Calibrating The Libor Market Model To Swaptions With An Extension For Illiquidity In South Africa

Leenesh Moodliyar

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African Institute Of Financial Markets And Risk Management
University of Cape Town
Cape Town

Declaration

I declare that this dissertation is my own, unaided work. It is being submitted for the Degree of Master of Philosophy in the University of the Cape Town. It has not been submitted before for any degree or examination in any other University.

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Abstract

The popularity of the LIBOR Market Model (LMM) in interest rate modelling is a result of its consistency with market practice of pricing interest rate derivatives. In the context of a life insurance company, the LMM is calibrated to swaptions as they are actively traded for a wide variety of maturities and they serve as the natural hedge instruments for many of the long dated maturity products with embedded options. Before calibrating the model we extend the calibration process to address the issue of illiquidity in the South African swaption market. The swaption surface used in calibrating the model is generated with market implied quotes for the hedgeable component and thereafter using historical volatilities for the unhedgeable or illiquid component. Rebonato's 3 parameter correlation function proposed by Rebonato (2005) provides the best fit to historical data. We assume a general piecewise constant parameterisation for the instantaneous forward rate volatilities. These volatilities are then determined analytically using the Rectangular Cascade Calibration Algorithm from Brigo and Morini (2006). The calibration generates a stable volatility term structure with the instantaneous forward rate volatilities being positive and real. Through an extension of the calibration we are able to capture the benefits of a pure replication component and accommodate a large unhedgeable component in the price faced by life insurance companies in South Africa.

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Chapter 1

LIBOR Market Model

1.1 Introduction

Interest rate caps/floors and European swaptions are the most popular over-the-counter interest rate derivatives. The standard market models used in valuing these instruments are versions of Black's (1976) model. The LIBOR Market Model (LMM) is well liked due to its consistency with market practice of pricing interest rate derivatives. The model was originally developed by Miltersen, Sandmann and Sondermann (1997), Jamshidian (1997) and Brace, Musiela and Gatarek (1997). Prior to the LMM, interest rate models referred to as short rate models meant that the dynamics of all interest rates was determined by the dynamics of the instantaneous rate. The next breakthrough came from Heath, Jarrow and Morton (1992) with the use of the HJM model which moved away from modelling the short rate to modelling the whole term structure. The LMM was one more shift in the development moving from instantaneous forward rates to forward rates with market compounding. Attractive features of the LMM include forward rates that are directly observed in the market and easier calibration to prices of market instruments.

The LMM is calibrated to caps, swaptions or a combination of caps and swaptions. Focusing the context at a life insurance company where the products are long dated we choose to calibrate the LMM to swaptions. Swaptions are available for a wide variety of maturities and bear close resemblance to the product/liability we wish to hedge or price. Calibrating the LMM to swaptions is typically performed to a full set of market quotes. Using the available market information, the parameters for the instantaneous forward rate volatilities are obtained through optimization (Gatarek, Bachert and Maksymiuk (2007)) or analytically with the algorithm from Brigo and Morini (2006). The product/liability is then priced using Monte Carlo simulation.

Unfortunately the illiquidity of swaptions, particularly beyond the 10 year maturity, in the South African market presents a major problem for calibrating the LMM.

Interpolation techniques used in the calibration to assist with the missing swaption quotes vary from simple linear interpolation (Rebonato and Joshi (2002)), functional fitting forms (Brigo and Mercurio (2001)) to changes in the calibration algorithm incorporating assumptions on the forward rate volatilities (Brigo and Morini (2006)). Making use of the available market information, the interpolation techniques attempt to directly fill the missing swaption volatilities or indirectly fill the forward rate volatilities linked to the missing swaption volatilities. Illiquidity in the South African swaption market significantly reduces the quotes available for calibration and the use of these interpolation techniques are limited in filling the target swaption surface. The long dated nature of the products also requires extending the target swaption surface beyond the quoted maturities.

We propose our own extension to the calibration process to address the issue of illiquidity in the South African swaption market. The new calibration setup consists of a combination of market implied volatilities for the hedgeable component and historical volatilities for the unhedgeable or illiquid component. The historical volatilities are calculated by assuming that the log returns of the swap rate with fixed maturities are normally distributed. We then smooth the target swaption surface using the parametric form from Brigo, Mercurio and Morini (2005). Through our extension we are able to capture the benefits of a pure replication component and address the substantial unhedgeable component in the price faced by life insurance companies in South Africa.

For the remaining part of Chapter 1 we will introduce some of the theorems and definitions used in setting up the LMM before ending with a derivation of the forward rate dynamics under the Terminal and Spot measure.

In Chapter 2 we discuss the instruments used in calibrating the LMM. We then look at the possible forms of modelling the covariance structure from Rebonato (2005), after which we will go through popular specifications for the instantaneous correlation and forward rate volatility. Two approximation formulas by Jackel and Rebonato (2003) for the instantaneous swap rate volatility are covered. The formulas allow for the LMM to be calibrated without the need for Monte Carlo simulation. We then introduce a common analytical calibration algorithm from Brigo and Morini (2006) that is able to match the target swaption surface exactly. We end Chapter 2 with a description of our extension to the calibration process.

In Chapter 3 we calibrate the LMM to European swaption volatilities using South African market data. We start the Chapter by fitting the correlation functions from Rebonato (2005) and Schoenmakers and Coffey (2000) to historical one year forward rates. Following a setup similar to Jackel and Rebonato (2003), we then investigate the accuracy of the instantaneous approximation formulas for various maturities using: our yield curve, the Rebonato (2005) correlation function and a parametric specification for the forward rate instantaneous volatilities. We then turn to the cascade calibration algorithm from Brigo and Morini (2006) to calculate the forward rate volatilities. Before calibrating the LMM, we create a new swaption surface consisting of a combination of market and historical volatilities. We assign a weight to the market and historical volatilities before smoothing the swaption surface using the parametric form from Brigo, Mercurio and Morini (2005). We then assess the calibration after stressing the market volatilities and with missing market quotes.

1.2 General Framework And Pricing

We start with a brief outline of the theory underlying the mathematics used in setting up the LMM. The rest of this chapter follows largely from the 1^{st} chapter of Gatarek, Bachert and Maksymiuk (2007) and the $8^{th}, 11^{th}$ and 12^{th} chapter of Björk (2004). The probability triple (Ω, Σ, P) consists of: a sample space Ω containing a set of elementary outcomes, a $\sigma-$algebra Σ of possible events (subsets of Ω) and a probability function $P : \Sigma \to [0, 1]$ that assigns probabilities to the events in Σ. Let $F_t \subset \Sigma$ be a family of increasing $\sigma-$algebras. A stochastic process is an indexed collection of F_t-measurable random variables $X(t)$. Each random variable is defined on the same probability triple (Ω, Σ, P). We define the expected value $E[X]$ of the random variable X as the integral

$$E[X] = \int_{\Omega} X(\omega)P(d\omega) \tag{1.1}$$

and the conditional expectation $E(X|\Xi)$ of a random variable X with respect to a $\sigma-$ algebra $\Xi \subset \Sigma$ as the only Ξ-measurable random variable such that for all $B \in \Xi$

$$\int_B X(\omega)P(d\omega) = \int_B E(X|\Xi)(\omega)P(d\omega). \tag{1.2}$$

From Gatarek, Bachert and Maksymiuk (2007), *"a particular stochastic process is determined by specifying the joint probability distributions of the various random variables $X(t)$"*. If we consider a predictable stochastic process X such that $P(\int_0^t |X(s)|^2 ds < \infty) = 1$ we may then define the stochastic integral with respect to the Wiener process $W(t)$ to be $C(t) = \int_0^t X(s)dW(s)$. If the process X is deterministic then C is Gaussian with independent increments. The stochastic integral has the following properties: $E[C(t)] = 0$ and $E[C^2(t)] = E[\int_0^t |X(s)|^2 ds]$. We can say that process X satisfies the Itô stochastic differential equation

$$dX(t) = a(t, X(t))dt + b(t, X(t))dW(t) \text{ and } X(0) = x \tag{1.3}$$

if

$$X(t) = X(0) + \int_0^t a(s, X(s))ds + \int_0^t b(s, X(s))dW(s). \tag{1.4}$$

In addition, from Gatarek, Bachert and Maksymiuk (2007) *"process X is a Markov process if a and b are deterministic functions with properties that ensure uniqueness of solution"*. Let $Z(t, X(t))$ be a smooth function. Then the process $Z(t, X(t))$ satisfies the stochastic differential equation

$$dZ(t, X(t)) = \frac{\partial Z}{\partial t}dt + \frac{\partial Z}{\partial x}dX + \frac{1}{2}\frac{\partial^2 Z}{\partial x^2}(dX)^2. \tag{1.5}$$

The Geometric Wiener process is used in modelling of asset prices and satisfies the following stochastic equation

$$dX(t) = \mu X(t)dt + \sigma X(t)dW(t) \tag{1.6}$$

where the coefficient μ is called the drift and the coefficient σ is called the volatility. An $N-$dimensional stochastic process $M(t)$ is a martingale with respect to F_t if $E[|M(t)|] < \infty$ and the following property also holds

$$M(t) = E[M(T)|F_t] \qquad \text{for} \qquad t \leq T. \tag{1.7}$$

Every stochastic integral with respect to a Wiener process is a martingale and hence any Wiener process is a martingale. However a Geometric Wiener process is a martingale only if $\mu = 0$. Any continuous martingale M with respect to the filtration generated by a Wiener process can be represented as an Itô integral $M(t) = \int_0^t X(s)dW(s)$ for some predictable process X. A martingale is considered as a model for a fair game and therefore well liked as an appropriate model in the financial markets. We start with a short list of definitions and theorems that will be used in the construction of the forward LIBOR process.

We consider a market model consisting of the asset price processes $S_0, S_1, ..., S_N$ on the time interval $[0, T]$. Portfolios or trading strategies that are of interest to us are those that are said to be self-financing. From Björk (2004), these are *"trading strategies with no exogenous infusion or withdrawal of money, i.e. a strategy where the purchase of a new asset is financed solely by the sale of assets already in the portfolio".*

Definition 1.1 (Björk (2004)): *Let the $N-$dimensional price process $\{S(t); t \geq 0\}$ be given.*

- *A portfolio is any F_t^S-adapted $N-$dimensional process $\{h(t); t \geq 0\}$.*

- *The value process V^h corresponding to the portfolio h is given by*

$$V^h(t) = \sum_{i=1}^{N} h_i(t)S_i(t). \tag{1.8}$$

- *A portfolio h is called self-financing if the value V^h satisfies the condition*

$$dV^h(t) = \sum_{i=1}^{N} h_i(t)dS_i(t). \tag{1.9}$$

Definition 1.2 (Björk (2004)): *A numeraire is a price process $S_0(t)$ almost strictly positive for each $t \in [0, T]$.*

Before we look at the fair price of a contingent claim we will first introduce the notion of arbitrage. We interpret arbitrage as a serious case of mispricing, from Björk (2004) *"the possibility of making a positive amount of money out of nothing without taking any risk"*.

Definition 1.3 (Björk (2004)): *An arbitrage possibility in a financial market is a self-financed portfolio h such that*

$$V^h(0) = 0, \tag{1.10}$$

$$P(V^h(T) \geq 0) = 1, \tag{1.11}$$

$$P(V^h(T) > 0) > 0. \tag{1.12}$$

We say that the market is arbitrage free if there are no arbitrage possibilities.

Referring to Theorem 10.22 from Björk (2004) we have:

Theorem 1.1 (Björk (2004)): *The market model is free of arbitrage if and only if there exists a martingale measure, i.e a measure $Q \sim P$ such that the processes $\frac{S_0(t)}{S_0(t)}, \frac{S_1(t)}{S_0(t)}, ..., \frac{S_N(t)}{S_0(t)}$ are (local) martingales under Q.*

A fundamental idea behind arbitrage-free pricing is the possibility of replicating the payoff of a contingent claim in terms of a portfolio based on the underlying assets.

Definition 1.4 (Björk (2004)): *We say that a claim $\mathcal{X}$ at time T can be replicated or hedged if there exists a self-financing portfolio h such that*

$$V^h(T) = \mathcal{X}, \;\; P\text{-}a.s. \tag{1.13}$$

In this case we call h the replicating or hedging portfolio. If every contingent claim is replicable we say that the market is complete.

Referring to Theorem 10.18 and Theorem 10.24 from Björk (2004) we have:

Theorem 1.2 (Björk (2004)): *Assuming absence of arbitrage, the market model is complete if and only if the martingale measure Q is unique.*

Theorem 1.3 (Björk (2004)): *The arbitrage free t-price for a claim $\mathcal{X}$ with maturity T is given by*

$$Price(t, \mathcal{X}) = S_0(t)E^Q\left[\frac{\mathcal{X}}{S_0(T)}|\mathcal{F}_t\right] \tag{1.14}$$

where Q is the martingale measure with S_0 as the numeraire.

In particular, if we assume that S_0 in Theorem 1.3 is the money account given by

$$S_0(t) = S_0(0)e^{\int_0^t r(s)ds} \tag{1.15}$$

where r is the short rate we obtain the Risk Neutral Valuation Formula. We now look at the Girsanov Theorem, Theorem 11.3 from Björk (2004), which will provide the control and determine the effect a measure transformation will have on a Wiener process.

Theorem 1.4 (Björk (2004)): *Let W^P be a d-dimensional standard P-Wiener process on $(\Omega, \mathcal{F}, P, \Sigma)$ and let φ be any d-dimensional adapted column vector process. Choose a fixed T and define the process L on $[0, T]$ by*

$$dL_t = \varphi_t L_t dW_t^P \quad \textit{and} \quad L_0 = 1. \tag{1.16}$$

Assume that $E^P[L_T] = 1$ *and define the new probability measure* Q *on* $\mathcal{F}_T$ *by*

$$L_T = \frac{dQ}{dP} \quad on \quad \mathcal{F}_T. \tag{1.17}$$

Then

$$dW_t^P = \varphi_t dt + dW_t^Q \tag{1.18}$$

where W^Q *is a* Q*-Wiener process. The process* φ *is often referred to as the Girsanov kernel of the measure transformation.*

Suppose we want to change the numeraire from S_0 to S_1, a key part of the problem is how to find the appropriate Girsanov transformation which will take us from Q^0 to Q^1 where Q^1 is the martingale measure corresponding to the numeraire S_1.

Proposition 1.1 (Björk (2004)): *Assume that* Q^0 *is a martingale measure for the numeraire* S_0 *on filtration* $\mathcal{F}_T$ *and assume that* S_1 *is a positive asset price process such that* $\frac{S_1(t)}{S_0(t)}$ *is a true* Q^0 *martingale. Define* Q^1 *on* $\mathcal{F}_T$ *by the likelihood process*

$$L_0^1(t) = \frac{S_0(0)}{S_1(0)} \frac{S_1(t)}{S_0(t)}, \qquad 0 \leq t \leq T. \tag{1.19}$$

Then Q^1 *is a martingale measure for* S_1*.*

We can conclude this section with the following two points from Björk's chapter in Biais and Runggaldier (1997):

- *"A contingent claim should be priced in such a way that it should not introduce arbitrage possibilities in the market. This requirement is reflected by the fact that all derivatives must be priced by Theorem 1.3 where the same Q is used for all derivatives".*
- *"In an incomplete market the requirement of no arbitrage is no longer sufficient to determine a unique price for a derivative. There are several martingale measures - all of which can be used to price a contingent claim consistent with no arbitrage".* From Björk (2004), the *"price in these markets is also partly determined by aggregate supply and demand, liquidity considerations and other*

factors. These aspects are aggregated into the particular martingale measure that is chosen by the market".

1.3 Constructing The Forward LIBOR Process

Consider a fixed set of increasing maturities $T_{-1}, ..., T_N$ with the current time or valuation date being $T_{-1} = 0$. We define the simple compounded forward rate $F(t, T_{i-1}, T_i)$ or $F_i(t)$ resetting at T_{i-1} and with a maturity T_i for $i = 1, ..., N$. Let $P(t, T_i)$ or $P_i(t)$ denote the time-t value of a zero coupon bond maturing at the payment time of the i^{th} forward rate $F_i(t)$ and let δ_i be the corresponding accrual factor for the period $[T_{i-1}, T_i]$. The notation is illustrated in Figure 1.1. We consider an m-dimensional Q^N-Wiener process W^N. For each $i = 1, ..., N$, we define W^i as the m-dimensional Q^i-Wiener process generated by W^N under the Girsanov transformation $Q^N \rightarrow Q^i$.

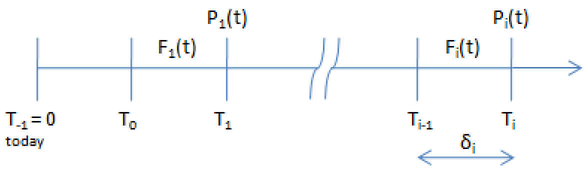

Fig. 1.1: Notation Used In Construction Of Forward LIBOR Process

Definition 1.5 (Björk (2004)): *Let $F_i(t)$ denote the LIBOR forward rate contracted at time t for the period $[T_{i-1}, T_i]$, i.e.*

$$F_i(t) = \frac{1}{\delta_i}\frac{P_{i-1}(t) - P_i(t)}{P_i(t)}, \qquad i = 1, ..., N. \tag{1.20}$$

Let us consider an asset given by $V_i(t) = \delta_i[P_{i-1}(t) - P_i(t)]$ in a no-arbitrage framework. Then there exists a martingale measure for this asset, using Theorem 1.1. Lets us call this measure Q^A and its associated numeraire A_t. Thus

$$\frac{V_i(t)}{A_t} = E^{Q^A}\left[\frac{V_i(T)}{A_T}\right]. \tag{1.21}$$

So under a change of numeraire from A_t to $P_i(t)$ we have

$$\frac{V_i(t)}{P_i(t)} = E^{Q^i}\left[\frac{V_i(T)}{P_i(T)}\right] \tag{1.22}$$

where the measure associated with numeraire $P_i(t)$ is Q^i with Brownian motion W^i. Using the earlier definition of the i^{th} forward we see that F_i is a martingale under measure Q^i. So for every $i = 1, ..., N$ the LIBOR process F_i is a martingale under the corresponding forward measure Q^i on the interval $[0, T_{i-1}]$.

Definition 1.6 (Björk (2004)): *If the LIBOR forward rates have the dynamics*

$$dF_i(t) = F_i(t)\sigma_i(t)dW^i(t), \qquad i = 1, ..., N \tag{1.23}$$

where W^i is a Q^i Wiener process, then we say that we have a discrete tenor LIBOR market model with volatilities $\sigma_1, ..., \sigma_n$.

We will be changing to the following form of the LIBOR market model to point out explicitly the correlation between the Wiener processes

$$dF_i(t) = F_i(t)\sigma_i(t)dW_i(t), \qquad i = 1, ..., N \tag{1.24}$$

where $\sigma_i(t)$ is a scalar deterministic function and W_i is a scalar Q_i Wiener process with a given correlation structure $dW_i(t)dW_j(t) = \rho_{ij}$. We have seen that the LIBOR process F_i is a martingale under the corresponding forward measure Q^i. However when simulating in Monte Carlo we have to specify all LIBOR rates $F_1, ..., F_N$ under one common measure. The Spot and Terminal measures are commonly used. These probability measure changes are associated with a change of numeraire, the discretely rebalanced bank account is used as the numeraire for the Spot measure and the T_N bond $P_N(t)$ as the numeraire for the Terminal measure. A change of numeraire will impact the drift term but not the volatility of the asset. One can expect the LIBOR rates under this common measure Q^A to have dynamics of the form

$$dF_i(t) = F_i(t)\mu_i(t, F(t))dt + F_i(t)\sigma_i(t)dW^A(t) \tag{1.25}$$

for $i = 1, ..., N$ where μ_i is some deterministic function. From Jamshidian (1997), *"the no arbitrage condition naturally translates into a constraint on the forward LIBOR rates, an equation relating the drifts and covariance matrix of the forward rates"*.

Proposition 1.1 is used to find the appropriate Girsanov transformation when changing the numeraire. The density $L_i^{i-1}(t)$ used in relating the measures Q^i and Q^{i-1} is given by

$$L_i^{i-1}(t) = \frac{P_i(0)}{P_{i-1}(0)} \frac{P_{i-1}(t)}{P_i(t)} = \frac{P_i(0)}{P_{i-1}(0)} [1 + \delta_i F_i(t)]. \tag{1.26}$$

Using this result we obtain the dynamics for L_i^{i-1} under Q^i as

$$dL_i^{i-1}(t) = \frac{P_i(0)}{P_{i-1}(0)} \delta_i dF_i(t), \tag{1.27}$$

assuming that the dynamics for F_i is given by Definition 1.6 we have

$$dL_i^{i-1}(t) = \frac{P_i(0)}{P_{i-1}(0)} \delta_i F_i(t) \sigma_i(t) dW^i(t) \tag{1.28}$$

or

$$\frac{dL_i^{i-1}(t)}{L_i^{i-1}(t)} = \frac{\delta_i F_i(t) \sigma_i(t) dW^i(t)}{1 + \delta_i F_i(t)}. \tag{1.29}$$

From the Girsanov Theorem 1.4 we have

$$dW^i(t) = \frac{\delta_i F_i(t) \sigma_i(t)}{1 + \delta_i F_i(t)} dt + dW^{i-1}(t), \tag{1.30}$$

so under the Terminal measure Q^N where $W^N(t)$ is an m-dimensional Q^N-Wiener process we generate

$$dW^N(t) = \frac{\delta_N F_N(t) \sigma_N(t)}{1 + \delta_N F_N(t)} dt + dW^{N-1}(t). \tag{1.31}$$

Applying this inductively we obtain

$$dW^i(t) = -\sum_{k=i+1}^{N} \frac{\delta_k F_k(t) \sigma_k(t)}{1 + \delta_k F_k(t)} dt + dW^N(t). \tag{1.32}$$

Finally we can substitute this into Definition 1.6 to obtain the dynamics of F_i under the Terminal measure

$$dF_i(t) = -F_i(t) \sum_{k=i+1}^{N} \frac{\delta_k F_k(t)}{1 + \delta_k F_k(t)} \sigma_i(t)\sigma_k(t)\rho_{ik} dt + F_i(t)\sigma_i(t) dW_i^N(t). \tag{1.33}$$

Denoting the spot measure by Q^B, we construct a discretely rebalanced bank account numeraire to tie into the discrete forward rate environment. The bank account is a self-financing portfolio that is rebalanced only at maturities on the discrete tenor structure. This is achieved by rolling over the bond with the shortest remaining maturity. So if we are at T_0 we will invest one unit of money into the T_1 bond. At T_1 we sell the T_1 bond and invest everything into the T_2 bond and then repeat the procedure until T_N. So the value of this self-financing portfolio is given by

$$B(t) = P_q(t)B(T_q) \quad for \quad T_{q-1} \leq t < T_q \tag{1.34}$$

where

$$B(T_q) = \prod_{i=1}^{q} (1 + \delta_i F_i(T_{i-1})). \tag{1.35}$$

From Volume 2 of Andersen and Piterbarg (2010), portfolio B is locally risk free so at any time t the value of $B(T_q)$ is known and the only random part of the numeraire is the discount bond $P_q(t)$. So we only need to establish the dynamics under the measure Q^q. Applying the iteration idea shown above to equation (1.30) we have

$$dW^i(t) = \sum_{k=q+1}^{i} \frac{\delta_k F_k(t)\sigma_k(t)}{1 + \delta_k F_k(t)} dt + dW^q(t). \tag{1.36}$$

So under the Spot measure Q^B where W^B is an m-dimensional Q^B-Wiener process, the dynamics of F_i is given by

$$dF_i(t) = F_i(t) \sum_{k=q+1}^{i} \frac{\delta_k F_k(t)}{1 + \delta_k F_k(t)} \sigma_i(t)\sigma_k(t)\rho_{ik} dt + F_i(t)\sigma_i(t) dW_i^B(t). \tag{1.37}$$

Chapter 2

Calibration

The LMM is calibrated to caps, swaptions or a combination of caps and swaptions. We start the chapter by introducing these instruments and point out the conditions that need to be satisfied by the LMM for consistency with the Black-76 formula. We then look at the covariance structure of the LMM and choose a formulation that will result in the recovery of swaption prices implied by a set of instantaneous forward rate volatilities and correlation. We look at some popular specifications for the instantaneous forward rate volatilities and correlation. The choice of specification impacts the terminal correlation and the term structure of volatility. Two approximation formulas by Jackel and Rebonato (2003) for the instantaneous swap rate volatility are covered, these formulas will be used in calibrating the LMM to entire swaption matrix of market volatilities. We then introduce a common analytical calibration algorithm from Brigo and Morini (2006) that is able to match the input/target swaption surface exactly. Lastly, we end this chapter with our extension to the calibration process to address the issue of illiquidity in the South African swaption market.

2.1 Market Instruments

[1]Market practice is to use the Black-76 formula for the pricing of caps, floors and swaptions. A cap is a contract at time T_i that gives the holder of the cap the amount $X_i = \delta_i \max\left[F_i(T_{i-1}) - R, 0\right]$ for each $i = 1, ..., N$ where R is the cap rate at

[1] Section 2.1 follows chapter 27 of Björk (2004).

resettlement dates $T_0, ..., T_N$. We see that a cap is a portfolio of individual caplets $X_1, ..., X_N$ and the caplet X_i is just a call option on the underlying spot rate.

Definition 2.1 (Björk (2004)): *The Black-76 formula for the caplet* $X_i = \delta_i \max\left[F_i(T_{i-1}) - R, 0\right]$ *is given by the expression*

$$Caplet\ Price_i(t) = \delta_i P_i(t)[F_i(t)N(d_1) - RN(d_2)],\ \ i = 1,\ ...,\ N \tag{2.1}$$

where

$$d_1 = \frac{1}{\hat{\sigma}_i \sqrt{T_i - t}} \left[\ln\left(\frac{F_i(t)}{R}\right) + \frac{1}{2}\hat{\sigma}_i^2(T_i - t) \right] \tag{2.2}$$

$$d_2 = d_1 - \hat{\sigma}_i \sqrt{T_i - t} \tag{2.3}$$

and the constant $\hat{\sigma}_i$ *is known as the Black volatility for* $Caplet\ Price_i(t)$.

Given that F_i, see Equation (1.23), is a Geometric Brownian motion we have

$$F_i(T) = F_i(t) e^{\int_t^T \sigma_i(s) dW^i(s) - \frac{1}{2}\int_t^T \left\|\sigma_i(s)\right\|^2 ds}. \tag{2.4}$$

We can then write

$$F_i(T) = F_i(t) e^{X_i(t,T)} \tag{2.5}$$

where $X_i(t, T)$ is normally distributed with the mean $m_i(t, T)$ and the variance $v_i^2(t, T)$ are defined respectively as

$$m_i(t, T) = -\frac{1}{2}\int_t^T \left\|\sigma_i(s)\right\|^2 ds \tag{2.6}$$

and

$$v_i^2(t, T) = \int_t^T \left\|\sigma_i(s)\right\|^2 ds. \tag{2.7}$$

Using Theorem 1.3 under the T_i forward measure, the price for caplet i is given by

$$Caplet\ Price_i^{LMM}(t) = \delta_i P_i(t) E^{T_i}[\max\left[F_i(T_{i-1}) - R, 0\right] | F_t],\ \text{i} = 1,\ ...,\ \text{N}. \tag{2.8}$$

After calculating the expectation of Equation (2.8), see Theorem 10.4.2 from Shreve (2004) for the full calculation, we obtain a caplet price given by the Black type formula.

Proposition 2.1 (Björk (2004)): *Using the LMM, the price for caplet* $X_i = \delta_i \max\left[F_i(T_{i-1}) - R, 0\right]$ *is given by*

$$\text{Caplet Price}_i^{LMM}(t) = \delta_i P_i(t)[F_i(t)N(d_1) - RN(d_2)],\ \ i = 1,\ \ldots,\ N \tag{2.9}$$

where

$$d_1 = \frac{1}{v_i(t, T_{i-1})}\left[\ln\left(\frac{F_i(t)}{R}\right) + \frac{1}{2}v_i^2(t, T_{i-1})\right] \tag{2.10}$$

$$d_2 = d_1 - v_i(t, T_{i-1}). \tag{2.11}$$

Analogously to an interest rate cap, *"an interest rate floor is a portfolio of put options on interest rates. Each of the individual options comprising a floor is known as a floorlet"* Hull (2009).

Proposition 2.2: *Using the LMM, the price for floorlet* $X_i = \delta_i \max\left[R - F_i(T_{i-1}), 0\right]$ *is given by*

$$\text{Floorlet Price}_i^{LMM}(t) = \delta_i P_i(t)[RN(-d_2) - F_i(t)N(-d_1)],\ \ i = 1,\ \ldots,\ N \tag{2.12}$$

where

$$d_1 = \frac{1}{v_i(t, T_{i-1})}\left[\ln\left(\frac{F_i(t)}{R}\right) + \frac{1}{2}v_i^2(t, T_{i-1})\right] \tag{2.13}$$

$$d_2 = d_1 - v_i(t, T_{i-1}). \tag{2.14}$$

Before we look at the definition of a swaption we will first consider the specification of its underlying instrument, the swap. From Hull (2009), *"a swap is an agreement between two companies to exchange cash flows in the future".* In particular, an interest rate swap exchanges a *"fixed rate of interest on a certain notional amount for a floating rate of interest on the same notional amount"* Hull (2009). The holder of a receiver swap with tenor $T_\beta - T_\alpha$ will at dates $T_{\alpha+1}, ..., T_\beta$ receive the fixed leg and pay the floating leg. The payments move in the other direction for a payer swap.

Definition 2.2 (Björk (2004)): *The payments in a* $T_\alpha \times (T_\beta - T_\alpha)$ *payer swap are as follows*

- *Payments will be made and received at* $T_{\alpha+1}, T_{\alpha+2}, ..., T_\beta$.
- *For every elementary period* $[T_i, T_{i+1}]$, $i = \alpha, ..., \beta-1$, *the LIBOR rate* $F_{i+1}(T_i)$ *is set at time* T_i *and the floating leg* $\delta_{i+1}F_{i+1}(T_i)$ *is received at* T_{i+1}.
- *For the same period the fixed leg* $\delta_{i+1}K$ *is paid at* T_{i+1}.

The net value of the $T_\alpha \times (T_\beta - T_\alpha)$ payer swap at time $t < T_\alpha$ is given by

$$Swap\ Value(t) = \sum_{i=\alpha+1}^{\beta} \delta_i P_i(t) F_i(t) - K \sum_{i=\alpha+1}^{\beta} \delta_i P_i(t). \tag{2.15}$$

Using Equation (1.20), we can rewrite the total value of the floating side at time t for $t \leq T_\alpha$ as

$$\sum_{i=\alpha+1}^{\beta} (P_{i-1}(t) - P_i(t)) = P_\alpha(t) - P_\beta(t). \tag{2.16}$$

So the

$$Swap\ Value(t) = P_\alpha(t) - P_\beta(t) - K \sum_{i=\alpha+1}^{\beta} \delta_i P_i(t). \tag{2.17}$$

Definition 2.3 (Björk (2004)): *The par or forward swap rate* $S_\alpha^\beta(t)$ *of the* $T_\alpha \times (T_\beta - T_\alpha)$ *swap is the value of* K *for which Swap Value(t)* $= 0$, *i.e.*

$$S_\alpha^\beta(t) = \frac{P_\alpha(t) - P_\beta(t)}{\sum_{i=\alpha+1}^{\beta} \delta_i P_i(t)}. \tag{2.18}$$

From Hull (2009), *"Swaptions are options on interest rate swaps that give the holder the right to enter into a certain interest rate swap at a certain time in the future"*. Market practice is to compute swaption prices using a formal extension of the Black-76 formula. Prices quoted in the market are typically for par swap rates in terms of the implied Black volatilities.

Definition 2.4 (Björk (2004)): *A European payer swaption* $T_\alpha \times (T_\beta - T_\alpha)$ *with strike* K *is a contract which at the exercise date* T_α *gives the holder the right but not the obligation to enter into a* $T_\alpha \times (T_\beta - T_\alpha)$ *swap with the fixed swap rate* K.

The payer swaption is a contingent T_α claim defined by

$$\text{Swaption Payoff} = \max[\textit{Swap Value(t)}, 0] \tag{2.19}$$

and using Definition 2.3 we have

$$\text{Swaption Payoff} = \sum_{i=\alpha+1}^{\beta} \delta_i P_i(T_\alpha) \max[S_\alpha^\beta(T_\alpha) - K, 0]. \tag{2.20}$$

When expressed in the numeraire $\sum_{i=\alpha+1}^{\beta} \delta_i P_i(T_\alpha)$ the swaption is a call option on S_α^β with strike price K.

Definition 2.5 (Björk (2004)): *The Black-76 formula for a $T_\alpha \times (T_\beta - T_\alpha)$ payer swaption with strike K is defined as*

$$\textit{Swaption Price}_{\alpha,\beta}(t) = \sum_{i=\alpha+1}^{\beta} \delta_i P_i(t)[S_\alpha^\beta(t)N(d_1) - KN(d_2)] \tag{2.21}$$

where

$$d_1 = \frac{1}{\hat{\sigma}_{\alpha,\beta}^{swap}\sqrt{T_\alpha - t}}\left[\ln\left(\frac{S_\alpha^\beta(t)}{K}\right) + \frac{1}{2}(\hat{\sigma}_{\alpha,\beta}^{swap})^2(T_\alpha - t)\right] \tag{2.22}$$

$$d_2 = d_1 - \hat{\sigma}_{\alpha,\beta}^{swap}\sqrt{T_\alpha - t} \tag{2.23}$$

and the constant $\hat{\sigma}_{\alpha,\beta}^{swap}$ is the implied Black volatility.

We now turn our attention to the Swap Market Model (SMM). Unlike the LMM, each forward swap rate S_n^k is modelled as a Geometric Brownian motion under its own martingale measure Q_n^k corresponding to the numeraire $\sum_{i=n+1}^{k} \delta_i P_i(t)$.

Definition 2.6 (Björk (2004)): *Consider a fixed subset $\mathcal{N}$ of all positive integer pairs (n, k) such that $0 \leq n < k \leq N$ and a deterministic function of time $\sigma_{n,k}^{swap}$ for each $(n, k) \in \mathcal{N}$. A swap market model with the volatilities $\sigma_{n,k}^{swap}$ is then specified by assuming that the par swap rates have dynamics of the form*

$$dS_n^k(t) = S_n^k(t)\sigma_{n,k}^{swap}(t)dW^{n,k}(t) \tag{2.24}$$

where $W^{n,k}$ is a Wiener under $Q^{n,k}$.

Given that S_α^β is a Geometric Brownian motion we have

$$S_\alpha^\beta(T_\alpha) = S_\alpha^\beta(t) e^{\int_t^{T_\alpha} \sigma_{\alpha,\beta}^{swap}(s) dW^{\alpha,\beta}(s) - \frac{1}{2}\int_t^{T_\alpha} \left\| \sigma_{\alpha,\beta}^{swap}(s) \right\|^2 ds}. \tag{2.25}$$

We can then write

$$S_\alpha^\beta(T_\alpha) = S_\alpha^\beta(t) e^{X_{\alpha,\beta}(t,T_\alpha)} \tag{2.26}$$

where $X_{\alpha,\beta}(t,T_\alpha)$ is normally distributed with mean $m_{\alpha,\beta}(t,T_\alpha)$ and variance $v_{\alpha,\beta}^2(t,T_\alpha)$ defined as

$$m_{\alpha,\beta}(t,T_\alpha) = -\frac{1}{2}\int_t^{T_\alpha} \left\| \sigma_{\alpha,\beta}^{swap}(s) \right\|^2 ds \tag{2.27}$$

and

$$v_{\alpha,\beta}^2(t,T_\alpha) = \int_t^{T_\alpha} \left\| \sigma_{\alpha,\beta}^{swap}(s) \right\|^2 ds. \tag{2.28}$$

Performing a similar calculation, shown earlier for Proposition 2.1, we can obtain a swaption price given by the Black type formula.

Proposition 2.3 (Björk (2004)): *Using the SMM, the price for a $T_\alpha \times (T_\beta - T_\alpha)$ payer swaption with strike K is defined as*

$$Swaption\ Price_{\alpha,\beta}^{SMM}(t) = \sum_{i=\alpha+1}^{\beta} \delta_i P_i(t)[S_\alpha^\beta(t) N(d_1) - K N(d_2)] \tag{2.29}$$

where

$$d_1 = \frac{1}{v_{\alpha,\beta}(t,T_\alpha)}\left[\ln\left(\frac{S_\alpha^\beta(t)}{R}\right) + \frac{1}{2} v_{\alpha,\beta}^2(t,T_\alpha)\right] \tag{2.30}$$

$$d_2 = d_1 - v_{\alpha,\beta}(t,T_\alpha). \tag{2.31}$$

2.2 Forward Rate Covariance Structure

We saw earlier that the stochastic differential equation for the forward rate is of the form

$$\frac{dF_i}{F_i} = \mu_i dt + \sigma_i dW_i \tag{2.32}$$

where the drift μ_i reflects the no-arbitrage condition and is a function of the forward rates, the instantaneous volatilities of the forward rates and the correlation among the forward rates. From this observation we see that the core of the LMM calibration comes down to the simultaneous specification of the time-dependent volatilities and correlations. Following Rebonato (2005), we consider three forms of the forward rate covariance structure. We rewrite the stochastic differential equation in matrix notation and consider the scenario where n forward rates describe the yield curve.

Modelling Each Forward Rate

$$\frac{d\bar{F}}{\bar{F}} = \bar{\mu}dt + \bar{v}d\bar{W} \tag{2.33}$$

where the matrix $\bar{v}$ is a time dependent diagonal matrix whose ii^{th} element is given by the instantaneous volatility of the ith forward rate

$$\bar{v} = \begin{bmatrix} \sigma_1 & 0 & ... & 0 \\ 0 & \sigma_2 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & \sigma_n \end{bmatrix}.$$

The exogenous correlation structure between the forward rates can be recovered by requiring that

$$\bar{\rho}dt = d\bar{W}d\bar{W}^T \tag{2.34}$$

where $\bar{\rho}$ is the desired correlation matrix.

Using The Covariance Matrix

$$\frac{d\bar{F}}{\bar{F}} = \bar{\mu}dt + \bar{\sigma}d\bar{Z} \tag{2.35}$$

Let us reduce the dimensionality of the problem to only m independent Brownian shocks

$$d\bar{Z}d\bar{Z}^T = \bar{I}dt \tag{2.36}$$

where $d\bar{Z}$ is a vector whose elements are increments of m orthogonal Brownian motions and $\bar{I}$ is the identity matrix. The $(j,k)^{th}$ element of matrix $\bar{\sigma}$,

$$\bar{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \dots & \sigma_{1m} \\ \sigma_{21} & \sigma_{22} & \dots & \sigma_{2m} \\ \dots & \dots & \dots & \dots \\ \sigma_{n1} & \sigma_{n2} & \dots & \sigma_{nm} \end{bmatrix}$$

contains the responsiveness of the j^{th} forward rate to a random shock from the k^{th} factor. The covariance matrix is a crucial part of the LMM calibration with the covariance matrix elements determining both the drift and stochastic components of the forward rate evolution. The covariance matrix Σ between the forward rates is given by

$$\Sigma = \bar{\sigma}\bar{\sigma}^T. \tag{2.37}$$

Separating Volatility And Correlation

The previous formulation made direct use of the forward rate covariance elements, we now separate the volatility and correlation information from the covariance elements. Brigo and Morini (2006) argue that through an exogenous specification of the instantaneous correlation a number of desirable properties can be obtained for the correlation matrix. These additional properties are based both on intuition and empirical observation. These characteristics may include a reduction in correlation as the distance between forward rate maturities increases or the increase in interdependency between equally spaced forward rates as their maturities increase. We start with the same stochastic differential equation from the previous formulation

$$\frac{d\bar{F}}{\bar{F}} = \bar{\mu}dt + \bar{\sigma}d\bar{Z}. \tag{2.38}$$

We revert back to the scalar notation to simplify the workings:

$$\frac{dF_i}{F_i} = \mu_i dt + \sum_{k=1}^{m} \sigma_{ik} dZ_k. \tag{2.39}$$

Multiply and divide each loading σ_{ik} by the volatility σ_i of the ith forward rate

$$\frac{dF_i}{F_i} = \mu_i dt + \sigma_i \sum_{k=1}^{m} \frac{\sigma_{ik}}{\sigma_i} dZ_k. \tag{2.40}$$

Using the caplet-pricing condition which links the instantaneous volatility to the Black caplet volatility

$$\sum_{k=1}^{m} {\sigma_{ik}}^2 = {\sigma_i}^2, \tag{2.41}$$

we have

$$\frac{dF_i}{F_i} = \mu_i dt + \sigma_i \sum_{k=1}^{m} \frac{\sigma_{ik}}{\sqrt{\sum_{k=1}^{m} {\sigma_{ik}}^2}} dZ_k. \tag{2.42}$$

By defining $b_{ik} = \frac{\sigma_{ik}}{\sqrt{\sum_{k=1}^{m} {\sigma_{ik}}^2}}$, we can reduce the stochastic differential equation to

$$\frac{dF_i}{F_i} = \mu_i dt + \sigma_i \sum_{k=1}^{m} b_{ik} dZ_k. \tag{2.43}$$

If we denote $\bar{b}$ as the matrix of elements b_{jk}, it can be shown that $\bar{b}\bar{b}^T = \bar{\rho}$.

2.3 Calibrating To Swaptions

The focus of this and the next section will be the recovery of swaption prices implied by a set of forward rate volatilities and correlations. We saw earlier in the derivation of the Black formula for swaptions that under the measure Q_α^β corresponding to the numeraire $\sum_{i=\alpha+1}^{\beta} \delta_i P_i(T_\alpha)$ the swap rate $S_\alpha^\beta(t)$ is a martingale and assumes the following lognormal dynamics

$$dS_\alpha^\beta(t) = \sigma_{\alpha,\beta}^{swap} S_\alpha^\beta(t) dW^{\alpha,\beta}(t) \tag{2.44}$$

where $W^{\alpha,\beta}$ is the standard Brownian motion under Q_α^β. We showed earlier with Definition 2.3 that the swap rate is given by

$$S_\alpha^\beta(t) = \frac{P_\alpha(t) - P_\beta(t)}{\sum_{i=\alpha+1}^{\beta} \delta_i P_i(t)}, \tag{2.45}$$

using Equations (1.20) and (2.16) we now have

$$S_\alpha^\beta(t) = \frac{\sum_{i=\alpha+1}^{\beta} F_i \delta_i P_i(t)}{\sum_{i=\alpha+1}^{\beta} \delta_i P_i(t)}. \tag{2.46}$$

Thus the swap rate can be written as a linear combination of the forward rates

$$S_\alpha^\beta(t) = \sum_{i=\alpha+1}^{\beta} w_i(t) F_i(t) \tag{2.47}$$

where

$$w_i(t) = \frac{\delta_i P_i(t)}{\sum_{i=\alpha+1}^{\beta} \delta_i P_i(t)}. \tag{2.48}$$

After applying Itô's lemma to the swap rate we can then show

$$(\sigma_{\alpha,\beta}^{swap} S_\alpha^\beta)^2 = \sum_{j=\alpha+1}^{\beta} \sum_{k=\alpha+1}^{\beta} \frac{\partial S_\alpha^\beta}{\partial F_j} \frac{\partial S_\alpha^\beta}{\partial F_k} \sigma_j F_j \sigma_k F_k \rho_{jk}. \tag{2.49}$$

From Jackel and Rebonato (2003), we see that even though the partial derivatives are rather involved, as the weights w indirectly depend on the forward rates, a very good approximation can be made for short to medium option expiries with

$$\frac{\partial S_\alpha^\beta}{\partial F_j} \approx w_j. \tag{2.50}$$

The full derivative can still be worked out but w_j would still constitute the leading term with the other additional or shape correction terms being non-zero only if the yield curve is not flat. Let us indicate the correct derivative with the term w'_j, we can now write the instantaneous volatility for the swap rate as

$$\sigma_{\alpha,\beta}^{swap}(t) = \sqrt{\sum_{j=\alpha+1}^{\beta} \sum_{k=\alpha+1}^{\beta} \zeta_{jk}(t) \rho_{jk}(t) \sigma_j(t) \sigma_k(t)} \tag{2.51}$$

where

$$\zeta_{jk}(t) = \frac{w'_j(t)F_j(t)w'_k(t)F_k(t)}{\left(\sum_{i=\alpha+1}^{\beta} w_i(t)F_i(t)\right)^2}. \tag{2.52}$$

Jackel and Rebonato (2003) reach the following two conclusions:

- *"the coefficients (ζ) are very mildly dependent on the path realizations for movements in the forward curves that are dominated by a parallel shock"*
- *"the expectation of the average Black volatility is very close to the value obtainable by integrating the swap rate instantaneous volatilities calculated using* today's *values for the coefficients (ζ) and the forward rates".* This is the case even when the forward curve experiences more complex changes (shocks to slope and curvature).

The implied Black volatility is linked to the instantaneous swap rate volatility with the following equation

$$(\hat{\sigma}^{swap}_{\alpha,\beta})^2 T_{exp} = \int_0^{T_{exp}} \sigma^{swap}_{\alpha,\beta}(u)^2 du \tag{2.53}$$

where T_{exp} is the swaption expiry. There are a number of issues arising from this equation, recalling that the quantity $\int_0^{T_{exp}} \sigma^{swap}_{\alpha,\beta}(u)^2 du$ is a path dependent integral. Rebonato (2005) points out that not only is there no exact solution but more importantly there is *"no single deterministic Black swap rate volatility associated with a set of deterministic Black forward rate volatilities".* The issue that distributions of the lognormal forward rates are incompatible with the lognormal swap rates is relatively minor.

Jackel and Rebonato (2003) show that if the following terms F, w, w'_j are deterministic and calculated using today's yield curve we have the following deterministic approximation for the swap rate volatility

$$\sigma^{swap}_{\alpha,\beta}(t) \approx \sqrt{\sum_{j=\alpha+1}^{\beta} \sum_{k=\alpha+1}^{\beta} \zeta_{jk}(0)\rho_{jk}(t)\sigma_j(t)\sigma_k(t)}. \tag{2.54}$$

This approximation together with the Black formula for swaptions provides a result that is very close to one where Monte Carlo simulation with the correct volatilities is being used. We are now able to calibrate with great ease to an entire swaption matrix as not a single Monte Carlo simulation is carried out. Jackel and Rebonato (2003) provide the following two approximations:

Constant Weight Approximation

$$\sigma_{\alpha,\beta}^{swap}(t) \approx \sqrt{\sum_{j=\alpha+1}^{\beta} \sum_{k=\alpha+1}^{\beta} \zeta_{jk}^{cwa}(0)\rho_{jk}(t)\sigma_j(t)\sigma_k(t)} \tag{2.55}$$

where

$$\zeta_{jk}^{cwa}(0) = \frac{w_j(0)F_j(0)w_k(0)F_k(0)}{\left(\sum\limits_{i=\alpha+1}^{\beta} w_i(0)F_i(0)\right)^2}. \tag{2.56}$$

Refined Weight Approximation

$$\sigma_{\alpha,\beta}^{swap}(t) \approx \sqrt{\sum_{j=\alpha+1}^{\beta} \sum_{k=\alpha+1}^{\beta} \zeta_{jk}^{rwa}(0)\rho_{jk}(t)\sigma_j(t)\sigma_k(t)} \tag{2.57}$$

where

$$\zeta_{jk}^{rwa}(0) = \left[\frac{P_{i+1}(0)F_i(0)\delta_i}{A_1(0)} + \frac{\left[A_1(0)B_i(0) - A_i(0)B_1(0)\right] F_i(0)\delta_i}{A_1(0)B_1(0)\left[1 + F_i(0)\delta_i\right]}\right] \times$$

$$\left[\frac{P_{j+1}(0)F_j(0)\delta_j}{A_1(0)} + \frac{\left[A_1(0)B_j(0) - A_j(0)B_1(0)\right] F_j(0)\delta_j}{A_1(0)B_1(0)\left[1 + F_j(0)\delta_j\right]}\right], \tag{2.58}$$

$$A_i(0) = \sum_{j=i}^{\beta} P_{j+1}(0)F_j(0)\delta_j, \tag{2.59}$$

$$B_i(0) = \sum_{j=i}^{\beta} P_{j+1}(0)\delta_j. \tag{2.60}$$

2.4 Instantaneous Correlation

We have seen earlier that the covariance terms drive both the deterministic and the stochastic parts of the LMM evolution

$$Cov(i,j,k) = \int_{T_k}^{T_{k+1}} \sigma_i(u)\sigma_j(u)\rho_{ij}(u)du \tag{2.61}$$

where $\sigma_i(t)$ is the instantaneous volatility of the i^{th} forward rate and $\rho_{ij}(t)$ is the instantaneous correlation between the i^{th} and j^{th} forward rate. Without an explicit time dependence for the correlation function we may rewrite the covariance elements as

$$Cov(i,j,k) = \rho_{ij} \int_{T_k}^{T_{k+1}} \sigma_i(u)\sigma_j(u)du. \tag{2.62}$$

This assumption not only eases computation but allows for a simpler correlation function. We may question if this assumption has a serious pricing impact. An important feature of the covariance structure is the reduction in correlation between two forward rates as the difference between their maturities increases, we refer to this as decorrelation. Rebonato (2005) shows that if the degree of decorrelation is on average correctly recovered then the details of the shape of the correlation function are relatively unimportant. Following Rebonato (2005), in a time homogeneous world the decorrelation is dependent on two quantities. The first quantity is the difference between the expiries of the two forward rates, as we might expect the forward rates move less in step the further apart they are. The second quantity is the first expiry of the two rates. Understanding the dependence of correlation on this quantity is not as obvious. As we might expect that decorrelation will be greater the earlier the first expiry, it is uncertain whether the rate of decorrelation is monotonic. Our first response to determining the correlation would be trying to extract it from observed instrument prices. Referring to the terminal correlation

$$Terminal\ Correlation_{i,j}(t,T) = \frac{\int_t^T \sigma_i(u)\sigma_j(u)\rho_{i,j}(u)dt}{\sqrt{\int_t^T \sigma_i(u)^2 dt}\sqrt{\int_t^T \sigma_j(u)^2 dt}} \tag{2.63}$$

with $T \leq min(T_i, T_j)$, extracting the correlation from the observed prices would only be possible if the assumption on the time dependence of the forward rate instantaneous volatilities is correct. The extraction of correlation information from swaption data would also imply a great amount of faith in the joint informational efficiency of the caplet and swaption markets.

We now turn to historical data for an estimation of the forward rate correlation matrix. The historical estimation is based on a time series of past interest rate curves observed under the real world measure. We see that using the real world measure is made possible by referring back to Girsanov's theorem. The instantaneous correlations between the driving Brownian motions for the forward rate dynamics do not depend on the probability measure under which they are specified. Following Brigo and Morini (2006), it is important to remember that the maturities in the LMM are fixed, whereas the market data is commonly quoted with a fixed time-to-maturity. So we move away from discount factors of the form

$$P(t, t+Z), P(t+1, t+1+Z), ..., P(t+d, t+d+Z)$$

to using interpolation and obtaining discount factors of the form

$$P(t, t+T), P(t+1, t+T), ..., P(t+d, t+T)$$

where Z being the time-to-maturity and T being the maturities in the tenor structure. Daily log returns of the forward rates are extracted and our estimates are based on the following multivariate Gaussian approximation

$$\ln\left(\frac{F_1(t+\triangle t)}{F_1(t)}\right), ..., \ln\left(\frac{F_n(t+\triangle t)}{F_n(t)}\right) \sim N_n(\mu, \Sigma).$$

Direct calibration to historical data is usually filled with its own problems. These problems include outliers, non-synchronous data and discontinuities in the correlation surface. We see by fitting a suitable parametric form to the historical correlation data we end up with a correlation matrix that is smooth, regular and contains other desirable properties of the parametric form. We now look at the following three parameter correlation function proposed by Rebonato (2005)

$$\rho_{ij} = \rho_0 + (1 - \rho_0) e^{-B_1 e^{-B_2 \min(T_i, T_j)} |T_i - T_j|} \tag{2.64}$$

with T_i and T_j being the expiries of the i^{th} and j^{th} forward rates. B_1, B_2 and ρ_0 are the decay constants. The correlation function has the following desirable properties:

- the further apart the forward rates the greater the decorrelation
- the decorrelation is a function of the earliest expiring forward rate
- as the difference between maturities increases the decorrelation asymptotically reaches some finite value ρ_0
- results in a correlation matrix that is always real, symmetric and with positive eigenvalues.

Another approach that is commonly used is that by Schoenmakers and Coffey (2000). They consider a series of coefficients $\{d_i\}$, $i = 1, ..., M$ such that

$$d_1 = 1, \tag{2.65}$$

$$d_i > d_j \quad \text{for} \quad i > j, \tag{2.66}$$

$$\frac{d_i}{d_{i+1}} > \frac{d_j}{d_{j+1}} \quad \text{for} \quad j > i \tag{2.67}$$

and assuming

$$\rho_{i,j} = \frac{c_i}{c_j} \quad i \leq j \quad i, j = 1, ..., M. \tag{2.68}$$

Using this sequence one can construct a possible correlation matrix. An interesting feature of the model is that it assumes that the function $\rho_{i,i+k}$ should be an increasing function of i for a fixed k. One does need to check the validity of this assumption with empirical work. The structure has much of the same desirable properties as the structure above, see Figure 2.1. The degrees of freedom are reduced further by assigning a parametric form to the function d. Schoenmakers and Coffey (2000) provide the following three-parameter form

$$\begin{aligned}\rho_{i,j} = e^{[-|i-j|(\beta-\frac{\alpha_2}{6M-18}(i^2+j^2+ij-6i-6j-3M^2+15M-7)}&\\ {}^{+\frac{\alpha_1}{6M-18}(i^2+j^2+ij-3Mi-3Mj+3i+3j+3M^2-6M+2))]}&\end{aligned} \tag{2.69}$$

with the parameters α_1, α_2 and β being non-negative.

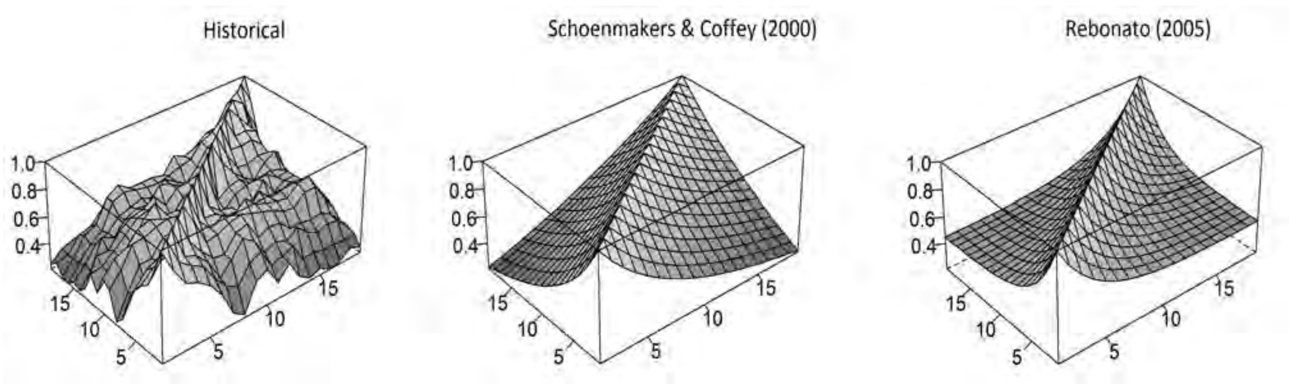

Fig. 2.1: Fitting To Historical Correlation

2.5 Rank Reduction

There may be a need to reduce the dimensionality of the problem to ease the computation when pricing more exotic products, as is the case when pricing Bermudan swaptions. This can be achieved using rank reduction techniques such as Rebonato's geometric construction or eigenvalue zeroing.

2.5.1 Rebonato's Geometric Construction

We refer to the stochastic differential equation derived earlier for the forward rate dynamics using only m factors to shock the yield curve

$$\frac{dF_i}{F_i} = \mu_i dt + \sigma_i \sum_{k=1}^{m} b_{ik} dZ_k. \tag{2.70}$$

The loadings b_{ik}, which are sensitivities of the i^{th} forward to the k^{th} shock, need to to satisfy the following condition

$$\sum_{k=1}^{m} b_{ik}^2 = 1. \tag{2.71}$$

Rebonato provides a geometric interpretation of this condition as the coordinates of a point on the surface of a hypersphere of radius 1. Making use of polar coordinates one can define the point by

$$b_{ik} = \cos\theta_{ik} \prod_{j=1}^{k-1} \sin\theta_{ij}, \qquad k = 1, 2, ..., m-1 \tag{2.72}$$

$$b_{ik} = \prod_{j=1}^{k-1} \sin\theta_{ij}, \qquad k = m. \tag{2.73}$$

This formulation is computationally useful as we can cast a constrained optimization problem in terms of an equivalent unconstrained one. The loadings constraint will always be automatically satisfied. The optimization involves varying the coefficients b_{jk} such that the distance (we will use the χ^2 measure) between the target and model is minimized

$$\chi^2 = \sum_{j,k} (\rho_{model\,jk} - \rho_{market\,jk})^2 \tag{2.74}$$

where

$$\rho_{model\,jk} = [bb^T]_{jk} = \sum_{r=1}^{m} b_{jr} b_{rk}. \tag{2.75}$$

For a large number of forward rates the optimization may no longer be quick and efficient. The solution may also bear no resemblance to the familiar modes of deformation of level, slope and curvature that we commonly use for interpretation.

2.5.2 Eigenvalue Zeroing

Given that ρ is a positive definite symmetric $n \times n$ matrix we can write

$$\rho = PHP^T \tag{2.76}$$

where P is a real orthogonal matrix and H is a diagonal matrix containing the positive eigenvalues of ρ. The columns of matrix P contain the eigenvectors associated to the eigenvalues. Let Λ be the diagonal matrix whose entries are the square roots of the corresponding entries of H so that if we set $A := P\Lambda$ we have

$$AA^T = \rho \tag{2.77}$$

and

$$A^T A = H. \tag{2.78}$$

Following Brigo and Morini (2006), we can try to mimic the decomposition $\rho = AA^T$ by means of a suitable m-rank (with dimensions $n \times m$ with typically $m << n$) matrix B such that BB^T is an m-rank correlation matrix. Instead of parameterising B in terms of angles as we have seen in Rebonato's Geometric Construction, we construct the diagonal matrix Λ^* by removing the smallest diagonal elements. Matrix B is now given by

$$B := P^* \Lambda^* \tag{2.79}$$

where matrix P^* contains the eigenvectors corresponding to the eigenvalues in matrix Λ^*. The correlation matrix can now be defined by

$$\rho^*_{model} = BB^T. \tag{2.80}$$

We see that matrix ρ^*_{model} is positive semi-definite. However the diagonal elements are not equal to one. This can be resolved by interpreting ρ^*_{model} as the covariance matrix. The correlation matrix is then defined by

$$\rho_{model\,ij} = \frac{\rho^*_{model\,ij}}{\sqrt{\rho^*_{model\,ii}\rho^*_{model\,jj}}}. \tag{2.81}$$

2.6 Instantaneous Volatility Structure

Calibrating the LMM to caplets or European swaptions is a highly underdetermined system. We make some structural assumptions about the shape of the forward rate volatility functions to reduce the number of unknown variables. Following Brigo and Morini (2006), we assess the impact of the assumptions using the term structure of volatility and the terminal correlation. The term structure of volatility is a graph of expiry times against average volatilities (V) of the forward rates up to that expiry time, see Figure 2.2. So at time $t = T_j$ the volatility term structure is the graph of the points

$$\{[T_{j+1}, V(T_j, T_{j+1})], [T_{j+2}, V(T_j, T_{j+2})], ..., [T_{M-1}, V(T_j, T_{M-1})]\}$$

where for $h > j + 1$ we have

$$V^2(T_j, T_{h-1}) = \frac{1}{T_{h-1} - T_j} \int_{T_j}^{T_{h-1}} \sigma_h(t)^2 dt. \tag{2.82}$$

Ideally the calibration of the LMM should feature a generally realistic, smooth

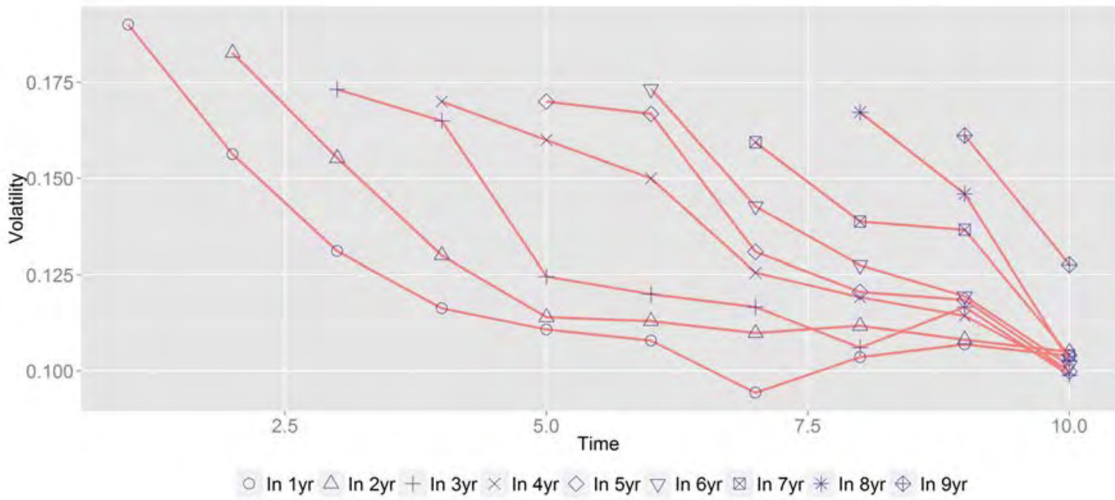

Fig. 2.2: Evolution Of The Term Structure Of Volatility

and qualitatively stable evolution of the term structure. Some of the more common specifications for $\sigma_i(t)$ corresponding to forward rate $F_i(t)$ where $i = 1, ..., N$ are shown in Table 2.1.

Tab. 2.1: Popular Instantaneous Volatility Specifications

Specification	Description
A	Assume volatility is constant in time
	$\sigma_i(t) = \sigma_i$ for $0 \leq t \leq T_{i-1}$
B	Assume volatility is piecewise constant
	$\sigma_i(t) = \sigma_{ij}$ for $T_{j-1} \leq t \leq T_j$, $j = 0, ..., i-1$
C	Assume volatility depends only on time to maturity
	$\sigma_i(t) = \eta_{i-j}$ for $T_{j-1} \leq t \leq T_j$, $j = 0, ..., i-1$
D	Assume volatility depends on maturity and time to maturity
	$\sigma_i(t) = \phi_i \psi_{i-j}$ for $T_{j-1} \leq t \leq T_j$, $j = 0, ..., i-1$
E	Assume volatility follows a functional parameterised form
	$\sigma_i(t) = \phi_i([a(T_{i-1} - t) + d]e^{-b(T_{i-1}-t)} + c)$

Specification B in Table 2.1 is often referred to as the general piecewise constant parameterisation, under this assumption the instantaneous volatilities can be structured as per Table 2.2.

Tab. 2.2: General Piecewise Constant Parameterisation

Forward Rate	$t\epsilon(0, T_0]$	$(T_0, T_1]$	$(T_1, T_2]$	$\cdots$	$(T_{M-1}, T_{M-2}]$
$F_1(t)$	$\sigma_{1,1}$	expired	expired	$\cdots$	expired
$F_2(t)$	$\sigma_{2,1}$	$\sigma_{2,2}$	expired	$\cdots$	expired
$\vdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$F_M(t)$	$\sigma_{M,1}$	$\sigma_{M,2}$	$\sigma_{M,3}$	$\cdots$	$\sigma_{M,M}$

Specification D is a product of two structures and provides a significant reduction in parameters from specification B. The first structure depends on the time to maturity (ψ) and the second depends only on the maturity (ϕ). We see that if the ϕ's are constant then the term structure of volatility remains the same in time but on the contrary if all the ψ's are the same the structure changes. Specification E has a similar form consisting of a parametric core depending on the time to maturity that is locally altered for each maturity with parameter ϕ. The functional form being a combination of a linear and an exponential term allows for a hump in the curve with the instantaneous volatility tending asymptotically to a finite value c. The parameters for specification D and E are typically solved through optimisation. With our focus being on pricing and hedging of long dated maturity products, the large number of forward rates makes solving for these parameters through optimisation slow and inefficient. We start the next section with an algorithm from Brigo and Morini (2006) that solves analytically for the parameters of specification B.

2.7 Rectangular Cascade Calibration With Endogenous Interpolation Algorithm

The algorithm provides a quick and analytical procedure that does not require the use of simulation or optimization. A key part of the setup is the use of the Constant Weight Approximation formula, Equation 2.55, together with the general piecewise constant parameterisation for the forward rate volatility specification. By inverting the approximation formula the exact or quasi-exact input swaption prices are retrieved without calibration error. An interesting feature of the algorithm is that it allows for a one-to-one correspondence between the instantaneous forward rate volatilities and the market swaption volatilities. This is shown in Table 2.3.

Tab. 2.3: Swaption And Forward Rate Volatilities In Cascade Calibration

	Length		
Maturity	1 yr	2 yr	3 yr
1 yr	$\hat{\sigma}^{swap}_{0,1}$	$\hat{\sigma}^{swap}_{0,2}$	$\hat{\sigma}^{swap}_{0,3}$
	$\boxed{\sigma_{1,1}}$	$\sigma_{1,1}, \boxed{\sigma_{2,1}}$	$\sigma_{1,1}, \sigma_{2,1}, \boxed{\sigma_{3,1}}$
2 yr	$\hat{\sigma}^{swap}_{1,2}$	$\hat{\sigma}^{swap}_{1,3}$	$\hat{\sigma}^{swap}_{1,4}$
	$\sigma_{2,1}, \boxed{\sigma_{2,2}}$	$\sigma_{2,1}, \sigma_{2,2}, \sigma_{3,1}$	$\sigma_{2,1}, \sigma_{2,2}, \sigma_{3,1}$
		$\boxed{\sigma_{3,2}}$	$\sigma_{3,2}, \boxed{\sigma_{4,1}, \sigma_{4,2}}$
3 yr	$\hat{\sigma}^{swap}_{2,3}$	$\hat{\sigma}^{swap}_{2,4}$	$\hat{\sigma}^{swap}_{2,5}$
	$\sigma_{3,1}, \sigma_{3,2}, \boxed{\sigma_{3,3}}$	$\sigma_{3,1}, \sigma_{3,2}, \sigma_{3,3}$	$\sigma_{3,1}, \sigma_{3,2}, \sigma_{3,3}$
		$\sigma_{4,1}, \sigma_{4,2}, \boxed{\sigma_{4,3}}$	$\sigma_{4,1}, \sigma_{4,2}, \sigma_{4,3}$
			$\boxed{\sigma_{5,1}, \sigma_{5,2}, \sigma_{5,3}}$

Starting with the first market swaption volatility $\hat{\sigma}^{swap}_{0,1}$ we obtain the corresponding forward rate volatility directly from inverting the approximation formula

$$\left(\hat{\sigma}^{swap}_{\alpha,\beta}\right)^2 \approx \sum_{j=\alpha+1}^{\beta} \sum_{i=\alpha+1}^{\beta} \frac{w_i(0)w_j(0)F_i(0)F_j(0)\rho_{i,j}}{T_\alpha S^\beta_\alpha(0)^2} \sum_{h=0}^{\alpha}(T_h - T_{h-1})\sigma_{i,h+1}\sigma_{j,h+1}. \tag{2.83}$$

After simplifying

$$\left(\hat{\sigma}^{swap}_{0,1}\right)^2 \approx {\sigma_{1,1}}^2. \tag{2.84}$$

Moving to the next swaption volatility $\hat{\sigma}^{swap}_{0,2}$ we have

$$\begin{aligned} S^2_0(0)^2 \left(\hat{\sigma}^{swap}_{0,2}\right)^2 &\approx w_1(0)^2 F_1(0)^2 {\sigma_{1,1}}^2 + w_2(0)^2 F_2(0)^2 {\sigma_{2,1}}^2 \\ &+ 2\rho_{1,2} w_1(0) F_1(0) w_2(0) F_2(0) \sigma_{1,1}\sigma_{2,1}. \end{aligned} \tag{2.85}$$

The only unknown $\sigma_{2,1}$ can be recovered analytically by solving the quadratic equation. The first part of the setup called the Cascade Calibration Algorithm (CCA) addresses the quantities in the upper triangular part of the matrix, shown in grey, by solving the quadratic equation of the form

$$A_{\alpha,\beta}{\sigma_{\beta,\alpha+1}}^2 + B_{\alpha,\beta}\sigma_{\beta,\alpha+1} + C_{\alpha,\beta} = 0. \tag{2.86}$$

The larger of the two roots is assigned to the forward rate volatility $\sigma_{\beta,\alpha+1}$. The solution may not always be positive or even real. As we move through the table from left to right and from top to bottom there will be positions in the table that will have more than one unknown. This is the case when we reach market volatility $\hat{\sigma}_{1,4}$ in the matrix above. At each position of the matrix the unknown forward rate volatilities are outlined with a rectangular block. The second part of the setup addresses this problem by equating the unknowns to each other. The algorithm is now able to extend to the entire swaption matrix. The name of the algorithm now changes to Rectangular Cascade Calibration Algorithm (RCCA). In the case where one reaches the last column, $\beta = s + \alpha$ where s is the number of columns in the matrix, the algorithm assumes all the unknowns are equal to the standard unknown $\sigma_{\beta,\alpha+1}$

$$\sigma_{\beta,\alpha+1} = \sigma_{\beta,\alpha} = ... = \sigma_{\beta,1}. \tag{2.87}$$

Applying this to the matrix above results in

$$\sigma_{4,1} = \sigma_{4,2}. \tag{2.88}$$

So using the entire swaption matrix we can now solve for $\sigma_{\beta,\alpha+1}$ with the new quadratic equation of the form

$$A^*_{\alpha,\beta}{\sigma_{\beta,\alpha+1}}^2 + B^*_{\alpha,\beta}\sigma_{\beta,\alpha+1} + C^*_{\alpha,\beta} = 0. \tag{2.89}$$

So far the algorithm required a complete swaption matrix featuring market swaption volatilities for every position of maturity and length. However this may not always be the case due to non-traded maturities resulting in a number of rows missing in the matrix. The final part of the setup referred to Rectangular Cascade Calibration with Endogenous Interpolation Algorithm (RCCAEI) addresses the issue of missing maturities by moving to the next quoted maturity and using information/volatilities from the quotations to continue the calibration. Referring to the above matrix, if the 2yr maturity is missing we are unable to calculate $\sigma_{3,2}$. The algorithm will then move to the next diagonal element on the 3yr maturity. After assuming $\sigma_{3,2} = \sigma_{3,3}$, the value will be calculated using the swaption volatility ${\hat{\sigma}_{2,3}}^{swap}$.

2.8 Extension For Illiquidity In South African Market

Focusing the context on a life insurance company, we extend the calibration of the LMM to accommodate illiquidity in the South African swaption market. By extending the calibration we are now able to price and hedge many of the products created for the policyholder that will provide protection on interest rates, principal and yield. We focus on the long dated maturity products with guarantees or embedded options. A typical product is the Guaranteed Annuity Option (GAO), from Pelsser (2003) *"the insurer guarantees to convert a policyholder's accumulated funds to a life annuity at the better of the market rate prevailing at the time of conversion and a guaranteed rate".* These products do not have observed market prices and require models to create a price given a set of market inputs. A replicating portfolio of traded securities is set up and hedged over time so that at maturity the market value of the portfolio corresponds to the liability/product. Depending on the nature of the product, traders will typically vega and gamma hedge these products using European swaptions with the delta cancelling out to a large extent between the product and the hedge. Looking at the South African interest rate environment, swaptions are actively traded for a wide variety of maturities with trades being executed for large notional amounts. Swaptions bear close resemblance to many of these structured products and are commonly used in hedging them. Hence the reason for calibrating the LMM to swaptions is because they serve as the natural hedge instruments. Unfortunately illiquidity of swaptions is a major problem beyond 10 years given that many of these products have maturities far beyond this.

Following Rebonato (2005), the implied or relative pricing route requires either the volatilities and correlations to be deterministic and perfectly known (which is certainly not the case) or the input functions are deterministic and not known by the market but with an additional liquid benchmark instrument to "lock-in" the covariance elements that are implied by the market. The latter is not the case for the unhedgeable part of the calibration. The cap market provides no alternative given that it is both illiquid and only available over a short range of maturities. Under the assumption of informational efficiency the historical volatilities provide an unbiased estimate that can be used in the pricing and hedging of the undiversifiable risk. The failure of informational efficiency requires for a systematic imbalance in supply and demand for the derivative and at the same time there needs to be limitations to the actions of pseudo-arbitrageurs.

We propose an extension to the LMM calibration process. The new calibration setup will consist of a combination of market implied quotes for the hedgeable part and historical data for the unhedgeable part. This is achieved by generating a target swaption surface consisting of market implied volatilities for the hedgeable component and thereafter using historical volatilities for the unhedgeable or illiquid component. The historical volatilities are calculated by assuming that the log returns of the swap rate with fixed maturities are normally distributed. The use of historical information in the calibration is not uncommon, as we have shown earlier with the instantaneous correlation being typically obtained through fitting of historical data. The usual problems encountered with historical data of outliers, non-synchronous and stale data will be addressed by smoothing the data with a suitable parametric form. Before solving for these parameters through optimisation we assign weights to the implied and historical data points. The weights are impacted by a number of factors that include maturity of product, bid-ask spreads of the implied quotes and even incorporating a view from the trader. The extension allows the calibration of the LMM to move away from just calibrating to market instruments to one that is setup to facilitate a hedge strategy for a long dated maturity product.

We are reminded that a suitably calibrated model will ensure:

- Correct recovery of future swaptions prices used in hedging
- Little future re-estimation of model parameters
- Stable calibration.

We conclude this section with a quote from Jamshidian (1997) *"The very principle of arbitrage by dynamic trading can be questioned as a practical proposition. This had significant bearing on our attitude to modelling, and leads us to treat the theory only as a guide, making sensible improvisations when useful".*

Chapter 3

Calibration Results To South African Market Data

In this chapter we calibrate the LMM to European swaption volatilities using South African market data. The market data includes quotes used in constructing the yield curve from December 2010 to December 2014 and mid at-the-money European swaption volatilities for December 2014.[1] The quotes used in bootstrapping the yield curve consists of deposits for the first 3 months, forward rate agreements (FRAs) for the next 21 months and then swaps from 3 to 30 years. Extending the yield curve, in our case beyond 30 years, is typically performed using an extrapolation technique that is compatible with hedging the long term interest rate risk. Extrapolation of the yield curve falls outside the scope of the report and as a result we only consider swaption maturities and tenors that fall within 30 years from December 2010. We recall from Section 2.3, the link to the implied Black swaption volatility ($\hat{\sigma}_{\alpha,\beta}^{swap}$) is given by

$$(\hat{\sigma}_{\alpha,\beta}^{swap})^2 T_{exp} = \int_0^{T_{exp}} \sigma_{\alpha,\beta}^{swap}(u)^2 du \tag{3.1}$$

where $\sigma_{\alpha,\beta}^{swap}(t)$ is the instantaneous swap rate volatility. The forward rate covariance elements, instantaneous volatility (σ_j) and correlation (ρ_{jk}) are linked to the

[1] All market data used was provided by a life insurance company in South Africa and will be available upon request.

instantaneous swap rate volatility with (see Section 2.3)

$$\sigma_{\alpha,\beta}^{swap} = \sqrt{\sum_{j=\alpha+1}^{\beta}\sum_{k=\alpha+1}^{\beta}\zeta_{jk}\rho_{jk}\sigma_j\sigma_k}. \tag{3.2}$$

We start by fitting the correlation functions from Rebonato and Schoenmakers-Coffey, described in Section 2.4, to historical data. The correlation function with a closer fit to the data will be used in Equation 3.2.

3.1 Fitting Correlation Function To Historical Data

Using daily data from December 2010 to December 2014, we calculate the historical correlation for one year forward rates with the last forward rate expiring in 30 years time from December 2010. We then fit the data with the correlation functions from Rebonato (2005) and Schoenmakers and Coffey (2000) using MATLAB's Nelder-Mead algorithm. The optimization parameters are shown in Table 3.1.

Tab. 3.1: Parameters Of Correlation Function After Fitting To Historical Data

Rebonato (2005)	$\rho_0 = 0.1135$	$B_1 = 0.9595$	$B_2 = 0.0223$
Schoenmakers-Coffey (2000)	$\alpha_1 = 1.2436 \times 10^{-8}$	$\alpha_2 = 4.8365 \times 10^{-8}$	$\beta = 0.5003$

Even though the Rebonato (2005) correlation function provides a better fit, it still fails to capture the correlation among the forward rates at the far end of the yield curve. The difference between the historical correlation and Rebonato (2005) is shown in Figure 3.1.

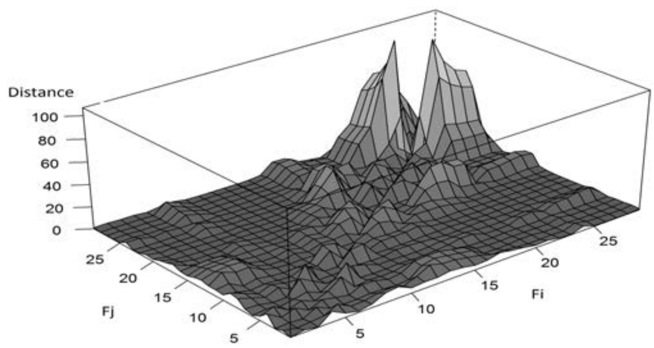

Fig. 3.1: Comparing Rebonato's Correlation Function To Historical Data

Making use of principle component analysis we are able to determine the factor contributions and the minimum number of factors needed to capture the variability across the forward rates. Looking at Table 3.2, we see that only 3 factors are needed to account for 95 % of the variability.

Tab. 3.2: Factor Contributions

	Principal Component					
	1	2	3	4	5	6
Proportion of Variance %	83	9.7	2.9	1.9	1	0.6
Cumulative Proportion %	83	92.7	95.6	97.5	98.5	99.1

Referring to Equation 3.2 and Section 2.3, the swaption price can be calculated without the need for Monte Carlo simulation using the approximation formula from Jackel and Rebonato (2003). We investigate the accuracy of the approximation formula for various maturities using: our yield curve, Rebonato (2005) correlation function and a parametric specification for the forward rate instantaneous volatilities.

3.2 Accuracy Of Approximation Formula

Following a setup similar to Jackel and Rebonato (2003), we investigate the accuracy of the approximation Formulas 2.55 and 2.57 of Section 2.3. We will be answering the following questions: how does the accuracy vary with maturity, is there a need to use the more complex Refined Weight Approximation formula and at which maturities are our pricing errors the largest? We priced a series of co-terminal at-the-money swaptions with the last payment being made in 20.5 years time. We assume the instantaneous volatility is given by specification E of Table 2.1 with ϕ_i being a constant and equal to 1, $a = 0.5$, $b = 1.5$, $c = 0.15$ and $d = -0.05$. The prices from the two approximation formulas are compared to the price from a Monte Carlo simulation performed in MATLAB. The comparison is done for both a constant and non-constant yield curve, see Figures 3.2 and 3.3.

Fig. 3.2: Pricing Error With Approximation Formula Using Constant Yield Curve

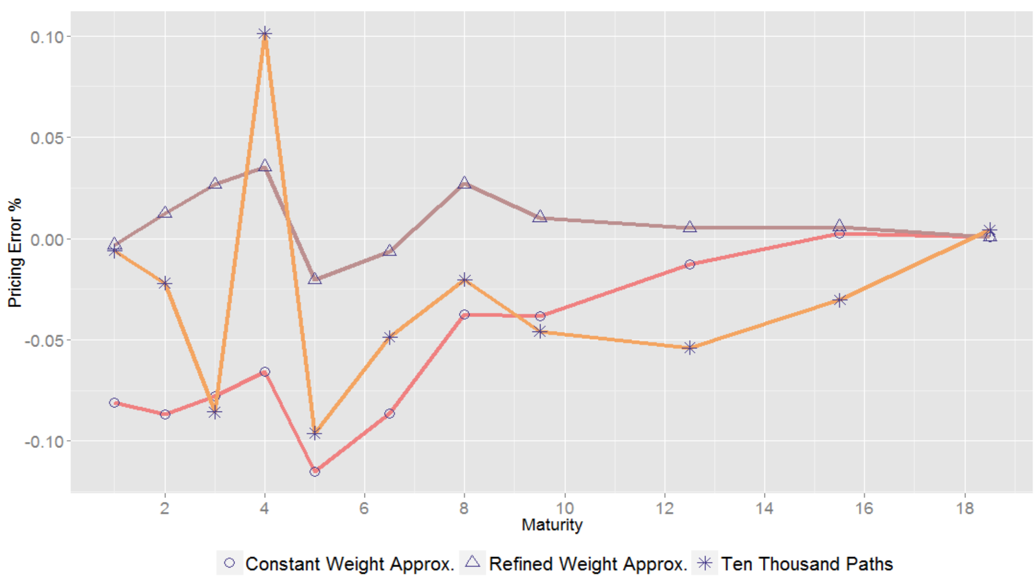

Fig. 3.3: Pricing Error With Approximation Formula Using Non-Constant Yield Curve

The input to the Monte Carlo simulation include 100 000 paths, simulation under the spot measure, discretization using the Euler scheme and variance reduction with antithetic variates. As expected for the constant yield curve, the pricing error for the

Constant and Refined Weight Approximation formulas are identical. We see that the pricing error for the Constant Weight Approximation formula is less than 15 basis points across all maturities, with the error being the largest for the earlier maturities. Given that the pricing error is relatively small in comparison to the observed bid-ask spreads we continue to use the Constant Weight Approximation formula for the rest of the results section. Using the Constant Weight Approximation formula and assuming a general piecewise constant parameterisation for the forward rate volatilities, we can now turn to the RCCAEI algorithm from Brigo and Morini (2006) to provide a quick and analytical calibration algorithm to calculate the forward rate volatilities.

3.3 Implied And Historical Swaption Volatilities

The market volatilities for at-the-money swaptions in December 2014 are shown in Table 3.3. We see that the swaption volatilities are only quoted to the 10 year

Tab. 3.3: Market Quotes For At-The-Money Swaptions

	Swap Length		
Maturity	5	10	15
1	$\hat{\sigma}^{swap}_{0,5} = 0.188$	$\hat{\sigma}^{swap}_{0,10} = 0.185$	$\hat{\sigma}^{swap}_{0,15} = 0.190$
5	$\hat{\sigma}^{swap}_{4,9} = 0.185$		
10		$\hat{\sigma}^{swap}_{9,19} = 0.178$	

maturity. Before calibrating the LMM, we extend the swaption surface following the approach in Section 2.8. We firstly determine the historical lognormal swap rate volatilities. Swap rates with fixed maturities are calculated from the daily yield curves using historical data from December 2011 to December 2014. We then determine the standard deviation of the log returns for the various swap rates. The annualised historical volatilities are shown in Table 3.4.

Tab. 3.4: Historical Volatilities For At-The-Money Swaptions

	Swap Length							
Maturity	3	5	7	10	12	15	17	20
1	0.177	0.162	0.151	0.146	0.146	0.147	0.149	0.149
3	0.163	0.150	0.145	0.145	0.145	0.148	0.148	0.158
5	0.169	0.152	0.152	0.148	0.149	0.149	0.149	0.151
7	0.186	0.175	0.165	0.163	0.164	0.161	0.161	
10	0.234	0.188	0.189	0.188	0.183	0.178		
12	0.219	0.242	0.240	0.209	0.201			
15	0.342	0.301	0.260	0.230				
17	0.354	0.345	0.332					
20	0.588	0.481						
22	0.421							

The lower triangular part of Table 3.4 can be filled by extending the yield curve beyond 30 years with a suitable extrapolation technique. We create a new swaption surface consisting mainly of market volatilities for the first 10 years and historical volatilities for the remaining maturities and tenors. We assign a weight to the market and historical volatilities before smoothing the swaption surface. In discussing the results below, the symbol $W_{MKT/HIST} = z$ would indicate that the market volatilities are weighted z times more than the historical volatilities. We then smooth the volatilities using the parametric form from Brigo, Mercurio and Morini (2005)

$$\textit{Volatility } (M,T) = F_1(M) + \left(\frac{e^{x_6 \cdot \ln(T)}}{x_5 \cdot M} + F_2(M) \right) e^{-x_7 \cdot e^{x_{12} \cdot \ln(T)}} \tag{3.3}$$

where

$$F_1(M) = x_3 + (e^{x_{10} \cdot \ln(M)} \cdot x_1 + x_4) \cdot e^{-x_2 \cdot e^{x_{11} \cdot \ln(M)}} \tag{3.4}$$

$$F_2(M) = x_8 + (e^{x_9 \cdot \ln(M)} \cdot x_{13} + x_{14}) \cdot e^{-x_{15} \cdot e^{x_{16} \cdot \ln(M)}} \tag{3.5}$$

with m and t indicating the swaption maturity and tenor, respectively. The optimization parameters, using MATLAB's Nelder-Mead algorithm, for the parametric form are shown in Table 3.5.

Tab. 3.5: Smoothing Parameters After Fitting To Target Swaption Surface

Parameter	$W_{MKT/HIST} = 1$	$W_{MKT/HIST} = 10$	$W_{MKT/HIST} = 100$
x_1	0.100	0.100	0.100
x_2	53.992	53.999	54.006
x_3	0.184	0.188	0.189
x_4	1.488	0.1663	1.785
x_5	344.679	344.679	344.679
x_6	-2.729	-2.729	-2.730
x_7	39.415	39.425	39.436
x_8	5.380	5.552	5.626
x_9	0.000	-0.004	-0.003
x_{10}	0.128	0.285	0.382
x_{11}	-1.133	-1.104	-1.087
x_{12}	-2.456	-2.153	-1.872
x_{13}	-28.299	-28.531	-28.696
x_{14}	21.953	21.858	21.840
x_{15}	0.161	0.182	0.197
x_{16}	-0.010	-0.118	-0.110

The difference between the smooth swaption and market volatilities are shown in Table 3.6.

Tab. 3.6: 100 $\times$ Absolute Difference Between Smooth And Market Volatilities

Swaption Volatility	$W_{MKT/HIST} = 1$	$W_{MKT/HIST} = 10$	$W_{MKT/HIST} = 100$
$\hat{\sigma}^{swap}_{0,5}$	1.47	0.19	0.04
$\hat{\sigma}^{swap}_{0,10}$	2.11	0.22	0.25
$\hat{\sigma}^{swap}_{0,15}$	2.79	0.81	0.30
$\hat{\sigma}^{swap}_{4,9}$	1.80	0.95	0.29
$\hat{\sigma}^{swap}_{9,19}$	0.00	0.01	0.05

We see that for $W_{MKT/HIST} = 10$ the difference is less than 100 basis points. The market portion of the swaption surface for $W_{MKT/HIST} = 10$ is shown in Figure 3.4.

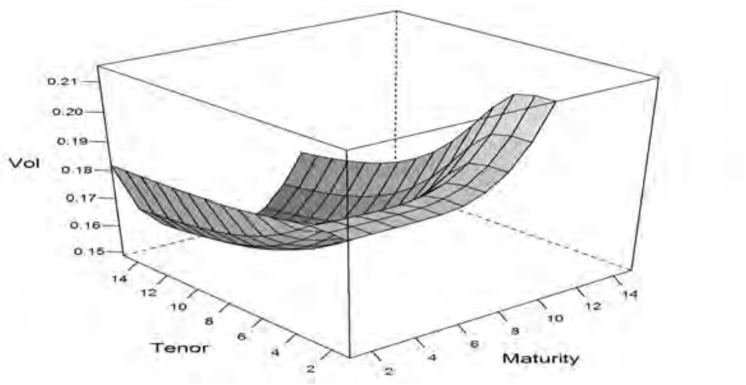

Fig. 3.4: Market Portion Of Swaption Surface For $W_{MKT/HIST} = 10$

We then solve for the forward rate volatilities using the RCCAEI algorithm. The forward rate volatilities for $W_{MKT/HIST} = 10$ are shown in Table 3.7.

Tab. 3.7: Forward Rate Volatilities For $W_{MKT/HIST} = 10$

σ_{ij}	1	2	3	4	5	6	7	8	9	10
1	0.188									
2	0.252	0.086								
3	0.289	0.212	0.086							
4	0.318	0.254	0.212	0.086						
5	0.339	0.261	0.055	0.212	0.086					
6	0.359	0.249	0.315	0.055	0.212	0.086				
7	0.379	0.245	0.244	0.301	0.055	0.212	0.086			
8	0.395	0.252	0.224	0.25	0.311	0.055	0.212	0.086		
9	0.411	0.261	0.221	0.211	0.258	0.046	0.055	0.212	0.086	
10	0.425	0.273	0.23	0.203	0.207	0.358	0.345	0.055	0.212	0.086
11	0.304	0.304	0.423	0.203	0.192	0.207	0.307	0.443	0.121	0.212
12	0.234	0.234	0.234	0.601	0.174	0.185	0.206	0.346	0.518	0.121
13	0.205	0.205	0.205	0.205	0.728	0.161	0.203	0.242	0.393	0.291
14	0.191	0.191	0.191	0.191	0.191	0.855	0.175	0.248	0.293	0.516
15	0.189	0.189	0.189	0.189	0.189	0.189	0.902	0.211	0.299	0.389
16	0.188	0.188	0.188	0.188	0.188	0.188	0.188	1.007	0.282	0.386
17	0.194	0.194	0.194	0.194	0.194	0.194	0.194	0.194	1.106	0.368
18	0.207	0.207	0.207	0.207	0.207	0.207	0.207	0.207	0.207	1.245
19	0.228	0.228	0.228	0.228	0.228	0.228	0.228	0.228	0.228	0.228

Looking at Table 3.7, we see that the forward rate volatilities are positive and real. The volatility term structure for $W_{MKT/HIST} = 10$ is shown in Figure 3.5.

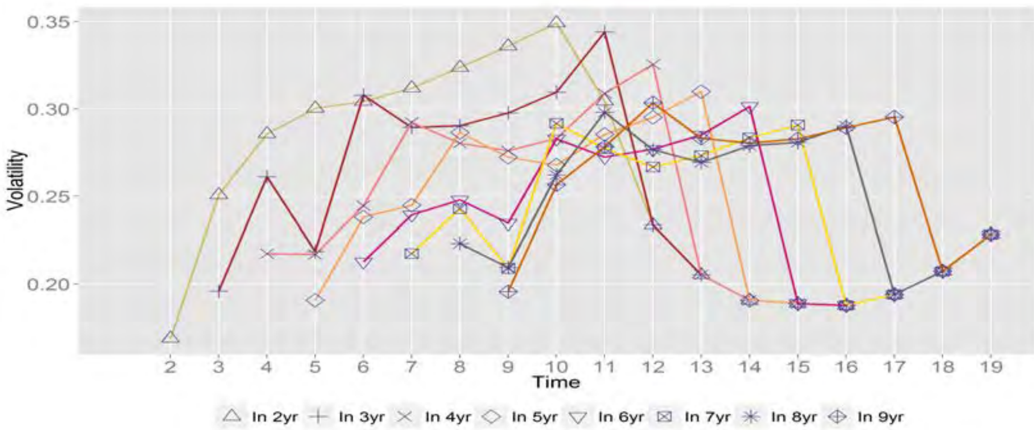

Fig. 3.5: Volatility Term Structure For $W_{MKT/HIST} = 10$

We then assess the calibration under a stress to the market volatilities of 100 basis points up and down. The term structure of volatility shown in Figures 3.6 and 3.7 remains fairly stable under both market stress scenarios.

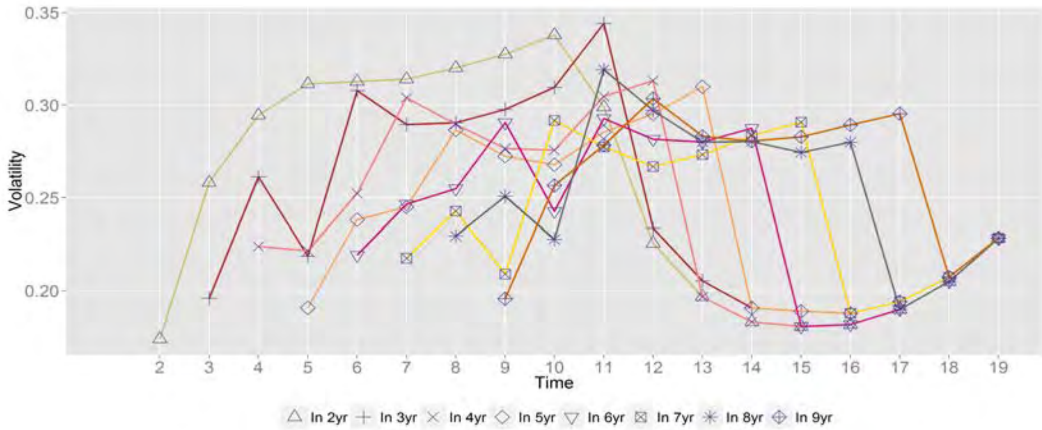

Fig. 3.6: Volatility Term Structure For $W_{MKT/HIST} = 10$: 1% Stress

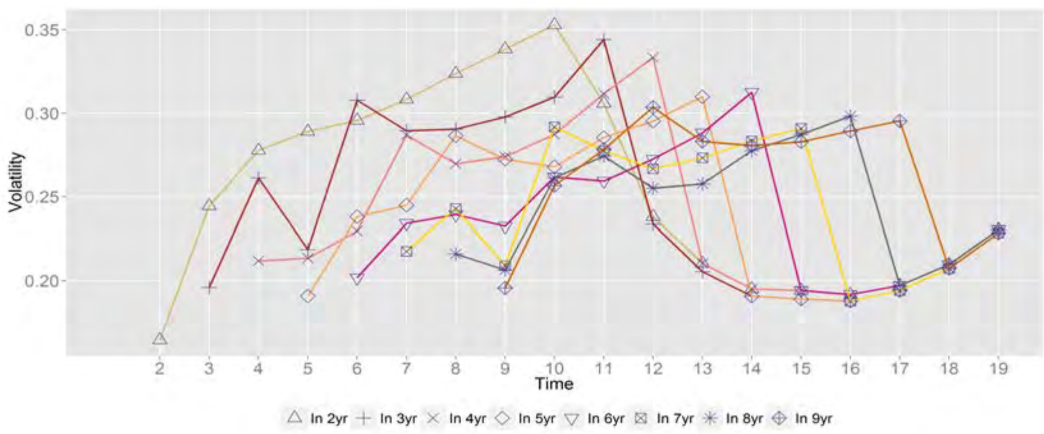

Fig. 3.7: Volatility Term Structure For $W_{MKT/HIST} = 10$: -1% Stress

Referring to Tables 3.8 and 3.9, the forward rate volatilities for both stress scenarios are real and positive.

Tab. 3.8: For $W_{MKT/HIST} = 10$ After Bumping Market Volatilities By 1%

σ_{ij}	1	2	3	4	5	6	7	8	9	10
1	0.194									
2	0.259	0.089								
3	0.299	0.218	0.089							
4	0.331	0.258	0.218	0.089						
5	0.357	0.266	0.046	0.218	0.089					
6	0.379	0.246	0.338	0.046	0.218	0.089				
7	0.401	0.227	0.264	0.324	0.046	0.218	0.089			
8	0.418	0.222	0.23	0.288	0.325	0.046	0.218	0.089		
9	0.435	0.22	0.213	0.239	0.312	0.325	0.046	0.218	0.089	
10	0.45	0.226	0.21	0.216	0.251	0.102	0.319	0.046	0.218	0.089
11	0.299	0.299	0.416	0.205	0.225	0.313	0.383	0.414	0.046	0.218
12	0.226	0.226	0.226	0.576	0.196	0.241	0.272	0.415	0.527	0.046
13	0.197	0.197	0.197	0.197	0.693	0.201	0.255	0.305	0.45	0.524
14	0.183	0.183	0.183	0.183	0.183	0.809	0.22	0.298	0.352	0.539
15	0.181	0.181	0.181	0.181	0.181	0.181	0.853	0.258	0.355	0.435
16	0.182	0.182	0.182	0.182	0.182	0.182	0.182	0.967	0.327	0.432
17	0.19	0.19	0.19	0.19	0.19	0.19	0.19	0.19	1.079	0.408
18	0.205	0.205	0.205	0.205	0.205	0.205	0.205	0.205	0.205	1.213
19	0.228	0.228	0.228	0.228	0.228	0.228	0.228	0.228	0.228	0.228

Tab. 3.9: For $W_{MKT/HIST} = 10$ After Bumping Market Volatilities By -1%

σ_{ij}	1	2	3	4	5	6	7	8	9	10
1	0.183									
2	0.245	0.084								
3	0.281	0.208	0.084							
4	0.303	0.252	0.208	0.084						
5	0.32	0.258	0.067	0.208	0.084					
6	0.338	0.253	0.291	0.036	0.208	0.084				
7	0.357	0.26	0.228	0.303	0.048	0.208	0.084			
8	0.373	0.275	0.221	0.21	0.308	0.048	0.208	0.084		
9	0.388	0.289	0.231	0.188	0.199	0.1	0.048	0.208	0.084	
10	0.401	0.304	0.248	0.196	0.164	0.258	0.348	0.171	0.208	0.084
11	0.306	0.306	0.429	0.206	0.165	0.144	0.232	0.406	0.245	0.208
12	0.238	0.238	0.238	0.62	0.159	0.142	0.133	0.273	0.459	0.245
13	0.21	0.21	0.21	0.21	0.759	0.129	0.157	0.175	0.332	0.367
14	0.195	0.195	0.195	0.195	0.195	0.9	0.139	0.206	0.231	0.415
15	0.194	0.194	0.194	0.194	0.194	0.194	0.958	0.176	0.255	0.31
16	0.192	0.192	0.192	0.192	0.192	0.192	0.192	1.044	0.251	0.34
17	0.197	0.197	0.197	0.197	0.197	0.197	0.197	0.197	1.135	0.338
18	0.21	0.21	0.21	0.21	0.21	0.21	0.21	0.21	0.21	1.263
19	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23

Lastly, we come back to the issue of illiquidity and investigate how well the calibration procedure handles missing volatilities from the market. We achieve this by removing one of the market volatilities and then recalibrate the model. The difference between the target volatilities and the correct market volatilities for $W_{MKT/HIST} = 10$ are shown in Table 3.10.

Tab. 3.10: 100 × Absolute Difference Between Smooth And Market Volatilities

	Missing Market Volatility				
Smooth Swaption Volatility	$\hat{\sigma}^{swap}_{0,5}$	$\hat{\sigma}^{swap}_{0,10}$	$\hat{\sigma}^{swap}_{0,15}$	$\hat{\sigma}^{swap}_{4,9}$	$\hat{\sigma}^{swap}_{9,19}$
$\hat{\sigma}^{swap}_{0,5}$	0.43	0.30	0.37	0.38	0.19
$\hat{\sigma}^{swap}_{0,10}$	0.29	0.52	1.09	0.21	0.22
$\hat{\sigma}^{swap}_{0,15}$	0.85	1.16	1.87	0.73	0.81
$\hat{\sigma}^{swap}_{4,9}$	1.06	0.95	0.79	2.22	0.94
$\hat{\sigma}^{swap}_{9,19}$	0.00	0.01	0.01	0.03	0.03

The largest difference is 222 basis points under the case where we are missing market volatility $\hat{\sigma}^{swap}_{4,9}$.

Chapter 4

Conclusion

Swaptions are typically used in hedging many of the long dated maturity products with embedded options at a life insurance company. The pricing and hedging of these products is facilitated with a market model such as the LMM. Unfortunately, in South Africa, the illiquidity of swaptions with maturities greater than 10 years is a problem for the calibration. We address this issue by extending the calibration to involve a combination of market implied quotes up to 10 years for the hedgeable part and thereafter making use of historical volatilities for maturities beyond 10 years. The historical volatilities are calculated by assuming that the log returns of the swap rates with fixed maturities are normally distributed. We then calibrated the LMM to the extended swaption surface using the Rectangular Cascade Calibration Algorithm by Brigo and Morini (2006). The resulting term structure of volatilities is fairly smooth and stable with the instantaneous forward rate volatilities being positive and real. We recommend evaluating the performance and hedging cost of the extended calibration using an out-of-sample analysis. The extended calibration allows us to capture the benefits of pure replication pricing approach and address the substantial unhedgeable component faced by life insurance companies in South Africa.

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