Internal monoid actions in a cartesian closed category and higher-dimensional group automorphisms

Pako Ramasu

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Supervisor: Professor George Janelidze

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Abstract

The notion of cat^1-group which was introduced by Loday is equivalent to the notions of crossed module and of internal category in the category of groups. This notion of cat^1-groups and their morphisms admits natural generalization to cat^n-groups, which give rise to n-fold categories in the category of groups. There is also a characterization of cat^n-groups in terms of crossed n-cubes which was given by Ellis and Steiner.

The category $\mathbf{Cat}^n(\mathbf{Groups})$ of internal n-fold categories in the category of groups is a cartesian closed category, however given an object X in $\mathbf{Cat}^n(\mathbf{Groups})$, calculating corresponding action representing object $Aut(X)$ directly would require an enormous calculations. The main purpose of the thesis is to describe that object avoiding such calculations as much as possible. The main tool used in the thesis, apart from the theory of cartesian closed categories, is Loday's theory of cat^n-groups. We define a cat^n-group X as an additive M_n-group X, and then construct the corresponding $Aut(X)$, where M_n is a monoid. Since the category of cat^n-groups is equivalent to $\mathbf{Cat}^n(\mathbf{Groups})$ and since the cartesian closed category $\mathbf{Sets}^{M_n}$ of M_n-sets is much easier to handle than the cartesian closed category of n-fold categories, we shall work just with cat^n-groups.

To assert that, $Aut(X)$ is an action representing object in $\mathbf{Sets}^{M_n}$, is to assert that, there is a canonical bijection between B-actions of cat^n-group B on X and the internal group homomorphism $B \to Aut(X)$. Thus, we confirm the construction of $Aut(X)$ by establishing that bijection.

Finally, as one of the results of this work, we give the comparison between our cat^1-group $Aut(X)$ and Norrie's actor crossed module $(D(G, Z), Aut(Z, G, \rho), \omega)$ of a crossed module (Z, G, ρ) in dimension one.

Contents

Introduction

Algebraic and categorical structures such as monoids, groups, reflexive graphs and groupoids can be described within a category with their algebraic identities expressed by commutative diagrams as in S. Mac Lane [27]. These internalized structures admit equivalent descriptions and natural generalization to higher dimensions (see [13] and [17]). In particular, the following higher-dimensinal structures form equivalent categories: n-fold category in category of groups, crossed n-cube in the sense of G. Ellis and R. Steiner [17], and cat^n-group (n-categorical group) described in J.-L. Loday [25].

It is very important to note that, given an object X with an algebraic structure in any cartesian closed category **C**, we can form the corresponding action representing object $Aut(X)$ of invertible elements of the internal monoid $Hom(X, X)$. The main purpose of this thesis is to describe the group $Aut(X)$, in the case of a cat^n-group X, as the subgroup of the corresponding internal automorphism group in the category of M-sets, for a suitable monoid M. This notion of representable action which provides categorical description for automorphism group $Aut(X)$ was introduced by F. Borceux, G. Janelidze and G.M. Kelly (see [9]). They indicated that, if **C** is a cartesian closed category, then the B-actions of internal group B on a internal group X determines the internal group homomorphisms from B to $Aut(X)$. Moreover, these actions are equivalent to split extensions, with this equivalence obtained via the classical semidirect product in the category of groups in the sense of [12].

The study of automorphisms of a crossed module (Z, G, ρ) was initiated by J.H.C. Whitehead (see [35]), when he showed that the set $Der(G, Z)$ of derivations from G to Z has a natural monoid structure, and he further described the group $D(G, Z)$ of invertible elements of $Der(G, Z)$ called the Whitehead group of regular derivatives. His work was continued by K.J. Norrie [29] and A.S.-T. Lue [26] when they constructed the actor crossed module $(D(G, Z), Aut(Z, G, \rho), \omega)$. On the other hand, still in [9], it has been observed that various (but not all) kinds of actors can be defined as suitable action representing objects. In particular, so are the actors in the categories of groups and of Lie algebras; in the Lie algebra case the actor of X is the Lie algebra $Der(X)$ of derivations of X.

As suggested, in a sense, by the work of the above mentioned and various previous authors, the objectives of this thesis can be described as follows:

(a) To construct the internal-group-automorphism M_n-group $Aut(X)$, for a cat^n-group X, as a subgroup of the similar group defined for X considered as an M_n-group, where M_n denotes the monoid $\{1, s_i, t_i, s_it_j, s_is_j, t_it_j\}$ such that for $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n$ we have:

- **(i)** $s_is_j = s_js_i$, $t_it_j = t_jt_i$ and $s_it_j = t_js_i$, $i \neq j$;
- **(ii)** $s_ix = 0 = t_iy$ implies $x + y = y + x$ for $x, y \in X$;
- **(iii)** $s_it_i = t_i$ and $t_is_i = s_i$.

(b) To prove that, for every cat^n-group B, every action $B \times X \to X$ determines a morphism $B \to Aut(X)$.

(c) To use the universal property that defines our cat^1-group $Aut(X)$ (for dimension one), to prove that Norrie's actor crossed module $(D(G, Z), Aut(Z, G, \rho), \omega)$ and our cat^1-group $Aut(X)$ are essentially the same thing, where X is an M_1-group in $\mathbf{Sets}^{M_1}$ and M_1 denotes the monoid $\{1, s_1, t_1\}$, with $s_1t_1 = t_1$, $t_1s_1 = s_1$ such that, if $s_1x = 0 = t_1y$ then $x + y = y + x$ for $x, y \in X$.

0.1 Internal categorical structures

It is know that, the category $\mathbf{Cat}^n(\mathbf{Groups})$ of internal n-fold categories ($n = 1, 2, ...$) in the category of groups is exact, has finite coproducts, and is isomorphic to the category of internal groups in cartesian closed category of n-fold categories. Therefore it is action representable in the sense of [9] by Theorem 4.4 in that paper. However, given an object X in $\mathbf{Cat}^n(\mathbf{Groups})$, calculating the corresponding action representing object $Aut(X)$ directly would require an enormous calculations, and the main purpose of the thesis is to describe that object avoiding such calculations as much as possible.

The main tool used in the thesis, apart from the theory of cartesian closed category, is J.-L. Loday's theory of n-categorical groups, also called cat^n-groups for short (see [25]). Recall that, a cat^n-group is an additive group X equipped with endomorphisms $s_1, s_2, ..., s_n, t_1, t_2, ..., t_n$ with

$$s_is_j = s_js_i, t_it_j = t_jt_i, s_it_j = t_js_i \quad \text{(i,j = 1,2, ..., n;} \quad i \neq j), \tag{0.1}$$

$$s_it_i = t_i, t_is_i = s_i, [ker(s_i), ker(t_i)] = 0 \quad \text{(i= 1,2,..., n).} \tag{0.2}$$

Equivalently, we could begin with the monoid M_n generated by the set

$$\{s_1, s_2, ..., s_n, t_1, t_2, ..., t_n\}$$

satisfying the equalities (0.1) and the first two equalities of (0.2), and define a cat^n-group X as an additive M_n-group X with

$$s_ix = 0 = t_iy \Longrightarrow x + y = y + x \tag{0.3}$$

for all $i = 1, 2, ..., n$ and $x, y \in X$. When $n = 1$, such an M_n-group is also called a categorical group, (see Section 3.1). Since the category of cat^n-groups is equivalent to $\mathbf{Cat}^n(\mathbf{Groups})$ and since the cartesian closed category $\mathbf{Sets}^{M_n}$ of M_n-sets is much easier to handle than the category of n-fold categories, we shall work just with cat^n-groups.

Then we construct the action representing object $Aut(X)$ as follows:

Take $M_1 = \{1, s_1, t_1\}$ satisfying the equalities (0.3) and the first two equalities of (0.2); then the category $\mathbf{Sets}^{M_1}$ can be identified with the category of reflexive graphs, admitting therefore a faithful product-preserving functor

$$\mathbf{U}_1 : \mathbf{Cat} \to \mathbf{Sets}^{M_1}$$

(where $\mathbf{Cat} = \mathbf{Cat}^1$ is the category of categories) defined as follows: For any category X in $\mathbf{Cat}$, the underlying set of $\mathbf{U}_1(X)$ is the set X_1 of morphisms of X, for $x \in \mathbf{U}_1(X)$, $s_1 x$ and $t_1 x$ are the identity morphisms of the domain of x and of the codomain of x, respectively. This functor being product preserving sends monoids to monoids, so that, for an internal monoid X^X in the category $\mathbf{Cat}$, there is an internal monoid homomorphism

$$\theta_X^X : \mathbf{U}_1(X^X) \to \mathbf{U}_1(X)^{\mathbf{U}_1(X)},$$

in $\mathbf{Sets}^{M_1}$ making the diagram

$$\begin{array}{ccc} \mathbf{U}_1(\underline{Hom}(X,X)) & \xrightarrow{\mathbf{U}_1(i_{X,X})} & \mathbf{U}_1(X^X) \\ {\scriptstyle \Theta_X^X} \downarrow & & \downarrow {\scriptstyle \theta_X^X} \\ Hom(\mathbf{U}_1(X), \mathbf{U}_1(X)) & \xrightarrow{i_{\mathbf{U}_1(X),\mathbf{U}_1(X)}} & \mathbf{U}_1(X)^{\mathbf{U}_1(X)} \end{array} \qquad (0.4)$$

a pullback, where $\mathbf{U}_1(i_{X,X})$ and $i_{\mathbf{U}_1(X),\mathbf{U}_1(X)}$ are inclusion maps and $\underline{Hom}(X, X)$ is the subcategory of the functor category X^X whose objects are homomorphic functors and morphisms are homomorphic natural transformations (see Theorem 2.4.4 and Definition 2.4.5 for more details).

As follows from the commutative diagram (0.4), we can calculate $Aut(X)$ using $Aut(\mathbf{U}_1(X))$. In particular, the image $\theta_X^X(\mathbf{U}_1(X^X))$ of $\mathbf{U}_1(X^X)$ in $\mathbf{U}_1(X)^{\mathbf{U}_1(X)}$ satisfies some additional sufficient conditions that make the maximal subgroup $Aut(X)$ of $Hom(X, X)$ a cat^1-group. Then, we describe $Aut(X)$ for an internal group X in $\mathbf{Cat}^n$ for $n > 1$ by analogy as follows:

0.2 The results

The main result is:
Using the language of cat^n-groups, $Aut(X)$, constructed by the afore-mentioned analogy, consists of all the maps $\alpha : M_n \times X \to X$ satisfying

$$m\alpha(m', x) = \alpha(mm', mx), \tag{0.5}$$

$$\alpha(m, x_1 + x_2) = \alpha(m, x_1) + \alpha(m, x_2), \tag{0.6}$$

$$\alpha(m, -) : X \to X \quad \text{is a bijection,} \tag{0.7}$$

$$\alpha(m, x) = \alpha(m, t_i x) - \alpha(s_i m, t_i x) + \alpha(s_i m, x) \quad \text{(i=1, 2, ...,n),} \tag{0.8}$$

$$\alpha(m, x) = \alpha(t_i m, x) - \alpha(t_i m, s_i x) + \alpha(m, s_i x) \quad \text{(i=1, 2, ...,n),} \tag{0.9}$$

for all $m, m' \in M_n$ and $x, x_1, x_2 \in X$; its structure is defined by

$$(\alpha + \beta)(m, x) = \alpha(m, \beta(m, x)), \qquad (m'\alpha)(m, x) = \alpha(mm', x), \tag{0.10}$$

and the role of 0 in it is played by the second projection $\pi_2 : M_n \times X \to X$ (see Section 3.1 for more details).

We show in Proposition 3.1.11 that, $Aut(X)$ as a cat^n-group satisfies

$$s_i\alpha = \pi_2 = t_i\beta \Longrightarrow \alpha(m, \beta(m, x)) = \beta(m, \alpha(m, x)) \ (i = 1, 2, ..., n)$$

for all $\alpha, \beta \in Aut(X)$ and $(m, x) \in M_n \times X$.

In Section 3.2 we prove that for every cat^n-group B, every action $B \times X \to X$ determines a morphism $B \to Aut(X)$.

0.2.1 Remark. *Lastly, we give the connection between the actor crossed module* $(D(G, Z), Aut(Z, G, \rho), \omega)$ *in the sense of K.J. Norrie* [29] *and our cat^1-group* $Aut(X)$ *in dimension one, with* (Z, G, ρ) *being the crossed module corresponding to cat^1-group* X*.*

0.3 Structure of the thesis

The thesis consists of the Introduction and three Chapters. Chapter 1 "Preliminaries and notation", devoted to known material needed in the next chapters, has the following sections:

1. Adjunctions

2. Cartesian closed categories

3. Internal structures in general categories

4. Split extensions and actions of groups

5. Crossed modules and actors

6. Cat^n-groups

Chapter 2 "Internal algebraic structures in cartesian closed categories", first describes/recalls exponents and internal hom objects for internal monoids in the cartesian closed category of M-sets (for an arbitrary monoid M). Since these descriptions will be used to obtain similar ones in the cartesian closed category of all ('small') categories, the next step is to consider a finite product preserving functor $\mathbf{U} : \mathbf{C} \to \mathbf{D}$ between abstract cartesian closed categories and use it to compare the exponents and internal hom objects in $\mathbf{C}$ with those in $\mathbf{D}$. Accordingly the sections of Chapter 2 are:

1. The internal monoid X^X

2. Internal Hom objects

3. The internal monoids $\mathbf{U}(X^X)$ and $\mathbf{U}(X)^{\mathbf{U}(X)}$

4. Applying a product preserving functor

Chapter 3 "Higher-dimensional group automorphisms" has the following sections:

1. The categorical group $Aut(X)$

2. Action representability

3. The connection with Norrie's actors

This chapter begins with the one-dimensional case, where the object $Aut(X)$ is constructed using the results of Chapter 2, then introduces its higher-dimensional version of $Aut(X)$ 'by analogy', and then proves that it is the right construction as described in the first section of this note. It ends by indicating the connection with Norrie's actors [29], again in dimension one, since the higher dimensional case was not developed in [29].

Chapter 1

Preliminaries and notation

1.1 Adjunctions

In this section, we recall some general theory of adjunctions, see [27].

1.1.1 Definition. *An **adjunction***

$$(F, G, \varphi) : \mathbf{D} \longrightarrow \mathbf{C}$$

from a category $\mathbf{D}$ *to a category* $\mathbf{C}$ *is given by a pair of functors*

$$\mathbf{C} \underset{F}{\overset{G}{\rightleftarrows}} \mathbf{D}\,,$$

and, for each object C *in* $\mathbf{C}$*,* D *in* $\mathbf{D}$*, a bijection*

$$\varphi_{D,C} : hom(F(D), C) \longrightarrow hom(D, G(C)), \tag{1.1}$$

is natural in C *and* D*.*

We have $\varphi_{D,C}(h) = G(h)\eta_D$*, for* h *in* $hom(F(D), C)$ *and the map*

$\eta_D : D \longrightarrow GF(D)$ defined as follows; given $\varphi_{D,C}$ as in (1.1)*, set $C = F(D)$, then the left-hand hom-set of* (1.1) *contains the identity $1_{F(D)} : F(D) \to F(D)$ and its $\varphi_{D,F(D)}$-image gives us the morphism*

$$\eta_D = \varphi_{D,F(D)}(1_{F(D)}) : D \longrightarrow GF(D)$$

called the (*the D-component of the*) *unit of the adjunction.*

The inverse $\varphi_{D,C}^{-1}$ of the map $\varphi_{D,C}$ will be denoted by

$$\psi_{D,C} : hom(D, G(C)) \longrightarrow hom(F(D), C), \tag{1.2}$$

and we have $\psi_{D,C}(f) = \varepsilon_C F(f)$, for f in $hom(D, G(C))$, with the map $\varepsilon_C : FG(C) \longrightarrow C$ described as follows: we let $D = G(C)$ in (1.2)*, then the identity element $1_{G(C)}$ is now our f, its image under $\psi_{G(C),C}$ is called the counit of the adjunction ε_C, that is*

$$\varepsilon_C = \psi_{G(C),C}(1_{G(C)}) : FG(C) \longrightarrow C.$$

The naturality of $\varphi_{D,C}$ means that for all $v : C \longrightarrow C'$ in **C** *and $u : D' \longrightarrow D$ in* **D** *we have*

$$\varphi_{D,C'}(v \circ h) = G(v) \circ \varphi_{D,C}(h),$$

for $h \in hom(F(D), C)$ and

$$\varphi_{D',C}(h \circ F(u)) = \varphi_{D,C}(h) \circ u,$$

and the same can be done for $\psi_{D,C}$.

1.2 Cartesian closed categories

In this section we recall the definitions and some examples of cartesian closed categories. For reference see [28].

1.2.1 Definition. *A category* $\mathbf{C}$ *is called* ***cartesian closed category*** *if it has finite products (that is, a terminal object and binary products) and if all objects of* $\mathbf{C}$ *are exponentiable. This means that, for a fixed object* X *in* $\mathbf{C}$*, the functor*

$$(-) \times X : \mathbf{C} \longrightarrow \mathbf{C} \tag{1.3}$$

has a right adjoint denoted by

$$(-)^X : \mathbf{C} \longrightarrow \mathbf{C}. \tag{1.4}$$

If, more generally, such a right adjoint exists for the given X *but not necessarily for all objects in* $\mathbf{C}$*, then* X *is said to be an exponentiable object of the category* $\mathbf{C}$*, while the value* $(-)^X(B) = B^X$ *of* (1.4) *for object* B *in* $\mathbf{C}$ *is called the exponential of* B *and* X*.*

We shall use the following notation:

- *Instead of* $(B) \times X$ *we shall write* $B \times X$ *and we shall write* B^X *instead of* $(B)^X$*.*
- *The unit component* $B \to (B \times X)^X$ *will be denoted by* η_B^X*.*
- *The counit component* $B^X \times X \to B$ *will be denoted by* ε_B^X*.*
- *Accordingly, the bijection*

$$\varphi_{A,B} : hom(A \times X, B) \longrightarrow hom(A, B^X) \tag{1.5}$$

and its inverse

$$\psi_{A,B} : hom(A, B^X) \longrightarrow hom(A \times X, B) \tag{1.6}$$

are defined by $\varphi_{A,B}(h) = h^X \eta_A^X$ *and* $\psi_{A,B}(f) = \varepsilon_B^X(f \times X)$*, respectively, for* $h \in hom(A \times X, B)$ *and* $f \in hom(A, B^X)$*.*

1.2.2 Definition. *An* ***action*** *of a monoid M on a set X is defined as a triple (M, X, h), in which M is a monoid, X is a set and $h : M \times X \longrightarrow X$ a map written as $(m, x) \longmapsto mx$ and satisfying the identities*

$$0x = x, \quad (m_1 + m_2)x = m_1(m_2x)$$

for $x \in X$ and $m, m_1, m_2 \in M$.

The examples of cartesian closed categories considered below in this section will be used in Chapter 2.

1.2.3 Example. *Let M be a monoid, and $\mathbf{Sets}^M$ be the category of M-sets. An object X in $\mathbf{Sets}^M$ is an action $M \times X \to X$ on the set X. If Y is another object in $\mathbf{Sets}^M$, $hom(X, Y)$ is the set of M-morphisms, that is, of all the maps w from X to Y with $w(mx) = mw(x)$ for all x in X and m in M . The category $\mathbf{Sets}^M$ is cartesian closed and, for objects A, B and X in $\mathbf{Sets}^M$, we have:*

(a) $B^X = \{\alpha : M \times X \to B \mid m\alpha(m', x) = \alpha(mm', mx)$ *for all* $m, m' \in M$ *and* $x \in X\}$, *with M acting on B^X by* $(m\alpha)(m', x) = \alpha(m'm, x)$;

(b) *for a morphism $u : B \to B'$, the induced morphism $u^X : B^X \to B'^X$ is defined by $u^X(\alpha) = u\alpha$ for $\alpha \in B^X$;*

(c) $\eta_A^X : A \to (A \times X)^X$ *is defined by* $(\eta_A^X(a))(m, x) = (ma, x)$ *for $a \in A$, $m \in M$ and $x \in X$;*

(d) $\varphi_{A,B} : hom(A \times X, B) \longrightarrow hom(A, B^X)$ *is defined by*
$\varphi_{A,B}(h)(a)(m, x) = h^X(\eta_A^X(a))(m, x)$ *(see* (1.1))
$= h\eta_A^X(a)(m, x)$ *(see* (b))
$= h(ma, x)$ *for $h \in hom(A \times X, B)$, $a \in A$, $m \in M$ and $x \in X$;*

(e) $\varepsilon_B^X : B^X \times X \to B$ *is defined by*
$\varepsilon_B^X(\alpha, x) = \alpha(1, x)$ *for $\alpha \in B^X$ and $x \in X$;*

(f) $\psi_{A,B}: hom(A, B^X) \longrightarrow hom(A \times X, B)$ *is defined by*

$\psi_{A,B}(f)(a, x) = \varepsilon_B^X(f \times X)(a, x)$ *(see (1.2))*

$= \varepsilon_B^X(f(a), x)$

$= f(a)(1, x)$ *for* $f \in hom(A, B^X)$, $a \in A$ *and* $x \in X$ *(see (e))* .

1.2.4 Example. *The category* **Cat** *of small categories is cartesian closed. We use the following notation:*

The morphisms in categories A, B and X will be denoted with the same but lowercase letters, possibly with primes, while the objects again in the same way, but underlined.

(a) B^X *is the category of all functors* $X \to B$*;*

(b) *for a functor* $U : B \to B'$*, the induced functor* $U^X : B^X \to B'^X$ *carries a natural transformation* $\tau : V \to V'$ *to the natural transformation* $U\tau = 1_V\tau : UV \to UV'$ *defined by* $(U\tau)_{\underline{x}} = U(\tau_{\underline{x}})$*, where* $\underline{x}$ *is an object in* X*;*

(c) $\eta_A^X : A \to (A \times X)^X$ *is the functor that carries an object* $\underline{a}$ *in A to the functor* $\eta_A^X(\underline{a}) : X \to A \times X$ *that carries* $x : \underline{x} \to \underline{x}'$ *to* $(1_{\underline{a}}, x) : (\underline{a}, \underline{x}) \to (\underline{a}, \underline{x'})$*, and carries a morphism* $a : \underline{a} \to \underline{a'}$ *to the natural transformation* $\eta_A^X(a) : \eta_A^X(\underline{a}) \to \eta_A^X(\underline{a'})$ *defined by* $(\eta_A^X(a))_{\underline{x}} = (a, 1_{\underline{x}}) : (\underline{a}, \underline{x}) \to (\underline{a'}, \underline{x})$*;*

(d) $\varphi_{A,B}: hom(A \times X, B) \longrightarrow hom(A, B^X)$ *carries a functor* $H : A \times X \to B$ *in* $hom(A \times X, B)$ *to the functor* $\varphi_{A,B}(H) : A \to B^X$ *in* $hom(A, B^X)$ *that carries an object* $\underline{a}$ *in A to the functor* $H(\underline{a}, -) : X \to B$*, and carries a morphism* $a : \underline{a} \to \underline{a'}$ *to the natural transformation* $H(a, -) : H(\underline{a}, -) \to H(\underline{a'}, -)$ *defined by* $H(a, -)_{\underline{x}} = H(a, \underline{x})$*;*

(e) $\varepsilon_B^X : B^X \times X \to B$ *carries a morphism* $(\tau, x) : (V, \underline{x}) \to (V', \underline{x'})$ *in* $B^X \times X$ *to the morphism* $\tau_{\underline{x'}}V(x) = V'(x)\tau_{\underline{x}} : V(\underline{x}) \to V'(\underline{x'})$ *in B;*

(f) $\psi_{A,B}: hom(A, B^X) \longrightarrow hom(A \times X, B)$ *carries a functor* $F : A \to B^X$ *in* $hom(A, B^X)$ *to the functor* $\psi_{A,B}(F) : A \times X \to B$ *in* $hom(A \times X, B)$ *that* *carries a morphism* $(a,x) : (\underline{a},\underline{x}) \to (\underline{a'},\underline{x'})$ *in* $A \times X$ *to the morphism* $F(a)_{\underline{x'}}(F(\underline{a})(x)) = (F(\underline{a'})(x))F(a)_{\underline{x}}$ *in* B.

1.2.5 Example. *Generalizing Example* 1.2.3*, consider the functor category* $\mathbf{Sets}^{\mathbf{C}^{op}}$*, where* $\mathbf{C}$ *is a locally small category; the objects of this cartesian closed category are set-valued functors and the morphisms are natural transformations. For* C *in* $\mathbf{C}$ *and* P, Q *in* $\mathbf{Sets}^{\mathbf{C}^{op}}$ *we have:*

(a) *the exponential* Q^P *is defined by* $Q^P(C) = Nat(hom(-,C) \times P, Q)$ *of all natural transformations from* $hom(-,C) \times P$ *to* Q*;*

(b) *given natural transformation* $U : Q \to Q'$*, then* $U^P(C) : Nat(hom(-,C) \times P, Q) \to Nat(hom(-,C) \times P, Q')$ *is given by* $(U^P(C)(\varrho))_C(1_C, p) = (U\varrho)_C(1_C, p)$*, where* $\varrho \in Nat(hom(-,C) \times P, Q)$ *and* $p \in P(C)$*;*

(c) $\eta_R^P : R \to (R \times P)^P$ *is a natural transformation which carries object* r *in* $R(C)$ *to* $(\eta_R^P)_C(r) : hom(-,C) \times P \to R \times P$ *defined by* $((\eta_R^P)_C(r))_{C'}(c, p') = (R(c)(r), p')$ *for* c *in* $hom(C', C)$*,* p' *in* $P(C')$ *and* $R(c)$ *is a morphism from* $R(C)$ *to* $R(C')$*;*

(d) $\varphi_{R,Q} : Nat(R \times P, Q) \longrightarrow Nat(R, Q^P)$ *carries a natural transformation* $H : R \times P \to Q$ *to the natural transformation* $\varphi_{R,Q}(H) : R \to Q^P$ *that carries an object* $r \in R(C)$ *to* $(\varphi_{R,Q}(H))_C(r) : hom(-,C) \times P \to Q$ *defined by* $((\varphi_{R,Q}(H))_C(r))_{C'}(c, p') = H(R(c)(r), p')$ *for* $C' \in \mathbf{C}$*,* $p' \in P(C')$ *and* c *is a morphism from* C' *to* C *;*

(e) *the evaluation morphism* $\varepsilon_Q^P : Q^P \times P \to Q$ *is defined as* $(\varepsilon_Q^P)_C(\varrho, p) = \varrho_C(1_C, p) \in Q(C)$ *for* $C \in \mathbf{C}$*,* $\varrho \in Nat(hom(-,C) \times P, Q)$ *and* $p \in P(C)$*;*

(f) $\psi_{R,Q} : Nat(R, Q^P) \longrightarrow Nat(R \times P, Q)$ *carries a natural transformation* $F : R \to Q^P$ *to a natural transformation* $\psi_{R,Q}(F) : R \times P \to Q$ *defined by* $(\psi_{R,Q}(F))_C(r, p) = F(r)(1_C, p)$ *for* $C \in \mathbf{C}$*,* $r \in R(C)$ *and* $p \in P(C)$*.*

1.3 Internal structures in general categories

For reference see [27].

1.3.1 Definition. *A **monoid** in a category* $\mathbf{C}$ *with finite products and a terminal object* 1 *is a triple* $M = (M, e_M, w_M)$*, in which* M *is an object in* $\mathbf{C}$*,* $e_M : 1 \to M$ *and* $w_M : M \times M \to M$ *are morphisms in* $\mathbf{C}$ *such that the diagrams*

$$\begin{array}{ccccc}
M \times (M \times M) & \xrightarrow{\Gamma} & (M \times M) \times M & \xrightarrow{w_M \times 1_M} & M \times M \\
{\scriptstyle 1_M \times w_M}\downarrow & & & & \downarrow{\scriptstyle w_M} \\
M \times M & & \xrightarrow{\qquad\quad w_M \qquad\quad} & & M
\end{array}$$

and

$$\begin{array}{ccccc}
1 \times M & \xrightarrow{e_M \times 1_M} & M \times M & \xleftarrow{1_M \times e_M} & M \times 1 \\
{\scriptstyle \lambda_M}\downarrow & & \downarrow{\scriptstyle w_M} & & \downarrow{\scriptstyle \varpi_M} \\
M & = & M & = & M
\end{array}$$

commute, where Γ*,* λ_M *and* ϖ_M *are canonical isomorphisms.*

1.3.2 Definition. *A **group** in a category* $\mathbf{C}$ *with finite products and a terminal object* 1 *is a monoid* (M, e_M, w_M) *together with a morphism* $\zeta : M \to M$ *in* $\mathbf{C}$ *which makes the diagram*

$$\begin{array}{ccccc}
M & \xrightarrow{<1_M, 1_M>} & M \times M & \xrightarrow{1_M \times \zeta} & M \times M \\
{\scriptstyle !}\downarrow & & & & \downarrow{\scriptstyle w_M} \\
1 & & \xrightarrow{\qquad\quad e_M \qquad\quad} & & M
\end{array}$$

commute. Internal groups are usually called group objects.

1.3.3 Example. *In* **Sets**, *group objects are just groups in the classical sense.*

1.3.4 Example. *In the category* **Top** *of topological spaces and continuous functions an internal group is just a topological group.*

1.3.5 Example. *The category of internal groups in the category of small categories is equivalent to the category of crossed modules.*

1.3.6 Example. *A group internal to the category of groups and group homomorphisms is an abelian group.*

1.3.7 Definition. *An* ***internal preorder*** *in a category* **C** *with finite products is a pair* $C = (C_0, C_1)$ *in* **C**, *where* C_0 *is an object in* **C** *and* C_1 *is a subobject in* $C_0 \times C_0$ *which is a reflexive and transitive relation on* C_0.

1.3.8 Definition. *An* ***internal reflexive graph*** *in* **C** *is given by the diagram*

$$\xymatrix{C_1 \ar@/^1.5pc/[rr]^{d} \ar@/_1.5pc/[rr]_{c} && C_0 \ar[ll]_{e}}$$

where C_0 *is called the object of objects,* C_1 *the object of morphisms,* d *the domain,* c *the codomain and* e *the identity, with* $de = 1_{C_0} = ce$.

1.3.9 Definition. *An* ***internal category*** *in category* ***C*** *with pullbacks is a system* $C = (C_0, C_1, e, d, c, m)$ *which consists of two objects* C_0 *and* C_1 *of* ***C***, *called the object of objects and the object of morphisms, respectively, together with four maps in* ***C***, *displayed as*

$$\xymatrix{C_2 = C_1 \times_{C_0} C_1 \ar[r]^-{m} & C_1 \ar@/^1.5pc/[r]^{d} \ar@/_1.5pc/[r]_{c} & C_0 \ar[l]_{e}} \;, \tag{1.7}$$

where d is called the domain again, c the codomain, e the identity and m the composition; m is defined using the following pullback

$$\begin{array}{ccc} C_1 \times_{C_0} C_1 & \xrightarrow{\pi_2} & C_1 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle c} \\ C_1 & \xrightarrow{d} & C_0. \end{array} \tag{1.8}$$

The following conditions are required:

(a) $de = 1_{C_0} = ce$;

(b) $c\pi_1 = cm$, $dm = d\pi_2$;

(c) $\pi_2 = m(ed \times_{C_0} 1_{C_1})$, $m(1_{C_1} \times_{C_0} ec) = \pi_1$;

(d) $m(1_{C_1} \times_{C_0} m) = m(m \times_{C_0} 1_{C_1})$.

1.3.10 Example. *Suppose* **C** *is the category of groups. Here C_0 and C_1 above become groups and our structural morphisms m, d, c and e become group homomorphisms; this implies that the composite of composable pair of morphisms*

$$\underline{y} \xrightarrow{y} \underline{x} \xrightarrow{x} \underline{x'}$$

is uniquely determined by

$$m(x, y) = m((x, ed(x)) - (ed(x), ed(x)) + (ed(x), y)) = x - ed(x) + y. \tag{1.9}$$

In other words, the categorical composition m is determined by the group structure of $C = (C_0, C_1, e, d, c, m)$. Alternatively, we also have

$$m(x, y) = y - ed(x) + x. \tag{1.10}$$

Hence the group operation satisfies

$$x - ed(x) + y = y - ed(x) + x.$$

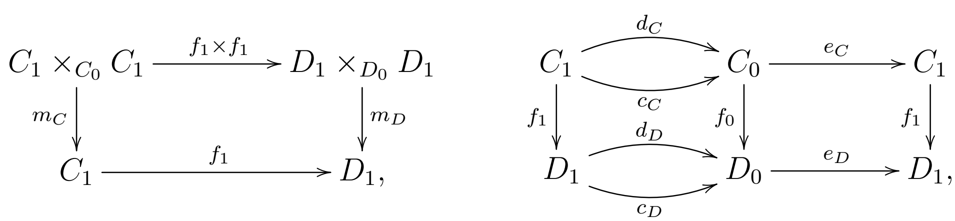
In particular if $d(x)$ is the unit element in C_0, then $ed(x)$ is the unit element in C_1, and it follows from (1.9) and (1.10) that

$$x + y = y + x \text{ when } x \in ker(d) \text{ and } y \in ker(c). \tag{1.11}$$

1.3.11 Remark. *Again, from (1.9) we observe that each element x in C_1 has an inverse for the composition, given by $x^{-1} = ed(x) - x + ec(x)$, so that every internal category in the category of groups is an internal groupoid.*

One can speak of internal categories in **Groups** and internal groups in **Cat** as essentially the same thing. This suggest that, there is an equivalence of categories between these two categories of internal structures.

1.3.12 Definition. *An* ***internal functor*** *$f : C \longrightarrow D$ between two internal categories C and D in the same category $\boldsymbol{C}$ is defined as a pair of maps $f_0 : C_0 \longrightarrow D_0$ and $f_1 : C_1 \longrightarrow D_1$ of $\boldsymbol{C}$ making the following diagrams commute:*



in obvious notation, where by the commutativity of the right hand diagram we mean $f_0 d_C = d_D f_1$, $f_0 c_C = c_D f_1$ and $f_1 e_C = e_D f_0$.
We will denote the category of internal categories and internal functors in the category $\mathbf{C}$ by $\mathbf{Cat}(\mathbf{C})$.

1.3.13 Definition. *The category $\mathbf{Cat}^n(\mathbf{C})$ of n-fold categories internal in $\mathbf{C}$ is defined inductively as follows: $\mathbf{Cat}^1(\mathbf{C}) = \mathbf{Cat}(\mathbf{C})$ and, for $n \geq 2$,*
$\mathbf{Cat}^n(\mathbf{C}) = \mathbf{Cat}(\mathbf{Cat}^{n-1}(\mathbf{C}))$.

1.3.14 Lemma. *Let M and M_1 be monoids both acting on an object X in an arbitrary category $\mathbf{C}$. The following conditions are equivalent:*

(a) *the two actions commute with each other in the sense that* $m(m'x) = m'(mx)$ *for every object* C *in* $\mathbf{C}$ *and morphisms* $m : C \to M$, $m' : C \to M_1$, $x : C \to X$;

(b) *the two actions determine an* $M_1 \times M$*-action on* X *by* $(m', m)x = m'(mx)$ *for all* m, m' *and* x *as above.*

Proof. Just note that, say, an M-action on X is nothing but a monoid homomorphism $M \to End(X)$ (where $End(X)$ is the endomorphism monoid of X). $\square$

Using this lemma in the case $M_1 = \{1, s_1, t_1\}$ with $s_1t_1 = t_1$ and $t_1s_1 = s_1$, it is easy to prove the following (known) fact:

1.3.15 Theorem. *The category* $\mathbf{Cat}^n(\mathbf{Groups})$ *is equivalent to the full subcategory of* $\mathbf{Groups}^{\overbrace{\{1,s_1,t_1\}\times\{1,s_1,t_1\}\times ... \times\{1,s_1,t_1\}}^{n \quad times}}$ *with objects all* $\overbrace{\{1,s_1,t_1\}\times\{1,s_1,t_1\}\times ... \times\{1,s_1,t_1\}}^{n \quad times}$*-groups* X *with* $s_1t_1 = t_1, t_1s_1 = s_1$ *and* $(\overbrace{1,1,...,1}^{k \quad times}, s_1, 1, ..., 1)x = 0 = (\overbrace{1,1,...,1}^{k \quad times}, t_1, 1, ..., 1)y \Rightarrow x + y = y + x$, *for all* $x, y \in X$ *and for all* $k = 0, 1, ..., n-1$.

1.4 Split extensions and actions of groups

Let B and X be groups. It is well known, and mentioned in [9] and [10] is a more general situation, that giving a group homomorphism $B \to Aut(X)$ is equivalent to giving a split extension of B with kernel X. Let us recall the details:

1.4.1 Definition. *A* ***split extension*** *of groups* A, B *and* X *is a diagram of the form*

$$X \xrightarrow{\kappa} A \underset{\nu}{\overset{\delta}{\rightleftarrows}} B \qquad (1.12)$$

in which $\delta\nu = 1_B$ *and* (X, κ) *is a kernel of* δ.
Note also that for such a split extension we always have

$$\nu(B) \bigcap \kappa(X) = 0$$

since, for $a \in \nu(B) \bigcap \kappa(X)$*, we have*

$$a = \nu\delta(a) = \nu(0) = 0.$$

1.4.2 Proposition. *Let* (1.12) *be a split extension. Then the maps* $\xi : B \times X \longrightarrow A$ *and* $\omega : A \longrightarrow B \times X$ *defined by*

$$\xi(b, x) = \kappa(x) + \nu(b) \tag{1.13}$$

and

$$\omega(a) = (\delta(a), \kappa^{-1}(a - \nu\delta(a))) \tag{1.14}$$

respectively, are bijections, inverse to each other. In particular

$$\kappa(X) + \nu(B) = A,$$

where the left-hand side is defined as $\kappa(X)+\nu(B) = \{\kappa(x)+\nu(b) \mid x \in X, b \in B\}$.

Proof. Given a in A, we have

$$\xi\omega(a) = \xi(\delta(a), \kappa^{-1}(a - \nu\delta(a))) = a - \nu\delta(a) + \nu\delta(a) = a.$$

That is, $\xi\omega = 1_A$.
Conversely, for any $b \in B$ and $x \in X$, we get

$$\omega\xi(b, x) = \omega(\kappa(x) + \nu(b)) = (\delta(\kappa(x) + \nu(b)), \kappa^{-1}(\kappa(x) + \nu(b) - \nu\delta(\kappa(x) + \nu(b)))$$
$$= (0 + b, \kappa^{-1}(\kappa(x) - \nu(b) + \nu(b))) = (b, x).$$

This shows that, indeed $\omega\xi = 1_{B\times X}$. $\square$

1.4.3 Construction. *Consider a split extension* (1.12) *of groups. If we let* ξ *and* ω *be defined as in Proposition* 1.4.2 *to be group isomorphisms, then the "new" group structure on* $B \times X$ *will be given as follows:*

$$(b_1, x_1) + (b_2, x_2) = \omega(\xi(b_1, x_1) + \xi(b_2, x_2)) = \omega((\kappa(x_1) + \nu(b_1)) + (\kappa(x_2) + \nu(b_2)))$$
$$= \omega(\kappa(x_1) + (\nu(b_1) + \kappa(x_2) + \nu(-b_1)) + \nu(b_1) + \nu(b_2))$$
$$= \omega(\kappa(x_1) + (\nu(b_1) + \kappa(x_2) + \nu(-b_1)) + \nu(b_1 + b_2)).$$

In order to simplify this expression further, we need:

1.4.4 Definition. *An* ***action*** *of a group* B *on a group* X *is defined as a triple* (B, X, h)*, in which* B *and* X *are groups and* $h : B \times X \longrightarrow X$ *a map written as* $(b, x) \longmapsto bx$ *and satisfying the identities*

$$0x = x,\ (b_1 + b_2)x = b_1(b_2x),\ b(x_1 + x_2) = bx_1 + bx_2$$

for $x, x_1, x_2 \in X$ *and* $b, b_1, b_2 \in B$*.*
A group X equipped with such an action is also called a B-group.

1.4.5 Lemma. *In the notation above, for every* $b \in B$ *and* $x \in X$*, there exist a unique* $y \in X$ *with* $\kappa(y) = \nu(b) + \kappa(x) + \nu(-b)$*.*

Proof. This follows from the fact that (X, κ) is a kernel of δ and

$$\delta(\nu(b) + \kappa(x) + \nu(-b) = \delta\nu(b) + 0 + \delta\nu(-b) = 0.$$

The element y will be denoted by bx. That is, we have

$$\kappa(bx) = \nu(b) + \kappa(x) + \nu(-b),$$

or, equivalently,

$$bx = \kappa^{-1}(\nu(b) + \kappa(x) + \nu(-b)). \tag{1.15}$$

Using this, our calculations in Construction 1.4.3 gives

$$(b_1, x_1) + (b_2, x_2) = \omega(\kappa(x_1) + \kappa(b_1x_2) + \nu(b_1 + b_2)) = \omega\xi(b_1 + b_2, x_1 + b_1x_2),$$

and so

$$(b_1, x_1) + (b_2, x_2) = (b_1 + b_2, x_1 + b_1x_2).\square \tag{1.16}$$

1.4.6 Definition. *Given (B, X, h) above, the set $B \times X$ equipped with the group structure defined by* (1.16) *is called the **semi-direct product** of B and X denoted by $B \ltimes X$.*

1.4.7 Proposition. *Given groups B and X and a map $h : B \times X \to X$ written as $h(b, x) = bx$, the following conditions are equivalent:*

(a) *The addition defined by* (1.16) *makes $B \times X$ a group;*

(b) *The addition defined by* (1.16) *makes $B \times X$ a monoid;*

(c) *X is a B-group.*

- An object in the category of group actions on groups **Act**(**Groups**), is given by a triple (B, X, h) (Definition 1.4.4). A morphism $(B, X, h) \longrightarrow (B', X', h')$ in this category is a pair (u, v) of group homomorphisms $u : B \longrightarrow B'$ and $v : X \longrightarrow X'$ with

$$v(bx) = u(b)v(x) \tag{1.17}$$

for all $b \in B$ and $x \in X$.

- An object in the category $\mathbf{Split}_{ext}(\mathbf{Groups})$ is given by a split extension of groups of the form (1.12), and a morphism from a split extension (1.12) to another split extension displayed as the bottom row in the diagram

$$
\begin{array}{ccccc}
X & \xrightarrow{\kappa} & A & \underset{\nu}{\overset{\delta}{\rightleftarrows}} & B \\
\downarrow{\scriptstyle v} & & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle u} \\
X' & \xrightarrow{\kappa'} & A' & \underset{\nu'}{\overset{\delta'}{\rightleftarrows}} & B'
\end{array}
\tag{1.18}
$$

  is a triple (u, t, v) of group homomorphisms, as in the diagram, with $u\delta = \delta' t$, $t\nu = \nu' u$ and $t\kappa = \kappa' v$.

1.4.8 Theorem. *There is a category equivalence between* $\mathbf{Split}_{ext}(\mathbf{Groups})$ *and* $\mathbf{Act}(\mathbf{Groups})$.

Proof. We define functors $U : \mathbf{Act}(\mathbf{Groups}) \longrightarrow \mathbf{Split}_{ext}(\mathbf{Groups})$ and $V : \mathbf{Split}_{ext}(\mathbf{Groups}) \longrightarrow \mathbf{Act}(\mathbf{Groups})$ as follows:

- For a morphism $(u, v) : (B, X, h) \longrightarrow (B', X', h')$ in $\mathbf{Act}(\mathbf{Groups})$, $U(u, v)$ is the morphism

$$
\begin{array}{ccccc}
X & \xrightarrow{<0,1>} & B \times X & \underset{<1,0>}{\overset{\pi_1}{\rightleftarrows}} & B \\
\downarrow{\scriptstyle v} & & \downarrow{\scriptstyle u\times v} & & \downarrow{\scriptstyle u} \\
X' & \xrightarrow{<0,1>} & B' \times X' & \underset{<1,0>}{\overset{\pi_1}{\rightleftarrows}} & B'
\end{array}
\tag{1.19}
$$

in which the group structure on $B \times X$ and on $B' \times X'$ are defined by (1.16).

- For a morphism (1.18) in $\mathbf{Split}_{ext}(\mathbf{Groups})$, $V(u, t, v) = (u, v) : (B, X, h) \longrightarrow (B', X', h')$ with h and h' both defined by (1.15).

Proposition 1.4.7 $(c) \Rightarrow (a)$, tells us that the functor U is well-defined on objects. After that, it is easy to see that it is well-defined on morphisms. We only need to check that $(u \times v) : B \times X \longrightarrow B^{'} \times X^{'}$ preserves the addition defined by (1.16), and observe that the relevant parts of (1.19) obviously commute.
Proposition 1.4.7 $(a) \Rightarrow (c)$, together with the isomorphism between A and $B \times X$ (recall that bijection of Proposition 1.4.2 becomes an isomorphism when $B \times X$ is equipped with the addition defined by (1.16)), tells us that the functor V is well-defined on objects. After that, to see that it is well-defined on morphisms, we only need to check the identity (1.17), when the actions are defined by (1.15) using the split extensions displayed in (1.18).
A routine calculation shows that $UV = 1_{\mathbf{Split}_{ext}(\mathbf{Groups})}$ and $VU \cong 1_{\mathbf{Act(Groups)}}$.

1.5 Crossed modules and actors

In this section, we recall some standard results on crossed modules which were introduced by J.H.C. Whitehead [34] in 1946. They play a crucial role in homotopy theory. As recalled below, these objects correspond to internal categories in the category of groups.

1.5.1 Definition. *A* ***crossed module*** (Z, G, ρ) *consists of a group homomorphism $\rho : Z \to G$, together with an action $(g, z) \mapsto gz$ of G on Z satisfying the following conditions:*

(a) $\rho(gz) = g + \rho(z) - g$;

(b) $\rho(z_1)z_2 = z_1 + z_2 - z_1$,

for $z_1, z_2 \in Z$ and $g \in G$.
A crossed module morphism $(f_0, f_1) : (Z, G, \rho) \to (Z', G', \rho')$ is a pair of group homomorphisms $f_0 : Z \to Z'$, $f_1 : G \to G'$ such that $f_1\rho = \rho' f_0$ and

$f_0(gz) = f_1(g)f_0(z)$ for all $z \in Z$ and $g \in G$.

We denote by **CM**(**Groups**) the category of crossed modules and morphisms between them.

Below we give a few examples of crossed modules.

1.5.2 Example. *(N, G, i), the inclusion of a normal subgroup N of a group G equipped with the conjugation action of G on N.*

1.5.3 Example. *$(G, Aut(G), \rho)$, where G is any group and $\rho(g)$ is the inner automorphism of a group G determined by $g \in G$, together with an action defined by $(\ell, g) \mapsto \ell(g)$ for $\ell \in Aut(G)$ and $g \in G$.*

1.5.4 Example. *Suppose G is a group and Z is a left G-module; let $0 : Z \to G$ be the trivial map sending everything in Z to the identity element of G, then $(Z, G, 0)$ is a crossed module.*

The study of automorphism of a crossed module (Z, G, ρ) was initiated by J.H.C. Whitehead (see [35]), when he showed that the set $Der(G, Z)$ of derivations from G to Z has a natural monoid structure, and he further described a group of invertible elements of $Der(G, Z)$ called the Whitehead group of regular derivatives. This work was continued by A.S.-T. Lue [26] and K.J. Norrie [29] when they introduced the notion of actor crossed module.

1.5.5 Construction. *Given a crossed module (Z, G, ρ), $Der(G, Z)$ will denote the set of all derivations from G to Z, that is, the set of all the maps $\Delta : G \to Z$, such that for all $g_1, g_2 \in G$ we have*

$$\Delta(g_1 + g_2) = \Delta(g_1) + g_1\Delta(g_2).$$

Each derivation Δ defines endormophisms $\ell(= \ell_\Delta)$ and $\sigma(= \sigma_\Delta)$ of G and Z respectively, defined by

$$\ell_\Delta(g) = \rho\Delta(g) + g \text{ and } \sigma_\Delta(z) = \Delta\rho(z) + z,$$

such that $\ell_\Delta\rho(z) = \rho\sigma_\Delta(z)$, $\sigma_\Delta\Delta(g) = \Delta\ell_\Delta(g)$, $\sigma_\Delta(gz) = \ell_\Delta(g)\sigma_\Delta(z)$, *for all* $g \in G$ *and* $z \in Z$.

The Whitehead binary operation of derivatives Δ_1 *and* Δ_2 *in* $Der(G, Z)$ *is defined by*

$$(\Delta_1 \circ \Delta_2)(g) = \Delta_1\ell_{\Delta_2}(g) + \Delta_2(g)(= \sigma_{\Delta_1}\Delta_2(g) + \Delta_1(g)).$$

This operation is associative and it turns $Der(G, Z)$ *into a monoid, with identity element the derivative which maps each element of* G *into the identity element of* Z.

1.5.6 Definition. *The* ***Whitehead group*** $D(G, Z)$ *is a group of invertible elements of* $Der(G, Z)$*; the elements of* $D(G, Z)$ *are called regular derivatives.*

1.5.7 Proposition. (*see* [35]) *The following statements are equivalent:*

(a) $\Delta \in D(G, Z)$;

(b) $\ell \in Aut(G)$;

(c) $\sigma \in Aut(Z)$.

1.5.8 Construction. *Let* $Aut(Z, G, \rho)$ *be the automorphism group of the crossed module* (Z, G, ρ). *Its elements are pairs* (γ, μ) *with* γ *in* $Aut(Z)$ *and* μ *in* $Aut(G)$ *satisfying*

$$\mu\rho = \rho\gamma \text{ and } \gamma(gz) = \mu(g)\gamma(z).$$

Then there is a group homomorphism $\omega : D(G, Z) \to Aut(Z, G, \rho)$ *defined by* $\omega(\Delta) =< \sigma, \ell >$ *and there is an action of* $Aut(Z, G, \rho)$ *on* $D(G, Z)$ *given by* $(\gamma, \mu)\Delta = \gamma\Delta\mu^{-1}$, *which makes* $(D(G, Z), Aut(Z, G, \rho), \omega)$ *a crossed module called the actor crossed module.*

There is a morphism of crossed modules

$$(f_0, f_1) : (Z, G, \rho) \longrightarrow (D(G, Z), Aut(Z, G, \rho), \omega),$$

where $f_0 : Z \to D(G, Z)$ is a group homomorphism that carries an element z in Z to the derivation $f_0(z) : G \to Z$ defined by $f_0(z)(g) = z - gz$,
and $f_1 : G \to Aut(Z, G, \rho)$ carries an element g in G to the pair (γ_g, μ_g), defined by $\gamma_g(z) = gz$ and $\mu_g(g') = g + g' - g$, respectively (see [29]).

1.5.9 Example. *If N is a normal subgroup of G with inclusion $i : N \hookrightarrow G$ then $(D(G, N), T, w)$ is our actor crossed module where T is isomorphic to the subgroup of $Aut(G)$ consisting of those automorphisms which restrict to automorphisms of N.*

1.5.10 Example. *As a special case of Example 1.5.9 when $N = 1$ or $N = G$ we see that actor crossed module $(D(G, 1), Aut(1, G, \rho))$ is isomorphic to crossed module $(1, Aut(G))$ and that $(D(G, G), Aut(G, G, \rho), 1)$ is isomorphic to $(Aut(G), Aut(G), 1)$.*

1.5.11 Example. *Let Z be a G-module. Then $(D(G, Z), Aut(Z, G, 0), 0)$ is our actor crossed module where 0 is a trivial homomorphism.*

1.6 Cat^n-groups

In 1982 J.-L. Loday [25] introduced *cat^n*-groups which are equivalent to internal n-fold categories in category of groups. The purpose of this section is to recall this equivalence in details.

1.6.1 Definition. *A **cat^1-group** is a triple (X, s_1, t_1), where X is a group, and $s_1, t_1 : X \to X$ are group homomorphisms with*

(a) $s_1 t_1 = t_1$, $t_1 s_1 = s_1$;

(b) $[ker(s_1), ker(t_1)] = 1$.

A morphism of cat^1*-groups* $(X, s_1, t_1) \to (Y, s'_1, t'_1)$ *consists of a group homomorphism* $f : X \to Y$ *such that* $fs_1 = s'_1 f$ *and* $ft_1 = t'_1 f$.
We shall denote the category of cat^1-groups by $\mathbf{Cat^1}$**-Groups**.

That is, an M_1-group X defined in Lemma 1.3.14 is essentially the same as cat^1-group (X, s_1, t_1), and we are simply using the same notation for the non-identity elements of M_1 and the corresponding endomorphisms of X.

1.6.2 Theorem. *The category* $\mathbf{Cat}(\mathbf{Groups})$ *of internal categories in the category* **Groups** *of groups is equivalent to the category* $\mathbf{Cat^1}$**-Groups** *of* cat^1*-groups.*

Proof. Let us just indicate how to construct the functors
$U : \mathbf{Cat}(\mathbf{Groups}) \to \mathbf{Cat^1}$**-Groups** and $V : \mathbf{Cat^1}$**-Groups** $\to \mathbf{Cat}(\mathbf{Groups})$ that are inverse to each other up to isomorphism. For an object
$C = (C_1, C_0, d, c, e, m)$ in $\mathbf{Cat}(\mathbf{Groups})$, we take

$$U(C) = (C_1, ed, ec),$$

and for a morphism $f = (f_1, f_0) : C \to C'$

$$U(f) = f_1.$$

For an object $X = (X, s_1, t_1)$ in $\mathbf{Cat^1}$**-Groups**, we take

$$V(X) = (X, s_1(X), \overline{s_1}, \overline{t_1}, i, m),$$

where $\overline{s_1}$ and $\overline{t_1}$ are induced by s_1 and t_1 respectively, i is the inclusion map, and m is defined by

$$m(x,y) = x - s_1(x) + y = y - s_1(x) + x,$$

and for a morphism $f : X \to X'$

$$V(f) = (f, \overline{f}),$$

where $\overline{f}$ is induced by f. □

It will be shown in Theorem 1.6.7 that this equivalence can be extended to higher-dimensional categorical structures.

1.6.3 Lemma. *The following data are equivalent:*

(a) *a crossed module* (Z, G, ρ)*;*

(b) *a cat*^{1}*-group* (G, s_1, t_1)*;*

Proof. See [25]. □

1.6.4 Remark. *Actually, suitable internal* n*-dimensional structures admit several equivalent descriptions:* n*-fold categories of groups give rise to cat*^{n}*-groups as shown in Theorem* 1.6.7*, and there is also another characterization of these structures in terms of crossed* n*-cubes* (*see* [17] *for details*).

1.6.5 Definition. *A* **cat**$^{\mathbf{n}}$***-group*** *consists of a group* X *together with* $2n$ *endomorphisms* $s_i, t_i : X \to X$, $i = 1, 2, ..., n$, *such that for* $1 \leq i, j \leq n$,

(i) $s_i t_i = t_i$, $t_i s_i = s_i$,

(ii) $s_it_j = t_js_i$, $s_is_j = s_js_i$, $t_it_j = t_jt_i$, $i \neq j$,

(iii) $[ker(s_i), ker(t_i)] = 1$.

A morphism of cat^n*-groups* $(X, s_i, t_i) \to (Y, s'_i, t'_i)$ *is a group homomorphism* $f : X \to Y$ *such that* $fs_i = s'_if$ *and* $ft_i = t'_if$ *for* $i = 1, 2, ..., n$.
By convention cat^0*-group is just a group.* We will denote by $\mathbf{Cat}^n$**-Groups** the category of cat^n-groups.

1.6.6 Remark. *The categories* $\mathbf{Cat}^1$-($\mathbf{Cat}^n$**-Groups**) *and* $\mathbf{Cat}^{n+1}$**-Groups** *are isomorphic, where an object in* $\mathbf{Cat}^1$-($\mathbf{Cat}^n$**-Groups**) *consists of a triple* $(\mathcal{X}, s_1, t_1)$ *in which* $\mathcal{X}$ *is a* cat^n*-group* (X, s_i, t_i) $(i = 1, 2, ..., n)$ *in* $\mathbf{Cat}^n$**-Groups**, *and* $s_1, t_1 : \mathcal{X} \to \mathcal{X}$ *are morphisms in* $\mathbf{Cat}^n$**-Groups** *satisfying the usual conditions*

$$s_1t_1 = t_1, t_1s_1 = s_1 \text{ and } [ker(s_1), ker(t_1)] = 1.$$

1.6.7 Theorem. *The category* $\mathbf{Cat}^n$**-Groups** *of* cat^n*-groups and the category* $\mathbf{Cat}^n$(**Groups**) *of* n*-fold internal categories in category of groups are equivalent.*

Proof. We prove by induction. The case $n = 1$ is given by Theorem 1.6.2. Assume that the categories $\mathbf{Cat}^{n-1}$**-Groups** and $\mathbf{Cat}^{n-1}$(**Groups**) are equivalent. Then by Theorem 1.6.2 and Remark 1.6.6 $\mathbf{Cat}^n$**-Groups** is equivalent to **Cat**-($\mathbf{Cat}^{n-1}$**-Groups**). It follows from the induction hypothesis that, **Cat**-($\mathbf{Cat}^{n-1}$**-Groups**) is equivalent to $\mathbf{Cat}(\mathbf{Cat}^{n-1}(\mathbf{Groups})) = \mathbf{Cat}^n(\mathbf{Groups})$. □

Chapter 2

Internal algebraic structures in cartesian closed categories

Recall that, the category $\mathbf{Sets}^{M_1}$ of M_1-sets is a cartesian closed category, with the exponent $X^X = \{\alpha : M_1 \times X \to X \mid m\alpha(m', x) = \alpha(mm', mx)$ for all $m, m' \in M_1,\ x \in X\}$, where X is an M_1-set in $\mathbf{Sets}^{M_1}$, with M_1 acting on X^X by $(m\alpha)(m', x) = \alpha(m'm, x)$, and M_1 denotes the monoid $\{1, s_1, t_1\}$, such that, $s_1t_1 = t_1$ and $t_1s_1 = s_1$. Similarly, the category **Cat** of small categories is a cartesian closed category, with the exponent X^X the functor category for X in **Cat**.

To give an explicit description of the internal automorphism group of group X as cat^1-group in $\mathbf{Sets}^{M_1}$ we need the forgetful functor $\mathbf{U}_1 : \mathbf{Cat} \to \mathbf{Sets}^{M_1}$ defined as follows. For any category X in **Cat**, the underlying set of $\mathbf{U}_1(X)$ is the set X_1 of morphisms of X, and, $x \in \mathbf{U}_1(X)$, where s_1x and t_1x are the identity morphisms of the domain of x and of the codomain of x, respectively. This functor being product preserving sends monoids to monoids, so that, for the internal monoid X^X decribed in Construction 2.1.1 in the category **Cat**, there is an internal monoid homomorphism $\theta_X^X : \mathbf{U}_1(X^X) \to \mathbf{U}_1(X)^{\mathbf{U}_1(X)}$ in $\mathbf{Sets}^{M_1}$ constructed in Section 2.3, making the diagram

$$\begin{array}{ccc} \mathbf{U}_1(\underline{Hom}(X,X)) & \xrightarrow{\mathbf{U}_1(i_{X,X})} & \mathbf{U}_1(X^X) \\ {\scriptstyle \Theta_X^X} \downarrow & & \downarrow {\scriptstyle \theta_X^X} \\ Hom(\mathbf{U}_1(X),\mathbf{U}_1(X)) & \xrightarrow{i_{\mathbf{U}_1(X),\mathbf{U}_1(X)}} & \mathbf{U}_1(X)^{\mathbf{U}_1(X)} \end{array}$$

to be a pullback, where $\mathbf{U}_1(i_{X,X})$ and $i_{\mathbf{U}_1(X),\mathbf{U}_1(X)}$ are inclusion maps and $\underline{Hom}(X,X)$ is the subcategory of the functor category X^X whose objects are homomorphic functors and morphisms are homomorphic natural transformations. The idea here is to show that, if we take a categorical group X in **Cat**, the image $\theta_X^X(\mathbf{U}_1(X^X))$ of $\mathbf{U}_1(X^X)$ in $\mathbf{U}_1(X)^{\mathbf{U}_1(X)}$ satisfies some additional sufficient conditions that make the maximal subgroup $Aut(\mathbf{U}_1(X))$ of $Hom(\mathbf{U}_1(X),\mathbf{U}_1(X))$ a cat^1-group. Then, we describe $Aut(X)$ using $Aut(\mathbf{U}_1(X))$ for an internal group X in $\mathbf{Cat}^n$ for $n > 1$ by analogy in Chapter 3.

2.1 The internal monoid X^X

It was observed in [8] that, for any object X in a cartesian closed category C, the object X^X has a canonical internal monoid structure, that is, the object X^X is equipped with identity $e_{X^X} : 1 \to X^X$ and multiplication $w_{X^X} : X^X \times X^X \to X^X$ morphisms defined as follows:

2.1.1 Construction. (a) *The identity*

$$e_{X^X} : 1 \longrightarrow X^X \tag{2.1}$$

for the canonical monoid structure on X^X corresponds via the canonical *bijection* (1.5) *to the canonical isomorphism*

$$\lambda_X : 1 \times X \longrightarrow X. \tag{2.2}$$

That is, given the second projection λ_X*, we have*

$$e_{X^X} = \varphi_{1,X}(\lambda_X) = (\lambda_X)^X \eta_1^X. \tag{2.3}$$

(b) *The multiplication*

$$w_{X^X} : X^X \times X^X \longrightarrow X^X \tag{2.4}$$

for the canonical monoid structure on X^X *corresponds via the canonical bijection* (1.5) *to the composite*

$$X^X \times X^X \times X \xrightarrow{X^X \times \epsilon_X^X} X^X \times X \xrightarrow{\epsilon_X^X} X\ , \tag{2.5}$$

that is,

$$w_{X^X} = \varphi_{X^X \times X^X, X}(\epsilon_X^X(X^X \times \epsilon_X^X)) = (\epsilon_X^X(X^X \times \epsilon_X^X))^X \eta_{X^X \times X^X}^X. \tag{2.6}$$

2.1.2 Example. *Recall that, if M is a monoid, the category* $\mathbf{Sets}^M$ *is a cartesian closed category, with the M-set* X^X *defined as follows:*
$X^X = \{\alpha : M \times X \to X \mid m\alpha(m', x) = \alpha(mm', mx)$ *for all* $m, m' \in M$ *and* $x \in X\}$*, with* M *acting on* X^X *by* $(m\alpha)(m', x) = \alpha(m'm, x)$ *(see Example* 1.2.3*).*

The M*-set* X^X *has a canonical monoid structure defined below according to Construction 2.1.1:*

(a) *The identity element* u *of* X^X *is defined by*

$u(m, x) = e_{X^X}(*)(m, x) = (((\lambda_X)^X \eta_1^X)(*))(m, x)$ *(by* 2.3*)*
$= \lambda_X(m*, x)$ *(by Example* 1.2.3 *(b) and (c))*

$= x,$

where $*$ is the unique element in the one-element set 1*. To make this clear, recall that since 'the' terminal object in* $\mathbf{Sets}^M$ *is a one-element set* $1(=\{*\})$ *equipped with the trivial action on* M*, a morphism* $\{*\} \to X^X$ *(such as* e_{X^X}*) can be identified with an element* u *of* X^X *with* $mu = u$ *for each* m *in* M*.*

(b) *The multiplication on* X^X *is defined by*

$$(\alpha\beta)(m,x) = w_{X^X}(\alpha,\beta)(m,x) = \alpha(m,\beta(m,x)).$$

Indeed, from Example 1.2.3 *(e) and* (2.6)*, we have*

$$\begin{aligned}
&w_{X^X}(\alpha,\beta)(m,x) = \varphi_{X^X\times X^X,X}(\epsilon_X^X(X^X\times\epsilon_X^X))(\alpha,\beta)(m,x)\\
&= (\epsilon_X^X(X^X\times\epsilon_X^X))(m\alpha,m\beta,x)\\
&= \epsilon_X^X(m\alpha,(m\beta)(1,x)) = \epsilon_X^X(m\alpha,\beta(m,x))\\
&= (m\alpha)(1,\beta(m,x)) = \alpha(m,\beta(m,x)),
\end{aligned}$$

for all α *and* β *in* X^X*,* x *in* X *and* m *in* M*.* $\square$

Note that, in the case where M *is a group, an element* $\partial \in X^X$ *of the exponential* X^X *of* M*-set* X *is a function* $\partial : X \to X$*, and the action of an element* $m \in M$ *on this* ∂ *is defined by conjugation as*

$$(m\partial)(x) = m(\partial(-mx)).$$

2.2 Internal Hom objects

The object B^X in a cartesian closed category $\mathbf{C}$ is functorial in both B and X, yielding a bifunctor $\mathbf{C}^{op} \times \mathbf{C} \longrightarrow \mathbf{C}$, but since the functors $(-)^X : \mathbf{C} \to \mathbf{C}$ for each X in $\mathbf{C}$, were already introduced, we only need to describe the functors $B^{(-)} : \mathbf{C}^{op} \longrightarrow \mathbf{C}$, for each B in $\mathbf{C}$.

2.2.1 Definition. *Given a morphism $p : Y \longrightarrow X$, we define $B^p : B^X \longrightarrow B^Y$ as the morphism corresponding, via the bijection $hom(B^X, B^Y) \cong hom(B^X \times Y, B)$, to the composite $\varepsilon_B^X(B^X \times p) : B^X \times Y \longrightarrow B^X \times X \longrightarrow B$, where B, X and Y are internal magmas in $\mathbf{C}$; that is,*

$$B^p = \varphi_{B^X,Y}(\varepsilon_B^X(B^X \times p)) = (\varepsilon_B^X(B^X \times p))^Y \eta_{B^X}^Y \tag{2.7}$$

(*see* (1.6)).
The fact that this indeed yields a bifunctor $\mathbf{C}^{op} \times \mathbf{C} \longrightarrow \mathbf{C}$, sending (X, B) to B^X, is a long but routine calculation.

2.2.2 Example. *As it was observed in* [8]*, when objects (B, w_B, e_B) and (X, w_X, e_X) are monoids, or, more generally unitary magmas, a homomorphism $f : X \to B$ is a map $f : X \to B$ making the diagram*

$$\begin{array}{ccccc} X \times X & \xrightarrow{w_X} & X & \xleftarrow{e_X} & 1 \\ \downarrow{\scriptstyle f\times f} & & \downarrow{\scriptstyle f} & & \| \\ B \times B & \xrightarrow[w_B]{} & B & \xleftarrow[e_B]{} & 1 \end{array},$$

commute, where w_B and w_X are the binary operations of the internal magmas B and X respectively. Equivalently, we could say that a map $f : X \to B$ is an unitary magma homomorphism if f is an element of hom object $Hom(X, B)$ which is defined via a suitable equalizer diagram

$$Hom(X,B) \xrightarrow{i_{B,X}} B^X \underset{B^{w_X}}{\overset{j_{B,X}}{\rightrightarrows}} B^{X \times X}, \tag{2.8}$$

in which $j_{B,X}$ *is the composite*

$$(w_B)^{X\times X} < (\pi_1^{'})^{X\times X}, (\pi_2^{'})^{X\times X} >^{-1} < B^{\pi_1}, B^{\pi_2} > \qquad (2.9)$$

where π_1 *and* π_2 *are the projections for* $X \times X$*, and* $\pi_1^{'}$ *and* $\pi_2^{'}$ *are the projections for* $B \times B$*.*

Consider the diagram (2.8) *in the case when the ground category* $\mathbf{C}$ *is the category of* M*-sets, and for simplicity,* B *and* X *are internal magmas in that category. The exponential* B^X *is given by:*

$B^X = \{\alpha : M \times X \to B \mid m\alpha(m', x) = \alpha(mm', mx)$ *for all* $m, m' \in M$ *and* $x \in X\}$*, with* M *acting on* B^X *by* $(m\alpha)(m', x) = \alpha(m'm, x)$*, after which we need some calculations:*
First, given a morphism $p : Y \longrightarrow X$ *of* M*-sets,* $m \in M$*,* $y \in Y$ *and* $\wp \in (B^X \times Y)^Y$ *we observe :*

- *the map* $\eta^Y_{B^X} : B^X \longrightarrow (B^X \times Y)^Y$ *is defined by* $(\eta^Y_{B^X}(\alpha))(m, y) = (m\alpha, y)$ *(see Example* 1.2.3(*c*)*);*

- *the map* $B^X \times p : B^X \times Y \longrightarrow B^X \times X$ *is defined by*

$$(B^X \times p)(\alpha, y) = (\alpha, p(y)),$$

and so the map

$$(B^X \times p)^Y : (B^X \times Y)^Y \longrightarrow (B^X \times X)^Y$$

is defined by

$$((B^X \times p)^Y(\wp))(m,y) = ((B^X \times p)\wp)(m,y);$$

- *as follows from the previous observations, the map*

$$(B^X \times p)^Y \eta^Y_{B^X} : B^X \longrightarrow (B^X \times X)^Y$$

is defined by

$$(((B^X \times p)^Y \eta^Y_{B^X})(\alpha))(m,y) = (m\alpha, p(y));$$

- *therefore the map* $B^p = (\varepsilon^X_B(B^X \times p))^Y \eta^Y_{B^X} : B^X \longrightarrow B^Y$ *is defined by* $(B^p(\alpha))(m,y) = (((\varepsilon^X_B(B^X \times p))^Y \eta^Y_{B^X})(\alpha))(m,y) = \varepsilon^X_B(m\alpha, p(y)) = (m\alpha)(1, p(y))$ $= \alpha(m, p(y))$.

This simple description of B^p*, namely*

$$(B^p(\alpha))(m,y) = \alpha(m, p(y)), \tag{2.10}$$

should have been expected of course.
Let us now calculate $j_{B,X} : B^X \longrightarrow B^{X\times X}$ *which is the composite*

$$(w_B)^{X\times X} < (\pi_1')^{X\times X}, (\pi_2')^{X\times X} >^{-1} < B^{\pi_1}, B^{\pi_2} >,$$

that is

$(j_{B,X}(\alpha))(m,x_1,x_2) = (((w_B)^{X\times X} < (\pi_1')^{X\times X}, (\pi_2')^{X\times X} >^{-1} < B^{\pi_1}, B^{\pi_2} >)(\alpha))$
(m, x_1, x_2)*:*

First, we have;

$((< (\pi_1^{'})^{X\times X}, (\pi_2^{'})^{X\times X} >^{-1}< B^{\pi_1}, B^{\pi_2} >)(\alpha))(m, x_1, x_2)$
$= (< (\pi_1^{'})^{X\times X}, (\pi_2^{'})^{X\times X} >^{-1} (B^{\pi_1}(\alpha), B^{\pi_2}(\alpha)))(m, x_1, x_2)$
$=< B^{\pi_1}(\alpha), B^{\pi_2}(\alpha) > (m, x_1, x_2)$
$= ((B^{\pi_1}(\alpha))(m, x_1, x_2), (B^{\pi_2}(\alpha))(m, x_1, x_2)) = (\alpha(m, \pi_1(x_1, x_2)), \alpha(m, \pi_2(x_1, x_2))),$
by (2.10),

which implies

$$(j_{B,X}(\alpha))(m, x_1, x_2) = w_B(\alpha(m, \pi_1(x_1, x_2)), \alpha(m, \pi_2(x_1, x_2))) \\ = w_B(\alpha(m, x_1), \alpha(m, x_2)). \quad (2.11)$$

Next, we need to calculate $B^{w_X} : B^X \longrightarrow B^{X\times X}$*, but* (2.10) *immediately gives*

$$(B^{w_X}(\alpha))(m, x_1, x_2) = \alpha(m, w_X(x_1, x_2)), \quad (2.12)$$

for all $x_1, x_2 \in X$ *and* $m \in M$.
From (2.11) *and* (2.12) *we conclude:*

2.2.3 Proposition. *Let* (B, w_B, e_B) *and* (X, w_X, e_X) *be internal magmas in* $\mathbf{Sets}^M$*, then the* M*-set* $Hom(X, B)$ *can be described as*
$Hom(X, B) = \{\alpha \in B^X \mid \alpha(m, w_X(x_1, x_2)) = w_B(\alpha(m, x_1), \alpha(m, x_2)),\ \forall m, m' \in M \text{ and } \forall x_1, x_2 \in X\}$
with $i_{B,X} : Hom(X, B) \longrightarrow B^X$ *being the inclusion map and* M*-action on* $Hom(X, B)$ *is given by* $(m\alpha)(m', x) = \alpha(m'm, x)$.

2.2.4 Remark. *Let* $B = X$*, then we observe that the object*
$Hom(X, X) = End(X)$ *becomes a submonoid of* X^X*. In particular, the object* $Aut(X)$ *of invertible elements of* $End(X)$*, is the maximal subgroup of* $End(X)$ *in the category* $\mathbf{Sets}^M$*. This subgroup is given by the following pullback:*

$$\begin{array}{ccc} Aut(X) & \longrightarrow & 1 \\ \downarrow & & \downarrow {\scriptstyle <e_{End(X)}, e_{End(X)}>} \\ End(X) \times End(X) & \xrightarrow{<w_{End(X)}, w^{op}_{End(X)}>} & End(X) \times End(X) \end{array}$$

with $w^{op}_{End(X)} = w_{End(X)}\tilde{t}$, *where* $\tilde{t}$ *is the twisting isomorphism of the binary operation, and* $e_{End(X)}$ *is the identity morphism of* $End(X)$, *(see* [8] *for more details).*

As follows from Remark 2.2.4 and Proposition 2.2.3 which gives description of internal monoid $End(X)$ when $B = X$ we can now describe the internal automorphism group $Aut(X)$ of X in the category $\mathbf{Sets}^M$ as follows:

2.2.5 Lemma. $Aut(X) = \{(\alpha : M \times X \to X) \in End(X) \mid \alpha(m, -)$ *is a bijection for all* $m \in M\}$, *with* M *acting on* $Aut(X)$ *by* $(m\alpha)(m', x) = \alpha(m'm, x)$ *for all* $m, m' \in M$ *and* $x \in X$.

2.3 The internal monoids $\mathbf{U}(X^X)$ and $\mathbf{U}(X)^{\mathbf{U}(X)}$

2.3.1 Construction. *If we take two adjunctions*

$$((-) \times X, (-)^X, \eta^X, \epsilon^X) : \mathbf{Cat} \to \mathbf{Cat}$$

where X *is an object in* $\mathbf{Cat}$, *and*

$$((-) \times \mathbf{U_1}(X), (-)^{\mathbf{U_1}(X)}, \eta^{\mathbf{U_1}(X)}, \epsilon^{\mathbf{U_1}(X)}) : \mathbf{Sets}^{M_1} \to \mathbf{Sets}^{M_1},$$

where the functor $\mathbf{U}_1 : \mathbf{Cat} \longrightarrow \mathbf{Sets}^{M_1}$ *is defined as follows:*

(a) *for a category* X, *the underlying set of* $\mathbf{U}_1(X)$ *is the set* X_1 *of morphisms in* X;

(b) *for* x *in* $\mathbf{U}_1(X)$, sx *and* tx *are the identity morphisms of the domain of* x *and of the codomain of* x, *respectively;*

(c) *for a functor $F : X \longrightarrow Y$, $\mathbf{U}_1(F)$ is the corresponding map from X_1 to Y_1.*

Then, since the functor $\mathbf{U}_1 : \mathbf{Cat} \longrightarrow \mathbf{Sets}^{M_1}$ obviously preserves finite products, that is, for any product $B \times X$ of objects B and X in $\mathbf{Cat}$ we have $\mathbf{U}_1(B \times X) \cong \mathbf{U}_1(B) \times \mathbf{U}_1(X)$ (canonically) in $\mathbf{Sets}^{M_1}$, we shall look at it as a special case of an abstract finite product preserving functor $\mathbf{U} : \mathbf{C} \longrightarrow \mathbf{D}$ between cartesian closed categories.

For $\mathbf{U} : \mathbf{C} \longrightarrow \mathbf{D}$ above, and objects X and B in $\mathbf{C}$, we can compare $\mathbf{U}(B^X)$ with $\mathbf{U}(B)^{\mathbf{U}(X)})$ as follows:

By the universal property of $\varepsilon^{\mathbf{U}(X)}_{\mathbf{U}(B)}$, there exists a unique morphism $\theta^X_B : \mathbf{U}(B^X) \longrightarrow \mathbf{U}(B)^{\mathbf{U}(X)}$ making the diagram

$$\begin{array}{ccc} \mathbf{U}(B^X) \times \mathbf{U}(X) & \xrightarrow{\theta^X_B \times \mathbf{U}(X)} & \mathbf{U}(B)^{\mathbf{U}(X)}) \times \mathbf{U}(X) \\ {\scriptstyle <\mathbf{U}(\underline{\pi_1}),\mathbf{U}(\underline{\pi_2})>^{-1}} \downarrow & & \downarrow {\scriptstyle \varepsilon^{\mathbf{U}(X)}_{\mathbf{U}(B)}} \\ \mathbf{U}(B^X \times X) & \xrightarrow{\mathbf{U}(\varepsilon^X_B)} & \mathbf{U}(B) \end{array} \tag{2.13}$$

commute. The morphism $< \mathbf{U}(\underline{\pi_1}), \mathbf{U}(\underline{\pi_2}) >^{-1}$ is the inverse of the canonical isomorphism. Equivalently, θ^X_B can be defined as the image of $\mathbf{U}(\varepsilon^X_B)$ under the composite

$$hom(\mathbf{U}(B^X \times X), \mathbf{U}(B)) \cong hom(\mathbf{U}(B^X) \times \mathbf{U}(X), \mathbf{U}(B)) \cong hom(\mathbf{U}(B^X), \mathbf{U}(B)^{\mathbf{U}(X)}) \tag{2.14}$$

of canonical bijections, that is

$$\varphi_{\mathbf{U}(B^X),\mathbf{U}(B)}(\mathbf{U}(\varepsilon^X_B) < \mathbf{U}(\underline{\pi_1}), \mathbf{U}(\underline{\pi_2}) >^{-1}) = \theta^X_B,$$

or, equivalently

$$\psi_{\mathbf{U}(B^X),\mathbf{U}(B)}(\theta_B^X) = \mathbf{U}(\varepsilon_B^X) < \mathbf{U}(\underline{\pi_1}), \mathbf{U}(\underline{\pi_2}) >^{-1}.$$

Let us now calculate θ_B^X in the case of functor $\mathbf{U}_1 : \mathbf{Cat} \longrightarrow \mathbf{Sets}^{M_1}$.

Step1 *Description of $\mathbf{U}_1(\varepsilon_B^X)$, and its image in hom$(\mathbf{U}_1(B^X) \times \mathbf{U}_1(X), \mathbf{U}_1(B))$ under the first canonical bijection in* (2.14).

Here the morphism ε_B^X is as in Example 1.2.4(*e*). *That is, it is a functor $B^X \times X \longrightarrow B$ that carries a morphism $(\tau, x) : (V, \underline{x}) \longrightarrow (V', \underline{x'})$ in $B^X \times X$ to the morphism $\tau_{\underline{x'}} V(x) = V'(x)\tau_{\underline{x}} : V(\underline{x}) \longrightarrow V'(\underline{x'})$ in B. According to the definition of $\mathbf{U}_1$, we can simply write*

$$\mathbf{U}_1(\varepsilon_B^X)(\tau, x) = \tau_{\underline{x'}} V(x) = V'(x)\tau_{\underline{x}}, \tag{2.15}$$

and say that the image of $\mathbf{U}_1(\varepsilon_B^X)$ in the set hom$(\mathbf{U}_1(B^X) \times \mathbf{U}_1(X), \mathbf{U}_1(B))$ is the map $h : \mathbf{U}_1(B^X) \times \mathbf{U}_1(X) \longrightarrow \mathbf{U}_1(B)$. As follows from (2.14), *we have*

$$h = hom(< \mathbf{U}_1(\underline{\pi_1}), \mathbf{U}_1(\underline{\pi_2}) >^{-1}, \mathbf{U}_1(B))(\mathbf{U}_1(\varepsilon_B^X))$$
$$= \mathbf{U}_1(\varepsilon_B^X) < \mathbf{U}_1(\underline{\pi_1}), \mathbf{U}_1(\underline{\pi_2}) >^{-1},$$

such that h is defined by

$$h(\tau, x) = \tau_{\underline{x'}} V(x) = V'(x)\tau_{\underline{x}}, \tag{2.16}$$

where $\underline{x}$ and $\underline{x'}$ are the domain and the codomain of x, while V and V' are the domain and the codomain of τ, respectively.

Step2 *Description of the image of the map $h : \mathbf{U}_1(B^X) \times \mathbf{U}_1(X) \longrightarrow \mathbf{U}_1(B)$ above under the second canonical bijection in* (2.14).

For the desired map $\theta_B^X : \mathbf{U}_1(B^X) \longrightarrow \mathbf{U}_1(B)^{\mathbf{U}_1(X)}$, according to Example 1.2.3(*d*), *we have*

$$\theta_B^X(\tau)(m, x) = h(m\tau, x), \tag{2.17}$$

which, according to (2.16), *gives:*

2.3.2 Proposition. *For the morphism* $\theta_B^X : \mathbf{U}_1(B^X) \longrightarrow \mathbf{U}_1(B)^{\mathbf{U}_1(X)}$ *we have:*

$$\theta_B^X(\tau)(1,x) = \tau_{\underline{x'}} V(x) = V^{'}(x)\tau_{\underline{x}}, \tag{2.18}$$

$$\theta_B^X(\tau)(s,x) = V(x), \tag{2.19}$$

$$\theta_B^X(\tau)(t,x) = V^{'}(x), \tag{2.20}$$

for every $(\tau : V \longrightarrow V^{'}) \in \mathbf{U}_1(B^X)$ *and for every* $(x : \underline{x} \longrightarrow \underline{x}^{'}) \in \mathbf{U}_1(X)$*.*

2.3.3 Remark. *It is then easy to check that the morphism* $\theta_B^X : \mathbf{U}_1(B^X) \longrightarrow \mathbf{U}_1(B)^{\mathbf{U}_1(X)}$ *is injective.*

2.3.4 Example. *Let* B *and* X *be additive (but not necessarily commutative) monoids, considered as one-object categories. In this case:*

(a) $\mathbf{U_1}(B)$ *and* $\mathbf{U}_1(X)$ *are the same as* B *and* X*, respectively, considered as* $\{1, s_1, t_1\}$*-sets, with* $s_1 b = 0 = t_1 b$ *for any* $b \in B$*, and, similarly,* $s_1 x = 0 = t_1 x$ *for any* $x \in X$*;*

(b) $\mathbf{U}_1(B)^{\mathbf{U}_1(X)} = \{\alpha : \{1, s_1, t_1\} \times X \longrightarrow B \mid \alpha(s_1, 0) = 0 = \alpha(t_1, 0)\}$*. Equivalently, this set can be described as the set of triples* $(\alpha_1, \alpha_{s_1}, \alpha_{t_1})$ *of maps* $X \longrightarrow B$ *such that* $\alpha_{s_1}(0) = 0 = \alpha_{t_1}(0)$*, and so it has* $\mid B \mid^{|X|} \mid B \mid^{|X|-1} \mid B \mid^{|X|-1} = \mid B \mid^{3|X|-2}$ *elements (where* $\mid B \mid$ *denotes the number of elements in* B*, etc.);*

(c) $\mathbf{U}_1(B^X)$ *can be described as a set of triples* $(v, V, V^{'})$*, in which* $V, V^{'} : X \longrightarrow B$ *are monoid homomorphisms and* v *is an element of* B *with* $v + V(x) = V^{'}(x) + v$ *for each* $x \in X$*; the monoid* $\{1, s_1, t_1\}$ *acts on it via* $s_1(v, V, V^{'}) = (0, V, V)$ *and* $t_1(v, V, V^{'}) = (0, V^{'}, V^{'})$*;*

(d) *using the notation of (b), (c) and Proposition* 2.3.2*, we can write*

$$\theta_B^X(v, V, V^{'}) = (\alpha_1, \alpha_{s_1}, \alpha_{t_1}) \Longleftrightarrow (\alpha_1 = v + V = V^{'} + v, \alpha_{s_1} = V, \alpha_{t_1} = V^{'}), \tag{2.21}$$

where $v{+}V = V'{+}v$ *is the map* $X \longrightarrow B$ *carrying* x *to* $v{+}V(x) = V'(x){+}v$, *for each* $x \in X$.

(e) *in particular, if* B *is a group, then* v *and* V *above determine* V' *uniquely, and so the set* $\mathbf{U}_1(B^X)$ *becomes bijective to* $hom(X, B) \times B$.
For instance, when $B = \mathbb{Z}/n\mathbb{Z} = X$, *it is easy to see that* $\mathbf{U}_1(B^X)$ *and* $\mathbf{U}_1(B)^{\mathbf{U}_1(X)}$ *have* n^2 *and* n^{3n-2} *elements, respectively.*

2.3.5 Example. *Let* B *and* X *be preorders, considered as categories. In this case* $\mathbf{U_1}(B) = B_1$ *and* $\mathbf{U_1}(X) = X_1$ *are the preorders on* B *and* X, *respectively, considered as* $\{1, s_1, t_1\}$*-sets, with* $s_1(b, b') = (b, b)$ *and* $t_1(b, b') = (b', b')$ *for* $b, b' \in B$, *and, similarly,* $s_1(x, x') = (x, x)$ *and* $t_1(x, x') = (x', x')$ *for* $x, x' \in X$. *Since we are considering* B *as a category, let us write* $d(b, b') = b$ *and* $c(b, b') = b'$ *for any* (b, b') *in* B_1, *and let us do the same with* X. *We observe:*

(a) $d\alpha(s_1, (x, x)) = c\alpha(s_1, (x, x))$ *and* $d\alpha(t_1, (x, x)) = c\alpha(t_1, (x, x))$ *for every* $x \in X$. *Indeed, we have*

$$d\alpha(s_1, (x, x)) = d\alpha(s_1 1, s_1(x, x)) = ds_1\alpha(1, (x, x)) = cs_1\alpha(1, (x, x)) = \\ c\alpha(s_1 1, s_1(x, x)) = c\alpha(s_1, (x, x)),$$

$$d\alpha(t_1, (x, x)) = d\alpha(t_1 1, t_1(x, x)) = dt_1\alpha(1, (x, x)) = ct_1\alpha(1, (x, x)) = \\ c\alpha(t_1 1, t_1(x, x)) = c\alpha(t_1, (x, x)).$$

(b) *For* $(\alpha : \{1, s_1, t_1\} \times X_1 \longrightarrow B_1) \in \mathbf{U_1}(B)^{\mathbf{U_1}(X)}$, *we define maps* $\alpha_{s_1}, \alpha_{t_1} : X \longrightarrow B$ *by*

$$\alpha_{s_1}(x) = d\alpha(s_1, (x, x)) = c\alpha(s_1, (x, x)), \alpha_{t_1}(x) = d\alpha(t_1, (x, x)) = c\alpha(t_1, (x, x)). \tag{2.22}$$

Then:

$d\alpha(m,(x,x^{'})) = ds_1\alpha(m,(x,x^{'})) = d\alpha(s_1m, s_1(x,x^{'})) = d\alpha(s_1m,(x,x)) = \alpha_{s_1m}(x)$,

$c\alpha(m,(x,x^{'})) = ct_1\alpha(m,(x,x^{'})) = c\alpha(t_1m, t_1(x,x^{'})) = c\alpha(t_1m,(x^{'},x^{'}))$
$= \alpha_{t_1m}(x^{'})$,
and so

$$\alpha(m,(x,x^{'})) = (\alpha_{s_1m}(x), \alpha_{t_1m}(x^{'})). \tag{2.23}$$

(c) $(2,23)$ *tells us that*

$$\alpha_{s_1m}(x) \leqslant \alpha_{t_1m}(x^{'}), \tag{2.24}$$

and, required for every $m \in \{1, s_1, t_1\}$ *and all* $x \leqslant x^{'}$ *in* X*, it is obviously the same as*

$$\alpha_{s_1}(x) \leqslant \alpha_{s_1}(x^{'}),\ \alpha_{t_1}(x) \leqslant \alpha_{t_1}(x^{'}),\ \alpha_{s_1}(x) \leqslant \alpha_{t_1}(x),$$

required for all $x \leqslant x^{'}$ *in* X*. This makes* $\alpha_{s_1}, \alpha_{t_1} : X \longrightarrow B$ *order-preserving maps with* $\alpha_{s_1} \leqslant \alpha_{t_1}$*.*

(d) *To give* $(\tau : V \longrightarrow V^{'}) \in \mathbf{U_1}(B^X)$ *is just to give order-preserving maps* $V, V^{'} : X \longrightarrow B$ *with* $V \leqslant V^{'}$*. Moreover, in the notation of Proposition* 2.3.2*, we have*

$$\begin{gathered}
d\theta_B^X(\tau)(1,(\underline{x},\underline{x^{'}})) = d\theta_B^X(\tau)(1,x) = d(\tau_{\underline{x^{'}}}V(x)) = dV(x) = V(\underline{x}),\\
d\theta_B^X(\tau)(s_1,(\underline{x},\underline{x^{'}})) = d\theta_B^X(\tau)(s_1,x) = dV(x) = V(\underline{x}),\\
d\theta_B^X(\tau)(t_1,(\underline{x},\underline{x^{'}})) = d\theta_B^X(\tau)(t_1,x) = dV^{'}(x) = V^{'}(\underline{x}),\\
c\theta_B^X(\tau)(1,(\underline{x},\underline{x^{'}})) = c\theta_B^X(\tau)(1,x) = c(V^{'}(x)\tau_{\underline{x}}) = cV^{'}(x) = V^{'}(\underline{x^{'}}),\\
c\theta_B^X(\tau)(s_1,(\underline{x},\underline{x^{'}})) = c\theta_B^X(\tau)(s_1,x) = cV(x) = V(\underline{x^{'}})\\
c\theta_B^X(\tau)(t_1,(\underline{x},\underline{x^{'}})) = c\theta_B^X(\tau)(t_1,x) = cV^{'}(x) = V^{'}(\underline{x^{'}}).
\end{gathered}$$

From these equalities we obtain

$$\theta_B^X(\tau)(1,(\underline{x},\underline{x^{'}})) = (V(\underline{x}), V^{'}(\underline{x^{'}})), \tag{2.25}$$

$$\theta_B^X(\tau)(s_1, (\underline{x}, \underline{x}^{'})) = (V(\underline{x}), V(\underline{x}^{'})), \tag{2.26}$$

$$\theta_B^X(\tau)(t_1, (\underline{x}, \underline{x}^{'})) = (V^{'}(\underline{x}), V^{'}(\underline{x}^{'})). \tag{2.27}$$

Note that, if we compare these formulas with (2.23), then we can conclude that θ_B^X is a bijection.

For a finite product preserving functor $\mathbf{U} : \mathbf{C} \longrightarrow \mathbf{D}$ between cartesian closed categories, $\mathbf{U}$ sends monoids to monoids, and, for $\theta_X^X : \mathbf{U}(X^X) \longrightarrow \mathbf{U}(X)^{\mathbf{U}(X)}$ defined as in the previous section (for $B = X$), we have:

2.3.6 Theorem. *The morphism* $\theta_X^X : \mathbf{U}(X^X) \longrightarrow \mathbf{U}(X)^{\mathbf{U}(X)}$ *is a homomorphism of internal monoids.*

Proof. We have to show that the diagram

$$\begin{array}{ccccc}
\mathbf{U}(X^X) \times \mathbf{U}(X^X) & \xrightarrow{w_{\mathbf{U}(X^X)}} & \mathbf{U}(X^X) & \xleftarrow{e_{\mathbf{U}(X^X)}} & 1 \\
\Big\downarrow{\scriptstyle \theta_X^X \times \theta_X^X} & & \Big\downarrow{\scriptstyle \theta_X^X} & & \Big\| \\
\mathbf{U}(X)^{\mathbf{U}(X)} \times \mathbf{U}(X)^{\mathbf{U}(X)} & \xrightarrow[w_{\mathbf{U}(X)^{\mathbf{U}(X)}}]{} & \mathbf{U}(X)^{\mathbf{U}(X)} & \xleftarrow[e_{\mathbf{U}(X)^{\mathbf{U}(X)}}]{} & 1
\end{array} \tag{2.28}$$

whose top row is the monoid structure of $\mathbf{U}(X^X)$, commutes. Since the monoid structure of $\mathbf{U}(X^X)$ is obtained by applying the functor $\mathbf{U}$ to the monoid structure of X^X described in Construction 2.1.1, to prove that (2.28) commutes is to prove that

$$\begin{array}{ccccc}
\mathbf{U}(X^X \times X^X) & \xrightarrow{\mathbf{U}(w_{X^X})} & \mathbf{U}(X^X) & \xleftarrow{\mathbf{U}(e_{X^X})} & \mathbf{U}(1) \\
\Big\downarrow{\scriptstyle <\theta_X^X \mathbf{U}(\pi_1), \theta_X^X \mathbf{U}(\pi_2)>} & & \Big\downarrow{\scriptstyle \theta_X^X} & & \Big\downarrow{\scriptstyle !_{\mathbf{U}(1)}} \\
\mathbf{U}(X)^{\mathbf{U}(X)} \times \mathbf{U}(X)^{\mathbf{U}(X)} & \xrightarrow[w_{\mathbf{U}(X)^{\mathbf{U}(X)}}]{} & \mathbf{U}(X)^{\mathbf{U}(X)} & \xleftarrow[e_{\mathbf{U}(X)^{\mathbf{U}(X)}}]{} & 1
\end{array} \tag{2.29}$$

(where $\pi_i : X^X \times X^X \longrightarrow X^X (i = 1, 2)$ are the product projections) does commute. The commutativity of the first square is the equality

$$w_{\mathbf{U}(X)^{\mathbf{U}(X)}} < \theta_X^X \mathbf{U}(\pi_1), \theta_X^X \mathbf{U}(\pi_2) >= \theta_X^X \mathbf{U}(w_{X^X}) \tag{2.30}$$

in $hom(\mathbf{U}(X^X \times X^X), \mathbf{U}(X)^{\mathbf{U}(X)})$, which is equivalent to the equality

$$\varepsilon_{\mathbf{U}(X)}^{\mathbf{U}(X)}(w_{\mathbf{U}(X)^{\mathbf{U}(X)}} < \theta_X^X \mathbf{U}(\pi_1), \theta_X^X \mathbf{U}(\pi_2) > \times \mathbf{U}(X)) = \varepsilon_{\mathbf{U}(X)}^{\mathbf{U}(X)}(\theta_X^X \mathbf{U}(w_{X^X}) \times \mathbf{U}(X))$$

in $hom(\mathbf{U}(X^X \times X^X) \times \mathbf{U}(X), \mathbf{U}(X))$ (see (1.6)), and so it is also equivalent to

$$\begin{aligned} &\varepsilon_{\mathbf{U}(X)}^{\mathbf{U}(X)}(w_{\mathbf{U}(X)^{\mathbf{U}(X)}} \times \mathbf{U}(X))(< \theta_X^X \mathbf{U}(\pi_1), \theta_X^X \mathbf{U}(\pi_2) > \times \mathbf{U}(X)) \\ &\qquad = \varepsilon_{\mathbf{U}(X)}^{\mathbf{U}(X)}(\theta_X^X \times \mathbf{U}(X))(\mathbf{U}(w_{X^X}) \times \mathbf{U}(X)), \end{aligned} \tag{2.31}$$

since $(-) \times \mathbf{U}(X)$ is a functor.
We observe:

- $w_{\mathbf{U}(X)^{\mathbf{U}(X)}}$ is defined as the morphism $\mathbf{U}(X)^{\mathbf{U}(X)} \times \mathbf{U}(X)^{\mathbf{U}(X)} \longrightarrow \mathbf{U}(X)^{\mathbf{U}(X)}$ corresponding to the morphism $\varepsilon_{\mathbf{U}(X)}^{\mathbf{U}(X)}(\mathbf{U}(X)^{\mathbf{U}(X)} \times \varepsilon_{\mathbf{U}(X)}^{\mathbf{U}(X)})$ under the canonical bijection

$$\begin{aligned} &hom(\mathbf{U}(X)^{\mathbf{U}(X)} \times \mathbf{U}(X)^{\mathbf{U}(X)}, \mathbf{U}(X)^{\mathbf{U}(X)}) \cong \\ &\qquad hom(\mathbf{U}(X)^{\mathbf{U}(X)} \times \mathbf{U}(X)^{\mathbf{U}(X)} \times \mathbf{U}(X), \mathbf{U}(X)), \end{aligned} \tag{2.32}$$

and so

$$\varepsilon_{\mathbf{U}(X)}^{\mathbf{U}(X)}(w_{\mathbf{U}(X)^{\mathbf{U}(X)}} \times \mathbf{U}(X)) = \varepsilon_{\mathbf{U}(X)}^{\mathbf{U}(X)}(\mathbf{U}(X)^{\mathbf{U}(X)} \times \varepsilon_{\mathbf{U}(X)}^{\mathbf{U}(X)}) \tag{2.33}$$

(see (1.6) again).

- θ_X^X is defined as the image of $\mathbf{U}(\varepsilon_X^X)$ under the composite

$hom(\mathbf{U}(X^X \times X), \mathbf{U}(X)) \cong hom(\mathbf{U}(X^X) \times \mathbf{U}(X), \mathbf{U}(X))$
$\cong hom(\mathbf{U}(X^X), \mathbf{U}(X)^{\mathbf{U}(X)})$,

and so $\varepsilon^{\mathbf{U}(X)}_{\mathbf{U}(X)}(\theta^X_X \times \mathbf{U}(X))$ corresponds to $\mathbf{U}(\varepsilon^X_X)$ under the first of these bijections. That is,

$$\varepsilon^{\mathbf{U}(X)}_{\mathbf{U}(X)}(\theta^X_X \times \mathbf{U}(X)) = \mathbf{U}(\varepsilon^X_X) < \mathbf{U}(\pi'_1), \mathbf{U}(\pi'_2) >^{-1}, \tag{2.34}$$

where $\pi'_1 : X^X \times X \longrightarrow X^X$ and $\pi'_2 : X^X \times X \longrightarrow X$ are the product projections.

• After that, to prove (2.31) is to prove that

$$\varepsilon^{\mathbf{U}(X)}_{\mathbf{U}(X)}(\mathbf{U}(X)^{\mathbf{U}(X)} \times \varepsilon^{\mathbf{U}(X)}_{\mathbf{U}(X)})(< \theta^X_X\mathbf{U}(\pi_1), \theta^X_X\mathbf{U}(\pi_2) > \times \mathbf{U}(X))$$
$$= \mathbf{U}(\varepsilon^X_X) < \mathbf{U}(\pi'_1), \mathbf{U}(\pi'_2) >^{-1} (\mathbf{U}(w_{X^X}) \times \mathbf{U}(X)). \tag{2.35}$$

• Let $p_i : X^X \times X^X \times X \longrightarrow X^X (i = 1, 2)$ and $p_3 : X^X \times X^X \times X \longrightarrow X$ be the product projections. Since

$$< \mathbf{U}(< p_1, p_2 >), \mathbf{U}(p_3) >: \mathbf{U}(X^X \times X^X \times X) \longrightarrow \mathbf{U}(X^X \times X^X) \times \mathbf{U}(X)$$

is an isomorphism, to prove (2.35) is to prove that

$$\varepsilon^{\mathbf{U}(X)}_{\mathbf{U}(X)}(\mathbf{U}(X)^{\mathbf{U}(X)} \times \varepsilon^{\mathbf{U}(X)}_{\mathbf{U}(X)})(< \theta^X_X\mathbf{U}(\pi_1), \theta^X_X\mathbf{U}(\pi_2) > \times \mathbf{U}(X)) < \mathbf{U}(< p_1, p_2 >),$$
$$\mathbf{U}(p_3) >= \mathbf{U}(\varepsilon^X_X) < \mathbf{U}(\pi'_1), \mathbf{U}(\pi'_2) >^{-1} (\mathbf{U}(w_{X^X}) \times \mathbf{U}(X)) < \mathbf{U}(< p_1, p_2 >), \mathbf{U}(p_3) > . \tag{2.36}$$

• We have
$\varepsilon^{\mathbf{U}(X)}_{\mathbf{U}(X)}(\mathbf{U}(X)^{\mathbf{U}(X)} \times \varepsilon^{\mathbf{U}(X)}_{\mathbf{U}(X)})(< \theta^X_X\mathbf{U}(\pi_1), \theta^X_X\mathbf{U}(\pi_2) > \times \mathbf{U}(X)) < \mathbf{U}(< p_1, p_2 >), \mathbf{U}(p_3) >= \varepsilon^{\mathbf{U}(X)}_{\mathbf{U}(X)}(\mathbf{U}(X)^{\mathbf{U}(X)} \times \varepsilon^{\mathbf{U}(X)}_{\mathbf{U}(X)})(\mathbf{U}(X)^{\mathbf{U}(X)} \times \theta^X_X \times \mathbf{U}(X)) < \theta^X_X\mathbf{U}(p_1), \mathbf{U}(p_2),$
$\mathbf{U}(p_3) > = \varepsilon^{\mathbf{U}(X)}_{\mathbf{U}(X)}(\mathbf{U}(X)^{\mathbf{U}(X)} \times \mathbf{U}(\varepsilon^X_X) < \mathbf{U}(\pi'_1), \mathbf{U}(\pi'_2) >^{-1}) < \theta^X_X\mathbf{U}(p_1), \mathbf{U}(p_2),$
$\mathbf{U}(p_3) >$ (by (2.34))
$= \varepsilon^{\mathbf{U}(X)}_{\mathbf{U}(X)}(\theta^X_X \times \mathbf{U}(X)) < \mathbf{U}(p_1), \mathbf{U}(\varepsilon^X_X)\mathbf{U}(< p_2, p_3 >) >$
$= \mathbf{U}(\varepsilon^X_X) < \mathbf{U}(\pi'_1), \mathbf{U}(\pi'_2) >^{-1} < \mathbf{U}(p_1), \mathbf{U}(\varepsilon^X_X)\mathbf{U}(< p_2, p_3 >) >$ (by (2.34))
$= \mathbf{U}(\varepsilon^X_X(X^X \times \varepsilon^X_X)$
and

$\mathbf{U}(\varepsilon_X^X) < \mathbf{U}(\pi_1^{'}), \mathbf{U}(\pi_2^{'}) >^{-1} (\mathbf{U}(w_{X^X}) \times \mathbf{U}(X)) < \mathbf{U}(< p_1, p_2 >), \mathbf{U}(p_3) >$
$= \mathbf{U}(\varepsilon_X^X(w_{X^X} \times X)) = \mathbf{U}(\varepsilon_X^X(X^X \times \varepsilon_X^X),$
using again (1.6). This proves (2.36), and so proves the commutativity of the first square of (2.29).

To prove the commutativity of the second square of (2.29), that is, to prove the equality

$$e_{\mathbf{U}(X)^{\mathbf{U}(X)}} !_{\mathbf{U}(1)} = \theta_X^X \mathbf{U}(e_{X^X}) \tag{2.37}$$

in $hom(\mathbf{U}(1), \mathbf{U}(X)^{\mathbf{U}(X)})$, again from (1.6) this is equivalent to the equality

$$\psi_{\mathbf{U}(1),\mathbf{U}(X)}(e_{\mathbf{U}(X)^{\mathbf{U}(X)}} !_{\mathbf{U}(1)}) = \psi_{\mathbf{U}(1),\mathbf{U}(X)}(\theta_X^X \mathbf{U}(e_{X^X})),$$

this clearly gives us

$$\varepsilon_{\mathbf{U}(X)}^{\mathbf{U}(X)}(e_{\mathbf{U}(X)^{\mathbf{U}(X)}} !_{\mathbf{U}(1)} \times \mathbf{U}(X)) = \varepsilon_{\mathbf{U}(X)}^{\mathbf{U}(X)}(\theta_X^X \mathbf{U}(e_{X^X}) \times \mathbf{U}(X))$$

in $hom(\mathbf{U}(1) \times \mathbf{U}(X), \mathbf{U}(X))$, and so also equivalent to

$$\varepsilon_{\mathbf{U}(X)}^{\mathbf{U}(X)}(e_{\mathbf{U}(X)^{\mathbf{U}(X)}} \times \mathbf{U}(X))(!_{\mathbf{U}(1)} \times \mathbf{U}(X)) = \varepsilon_{\mathbf{U}(X)}^{\mathbf{U}(X)}(\theta_X^X \times \mathbf{U}(X))(\mathbf{U}(e_{X^X}) \times \mathbf{U}(X)). \tag{2.38}$$

We observe:

• $e_{\mathbf{U}(X)^{\mathbf{U}(X)}}$ is a morphism $1 \longrightarrow \mathbf{U}(X)^{\mathbf{U}(X)}$ corresponding to the morphism $\lambda_{\mathbf{U}(X)}$ under the canonical bijections

$$hom(1, \mathbf{U}(X)^{\mathbf{U}(X)}) \cong hom(1 \times \mathbf{U}(X), \mathbf{U}(X)), \tag{2.39}$$

and so

$$\varepsilon_{\mathbf{U}(X)}^{\mathbf{U}(X)}(e_{\mathbf{U}(X)^{\mathbf{U}(X)}} \times \mathbf{U}(X)) = \lambda_{\mathbf{U}(X)}. \tag{2.40}$$

• From (2.34) and (2.40), we conclude that to prove (2.38) is to prove

$$\lambda_{\mathbf{U}(X)}(!_{\mathbf{U}(1)} \times \mathbf{U}(X)) = \mathbf{U}(\varepsilon_X^X) < \mathbf{U}(\pi_1^{'}), \mathbf{U}(\pi_2^{'}) >^{-1} (\mathbf{U}(e_{X^X}) \times \mathbf{U}(X)). \tag{2.41}$$

• Since $< \mathbf{U}(!_X), 1_{\mathbf{U}(X)} >: \mathbf{U}(X) \longrightarrow \mathbf{U}(1) \times \mathbf{U}(X)$ is an isomorphism, to prove (2.41) is to prove

$$\begin{aligned} &\lambda_{\mathbf{U}(X)}(!_{\mathbf{U}(1)} \times \mathbf{U}(X)) < \mathbf{U}(!_X), 1_{\mathbf{U}(X)} > \\ &\quad = \mathbf{U}(\varepsilon_X^X) < \mathbf{U}(\pi_1'), \mathbf{U}(\pi_2') >^{-1} (\mathbf{U}(e_{X^X}) \times \mathbf{U}(X)) < \mathbf{U}(!_X), 1_{\mathbf{U}(X)} > . \end{aligned} \tag{2.42}$$

We have

$$\lambda_{\mathbf{U}(X)}(!_{\mathbf{U}(1)} \times \mathbf{U}(X)) < \mathbf{U}(!_X), 1_{\mathbf{U}(X)} >= 1_{\mathbf{U}(X)}$$

and

$$\begin{aligned} &\mathbf{U}(\varepsilon_X^X) < \mathbf{U}(\pi_1'), \mathbf{U}(\pi_2') >^{-1} (\mathbf{U}(e_{X^X}) \times \mathbf{U}(X)) < \mathbf{U}(!_X), 1_{\mathbf{U}(X)} > \\ &\quad = \mathbf{U}(\varepsilon_X^X(e_{X^X} \times X) <!_X, 1_X >) = \mathbf{U}(\lambda_X <!_X, 1_X >) = 1_{\mathbf{U}(X)}. \end{aligned}$$

This proves (2.42), and so proves the commutativity of the second square of (2.29). □

2.4 Applying a product preserving functor

Recall from Example 1.2.4 that, the category **Cat** of small categories is a cartesian closed category, that is, it has a terminal object 1 and binary products $B \times X$, as well as exponentials B^X, for all objects B and X in **Cat**, where exponent B^X is the functor category. In this section we assume that objects B and X are internal magmas in **Cat**. We shall use the same notation as in Section 2.2.

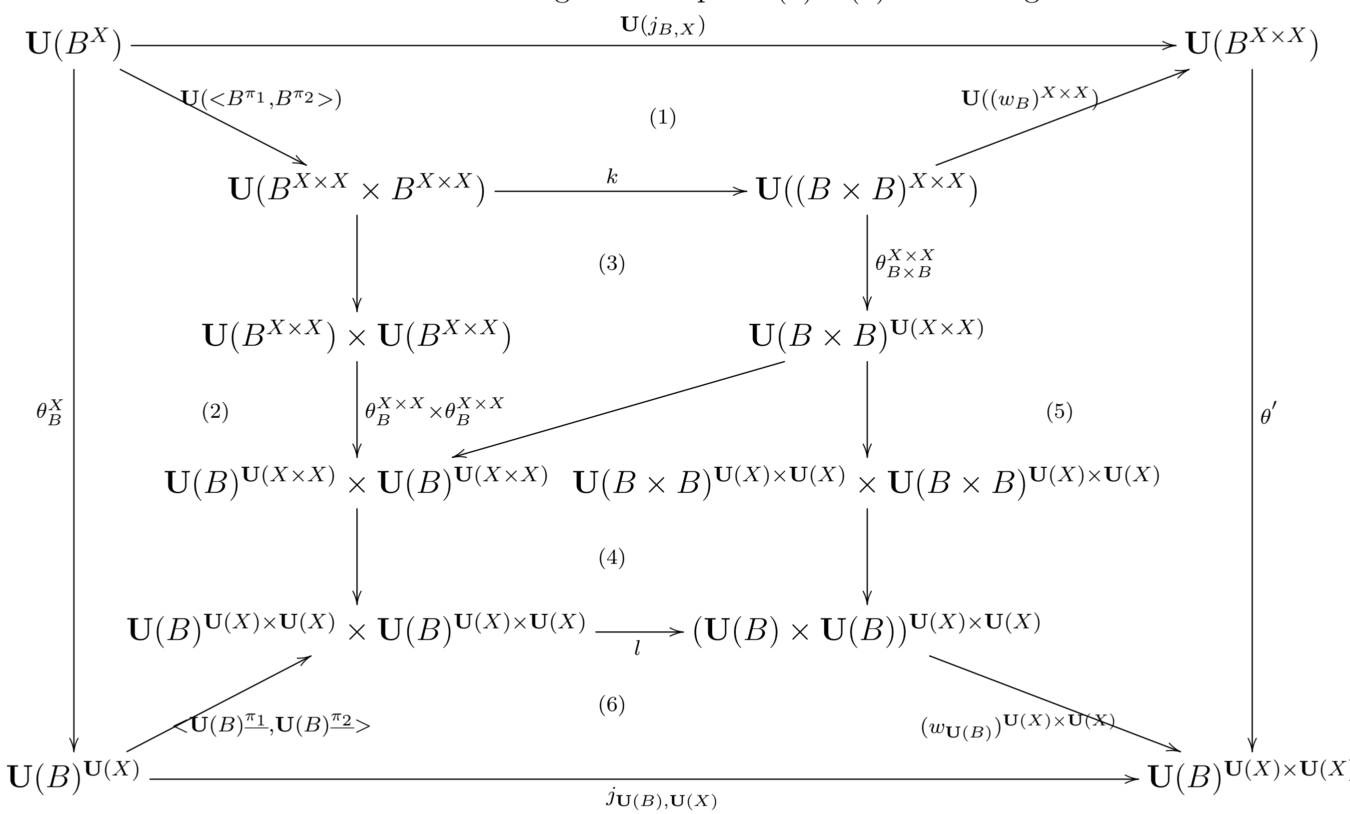
As follows from (2.8) we have:

2.4.1 Lemma. *The diagram*

$$\begin{array}{ccc} \mathbf{U}(B^X) & \xrightarrow{\mathbf{U}(j_{B,X})} & \mathbf{U}(B^{X\times X}) \\ \Big\downarrow{\scriptstyle \theta_B^X} & & \Big\downarrow{\scriptstyle \mathbf{U}(B)^{<\mathbf{U}(\pi_1),\mathbf{U}(\pi_2)>^{-1}}\theta_B^{X\times X}} \\ \mathbf{U}(B)^{\mathbf{U}(X)} & \xrightarrow[j_{\mathbf{U}(B),\mathbf{U}(X)}]{} & \mathbf{U}(B)^{\mathbf{U}(X)\times\mathbf{U}(X)} \end{array} \tag{2.43}$$

commutes.

Proof. It is a routine calculation checking that the parts $(1)-(6)$ of the diagram



in which $k = \mathbf{U}(<(\pi_1')^{X\times X}, (\pi_2')^{X\times X}>^{-1})$, $l = <(\pi_1')^{\mathbf{U}(X)\times\mathbf{U}(X)}, (\pi_2')^{\mathbf{U}(X)\times\mathbf{U}(X)}>^{-1}$, $w_{\mathbf{U}(B)}$ is the operation on $\mathbf{U}(B)$ induced by w_B, $\theta' = \mathbf{U}(B)^{<\mathbf{U}(\pi_1),\mathbf{U}(\pi_2)>^{-1}}\theta_B^{X\times X}$ and the unlabelled arrows are the appropriate canonical isomorphisms, commute.

□

2.4.2 Remark. *As follows from Definition* 2.2.1*, it can be shown that,* $\theta_B^X : \mathbf{U}(B^X) \to \mathbf{U}(B)^{\mathbf{U}(X)}$ *is natural in* B *and* X*, where* $\mathbf{U}$ *is defined as in Construction* 2.3.1*.*

2.4.3 Lemma. *The diagram*

$$\begin{array}{ccc} \mathbf{U}(B^X) & \xrightarrow{\mathbf{U}(B^{w_X})} & \mathbf{U}(B^{X\times X}) \\ \downarrow{\scriptstyle \theta_B^X} & & \downarrow{\scriptstyle \mathbf{U}(B)^{<\mathbf{U}(\pi_1),\mathbf{U}(\pi_2)>^{-1}}\theta_B^{X\times X}} \\ \mathbf{U}(B)^{\mathbf{U}(X)} & \xrightarrow[\mathbf{U}(B)^{w_{\mathbf{U}(X)}}]{} & \mathbf{U}(B)^{\mathbf{U}(X)\times\mathbf{U}(X)} \end{array} \tag{2.44}$$

in which $w_{\mathbf{U}(X)}$ *is the operation on* $\mathbf{U}(X)$ *induced by* w_X *and* $\pi_i : X \times X \to X$ $(i = 1, 2)$ *are the product projections, commutes.*

Proof. Using the commutativity of

$$\begin{array}{ccc} \mathbf{U}(B^X) & \xrightarrow{\mathbf{U}(B^{w_X})} & \mathbf{U}(B^{X\times X}) \\ \downarrow{\scriptstyle \theta_B^X} & & \downarrow{\scriptstyle \theta_B^{X\times X}} \\ \mathbf{U}(B)^{\mathbf{U}(X)} & \xrightarrow[\mathbf{U}(B)^{\mathbf{U}(w_X)}]{} & \mathbf{U}(B)^{\mathbf{U}(X\times X)} \end{array} \tag{2.45}$$

which follows from the naturality of θ, and the definition of $w_{\mathbf{U}(X)}$, which is nothing but

$$w_{\mathbf{U}(X)} = \mathbf{U}(w_X) < \mathbf{U}(\pi_1), \mathbf{U}(\pi_2) >^{-1},$$

we obtain
$\mathbf{U}(B)^{w_{\mathbf{U}(X)}}\theta_B^X = \mathbf{U}(B)^{<\mathbf{U}(\pi_1),\mathbf{U}(\pi_2)>^{-1}}\mathbf{U}(B)^{\mathbf{U}(w_X)}\theta_B^X$
$= \mathbf{U}(B)^{<\mathbf{U}(\pi_1),\mathbf{U}(\pi_2)>^{-1}}\theta_B^{X\times X}\mathbf{U}(B^{w_X})$. □

2.4.4 Theorem. *The left-hand square of the diagram*

$$\begin{array}{ccccc}
\mathbf{U}(\underline{Hom}(X,B)) & \xrightarrow{\mathbf{U}(i_{B,X})} & \mathbf{U}(B^X) & \underset{\mathbf{U}(B^{w_X})}{\overset{\mathbf{U}(j_{B,X})}{\rightrightarrows}} & \mathbf{U}(B^{X\times X}) \\
{\scriptstyle \Theta_B^X}\Big\downarrow & & {\scriptstyle \theta_B^X}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathbf{U}(B)^{<\mathbf{U}(\pi_1),\mathbf{U}(\pi_2)>^{-1}}\theta_B^{X\times X}} \\
Hom(\mathbf{U}(X),\mathbf{U}(B)) & \xrightarrow{i_{\mathbf{U}(B),\mathbf{U}(X)}} & \mathbf{U}(B)^{\mathbf{U}(X)} & \underset{\mathbf{U}(B)^{w_{\mathbf{U}(X)}}}{\overset{j_{\mathbf{U}(B),\mathbf{U}(X)}}{\rightrightarrows}} & \mathbf{U}(B)^{\mathbf{U}(X)\times\mathbf{U}(X)}
\end{array} \tag{2.46}$$

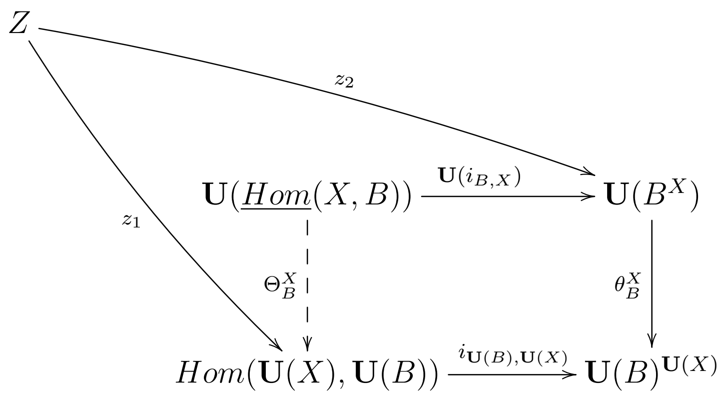
is a pullback, in which $\theta_B^{X\times X}$ *is a monomorphism and* $\underline{Hom}(X,B)$ *is an internal Hom object in* **Cat** (*G. Janelidze, unpublished*).

Proof. Observe that, the right-hand squares of (2.46) are formed by diagrams (2.43) and (2.44). Using the fact that, $i_{\mathbf{U}(B),\mathbf{U}(X)}$ is an equalizer of the pair of arrows $j_{\mathbf{U}(B),\mathbf{U}(X)}$ and $\mathbf{U}(B)^{w_{\mathbf{U}(X)}}$, we have

$j_{\mathbf{U}(B),\mathbf{U}(X)}\theta_B^X\mathbf{U}(i_{B,X}) = \mathbf{U}(B)^{<\mathbf{U}(\pi_1),\mathbf{U}(\pi_2)>^{-1}}\theta_B^{X\times X}\mathbf{U}(j_{B,X})\mathbf{U}(i_{B,X})$ (by (2.43))
$= \mathbf{U}(B)^{<\mathbf{U}(\pi_1),\mathbf{U}(\pi_2)>^{-1}}\theta_B^{X\times X}\mathbf{U}(B^{w_X})\mathbf{U}(i_{B,X})$ (since $i_{B,X}$ is an equalizer of $j_{B,X}$ and B^{w_X})
$= \mathbf{U}(B)^{w_{\mathbf{U}(X)}}\theta_B^X\mathbf{U}(i_{B,X})$ (by (2.44)),

thus, there exists a unique morphism $\mathbf{U}(\underline{Hom}(X,B)) \overset{\Theta_B^X}{\dashrightarrow} Hom(\mathbf{U}(X),\mathbf{U}(B))$ such that the left-hand square commutes in (2.46).

Consider the diagram



(2.47)

in which $i_{\mathbf{U}(B),\mathbf{U}(X)}z_1 = \theta_B^X z_2$. We have

$$\mathbf{U}(B)^{<\mathbf{U}(\pi_1),\mathbf{U}(\pi_2)>^{-1}}\theta_B^{X\times X}\mathbf{U}(j_{B,X})z_2 = j_{\mathbf{U}(B),\mathbf{U}(X)}\theta_B^X z_2 \text{ (by (2.43))}$$
$$= j_{\mathbf{U}(B),\mathbf{U}(X)}i_{\mathbf{U}(B),\mathbf{U}(X)}z_1$$
$$= \mathbf{U}(B)^{w_{\mathbf{U}(X)}}i_{\mathbf{U}(B),\mathbf{U}(X)}z_1 \text{ (since } i_{\mathbf{U}(B),\mathbf{U}(X)} \text{ equalizes } j_{\mathbf{U}(B),\mathbf{U}(X)} \text{ and } \mathbf{U}(B)^{w_{\mathbf{U}(X)}})$$
$$= \mathbf{U}(B)^{w_{\mathbf{U}(X)}}\theta_B^X z_2$$
$$= \mathbf{U}(B)^{<\mathbf{U}(\pi_1),\mathbf{U}(\pi_2)>^{-1}}\theta_B^{X\times X}\mathbf{U}(B^{w_X})z_2 \text{ (by (2.43))},$$

whence

$$\mathbf{U}(j_{B,X})z_2 = \mathbf{U}(B^{w_X})z_2 \tag{2.48}$$

since $\theta_B^{X\times X}$ is monic and $\mathbf{U}(B)^{<\mathbf{U}(\pi_1),\mathbf{U}(\pi_2)>^{-1}}$ is an isomorphism. Consequently, there exists a unique morphism $Z \overset{\Upsilon}{\dashrightarrow} \mathbf{U}(\underline{Hom}(X,B))$ such that

$$\mathbf{U}(i_{B,X})\Upsilon = z_2. \tag{2.49}$$

Moreover

$$i_{\mathbf{U}(B),\mathbf{U}(X)}\Theta_B^X\Upsilon = \theta_B^X\mathbf{U}(i_{B,X})\Upsilon = \theta_B^X\ z_2 \text{ (by (2.49))} = i_{\mathbf{U}(B),\mathbf{U}(X)}z_1,$$

which implies that

$$\Theta_B^X\Upsilon = z_1 \tag{2.50}$$

since $i_{\mathbf{U}(B),\mathbf{U}(X)}$ is monic. Consequently, from (2.49) and (2.50) we conclude that, the commutative left-hand square in (2.46) is indeed a pullback. □

One may then describe automorphism group $\underline{Aut}(X)$ in **Cat** for X in **Cat** and $\mathbf{U} = \mathbf{U}_1 : \mathbf{Cat} \to \mathbf{Sets}^{M_1}$ (as in Construction 2.3.1). First, note that, as follows from Proposition 2.2.3 and Theorem 2.4.4, we can write

$\mathbf{U}_1(\underline{Hom}(X, B)) = \{(\tau : V \longrightarrow V') \in \mathbf{U}_1(B^X) \mid \theta_B^X(\tau) \in Hom(\mathbf{U}_1(X), \mathbf{U}_1(B))\} =$
$\{(\tau : V \longrightarrow V') \in \mathbf{U}_1(B^X) \mid \forall m \in M_1\ \forall x_1, x_2 \in \mathbf{U_1}X\ (\theta_B^X(\tau))(m, w_X(x_1, x_2))\} =$
$\{w_B((\theta_B^X(\tau))(m, x_1), (\theta_B^X(\tau))(m, x_2))\}$,

and, using the explicit description of the (injective) map
$\theta_B^X : \mathbf{U}_1(B^X) \longrightarrow \mathbf{U}_1(B)^{\mathbf{U}_1(X)}$ in Proposition 2.3.2, we obtain
$\mathbf{U}_1(\underline{Hom}(X, B)) = \{(\tau : V \longrightarrow V') \in \mathbf{U}_1(B^X) \mid \forall x_1, x_2 \in X\ P_1(\tau, x_1, x_2)\ \&$
$P_2(\tau, x_1, x_2)\ \&\ P_3(\tau, x_1, x_2)\}$,
where $P_i(\tau, x_1, x_2) (i = 1, 2, 3)$ are the following three conditions respectively:

- $P_1(\tau, x_1, x_2) : \forall_{(x_1 : \underline{x}_1 \to \underline{x'}_1, x_2 : \underline{x}_2 \to \underline{x'}_2) \in X}\ \tau_{w_X(\underline{x'}_1, \underline{x'}_2)} V(w_X(x_1, x_2)) = w_B(\theta_B^X(\tau)(1, x_1), \theta_B^X(\tau)(1, x_2))$
  $= w_B(\tau_{\underline{x'}_1} V(x_1), \tau_{\underline{x'}_2} V(x_2))$,

- $P_2(\tau, x_1, x_2) : \forall x_1, x_2 \in X\ V(w_X(x_1, x_2)) = w_B(\theta_B^X(\tau)(s, x_1), \theta_B^X(\tau)(s, x_2)) =$
  $w_B(V(x_1), V(x_2))$,

- $P_3(\tau, x_1, x_2) : \forall x_1, x_2 \in X\ V'(w_X(x_1, x_2)) = w_B(\theta_B^X(\tau)(t, x_1), \theta_B^X(\tau)(t, x_2)) =$
  $w_B(V'(x_1), V'(x_2))$.

In order to make these conditions more natural, let us introduce:

2.4.5 Definition. **(a)** *A functor* $V : X \longrightarrow B$ *is said to be* ***homomorphic****, if, for every two morphisms* $x_1 : \underline{x}_1 \longrightarrow \underline{x}'_1$ *and* $x_2 : \underline{x}_2 \longrightarrow \underline{x}'_2$ *in* X*, the morphism*

$$V(w_X(x_1, x_2)) : V(w_X(\underline{x}_1, \underline{x}_2)) \longrightarrow V(w_X(\underline{x}'_1, \underline{x}'_2))$$

coincides with the morphism

$$w_B(V(x_1), V(x_2)) : w_B(V(\underline{x_1}), V(\underline{x_2})) \longrightarrow w_B(V(\underline{x'_1}), V(\underline{x'_2})).$$

(b) *A natural transformation* $\tau : V \longrightarrow V'$ *between homomorphic functors is said to be* ***homomorphic****, if, for every two objects* $\underline{x_1}$ *and* $\underline{x_2}$ *in* X*, the morphism*

$$\tau_{w_X(\underline{x_1}, \underline{x_2})} : V(w_X(\underline{x}_1, \underline{x}_2)) \longrightarrow V'(w_X(\underline{x}_1, \underline{x}_2))$$

coincides with the morphism

$$w_B(\tau_{\underline{x}_1}, \tau_{\underline{x}_2}) : w_B(V(\underline{x_1}), V(\underline{x_2})) \longrightarrow w_B(V'(\underline{x_1}), V'(\underline{x_2})).$$

2.4.6 Remark. *The two notions introduced in Definition* 2.4.5 *agree, as much as the present context allows of course, with the notions of strict monoidal functor and of monoidal natural transformation (of strict monoidal functors), respectively.*

Our description of $\mathbf{U}_1(\underline{Hom}(X, B))$ easily gives:

2.4.7 Theorem. *In the context above,* $\underline{Hom}(X, B)$ *in* **Cat** *can be described as the subcategory of the functor category* B^X *with objects all homomorphic functors and morphims all homomorphic natural transformations. Moreover, the functor* $i_{B,X} : \underline{Hom}(X, B) \to B^X$ *can be then identified with the corresponding inclusion functor.*

Next, the category $\underline{End}(X) = \underline{Hom}(X, X)$ $(B = X)$ has an obvious internal monoid structure in **Cat**, and $\underline{Aut}(X)$ can be obtained as associated 'internal group of invertible elements'. This gives

2.4.8 Lemma. $\underline{Aut}(X) = \{(\tau : V \to V') \in \underline{Hom}(X, X) \mid V$ *and* V' *are isomorphisms, and there exists* $\bar{\tau} : V^{-1} \to V'^{-1}$ *such that* $\bar{\tau}\tau = 1_{1_X} = \tau\bar{\tau}\}$.

From this lemma we easily conclude:

2.4.9 Theorem. *For every category* X*, the category* $\underline{Aut}(X)$ *is the full subcategory of* X^X *with objects all homomorphic functors that are isomorphisms and morphisms all homomorphic natural transformations that are isomorphisms.*

Chapter 3

Higher-dimensional group automorphisms

3.1 The categorical group Aut(X)

The aim of this section is to describe the internal-group-automorphism M_1-group $Aut(X)$ in $\mathbf{Sets}^{M_1}$ as cat^n-group, by describing objects involved in the left-hand square of diagram (2.46) in the case where $\mathbf{U} = \mathbf{U}_n : \mathbf{Cat}^n \to \mathbf{Sets}^{M_n}$ and X is a cat^n-group corresponding to an internal group $cat(X)$ in $\mathbf{Cat}^n$.

3.1.1 Definition. *An **internal** M_1**-group** X in the category $\mathbf{Sets}^{M_1}$ is an M_1-set X equipped with a group structure.*
Such an M_1-group is said to be a categorical group if it satisfies the implication

$$(s_1x = 0 = t_1y) \Longrightarrow x + y = y + x, \tag{3.1}$$

for all $x, y \in X$ and $s_1, t_1 \in M_1$.

3.1.2 Lemma. *The following statements are equivalent:*

(a) (C_0, C_1, d, c, e, m) *is an internal group in* **Cat***;*

(b) X *is a categorical group in* $\mathbf{Sets}^{M_1}$.

Proof. For an internal group $C = (C_0, C_1, d, c, e, m)$ in **Cat** which is equivalent to an internal category in the category of groups, we take (C_1, ed, ec) to be our categorical group.
Conversely, given a categorical group X in $\mathbf{Sets}^{M_1}$, the internal group $cat(X)$ in **Cat** corresponding to X is constructed as follows:

- the group $cat(X)_0$ of objects of $cat(X)$ is defined by

$$cat(X)_0 = \{x \in X \mid s_1 x = x\} (= \{x \in X \mid t_1 x = x\}) \tag{3.2}$$

  and the identity morphism of any $x \in cat(X)_0$ is x itself.

- the group $cat(X)_1$ of morphisms of $cat(X)$ is X itself;

- the domain and codomain of $x \in cat(X)_1 = X$ are $s_1 x$ and $t_1 x$ respectively. □

Recall that this notion of an internal group in **Cat** is equivalent to that of internal category in **Groups**.
As follows from (1.10), given x and y with $s_1 x = t_1 y$, then the composite xy is uniquely defined by

$$xy = x - s_1 x + y (= x - t_1 y + y). \tag{3.3}$$

3.1.3 Remark. (i) *As it easily follows from* (1.11)*, we can also replace* (3.3) *with*

$$xy = y - s_1 x + x (= y - t_1 y + x). \tag{3.4}$$

(ii) *Again as follows from* (3.3) *every morphism* x *has an inverse for composition given by*

$$x^{-1} = s_1 x - x + t_1 x (= t_1 x - x + s_1 x) \tag{3.5}$$

such that

$$xx^{-1} = x - s_1x + s_1x - x + t_1x = t_1x$$

is identity morphism of the codomain of x and

$$x^{-1}x = s_1x - x + t_1x - t_1x + x = s_1x,$$

is identity morphism of the domain of x, so every internal category $cat(X)$ in **Cat** *is in fact an internal groupoid.*

3.1.4 Construction. *For us to construct categorical group $Aut(X)$ in $\mathbf{Sets}^{M_1}$ we need to describe objects involved in the left-hand square of diagram (2.46) in the case where $\mathbf{U} = \mathbf{U}_1 : \mathbf{Cat} \to \mathbf{Sets}^{M_1}$, and the role of B and X is played by categorical groups $cat(B)$ and $cat(X)$ as defined in Lemma 3.1.2 ($cat(B)$ and $cat(X)$ can of course be considered as internal magmas in* **Cat**).
We display our diagram simply as

$$\begin{array}{ccc} \mathbf{U}_1(\underline{Hom}(X,B)) & \xrightarrow{\mathbf{U}_1(i_{B,X})} & \mathbf{U}_1(B^X) \\ {\scriptstyle \Theta_B^X} \downarrow & & \downarrow {\scriptstyle \theta_B^X} \\ Hom(\mathbf{U}_1(X), \mathbf{U}_1(B)) & \xrightarrow{i_{\mathbf{U}_1(B),\mathbf{U}_1(X)}} & \mathbf{U}_1(B)^{\mathbf{U}_1(X)}. \end{array} \tag{3.6}$$

Recall that (3.6) is in fact a pullback. Our first step is to describe the image $\theta_B^X(\mathbf{U}_1(B^X))$ of $\mathbf{U}_1(B^X)$ in $\mathbf{U}_1(B)^{\mathbf{U}_1(X)}$. We observe that:

(a) *according to Example 1.2.3(a), and having in mind that we identified B with $cat(B)$ and X with $cat(X)$, we can write*
$\mathbf{U}_1(B)^{\mathbf{U}_1(X)} = \{\alpha : M_1 \times X \to B \mid m\alpha(m', x) = \alpha(mm', mx)$ *for all* $m, m' \in M_1$ *and* $x \in X\}$, *with M_1 acting on B^X by $(m\alpha)(m', x) = \alpha(m'm, x)$;*

(b) *if $\alpha : M_1 \times X \to B$ is the θ_B^X-image of $\tau : V \to V'$, then, according to Proposition 2.3.2, we have*

$$V(x) = \alpha(s_1, x), V'(x) = \alpha(t_1, x), \tau_y = \alpha(1, y), \tag{3.7}$$

for every $x \in X$ *and every* $y \in cat(X)_0$*. On the other hand, if* $(\alpha : M_1 \times X \to B) \in \mathbf{U}_1(B)^{\mathbf{U}_1(X)}$ *and* $(\tau : V \to V') \in \mathbf{U}_1(B^X)$ *satisfy* (3.7)*, then we immediately obtain from* (2.19) *and* (2.20) *that,*

$$\theta_B^X(s_1, x) = \alpha(s_1, x)$$

and

$$\theta_B^X(t_1, x) = \alpha(t_1, x).$$

For each $(x : \underline{x} \to \underline{x'}) \in \mathbf{U}_1(X)$*, we will have*

$$\theta_B^X(\tau)(1, x) = \tau_{\underline{x'}} V(x) = \alpha(1, \underline{x'})\alpha(s_1, x) = \alpha(1, t_1 x)\alpha(s_1, x)$$
$$= \alpha(1, t_1 x) - s_1\alpha(1, t_1 x) + \alpha(s_1, x).$$

That is, $(\alpha : M_1 \times X \to B) \in \mathbf{U}_1(B)^{\mathbf{U}_1(X)}$ *is the* θ_B^X*-image of* $(\tau : V \to V') \in \mathbf{U}_1(B^X)$ *if and only if* (3.7) *holds and*

$$\alpha(1, x) = \alpha(1, t_1 x) - s_1\alpha(1, t_1 x) + \alpha(s_1, x), \tag{3.8}$$

for every $x \in X$*;*

(c) *to complete our description , we need to find a necessary and sufficient condition on* $(\alpha : M_1 \times X \to B) \in \mathbf{U}_1(B)^{\mathbf{U}_1(X)}$ *satisfying* (3.8)*, under which* V*,* V' *and* τ *defined by* (3.7) *form a triple in which* V *and* V' *are functors from* X *to* B*, and* τ *is a natural transformation from* V *to* V'*. This desired condition should therefore be equivalent to the conjunction of the following conditions:*

- V *preserves domains, which can be written as:*

$$s_1\alpha(s_1, x) = \alpha(s_1, s_1 x), \tag{3.9}$$

for each x *in* X*;*

- *V preserves codomains, which can be written as:*

$$t_1\alpha(s_1, x) = \alpha(s_1, t_1x), \tag{3.10}$$

  for each x in X;

- *V preserves identity morphisms, which follows from* (3.9) (*and also follows from* (3.10))*;*

- *V preserves composition, which can be written as:*

$$\alpha(s_1, x - s_1x + y) = \alpha(s_1, x) - s_1\alpha(s_1, x) + \alpha(s_1, y), \tag{3.11}$$

  for all $x, y \in X$ with $s_1x = t_1y$;

- *V' preserves domains, which can be written as:*

$$s_1\alpha(t_1, x) = \alpha(t_1, s_1x), \tag{3.12}$$

  for each x in X;

- *V' preserves codomains, which can be written as:*

$$t_1\alpha(t_1, x) = \alpha(t_1, t_1x), \tag{3.13}$$

  for each x in X;

- *V' preserves identity morphisms, which follows from* (3.12) (*and also follows from* (3.13))*;*

- *V' preserves composition, which can be written as:*

$$\alpha(t_1, x - s_1x + y) = \alpha(t_1, x) - s_1\alpha(t_1, x) + \alpha(t_1, y), \tag{3.14}$$

  for all $x, y \in X$ with $s_1x = t_1y$;

- *the components of τ have the right domains, which can be written as:*

$$s_1\alpha(1, x) = \alpha(s_1, x), \tag{3.15}$$

  for each $x \in cat(X)_0$;

- *the components of τ have the right codomains, which can be written as:*

$$t_1\alpha(1,x) = \alpha(t_1,x), \tag{3.16}$$

for each $x \in cat(X)_0$;

- *τ is natural, which can be written as:*

$$\alpha(1,t_1x) - s_1\alpha(1,t_1x) + \alpha(s_1,x) = \alpha(t_1,x) - s_1\alpha(t_1,x) + \alpha(1,s_1x), \tag{3.17}$$

for each x in X.

(d) *However, being an element of $\mathbf{U}_1(B)^{\mathbf{U}_1(X)}$, α satisfies the equality $m\alpha(m',x) = \alpha(mm',mx)$ for all $m,m' \in M_1$ and $x \in X$, which implies* (3.9), (3.10), (3.12), (3.13), (3.15) *and* (3.16).

This gives us :

3.1.5 Proposition. *The image $\theta_B^X(\mathbf{U}_1(B^X))$ of the map*

$$\theta_B^X : \mathbf{U}_1(B^X) \to \mathbf{U}_1(B)^{\mathbf{U}_1(X)}$$

consists of all α in $\mathbf{U}_1(B)^{\mathbf{U}_1(X)}$ satisfying (3.8), (3.11), (3.14) *and* (3.17).

Our next step is to describe the image $i_{\mathbf{U}_1(B),\mathbf{U}_1(X)}(Hom(\mathbf{U}_1(X),\mathbf{U}_1(B)))$ in $\mathbf{U}_1(B)^{\mathbf{U}_1(X)}$. However, is already done in Proposition 2.2.3, which tells us that this image consists of all α in $\mathbf{U}_1(B)^{\mathbf{U}_1(X)}$ satisfying

$$\alpha(m, w_X(x_1,x_2)) = w_B(\alpha(m,x_1),\alpha(m,x_2))$$

for all $m \in \{1,s_1,t_1\}$ and $x_1,x_2 \in X$, or, written in the notation we are using now, satisfying

$$\alpha(m,x_1+x_2) = \alpha(m,x_1) + \alpha(m,x_2), \tag{3.18}$$

for all $m \in \{1, s, t\}$ and $x_1, x_2 \in X$.
It remains to describe $\mathbf{U}_1(\underline{Hom}(X, B))$, but since (3.6) is a pullback (by Theorem 2.4.4), it is canonically isomorphic to the intersection

$$\theta_B^X(\mathbf{U}_1(B^X)) \cap i_{\mathbf{U}_1(B),\mathbf{U}_1(X)}(Hom(\mathbf{U}_1(X), \mathbf{U}_1(B))) \tag{3.19}$$

of the images considered above. According to our previous remarks, this intersection (3.19) consists of all $\alpha \in \mathbf{U}_1(B)^{\mathbf{U}_1(X)}$ satisfying (3.8), (3.11), (3.14), (3.17) and (3.18). However, it is easy to see that when $\alpha \in \mathbf{U}_1(B)^{\mathbf{U}_1(X)}$, (3.18) implies (3.11) and (3.14); thus we obtain:

3.1.6 Proposition. $\mathbf{U}_1(\underline{Hom}(X, B))$ *is canonically isomorphic to the* M_1*-set for all* $\alpha \in \mathbf{U}_1(B)^{\mathbf{U}_1(X)}$ *satisfying* (3.8)*,* (3.17) *and* (3.18)*.*

When X is merely an M_1-set, we already know that

$B^X = \{\alpha : M_1 \times X \to B \mid m\alpha(m', x) = \alpha(mm', mx)$ for all $m, m' \in M_1$ and $x \in X\}$, with M_1 acting on B^X by $(m\alpha)(m', x) = \alpha(m'm, x)$ (see Example 1.2.3(a)).

Moreover, when $B = X$, it has been shown in Example 2.1.2(b) that the monoid structure on X^X has the explicit form

$$(\alpha\beta)(m, x) = \alpha(m, \beta(m, x)).$$

When X is a categorical group, the same formula will describe the composition on the internal-group-endomorphism M_1-monoid $End(X)$. This composition makes the internal-group-automorphism M_1-group $Aut(X)$ a categorical group. However, since we decided to use the additive notation for the categorical groups, we should now write

$$(\alpha + \beta)(m, x) = \alpha(m, \beta(m, x)). \tag{3.20}$$

Having in mind that the role of X is played by categorical group $cat(X)$ (see Construction 3.1.4), then from Lemma 2.2.5 and Proposition 3.1.6, we obtain:

3.1.7 Theorem. *Let M_1 be as in Construction 2.3.1, and X be a categorical group. The categorical group $Aut(X)$ can be described as (that is, is canonically isomorphic to) the M_1-group of all the maps $\alpha : M_1 \times X \to X$ satisfying the following conditions:*

(a) $m\alpha(m', x) = \alpha(mm', mx)$ *for all* $m, m' \in M_1$ *and* $x \in X$*;*

(b) $\alpha(1, x) = \alpha(1, t_1 x) - s_1\alpha(1, t_1 x) + \alpha(s_1, x)$ *for every* $x \in X$*;*

(c) $\alpha(1, t_1 x) - s_1\alpha(1, t_1 x) + \alpha(s_1, x) = \alpha(t_1, x) - s_1\alpha(t_1, x) + \alpha(1, s_1 x)$ *for each* $x \in X$*;*

(d) $\alpha(m, x_1 + x_2) = \alpha(m, x_1) + \alpha(m, x_2)$, *for all* $x_1, x_2 \in X$*;*

(e) α *is invertible, that is, for every* $m \in M_1$*, the map* $\alpha(m, -) : X \to X$ *is a bijection.*

The group operation on $Aut(X)$ is defined by (3.20), with M_1 acting on $Aut(X)$ by $(m\alpha)(m', x) = \alpha(m'm, x)$ for all $m, m' \in M_1$ and $x \in X$.

Although Theorem 3.1.7 follows from previous results, it is interesting to check directly that $Aut(X)$ described as in Theorem 3.1.7 satisfies (3.1). We observe the following:

• *To check that $Aut(X)$ satisfies (3.1), is to check that $\alpha + \beta = \beta + \alpha$ provided that α and β (are in $Aut(X)$ and) satisfy*

$$s_1\alpha = \pi_2 = t_1\beta, \tag{3.21}$$

where $\pi_2 : M_1 \times X \to X$ is the second projection map, which obviously plays the role of $0 \in Aut(X)$.

• *We have:*

$$(\alpha+\beta)(s_1,x) = \alpha(s_1,\beta(s_1,x)) = (s_1\alpha)(1,\beta(s_1,x)) = \pi_2(1,\beta(s_1,x)) = \beta(s_1,x),$$

$$(\beta+\alpha)(s_1,x) = \beta(s_1,\alpha(s_1,x)) = \beta(s_1,(s_1\alpha)(1,x)) = \beta(s_1,\pi_2(1,x)) = \beta(s_1,x),$$

and so $(\alpha+\beta)(s_1,x) = (\beta+\alpha)(s_1,x)$*;*

- *Similarly*

$$(\alpha+\beta)(t_1,x) = \alpha(t_1,\beta(t_1,x)) = \alpha(t_1,(t_1\beta)(1,x)) = \alpha(t_1,\pi_2(1,x)) = \alpha(t_1,x),$$

$$(\beta+\alpha)(t_1,x) = \beta(t_1,\alpha(t_1,x)) = (t_1\beta)(1,\alpha(t_1,x)) = \pi_2(1,\alpha(t_1,x)) = \alpha(t_1,x),$$

and so $(\alpha+\beta)(t_1,x) = (\beta+\alpha)(t_1,x)$*;*

- *It remains to show that* $(\alpha+\beta)(1,x) = (\beta+\alpha)(1,x)$*. We have*

$(\alpha+\beta)(1,x) = \alpha(1,\beta(1,x))$
$= \alpha(1,t_1\beta(1,x)) - s_1\alpha(1,t_1\beta(1,x)) + \alpha(s_1,\beta(1,x))$ *(by* (3.8)*)*
$= \alpha(1,\beta(t_1,t_1x)) - s_1\alpha(1,\beta(t_1,t_1x)) + \alpha(s_1,\beta(1,x))$ *(by Theorem* 3.1.7*(a) applied to* β*)*
$= \alpha(1,\beta(t_1,x)) - s_1\alpha(1,\beta(t_1,x)) + \alpha(s_1,\beta(1,x))$ *(since* $t_1x = x$*)*
$= \alpha(1,(t_1\beta)(1,t_1x)) - s_1\alpha(1,(t_1\beta)(1,t_1x)) + (s_1\alpha)(1,\beta(1,x))$
$= \alpha(1,x) - s_1\alpha(1,x) + \beta(1,x)$ *(by* (3.21)*).*

Similarly,

$(\beta+\alpha)(1,x) = \beta(1,\alpha(1,x))$
$= \beta(1,t_1\alpha(1,x)) - s_1\beta(1,t_1\alpha(1,x)) + \beta(s_1,\alpha(1,x))$*(by* (3.8) *applied to* β *and* $\alpha(1,x)$*)*
$= \beta(t_1,\alpha(1,x)) - s_1\beta(t_1,\alpha(1,x)) + \beta(1,s_1\alpha(1,x))$*(by* (3.17) *applied to* β *and*

$\alpha(1, x))$
$= (t_1\beta)(1, \alpha(1, x)) - s_1(t_1\beta)(1, \alpha(1, x)) + \beta(1, (s_1\alpha)(1, x))$(*see again the last formula in Example* 1.2.3(*a*)*, and note that* $s_1\alpha(1, x) = (s_1\alpha)(1, x)$ *since* $s1x = x$)
$= \alpha(1, x) - s_1\alpha(1, x) + \beta(1, x)$(*by the second equality in* (3.21)),

which gives the desired equality $(\alpha + \beta)(1, x) = (\beta + \alpha)(1, x)$. $\square$

3.1.8 Remark. *The intermediate of the calculations above, namely the equalities*

$$(\alpha + \beta)(1, x) = \alpha(1, x) - s\alpha(1, x) + \beta(1, x) = (\beta + \alpha)(1, x),$$

$$(\alpha + \beta)(s_1, x) = \beta(s_1, x) = (\beta + \alpha)(s_1, x),$$

$$(\alpha + \beta)(t_1, x) = \alpha(t_1, x) = (\beta + \alpha)(t_1, x),$$

are also useful.

Recall that, cat^n-group is an additive group X equipped with endomorphisms $s_1, s_2, ..., s_n, t_1, t_2, ..., t_n$ with

$$s_i s_j = s_j s_i, t_i t_j = t_j t_i, s_i t_j = t_j s_i \quad \text{(i,j= 1,2, ..., n;} \quad i \neq j), \tag{3.22}$$

$$s_i t_i = t_i, t_i s_i = s_i, [ker(s_i), ker(t_i)] = 0, \quad \text{(i= 1,2,..., n)} \quad . \tag{3.23}$$

Equivalently, we could begin with the monoid M_n generated by the set

$$\{s_1, s_2, ..., s_n, t_1, t_2, ..., t_n\}$$

and the equalities (3.22) and the first two equalities of (3.23), and define a cat^n-group X as an additive M_n-group X with

$$s_i x = 0 = t_i y \Longrightarrow x + y = y + x \tag{3.24}$$

for all $i = 1, 2, ..., n$ and $x, y \in X$.

Let X be cat^n-group identified with its corresponding n-fold category of groups, such that, our product preserving functor $\mathbf{U_1}$ as in Construction 2.3.1 is given as $\mathbf{U}_n : \mathbf{Cat}^n \to \mathbf{Sets}^{M_n}$. We give the analogous description of $Aut(X)$ as follows:

3.1.9 Theorem. *The cat^n-group $Aut(X)$ can be described as (that is, is canonically isomorphic to) the M_n-group of all the maps $\alpha = M_n \times X \to X$ satisfying*

$$m\alpha(m', x) = \alpha(mm', mx), \tag{3.25}$$

$$\alpha(m, x_1 + x_2) = \alpha(m, x_1) + \alpha(m, x_2), \tag{3.26}$$

$$\alpha(m, -) : X \to X \quad \textit{is a bijection,} \tag{3.27}$$

$$\alpha(m, x) = \alpha(m, t_i x) - \alpha(s_i m, t_i x) + \alpha(s_i m, x) \qquad \text{(i= 1,2,..., n)}, \tag{3.28}$$

$$\alpha(m, x) = \alpha(t_i m, x) - \alpha(t_i m, s_i x) + \alpha(m, s_i x) \qquad \text{(i= 1,2,..., n)}, \tag{3.29}$$

for all $m, m' \in M_n$ and $x, x_1, x_2 \in X$; its structure is defined by

$$(\alpha + \beta)(m, x) = \alpha(m, \beta(m, x)), \quad (m'\alpha)(m, x) = \alpha(mm', x), \tag{3.30}$$

and the role of 0 *in it is played by the second projection $\pi_2 : M_n \times X \to X$.*

If instead of cat^n-groups we were considering M_n-groups (or, more generally M-groups for an arbitrary monoid M), then $Aut(X)$ would be constructed similarly but with (3.25) – (3.27) instead of (3.25) – (3.29). Therefore in order to prove that $Aut(X)$ constructed with (3.25) – (3.29) (and with the structure given by (3.30)) is a cat^n-group we only need to prove that it is closed in the M_n-group of all the maps $M_n \times X \to X$ satisfying (3.25) – (3.27) under the operations defined by (3.30), and that it satisfies (3.24). And to say that it satisfies (3.24) is to say that

$$s_i\alpha = \pi_2 = t_i\beta \Longrightarrow \alpha(m, \beta(m, x)) = \beta(m, \alpha(m, x)) \quad \text{(i= 1, 2, ...,n)}, \tag{3.31}$$

for all $\alpha, \beta \in Aut(X)$ and $(m, x) \in M_n \times X$. These proofs are given below in the form of two propositions.

3.1.10 Proposition. *$Aut(X)$ defined via* (3.25)$-$(3.29) *is closed in the M_n-group of all the maps $M_n \times X \to X$ satisfying* (3.25)$-$(3.27) *under the operations defined by* (3.30).

Proof. Suppose α and β satisfy (3.25) $-$ (3.29).
Then, for each $i = 1, 2, ..., n$, we have

$(\alpha+\beta)(m,x) = \alpha(m,\beta(m,x)) = \alpha(m,t_i\beta(m,x)) - \alpha(s_im,t_i\beta(m,x)) +$
$\alpha(s_im,\beta(m,x)) = \alpha(m,\beta(t_im,t_ix)) - \alpha(s_im,\beta(t_im,t_ix)) + \alpha(s_im,\beta(m,x))$
$= \alpha(m,\beta(t_im,t_ix)) - \alpha(s_im,\beta(t_im,t_ix)) + \alpha(s_im,\beta(m,t_ix)) - \alpha(s_im,\beta(s_im,t_ix)) +$
$\alpha(s_im,\beta(s_im,x))$,

$(\alpha+\beta)(m,t_ix) - (\alpha+\beta)(s_im,t_ix) + (\alpha+\beta)(s_im,x)$
$= \alpha(m,\beta(m,t_ix)) - \alpha(s_im,\beta(s_im,t_ix)) + \alpha(s_im,\beta(s_im,x))$
$= \alpha(m,t_i\beta(m,t_ix)) - \alpha(s_im,t_i\beta(m,t_ix)) + \alpha(s_im,\beta(m,t_ix)) - \alpha(s_im,\beta(s_im,t_ix)) +$
$\alpha(s_im,\beta(s_im,x))$
$= \alpha(m,\beta(t_im,t_ix)) - \alpha(s_im,\beta(t_im,t_ix)) + \alpha(s_im,\beta(m,t_ix)) - \alpha(s_im,\beta(s_im,t_ix)) +$
$\alpha(s_im,\beta(s_im,x))$,

and so $(\alpha+\beta)(m,x) = (\alpha+\beta)(m,t_ix) - (\alpha+\beta)(s_im,t_ix) + (\alpha+\beta)(s_im,x)$, which means that $\alpha+\beta$ satisfies (3.28).

We also have

$(\alpha+\beta)(m,x) = \alpha(m,\beta(m,x)) = \alpha(t_im,\beta(m,x)) - \alpha(t_im,s_i\beta(m,x)) +$
$\alpha(m,s_i\beta(m,x)) = \alpha(t_im,\beta(m,x)) - \alpha(t_im,\beta(s_im,s_ix)) + \alpha(m,\beta(s_im,s_ix))$
$= \alpha(t_im,\beta(t_im,x)) - \alpha(t_im,\beta(t_im,s_ix)) + \alpha(t_im,\beta(m,s_ix)) - \alpha(t_im,\beta(s_im,s_ix)) +$
$\alpha(m,\beta(s_im,s_ix))$,

$(\alpha+\beta)(t_im,x) - (\alpha+\beta)(t_im,s_ix) + (\alpha+\beta)(m,s_ix)$

$= \alpha(t_i m, \beta(t_i m, x)) - \alpha(t_i m, \beta(t_i m, s_i x)) + \alpha(m, \beta(m, s_i x))$
$= \alpha(t_i m, \beta(t_i m, x)) - \alpha(t_i m, \beta(t_i m, s_i x)) + \alpha(t_i m, \beta(m, s_i x)) - \alpha(t_i m, s_i \beta(m, s_i x)) + \alpha(m, s_i \beta(m, s_i x))$
$= \alpha(t_i m, \beta(t_i m, x)) - \alpha(t_i m, \beta(t_i m, s_i x)) + \alpha(t_i m, \beta(m, s_i x)) - \alpha(t_i m, \beta(s_i m, s_i x)) + \alpha(m, \beta(s_i m, s_i x))$

and so $(\alpha + \beta)(m, x) = (\alpha + \beta)(t_i m, x) - (\alpha + \beta)(t_i m, s_i x) + (\alpha + \beta)(m, s_i x)$, which means that $\alpha + \beta$ satisfies (3.29).

That is $Aut(X)$ is closed under the addition defined by (3.30).

Next suppose α satisfies (3.25) – (3.29) and $\alpha + \beta = 0$, that is, $\beta(m, -) : X \to X$ is the inverse map of $\alpha(m, -) : X \to X$, for each $m \in M_n$. Then

$\alpha(m, \beta(m, t_i x)) - \beta(s_i m, t_i x) + \beta(s_i m, x) = \alpha(m, \beta(m, t_i x)) - \alpha(m, \beta(s_i m, t_i x)) + \alpha(m, \beta(s_i m, x))$
$= t_i x - (\alpha(m, t_i \beta(s_i m, t_i x)) - \alpha(s_i m, t_i \beta(s_i m, t_i x)) + \alpha(s_i m, \beta(s_i m, t_i x))) + \alpha(m, t_i \beta(s_i m, x)) - \alpha(s_i m, t_i \beta(s_i m, x)) + \alpha(s_i m, \beta(s_i m, x))$
$= t_i x - (\alpha(m, \beta(s_i m, t_i x)) - \alpha(s_i m, \beta(s_i m, t_i x)) + t_i x) - \alpha(m, \beta(s_i m, t_i x)) - \alpha(s_i m, \beta(s_i m, t_i x)) + x$
$= t_i x - (\alpha(m, \beta(s_i m, t_i x)) - t_i x + t_i x) + \alpha(m, \beta(s_i m, t_i x)) - t_i x + x = x = \alpha(m, \beta(m, x))$,

$\alpha(m, \beta(t_i m, x)) - \beta(t_i m, s_i x) + \beta(m, s_i x) = \alpha(m, \beta(t_i m, x) - \alpha(m, \beta(t_i m, s_i x)) + \alpha(m, \beta(m, s_i x))$
$= \alpha(t_i m, \beta(t_i m, x)) - \alpha(t_i m, s_i \beta(t_i m, x)) + \alpha(m, s_i \beta(t_i m, x)) - (\alpha(t_i m, \beta(t_i m, s_i x)) - \alpha(t_i m, s_i \beta(t_i m, s_i x)) + \alpha(m, s_i \beta(t_i m, s_i x))) + s_i x$
$= x - \alpha(t_i m, \beta(t_i m, s_i x)) + \alpha(m, \beta(t_i m, s_i x)) - (s_i x - \alpha(t_i m, \beta(t_i m, s_i x)) + \alpha(m, \beta(t_i m, s_i x))) + s_i x$
$= x - s_i x + \alpha(m, \beta(t_i m, s_i x)) - (s_i x - s_i x + \alpha(m, \beta(t_i m, s_i x))) + s_i x = x =$

$\alpha(m, \beta(m, x))$,

and since $\alpha(m, -)$ is a bijection, this implies that β satisfies (3.28) and (3.29). That is, $Aut(X)$ is closed under the inverses.

Again, assuming that α satisfies (3.25) – (3.29), we have

$$(m'\alpha)(m, x) = \alpha(mm', x) = \alpha(mm', t_i x) - \alpha(s_i mm', t_i x) + \alpha(s_i mm', x)$$
$$= (m'\alpha)(m, t_i x) - (m'\alpha)(s_i m, t_i x) + (m'\alpha)(s_i m, x),$$

$$(m'\alpha)(m, x) = \alpha(mm', x) = \alpha(t_i mm', x) - \alpha(t_i mm', s_i x) + \alpha(mm', s_i x)$$
$$= (m'\alpha)(t_i m, x) - (m'\alpha)(t_i m, s_i x) + (m'\alpha)(m, s_i x),$$

and so $m'\alpha$ satisfies (3.28) and (3.29), which completes the proof. □

3.1.11 Proposition. (3.31) *holds for all* $\alpha, \beta \in Aut(X)$ *and* $(m, x) \in M_n \times X$.

Proof. We begin with a special case where the role of m is played by ms_i. If $s_i\alpha = \pi_2$, we have:

$$\alpha(ms_i, \beta(ms_i, x)) = (s_i\alpha)(m, \beta(ms_i, x)) = \beta(ms_i, x),$$

$$\beta(ms_i, \alpha(ms_i, x)) = \beta(ms_i, (s_i\alpha)(m, x)) = \beta(ms_i, x)$$

and so

$$\alpha(ms_i, \beta(ms_i, x)) = \beta(ms_i, \alpha(ms_i, x)). \tag{3.32}$$

Next, suppose the role of m is played by mt_i. If $t_i\beta = \pi_2$, we have:

$$\alpha(mt_i, \beta(mt_i, x)) = \alpha(mt_i, (t_i\beta)(m, x)) = \alpha(mt_i, x),$$

$$\beta(mt_i, \alpha(mt_i, x)) = (t_i\beta)(m, \alpha(mt_i, x)) = \alpha(mt_i, x),$$

and so

$$\alpha(mt_i, \beta(mt_i, x)) = \beta(mt_i, \alpha(mt_i, x)).$$

It remains to prove the right-hand equality of (3.31) when m can neither be presented as ms_i, nor as mt_i. Therefore it remains to prove the right-hand equality of (3.31) when m commutes with s_i and with t_i.

First we observe that
$\alpha(m, \beta(m, x)) = (\alpha+\beta)(m, x) = (\alpha+\beta)(m, t_ix) - (\alpha+\beta)(s_im, t_ix) + (\alpha+\beta)(s_im, x)$
$= (\alpha + \beta)(m, t_ix) - s_i(\alpha + \beta)(m, t_ix) + (\alpha + \beta)(s_im, x)$
$= \alpha(m, \beta(m, t_ix)) - s_i\alpha(m, \beta(m, t_ix)) + \alpha(s_im, \beta(s_im, x))$

and similarly

$$\beta(m, \alpha(m, x)) = \beta(m, \alpha(m, t_ix)) - s_i\beta(m, \alpha(m, t_ix)) + \beta(s_im, \alpha(s_im, x)),$$

and so, using (3.32), we conclude that it suffices to prove

$$\alpha(m, \beta(m, t_ix)) = \beta(m, \alpha(m, t_ix)).$$

(for any $i = 1, 2, ..., n$). But we have (using the fact that m commutes with s_i and with t_i):

$\alpha(m, \beta(m, t_ix)) = \alpha(m, t_i\beta(m, t_ix)) - \alpha(s_im, t_i\beta(m, t_ix)) + \alpha(s_im, \beta(m, t_ix))$
$= \alpha(m, \beta(t_im, t_ix)) - \alpha(s_im, \beta(t_im, t_ix)) + \alpha(s_im, \beta(m, t_ix))$

$= \alpha(m, \beta(mt_i, t_i x)) - \alpha(ms_i, \beta(mt_i, t_i x)) + \alpha(ms_i, \beta(m, t_i x))$
$= \alpha(m, (t_i\beta)(m, t_i x)) - (s_i\alpha)(m, (t_i\beta)(t_i m, t_i x)) + (s_i\alpha)(m, \beta(m, t_i x))$
$= \alpha(m, t_i x) - t_i x + \beta(m, t_i x),$

$\beta(m, \alpha(m, t_i x)) = \beta(t_i m, \alpha(m, t_i x)) - \beta(t_i m, s_i\alpha(m, t_i x)) + \beta(m, s_i\alpha(m, t_i x))$
$= \beta(t_i m, \alpha(m, t_i x)) - \beta(t_i m, \alpha(s_i m, t_i x)) + \beta(m, \alpha(s_i m, t_i x))$
$= \beta(mt_i, \alpha(m, t_i x)) - \beta(mt_i, \alpha(ms_i, t_i x)) + \beta(m, \alpha(ms_i, t_i x))$
$= (t_i\beta)(m, \alpha(m, t_i x)) - (t_i\beta)(m, (s_i\alpha)(m, t_i x)) + \beta(m, (s_i\alpha)(m, t_i x))$
$= \alpha(m, t_i x) - t_i x + \beta(m, t_i x),$

which proves it. □

3.2 Action representability

The purpose of this section is to show that, for every cat^n-group B, every action $B \times X \to X$ determines a morphism $B \to Aut(X)$ (see [9]). That is, we need to prove that given a cat^n-group B and a map $B \times X \to X$ written as $(b, x) \mapsto bx$ satisfying

$$m(bx) = (mb)(mx), \tag{3.33}$$

$$0x = x, \tag{3.34}$$

$$b_1(b_2 x) = (b_1 + b_2)x, \tag{3.35}$$

$$b(x_1 + x_2) = bx_1 + bx_2 \tag{3.36}$$

(in obvious notation) for $b, b_1, b_2 \in B$ and $x, x_1, x_2 \in X$, the map $\alpha : M_n \times X \to X$, defined by

$$\alpha(m, x) = (mb)x, \tag{3.37}$$

satisfies conditions (3.25) – (3.29) (here again, we use the corresponding results of M_n-groups).

In particular, given internal groups B and X in a cartesian closed category, there is canonical bijection Φ_B^X between the set $Act(B, X)$ of internal group actions of B on X and the set $Hom(B, Aut(X))$ of internal group homomorphisms from B to $Aut(X)$. "Canonical" refers to the fact that the diagram

$$\begin{array}{ccc} Act(B,X) & \xrightarrow{\Phi_B^X} & Hom(B,Aut(X)) \\ {\scriptstyle inclusion}\downarrow & & \downarrow{\scriptstyle inclusion} \\ hom_{\mathbf{C}}(B \times X, X) & \xrightarrow{\varphi_{B,X}} & hom_{\mathbf{C}}(B, X^X) \end{array} \qquad (3.38)$$

commutes; here we write $\varphi_{B,X}$ for the bijection (1.5) with B and X playing the roles of A and B, respectively. In other words, "canonical" refers to the fact that Φ_B^X is induced by $\varphi_{B,X}$. Later in this section we consider the case of $\mathbf{C} = \mathbf{Cat^n}$, B and X as cat^n-groups corresponding to n-fold categories of groups, and our aim will be:

• to write down the explicit formulas for Φ_B^X and its inverse, which will be denoted by Ψ_B^X, and

• prove directly that those formulas indeed define maps from $Act(B, X)$ to $Hom(B, Aut(X))$ and from $Hom(B, Aut(X))$ to $Act(B, X)$ respectively.

3.2.1 Lemma. *Let $\mathbf{U} : \mathbf{C} \to \mathbf{D}$ be a finite product preserving functor between cartesian closed categories, and objects X and B in $\mathbf{C}$. The diagram*

$$\begin{array}{ccc} hom_{\mathbf{C}}(B \times X, X) & \xrightarrow{\varphi_{B,X}} & hom_{\mathbf{C}}(B, X^X) \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ hom_{\mathbf{D}}(\mathbf{U}(B) \times \mathbf{U}(X), \mathbf{U}(X)) & \xrightarrow{\varphi_{\mathbf{U}(B),\mathbf{U}(X)}} & hom_{\mathbf{D}}(\mathbf{U}(B), \mathbf{U}X)^{\mathbf{U}(X)}) \end{array}$$

(3.39)

commute, in which u and v are defined by $u(h) = \mathbf{U}(h) < \mathbf{U}(\pi_1), \mathbf{U}(\pi_2) >^{-1}$ (where $\pi_1 : B \times X \to B$ and $\pi_2 : B \times X \to X$ are the product projections) *and* $v(f) = \theta_X^X \mathbf{U}(f)$, *respectively, in which* $h \in hom_{\mathbf{C}}(B \times X, X)$ *and* $f \in hom_{\mathbf{C}}(B, X^X)$.

Proof. For a given morphism $h : B \times X \to X$ which corresponds to $f : B \to X^X$ under the canonical bijection (see (1.6)), we consider the commutative diagram

$$\begin{array}{ccccc} hom_{\mathbf{D}}(\mathbf{U}(X^X \times X), \mathbf{U}(X)) & \longrightarrow & hom_{\mathbf{D}}(\mathbf{U}(X^X) \times \mathbf{U}(X), \mathbf{U}(X)) & \longrightarrow & hom_{\mathbf{D}}(\mathbf{U}(X^X), \mathbf{U}(X)^{\mathbf{U}(X)}) , \\ \Big\downarrow {\scriptstyle hom_{\mathbf{D}}(\mathbf{U}(f \times X), \mathbf{U}(X))} & & \Big\downarrow {\scriptstyle hom_{\mathbf{D}}(\mathbf{U}(f) \times \mathbf{U}(X), \mathbf{U}(X))} & & \Big\downarrow {\scriptstyle hom_{\mathbf{D}}(\mathbf{U}(f), \mathbf{U}(X)^{\mathbf{U}(X)})} \\ hom_{\mathbf{D}}(\mathbf{U}(B \times X), \mathbf{U}(X)) & \longrightarrow & hom_{\mathbf{D}}(\mathbf{U}(B) \times \mathbf{U}(X), \mathbf{U}(X)) & \longrightarrow & hom_{\mathbf{D}}(\mathbf{U}(B), \mathbf{U}(X)^{\mathbf{U}(X)}) \end{array} \tag{3.40}$$

whose horizontal arrows are canonical isomorphisms. We know that θ_X^X is the image of $\mathbf{U}(\varepsilon_X^X)$ under the top composite (see (2.14)). Therefore $v(f)$ is the image of

$$hom_{\mathbf{D}}(\mathbf{U}(f \times X), \mathbf{U}(X))(\mathbf{U}(\varepsilon_X^X)) = \mathbf{U}(\varepsilon_X^X)\mathbf{U}(f \times X)$$

under the bottom composite. Explicitly,
$v(f) = \varphi_{\mathbf{U}(B),\mathbf{U}(X)}(hom_{\mathbf{D}}(< \mathbf{U}(\pi_1), \mathbf{U}(\pi_2) >^{-1}, \mathbf{U}(X))(\mathbf{U}(\varepsilon_X^X)\mathbf{U}(f \times X)))$
$= \varphi_{\mathbf{U}(B),\mathbf{U}(X)}(\mathbf{U}(\varepsilon_X^X)\mathbf{U}(f \times X) < \mathbf{U}(\pi_1), \mathbf{U}(\pi_2) >^{-1})$
$= \varphi_{\mathbf{U}(B),\mathbf{U}(X)}(\mathbf{U}(h) < \mathbf{U}(\pi_1), \mathbf{U}(\pi_2) >^{-1})$,
and from the top arrow of (3.39), we have

$$\psi_{B,X}(f) = \varepsilon_X^X(f \times X) = h \text{ for } f \in hom_{\mathbf{C}}(B, X^X).$$

Therefore, we have

$$\varphi_{\mathbf{U}(B),\mathbf{U}(X)}(u(h)) = \varphi_{\mathbf{U}(B),\mathbf{U}(X)}(\mathbf{U}(h) < \mathbf{U}(\pi_1), \mathbf{U}(\pi_2) >^{-1}) = v(f) = v(\varphi_{B,X}(h)),$$

for every $h : B \times X \to X$, as desired. $\square$

3.2.2 Theorem. *Let B and X be cat^n-groups identified with their corresponding n-fold categories as in Section* 3.1*, and M_n be a monoid as in Theorem* 3.1.9*. Then the maps*

$$\Phi_B^X : Act(B, X) \to Hom(B, Aut(X))$$

and

$$\Psi_B^X : Hom(B, Aut(X)) \to Act(B, X)$$

are bijections.

Proof. We apply Lemma 3.2.1 to the functor $\mathbf{U}_n : \mathbf{Cat}^n \to \mathbf{Sets}^{M_n}$ of Construction 2.3.1. Since in this case u and v are nothing but straight forward inclusions, from Example 1.2.3(d), we obtain

$$\varphi_{B,X}(h)(b)(m, x) = h(mb, x), \tag{3.41}$$

for each $h : B \times X \to X$, $b \in B$, $m \in M_n$, and $x \in X$. Similarly, we have

$$\Phi_B^X(h)(b)(m, x) = h(mb, x) = (mb)x, \tag{3.42}$$

for each $h \in Act(B, X)$, $b \in B$, $m \in M_n$, and $x \in X$. According to Example 1.2.3(f) the inverse Ψ_B^X of this map is explicitly defined by

$$\Psi_B^X(f)(b, x) = f(b)(1, x), \tag{3.43}$$

for each $f \in Hom(B, Aut(X))$, $b \in B$, and $x \in X$.
Now we are going to prove directly that (3.42) and (3.43) indeed define maps from $Act(B, X)$ to $Hom(B, Aut(X))$ and from $Hom(B, Aut(X))$ to $Act(B, X)$ respectively. For, we take $h \in hom_{\mathbf{Cat}^n}(B \times X, X)$ and $f \in hom_{\mathbf{Cat}^n}(B, X^X)$ corresponding to each other under the bijection $\varphi_{B,X}$ and we are going to prove directly that h is in $Act(B, X)$ if and only if f is in $Hom(B, Aut(X))$.

Suppose h is in $Act(B, X)$. Let $\Phi_B^X(h) = f$. Then according to Theorem 3.1.9, and having in mind that f should be an internal group homomorphism, we have to prove that

$$mf(b)(m', x) = f(b)(mm', mx), \tag{3.44}$$

$$f(b)(m, x_1 + x_2) = f(b)(m, x_1) + f(b)(m, x_2), \tag{3.45}$$

$$\text{the map} \quad f(b)(m, -) : X \to X \quad \text{is a bijection,} \tag{3.46}$$

$$f(b)(m, x) = f(b)(m, t_i x) - f(b)(s_i m, t_i x) + f(b)(s_i m, x) \quad \text{(i=1,2,...,n)}, \tag{3.47}$$

$$f(b)(m, x) = f(b)(t_i m, x) - f(b)(t_i m, s_i x) + f(b)(m, s_i x) \quad \text{(i=1,2,...,n)}, \tag{3.48}$$

$$f(b_1 + b_2)(m, x) = (f(b_1) + f(b_2))(m, x) = f(b_1)(m, f(b_2)(m, x)), \tag{3.49}$$

for all: $b, b_{1,}, b_2$ in B; $x, x_{1,}, x_2$ in X; m, m' in M_n and $i = 1, 2, ..., n$.

Note that, (3.44) follows from the fact that, $B \times X \to X$ is an M_n-morphism, that is, $m(m'x) = (mm')(mx)$.
Proving (3.45) and (3.49) is straight forward, (using the action axioms):

$$f(b)(m, x_1 + x_2) = (mb)(x_1 + x_2) = (mb)x_1 + (mb)x_2 = f(b)(m, x_1) + f(b)(m, x_2),$$

$$f(b_1 + b_2)(m, x) = (m(b_1 + b_2))x = (mb_1 + mb_2)x = (mb_1)((mb_2)x)$$
$$= f(b_1)(m, f(b_2)(m, x)),$$

while (3.46) follows from (3.49), since (3.49) makes the maps $f(b)(m, -)$ and $f(-b)(m, -)$ inverse of each other.

Next we have to prove that the map $\alpha : M_n \times X \to X$, defined by (3.37) satisfies (3.47) and (3.48). We have

$$\alpha(m, t_i x) - \alpha(s_i m, t_i x) + \alpha(s_i m, x) = (mb)(t_i x) - ((s_i m)b)(t_i x) + ((s_i m)b)x$$
$$= (mb)(t_i x) - (s_i(mb))(t_i x) + (s_i(mb))x,$$

and to prove that α satisfies (3.47) we need to prove that what we have above is equal to $(mb)x$. Since mb could be any element of B, this is the same as to prove that

$$b(t_i x) - (s_i b)(t_i x) + (s_i b)x = bx \quad \text{(i=1,2,...,n)}, \tag{3.50}$$

for all $b \in B$ and $x \in X$. Similarly, we have

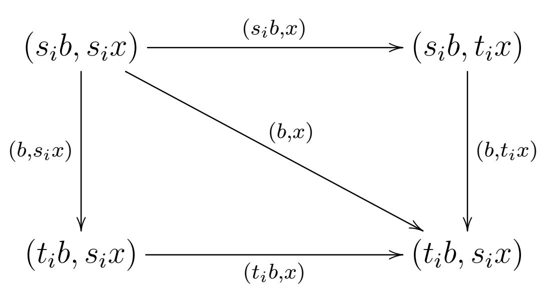
$$\alpha(t_i m, x) - \alpha(t_i m, s_i x) + \alpha(m, s_i x) = ((t_i m)b)x - ((t_i m)b)(s_i x) + (mb)(s_i x)$$
$$= (t_i(mb))x - (t_i(mb))(s_i x) + (mb)(s_i x),$$

and to prove that α satisfies (3.48) we need to prove that what we have above is equal to $(mb)x$. Again, since mb could be any element of B, this is the same as to prove that

$$(t_i b)x - (t_i b)(s_i x) + b(s_i x) = bx \quad \text{(i=1,2,...,n)}, \tag{3.51}$$

for all $b \in B$ and $x \in X$.

In order to prove (3.50) and (3.51), we shall use the commutative diagram



of $(n-1)$-dimensional morphisms in the n-fold category $B \times X$. According to the way morphisms are composed in $B \times X$ along the direction i, we have

$$(b, t_i x) - (s_i b, t_i x) + (s_i b, x) = (b, x) = (t_i b, x) - (t_i b, s_i x) + (b, s_i x),$$

and since the action $B \times X \to X$ preserves composition, this gives (3.50) and (3.51), as desired.

Conversely, suppose f is in $Hom(B, Aut(X))$ and $\alpha = f(b)$. We have to prove that

$$m(bx) = (mb)(mx), \tag{3.52}$$

$$0x = x, \tag{3.53}$$

$$b_1(b_2x) = (b_1 + b_2)x, \tag{3.54}$$

$$b(x_1 + x_2) = bx_1 + bx_2 \tag{3.55}$$

for all b, b_1, b_2 in B and x, x_1, x_2 in X. However, (3.52) follows from (3.44), (3.53) follows from (3.46) since $Aut(X)$ is a group, while proving (3.54) and (3.55) is a straight forward calculation using the fact that f is a group homomorphism and using (3.18) (for $\alpha = f(b)$), respectively.
Lastly, we have to show that Φ_B^X and Ψ_B^X are inverse to each other.

$$\Psi_B^X \Phi_B^X(h)(b, x) = \Phi_B^X(h)(b)(1, x) = h(b, x),$$

that is $\Psi_B^X \Phi_B^X = 1_{Act(B,X)}$.
Conversely, we have

$$\Phi_B^X \Psi_B^X(f)(b)(m, x) = \Psi_B^X(f)(mb, x) = f(mb)(1, x) = h(mb, x) = f(b)(m, x),$$

the last equality is due to the M_n action on $Aut(X)$ and so $\Phi_B^X \Psi_B^X = 1_{Hom(B,Aut(X))}$. This proves that Φ_B^X and Ψ_B^X are bijective. □

3.3 The connection with Norrie's actors

3.3.1 Remark. *The crossed module corresponding to our* cat^1*-group* $Aut(X)$ *described in Theorem* 3.1.7 *is given as*
$(\{\alpha \in Aut(X) \mid \alpha(s_1, x) = x, \forall x \in X \text{ and } s_1 \in M_1\}, \{\alpha \in Aut(X) \mid \alpha(1, x) = \alpha(s_1, x) = \alpha(t_1, x), \forall x \in X \text{ and } s_1, t_1 \in M_1\}, \tilde{\rho}),$
with
$\tilde{\rho} : \{\alpha \in Aut(X) \mid \alpha(s_1, x) = x, \forall x \in X \text{ and } s_1 \in M_1\} \to \{\alpha \in Aut(X) \mid \alpha(1, x) = \alpha(s_1, x) = \alpha(t_1, x), \forall x \in X \text{ and } s_1, t_1 \in M_1\}$
defined by $\tilde{\rho}(\alpha) = t_1\alpha$ *for for* $t_1 \in M_1$ *and the action is conjugation (see Lemma* 1.6.4*), with axioms of our crossed module shown below:*

(a) *take* $s_1\alpha \in \{\alpha \in Aut(X) \mid \alpha(1, x) = \alpha(s_1, x) = \alpha(t_1, x), \forall x \in X \text{ and } 1, s_1, t_1 \in M_1\}$ *and* $\beta \in \{\alpha \in Aut(X) \mid \alpha(s_1, x) = x, \forall x \in X \text{ and } s \in M_1\}$, *then we have*
$\tilde{\rho}(s_1\alpha(\beta - s_1\beta)) = t_1(s_1\alpha + (\beta - s_1\beta) - s_1\alpha)$
$= s_1\alpha + t_1(\beta - s_1\beta) - s_1\alpha$
$= s_1\alpha + \tilde{\rho}(\beta - s_1\beta) - s_1\alpha;$

(b) *given* $\beta, \alpha - s_1\alpha$ *in* $\{\alpha \in Aut(X) \mid \alpha(s_1, x) = x, \forall x \in X \text{ and } s_1 \in M_1\}$ *and* $-t_1\beta + \beta$ *in* $\{\alpha \in Aut(X) \mid \alpha(t_1, x) = x, \forall x \in X \text{ and } t_1 \in M_1\}$, *then we have*
$(\alpha - s_1\alpha) + (-t_1\beta + \beta) = (-t_1\beta + \beta) + (\alpha - s_1\alpha)$ *(by* (3.1)*).*

such that
$t_1\beta + (\alpha - s_1\alpha) - t_1\beta = \beta + (\alpha - s_1\alpha) - \beta.$

From the equality above, we obtain
$\tilde{\rho}(\beta)(\alpha - s_1\alpha) = t_1\beta(\alpha - s_1\alpha) = t_1\beta + (\alpha - s_1\alpha) - t_1\beta = \beta + (\alpha - s_1\alpha) - \beta.$

Similarly, the crossed module corresponding to our categorical group X with identity element 0 is given by $(\{x \in X \mid s_1x = 0, s_1 \in M_1\}, s_1X, \bar{\rho})$, where $\bar{\rho} = t_1 \mid_{\{x\in X|s_1x=0,s_1\in M_1\}}$ and the action is conjugation .

3.3.2 Construction. *Recall that, for a crossed module (Z, G, ρ) the actor crossed module $(D(G, Z), Aut(Z, G, \rho), \omega)$ can be constructed as shown in Section 1.5, where $D(G, Z)$ is a group of invertible elements of the monoid $Der(G, Z)$ of all derivations $\Delta : G \to Z$ defined by $\Delta(g_1 + g_2) = \Delta(g_1) + g_1\Delta(g_2)$, for $g_1, g_2 \in G$.*

3.3.3 Remark. *We show that, this actor crossed module is essentially the same as cat^1-group $Aut(X)$ described in Theorem 3.1.7.*
We only need to show that, given the crossed module
$(\{\alpha \in Aut(X) \mid \alpha(s_1, x) = x, \forall x \in X \text{ and } s_1 \in M_1\}, \{\alpha \in Aut(X) \mid \alpha(1, x) = \alpha(s_1, x) = \alpha(t_1, x), \forall x \in X \text{ and } s_1, t_1 \in M_1\}, \tilde{\rho})$
corresponding to $Aut(X)$ the monoid $Der(G, Z)$ is isomorphic to
$\{\alpha \in Aut(X) \mid \alpha(s_1, x) = x, \forall x \in X \text{ and } s_1 \in M_1\}$.
Suppose that the crossed module (Z, G, ρ) corresponds to cat^1-group X. As follows from Remark 3.3.1, the monoid of derivations in this case will be $Der(s_1X, \{x \in X \mid s_1x = 0, s_1 \in M_1\})$, such that for each derivative

$$\bar{\Delta} : s_1X \to \{x \in X \mid s_1x = 0, s_1 \in M_1\},$$

we have

$$\bar{\Delta}(x_1 + x_2) = \bar{\Delta}(x_1) + x_1 + \bar{\Delta}(x_2) - x_1. \tag{3.56}$$

It is clear that, if $\alpha(s_1, x) = x$, then $s_1\alpha(1, t_1x) = t_1x$, and as follows from Theorem 3.1.7(b), we have

$$\alpha(1, x) = \alpha(1, t_1x) - t_1x + x. \tag{3.57}$$

Again, by Theorem 3.1.7(c), we get

$$\alpha(1, t_1x) - t_1x + x = \alpha(t_1, x) - \alpha(t_1, s_1x) + \alpha(1, s_1x),$$

and so

$$\alpha(1, t_1x) - t_1x + x = \alpha(t_1, x) - t_1\alpha(1, s_1x) + \alpha(1, s_1x).$$

Therefore, we have

$$\alpha(t_1, x) = \alpha(1, t_1x) - t_1x + x - \alpha(1, s_1x) + -t_1\alpha(1, s_1x). \tag{3.58}$$

Now we define the map
$\Re : \{\alpha \in Aut(X) \mid \alpha(s_1, x) = x, \forall x \in X \text{ and } s_1 \in M_1\} \longrightarrow Der(s_1X, \{x \in X \mid s_1x = 0, s_1 \in M_1\})$
by

$$\Re(\alpha)(x) = \alpha(1, x) - x, \tag{3.59}$$

for all $x \in s_1X$.
As follows from (3.56), *we have to check that,*

$$\Re(\alpha)(x_1 + x_2) = \Re(\alpha)(x_1) + x_1 + \Re(\alpha)(x_2) - x_1,$$

and that

$$s_1(\alpha(1, x) - x) = 0,$$

for all $x_1, x_2 \in s_1X$ *and* $\alpha \in \{\alpha \in Aut(X) \mid \alpha(s_1, x) = x, \forall x \in X,\ s_1 \in M_1\}$.
Indeed, we have
$\Re(\alpha)(x_1 + x_2) = \alpha(1, x_1 + x_{2)}) - (x_1 + x_2)$ (*by* (3.59))
$= \alpha(1, x_1) + \alpha(1, x_2) - x_2 - x_1$ (*by Theorem* 3.1.7(*d*))
$= \Re(\alpha)(x_1) + x_1 + \Re(\alpha)(x_2) - x_1$ (*by* (3.59)),
and

$$s_1(\alpha(1, x) - x) = \alpha(s_1, s_1x) - s_1x = s_1x - s_1x = 0.$$

Now we need to show that $\Re$ *is a monoid homomorphism. It is easy to see that, since* π_2 *is actually the identity element in*
$\{\alpha \in Aut(X) \mid \alpha(s_1, x) = x, \forall x \in X \text{ and } s_1 \in M_1\}$*, we have*

$$\Re(0)(x) = \Re(\pi_2)(x) = \pi_2(1, x) - x = x - x = 0,$$

such that, $\Re(0) = 1$*, where* 1 *is the identity element in*
$Der(s_1X, \{x \in X \mid s_1x = 0, s_1 \in M_1\}$*, defined by* $1(s_1X) = 0$*,* 0 *is the identity element in* $\{x \in X \mid s_1x = 0, s_1 \in M_1\}$
For each $\alpha, \beta \in \{\alpha \in Aut(X) \mid \alpha(s_1, x) = x, \forall x \in X \text{ and } s_1 \in M_1\}$*, we have*
$\Re(\alpha + \beta)(x) = (\alpha + \beta)(x) - x = \alpha(1, \beta(1, x)) - x$
$= \alpha(1, \beta(t_1, x)) - \beta(t_1, x) + \beta(1, x) - x$ *(by* (3.57)*),*
while

$(\Re(\alpha) \circ \Re(\beta))(x) = \Re(\alpha)(t_1\Re(\beta)(x) + x) + \Re(\beta)(x)$ *(see Construction* 1.5.5*))*
$= \Re(\alpha)(t_1\beta(1, x) - t_1x + x) + \beta(1, x) - x$ *(by* (3.59)*)*
$= \Re(\alpha)(\beta(t_1, x)) + \beta(1, x) - x$ *(since* $\in s_1X$*)*
$= \alpha(1, \beta(t_1, x)) - \beta(t_1, x) + \beta(1, x) - x,$
therefore

$$\Re(\alpha + \beta)(x) = (\Re(\alpha) \circ \Re(\beta))(x),$$

that is, $\Re$ *is a monoid homomorphism.*
To prove that $\Re$ *is injective, it suffices to prove that if* $\Re(\alpha) = 1$*, then* $\alpha = \pi_2$*, since* $\Re$ *is a group homomorphism. Assume that,* $\Re(\alpha) = 1$*, then as follows from* (3.59) $\alpha(1, x) = x$ *for each* $x \in X$*, which gives*

$$\alpha(1, x) = \alpha(1, t_1x) - t_1x + x = t_1x - t_1x + x = x,$$

and
$\alpha(t_1, x) = \alpha(1, t_1x) - t_1x + x - \alpha(1, s_1x) + t_1\alpha(1, s_1x) = t_1x - t_1x + x - s_1x + s_1x = x.$

We have proved that, $\alpha = \pi_2$ *whenever* $\Re(\alpha) = 1$ *as desired.*

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