Finitely Generated Function Algebras

by

J. Sacks.

A thesis prepared under the supervision of Dr. W. Kotzé, in partial fulfilment of the requirements for the degree of Master of Science in Mathematics.

Copyright by the University of Cape Town.

1970.
The copyright of this thesis vests in the author. No quotation from it or information derived from it is to be published without full acknowledgement of the source. The thesis is to be used for private study or non-commercial research purposes only.

Published by the University of Cape Town (UCT) in terms of the non-exclusive license granted to UCT by the author.
# CONTENTS

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Introduction</td>
<td></td>
<td>(i)</td>
</tr>
<tr>
<td>Chapter I</td>
<td>Finitely generated function algebras and analytic functions</td>
<td>1</td>
</tr>
<tr>
<td>Chapter II</td>
<td>Boundaries of finitely generated function algebras</td>
<td>27</td>
</tr>
<tr>
<td>Chapter III</td>
<td>Extensions of finitely generated function algebras</td>
<td>49</td>
</tr>
<tr>
<td>Chapter IV</td>
<td>Function algebras as direct limits of their finitely generated sub-function algebras</td>
<td>67</td>
</tr>
<tr>
<td>Bibliographical notes</td>
<td></td>
<td>86</td>
</tr>
<tr>
<td>Bibliography</td>
<td></td>
<td>90</td>
</tr>
</tbody>
</table>
INTRODUCTION

The theory of function algebras has been an active field of research over the past two decades and its coming of age has been heralded by the appearance within the last twelve months of three textbooks devoted entirely to them, namely the books by Browder [6], Leibowitz [4] and Gamelin [9]. One of the attractive features of the theory of function algebras is that it draws on diverse specialities like the theory of Banach algebras, harmonic analysis and the theory of analytic functions of several complex variables.

The last mentioned theory has led to some of the most powerful results in the theory of function algebras. Not surprisingly, many of these results, for example Rossi's local maximum modulus principle theorem 2.24, were first proved for finitely generated and then extended to arbitrary function algebras. This observation, together with the fact that there has been no systematic study of finitely generated function algebras, led to the writing of this thesis. We have made use of some of the results of the theory of analytic functions of several complex variables, though we have not specifically used the methods thereof. What we have looked for is ways in which the functions of finitely generated function algebras behave like analytic functions and then tried to see if
arbitrarily generated function algebras behave in a similar way.

This angle of approach is justified in Chapter I. Firstly it is shown how every finitely generated function algebra is metrically and algebraically equivalent to $P(K)$, the uniform closure of the algebra of polynomials on a certain subset $K$, which depends on the generators, of $\mathbb{C}^n$, where $n$ is the number of generators. Moreover the maximal ideal space of such a function algebra is homeomorphic to $K$. What is surprising is that $P(K)$ is equal to the uniform closure of the algebra of all functions holomorphic in a neighbourhood of $K$, a deep result from the theory of analytic functions of several complex variables. That the maximal ideal space of a finitely generated function algebra is homeomorphic to a subset of $\mathbb{C}^n$ led us to the conclusion that every finitely generated function algebra must necessarily be defined on a metrizable space, a fact which is made use of in Chapter II. In 1.20 an interesting example of a finitely generated function algebra is given. The remainder of Chapter I is devoted to a discussion of Rickart's recent paper [8] on holomorphic functions of an infinite number of complex variables and use is made of his generalized Oka polynomial approximation theorem to extend the equivalence of finitely generated function algebras and algebras of holomorphic functions to the arbitrarily generated case.
In Chapter II "boundary" phenomena of function algebras are discussed, with the emphasis on boundaries of finitely generated function algebras. There are certain subsets, called boundaries, of the maximal ideal space of a function algebra on which every member achieves its maximum modulus. This is another "analytic" characteristic of function algebras. We have also included an exposition of Rossi's local maximum modulus principle, emphasizing the role of finitely generated function algebras.

Chapter III is an attempt to explore the connection between analytic function theory and finitely generated function algebras from another direction. We first establish under what conditions adjoining a continuous function to a function algebra leaves its maximal ideal space (and its Shilov boundary) unaltered. We then apply these results to obtain a continuous algebraic epimorphism from certain finitely generated function algebras onto the uniform closure of the space of continuous functions on a compact subset of the plane which are analytic in the interior of this subset.

Chapter IV was inspired by Royden [16], whose treatment of algebras as direct limits of their finitely generated subalgebras led us to investigate what properties of function algebras are inherited from their finitely generated sub-function algebras. Unfortunately we had to restrict ourselves to function algebras containing a 1:1 function, but with this
restriction managed to obtain results relating direct systems of finitely generated sub-function algebras and the inverse systems of their spectra to the corresponding objects in analytic function theory. We also found that analyticity (in the function algebraic sense of the word) is preserved under the taking of direct limits and that function algebras containing a 1:1 function which are defined on a connected space imitate the behaviour of analytic functions in that any member of such a function algebra which vanishes on an open subset, vanishes identically.

To keep the length of the thesis within reasonable bounds we had to omit the proofs of some results. In every case where this is done a reference is given to a textbook in which the result in question and its proof can be found. The reader is referred to the bibliographical notes for a detailed account of to whom the results contained in the thesis can be attributed.

In conclusion I wish to express my deep indebtedness to my supervisor, Dr. W. Kotzé, for his many valuable suggestions and constant encouragement.
CHAPTER I.
FINITELY GENERATED FUNCTION ALGEBRAS AND
ANALYTIC FUNCTIONS.

In general we shall be concerned with certain sub-algebras of \( C(X) \), the space of all continuous complex-valued functions on a compact Hausdorff space \( X \). If we define addition, scalar multiplication and multiplication as these operations applied pointwise to the members of \( C(X) \), then the latter is an algebra.

Putting
\[
\| f \| = \sup \{ |f(x)| : x \in X \}
\]
for each \( f \) in \( C(X) \), we furthermore obtain that \( C(X) \) is a Banach algebra.

A subset \( A \) of \( C(X) \) is called a function algebra if \( A \) is a subalgebra of \( C(X) \) satisfying:

1. If \( x, y \) are in \( X \) and \( x \neq y \), then there is an \( f \) in \( A \) such that \( f(x) \neq f(y) \), i.e. \( A \) separates the points of \( X \).
2. The constant function \( 1 \), and hence every constant function, belongs to \( A \).
3. \( A \) is closed with respect to the uniform norm; thus \( A \) itself is a Banach algebra.

Definition 1.1: Let \( B \) be any Banach algebra. A map
m : B → C satisfying:
1. m is a non-zero linear functional.
2. m(xy) = m(x)m(y) for all x, y ∈ B
is called a complex homomorphism of B. We denote the set of all complex homomorphisms of B by Spec B.

The following proposition shows that every complex homomorphism of a Banach algebra with an identity is bounded and has norm equal to one. (Note that we can assume that the identity has norm equal to one).

**Proposition 1.2:**

If B is a Banach algebra with identity e of norm one, then m(e) = 1 and ||m|| = 1 for all m in Spec B.

**Proof:** Let m belong to Spec B. Since m ≠ 0, there is an x in B such that m(x) ≠ 0. Now m(x) = m(xe) = m(x)m(e). Therefore m(e) = 1 and ||m|| ≥ 1. For any y ∈ B we have

|m(y)|^n = |[m(y)]|^n = |m(y^n)| ≤ ||m|| ||y^n|| ≤ ||m|| ||y||^n.

Therefore |m(y)| ≤ ||m||^{1/n} ||y||, which implies that |m(y)| ≤ ||y||. Hence ||m|| ≤ 1 and so ||m|| = 1.

**Note 1.3:** It is well-known from the General Theory of Banach algebras that if B is a commutative Banach algebra with an identity (as is every function algebra) then there is a 1:1 correspondence between Spec B and the set of all maximal
ideals of $B$, usually denoted $M_B$. This correspondence is obtained by mapping each complex homomorphism onto its kernel, which is a maximal ideal. See e.g. [2].

**Theorem 1.4:**
Let $A$ be a function algebra. Define for each $f$ in $A$ and each $m$ in Spec $A$

$$\hat{f}(m) = m(f).$$

Then there exists a unique Hausdorff topology on Spec $A$ such that Spec $A$ is compact and each $\hat{f}$ is continuous on Spec $A$.

**Proof:** The theorem is just a restatement of the well-known theorem in the General Theory of Banach algebras dealing with the Gelfand Representation of a commutative Banach algebra with an identity. In that setting, for fixed $m$ in Spec $A$ and $f$ in $A$, we have the following commutative diagram yielding the Gelfand Representation.

\[
\begin{array}{c}
A \\
\downarrow v \\
A/M \\
\uparrow \psi \\
\end{array} \quad \begin{array}{c}
\downarrow \phi \\
\end{array} \quad \begin{array}{c}
\downarrow m \\
\end{array} \\
\end{array}
\]

\[
\begin{array}{c}
A \\
\end{array} \quad \begin{array}{c}
\rightarrow \\
\rightarrow \\
\rightarrow \\
\end{array} \quad \begin{array}{c}
\mathbb{C} \\
\end{array}
\]
where $M$ is the kernel of $m$, $v$ is the canonical map from $A$ onto the quotient $A/M$ and $\phi$ is the isomorphism existing by virtue of the Gelfand-Mazur theorem. Since we have identified $m$ with $M$, $\hat{f}(m) = \hat{f}(M) = m(f)$, and the result follows. Explicitly, the topology for $\text{Spec } A$ is the weakest one which makes each $\hat{f}$ continuous.

In view of the previous theorem, we shall refer to $\hat{f}$ as the \textit{Gelfand transform} of $f$.

\textbf{Theorem 1.5:}

There is a map $i : X \to \text{Spec } A$ which embeds $X$ homeomorphically as a closed subset of $\text{Spec } A$. For each $f \in A$, $f \circ i^{-1}$ coincides with $\hat{f}$ on the image of $X$ under $i$. Moreover, for each $f$ in $A$ and each $m$ in $\text{Spec } A$

$$|f(m)| \leq \sup\{|f(x)| : x \in X\} = \sup\{|\hat{f}(i(x))| : x \in X\}$$

and so $\|\hat{f}\| = \|f\|$.

\textbf{Proof:} For each $x$ in $X$, define $m_x$ in $\text{Spec } A$ by $m_x(f) = f(x)$. The required map is then given by $i(x) = m_x$. Suppose $x \neq y$. Then there is a function $g$ in $A$ such that $g(x) \neq g(y)$. Hence $m_x(g) \neq m_y(g)$ and so $i$ is $1:1$. To show that $i$ is continuous as a map from $X$ onto $i(X)$ (provided with the relative topology induced by $\text{Spec } A$), consider the open neighbourhood of $m_x$ in $i(X)$ defined by
\[
U(m_{x_0}; \varepsilon ; f_1, \ldots, f_r) = \{ m_x \in i(X) : |\hat{f}_i(m_x) - \hat{f}_i(m_{x_0})| < \varepsilon \text{ for } i = 1, \ldots, r \}
\]
\[
= \{ m_x \in i(X) : |m_x(f_i) - m_{x_0}(f_i)| < \varepsilon \text{ for } i = 1, \ldots, r \}
\]
\[
= \{ m_x \in i(X) : |f_i(x) - f_i(x_0)| < \varepsilon \text{ for } i = 1, \ldots, r \}
\]
Since each \( f_i \) is continuous there is a neighbourhood \( V_i \) of \( x_0 \) for each \( i = 1, \ldots, r \) such that
\[
x \in V_i \Rightarrow |f_i(x) - f_i(x_0)| < \varepsilon .
\]
Clearly \( V = \cap_{i=1}^r V_i \) is a neighbourhood of \( x_0 \) such that
\[
x \in V \Rightarrow i(x) \in U(m_{x_0}; \varepsilon ; f_1, \ldots, f_r).
\]
The subspace \( i(X) \) of \( \text{Spec } A \) is Hausdorff. Thus \( i \) is a 1:1 map from a compact space onto a Hausdorff space, and hence a homeomorphism. Being the continuous image of the compact space \( X \), \( i(X) \) is itself compact and is thus a closed subset of \( \text{Spec } A \).

For \( i(x) \in \text{Spec } A \) we have
\[
\hat{f}(i(x)) = \hat{f}(m_x) = m_x(f) = f(x) = f(i^{-1}(i(x)))
\]
and hence \( f \circ i^{-1} \) coincides with \( \hat{f} \) on \( i(X) \).

For \( f \) in \( A \) and \( m \) in \( \text{Spec } A \) we have that
\[
|\hat{f}(m)| = |m(f)| \leq \| m \| \| f \| , \text{ by 1.2.}
\]
But \( \| f \| = \sup \{ |f(x)| : x \in X \} = \sup \{ |\hat{f}(m_x)| : x \in X \} \)
Hence \( \hat{f} \) achieves its maximum modulus on \( i(X) \) and \( \| \hat{f} \| = \| f \| . \)

If we let \( \hat{A} = \{ \hat{f} : f \in A \} \), then it follows from the two previous theorems that \( A \) is isometrically isomorphic to \( \hat{A} \).
Definition 1.6:
For n a positive integer, \( \mathbb{C}^n \) denotes the cartesian product of n copies of the complex plane \( \mathbb{C} \).
i.e. \( z \in \mathbb{C}^n \) iff \( z = (z_1, \ldots, z_n) \) where \( z_i \in \mathbb{C} \) for \( i = 1, \ldots, n \).
We provide \( \mathbb{C}^n \) with the metric \( |z| = \max \{|z_i| : i = 1, \ldots, n\} \).
It can be shown that this metric is equivalent to the Euclidean metric.

A polynomial on \( \mathbb{C}^n \) is simply a polynomial in n complex variables. Such a polynomial will be a finite sum of the form \( \sum a_{r_1} \cdots z_1^{r_1} \cdots z_n^{r_n} \), where the \( a_{r_1} \cdots r_n \) are complex constants.

If \( K \) is a compact subset of \( \mathbb{C}^n \) we denote by \( P_0(K) \) the algebra of polynomials on \( \mathbb{C}^n \) restricted to \( K \). The uniform closure of \( P_0(K) \) will be denoted by \( P(K) \).

Definition 1.7: Let \( K \) be a compact subset of \( \mathbb{C}^n \). The set
\[ \text{hull} (K) = \{ z \in \mathbb{C}^n : |p(z)| < \|p\|_K \text{ for each polynomial } p \} \]
where \( \|p\|_K = \sup \{|p(z)| : z \in K\} \),
is called the polynomially convex hull of \( K \).
We say that \( K \) is polynomially convex if \( \text{hull} (K) = K \).

Proposition 1.8:
If \( K \) is a compact subset of \( \mathbb{C}^n \) then so is \( \text{hull} (K) \), and \( \text{hull} (K) \) is the largest set containing \( K \) to which every element of
P(K) can be extended with preservation of the norm.

Proof: Since hull (K) is an intersection of closed sets it is a closed subset of the space $\mathbb{C}^n$. It is also bounded, for if $z_i$ denotes the i-th coordinate function on $\mathbb{C}^n$ and $a_i = \sup \{ |z_i| : z \in K \}$, then any point $w = (w_1, \ldots, w_n)$ which is such that $|w_i| > a_i$ for some i cannot lie in hull (K).

Hence hull (K) is a closed and bounded subset of $\mathbb{C}^n$, and since the metric we have defined on $\mathbb{C}^n$ is equivalent to the Euclidean metric, it follows that hull (K) is compact.

Suppose that $f \in P(K)$. Then there is a sequence $\{p_n\}$ of polynomials converging uniformly on K to f. By the definition of hull (K), $\{p_n\}$ also converges uniformly on hull (K).

Thus we can define an extension $\hat{f}$ of f to hull (K) by setting $\hat{f}(z) = \lim_{n \to \infty} p_n(z)$. It is a routine matter to check that $\hat{f}$ is well-defined.

Hence hull (K) is the largest set containing K to which each member of P(K) can be continuously extended with preservation of the norm. We shall soon see that hull (K) = Spec P(K) and that $\hat{f}$ as we have just defined it corresponds with the Gelfand transform of f for f in P(K).

Definition 1.9: Let A be a function algebra. If there is a subset $B \subset A$ such that A is the uniform closure of the set of all polynomials in finitely many members of B then we say
that \( A \) is generated by \( B \). If \( B = \{f_1, \ldots, f_n\} \) is a finite subset of \( A \) then \( A \) is said to be \emph{finitely generated}.

If \( \{f_1, \ldots, f_n\} \) generate \( A \) we write \( A = [f_1, \ldots, f_n] \).

\textbf{Theorem 1.10:}

Let \( A = [f_1, \ldots, f_n] \). Then there is a compact polynomially convex subset \( \mathfrak{K} \) of \( \mathbb{C}^n \) such that \( \text{Spec} \ A \) is homeomorphic to \( \mathfrak{K} \) and \( A \) is isometrically isomorphic to \( \mathbb{P}(\mathfrak{K}) \).

\textbf{Proof:} Define the map \( F : \text{Spec} \ A \to \mathbb{C}^n \) by
\[
F(m) = (\hat{f}_1(m), \ldots, \hat{f}_n(m))
\]
for each \( m \) in \( \text{Spec} \ A \). \( F \) thus defined is continuous since the composition of \( F \) with the respective projections from \( \mathbb{C}^n \) into \( \mathbb{C} \) simply yields the Gelfand transform of each of the generators of \( A \), which are all continuous.

To show that \( F \) is \( 1:1 \) let \( m \) and \( m' \) in \( \text{Spec} \ A \) be such that \( m \neq m' \). Then there exists an \( f \) in \( A \) such that \( m(f) \neq m'(f) \). Since \( f \) is a uniform limit of polynomials in the \( \hat{f}_i \), \( i = 1, \ldots, n \), for some \( j \) with \( 1 \leq j \leq n \), we must have that \( m(\hat{f}_j) \neq m'(\hat{f}_j) \). Hence \( \hat{f}_j(m) \neq \hat{f}_j(m') \) and so \( F(m) \neq F(m') \).

We denote the image of \( \text{Spec} \ A \) under \( F \) i.e. the set
\[
\bigcap_{i=1}^n \mathbb{R}(\hat{f}_i),
\]
where \( \mathbb{R}(\hat{f}_i) \) is the range of \( \hat{f}_i \), by \( K \).

Since \( \mathbb{C}^n \) is Hausdorff, so is \( K \) and thus \( F \) is a continuous \( 1:1 \) map from a compact space onto a Hausdorff space. Hence \( F \) is a homeomorphism.

To show that \( K \) is polynomially convex it is sufficient
to show that \( \text{hull}(K) \subset K \). Suppose that \( z \in \text{hull}(K) \). The map \( m_z : \text{p}(f_1, \ldots, f_n) \to \text{p}(z) \) which sends elements of \( A \) expressible as polynomials in the \( f_i \) onto \( \mathbb{C} \) is a complex homomorphism on such elements. It is also bounded since

\[
|p(z)| \leq \|p\|_\text{hull}(K) = \|p(f_1, \ldots, f_n)\|_{\text{Spec } A} = \|p(f_1, \ldots, f_n)\|_X,
\]

and so can be extended by continuity to the whole of \( A \) to become an element \( m \in \text{Spec } A \).

Hence \( p(z) = m(p(f_1, \ldots, f_n)) = p(m(f_1), \ldots, m(f_n)) \)

\[
= p(f_1(m), \ldots, f_n(m)) = p(z')
\]

for some \( z' \) in \( K \). Therefore \( z = z' \) and so \( z \) is in \( K \).

Clearly there is an isomorphism between the set of all uniform limits of polynomials in the \( f_i \) and the set of all uniform limits of polynomials in \( n \) complex variables on \( K \). This isomorphism is norm-preserving, since the norm on \( P(K) \) is the uniform norm. Since \( A \) is isometrically isomorphic to \( \hat{A} \), it follows that \( A \) is isometrically isomorphic to \( P(K) \).

**Proposition 1.11:**

If \( K \) is a compact subset of \( \mathbb{C}^n \), then \( \text{Spec } P(K) \) is homeomorphic to \( \text{hull}(K) \).

**Proof:** Suppose \( z \in \text{hull}(K) \). Then the map \( m_z : \text{p}_0(K) \to \mathbb{C} \) defined by \( m_z(p) = p(z) \) is a complex homomorphism on \( \text{p}_0(K) \). Also

\[
\|m_z\| = \sup \{ |p(z)|/\|p\|_K : p \in \text{p}_0(K) \} \leq 1.
\]

In fact
$\| m_z \| = 1$ since 1 is in $P_0(K)$.
Conversely, if $m_z$ is a complex homomorphism on $P_0(K)$ such that $\| m_z \| = 1$, then $z \in \text{hull } (K)$.
Hence $z \in \text{hull } (K)$ iff $\| m_z \| = 1$. But $\| m_z \| = 1$ iff $m_z$ can be extended by continuity to $P(K)$. Thus $z \in \text{hull } (K)$ iff there exists a complex homomorphism $m_z$ in Spec $P(K)$ such that $m_z(p) = p(z)$ for every polynomial $p$ on $\mathbb{C}^n$.

The map $F : \text{Spec } P(K) \rightarrow \text{hull } (K)$ defined by

$$F(m_z) = (m_z(z_1), \ldots, m_z(z_n)) = (zp, \ldots, z_1)$$

is clearly a homeomorphism from Spec $P(K)$ onto hull $(K)$.

**Corollary 1.12:** If $K$ is a compact polynomially convex subset of $\mathbb{C}^n$, then Spec $P(K)$ is homeomorphic to $K$.

**Proof:** Immediate from the previous proposition.

Thus we have shown that a finitely generated function algebra $A = [f_1, \ldots, f_n]$ is isometrically isomorphic to $P(K)$ where $K = \bigcap_1^n \overline{R(f_i)}$ and that Spec $A$ is homeomorphic to $K$ which can be identified with Spec $P(K)$.

**Definition 1.13:**

Let $D \subseteq \mathbb{C}^n$ be open. A complex-valued function $f$ defined on $D$ is said to be **holomorphic in $D$** if for each point $w \in D$ there is an open neighbourhood $U$ of $w$ contained in $D$, such that $f$ has a power series expansion
\[ f(z) = \frac{r_1 \cdots z}{r_n} + a r_1 \cdots r_n (z_1 - w_1) \cdots (z_n - w_n)^{r_n} \]

which converges for all \( z \in U \)

**Definition 1.14:** Let \( K \) be a compact polynomially convex subset of \( \mathbb{C}^n \). We denote by \( A(K) \) the uniform closure of the algebra of all functions holomorphic in a neighbourhood of \( K \).

**Theorem 1.15:** (Oka's polynomial approximation theorem).

If \( K \) is a compact polynomially convex subset of \( \mathbb{C}^n \), then \( A(K) = P(K) \).

**Proof:** See e.g. [3]. p.56.

**Corollary 1.16:** A finitely generated function algebra \( A = \left[ f_1, \ldots, f_n \right] \) is isometrically isomorphic to \( A(K) \) where \( K = \bigcap_{i=1}^{n} R(f_i) \) and \( \text{Spec } A \) is homeomorphic to \( K \).

**Proof:** The assertion follows from 1.10 and 1.15.

The fact that \( \text{Spec } A \) is homeomorphic to a subset of \( \mathbb{C}^n \) when \( A \) is finitely generated forces us to the conclusion that \( \text{Spec } A \), and hence \( X \), is metrizable. This can be proved independently, as we now show.

**Theorem 1.17:** (Urysohn's metrization theorem).
A $T_3$ topological space whose topology has a countable base is metrizable.

**Proof:** See e.g. [5] p.125

**Proposition 1.18:**
If $A$ is a function algebra on the compact Hausdorff space $X$, then the topology on $X$ is the weakest topology for which every function in $A$ is continuous.

**Proof:** Let $\mathcal{J}$ be the given topology on $X$ and $\mathcal{W}$ the weak $A$ topology on $X$. Then the identity map $(X,\mathcal{J}) \to (X,\mathcal{W})$ is continuous by definition of the weak topology. Since $A$ separates the points of $X$, $(X,\mathcal{W})$ must be Hausdorff. Hence the identity map is a continuous map from the compact space $(X,\mathcal{J})$ onto the Hausdorff space $(X,\mathcal{W})$, and thus a homeomorphism.

**Theorem 1.19:** If $A$ is a finitely generated function algebra on a compact Hausdorff space $X$, then $X$ is metrizable.

**Proof:** Let $A = [f_1, \ldots, f_n]$.

Put $B = \{ \sum_{j=1}^{k} a_j f_1^{m_1} f_2^{m_2} \cdots f_n^{m_n} : a_j \in \mathbb{C} \text{ and } a_j \text{ has rational real and imaginary parts} \}$. Clearly, $\overline{B} = A$ and so $A$ is separable since $B$ is countable. By 1.17 it is sufficient for us to show that the topology on $X$ is countably based. We show in fact that the set

$$\{ f^{-1}(U) : f \in B \text{ and } U \text{ is an open disc in the complex plane} \}$$
with rational radius and centre} forms a countable base for the topology on $X$.

Let $\mathcal{O}$ be the family of all open discs in the complex plane with rational radius and with centre having rational real and imaginary parts. $\mathcal{O}$ forms a countable base for the topology of the plane. Since for any family of subsets $\mathcal{F}$,

$$f^{-1}(\cup \{F : F \in \mathcal{F}\}) = \cup \{f^{-1}(F) : F \in \mathcal{F}\},$$

it is clear that the set $\{f^{-1}(U) : f \in A, U \in \mathcal{O}\}$ forms a base for the topology of $X$, as the latter topology consists in all possible pre-images of open sets in the plane. We make the even stronger claim that the countable set $\{f^{-1}(U) : f \in B, U \in \mathcal{O}\}$ forms a base for the topology on $X$, in which case the theorem is proved.

Let $f \in A$ and $U \in \mathcal{O}$. Since $B = A$, there is a sequence $\{f_n\} \subset B$ such that $f_n$ converges to $f$ uniformly on $X$. Now $x \in f^{-1}(U)$ implies $f(x) \in U$. For $n$ large enough, $f_n(x) \in U$ i.e. $x \in f_n^{-1}(U)$. Hence $x \in \cup f_n^{-1}(U)$, and so $f^{-1}(U) \subset \cup f_n^{-1}(U)$.

In view of the preceding theorem, we ask whether it is possible to produce a metric for $\text{Spec} A$ when $A$ is a finitely generated function algebra. We can do so, using the homeomorphism

$$F : \text{Spec} A \rightarrow \prod \hat{\mathbb{R}}(f_i) = K \subset \mathbb{C}^n.$$
Two points m and m' in Spec A correspond to the points 
\((\hat{f}_1(m), \ldots, \hat{f}_n(m))\) and 
\((\hat{f}_1(m'), \ldots, \hat{f}_n(m'))\) in K respectively.
The distance between the latter two points is given by
\[ \max \{ |\hat{f}_i(m) - \hat{f}_i(m')| : i = 1, \ldots, n \}. \]

We can define the metric d on Spec A by setting
\[ d(m, m') = \max \{ |\hat{f}_i(m) - \hat{f}_i(m')| : i = 1, \ldots, n \}. \]

This metric will be consistent with the topology on Spec A, since Spec A and K are homeomorphic. Since X is homeomorphically embedded in Spec A, d restricted to X is given by
\[ d(x, y) = \max \{ |f_i(x) - f_i(y)| : i = 1, \ldots, n \}. \]

Example of a finitely generated function algebra \(1.20\).

Definition: Let X be a compact subset of \(\mathbb{C}^n\). We denote by 
\(R(X)\) the uniform closure of the set \(R_e(X)\) of all rational 
functions on X i.e. the uniform closure on X of the set of all 
quotients \(p/q\) where \(p\) and \(q\) are polynomials and \(q\) does not 
vanish on X.

The rationally convex hull of X is the set 
\[ R - \text{hull}(X) = \{ z \in \mathbb{C}^n : \|g(z)\| \leq \|g\|_X \text{ for all } g \text{ rational on } X \}. \]

\(P(X) = [z_1, \ldots, z_n]\), the function algebra generated by the 
n coordinate functions. Clearly \(P(X) \subseteq R(X)\). It turns out 
that \(R(X)\) is also finitely generated.
**Lemma:** If $A$ is a Banach algebra with an identity $e$, then $\|x-e\| < 1$ implies that $x$ has an inverse.

**Proof:** See [7] p.177.

**Theorem:**
Let $X$ be a compact set in $\mathbb{C}^n$. Then there is a function $f$ in $R(X)$ such that $R(X) = [z_1, \ldots, z_n, f]$.

**Proof:** Let $Q_1$ be the family of all rational functions of the form $1/p$ where $p$ is a polynomial vanishing nowhere on $X$.

Then $z_1, \ldots, z_n$ and $Q_1$ generate $R(X)$. Let $P_1$ be the family of all polynomials $p$ on $\mathbb{C}^n$ such that $p$ vanishes nowhere on $X$ and all the coefficients in $p$ have rational real and imaginary parts. $P_1$ is countable. Enumerate its members as $p_1, p_2, \ldots$. Since $X$ is compact, the family $Q = \{1/p : p \in P_1\}$ is dense in $Q_1$. So $R(X)$ is generated by the countable set $Q \cup \{z_1, \ldots, z_n\}$.

Choose a sequence $\{a_i\}$ of positive numbers as follows:

Let $a_1 = 1$ and

$$a_{k+1} = 1/2^{k+2} \|1/p_{k+1}\|^{-1} \min\{a_i/\|p_i\|, \ldots, a_k/\|p_k\|\}$$

for $k = 1, 2, \ldots$.

Then $\sum_{i=k+1}^\infty a_i/\|p_i\| \leq a_k/\|p_k\| \sum_{i=k+1}^\infty 1/2^{i+1} \|1/p_i\|^{-1} \|1/p_i\| < a_k/\|p_k\|$. 


Put \( f = \sum_{i=1}^{\infty} a_i/p_i \). Since \( \sum_{i=1}^{\infty} a_i \| 1/p_i \| < \| p_i \| < \infty \) and \( a_i/p_i \in R(X) \) for each \( i \), it follows that \( f \in R(X) \).

We prove that in fact \( R(X) = [z_1, \ldots, z_n, f] \).

To do this it is sufficient to prove that
\[
\frac{1}{p_i} \in \{ z_1, \ldots, z_n, f \} = A \quad \text{for each} \quad i = 1, 2, \ldots
\]

Since \( \| p_i f - 1 \| = \| \sum_{i=1}^{\infty} p_i a_i/p_i - 1 \| = \| \sum_{i=2}^{\infty} p_i a_i/p_i \| \leq \sum_{i=2}^{\infty} \| p_i \| a_i/p_i \| < 1 \)

it follows from the previous lemma that \( p_i f \) is invertible in \( A \).

But then \( p_i^{-1} = f(p_i f)^{-1} \in A \). Now assume that
\[
\frac{1}{p_1}, \ldots, \frac{1}{p_k} \in A.
\]

Then
\[
\| p_{k+1} a_{k+1} (f - \sum_{i=1}^{k} a_i/p_i) - 1 \| = \| p_{k+1} a_{k+1} \sum_{i=k+1}^{\infty} a_i/p_i - 1 \|
\]

and so \( p_{k+1} a_{k+1}^{-1} (f - \sum_{i=1}^{k} a_i/p_i) \) is an invertible element of \( A \).

Put \( g_k = a_{k+1}^{-1} (f - \sum_{i=1}^{k} a_i/p_i) \). Then \( g_k \in A \) by hypothesis.

Therefore \( p_{k+1}^{-1} = g_k (p_{k+1} g_k)^{-1} \in A \).

By induction, every \( 1/p_i \) is in \( A \), so the theorem is proved.

**Theorem:**

Let \( X \) be a compact subset of \( \mathbb{Q}^n \). Then

(a) \( z \in R - \text{hull} (X) \) iff \( p(z) \in p(X) \) for every polynomial \( p \) on \( \mathbb{Q}^n \).

(b) \( \text{Spec} \ R(X) = R - \text{hull} (X) \).

**Proof:** (a) If there is a polynomial \( p \) such that \( p(z) \in p(X) \)
then \( p - p(z) \) is a polynomial with no zeros on \( X \), so the rational function \( (p - p(z))^{-1} \) is finite on \( X \). Since this function has value \( \infty \) at \( z \), \( z \in R\text{-hull}(X) \). Conversely, if \( z \in R\text{-hull}(X) \), then there are polynomials \( p \) and \( q \) such that \( |q| > 0 \) on \( X \) and \( |p(z)/q(z)| > \|p/q\|_X \). Multiplying by a suitable constant we can assume that \( p(z)/q(z) = 1 \). Then \( r = p - q \) is a polynomial, and \( r(z) = 0 \in r(X) \).

(b) If \( z \in R\text{-hull}(X) \), then \( g \to g(z) \) is a continuous algebra homomorphism from \( R_0(X) \) onto \( \mathbb{C} \), so it extends to a complex homomorphism \( m_z \) of \( R(X) \).

Conversely, let \( m \in \text{Spec } R(X) \). Then the restriction of \( m \) to \( P(X) \) is a complex homomorphism of \( P(X) \), so there is a point \( z \in \text{hull}(X) \) such that \( m(p) = p(z) \) for every polynomial \( p \). Given \( g \in R_0(X) \), \( g = p/q \) where \( p \) and \( q \) are polynomials and \( q \) vanishes nowhere on \( X \). Since \( q \) is an invertible element of \( R(X) \), \( q(z) \neq 0 \) (otherwise \( q \) would belong to the maximal ideal \( M = \{g \in R(X): g(z) = 0 \} \)). Thus \( p = qg \) and so \( p(z) = q(z)m(g) \) and \( m(g) = p(z)/q(z) \).

Note that condition (a) has the following equivalent form: \( z \in R(X) \) iff every polynomial which vanishes at \( z \) also vanishes somewhere on \( X \). So we see that \( z \in R\text{-hull}(X) \), and \( m \) agrees with \( m_z \) on a dense subalgebra. Hence \( m = m_z \).

**Corollary:** Let \( X \) be a compact subset of \( \mathbb{C}^n \). Then
\[ R(X) = [z_1, \ldots, z_n, f] \] is isometrically isomorphic to \( P(K) \), where \( K = R-\text{hull}(X) \times f(R-\text{hull}(X)) \subset \mathbb{C}^{n+1} \).

We have shown that a finitely generated function algebra \( A = [f_1, \ldots, f_n] \) is isometrically isomorphic to the uniform closure of the algebra of all holomorphic functions about the compact polynomially convex set \( F(\text{Spec } A) \subset \mathbb{C}^n \), where \( F(m) = (f_1(m), \ldots, f_n(m)) \). The question arises whether one can construct a similar equivalence for any function algebra \( A \). It is in fact possible to do this, but first we have to define the concept of a holomorphic function of an infinite number of complex variables, a notion due to Rickart [8].

**Holomorphic functions of an infinite number of complex variables.**

**Definition 1.21:**

\[ W^\Omega = \prod_{\omega \in \Omega} W^\omega, \Omega \text{ any indexing set, provided with the usual product topology. Thus if } w = \{w_\omega : \omega \in \Omega\} \text{ is in } W^\Omega \text{ then a basic system of neighbourhoods for } w \text{ consists in sets of the form } \{z \in W^\Omega : |z_\omega - w_\omega| < \varepsilon \text{ for } \omega \in \pi\} \text{ where } \varepsilon > 0 \text{ and } \pi \text{ is a finite subset of } \Omega. \] Hereafter \( \pi \) will always denote a finite subset of \( \Omega \).

For each finite subset \( \pi \subset \Omega \), we define the
projection $p_{\pi}:\mathbb{C}^\Omega \rightarrow \mathbb{C}^\pi$ by

$$p_{\pi}\{z_\omega: \omega \in \Omega\} = \{z_\omega: \omega \in \pi\}.$$ 

A set $X$ contained in $\mathbb{C}^\Omega$ is said to be determined by $\pi$, or simply finitely determined, if $X = p_{\pi}^{-1}(p_{\pi}(X))$, which means that $z \in \mathbb{C}^\Omega$ being in $X$ depends only on the values of $z_\omega$ for $\omega \in \pi$.

Suppose that $f$ is a function with domain $D$ in $\mathbb{C}^\Omega$. We say that $f$ is determined by $\pi$ or is finitely determined if $D$ is determined by the finite subset $\pi$ of $\Omega$ and $f(z) = f(w)$ for all $z, w$ in $D$ such that $p_{\pi}(z) = p_{\pi}(w)$.

If $f$ is determined by $\pi$ we can unambiguously define the projection $f_{\pi}$ of $f$ into $\mathbb{C}^\pi$ by setting $f_{\pi}(p_{\pi}(z)) = f(z)$ for all $z$ in $D$.

Definition 1.22: By a polynomial on $\mathbb{C}^\Omega$ we mean an ordinary polynomial in a finite number of the complex variables $z_\omega$, $\omega \in \Omega$. Every polynomial on $\mathbb{C}^\Omega$ is thus finitely determined and continuous.

Let $X$ be a compact subset of $\mathbb{C}^\Omega$. The set $\text{hull}(K) = \{z \in \mathbb{C}^\Omega: |p(z)| \leq \|p\|_K \text{ for all polynomials } p \text{ on } \mathbb{C}^\Omega\}$ is called the polynomially convex hull of $K$ in $\mathbb{C}^\Omega$. $K$ is said to be polynomially convex if $K = \text{hull}(K)$.

As in the finite case we let $P_0(K)$ be the algebra of all polynomials on $K$ and $P(K)$ the uniform closure of $P_0(K)$. 
Now let $K$ be any compact polynomially convex set in $\mathfrak{D}^\Omega$ and let $\pi$ be a finite subset of $\Omega$. We denote by $K_\pi$ the set

\[ \text{hull}(p_\pi K) \subset \mathfrak{D}^\pi. \]

(Note that in general $p_\pi K$ is not polynomially convex.)

**Lemma 1.23**: ([8])

(1) If $\pi \subset \pi'$ then $p_\pi^{-1}(K_{\pi'}) \subset p_\pi^{-1}(K_\pi)$

(2) For arbitrary $\pi_0$, $K = \bigcap_{\pi \supset \pi_0} p_\pi^{-1}(K_\pi)$

(3) Suppose $K \subset G \subset \mathfrak{D}^\Omega$, $G$ open. Then for arbitrary $\pi_0$ there exists $\pi \supset \pi_0$ such that $K_\pi \subset p_\pi G$.

**Proof**: (1) For $\pi \subset \pi'$ we have

\[ p_{\pi'}^{-1}(K_{\pi'}) \subset p_{\pi'}^{-1}(K_{\pi}). \]

Since $K_\pi \times \mathfrak{D}^{\pi'-\pi}$ is clearly polynomially convex in $\mathfrak{D}^{\pi'}$ we have that $K_\pi \subset K_\pi \times \mathfrak{D}^{\pi'-\pi}$. Hence

\[ p_{\pi'}^{-1}(K_{\pi'}) \subset p_{\pi'}^{-1}(K_{\pi} \times \mathfrak{D}^{\pi'-\pi}) = p_{\pi'}^{-1}(K_{\pi}). \]

(2) since $K \subset \bigcap_{\pi \supset \pi_0} p_\pi^{-1}(K_\pi)$. To obtain the reverse inclusion, suppose that $z$ is in $\mathfrak{D}^\Omega - K$. Since $K$ is polynomially convex there is a polynomial $P$ on $\mathfrak{D}^\Omega$ such that $|P(z)| > ||P||$. Let $\pi$ be any finite subset of $\Omega$ which contains $\pi_0$ along with all $\omega \in \Omega$ such that $P$ involves the variables $z_\omega$. Then $P$ is determined by $\pi$ and hence $P_\pi$, the projection of $P$ into $\mathfrak{D}^\pi$, satisfies

\[ |P_\pi(p_\pi(z))| = |P(z)| > ||P||_K = ||P_\pi||_{p_\pi K}. \]
Since \( P_\pi \) is a polynomial on \( \mathbb{C}^\pi \), it follows that
\[
P_\pi(z) \notin \text{hull}(P_\pi K) = K_\pi,
\]
that is, \( z \notin P_\pi^{-1}(K_\pi) \) and hence \( z \notin K \cap \bigcap_{\pi \in \pi_\pi_\pi} P_\pi^{-1}(K_\pi) \).

\[(3) \quad \text{Let } L = \{ z \in \mathbb{C}^\Omega : |z| \leq \sup_{w \in \Omega} |w|, w \in K \}\]

is a compact polydisc containing \( K \). Since \( K \subseteq L \cap G \),
\( K \cap (L - G) = \emptyset \). But by (2), \( K = \bigcap_{\pi \in \pi_\pi_\pi} P_\pi^{-1}(K_\pi) \).

Hence \( (L - G) \cap \bigcap_{\pi \in \pi_\pi_\pi} P_\pi^{-1}(K_\pi) = \emptyset \).

By the compactness of \( K \) there are \( \pi_1, \ldots, \pi_k \) with \( \pi_i \supset \pi_\pi_\pi, i = 1, \ldots, k \), such that
\[
L \cap P_\pi^{-1}(K_{\pi_1}) \cap \ldots \cap P_\pi^{-1}(K_{\pi_k}) \subseteq G.
\]

Let \( \pi = \pi_1 \cup \ldots \cup \pi_k \). Then by (1) we have that
\[
P_\pi^{-1}(K_\pi) \subseteq \bigcap_{i=1}^{k} P_\pi^{-1}(K_{\pi_i}).
\]

Hence \( P_\pi^{-1}(K_\pi) \subseteq \bigcap_{i=1}^{k} P_\pi^{-1}(K_{\pi_i}) \) and so
\[
L \cap P_\pi^{-1}(K_\pi) \subseteq G.
\]

This implies that \( P_\pi L \cap K_\pi \subseteq P_\pi G \). The set \( P_\pi L \), being a polydisc in \( \mathbb{C}^\pi \), is polynomially convex. Since \( P_\pi K \subseteq P_\pi L \) it follows that
\[
K_\pi \subseteq P_\pi L, \text{ and hence that } K_\pi \subseteq P_\pi G.
\]

**Definition 1.24**: A complex-valued function \( h \) is said to be \( \text{holomorphic in } \mathbb{C}^\Omega \) if it is defined in an open subset of \( \mathbb{C}^\Omega \)
and is locally a uniform limit of polynomials. Clearly, if \( \Omega \) is finite, then a holomorphic function in \( \mathbb{C}^\Omega \) is an ordinary holomorphic function of several complex variables.
The following lemma enables us to reduce questions concerning holomorphic functions in $\mathcal{U}$ to the finite dimensional case.

**Lemma 1.25:** ([8])

Let $h$ be a function which is holomorphic in some neighbourhood $G$ of a compact set $K$ in $\mathcal{U}$. Then there exists a neighbourhood $G_0$ of $K$ contained in $G$ on which $h$ is finitely determined.

**Proof:** For each $w$ in $K$ we can find a basic neighbourhood $U$ of $w$ contained in $G$, such that $h$ restricted to $U$ is bounded and a uniform limit of polynomials. Since $K$ is compact, it is covered by a finite number of such neighbourhoods, say $U_1, \ldots, U_k$. Let $G_0 = U_1 \cup \cdots \cup U_k$. Then

$$K \subset G_0 \subset G.$$  

To show that $h$ is finitely determined on $G_0$ we note first that each $U_i$, being a basic neighbourhood, is finitely determined. Suppose

$$U_i = \{z \in \mathcal{U} : |w_\omega - z_\omega| < \varepsilon; \omega \in \pi_i \}.$$  

Then $G_0$ is determined by $\pi = \bigcup_{i=1}^{k} \pi_i$ and is hence also finitely determined.

Now the function $h$ is bounded on $U_i$ for each $i$, so the polynomials which approximate $h$ on $U_i$ are also bounded. But since $U_i = p_{\pi}^{-1}(p_{\pi} U_i)$, a polynomial which is bounded on $U_i$ obviously cannot depend on any of the variables $z_\omega$ for $\omega \notin \pi$. It follows from this that the restriction of $h$ to $U_i$ is determined by $\pi$. 

Finally, suppose $z$ and $w$ are any two points in $G_0$ such that $p_\pi w = p_\pi z$. For some $i$, $w \in U_i$. Then $z \in p_\pi^{-1}(p_\pi U_i) = U_i$.

But we have just shown that $h$ restricted to $U_i$ is determined by $\pi$, hence $h(z) = h(w)$. Thus on $G_0$, $h$ is finitely determined.

**Theorem 1.26:** ([8]) (Generalized Oka Polynomial Approximation Theorem).

If $f$ is a function holomorphic in a neighbourhood of the compact polynomially convex set $K$ in $\mathbb{C}^\Omega$, then $f$ is a uniform limit on $K$ of polynomials.

**Proof:** By the above lemma we may assume that there is an open set $G$ containing $K$ in which $f$ is holomorphic and determined by some finite subset $\pi_0$ of $\Omega$. By 1.23 (3) there exists a finite set $\pi$ containing $\pi_0$ such that $K_\pi \subset p_\pi G_0$. Since $f$ is finitely determined on $G_0$, we can define the projection $f_\pi$ of $f$ into $\mathbb{C}^\pi$ by $f_\pi(p_\pi(z)) = f(z)$, in which case $f_\pi$ is a holomorphic function in the ordinary sense on the open set $p_\pi G_0$ which contains the compact polynomially convex set $K_\pi$. Thus we can apply Oka's Polynomial Approximation Theorem for finite dimensions to obtain $f_\pi$ as a uniform limit on $K_\pi$ of polynomials in the variables $z_\omega$, $\omega \in \pi$. Hence $f$ is a uniform limit on $p_\pi^{-1} K_\pi$ of polynomials on $\mathbb{C}^\Omega$. But $K \subset p_\pi^{-1} K_\pi$, and so $f$ is a uniform limit also on $K$ of polynomials on $\mathbb{C}^\Omega$. 
The preceding results obtained by Rickart on holomorphic functions of an infinite number of complex variables enable us to generalize certain of our earlier results to function algebras which are not necessarily finitely generated.

**Proposition 1.27:** Let $K$ be a compact polynomially convex subset of $\mathbb{C}^n$. Then $\text{Spec } P(K)$ is homeomorphic to $\text{hull } (K)$.

**Proof:** Suppose $z \in \text{hull } (K)$. Then the map 
$$m_z : P_0(K) \to \mathbb{C}$$ defined by $m_z(p) = p(z)$ is a complex homomorphism on $P_0(K)$. (Note that $p$ involves only finitely many of the components of $z$). Also 
$$\|m_z\| = \sup \{|p(z)|/\|p\|_K : p \in P_0(K)\} < 1.$$ In fact 
$$\|m_z\| = 1,$$ since $1$ is in $P_0(K)$.

Conversely, if $m_z$ is a complex homomorphism on $P_0(K)$ such that $\|m_z\| = 1$, then $z \in \text{hull } (K)$.

Hence $z \in \text{hull } (K)$ if and only if $\|m_z\| = 1$.

But $\|m_z\| = 1$ if and only if there exists a complex homomorphism $m_z$ in $\text{Spec } P(K)$ such that $m_z(p) = p(z)$ for every polynomial $p$ on $\mathbb{C}^n$.

The map $F : \text{Spec } P(K) \to \text{hull } (K)$ defined by 
$$F(m_z) = \{m_z(z_\omega) : \omega \in \Omega\} = \{z_\omega : \omega \in \Omega\}$$ is clearly a homeomorphism from $\text{Spec } P(K)$ onto $\text{hull } (K)$.

**Corollary 1.28:** If $K$ is a compact polynomially convex subset of $\mathbb{C}^n$ then $\text{Spec } P(K)$ is homeomorphic to $K$. 

Theorem 1.29: Let $A$ be a function algebra and suppose that $B \subseteq A$ is a set of generators for $A$. Then there is a compact polynomially convex subset $K \subseteq \mathbb{C}^B$ such that $\text{Spec } A$ is homeomorphic to $K$ and $A$ is isometrically isomorphic to $\mathcal{P}(K)$ (and hence to the uniform closure of the algebra of functions holomorphic in a neighbourhood of $K$). Moreover, $\text{Spec } \mathcal{P}(K)$ is homeomorphic to $K$.

Proof: Define the map $F: \text{Spec } A \to \mathbb{C}^B$ by

$$F(m) = \{f(m) : f \in B\}.$$  

As before, $\text{Spec } A$ is homeomorphic to the set $K = F(\text{Spec } A) = \bigcap_{f \in B} R(\hat{f}) \subseteq \mathbb{C}^B$.

Hence $K$ is compact. $A$ is clearly isometrically isomorphic to $\mathcal{P}(K)$ (and hence to the uniform closure of the algebra of functions holomorphic in a neighbourhood of $K$, by the Generalized Oka Polynomial Approximation Theorem).

To show that $K$ is polynomially convex it is sufficient to show that $\text{hull } (K) \subseteq K$. Suppose that $z \in \text{hull } (K)$. Then the map

$$m_z : p(f_1, \ldots, f_k) \to p(z)$$

is a complex homomorphism of $\mathcal{P}_0(K)$, where $\{f_1, \ldots, f_k\} \subseteq B$.

It is also bounded since

$$|p(z)| \leq \|p\|_{\text{hull } (K)} = \|p(\hat{f}_1, \ldots, \hat{f}_k)\|_{\text{Spec } A} = \|p(f_1, \ldots, f_k)\|_X$$

and so can be extended by continuity to the whole of $A$ to become an element $m \in \text{Spec } A$. 
Hence $p(z) = m(p(f_1, \ldots, f_k)) = p(m(f_1), \ldots, m(f_k))$

$= p(f_1(m), \ldots, f_k(m)) = p(z')$ for some $z' \in K$.

Hence $zz = z'$ and so $z \in K$.

Every function algebra has, of course, at least one set of generators, namely itself, so that 1.29 is always true for $B = A$. 
We saw earlier (Chapter I, 1.5) that for any function algebra \( A \), if \( f \) is a member of \( A \) then \( \hat{f} \) achieves its maximum modulus on \( X \) regarded as a subset of \( \text{Spec} \ A \). In this sense \( f \) behaves like an analytic function, if we consider \( X \) as a "boundary" of \( \text{Spec} \ A \). The question arises, are there any other subsets of \( \text{Spec} \ A \) which behave like "boundaries", and if so, is there a unique minimal one amongst them?

**Definition 2.1:** Let \( A \) be a function algebra on \( X \). A boundary of \( \text{Spec} \ A \) for \( A \) is a subset \( E \) of \( \text{Spec} \ A \) such that for each \( f \) in \( A \) there is a point \( m \) in \( E \) such that

\[ |\hat{f}(m)| = \| \hat{f} \| . \]

Clearly, according to this definition, \( X \) is a boundary of \( \text{Spec} \ A \) for \( A \).

**Theorem 2.2:** For every function algebra \( A \) there is a unique minimal closed boundary of \( \text{Spec} \ A \).

**Proof:** See e.g. [9], p. 9.
Definition 2.3: The unique minimal closed boundary of \( \text{Spec } A \) which exists by virtue of 2.2 is called the Shilov boundary of \( A \), and is denoted by \( bA \).

Definition 2.4: A point \( m_0 \) in \( \text{Spec } A \) is said to be a peak point for \( A \) if there is an \( f \) in \( A \) such that 

\[ f(m_0) = 1 \quad \text{and} \quad |f(m)| < 1 \quad \text{for all} \quad m \neq m_0. \]

Note that in general peak points for \( A \) need not exist, even if \( A = C(X) \). However, in the case that \( X \) is compact metrizable, Bishop proved the following.

Theorem 2.5: If \( A \) is a function algebra on a compact metrizable space \( X \) then the set of peak points for \( A \) is dense in \( bA \).
Furthermore, the set of peak points for \( A \) is the unique minimal (not necessarily closed) boundary for \( A \).

Proof: See [10].

We denote the set of peak points of a function algebra \( A \) by \( M(A) \). If \( A \) is defined on a metrizable space we shall refer to \( M(A) \) as the minimal boundary for \( A \).

Proposition 2.6: Let \( A \) be a function algebra on \( X \). Then

\[ M(A) \subset bA \subset X \subset \text{Spec } A. \]

Corollary 2.7: If \( A \) is a finitely generated function algebra
on $X$, then the Shilov boundary of $A$ is the closure in the weak topology of the minimal boundary for $A$.

**Proof:** By 1.19, $X$ is metrizable. Hence by 2.5, $M(A)$ is dense in $\partial A$ and is the unique minimal boundary for $A$.

Thus when dealing with the Shilov boundary of a finitely generated function algebra we need only essentially consider the peak points for $A$.

In the same paper [10], Bishop also proved the following characterization of the minimal boundary.

**Theorem 2.8:** Let $A$ be a function algebra on the compact metrizable space $X$, and let $d$ be a metric on $X$. For each positive integer $n$ and each $x$ in $X$ let

$D_n(x) = \{y \in X : d(x,y) > 1/n\}$. Let

$U_n = \{x \in X : \exists f \text{ in } A \text{ with } \|f\| \leq 1, \|f(x)\| > 3/4 \text{ and } \|f(y)\| < 1/4 \text{ for all } y \in D_n(x)\}$.

Then $U_n$ is open and $\cap U_n = M(A)$.

**Proof:** For each $f$ in $A$ let

$\sigma_n(f) = \{x \in X : \|f(x)\| > 3/4 \text{ and } \|f(y)\| < 1/4 \text{ for all } y \in D_n(x)\}$.

The set $\sigma_n(f)$ is open, for if $x_0 \in \sigma_n(f)$, we can find a neighbourhood of $x_0$ on which $|f| > 3/4$ and which is contained in
$X - D_n(x_0)$ i.e. each point of such a neighbourhood also belongs to $\sigma_n(f)$.

Now $U_n = \bigcup \{\sigma_n(f) : f \in \Lambda \text{ and } \|f\| < 1\}$, and it is therefore open. Let $x \in M(\Lambda)$ i.e. $x$ is a peak point for some function $f$ in $\Lambda$. We can assume that $\|f\| = 1$ and hence that $|f(x)| = 1$. For each $n$, $D_n(x)$ is a closed and therefore compact set. But

$|f(y)| < 1$ for $y \in D_n(x)$ and thus for some positive integer $p_n$,

$|f(y)|^{p_n} < 1/4$ for $y \in D_n(x)$.

Thus $x \in \sigma_n(f^{p_n})$ and so $x \in U_n$. Since this is true for each $n$, it follows that $x \in \cap U_n$. Therefore $M(\Lambda) \subset \cap U_n$.

To prove the reverse inclusion, let $x$ be a fixed element of $\cap U_n$. We construct by induction a sequence $\{g_n\} \subset \Lambda$ satisfying the following conditions.

1. $\|g_{n+1} - g_n\| < 2^{-n+1}$
2. $\|g_n\| < 3(1 - 2^{-n-1})$
3. $g_n(x) = 3(1 - 2^{-n})$
4. $|g_{n+1}(y) - g_n(y)| < 2^{-n-1}$ for $y \in D_n(x)$

We start by constructing $g_1$. Since $x \in U_1$, there is a function $f$ in $\Lambda$ such that $\|f\| = 1$ and $x \in \sigma_1(f)$. Let

$$g_1 = 3/2 \frac{[f(x)]^{-1}}{f}.$$  

Since $|f(x)| > 3/4$, it follows that

$\|g_1\| = 3/2 \cdot 4/3 \|f\| < 3/2 \cdot 4/3 < 2 < 3(1 - 2^{-2})$

so that $g_1$ satisfies (2). Also

$$g_1(x) = 3/2 \frac{[f(x)]^{-1}f(x)}{f} = 3/2 = 3(1 - 2^{-1}).$$
Hence \( g_1(x) \) satisfies all the relevant conditions.

Assume now that \( g_1, \ldots, g_k \) have been chosen to satisfy (1) - (4). Since \( g_k(x) = 3(1 - 2^{-k}) \) and \( g_k \) is continuous, there exists an integer \( j > k \) such that

\[
|g_k(y)| < 3(1 - 2^{-k}) + 2^{-k-2} \quad \text{for } d(x,y) < 1/j
\]
i.e. for \( y \in X - D_j(x) \). Since \( x \in U_j \), there is an \( f \) in \( A \) such that \( ||f|| \leq 1 \) and \( x \in \sigma_j(f) \). Let

\[
h = 3 \cdot 2^{-k-1} [f(x)]^{-1} f. \quad \text{Then } \quad h(x) = 3 \cdot 2^{-k-1}.
\]

Since \( ||f|| \leq 1 \) and \( ||f(x)|| > 3/4 \), \( ||h|| = 3 \cdot 2^{-k-1} \cdot 4/3 = 2^{-k+1} \).

Since also \( ||f(y)|| < 1/4 \) for \( y \in D_j(x) \) it follows that

\[
|h(y)| < 3 \cdot 2^{-k-1} \cdot 4/3 \cdot 1/4 = 2^{-k-1} \quad \text{for } y \in D_j(x).
\]

Let \( g_{k+1} = g_k + h \).

Then \( ||g_{k+1} - g_k|| = ||g_k + h - g_k|| = ||h|| \leq 2^{-k+1} \) \( \ldots \) (1)

Now \( j > k \Rightarrow D_k(x) \subset D_j(x) \). Hence

\[
|g_{k+1}(y) - g_k(y)| = |h(y)| < 2^{-k-1} \quad \text{for } y \in D_k(x) \) \( \ldots \) (4)

Also, \( g_{k+1}(x) = g_k(x) + h(x) = 3(1 - 2^{-k}) + 3 \cdot 2^{-k-1} = 3 - 3(2^{-k} - 2^{-k-1}) = 3(1 - 2^{-k-1}) \) \( \ldots \) (3)

If \( y \in D_j(x) \), then

\[
|g_{k+1}(y)| = |g_k(y) + h(y)| \leq |g_k(y)| + |h(y)|
\]
\[
< ||g_k|| + 2^{-k-1} \leq 3(1 - 2^{-k-1}) + 2^{-k-1} = 3 - 2^{-k} < 3(1 - 2^{-k-2})
\]
since \( 2^{-k} > 3 \cdot 2^{-k-2} \).
If \( y \in X - D_j(x) \), then
\[
|g_{k+1}(y)| \leq |g_k(y)| + |h(y)|
\]
\[
< 3(1 - 2^{-k}) + 2^{-k-2} + ||h||
\]
\[
\leq 3(1 - 2^{-k}) + 2^{-k-2} + 2^{-k+1}
\]
\[
= 3(1 - 2^{-k-2}).
\]
Hence \( \|g_{k+1}\| \leq 3(1 - 2^{-k-2}) \) \( \ldots \) (2)

Thus \( g_{k+1} \) satisfies (1) - (4). We have thus constructed the sequence \( \{g_n\} \) by induction.

By condition (1), \( \{g_n\} \) converges uniformly on \( X \) to a function \( g \) in \( A \).

By (2), \( \|g\| \leq 3. \)

By (3), \( g_n(x) = 3. \)

If \( y \in D_n(x) \), then
\[
|g(y)| \leq \|g_n\| + \sum_{k=n}^{\infty} |g_{k+1}(y) - g_k(y)|
\]
\[
< 3(1 - 2^{-n-1}) + \sum_{k=n}^{\infty} 2^{-k-1}, \text{ by (4)}
\]
\[
< 3
\]
Hence \( x \) is a peak point for \( g \) and so \( x \in M(A) \), as was to be proved.

Recalling that if \( A \) is the finitely generated function algebra \([f_1, \ldots, f_n]\), then \( X \) must be metrizable and the metric on \( X \) can be taken to be
\[
d(x, y) = \max\{|f_i(x) - f_i(y)|: i = 1, \ldots, n\},
\]
we can rewrite Bishop's characterization of the minimal boundary as follows.
Corollary 2.9: Let \( A = [f_1, \ldots, f_n] \). For each positive integer \( n \) and each \( x \in X \) let
\[
D_n(x) = \{ y \in X : \max_i |f_i(x) - f_i(y)| > 1/n \}.
\]
Let
\[
U_n = \{ x \in X : \exists f \in A \text{ with } \|f\| < 1, \ |f(x)| > 3/4 \text{ and } |f(y)| < 1/4 \text{ for all } y \in D_n(x) \}.
\]
Then \( U_n \) is open and \( \bigcap_{n} U_n = M(A) \).

We can obtain a very similar result for arbitrary function algebras by applying some of the techniques developed by Rickart in his paper [8], which we discussed in Chapter I.

Theorem 2.10: Let \( A \) be a function algebra on a (not necessarily metrizable) compact Hausdorff space \( X \).

Then \( m \in \text{Spec } A \) is a peak point for \( A \) if there exists some finite subset \( \pi \subset A \) such that
\[
m \in F^{-1}[\bigcap_n p_{\pi}^{-1}(U_n)],
\]
where
\[
F: \text{Spec } A \rightarrow K = \text{tr } R(f) \quad \text{and}
\]
\[
U_n = \{ z \in p_{\pi}K : \exists \text{ a function } f \in A(p_{\pi}K) \text{ such that } \|f\| < 1, \ |f(z)| > 3/4, \ |f(w)| < 1/4 \text{ for } w \in D_n(z) \},
\]
where
\[
D_n(z) = \{ w \in p_{\pi}K : |z - w| > 1/n \}
\]

Proof: Suppose that \( m \in \text{Spec } A \) is a peak point for \( A \). We showed in Chapter I that \( A \) is isometrically isomorphic to \( A(K) \). Therefore \( F(m) \in K \) is a peak point for some function \( g \) which is holomorphic in a neighbourhood \( G \) of \( K \). By 1.25
there is an open set \( G_0 \) containing \( K \) and contained in \( G \) on which \( g \) is determined by some finite subset \( \pi \subset A \). Now \( g_\pi \), the projection of \( g \) into \( \mathbb{C}_\pi \), is a holomorphic function of a finite number of complex variables on the open set \( p_\pi G_0 \). Since \( g_\pi (p_\pi (z)) = g(z) \) for \( z \in G_0 \), it follows that \( p_\pi (F(m)) \) is a peak point for \( A(p_\pi K) \), which is a function algebra on the compact metric space \( p_\pi K \). Hence by Bishop's characterization,

\[
p_\pi (F(m)) \in \cap U_n.
\]

But

\[
F(m) \in p_\pi^{-1}(\cap U_n) = \cap p_\pi^{-1}(U_n),
\]

and hence

\[
m \in F^{-1}[\cap p_\pi^{-1}(U_n)].
\]

Conversely, suppose that for some finite subset \( \pi \subset A \),

\[
p_\pi (F(m)) \in \cap U_n.
\]

Then by Bishop's characterization

\[
p_\pi (F(m)) \text{ is a peak point for } A(p_\pi K).
\]

Hence there exists an open set \( G^\pi \supset p_\pi K \) and a function \( g_\pi \) holomorphic in \( G^\pi \) such that \( p_\pi (F(m)) \) is a peak point for \( g_\pi \). The function

\[
g \in A(K) \text{ defined by } g(z) = g_\pi (p_\pi (z)) \text{ for } z \in p_\pi^{-1}(G^\pi)
\]

is holomorphic in the open set \( p_\pi^{-1}(G^\pi) \) containing \( p_\pi^{-1}(p_\pi K) \supset K \), and clearly \( F(m) \) is a peak point for \( A(K) \). Hence \( m \) is a peak point for \( A \).

The Local Maximum Modulus Principle.

We shall be taking Stolzenberg's path to Rossi's result.

A result which we shall need, and which often crops up in other contexts, is the following theorem due to Shilov.

**Theorem 2.11:** (Shilov's idempotent theorem) ([13])

Let $A$ be a finitely generated function algebra on $X$. If $\text{Spec } A = S \cup T$ where $S$ and $T$ are nonempty disjoint closed subsets, then there is an $f$ in $A$ such that $
abla f = 0$ on $S$ and $\nabla f = 1$ on $T$.

**Proof:** $A$ is isometrically isomorphic to $P(K)$, as was shown in Chapter I, 1.10, and $\text{Spec } A$ is homeomorphic to $K$. Let $S_1$ and $T_1$ be the subsets of $K$ corresponding to the $S$ and $T$ under the homeomorphism $F$. Thus $S_1$ and $T_1$ are nonempty disjoint closed subsets of $K$ whose union is $K$. Let $W_1$ and $W_2$ be disjoint neighbourhoods of $S_1$ and $T_1$ in $\mathbb{D}^n$. Then $W = W_1 \cup W_2$ is a neighbourhood of $K$ and the function $g$ defined as $0$ in $W_1$ and as $1$ in $W_2$ is holomorphic in $W$. By Oka's polynomial approximation theorem, the restriction of $g$ to $K$ lies in $P(K)$. The function in $A$ which corresponds to $g$ is then the desired element $f$.

**Theorem 2.12:**

Let $A$ be any function algebra on $X$. If $\text{Spec } A = S \cup T$ where $S$ and $T$ are nonempty disjoint closed subsets, then there is an
f in A such that \( f = 0 \) on \( S \) and \( f = 1 \) on \( T \).

**Proof:** Using 1.29 and 1.26 we can repeat the proof of 2.11.

An interesting consequence of Shilov's theorem is the following

**Corollary 2.13:** Let \( A \) be a function algebra on \( X \). If \( X \) is connected, then so is \( \text{Spec} \ A \). In particular, if \( X \) is a connected compact set in \( \mathbb{C}^n \), then \( \text{hull}(X) \) is also connected.

**Proof:** If \( \text{Spec} \ A \) is not connected, let \( S \) and \( T \) be nonempty disjoint closed subsets of \( \text{Spec} \ A \) such that \( \text{Spec} \ A = S \cup T \).

By the preceding theorem, we can find an \( f \) in \( A \) such that \( \hat{f} = 0 \) on \( S \) and \( \hat{f} = 1 \) on \( T \). Thus \( f \) restricted to \( X \) takes only the values 0 or 1. Since \( X \) is connected, either \( f = 0 \) or \( f = 1 \). Neither possibility is consistent with the properties of \( f \). Hence \( \text{Spec} \ A \) is connected.

**Definition 2.14:** ([14])

Let \( D \) be an open set in \( \mathbb{C}^n \) and \( z \in D \). Consider functions which are each defined in some neighbourhood of \( z \) in \( D \). Call two such functions equivalent if they coincide on a neighbourhood of \( z \). This defines an equivalence relation, and the equivalence class of a function \( f \) at \( z \), denoted by \([f]_z\), is called the germ of \( f \) at \( z \).

If \( f \) is a holomorphic function, then \([f]_z\) amounts to a convergent power series.
Germs at \( z \) form a ring with the obvious definition of addition and multiplication.

The ring \( \mathcal{O}_z \) of germs of holomorphic functions is a commutative integral domain with identity and unique factorization. Topologize the space of germs \( \mathcal{O} = \bigcup_{z \in D} \mathcal{O}_z \) by defining the following basis for the open sets:

let \( k \in \mathcal{O} \), then \( k \in \mathcal{O}_{z_0} \) and so \( k = [f]_{z_0} \), where \( f \) is defined in an \( \varepsilon \)-neighbourhood \( N_{z_0} \) of \( z_0 \) in \( D \). At each \( z \in N_{z_0} \), take that class in \( \mathcal{O}_z \) containing the direct analytic continuation of \( f \) i.e. take \( [f]_z \). Then define \( \bigcup_{z \in N_{z_0}} [f]_z \) to be an open set and the collection of such sets to be the basis of open sets.

A holomorphic function \( f \) in \( D \) amounts to a continuous mapping \( f : D \to \mathcal{O} \) which assigns to each point in \( D \) a holomorphic germ over that point.

Now form the quotient field \( M_z \) of \( \mathcal{O}_z \) for each \( z \) in \( D \). Topologize \( M = \bigcup_{z \in D} M_z \) as follows: let \( b \in M \), then \( b \in M_{z_0} \) and

\[ b = [a/\beta]_{z_0} \]

and is represented by \( [f_1]_{z_0}/[f_2]_{z_0} \), where

\( f_1 \) and \( f_2 \) are holomorphic functions at \( z_0 \), and because of unique factorization we may take \( f_1 \) and \( f_2 \) to be coprime at \( z_0 \).

Let \( N_{z_0} \) be a neighbourhood of \( z_0 \) in which \( f_1 \) and \( f_2 \) are defined and are still coprime. At each \( w \in N_{z_0} \), take that class in \( M_w \) represented by \( [f_1]_w/[f_2]_w \). The union over \( N_{z_0} \) of these
classes we define as an open set and the collection of all such sets we take as the basis for the topology.

The elements of $M_z$ are called the germs of meromorphic functions over $z$.

A meromorphic function in $D$ is a continuous mapping which assigns to each point of $D$ a meromorphic germ over that point. At a point $z_0 \in D$, a meromorphic function $g$ is defined by the quotient of two functions $f_1$ and $f_2$ coprime and holomorphic at $z_0$.

(a) If $f_2(z_0) \neq 0$ i.e. $f_2$ is a unit, then $f_1/f_2$ is holomorphic there and hence $g$ is holomorphic at $z_0$ and therefore in a neighbourhood of $z_0$. The point $z_0$ is called a regular point of $g$.

(b) If $f_2(z_0) = 0$ and $f_1(z_0) \neq 0$, then $g$ is said to have a pole at $z_0$.

(c) If $f_2(z_0) = 0$ and $f_1(z_0) = 0$, then $z_0$ is called a point of indeterminacy of $g$.

Definition 2.15:

A polynomial polyhedron in $\mathbb{C}^n$ is a compact set $K$ of the form

$$K = \{z \in U : |f_j(z)| \leq k_j, \ j = 1, \ldots, r\},$$

where $U$ is an open subset of $\mathbb{C}^n$, the $f_j$ are polynomials on $U$ and the $k_j$ are non-negative constants.

By an open polynomial polyhedron we mean the interior of a polynomial polyhedron.
Lemma 2.16: Let $X$ be a compact subset of $\mathbb{C}^n$. Then $\text{hull}(X)$ can be represented as the intersection of a descending chain of open polynomial polyhedra $O_i$, with $O_i \supset O_{i+1}$.

Proof: Let $p$ be any polynomial on $\mathbb{C}^n$. Let
$$O_i = \{ z \in \mathbb{C}^n : |p(z)| < \|p\| + \frac{1}{i} \}$$

Theorem 2.17: (Solution of the Cousin 1 problem on an open polynomial polyhedron)

Let $O$ be an open polynomial polyhedron. Then every Cousin 1 problem on $O$ admits a solution i.e. if $\{O_i\}$ is an open cover of $O$ and we assign meromorphic functions $g_i$ on $O_i$ such that $g_i - g_j$ is holomorphic on $U_i \cap U_j$ for all $i, j$, then there exists a function $g$ meromorphic on $O$ such that for each $i$, $g - g_i$ is holomorphic on $O_i$.

Proof: See [15]

Definition 2.18:
A point $m_o$ in $\text{Spec } A$ is a local peak point for the function algebra $A$ iff there is an open neighbourhood $U$ of $m_o$ in $\text{Spec } A$ and a function $f$ in $A$ such that $\hat{f}(m_o) = 1$ and $|\hat{f}| < 1$ on $U - \{m_o\}$.

Theorem 2.19: ([12])

Let $X$ be a compact subset of $\mathbb{C}^n$. Then every local peak point
for \( P(X) \) is actually a peak point for \( P(X) \).

Proof: Let \( m_0 \) be a local peak point for \( P(X) \). Then there is a neighbourhood \( U \) of \( m_0 \) in \( \text{hull}(X) \) and an \( f \) in \( P(X) \) such that 
\[
1 = \hat{f}(m_0) > |\hat{f}(m)| \quad \text{for all } m \in U - \{m_0\}.
\]
By shrinking \( U \), we can assume that in fact
\[
1 = \hat{f}(m_0) > \sup\{|\hat{f}(m)| : m \in \overline{U} - \{m_0\}\}.
\]

Let \( F : \text{hull}(X) \to \mathbb{C}^{n+1} \) be defined by
\[
F(z_1, \ldots, z_n) = (z_1, \ldots, z_n, f(z_1, \ldots, z_n)).
\]
From the fact that \( \text{hull}(X) \) is the maximal ideal space of \( P(X) \), it follows easily that \( F(\text{hull}(X)) = \text{hull}(F(X)) \).

Therefore we can assume that \( \text{hull}(X) \subset \mathbb{C}^{n+1} \) and that \( f = z_n^{n+1} \), the \((n+1)\)-st coordinate function. By Bishop's characterization of the minimal boundary, it is sufficient to show that for each neighbourhood \( V \) of \( m_0 \) in \( \text{hull}(X) \) there is a \( g \) in \( P(X) \) such that \( |\hat{g}| \leq 1 \) on \( \text{hull}(X) \), \( |\hat{g}| < 1/4 \) on \( \text{hull}(X) - V \), and \( |\hat{g}(m_0)| > 3/4 \). By Oka's Polynomial Approximation Theorem, it is sufficient to find such a \( g \) which is holomorphic in a neighbourhood of \( \text{hull}(X) \).

Choose a compact subset \( C \) of \( \mathbb{C}^n \) such that
\[
C \cap \text{hull}(X) = \overline{U}, \quad m_0 \in \text{int} C \cap \text{hull}(X) \subset U, \quad \text{and } \hat{f} \text{ is holomorphic in a neighbourhood of } C.
\]
Then \( f \) is holomorphic and non-constant on the component of \( \text{int} C = U_0 \) containing \( m_0 \) and so \( f(U_0) \) contains a neighbourhood of \( 1 = \hat{f}(m_0) \).

Hence for a sufficiently small \( \varepsilon > 0 \), \( f(U_0) \) contains the
interval $[1, 1 + \varepsilon]$. Let $T$ be an open polynomial polyhedron containing $[1, 1 + \varepsilon]$.

On $T \times U_0$ define

$$H_t = \{ z \in U_0 : z + t = 0 \}.$$

Express $\text{hull}(X)$ as the intersection of a descending chain of relatively compact open sets $O_i$ i.e. open polynomial polyhedra such that $O_i \supseteq O_i - t$ (this is possible in view of 2.16). We claim that if $i$ is large enough, then every $H_t \cap O_i$ is closed in $O_i$. Suppose not. Then for each $i$ there exists a $t_i$ in $[1, 1 + \varepsilon]$ and a point $s_i$ which belongs to the closure of $H_{t_i} \cap O_i$, but not to $H_{t_i}$. Then $s_i$ belongs to the compact set $C$, but not to its interior $U_0$, because $H_{t_i}$ is closed in $U_0$. Also, by continuity, $f(s_i) = t_i$. By passing to subsequences we can arrange that $t_i$ converges to $t^*$ in $[1, 1 + \varepsilon]$ and $s_i$ converges to $s^*$ on the boundary of $C$, since $[1, 1 + \varepsilon]$ and $\text{bdry} \ C$ are compact. Then

$$f(s^*) = t^* > 1.$$  

also,

$$s^* \in \bigcap_{O_i} \text{hull}(X).$$

Therefore $s^* \in \text{hull}(X) \cap \text{bdry} \ C = \text{bdry} \ U$ and $f(s^*) > 1$, which contradicts the fact that $\hat{f}(m_0) > \sup \{ |\hat{f}(m)| : m \in U - \{m_0\} \}$.

Hence for $i$ large enough, every $H_t \cap O_i$ is closed in $O_i$. Choose $i = i_0$ that large and let $O = O_{i_0}$.

Let $W = \{(t, z) \in T \times U_0 : f(z) - t = 0 \}$. Then $W$ is closed in $T \times O$. Cover $T \times O$ by the open sets $T \times U_0$. 
and \((T \times O) - W\) and assign the meromorphic functions

\[\frac{1}{z_{n+1}^{-1}} - t\] on \(T \times U_0\) and \(0\) on \((T \times O) - W\). Since \(T \times O\) is again an open polynomial polyhedron we can solve the Cousin 1 problem on it to obtain a meromorphic function \(G\) on all of \(T \times O\) such that \(G\) is holomorphic off \(W\) and on \(T \times U_0\),

\[G = \frac{1}{z_{n+1}^{-1}} - t + E,\]

where \(E\) is holomorphic on \(T \times U_0\).

Choose a neighbourhood \(V\) of \(m_0\) in \(hull(X)\) such that \(\bar{V} \subset U_0 \cap hull(X)\). Since the pole set of \(G\) intersects \([1, 1 + \varepsilon] \times hull(X)\) only in \(\{(1, m_0)\}\), \(|G|\) is bounded on \([1, 1 + \varepsilon] \times (hull(X) - V)\), and \(G(t, z)\) is holomorphic about \(hull(X)\) for each (fixed) \(t > 1\). Also \(\sup\{G(t, z)\} \) on \(hull(X)\) tends to infinity as \(t\) converges to 1 from above. Hence there is a \(\delta, 0 < \delta < \varepsilon\), such that for each \(t \in (1, 1 + \delta]\), \(|G(t, z)|\) attains its maximum over \(hull(X)\) only on \(V\). Also there is a constant \(c > 0\) such that \(|E| < c\) on \([1, \delta] \times \bar{V}\). Consider the ratio

\[R_t = \frac{|G(t, m_0)|}{\sup\{|G(t, z)| : z \in hull(X)\}}\] for each \(t \in (1, 1 + \delta]\). Then \(R_t \leq 1\), and

\[R_t = (1 - t)^{-1} + E(t, m_0)/\sup_{\bar{V}}(z_{n+1}^{-1} - t)^{-1} + E(t, z)\]

\[> [(t - 1) - c]/[\sup_{\bar{V}}(z_{n+1}^{-1} - t^{-1})] + c\]

However on \(\bar{V}\), \(|z_{n+1}^{-1}| \leq 1\) and \(z_{n+1}^{-1} = 1\) precisely at \(m_0\).

Therefore, for \(t \in (1, 1 + \delta]\), \(\sup_{\bar{V}}|z_{n+1}^{-1} - t|^{-1} = (t - 1)^{-1}\).
Also, for \( t > 1 \), close enough to 1, we have \( (t - 1)^{-1} > c \).

So \( 1 > R_t > \frac{(t - 1)^{-1} - c}{(t - 1)^{-1} + c} \to 1 \) as \( t \to 1 \) from above. Hence for \( t \in (1, 1 + \delta] \) sufficiently close to 1, if we define

\[
g(z) = \frac{G(t,z)}{\sup \{ |G(t,z)| : z \in \text{hull}(X) \}} \text{ on } \Omega
\]

then \( g \) will be holomorphic on \( \Omega \) and we will have

\[
|g| < 1 \text{ on hull } (X), \quad |g| < 1/4 \text{ on hull } (X) - V \text{ and } |g(m_0)| > 3/4.
\]

**Remark:** Stolzenberg proved his result for \( P(X), X \) a compact subset of \( \mathbb{C}^n \). His result clearly implies the local peak point theorem for finitely generated function algebras i.e. if \( A \) is a finitely generated function algebra then every local peak point for \( A \) is also a peak point for \( A \). We can extend Stolzenberg's result to the following, again using Rickart's basic lemma 1.25.

**Theorem 2.20:** Let \( A \) be any function algebra. If \( m_0 \) is a local peak point for \( A \), then \( m_0 \) is in fact a peak point for \( A \).

**Proof:** By 1.29, \( A \) is isometrically isomorphic to the function algebra \( A(K) \), \( K \) a compact polynomially convex subset of \( \mathbb{C}^n \).

Let \( w \) be a local peak point for \( A(K) \). Then there exists a neighbourhood \( U \) of \( w \) and a member \( f \) of \( A(K) \) such that \( f(w) = 1 \) and \( |f| < 1 \) on \( U - \{w\} \). Now \( f \) is holomorphic in a neighbourhood \( G \) of \( K \) and so by 1.25 there is an open set \( G \)
containing $K$ and contained in $G$ on which $f$ is determined by some finite subset $\pi_0 \subset A$. By 1.23(3) there is a finite set $\pi \supset \pi_0$ such that $K_\pi \subset p_\pi G$. Now $f_\pi$, the projection of $f$ into $\mathbb{C}^\pi$, is a holomorphic function of a finite number of complex variables on the open set $p_\pi G$. Since $f_\pi (p_\pi (z)) = f(z)$ for $z \in G$ it follows that $p_\pi (w) \in p_\pi K \subset K_\pi \subset p_\pi G$ is a local peak point for $A(K_\pi) = P(K_\pi)$. Now $K_\pi$ is a compact subset of $\mathbb{C}^\pi$ and so by 2.19 it follows that $p_\pi (w)$ is in fact a peak point for $P(K_\pi)$ i.e. $\exists$ a function $g$ on $K_\pi$ such that $g((p_\pi (w))) = 1$ and $|g| < 1$ on $K_\pi$. Define the function $h$ on $p_\pi^{-1}(K_\pi)$ by $h(z) = g(p_\pi (z))$ then $h$ is finitely determined and $h_\pi = g$. Since $p_\pi^{-1}(K_\pi)$ contains $K$, it follows that $h(w) = 1$ and $|h| < 1$ on $K$, and hence that $w$ is a peak point for $A(K)$.

**Definition 2.21:**

A compact set $K \subset \text{Spec } A$ is called a **local peak set** for the function algebra $A$ if there is an open set $U$ in $\text{Spec } A$ containing $K$ and an element $f \in A$ such that $f|K = 1$ and $|f| < 1$ on $U - K$. $K$ is a **peak set** for $A$ if $U$ can be taken to be $\text{Spec } A$.

The next theorem generalizes 2.20 to local peak sets.

**Theorem 2.22:**

Let $A$ be any function algebra. If $K$ is a local peak set for $A$
then $K$ is in fact a peak set for $A$.

**Proof:** Choose an open set $U$ containing $K$ and an element $f$ in $A$ such that $\hat{f} = 1$ on $K$ and $|\hat{f}| < 1$ on $U - K$.

Let $H = \{ m \in \text{Spec } A : \hat{f}(m) = 1 \}$. By hypothesis $K = H \cap U = H \cap \overline{U}$ and so $K$ is open and closed in $H$. Suppose $m$ is not a member of $H$. Then the function $g = 1 - f$, which is in $A$, is such that $\hat{g}(m) \neq 0$ but $\hat{g} = 0$ on $H$. So $H$ is the set of common zeros in $\text{Spec } A$ of its kernel $I = \{ g \in A : \hat{g} = 0 \text{ on } H \}$.

Hence $H$ is the maximal ideal space of the Banach algebra $B = A/I$, and the Gelfand map identifies $B$ with $\hat{A}$ restricted to $H$.

Now $K$ is an open-closed subset of $H = \text{Spec } B$. Therefore by Shilov's theorem on idempotents, $\exists x \in B$ such that $\hat{x} = 0$ on $K$ and $\hat{x} = 1$ on $H - K$. Thus $K$ is the zero-set of its kernel $J = \{ h \in A : \hat{h} = 0 \text{ on } K \}$. (Consider any $m \in \text{Spec } A - K$.

If $m \not\in H$, then $\exists g \in A$ such that $\hat{g}(m) \neq 0$ but $\hat{g} = 0$ on $H$, which contains $K$. If $m \in H$, then $m \in H - K$ and so $\hat{x}(m) = 1$, $\hat{x} = 0$ on $K$ in either case $m$ is not in the zero-set of $J$.)

Let $A_K = \{ g \in A : \hat{g} \text{ is constant on } K \}$

Then $J$ is a closed ideal of $A$ and $A_K$ is the closed subalgebra of $A$ which is just $J$ with an identity adjoined. So $\text{Spec } A_K$ is the one-point compactification of $\text{Spec } J$. Since $J$ is a closed ideal of $A$ and $K = \text{zero-set } (J)$, $\text{Spec } J = \text{Spec } A - K$.

Hence $\text{Spec } A_K$ is $\text{Spec } A$ with $K$ identified to a point

(If $m \in \text{Spec } A_K$ and $m(J) = \{ 0 \}$, then $m(g + a) = a$, so there is
only one such homomorphism. If \( m(J) \neq \{0\} \), extend \( m \) uniquely to a complex homomorphism \( \tilde{m} \) of \( A \) by setting
\[
\tilde{m}(h) = m(gh)/m(g),
\]
where \( g \) is any member of \( J \) such that \( m(g) \neq 0 \). Thus we have \( \text{Spec } A_K \) equal to the one-point compactification of \( \text{Spec } J \). Since \( f = 1 \) on \( K \), \( f \in A_K \). If \( m_0 \) is the point \( \{K\} \) in \( \text{Spec } A_K \), then \( \hat{f}(m_0) = 1 \). Also, if \( m \in \text{Spec } A_K \cap U \) and \( |f(m)| = 1 \), then
\[
\hat{f}(m) = 1 \text{ and so } m \in H \cap U = K; \text{ hence } m = m_0.
\]
Thus on \( \text{Spec } A_K \), \( \hat{f} \) peaks within \( U \) exactly at \( m_0 \) i.e. \( m_0 \) is a local peak point for \( A_K \). Therefore by 2.20, \( m_0 \) is a peak point for \( A_K \). Thus \( \exists h \in A \) such that \( \hat{h} = 1 \) on \( K \) and \( |h| < 1 \) on \( \text{Spec } A - K \). Hence \( K \) is a peak set for \( A \).

Theorem 2.23:
(Local Maximum Modulus Principle (Version 1))
Let \( U \) be an open set in \( \text{Spec } A \). Suppose that for some \( f \in A \) (1) \( |f| < 1 \) on \( U \), (2) \( |f| < 1 \) on \( \text{bdry } U \) and (3) \( \hat{f}(m_0) = 1 \) for some \( m_0 \in U \). Then there is an element \( g \in A \) such that
\[
(a) \quad |g| < 1 \text{ on } \text{Spec } A;
(b) \quad \{m \in \text{Spec } A : |\hat{g}(m)| = 1\} = \{m : \hat{g}(m) = 1\} = \{m : \hat{f}(m) = 1\} \cap U
\]

Proof: Replace \( f \) by \( \frac{1}{2}(1 + f) \). Then if \( m \in U \), \( |\hat{f}(m)| = 1 \) iff \( \hat{f}(m) = 1 \). Let \( K = \{m \in U : \hat{f}(m) = 1\} \). By hypothesis (3), \( K \) is nonempty and by (2), \( K = \{m \in \overline{U} : \hat{f}(m) = 1\} \), so
K is compact. Since \( f' = 1 \) on \( K \) and \( |f'| < 1 \) on \( U - K \), \( K \) is a local peak set for \( A \). By 2.22, \( K \) is a peak set for \( A \), so there is some \( g \in A \) such that \( \hat{g} = 1 \) on \( K \) and \( |g| < 1 \) on \( \text{Spec} \ A - K \).

Since \( \hat{g} = 1 \) precisely on \( K \), (a) and (b) hold.

**Theorem 2.24:**

(Local Maximum Modulus Principle (Version I))

Let \( U \) be an open set in \( \text{Spec} \ A \) disjoint from the Shilov boundary of \( A \). Then

(a) \( \text{bdry} \ U \) is nonempty.

(b) \( \sup |f'| = \sup_{\text{bdry} \ U} |f'| \) for every \( f \in A \).

If we let \( A_U \) denote the closure of \( A \cup U \) in \( C(\overline{U}) \) then \( bA_U \subset \text{bdry} \ U \).

**Proof:** \( U \) is not in \( \text{Spec} \ A \), for if \( U \) were open and closed then Shilov's theorem would imply that there is an \( f \) in \( A \) such that \( \hat{f} = 1 \) on \( U \), \( \hat{f} = 0 \) on \( \text{Spec} \ A - U \), which contains \( bA \). This is impossible. Hence \( \text{bdry} \ U \) is nonempty.

Assume that \( \sup |f'| \neq \sup_{\text{bdry} \ U} |f'| = \max |\hat{f}'| \) for some \( f \) in \( A \). If \( \max |\hat{f}'| > a = \sup_{\text{bdry} \ U} |f'| \), then \( |\hat{f}(m)| > a \) for some \( m \in \text{bdry} \ U \). By continuity, \( |\hat{f}'| > a \) on a neighbourhood of \( m \) and this neighbourhood must contain a point of \( U \), so we have a contradiction. If \( \max |\hat{f}'| < a \), we can multiply by a scalar to get \( |\hat{f}'| < 1 = \sup |\hat{f}'| \) on \( \text{bdry} \ U \). Since \( \hat{f} \) is continuous and \( U \) has compact closure, we have
sup |f| = |f(m_o)| = 1 for some m_o ∈ U. Replace f by e^{-iθ}f, where f(m_o) = e^{iθ}. Then f satisfies the hypothesis of Version 1, so there is a g in A with |g| ≤ 1 on Spec A and
\{m : |g(m)| = 1\} ⊂ U. Since U is disjoint from bA and g attains its maximum modulus only within U, we have a contradiction once more. Hence sup |f| = sup |f|.
U bdry U
CHAPTER III.
EXTENSIONS OF FINITELY GENERATED
FUNCTION ALGEBRAS.

Let \( A \) be a function algebra on \( \text{Spec} \ A \). If \( f \in C(\text{Spec} A) \), \([A,f]\) will denote the function algebra generated by \( A \) and \( f \). By 2.6 it follows that \( b[A,f] \subset \text{Spec} A \subset \text{Spec}[A,f] \). It is natural to ask under what conditions \( \text{Spec} A = \text{Spec}[A,f] \) and \( bA = b[A,f] \).

We firstly prove some theorems describing conditions on \( f \) which ensure that \( \text{Spec} A = \text{Spec}[A,f] \) and \( bA = b[A,f] \). Many of these results depend on the local maximum modulus principle, in particular those results in which \( f \) behaves in a sense like a holomorphic function.

Having done this, we attempt to apply this theory of extensions of function algebras to the finitely generated case. Again we find a close connection with the theory of analytic functions.

Definition 3.1: Let \( A \) be a function algebra. We say that \( f \in C(\text{Spec} A) \) is \( A \)-holomorphic at a point \( m \in \text{Spec} A \) if there is a neighbourhood \( U \) of \( m \) such that \( f \) can be approximated uniformly on \( U \) by functions in \( A \). A function \( f \) is \( A \)-holomorphic on a subset \( S \) of \( \text{Spec} A \) if \( f \) is \( A \)-holomorphic at every point of \( S \).
Lemma 3.2: Let $A$ be a function algebra, $U$ an open subset of $\text{Spec } A$ and $f \in C(\text{Spec } A)$ a function which is $A$-holomorphic on $U$. Then

$$b[A,f] \subset bA \cup (\text{Spec } A - U).$$

Proof: Suppose $m \in U - bA$. There is an open set $V$ containing $m$ such that $V$ and $bA$ are disjoint, while $f$ can be approximated uniformly on $V$ by functions in $A$. Hence the closure of $[A,f]$ in $C(\overline{V})$ coincides with the closure $(A|V)^{-}$ of $A$ in $C(\overline{V})$.

By the local maximum modulus principle 2.24, the Shilov boundary of $(A|V)^{-}$ is contained in $\text{bdry } V$. Thus $b[A,f]$ cannot meet $V$. Hence $b[A,f]$ is disjoint from $U - bA$.

Definition 3.3: Let $A$ be a function algebra on $X$. The \textbf{$A$-convex hull} of a closed subset $E$ of $X$ is the set

$$A\text{-hull } (E) = \{ m \in \text{Spec } A : |f(m)| \leq \| f \|_E \text{ for all } f \in A \}.$$ 

$E$ is said to be \textbf{$A$-convex} iff $E = A\text{-hull } (E)$. Equivalently we can define $A\text{-hull } (E)$ as the set of all complex homomorphisms which extend continuously to $(A|E)^{-}$. Since every complex homomorphism of $(A|E)^{-}$ yields a member of $\text{Spec } A$, we have the following

Theorem 3.4: Let $A$ be a function algebra on $X$, and $E$ a closed subset of $X$. Then $A\text{-hull}(E) = \text{Spec } (A|E)^{-}$. 

**Theorem 3.5: (Glicksberg's Lemma)**

Let $A$ be a function algebra on $X$, and let $U$ be a nonempty open subset of $X$. Every function $f \in A$ which vanishes on $U$ also vanishes on $\text{Spec } A - A\text{-hull}(X - U)$. If $X = bA$, then $V = \text{Spec } A - A\text{-hull}(X - U)$ is an open subset of $\text{Spec } A$ and $V \cap U$ is dense in $U$.

**Proof:** See [9] p.39.

**Theorem 3.6:** Let $A$ be a function algebra. Let $f \in C(\text{Spec } A)$ be such that $f$ is $A$-holomorphic on $\text{Spec } A - f^{-1}(0)$. Then $\text{Spec } [A,f] = \text{Spec } A$ and $b[A,f] = bA$.

**Proof:** Let $B = [A,f]$. By 3.2, $bB \subset bA \cup f^{-1}(0)$.

Suppose $bB \neq bA$. Then $bB - bA$ is a nonempty relatively open subset of $bB$ on which $f$ vanishes. By Glicksberg's lemma, $f$ vanishes on an open subset $U$ of $\text{Spec } A$ which meets $bB - bA$. But then $A \cap U = B \cap U$ and therefore

$$U \cap (bB - bA) \subset b(B \cap U) = b(A \cap U) \subset b\text{dry } U \cup bA.$$ 

This is a contradiction and hence $bB = bA$.

Let $r$ be the projection of $\text{Spec } B$ onto $\text{Spec } A$ i.e. if $m \in \text{Spec } B$ then $r(m)$ is the restriction of $m$ to $A$. We must show that every fiber $r^{-1}(m')$ consists of exactly one point i.e. that every $m' \in \text{Spec } A$ extends uniquely to $B$.

Let $g = f \circ r \in C(\text{Spec } B)$. Then $g$ is constant on every fiber $r^{-1}(m')$. Now $g$ is $B$-holomorphic on $\text{Spec } B - g^{-1}(0)$. 
By the first part of the theorem, \( b[B,f] = bB = bA \). However, \( g \) coincides with \( f \) on \( \text{Spec} A \) and, in particular, on \( bA \). Therefore \( g = f \) and \( f \) is constant on every fiber \( r^{-1}(m') \).
It follows that \( r^{-1}(m') \) consists of no more than one point. Hence \( r \) is in fact a homeomorphism and \( \text{Spec} B = \text{Spec} A \).

Corollary 3.7: Let \( A \) be a function algebra. Let \( B \) be a function algebra on \( \text{Spec} A \) such that \( A \subseteq B \) and such that every \( g \in B \) is \( A \)-holomorphic on \( \text{Spec} A \). Then
\[ \text{Spec} B = \text{Spec} A \text{ and } bB = bA. \]

Proof: By 3.2 every function in \( B \) attains its maximum modulus on \( bA \). Therefore \( bB = bA \). By 3.6, every \( m \in \text{Spec} A \) extends uniquely to each \( g \in B \). Hence \( \text{Spec} A = \text{Spec} B \).

Theorem 3.8:
Let \( A \) be a function algebra. Suppose \( f \in C(\text{Spec} A) \) satisfies a relation of the form
\[ f^n + g_{n-1}f^{n-1} + \cdots + g_1f + g_0 = 0, \]
where \( g_0, \ldots, g_{n-1} \in A \). Then \( \text{Spec}[A,f] = \text{Spec} A \) and
\[ b[A,f] = bA. \]

Proof: We can assume, by induction, that the theorem is true for all function algebras and all continuous roots of monic polynomials of degree \( n - 1 \). Consider the formal derivative
\[ h = nf^{n-1} + (n - 1)g_{n-1}f^{n-2} + \cdots + g_1. \]
In a neighbourhood of a point \( m \) at which \( h(m) \neq 0 \), the function \( f \) can be expressed as a convergent power series in the coefficients \( g_0, \ldots, g_{n-1} \). Hence \( f \) is \( \mathcal{A} \)-holomorphic off the set \( h^{-1}(0) \). It follows that \( h \) is \( \mathcal{A} \)-holomorphic off the set \( h^{-1}(0) \). By 3.6,

\[
b[A, h] = bA \quad \text{and} \quad \text{Spec}[A, h] = \text{Spec} A.
\]

Now \( f \) satisfies a monic polynomial of degree \( n - 1 \) with coefficients in \([A, h]\), namely

\[
 f^{n-1} + \frac{(n - 1)}{n}g_{n-1}f^{n-2} + \cdots + \frac{(g_1 - h)}{n} = 0.
\]

By the induction assumption, \( \text{Spec}[A, h, f] = \text{Spec} A \) and \( b[A, h, f] = bA \). But \([A, h, f] = [A, f]\) and hence \( \text{Spec}[A, f] = \text{Spec} A \) and \( b[A, f] = bA \).

**Corollary 3.9:** Let \( A \) be a function algebra, and let \( f \in C(\text{Spec} A) \).

Let \( S_j, 1 \leq j \leq n, \) be subsets of \( \text{Spec} A \) such that \( f|S_j \in A|S_j, 1 \leq j \leq n, \) while \( \bigcup_{j=1}^{n} S_j = \text{Spec} A. \) Then \( \text{Spec}[A, f] = \text{Spec} A \) and \( b[A, f] = bA. \)

**Proof:** If \( g_j \in A \) coincides with \( f \) on \( S_j \), then

\[
\prod_{j=1}^{n}(f - g_j) = 0. \quad \text{The result follows on application of 3.8.}
\]

**Theorem 3.10:** (Baire Category Theorem)

Let \( X \) be a compact topological space. If \( \{E_j\}_{j=1}^{\infty} \) is any countable closed covering of \( X \) then for at least one \( m, \)
The above theorem due to R. Baire is in fact true for a wider class of spaces called Baire spaces.

Theorem 3.11: Let $A$ be a function algebra on $X$ and let

$\{E_j\}_{j=1}^{\infty}$ be a countable closed cover of $\text{Spec } A$.

If $f \in C(\text{Spec } A)$ satisfies $f|_{E_j} \in A|_{E_j}$, $1 \leq j < \infty$, then


Proof: We show first that $b[A,f] = bA$. Suppose not. Then by the Baire category theorem 3.10, there is an index $m$ and a nonempty relatively open subset $U$ of $b[A,f] - bA$ such that $U \subset E_m$. Choose $f_m \in A$ such that $f = f_m$ on $E_m$. By Glicksberg's lemma 3.5, $f - f_m$ vanishes on an open subset $V$ of $\text{Spec } A$ such that $V \cap U$ is dense in $U$. By the local maximum modulus principle 2.24, $b[A,f]$ cannot meet $V - bA$. This contradicts the fact that $V - bA$ contains the nonempty subset $U \cap V$ of $b[A,f]$. Hence $b[A,f] = bA$.

To show that $\text{Spec } [A,f] = \text{Spec } A$, we employ an argument due to Quigley (unpublished). Let $r : \text{Spec } [A,f] \to \text{Spec } A$ be the natural projection which restricts the members of $\text{Spec } [A,f]$ to $A$. If $g \in A$, then $\hat{g} = g \circ r$. Now $f \circ r \in [A,f]$ on each set of the closed cover $[r^{-1}(E_j)]_{j=1}^{\infty}$ of $\text{Spec } [A,f]$.

Applying what we have already proved to the algebras
[A, f] and [A, f, f ◦ r], considered as function algebras on Spec [A, f], we find that b[A, f, f ◦ r] = b[A, f] = bA.

Since f - f ◦ r vanishes on b[A, f, f ◦ r], f - f ◦ r must vanish on Spec [A, f, f ◦ r] ⊃ Spec[A, f]. Consequently, f = f ◦ r, and f is constant on each fiber r⁻¹(m). It follows that r is 1:1, and hence that Spec[A, f] = Spec A.

**Theorem 3.12:** Let A be a function algebra on Spec A and let f ∈ C(Spec A). Let B = [A, f]. For z ∈ R(f), let

\[ T_z = \{ m ∈ Spec A : f(m) = z \}. \]

Then Spec B = Spec A → \( T_z \) is A-convex for each z ∈ R(f).

**Proof:** Put \( M_z = A\)-hull (\( T_z \)) (taken in Spec A).

Let \( K \subseteq Spec A \times \mathbb{C} \) be defined by

\[ K = \{(m, f(m)) : m ∈ Spec A\}. \]

Then K is an imbedding of Spec A in Spec A x \( \mathbb{C} \). Let

\[ H = \{(m, w) ∈ Spec A \times \mathbb{C} : m ∈ M_w\}. \]

H can be imbedded in Spec B. Let r denote the natural projection of Spec B onto Spec A i.e.

\[ m = r(m') \iff g(m) = \hat{g}(m') \text{ for all } g ∈ A. \]

(Since A ⊂ B, the functions in A extend continuously to functions on Spec B).

Define a mapping \( s : Spec B → Spec A \times \mathbb{C} \) by \( s(m) = (r(m), f(m)) \).

We show that s is an imbedding of Spec B into Spec A x \( \mathbb{C} \) such that H ⊂ s(Spec B).
(i) \( s \) is 1:1. If \( s(m_1) = s(m_2) \), then \( r(m_1) = r(m_2) \), so that \( g(m_1) = g(r(m_1)) = g(r(m_2)) = g(m_2) \) for all \( g \in A \).
Also, \( f(m_1) = f(m_2) \). Hence \( h(m_1) = h(m_2) \) for all \( h \in B \) and so \( m_1 = m_2 \).

(ii) \( H \subset s(\text{Spec } B) \). For \( (m,w) \in H \) we define a complex homomorphism \( m'(m,w) \) on \( B \) as follows. Let \( B' \) be the dense subalgebra of \( B \) consisting of all polynomials in \( f \) with coefficients from \( A \). Define \( m'(m,w) \) on \( B' \) by
\[
    m'(m,w) \left( \sum g_i f^i \right) = \sum g_i (m) w^i, 
\]
for \( g_i \in A \).
Then \( m'(m,w) \) is clearly a complex homomorphism on \( B' \). To extend it to \( B \) it is sufficient to show that \( m'(m,w) \) is bounded on \( B' \). Now
\[
    \left| m'(m,w) \left( \sum g_i f^i \right) \right| = \left| \sum g_i (m) w^i \right| \leq \left\| \sum g_i w^i \right\|_T T_w = \left\| \sum g_i f^i \right\|_T_w
\]
since \( \sum g_i w^i \in A \) and \( (m,w) \in H \) i.e. \( m \in M_w \). Thus \( m'(m,w) \) extends to a complex homomorphism on \( B \). Also, for \( g \in A \), \( m'(m,w) (g) = g(m) \), which implies that \( r(m'(m,w)) = m \); and \( m'(m,w) (f) = w \). Therefore
\[
    s(m'(m,w)) = (m,w) \text{ and } H \subset s(\text{Spec } B).
\]

(iii) \( s(\text{Spec } A) = K \). For \( m \in \text{Spec } A \subset \text{Spec } B \), \( r(m) = m \).
Hence \( s(m) = (m,f(m)) \in K \).

If we now identify \( \text{Spec } A \) and \( \text{Spec } B \) with their imbeddings under \( s \) in \( \text{Spec } A \times \mathbb{C} \) we have
\[
    \text{Spec } A \subset H \subset \text{Spec } B.
\]
Lemma 3.13: Let $A$, $f$ and $B$ be as in the previous theorem. If $f(Spec\ B)$ properly contains $f(Spec\ A)$, then each point $w \in f(Spec\ B) - f(Spec\ A)$ lies in a bounded component of $\mathbb{C} - f(Spec\ A)$. Moreover, $f(Spec\ B)$ is the union of $f(Spec\ A)$ with those bounded components of $\mathbb{C} - f(Spec\ A)$ which meet $f(Spec\ B)$ (i.e. $f(Spec\ A)$ can only be enlarged to $f(Spec\ B)$ by completely filling in some holes).

Proof: Let $m \in Spec\ B$ be such that $f(m)$ is not in $f(Spec\ A)$. If $f(m)$ lies in the unbounded component of $\mathbb{C} - f(Spec\ A)$, then there is a polynomial $p$ in one complex variable such that $|p(f(m))| > \|p\|_{f(Spec\ A)}$. Since $g = p(f) \in B$, we have $|g(m)| > \|g\|_{Spec(A)}$. This contradicts the fact that $B$ as a function algebra on $Spec\ A$ has its Shilov boundary contained in $Spec\ A$.

Alternately, suppose that $f(m)$ lies in some bounded component $C$ of $\mathbb{C} - f(Spec\ A)$. If $C$ is not contained in $f(Spec\ B)$, there is a point $w$ on the boundary of $f(Spec\ B)$ and contained in $C - f(Spec\ A)$. Let $w = f(m_0)$.

Choose $a \in C - f(Spec\ B)$ such that $|a - w| < \inf\{|w - f(m')| : m' \in Spec\ A\}$.

Then $g(z) = \frac{1}{z - a}$ is analytic on a neighbourhood of $f(Spec\ B)$, so that $g(f) \in B$. But $|g(f(m_0))| > \|g(f)\|_{Spec\ A}$, which, as above, is impossible.
Theorem 3.14: (Lavrentiev)

Let \( K \) be a compact subset of the complex plane.

The conditions

(1) \( \text{int} \, K \) is empty
(2) the complement of \( K \) is connected

are necessary and sufficient for \( P(K) = C(K) \)


Theorem 3.15: Let \( H \) be as in 3.12 i.e.

\[ H = \{ (m, w) \in \text{Spec} \, A \times \mathbb{C} : m \in M_w \} \]

If \( f(\text{Spec} \, A) \) is a compact subset of the plane with connected complement and empty interior, then \( \text{Spec} \, B = H \).

Proof: By 3.13, \( f(\text{Spec} \, B) = f(\text{Spec} \, A) \). As in 3.12 consider the imbedding \( s : \text{Spec} \, B \to \text{Spec} \, A \times \mathbb{C} \) given by

\[ s(m) = (r(m), f(m)), \]

where \( r \) is the natural projection of \( \text{Spec} \, B \) onto \( \text{Spec} \, A \). Then

\[ s(\text{Spec} \, A) \subset H \subset s(\text{Spec} \, B) \subset \text{Spec} \, A \times f(\text{Spec} \, A) \]

and we wish to show that \( H = s(\text{Spec} \, B) \). If \( s(m) \in s(\text{Spec} \, B) - H \), from the definition of \( H \) there exists a function \( g \) in \( A \) such that

\[ 1 = |g(r(m))| > \| g \|_{M_f(m)} \]

Since \( s(m) \) is not in \( s(\text{Spec} \, A) \), we can choose a neighbourhood \( V \) of \( f(m) \) in \( \mathbb{C} \) such that \( (m', z') \in s(\text{Spec} \, A) \cap [\text{Spec} \, A \times (V \cap f(\text{Spec} \, A))] \Rightarrow \)

\[ |g(m')| < |g(r(m))|. \]
Also, since \( f(\text{Spec} \, A) \) is a compact subset of the plane with connected complement and empty interior, by Lavrentiev's theorem 3.14, every continuous function on \( f(\text{Spec} \, A) \) can be uniformly approximated on \( f(\text{Spec} \, A) \) by polynomials. In particular, if \( h \) is a function which peaks at \( f(m) \) i.e. 
\[ h(f(m)) = 1, \quad |h(z)| < 1 \quad \text{for } z \in f(\text{Spec} \, A), \quad z \neq f(m), \]
then \( h \) can be uniformly approximated by polynomials.

But \( f(\text{Spec} \, A) = f(\text{Spec} \, B) \Rightarrow h(f) \in B \). Choose an integer \( N \) such that 
\[ \|h^N\| f(\text{Spec} \, A) - V < 1/\|g\|. \]

Let \( h' = h^N(f)g \). Then \( h' \in B \), \( |h'(m)| = 1 \) and 
\[ \|h'\| f(\text{Spec} \, A) < 1, \] a contradiction as in 3.13.

**Corollary 3.16:** Let \( A \) be a function algebra on \( \text{Spec} \, A \). Let 
\[ H = \{(m, w) \in \text{Spec} \, A \times \mathbb{C} : m \in M_w\} \]. If \( f(\text{Spec} \, A) \) is a compact subset of the plane with connected complement and empty interior, and in addition \( T_w = M_w \) for each \( w \in f(\text{Spec} \, A) \) i.e. the set \( f^{-1}(w) \) is \( A \)-convex for each \( w \in f(\text{Spec} \, A) \), then \( \text{Spec} \, A = H \) and hence \( \text{Spec} \, A = \text{Spec} \, B \).

**Proof:** We had previously that \( \text{Spec} \, A \subseteq H \) if we identify \( \text{Spec} \, A \) with \( K = \{(m, f(m)) : m \in \text{Spec} \, A\} \). But 
\[ H = \bigcup \{M_w \times \{w\} : w \in f(\text{Spec} \, A)\} \]
\[ = \bigcup \{T_w \times \{w\} : w \in f(\text{Spec} \, A)\} \]
\[ = K. \]

Hence \( \text{Spec} \, A = H \). By 3.15, \( H = \text{Spec} \, B \) and so \( \text{Spec} \, A = \text{Spec} \, B \).
Remark: There is an interesting interpretation of the condition in 3.16 that \( T_w = f^{-1}(w) \) be A-convex for each \( w \in f(\text{Spec } A) \). It has been shown earlier (see 3.4) that if \( E \) is any closed subset of \( \text{Spec } A \), then \( \text{Spec}(A|E)^- = A\text{-hull}(E) \).

Since \( f \) is constant on each \( T_w \),

\[
([A,f]|T_w)^- = (A|T_w)^-.
\]

Hence

\[
\text{Spec}([A,f]|T_w)^- = \text{Spec}(A|T_w)^- = A\text{-hull}(T_w) = T_w
\]

for each \( w \in f(\text{Spec } A) \). Thus \( \text{Spec}[A,f] = \text{Spec } A \) if this is true for the restrictions of \([A,f]\) and \( A \) to each \( T_w \).

Theorem 3.12 stated that

\[
\text{Spec } A = \text{Spec}[A,f] \Rightarrow T_w \text{ is } A\text{-convex for all } w \in f(\text{Spec } A).
\]

What we have just proved in 3.16 is a partial converse to this, i.e. if \( T_w \) is A-convex for all \( w \in f(\text{Spec } A) \) and in addition \( f(\text{Spec } A) \) is a compact subset of the plane with connected complement and empty interior, then \( \text{Spec } A = \text{Spec}[A,f] \).

Gamelin and Wilken in [18] have produced the following counter-example to show that \( T_w \) is A-convex for each \( w \in \text{Spec } A \) does not necessarily imply that \( \text{Spec}[A,f] = \text{Spec } A \):

let \( f(z) = z^2 \overline{z} \), \( z \in \Delta \), where \( \Delta \) is the unit disc. The function \( f \) is a homeomorphism of \( \Delta \) and \( \text{Spec}[P(\Delta),f] \) is strictly larger than \( \Delta = \text{Spec } P(\Delta) \). Notice that \( f(\text{Spec } P(\Delta)) \) does not have empty interior.

Remark: We can extend 3.12, 3.13, 3.15 and 3.16 to the more
general situation in which a finite number of functions are adjoined to $A$. Then the statements of these results have to be formulated in terms of intersections of the level sets of the functions. The statements are the same except that instead of adjoining $f \in C(\text{Spec } A)$ to $A$, we adjoin
\[ \{f_1, \ldots, f_n\} \subset C(\text{Spec } A) \] and replace
\[ T_w \text{ by } T(w_1, \ldots, w_n) = \{m \in \text{Spec } A : f_i(m) = w_i, \ i = 1, \ldots, n\}, \]
\[ M_w \text{ by } M(w_1, \ldots, w_n) = A\text{-hull}(T(w_1, \ldots, w_n)) \] and
\[ H \text{ by } H' = \{(m, w_1, \ldots, w_n) \in \text{Spec } A \times \mathbb{C}^n : m \in M(w_1, \ldots, w_n)\}. \]

We now come to the application of the foregoing to finitely generated function algebras, but first we need to state a couple of theorems.

**Definition 3.17:** For $K$ a compact subset of the complex plane, $A_\circ(K)$ will denote the set of all $f \in C(K)$ which are analytic in the interior of $K$.

**Theorem 3.18:** (Mergelyan).
Let $K$ be a compact subset of the plane. If $K$ has connected complement, then $P(K) = A_\circ(K)$.

**Proof:** See [9] p.48.

**Theorem 3.19:** A compact subset $K$ of the plane is polynomially convex iff its complement is connected.

Now let $A = [g]$ be a singly generated function algebra on Spec $A$. Then $A$ is isometrically isomorphic to $P(K)$, where $K = R(\hat{g}) \subset \mathcal{H}$. But $K$ is compact and polynomially convex and hence by Mergelyan's theorem 3.18 and the characterization of polynomial convexity in the plane 3.19, $A$ is in fact isometrically isomorphic to $A_0(K)$. Suppose we adjoin to $A$ a function $f \in C(\text{Spec } A)$ which is such that $\text{Spec } A = \text{Spec } [A,f]$. Notice that $[A,f]$ is isometrically isomorphic to $P(K')$, where $K' = R(\hat{g}) \times R(\hat{f})$. Since $K$ is homeomorphic to Spec $A$ and $K'$ is homeomorphic to Spec $[A,f]$, it follows that $K$ is homeomorphic to $K'$. In particular, the homeomorphism $\varphi : K \to K'$ is given by $\varphi(z) = (z,w)$.

**Theorem 3.20:** Let $A = [g]$ be a singly generated function algebra on Spec $A$ and let $f \in C(\text{Spec } A)$ be such that Spec $A = \text{Spec } [A,f]$. Then there exists a continuous algebraic epimorphism $\Psi : [A,f] \twoheadrightarrow A_0(K)$.

**Proof:** Define the map $\Phi : P_0(K') \to P_0(K)$ by

$$\Phi(p(z,w)) = p(\varphi^{-1}(z,w)), \text{ where } p \in P_0(K')$$

i.e. $\Phi\left(\sum_{k=0}^{m} \sum_{i+j=k} a_{ij}z^iw^j\right) = \sum_{k=0}^{m} \sum_{i+j=k} a_{ij}z^k$

thus defined is clearly onto, but not necessarily 1:1.
\( \Phi \) is linear: let \( p_1 = \sum a_{i,j} z^i w^j \), \( p_2 = \sum b_{i,j} z^i w^j \) (we can assume for the sake of simplicity that \( z \) and \( w \) occur to the same powers). Then

\[
\Phi(p_1 + p_2) = \Phi(\sum a_{i,j} z^i w^j + \sum b_{i,j} z^i w^j) = \Phi(\sum (a_{i,j} + b_{i,j}) z^i w^j) = \sum (\Sigma a_{i,j} + b_{i,j}) z^k = \sum (\Sigma a_{i,j}) z^k + \sum (\Sigma b_{i,j}) z^k = \Phi(p_1) + \Phi(p_2)
\]

Similarly \( \Phi(a p) = a \Phi(p) \) for \( p \in P_0(K') \), \( a \in \mathbb{C} \), and \( \Phi(p_1 p_2) = \Phi(p_1) \Phi(p_2) \).

\( \Phi \) is continuous: let \( \varepsilon > 0 \) be given and let \( p_1 = \sum a_{i,j} z^i w^j \). Choose \( p_2 = \sum b_{i,j} z^i w^j \) such that

\[
\sum |a_{i,j} - b_{i,j}| = \sum |c_{i,j}| < \varepsilon/2^k|z|^k. \text{ Then } \sum |c_{i,j}| |z|^k \leq \sum |c_{i,j}| |z|^k < \varepsilon.
\]

Thus far we have shown that \( \Phi \) is a continuous algebraic epimorphism from \( P_0(K') \) onto \( P_0(K) \). It can be extended by continuity to \( \sim : P(K') \to P(K) \) as follows: if \( p \in P(K') \), define \( \sim(p) \) to be the uniform limit of \( \Phi(p_n) \), where \( p_n \to p \) uniformly. We now show that \( \Phi \) thus defined has the same properties as \( \Phi \).

Let \( f \in P(K) \). We want to show that \( f \in \sim(P(K')) \) i.e. that \( \Phi \) is onto. Now \( \forall \varepsilon > 0 \exists f_1 \in P_0(K) \) such that

\[ \| f - f_1 \| < \varepsilon. \] Since \( \Phi \) is onto, \( \exists g_1 \in P_0(K') \) such that
\[ \Phi(g_1) = f_1. \] Hence \( \| f - \Phi(g_1) \| < \varepsilon. \) Therefore

\[ f \in \Phi(P_0(K')) = \Phi(P(K')) = \Phi(P(K')). \]

\[ \Phi \] has the same algebraic properties as \( \Phi. \) Let

\[ f_1, f_2 \in P(K') \] and \( \alpha, \beta \in \mathbb{C}. \) We must show that

\[ \Phi(\alpha f_1 + \beta f_2) = \alpha \Phi(f_1) + \beta \Phi(f_2), \] or equivalently that for any \( \varepsilon > 0 \) we can get

\[ \| \Phi(\alpha f_1 + \beta f_2) - \alpha \Phi(f_1) - \beta \Phi(f_2) \| < \varepsilon. \]

Now

\[ \| \Phi(\alpha f_1 + \beta f_2) - \alpha \Phi(f_1) - \beta \Phi(f_2) \| \]

\[ = \| \Phi(\alpha f_1 + \beta f_2) - \Phi(\alpha f_1 + \beta f_2) + \Phi(\alpha f_1 + \beta f_2) - \alpha \Phi(f_1) - \beta \Phi(f_2) \| \]

\[ = \| \Phi(\alpha f_1 + \beta f_2) - \Phi(\alpha f_1 + \beta f_2) + \Phi(\alpha f_1 + \beta f_2) - \alpha \Phi(f_1) - \beta \Phi(f_2) \| \]

\[ = \| \Phi(\alpha f_1 + \beta f_2) - \Phi(\alpha f_1 + \beta f_2) + \alpha (\Phi(f_1) - \Phi(f_1)) + \beta (\Phi(f_2) - \Phi(f_2)) \| \]

\[ \leq \| \Phi(\alpha f_1 + \beta f_2) - \Phi(\alpha f_1 + \beta f_2) \| + |\alpha| \| \Phi(f_1) - \Phi(f_1) \| + |\beta| \| \Phi(f_2) - \Phi(f_2) \| \]

\[ < \varepsilon' + |\alpha| \varepsilon' + |\beta| \varepsilon' < \varepsilon \]

for \( f_1 \) and \( f_2 \in P_0(K') \) chosen suitably close to \( f_1 \) and \( f_2 \) respectively.

Similarly, \( \Phi(f_1 f_2) = \Phi(f_1) \Phi(f_2). \)

Thus we have a continuous algebraic epimorphism

\[ \tilde{\Phi} : P(K') \to P(K) = A_0(K). \]

But \([A,f]\) is isometrically isomorphic to \( P(K') \) and hence there is a continuous algebraic epimorphism

\[ \tilde{\psi} : [A,f] \to A_0(K). \]
We can strengthen 3.20 if, in addition \( b[A,f] = bA \).

To do so, we need a couple of further results.

**Theorem 3.21:** Let \( K \) be a compact subset of the plane. Then the Shilov boundary of \( \mathcal{P}(K) \) is the topological boundary of \( \text{hull}(K) \).


**Proposition 3.22:** Let \( A = [g] \) be a singly generated function algebra. Then \( bA = \text{Spec } A \Rightarrow A = C(X) \).

**Proof:** We prove the equivalent assertion that \( \mathcal{P}(K) = C(K) \), where \( K = R(g) \).

Now \( bA = \text{Spec } A \Rightarrow \text{bdry } K = K \), by 3.21

\[
\Rightarrow K = \text{int } K \Rightarrow K = K
\]

\( \Rightarrow \text{int } K \) is empty, since \( K = K \),

\( \Rightarrow \text{K is a compact subset of the plane} \n\)

with empty interior and connected complement, by 3.19.

**Corollary 3.23:** Let \( A = [g] \) be a singly generated function algebra on \( \text{Spec } A \) such that \( bA = \text{Spec } A \). Let \( f \in C(\text{Spec } A) \) be such that \( \text{Spec}[A,f] = \text{Spec } A \) and \( bA = b[A,f] \). Then there exists a continuous algebraic epimorphism

\[
\Psi : [A,f] \rightarrow C(K).
\]

**Proof:** The result follows immediately from 3.20 and 3.22.
Remarks: (i) The function $f \in C(\text{Spec } A)$ in 3.20 could be such that it satisfies the conditions of any one of 3.6 - 3.9, 3.11 and 3.16. The function $f \in C(\text{Spec } A)$ in 3.23 could be such that it satisfies the conditions of any one of 3.6 - 3.9 and 3.11.

(ii) Theorem 3.20 can be extended to the case where a finite number of functions, say $\{f_1, \ldots, f_n\}$, are adjoined to $A$, provided that $\text{Spec } [A, f_1, \ldots, f_n] = \text{Spec } A$. e.g. if $\{f_1, \ldots, f_n\}$ are $A$-holomorphic on $\text{Spec } A$ (by corollary 3.7) or if $\{f_1, \ldots, f_n\}$ satisfy the conditions of corollary 3.16 (see remark preceding definition 3.17).

(iii) The definition of an $A$-holomorphic function (see Definition 3.1) is identical to that of a function which is locally approximable by the functions of a function algebra $A$. Moreover, functions locally approximable by the functions of $A$ are also locally in $A$. Thus all results flowing from theorem 3.6 are true for functions locally approximable by functions in $A$ and functions locally in $A$. For a discussion of the adjunction of such functions to a function algebra see the M.Sc. thesis of J.P.G. Ewer, "Approximation Properties of Function Algebras" [35], done at the University of Cape Town.
CHAPTER IV.
FUNCTION ALGEBRAS AS DIRECT LIMITS
OF THEIR
FINITELY GENERATED SUB-FUNCTION ALGEBRAS.

In the case that a function algebra $A$ contains a 1:1 function, we can express it as the direct limit of its finitely generated sub-function algebras. The value of this procedure will be seen later on when we shall show how information about such a function algebra can be obtained from the characteristics of its finitely generated sub-function algebras and vice versa.

Definition 4.1: A directed set $D$ is a set with a partial order relation $\leq$ such that for $a, \beta \in D$ there is a $\gamma \in D$ with $a \leq \gamma$ and $\beta \leq \gamma$.

A direct system of algebras and algebra homomorphisms
\{\(A^\alpha, h^\beta_\alpha\)\} consists in a collection \{\(A^\alpha\)\} of algebras indexed by a directed set $D$, and a collection of algebra homomorphisms
\(h^\beta_\alpha : A^\alpha \rightarrow A^\beta\) for every $a, \beta \in D$ with $a \leq \beta$ such that
(a) \(h^a_\alpha = \text{identity of } A^\alpha\)
(b) \(h^\gamma_\alpha = h^\gamma_\beta h^\beta_\alpha : A^\alpha \rightarrow A^\beta\) for $a \leq \beta \leq \gamma$ in $D$.

A direct limit of a direct system of algebras and algebra homomorphisms \{\(A^\alpha, h^\beta_\alpha\)\} is an algebra $A$ together with a family of algebra homomorphisms \(h_\alpha : A^\alpha \rightarrow A(\alpha \in D)\) satisfying the
conditions

(1) \( h_\beta \circ h^\beta_\alpha = h_\alpha \) if \( \alpha \leq \beta \) in \( D \), and

(2) for any algebra \( A' \) and any collection of algebra homomorphisms \( h'_\alpha : A^\alpha \to A' \) such that \( h'_\beta \circ h^\beta_\alpha = h'_\alpha \) if \( \alpha \leq \beta \) in \( D \), there exists a unique isomorphism \( h' : A \to A' \) such that \( h' \circ h_\alpha = h'_\alpha \) for all \( \alpha \in D \). The situation is summed up in the following diagrams in which all the triangles commute.

\[
\begin{array}{c}
A^\alpha \\
| \downarrow h^\beta_\alpha \\
A^\beta \\
| \downarrow h_\beta \\
A \\
| \downarrow h \\
A' \\
\end{array}
\]

\[
\begin{array}{c}
A^\alpha \\
| \downarrow h^\beta_\alpha \\
A^\beta \\
| \downarrow h_\beta \\
A \\
| \downarrow h \\
A' \\
\end{array}
\]

A is also denoted by \( \text{lim}. \text{dir. } A^\alpha \).

The direct limit of a direct system \( \{ A^\alpha, h^\beta_\alpha \} \) has a canonical form, which can be constructed as follows.

Let \( \Sigma A^\alpha \) be the direct sum of the \( A^\alpha \) i.e. \( \Sigma A^\alpha \) consists of all \( f \in \prod_{\alpha \in \mathcal{D}} A^\alpha \) such that only a finite number of coordinates of \( f \) are not zero. For each \( \alpha \in \mathcal{D} \) there is an embedding
i_a : ΣA^a → ΣA^a; given f^a in A^a, i_a(f^a) is the element of ΣA^a with a-th coordinate f^a and all others zero.

Let S be the submodule of ΣA^a generated by the ranges

(i_β h_α^β - i_α)(A^a) for all (α, β) ∈ D x D with α ≤ β.

Then ΣA^a/S is a direct limit for \{A^a, h_α^β\} where

h_α : ΣA^a → ΣA^a/S is the composite of the canonical map from ΣA^a onto ΣA^a/S and i_α. (see [4], p.211.)

**Definition 4.2:** An inverse system of topological spaces and continuous maps \{X_α, r_α^β\} consists in a collection \{X_α\} of topological spaces indexed by a directed set D, and a collection of continuous functions r_α^β : X_β → X_α for α ≤ β in D such that

(a) r_α^α = identity of X_α
(b) r_α^γ = r_α^β r_β^γ : X_γ → X_α for α ≤ β ≤ γ in D.

An inverse limit of an inverse system of topological spaces and continuous maps is a topological space X together with a family of continuous maps r_α : X → X_α, α ∈ D, satisfying the following conditions

(1) r_α^β r_β^α = r_α
(2) for any topological space X' and any family of continuous maps r_α : X' → X such that

r_α^β r_β^α = r_α for (α, β) ∈ D x D with α ≤ β,

there exists a unique continuous map r : X' → X such that
$r_\alpha \circ r = r'_\alpha$ for all $\alpha \in D$. We can sum up by saying that all triangles in the following diagrams commute.

The inverse limit is also denoted by $\lim \text{inv.} \{X_\alpha, r_\alpha^\beta\}$.

We can construct an inverse limit of an inverse system of topological spaces and continuous maps as follows:

Define a thread to be an element $x$ of the topological product $\prod_{\alpha \in D} X_\alpha$ such that $r_\alpha^\beta(x_\beta) = x_\alpha$ whenever $\alpha \leq \beta$.

Let $X$ be the set of all threads provided with the relative topology induced by $\prod X_\alpha$, and let $r_\alpha(x) = x_\alpha$ for $x \in X$.

i.e. $r_\alpha$ is the restriction to $X$ of the projection from $\prod X_\alpha$ onto $X_\alpha$. Then $\{X, r_\alpha\}$ is an inverse limit of the inverse system $\{X_\alpha, r_\alpha^\beta\}$. For if $\{X', r'_\alpha\}$ is such that $r'_\alpha = r_\alpha^\beta \circ r_\alpha'$ for all $(\alpha, \beta) \in D \times D$ such that $\alpha \leq \beta$, then each element $x'$ of $X'$ defines a thread $(r'_\alpha(x')) \alpha \in D$. The map $r : X' \to X$
which sends \( x' \in X' \) onto its corresponding thread is easily verified to be a commuting homeomorphism and is unique by definition. Henceforth we shall refer to \( \{X, r_a\} \) as just defined as the inverse limit of the inverse system.

**Proposition 4.3:**

Let \( \{X_a, r^a_b\} \) be an inverse system of compact Hausdorff spaces and continuous maps. Then the inverse limit of the system is also compact.

**Proof:** See e.g. [4]. p.210.

Now suppose we are given a function algebra \( A \) containing a 1:1 function \( g \). Let \( D \) be the family of finite subsets of \( A \) which include \( g \) as a member i.e. \( \alpha \in D \) iff \( \alpha = \{g, f_1, \ldots, f_k\} \) where the \( f_i \) and \( g \) belong to \( A \). If we direct \( D \) by inclusion, then \( D \) is clearly a directed set. Put \( A^\alpha = \{g, f_1, \ldots, f_k\} \). Clearly, \( A = \bigcup A^\alpha \). Let \( i^\beta_\alpha : A^\alpha \to A^\beta \) be the inclusion map for \( \alpha \leq \beta \). Then \( \{A^\alpha, i^\beta_\alpha\} \) forms a direct system of algebras and algebra homomorphisms, having as direct limit \( A = \bigcup A^\alpha \), with the \( i^\alpha_\alpha : A^\alpha \to A \) taken as the inclusion map. For suppose \( \{A', i'_\alpha\} \) is such that \( i'_\beta \circ i'^\beta_\alpha = i'_\alpha \) for all \( (\alpha, \beta) \in (D \times D) \) with \( \alpha \leq \beta \). We can define an algebra homomorphism \( i : A \to A' \) as follows: if \( f \in A \) then there is an \( \alpha \in D \) such that \( f \in A^\alpha \). Let \( i(f) = i'_\alpha(i^{-1}_\alpha(f)) \), which is well defined since \( i_\alpha \) is the
inclusion map. It is easy to show that i has the required properties.

Note that if we were to define D simply as the family of all finite subsets of A we would have no guarantee that the subalgebras generated by these finite subsets would in fact be function algebras, since they may fail to separate points, even though A does so. Purely as an algebra, however, a function algebra is the direct limit of its finitely generated subalgebras. Indeed, if A did not contain a 1:1 function we could not even use our somewhat artificial tactic to ensure that A is the direct limit of certain of its finitely generated sub-function algebras. That A contains a 1:1 function is a sufficient condition for A to be so expressed. We do not know if it is necessary. Henceforth, whenever we discuss function algebras as direct limits we shall tacitly assume that A contains a 1:1 function.

Now let A be a function algebra containing a 1:1 function g and let D be the directed set described above.

Define \( r^\beta_\alpha : \text{Spec } A^\beta \to \text{Spec } A^\alpha \) for \( \alpha \leq \beta \) in D as follows:

if \( m \in \text{Spec } A^\beta \), then \( r^\beta_\alpha(m) = m \) restricted to \( A^\alpha \).

Then \( \{ \text{Spec } A^\alpha, r^\beta_\alpha \} \) is an inverse system of compact Hausdorff spaces and continuous maps. Let

\( r^\alpha_\beta : \text{Spec } A \to \text{Spec } A^\alpha \) be the map sending each element of \( \text{Spec } A \) onto its restriction to \( A^\alpha \).
Theorem 4.4: (Royden, [16])

{ Spec A, r^'} is an inverse limit of the inverse system
{ Spec A^a, r^_a^}.  

Proof: Clearly, \( \lim. \text{inv.} \{ \text{Spec } A^a, r^a_\alpha \} = \prod_a \text{Spec } A^a \), since the \( r^a_\alpha \) are simply restriction maps, and

\[ r_\alpha : \prod_a \text{Spec } A^a \to \text{Spec } A^a \text{ is simply the } \alpha \text{-th projection map.} \]

Define \( r : \text{Spec } A \to \prod_a \text{Spec } A^a \) by letting the \( \alpha \)-th coordinate of \( r(m) \) be the restriction of \( m \) to \( A^a \), for each \( m \) in \( \text{Spec } A \). This map is easily seen to be a homeomorphism which makes the above diagram commutative. Since the inverse limit of an inverse system of topological spaces and continuous maps is unique up to commuting homeomorphism, we can consider

\( \{ \text{Spec } A, r^' \} \) as the inverse limit of \( \{ \text{Spec } A^a, r^a_\alpha \} \).
We now show how we can replace the scheme

\[ i_{\alpha} : \Lambda \rightarrow \Lambda^\alpha \]

by the equivalent scheme

\[ i_{\beta} : \Lambda \rightarrow \Lambda^\beta \]

\[ r_{\alpha} : \text{Spec}(A) \rightarrow \text{Spec} A^\alpha \]

\[ r_{\beta} : \text{Spec}(A) \rightarrow \text{Spec} A^\beta \]

\[ \Lambda(K_{\alpha}) \rightarrow \Lambda^{\alpha}(K_{\alpha}) \]

\[ \Lambda(K) \rightarrow \Lambda^{\beta}(K_{\beta}) \]

\[ K \rightarrow K^{\alpha} \]

\[ K \rightarrow K^{\beta} \]

where \( K_{\alpha} = F(\text{Spec } A^\alpha) \), \( K = F(\text{Spec } A) \)

\( A(K_{\alpha}) = \) uniform closure of the algebra of functions holomorphic in a neighbourhood of \( K_{\alpha} \) and

\( A(K) = \) uniform closure of the algebra of functions holomorphic in a neighbourhood of \( K \).
**Definition 4.5:**
Let \( \{ A^a, h^\beta_a \} \) and \( \{ B^a, k^\beta_a \} \) be two direct systems of function algebras and algebra homomorphisms with the same directed set \( D \).
An isomorphism \( \{ \lambda_a \} \) from \( \{ A^a, h^\beta_a \} \) onto \( \{ B^a, k^\beta_a \} \) is a collection of isomorphisms
\[
\lambda_a : A^a \to B^a
\]
such that for \((a, \beta) \in (D \times D)\) with \( a \leq \beta \)
the following diagram is commutative

\[
\begin{array}{ccc}
A^a & \xrightarrow{\lambda_a} & B^a \\
\downarrow{h^\beta_a} & & \downarrow{k^\beta_a} \\
A^\beta & \xrightarrow{\lambda_\beta} & B^\beta \\
\end{array}
\]

**Proposition 4.6:** Let \( \{ A^a, h^\beta_a \} \) be a direct system of algebras and algebra homomorphisms with direct limit \( \{ A, h_a \} \).
Suppose that \( f^a \in A^a \) and \( f^\beta \in A^\beta \). Then \( h_a(f^a) = h_\beta(f^\beta) \)
iff \( h_\gamma(f^a) = h_\beta(f^\beta) \) for some \( \gamma \) in \( D \) such that
\( a \leq \gamma \) and \( \beta \leq \gamma \).

**Proof:** See [4]. p.212.

**Proposition 4.7:**
Let \( \{ A^a, h^\beta_a \} \) and \( \{ B^a, k^\beta_a \} \) be two direct systems of algebras
and algebra homomorphisms with the same directed set $D$ such that
the $h^\beta_a$ and the $k^\beta_a$ are monomorphisms.

Then any isomorphism \( \{ l_a \} : \{ A^a, h^\beta_a \} \to \{ B^a, k^\beta_a \} \) induces an
isomorphism $l : \lim \text{ dir.} \{ A^a, h^\beta_a \} \to \lim \text{ dir.} \{ B^a, k^\beta_a \}$
which makes the following diagram commutative for all $a \in D$.

\[
\begin{array}{ccc}
A^a & \xrightarrow{l_a} & B^a \\
\downarrow h_a & & \downarrow k_a \\
\lim \text{ dir.} \{ A^a \} & \xrightarrow{l} & \lim \text{ dir.} \{ B^a \}
\end{array}
\]

**Proof:** Clearly, the $h_a$ and the $k_a$ are monomorphisms.

Let $a \in \lim \text{ dir.} \{ A^a, h^\beta_a \}$. Then for some $a$, $h^{-1}_a(a) \in A^a$.

Define $l(a) = (k_a \circ l_a \circ h^{-1}_a)(a)$. Such a definition clearly
makes the diagram of the statement commutative. We must show,
however, that $l$ is well-defined i.e. that $l(a)$ is independent
of the choice of $a$. 

Suppose $h^{-1}_\alpha(a) \in A^\alpha$ and $h^{-1}_\beta(a) \in A^\beta$. We must show that

$$(k_{\alpha} \circ l_{\alpha} \circ h_{\alpha}^{-1})(a) = (k_{\beta} \circ l_{\beta} \circ h_{\beta}^{-1})(a).$$

By 4.6, there is a $\gamma \in D$ with $\alpha \leq \gamma$ and $\beta \leq \gamma$ such that

$$(h_{\gamma} \circ h_{\alpha}^{-1})(a) = (h_{\gamma} \circ h_{\beta}^{-1})(a).$$

Therefore

$$(l_{\gamma} \circ h_{\alpha}^{-1})(a) = (l_{\gamma} \circ h_{\beta}^{-1})(a)$$

and so

$$(k_{\alpha} \circ l_{\alpha} \circ h_{\alpha}^{-1})(a) = (k_{\beta} \circ l_{\beta} \circ h_{\beta}^{-1})(a).$$

Hence $(l_{\alpha} \circ h_{\alpha}^{-1})(a) \in B^\alpha$ and $(l_{\beta} \circ h_{\beta}^{-1}) \in B^\beta$

have a common successor in $B^\gamma$, and therefore

$$(k_{\alpha} \circ l_{\alpha} \circ h_{\alpha}^{-1})(a) = (k_{\beta} \circ l_{\beta} \circ h_{\beta}^{-1})(a),$$

which was to be shown.

The map $l$ is thus well-defined and is clearly an isomorphism.
Corollary 4.8: Let $A$ be a function algebra containing a 1:1 function $g$ and let $\{A^{\alpha}, i^\beta_{\alpha}\}$ be the direct system of subfunction algebras described earlier. Then $\{P(K^\alpha_{\alpha}), i^\beta_{\alpha}\}$ is also a direct system ($i^\beta_{\alpha}$ denotes inclusion in both cases) and $A$ is isometrically isomorphic to $\lim\text{ dir.} \{P(K^\alpha_{\alpha})\}$.

Proof: The collection $A^\alpha \rightarrow P(K^\alpha_{\alpha})$ is an isomorphism, in fact an isometric isomorphism. The result then follows from the previous proposition.

If we set $K = \bigcap_{f \in A} R(f)$ then $\lim\text{ dir.} \{P(K^\alpha_{\alpha}), i^\beta_{\alpha}\}$ is isometrically isomorphic to $P(K)$, so that 4.8 agrees with our general result that any function algebra $A$ is isometrically isomorphic to $P(K)$.

We now turn to the dual situation of the inverse system
Definition 4.9: Let \( \{X_\alpha, r^\beta_\alpha\} \) and \( \{Y_\alpha, s^\beta_\alpha\} \) be two systems of compact Hausdorff spaces and continuous maps with the same directed set \( D \). A homeomorphism \( \{e_\alpha\} \) from \( \{X_\alpha, r^\beta_\alpha\} \) onto \( \{Y_\alpha, s^\beta_\alpha\} \) is a collection of homeomorphisms \( g_\alpha : X_\alpha \to Y_\alpha \) such that the following diagram is commutative for all \( (\alpha, \beta) \in D \times D \) with \( \alpha \leq \beta \).

\[
\begin{array}{ccc}
X_\alpha & \xrightarrow{g_\alpha} & Y_\alpha \\
\uparrow r^\beta_\alpha & & \uparrow s^\beta_\alpha \\
X_\beta & \xrightarrow{g_\beta} & Y_\beta
\end{array}
\]

The proofs on direct systems are easily dualized to yield the following assertions, whose proofs we omit.

Proposition 4.10: Let \( \{X_\alpha, r^\beta_\alpha\} \) and \( \{Y_\alpha, s^\beta_\alpha\} \) be two inverse systems of compact Hausdorff spaces such that the \( r^\beta_\alpha \) and the \( s^\beta_\alpha \) are homeomorphisms. Then any homeomorphism \( \{e_\alpha\} : \{X_\alpha, r^\beta_\alpha\} \to \{Y_\alpha, s^\beta_\alpha\} \) induces a homeomorphism \( g : \lim \text{inv. } \{X_\alpha\} \to \lim \text{inv. } \{Y_\alpha\} \) which makes the following diagram commutative for all \( \alpha \in D \).

\[
\begin{array}{ccc}
X_\alpha & \xrightarrow{g_\alpha} & Y_\alpha \\
\uparrow r^\beta_\alpha & & \uparrow s^\beta_\alpha \\
\lim \text{inv.} \{X_\alpha\} & \xrightarrow{g} & \lim \text{inv.} \{Y_\alpha\}
\end{array}
\]
Theorem 4.11: Let $K_a = F(\text{Spec } A^a)$ and $r^\beta_a : K_\beta \to K_a$ be the projection onto $K_a$ for $a < \beta$. Then $\{K_a, r^\beta_a\}$ is an inverse system of compact Hausdorff spaces and continuous maps, and \( \lim \text{ inv. } \{K_a, r^\beta_a\} \) is homeomorphic to Spec $A$.

Topologies on the spectra of a direct system.

We started off with a function algebra $A$ containing a 1:1 function $g$ on a compact Hausdorff space $X$, and then formed the direct system of finitely generated sub-function algebras containing $g$, $\{A^a, i^\beta_a\}$. Since $A^a$ is defined on $X$, Spec $A^a$ contains $X$ for each $a \in D$. However, we know that Spec $A^a$ has the weak topology induced on it by $A^a$, and so the relative topology on $X$ changes as $a$ changes. Since the topology on Spec $A$, which is homeomorphic to $\lim \text{ inv. } \{\text{Spec } A^a\}$, is the weak topology induced by $A$, which is the direct limit of the system $\{A^a, i^\beta_a\}$, and the latter equals $\cup A^a$, it follows that the open sets of Spec $A$ determine the topology of $X$ in such a way that the topology on $X$ is just the union of all the topologies induced on $X$ by the Spec $A^a$, $a \in D$.

Definition 4.12: Let $A$ be a function algebra on a compact Hausdorff space $X$.

$A$ is antisymmetric iff $f$ is in $A$ and $f$ is real-valued imply that $f$ is constant.
A is pervasive iff \((A\cap F)^\prime = C(F)\) for every proper closed subset \(F\) of \(X\).

A is analytic iff the only element of \(A\) which vanishes on a nonvoid open set in \(X\) is the zero function.

A is essential iff \(A\) contains no non-zero ideal of \(C(X)\) (this is equivalent to the assertion that there is no proper closed subset \(F\) of \(X\) such that every continuous function vanishing on \(F\) belongs to \(A\)).

A is a maximal subalgebra of \(C(X)\) iff \(A \neq C(X)\) and \(A\) is contained properly in no proper closed subalgebra of \(C(X)\).

We have the following theorem relating the above properties of a function algebra.

**Theorem 4.13:**
For a maximal subalgebra \(A\) of \(C(X)\), the following are equivalent.

1. \(A\) is pervasive
2. \(A\) is analytic
3. \(A\) is an integral domain
4. \(A\) is essential.

**Proof:** See [4] p.175.

**Theorem 4.14:** Let \(A\) be a function algebra on \(X\) containing
a 1:1 function \( g \). Then \( A \) is analytic iff each of its sub-function algebras of the form \( A^a = [g, f_1, \ldots, f_k] \) is analytic.

Proof: Suppose that \( A \) is analytic and that \( f \in A^a \) vanishes on the open set \( U \), which is open in the weak topology on \( X \) induced by \( A^a \). By the above discussion on the topologies on the spectra of the direct system \( \{A^a, i_a^\beta \} \) it follows that \( U \) is also open with respect to the topology on \( X \) induced by \( A \). Hence \( f \) vanishes on \( X \).

Conversely, suppose that \( A^a \) is analytic for each \( a \in D \).

Let \( f \in A \) vanish on the set \( U \) open in the weak \( A \)-topology. Then for some \( a \in D \), \( U \) is open in the weak \( A^a \)-topology on \( X \) and hence \( f \) vanishes identically on \( X \).

**Corollary 4.15:**

If \( A \) is a maximal subalgebra of \( C(X) \) containing a 1:1 function \( g \), then the following are equivalent.

1. \( A \) is pervasive
2. \( A \) is analytic
3. \( A \) is an integral domain
4. \( A \) is essential
5. \( A^a \) is analytic for each \( a \in D \).


**Theorem 4.16:** (Identity theorem)

If \( f(z) \) and \( g(z) \) are holomorphic functions in a connected open
set $D$ contained in $\mathbb{C}^n$ and if $f(z) = g(z)$ for all points $z$ in a nonempty subset $U$ contained in $D$, then $f(z) = g(z)$ for all points $z$ in $D$.


**Corollary 4.17:** Let $D$ be an open connected subset of $\mathbb{C}^n$, and let $f(z)$ be holomorphic in $D$. If $f$ vanishes on an open subset $U$ of $D$ then $f$ vanishes identically on $D$.

**Theorem 4.18:** Let $X$ be any topological space. The union of any family of connected subsets having at least one point in common is also connected.

**Proof:** See [17] p.108.

**Lemma 4.19:** Let $A = [f_1, \ldots, f_n]$ be a finitely generated function algebra on a connected compact Hausdorff space $X$. Then $K = F(\text{Spec } A)$ is contained in a connected neighbourhood of $\mathbb{C}^n$.

**Proof:** By 2.13 we saw that $K$ itself is a connected subset of $\mathbb{C}^n$. We can cover the topological boundary of $K$ with overlapping connected neighbourhoods whose union, say $L$, will be connected by 4.18. By the same theorem $K \cup L$ will be a connected neighbourhood of $K$. 
Theorem 4.20: If $A$ is a finitely generated function algebra on a connected compact Hausdorff space $X$, then $A$ is analytic.

Proof: Suppose that $A = [f_1, \ldots, f_n]$. Then $A$ is isometrically isomorphic to $A(K)$, the uniform closure of all functions holomorphic in a neighbourhood of $K = \bigcap_{i=1}^n \overline{R(f_i)}$. By the previous lemma $K$ is contained in a connected open set $L$. Suppose $f$ in $A$ vanishes on the open set $U$ in $X$. To $f$ in $A$ there corresponds a holomorphic function $h$ in $A(K)$ which vanishes on the corresponding set $U' = F(U)$ in $K$. By choosing a suitable $V$ open in $\mathbb{C}^n$ and intersecting it with $K$ we can assume that $U'$ is also open in $L$ and that by continuity, $h$ vanishes on $U'$. Hence by the Identity theorem, $h$ vanishes on $L$ and hence on $K$. Thus $f$ vanishes on $X$.

The previous theorem can be generalized to function algebras containing a $1:1$ function.

Theorem 4.21: Let $A$ be a function algebra containing a $1:1$ function $g$ on the compact Hausdorff space $X$. If $X$ is also connected, then $A$ is analytic.

Proof: $A$ is the direct limit of its finitely generated sub-function algebras $A^\alpha = [g, f_1, \ldots, f_k]$, and Spec $A$ is the inverse limit of the Spec $A^\alpha$. We claim that for every $\alpha$, Spec $A^\alpha$ is connected. Suppose not. Then for some $\alpha$ there exists an
open set \( U_\alpha \subset \text{Spec} \ A^\alpha \) such that \( \text{Spec} \ A^\alpha = U_\alpha \cup (\text{Spec} \ A^\alpha - U_\alpha) \).

By Shilov's theorem on idempotents, there exists an \( f_\alpha \in A^\alpha \) such that \( \hat{f}_\alpha = 0 \) on \( U_\alpha \) and \( \hat{f}_\alpha = 1 \) on \( \text{Spec} \ A^\alpha - U_\alpha \). But \( \hat{f}_\alpha \) also belongs to \( \hat{A} \) and \( U_\alpha \) is also a member of the topology on \( \text{Spec} \ A \). This would imply that \( \text{Spec} \ A \) is disconnected, a contradiction to the fact that by 2.13 \( \text{Spec} \ A \) must be connected.

By 4.20 we have that \( A^\alpha \) is analytic for each \( \alpha \) and by 4.14 it follows that \( A \) itself is analytic.
Chapter I: Gelfand in [19] and [20] originated the study of Banach algebras. Proposition 1.2 and theorems 1.4 and 1.5 are ultimately due to him, though we have followed Wermer's presentation of them in [1]. The early work on the correspondence between function algebras and algebras of polynomials in 1.8 - 1.12 was done by Shilov. See [13] and [1]. Polynomial and rational convexity were first thoroughly investigated by Stolzenberg in [12]. Oka's polynomial approximation theorem 1.15 is actually due to Shilov [13]. Proposition 1.18 is a bit of folklore - the proof we have given is to be found in [6] - and we have used it to prove theorem 1.19. The example of a finitely generated function algebra discussed in 1.20 is to be found in [4], though it appears to be based on a similar example in a more abstract setting due to Rossi [21].

Rickart in [6] recently defined the concept of a holomorphic function of an infinite number of complex variables and generalized many finite-dimensional results to the infinite-dimensional case. Lemmas 1.23 and 1.25 are due to him and he used them to prove his generalized Oka polynomial approximation theorem 1.26. We have used Rickart's results to obtain in proposition 1.27, corollary 1.28 and theorem 1.29 extensions
of earlier results on finitely generated function algebras to the arbitrarily generated case.

Chapter II: Theorem 2.2 was proved by Shilov in [22]. Peak points were first investigated by Bishop [10] and theorem 2.5 was the outcome. In the same paper Bishop gave an elegant constructive proof of his characterization of the minimal boundary, theorem 2.8, which we have rewritten in a different form for finitely generated function algebras in corollary 2.9. Using Bishop's characterization of the minimal boundary and Rickart's lemma 1.25 we have obtained theorem 2.10. Shilov's idempotent theorem 2.11 is another result from [13].

The solution of the Cousin I problem on an open polynomial polyhedron (theorem 2.17) is due to Oka [23] who based his work on the earlier efforts of Cousin [24]. The local peak point theorem 2.19 was first proved by Rossi in [11], though we have given Stolzenberg's proof of the same result for the finitely generated case in [12]. Theorem 2.20 is an extension to arbitrary function algebras of theorem 2.19 and was also first proved by Rossi in [11]. However, the proof we have given is based on Rickart's lemma 1.25. Theorems 2.22, 2.23 and 2.24 are the climax of Rossi's seminal paper, [11]. We have presented them as in [4].
Chapter III: Lemma 3.2 is one of the principle applications of the local maximum modulus principle, which is often invoked in the investigation of extensions of function algebras. Many authors have studied such extensions. Glicksberg in [25] provided another very useful tool, theorem 3.5. The statements about the Shilov boundaries in theorems 3.6 and 3.8 are also due to him. See [26]. The statement about the maximal ideal space in theorem 3.6 is due to Rickart [27], though the proof given here is an unpublished argument of Quigley. The statement about the maximal ideal space in theorem 3.8 appears to have been discovered first by Björk [28]. Corollary 3.9 is due to Stolzenberg [29]. We have followed Gamelin [9] in our presentation of the above-mentioned results. In a recent paper [18] Gamelin and Wilken extended Stolzenberg's corollary 3.9 to obtain theorem 3.11.

Wilken in [30] investigated under what conditions the maximal ideal space of an extended function algebra is strictly larger than the maximal ideal space of the original algebra and produced theorem 3.12, lemma 3.13 and theorem 3.15. Theorem 3.14 was proved by Lavrentiev [31]. Corollary 3.16 is a partial converse that we have obtained to theorem 3.12.

Theorem 3.18 is an approximation theorem due to Mergelyan [32] and theorem 3.19 was essentially proved by Runge [33]. We have used all the foregoing results of this
Chapter to obtain theorem 3.20, much of which was suggested by our supervisor Dr. W. Kotzé. Proposition 3.22 appears as problem 5.4 no. 5 in [4] and we have used it to obtain corollary 3.23.

Chapter IV. Except for proposition 4.3, theorem 4.4, proposition 4.6, theorem 4.13, theorem 4.16, corollary 4.17 and theorem 4.18, the results in this chapter are apparently new. Theorem 4.4 was essentially proved by Royden in [16] and theorem 4.13 is to be found in the paper by Hoffman and Singer [34].
BIBLIOGRAPHY.


17. Dugundji, J.D. Topology; Allyn and Bacon, Boston. 1966.


20. Gelfand, I. Normierte Ringe; Mat. Sbor. 9(1941), 3-24.


33. Runge, C. Zur theorie der eindeutigen analytischen Funktionen; Acta Math. 6(1885), 229 - 244.
