 0.66 | 0.72 | 0.67 |
| Momentum | 0.42 | 0.88 | 0.90 | 0.66 | 1.00 | 0.91 | 0.90 |
| Low-Volatility | -0.17 | 0.91 | 0.94 | 0.72 | 0.91 | 1.00 | 0.87 |
| Size | -0.99 | 0.91 | 0.90 | 0.67 | 0.90 | 0.87 | 1.00 |
| January 2010 - December 2014 | Average Benchmark Return: 8.9% | | | | | | |
| Factor | $IR$ | $\Theta$ | | | | | |
| Value | -1.01 | 1.00 | 0.88 | 0.60 | 0.80 | 0.79 | 0.91 |
| Earnings Yield | 1.38 | 0.88 | 1.00 | 0.74 | 0.84 | 0.86 | 0.88 |
| Dividend Yield | 1.08 | 0.60 | 0.74 | 1.00 | 0.67 | 0.76 | 0.66 |
| Momentum | 1.87 | 0.80 | 0.84 | 0.67 | 1.00 | 0.91 | 0.85 |
| Low-Volatility | 1.50 | 0.79 | 0.86 | 0.76 | 0.91 | 1.00 | 0.81 |
| Size | -1.19 | 0.91 | 0.88 | 0.66 | 0.85 | 0.81 | 1.00 |