THE ELASTIC RIGIDITIES OF RIBBED PLATES

by

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SYNOPSIS

This thesis deals with the flexural and twisting rigidities of orthotropic plates. Asymmetrically stiffened plates are examples of technically orthotropic plates and are found in civil engineering structures such as bridges and ribbed floor slabs. The formulae for the rigidities as found in the literature are reviewed.

The writer has devised an experimental method of determining the rigidities. This involves a bending test using the Moiré technique and a twisting test. In the writer's method only one model is needed to determine all the rigidities. This is compared to other methods for which two or three models are needed.

Sixteen different models of asymmetrically stiffened plates with ribs running in one direction only were made. The flexural rigidities of eight of these models and the twisting rigidities of all sixteen models were determined. The objective of the tests was to compare the test values with the theoretical values. More specifically the objective can be defined as follows.

a. To determine how well an asymmetrically stiffened plate behaves as an orthotropic plate, i.e. how far apart may the ribs be spaced before the plate becomes a system of plate elements and beam elements.

b. To determine the influence, if any, on the rigidities $D_x$, $D_1$ and $D_y$ of altering the spacing of the ribs of stiffened plates which have the same theoretical values of $D_x$. 
c. To determine whether there is any difference between the rigidities $D_{12}$ and $D_{21}$, which are the same for a true orthotropic plate and are usually assumed to be the same for an asymmetrically stiffened plate.

Conclusions are drawn on these points.
SYMBOLS

\[ a_{11}, a_{12}, \ldots, a_{ee} \] coefficients of deformation in \( X,Y,Z \) system

\[ a'_{11}, a'_{12}, \ldots, a'_{ee} \] coefficients of deformation in \( X',Y',Z' \) system

\[ a_1, a_3 \] constants in 8th order theory

\[ A_r \] area in Bredt formula

\[ A_y, A_z \] nodal spacings

\[ A_{11}, A_{12}, \ldots, A_{ee} \] moduli of elasticity in \( X,Y,Z \) system

\[ A'_{11}, A'_{12}, \ldots, A'_{ee} \] moduli of elasticity in \( X',Y',Z' \) system

\[ A \] rectangular plate length in \( X \)-direction

\[ B \] rectangular plate length in \( Y \)-direction

\[ b \] rib breadth

\[ C \] factor in the beam torsion constant

\[ D = \frac{Eh^3}{12(1-v^2)} \] flexural rigidity of isotropic plate

\[ D_{x}, D_{y}, D_{1}, D_{x}, D_{y}, D_{ij} \] orthotropic plate rigidities

\[ d \] rib depth below flange

\[ d_x, d_y, d_z \] infinitesimal distances in \( X,Y,Z \) system

\[ \frac{d^2w}{dx^2}, \frac{d^2w}{dy^2}, \frac{d^2w}{dz^2} \] etc. are all partial derivatives

\[ e_1, e_2, e_3, e_4 \] distances from a neutral axis

\[ E \] Young's modulus for an isotropic material

\[ E_x, E_y, E_z \] Young's moduli for an orthotropic material

\[ G = E/2(1+v) \] shear modulus for an isotropic material

\[ G_{xy}, G_{xz}, G_{yz} \] shear moduli for an orthotropic material

\[ H = D_1 + 2D_{xy} \] rigidity in the biharmonic equation

\[ h \] plate thickness

\[ I \] moment of inertia

\[ J \] beam torsion constant
\( k_x, k_y, k_{xy} \) correction factors for variation in dimensions

\( k_M \) Moiré method correction factor

\( M_x, M_y \) plate bending moments per unit length

\( M_{xy}, M_{yx} \) plate twisting moments per unit length

\( M_T \) beam twisting moment

\( p \) load

\( Q_x, Q_y \) plate shear forces per unit length

\( q \) distributed load per unit area

\( R = AY/AZ \) aspect ratio of nodal pattern

\( s \) rib spacing

\( s_x, s_y \) spacing between ribs in the X and Y directions respectively

\( u, v, w \) displacements in the X,Y,Z system

\( V \) elastic potential

\( X, Y, Z \) cartesian co-ordinate axes, where XOY is the plane of the plate, and X is the direction of greatest stiffness

\( X', Y', Z' \) cartesian co-ordinate axes, rotated with respect to X,Y,Z

\( \gamma_x, \gamma_y \) torsional rigidities in the formulae for \( M_{xy} \) and \( M_{yx} \)

\( \gamma_{xy}, \gamma_{xz}, \gamma_{yz} \) shear strains

\( \varepsilon_x, \varepsilon_y, \varepsilon_z \) normal strains

\( \sigma_x, \sigma_y, \sigma_z \) normal stresses

\( \tau_{xy}, \tau_{xz}, \tau_{yz} \) shear stresses

\( \nu \) Poisson's ratio for an isotropic material

\( \nu_x, \nu_y \) Poisson's ratios for an orthotropic plate
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CHAPTER 1

THE ORTHOTROPIC PLATE

1. DEFINITION OF AN ORTHOTROPIC PLATE

To develop theories of the structural performance of engineering materials, we assume that they possess certain idealised natural physical properties. It is assumed that the material is a perfectly elastic body which will resume its initial form after unloading. Mathematically this elastic property is described by Hooke's Law. An isotropic body is one which consists of a homogeneous material with the same mechanical properties in all directions. Such material bodies very seldom exist. Most structural materials such as steel are composed of crystals of various kinds, orientations, and elastic properties in all directions. However considering relatively small sizes of crystals and their random distribution it may be assumed that the elastic behaviour of one piece of material is expressed by the average of the elastic properties of all crystals. On the basis of this assumption the material is considered to be isotropic.

Some materials have definite differences in elastic properties in different directions. These materials are called anisotropic or aelotropic. For instance materials used in the aircraft industry such as delta wood, avia-sandwich plates, textolit and others are all anisotropic materials. Sometimes fabrication methods make it necessary to consider conditions of anisotropy for structural materials.
Some sheets of metal show a marked anisotropy depending on the direction of rolling.

In the case where a body possesses principally different elastic properties in three perpendicular or orthogonal directions, it is called orthogonally-anisotropic, or orthotropic for short. Natural timber is a typical example of such a material. The number of elastic constants needed to describe an isotropic body is two, namely, Young's Modulus, $E$, and Poisson's ratio, $\nu$. However, in an anisotropic body the number is higher with a maximum of 21.

The first and basic theoretical investigation in this field was by Cauchy who in 1828 gave generalised equations for the elasticity of anisotropic bodies. Gehring in 1860 investigated the thin anisotropic plate. Boussinesq considered equilibrium equations for anisotropic plates and bars in 1879. Voigt published a book in 1910 on the elastic properties of anisotropic crystals and found values of elastic constants. Geckeler in 1928 made further developments. All the above works were purely theoretical and were developed considering elements possessing properties of natural or true anisotropy.

A plate of constant thickness but with different elastic properties in mutually perpendicular directions is called naturally orthotropic. Structurally or technically orthotropic plates are those made usually of isotropic material but which are reinforced or stiffened in a particular direction or in mutually perpendicular directions. Examples are ribbed plates as in steel bridge decks, reinforced concrete flat and ribbed slabs as in bridges and large
buildings. The idea of the application of the theory of elasticity of orthotropic plates to reinforce concrete flat and ribbed slabs was proposed and developed by M.T. Huber in 1914.

Fig. 1.1 Examples of technically orthotropic plates.
The advantage of orthotropic plate theory used in bridge design is that maximum use is made of the strength of materials, resulting in an economical design. The progress in methods of structural analysis has assisted the development of orthotropic bridges. In a composite beam and slab system for example, provided that the elements of the grid are spaced closely enough together, the structure will act as a truly orthotropic plate, not as a system of individual elements. In steel bridges, the saving in weight by using orthotropic decks in general exceeds the extra cost of construction especially for longer spans.

Orthotropic bridges are likely to be more architecturally satisfying due to their lower weight and, therefore, cleaner lines. The desirability of building orthotropic bridges has been demonstrated, and therefore any advance in the theory is also desirable.

2. HUBER'S EQUATION

The methods of analysis of orthotropic plates in bending fall into two categories. These are numerical methods, namely, finite differences and finite elements, and mathematical methods. Huber's theory is the basis of them all although it is refined by some authors, as will be seen in Chapter 2.

Before deriving Huber's equation it is in order to list the various categories of plates as given in Ref. 2. Plates are classified as follows:

(a) Thick plates in which a triaxial state of stress is developed, which is defined by the complete set of
differential equations of the 3-dimensional theory of
elasticity. Plates for which the ratio of the thickness
h to the least dimension on plan exceeds 1/5 can be taken
as belonging to this class.

(b) Thin plates with small deflections, in which, under
deforption due to transverse loading, the membrane
stresses are very small in comparison with the flexural
stresses (up to 5%). This class may be taken to comprise
plates for which the ratio of the thickness h to the
span does not exceed 1/5 and the deflection w does not
become larger than h/5.

(c) Thin plates with large deflections (w/h > 1/5) are
caracterised by the fact that the flexural stresses are
accompanied by relatively large tensile or compressive
membrane stresses in the middle plane. These membrane
stresses significantly affect the bending moments.

(d) Membranes are plates in which, under the action of
transverse loading, bending moments are produced which
are so small that they are negligible in comparison
with the membrane stresses.

(e) Diaphragms are plate-like elements which are subjected
to loading which acts in the middle plane of the element
only.

Huber's equation only applies to thin plates with small
deflections, and is based on the following assumptions (Ref. 4):

i. The material of the plate is perfectly elastic, i.e.
the elastic response is independent of the load
intensity and the stress-strain relationship is given
by the law of Hooke.
ii. The material of the plate is perfectly homogeneous.

iii. Thickness is uniform and small in comparison with the other dimensions of the plate.

iv. Points of the plate lying initially on a normal to the plate middle-plane remain on the normal to the middle surface of the plate after bending.

v. Normal stresses in the direction transverse to the plane of the plate are negligible and the thickness of the plate does not undergo any deformation during bending.

vi. The deflections of the plate are small in comparison with its thickness; if the deflection parallel to the normal to the middle plane is denoted by $w$ then curvatures of the middle surface of the plate are given by the second partial derivatives of $w$.

vii. The deflections of the plate are such that there is no normal strain in planes tangent to the middle surface.

viii. The body forces are either disregarded or assumed to be a part of the external load.

ix. The direction of external load is perpendicular to the plane of the plate.

Consider an orthotropic plate of uniform thickness.

Let the principal directions of orthotropy lie along the $X$ and $Y$ axes of a rectangular system of coordinates $X, Y, Z$. Let the displacements of a point in the plate after bending be $u, v, w$, corresponding to the axes $X, Y, Z$. 
Fig. 1.2

With the notation in Fig. 1.2, point O' lies below point O due to assumption vii. Also if the angle through which the normal OM turns is small, we may say that

\[ u = -z \frac{dw}{dx} \quad ; \quad v = -z \frac{dw}{dy} \]
where \( G_{xy} \) is the shear modulus in the \( XY \) plane. Solving Eqs 1.3 for \( \sigma_x \) and \( \sigma_y \), we have

\[
\sigma_x = \frac{E_x \epsilon_x}{(1-\nu_x \nu_y)} + \frac{\nu_y E_x \epsilon_y}{(1-\nu_x \nu_y)}
\]

\[
\sigma_y = \frac{E_y \epsilon_y}{(1-\nu_x \nu_y)} + \frac{\nu_x E_y \epsilon_x}{(1-\nu_x \nu_y)} \quad \text{Eqs 1.4}
\]

and \( \tau_{xy} = G_{xy} \gamma_{xy} \)

substituting Eqs 1.1 into Eqs 1.4 we have

\[
\sigma_x = -z \frac{E_x}{(1-\nu_x \nu_y)} \frac{d^2 w}{dx^2} - z \frac{\nu_y E_x}{(1-\nu_x \nu_y)} \frac{d^2 w}{dy^2}
\]

\[
\sigma_y = -z \frac{E_y}{(1-\nu_x \nu_y)} \frac{d^2 w}{dy^2} - z \frac{\nu_x E_y}{(1-\nu_x \nu_y)} \frac{d^2 w}{dx^2} \quad \text{Eqs 1.5}
\]

\[
\tau_{xy} = -z 2G_{xy} \frac{d^2 w}{dx dy}
\]

The stress distribution over an element of plate is shown in Fig. 1.4.
Taking any slice of thickness \(dz\) of the block in Fig. 1.4 it can be seen that for equilibrium

\[
\tau_{xy} = \tau_{yx}
\]

summing the moments due to the stresses acting on an element of the block of plate, we find the bending and twisting moments per unit length of the plate.

\[
M_x = \int_{-h/2}^{h/2} \sigma_x z \, dz
\]

\[
= -\frac{E_x}{(1-\nu_x \nu_y)} \left[ \frac{d^2w}{dx^2} + \nu_y \frac{d^2w}{dy^2} \right] \int_{-h/2}^{h/2} z^2 \, dz
\]

\[
= -\frac{E_x}{(1-\nu_x \nu_y)} \frac{h^3}{12} \left[ \frac{d^2w}{dx^2} + \nu_y \frac{d^2w}{dy^2} \right]
\]

\[
= -D_x \left[ \frac{d^2w}{dx^2} + \nu_y \frac{d^2w}{dy^2} \right]
\]

\[\text{Eq 1.6}\]

Similarly

\[
M_y = -D_y \left[ \frac{d^2w}{dy^2} + \nu_x \frac{d^2w}{dx^2} \right]
\]

\[\text{Eq 1.7}\]

where

\[
D_y = \frac{E_y h^3}{12(1-\nu_x \nu_y)}
\]

\[
M_{xy} = -\int_{-h/2}^{h/2} \tau_{xy} \, dz = M_{yx}
\]

\[
= 2G_{xy} \frac{d^2w}{dxdy} \int_{-h/2}^{h/2} z^2 \, dz
\]

\[
= 2G_{xy} \frac{h^3}{12} \frac{d^2w}{dxdy}
\]
The equations of equilibrium of an element of plate can now be established from Fig. 1.5. $Q_x$ and $Q_y$ are the shear forces per length and $q$ is the distributed load on the element. Taking moments about the $Y$ axis, and neglecting small terms

$$Q_x = \frac{dM_x}{dx} - \frac{dM_{yx}}{dy}$$

Eq 1.9

Taking moments about the $X$ axis,

$$Q_y = \frac{dM_y}{dy} - \frac{dM_{xy}}{dx}$$

Eq 1.10

And summing the forces in the $Z$ direction

$$\frac{dQ_x}{dx} + \frac{dQ_y}{dy} + q = 0$$

Eq 1.11

Differentiating Eqs 1.9 and 1.10 and substituting into Eq 1.11 gives
\[ \frac{d^2 M_x}{dx^2} - \frac{d^2 M_y}{dxdy} + \frac{d^2 M_y}{dy^2} - \frac{d^2 M_{xy}}{dxdy} + q = 0 \quad \text{Eq 1.12} \]

\[ \therefore D_x \left[ \frac{d^4 w}{dx^4} + \nu \frac{d^4 w}{dx^2dy^2} \right] + 4D_{xy} \frac{d^4 w}{dxdy^2} + D_y \left[ \frac{d^4 w}{dy^4} + \nu \frac{d^4 w}{dx^2dy^2} \right] = q \]

\[ \therefore D_x \frac{d^4 w}{dx^4} + \left[ \nu D_x + \nu D_y + 4D_{xy} \right] \frac{d^4 w}{dx^2dy^2} + D_y \frac{d^4 w}{dy^4} = q \quad \text{Eq 1.13} \]

\[ \therefore \quad D_x = D_1 + 2D_{xy} \]

\[ \text{where} \quad H = D_1 + 2D_{xy} \]

\[ \text{and} \quad D_1 = \nu D_y = \nu D_x \]

Equation 1.13 is the differential equation of a bent orthotropic plate and is due to Huber. It can be expressed in finite difference form, or used to find a mathematical solution to any particular plate problem. The mathematical solution however is limited to a rectangular plate and has been solved for various boundary conditions. These may be clamped, simply supported, elastically supported, or free edges in various combinations. A finite difference or finite element solution can be found for a plate of any plan shape.

For all solutions it is necessary to know the values of the rigidities \( D_x, D_y, D_1 \) and \( D_{xy} \). For a true orthotropic plate this amounts to knowing the values of \( E_x, E_y, G_{xy} \) and either \( \nu_x \) or \( \nu_y \). For a technically orthotropic plate it is the geometry which changes with a rotation in the plane of the plate, not the material properties, and therefore the
problem is different.

3. **THE EQUATIONS OF THE ISOTROPIC PLATE**

For an isotropic plate of thickness h we have

\[ E_x = E_y = E \]
\[ v_x = v_y = v \]

and \[ G_{xy} = G = \frac{E}{2(1+v)} \].

Substituting these values into the moment equations 1.6, 1.7, 1.8 we have

\[ M_x = -\frac{Eh^3}{12(1-v^2)} \left[ \frac{d^2w}{dx^2} + \nu \frac{d^2w}{dy^2} \right] \]

\[ = -D \left[ \frac{d^2w}{dx^2} + \nu \frac{d^2w}{dy^2} \right] \]

\[ M_y = -D \left[ \frac{d^2w}{dy^2} + \nu \frac{d^2w}{dx^2} \right] \]

and \[ M_{xy} = 2G \frac{h^3}{12} \frac{d^2w}{dxdy} \]

\[ = 2 \frac{E}{2(1+v)} \frac{h^3}{12} \frac{d^2w}{dxdy} \]

\[ = \frac{Eh^3}{12(1-v^2)} \cdot (1-v) \cdot \frac{d^2w}{dxdy} \]

\[ = D(1-v) \cdot \frac{d^2w}{dxdy} \]

\[ = M_{yx} \]

Here \[ D = \frac{Eh^3}{12(1-v^2)} \]

is known as the flexural rigidity of the isotropic plate.
The differential equation of the isotropic plate becomes

\[
\frac{d^4w}{dx^4} + 2 \frac{d^4w}{dx^2 dy^2} + \frac{d^4w}{dy^4} = \frac{q}{D}
\]

since \( D_x = D_y = D \)

and

\[
H = D_1 + 2D_{xy} = vD + D(1-v)
\]

\( = D \)

The isotropic plate is thus a special case of the orthotropic plate. For a solution of an isotropic plate only two elastic constants must be known, namely \( E \) and \( v \). For an orthotropic plate however there are four unknowns, namely \( D_x, D_y, D_1 \) and \( D_{xy} \). Formulae for these for various technically orthotropic plates due to different authors are given in Chapter 2.
CHAPTER 2

THEORETICAL VALUES OF THE ORTHOTROPIC PLATE CONSTANTS

1. GENERAL

In this chapter recommended formulae for the rigidities \( D_x, D_y, D_1, D_{xy} \) and \( H \) are surveyed. These depend on the various authors' opinions and on the different types of technically orthotropic plates which they are discussing. Although only one type of plate was tested experimentally in this thesis, rigidity formulae are given below for other types as well since they are often relevant. Formulae are given under the author's name, and a summary appears at the end of the chapter. Formulae will be translated into the writer's symbols, and the \( X \) direction is always taken as the direction of greatest flexural stiffness, \( D_x \).

Summarizing the formulae for a true orthotropic plate of uniform thickness \( h \), obtained in Chapter 1,

\[
D_x = \frac{E_x I}{(1-v_x v_y)}
\]

\[
D_y = \frac{E_y I}{(1-v_x v_y)}
\]

\[
D_1 = v_x D_y = v_y D_x
\]

\[
D_{xy} = G_{xy} \frac{h^3}{12}
\]

Eqs 2.1

where \( I = \frac{h^3}{12} \) = the moment of inertia for a beam of
rectangular section, depth h, and unit width.

In general for technically orthotropic plates, $v_x$ and $v_y$ are not true Poisson's ratios, since they depend on the geometry of the structure as well as the material properties. They are defined as

$$v_x = \frac{D_{12}}{D_y}; \quad v_y = \frac{D_{21}}{D_x}$$

where $D_{12}$ and $D_{21}$ are the rigidities in the formulae

$$M_x = -D_x \frac{d^2w}{dx^2} - D_{12} \frac{d^2w}{dy^2}$$
$$M_y = -D_y \frac{d^2w}{dy^2} - D_{21} \frac{d^2w}{dx^2}$$

Most authors assume that

$$D_1 = D_{12} = D_{21}$$

which is true for a true orthotropic plate.

It is necessary to define two more rigidities, namely $\gamma_x$ and $\gamma_y$, in order to avoid confusion in this chapter. They are defined by

$$M_{xy} = \gamma_x \frac{d^2w}{dx dy} \quad \text{Eqs 2.2}$$
$$M_{yx} = \gamma_y \frac{d^2w}{dx dy}$$

For a true orthotropic plate

$$\gamma_x = \gamma_y = 2D_{xy} = 2G_{xy} \frac{h^3}{12}$$

For a technically orthotropic plate, in general

$$\gamma_x \neq \gamma_y$$

Therefore, from Eqs 1.12, 1.13

$$2H = 2D_1 + \gamma_x + \gamma_y \quad \text{Eq 2.3}$$
and we now define $D_{xy}$ for a technically orthotropic plate as

$$2D_{xy} = \frac{\gamma_x + \gamma_y}{2}$$

Eq 2.4

This distinction between $D_{xy}$, $\gamma_x$ and $\gamma_y$ is not always drawn in the literature; sometimes $2D_{xy}$ is used when $\gamma_x$ is meant.

2. FROM TIMOSHENKO AND WOINOWSKY-KRIEGER

Timoshenko and Woinowsky-Krieger (Ref. 1) only devote a short section to orthotropic plates, giving values for the rigidities and a mathematical solution for a rectangular plate simply supported on all four sides. All mathematical solutions are complex, and as an illustration, the solution in Ref. 1 is given below.

![Fig. 2.1](image)

The solution is for a uniformly distributed load, $q$, over the whole plate. Taking the axes as shown in Fig. 2.1, and representing the load by a double trigonometric series, Eq 1.13 becomes
The deflection is found to be

$$w = \frac{16}{\pi} q \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} \frac{1}{mn} \sin \frac{m\pi x}{A} \sin \frac{n\pi y}{B}$$

The curvatures, moments, etc. can be derived from this expression for any point in the plate. The impression that the writer has gained of these mathematical solutions is that sometimes the series converge slowly and sometimes not.

On the subject of rigidity formulae for various types of plate, Timoshenko says at the outset that, in particular, all values of torsional rigidity $D_{xy}$ based on purely theoretical considerations should be regarded as a first approximation, and a direct test is recommended to obtain more reliable values. Various types of plate mentioned are as follows:

Plates with many stiff ribs

A. Symmetrically stiffened plates

The following theory is taken from Leknitskii (Ref. 5)
For an isotropic plate reinforced on two sides by parallel stiff ribs

\[
D_x = \frac{E h^3}{12(1-v^2)} + \frac{E' I}{s}
\]

\[
D_y = \frac{E h^3}{12(1-v^2)} = D
\]

Eqs 2.5

\[H = D\]

where I is the moment of inertia of a pair of ribs about an axis in the middle plane and \(E'\) is the Young's modulus for the rib material. The formula for \(D_x\) comes from the equation for the moment per unit length, as follows:

\[M_x = \int_{-h/2}^{h/2} \sigma_x z \, dz + 2 \int_{h/2}^{H} \sigma_x \frac{h}{s} z \, dz\]

\[\therefore M_x = -\int_{-h/2}^{h/2} \frac{E}{(1-v^2)} \left[ \frac{d^2 w}{dx^2} + v \frac{d^2 w}{dy^2} \right] z^2 \, dz\]

\[-2 \int_{h/2}^{H} \frac{E b d^2 w}{s} \frac{dz}{dx^2} z^2 \, dz\]
In the Y-direction the stiffeners are assumed to have no effect on the bending. The \( M_y \) equation is therefore the same as for an isotropic plate.

\[
M_y = -\frac{Eh^3}{12(1-v^2)} \frac{d^2w}{dx^2} - \frac{vEh^3}{12(1-v^2)} \frac{d^2w}{dy^2}
\]

\[
= -D_x \frac{d^2w}{dx^2} - D_{12} \frac{d^2w}{dy^2}
\]

Here \( D_{12} = D_{21} = D_1 = ud \)

\( H = D_1 + 2D_{xy} \)

\( = ud + D(1-v) = D \)

Notice that he is assuming that \( D_{xy} \) is the torsional stiffness of the plate alone; the stiffness of the ribs is ignored.

For plates reinforced by two sets of mutually perpendicular ribs, symmetrically distributed on both sides of the middle plane, the following formulae are given

\[
D_x = \frac{Eh^3}{12(1-v^2)} + \frac{E'I_x}{s_x}
\]

\[
D_y = \frac{Eh^3}{12(1-v^2)} + \frac{E''I_y}{s_y}
\]

\( H = \frac{Eh^3}{12(1-v^2)} = D \)
It is assumed that the axes of the ribs are parallel to the principal directions; $E'$ and $E''$ are the Young's moduli for the ribs parallel to $X$ and to $Y$; $I_x$ and $I_y$ are the moments of inertia of the ribs about lines in the middle plane of the plate; $s_x$ and $s_y$ are the rib spacings.

B. Asymmetrically stiffened plates

We now consider a slab reinforced asymmetrically by equidistant ribs running in one direction only. Timoshenko states that the theory as given in Chapter 1 can only give a rough idea of the actual state of stress and strain in the slab. This is because confusion arises over the position of a neutral plane. Before proceeding to the formulae for the rigidities as given in Timoshenko, some comments on the position of neutral planes are in order.

Consider a moment $M_x$ acting on the T-section shown. If the T-section were a beam, it would bend about a neutral axis passing through the centroid of the T-section.
imagine a moment $M_y$ acting in the YZ plane. The plate part of the structure will want to bend about a neutral surface which is the same as the middle plane of the plate. Where the plate lies on top of and is part of the beam, the neutral surface will move down in some manner. However, the bending in the X and Y directions is not independent.

Consider, for example, an element of material at the level of the middle plane of the plate. $M_x$ would apply a stress $\sigma_x$ on the element causing a compression $\varepsilon_x = \frac{\sigma_x}{E}$ and an extension $\varepsilon_y = -v \frac{\sigma_x}{E}$ if the element were unrestrained in the Y direction. The strain $\varepsilon_y$ affects the neutral surface for bending in the YZ plane. Similarly $M_y$ affects the neutral axis for bending in the XZ plane. The true position of the neutral surfaces might be something like those shown in Fig. 2.4.

It is not necessary that the two neutral surfaces be the same. This would mean that at any point in the structure where the strain $\varepsilon_x = 0$, the strain $\varepsilon_y$ is also zero. If the surfaces were the same, it would imply that $D_x = D_y$ as is proved below.
Consider a piece of irregular plate (Fig. 2.5) which has its surface bounded by lines parallel to the X direction. Consider a length A in the Y direction and unit length in the X direction. Give it an assumed neutral surface.

![Diagram of an irregular plate](image)

**Fig. 2.5**

Divide it into elementary strips of width dy, depth h, and let the distance from the centroid of a strip to the neutral surface be z. Then,

\[ I_x = \int_0^A \left[ \frac{h^3}{12} + h z^2 \right] dy \]

\[ \therefore \frac{I_x}{A} = \frac{1}{A} \int_0^A \left[ \frac{h^3}{12} + h z^2 \right] dy = \bar{I}_x \]

= a moment of inertia per unit length of the cross section

Also \( I_y = 1 \left[ \frac{h^3}{12} + h z^2 \right] \)
\[ I_y = \frac{1}{A} \int_A \left[ \frac{h^3}{12} + hz^2 \right] \, dy \]

= average moment of inertia in the Y direction, per unit length

= \bar{I}_x

\[ \therefore \quad \bar{I}_y = \bar{I}_x \]

A common neutral surface implies equal moments of inertia in the X and Y directions, i.e. \( D_x = D_y \).

By experiment \( D_x \neq D_y \), therefore the neutral surfaces are not the same.

This also implies that one cannot estimate the value of \( D_y \) of the section shown in Fig. 2.6, for example, by taking sections parallel to the XZ plane and averaging the results. Formulae for \( D_x \) and \( D_y \) of sections such as this are given in Ref. 4.

Fig. 2.6

To return to Timoshenko, the rigidity formulae (proposed by Huber) for an asymmetrically stiffened plate are as follows
Let $I_x$ be the moment of inertia of a T-section of width $s$, and $\alpha = h/(h+d)$.

Then $D_x = \frac{EI_x}{s}$

$$D_y = \frac{Eh^3}{12} \cdot \frac{s}{(s-b+\alpha^3b)}$$

$D_1 = 0$ \hspace{1cm} \text{Eqs 2.7}

$$2D_{xy} = D(1-\nu) + \frac{GJ}{2s}$$

In the formula for $D_x$, the plate contribution to $I_x$ is not divided by $(1-\nu^2)$. The explanation for $D_y$ is given in Section 2 on Huber. In general the formulae for $D_x$, $D_y$ and $D_1$ neglect the effect of Poisson's ratio.

In the formula for $D_{xy}$, $J$ is the torsion constant of one rib. The slab and the rib are assumed to resist the torsional moment separately. No additional stiffness due to their continuity is allowed. $J$ is defined as in the usual theory of twisting of beams (see for example Ref. 6) where, if $\phi$ = the angle of twist per unit length of the beam and $M_T$ = the twisting moment.
Fig. 2.8

\[ M_T = G.J.e \]  \hspace{1cm} \text{Eq. 2.8}

For a rectangular beam cross-section

\[ J = b^3d.C \quad (b < d) \]
\[ J = bd^3.C \quad (d < b) \]

where \( C \) is a constant depending on the ratio \( b : d \). When \( b = d \), \( C = 0.1406 \). As the ratio \( b/d \) increases, \( C \) increases, asymptotically approaching the value \( C = 1/3 \) as \( b/d \) tends to infinity.

C. **Reinforced Concrete Slabs** (due to M.T. Huber)

Let \( E_s \) = Young's modulus of steel
\[ E_c = \text{Young's modulus of concrete} \]
\[ \nu_c = \text{Poisson's ratio of concrete} \]
\[ n = \frac{E_s}{E_c} \]

For a slab with two way reinforcement in the directions \( X \) and \( Y \) it is assumed that
\[ D_x = \frac{E_c}{(1-\nu_c^2)} \left[ I_{cx} + (n-1)I_{sx} \right] \]
\[ D_y = \frac{E_c}{(1-\nu_c^2)} \left[ I_{cy} + (n-1)I_{sy} \right] \]

Eqs 2.9

\[ D_1 = \nu_c \cdot \sqrt{D_x D_y} \]

\[ D_{xy} = \frac{(1-\nu_c)}{2} \cdot \sqrt{D_x D_y} \]

where \( I_{cx} \) is the moment of inertia of the slab material
\( I_{sx} \) that of the reinforcement taken about the neutral
axis in the section \( x = \text{constant} \)

\( I_{cy}, I_{sy} \) are the respective values for the section
\( y = \text{constant} \).

\[ H = D_1 + 2D_{xy} = \sqrt{D_x D_y} \]

Eq 2.10

Eq 1.13 now becomes

\[ D_x \frac{d^4w}{dx^4} + 2 \sqrt{D_x D_y} \frac{d^4w}{dx^2dy^2} + D_y \frac{d^4w}{dy^4} = q \]

By introducing a new variable

\[ y_1 = y \sqrt{D_x / D_y} \]

Eq 1.13 reduces to

\[ \frac{d^4w}{dx^4} + 2 \frac{d^4w}{dx^2 dy_1^2} + \frac{d^4w}{dy_1^4} = \frac{q}{D_x} \]

which can be solved as an isotropic plate. There was
therefore an end in view in choosing the formulae for \( D_1 \) and \( D_{xy} \). The explanation for their choice, which is not a strictly theoretical derivation, is as follows.

For an isotropic plate

\[
M_{xy} = D(1-v) \cdot \frac{d^2w}{dx dy} = M_{yx}
\]

For a technically orthotropic plate, however, in general \( M_{xy} \neq M_{yx} \) since the torsional rigidities will be different in the \( X, Y \) directions. As a reasonable approximation for the rigidities consider the expression for an isotropic plate and substitute for the values of \( D \) and \( v \) by certain middle values of \( D_x, D_y \) and \( \nu_x, \nu_y \).

Choose these values as

\[
D = \sqrt{D_x D_y}
\]

\[
v = \sqrt{\nu_x \nu_y}
\]

Then \( M_{xy} = (1-\sqrt{\nu_x \nu_y}) \sqrt{D_x D_y} \cdot \frac{d^2w}{dx dy} \)

or

\[
2D_{xy} = (1-\sqrt{\nu_x \nu_y}) \sqrt{D_x D_y}
\]

and

\[
D_1 = \sqrt{\nu_x \nu_y} \cdot \sqrt{D_x D_y}
\]

\[
\therefore H = D_1 + 2D_{xy} = \sqrt{D_x D_y}
\]

and it is assumed that

\[
\nu_c = \sqrt{\nu_x \nu_y}
\]

On the subject of the values of \( D_x \) and \( D_y \) in Eqs 2.9, it is clear that they are derived by transforming the section
to an equivalent concrete section and again assuming that
\[ v_c = \sqrt{v_x v_y} \]

However Timoshenko notes that Eqs 2.9 are not independent of
the state of the concrete. Any difference, for instance,
in the reinforcement in the directions \( x \) and \( y \) will affect
the ratio \( D_x/D_y \) much more after cracking of the concrete
than before.

D. Plywood plates

A table of elastic constants for five types of plywood
is given. This is useful as a test of a formula for \( G_{xy} \)
for true orthotropic plates developed by Klöppel and Yamada
(Ref 7).

\[ G_{xy} = \frac{E_x E_y}{E_x + (1+2v_x)E_y} \quad \text{Eq 2.11} \]

This is based on purely theoretical considerations. In the
table below the test and theoretical values are compared.

Unit = \( 10^6 \) psi

<table>
<thead>
<tr>
<th>Material</th>
<th>( E_x )</th>
<th>( E_y )</th>
<th>( v_x E_y )</th>
<th>( G_{xy} )</th>
<th>( G_{xy} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maple, 5-ply</td>
<td>1.87</td>
<td>0.60</td>
<td>0.073</td>
<td>0.159</td>
<td>0.429</td>
</tr>
<tr>
<td>Afara, 3-ply</td>
<td>1.96</td>
<td>0.165</td>
<td>0.043</td>
<td>0.110</td>
<td>0.146</td>
</tr>
<tr>
<td>Gaboon (Okoume), 3-ply</td>
<td>1.28</td>
<td>0.11</td>
<td>0.014</td>
<td>0.085</td>
<td>0.099</td>
</tr>
<tr>
<td>Birch, 3- and 5-ply</td>
<td>2.00</td>
<td>0.167</td>
<td>0.077</td>
<td>0.17</td>
<td>0.144</td>
</tr>
<tr>
<td>Birch, with bakelite membranes</td>
<td>1.70</td>
<td>0.85</td>
<td>0.061</td>
<td>0.10</td>
<td>0.542</td>
</tr>
</tbody>
</table>
The agreement in general is seen to be bad. The theoretical $G_{xy}$ varies from more than five times too small to 1.2 times too big. It is not known, however, how accurate the tests were. Klöppel and Yamada probably would not claim that their formula is applicable to plywood plates as they are not true orthotropic plates. They are plates of constant thickness, however, and the table does suggest that the formula will not be valid for any technically orthotropic plates.

E. Gridworks

Orthotropic plate theory may also be applied to gridworks if the distances between the beams in both directions is small compared with the dimensions of the grid.

The beams are supported at their ends and the load is applied normal to the XY plane. If $EI_x$, $EI_y$ are the flexural
rigidities of beams parallel to the X,Y axes then

\[ D_x = \frac{EI_x}{s_x} \quad ; \quad D_y = \frac{EI_y}{s_y} \]

\( D_1 \) is taken as zero since there is no slab to provide Poisson's ratio effects. For twisting we have

\[ M_{xy} = \frac{GJ_x}{s_x} \frac{d^2w}{dx dy} \]

\[ M_{yx} = \frac{GJ_y}{s_y} \frac{d^2w}{dx dy} \]

where \( J_x, J_y \) are the torsion constants of the beams in the X,Y directions.

Substituting these expressions into the equation of equilibrium Eq 1.12

\[ \frac{d^2M_x}{dx^2} - \frac{d^2M_{yx}}{dy^2} + \frac{d^2M_{xy}}{dx dy} + q = 0 \]

We find that the D.E. for a gridwork is

\[ D_x \frac{d^4w}{dx^4} + \left[ \frac{GJ_x}{s_x} + \frac{GJ_y}{s_y} \right] \frac{d^4w}{dx^2dy^2} + D_y \frac{d^4w}{dy^4} = q \]

i.e. \( 2H = \frac{GJ_x}{s_x} + \frac{GJ_y}{s_y} \)

\[ = 4D_{xy} \]

\[ \therefore D_{xy} = \frac{GJ_x}{4s_x} + \frac{GJ_y}{4s_y} \]
F. Corrugated Plates

Plates made from isotropic material which have been given waves in one direction may be considered as approximately uniform and orthotropic if the number of waves is sufficiently large, or if the chord of the corrugations is small by comparison with the length of a side of the plate. The calculation of stiffness reduces, as for plates with parallel stiffeners, to the calculation of moments of inertia per unit length for the basic sections.

Assuming the waves to be sinusoidal

\[ z = f \sin \frac{\pi y}{L} \]

Let the arc length be \( s \), the plate thickness \( h \).

Then

\[ D_x = EI \]

\[ D_y = \frac{L}{s} \frac{Eh^3}{12(1-v^2)} \]

\[ D_1 = 0 \]

\[ H = 2D_{xy} = \frac{s}{L} \frac{Eh^3}{12(1+v)} \]

in which approximately
3. **FROM M.T. HUBER**

Huber's work is summarised in his books (Ref. 14, 15) where he develops his well-known theory and derives various mathematical solutions to problems. His rigidities for asymmetrically stiffened plates were given under Timoshenko (section 2.) where $\nu$ was taken as zero as a first approximation. This assumption is not so bad for concrete as for steel ($\nu = 0.3$). However, the writer has seen the following ranges for $\nu$ of concrete given, namely $0.1 - 0.2; 0.15 - 0.25; 0.2 - 0.3$. These are substantial compared to steel. Rowe (Ref. 22) states that considerable errors occur in the calculation of the transverse moments in concrete bridges if $\nu$ is not taken into account.

In Ref. 14 appear the only formulae for $\nu_x$ and $\nu_y$ with a rational explanation which the writer has seen.

Fig. 2.11

As an introduction consider a true orthotropic plate.
Imagine the plate to be divided into X-strips and Y-strips of unit width, and denote the bending stiffnesses by 
\[ D'_x = E_x I \] and \[ D'_y = E_y I \]. Then we get for the curvatures of the two strips the following equations

\[ \frac{1}{r_x} = \frac{M_x}{D'_x} \quad ; \quad \frac{1}{r_y} = \frac{M_y}{D'_y} \]

where \( r_x \) and \( r_y \) are the radii of curvature. The interdependence of the strips and the plate prohibit the free lateral strain, and hence there is a change in curvature. The moment \( M_x \) has a reducing influence on the curvature of the Y-strips, and similarly \( M_y \) reduces the curvature of the X-strips. Apparently we may write by analogy with the isotropic plate

\[ \frac{1}{r_x} = \frac{M_x}{D'_x} - v_y \frac{M_y}{D'_y} \]
\[ \frac{1}{r_y} = \frac{M_y}{D'_y} - v_x \frac{M_x}{D'_x} \]

Eqs 2.12

Now consider a plate made of isotropic material but which is stiffened by ribs

[Diagram of a stiffened plate]

Fig. 2.12
Denote the bending stiffness of a unit cross-section width of an imaginary plate formed by putting ribs together by

\[ D_{y}^{r} = \frac{E(h+d)^3}{12(1-\nu^2)} \]

Denote the stiffness of the plate alone by

\[ D_{y}^{p} = \frac{Eh^3}{12(1-\nu^2)} \]

From Fig. 2.13

\[ \alpha = \frac{s}{r_{y}} \quad \text{and} \quad \frac{1}{r_{y}} = \frac{M_{y}}{D_{y}} \]

Then for a constant moment \( M_{y} = 1 \)

\[ \alpha = \frac{s}{D_{y}} = \frac{s-b}{D_{y}^{p}} + \frac{b}{D_{y}} \]
\[
D_y = D_y^P \frac{s}{D_y^P (s-b) + \frac{V}{D_y}}
\]

Eq 2.13

This is one of Eqs 2.7 as in Timoshenko (section 2.)

When \(\frac{h^3}{(h+d)3}\) is small

\[
D_y = D_y^P \frac{s}{(s-b)}
\]

Eq 2.14

Now consider the ribbed plate to be divided into X- and Y-strips as before. Due to the action of moment \(M_Y\) alone onto an element of the plate there should be caused in the Y-direction the curvature \(\frac{M_Y}{D_y^P}\)

where \(D_y^P = \frac{Eh^3}{12}\) is the beam bending stiffness of the Y-strip of plate. All stiffnesses marked with a ' are the beam bending stiffnesses. In the X-direction there should be

caused a curvature of \(-\nu \frac{M_Y}{D_x^P}\)

Now imagine the ribs to be added to the plate. In the Y-direction the curvature is reduced in the ratio \(D_y^P : D_y\), i.e. it becomes \(\frac{M_Y}{D_y}\)

In the X-direction the curvature is affected to a greater extent by the ribs, and will be reduced in ratio \(D_x^P : D_x\).

It becomes \(-\nu \frac{D_x^P}{D_x} \frac{M_Y}{D_y}\).

Using Eq 2.14 we may write it as
\[
\frac{D'_X}{D'_X} \cdot \frac{s}{s-b} \cdot \frac{M_Y}{D'_Y} 
\]

Hence \( \nu_Y = \nu \cdot \frac{D'_X}{D'_X} \cdot \frac{s}{s-b} \) \hspace{1cm} \text{Eq 2.15}

To derive a value for \( \nu_X \), imagine a moment \( M_X \) alone to act. This causes in the X-direction a curvature of \( \frac{M_X}{D'_X} \) and in the Y-direction a curvature of \(-\nu \frac{M_X}{D'_X}\)

where \( D'_X = \frac{EI}{s} \).

The curvature in the Y-direction is now reduced in the ratio of \((s-b):s\) to allow for the stiffening effect of the ribs, and becomes \(-\nu \frac{s-b}{s} \cdot \frac{M_X}{D'_X}\)

Hence \( \nu_X = \nu \cdot \frac{(s-b)}{s} \) \hspace{1cm} \text{Eq 2.16}

Eq 2.15 can be rewritten as

\[
\nu_Y = \nu \cdot \frac{h^3 s}{12 I_X} \cdot \frac{s}{(s-b)} \] \hspace{1cm} \text{Eq 2.17}

where \( I_X \) is the moment of inertia of the T-section.

Huber’s values for \( D_{12} \) and \( D_{21} \) would become

\[
D_{21} = \nu \cdot \frac{h^3 s}{12 I_X} \cdot \frac{s}{(s-b)} \cdot D_X = \nu D \frac{s}{(s-b)} \] \hspace{1cm} \text{Eq 2.18}

\[
D_{12} = \nu \cdot \frac{s-b}{s} \cdot D_Y \neq \nu D \] \hspace{1cm} \text{Eq 2.19}
Cornelius (Ref. 16) recommends that for bridges with concrete slabs and relatively small steel ribs

\[ \frac{v_x}{v_y} = \frac{D_x}{D_y} \text{ with } v_{\text{max}} = v_{\text{concrete}} \]

when the ribs become large \( v_x = v_y = 0 \).

4. FROM TROITSKY

Troitsky's book (Ref. 3) deals with the theory and design of steel bridge decks. Several methods of solution are given, one of which is to replace the deck by a theoretically orthotropic plate, which he terms the method of elastic equivalence. This is based on Huber's equation. The structural geometry of the deck may have many forms. The deck plate may be stiffened by torsionally weak ribs such as rectangular, I, channel, or angle sections, or by torsionally stiff ribs which are hollow sections of various shapes. Often the deck will have stiff ribs in one direction and weak ribs in the perpendicular direction. The deck plate may also be a concrete slab instead of a steel plate.

The way of comparing weak and stiff decks is by calculating the value of \( \alpha \), which is equal to 1 for an isotropic plate.

\[ \alpha = \frac{H}{\sqrt{D_x D_y}} \]

Examples of relatively stiff and weak decks are given in Fig. 2.14. An asterisk signifies a possible simplifying assumption in the calculations, e.g. \( D_y = 0 \) may be used if
the ratio of $D_x$ to $D_y$ is very large.

In Fig. 2.15 taken from Ref. 3, Troitsky's formulae for rigidities and moments are given.

<table>
<thead>
<tr>
<th>Cases</th>
<th>$D_x \frac{d^4w}{dx^4} + 2H \frac{d^4w}{dx^2dy^2} + D_y \frac{d^4w}{dy^4} = q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H &gt; \sqrt{D_x D_y}$</td>
<td></td>
</tr>
<tr>
<td>* $D_y = 0$</td>
<td></td>
</tr>
<tr>
<td>$H = \sqrt{D_x D_y}$</td>
<td></td>
</tr>
<tr>
<td>$H &lt; \sqrt{D_x D_y}$</td>
<td></td>
</tr>
<tr>
<td>* $H = 0$</td>
<td></td>
</tr>
<tr>
<td>* $D_y = H = 0$</td>
<td></td>
</tr>
</tbody>
</table>

Fig. 2.14
Fig. 2.15

\[
D_x = \frac{EI_x}{s_x}
\]

\[
D_y = \frac{EI_y}{s_y}
\]

\[
2D_{xy} = \frac{G}{6} \left[ \frac{b^3d}{s_x} + \frac{b_1^3d_1}{s_y} \right]
\]

\[
H = \frac{Eh^3}{12(1-\nu^2)} + 2D_{xy}
\]

\[
M_x = -D_x \left[ \frac{d^2w}{dx^2} + \nu \frac{d^2w}{dy^2} \right]
\]

\[
M_y = -D_y \left[ \frac{d^2w}{dy^2} + \nu \frac{d^2w}{dx^2} \right]
\]

\[
M_{xy} = 2D_{xy} \frac{d^2w}{dxdy}
\]

Notice that in \(D_x\) and \(D_y\) the plate portion is not divided by \((1-\nu^2)\), and that in \(M_x\) and \(M_y\), \(\nu\) is used instead of \(\nu_x\) and \(\nu_y\). In the formula for \(D_{xy}\) the torsion constant \(J\) is taken as

\[
J = \frac{1}{3} b^3d + \frac{1}{3} b_1^3d_1
\]
The constant \( C \) is therefore always taken as \( 1/3 \). This is an approximation due to Föppl (Ref. 8) which is meant to increase the stiffness to allow for the continuity between rib and slab or between the elements of the rib if, for example, the rib is an I-section. It should only be applicable to thin steel sections.

Notice also that in the \( M_{xy} \) equation the stiffness of the plate is ignored.

**Hollow sections**

![Diagram of a hollow section](image)

Fig. 2.16

For hollow sections the value of \( H \) is very different from the formula for open sections.

\[
D_x = \frac{EI_x}{s_x}, \quad D_y = \frac{Eh^3}{12(1-v^2)}
\]

\[
2H = \frac{1}{G} \sum \frac{A_x^2}{ds} \quad \text{(from Bredt formula)}
\]

\( \delta \) is the variable thickness of the wall.

\( A_r \) is the mean of the areas enclosed by the outer and inner boundaries of the cross-section of the wall.
\( s_p \) is the total perimeter of the wall measured at the centre line.

Note that again the plate portion of the value of \( D_x \) is not divided by \((1-\nu^2)\). The rib is assumed not to increase the value of \( D_y \).

**Exact theory of stiffened plates**

Orthogonally stiffened plates may be replaced by an equivalent orthotropic plate of constant thickness when the ribs are disposed symmetrically with respect to the middle plane of the plate. When the stiffeners are located only on one side of the plate the unknown location of the neutral surfaces of the bending stresses increases the complexity of determining the rigidities. The analysis of the problem can be extended to include the effect of the strain in the middle plane of the plate which produces additional shear stresses disregarded in Huber's method.

Therefore Huber's theory presents only an approximate solution of the problem. Investigations by Pfluger, Trenks, Giencke, and Massonnet resulted in the development of the exact theory of orthogonally stiffened plates. They find that the governing D.E. is of the 8th order.

\[
D_1 \frac{d^8 w}{dx^8} + D_2 \frac{d^8 w}{dx^5 dy^3} + D_3 \frac{d^8 w}{dx^3 dy^5} + D_4 \frac{d^8 w}{dx dy^7} + D_5 \frac{d^8 w}{dy^8} = q
\]

where the constants \( D_1, D_2, \ldots D_5 \) are expressed by the geometric and elastic characteristics of the plate and ribs. However the solution of this equation requires a very
laborious calculation which seems to be almost impossible.

Giencke attempted to determine rigorously the effective torsional rigidity of the stiffened plate. He found that Huber's formula

\[ 2H = \nu_y D_y + \nu_x D_x + 4D_{xy} \]

should contain additional terms to show the influence of the eccentricity of the ribs with respect to the middle plane of the plate, and called this "the apparent torsional rigidity". However, Massonnet found that this value proposed by Giencke was excessive. This is expanded on later in this chapter.

Generally, the stresses obtained by rigorous methods are somewhat lower than those obtained from Huber's approximate theory. The computation procedures involved in working with the above 8th order D.E. are considered by Troitsky to be far too complicated to be considered for practical application in design at this stage of the development of the exact theory. The simplified theory has proved its application in the practical design of bridges.

5. FROM BARES AND MASSONNET

Ref. 4 is concerned with the theory and design of concrete beam-and-slab and hollow section bridges. Elastic theory is used to derive the Guyon-Massonnet-Bares method of lateral distribution coefficients. The underlying solution is a mathematical one, resulting in tables which are used for design. To use the tables two stiffness parameters are needed. These are the parameter of lateral stiffness \( \beta \), and the torsional parameter \( \alpha \). They are defined as follows.
\[ \beta = \frac{B}{A} \cdot \frac{D_x}{D_y} \]
\[ \alpha = \frac{H}{\sqrt{D_x D_y}} \]

For grillages \( v \) is always taken as zero, but for composite beam-and-slab sections the portion of the rigidity caused by the slab should be divided by the factor \( (1-v^2) \). Their formulae are therefore
\[ D_x = \frac{E}{s_x} b d_1^3 + \frac{E}{s_x} b d e_2^2 + \frac{E}{(1-v^2)} \left[ \frac{h_3^3}{12} + h e_1^2 \right] \]

\[ D_y = \frac{E}{(1-v^2)} \frac{h^3}{12} = D \]

\[ D_1 = \nu D \]

\[ H = D_1 + 2D_{xy} = \nu D + D(1-\nu) + \frac{GJ_x}{2s_x} \]

\[ = D + \frac{GJ_x}{2s_x} \]

Notice that \( D_1 \) is not considered negligible and put equal to zero. As in Timoshenko the torsional stiffness is obtained by summing the stiffnesses of the slab and the beam acting separately.

If there are beams running in both the X and Y directions then the formulae become

\[ D_x \text{ as before} \]

\[ D_y = \frac{E}{s_y} b_1 \frac{d_1^3}{12} + \frac{E}{s_y} b_1 d_1 e_4^2 + \frac{E}{(1-v^2)} \left[ \frac{h_3^3}{12} + h e_3^2 \right] \]
At this stage it should be pointed out that there is an anomaly in the formulae for the torsional stiffness of a beam and that for a plate. This is mentioned in the literature but the writer has not come across an explanation. Broadly speaking, the anomaly lies in the fact that the stiffness of a beam is twice that of an equivalent plate section.

Consider an element of isotropic plate, thickness h.

Fig. 2.20
Isotropic plate element.
As in Chapter 1, we have

\[ \tau_{xy} = G \gamma_{xy} \]

\[ \gamma_{xy} = \frac{du}{dy} + \frac{dv}{dx} \]

\[ = -2z \frac{d^2w}{dx dy} \]

\[ M_{xy} = - \int_{-h/2}^{h/2} \tau_{xy} z \, dz \]

\[ = + \int_{-h/2}^{h/2} G.2 \frac{d^2w}{dx dy} \cdot z^2 \, dz \]

\[ = 2G \frac{h^3}{12} \frac{d^2w}{dx dy} = D(l-v) \frac{d^2w}{dx dy} \]

\[ \therefore 2D_{xy} = 2G \frac{h^3}{12} = D(l-v) \]

or \[ D_{xy} = \frac{Gh^3}{12} \]

Now consider a beam being twisted by a moment \( M_T \) through an angle \( \theta \) per unit length.

![Diagram](image-url)
From the classical theory of torsion

\[ M_T = G J \phi \]

where \( J = \frac{bh^3}{3} \) as \( b/h \to \infty \)

\( \phi \) corresponds exactly to \( \frac{d^2w}{dx dy} \)

\( M_{xy} \) is a moment per unit length in the Y direction and has the units of force; \( M_T \) is a moment with units of force x distance. Let the ratio \( b/h \) of the beam become very large and consider an element of the beam of unit length in the Y direction. To compare the two expressions for moment \( M_T \) must be divided by \( b \):

\[ \frac{M_T}{b} = \frac{G b h^3}{3b} \phi = \frac{G h^3}{3} \frac{d^2w}{dx dy} \]

\[ M_{xy} = 2 \frac{G h^3}{12} \frac{d^2w}{dx dy} = \frac{G h^3}{6} \frac{d^2w}{dx dy} \]

The conclusion is that to find the torsional stiffness of a slab or of the slab part of a beam-and-slab structure the equivalent beam stiffness must be divided by 2.

6. FROM HUFFINGTON

Huffington (Ref. 9) has done a deep study of the rigidities of symmetrically and asymmetrically stiffened plates. His application is to plates with relatively small stiffeners, i.e. the value of \( D_x/D_y \) is low. These are more common in aircraft and ship building. When we say that a technically orthotropic plate is replaced by an equivalent orthotropic plate of constant thickness, the equivalence can
be defined in several ways. It could mean that either the deflection or one of the strain components of the actual and equivalent plates be the same at some point. It could mean that the mean difference of deflections or strain components of the two plates be zero. Buffington uses a third form on which to base equivalence, namely, that the total strain energies of the two plates be equal.

A. Symmetrically stiffened plates

Determination of $D_x$

![Diagram of symmetrically stiffened plates](image)

Fig. 2.22

Considering a stiffened plates simply supported at $x = 0$ and $x = A$, and infinitely long in the $Y$ direction, loaded by a U.D.L., Buffington derives a very large expression for $D_x$. This depends on the elastic and geometric properties of the section, and on the span in the $X$ direction, $A$. However as $A$ is increased from zero, $D_x$ swiftly approaches an asymptotic value. This is

$$D_x = D + \frac{EI_s S}{S_x}$$

where $I_s$ is the moment of inertia of the pair of stiffeners with respect to the middle plane of the plate. This value of $D_x$ is the same as given by Leknitskii and others.
Determination of $D_y$

To determine $D_y$ consider the stiffened plate as being simply supported at $y = 0$ and $y = B$, and infinitely long in the $X$-direction.

![Fig. 2.23](image)

The moment-curvature relationship is

$$M_y = - D(y) \frac{d^2 w}{dy^2} = \frac{q}{2} (By - y^2)$$

where $D(y)$ is the variable flexural rigidity of the stiffened plate. A lower bound to $D_y$ is found by putting $D(y) = D$. This is a common value used by others. An upper bound to $D_y$ is found from

$$D(y) = \frac{E h(y)^3}{12(1-\nu^2)}$$

where $h(y)$ is the total thickness of the plate-stiffener combination, subject to the following limitation. When the ratio of stiffener height to stiffener width is greater than unity, a reduced height of the order of the stiffener width should be employed.

By strain energy considerations the value of $D_y$ becomes

$$D_y = \frac{bs}{30 \int_0^B \frac{(By-y^2)^2}{D(y)} dy}$$

**Eq 2.24**
The evaluation of the integral in the above equation can be tedious when the number of stiffeners is large. However $D_y$ rapidly approaches an asymptotic value as the number of stiffeners increases, as expected.

**Determination of $D_{xy}$**

The method given for this is really a reference to another method. It is not clear what is done. The formula for $D_{xy}$ is given as

$$2D_{xy} = \frac{GK}{2}$$

where $K = $ torsion constant obtained by the membrane analogy, relaxation methods, or some empirical method.

![Diagram of a section with stiffeners](image)

**Fig. 2.24**

It is not clear whether the section in Fig. 2.24 is regarded as a beam and $K$ found accordingly. The anomaly between the stiffness of beams and slabs causes most of the confusion.

**Determination of $D_1$**

A very large expression is derived for $D_1$, in which the span dimension, $A$, appears. As $A$ increases to $\infty$, $D_1$ takes on the limiting value.
\[ D_1 = vD \]

This value has been recommended by others.

B. Asymmetrically stiffened plates

For these plates, \( D_x \) may be determined as in section 5 (p. 45). For \( D_y \) Huffington says that a judicious selection of continuous functions to represent the neutral surface and the effective thickness \( h(y) \) must be made, otherwise the method is as for symmetrically stiffened plates. \( D_{xy} \) is determined as in section 1. above. \( D_1 \) requires modification but it is not stated whether the limiting case

\[ D_1 = vD \]

is materially affected.

7. FROM GIECKE

Giencke (Ref. 11) was involved in the derivation of the more exact 8th order theory already mentioned.

Fig. 2.25
Basically, it involves considering the in-plane forces in
the slab or plate as well as the other forces, $M_x$, $M_y$, $Q_x$, $Q_y$, $M_{xy}$, $M_{yx}$. Three differential equations in the
displacements $u$, $v$, and $w$ are found. These are combined
into one 8th order D.E. in terms of $w$ only. Using this
theory, Giencke proposes corrections to the 4th order theory.
The moment equations become

$$
M_x^* = -D_x \frac{d^2w}{dx^2} - \nu \left[ D + e_1 e_3 \frac{Eh}{(1-\nu^2)} \right] \frac{d^2w}{dy^2} \quad \text{Eq 2.25}
$$

$$
M_y^* = -D_y \frac{d^2w}{dy^2} - \nu \left[ D + e_1 e_3 \frac{Eh}{(1-\nu^2)} \right] \frac{d^2w}{dx^2} \quad \text{Eq 2.26}
$$

$$
M_{xy}^* = \left[ D(1-\nu) + \frac{GJ_x}{s_x} + \frac{e_1(e_1+e_3)}{2} \frac{Eh}{(1+\nu)} \right] \frac{d^2w}{dx dy} \quad \text{Eq 2.27}
$$

$$
M_{yx}^* = \left[ D(1-\nu) + \frac{GJ_y}{s_y} + \frac{e_3(e_1+e_3)}{2} \frac{Eh}{(1+\nu)} \right] \frac{d^2w}{dy dx} \quad \text{Eq 2.28}
$$

where $E$ is the modulus of the plate material if the rib and plate materials are different.

From the above equations

$$
H^* = D + \frac{GJ_x}{2s_x} + \frac{GJ_y}{2s_y} + \nu e_1 e_3 \frac{Eh}{(1-\nu^2)} + \frac{(e_1+e_3)^2 Eh}{4(1+\nu)} \quad \text{Eq 2.29}
$$

Massonnet (Ref. 12, 4) finds that Giencke's values are
too large. He finds that the values $e_1$, $e_3$ in the above
formulae should be replaced by values ranging between $e_1$, $e_3$
and zero. The actual values depend on the boundary conditions
and the loading. His values for the rigidities are (with the
notation in Figs. 2.19, 2.25)

\[ D_x^* = D_x + (e_1 - a_1)^2 E \left[ h + \frac{bd}{s_x} \right] \quad \text{Eq 2.30} \]

\[ D_y^* = D_y + (e_3 - a_3)^2 E \left[ h + \frac{b_1d_1}{s_y} \right] \quad \text{Eq 2.31} \]

\[ D_1^* = v \left[ D + a_1a_3 \frac{Eh}{(1-v^2)} \right] \quad \text{Eq 2.32} \]

\[ \gamma_x^* = D(1-v) + \frac{GJ_x}{s_x} + \frac{a_1(a_1+a_3)}{2} \frac{Eh}{(1+v)} \quad \text{Eq 2.33} \]

\[ \gamma_y^* = D(1-v) + \frac{GJ_y}{s_y} + \frac{a_3(a_1+a_3)}{2} \frac{Eh}{(1+v)} \quad \text{Eq 2.34} \]

\[ H^* = D + \frac{GJ_x}{s_x} + \frac{GJ_y}{s_y} + ve_1e_3 \frac{Eh}{(1-v^2)} + \]

\[ (a_1+a_3)^2 \frac{Eh}{4(1+v)} \quad \text{Eq 2.35} \]

where \(0 < a_1 < e_1\) and \(0 < a_3 < e_3\)

and \(a_1, a_3\) depend on the boundary conditions and the loading.

Unfortunately, without going through the 8th order theory it is impossible to give meanings to \(a_1, a_3\); they must just be accepted as terms in the equations.

For cylindrical (pure) bending in the \(X\) direction

\((M_x = \text{const}, M_y = 0)\)

\[ D_x^* = D_x \quad \quad D_y^* = D_y = D \]

\[ H^* = D + \frac{GJ_x}{2s_x} + \frac{GJ_y}{2s_y} + ve_1e_3 \frac{Eh}{(1-v^2)} + \quad \text{Eqs 2.36} \]

\[ (e_1+e_3)^2 \frac{Eh}{4(1+v)} \]
For pure twisting, i.e. \( \frac{d^2w}{dx^2} = \frac{d^2w}{dy^2} = 0 \)

and \( \frac{d^2w}{dxdy} \neq 0 \) at all points.

\[ a_1 = a_3 = 0 \]

\[ D_x^* = D_x + e_1^2 E \left[ h + \frac{bd}{s_x} \right] \]

\[ D_y^* = D_y + e_3^2 E \left[ h + \frac{b_d}{s_x} \right] \]

\[ H^* = D + \frac{GJ_x}{2s_x} + \frac{GJ_y}{2s_y} = H \]

Eqs 2.37

In general, the rigidities obtained from the ordinary 4th order theory are always the minimum values.

8. FROM ADOTTE

Adotte (Ref. 10) deals with asymmetrically stiffened plates of any kind, steel or concrete, and gives what he terms "a second order" theory for them. This second order theory takes into account the in-plane forces in the plate or slab part of the structure, and so has the same base as the 8th order theory. In his method the bending and in-plane forces are found separately. Finite differences are used to obtain solutions. Basically, two general operator patterns are derived which are applied to the nodes in the structure. There are two sets of simultaneous equations to be solved. The unknowns at each node are the deflection, \( w \), and Airy's stress function, \( F \). By choosing values of \( F \) and then solving for the values of \( w \), and then substituting in the values of \( w \) to recalculate the values of \( F \), and so
on, the plate is solved.

Huber's equation (1.13) forms part of one of the general operators so, as with most other methods, values are needed for $D_x$, $D_y$ and $H$. These are given by Adotte as

$$D_x = \frac{E I_x}{s_x (1-\nu^2)} \quad D_y = \frac{E I_y}{s_y (1-\nu^2)}$$

$$H = \frac{1}{2} (D_x + D_y) \quad \text{Eqs 2.38}$$

where $I_x, I_y$ are the moments of inertia of the repeating sections in the $X,Y$ directions. These are said to come from the standard theory of elasticity. Dividing the whole of $I_x, I_y$ by $(1-\nu^2)$ is likely to overestimate $D_x, D_y$. The only place where the writer has seen the formula for $H$ given above is in Marcus's equation

$$D_x \frac{d^4 w}{dx^4} + (D_x + D_y) \frac{d^4 w}{dx^2 dy^2} + D_y \frac{d^4 w}{dy^2} = q$$

which was discounted and replaced by Huber.

9. OTHER AUTHORS

In this section references on the subject of orthotropic plates are given, but which contain no new ideas on the values of the rigidities.

Zienkiewicz and Cheung (Ref. 17) among others produced an orthotropic rectangular element stiffness matrix for use in a finite element analysis. Values of $D_x, D_y, D_1$ and $D_{xy}$ are needed for substitution into this matrix. An alternative method is to use isotropic plate elements and beam elements.
Bares (Ref. 2) gives solutions in tabular form for various values of $D_x/D_y$ and various load and support conditions. In Ref. 20 he presents his version of the more exact theory and describes some experimental work on an orthotropic slab.

Leknitskii (Ref. 5) is concerned with mathematical solutions of anisotropic plates with various shapes, support conditions, and loadings. He gives rigidity formulae for plates with many stiff ribs, plywood plates, corrugated plates, and plates with the modulus $E$ varying generally through the thickness.

Some mathematical solutions for bridge decks using the more exact theory appear in Ref. 13, 18, 19. The views on rigidity formulae follow those of Cornelius and Giencke.

10. **SUMMARY**

In the experimental part of this work tests are done on homogeneous, asymmetrically stiffened plates with ribs running in one direction only. In the summary below are the rigidities for this type of orthotropic plate only.
A. Values for $D_x$

\[ D_x = \frac{EI}{s} \]  
\[ D_x^* = D_x + (e_1 - a_1)^2 \cdot E \left[ h + \frac{bd}{s} \right] \]

where $D_x$ is as in (i) above, and $0 < a_1 < e_1$.

This is Massonnet's 8th order correction.

\[ D_x = \frac{EI_x}{s} \]  
\[ D_x = \frac{EI_x}{s(1-v^2)} \]

where $I_x$ is the moment of inertia of the T-section about its neutral axis, are approximations which can be used if
a. \( v \) is very small

b. the plate contribution to \( I_x \) is small for formula (iii)

c. the beam contribution to \( I_x \) is small for formula (iv)

B. Values of \( D_y \)

i. \[
D_y = \frac{Eh^3}{12(1-v^2)} = D \quad \text{Eq 2.43}
\]

This is a safe formula which some consider conservative.

ii. \[
D_y = \frac{E h^3 s}{12(s-b+a^3b)} \quad \text{Eq 2.44}
\]

where \( a = h/(h+d) \)

This is Huber's formula

iii. \[
D < D_y < \frac{B}{30 \int_0^B \frac{(By-y^2)^2}{D(y)} \, dy} \quad \text{Eq 2.45}
\]

where \( B \) is the span in the \( Y \) direction

where \( D(y) = \frac{E[h(y)]^3}{12(1-v^2)} \)

where \( h(y) \) is the total thickness of the plate stiffener combination subject to the following limitation. When the ratio of stiffener height to stiffener width is greater than unity a reduced height of the order of the stiffener width should be used.
This was Huffington's recommendation for symmetrically stiffened plates, and was really applied to plates with simply supported edges.

C. Values of $D_1$

i. $D_1 = \nu D$  
   \[ \text{Eq 2.46} \]

   This is the commonly accepted one and seems the most reasonable.

ii. $D_1 = \nu D + \nu \epsilon_1 \epsilon_3 \frac{Eh}{(1-\nu^2)} = \nu D$  
    \[ \text{Eq 2.47} \]

   This is Giencke's approximate formula from 8th order theory, and is put in to show that no change is made to the value of $D_1$ by this theory when the ribs run in one direction only ($\epsilon_3 = 0$).

iii. $D_1 = 0$.  
    \[ \text{Eq 2.48} \]

   This is a first approximation used if $\nu$ is small or if the contribution of the plate to $D_x$ is small.

D. Values of $\gamma_x$, $\gamma_y$, and $D_{xy}$

i. $\gamma_x = D(1-\nu) + G\frac{J}{s}$

   $\gamma_y = D(1-\nu)$  
   \[ \text{Eqs 2.49} \]

   $2D_{xy} = D(1-\nu) + G\frac{J}{2s}$

   This is the usual set of formulae. $\gamma_x$ represents the
sum of the torsional rigidities of the plate and the beam taken separately. There is no formula which allows for their additional strength due to continuity, except for ii. below, which is really for thin steel sections.

For steel ribs joined to a concrete slab by shear connectors or joined to a steel deck plate by bolts there may well be a negligible continuity effect.

ii. \[ \gamma_x = D(1-v) + G \frac{bd^3}{3s} \]
\[ \gamma_y = D(1-v) \]
\[ 2D_{xy} = D(1-v) + G \frac{bd^3}{6s} \]

In the formula for \( \gamma_x \) the beam contribution is calculated with \( c = 1/3 \), irrespective of the ratio \( d/b \).

iii. \[ \gamma_x = D(1-v) + \frac{e_1^2 Eh}{2(1+v)} + \frac{GJ}{s} \]
\[ \gamma_y = D(1-v) \]
\[ 2D_{xy} = D(1-v) + \frac{e_1^2 Eh}{4(1+v)} + \frac{GJ}{2s} \]

These formulae come from the 8th order approximation due to Giencke. For Massonnet's formulae replace \( e_1 \) by \( a_1 \), where \( 0 < a_1 < e_1 \), in the above equations.
GENERAL ANISOTROPIC ELASTIC THEORY
APPLIED TO ORTHOTROPIC PLATES

Statements such as "The bending of orthotropic plates is characterised by four independent elastic constants $D_x$, $D_y$, $D_1$ and $D_{xy}$", are sometimes made in the literature. This can be misleading because there are actually nine elastic constants which characterise an orthotropic plate, but usually only four are needed in a bending analysis.

In Chapter 1 the relationships between stress and strain in an orthotropic plate were stated in Eqs 1.3. There the co-ordinate system $X,Y,Z$ corresponded to the directions of principal stiffness. It will be seen that the stress-strain relationships are more complicated when this is not the case, and hence the moment equations (1.6, 1.7, 1.8) acquire extra terms. The method of finding the moments in directions other than the directions of principal stiffness will be given. One application of this is that it can be used to determine the rigidities of orthotropic plates by experiment.

The basic theory in this chapter comes from Ref. 23. We begin by considering the state of stress and strain in a continuous anisotropic body.

1. The State of Stress in a Continuous Solid Body

The state of stress at any given point of a continuous body is determined entirely by the components of stress in 3
mutually perpendicular planes which pass through the chosen point. Choosing a mutually perpendicular set of axes \(X, Y, Z\) the stress components are defined as in Fig. 3.1.

Fig. 3.1

We can find by projection the components of stress in a new orthogonal system \(X', Y', Z'\). The position of the new system with respect to the first system is determined by Table 3.1 of direction cosines. In the table \(\alpha_1 = \cos (XOX')\)
\(\gamma_2 = \cos (ZOX')\) and so forth.

<table>
<thead>
<tr>
<th>TABLE 3.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Direction cosines</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>(X)</th>
<th>(Y)</th>
<th>(Z)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(X')</td>
<td>(\alpha_1)</td>
<td>(\beta_1)</td>
<td>(\gamma_1)</td>
</tr>
<tr>
<td>(Y')</td>
<td>(\alpha_2)</td>
<td>(\beta_2)</td>
<td>(\gamma_2)</td>
</tr>
<tr>
<td>(Z')</td>
<td>(\alpha_3)</td>
<td>(\beta_2)</td>
<td>(\gamma_3)</td>
</tr>
</tbody>
</table>

The stresses in the new coordinate system (i.e. on the
areas normal to the \(X'\), \(Y'\) and \(Z'\) axes) are

\[
\sigma_x' = \sigma_x \alpha_1^2 + \sigma_y \beta_1^2 + \sigma_z \gamma_1^2 + 2 \tau_{yz} \beta_1 \gamma_1 + 2 \tau_{xz} \alpha_1 \gamma_1 + 2 \tau_{xy} \alpha_1 \beta_1
\]

\[
\tau_{yz}' = \sigma_x \alpha_2 \alpha_3 + \sigma_y \beta_2 \beta_3 + \sigma_z \gamma_2 \gamma_3 + \tau_{yz} (\beta_2 \gamma_3 + \beta_3 \gamma_2)
\]

\[
+ \tau_{xz} (\alpha_2 \gamma_3 + \alpha_3 \gamma_2) + \tau_{xy} (\alpha_2 \beta_3 + \alpha_3 \beta_2)
\]

Eqs 3.1

The expressions for \(\sigma_y'\), \(\sigma_z'\), \(\tau_{xz}'\), \(\tau_{xy}'\) are obtained by cyclic permutation of the indices of \(\alpha\), \(\beta\) and \(\gamma\). The formulae for the transition from the system \(X', Y', Z'\) to \(X, Y, Z\) have the form:

\[
\sigma_x = \sigma_x' \alpha_1^2 + \sigma_y' \alpha_2^2 + \sigma_z' \alpha_3^2 + 2 \tau_{yz} \alpha_2 \alpha_3 + 2 \tau_{xz} \alpha_1 \alpha_3 + 2 \tau_{xy} \alpha_1 \alpha_2
\]

\[
\tau_{yz} = \sigma_x' \beta_1 \gamma_1 + \sigma_y' \beta_2 \gamma_2 + \sigma_z' \beta_3 \gamma_3 + \tau_{yz} (\beta_2 \gamma_3 + \beta_3 \gamma_2)
\]

\[
+ \tau_{xz}' (\beta_1 \gamma_3 + \beta_3 \gamma_1) + \tau_{xy}' (\beta_1 \gamma_2 + \beta_2 \gamma_1)
\]

Eqs 3.2

As before, cyclic permutation of \(\alpha\), \(\beta\)

and \(\gamma\) gives the remaining stress component formulae.

2. Strain

The state of deformation in the neighbourhood of a given point of a continuous body is determined by six components of strain: 3 components of relative normal strain which we denote by \(\epsilon\) with one index which indicates the direction of the strain; 3 components of relative shear strain which we denote by \(\gamma\) with 2 indices.

For the transition from one orthogonal coordinate system \(X, Y, Z\) to another \(X', Y', Z'\), the components of strain transform according to formulae of type (3.1). Here we substitute
\[ \varepsilon_x' = \varepsilon_x \alpha_1 + \varepsilon_y \beta_1 + \varepsilon_z \gamma_1 + \gamma_{yz} \alpha_1 \beta_1 + \gamma_{xz} \alpha_1 \beta_1 \]

\[ \gamma_{yz} = 2\varepsilon_x \alpha_2 \beta_3 + 2\varepsilon_y \beta_2 \beta_3 + 2\varepsilon_z \gamma_2 \gamma_3 + \gamma_{yz} (\beta_2 \gamma_3 + \beta_3 \gamma_2) \]

\[ + \gamma_{xz} (\alpha_2 \gamma_3 + \alpha_3 \gamma_2) + \gamma_{xy} (\alpha_2 \beta_3 + \alpha_3 \beta_2) \]

Eqs 3.3

and conversely,

\[ \varepsilon_x = \varepsilon_x' \alpha_1 + \varepsilon_y' \alpha_2 + \varepsilon_z' \alpha_3 + \gamma_{yz} \alpha_2 \beta_3 + \gamma_{xz} \alpha_1 \beta_2 + \gamma_{xy} \alpha_1 \alpha_2 \]

\[ \gamma_{yz} = 2\varepsilon_x' \beta_1 \gamma_1 + 2\varepsilon_y' \beta_2 \gamma_2 + 2\varepsilon_z' \beta_3 \gamma_3 + \gamma_{yz} (\beta_2 \gamma_3 + \beta_3 \gamma_2) \]

\[ + \gamma_{xz} (\beta_1 \gamma_3 + \beta_3 \gamma_1) + \gamma_{xy} (\beta_1 \gamma_2 + \beta_2 \gamma_1) \]

Eqs 3.4

We denote the components of displacement of the particles of a continuous body along the axes of the Cartesian coordinate system \( X, Y, Z \) by \( u, v, w \). In the case of small displacements of a continuous body, it is possible to neglect the squares and products of the derivatives of the displacements. We have:

\[ \varepsilon_x = \frac{du}{dx} \quad \varepsilon_y = \frac{dv}{dy} \quad \varepsilon_z = \frac{dw}{dz} \]

\[ \gamma_{yz} = \frac{dv}{dz} + \frac{dw}{dy} \quad \gamma_{xz} = \frac{dw}{dx} + \frac{du}{dz} \]

\[ \gamma_{xy} = \frac{du}{dy} + \frac{dv}{dx} \]

Eqs 3.5

Formulae and equations given above are valid for both elastic and inelastic continuous bodies; their derivations can be found in any text on the theory of elasticity.
3. **The Generalized Hooke's Law**

We assume that the components of strain are linear functions of the components of stress. In other words, it is assumed that a continuous body satisfies the generalized Hooke's Law. When no elements of elastic symmetry are present in the general case of a homogeneous anisotropic body, the equations which express the generalized Hooke's Law in Cartesian coordinates X,Y,Z have the form:

\[
\varepsilon_x = a_{11}\sigma_x + a_{12}\sigma_y + a_{13}\sigma_z + a_{14}\tau_{yz} + a_{15}\tau_{xz} + a_{16}\tau_{xy}
\]

\[
\varepsilon_y = a_{21}\sigma_x + a_{22}\sigma_y + a_{23}\sigma_z + a_{24}\tau_{yz} + a_{25}\tau_{xz} + a_{26}\tau_{xy}
\]

......

\[
\gamma_{xy} = a_{61}\sigma_x + a_{62}\sigma_y + \ldots \quad \ldots + a_{66}\tau_{xy}
\]

Eqs 3.6

These equations contain 36 coefficients \(a_{ij}\), called the coefficients of deformation. By solving the above equations for the stress components \(\sigma_x, \sigma_y, \ldots, \tau_{xy}\) we obtain an equivalent form for the equations of the generalized Hooke's Law:

\[
\sigma_x = A_{11}\varepsilon_x + A_{12}\varepsilon_y + A_{13}\varepsilon_z + A_{14}\gamma_{yz} + A_{15}\gamma_{xz} + A_{16}\gamma_{xy}
\]

\[
\sigma_y = A_{21}\varepsilon_x + A_{22}\varepsilon_y + A_{23}\varepsilon_z + A_{24}\gamma_{yz} + A_{25}\gamma_{xz} + A_{26}\gamma_{xy}
\]

......

\[
\tau_{xy} = A_{61}\varepsilon_x + A_{62}\varepsilon_y + \ldots \quad \ldots + A_{66}\gamma_{xy}
\]

Eqs 3.7

The constants \(A_{ij}\) are called the moduli of elasticity. If an elastic potential exists, the number of elastic constants in the most general case of anisotropy is reduced to 21. Such
an elastic potential exists when the variation of the body under deformation occurs isothermally or adiabatically. We shall assume that the variations of deformation occur isothermally.

Denoting the elastic potential by $\bar{V}$ (equal to the potential energy of deformation per unit of volume), we have the following relations:

$$\sigma_x = \frac{d\bar{V}}{d\varepsilon_x}, \quad \sigma_y = \frac{d\bar{V}}{d\varepsilon_y}, \ldots, \quad \tau_{xy} = \frac{d\bar{V}}{d\gamma_{xy}}.$$ 

By differentiating the above with respect to $\varepsilon_x', \varepsilon_y', \ldots, \gamma_{xy}'$, we obtain

$$\frac{d\sigma_x}{d\varepsilon_y} = \frac{d\sigma_y}{d\varepsilon_x}, \quad \frac{d\sigma_x}{d\varepsilon_z} = \frac{d\sigma_z}{d\varepsilon_x}$$

and so forth.

Hence it follows that

$$A_{21} = A_{12}, \quad A_{31} = A_{13}, \ldots, \quad A_{65} = A_{56},$$
or, in general $A_{ij} = A_{ji}$ $(i, j = 1, 2, \ldots, 6)$

Similar relations hold for the coefficients of deformation, i.e. $a_{ij} = a_{ji}$ $(i, j = 1, 2, \ldots, 6)$.

4. The Elastic Potential

In the general case the expression for the elastic potential has the form:

$$\bar{V} = \frac{1}{2}a_{11}\sigma_x^2 + a_{12}\sigma_x\sigma_y + a_{13}\sigma_x\sigma_z + a_{14}\sigma_x\sigma_{yz} + a_{15}\sigma_x\tau_{xz}$$
$$+ a_{16}\sigma_x\tau_{xy} + \frac{1}{2}a_{22}\sigma_y^2 + a_{23}\sigma_y\sigma_z + a_{24}\sigma_y\tau_{yz}$$
$$+ a_{25}\sigma_y\tau_{xz} + a_{26}\sigma_y\tau_{xy} + \frac{1}{2}a_{33}\sigma_z^2 + a_{34}\sigma_z\tau_{yz}.$$
Or, more briefly,
\[ \bar{V} = \frac{1}{2}(\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \ldots + \tau_{xy} \gamma_{xy}) \]  
Eq 3.9

Or, in terms of the strains alone
\[ \bar{V} = \frac{1}{2}A_{11}\varepsilon_x^2 + A_{12}\varepsilon_x \varepsilon_y + \ldots + A_{1s}\varepsilon_x \gamma_{xy} \]
\[ + \frac{1}{2}A_{22}\varepsilon_y^2 + \ldots + A_{2s}\varepsilon_y \gamma_{xy} + \ldots + \frac{1}{2}A_{ss}\gamma_{xy}^2 \]
Eq 3.10

We find the potential energy of deformation for the whole body by means of integration over its volume \( w \):
\[ V = \iiint w \, d\bar{V} \]

5. The Technical Elastic Constants

The coefficients of deformation, \( a_{ij} \), can be written in terms of the Young's moduli, Poisson coefficients and shear moduli

I. \( a_{11} = \frac{1}{E_{xx}} \) \( a_{22} = \frac{1}{E_{yy}} \) \( a_{33} = \frac{1}{E_{zz}} \)

II. \( a_{12} = -\frac{\nu_{yx}}{E_{yy}} = -\frac{\nu_{xy}}{E_{xx}} \), \( a_{23} = -\frac{\nu_{yz}}{E_{zz}} = -\frac{\nu_{zy}}{E_{yy}} \)

\[ a_{13} = -\frac{\nu_{xz}}{E_{xx}} = -\frac{\nu_{zx}}{E_{zz}} \]

III. \( a_{44} = \frac{1}{G_{yz}} \) \( a_{55} = \frac{1}{G_{xz}} \) \( a_{ss} = \frac{1}{G_{xy}} \)
IV. \[ a_{56} = \frac{\mu_{xy,zz}}{G_{xz}} = \frac{\mu_{zx,xy}}{G_{xy}} \]
\[ a_{46} = \frac{\mu_{yx,yz}}{G_{yz}} = \frac{\mu_{yz,xy}}{G_{xy}} \]
\[ a_{45} = \frac{\mu_{zx,xy}}{G_{yz}} = \frac{\mu_{yz,zx}}{G_{xz}} \]

V. \[ a_{14} = \frac{\eta_{yz,x}}{E_{xx}} = \frac{\eta_{x,yz}}{E_{yz}} \]
\[ a_{25} = \frac{\eta_{zx,y}}{E_{yy}} = \frac{\eta_{y,zx}}{G_{xz}} \]
\[ a_{38} = \frac{\eta_{xy,z}}{E_{zz}} = \frac{\eta_{z,xy}}{G_{xy}} \]

VI. \[ a_{24} = \frac{\eta_{y,z}}{E_{yy}} = \frac{\eta_{y,z}}{E_{yy}} \]
\[ a_{18} = \frac{\eta_{xy,x}}{E_{xx}} = \frac{\eta_{x,xy}}{G_{xy}} \]
\[ a_{34} = \frac{\eta_{yz,z}}{E_{zz}} = \frac{\eta_{z,yz}}{G_{yz}} \]
\[ a_{15} = \frac{\eta_{zx,x}}{E_{xx}} = \frac{\eta_{x,zx}}{G_{xz}} \]
\[ a_{26} = \frac{\eta_{xy,y}}{E_{yy}} = \frac{\eta_{y,xy}}{G_{xy}} \]

Here \( E_{xx}, E_{yy}, E_{zz} \) are the Young's moduli (for tension-compression with respect to the directions \( x,y,z \)); \( G_{yz}, G_{xz}, G_{xy} \) are the shear moduli for planes which are parallel to the coordinates \( yoz, xoz, xoy \); \( \nu_{yx}, \nu_{zy}, \nu_{xz}, \nu_{xy}, \nu_{yz}, \nu_{zx} \) are the Poisson coefficients which characterize the transverse compression for tension in the direction of the axis of the first coordinate (thus, \( \nu_{yx} \) is a coefficient which characterizes the decrease in the \( x \) direction for tension in the \( y \) direction, and so on).

\( \mu_{zx,yz}, \mu_{xy,yz}, \ldots, \mu_{yz,xy} \) we call the coefficients of Chentsov. They characterize the shear in the planes which are parallel to the coordinates and which induce tangential stresses parallel to the other coordinate planes.
Thus for example \( \mu_{ZX,YZ} \) characterizes the shear in the plane parallel to the \( yz \) plane which induces the stress \( \tau_{ZX}' \) and so forth.

Finally, \( \eta_{YZ,X'} \), \( \eta_{ZX,X'} \), \ldots, \( \eta_{XY,Z} \) are called the coefficients of mutual influence of the first kind, and \( \eta_{X,YZ} \), \( \eta_{Y,YZ} \), \ldots, \( \eta_{Z,XY} \) are called the coefficients of mutual influence of the second kind. The first characterize the stretching in the directions parallel to the axes which are induced by the tangential stresses; the second are shears in the planes parallel to the coordinates under the influence of normal stresses.

6. Elastic Symmetry

If the internal composition of a material possesses symmetry of any kind, then symmetry can be observed in its elastic properties. This can be extended to bodies with structural symmetry. If an anisotropic body possesses an elastic symmetry then the equations of the generalized Hooke's Law are simplified.

We find the simplifications by applying the following method. We refer the body to the first coordinate system \( X,Y,Z \), and then to the second coordinate system \( X',Y',Z' \), which is symmetric to the first. Since the directions of similar axes of both systems are equivalent with respect to elastic properties, the equations of the generalized Hooke's Law and the expression of elastic potential will have the same form in both the first and second systems; the corresponding elastic constants entering into their composition also will be identical.
For the first system \((X,Y,Z)\)

\[
\bar{V} = b_{11} \sigma_x^2 + a_{12} \sigma_x \sigma_y + \ldots + b_{sn} \tau_{xy}^2
\]

For the second coordinate system \((X',Y',Z')\)

\[
\bar{V} = b_{11} \sigma'_x^2 + a_{12} \sigma'_x \sigma'_y + \ldots + b_{sn} \tau'_{xy}^2
\]

Since one and the same quantity is being discussed, we have

\[
\begin{align*}
&b_{11} \sigma_x^2 + a_{12} \sigma_x \sigma_y + \ldots + b_{sn} \tau_{xy}^2 \\
= &b_{11} \sigma'_x^2 + a_{12} \sigma'_x \sigma'_y + \ldots + b_{sn} \tau'_{xy}^2
\end{align*}
\]

We express \(\sigma'_x, \sigma'_y, \ldots, \tau'_{xy}\) in terms of \(\sigma_x, \sigma_y, \ldots, \tau_{xy}\) using formulae of type (3.1). Equating coefficients for \(\sigma'_x, \sigma'_y, \ldots, \tau'_x\) in the left and right sides, we find that some coefficients \(a_{ij}\) are equal to zero and others are connected by definite relations. As a result, bodies possessing elastic symmetry have less than 21 independent elastic constants. Considering the elastic potential expressed in terms of the components of strain, we can find the relationship between the moduli \(A_{ij}\) in the same way.

7. Elastic Symmetry in an Orthotropic Material

![Fig. 3.2](image)
Let $X$ and $Y$ be the principal directions in the orthotropic plate as in Fig. 3.2(a). It can be seen that the axis system $(X,Y,Z)$ can be rotated in 3 possible ways (Fig. 3.2(b), (c), (d)) so that the new coordinate system $(X',Y',Z')$ is equivalent with respect to elastic properties.

i. For the rotation from system 3.2(a) to 3.2(b) the table of direction cosines is:

![Diagram](image)

<table>
<thead>
<tr>
<th></th>
<th>X</th>
<th>Y</th>
<th>Z</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X'$</td>
<td>-1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$Y'$</td>
<td>0</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>$Z'$</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Fig. 3.3

From Eqs 3.1 we have

\[
\begin{align*}
\sigma'_x &= \sigma_x & \tau'_{yz} &= -\tau_{yz} \\
\sigma'_y &= \sigma_y & \tau'_{xz} &= -\tau_{xz} \\
\sigma'_z &= \sigma_z & \tau'_{xy} &= \tau_{xy}
\end{align*}
\]

Equating $\vec{V}$ for the two systems

\[
\vec{V} = \beta a_{11} \sigma_x^2 + a_{12} \sigma_x \sigma_y + a_{13} \sigma_x \sigma_z + a_{14} \sigma_x \tau_{yz} \\
+ a_{15} \sigma_x \tau_{xz} + a_{16} \sigma_x \tau_{xy} + \frac{1}{2} a_{22} \sigma_y^2 + a_{23} \sigma_y \sigma_z \\
+ a_{24} \sigma_y \tau_{yz} + a_{25} \sigma_y \tau_{xz} + a_{26} \sigma_y \tau_{xy} + \frac{1}{2} a_{33} \sigma_z^2 \\
+ a_{34} \sigma_z \tau_{yz} + a_{35} \sigma_z \tau_{xz} + a_{36} \sigma_z \tau_{xy} + \frac{1}{2} a_{44} \tau_{yz}^2 \\
+ a_{45} \tau_{yz} \tau_{xz} + a_{46} \tau_{yz} \tau_{xy} + \frac{1}{2} a_{55} \tau_{xz}^2 + a_{56} \tau_{xz} \tau_{xy} \\
+ \frac{1}{2} a_{66} \tau_{xy}^2
\]
Equating coefficients of like terms:

\[ a_{14} = a_{15} = a_{24} = a_{25} = a_{34} = a_{35} = a_{45} = a_{56} = 0 \]

ii. For the rotation from system 3.2(a) to 3.2(c) the table of direction cosines is:

\[
\begin{array}{ccc}
X' & Y' & Z' \\
X' & 1 & 0 & 0 \\
Y' & 0 & -1 & 0 \\
Z' & 0 & 0 & -1 \\
\end{array}
\]

Fig. 3.4

From Eqs 3.1 we have

\[
\begin{align*}
\tau'_{xy} &= \tau_{xy} \\
\tau'_{xz} &= \tau_{xz} \\
\tau'_{yz} &= \tau_{yz} \\
\sigma'_x &= \sigma_x \\
\sigma'_y &= \sigma_y \\
\sigma'_z &= \sigma_z \\
\end{align*}
\]

Proceeding as in i. above this yields:

\[ a_{15} = a_{16} = a_{25} = a_{26} = a_{35} = a_{36} = a_{45} = a_{46} = 0 \]
iii. For the rotation from 3.2(a) to 3.2(d) no new information emerges because iii. can be seen to be the sum of rotations i. and ii.

For an orthotropic material in which the axis system (X,Y,Z) corresponds to the directions of principal stiffness the matrix of coefficients of deformation therefore simplifies to:

\[
[a_{ij}] = \begin{bmatrix}
an_{11} & an_{12} & an_{13} & 0 & 0 & 0 \\
an_{22} & an_{22} & an_{23} & 0 & 0 & 0 \\
an_{33} & 0 & an_{33} & 0 & 0 & 0 \\
ap_{44} & 0 & 0 & an_{44} & 0 & 0 \\
\text{SYM} & an_{55} & 0 & 0 & an_{55} & 0 \\
ap_{66} & \end{bmatrix}
\]

The moduli of elasticity, \( A_{ij} \), can be found in the same manner as \( a_{ij} \). The elastic potential is expressed in terms of the components of strain instead of stress; this is the only difference. The simplified matrix of moduli of elasticity becomes:

\[
[A_{ij}] = \begin{bmatrix}
A_{11} & A_{12} & A_{13} & 0 & 0 & 0 \\
A_{22} & A_{22} & A_{23} & 0 & 0 & 0 \\
A_{33} & 0 & A_{33} & 0 & 0 & 0 \\
A_{44} & 0 & 0 & A_{44} & 0 & 0 \\
\text{SYM} & A_{55} & 0 & 0 & A_{55} & 0 \\
A_{66} & \end{bmatrix}
\]
8. The Transformation of Elastic Constants under a transformation of the coordinate system

The elastic constants which enter into equations of the generalized Hooke's Law of an anisotropic body depend on the direction of the axes of the coordinate system. If the direction of the axes varies, then the elastic constants vary. Only in the case of an isotropic body are the constants invariant in any orthogonal coordinate system. When studying the state of stress of an anisotropic body the following question arises: if we know the elastic constants expressed in one coordinate system, what are the corresponding constants in another more convenient system?

This question can be answered by applying the method already used for the simplification of the expressions of the elastic potential and for the equations of the generalized Hooke's Law when elastic symmetry is present. Let the elastic constants for the system \(X,Y,Z\) be known. It is required to determine the constants in the new system \(X',Y',Z'\). The position of the new system with respect to the old is defined by the table of direction cosines. We write the expressions for the elastic potential \(\bar{V}\) expressed in the systems \(X,Y,Z\) and \(X',Y',Z'\). By equating these expressions, we obtain an equation on the left side of which are the components of stress or strain in the old system; on the right side are the same quantities for the new system. We express the components of stress or strain in the old coordinate system in terms of the components of stress or strain in the new coordinate system. Comparing the coefficients of the squares and products of the components
of stress or strain referred to the new system we obtain the desired formulae for the new elastic constants on the left and right sides of the equation.

We are primarily interested in the transformation of the moduli of elasticity, $A_{ij}'$, since they enter into the equations for bending and twisting moments. The transformation of $a_{ij}$ will be given here but not derived.

Let the system $X, Y, Z$ be that of the principal stiffnesses. In this system the planes $XZ$, $XY$ and $YZ$ are all planes of elastic symmetry. In the new system $X', Y', Z'$, where $Z'$ corresponds to $Z$, there is only one plane of elastic symmetry, which is the $XY$ plane. The matrix of the moduli of elasticity is that corresponding to rotation $i.$ in Section 8 above, and is:

\[
\begin{bmatrix}
A_{11}' & A_{12}' & A_{13}' & 0 & 0 & A_{16}' \\
A_{21}' & A_{22}' & A_{23}' & 0 & 0 & A_{26}' \\
A_{31}' & A_{32}' & A_{33}' & 0 & 0 & A_{36}' \\
& & & A_{44}' & A_{45}' & 0 \\
& & & SYM & A_{55}' & 0 \\
& & & & & A_{66}'
\end{bmatrix}
\]

The table of direction cosines for the rotation about the $Z$ axis is:

\[
\begin{array}{c|ccc}
& x & y & z \\
\hline
x' & \cos \alpha & \sin \alpha & 0 \\
y' & -\sin \alpha & \cos \alpha & 0 \\
z' & 0 & 0 & 1 \\
\end{array}
\]

Fig. 3.5
We have

\[
\vec{V} = \frac{1}{2}A_{11}c_x^2 + A_{12}c_xc_y + A_{13}c_xc_z + \frac{1}{2}A_{22}c_y^2 + A_{23}c_yc_z + \frac{1}{2}A_{33}c_z^2 + \frac{1}{2}A_{44}c_{yz}^2 + \frac{1}{2}A_{55}c_{xz}^2 + \frac{1}{2}A_{66}c_{xy}^2
\]

From the table of direction cosines

\[
\begin{align*}
\epsilon_x &= \epsilon_x' \cos^2 \alpha + \epsilon_y' \sin^2 \alpha - \gamma_{xy}' \cos \alpha \sin \alpha \\
\epsilon_y &= \epsilon_x' \sin^2 \alpha + \epsilon_y' \cos^2 \alpha + \gamma_{xy}' \cos \alpha \\
\epsilon_z &= \epsilon_z' \\
\gamma_{yz} &= \gamma_{yz}' \cos \alpha + \gamma_{xz}' \sin \alpha \\
\gamma_{xz} &= -\gamma_{yz}' \sin \alpha + \gamma_{xz}' \cos \alpha \\
\gamma_{xy} &= 2\epsilon_x' \cos \alpha \sin \alpha - 2\epsilon_y' \cos \alpha \sin \alpha + \gamma_{xy}' (\cos^2 \alpha - \sin^2 \alpha)
\end{align*}
\]

Substituting these into the expression for \(\vec{V}\), we get the following relationships between \(A_{ii}'\) and \(A_{ij}\):

\[
\begin{align*}
A_{11} &= A_{11}\cos^4 \alpha + (2A_{12} + 4A_{36})\cos^2 \alpha \sin^2 \alpha + A_{22}\sin^4 \alpha \\
A_{12} &= A_{12}(\cos^4 \alpha + \sin^4 \alpha) + (A_{11} - 4A_{36} + A_{22})\cos^2 \alpha \sin^2 \alpha \\
A_{13} &= A_{13}\cos^2 \alpha + A_{23}\sin^2 \alpha \\
A_{16} &= \left[ -A_{11}\cos^2 \alpha + A_{22}\sin^2 \alpha + (A_{12} + 2A_{36})\cos 2\alpha \right] \frac{\sin 2\alpha}{2} \\
A_{22} &= A_{11}\sin^4 \alpha + (2A_{12} + 4A_{36})\cos^2 \alpha \sin^2 \alpha + A_{22}\sin^4 \alpha
\end{align*}
\]
\[ A_{23} = A_{13} \sin^2 \alpha + A_{23} \cos^2 \alpha \]
\[ A_{2e} = \left[ -A_{11} \sin^2 \alpha + A_{22} \cos^2 \alpha - (A_{12} + 2A_{ee}) \cos 2\alpha \right] \frac{\sin 2\alpha}{2} \]
\[ A_{43} = A_{33} \]
\[ A_{4e} = (A_{23} - A_{13}) \cos \alpha \sin \alpha \]
\[ A_{44} = A_{44} \cos^2 \alpha + A_{55} \sin^2 \alpha \]
\[ A_{45} = (A_{44} - A_{55}) \cos \alpha \sin \alpha \]
\[ A_{55} = A_{44} \sin^2 \alpha + A_{55} \cos^2 \alpha \]
\[ A_{6e} = A_{66} \cos^2 2\alpha + (A_{11} - 2A_{12} + A_{22}) \cos^2 \alpha \sin^2 \alpha \]

Eqs 3.11

The relationships between the coefficients of deformation, \( a'_{ij} \), in the system \( X', Y', Z' \) and \( a_{ij} \) in the system \( X, Y, Z \) can be derived as for the moduli of elasticity \( A_{ij} \). The only difference is that \( V \) must be expressed in terms of the components of stress instead of strain. The transformation rules are as follows:

\[ a_{11} = a_{11} \cos^4 \alpha + (2a_{12} + a_{66}) \cos^2 \alpha \sin^2 \alpha + a_{22} \sin^4 \alpha \]
\[ a_{12} = a_{12} (\cos^4 \alpha + \sin^4 \alpha) + (a_{11} + a_{22} - a_{66}) \cos^2 \alpha \sin^2 \alpha \]
\[ a_{13} = a_{13} \cos^2 \alpha + a_{23} \sin^2 \alpha \]
\[ a_{1e} = \left[ -2a_{11} \cos^2 \alpha + 2a_{22} \sin^2 \alpha + (2a_{12} + a_{66}) \cos 2\alpha \right] \frac{\sin 2\alpha}{2} \]
\[ a_{22} = a_{11} \sin^4 \alpha + (2a_{12} + a_{66}) \cos^2 \alpha \sin^2 \alpha + a_{22} \cos^4 \alpha \]
\[ a_{23} = a_{13} \sin^2 \alpha + a_{23} \cos^2 \alpha \]
The moduli of elasticity $A'_{ij}$ correspond to a general set of axes $X', Y', Z'$ but where $X'$ and $Y'$ always lie in the plane of the plate. It can be seen from Eqs 3.11 that in the special case where $X'$ and $Y'$ are the principal elastic directions, then $\alpha = 0$ and $A_{13} = A_{23} = A_{33} = 0$ and for the other moduli $A'_{ij} = A_{ij}$.

9. Moments in thin orthotropic plates

In the bending theory of thin orthotropic plates, the stress $C_Z$ is taken as zero since it must be zero on the surface of the plate, and is assumed not to build up to an amount which will affect the strains $\varepsilon_x$, $\varepsilon_y$, and $\gamma_{xy}$. This means that the values of $a_{13}$, $a_{23}$, $a_{33}$ are not used in analysis, and this in turn means that the values of $A_{13}$, $A_{23}$, $A_{33}$ are not used. It also means that $A_{11}$, $A_{12}$, $A_{22}$ have their values altered. To show this consider the $X, Y, Z$
system for which the alternative matrices of $a_{ij}$ are

$$\begin{bmatrix}
a_{11} & a_{12} & a_{13} & 0 & 0 & 0 \\
a_{22} & a_{23} & 0 & 0 & 0 & 0 \\
a_{33} & 0 & 0 & 0 & 0 & 0 \\
a_{44} & 0 & 0 & 0 & 0 & 0 \\
\text{SYM} & a_{55} & 0 & 0 & 0 & 0 \\
a_{66}
\end{bmatrix}$$

Here the values of $a_{ij}$ are as given in section 5. The values of $A_{ij}$ are now found by $[A_{ij}] = [a_{ij}]^{-1}$. Comparing values of $A_{ij}$ for the alternatives:

**Thin plate**

- $A_{11} = \frac{E_x}{(1-v_{yk}v_{yx})}$
- $A_{12} = \frac{E_y v_{yx}}{(1-v_{yk}v_{yx})}$
- $A_{22} = \frac{E_y}{(1-v_{yk}v_{yx})}$
- $A_{66} = G_{xy}$

**Orthotropic body**

- $A_{11} = \frac{E_x (1-v_{yk}v_{zy})}{(1-v_{yk}v_{zy}-v_{yx}v_{yx}-v_{yk}v_{yx}-v_{yk}v_{yx}-2v_{yx}v_{yx}v_{yx})}$
- $A_{12} = \frac{E_y (v_{yx}-v_{yx}v_{yx})}{(1-v_{yk}v_{zy}-v_{yx}v_{yx}-v_{yk}v_{yx}-v_{yk}v_{yx}-2v_{yx}v_{yx}v_{yx})}$
- $A_{22} = \frac{E_y (1-v_{yk}v_{zx})}{(1-v_{yk}v_{zy}-v_{yx}v_{yx}-v_{yk}v_{yx}-v_{yk}v_{yx}-2v_{yx}v_{yx}v_{yx})}$

In the thin plate theory the stresses become

$$\sigma_x = A_{11} \epsilon_x + A_{12} \epsilon_y$$
$$\sigma_y = A_{12} \epsilon_x + A_{22} \epsilon_y$$
$$\tau_{xy} = A_{66} \gamma_{xy}$$

The moments become
\[
M_x = \int_{-h/2}^{h/2} \sigma_x z \, dz = -\left( D_{11} \frac{d^2w}{dx^2} + D_{12} \frac{d^2w}{dy^2} \right)
\]
\[
M_y = \int_{-h/2}^{h/2} \sigma_y z \, dz = -\left( D_{22} \frac{d^2w}{dy^2} + D_{12} \frac{d^2w}{dx^2} \right)
\]
\[
M_{xy} = \int_{-h/2}^{h/2} \tau_{xy} z \, dz = 2D_{66} \frac{d^2w}{dxdy}
\]

where \( D_{ij} = A_{ij} \frac{h^3}{12} \)

The assumption that \( \sigma_z \) is zero therefore not only has the
effect of removing a term from the equations for \( M_x \) and \( M_y \),
but also changes the values of \( D_{11} \), \( D_{12} \) and \( D_{22} \). In general
the Poisson's ratios \( \nu_{xz} \), \( \nu_{zx} \), \( \nu_{yz} \) and \( \nu_{zy} \) are not equal to
zero. One would expect the terms \( \nu_{yz}\nu_{zy} \) and \( \nu_{xz}\nu_{zx} \) to be
of the same order as \( \nu_{xy}\nu_{yx} \). Provided that none of the
Poisson's ratios is negative, it can be seen that the thin
plate theory underestimates the values of \( D_{11} \), \( D_{12} \) and \( D_{22} \).
This is an error on the safe side if one is calculating the
deflections.

The moment equations in the system \( X', Y', Z' \) are derived
as follows in the thin plate theory

\[
\sigma_x' = A_{11} \epsilon_x' + A_{12} \epsilon_y' + A_{16} \gamma_{xy}'
\]
\[
\sigma_y' = A_{12} \epsilon_x' + A_{12} \epsilon_y' + A_{16} \gamma_{xy}'
\]
\[
\tau_{xy}' = A_{16} \epsilon_x' + A_{16} \epsilon_y' + A_{66} \gamma_{xy}'
\]

\[ \therefore \] with the sign conventions as shown in Figs. 1.4, 1.5
\[ M'_x = \int_{-h/2}^{h/2} \sigma' x z \, dz = -\left( D_{12} \frac{d^2w}{dx^2} + D_{12} \frac{d^2w}{dy^2} + 2D_{12} \frac{d^2w}{dx\,dy} \right) \]

\[ M'_y = \int_{-h/2}^{h/2} \sigma' y z \, dz = -\left( D_{22} \frac{d^2w}{dy^2} + D_{12} \frac{d^2w}{dx^2} + 2D_{12} \frac{d^2w}{dx\,dy} \right) \]

\[ M'_{xy} = \int_{-h/2}^{h/2} \tau'_{xy} z \, dz = D_{12} \frac{d^2w}{dx^2} + D_{12} \frac{d^2w}{dy^2} + 2D_{12} \frac{d^2w}{dx\,dy} \]

\[ = M'_{yx} \]

10. Mohr's circle for moments

The stresses in the system X', Y', Z' are from Eqs 3.1

\[ \sigma'_x = \sigma_x \cos^2 \alpha + \sigma_y \sin^2 \alpha + 2\tau_{xy} \cos \alpha \sin \alpha \]

\[ \sigma'_y = \sigma_x \sin^2 \alpha + \sigma_y \cos^2 \alpha - 2\tau_{xy} \cos \alpha \sin \alpha \]

\[ \tau'_{xy} = -\sigma_x \cos \alpha \sin \alpha + \sigma_y \cos \alpha \sin \alpha + \tau_{xy} (\cos^2 \alpha - \sin^2 \alpha) \]

These can be rewritten as

\[ \sigma'_x = \frac{\sigma_x + \sigma_y}{2} + \frac{(\sigma_x - \sigma_y)}{2} \cos 2\alpha + \tau_{xy} \sin 2\alpha \]

\[ \sigma'_y = \frac{\sigma_x + \sigma_y}{2} - \frac{(\sigma_x - \sigma_y)}{2} \cos 2\alpha - \tau_{xy} \sin 2\alpha \]

\[ \tau'_{xy} = -\frac{(\sigma_x - \sigma_y)}{2} \sin 2\alpha + \tau_{xy} \cos 2\alpha \]

From the above the moments become

\[ M'_x = \frac{(M_x + M_y)}{2} + \frac{(M_x - M_y)}{2} \cos 2\alpha - M_{xy} \sin 2\alpha \]
The above moment equations have certain differences in the signs from the stress equations because of the way the moments and stresses were defined in Figs. 1.4, 1.5. These moment formulae can be represented by a Mohr circle construction as shown in Fig. 3.6. The form of $M'_x$, $M'_y$, and $M'_{xy}$ can be seen to be satisfied by rotating triangle OAB through an angle of $2\alpha$ to position OA'B'.

![Fig. 3.6](image-url)
When $M'_{xy} = 0$, then $M'_x, M'_y$ are the principal bending moments, and the angle $\alpha$ is defined by

$$\tan 2\alpha = -\frac{2M_{xy}}{(M_x - M_y)}$$

The principal bending moments can be found from

$$M_{max}, M_{min} = \frac{M_x + M_y}{2} \pm \sqrt{\left(\frac{M_x - M_y}{2}\right)^2 + (M_{xy})^2}.$$
CHAPTER 4

EXPERIMENTAL METHODS OF FINDING THE RIGIDITIES

From a study of the literature it appears that only one method of finding the rigidities of an orthotropic plate has been used. This involves a twisting test and a minimum of two bending tests, and was developed by Bergsträsser (Ref. 24) in 1927. A minimum of three models is needed to determine all the rigidities $D_x$, $D_y$, $D_1$ and $D_{xy}$. This method will be described below.

The time taken to make a model is considerable if it is to be well made. The writer accordingly has devised a method for which only one model is needed to measure all the rigidities. This method, however, also has its limitations as will be seen.

1. Bergsträsser's Method

This method is described by Hearmon and Adams (Ref. 25) who used it for tests on plywood plates.

Fig. 4.1
Consider a plate ABCD cut from orthotropic material. Suppose the plate is subjected to uniform bending moments $M_x$ per unit length of side along AD,BC, $M_y$ per unit length along AB,CD and a uniform twisting moment $M_{xy}$ per unit length along all four sides. The general expression for the deflection of a thin plate subject to these conditions is

$$
h^3w = 6M_x (a_{11}x^2 + a_{12}y^2 + a_{1e}xy) + 6M_y (a_{21}x^2 + a_{22}y^2 + a_{2e}xy) + 6M_{xy}(a_{1e}x^2 + a_{2e}y^2 + a_{e}exy)\tag{4.1}
$$

where $w$ is the small deflection, normal to the plane of the plate, of the point $x,y$ relative to the origin of coordinates, where $h$ is the plate thickness, and where $a_{ij}$ are as defined in Eqs 3.12.

When $\theta = 0$ ($a_{1e} = a_{2e} = 0$) and when $M_{xy} = 0$, the deflection is symmetrical about the lines $x = 0$, $y = 0$, and antisymmetrical about these lines when $M_x = M_y = 0$, and $M_{xy} \neq 0$. The first case corresponds with pure bending, and the second with pure torsion.

When $\theta \neq 0$, then $a_{1e} \neq 0$, $a_{2e} \neq 0$ and, if $M_{xy} = 0$, some twisting will occur owing to the existence of the terms $a_{1e}xy$ and $a_{2e}xy$ in Eq 4.1. Similarly, if $M_x = M_y = 0$, some symmetrical bending will occur owing to the presence of the terms $a_{1e}x^2$ and $a_{2e}y^2$. 
The Bending Test

Fig. 4.2

The plate and its loading and support points are shown in Fig. 4.2. Ball-bearings provide support at the points S, and the load is applied at points L by the method shown in Fig. 4.3.

Fig. 4.3

This loading is considered to provide a uniform moment per
unit length $M_x = \frac{P \cdot 3}{2 \cdot 6} = \frac{P}{4}$ in the centre of the plate. The 
moments $M_y$ and $M_{xy}$ are considered to be zero in the same 
region. Deflections are measured along the lines 1., 2., 3., 4. 
by means of the apparatus shown in Fig. 4.4. The foot of 
the gauge B is placed

![Fig. 4.4](image)

on the central point of the plate; the feet A and C are 
placed along the required line; and foot D is used to steady 
the platform.

Now if a bending moment $M_x (= P/4)$ only is applied to 
the plate ($M_y = M_{xy} = 0$) Eq 4.1 becomes

$$h^3w = 6M_x (a_{11}x^2 + a_{12}y^2 + a_{13}xy)$$

Along line 1.

$$x = 2.75\text{ in}, \ y = 0 \quad \therefore \ a_{11} = \frac{wh^3}{11.35P}$$

Along line 2.

$$x = 0, \ y = 2.75\text{ in} \quad \therefore \ a_{12} = \frac{wh^3}{11.35P}$$

Along line 3.
Along line 4:

\[ x = y = 2.75\sqrt{2} \]

\[ \therefore a_{11} + a_{12} + a_{1e} = \frac{wh^3}{5.67P} \]

Along line 4:

\[ x = y = 2.75\sqrt{2} \]

\[ \therefore a_{11} + a_{12} - a_{1e} = \frac{wh^3}{5.67P} \]

In the case \( \theta = 0^\circ \)

\[ a_{11} = a_{11} ; a_{12} = a_{12} ; a_{1e} = 0 \]

In the case \( \theta = 90^\circ \)

\[ a_{11} = a_{22} ; a_{12} = a_{12} ; a_{1e} = 0 \]

Notice that to find values of \( a_{11}, a_{12} \) and \( a_{22} \) a minimum of two models is needed. The simplest two values of \( \theta \) to use are \( \theta = 0^\circ \) and \( \theta = 90^\circ \). Measurements along lines 3. and 4. are redundant; they can be taken to give better average results but are not necessary.

A strong recommendation for this bending test is that the plate is in pure bending when \( \theta = 0^\circ \) or \( 90^\circ \). There are thus no extra terms in the theoretical formulae for \( D_x, D_y \) and \( D_1 \) according to the 8th order theory of Massonnet (Eqs 2.30 to 2.32), when asymmetrically stiffened plates are tested.

Hearmon and Adams tested three types of plywood using \( \theta = 0^\circ, 22.5^\circ, 45^\circ, 67.5^\circ \) and \( 90^\circ \) for each. They measured deflections along all four lines 1., 2., 3., 4. There were thus many measurements and the redundant ones along lines 3. and 4. agreed quite well with the results for lines 1. and 2. They estimate the uncertainty in the average results to be a maximum of 3%.

Note that the flexural rigidities can be calculated from
the constants $a_{11}$, $a_{12}$ and $a_{22}$ by

$$A_{11} = \frac{a_{22}}{a_{11}a_{22} - a_{12}^2}; \quad A_{12} = \frac{-a_{12}}{a_{11}a_{22} - a_{12}^2}; \quad A_{22} = \frac{a_{11}}{a_{11}a_{22} - a_{12}^2}$$

and $D_{ij} = A_{ij} \frac{h^3}{12}$.

Hoppmann (Ref. 21) did tests on three types of asymmetrically stiffened plates using the method described above. He found $a_{12}$ and $a_{21}$ separately, and they compare very badly. To show the relative discrepancies his results are given below in Table 4.1.

<table>
<thead>
<tr>
<th>Type of stiffener</th>
<th>$\frac{a_{11}}{h^3} \times 10^6$</th>
<th>$\frac{a_{12}}{h^3} \times 10^6$</th>
<th>$\frac{a_{21}}{h^3} \times 10^6$</th>
<th>$\frac{a_{22}}{h^3} \times 10^6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Brass rod</td>
<td>9.7</td>
<td>0.06</td>
<td>-4.9</td>
<td>13.9</td>
</tr>
<tr>
<td>Grooved steel plate</td>
<td>24.5</td>
<td>-8.55</td>
<td>-9.3</td>
<td>33.5</td>
</tr>
<tr>
<td>Grooved duralumin</td>
<td>90</td>
<td>-12.8</td>
<td>-39.2</td>
<td>612</td>
</tr>
</tbody>
</table>

**Table 4.1**

Here $h$ is the thickness of an equivalent true orthotropic plate, which need not be defined. Units are (lb in.)$^{-1}$.

However in a later paper (Ref. 26) in tests on two more plates he obtained much closer values of $a_{12}$ and $a_{21}$. These are shown in Table 4.2.
The results in Table 4.1 do suggest that $a_{12}$ and $a_{21}$ or $D_{12}$ and $D_{21}$ are not as well defined in this test as $a_{11}$ and $a_{22}$ or $D_x$ and $D_y$ are. In other words the test is not as sensitive to these elastic constants as one would like.

The Twisting Test

The method used is illustrated in Fig. 4.5. The test plates are square measuring 6 x 6 in. The deflections are measured exactly as in the bending test.

![Fig. 4.5](image)

The plate is loaded on diagonally opposite corners and supported on ball-bearings at the other two corners. The theory of this method of loading is given by Timoshenko (Ref. 1) and shows that the concentrated loads $P/2$ at each
corner produce a torsional moment \( M_{XY} \) per unit length along each side equal to \( P/4 \). Hence from Eq. 4.1

\[
2h^3w = 3P \left( a_1 e x^2 + a_2 e y^2 + a_3 e x y \right)
\]

Along line 1.: \( a_1 e = \frac{wh^3}{11.35P} \)

Along line 2.: \( a_2 e = \frac{wh^3}{11.35P} \)

Along line 3.: \( a_1 e + a_2 e + a_3 e = \frac{wh^3}{5.67P} \)

Along line 4.: \( a_1 e + a_2 e - a_3 e = \frac{wh^3}{5.67P} \)

For the case \( \theta = 0^\circ \) or \( 90^\circ \), there should be no deflection along lines 1. and 2.

Along line 3.: \( a_{3e} = \frac{wh^3}{5.67P} \)

Along line 4.: \( a_{3e} = -\frac{wh^3}{5.67P} \)

It can be seen that a minimum of one measurement is needed to find \( a_{3e} \), and obviously only one plate is needed.

Then \( A_{3e} = \frac{1}{a_{3e}} \)

and \( 2D_{xy} = A_{3e} \frac{h^3}{12} \)

In general this is an accurate, simple test. It also has the advantage that, by the 8th order theory of Massonnet, the theoretical formula for \( D_{xy} \) is the same as that for the 4th order theory (see Eq. 2.37). In other words, for an asymmetrically stiffened plate, the eccentricity of the ribs does not cause extra terms to be added to the theoretical formula for \( D_{xy} \).
2. **Another Method**

The method given below was developed by the writer who was trying to find a way of determining the rigidities using only one model instead of three. This was done eventually and used for the experimental work in this thesis, and is described in section 3. of this chapter. Other investigators might be tempted to pursue an approach described in this section (section 2.) which appears promising at first. It is shown, however, that this approach still requires more than one model for testing.

First of all it should be pointed out that, in the method in section 1. above, it is not necessary to determine the constants $a_{11}$, $a_{12}$, $a_{21}$ and $a_{22}$ and then to calculate $D_x$, $D_{12}$, $D_{21}$ and $D_y$ from them. The rigidities can be found more directly as follows. Considering an asymmetrically stiffened plate, in Fig. 4.2 let the stiffeners run parallel to line 1. (i.e. $\theta = 0$). Then at the centre of the plate we can say

$$
M_x = \frac{P}{4} = D_x \frac{d^2w}{dx^2} + D_{12} \frac{d^2w}{dy^2}
$$

$$
M_y = 0 = D_y \frac{d^2w}{dy^2} + D_{21} \frac{d^2w}{dx^2}
$$

The values of $\frac{d^2w}{dx^2}$ and $\frac{d^2w}{dy^2}$ can be measured experimentally, and so we have two equations in the four unknowns $D_x$, $D_{12}$, $D_{21}$ and $D_y$.

Now let the stiffeners run parallel to the $y$ direction in Fig. 4.2 (i.e. $\theta = 90$), and we can say

$$
M_y = \frac{P}{4} = D_y \frac{d^2w}{dy^2} + D_{21} \frac{d^2w}{dx^2}
$$
\[ M_x = 0 = D_x \frac{d^2 w}{dx^2} + D_{12} \frac{d^2 w}{dy^2} \]

The values of \( \frac{d^2 w}{dx^2} \) and \( \frac{d^2 w}{dy^2} \) are again measured, and now we have four independent equations in the four unknowns, which can be solved. The only experimental difference between the method of section 1. and this method is that deflections are measured in the former and curvatures in the latter.

An approach to a method of finding the rigidities is seen by considering the twisting test (Fig. 4.5). Although it is used in section 1. to measure \( a_{yy} \) (or \( D_{xy} \)) only, the plate does bend as well. If the principal directions of stiffness \((X, Y)\) coincide with the axes \((x, y)\) in Fig. 4.5 (i.e. \( \theta = 0^\circ \) or \( 90^\circ \)) then along lines 1. and 2. there is no bending, only twisting; along lines 3. and 4. there is pure bending and no twisting. Since this bending occurs, it would seem that by making some sort of measurements, some information about the bending rigidities \( D_x, D_{12}, D_{21} \) and \( D_y \) should be able to be found. What can be found is discussed below.

![Diagram](image)

Fig. 4.6
Consider a square plate (Fig. 4.6) loaded as shown at the corners by a force $P$. Let the stiffeners run in the general direction $X$ at an angle $\theta$ to the diagonal $BD$. As was shown in Chapter 3 (Eqs 3.13) the moments in the plate at any point can be represented by a Mohr's circle (Fig. 4.7). From statics $M_{\text{max}} = P/2$; $M_{\text{min}} = -P/2$ and the origin of the coordinates in Fig. 4.7 corresponds to the centre of the circle. It can be seen that

\[ M_x = \frac{P}{2} \cos 2\theta = -M_y \]  
\[ M_{xy} = \frac{P}{2} \sin 2\theta \]  

Eqs 4.2

Now choose another general rectangular pair of axes $X', Y'$ such that $\hat{X}OX' = \alpha$. Using Eqs 3.13 to express $M'_x, M'_y$ and $M'_{xy}$ in terms of $M_x, M_y$ and $M_{xy}$, we have

\[ M'_x = \frac{(M_x + M_y)}{2} + \frac{(M_x - M_y)}{2} \cos 2\alpha - M_{xy} \sin 2\alpha \]
\[ M'_y = \frac{(M_x + M_y)}{2} - \frac{(M_x - M_y)}{2} \cos 2\alpha + M_{xy} \sin 2\alpha \]

\[ M'_{xy} = \frac{(M_x - M_y)}{2} \sin 2\alpha + M_{xy} \cos 2\alpha \]

Eqs 4.3

From Fig. 4.7

\[ M'_x = \frac{P}{2} \cos (2\theta + 2\alpha) = -M'_y \]

Eqs 4.4

\[ M'_{xy} = \frac{P}{2} \sin (2\theta + 2\alpha) \]

Eqs 4.3 can now be used to give relationships between the rigidities, since for

\[ M_x \] we substitute \( D_x \frac{d^2w}{dx^2} + D_{12} \frac{d^2w}{dy^2} \)

\[ M_y \] we substitute \( D_y \frac{d^2w}{dy^2} + D_{21} \frac{d^2w}{dx^2} \)

\[ M_{xy} \] we substitute \( 2D_{xy} \frac{d^2w}{dxdy} \)

where \( \frac{d^2w}{dx^2} \), \( \frac{d^2w}{dy^2} \) and \( \frac{d^2w}{dxdy} \) are measured by experiment. \( \theta \) is fixed for a particular model.

In the general case where \( \theta \) lies between 0° and 45°, we have three basic equations

\[ M'_x = \frac{P}{2} \cos (2\theta + 2\alpha) = \frac{(M_x - M_y)}{2} \cos 2\alpha - M_{xy} \sin 2\alpha \quad (1) \]

\[ M'_{xy} = \frac{P}{2} \sin (2\theta + 2\alpha) = \frac{(M_x - M_y)}{2} \sin 2\alpha + M_{xy} \cos 2\alpha \quad (2) \]

\[ \frac{(M_x + M_y)}{2} = 0 \quad (3) \]
Let us see what happens when we choose different values of $\alpha$ and substitute them into the above equations.

For equation (1):

$$\alpha = 0^\circ : \frac{P}{2} \cos 2\theta = \frac{(M_x - M_y)}{2}$$  \hspace{1cm} (1A)

$$\alpha = 45^\circ : -\frac{P}{2} \sin 2\theta = -M_{xy}$$  \hspace{1cm} (1B)

$$\alpha = 90^\circ : -\frac{P}{2} \cos 2\theta = -\frac{(M_x - M_y)}{2}$$  \hspace{1cm} (1C) = (1A)

$$\alpha = 30^\circ : \frac{P}{2} \cos 2\theta \frac{1}{2} - \frac{P}{2} \sin 2\theta \frac{\sqrt{3}}{2}$$

$$= \frac{(M_x - M_y)}{2} \frac{1}{2} - M_{xy} \frac{\sqrt{3}}{2}$$  \hspace{1cm} (1D)

$$= \frac{1}{2} (1A) + \frac{\sqrt{3}}{2} (1B)$$

$$\alpha = 60^\circ : -\frac{P}{2} \cos 2\theta \frac{1}{2} - \frac{P}{2} \sin 2\theta \frac{\sqrt{3}}{2}$$

$$= -\frac{(M_x - M_y)}{2} \frac{1}{2} - M_{xy} \frac{\sqrt{3}}{2}$$  \hspace{1cm} (1E)

$$= -\frac{1}{2} (1A) + \frac{\sqrt{3}}{2} (1B)$$

For equation (2):

$$\alpha = 0^\circ : \frac{P}{2} \sin 2\theta = M_{xy}$$  \hspace{1cm} (2A) = (1B)

$$\alpha = 45^\circ : \frac{P}{2} \cos 2\theta = \frac{(M_x - M_y)}{2}$$  \hspace{1cm} (2B) = (1A)
\[
\alpha = 30^\circ: \frac{P}{2} \cos 2\theta \sqrt{3} + \frac{P}{2} \sin 2\theta \frac{1}{2} \]

\[
= \frac{(M_x - M_y)}{2} \sqrt{3} \frac{1}{2} + M_{xy} \frac{1}{2}
\]

(2C)

\[
(2C) = \frac{\sqrt{3}}{2} (1A) - \frac{1}{2} (1B)
\]

It can be seen that equations (1) and (2) yield no more than two independent equations, namely (1A) and (1B). The conclusion is that, when the stiffeners are placed in a general direction at an angle \( \theta \) to the diagonal, the total possible number of independent equations relating the rigidities that can be found is three, namely (1A), (1B) and (3).

**In the case \( \theta = 0^\circ \)**

We can find the following equations

\[
\frac{(M_x + M_y)}{2} = 0
\]

(3)

\[
\frac{P}{2} = \frac{(M_x - M_y)}{2}
\]

(1A)

\[
0 = M_{xy}
\]

(1B)

However, \( \frac{\partial^2 w}{\partial x \partial y} = 0 \) in this case and equation (1B) therefore gives no information. Therefore only two independent equations may be found by experiment when \( \theta = 0 \).
In the case \( \theta = 45^\circ \)

Here the stiffeners are parallel to the sides AB,CD of the plate, as in the twisting test in section 1. We have the following equations:

\[
\frac{P}{2} = M_{xy} \quad (1B)
\]

\[
\frac{M_x + M_y}{2} = 0 \quad (3)
\]

\[
\frac{M_x - M_y}{2} = 0 \quad (1A)
\]

However when \( \theta = 45^\circ \) we have

\[
\frac{d^2w}{dx^2} = \frac{d^2w}{dy^2} = 0
\]

Every term in the expressions for \( M_x \) and \( M_y \) has either \( \frac{d^2w}{dx^2} \) or \( \frac{d^2w}{dy^2} \) as a factor, and therefore equations (3) and (1A) give no information. The only information one can possibly find from a twisting test when \( \theta = 45^\circ \) is from

\[
\frac{P}{2} = M_{xy} = 2D_{xy} \frac{d^2w}{dx\,dy} \quad (1B)
\]

From the above we conclude that, if one wants to find all the rigidities \( D_x, D_{12}, D_{21}, D_y \) and \( D_{xy} \) by means of a twisting test as in Fig. 4.6, one must use more than one plate model. This is because \( \theta \) must have more than one value. If one of the models has \( \theta = 45^\circ \), then two other models must be used (say \( \theta = 0^\circ, 22\frac{1}{2}^\circ \)). If none of the models has \( \theta = 45^\circ \), then two models is the total minimum
number which must be used.

3. **The Writer's Method**

It was decided that tests would be done on a number of technically orthotropic plates to determine the values of the rigidities $D_x$, $D_{12}$, $D_{21}$, $D_y$ and $D_{xy}$. These values were to be compared with the values given by theoretical formulae. The type of orthotropic plate chosen was the asymmetrically stiffened plate with stiffeners running in one direction only. As has been stated, a major consideration was to find a method of determining the rigidities which involved making only one model of each orthotropic plate. The method used is described below.

**The Twisting Test**

The torsional rigidity, $D_{xy}$, is determined by means of a twisting test similar to that described in section 1. The stiffeners are made parallel to a side of the plate.

![Diagram](image.png)

Fig. 4.8

Points B and D of the plate are supported on screws, and point A is held down by another fixed screw. The applied
load, $P$, is transmitted to the plate through a deflection gauge (1 div. = 0.01 mm).
The fixed-screws are attached to a model-holding-board, as in Fig. 4.9, and the deflection gauge is fixed into the board. The model creeps under load, as will be discussed in Chapter 5, and the model-holding-board must be tapped lightly and continuously to keep the hand of the gauge moving steadily. The friction in the gauge is small, but it does exist and the board must be tapped for accurate results.

\[ D_{xy} \] is found as follows:

\[ M_{xy} = 2D_{xy} \frac{d^2w}{dxdy} \]

\( \frac{d^2w}{dxdy} \) is the same at all points of the plate and is the change of slope in the X direction per unit distance in the Y direction. If \( w \) is the deflection of the corner point C in Fig. 4.8 (A,B,D being fixed), and if \( \ell \) is the length of side of the square plate, then

\[ \frac{d^2w}{dxdy} = \frac{w}{\ell^2} \]

\[ \therefore \frac{P}{2} = 2D_{xy} \frac{w}{\ell^2} \]

\[ \therefore D_{xy} = \frac{P\ell^2}{4w} \quad \text{Eq 4.4} \]

The Bending Test

The same square model used in the twisting test is used here. It is supported on its four corner points on screws which are fixed to the model-holding-board. This board is the same one as that used in the twisting test. The conversion to the bending test requires the replacement of
the downward screw at A and the deflection gauge at C by upward fixed screws.

Fig. 4.10

Fig. 4.11
The model is loaded in the positions shown in Fig. 4.11, and the whole apparatus is shown in Fig. 4.12. The load is applied through deflection gauges as in the calibrating test. The friction in the deflection gauges is considered to be less than the friction in a system using a plunger and platform (see Fig. 4.13).

![Diagram](image)

**Fig. 4.13**

Consider the loaded plate (Fig. 4.11). The reactions at the four corners can be found from statics. In each load case equations are found for the total statical moment at midspan. If \( P = 1 \text{N} \), for case (a), at section YY: \( M_x = 2 \frac{3}{8} P \) 80 N mm

\[ M_x = 60 \text{ N mm} \]

and at section XX: \( M_y = 60 \text{ P N mm} \)

\[ M_y = 60 \text{ N mm} \]

Similarly the moments are found for cases (b) and (c). The moments could be found at any section, not necessarily at midspan. If the section is too close to the loads however, the curvatures are difficult to determine accurately.
These total statical moments can also be expressed in terms of the bending rigidities. Consider the $M_x$ equation and consider the midspan section YY (length $160\ mm$) divided into many small divisions, each of size $dy$.

![Diagram](image)

**Fig. 4.14**

At the centre point of this element there exist curvatures $\frac{d^2w}{dx^2}$ and $\frac{d^2w}{dy^2}$.

The moment per unit length at this point is

$$M_x = D_x \frac{d^2w}{dx^2} + D_{12} \frac{d^2w}{dy^2}$$

The units are

$$N\frac{mm}{mm} = N\ mm\ \frac{1}{mm} + N\ mm\ \frac{1}{mm}$$

The total statical moment over the whole length of the midspan section is

$$\int_{-80}^{80} M_x\ dy = D_x \int_{-80}^{80} \frac{d^2w}{dx^2}\ dy + D_{12} \int_{-80}^{80} \frac{d^2w}{dy^2}\ dy \quad \text{Eq 4.5}$$

The units are now

$$N\frac{mm}{mm}\ mm = (N\ mm)\ \frac{1}{mm}\ mm + (N\ mm)\ \frac{1}{mm}\ mm$$

The integrated curvature terms are found as follows. The
test is done in the Moiré apparatus (see Chapter 5). From Moiré photographs by taking lines parallel to the X direction at regularly spaced intervals, the values of $\frac{d^2w}{dx^2}$ at nine points are obtained.

![Diagram of Moiré photographs]

**Fig. 4.15**

The value of $\int_{-80}^{80} \frac{d^2w}{dx^2} \, dy$ is now found by Simpson's Rule. A point to notice is that it does not matter whether the plate is resting equally on all four supports. It may rest on only three yet it will not affect the moment equation. The values of $\frac{d^2w}{dx^2}$ will be different but the moment equation will still be valid.

The value of the term $\int_{-80}^{80} \frac{d^2w}{dy^2} \, dy$ is simple to obtain since

$$\int_{-80}^{80} \frac{d^2w}{dy^2} \, dy = \left( \frac{dw}{dy} \right)_y = 80 - \left( \frac{dw}{dy} \right)_y = -80$$
No curve plotting is necessary. The required values can be estimated from the enlarged Moiré photograph with reasonable accuracy.

The moment equation in the $Y$ direction is derived similarly.

\[
\int_{-80}^{80} M_y \, dx = D_y \int_{-80}^{80} \frac{d^2 w}{dy^2} \, dx + D_{21} \int_{-80}^{80} \frac{d^2 w}{dx^2} \, dx \quad \text{Eq 4.6}
\]

For each of the load cases in Fig. 4.11 there are two moment equations, namely for $M_x$ and $M_y$ at midspan. Since there are four unknowns $D_x$, $D_{12}$, $D_{21}$ and $D_y$ to be found, only two load cases are necessary for their solution. The third load case acts as a check and gives extra information resulting in better average values of the unknowns.

The positions of the loads were chosen with three considerations in mind.

1. The loads must not be too close to the points at which the curvatures are required as this makes the curvatures difficult to determine accurately.

2. The loading points must be on or between the ribs of all four different plate rib spacings (see Chapter 5). In other words a load cannot be applied right on the edge or just off a rib.

3. The loads must bend the plate in modes which are as different as possible. This gives well-conditioned equations which produce accurate values for the rigidities.

The limitations of this bending test are as follows.

The plate is not in pure bending. According to Massonnet
this means that the theoretical formulae for the bending rigidities must have extra terms in them due to the eccentricity of the ribs. It must be borne in mind, therefore, that values of rigidities found by experiment should be larger than the theoretical values calculated on the basis of the 4th order theory. For the twisting test however the experimental values should coincide with those calculated by the 4th order theory. The other limitation is that, when the ratio of $D_x$ to $D_y$ is large, the plate tends to bend only in the $Y$-direction no matter where the point loads are put on the plate. In other words, before an appreciable curvature in the $X$-direction occurs, the plate may be deforming plastically in the $Y$-direction. The solution would be for the plate to be simply supported continuously along two opposite edges rather than on four corner support points. In the bending tests performed by the writer the model was only supported on the four corner points.
1. **The Models**

It was decided to make 16 different orthotropic plate models for testing. These were asymmetrically stiffened plates with ribs running in one direction only. This is a common civil engineering application which is seen in concrete bridges and floor slabs of large buildings. The 16 models fall into 4 sets of 4. In each set of 4 the moment of inertia in the X (stiff) direction is the same. In other words the theoretical value of $D_x$ is the same. The plates are shown in full scale in Fig. 5.1. In all the plates the ratio of rib spacing to rib width, $s/b$, is constant at 4 : 1. This is necessary to keep $D_x$ the same in each set of 4 plates. The different rib depths give approximate ratios of $D_x/D$ of 1.5, 3, 10 and 25.

The objective of the tests is to compare the test values with the theoretical values of the rigidities. However the objective may be more specifically defined as follows:
Plate

4/15
4/3
4/6
4/9
6/15
6/3
6/6
6/9
8/15
8/3
8/6
8/9
12/15
12/3
12/6
12/9

Scale: full size
Dimensions in mm

Fig 5.1.
(a) To determine how well an asymmetrically stiffened plate behaves as an orthotropic plate, i.e. how far apart may the ribs be spaced before the plate becomes a system of plate elements and beam elements.

(b) To determine the influence, if any, on the rigidities $D_x$, $D_1$, $D_y$, of altering the spacing of the ribs of stiffened plates, which have the same theoretical values of $D_x$.

(c) To determine whether there is any difference between the rigidities $D_{12}$ and $D_{21}$, which are the same for a true orthotropic plate and are usually assumed to be the same for an asymmetrically stiffened plate.

The bending test was done in the Moiré apparatus, and therefore the models had to have a reflecting surface. They were therefore made of black perspex which is a good reflecting material commonly used in Moiré work. Black perspex is obtainable in sheets of nominal thickness of $1/8"$, $3/16"$ and $1/4"$. In metric units these sizes are approximately 3.2, 4.8 and 6.4 mm. The thickness of the sheets varies considerably and the material was milled down to the required size in the case of the thinner plates. For the thicker plates clear perspex which comes in thicker sheets was used, also being milled to size.

It is not possible to make the plate and ribs separately and then to glue them together as the glue warps the plate which is only 3 mm thick. Models 4/6 and 12/9 were made by glueing two $1/4"$ sheets of black perspex together and then milling out the plate. It was found however by glueing a sheet of black to a sheet of clear perspex that air bubbles
are left between the sheets when the glue dries. These discontinuities should in general decrease the strength of the resulting plate. Model 12/9 was also made out of clear perspex. The bending test was only performed on the thinner 8 plates. The thicker models made of clear perspex can also be tested if they are painted or given a reflective coating in some way. The best method seems to be to dip the models in a fairly thin solution of high gloss black paint. More than one coat may be needed. Other methods considered were sticking a transfer on the surface or spray painting or electroplating it. No suitable transfer could be found and electroplating of this type is not done by any firm in Cape Town. The painted surface is not as flat nor as reflective as the black perspex surface and the results can not be expected to be as good with this method.

2. Variation in dimensions

The variation of the dimensions of the models from those specified is not negligible since the plates are thin. A small absolute deviation may therefore be a relatively large one. Correction factors were applied to the stiffnesses of the models to compensate for this variation. These factors are defined as follows:

\[
\begin{align*}
D_x \text{ (actual)} &= k_x \cdot D_x \text{ (specified)} \\
D_{12} \text{ (actual)} &= k_x \cdot D_{12} \text{ (specified)} \\
D_y \text{ (actual)} &= k_y \cdot D_y \text{ (specified)} \\
D_{21} \text{ (actual)} &= k_y \cdot D_{21} \text{ (specified)}
\end{align*}
\]

In order to calculate a correction factor a theoretical
formula for the rigidity in question must be assumed. The following were the formulae taken:

\[
D_y = D = \frac{Eh^3}{12(1-\nu^2)} \quad \text{Eq 2.43}
\]

\[
D_x = \frac{EI_x}{s} \quad \text{Eq 2.41}
\]

The dimensions were measured at the points shown in Fig. 5.3. The symbols used for the dimensions are shown in Fig. 5.2.
Measurements are made on each rib and between each rib.

The thickness of the plates was measured at the positions
shown by means of a dial gauge set above a base which was surface-ground to make it as smooth as possible. The dimension b was measured with a vernier.

Fig. 5.4

Each thickness value was an estimated average of the thickness in the vicinity of the points shown, obtained by moving the plate around on the base.

Calculation of $k_y$

Since the equation for $M_y$ in the bending test is applied at the midspan section XX, the thickness measurements along this line are used (see Fig. 5.3). For each plate 7 equally spaced points are used. Since $D_y$ depends on the cube of the thickness $h$, the average value of $h$ can be calculated as

$$h_m = \sqrt[3]{\frac{\sum h_i^3}{7}}$$
It is not necessary, however, to calculate the cubes of each measurement as an approximate method is sufficiently accurate. The specified plate thickness is 3 mm for all models which, by chance, also simplifies the calculation as shown below.

<table>
<thead>
<tr>
<th></th>
<th>$h$</th>
<th>$h^3$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$3 + \Delta_1$</td>
<td>$3^3 + 3 \cdot 3^2 \Delta_1 + 3 \cdot 3 \cdot \Delta_1^2 + \Delta_1^3$</td>
</tr>
<tr>
<td></td>
<td>$3 + \Delta_2$</td>
<td>$3^3 + 3 \cdot 3^2 \Delta_2 + 3 \cdot 3 \cdot \Delta_2^2 + \Delta_2^3$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td></td>
<td>$3 + \Delta_n$</td>
<td>$3^3 + 3 \cdot 3^2 \Delta_n + 3 \cdot 3 \cdot \Delta_n^2 + \Delta_n^3$</td>
</tr>
<tr>
<td></td>
<td>$3n + \Sigma \Delta$</td>
<td>$3^3 \cdot \Sigma \Delta + 3 \cdot 3 \cdot \Sigma \Delta^2 + \Sigma \Delta^3$</td>
</tr>
<tr>
<td>mean</td>
<td>$3 + \frac{\Sigma \Delta}{n}$</td>
<td>$3^3 + 3 \cdot 3^2 \frac{\Sigma \Delta}{n} + 3 \cdot 3 \cdot \frac{\Sigma \Delta^2}{n} + \frac{\Sigma \Delta^3}{n}$</td>
</tr>
</tbody>
</table>

From the above, the geometric mean is defined from

$$h_m^3 = 3^3 + 3 \cdot 3^2 \frac{\Sigma \Delta}{n} + 3 \cdot 3 \cdot \frac{\Sigma \Delta^2}{n} + \frac{\Sigma \Delta^3}{n}$$

The arithmetic mean is

$$h_a = 3 + \frac{\Sigma \Delta}{n}$$

$$h_a^3 = 3^3 + 3^3 \frac{\Sigma \Delta}{n} + 3^2 \left( \frac{\Sigma \Delta}{n} \right)^2 + \left( \frac{\Sigma \Delta}{n} \right)^3$$

Since $\Delta$ is small, the last two terms in $h_m$ and $h_a$ can be neglected, giving

$$h_a = h_m$$

The correction factor $k_y$ is given by

$$k_y = \frac{h_m^3}{3^3}$$
since the specified plate thickness is 3 mm.

\[ k_y = \frac{3^3 + 3 \sum \Delta}{3^3} = 1 + \frac{\sum \Delta}{n} \]

Therefore \( k_y \) can be found from the arithmetic mean of \( h \).

For example, if \( h_a = 3.047 \) mm

then \( k_y = 1.047 \).

This is quite accurate enough for the purpose of these experiments.

It is important to know how close the true mean value of \( h \) is to the calculated mean value \( h_a \). For this reason we calculate standard deviations. The standard deviation of a single observation of \( h \) is

\[ SDSOH = \frac{\sum (h_i - h_a)^2}{n} \]

The standard deviation of the mean of \( h, h_a \), is

\[ SDMNH = \frac{SDSOH}{\sqrt{n}} \]

The standard deviation of a single observation of \( h, SDSOH \), describes the accuracy with which we know the value of \( h \) at any point along section XX. For example, if \( SDSOH = 0.04 \) mm, then we can say with 68% confidence that at any particular point of the plate along the section XX the measured value of \( h \) will be in the range \( h_a \pm 0.04 \) mm. If \( SDSOH = 0.04 \) mm, then \( SDMNH = \frac{0.04}{\sqrt{7}} = 0.015 \) mm. We can now say with 68% confidence that the mean value of \( h \) lies in the range \( h_a \pm 0.015 \) mm. Consider Fig. 5.5.
In the above calculations it has been assumed that \( h \) is normally distributed. This should be so from the Central Limit Theorem of statistics because each measured value of \( h \) was an average value for its surrounding area. We would expect \( h \) to be normally distributed anyway in the absence of evidence to the contrary. By cubing each value of \( h \), we obtain the distribution of \( h^3 \). Strictly speaking it will be log-normally distributed, but the proof that \( h_a = h_m \) implies that it is approximately normally distributed too.

Consider a value of \( h \) of \( (h_a + \Delta) \). Cubing this we get

\[
(h_a + \Delta)^3 = h_a^3 + 3h_a^2 \Delta + \ldots
\]

Now let \( \Delta = \text{SDSOH} \).

Then we may say that the standard deviation of \( h^3 \) is

\[
h_a^3 + 3h_a^2 \Delta - h_a^3 = 3h_a^2 \Delta
\]

\[
= 3h_a^2 \text{SDSOH}
\]

Now the standard deviation of the mean of \( h^3 \) is \( 3h_a^2 \text{SDMNH} \).

The relative standard deviation of the mean of \( h^3 \) is
RSDMNH3 = \frac{3h_a^2 SDMNH}{h_a^3}

Since \( h_a \neq 3 \) mm

\[ RSDMNH3 \div SDMNH \]

As an example of what this means, if SDMNH = 0.015 mm, then we may say with 68% confidence that the mean of \( h_a^3 \) lies within the range

\[ h_a^3 (1 \pm RSDMNH3) \]

or \( h_a^3 (1 \pm 0.015) \)

In other words we are 68% confident that the true value of the mean of \( h_a^3 \) lies within 1.5% of our calculated value \( h_a^3 \).

Note that RSDMNH3 is dimensionless but SDMNH is in mm.

Calculation of \( k_x \)

For the calculation of \( k_x \) the measurements along the midspan section YY must be used (see Fig. 5.3). First of all the quantities \( h_a \), SDSOH and SDMNH are calculated. Then the values of \( d_i \) are obtained from

\[ d_i = (h + d)_i - h_a \]

Then \( d_a \), SDSOD and SDMND are found. Next \( b_a \), SDSOB and SDMNB are found. The notation is the same as for \( h \) in the calculation of \( k_y \). The theoretical formula used for \( D_x \) is

\[ D_x = \frac{EI_x}{s} \]

where \( \frac{I_x}{s} \) is the moment of inertia per unit width of plate.
For the T-section

\[ I_x = \frac{h^3}{12} + \frac{bd^3}{12s} + \frac{b2h(d+h)^2}{4(sh+bd)} \]

By substituting \( h_a \) for \( h \), \( d_a \) for \( d \), \( b_a \) for \( b \) we find \( I_x \) (model). Then

\[ k_x = \frac{I_x(\text{model})}{I_x(\text{specified})} \]

The relative standard deviation of the mean of \( I_x \), \( RSDMNX \), can also be calculated. From the theory of errors if

\[ F = f(a,b, \ldots , n) \]

and the standard deviations of the dependent variables are \( SDA, SDB, \ldots , SDN \), then the standard deviation of \( F \) is

\[ SD_F = \sqrt{\left(\frac{dF}{da}\right)^2 SDA^2 + \left(\frac{dF}{db}\right)^2 SDB^2 + \ldots + \left(\frac{dF}{dn}\right)^2 SDN^2} \]

Applying this to the formula for \( I_x \) and taking SDMNS = 0,

\[ SDMNX = \sqrt{\left(\frac{dI_x}{dh}\right)^2 SDMNH^2 + \left(\frac{dI_x}{dd}\right)^2 SDMND^2 + \ldots + \left(\frac{dI_x}{dn}\right)^2 SDMNB^2} \]

Then \( RSDMNX = \frac{SDMNX}{I_x} \)

In Appendix A the computer program which calculates \( k_x \), \( k_y \), \( RSDMNH3 \) and \( RSDMNX \) will be found.

3. Creep

A factor which has to be taken into account in the bending and twisting tests is the phenomenon of creep.
When a plate is loaded it does not immediately bend to its final deflected position. It deflects rapidly at first, say over the first minute, slowing down all the time. It continues to deflect appreciably for at least the first 12 minutes after loading. Fig. 5.6 shows a graph of deflection against time obtained in a preliminary test. This was a T-beam model made of black perspex. It can be seen that the creep rate is still appreciable at 12 minutes. Presumably as the time tends to infinity the creep rate falls off to zero, and it may be unnoticeable after say 30 minutes. One cannot wait for this during the experiments so the deflections must be measured at a certain known time after the load is applied. The times used by the writer were 2 minutes or 5 minutes. It can be seen that the creep rate is not vastly different at these two times, but other investigators seem to use 4 or 5 minute creep times. It can be seen in Fig. 5.6 that when the model is unloaded, the creep rate falls off more quickly than when the model is loaded. This is because there is an initial load still on the model causing it to creep in the opposite direction. It is also possible that gravity is resisting the upward movement of the model.

In the next section will be found the writer's method of determining $E$ and $\nu$ for black perspex. It will be realised that $E$ must be defined as the $E$ at a particular creep time, because as a model deflects its strength apparently decreases. Since the rigidities are calculated from $E$ among other factors, they are also defined as the rigidities at a particular creep time.
The twisting test was performed on the thinner 8 models using a creep time of 5 minutes. The deflections were measured at 1 minute intervals, and so it was possible to calculate the ratio between $E_2$ and $E_5$, where the subscript denotes the creep time in minutes.

$$\frac{E_2}{E_5} = \frac{w_5}{w_2}$$

where $w$ is the deflection of the corner of the plate. This ratio was calculated for 27 applications of a load and averaged to find $E_2/E_5 = 1.018$ with a standard deviation of 0.005. This value should be independent of temperature or the applied load. The writer has not found any of the other ratios corresponding to different creep times, but a rough estimate is that $w_{12}/w_2 = 1.05$.

In other words the strain increases by 5% between 2 and 12 minutes after loading.

4. The Moiré Method

The Moiré apparatus was used to determine the curvatures required in the bending test, and is shown in Fig. 5.8. The application of the Moiré method to plate bending analysis was first described by Ligtenberg (Ref. 27) and will not be discussed here.

![Diagram of Moiré apparatus](attachment:image.png)
The constant for the apparatus which is needed to calculate the curvatures is, with the notation in Fig. 5.7, the value of $d/2a$. In the experiments values were

$$d = 2.14 \text{ mm}$$
$$a = 795 \text{ mm}$$
$$d/2a = 0.001344$$

$= \text{ the change in slope between two adjacent fringes.}$

The lighting is provided by four 500 W lights. The heat generated by these is considerable, and the model warms up quickly when they are switched on. As will be seen in Chapter 6, the values of elastic constants determined in the

![Fig. 5.8](image)

Moiré apparatus differ appreciably from those determined in a test done at room temperature. This effect is allowed
for when comparing twisting test with bending test results.

The apparatus is enclosed by a roof and curtains when in operation. This is because the exposure times for the film would otherwise vary considerably depending on the time of day or night when photographs are taken. The screen can be rotated about a vertical axis so that the curvatures in any direction in the model may be found. The camera used is a Linhof and Professional Line copy film is used for good contrast. The negatives can be enlarged to full scale in an enlarger in the dark room.

The method used in taking the photographs using a creep time of 2 minutes is as follows:

a. The initial load due to the spring force of the deflection gauges is applied.

b. The plate is loaded through the deflection gauges with the force $P$ required to produce sufficient fringes. $P$ is applied for 2 minutes.

c. The plate is unloaded for 2 minutes. Steps b. and c. make up the first loading cycle during which no photographs are taken. This is to allow the model to settle in on its supports.

d. The first exposure is taken and the gauges are read.

e. The load is applied for 2 minutes. The model-holding-board is tapped at intervals to free the deflection gauge spindles of friction.

f. The second exposure is taken and the gauges are read. The deflection difference between the loaded and unloaded states is used to correct $P$ since the force exerted by the deflection gauge spring decreases with
extrusion of the spindle. This change in force is constant at 3 g wt per mm extrusion, and is almost negligible in the bending test but not in the twisting test. In all bending tests the deflection of the plate required to produce sufficient fringes was less than 1/3 of the thickness of the plate (i.e. less than 1 mm).

g. Steps c., d., e. and f. are repeated twice so that 3 photographs are obtained for the particular load case and screen orientation. For the plates with 3 mm deep ribs only one photograph was taken, and a creep time of 5 minutes was used.

Each photograph is developed, the negative enlarged to full scale, and the fringes are traced off. Graphs of slope versus distance along the required lines are plotted.

The curvature at the required point is found by measuring the slope of the curve. Accurate graph paper and an accurate protractor are used.
An alternative to the graphical determination of the curvatures is to fit a polynomial to the curve by the method of least squares using a computer. The polynomial has the form

$$\frac{dw}{dx} = C_1 + C_2 x + C_3 x^2 + \ldots + C_n x^n$$

The degree, \( n \), of the polynomial can be chosen and it must be a maximum of one less than the number of points on the curve. The writer wrote and ran a programme but the results were not as satisfactory as those of the graphical method. The reasons for this were as follows. The difference between results using 3rd, 4th and 5th degree polynomials was not as small as the writer would have liked, although an average result was acceptable. Also, when a point load is on or near the line in question the curve is not representable by a polynomial. The graphical method is definitely superior in this case to the method of averaging polynomial results. Using the graphical method one can see the effects of the loads more clearly; and is safer.

5. **Calculation of \( E \) and \( \nu \)**

The values of \( E \) and \( \nu \) of black perspex are required for the calculation of theoretical rigidities for comparison with the experimentally determined rigidities. The writer has used the bending test on a black perspex plate of uniform thickness to determine \( E \) and \( \nu \). The plate was 3 mm thick being milled on one face down to size. The tests used were those shown in Fig. 4.11 (a) and (b). The one shown in Fig. 4.11 (c) is the same as Fig. 4.11 (b).
A creep time of 2 minutes was used, and 3 photographs were taken per load case and screen orientation.

The test provides 4 equations in $D$ and $D_1$. Equations $A_1$ and $B_1$ are of the form

$$M_x = k_x (D \int \frac{d^2w}{dx^2} dy + D_1 \int \frac{d^2w}{dy^2} dy)$$

Equations $A_2$ and $B_2$ are of the form

$$M_y = k_y (D \int \frac{d^2w}{dy^2} dx + D_1 \int \frac{d^2w}{dx^2} dx)$$

Correction factors for thickness were calculated as in section 2. above.

$$k_x = 1.047 \quad k_y = 1.076$$

The set of equations was

$$A_1 \quad 8600 = D + 0.355 D_1$$
$$A_2 \quad 8570 = D + 0.343 D_1$$
$$B_1 \quad 7440 = D - 0.0803 D_1$$
$$B_2 \quad 14700 = D + 2.70 D_1$$

where $D$ and $D_1$ are in (N mm).

Equations $A_1$ and $A_2$ are not solved together being very similar. All combinations of the 4 equations, with this exception, are solved. Means are found for $D$ and $D_1$. $\nu$ is then calculated as $\nu = D_1/D$.

The value of $D(1-\nu)/2$ is then calculated for comparison with the result of the twisting test. $E$ is calculated from

$$E = D.12.(1-\nu^2)/h^3$$
Note that this value of $E$ is the value only for a creep time of 2 minutes and at the Moiré apparatus temperature.

Solving equations:

- $A_1$ and $B_1$ : $D_1 = 2570 \text{ N mm}$ \quad $D = 7640 \text{ N mm}$
- $A_1$ and $B_2$ : $D_1 = 2600 \text{ N mm}$ \quad $D = 7680 \text{ N mm}$
- $A_2$ and $B_1$ : $D_1 = 2670 \text{ N mm}$ \quad $D = 7650 \text{ N mm}$
- $A_2$ and $B_2$ : $D_1 = 2600 \text{ N mm}$ \quad $D = 7680 \text{ N mm}$
- $B_1$ and $B_2$ : $D_1 = 2610 \text{ N mm}$ \quad $D = 7650 \text{ N mm}$

Mean $D_1 = 2610 \text{ N mm}$
Mean $D = 7660 \text{ N mm}$
$\therefore \nu = 0.34(1)$
$D(1-\nu)/2 = 2520 \text{ N mm}$
$E_2 = 3.01 \text{ GPa}$

The agreement between the experimental values for $D$ and for $D_1$ can be seen to be good. The variation in $D_1$ between the highest and lowest values for $D_1$ is 4%, and for $D$ is 0.5%.

The value for $E_2$ compares well with that obtained in a preliminary beam test (see section 3.) where the value obtained was $E_2 = 3.00 \text{ GPa}$, after correcting for the temperature difference.

6. A study of the errors in the experiments

The statement 'The error in the Moiré method is 5%' has two interpretations. One is that if a single Moiré photograph is taken and the curve is plotted along a particular line and the curvature is measured at a particular point, then we expect our result to be within 5% of the true curvature (with say 95% confidence). This is the most
likely meaning. The percentage error will depend on the following sources of error:

a. Error in \( \frac{d}{2a} \).

b. Model not perpendicular to camera axis.

c. Error inherent in the curvature of the screen. The radius of the screen is made equal to \( 3.5a \) to minimise this error.

d. Error in enlarging the negative.

e. Error in tracing the fringes.

f. Error in the graph paper.

\( g. \) Error in the protractor.

h. Error in estimating the slope with the protractor.

\( i. \) Error due to temperature difference in the model when taking more than one photograph.

The other interpretation of the statement applies when more than one photograph is taken and results are averaged. Besides the expected error in the mean result calculated by the standard deviation of the mean, there is also a constant error which is caused by the Moiré method. The second interpretation is that this constant error is 5%.

In these experiments the second interpretation is used. In order to compare theoretical and experimental results a study of the errors in the results is necessary to determine how confident one is of one's conclusions.

Consider the sources of error listed above. In the experiments the constant error in the Moiré method is caused by errors a. to d. It is assumed that the enlarger was set up once and not reset at all. It was in fact reset occasionally, but in the test for E and \( \gamma \) it probably was not.
Errors e. to i. are not constant but vary from photograph to photograph. They are the main source of the standard deviation of the mean of the result found.

Suppose that we are measuring $D$. Then

$$D_{\text{true}} = k_m D_{\text{mean}}$$

$D_{\text{mean}}$ is the experimental mean

$k_m$ is the constant error in the Moiré method and can be expressed as

$$k_m = k_a k_b k_c k_d$$

All $k$ are close to unity.

From the theory of errors

$$(TSDMND)^2 = (D, SDKM)^2 + (k_m, SDMND)^2$$

$SDMND$ is the standard deviation of the mean of $D$ calculated from the experimental results.

$SDKM$ is the standard deviation of $k_m$.

$TSDMND$ is the true standard deviation of the mean of $D$.

Also

$$(SDKM)^2 = (k_b k_c k_d SDKA)^2 + \ldots + (k_a k_b k_c SDKD)^2$$

The writer estimates $SDKA$, $\ldots$, $SDKD$ as the following values.

Errors a. and b. can be taken together as $k_a$ and $k_b$ left out.

$SDKA \div 2/800 = 0.0025$

$SDKC \div 0$

$SDKD \div 1/160 = 0.0063$

$\therefore (SDKM)^2 \div (1.0, 0.0025)^2 + (1.0, 0.0063)^2$

$$= (0.0068)^2$$
In other words it is estimated with 68% confidence that, due to the constant error in the Moiré method, the mean value of D is within 0.7% of the calculated mean of D. Or, it is estimated with 95% confidence that it is within 1.4% of the calculated mean of D.

The error due to the variation in the dimensions

It will be realised that the variation of the model dimensions is a source of uncertainty in the results. More specifically, when a correction factor $k_x$ or $k_y$ is calculated, there is a degree of uncertainty in the value obtained. However, it is not necessary to do extra calculations to allow for this as the uncertainty is taken into the individual results for D, $D_x$, $D_y$, or $D_1$. When these results are averaged, part of the standard deviation is due to the uncertainty in the dimensions. There is no constant error like that of the Moiré method. RSDMNIX and RSDMNH3 as described in section 2. above are only used to show that the number of dimension measurements is sufficient; they are not used for further calculation.

Standard deviations of D, $v$, and E

Using the results in section 5. above, the following values have been calculated:

- $\text{Mean } D = 7660 \text{ N mm}$
- $\text{SDMND} = 9.4 \text{ N mm}$
- $\text{TSDMND} = 53 \text{ N mm}$
- $\text{RSDMND} = \frac{TSDMND}{D} = 0.007$
Mean $\nu = 0.341$
SDMN $\nu = 0.0024$
RSDMN $\nu = 0.007$

$E = 3.01$ GPa
SDMNE = 0.021 GPa
RSDMNE = 0.007

In the calculation of these values allowance has been made for variation in experimental results and for the constant error in the Moiré method. The conclusions may be stated as follows:

a. We can say with 95% confidence that $D$ of the black perspex plate 3 mm thick lies in the range $D = 7660 (1 \pm 0.014)$ N mm.

b. We can say with 95% confidence that $\nu$ of the black perspex plate 3 mm thick lies in the range $\nu = 0.341 \pm 0.005$.

c. We can say with 95% confidence that $E$ of the black perspex plate 3 mm thick lies in the range $E = 3.01 \pm 0.04$ GPa.

Notice that the relative standard deviations of the three variables are all 0.007. This is the same as SDKM. This means that the variation in experimental results was so small that the constant error in the Moiré method is the overriding source of error.

**Standard deviations of DX1, DX2, DX3, DX4, DY1, DY2, DY3**

In Appendix A will be found the theoretical values for $D_x$ and $D_y$ which are compared to the experimental values.
These are:

\[ DX_1 = \frac{E I_x}{s} \]  
Eq 2.41

\[ DX_2 = \frac{E}{s} \left( \frac{bd^3}{12} + bde_2^2 \right) + \frac{E}{(1-v^2)} \left( \frac{h^3}{12} + he_1^2 \right) \]  
Eq 2.39

\[ DX_3 = \frac{E I_x}{s(1-v^2)} \]  
Eq 2.42

\[ DX_4 = DX_2 + e_1^2 E \left( h + \frac{bd}{s} \right) \]  
Eq 2.40

\[ DY_1 = D \]  
Eq 2.43

\[ DY_2 = \frac{E h^3}{12} \frac{s}{(s-b+a^2 b)} \]  
Eq 2.44

\[ DY_3 \text{ is from Eq 2.45.} \]

Since \( E, D \) and \( v \) are used to calculate these values, there is an uncertainty in them. We now find values for the uncertainties. With the same notation as before

\[ \text{RSDMND}_{DX_1} = \text{RSDMNE} = 0.007 \]
\[ \text{RSDMND}_{DX_3} = \text{RSDMND} = 0.007 \]

Since \( \text{RSDMND}_{DX_2} \) must lie between \( \text{RSDMND}_{DX_1} \) and \( \text{RSDMND}_{DX_3} \):

\[ \text{RSDMND}_{DX_2} = 0.007 \]
\[ \text{RSDMND}_{DX_4} = \text{RSDMND}_{DX_2} = 0.007 \]
\[ \text{RSDMND}_{DY_1} = \text{RSDMND} = 0.007 \]
\[ \text{RSDMND}_{DY_2} = \text{RSDMNE} = 0.007 \]
\[ \text{RSDMND}_{DY_3} = \text{RSDMND} = 0.007 \]
For all the above variables we may therefore say with 95% confidence that the values lie within 1.4% of the calculated values.

7. **Results for the Stiffened Plates**

The bending test on the stiffened plates is conducted in exactly the same way as the test on the plate of uniform thickness described in section 5. above. All three loading cases as shown in Fig. 4.11 are used. A typical set of the 6 different photographs per plate are shown in Fig. 5.10. The photographs are those for the 4/1.5 plate. Fig. 5.11 shows the curves plotted for the two photographs in Fig. 5.10 (a).

The derived equations for the plates 4/1.5 and 4/3 are set out on page 142. The equations for the 3 other 1.5 mm deep ribbed plates are similar to those for the 4/1.5 plate, and the equations for the 3 other 3 mm deep ribbed plates are similar to those for the 4/3 plate.
Fig. 5.10 (a) corresponding to Fig. 4.11 (a)

Photo A1

Photo A2
Photo B1

Fig. 5.10 (b) corresponding to Fig. 4.11 (b)

Photo B2
Fig. 5.10 (c) corresponding to Fig. 4.11 (c)

Photo C1

Photo C2
Fig. 5.11

Change of slope between fringes

\[ \frac{d}{2a} = 0.1344 \times 10^{-2} = 0.2 \text{ inches} \]

<table>
<thead>
<tr>
<th>Line</th>
<th>( \angle )</th>
<th>( \tan \angle )</th>
<th>SM</th>
<th>Prod.</th>
</tr>
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<td>1</td>
<td>0.520</td>
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<td>0.330</td>
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<td>0.660</td>
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<tr>
<td>6</td>
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<td>0.334</td>
<td>4</td>
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<td>9</td>
<td>28.1</td>
<td>0.534</td>
<td>1</td>
<td>0.534</td>
</tr>
</tbody>
</table>

\[ \int_{-80}^{80} \frac{\partial^2 u}{\partial x^2} \, dy = 9.612 \times \frac{20}{3} \times 0.001344 = 16.97 \] (dimensionless)
4 ribs, 1.5 mm deep

The equations are:

A1. \[ 14900 = D_x + 0.628 D_{12} \]
B1. \[ 13000 = D_x + 0.015 D_{12} \]
C1. \[ 24300 = D_x + 3.77 D_{12} \]

A2. \[ 8210 = D_y + 0.178 D_{21} \]
B2. \[ 10600 = D_y + 1.00 D_{21} \]
C2. \[ 7900 = D_y - 0.094 D_{21} \]

4 ribs, 3 mm deep

A1. \[ 31600 = D_x + 1.53 D_{12} \]
B1. \[ 26180 = D_x + 0.302 D_{12} \]
C1. \[ 55400 = D_x + 8.14 D_{12} \]

A2. \[ 8800 = D_y + 0.058 D_{21} \]
B2. \[ 8550 = D_y + 0.302 D_{21} \]
C2. \[ 8050 = D_y - 0.095 D_{21} \]

The solutions of the equations are as follows:
<table>
<thead>
<tr>
<th>Ribs</th>
<th>Rib depth</th>
<th>$D_x$ N mm</th>
<th>$D_{12}$ N mm</th>
<th>$D_{21}$ N mm</th>
<th>$D_y$ N mm</th>
<th>Eqs</th>
</tr>
</thead>
<tbody>
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<td>(1160)</td>
<td>8000</td>
<td>A,C</td>
</tr>
<tr>
<td></td>
<td></td>
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<td>3010</td>
<td>2480</td>
<td>8120</td>
<td>B,C</td>
</tr>
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<td>1,5</td>
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<td>2980</td>
<td>8350</td>
<td>A,B</td>
</tr>
<tr>
<td></td>
<td></td>
<td>13190</td>
<td>2980</td>
<td>3540</td>
<td>8250</td>
<td>A,C</td>
</tr>
<tr>
<td></td>
<td></td>
<td>13020</td>
<td>3020</td>
<td>3130</td>
<td>8210</td>
<td>B,C</td>
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<tr>
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<td>3760</td>
<td>3580</td>
<td>7760</td>
<td>A,B</td>
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<td></td>
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<td>2980</td>
<td>2700</td>
<td>8640</td>
<td>B,C</td>
</tr>
<tr>
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<td>13260</td>
<td>3420</td>
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<td>8390</td>
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<td>4220</td>
<td>8170</td>
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<tr>
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<tr>
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<td></td>
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<td>2730</td>
<td>8550</td>
<td>A,B</td>
</tr>
<tr>
<td></td>
<td></td>
<td>26950</td>
<td>2780</td>
<td>3960</td>
<td>8480</td>
<td>A,C</td>
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<td></td>
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<td></td>
<td>25300</td>
<td>3200</td>
<td>1800</td>
<td>8260</td>
<td>B,C</td>
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</tbody>
</table>

Table 5.1

The values in brackets have been ignored in taking the means.

The mean results for each plate are as follows:
Table 5.2

<table>
<thead>
<tr>
<th>Ribs</th>
<th>Rib depth</th>
<th>$D_x$ N mm</th>
<th>$D_{12}$ N mm</th>
<th>$D_{21}$ N mm</th>
<th>$D_y$ N mm</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>1.5</td>
<td>12980</td>
<td>3030</td>
<td>2690</td>
<td>7940</td>
</tr>
<tr>
<td>6</td>
<td>1.5</td>
<td>13080</td>
<td>3070</td>
<td>3220</td>
<td>8270</td>
</tr>
<tr>
<td>8</td>
<td>1.5</td>
<td>13290</td>
<td>3190</td>
<td>3140</td>
<td>8260</td>
</tr>
<tr>
<td>12</td>
<td>1.5</td>
<td>13390</td>
<td>3090</td>
<td>3460</td>
<td>8210</td>
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<tr>
<td>Mean</td>
<td></td>
<td>13180</td>
<td>3100</td>
<td>3130</td>
<td>8170</td>
</tr>
</tbody>
</table>

| SDMN | 74 | 75 | 160 | 77 |

| RSDMN | 0.006 | 0.024 | 0.051 | 0.009 |

| True RSDMN, $\sigma$ | 0.009 | 0.025 | 0.051 | 0.011 |

Table 5.3

<table>
<thead>
<tr>
<th>Ribs</th>
<th>Rib depth</th>
<th>$D_x$ N mm</th>
<th>$D_{12}$ N mm</th>
<th>$D_{21}$ N mm</th>
<th>$D_y$ N mm</th>
</tr>
</thead>
<tbody>
<tr>
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<td>25780</td>
<td>3980</td>
<td>(920)</td>
<td>8710</td>
</tr>
<tr>
<td>6</td>
<td>3</td>
<td>26860</td>
<td>3060</td>
<td>(3260)</td>
<td>8630</td>
</tr>
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<td>3</td>
<td>25570</td>
<td>3960</td>
<td>(2240)</td>
<td>8270</td>
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<td>3</td>
<td>25640</td>
<td>3190</td>
<td>(1800)</td>
<td>8350</td>
</tr>
<tr>
<td>Mean</td>
<td></td>
<td>25960</td>
<td>3550</td>
<td>-</td>
<td>8490</td>
</tr>
</tbody>
</table>

| SDMN | 262 | 178 | -   | 69 |

| RSDMN | 0.010 | 0.051 | - | 0.008 |

| True RSDMN, $\sigma$ | 0.012 | 0.051 | - | 0.011 |

In the table above for the 3 mm deep ribbed plates the means are for a 2 minute creep time, found by multiplying the actual means by 1.018 (see section 3. above). SDMN is calculated as before and
RSDMN = \frac{SDMN}{Mean}

The constant error in the Moiré method must be taken into account

\[ \text{(True RSDMN)}^2 = \sigma^2 = (\text{RSDMN})^2 + (\text{SDKM})^2 \]

\( \sigma \) represents the true uncertainty with which the variable in question is known.

8. Discussion

a. \( D_x \) results

For the thinner 4 plates it is seen that the experimental value for \( D_x \) increases with the number of ribs. However the total increase is only 3%. It may be said that \( D_x \) does not vary in the range of rib numbers tested. For the thicker 4 plates \( D_x \) can not be said to either increase or decrease with the number of ribs.

The 4 theoretical values of \( D_x \) have been calculated in Appendix A. They are compared to the mean experimental values below.

<table>
<thead>
<tr>
<th></th>
<th>1,5 mm rib</th>
<th>( \frac{(DX - D_x)}{D_x} )</th>
<th>3,0 mm rib</th>
<th>( \frac{(DX - D_x)}{D_x} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>DX1</td>
<td>12060</td>
<td>-0,085</td>
<td>24720</td>
<td>-0,048</td>
</tr>
<tr>
<td>DX2</td>
<td>13030</td>
<td>-0,011</td>
<td>26040</td>
<td>0,003</td>
</tr>
<tr>
<td>DX3</td>
<td>13650</td>
<td>0,036</td>
<td>27970</td>
<td>0,077</td>
</tr>
<tr>
<td>DX4</td>
<td>13660</td>
<td>0,036</td>
<td>30100</td>
<td>0,160</td>
</tr>
<tr>
<td>( D_x )</td>
<td>13180</td>
<td>-</td>
<td>25960</td>
<td>-</td>
</tr>
</tbody>
</table>


The uncertainties in DX were calculated in section 6. To show the comparison between experimental and theoretical results most clearly the frequency distributions of the variables are shown below.

\begin{align*}
\text{DX1} &= \frac{EI_x}{s} & \text{Eq 2.41} \\
\text{DX2} &= \frac{E}{s} \left( \frac{bd^3}{12} + bde_2^2 \right) + \frac{E}{(1-v^2)} \left( \frac{h^3}{12} + he_1^2 \right) & \text{Eq 2.39} \\
\text{DX3} &= \frac{EI_x}{(1-v^2)} & \text{Eq 2.42} \\
\text{DX4} &= \text{DX2} + e_1^2 E \left( h + \frac{bd}{s} \right) & \text{Eq 2.40}
\end{align*}
It can be seen from Figs. 5.12, 5.13 that the formulae for DX1, DX3 and DX4 are highly unlikely to be good estimates of DX for the range of plates tested. The probability that DX2 is the true formula is high. We may say that DX is best expressed by DX2.

As the size of the rib with respect to the plate diminishes, DX2 and DX3 converge. As the size of the rib with respect to the plate increases, DX1 and DX2 converge. Thus other formulae besides DX2 will give good approximate values in special circumstances.

The 8th order theory of Massonnet predicts that DX lies between DX2 and DX4. For the thinner 4 plates DX lies comfortably inside this range, but closer to DX2 (see Fig. 5.12). For the thicker 4 plates the value of DX lies just 0.3% below DX2 (see Fig. 5.13). The conclusion is that the true value of DX can be very little greater than the value given by DX2.

b. D_y results

The spread of the results is small as shown by the value of \( \sigma \) of 0.011 obtained for both the 1.5 mm and 3 mm rib plates. The test therefore determines D_y well. The value of D_y shows no significant increase or decrease with increasing rib numbers.

The theoretical formulae for D_y were:

\[
D_y^1 = D \\
D_y^2 = \frac{Eh^3 s}{12(s-b+\alpha^2 b)}
\]

Eq 2.43

Eq 2.44
DY3 = \frac{B^5}{30\int_0^B \frac{(By - y^2)^2}{D(y)} dy}

where B is the span in the Y direction

where \( D(y) = \frac{E[h(y)]^3}{12(1-\nu^2)} \)

where \( h(y) \) is the total thickness of the plate stiffener combination subject to the following limitation. When the ratio of stiffener height to stiffener width is greater than unity a reduced height of the order of the stiffener width is used.

By the 8th order theory \( D_y = D_Y \), there is no increase in \( D_y \) due to the eccentric ribs. In Appendix A the theoretical values are given. They are compared below.

<table>
<thead>
<tr>
<th></th>
<th>1.5 mm rib</th>
<th>( \frac{(DY-D_y)}{D_Y} )</th>
<th>3 mm rib</th>
<th>( \frac{DY-D_y}{D_Y} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>DY1</td>
<td>7660</td>
<td>-0.062</td>
<td>7660</td>
<td>-0.098</td>
</tr>
<tr>
<td>DY2</td>
<td>8930</td>
<td>0.093</td>
<td>8990</td>
<td>0.059</td>
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<tr>
<td>DY3</td>
<td>9300</td>
<td>0.138</td>
<td>9810</td>
<td>0.162</td>
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<tr>
<td>( D_y )</td>
<td>8170</td>
<td></td>
<td>8490</td>
<td></td>
</tr>
</tbody>
</table>

The frequency distributions are as follows:

Fig. 5.15
The formula $D_{Y2}$ and $D_{Y3}$ can be rejected confidently. The mean value of $D_y$ is very significantly larger than $D_{Y1}$. The cause is easily seen when tracing the curves of slope versus distance from the Moiré fringe patterns. As an example consider a 4-ribbed plate.

Fig. 5.17

The section taken is general and the form of the curve is a common one. The actual deflection of the plate is
represented by the dotted 'wobbling' line. The full smooth line is the one used to find the curvature at midspan for eventual use in the system of equations. The 'wobble' for an actual plate can be seen in Fig. 5.11 (Photo A2). Thus \( D_y \) is calculated by pretending that the plate is smoothly bent, or, in other words, is behaving as its equivalent true orthotropic plate would. This gives a low value to the curvature at midspan and thus a high value to \( D_y \).

This effect is most noticeable in plates with low numbers of ribs. For high numbers of ribs not enough Moiré fringes can be obtained to see whether the 'wobble' is as pronounced as in the low rib numbered plates. Thus \( D_y \) is an arbitrary value. If the deflection of the plate is required at a point well away from the supports, then the high value of \( D_y \) may be used (\( D_y = 8170 \) or \( 8490 \, N \, mm \)). If the actual curvature at a point in the plate is required, then the low value (\( D_y = D \)) should be used.

c. \( D_{12} \) and \( D_{21} \) results

Values of \( D_{12} \) and \( D_{21} \) can not be said to either increase or decrease with increasing rib numbers. \( D_{12} \) is defined more precisely by the experiments than \( D_{21} \). This is seen from the thinner plates where the values of \( \sigma \) were 0.025 and 0.051 for \( D_{12} \) and \( D_{21} \) respectively, and from the thicker plates where \( \sigma \) for \( D_{12} \) was 0.051 and \( D_{21} \) was very scattered. It was considered that it would be misleading to mean these values of \( D_{21} \) and draw conclusions from it. This scatter of results is partly due to the conditioning of the equations and partly due to the 'wobble' mentioned in the graphs in the
Y direction. This 'wobble' will interfere in any bending test devised.

For the thinner 4 plates the mean values of $D_{12}$ (3100) and $D_{21}$ (3130) agree very well. For these plates we may confidently take $D_{12} = D_{21}$. For the thicker 4 plates $D_{21}$ is not well enough defined to draw a conclusion.

The theoretical values of $D_1$ are

\[ D_1 = 0 \quad \text{Eq 2.48} \]

\[ D_1 = \nu D \quad \text{Eq 2.47} \]

Eq 2.47 applies for the 4th and 8th order theories. In section 5. above it was found that

\[ \nu D = 2610 \text{ N mm} \]

The frequency distributions of the variables are shown below.

![Diagram showing frequency distributions of variables](image)

Fig. 5.14

It can be seen that the theoretical value

\[ D_1 = \nu D \]
is well below the experimental values. There is no theoretical formula which satisfies the experimental results. Certainly $D_1 = 0$ is a poor approximation. The formula

$$D_{12} = D_{21} = vD$$

may be used as a conservative approximation.

8. Conclusions
a. The Method

It has been shown how the flexural rigidities of a stiffened plate can be determined from one model by the writer's method. It has been shown that the Moiré method is a convenient and accurate method for this purpose. Incorporated in the method is an accurate way of determining $E$ and $v$. The phenomenon of creep and the variation of the model dimensions from those specified has been catered for. A study of the errors in the experiments has been done in order to assess the confidence with which conclusions are drawn.

With the support and loading conditions used it was seen that, for the thinner 8 plates, $D_x$ and $D_y$ were well defined. $D_{12}$ was not as well defined and, for the 3 mm rib plates, $D_{21}$ was poorly defined. A survey of the literature shows that Bergsträsser's method does not give good results for $D_{12}$ and $D_{21}$ either.

For the plates with 6 and 9 mm deep ribs it was found that the plates were too stiff in the X direction for good results. In other words not enough fringes could be obtained in the X direction before the plate became heavily
bent in the Y direction. In order to get results for these stiffer plates the method must be modified. This may be done in several ways, which may be combined.

i. Instead of using a square plate a rectangular plate may be used. This plate must have the ribs running parallel to its long side. Only one model is necessary as the twisting test can be done on a rectangular plate. A rectangular plate will allow a greater curvature in the X direction for the same curvature in the Y direction. The conditioning of the equations may deteriorate using this method.

ii. Instead of 4 point supports two opposite sides of the plate may be simply supported. This has the same effect as i. above.

b. The Flexural Rigidities

For the range of rib numbers tested, it may be said that $D_x$, $D_{12}$, $D_{21}$ and $D_y$ do not vary. In this sense the plates behave as true orthotropic plates.

$D_x$ is best expressed by the formula for $D_{X2}$ (Eq 2.39). The formulae for $D_{X1}$, $D_{X3}$ and $D_{X4}$ may confidently be rejected as being the true formulae for $D_x$. The 8th order theory predicts that $D_x$ lies between $D_{X2}$ and $D_{X4}$. It is concluded that, for the loading and support conditions used, $D_x$ can be very little greater than $D_{X2}$.

$D_y$ may be given two different values. This is because the plate does not bend smoothly in the Y direction; the curvature is greatly reduced over the ribs. In this sense the plate does not behave as an orthotropic plate. If it
is required that the difference in deflections between the stiffened plate and the equivalent orthotropic plate be a minimum, then the value of $D_y$ to be used shows no agreement with any of the theoretical formulae. If it is required that the moments in the stiffened plate and the equivalent orthotropic plate be the same, then the formula $D_y = D$ should be used to find the curvatures. A high value of $D_y$ should then be used to calculate the moment at any point in order to be on the safe side.

$D_{12}$ may be said to be equal to $D_{21}$ for the thinner 4 plates. In this respect the plates behave as orthotropic plates. For the thicker 4 plates the results were not precise enough for a conclusion to be drawn. There is no theoretical formula which describes $D_{12}$ and $D_{21}$ well. $D_1 = 0$ is a very poor approximation. The formula $D_1 = uD$ may be used as a conservative approximation.
1. **General**

The method used was described in Chapter 4, section 3., and the models were described in Chapter 5, section 1. Some of the models were made from clear perspex and some from black perspex. It was considered prudent to test whether the properties of black and clear perspex are the same, and therefore a square model made of each was milled down to a uniform thickness of 3 mm. The twisting test was performed on each and the results for $D_{xy}$ compared. It was also considered prudent to test whether the milling of the models has any effect on the properties. Accordingly a square model of black perspex of nominal thickness 1/8 inch was cut and subjected to the twisting test. This model was not milled at all.

The variation in the dimensions of the models from those specified was allowed for by the calculation of a correction factor $k_{xy}$, as follows. The measurements made are shown in Fig. 5.3 for model 4/1,5. The measurements along the midspan line were given double the weight of the edge measurements. The mean of $h, h_a, \overline{h}$, was calculated and subtracted from each value of $(d + h)$ to find values of $d$. $d_a$ and $b_a$ were calculated. All means taken were the arithmetic means. The accuracy of this approximation was shown in Chapter 5. As was explained, a theoretical formula
for $D_{xy}$ must be assumed. The formula is

$$D_{xy} = \frac{D(1-\nu)}{2} + \frac{GJ}{4s}$$

Eq 2.49

Using the mean values of the dimensions, $k_{xy}$ is calculated as

$$k_{xy} = \frac{D_{xy} \text{ (actual)}}{D_{xy} \text{ (specified)}}$$

In Appendix B the values of $k_{xy}$ and the theoretical values of $D_{xy}$ are calculated. In addition the two components of $D_{xy}$ are given.

$$\text{TPC} = \text{theoretical plate contribution} = \frac{D(1-\nu)}{2}$$

$$\text{TBC} = \text{theoretical beam contribution} = \frac{GJ}{4s}$$

A creep time of 2 minutes was used in the experiments. Using $E$ and $\nu$ at 2 minutes creep time and at the Moiré apparatus temperature as found in Chapter 5, $G$ was calculated from

$$G = \frac{E}{2(1+\nu)}$$

The experiments were done at room temperature which varied between $16^\circ$ and $18^\circ$C, and is considered to be constant. The results were converted to the Moiré apparatus temperature which is appreciably higher than room temperature. The effect of the temperature difference was investigated by taking one plate and doing the twisting test in the Moiré apparatus with the lights on, as well as at room temperature. The comparison is given in the results in section 3.

The testing procedure was as follows.
a. Load the dial gauge with a weight sufficient to give a deflection of approximately $h/2$ to $2h/3$.
This deflection is larger than the usual limit of $h/5$ or $h/3$ of the small deflection theory. However the plate is supported in such a way that no horizontal support reactions are induced by large deflections. There will be small reactions caused by friction but these are considered to be negligible. The larger the deflection, the more precisely is it determined. The linearity of the load deflection graph was tested on two of the plates and found to be true within the limits of experimental error.

b. After 2 minutes unload the gauge.
This initial cycle is for the model to settle in on its supports.

c. After 2 minutes read the gauge and apply the load.

d. After 2 minutes read the gauge and remove the load.

Steps c. and d. are repeated until enough cycles have been performed to give a steady estimate of the differential downward deflection. This was usually 4 or 5 cycles. Throughout the test the model-holding-board is tapped at intervals to free the gauge spindle. Failure to do this can give erratic results, as the creep of the perspex is considerable.

As was mentioned in Chapter 5, the force exerted by the gauge spindle decreases linearly by 3 g wt per mm extrusion of the spindle. The nett force applied to the plate is calculated by subtracting the change in spindle force from the applied dead load. In the twisting test this correction is not negligible.
2. \( D_{xy} \) by Finite Differences

The theoretical formula for \( D_{xy} \) (Eq 2.49) cannot be expected to give the true value of \( D_{xy} \) for a ribbed plate. This is because the formula is the sum of the plate stiffness and the beam stiffness, acting as if they were separate members. The junction between the two must add strength to the combination.

The torsional rigidity of a T-beam, \( GJ_T \), can be found exactly by the method of finite differences as will be described. The question is whether this result can be used to find \( D_{xy} \) of a ribbed plate made up of these T-sections. We must assume that

\[
\frac{GJ_T}{s} = \gamma_x + \gamma_y = 4D_{xy}
\]

in order to calculate \( D_{xy} \). In comparing this value of \( D_{xy} \) with the experimental value, we are really testing the validity of this assumption. This was done for all the ribbed plates.

The analogy between the torsion of a beam and the deflection of a soap film under pressure is well known, and described in Ref. 6. The two problems are governed by similar differential equations. For torsion of a beam we have

\[
\frac{d^2 \phi}{dy^2} + \frac{d^2 \phi}{dz^2} = -2G\Theta \tag{Eq 6.1}
\]

where \( \phi(y,z) \) is a stress function and can be thought of as the deflection of a soap film stretched over the outline of
the cross-section, and where $\Theta$ is the rotation per unit length of the beam

\[
M_T = 2\int \phi(y,z) dydz
\]

\[
= 2 \times \text{volume under soap film}
\]

$D_{xy}$ is then calculated as

\[
D_{xy} = \frac{\text{(volume under soap film)}}{2s}
\]

The finite difference solutions were found using the computer program in Appendix C.

Fig. 6.1

Fig. 6.2

A Node Pattern
Only an L-section is analysed because of the symmetry. The differential equation (Eq 6.1) is applied in finite difference form at each node.

\[ \frac{d\phi}{dy} = 0 \]

General Operator Pattern

![Diagram](image)

Fig. 6.3

Along lines AB, CD, DE, EF the boundary condition is that \( \phi = 0 \). Along line BC \( \frac{d\phi}{dy} = 0 \). Along line AF when \( \phi = 0 \) the T-beam has 'closed ends' and when \( \frac{d\phi}{dy} = 0 \) it has 'open ends'.

\( D_{xy} \) is calculated twice giving a closed end and an open end solution. \( D_{xy} \) is then finally calculated as the weighted mean of the two results. For example, for a 4 ribbed plate

\[ D_{xy} = \frac{3D_{xy} \text{ (open)} + D_{xy} \text{ (closed)}}{4}. \]

In actual fact this refinement made very little difference to the results. It was found that

\[ D_{xy} \text{ (open)} - D_{xy} = 30 \text{ N mm} \]
for all ribbed plates.

The ratio of the node spacings, R, was a variable in the program. Different values of R were tested and the difference in results was negligible for ratios of between 4 and 1. The simultaneous equations were solved by Gauss diagonal reduction to save storage in the computer. This means that only the band width in the coefficient matrix of the set of equations was operated on. With this method roughly 500 nodes can be used without having to use files. The number of nodes which can be used depends on the aspect ratio, R, and the geometry of the cross-section. The university computer is a Univac 1106.

The accuracy of the solutions to the equations was checked by resubstituting the values of \( \phi \) found back into the system of equations. The maximum difference between the left and right hand sides of the equations was then found. This error was of the order of 0.003% for 500 nodes and, in general, increased with increasing node numbers.

Finally the values of \( \phi \) were used to calculate the volume under the imaginary membrane by Simpson's Rule. As a check the program was used to calculate \( D_{xy} \) of a uniform thickness plate 3 mm deep. The result was the same as that given by the theoretical formula.

3. Results

The results for the three uniform thickness plates were
It can be seen that it is very unlikely that black and clear perspex have different properties, or that milling has any effect on the properties. Meaning the results, it is concluded that $D_{xy}$ of a perspex plate at 16.3°C is 2720 N mm.

When the test was done in the Moiré apparatus with the lights on, the precision of the results was not as good. For the 3 mm milled black perspex plate values for $D_{xy}$ of 2480 and 2580 N mm were obtained. The temperature of the plate was estimated as 24°C. The value of $D_{xy}$ from the theoretical formula

$$D_{xy} = D(1-u)/2$$

using the values of $E$ and $u$ obtained in Chapter 5 is $D_{xy} = 2525$ N mm. The tests show therefore that the difference in temperature between that in the Moiré apparatus and outside it has a considerable effect on the properties of perspex. In order to correct the experimental results to the Moiré temperature a conversion factor of 2525/2720 was used. Assuming that $D_{xy}$ (i.e. $E$) varies linearly with change in temperature, the estimated decrease in $E$ per °C increase in temperature

$$= \frac{2720 - 2525}{24 - 16.3} \times \frac{100}{2620} \%$$
It must be emphasized that this is only an estimate. More tests must be done to find a more accurate value.

A typical result sheet for the twisting test is given below.

**TWISTING TEST READINGS**

Plate: 3 mm milled black perspex  
Initial load: Dial gauge  
Temp re: 16.6°C  
k_{xy}: 1.044

<table>
<thead>
<tr>
<th>P gwt</th>
<th>Time min.</th>
<th>w mm</th>
<th>$\Delta w_2$ ↓ mm</th>
<th>$\Delta w_2$ ↑ mm</th>
<th>P gwt</th>
<th>Time min.</th>
<th>w mm</th>
<th>$\Delta w_2$ ↓ mm</th>
<th>$\Delta w_2$ ↑ mm</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>9.963</td>
<td></td>
<td></td>
<td>111</td>
<td>17</td>
<td>7.500</td>
<td></td>
<td></td>
</tr>
<tr>
<td>111</td>
<td>2</td>
<td>7.568(2,395)</td>
<td></td>
<td></td>
<td>111</td>
<td>18</td>
<td>7.478</td>
<td>2,280</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>3</td>
<td>9.815</td>
<td></td>
<td>2.284</td>
<td>0</td>
<td>19</td>
<td>9.720</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>9.852</td>
<td></td>
<td></td>
<td>20</td>
<td>9.748</td>
<td>2,270</td>
<td></td>
<td></td>
</tr>
<tr>
<td>111</td>
<td>5</td>
<td>7.570</td>
<td></td>
<td></td>
<td>111</td>
<td>21</td>
<td>7.502</td>
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<tr>
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<td></td>
<td></td>
<td>111</td>
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<td>7.473</td>
<td>2,275</td>
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</tr>
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<td></td>
<td>2.290</td>
<td>0</td>
<td>23</td>
<td>9.715</td>
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</tr>
<tr>
<td>8</td>
<td></td>
<td>9.800</td>
<td></td>
<td></td>
<td>24</td>
<td>9.730</td>
<td>2,257</td>
<td></td>
<td></td>
</tr>
<tr>
<td>111</td>
<td>9</td>
<td>7.550</td>
<td></td>
<td></td>
<td>111</td>
<td>25</td>
<td>7.480</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>2.293</td>
<td>7.507</td>
<td></td>
<td></td>
<td>111</td>
<td>26</td>
<td>7.457</td>
<td>2,273</td>
<td></td>
</tr>
<tr>
<td>111</td>
<td>11</td>
<td>9.740</td>
<td></td>
<td></td>
<td>0</td>
<td>27</td>
<td>9.715</td>
<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td></td>
<td>9.794</td>
<td></td>
<td>2.287</td>
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<td>28</td>
<td>9.729</td>
<td>2,272</td>
<td></td>
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<tr>
<td>111</td>
<td>13</td>
<td>7.500</td>
<td></td>
<td></td>
<td>111</td>
<td>29</td>
<td>7.480</td>
<td></td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>2.317</td>
<td>7.477</td>
<td></td>
<td></td>
<td>111</td>
<td>30</td>
<td>7.450</td>
<td>2,279</td>
<td></td>
</tr>
<tr>
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<td></td>
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<tr>
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<td>9.758</td>
<td></td>
<td></td>
<td></td>
<td></td>
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<td></td>
<td></td>
</tr>
</tbody>
</table>

Mean $\Delta w_2$ ↓ = 2.286 mm

Nett weight, $P_N$, on plate = 111 - 3 (2.286) = 104.2 g wt
The experimental results for the ribbed plates are compared to the theoretical results for $D_{xy}$ below. $D_{xy}$ is from Eq 2.49 and is calculated in Appendix B. $D_{xy}$ is from the finite difference analysis in Appendix C.

\[
D_{xy} = \frac{6400, P_N . 9,806}{\Delta w_{el} k_{xy} \cdot 1000} = \frac{62,8,104,2}{2,286.1,044} = 2740 \text{ N mm}
\]

The difference between the results for the two plates 12/9 is small. It will be remembered that the black perspex one was made by glueing two sheets together and then milling
The results show that the glueing did not appreciably decrease the strength by the inclusion of air bubbles at the joint.

The $D_{xy}$ result for plate 6/1,5 can be seen to be unreliable. From theoretical considerations $D_{xy}$ of plate 4/1,5 must be greater than that of plate 6/1,5. It is considered that the error in $k_{xy}$ for this plate is probably large, causing most of the error.

The uncertainty in $D_{xy}$ is considered to be largely due to the error in assuming that Eq 2.49 is the true formula for $D_{xy}$. In other words, the uncertainty in $k_{xy}$ is the dominant uncertainty. Eq 2.49 underestimates the contribution of the rib to $D_{xy}$. Consequently the variation in rib dimensions is not given sufficient weight in the calculation of $k_{xy}$. In the bending test much more accurate theoretical formulae were available for $D_x$ and $D_y$ and so this source of uncertainty did not arise.

It can be seen that there is a large difference between the values of $D_{XY1}$ and $D_{xy}$. $D_{xy}$ is consistently higher than $D_{XY1}$. These differences are far too large to be accounted for by the uncertainties in $D_{XY1}$ and $D_{xy}$. The percentage differences were calculated on the basis of $D_{XY1}$ and follow a smooth pattern. The percentage difference can be seen to increase from zero when there is no rib to a maximum when the rib and plate depths are approximately equal. It then decreases as the rib depth increases. The percentage difference is larger for smaller numbers of ribs. The writer has not been able to find a satisfactory correction term to add to Eq 2.49 to bring it in line with
the experimental results. In the experiments the ratio of 
b : s was not varied, and this would limit the generality of
the correction term if it were to be found.

There is also a large difference between the results
for DXY2 and Dxy. DXY2 is consistently larger than Dxy. The percentage difference increases greatly with a decrease
in the number of ribs. The percentage difference increases
steadily as the ratio of the beam contribution to the plate
contribution to Dxy increases. DXY2 was found from the
rigidity of the T-beam, and it appears that this method does
not apply to ribbed plates.

4. Conclusions

It has been shown that it is very unlikely that black
and clear perspex have different properties, or that milling
has any effect on the properties. The models were therefore
satisfactorily made. The effect of temperature on the
strength of perspex was investigated. It was estimated
that the decrease in E per °C increase in temperature was 1%.

It has been shown that the equation

\[ D_{xy} = \frac{D(1-v)}{2} + \frac{GJ}{4s} \]  Eq 2.49

underestimates the torsional rigidities of asymmetrically
stiffened plates reinforced in one direction. This formula
gave values which were as much as 33% lower than the
experimental values of Dxy.

It has been shown that the assumption that Dxy can be
found from
\[ \text{D}_{XY} = \frac{GJ_T}{4s} \]

where \( GJ_T \) is the torsional rigidity of the T-section of which the plate is made up is incorrect. As the size of the rib increases with respect to the plate, the results appear to diverge.
BIBLIOGRAPHY


22. Rowe, R.E. - 'A load distribution theory for bridge slabs allowing for the effect of Poisson's ratio' - Mag. of Concrete Research, v.7, No. 20, July 1955.


APPENDIX A

KX AND KY FOR STIFFENED PLATES

DIMENSION HX(11),HY(12),HN(12),PN(12),TH(1)
REAL KX,KY,NU

10 FORMAT(H11,H12,H12,H12,H1,MT11)

100 FORMAT(H12,2X,H12,2X,H12,2X,H12,2X,H12,2X,H12,2X,H12,2X,H12,2X,H12,2X,H12,2X,H12)

110 FORMAT(1H11,2H12,2H12,2H12,2H12,2H12,2H12,2H12,2H12,2H12,2H12,2H12,2H12)

111 FORMAT(1H11,2H12,2H12,2H12,2H12,2H12,2H12,2H12,2H12,2H12,2H12,2H12,2H12)

DO 250 I=1,17
READ(5,16,NK,S,GA)
250 CONTINUE

DATA AND THEIR MEANS

E=301e6.
UX=0.31

READ(5,105)HMM(I),1=1,NM1
READ(5,105)HMM(I),1=1,NM2
READ(5,105)HMM(I),1=1,NM3

105 FORMAT(16,F5.2)

S4=0
DO 15 I=1,MC
15 S4=S4+HMM(I)**2.

C=(S4/15)**1/2.

DO 15 I=1,VC
15 DMM(I)=HMM(I)/C

S5=0
DO 20 I=1,MC
20 S5=S5+DMM(I)**2.

S6=S5/4**1/2.

N=SP/HR**1/2.

S1=(S6+1.)**1/2.

S2=(S6+2.)**1/2.

CALC OF DX AND KX

S1=(S1**2+2.*N**2*(D+D)**1/2.+5*N**2+D)**1/2.
S1T=(S1+5.*CA)**A**A/(2.*N**2*D**1/2)**1.5

PL=M**3./12.

P2=M**1.5.**2.
CALC OF STD DYS

SH=75
DO 25 I=1,SH
25 SH=SH+1:

CALC OF RSDNMX

C1D=SH**2/4+R+P*(D+P)*(D+P)*(D+P)*(D+P)/4,
(D+P)**2,
(C1D**2+D**2)*(SH+1.5+D+P)**2/2

CALC OF ND AND KX

RFAD=10*(T+1)**1/11.1
ST=0
DO 35 I=1,7
35 ST=ST+TM(1)**3,
TM=(ST/7)**1/3,
KX=ST/111.
DY1=E*Y**1./((12.-11.*N1**2.,1))
DY1=E*2.25/((1.-N1**2.,1))
Y**2=E*2.25/((1.-N1**2.,1)+(1.-(3.*DA)**3.11))
Z=150.,
SUM=1.,
R=(S+1A)/2.,
ZZ=FA
IF(DA.GT.R) DA=0
DO 39 1=2,M,2
P=P+R
SUM=(Z**2.,R**3./3.,-7.0**4./2.,R**5./5.,-Z**2.,R**3./3.1
1.7**4./2.,-R**5./5.)/1/(3.+DA)**3.*SUM
37 R=R+S
R=(S-DA)/2.,
P=P,
RR=P+R
DO 39 1=2,RR,2
SUM=SUM+(Z**2.,R**3./3.,-7.0**4./2.,R**5./5.,-Z**2.,R**3./3.1
1.7**4./2.,-R**5./5.)/17.
R=R+S
P=P-S+A
IF(R,E,100) R=0.,
90 CONTINUE
DA=27.
DY3=E(-160.*S/1720.)/(11.-N1**2.1)*SUM)
C CALC OF STD DEVS
C
GO TO 1=17
ST=(S**2.,-T1**2.,
SDSOT=(S**2.,1.)*0.5
SDMT=SDSOT/(7.0**0.5)
N=10,(5.15)NP,DA;DX1.;DX2T,DX3T,DX4T,XX,RSAMNI,DY1T,DY2T,DY3T,KY1
SDMT
115 FORMAT(1H0,3X,12,F11.1,3X,4F9.0,2F7.3,5X,3F7.0,2F6.3)
207 CONTINUE
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* indicates Black Perspex
C APPEJOIX B
C CALC OF DXY AND KXY
C
DIMENSION RH(13), DANDHM(13), RM(13), DH(36)
REAL DXY, KXY
RITE(5,100)
100 FORMAT(11,MO, RHMS, RHM, DEPTH, TPC, TBC, DXY, KXY*)
RITE(5,100)
110 FORMAT(11,*, RHMS, RHM, TPC, TBC, DXY, KXY*)
C DATA AND THEIR MEANS
C
Q=301N.
RH=.391
DD 35 11=1,N
READ(9,F5,N,S,MA,DN)
105 FORMAT(12,2F7.4,2F5.1)
F=1.0(N=N)
RP=308/F
READ(11,100)RM(i), 1=1,94
READ(9,100)RM(i), 1=1,94
READ(2,110)RM(i), 1=1,94
110 FORMA7AT(16F,21)
11=1+N
12=9*(M+1)
47=3*P
$=0
"=1.
66 1C 11=1,NH
$=S+M+RH(11)*
1F11=I1110,
?=2.
1F11=I2110,
1F11=1
CONTINUE
$=SH/(4.9NH/3.1)
DO 15 11=1,60
15 DRT(I)=DANDHM(I)-H
$H=H,
$R=0.
11=1.
DO 20 11=1,60
$D=SN+RM(1)*
$R=SR+RM(1)*
IF (I-III) 120,
  T=2,
  IF (I-K22120,)
  T=1.
20 CONTINUE
  D=SD/19.+MD/3.1
  E=SR/19.+ND/3.1
  G=(2.*1.+HH)
  TPC=G*2.25
  APC=G*HH*3./12.
  IF (D=-3.), 25
  TBC=G*DA*3.3*RA*C/(4.*S).
  ABC=G*D*3.3*B*C/(4.*S)
  GO TO 31
25 TBC=G*DA*RA*3.3*C/(4.*S)
  ABC=G*D*3.3*B*C/(4.*S)
  DXYT=TPC+TBC
  DXY=4PC+ABC
  KXY=DXY/DXYT
  WRITE (5,115) NR,DA,TPC,TBC,DXYT,KXY
115 FORMAT (5,15.5)
  CONTINUE
  CALL EXIT
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* INDICATES BLACK PERSPEX
APPENDIX C

BEAM TORSION BY FINITE DIFFERENCES

FOR CONNECT SIMPSON SUM
NL MUST BE EVEN. NH & NZ MUST BE ODD
FOR UNIFORM THICKNESS PLATE PUT NL>NY
DIMENSION 5745, 681, 5 (100), 5PH1 (765)
READ (5,1,NPL
4 FORMAT (13)
G=1.2,
DO 290 KJ=1,NPL
5 FORMAT (13)
9 FOR A1*(4/12)
6 WRITE(5,12)AL,KJ
10 FORMAT (11)
L=FA,2!, : M=FA,2!, : N=FA,2!), : D=FA,2!
11 WRITE(5,13)AL,KJ
A FORMAT
14 WRITE (5,13)AL,KJ
15 WRITE (5,13)AL,KJ
23 FORMAT *,*,RATIO OF MESH SIZES AY/AL=*,F10.5
24 FORMAT *,*,NO. OF NODES I5*,I5

C

SETTING UP THE EQUATIONS
C
DO 290 I=1,4
LI=1
29 DO 29 I=1,NO
A(I,J)=A(I,J)+1
20 CONTINUE
  DO 30 I=1,N
  DO 30 J=1,N
   A(I,J)=0
30 CONTINUE

  DO 40 I=1,N
  DO 40 J=1,N
   IF (I.GT.NM.AND.J.LT.NL) GO TO 40
   R=0.
   L=0.
   K=1;
   IF (I.LT.IO) GO TO 40
   IF (J.LE.IM) GO TO 39
   A(K0,NS)=1.
   GO TO 40
40 K=K+1
41 IF (K.GT.2) GO TO 40
42 IF (J.GE.IM.AND.J.LT.NM) A(K0,NS)=2
43 IF (J.GE.IM.AND.J.LT.HR) A(K0,NS)=3
44 IF (J.GE.IM.AND.J.LE.HR) A(K0,NS)=4
45 IF (J.LE.HR) A(K0,NS)=5
46 IF (J.LE.HR) A(K0,NS)=6
47 IF (K0.GT.0) A(K0,NS)=7
48 IF (K0.GT.1) A(K0,NS)=8
49 IF (K0.GT.2) A(K0,NS)=9
50 IF (K0.GT.3) A(K0,NS)=10
40 CONTINUE
   IF(LL.EQ.1)GO TO 90
   ERR=0.
   DO 60 I=1,N0
       IF=GF(I,NN)
       DO 66 J=1,N1
           JN=NY+J-1
           IF(JJ.LT.1.OR.JJ.GT.NH)GO TO 65
           GF=GF*AI(I,J)*PH(I,JJ)
   65 CONTINUE
   60 IF(DIF.ST.ERH) ERR=DF
      ER=EER/AY+2.
      WRITE(5,61)ER
      FORMAT(*,1X,MAX FRAC ERR=*,F14.10)
   GO TO 140

C        GAUSS DIAGONAL ELIMINATION

60 DO 115 I=1,N0
      IF(A(I,NP).EQ.1.0)
         DO 95 J=1,NN,1
   95 CONTINUE
      DO 105 J=1,NN,1
         A(J,J)=A(J,J)-A(J,K)*A(K,J)
   105 CONTINUE
   115 CONTINUE
   DO 120 I=1,NN,1
      PHI(I)=AI(I,JP)
      DO 117 J=1,NN,1
         IF(J+NP.GT.NH)GO TO 120
         PHI(I)=PH(I)-AI(I,J)*PH(J+J-1NP)
   117 CONTINUE
   120 CONTINUE
   WRITE(5,125) (PH(I),I=1,NN)
   125 FORMAT(5,I12)
      IF(IN.EQ.1)GO TO 130
      WRITE(5,130) (PH(I),I=1,NN)
   130 FORMAT(5,I12)
   135 LL=2
GO TO 25
C
C
C
CALC OF VOLUME UNDER MEMBRANE
C
140 GO 229 H=1, HH
MA=NY
MH=NA-1
LM=(H-1)+NY
MN=GC+1
NF=GC+2
NG=LC-2
S(N)=S(N)+(MO)
NY=NY/2
IF (H = 21,111, ...) DO 229 YM, H, 2
S(N)=S(N)++
200 CONTINUE
DO 229 YM, H, 2
S(N)=S(N)+
210 CONTINUE
GO TO 225
215 DO 220 YM, H, 2
S(N)=S(N)+
215 CONTINUE
DO 220 YM, H, 2
S(N)=S(N)+
220 CONTINUE
S(N)=S(N)++
225 CONTINUE
IF (H = 21, 1, 1) GO TO 247
IF (HM = NY, 2) GO TO 247
S(N)=0.
Y=NY+(Y+NY-11)*NY-NL+1
NK=MJ+1
NL=NM+2
MN=MJ+2
MO=MJ+2
NL=NM+2
MO=MJ+2
NL=NM+2
MO=MJ+2
NM=NY-NL+2
YM=NY-NL+2
IF (H = 21, 1, 1) DO 230 YM, H, 2
```
SMH = S(M) + 4. * PHI(I)
230 CONTINUE
DO 235 I = M + 1, N
SMH = S(M) + 2. * PHI(I)
235 CONTINUE
SMH = S(M) + PHI(MM)
DO 240 I = M + 1, N
SMH = S(M) + PHI(I)
240 CONTINUE
SMH = S(M) + 2. * PHI(MM)
245 CONTINUE
WRITE (*,245) S(M), M = 1, N
245 FORMAT (*)
45 FORMAT (15F1.2)
S5 = 0. DO 270 M = 1, R4 + 2
S5 = S5 + 5. * S(M)
270 CONTINUE
M = 2, R4 + 2
S5 = S5 + 2. * SMH
275 CONTINUE
WRITE (*,275) Y
275 FORMAT (*)
280 FORMAT (*.1,1X,Y = 1., F14.6, 1 N= M)
280 CONTINUE
CALL EXIT
END
```