

**UNIVERSITY OF CAPE TOWN**

**Department of Business Science**

**An Investigation into the Use of the Black-Scholes Model for  
Pricing Long Term Options, for the Purpose of Costing  
Maturity Guarantees**

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**This thesis is submitted in partial fulfilment of the requirements of the  
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## ABSTRACT

This thesis investigates the use of the Black-Scholes option pricing model for long term options for the purposes of costing long term maturity guarantees. The maturity guarantees concerned are typically given on endowment policies issued by life offices. These endowment policies have terms usually in excess of five years. The thesis investigates whether the assumptions underlying the Black-Scholes model, which was developed for pricing short term traded options, are still acceptable when applied to long term options, and if not, what adjustments need to be made. The paper focuses on the pricing of European put options which are equivalent to the payoff of a maturity guarantee.

The method of derivation of the Black-Scholes model is investigated. It is shown that the option value can be obtained by discounting at the risk-free rate the terminal payoff of the option, where the terminal payoff is calculated by letting the underlying asset drift upward at the risk-free rate. A framework is developed using this risk-neutral discounting approach and this is used to provide an understanding of the impact on the option price of deviations in the underlying assumptions.

The major assumptions underlying the Black-Scholes model, namely, the distribution of returns of the underlying asset, the volatility of returns of the underlying asset, and the risk-free interest rate, are investigated separately to determine how closely they fit reality and how sensitive the long term option price is to deviations from these assumptions. International evidence as well as empirical evidence for the South-African All-Share-Index are presented to establish the soundness of these assumptions. For all three of these assumptions recommendations are made as to how these variables should be modeled for the purposes of pricing long term options.

Investigation into the assumption for the distribution of returns of the underlying asset shows that mis-specification of the distribution leads to significant biases, and that these biases are exaggerated as the term of the option is extended. Evidence presented shows that the assumption of a log-normal distribution of returns underlying the Black-

Scholes model is not appropriate for short term returns as these exhibit significant leptokurtosis. The evidence also indicates that daily returns are not stable and that there is a reduction in the fatness of the tails of the distribution for longer term returns. The thesis recommends that for the purpose of pricing long term options, the log-normal be accepted as the conditional distribution of returns, conditional, that is, on the volatility of returns.

Investigation into the volatility of returns shows that volatility is significantly more important for long term options than short term options when pricing guarantees of a nominal rate of return per annum. Stochasticity of volatility is shown to lead to significant biases, the direction of which depend on the sign of the correlation between volatility and stock returns. Biases are increased with an increased term to expiry of the option, but mean reversion of volatility reduces the extent of this increase. Strong evidence for mean reversion is presented both internationally and for the South-African ALSI, however, evidence for a correlation between volatility and returns on All-Share-Index is not clear. The thesis recommends that for the purpose of pricing long term options volatility be modeled as an Ornstein-Uhlenbeck process which incorporates mean reversion.

Investigations into the interest rate assumption show that the option price is more sensitive to changes in the interest rate than to changes in volatility for long term options. Stochasticity of the interest rate introduces additional volatility into the risk-neutral distribution of the underlying asset and this leads to significant biases, the magnitude of which depends on the variance of the short term interest rate and the size of the correlation between bond and stock price changes. As with volatility, extension of the term of the option leads to an increase in the biases, but mean reversion of the interest rate reduces the extent of this increase. The thesis recommends the adoption of one of two international models for the interest rate process. Further research into the South-African term structure of interest rates is suggested.

In conclusion, the empirical evidence, theoretical derivations, and intuitive arguments presented in the thesis demonstrate that it is not sound to use the Black-Scholes model for the purposes of costing long term maturity guarantees, and that application of

Black-Scholes may lead to significant biases. Furthermore, these biases increase as the term of the option is lengthened. The thesis recommends that the long term options be valued by a risk-neutral discounting of the terminal payoff using a Monte Carlo simulation of the three stochastic processes, namely, for the stock price, its volatility, and the interest rate. The magnitude of the resultant bias due to the incorporation of all three stochastic processes recommended was not investigated in this thesis.

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# CHAPTER 1

## INTRODUCTION

The motivation for this thesis arises out of the problem of determining the cost of long term maturity guarantees given on single premium equity linked endowment policies. The thesis focuses on an option pricing approach to costing these long term maturity guarantees, and investigates the option pricing assumptions that can be used in the valuation of these long term options.

Numerous investment products marketed to both pension funds and individuals provide guarantees as to the minimum value of benefits that will be payable on maturity. Life offices, in particular, issue endowment policies which usually guarantee that at maturity the benefit payable will not be lower than the initial sum invested. In the South African market these policies are, for legislative reasons, issued with terms longer than five years. In addition to the capital guarantee they usually also guarantee that the value of the benefit will grow by typically 4% to 5% per annum.

With these products the life office has the task of determining what to charge the policyholder for the provision of these guarantees. Two approaches have been covered by the literature and used in practice to date for quantifying the value of these maturity guarantees, namely, what may be termed the conventional actuarial approach, and the immunization approach.

The conventional actuarial approach determines the minimum level of reserves required to reduce the 'probability of ruin' to an acceptable level, where 'ruin' is defined as a situation where the assets underlying the policy have a lower value than the level of the guarantee. The 'probability of ruin' is determined by simulating the investment performance of the underlying assets. The premium to be charged for the investment guarantee is determined as the cost of servicing these reserves. The most important

papers discussing this approach are those by Benjamin (1976) and the Maturity Guarantees Working Party (1980) (hereafter MGWP). In addition numerous earlier studies attempted to value asset guarantees using simulation techniques, for example, Turner (1969 and 1971), Leckie (1972), and Di Paolo (1969). Subsequent to the MGWP report various stochastic simulation models have been developed to model the investment performance of the underlying assets, the most significant of which has been that of Wilkie (1986a and 1986b).

The immunization approach is based on the recognition that the benefit payable under an equity linked endowment policy with a maturity guarantee, is equivalent to the current value of the equity portfolio plus the value of a put option on the portfolio, where the put option has an exercise price equal to the level of the guarantee. This is outlined algebraically below.

Let

$S(t)$  represent the value of the underlying assets of the policy at time  $t$ ,

$X(T)$  = the minimum guaranteed benefit payable at time  $T$ ,

$V(t)$  = the benefit payable to the policyholder at time  $t$ ,

$P(S, X, t)$  = the value of a put option on an asset with price  $S$ , exercise price  $X$ , and term to maturity  $t$ .

For a maturity guarantee, where maturity is at time  $T$ ,

$$V(T) = \max[S(T), X(T)]$$

which can be re-written as,

$$V(T) = S(T) + \max[X(T) - S(T), 0] \quad \dots(1)$$

The second term on the right hand side equation (1) is the payoff of a put option at time  $T$  with exercise price  $X(T)$ . The benefit to the policyholder can thus be expressed as the sum of the amount of the underlying assets at time  $T$  plus the payoff of a put option on these assets with the exercise price equal to the level of the guarantee. The guarantee can thus be secured by the purchase of a put option on the underlying

portfolio. Alternatively this can be viewed as the life office writing a put option on the underlying portfolio to the policyholder.

Expressing equation (1) in terms of values at the current time,  $t = 0$ ,

$$V(0) = S(0) + P(S(0), X(T), T).$$

For every one Rand of the policyholder's invested at time 0 in the underlying assets with price  $S(0)$ ,  $P(S(0), X(T), T)$  Rands must be used to purchase a put option on these assets. The value of this put option can then be taken as the fair price that the life office should charge the policyholder for the provision of the maturity guarantee. The price is fair in the sense that it will result in no abnormal profit or losses accruing to the office. Option pricing theory can then be applied to value the put option and so derive a value for the cost of the maturity guarantee.

Equation (1) can also be expressed in terms of a call option payoff as,

$$V(T) = X(T) + \max[S(T) - X(T), 0] .$$

In this case the benefit can be expressed as the sum of the guaranteed amount plus the payoff of a call option with exercise price equal to the level of the guarantee. The value of the call option does not represent the cost of the maturity guarantee, but rather the cost of obtaining exposure to (i.e. participating in the growth of) the underlying portfolio . Since this thesis is concerned with the costing of the maturity guarantee (and not with the underlying investment strategy), the focus will be on the valuation of the put option.

The immunization approach has been employed by Kahn (1971), Brennan and Schwartz (1976), Boyle and Schwartz (1977), Fagan (1977), Collins (1982), and Beenstock and Brasse (1986). Wilkie (1987) has also investigated the application of the option pricing approach to the bonus policy on with profit life assurance policies. Apart from Kahn (1971), all these papers used the option pricing model developed by Black and Scholes (1973) to price the options. Kahn applied Sprenkle's (1971) warrant valuation model to value the guarantee, as at that time the Black-Scholes model had not yet been developed.

The Black-Scholes model is used extensively in the investment market to price traded options which have terms of nine months and less. The model was not developed to price longer term guarantees. None of the studies listed above investigated the appropriateness of using the Black-Scholes model for pricing longer term options. The purpose of this thesis is to undertake such an investigation and determine whether it is sound to use the Black-Scholes model, and if not, what adaptations might be necessary.

In their report the Maturity Guarantee Working Party (1980, p. 112) commented that they "... spent some time studying the subject [of an immunization approach using options] and reached varying degrees of confidence that the mathematics were sound" and that "(i)n some cases the confidence was derived from the fact that nobody seems to have seriously challenged the underlying theory". This thesis attempts to provide such a challenge. It proposes that it is not appropriate to use the Black-Scholes model for pricing long term guarantees since application of Black-Scholes can lead to biases that are significant for long term options. The thesis focuses only on maturity guarantees, and thus is concerned with the pricing of European put options only. Surrender guarantees are more complicated and require valuation of American options.

The approach used in this thesis is to investigate the empirical soundness of the assumptions underlying the Black-Scholes model using both international evidence and evidence from the South African market, and to investigate the sensitivity of the option price to these assumptions. Attention is focused on how the impact, of deviations from the assumptions, varies as the term of the option is extended. Some quantification of the resultant pricing biases is also presented. Finally, a recommendation is made as to how the Black-Scholes assumptions might be adjusted and how the model might be adapted for the purposes of pricing long term options. Where pricing biases have been presented these have been calculated as the difference between the price of the adjusted model and the Black-Scholes model expressed as a percentage of the Black-Scholes price. The biases thus indicate the extent to which the

value of the option will increase or decrease from its current Black-Scholes value if valued by the adjusted model. Pricing calculations to arrive at a resultant option value for a specific underlying asset are not performed in this thesis, but rather a broad approach for valuation is recommended.

For the purposes of evidence of the South African equity market, the Johannesburg Stock Exchange (JSE) All-Share-Index (ALSI) has been used as a proxy. Data was obtained from the University of Cape Town database. Where evidence of the ALSI is presented, the index does not include an accumulation for dividends. The rationale for looking at dividend free data was twofold. Firstly, many of the guaranteed investment products emerging locally and internationally provide guarantees based on major equity indices exclusive of dividends. The analysis of the dividend free index data thus facilitates the pricing of guarantees given by such products. Secondly, it is not unreasonable to assume that for an index comprising a large number of shares, dividends are earned continuously. In such a case the only adjustment needed in the valuation of the option is to increase the drift rate of the index. In addition the option valuations considered in this thesis relate to options on an underlying index and not to options on the future contracts on the index, as are traded on the South African Futures Exchange (SAFEX).

The structure of the thesis is as follows. Chapter 2 explains the Black-Scholes model and its derivation. The chapter also provides a framework for understanding the impact on the option price of deviations in the underlying assumptions. Chapters 3, 4 and 5 investigate the major assumptions underlying the Black-Scholes model that impact on the long term option price. Chapter 3 looks at the assumption of the distribution of the underlying asset price, Chapter 4 looks at the volatility assumption, and Chapter 5 the interest rate assumption. Chapter 6 draws together the findings of Chapters 1 to 5 and suggests an approach to modeling the option price. A glossary has also been attached as an appendix explaining some of the more technical terms used in the thesis.

## CHAPTER 2

### THE BLACK-SCHOLES MODEL

#### 2.1 INTRODUCTION

The purpose of this chapter is to present the details and mechanics of the Black-Scholes model and to provide an understanding of the derivation of the model in order to facilitate the explanation of the impact of deviations in the underlying assumptions, which are discussed in Chapters 3 to 5.

The nature of the assumptions underlying the Black-Scholes model are explained using the alternative derivation of Cox and Ross (1976). The Black-Scholes model is then used to demonstrate how the price of the put option changes as the term to maturity is lengthened. An intuitive graphical explanation of the impact on the option price of deviations from the Black-Scholes assumptions is offered. This representation is used to explain the observed change in price of the Black-Scholes put option with outstanding term.

#### 2.2 BLACK-SCHOLES PRICING MODEL

The price,  $P$ , of a European put option as derived by Black and Scholes (1973) is given by,

$$P = X \cdot \exp(-rt)N(-d_2) - S \cdot N(-d_1)$$

Where,

$X$  = exercise price,

$S$  = spot price of the underlying asset,

$r$  = the instantaneous risk - free rate of interest,

$t$  = term to maturity of option,

$N(y)$  is the cumulative probability of the unit normal variable  $y$ ,

$$d_1 = \frac{\ln(S/X) + rt}{\sigma\sqrt{t}} + \frac{\sigma\sqrt{t}}{2}$$

$$d_2 = d_1 - \sigma\sqrt{t}$$

Where,

$\sigma$  = instantaneous standard deviation of the rate of return on the underlying asset.

Black and Scholes made the following assumptions in deriving their pricing formula:

- i) The distribution of the price of the underlying asset at the end of any finite interval is log-normal. This assumption is discussed in detail in Chapter 3.
- ii) The variance of the rate of return of the underlying asset is constant. This assumption is discussed in Chapter 4.
- iii) The risk-free rate of return is constant through time. This is discussed in Chapter 5.
- iv) The underlying asset pays no dividends or other distributions during the life of the option.
- v) There are no transaction costs or taxes.
- vi) There are no penalties to short selling of securities.
- vii) Security trading is continuous.

The interesting feature of the Black-Scholes pricing formula is that the expectation of drift in the underlying asset does not appear, that is, there is no specification in the equation to allow for the expected rate of growth of the asset over the term of the option. This feature is examined in more detail in the sections that follow.

## 2.3 BLACK-SCHOLES HEDGING ARGUMENT

The derivation of the Black-Scholes equation is based on a hedging argument. Under the Black-Scholes assumptions, the value of an option will depend only on the price of the underlying stock, the time to expiration and variables that are taken to be fixed constants, namely, the volatility of the returns of the underlying asset, and the risk-free rate of return. It is thus possible to create a hedged portfolio whose value does not depend on the price of the underlying stock. The hedged portfolio consists of a long position in the stock and a short position in the option and its value depends only on time and the value of the fixed constants.

As the price of the underlying stock changes and as time passes so the number of options to be sold to create a hedged position changes. If the hedge is maintained continuously then the return on the hedge position is completely independent of the change in the value of the stock, and hence since the return on the hedged position is certain, it must be equal to the risk-free rate. If this were not the case speculators would attempt to profit by borrowing money to set up such hedged portfolios. This increased demand will push up prices up to the point where there are no longer excess profits.

The creation of such a hedged portfolio has two important implications for the pricing of the options. Firstly, since the hedge position is riskless, the option premium at which the hedge portfolio yields a return equal to the risk free rate is the fair value of the option. Secondly, an investor who is perfectly hedged does not care whether the stock price rises or falls since the value of the hedge is independent of the direction of stock price moves. The consequence of this is that one does not need to attach utilities to the spectrum of terminal payoffs that the option may yield. The solutions to the equations are thus *preference-free*.

The properties of the hedge portfolio above can be used to derive the pricing formula. The derivation is covered in most standard option pricing text books, for example, Jarrow and Rudd (1983), a very brief summary of which is given below.

An equation is obtained for the change in the value of the hedge portfolio in a short interval of time using stochastic calculus. The equation is expressed in terms of the first order partial derivatives of the option price with respect to the stock price and time, and the second order partial derivative with respect to the stock price. Since the hedged portfolio is riskless this equation for the change in value of the portfolio is set equal to the risk-free rate of return. This partial differential equation is solved by transforming it into the heat equation of physics, and this yields the Black-Scholes pricing equation.

## 2.4 ALTERNATIVE THEORETICAL DERIVATIONS

Cox and Ross (1976) provide an alternative systematic technique for deriving the Black-Scholes equation. Their derivation provides deeper insight into the option valuation methodology and illustrates some important implications of the option valuation methodology which aren't always made clear in the option pricing literature.

From 2.3 above we have observed that the option price, through the construction of a hedged portfolio, does not depend directly on the structure of the investor's preferences. It follows that no matter what investor preferences are assumed, these will value the option identically. A convenient choice of preferences, as Cox and Ross (1976) observe, is risk-neutrality. In such a world, equilibrium requires that the expected returns on both the stock and the option equal the risk-free rate. This has two important implications. Firstly, the fair value of the option can be calculated by *discounting* the expected payoff of the option at maturity *at the risk-free rate*. Secondly, the payoff of the option at maturity should be calculated assuming that the *stock price has expected drift equal to the risk free rate*.

Algebraically, from the first implication,

$$P = \exp(-r(T-t)) \int h(S_T) \cdot dF(S_T / S_0) dS \quad \dots(1)$$

where,

$T$  is the term to maturity of the option,

$t$  is the elapsed term,

$X$  is the exercise price of the option,

$h(S_T) = \max \{(X - S_T); 0\}$  is the terminal value or payoff of the option, and

$F(S_T / S_0)$  is the cumulative probability distribution of the stock price at time  $T$  given the stock price at time 0.

They then use Kolmogorov's Backward equations to derive a partial differential equation for the diffusion process of the stock price. The equation is,

$$\frac{1}{2} \sigma^2 S^2 F_{SS} + \mu \cdot S F_S + F_t = 0 \quad \dots(2)$$

where,

$F_S, F_t$  are the first order partial derivatives of  $F$  with respect to  $S$  and  $t$  respectively,

$F_{SS}$  is the second order partial derivative of  $F$  with respect to  $S$ , and

$\mu$  = drift rate on the stock.

Finally, from the second implication above, the risk-free rate is then substituted for  $\mu$  into equation (2) above and this is transformed using equation (1) to give the Black-Scholes price.

The above derivation provides an important insight into the option valuation methodology, as described by Harrison and Kreps (1979, p. 406): "If a claim is priced by arbitrage in a world with one stock and one bond, then its value can be found by

first modifying the model so that the stock earns at the riskless rate and then computing the expected value of the claim".

Harrison and Kreps (1979) show that the drift rate adjustment (i.e. the replacing of the actual rate of return of the underlying asset by the risk-free rate of return) of Cox and Ross (1976) is equivalent to substitution of  $P^*$  for  $P$ , where  $P$  is the probability measure for the underlying asset and  $P^*$  the Unique Equivalent Martingale Measure under which the *stock has return equal to the risk-free rate*. As Harrison and Kreps (1979, p. 407) point out, "when Cox and Ross construct the preferences of the risk-neutral agent who gives the arbitrage value of claims, they are constructing an Equivalent Martingale Measure" .

While the Cox and Ross technique and the Harrison and Kreps insight might appear only of academic interest, they are in fact extremely important for a proper understanding of how the fair value of the option payoff is calculated. The Black-Scholes equation provides a very neat closed form solution to the valuation problem and is consequently widely used across the investment industry. The simplification of the partial differential equations into one equation however conceals a lot of information as to how the fair price is in fact derived. The Cox and Ross, and Harrison and Kreps approach has been used to develop a framework which provides an intuitive explanation of how deviations from the Black-Scholes assumptions affect the fair value of the option. This framework is used in section 2.6 below and in Chapters 3, 4 and 5.

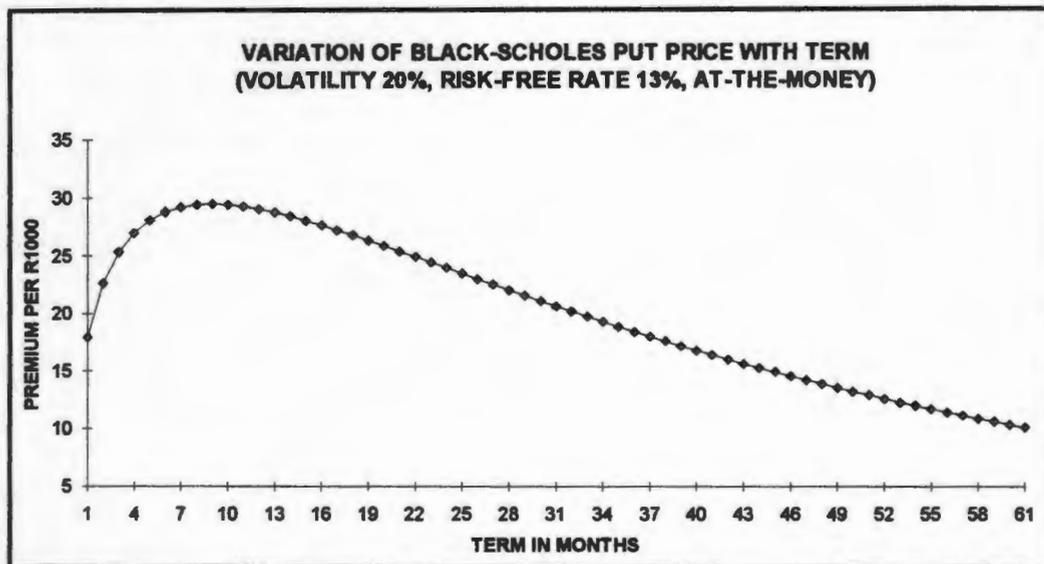
## **2.5 THE VARIATION OF BLACK-SCHOLES OPTION PRICE WITH TERM**

This paper is concerned with the pricing of long term options, hence we are interested in the change in the Black-Scholes price as the term to maturity is lengthened. The analysis which follows looks at this price/term relationship. The relationship which results, is that which would exist in a world subject to the limitations of the Black-

Scholes assumptions. Since the purpose of this paper is to investigate the soundness of, and if possible suggest improvements to, the Black-Scholes model, any alternative or adjusted model will need to be compared against this cost curve to assess how significantly prices under the new model differ.

Figure 2.1 below shows the price according to the Black-Scholes model, of an at-the-money European put option for various terms to maturity ranging from 1 month to 5 years (60 months). The prices have been calculated for an underlying stock price of R1000, assuming a volatility of 20% and risk-free rate of 13%.

Figure 2.1



As can be seen from the graph the Black-Scholes put price first increases with increasing term to exercise, reaches a peak, flattens and then reverses and declines with increasing term thereafter. This reversal of the trend in price is not evident in options that are traded on futures exchanges such as the South African Futures Exchange (SAFEX). This is in part due to the fact that trading is concentrated in short term options so that the reversal in price would not have set in. More importantly however, is the fact that options on the JSE equity indices that trade on SAFEX are not options on the index itself but rather options on the future contracts on the indices. The fair

value of the future converges to the spot price at maturity of the contract, so that if the future is priced at fair value then over its term its net carry cost will reduce to zero. The implication of this, ignoring dividends, is that, if the ALSI is assumed to have an expected drift rate equal to the risk-free rate (as it is in the Cox and Ross derivation) then the future will have an assumed drift rate of zero, since the upward drift of the ALSI will be balanced by the reduction in the net carry of the future. This zero drift rate assumption is evident in the Black model (Hull, 1989) which is used to price these options on future contracts.

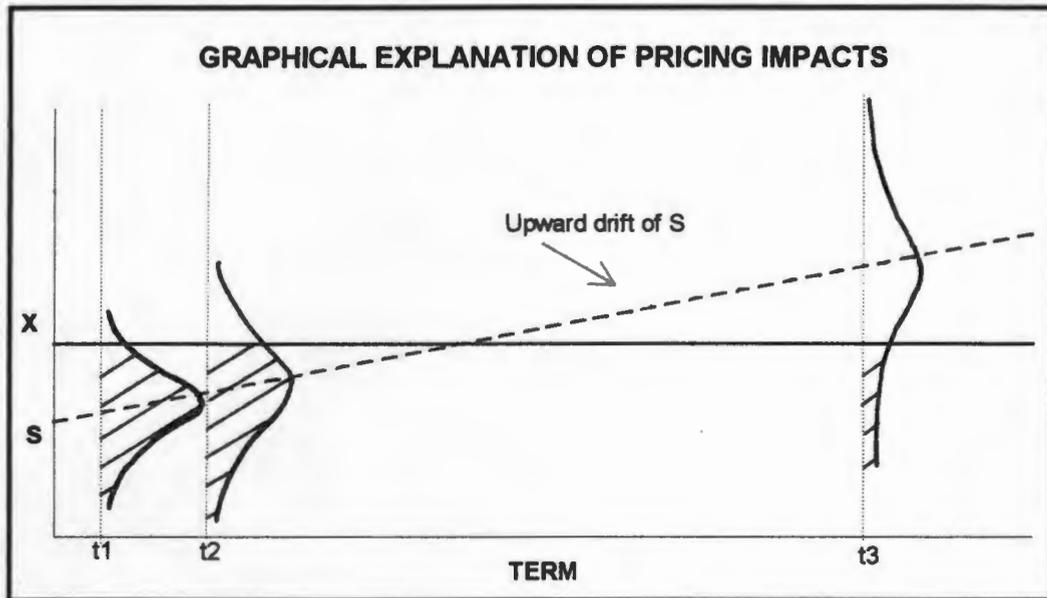
## 2.6 GRAPHICAL REPRESENTATION

Recall from 2.4, using the Cox and Ross methodology, that the fair value of an option is the discounted value (discounting at the risk-free rate) of the expected value at maturity of the payoff of the option. The expected value of the maturity payoff is calculated by multiplying the profit/loss of the option at each possible future stock price by the probability of such a price occurring. In the calculation of the latter, the underlying stock price is modified so that it has an expected upward drift rate equivalent to the risk free rate of return. This understanding of the pricing methodology produces a convenient graphical framework for investigation of the impacts on price of deviations from the underlying Black-Scholes assumptions. This framework is captured in Figure 2.2 overpage and is used to provide an intuitive explanation for the reversal in price trend observed in Figure 2.1.

The chart shows the potential payoffs of three put options maturing at times  $t_1$ ,  $t_2$ , and  $t_3$ , where  $t_2$  is only slightly longer than  $t_1$ , and  $t_3$  is significantly longer than both  $t_1$  and  $t_2$ . The option has been issued in-the-money with a strike price of  $X$  relative to a stock price of  $S$ , where  $X > S$ . At each of the maturity dates the distribution of possible stock prices is shown as assumed by Black-Scholes (log-normal distribution - discussed in Chapter 3). The options have positive payoffs at all stock prices below

X. The shaded areas of the distributions represent the probability weights attached to the positive payoffs of the options.

Figure 2.2



The effect of increasing the term from  $t_1$  to  $t_2$  is to increase the spread of the distribution around its mean, since the variance of the stock's distribution is assumed to be proportional to the square root of time (this is discussed in more detail in Chapter 3). This has the effect of reducing the probability weights attached to the small values of  $X - S$  and increasing the probability weights attached to the large values of  $X - S$ . The net effect of this is an increase in the option price when moving from  $t_1$  to  $t_2$  as reflected in the first part of the cost curve in Figure 2.1, that is, the positive price/term relationship.

When the term is increased substantially to say,  $t_3$ , another factor comes into play, namely the upward drift of the asset. The return on the asset has been assumed to be proportional to time, which might be expected in an environment of economic growth. The effect of this is to push the distribution of the stock price at time  $t_3$  further above the strike price so reducing the extent of the shaded area lying under the strike price. Where the increase in term is substantial, as is the case with  $t_3$  here, the effect of this

upward drift outweighs the increase spread effect which accounted for the increase in price in moving from  $t_1$  to  $t_2$ . The smaller shaded area at time  $t_3$  means that lower probability weight is attached to these positive payoffs and so the net effect is a reduction in the expected value of the maturity payoff and hence a lower option price.

### 2.6.1 ALGEBRAIC EXPLANATION

The impact of term on the Black-Scholes option price can be shown algebraically by looking at theta. Theta measures the rate of change of the option value with respect to time, keeping all other variables in the price unchanged. It is the first order partial derivative of the option price with respect to time. For a European put option, theta, measured with respect to outstanding term  $t$  (rather than elapsed time), is given by,

$$\Theta = \frac{S \cdot N'(d_1) \sigma}{2\sqrt{t}} - r \cdot X \cdot \exp(-rt) N(-d_2)$$

Where,

$N(-d_2)$  = the cumulative normal probability of  $-d_2$ ,

$N'(d_1) = \frac{1}{\sqrt{2\pi}} \exp(-d_1^2/2)$ , i.e. the normal probability density function of  $d_1$ .

The first term in theta accounts for the positive slope of the cost-curve. This term incorporates the effect of volatility,  $\sigma$ . The second term is responsible for the negative slope of the curve and incorporates the drift term  $r$ . As  $t$ , the term increases, both  $1/\sqrt{t}$  in the first term and  $\exp(-rt)$  in the second term decrease. The former however, decreases at a faster rate than the latter. The implication of this is that as the term to maturity increases the impact of the second term in theta, the "drift term", will become more significant than the first term, the "volatility term". This analysis is a simplification, and ignores the effect of the terms  $\sigma$ ,  $r$ , and  $t$  contained in  $N(-d_2)$ .

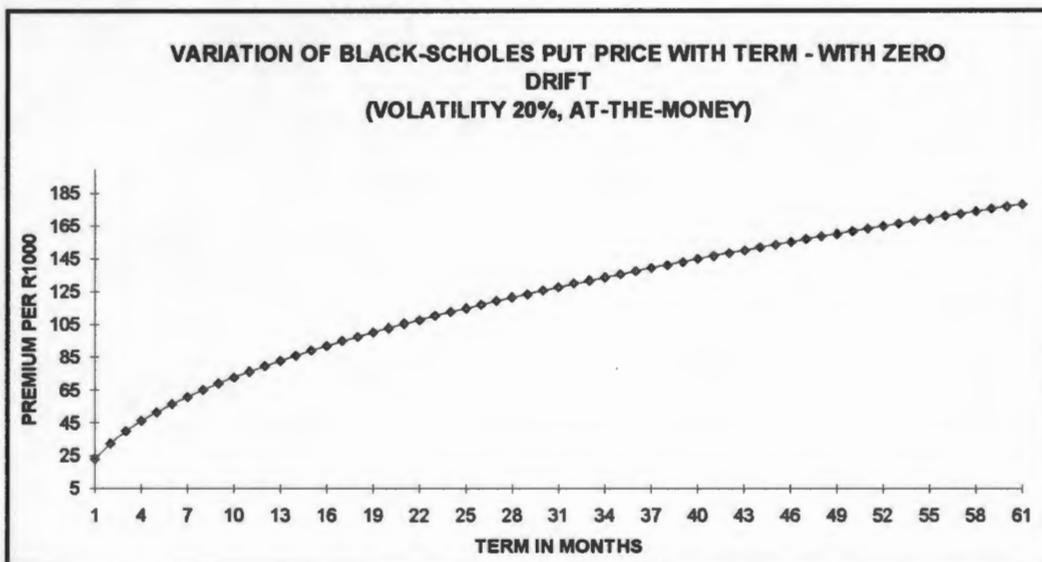
### 2.6.2 VOLATILITY VERSUS DRIFT

The explanations above have shown that for a small increase in term the volatility of

the option has a greater time to take effect, that is, there is greater likelihood of the option ending in-the-money - this increases the value of the option. However, with a significant increase in term, the drift of the underlying asset pushes the put increasingly out-the-money, so reducing the value of the option. The turning point between increasing price with term and declining price with term will thus be dictated by the magnitude of the drift rate of the underlying asset compared with the magnitude of the volatility of the asset.

For a given volatility, the lower the drift rate of the underlying asset the further out in time the turning point will be, and conversely the higher the drift rate of the asset the nearer the turning point will be. In particular if the drift rate is zero then the price curve should never reverse. If we set the risk-free rate equal to zero then it can be assumed that we have in effect, using the methodology explained in 2.4, removed the expected drift from the underlying asset. Figure 2.3 below plots the price of the same Black-Scholes put option as was valued in Figure 2.1 (volatility 20%), but with a risk-free rate of zero in this case.

**Figure 2.3**



The graph shows that the price of the put never peaks and reverses, but rather continues increasing with term throughout. This relationship of increasing cost with term is evident in options on SAFEX futures contracts. As explained in 2.5, the underlying assets in these options, namely the future contracts on the ALSI, are assumed in the derivation of the option price, to have zero upward drift, which accounts for this phenomenon of increasing cost with term.

## **2.7 PRACTICALITIES OF A HEDGED PORTFOLIO**

The assumptions of the Black-Scholes model, in particular assumptions v) through vii) in 2.2, enable the easy construction of a hedged portfolio. In practice however, these assumptions are not borne out, the implications of which are briefly discussed below.

In the Black-Scholes model transaction costs have been ignored. The hedge portfolio set up in the Black-Scholes derivation needs to be continuously rebalanced and this would involve incurring transaction costs. Brennan and Schwartz (1979) report that such transaction costs can lead to an understatement of about 3% of the price for a ten year option. Whether transaction costs should be included in the option price model or not depends on the objective of the exercise. The objective of this research is not to arrive at a price that the option would trade at in the market, but rather to provide a benchmark or basis to a life office for determining the value of a maturity guarantee.

In addition the Black-Scholes model also assumes that trading is continuous. In practice this is not possible, in particular the market is prone to sudden jumps or 'gaps' which makes rebalancing of the hedge portfolio of the Black-Scholes derivation difficult. In addition the life office will often wish to price guarantees on its internal portfolio, which might have a very different composition from any of the equity market indices for which options trade, and furthermore option contracts of the duration under consideration here are seldom traded.

It may be argued that since the underlying contracts are not traded, the option cannot be priced via the arbitrage free risk-neutral approach, since it is not possible to set up a hedged portfolio. This means that the option needs to be priced under investor preferences and utility functions. Although there is merit in this argument it is precisely in such debate that the option pricing approach differs most fundamentally from the conventional actuarial approach, in that the investor preference issue is not swept under the carpet. The conventional actuarial approach as used by the MGWP, dispenses with such preferences and just looks at expected values.

A possible response to the above criticism is that long term contracts are beginning to emerge in the market. Many of the products emerging internationally as well as locally provide long term maturity guarantees on major equity indices. Offices may thus be interested in pricing guarantees not only on their internal portfolios but also on the indices. In order to enable the providers of such products to hedge their exposures, long term option contracts (for example five year terms) are also beginning to emerge both on derivative exchanges and via over-the-counter markets. Furthermore, as mentioned before, the purpose of this thesis is not to arrive at a market price, but rather to employ the framework of option pricing theory to arrive at a theoretical value for guarantees. This is a theoretical valuation exercise in which the methodology of option pricing is used to give insight into the value of guarantees. Nevertheless, it appears that the market in long term options will grow to such an extent that long term option pricing will become commonplace.

## **2.8 CONCLUSION**

Investigation into the derivation of the Black-Scholes price has shown that the option price can be derived by discounting, at the risk-free rate, the expected payoff at maturity. The expected payoff at maturity is calculated by assuming that the underlying asset has an expected drift rate equal to the risk-free rate. Furthermore, as the Black-Scholes price is derived as part of a riskless hedge portfolio, investor

preferences do not enter into the calculation, and no utilities are attached to the terminal payoffs.

The resulting Black-Scholes put option price increases with term up to some point, after which the price decreases gradually with term. The graphical representation developed in this chapter shows that this trend in price is explained by the counteracting effects of volatility and drift of the underlying asset.

## CHAPTER 3

# DISTRIBUTION OF THE PRICE OF THE UNDERLYING ASSET

### 3.1 INTRODUCTION

This chapter investigates the assumption of the distribution of the returns and its effect on the price of the underlying asset in the Black-Scholes model. The assumption of the Black-Scholes model is explained briefly, and empirical data both international and for the South African ALSI is investigated to establish the soundness of the assumption. The chapter investigates what the impact on the option price is of a mis-specification of the distribution, and how this impact varies as the term of the option is lengthened. The chapter provides a recommendation, based on the findings of the investigations, as to how the distribution of the underlying asset should be incorporated into a long term option pricing model.

### 3.2 BLACK-SCHOLES ASSUMPTION

The basic assumption of the Black-Scholes model is that the underlying stock price process can be described by geometric Brownian motion, namely,

$$dS = \mu \cdot S \cdot dt + \sigma \cdot S \cdot dz,$$

where  $z$  is a Weiner process, i.e.  $z = \varepsilon \cdot \sqrt{dt}$ , and  $\varepsilon$  a unit normal random variable,

$S$  = price of underlying asset,

$\mu$  = instantaneous drift rate,

$\sigma$  = instantaneous standard deviation of returns on the underlying asset.

By application of Ito's Lemma (Hull, 1989), it can be shown that under this assumption the stock price is log-normally distributed, that is,

$$\ln(S_T/S_0) \sim N'[(\mu - \sigma^2/2)T, \sigma\sqrt{T}]$$

which can be re-written as,

$$\ln(S_T) \sim N'[\ln(S_0) + (\mu - \sigma^2/2)T, \sigma\sqrt{T}]$$

where,

$S_T$  = stock price at time  $T$ ;

$S_0$  = current stock price,

$N'[\cdot]$  is the normal probability density function .

Section 3.3 below investigates the theory and evidence of stock price behaviour to establish how well this log-normal assumption fits reality.

### **3.3 THEORY AND EVIDENCE OF STOCK PRICE BEHAVIOUR**

#### **3.3.1 A BRIEF REVIEW OF STOCK PRICE THEORY AND INTERNATIONAL EVIDENCE**

Fama (1965) provides a very comprehensive exposition of the development of theories on stock price behaviour up until the mid 1960's. Very briefly, the first complete random walk theory was developed by Bachelier at the turn of the century. His model, largely ignored at the time, was independently derived by Osborne over fifty years later. Even at that stage however the empirical evidence (Moore, 1962; Kendall, 1953) used to support theory showed signs of leptokurtosis, that is, too many values concentrated around the mean and too many values in the extreme tails. However, the evidence was still argued to be strong enough to support approximate normality or as in Kendall's study, the extreme observations were considered to be outliers and excluded from the statistical tests.

The hypothesis that returns are normally distributed was only first seriously questioned in the early 1960's by Benoit Mandelbrot. He argued that "academic research has too readily neglected the implications of the leptokurtosis usually observed in empirical distributions of price changes" (Fama, 1965, p. 42). The tails of the normal distribution are too small to account adequately for the number of large price changes that occur in practice. As Fama (1965, p. 42) comments "(t)he presence, in general, of leptokurtosis in the empirical distributions seems indisputable". It is somewhat surprising then to find, that over thirty years after this caution, that the investment community makes use of a pricing model based on the normality assumption.

In Fama's study (1965), daily data from 1957 to 1962 was used to test Mandelbrot's hypothesis. Fama compared the frequency distribution of the data (log of daily returns) to that which would be expected if the distributions were exactly normal. His results showed that the empirical distributions were more peaked in the centre and had longer tails than the normal distribution. Fama also looked at normal probability graphs of the data which plot  $z$  (the value which would be observed from a normal distribution) against  $u$  (the actual value observed). If the data did in fact follow a Gaussian distribution then a straight line should be produced (the derivation of these plots is described in more detail in Appendix A). His plots produced an elongated S shape with curvature at the top and the bottom indicating an excess of relative frequency in the tails of the empirical distribution.

### **3.3.2 EMPIRICAL EVIDENCE OF ALSI**

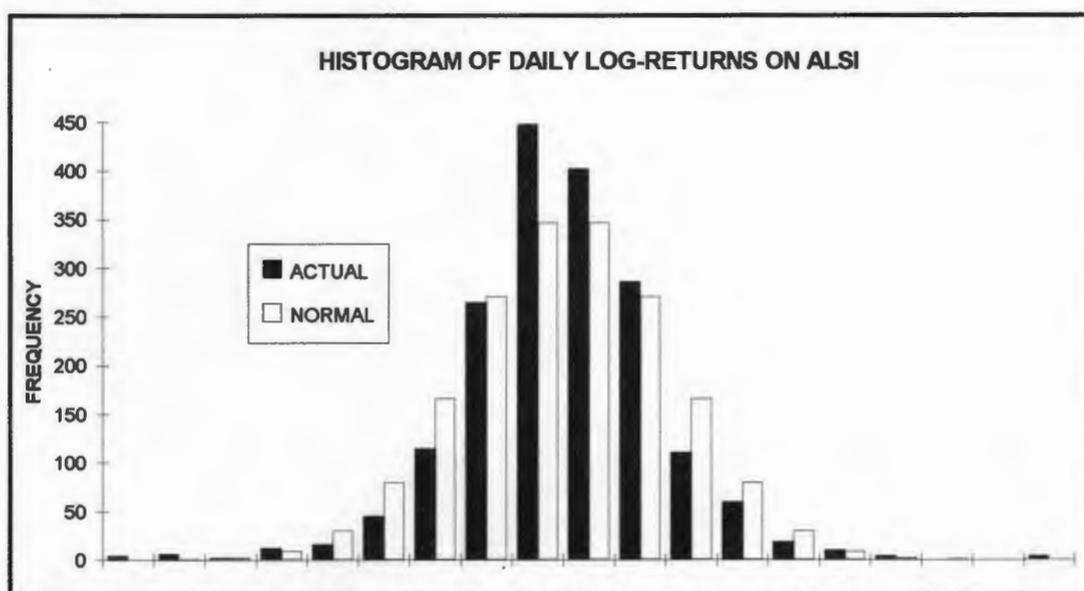
The methods used by Fama were applied to the daily log returns of prices of the ALSI for the period 9/11/87 to 1/02/95. The resulting frequency distribution of daily returns and normal probability plot are discussed below. The latter is used for comparison with the normal probability plot for five day returns presented in 3.6.1.2 .

#### **3.3.2.1 Frequency distribution of daily returns**

The histogram in Figure 3.1 shows the distribution of actual log returns against that

which would be expected from a normal probability distribution. On the x axis are the bands in which the observations are counted, and on the y axis, are the number of observations in each band out of the total of 1809 observations in the sample. The bands were chosen so as to cover 0.5, 1, 1.5, 2, 2.5, 3, 3.5, 4, and 4.5 standard deviations about the mean. (The mean and standard deviation of this sample were 14.5% and 4.6% per annum respectively).

**Figure 3.1**



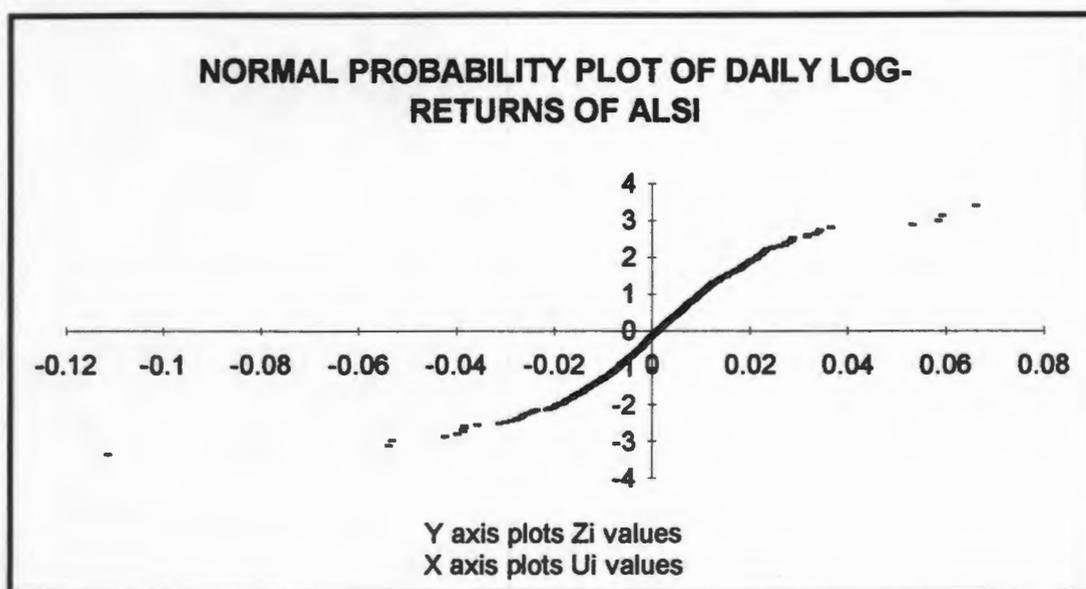
Four features can be identified from the graph. Firstly, the distribution of actual returns is more peaked in the centre, having a greater concentration of observations around the mean than the normal. Secondly the distribution of actual returns has thicker tails than the normal distribution, although the extent of this is difficult to observe in the graph due to small numbers in the tails. Thirdly, the distribution of actual returns is less bell shaped (or less concave/more convex around the mean) than the normal distribution - this is a consequence of being more peaked and having thicker tails. Finally, the distribution of actual returns is positively skewed, as compared to the symmetry of the normal distribution.

These results are consistent with those of Fama (1965) and provide some evidence that daily returns on the ALSI are not drawn from a population characterized by the normal distribution.

### 3.3.2.2 Normal probability plot

A normal probability plot of daily returns of the ALSI has been constructed using the methodology of Fama (1965) which is described in Appendix A. Figure 3.2 below shows the plot of observed log returns,  $U_i$ .

Figure 3.2



The  $U_i$ 's have an elongated S shape which indicates more observations in the tails of the distribution of actual returns than would be the case for a normal distribution. The tails of the S also have lower slopes than the central portion of the plot which indicates that the central bell of the actual distribution is higher than for a normal distribution. The S shape observed here is very similar to those presented by Fama (1965).

In conclusion, the international evidence as well as empirical evidence for the ALSI provide support for the non-normality of daily log-returns. Rather, the distribution of

daily returns appears to be more peaked and positively skewed and to have fatter tails than a normal distribution. Given that daily returns are non-normal the question that arises is what impact this has on option prices and in particular on long-term option prices. Section 3.4 describes the implications on option price of leptokurtosis in the short term, and section 3.5 considers how this effect might change for increased term to exercise.

### **3.4 IMPACT ON OPTION VALUATION OF MIS-SPECIFIED DISTRIBUTION**

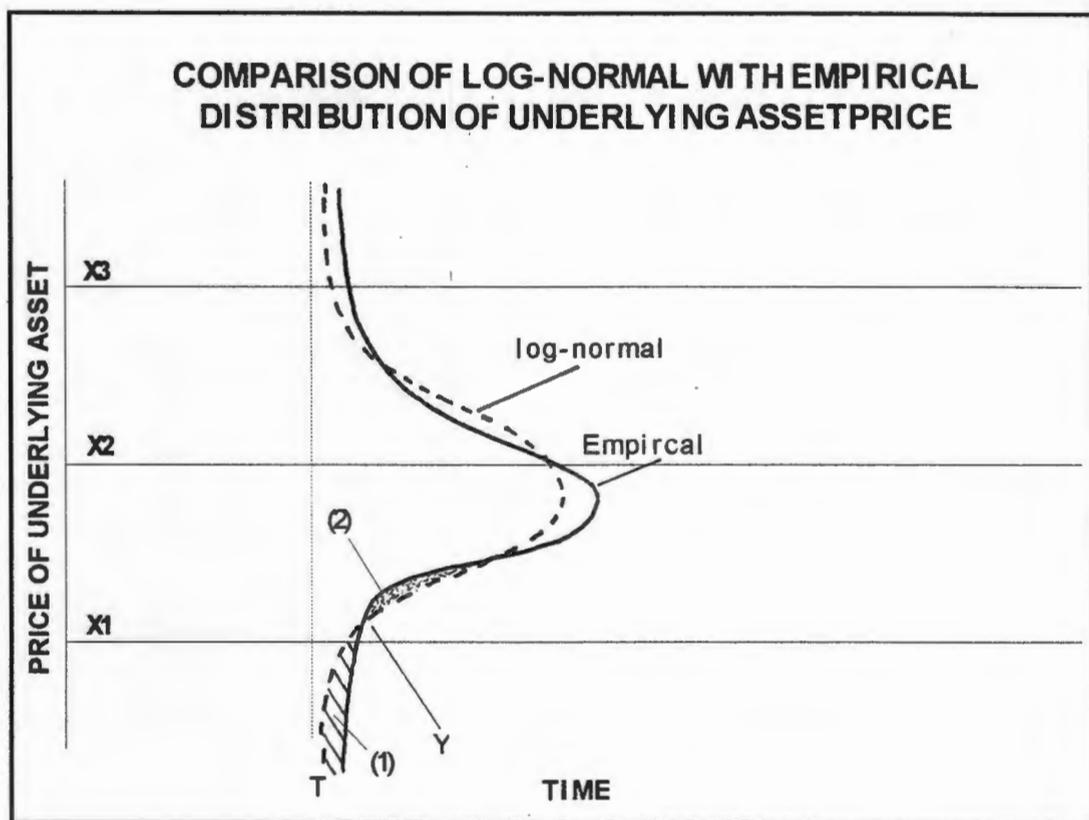
The approach here is first to give an intuitive explanation for the effect of deviations from normality and then to give some quantification of this effect.

#### **3.4.1 INTUITIVE EXPLANATION**

An intuitive explanation is provided by extending the framework used in Chapter 2. Figure 3.3 overpage shows the distribution of stock prices at maturity, time T, of a European put option. The figure compares the log-normal distribution with the empirical distribution that was indicated in 3.3.2.1.

Three different exercise prices are indicated by the horizontal lines X1, X2, and X3 and the distributions of possible stock prices at time T are shown. The broken line represents the log-normal distribution of prices and the solid line (denoted empirical distribution) representing the distribution of prices exhibiting leptokurtosis as indicated by the evidence in 3.3 above. It should be noted that the distributions have been drawn for prices and not log price ratios as in Figure 3.1. Recall from section 3.2 that,  $\ln(S_T/S_0) \sim N'[(\mu - \sigma^2/2)T, \sigma\sqrt{T}]$ , or,  $\ln(S_T) \sim N'[\ln(S_0) + (\mu - \sigma^2/2)T, \sigma\sqrt{T}]$ , which states that  $\ln(S_T)$  is normally distributed and  $S_T$  is log-normally distributed. It is  $S_T$  which has been plotted here as opposed to  $\ln(S_T/S_0)$  plotted in Figure 3.1.

Figure 3.3



For this example we define at-the-money *at maturity* as an option with exercise price  $X = S \cdot e^{rT}$ . In -the-money and out-the-money are then defined by  $X > S \cdot e^{rT}$  and  $X < S \cdot e^{rT}$  respectively. Since the stock price can be assumed to drift upward at the risk-free rate of return (as shown in the Cox and Ross derivation) it follows that, under this framework,  $X$  is the mean of the distribution of the stock price at maturity.

Consider first the put option with exercise price  $X1$ . This option is deeply out-of-the-money and has intrinsic value only at stock prices below  $X1$ . The shaded area (1) below  $X1$  represents the differences in probability (between the empirical distribution and the log-normal distribution) of the option being in-the-money at time  $T$ . As discussed in 3.3 the log-normal distribution understates the area in the tails of the distribution and hence the Black-Scholes model would attach a lower probability

weight to the payoffs of the option at stock prices below  $X_1$  than is suggested by the empirical distribution. This would result in the Black-Scholes undervaluing the option.

As one moves closer to the money so the steeper bell shape of the empirical distribution takes effect. Consider the option with exercise price  $X_2$  which is at-the-money *at maturity*, that is,  $X_2$  coincides with the mean of the distribution of  $S$  at time  $T$ . The option has intrinsic value for all stock prices below the line  $X_2$ . The undervaluation that was observed above is now compensated for by the broader bell shape of the log-normal which means that the Black-Scholes gives greater probability weight to the range of option payoffs below  $X_2$  but above  $X_1$  - this greater weight is represented by the shaded area (2). The effect of this is to reduce the extent of the undervaluation of Black-Scholes that occurs out-the-money.

As one moves deeper in-the-money, that is toward and beyond  $X_3$ , so the higher peak and the fatter *upper* tail of the empirical distribution take effect. The result is that Black-Scholes once again attaches too little probability weight to the option payoffs above  $X_2$ . The net effect is then that the extent of undervaluation increases as one moves deeper into the money i.e. as the exercise price moves further above  $X_2$  and  $X_3$ .

By inspection of the probability weightings of the two distributions in the above graph it would appear that the undervaluation in *percentage* terms, that is, expressed as a percentage of the corresponding option price, will be significantly greater for out-the-money than in-the-money options. At  $X_1$  the extent of under-weighting by Black-Scholes is indicated by the shaded area (1). At all other prices above the intersection  $Y$ , the extent of normal under-weighting is reduced by areas such as that passing through the shaded area (2), where the empirical probability weights are lower than the log-normal probability weights. The maximum undervaluation, in percentage terms, will thus appear to occur at some point around  $Y$ . It should be noted that in Rand terms the underpricing will be greatest in the higher value options, namely the in-the-money options.

The following simple algebraic example verifies, using Stoll's put/call parity theorem (Jarrow and Rudd, 1987), that the percentage undervaluation of out-the-money options is in fact greater than for in-the-money options.

From the put/call parity theorem,

$$P + S = C + X.e^{-rt}, \quad \dots(1)$$

where  $P$  and  $C$  are the Black-Scholes put and call values respectively.

Let  $P^\#$  and  $C^\#$  be the respective put and call values based on the actual distribution of the underlying asset, and consider the case where the call option is deeply out-the-money so that the exercise price is considerably higher than the stock price, for example,

$$X = 1.5.S.e^{rt}. \quad \dots(2)$$

From the above analysis (of Figure 3.3) we have that deep out-the-money options are greatly undervalued by Black-Scholes, so that, for example,

$$C^\# = 1.5 C.$$

Then by put/call parity of equation (1),

$$P^\# = P + 0.5 C$$

and

$$P = C + X.e^{-rt} - S.$$

Substituting (2) into this,

$$P = C + 0.5.S.$$

The percentage undervaluation for the call option is thus,

$$\frac{C - C^\#}{C^\#} = \frac{-0.5.C}{1.5.C} = -33\%$$

and for the put option is,

$$\frac{P - P^{\#}}{P^{\#}} = \frac{-0.5.C}{1.5.C + 0.5.S}$$

It can be seen that the latter is considerably less in absolute terms than 33, since the denominator includes the additional term  $0.5.S$ . In other words, the percentage undervaluation will be small for the in-the-money put as compared to the percentage undervaluation for the out-the-money call. The algebra will also show that for values of  $X$  lying between  $S$  and  $S.e^{rt}$ , the reverse will be the case, that is, the put will be undervalued by a greater extent than the call. The turning point of the extent of the undervaluation between the put and the call is for the case where  $X = S.e^{rt}$ , that is, where the option is at-the-money *at maturity*, as depicted by X2 in Figure 3.3. A similar analysis can be used to show that deep in-the-money calls will have a smaller percentage undervaluation than an out-the-money put.

Gemmill (1993) uses the fat tail property of empirical distributions to explain the undervaluation of deep out-the-money calls. He then uses the put/call parity theorem to infer that deep in-the-money puts will also be undervalued. It is of course more difficult to show undervaluation of in-the-money puts as argued in Figure 3.3 above due to the counterbalancing areas of the two distributions.

In conclusion, it has been shown that out-the-money calls and puts will be undervalued by Black-Scholes by a significant percentage, at-the-money calls and puts will be fairly priced by Black-Scholes if not slightly undervalued (as the impact of the fatter tails overshadows the slightly narrower bell shape of the empirical distribution), and in-the-money puts and calls will be undervalued, but to a much lesser extent, in percentage terms, than the out-the-money calls and puts.

### 3.4.2 QUANTIFICATION OF BIASES

Gastineau and Madansky (1988) attempted to establish how significant the impact of deviations from log-normality would be on option prices. Their approach was to

calculate the expected value of the option by numeric integration of the empirical distribution. The empirical distribution was obtained by studying four years of daily data for "optionable stocks". Their results are summarized in the Table 3.1 overpage. The prices are for a call option with a term of three months, and a strike price of \$40. The biases have been calculated as the difference in price expressed as a percentage of the Black-Scholes value and thus represent the extent to which the Black-Scholes price must be increased/decreased to equate to the value derived from the empirical distribution.

**Table 3. 1**

<b>Option Status</b>	<b>(1) Stock Price</b>	<b>(2) B/S value</b>	<b>(3) Empirical value</b>	<b>Bias {(3) - (2)} / (2)</b>
<b>Out-the-money</b>	\$28	0.13	0.25	92.31%
	\$32	0.60	0.70	16.67%
	\$36	1.72	1.70	-0.01%
<b>At-the-money</b>	\$40	3.67	3.68	0%
<b>In-the-money</b>	\$44	6.38	6.62	3.8%
	\$48	9.66	10.19	5.5%
	\$52	13.29	14.06	5.8%

The results in the last column, apart from the slight overvaluation for stock price at \$36, confirm the conclusions of the intuitive explanation given above, in particular, the extent of the Black-Scholes undervaluation for deep out-the-money options is extremely high.

### **3.4.3 AN ALGEBRAIC ANALYSIS**

Jarrow and Rudd (1987) derive an algebraic framework for determining how variations in the features of the distribution of the underlying asset will affect the option price. They use an Edgeworth series expansion to approximate the distribution of the asset

price. Using the log-normal as the approximating distribution they derive the value of an option as the Black-Scholes price plus adjustment terms. The adjustment terms take into account the discrepancies between moments of the log-normal distribution and the true stock price distribution. Jarrow and Rudd focus on the first three adjustment terms, with all else being explained by an error term. The three adjustment terms are:

- i) An addition for differing *variance* between the true distribution and approximating distribution.
- ii) A deduction for differing *skewness* between the true distribution and approximating distribution.
- iii) An addition for differing *kurtosis and variance* between the true distribution and approximating distribution.

The impact on the Black-Scholes option price of a deviation from log-normality will thus, according to Jarrow and Rudd's Edgeworth series expansion, depend on the extent to which the true distribution of the asset differs from the log-normal distribution on the counts of variance, skewness, and kurtosis. Each of these adjustment terms has a separate weighting factor in Jarrow and Rudd's option valuation equation, where the weighting factors are derived algebraically from the series expansion.

For the variance adjustment the weighting factor is always positive. This means that if the underlying distribution has a greater variance than the log-normal then an addition should be made to the Black-Scholes price as an adjustment for the variance of the true distribution. A higher variance thus inflates the resultant price above the Black-Scholes price.

For the skewness adjustment the weighting factor is negative for in-the-money, at-the-money and slightly out-the-money puts. This means, for these cases, that if the underlying distribution is more positively skewed than the log-normal then an addition (since the adjustment term,  $\eta$ ) above, is a deduction) should be made to the Black-Scholes price. The effect of skewness, for these options, is thus to inflate the resultant price above the Black-Scholes price. For deep out-the-money puts the weighting factor is positive, so that a deduction would need to be made from the Black-Scholes price for a positive skewness in the underlying distribution. The effect of skewness in this case is thus the opposite to the case of in-, at-, and slightly out-the-money puts, namely, a reduction in option price relative to Black-Scholes.

For the kurtosis and variance adjustment, the weighting factor is positive for deep in-the-money and deep out-the-money options. A higher kurtosis in the underlying distribution than log-normal would thus inflate the resultant price above the Black-Scholes price for these options. For all other options, that is, between the extremes of deep in-the-money and deep out-the-money, the weighting factor is negative, and the opposite to the above would result, namely, a higher kurtosis would reduce the resultant option price below the Black-Scholes price.

The sum of all the adjustment terms will have differing effects on the price of the option depending on the extent to which the option is in- or out-the-money. Jarrow and Rudd claim that in the case of a *log-normal* approximating distribution, for deep in- or out-the-money puts the *skewness* adjustment dominates while for at-the-money puts the *variance* and *kurtosis* terms will dominate. It is not, however, clear that this should in fact be the case, or at what prices these relationships hold. What the Jarrow and Rudd framework does provide is a means of determining algebraically what the impact on the option price is of deviations from the log-normal distribution. However, in order to calculate this impact it is necessary to know what the moments of the true underlying distribution are.

### 3.5 IMPACT OF A LONGER TERM TO MATURITY

One can use the intuitive framework of sections 2.6 and 3.4 to establish how the impact of mis-specifying the distribution will alter as one extends the term of the option.

For at- and out-the-money put options as the outstanding term increases so the drift term  $\mu \cdot dt$  will move the stock price further above the exercise price so shifting the positive payoff region of the option further into the tail of the distribution. The impact of this will be to increase the extent of undervaluation.

For in-the-money put options, the drift of the underlying asset price with term will first move the option towards at-the-money and then it will push the option out-the-money. The impact of this will thus be first to reduce the extent of the undervaluation, as the option moves to at-the money, and then to increase the extent of the undervaluation as the option moves out-the-money.

Call options are analogous. For at-the-money and in-the-money call options the drift with term will push the option deeper in-the-money and will thereby increase the extent of undervaluation. For out-the-money options, the drift will first push the option toward at-the-money and thereafter will move it in-the-money. The impact of increasing the term will thus be first to reduce the extent of the undervaluation, as the option is pushed from out-the-money to at-the-money, and then to increase the extent of the undervaluation as the option is pushed from at-the-money to in-the-money.

The impact of increasing term for out-the-money puts is the same as for in-the-money calls, and similarly, the impact for in-the-money puts is the same as for out-the-money calls - this is a consequence of the fact that for puts the upward drift in the underlying asset price pushes the option in the direction of out-the-money, whereas the impact of drift on calls is to push the option in the direction of in-the-money.

The turning point between the increasing effect and reducing effect of drift on the extent of undervaluation is the exercise price  $X = S.e^{rt}$ . This exercise price is equivalent to the expected value of the stock price at maturity under the risk-neutral setting of the Cox and Ross derivation (as described in section 2.4). Recall from 2.4 that in this risk-neutral world the stock is expected to drift upward at the risk-free rate.

While the above analyses does highlight the exaggeration effects of increased term of the option it makes the implicit and crucial assumption that long term returns exhibit the same deviations from normality as short term returns. Since we are concerned with pricing long term options, the focus of the next section will be to examine the distribution of long term returns.

## **3.6 LONG TERM DISTRIBUTION OF RETURNS**

### **3.6.1 STABILITY**

In his attempts to account for deviations from normality Mandelbrot (1963) wanted to keep the stationarity assumption which was fundamental to Bachelier's original random walk model and find a probability law similar to the normal distribution. He consequently focused on the stable distributions. Mandelbrot hypothesized that the distribution of stock returns is stable non-normal with a characteristic exponent,  $\alpha$ , of less than two. The characteristic exponent is a measure of the height of, or total probability contained in, the extreme tails of the stable distribution. The smaller the characteristic exponent the higher the tails. Where the characteristic exponent is less than two, the variance of the distribution is infinite. The normal distribution is a special case of the stable laws with characteristic exponent equal to two.

It is quite possible that a distribution drawn from the stable family with infinite variance will prove to be an improvement (on the normality assumption) for the purposes of option valuation. Although it is also possible that there is some fat-tailed

distribution with finite variance that could be used to adequately describe the data. Such a distribution would not, however, have the crucial property of the stable family, that of stability or invariance under addition. The decision whether to use a distribution from the stable family rests crucially on whether the assumption of stability is accepted. What follows then is evidence for and against the stability hypothesis.

### **3.6.1.1 International evidence of stability**

Fama (1965) was the first to conduct tests for stability. He used the method of comparing normal probability graphs of one day returns and sums of non-overlapping one-day returns. The normal probability plot for leptokurtic distributions has an elongated S shape with curvature in the tails, as highlighted in section 3.3.2. The degree of curvature in the extreme tails will be larger the smaller the value of alpha. If returns are drawn from a population which is stable then the sums of non-overlapping returns will have the same alpha as the distribution of the individual summands. The plot of sums of returns should, for a stable distribution, thus have an elongated S shape with roughly the same degree of curvature as the plot of the individual summands.

Fama plotted the normal probability graphs of sums of four day returns. His results showed that the graph of four day returns were almost indistinguishable from corresponding graph for daily returns. From this he concluded that the stability assumption seemed justified. Fama used this crude method of testing for stability as opposed to the more direct methods of observing alpha under addition, because the sampling period he covered was considered to be too short to reliably estimate alpha for returns over intervals greater than one day. The latter approach involves estimating alpha for daily returns and for increasing sums of daily returns and comparing them. If the sample is drawn from a stable distribution, then the two estimates should be approximately equal.

Following Fama (1965), a number of research studies were undertaken to investigate stability. Blume (1968) examining the distribution of monthly returns obtained results which supported the non-normal stable distribution hypothesis. Teichmoller (1971) examined distributions of daily returns and sums up to 10 days. He too concluded that the distribution belonged to the non-normal stable class of distributions, but he found somewhat fatter tails, that is, smaller  $\alpha$  than those reported by Fama and Blume.

According to Officer (1972, p. 807), "Fama (1965)... concluded that the distribution of monthly returns belonged to the non-normal member of the stable class of distributions". Fama, in fact, only concluded that *daily* returns belonged to the non-normal stable class and that (from the comparison of normal probability graphs) the stability assumption was justified. Fama drew no conclusion about *monthly returns*, although one can infer this from the two above mentioned conclusions. Officer (1972) on the other hand, looked at monthly returns and returns of sums up to five months. His results showed no tendency for the  $\alpha$  for larger sums to change from the  $\alpha$  for the summands. He omitted however, to report any  $\alpha$  values for these monthly and five monthly returns. From these results he suggested that, at least for sums up to five months, not much is lost by assuming stability of monthly returns. In the same study Officer looked at daily returns. His results showed a tendency for  $\alpha$  to increase for larger sums, from 1.51 for daily returns to 1.73 for twenty day returns.

Officer's results for daily and monthly returns may at first seem to contradict each other. Stability implies that if daily returns are stable then so too should be monthly returns. However, it does not follow that if daily returns are non-stable then so too should be monthly returns. Stability of monthly returns does not imply that daily returns need be stable - stability means invariance under addition not subtraction. Thus there may be no contradiction in Officer's evidence. The inferences that can be drawn from Officer's results are that daily returns are non-stable, and that the alpha exponent tends to increase as the sum size increases, but has some ceiling level toward which it approaches. The ceiling level appears to be attained by monthly returns and this level is fairly substantially less than 2 (1.73 for twenty day returns).

Officer (1972, p. 807) concludes that there is a "...tendency for longitudinal sums of daily stock returns to become "thinner-tailed" for larger sums, but not to the extent that a normal distribution approximates the distribution".

Blattberg and Gonedes (1974) used Monte Carlo techniques to simulate returns from a stable distribution and a non-stable student distribution. They then observed the effect on  $\alpha$  of these simulated distributions of moving from daily returns to sums of daily returns. Actual returns were then used to observe the effect on  $\alpha$  of the same increase in sums of daily returns and these results compared to those of the simulated distributions.

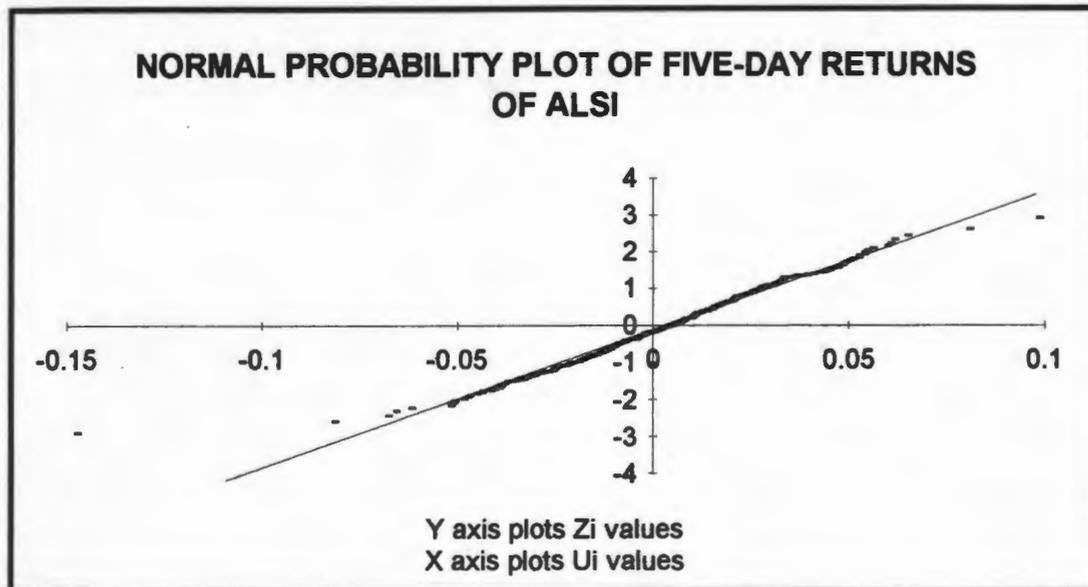
For simulated *stable* data with  $\alpha = 1.65$  the average estimate of  $\alpha$  for daily returns was 1.64 and for five day returns was 1.62. The slight decrease observed is consistent with the downward bias in the estimator of  $\alpha$  as the sample size decreases. For simulated *student* data with degrees of freedom set equal to 5, the average estimate of  $\alpha$  for daily returns was 1.7 and for five day returns was 1.8. For actual rates of return (daily) the average estimate of  $\alpha$  changed from 1.65 to 1.72 in moving from one day returns to five day returns. The increase in the average  $\alpha$  estimate was thus, they concluded, more consistent with student data (than stable) and so suggests that the process underlying stock returns is not stable.

### 3.6.1.2 ALSI evidence

Two approaches have been used in testing whether ALSI log returns are stable. Both make use of, or test for, the phenomenon that for a stable distribution, the sums of independent, identically distributed variables will, except for origin and scale, have the same distribution as the individual summands. The first test is the same as that performed by Fama (1965), namely, comparing normal probability graphs of one day, and sums of one day returns. The second test estimates alpha directly and observes the effect on alpha under addition as performed by Blattberg and Gonedes (1974) described above.

Figure 3.4 below shows the normal probability plot of five day returns on the ALSI, as constructed for one day returns in 3.3.2. The data used was (as in 3.3.2) log returns of the ALSI over the period 9/11/87 to 1/02/95. In the case of five day returns this gives a sample size of 361.

**Figure 3.4**



A straight line has been drawn to highlight the lack of curvature. It appears from this plot that five day returns do not depart significantly from normality - the plot of actual log-returns,  $U_i$ , is fairly straight and does not exhibit much S shaped curvature. Comparison with the plot for one day returns, Figure 3.2, shows quite clearly how the elongated S shape of the one day returns has flattened out for the five day returns. This evidence contradicts that of Fama (1965) who reported that one day and four day plots were indistinguishable. The evidence suggests that fat tails are less of a feature for long term returns than short term returns on the ALSI. This is evidence against the stability hypothesis.

For the purposes of estimating and observing the effect on alpha, the same daily data as above was used, and in addition data for weekly returns prior to 9/11/87 has been

used - the corresponding daily returns were not available for this period. Comparison of these returns would thus not strictly be a test of stability of sums of returns but also of stability of returns over time. These tests were nevertheless performed as they provide a larger sample size than in the case of sums of five day returns and provide more evidence of the behaviour of returns. For weekly returns data for the period 4/08/78 to 28/01/95 was used, giving a sample size of 859. The method used for calculating  $\alpha$  was initially proposed by Fama and Roll (1971) and is described in Appendix B. The calculations for  $\alpha$  for the ALSI have used  $f$  (see Appendix B) values of 0.95 and 0.97. The rationale for this choice of  $f$  value is described in Appendix C.

The  $\alpha$  estimates of daily returns, five day returns and weekly returns for the longer period to 1978 are presented in Table 3.2 below.

**Table 3.2**

	<b>Daily Returns</b>	<b>5 Day Returns</b>	<b>Weekly Returns to 1978</b>
$f = 0.95$	1.67	1.73	1.82
$f = 0.97$	1.72	1.82	1.78

The alpha estimate increases from 1.67 to 1.73 and from 1.72 to 1.82 for the two  $f$  values respectively when moving from one day to five day returns. The alpha estimates for the weekly data covering a longer period support the evidence of higher alpha values for five day returns.

The increases in alpha observed here are of a similar magnitude to those observed by Blattberg and Gonedes (1974) for their actual data as well as for their simulated student data. These results provide strong evidence that daily returns for the ALSI are not stable, but are probably more consistent with a student t-distribution than a normal distribution.

### 3.6.2 NORMALITY IN THE LONGER TERM

Gastinaeu (1988) reports that empirical work indicates that the longer the term the more closely the log-normal approximates the actual stock price distribution although he does not state the source of this empirical evidence. The evidence of Praetz (1972), Blattberg and Gonedes (1974) and that presented for the South African ALSI (in previous section), all of which demonstrated an increasing trend in alpha as the period of return is lengthened, provide some support for Gemmill's assertion. The evidence has shown that in moving from one day returns to four day or five day returns there is a reduction in the size of the fat tails. It is not possible to draw conclusions about returns for the length of periods consistent with maturity guarantees (such as one year, five years, and ten years) as the history of data is not sufficient to provide a large enough sample size for long term returns.

The contradiction between the empirical evidence and the hypothesized stable paretian distribution has stimulated much research into better explanations for market behaviour. Most research has focused on the independence assumption of the random walk theory. This thesis does not challenge the independence assumption but accepts it as a starting point, and focuses rather on arriving at an alternative probability density function.

The pioneering work done in this area is by Praetz (1972). Praetz quite rightly notes that the random walk models assume constant variance, yet in practice the market often has long intervals of relative activity followed by long periods of relative inactivity (see evidence of section 4.5.1.2). He argues that the Osborne model,

$$f(y) = \exp(-y^2 / 2\sigma^2 \tau) / (2\pi\sigma^2 \tau)^{0.5}$$

where,

$y$  is the stock price,

$f(y)$  is the probability of the stock price being equal to  $y$ , and

$\tau$  is the term over which the stock price is being predicted,

should be interpreted as a conditional distribution - conditional, that is, on the variance.

He retains the log-normal assumption for the conditional distribution of returns but also specifies the inverse gamma distribution for the variance as a random variable itself. The choice of this distribution for the variance is theoretically convenient as it is the natural conjugate for the normal distribution. In other words, if one combines the log-normal for returns conditional on the inverse gamma for volatility one gets a student t-distribution for returns. A Bayesian approach has recently (Jones, 1994; Russo, 1993) been used for estimating volatility for the purposes of option pricing. Both these authors use the inverse gamma as a prior distribution for volatility in their Bayesian estimation.

Praetz then proceeds to test this distribution against data and compares it to the log-normal, stable paretian of Mandelbrot, and the compound events model of Press. In testing the data Praetz used log returns, standardized by subtracting the sample mean and dividing by the sample standard deviation. The data was then grouped into 26 intervals. Parameters for the models were estimated by minimizing the Chi-square statistic. Praetz compares the models by comparing their Chi-square statistics. From the results of the Chi-square tests he concludes (p. 54) that "the scaled t-distribution has a far better fit, so that in every possible case, it is far better than the alternative distributions considered".

Praetz's approach has however been questioned by Blattberg and Gonedes (1974). They argue that Praetz's method of standardizing the data is inappropriate. They point out that simulation results in Fama and Roll (1971) suggest that the sample standard deviation is an extremely bad estimator of scale parameter for stable data. This they argue may explain why the stable distribution does not fit the data well in Praetz's tests and this casts serious doubts on Praetz's results.

Blattberg and Gonedes use two methods for discriminating between the stable and the student distributional fits, namely, a likelihood ratio test of the two distributions, and a

test for stability of returns, as described in 3.6.1.1 above. They proceed to show using more accurate techniques than Praetz, (in particular a more appropriate standardization of the data), that the student distribution still provides a better fit than the stable paretian distributions of Mandelbrot, so verifying the conclusions of Praetz's work. As opposed to Praetz's Chi-squared estimation of parameters Blattberg and Gonedes used maximum likelihood estimates for the student model and Fama and Roll estimators for the stable model.

Their results show that the likelihood ratio is strongly in favour of the student distribution, and that the characteristic exponent and degrees of freedom increase as the sum size (i.e. the length of the period over which returns are measured in terms of daily returns) increases. From this it is concluded that the data appear to converge to normality. The evidence for the ALSI is consistent with these results of Blattberg and Gonedes.

Blattberg and Gonedes do take cognizance of the fact that the log odds are inflated as a result of not using MLEs for the stable model. They resolve this by arguing that since the odds they observe are so strongly in favour of the student model (odds of about twenty to one in the lowest case), any reasonable adjustment (to take account of the inflation of the odds) would not alter their basic inference, namely that, "the Student model appears to describe the data better than does the stable model" (p. 275).

There is a fundamental difference in the properties of the student and stable distributions. Both student and stable models can account for fat tails in the distribution of daily returns. Under the student model, the distribution of returns converge to normality as the period over which returns are measured is extended. The stable on the other hand will not converge to normality but remains a stable distribution with the same characteristic exponent as the daily distribution.

### 3.7 CONCLUSION

This chapter has shown that mis-specification of the distribution of the underlying asset as normal when it should have leptokurtosis, leads to significant biases in the option price, especially so for out-the-money options. Furthermore, extension of the term of the option leads to exaggeration of these biases.

International evidence as well as evidence on the ALSI has shown that daily returns exhibit leptokurtosis with significant fat tails. There is a significant body of international evidence that shows that daily returns are not stable, and that the characteristic exponent increases as the length of the period over which returns are measured is extended. The evidence supports the hypothesis of a convergence of returns to normality in the longer term, and specific studies have demonstrated that the student t-distribution provides a better fit to data than a stable distribution. The evidence of the ALSI is consistent with these international studies. It is not, however, possible to investigate long term returns directly as the history of return price data is not long enough to constitute a significantly large sample size. It is thus difficult to draw firm conclusions about the distribution of long term returns.

The implication of this evidence is that for the purposes of pricing short term options normality of returns is not a good assumption, but as the term is extended, normality may become more acceptable. The fat tails observed in daily returns may be accounted for by appropriate modeling of the volatility of returns, as opposed to the constant volatility assumption made in Black-Scholes. It is recommended here, based on both the international and ALSI evidence presented, that log-normality be accepted as the distribution of returns *conditional* on volatility. The resulting unconditional distribution of returns will depend on the distribution adopted for volatility, and may coincide with the student distribution depending on what the evidence for volatility indicates, in which case the unconditional returns over the long term will be log-normally distributed. Volatility is discussed and investigated in more depth in Chapter 4.

## CHAPTER 4

# VOLATILITY

### 4.1 INTRODUCTION

This chapter investigates the volatility assumption in the Black-Scholes model. The significance of the volatility assumption for the option price is investigated, and attention is focused on how this significance changes as one moves from short term to longer term options. The chapter investigates the impact of the stochasticity of volatility, and models a number of features of the stochastic behaviour, once again focusing on the impact on longer term options. Evidence of volatility behaviour is presented from international studies, and some empirical work is conducted on the South African ALSI. Finally, the chapter makes various suggestions with respect to the incorporation of volatility into long term option pricing models. These suggestions are based on the evidence presented and the results of the investigations done into the impact of volatility on the option price.

### 4.2 BLACK-SCHOLES VOLATILITY

As outlined in 3.2 , the Black-Scholes model assumes that the stock price has the following diffusion process,

$$dS = \mu \cdot Sdt + \sigma \cdot Sdz,$$

where  $\sigma$  is the instantaneous volatility of returns.

The model thus assumes that the variance of returns of the underlying asset is constant over the term of the option. The volatility assumed in pricing an option using the

Black-Scholes model should thus be a forecast of the volatility of the underlying asset's returns over the future life (i.e. outstanding term) of the option.

In considering long term options or warrants, Black and Scholes claim (1973, p. 649) that in many cases their model "can be used as an approximation to give an estimate of the warrant value". They do, however, recognize that over a longer term, "the variance (of the) rate of return on the stock may be expected to change substantially" (1973, p. 648).

In the Black-Scholes model one is left with the problem of estimating what the volatility over the remaining life of the option will be, where, for the purpose of this thesis, the life of the option is very long. A number of methods have been used to estimate this volatility and are based on either historical returns or subjective forecasts (or a combination of these two).

When estimating future volatility from historical returns, attention needs to be given to the sampling horizon that should be used, as there is a trade-off in the length of the period chosen. The longer the period, the more data there will be and hence the lower the sampling error will be, on the other hand however, the data will also be more dated and hence less relevant to current market conditions since volatility may be unstable over time. It can be argued that in the case of long term options recent price information is less important than for short term options, and it has been suggested (Leong, 1991a) that as a rule of thumb, one should match the sampling horizon with the term to expiry of the option being priced. By doing this one will capture the effects of the behaviour of volatility which cause it to vary over time, in particular, such a method makes appropriate allowance for the effects of any mean reversion in volatility (discussed further in 4.5.1).

The constant volatility assumption of Black-Scholes is convenient in that it enables hedging argument to be used and together with this it leads to the derivation of a simple formula for the option price. The market volatility is not, however, constant,

and the extent of the deviation and its impact on option prices are examined in the sections below, with particular focus on the effect on longer term option pricing.

### **4.3 SIGNIFICANCE OF VOLATILITY FOR LONGER TERM OPTIONS**

Gastineau (1988) states that for long term options "volatility estimation is less critical, because it is considerably more stable over the long term than the short term". While there may be evidence to support the latter part of the statement, the former claim does not follow from this and no investigation into the sensitivity of the pricing of long term options to changes in volatility was presented to back up this claim. On the other hand Kuwahara and Marsh (1992 , p. 1611) argue that "...since warrants have much longer maturities than options...we would expect greater mispricing to result from an incorrect assumption that stock price volatility is constant at some historical level". These two assertions are in direct conflict and it is not intuitively obvious which is correct.

Part of the purpose of this section is to resolve the above conflict. This section thus investigates the sensitivity to volatility of a put option with increased term of the option. By assuming that we operate in a Black-Scholes setting, that is, by accepting all the Black-Scholes assumptions, we can determine the impact of a change in the volatility of the underlying asset on the price of the option. In this way we can establish the extent to which the Black-Scholes model mis-prices the option due to incorrect volatility assumptions. By looking at how this under- or over-pricing varies across term we can establish how the sensitivity of the option price to volatility changes with increased term of the option.

In the analysis which follows in this section the limitations of accepting the Black-Scholes assumptions must be borne in mind, so that the results obtained and deductions made are only strictly true in a Black-Scholes setting. In particular, the

effect of a stochastic variation in volatility is not considered here but is dealt with in 4.4, rather the constant volatility assumption is accepted. Nevertheless, the results, explanations and deductions which emerge here, contribute greatly to an understanding of the impact of volatility on the option price.

The sensitivity to volatility of a Black-Scholes put option is commonly measured by kappa (sometimes also referred to as vega or lambda). Kappa is the first order partial derivative of the option price with respect to the volatility of the underlying asset. Thus if kappa is large then the option is very sensitive to small changes in volatility and conversely, if kappa is small then small changes in volatility will have relatively little impact on the option price. Algebraically, using the same notation as in section 2.2,

$$\text{kappa} = S \sqrt{t} N'(d_1)$$

where,

$N'(d_1)$  is the normal probability density function of  $d_1$  viz.,

$$N'(d_1) = \frac{1}{\sqrt{2\pi}} \exp(-d_1^2/2), \text{ and}$$

$$d_1 = \frac{\ln(S/X) + rt}{\sigma\sqrt{t}} + \frac{\sigma\sqrt{t}}{2}.$$

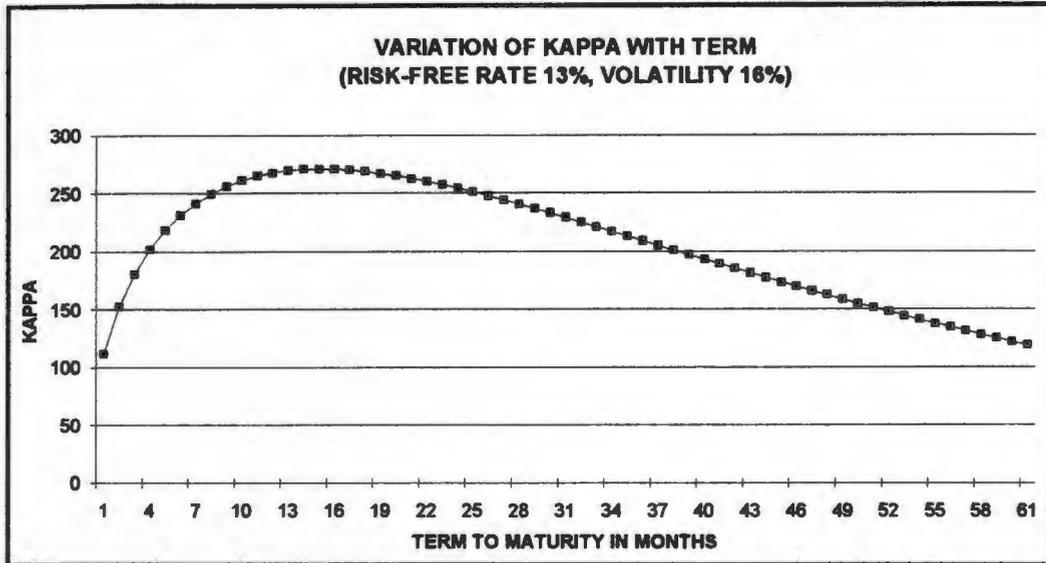
For all options kappa is positive since an increase in volatility increases the likelihood of the option ending up deep in-the-money.

Figure 4.1 overpage shows how kappa varies with term for an at-the-money put option (where the option is priced with a risk-free rate of 13% and volatility of 16%). The kappas calculated are for an option on a stock with a spot price of R1000.

As can be seen kappa first increases with term up to a point, levels off and then declines thereafter with increasing term. This means that the sensitivity of the option to volatility increases with term up to some point and thereafter the sensitivity declines. The graphical framework for explanation of the cost curve of Black-Scholes (as

detailed in 2.6) is used below to explain this relationship between sensitivity to changes in volatility and outstanding term.

Figure 4.1



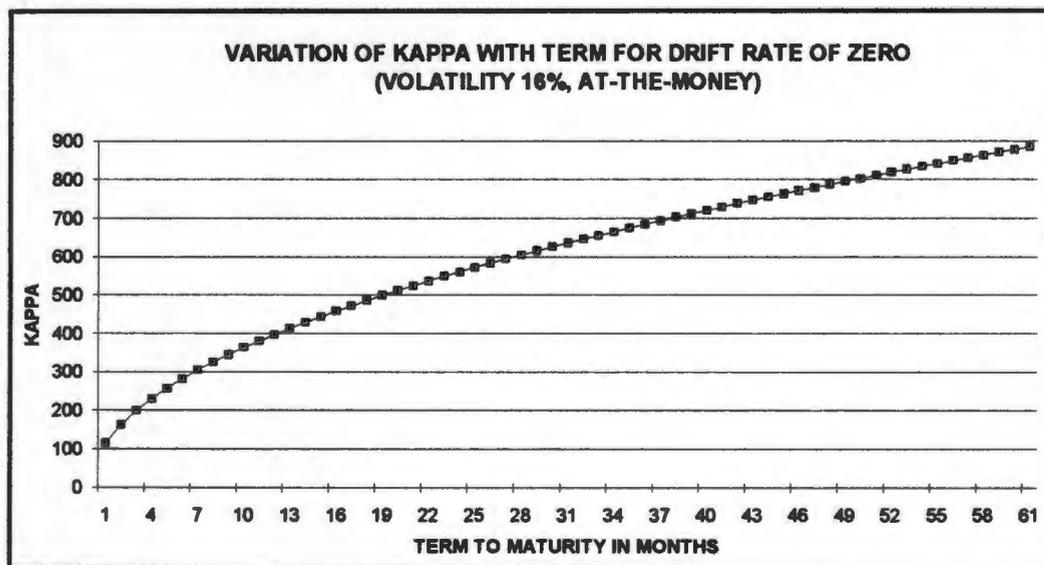
As was mentioned in 2.6 extending the term of the option has two conflicting effects. Firstly, since the volatility of the underlying log-normal distribution is proportional to the square root of time (that is, time elapsed since inception of the contract), an extension of term increases the spread of the distribution of possible stock prices at maturity. Secondly, since the drift rate of the asset is proportional to time (that is, time elapsed since inception), an extension of term increases the extent to which the underlying asset has drifted upward since inception of the option contract .

These two effects are reflected in the formula for kappa, both by the  $\sqrt{t}$  term, and by the  $N'(d_1)$  term. The increasing trend in kappa with term in the first part of the chart can thus be attributed to the effect of  $\sqrt{t}$ , namely, for a longer term option a given volatility will result in distribution of stock prices at maturity being more spread outwards than for a shorter term option. In effect, for a longer term option there is more time for the volatility to take effect, than for a shorter term option. Hence a longer term option is more sensitive to a change in volatility. The relationship between

kappa and term reverses though when the impact of the upward drift of the underlying asset, as reflected by the term  $N'(d_1)$ , begins to dominate. The upward drift pushes the positive payoff region of the option into the lower tail of the distribution where volatility changes will have less impact. Hence options with sufficiently long outstanding terms will be less sensitive to volatility than shorter term options.

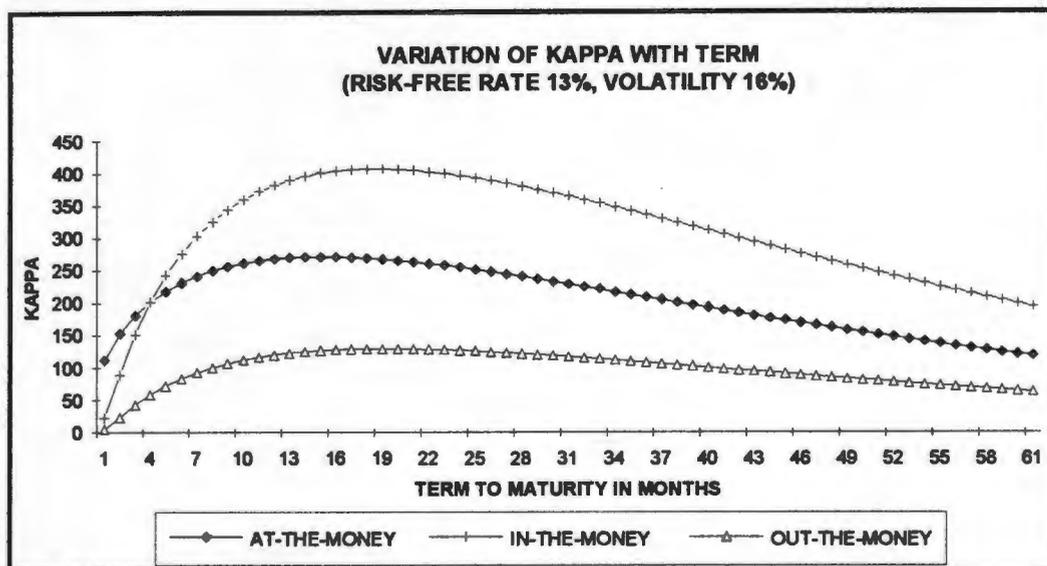
The conclusions and explanations given above for the relationship between term and sensitivity to volatility are corroborated by the trend of kappa where the risk-free rate is assumed to be zero. Since, in the Cox and Ross derivation (see 2.4), the asset is assumed to drift upward at the risk-free rate, a zero risk-free rate will mean that an at-the-money option remains at-the-money when the term is extended, since the asset has no upward drift. The impact of the effect of upward drift is thus removed. This is illustrated in Figure 4.2 which shows the kappa increasing with term throughout (volatility of 16%). This means that for an asset which has no upward drift, the option's sensitivity to volatility will increase without limit with increased term of the option.

Figure 4.2



It is also beneficial to examine how the sensitivity to volatility by term changes for in-the-money and out-the-money options. Figure 4.3 compares the variation of kappa with respect to term for an at-the-money, in-the-money and out-the-money put option (with an assumed drift rate of 13% and volatility of 16%). For the in-the-money puts the ratio of the exercise price to the spot price is somewhat arbitrarily set at 1.1 for all terms, and for out-the-money puts, the ratio is 0.9.

Figure 4.3



For an out-the-money put option the exercise price lies below the stock price at inception, so that the positive payoff region lies to a greater extent in the tail of the distribution of the asset price. The option is thus less sensitive to volatility than an at-the-money option and continues to be so over all terms. For in-the-money puts the exercise price lies above the stock price at inception, so that the drift of the stock will take longer to push the positive payoff region of the option into the tail of the distribution. The sensitivity of the option to volatility will thus be greater than for an at-the-money option. For very short terms however, the option will be less sensitive than an at-the-money option, since the payoff region will encompass part of the positive tail of the distribution (i.e. that part which lies above the mean of the distribution). The turning point in kappa will also be further out than for at-the-money

since it takes longer for the tail of the distribution to encompass the positive payoff region of the option.

The following conclusions can be drawn from the above analysis. Firstly, for an asset which is assumed to have positive drift, the sensitivity of the option price to volatility will first increase with term and then after some point, will decrease with term. Secondly, the lower the drift rate of the underlying asset the further out this turning point will be and conversely the higher the drift rate the earlier the turning point will be. Thirdly, the deeper in-the-money the put option is, the more sensitive to volatility it will be in the long term, and thus the greater the impact of extending the term will be. Lastly, the deeper out-the-money the put option is, the less sensitive to volatility it will be, and thus the less significant the impact of increased terms will be.

This paper is concerned with pricing maturity guarantees which will often, as pointed out in Chapter 1, be of the form of a return of capital plus some nominal rate of return, such as 5% per annum. The option required to secure these guarantees would thus be deep in-the-money, and would, in line with the third conclusion above, be more sensitive to volatility the longer the term of the option is. Figure 4.4 overpage shows the variation of kappa over term for put options which all secure a guaranteed return of 5% per annum (drift rate of 13%, and volatility of 16%). The chart shows that the sensitivity to volatility increases for a more significant period than before (compared with Figure 4.1), and remains very high after five years.

In all these graphs kappa is a measure of the Rand change in option price for a instantaneous change in volatility. A different picture would emerge if sensitivity was plotted as percentage (as opposed to absolute Rand amount). Since the price of the put option decreases with term beyond some point, the sensitivity expressed in terms of percentages of the option premium will increase with the term of the option throughout.

**Figure 4.4**

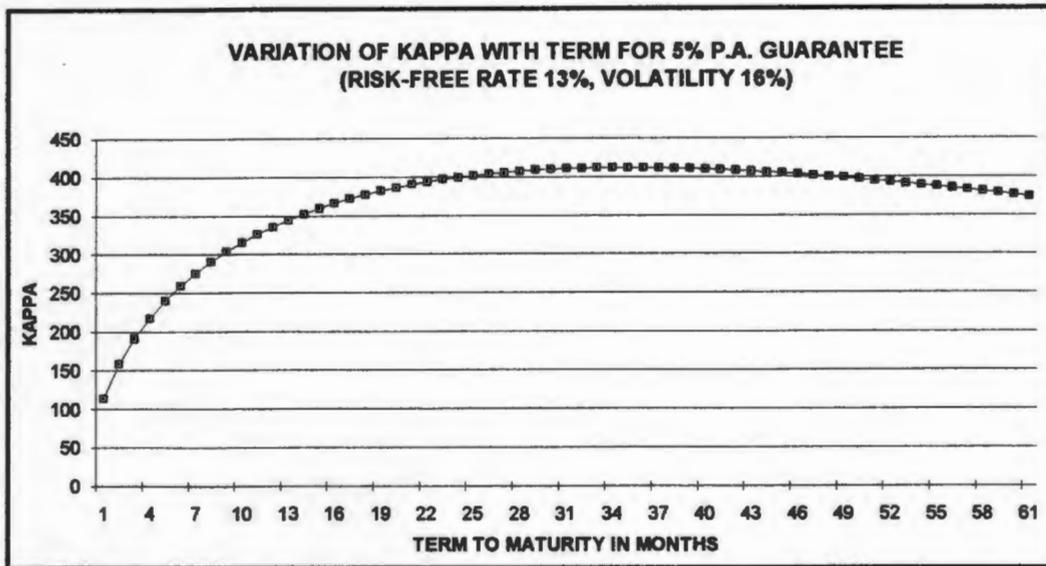


Table 4.1 below quantifies the impact on the option price of an error in the estimate of volatility. The prices calculated are for a five year option with a guaranteed return of 5% p.a. and risk-free rate of 13% p.a.. Assuming volatility is actually 16% the Black-Scholes price, for a stock price of R1000, should be R18.5. The errors are calculated as the price difference expressed as a percentage of this price.

**Table 4.1**

	<b>VOLATILITY</b>	<b>BLACK-SCHOLES PRICE</b>	<b>ERROR</b>
1%	17%	22.40	21.1%
ERROR	15%	14.89	-19.5%
2%	18%	26.57	43.6%
ERROR	14%	11.61	-37.26%

In other words, for this option, every 1% over- or under-estimation of volatility will thus result in approximately 20% under- or over-pricing respectively.

From the above analysis it can be concluded that, for the purpose of pricing maturity guarantees, in particular guarantees of positive nominal annual returns, accurate

volatility estimation is considerably more important than for traded short term options, and mis-estimation of volatility leads to significant errors in the option price.

## **4.4 IMPACT OF STOCHASTIC VOLATILITY**

Three aspects of stochasticity are considered in this section:

- i) The implications of stochasticity for the hedging argument.
- ii) The importance of the path followed by the stock price over the term of the option.
- iii) The impact of stochasticity on the option price.

### **4.4.1 HEDGING ARGUMENT**

If volatility is assumed to be stochastic, then Black-Scholes model no longer holds. You will recall from 2.3 that Black-Scholes in their derivation set up a hedged portfolio consisting of a long position in stock and a short position in the option on that stock. However, if we operate in a world with non-constant volatility, then the position is no longer perfectly hedged, since a change in volatility will affect the price of the option and the price of the stock with different magnitudes. Thus a riskless hedge cannot be formed from only one option and the stock, and consequently arbitrage alone cannot determine the hedge portfolio's excess expected return.

The above-mentioned problem is one faced by fund managers who run hedged positions in the market, and is referred to as kappa risk. The delta/gamma-neutral positions that such fund managers run, hedge the portfolio against small changes in the price of the stock, but are still vulnerable to changes in the underlying volatility.

Two approaches have been covered in the literature to deal with this problem, namely those based on equilibrium arguments and those based on complete markets. These are discussed separately below.

#### 4.4.1.1 Equilibrium Arguments

Garman (1976) and Cox, Ingersoll and Ross (1985a) have shown that a security with a price  $f$  that depends on state variables  $\theta_i$  will satisfy the differential equation,

$$\frac{df}{dt} + \sum_i \theta_i \cdot \frac{df}{d\theta_i} \cdot (m_i - \lambda_i s_i) + \frac{1}{2} \sum_{i,k} \rho_{i,k} \cdot s_i \cdot s_k \cdot \theta_i \cdot \theta_k \cdot \frac{d^2 f}{d\theta_i d\theta_k} = rf,$$

where,

$m_i$  is the expected rate of growth in  $\theta_i$ ,

$\lambda_i$  is the market price of risk of  $\theta_i$ ,

$s_i$  is the volatility of  $\theta_i$ ,

$\rho_{i,k}$  is the correlation between changes in and  $\theta_i$  and  $\theta_k$ , and

$r$  is the risk - free rate of interest.

According to the Capital Asset Pricing Model (CAPM), for a **traded** security

$$m_i = r + \lambda_i s_i .$$

When a variable  $\theta_i$  is not a traded security then one cannot assert that  $m_i = r + \lambda_i s_i$  or that  $m_i - \lambda_i s_i = r$ . In the case of option valuation, there are two variables  $S$ , the stock price, and  $V$ , the volatility of stock returns, of which only  $S$  is traded. The differential equation for the option thus contains the term

$$V \cdot \frac{df}{dV} \cdot (m_V - \lambda_V \cdot S_V)$$

on the left hand side, and it cannot be assumed that

$$m_V - \lambda_V \cdot S_V = r .$$

In order to solve this differential equation  $\lambda_V$  the market price of risk of volatility needs to be known. This requires knowledge of the investor's risk preferences. Rearranging the formula for  $m_V$  one gets

$$\lambda_v = \frac{m_v - r}{S_v}$$

What remains to be established from investor's risk preferences then, is thus the excess expected return per unit of risk.

Equilibrium arguments can be invoked where assumptions are made about the investor's preferences, and hence the investor's tolerance for bearing this risk (i.e. the risk of volatility) is known. Equilibrium based pricing models have been used by Dothan and Reisman (1985), Hull and White (1987), Scott (1987), Wiggins (1987) and Stein and Stein (1991). In particular, Hull and White (1987), recognizing the need for equilibrium arguments, assume that volatility is uncorrelated with aggregate consumption and thus that volatility risk is not priced. Scott (1987) assumes that the volatility risk can be diversified away and that volatility and stock returns are uncorrelated. Wiggins (1987) assumes that the investor has log utility. The algebra shows that for this utility function the price of risk is zero.

#### 4.4.1.2 Complete Markets

Where volatility is not a traded security, that is, where there is no traded asset whose price is equivalent to the instantaneous volatility of the asset underlying the option, then, as seen in the above section, a perfect hedge cannot be formed and excess risk remains. A market without such a traded security is described as incomplete. As a result, the option cannot be uniquely priced by arbitrage arguments alone and equilibrium pricing arguments are invoked to solve this non-uniqueness problem. If, however, we want to make use of the Equivalent Martingale Measure approach and follow the Cox and Ross, and Harrison and Kreps's method of risk-free discounting of the terminal payoffs of the option (section 2.4), then we must "complete" the market.

To "complete" the market an additional asset is introduced. Eisenberg and Jarrow (1994) derive the stochastic process that such an asset must satisfy and show that any

asset correlated with both the underlying stock and the volatility is acceptable. All the inter-temporal assumptions associated with the pricing of the volatility risk would be incorporated in the price of such an asset and a hedge could be constructed using stock, option and this  $\sigma$ -based asset. The differential equation would then be solved without hypothecation about investor risk preferences.

A complete market approach is used by Merton (1973), Cox and Ross (1976), Eisenberg (1987), Johnson and Shanno (1987), Scott (1987) and Eisenberg and Jarrow (1994).

The complete market approach has the advantage of *simplicity*, as opposed to the general equilibrium approach, where it is very difficult to identify a model which is both *realistic* and *practical* or usable. The complete market approach can be defended on the grounds that such  $\sigma$ -based assets (those used to "complete" the market) are beginning to emerge in some markets, for example, the Chicago Board Options Exchange has established a Volatility Index (Whaley, 1993) which is based on implied volatilities of a selection of options.

In order to avoid the difficulties associated with having to make assumptions about investors' risk preferences the approach below concentrates on the complete market approach only.

#### **4.4.2 PATH DEPENDENCE**

Since we are concerned with valuing European options, a question that arises is whether the path followed by volatility over time is of relevance, since the value of the option depends only on the terminal distribution of the stock price. If we know what the mean volatility is over the period is it necessary to know the path that volatility has taken?

The answer to this question is obtained by looking at the impact on the pricing formulae that a stochastic volatility assumption has. Hull and White (1987) provide a good explanation of the impact on the option valuation, and this is detailed in the sections which follow. As mentioned in 4.4.1.1 they used an equilibrium argument approach where risk preferences are removed from the partial differential equation by assuming that volatility is uncorrelated with aggregate consumption (so that there is no systematic or market related risk). The price of a European *call* option, denoted by  $f(S_t, \sigma_t^2, t)$ , is obtained by discounting the expected terminal payoff of the option at the risk-free rate. The same method of derivation can be used for the case of put options. Algebraically,

Define,

$f(S_t, \sigma_t^2, t)$  = the value of a call option at time  $t$ , with share price  $S_t$ , and instantaneous volatility  $\sigma_t^2$ .

$$= \exp(-r(T-t)) \int f(S_T, \sigma_T^2, T) p(S_T | S_t, \sigma_t^2) dS_T,$$

where,

$T$  = time at which the option matures,

$S_t$  = stock price at time  $t$ ,

$\sigma_t$  = instantaneous standard deviation at time  $t$ ,

$p(S_T | S_t, \sigma_t^2)$  = the conditional probability density of  $S_T$  given the stock price and variance at time  $t$ .

If  $\bar{V}$ , as the mean variance over the life of the option, is defined as,

$$\bar{V} = \frac{1}{T-t} \int_t^T \sigma_t^2 dt,$$

then the probability of  $S_T$  may be given as,

$$p(S_T | \sigma_t^2) = \int g(S_T | \bar{V}) \cdot h(\bar{V} | \sigma_t^2) d\bar{V},$$

where,

$g(S_T|\bar{V})$  = the conditional probability density of  $S_T$  given the mean volatility  $\bar{V}$ ,

$h(\bar{V}|\sigma_t^2)$  = the conditional probability density of  $\bar{V}$  given the volatility  $\sigma_t^2$  at time  $t$ .

Then,

$$f(S_t, \sigma_t^2, t) = \exp(-r(T-t)) \iint f(S_T, \sigma_t^2, t) \cdot g(S_T|\bar{V}) \cdot h(\bar{V}|\sigma_t^2) d\bar{V} dS_T$$

which can be re-written as,

$$f(S_t, \sigma_t^2, t) = \int [\exp(-r(T-t)) \int f(S_T) \cdot g(S_T|\bar{V}) dS_T] h(\bar{V}|\sigma_t^2) d\bar{V}. \quad \dots(1)$$

The arguments and conclusions as to whether the value of the option depends on the path taken by volatility differ for the case where stock returns and volatility are assumed to be correlated from that where they are assumed uncorrelated. These two cases are considered separately below:

#### 4.4.2.1 Zero correlation

If the instantaneous correlation between volatility and stock returns is zero then the term in equation (1) contained in square brackets, can be shown to be the Black-Scholes price for a call option on a security with mean variance  $\bar{V}$ . For this to be the case it is necessary to show that the probability density of  $S_T/S_0$  conditional on  $\bar{V}$ , that is  $g(S_T|\bar{V})$ , is log-normal, since in the Black-Scholes valuation the underlying asset's return's are assumed to be log-normally distributed. Hull and White, (1987, p. 285) show that if  $S$  and  $V$  are instantaneously uncorrelated then the terminal distribution of  $\log(S_T/S_0)$  is normal with mean  $rT - \bar{V}T/2$  and variance  $\bar{V}T$ , which depends only on the mean of the volatility. The distribution of  $S_T/S_0$  conditional on  $\bar{V}$  is thus log-normal. Furthermore, for an infinite number of paths of  $\sigma_t^2$ , where  $\sigma_t^2$  is stochastic, with the same mean volatility  $\bar{V}$ , all the paths produce the same terminal distribution of the stock price.

Thus (1) may be re-written as,

$$f(S_t, \sigma_t^2, t) = \int C(\bar{V})h(\bar{V}|\sigma_t^2)d\bar{V}, \quad \dots(2)$$

where,

$C(\bar{V})$  is the Black-Scholes price for a call with volatility estimate  $\bar{V}$ .

Thus for the purpose of the valuation of an option with stochastic volatility, in the case where volatility and stock returns are uncorrelated, knowledge of the path taken by volatility is not necessary. This does not, however, mean that the option can be valued purely on the estimate of the mean volatility, since the option price is an expectation taken over all possible values of  $\bar{V}$ . Knowledge of the distribution of the mean is thus needed. It can be shown (as detailed in 4.4.3.2) that the more confident one is in estimating the magnitude of  $\bar{V}$ , or in other words, the less spread the distribution of  $\bar{V}$  is about its mean, the closer  $f(S_t, \sigma_t^2, t)$  will be to the Black-Scholes value.

Eisenberg and Jarrow (1994) derive a similar result to (2) above using the complete market approach. They derive a solution for the case where volatility exhibits both mean-reversion and is instantaneously correlated to stock returns. Their equation shows the call value as a weighted average of Black-Scholes values, each with differing volatility where the weights are the probabilities of these volatilities occurring.

#### 4.4.2.2 Non-zero correlation

In order to simplify the analysis, and without loss of generality, Hull and White (1987) consider the discrete case where volatility is assumed to change at only  $n$  equally spaced times in the interval from 0 to  $T$ . Define  $S_i$  as the stock price at the end of the  $i$ th period and  $V_{i-1}$  as the volatility during the  $i$ th period. If the stock price and volatility are instantaneously correlated then both  $\log(S_i/S_{i-1})$  and  $\log(V_i/V_{i-1})$  are normal with correlation denoted by, say,  $\rho$ . Hull and White (1987) then show that the distribution of  $\log(S_T/S_0)$  conditional on  $V$  is normal with mean,

$$rT - \frac{\bar{V}T}{2} + \sum_i \frac{\rho\sqrt{V_{i-1}}}{\xi} \left[ \log\left(\frac{V_i}{V_{i-1}}\right) - \frac{\mu T}{n} + \frac{\xi^2 T}{2n} \right]$$

where,

$i$  refers to the  $i$ th period,  $i = 1$  to  $n$ ,

$V_i$  = the volatility during the  $i$ th period ,

$\mu$  = the drift rate of the volatility ,

$\xi$  = the variance of the volatility ,

and variance,

$$\bar{V}T(1 - \rho^2).$$

Thus the mean of the distribution is dependent on the attributes of the path followed by  $V$ .

It follows from this that in the case of correlated stock prices and volatility, one cannot just price the option by estimation of the mean volatility over the life of the option. Knowledge of the actual path taken by volatility over the term of the option is required.

#### 4.4.3 IMPACT ON OPTION PRICE

Most of the research into volatility to date has focused on the impact of two features of volatility behaviour, namely, the correlation between volatility and stock prices, and mean reversion of volatility (discussed in 4.5.1). Models have been developed incorporating these features, to establish whether the resultant prices explain some of the differences exhibited between Black-Scholes prices and actual market prices - this is discussed in more detail in 4.5.2.1.

Hull and White (1987) provide a theoretical derivation of the impact of stochastic volatility on the option price. They consider separately the case where volatility and underlying stock prices are uncorrelated and the case where volatility and stock prices are correlated. They also consider the impact of mean reversion. For each of these

they compare the Black-Scholes price with the price obtained from their model which incorporates that type of stochastic volatility. Their results, which are derived for call options, are discussed in the sections below. The results can easily be applied to put options using the put/call parity theorem.

#### 4.4.3.1 Zero Correlation

Hull and White conclude that for the case where volatility and stock prices are uncorrelated:

*Black-Scholes tends to - overprice "at-the-money" call options*

*- underprice deep in-the-money AND deep out-the-money calls*

where at-the-money is defined to be  $S_t = X \cdot \exp(-r(T - t))$ .

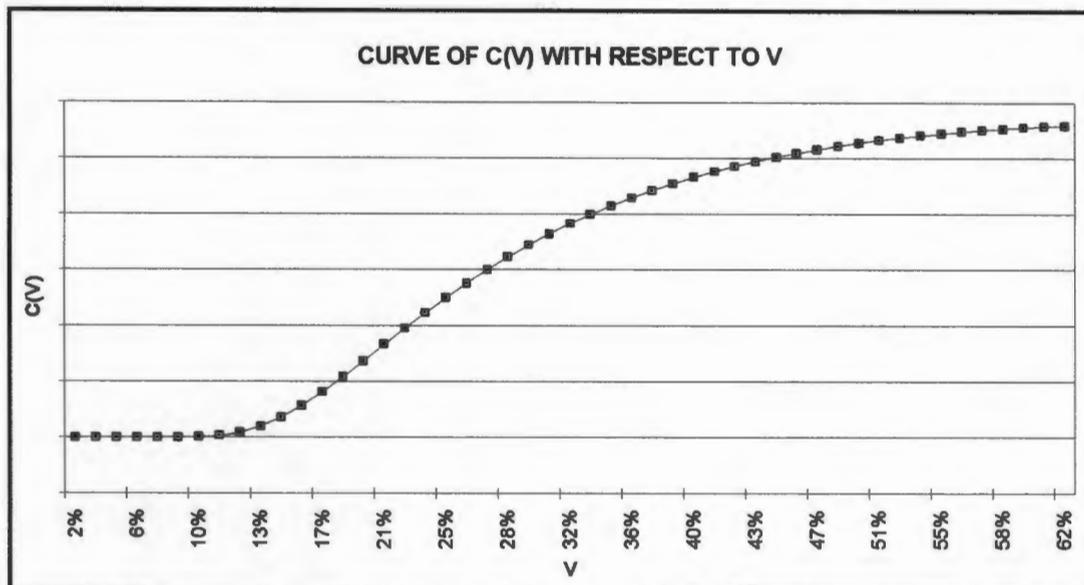
This pricing relationship will also apply to put options, which follows from the application of the put/call parity theorem.

The former result, namely, that the Black-Scholes price is greater than the stochastic volatility price may seem counter-intuitive. Intuitively one might expect that greater uncertainty about volatility increases uncertainty about the stock price and thus increases the call price. Hull and Whites' explanation of these pricing biases is given here to explain the impact of an extension of term on the bias.

In 4.4.2.1, for the case of zero correlation, it was shown that the option price is the average of Black-Scholes price where the averaging takes place over the distribution of the mean volatility. This is given algebraically in 4.4.2.1 as  $\int C(\bar{V})h(\bar{V}|\sigma_i^2)d\bar{V}$ , which will be re-written here as  $E[C(\bar{V})]$  where  $\bar{V}$  is the random variable. The Black-Scholes price would be represented by  $C(E(\bar{V}))$ . Thus a comparison of  $E[C(\bar{V})]$  with  $C(E(\bar{V}))$  would show the relative pricing of Black-Scholes and the stochastic volatility valuation. Figure 4.5 overpage plots a typical variation of  $C(\bar{V})$  with  $\bar{V}$  where the shape of the curve has been exaggerated to facilitate illustration of the concept and

accordingly no values have been attached to the y axis. (A risk-free rate of 12% and a term of 5 years have been used)

Figure 4.5



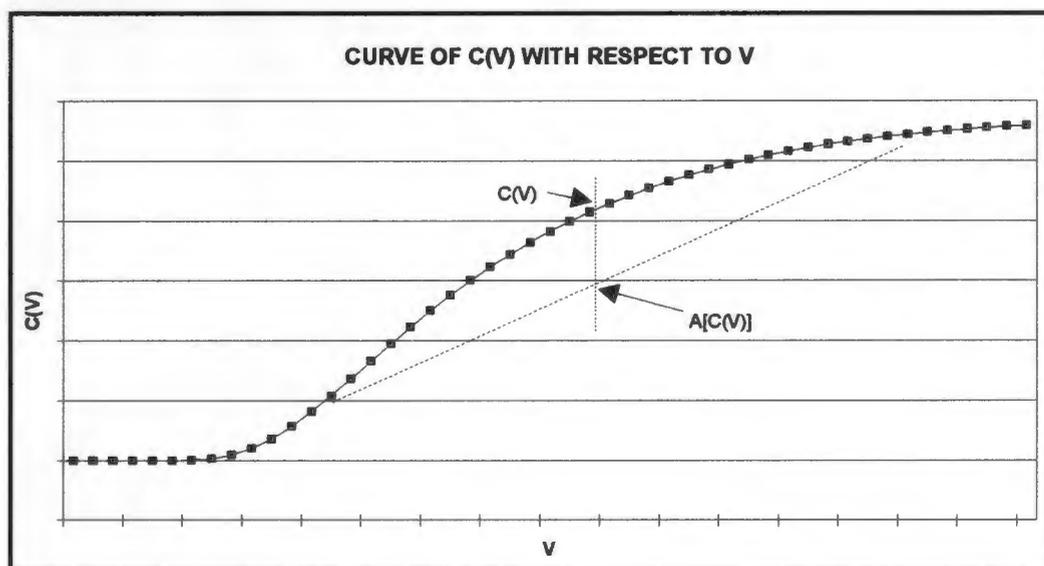
Where the curve is convex  $E[C(\bar{V})] > C(E(\bar{V}))$ , and where the curve is concave  $E[C(\bar{V})] < C(E(\bar{V}))$ . From the chart one sees that the former occurs for low values of  $\bar{V}$  and the latter occurs for high values of  $\bar{V}$ . Thus for low values of  $\bar{V}$  Black-Scholes underprices since  $C(E(\bar{V})) < E[C(\bar{V})]$  and for high values of  $\bar{V}$  Black-Scholes overprices since  $C(E(\bar{V})) > E[C(\bar{V})]$ . The turning point is at the point of inflection of the curve and is given by  $C''(\bar{V}) = 0$ . One finds (Hull and White, 1987) that when  $S_t = X \cdot \exp(-r(T-t))$  the point of inflection is at  $\bar{V} = 0$ , and thus the curve is always concave. It follows that Black-Scholes overprices "at-the-money" options. As  $\ln(S/X)$  approaches  $\pm \infty$  the point of inflection occurs at higher values of  $\bar{V}$  and so the convex part of the curve increases. It follows that Black-Scholes underprices options that are sufficiently in-the-money and out-the-money.

#### 4.4.3.2 Magnitude of the price biases for zero correlation

While the absolute magnitude of the price bias is small, as a percentage of the Black-Scholes value it is still quite significant. The extent of the bias depends on the current volatility and on  $\xi$ , the variance of the volatility itself. In Hull and White (1982, p. 293) the magnitude of the bias, for six month out-the-money options, is up to 5% for small values of  $\xi$ . For large values of  $\xi$  (e.g.  $\xi = 3$ ) the bias is as large as 20%.

Essentially the magnitude of the bias measures the vertical difference between  $C(\bar{V})$  and  $A[C(\bar{V})]$ , the average of  $C(\bar{V})$  over  $\bar{V}$ . This is shown pictorially in Figure 4.6 below.

Figure 4.6



The value of  $\bar{V}$  being considered lies in the concave part of the  $C(\bar{V})$  curve. The wider the distribution of  $\bar{V}$  is about its mean at this part of the curve, the lower the average of  $C(\bar{V})$  will be. This follows from the fact that for the concave section, a reduction in  $\bar{V}$  reduces the value of  $C(\bar{V})$  to a greater extent than an increase in  $\bar{V}$  increases  $C(\bar{V})$ . The lower the average of  $C(\bar{V})$ , the greater the vertical difference will be between  $C(\bar{V})$  and  $A[C(\bar{V})]$  and hence the greater the price bias will be.

Conversely the narrower the distribution of  $\bar{V}$  is about its mean the closer  $A[C(\bar{V})]$  will be to  $C(\bar{V})$ , that is, the smaller the vertical gap will be, and hence the smaller the price bias will be. For the convex part of the curve, the same analysis can be applied except that the bias will be in the other direction, that is,  $A[C(\bar{V})]$  will be greater than  $C(\bar{V})$ .

It follows from this that the greater  $\xi$ , the greater the price bias will be, since a greater  $\xi$  means a wider dispersion of  $\bar{V}$  about its mean.

Table 4.2 compares the Black-Scholes price for a put option at a given level of volatility with the value obtained from an averaging of Black-Scholes prices over a symmetric volatility distribution (as described above). The volatility used for averaging is a randomly generated normal distribution with mean of 18%. The impact of asymmetry in the distribution of volatility is thus ignored here. The values are calculated for a one year put option with a 5% guarantee and risk-free rate of 13%. The corresponding Black-Scholes price at 18% volatility would be R36.79 (per R1000). The biases are calculated as a percentage of the Black-Scholes price.

**Table 4.2**

<b>STD. DEVIATION OF VOLATILITY</b>	<b>STOCHASTIC PRICE</b>	<b>BIAS</b>
0.04	37.31	1.4%
0.07	38.35	4.2%
0.1	40.40	9.8%

The table shows clearly how the magnitude of the bias increases with an increase in the standard deviation of volatility.

#### **4.4.3.3 Mean reversion**

Hull and White (1987) also consider the impact of mean reversion of volatility (for the case where  $\rho=0$ ), where the drift of the volatility process is of the form,

$$\alpha.(\sigma^{\#} - \sigma_t).dt,$$

where,

$\sigma^{\#}$  is the long term value toward which the volatility stabilises , and  
 $\alpha$  is the speed of adjustment of the current volatility to its long term value .

Using a Monte Carlo simulation they show that this behaviour in volatility produces similar biases to above, but that the biases are less pronounced. The effect of mean reversion is to reduce the variance of  $V$ , since shocks to volatility have less persistence over time. The effect is similar to that of reducing  $\xi$  .

#### 4.4.3.4 Non-zero correlation

Using a Monte Carlo simulation Hull and White (1987) show that for the case where volatility and stock prices are **positively** correlated,

*Black-Scholes underprices out-the-money call options and,  
overprices "at-the-money" AND in-the-money call options.*

For put options, it follows from application of the put/call parity theorem that,

*Black-Scholes underprices in-the-money options and,  
overprices "at-the-money" AND out-the-money options.*

For the case where volatility and stock prices are **negatively** correlated, they show that the reverse of the above is true, viz.,

*Black-Scholes underprices in-the-money call options and,  
overprices "at-the-money" AND out-the-money call options.*

For put options application of put/call parity implies that,

*Black-Scholes underprices out-the-money options and,  
overprices "at-the-money" AND in-the-money options.*

Where at-the-money is once again taken to be  $S_t = X \cdot \exp(-r(T-t))$ .

These results can be seen intuitively by considering the impact of the correlation on the terminal distribution of prices. Where  $\rho > 0$ , very high stock prices are more probable, low stock prices are like absorbing states and so the distribution is more positively skewed than a log-normal distribution. For  $\rho < 0$ , very high and very low stock prices are less likely so that the distribution of prices is more peaked than a log-normal distribution. The impact of these two distributional changes can be analyzed using the graphical framework of 3.4.1 to obtain the pricing results above.

The size of the biases reported here by Hull and White (1987) were greater than those for the zero correlation case. For six month out-the-money options the biases ranged from about -50% to +50% for correlations from -1 to +1 respectively.

Wiggins (1987) uses an approximate numerical solution (hopscotch method) to his general equilibrium equation for stochastic volatility option prices and compares this to the prices obtained from Black-Scholes. His results are consistent with those of Hull and White for all the cases considered above, that is, zero, positive and negative correlation, as well as for mean reversion.

#### **4.4.4 EFFECT OF EXTENSION OF TERM**

Since this paper is concerned with pricing long term options the impact of increased term on the above analysis needs to be investigated. Hull and White (1987) argue that the effect of increased term is to exacerbate the magnitude of the bias. Their reasoning is that as the time to maturity increases the variance of the stock's volatility increases since the variance is proportional to the square root of time. The effect of this would be, as explained in section 4.4.3.2, to increase the bias.

On closer inspection, however, it appears that increasing the term has a *twofold* effect, namely:

- i) The effect on the curvature of the curve of  $C(\bar{V})$  with respect to  $\bar{V}$
- ii) The effect on the spread of  $V$

Hull and White's reasoning covers only the former. We consider both in more detail below.

#### 4.4.4.1 Effect on curvature of the $C(\bar{V})$ curve

Extending the term will have some effect on the extent of the convexity or concavity of the  $C(\bar{V})$  curve, which will impact on the extent of the bias or vertical difference between  $C(\bar{V})$  and  $A[C(\bar{V})]$  as described in 4.4.3.2 above. The impact on the curvature of an extension of term will depend on the degree to which the option is in- or out-the-money.

We consider first the case where the option is at-the-money at maturity. In order to obtain an accurate picture of the curvature one needs to look at the option kappas.

Figure 4.7

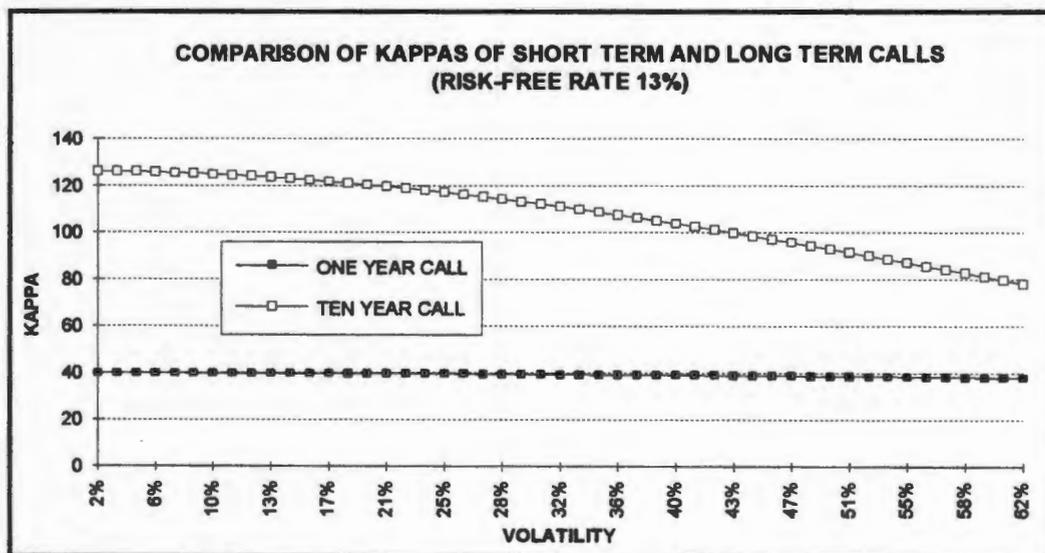
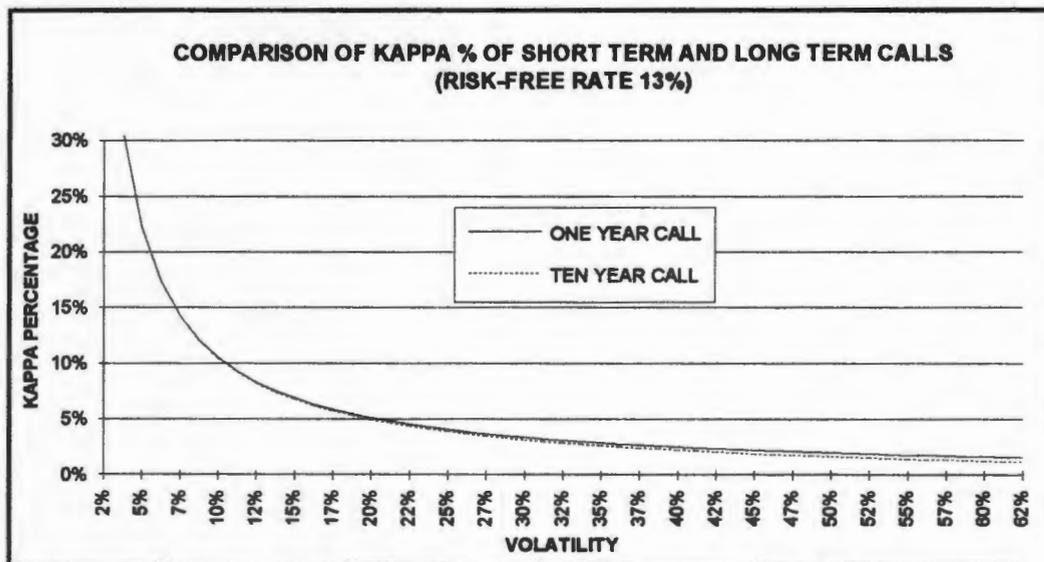


Figure 4.7 above compares the kappas (which represents the curvature of the option

price with respect to volatility) of a one year and a ten year call option (a drift rate of 13% has been assumed). Comparison of the kappas show quite clearly that the longer term option has a significantly higher curvature than the shorter term option. The implication of this is that the vertical difference between  $C(\bar{V})$  and  $A[C(\bar{V})]$  will be greater for the longer term option, and hence the bias between the Black-Scholes value and the stochastic value will be greater in *Rand terms* in the case of the longer term option.

Measurement in *Rand terms* is somewhat misleading however, since the price of the long term and short term calls will be significantly different, so that the bias in percentage terms may be very different to that observed above. In light of this, the kappas have been expressed as a percentage of the corresponding *put* option prices and these have been plotted in Figure 4.8 below. The chart shows that the kappa percentages are almost identical. This implies that the biases for short term and long term puts, in percentage terms, are close to equal for the case where the option is at-the-money at maturity.

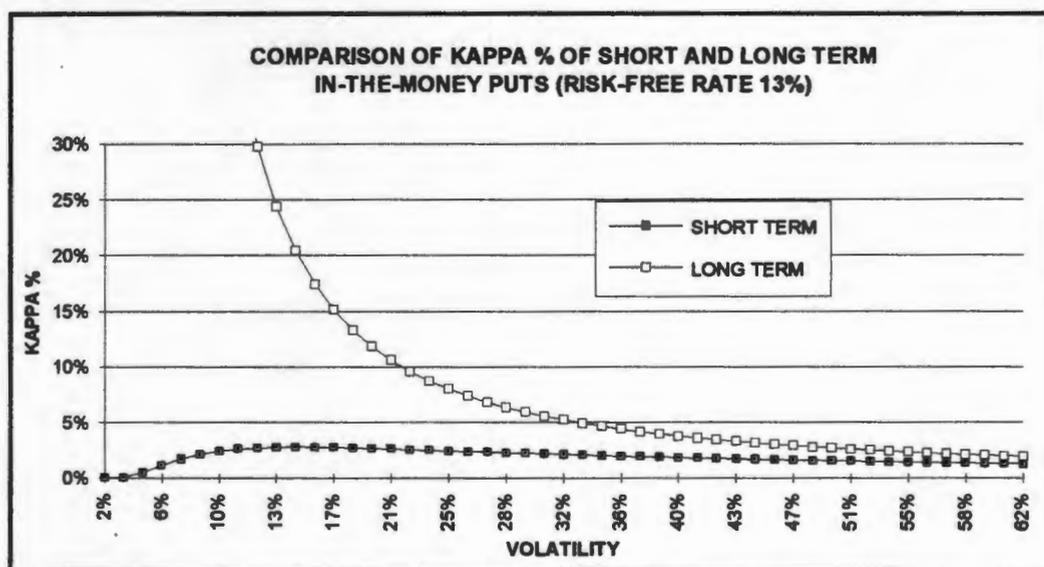
Figure 4.8



Now consider options that are in- or out-the-money. Figure 4.9 below plots the corresponding kappa percentages for a one year and a four year put option (a four year term has been chosen here, because it produces the most convenient graph - a ten year term will lead to the same conclusions but the results are more exaggerated) where  $X/S = 1.2$  in both and a drift rate of 13% has been assumed.

The chart shows that the curvature of the longer term option is significantly greater than for the short term option. This indicates that the biases will be significantly greater in the long term option. It can thus be concluded that, for options that are not at-the-money at maturity, an extension of term increases significantly the bias as a result of stochasticity of volatility.

Figure 4.9



#### 4.4.4.2 Effect on spread of $V$

Hull and Whites' argument is that as the term increases the variance of the volatility process increases. On the other hand, mean reversion has a similar effect to a reduction of  $\xi$ . The longer the term the greater the impact of the mean reversion effect since the rate of reversion of the instantaneous volatility to its long term mean is

proportional to time. The effect of an increased term on the spread of the distribution of  $V$  is thus dependent on which of the above two effects is the more powerful - the variance of volatility or mean reversion.

Table 4.3 below compares the Black-Scholes price for a put option with the value obtained allowing for stochastic volatility, for options with increased term. All the options considered are for guaranteed returns of 5% p.a. The stochastic price is derived, as discussed in 4.4.3.1, by averaging the Black-Scholes values for randomly generated normally distributed volatilities. A mean volatility of 18% has been used and a drift rate of 13% assumed, as in Table 4.2. Biases have been calculated as a percentage of the Black-Scholes price.

**Table 4.3**

<b>TERM</b>	<b>STD. DEV. OF VOLATILITY</b>	<b>BLACK-SCHOLES PRICE</b>	<b>STOCHASTIC PRICE</b>	<b>PRICE BIAS</b>
1 YEAR	9%	36.79	39.6	7.65%
2 YEAR	9%	37.03	42.27	14.14%
5 YEAR	9%	26.57	35.80	34.75%
10 YEAR	9%	12.74	22.68	78.04%

The table shows that there is a significant increase in bias as the term is extended. The effect of mean reversion is not captured in this table and thus the increase in bias is entirely attributable to the increase in variance of volatility. Table 4.4 overpage shows the same calculation but with allowance made for the effect of mean reversion. Mean reversion is incorporated through a standard deviation of volatility which decreases with term from 9% to 5% (this magnitude of decrease in standard deviation is indicated by the evidence presented in 4.5.1.2). The table shows that there is still a significant increase in bias with extension of term. The effect of mean reversion has thus been significantly overshadowed by the effect of the increase in variance of the volatility. The reduction in the bias due to mean reversion is greatest in the case of the

10 year option which indicates that, as expected, the impact of mean reversion is greater the longer the term of the option.

**Table 4.4**

<b>TERM</b>	<b>STD. DEV. OF VOLATILITY</b>	<b>BLACK-SCHOLES PRICE</b>	<b>STOCHASTIC PRICE</b>	<b>PRICE BIAS</b>
1 YEAR	9%	36.79	39.6	7.65%
2 YEAR	8%	37.03	41.05	10.84%
5 YEAR	6%	26.57	30.92	16.36%
10 YEAR	5%	12.74	16.35	28.4%

In conclusion, it has been shown that the net effect of an extension in term is to increase the bias found with short term options. The extent of the increase will depend on, the extent to which the option is in/out-the-money, the magnitude of the variance of the volatility, and the level of mean reversion.

## **4.5 EMPIRICAL EVIDENCE**

In this section we consider both international and ALSI evidence as to:

- i) The volatility of the returns of the underlying asset (section 4.5.1.1).
- ii) The volatilities implied by Black-Scholes option prices (section 4.5.1.2).

### **4.5.1 UNDERLYING ASSET**

#### **4.5.1.1 International Evidence**

There is "ample empirical evidence that variances exhibit stochastic behaviour through time" (Christie, 1982, p. 407). Christie (1982) also provides a good summary of the then state of theory concerning the behaviour of variance, which was largely explained

in terms of two arguments, namely, changes in variance due to information arrival, and changes in variance due to changes in the level of the stock price.

Papers by Press (1967), Beaver (1968), and Merton (1976a) have discussed variance changes in terms of information arrival, while papers by Cox (1975), Cox and Ross (1976), Geske (1979) related changes in variance to changes in the level of stock price. References to the association between changes in the level of stock price and variance have also been made by, Beckers (1980), Black (1976), Macbeth and Merville (1980), Schmalensee and Trippi (1978). However, in all these papers, apart from Black (1976), no attempt had been made to explain or understand the relationship between the changes in the level of stock price and changes in variance.

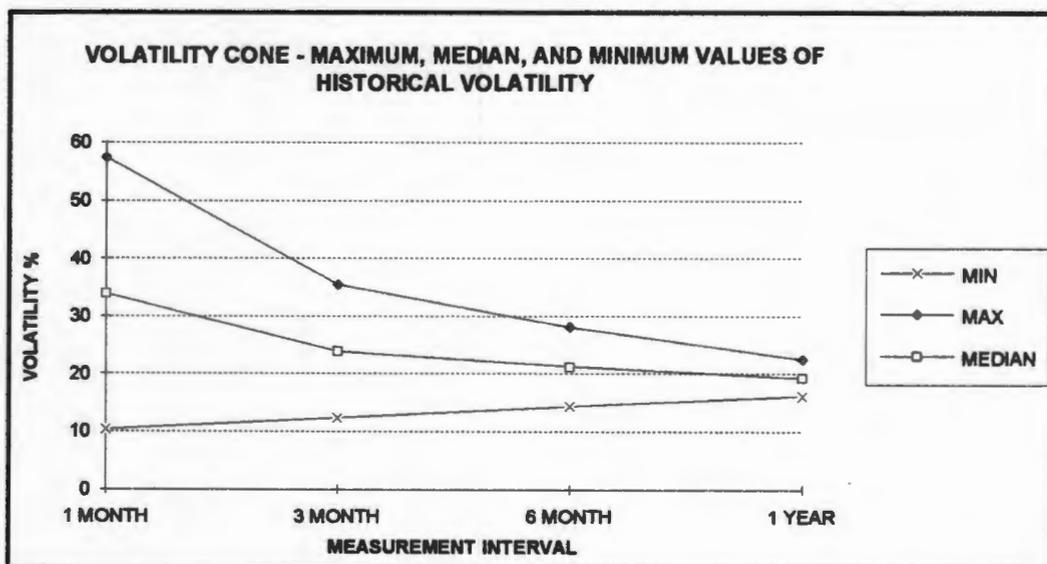
The first paper to provide evidence, and an explanation, of the relationship between variance and stock price was Christie (1982). In this paper Christie provides evidence of a negative correlation between the volatility of the return and the value of equity and shows that this is in substantial part attributable to financial leverage. Specifically, he demonstrates that volatility is an increasing function of financial leverage. These findings were based on data from a sample of 379 firms over the period 1962 to 1978 in the US. More recently, Wiggins (1987) calculated parameter estimates for the instantaneous correlation between volatility and stock returns for the S&P 500 index and also found strongly negative correlations.

While the above papers deal with the relationship between volatility and the price of the underlying asset, attention has also focused on the relationship of volatility with different time periods. Gemmill (1993) suggests that there may be two systematic or non-random components to volatility. The first is reversion to mean. The second is that long term volatilities may have a lower mean to which they revert than short term volatilities. Gemmill asserts that the former phenomenon is known to exist while the latter phenomenon is more controversial.

Evidence seems to support the proposition that volatility exhibits mean-reversion tendencies. The papers by Black (1976), Beckers (1983), Poterba and Summers (1986) and Wiggins (1987) provide evidence that shocks to volatility do not have great persistence over time. Merville and Piepta (1989) using implicit volatilities from Black-Scholes models find evidence consistent with the belief that volatilities follow a mean reverting diffusion process with noise. Stein and Stein (1991) ignore the noise term and model only the mean reverting component. They argue that the noise component could be due to mis-specification from using Black-Scholes to approximate volatility.

Burghardt and Lane (1991) construct a "volatility cone". They calculated historical volatilities for underlying contracts using one month, three month, six month and one year time horizons. For each of these volatilities the minimum, maximum, and median values were calculated. Figure 4.11 shows the resultant cone structure in volatility. The down-sloping upper line (max) and up-sloping lower line (min) provide evidence for mean reversion and indicate that there is lower uncertainty associated with long term volatility than short term volatility.

**Figure 4.11**

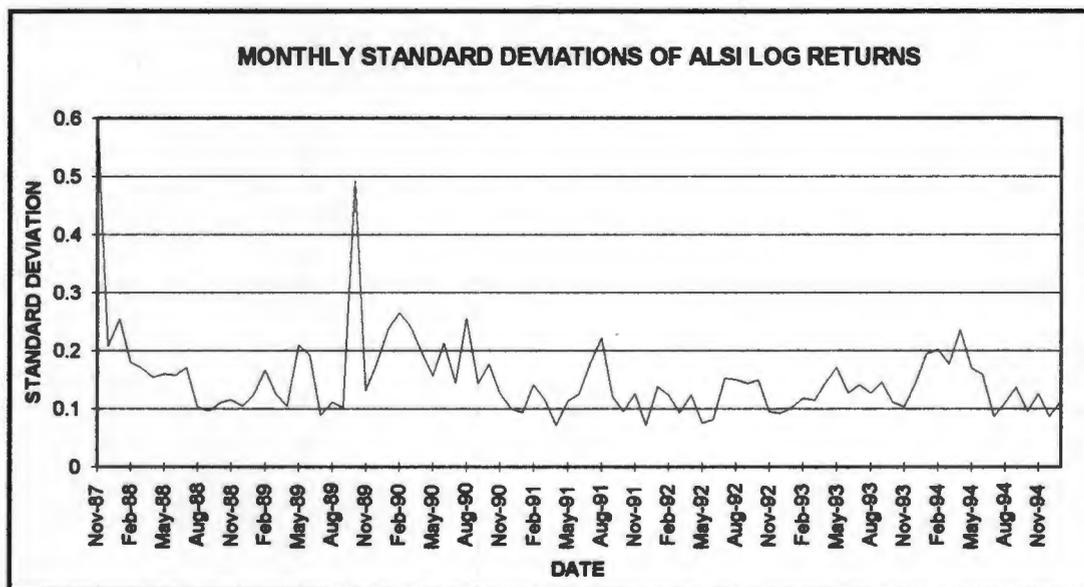


According to Leong (1991b, p. 47) "empirical evidence shows that short-term volatilities tend to be substantially higher than long-term". The evidence from the Burghardt and Lane study showed the mean volatility decreasing from 18.26% for the one month period to 17.86% for the one year period. Leong's conclusion and this evidence supports Gemmill's postulation above of higher short term volatility.

#### 4.5.1.2 Evidence on the ALSI

Figure 4.12 below plots the history of monthly standard deviations of daily log-returns of the ALSI over the period November 1987 to January 1995. The monthly standard deviations were calculated using non-overlapping samples of the returns. The graph shows that there has been substantial variation in volatility over time. It is also evident from the graph that while volatility may be high in one period (e.g. November 1987, October 1989), these highs do not persist.

Figure 4.12



A similar exercise to that undertaken by Leong (see 4.5.1.1) was performed on the ALSI. Standard deviations were calculated for 5 day, 10 day, one month, three month and six month intervals. The means and variances of the standard deviations for each

measurement interval were then calculated and plotted. This differs from the cone constructed by Leong where the minimums, medians, and maximums of the volatilities were plotted. Standard deviations were selected here as it was felt that these give a more accurate reflection of the dispersion of volatility than do the minimum, medians and maximums. Figure 4.13 below shows the cone structure indicated by the resultant figures for ALSI.

The centre line plots the means and the bottom and top lines plot one standard deviation below and above the mean respectively. While the slopes of the lines are not as steep as that of Leong, their convergent nature do indicate evidence of mean reversion.

Figure 4.13

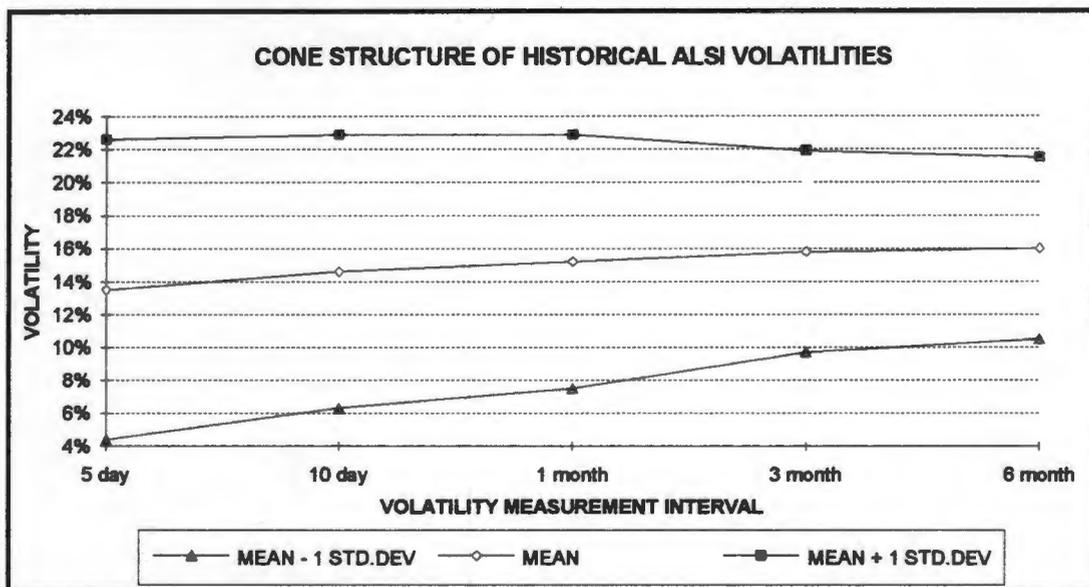


Table 4.5 overpage shows the corresponding figures for these lines where the means and standard deviations have been annualized. The standard deviation of volatility decreases from 9.1% for the 5 day measurement interval to 5.5% for the 6 month interval. This decrease in the standard deviation of volatility with term provides strong evidence of mean reversion. The corresponding means increase from 13.5% for 5 day to 16% for 6 month. The standard deviation of daily returns for the period as a whole,

which represents the mean of the maximum measurement interval for the period is 17%. This seems to indicate that long term volatility has a higher mean than short term volatility and is contrary to the postulation of Gemmill and the evidence of Leong in section 4.5.1.1.

**Table 4.5**

	5 Day	10 Day	1 Month	3 Month	6 Month
Mean	13.5	14.6	15.2	15.8	16
Std. deviation	9.1	8.3	7.7	6.1	5.5

Further evidence of the behaviour of volatility can be obtained by looking at the correlations of volatility lagged on itself - the autocorrelations. Table 5.6 below shows the autocorrelations calculated for 10 day volatilities of ALSI.

**Table 4.6**

	1 Lag	2 Lags	3 Lags	4 Lags	5 Lags
Correlation	0.338	0.199	0.176	0.19	0.145

While the first order autocorrelations are not as high as those reported by Scott (0.62 for S&P 500), Merville and Piepte (0.57 for S&P 500 futures) and French, Schwert and Stambaugh (0.7 for S&P composite portfolio) they do exhibit the same pattern, namely that as the lag decreases autocorrelations get larger. This evidence of intertemporal dependence in volatility provides further support for the hypothesis that volatility follows a mean reverting process.

Simple correlation tests were performed to identify any evidence of significant correlations between volatility and stock returns. Correlation coefficients were calculated for the series  $\{\sigma_t, \ln(\frac{s_t}{s_{t-1}})\}$  where  $\sigma_t$  is the standard deviation for the month t to t+1 and  $\ln(\frac{s_t}{s_{t-1}})$  is the return for the month t-1 to t. The correlation coefficient for the whole period November 1987 to January 1995 was -0.056. The

period was then split in half and correlation coefficients calculated for these two sub-periods. For the first sub-period the correlation was -0.168 and for the second sub-period it was 0.237. Thus it can be concluded that for the period as a whole there is no significant evidence of any correlation, however, there is evidence that the correlation changes significantly over time.

Christie (1982) performs simple empirical test to estimate the elasticity of volatility with respect to the stock price, based on the regression equation:

$$\ln(\sigma_t / \sigma_{t-1}) = \beta + \theta [\ln(S_t / S_{t-1})] + \varepsilon_t$$

where,

$\beta$  is a constant,

$\varepsilon_t$  = error term,

$\theta$  is the elasticity of volatility with respect to stock prices, and it is assumed that  $\theta$  is a constant.

Christie's results show a negative elasticity of volatility with respect to stock prices, namely,  $\theta = -0.22$ . The results of this regression performed on the ALSI do not support the negative elasticity that Christie finds and the  $R^2$  for the ALSI test was low (0.037) so that the results are not at all significant. Thus one can conclude that for the ALSI either there is little evidence of a strong relationship between volatility and stock returns or that the regression equation is mis-specified.

It is difficult to draw conclusions about the relationship between volatility and stock returns without knowledge of the diffusion process that volatility is assumed to follow. The procedure for estimating the correlation will depend on the form of the diffusion process assumed, a recommendation for which is described in 4.6.2.

## 4.5.2 IMPLIED VOLATILITY EVIDENCE

Options in the market are quoted in terms of their implied volatilities, that is, the level of volatility which in the Black-Scholes equation equates to the market price. Since the implied volatilities (IVs) are intended to be forecasts of volatility over the remaining life of the option, much information can be gleaned from observing the variation of implied volatility across exercise prices and across terms. The term "volatility smiles" commonly used in the derivatives market, in particular, refers to the variation of IVs across exercise prices. The biases observed in the market can then be weighed up against the hypothesis of Hull and White (1987) (as discussed in 4.4.3) and other stochastic volatility valuation models to establish whether there is some consistent explanation for these biases.

### 4.5.2.1 International evidence

The evidence of option pricing biases is quite extensive yet still not very conclusive. Evidence has been reported which supports each of the cases of negative correlation, zero correlation and positive correlation between volatility and stock returns as discussed in 4.4.3. A summary of this conflicting evidence is given below.

*Evidence consistent with the case of positive correlation between volatility and returns:*

Black (1975) reports that for call options on the Chicago Board Options Exchange (CBOE), deep in-the-money options have market prices below Black-Scholes values, and deep out-the-money options have market prices above Black-Scholes values. This is reflected in lower IVs for low exercise price calls than high exercise price calls.

Rubinstein (1985) performed non-parametric tests on option price data from CBOE which he split into two periods as follows:

Period 1: August 1976 to October 1977

Period 2: October 1977 to August 1978

His results for period 2 showed that the higher the strike price the higher the IV .

Kuwahara and Marsh (1992) investigated prices of Japanese warrants for the period 1988 to August 1990. They found that warrant prices were above their Black-Scholes values when out-the-money and below when in-the-money.

All three of these cases of higher IVs with higher strike price for calls are consistent with a positive correlation between volatility and returns, as outlined in 4.4.3.4.

*Evidence consistent with the case of zero correlation between volatility and returns:*

Merton (1976b) points out that practitioners observe Black-Scholes prices to be less than market prices for both deep in-the-money and deep out-the-money calls. This is consistent with a zero correlation between volatility and returns as outlined in 4.4.3.1.

*Evidence consistent with the case of negative correlation between volatility and returns:*

MacBeth and Merville (1979) found that for 3, 6 and 9 month options (on the CBOE) IVs declined with increasing exercise price. In-the-money options had market prices above Black-Scholes values and, out-the-money options had market prices below Black-Scholes values. Rubinstein's results for period 1 showed that the lower the strike price the higher the IV. The evidence for period 1 is conflicting with his results for period 2. Both these cases of higher IVs with lower exercise price for calls are consistent with a negative correlation between volatility and returns as outlined in 4.4.3.4.

Lauterbach and Schultz (1990) investigated daily prices of warrants listed on the New York and American stock exchanges for the period 1971 to 1980. The authors performed a regression of IVs against the extent to which the option is in/out-the-money and against the risk-free rate. The results were that the t-statistic of the interest rate was not significant, while the t-statistic of the extent in/out-the-money was highly

significant and the coefficient was significantly less than zero. The conclusion drawn was that the IVs are inversely related to the value of the underlying equity, which is consistent with a negative correlation between volatility and returns. Furthermore this relationship was found to be consistent over the whole period analyzed. This contrasts with the transitory biases reported by Rubenstein.

The authors concluded that models that allow for an inverse relation between equity and volatility such as the CEV (Constant Elasticity of Variance) model of Cox (1975) are promising alternatives to Black-Scholes. The authors also show that, based on their data, that the Square Root CEV (SRCEV) model was consistently more accurate predictor of market prices than Black-Scholes.

MacBeth and Merville (1979) also looked at the extent of biases across term and found that for in-the-money calls the shorter term options are underpriced by Black-Scholes to a smaller extent than longer term options, and for out-the-money calls shorter term options are overpriced to a lesser extent than longer term. This evidence of an exaggeration of bias with increased term is consistent with the Hull and White argument in 4.4.3.5. Rubinstein's results also indicated an increase in bias with increased term.

The above summary of evidence indicates that there are significant instances of biases in the Black-Scholes prices across exercise prices. The evidence is however conflicting - in some instances the biases are consistent with a positive correlation between volatility and returns, in some instances with a negative correlation, and in others with zero correlation. Furthermore, there is evidence that the direction of biases can change from period to period.

Wiggins (1987), Hull and White (1987), and Eisenberg and Jarrow (1994) argue that their stochastic volatility valuation models are consistent with the evidence of biases. While their models do produce "volatility smiles", the direction of these smiles in the market varies from period to period. It is thus difficult to postulate a specific model

that will be applicable at all times, as have Lauterbach and Schultz above. Possibly the only useful conclusion which can be drawn from the conflicting evidence is that the behaviour of volatility, in terms of correlations with the underlying stock price, varies from period to period, and consequently any long term option pricing model proposed should allow the user to decide on what correlation he wishes to incorporate.

It should further be recognized that the biases observed cannot be explained by the stochasticity of volatility alone - other market factors influence the extent and direction of these biases. One thus needs to be wary of drawing conclusions about the nature of the stochastic volatility based on implied volatility evidence. Other factors that might influence the bias are listed below.

1. Other distributional deviations from log-normality

The biases may be caused by other deviations in actual prices from log-normality, for example skewness (Chapter 3).

2. Liquidity effects

Liquidity is greatest for short term, at-the-money options. As one moves further away from the money and further out in term so one encounters a greater reluctance of market makers to write options. The reduction in supply means that prices are bid-up so that the preference free assumption of Black-Scholes is violated.

3. Dynamic hedging risks

The Black-Scholes price is based on the existence of a hedged portfolio. However, in practice as one moves further away from the money and further out in term so it becomes more difficult, in light of 2 above, to construct a hedged position. The pricing of a future contingent claim using the risk-neutral approach (see Section 2.3) may not be valid in these instances.

#### 4. Anticipation in direction of movement of market

Often the market has a strong sentiment of a move in either direction. While this will in part be factored into the price of the underlying asset, investors often have strong preferences for obtaining a particular equity position (i.e. long or short). The utility free approach of Black-Scholes is thus violated. The consistent discount (of around 6%) on the ALSI futures contract for a long period during the early 1990s is a case in point. Furthermore a significant portion of investors may be speculators rather than hedgers or arbitrageurs.

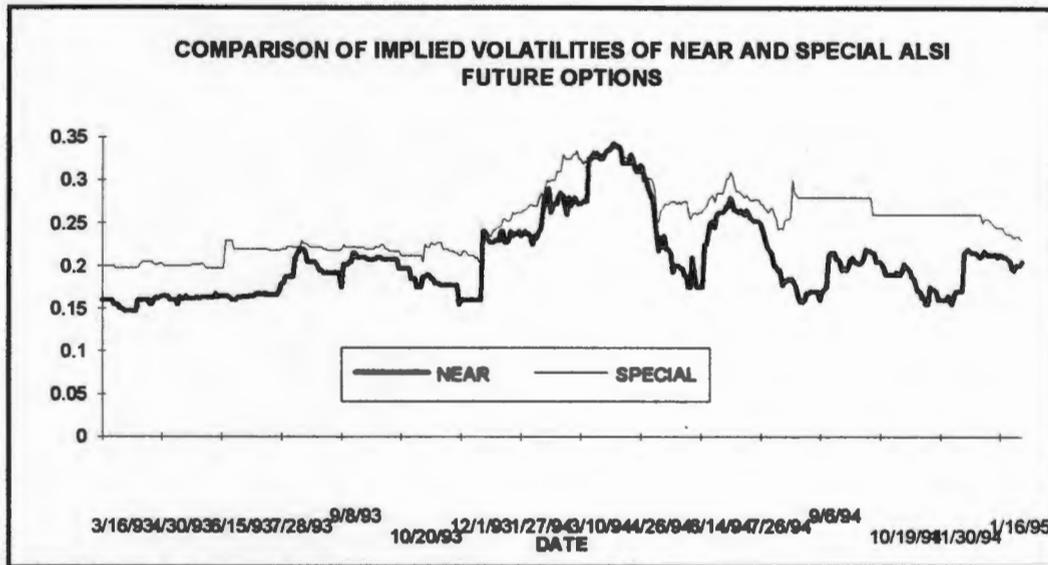
##### 4.5.2.2 ALSI Evidence

For the ALSI, records of volatility across exercise prices are not maintained by SAFEX. It is thus not possible to obtain a history of volatility smiles for ALSI. Record of IVs across the different term contracts are maintained by SAFEX and data for the period March 1993 to January 1995 was analysed. The volatilities are quoted separately for the near, medium, far and for special contracts. The near, medium and far contracts refer to the outstanding 3 month, 6 month and 9 month contracts respectively. The special contracts refer to those which are issued with terms in excess of one year - these contracts are special in that they are issued only for maturity in March of each year. For the purpose of comparison of short term volatilities with long term volatilities the implied volatilities of the near and special contracts have been plotted in Figure 4.14 overpage.

The flat or non-varying parts of the line for the special contracts reflect the lack of trading in this contract due to its long term (sometimes in excess of 18 months). The chart shows that the longer term contracts have for long periods had higher volatilities than the shorter term contracts. Although this is consistent with the evidence for the underlying asset that longer term volatility has a higher mean than shorter term volatility, as presented in 4.5.1.2 and Table 4.5, it is more likely that the higher implied volatilities are explained by the lack of liquidity, and the dynamic hedging risks for this

length of term as discussed in 4.5.2.1. Market makers are more reluctant to write options for these longer terms and consequently volatilities are bid upwards.

**Figure 4.14**



## 4.6 CONCLUSION

### 4.6.1 SUMMARY OF RESULTS

The discussions and results of this chapter have shown the following:

- i) Volatility is significantly more important for long term options than short term, and more so for deep in-the-money put options.
- ii) If volatility and returns are instantaneously uncorrelated then an option price allowing for stochastic volatility can be obtained by averaging Black-Scholes values where the averaging is over the distribution of volatility at maturity. If volatility and returns are instantaneously correlated then the path followed by prices is important, and time series of stock prices and volatilities need to be generated from their respective assumed diffusion processes.

- iii) The impact of stochastic volatility of returns on price depends on the size of the correlation between volatility and returns. For an in-the-money put option with positive correlation Black-Scholes overprices, and for other cases Black-Scholes underprices.
- iv) Mean reversion of volatility reduces the extent of the pricing biases above.
- v) Increased term to expiry increases the extent of biases. However, the impact of mean reversion increases with term and this reduces the extent of the biases. The former, however, outweighs the latter, so that the net effect of increased term, is still an increase in the bias.
- vi) There is strong international evidence suggesting negative correlations between volatility and returns, however, there is also conflicting evidence in implied volatilities as well as evidence of changes in the sign of correlations over time. There is no clear evidence of correlations for the South African ALSI.
- vii) There is a strong indication of mean reversion behaviour both internationally and for the South African ALSI.
- viii) The evidence for the South African ALSI indicates that long term volatility (annualized) is higher than short term volatility (annualized) which is contradictory to international evidence.

#### **4.6.2 RECOMMENDED APPROACH**

In light of the above findings, it is argued that any approach to modeling volatility in long term option pricing should bear the following in mind.

1. Mis-estimation of volatility is significantly more costly for long term options than short term options. There is thus strong justification for focusing more attention on modeling volatility.
2. The size and sign of the correlation between returns and volatility influence the extent and direction of price biases. It is thus necessary to have a process for volatility which allows for a correlation input.

3. If there is strong evidence that correlation is close to zero, or if there is no clear evidence of a significant positive or negative correlation, as in the case of ALSI for the period considered, then the modeling of a volatility diffusion process can be ignored. If furthermore the stock price is assumed to be log-normally distributed, then an averaging of Black-Scholes values can be used to obtain a price which allows for stochastic volatility. The requirement then is for a distributional assumption for volatility at maturity of the option. The distribution of short term volatility may be used as a guide and further research is required in this area. It is recommended that an Inverse Gamma distribution be considered a candidate for volatility, as has been used by Praetz (1972), Blattberg and Gonedes (1974), and Russo (1991). The use of the Inverse gamma for volatility when the conditional distribution of stock prices is assumed to be log-normal will result in a student-t distribution for the unconditional distribution of stock prices which is supported by the evidence cited in Chapter 3.
4. Mean reversion reduces the extent of pricing bias due to stochasticity, and the impact of mean reversion increases with term. For long term options mean reversion can significantly reduce the bias, and to ignore mean reversion will thus lead to an overstatement of the extent of increase in bias with term. Since there is evidence of mean reversion on the South African ALSI, it is essential to have a process which allows for mean reversion.
5. The mean reversion model should in the case of ALSI, allow the mean volatility to be higher for longer term than for shorter term options, so as to be consistent with the evidence of ALSI volatilities.

Further research into the ALSI is recommended, in particular, into the autocorrelations of volatility and the correlations between volatility and stock prices. For the former the evidence was less strong (lower autocorrelations) than that reported in international studies and for the latter, the evidence was not conclusive or supportive of international findings.

Taking all these into account, it is suggested that an Ornstein-Uhlenbeck process be used of the form:

$$d\sigma = f(\sigma).dt + \theta. dz$$

where,

$f(\sigma)$  allows for mean reversion behaviour and might be of the form

$$f(\sigma) = \beta.(\sigma - \bar{\sigma}) \text{ as used by Scott (1987),}$$

where,

$\bar{\sigma}$  is the long term mean of volatility ,

$\beta$  is the elasticity factor or speed of adjustment of the instantaneous volatility to its long term mean,

$\theta$  is the instantaneous standard deviation of volatility ,

$dz$  is a Weiner process, which has correlation  $\rho$  with the Weiner process driving the stock price diffusion process.

Alternately the mean reversion can be modeled on  $\ln \sigma$  , as done by Scott (1987) and Wiggins (1987). What remains then is an estimation of the parameters, which can be performed by method-of-moments (see Wiggins, 1987).

# CHAPTER 5

## INTEREST RATE

### 5.1 INTRODUCTION

This chapter investigates the interest rate component of the option pricing model, along similar lines to that for volatility in Chapter 4. For the purposes of the analysis of this chapter we assume that volatility is constant. Chapters 4 and 5 are thus comparative static analyses of the Black-Scholes model. The impacts of stochasticity of volatility and interest rates need to be combined in determining the net effect on option prices. This is considered in Chapter 6.

This chapter provides a brief description of the Black-Scholes interest rate assumption, and investigates the significance of this assumption for long term options. The impact of stochasticity of interest rates on the option price is investigated and the implications for long term options established. Some theory and evidence of interest rate behaviour is presented, and finally suggestions are made as to how interest rates could be incorporated into a long term option pricing model.

### 5.2 BLACK-SCHOLES ASSUMPTION

Black and Scholes (1973, p.40) assume that "the short term interest rate is known and is constant through time". Since the short term rate, that is the instantaneous rate, is assumed to be constant the resulting yield curve is assumed to be flat for the term of the option and it does not matter whether one invests in long or short maturity bonds when setting up a hedged portfolio for the purposes of the risk-neutral valuation as outlined in 2.3 and 2.4.

The interest rate assumption in the Black-Scholes model has a twofold impact. Firstly, it is the rate used to discount the maturity payoff, as reflected in the term  $X \cdot \exp(-rt)$  in the Black-Scholes price. Secondly, it is the expected rate at which the underlying asset can be assumed to drift upward over time using the methodology of section 2.4. A non-constant interest rate thus needs to be investigated to determine how the option price is affected through these two features.

### 5.3 SIGNIFICANCE OF INTEREST RATE FOR LONG TERM OPTIONS

Less attention appears to have been paid in option pricing literature to the stochasticity of interest rates than to stochasticity of volatility in pricing stock options. For long term options some authors have questioned the soundness of a constant interest rate assumption. For example, Beenstock and Brasse (1986, p. 153) suggest that, "(o)ver relatively short periods of time it may be safe to assume that  $r$  (the risk-free rate) is constant and deterministic... (h)owever, over the long time spans of maturity guarantees it might be more sensible to assume that interest rates are stochastic and variable", and Gastineau (1988, p. 228) claims that "in evaluating a 5 or 10 year warrant, interest rates are frequently more important than volatility". However, as with volatility (see 4.3), no investigation into interest rate sensitivity was reported and no comparisons of sensitivity with that of volatility were made. The purpose of this section is undertake such an investigation.

As in section 4.3, we assume that we are operating in a Black-Scholes setting, and we consider the impact on the option price of a change in the risk-free rate. The sensitivity of a Black-Scholes option to the risk-free rate is measured by rho. Rho is the first order partial derivative of the option price with respect to the risk-free rate.

For a put option, using the same notation as in section 2.2,

$$\rho = -X.t.e^{-rt} N(-d_2)$$

where,

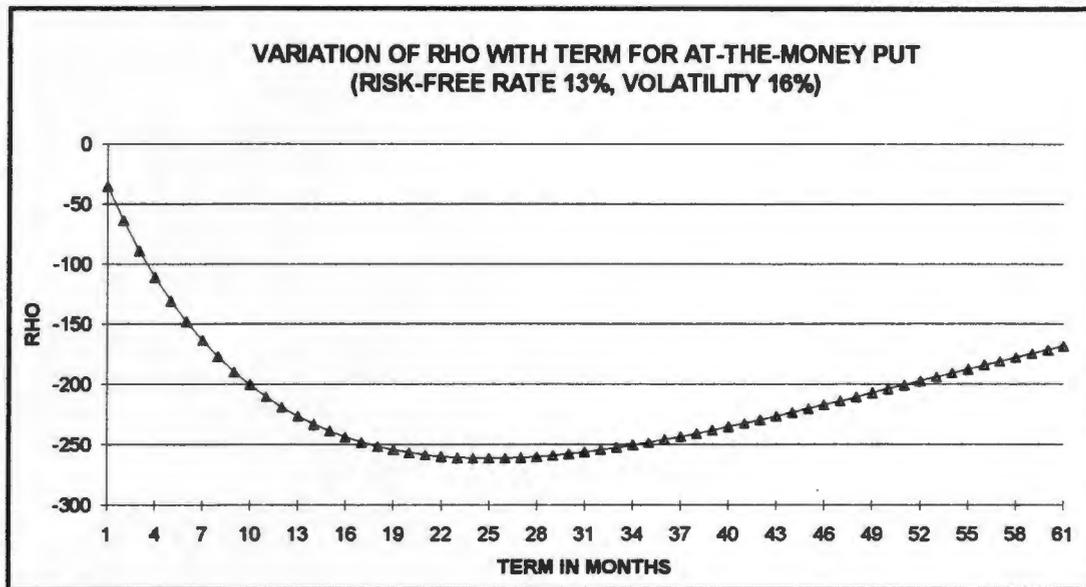
$$d_2 = [(\ln(S/X) + (r - \sigma^2/2)t)] / \sigma\sqrt{t}.$$

Rho is always negative for a put option. This may be explained in two ways. Firstly, an increase in the risk-free rate implies an increase in the cost of borrowing, or an increase in the return on holding money. Since a put option involves a delay in receipt of the agreed selling price of the stock, there is a loss of interest on money that could have been earned. The higher the risk-free rate the greater the extent of this loss and consequently the lower the put price is as a result of this reduction in income. Secondly, since the underlying asset can be assumed, without loss of generality, to have a drift rate equal to the risk-free rate (see section 2.3), an increase in the risk-free rate causes the asset to drift upward to a greater extent. This pushes the positive payoff region of the put more into the tail of the distribution of the stock price at maturity. This in turn reduces the present value of the terminal payoff of the option and hence the value of the option. These two effects explain the reduction in put price as the risk-free rate is increased.

Figure 5.1 overpage shows how rho varies with term for an at-the-money put option (volatility of 16%, risk-free rate of 13%). The rho has been calculated for an underlying asset price of R1000.

As with kappa (Figure 4.1), the interest rate sensitivity first increases with term (that is, it becomes more negative) up to a point, levels off and declines (becomes less negative) thereafter with increased term.

Figure 5.1.



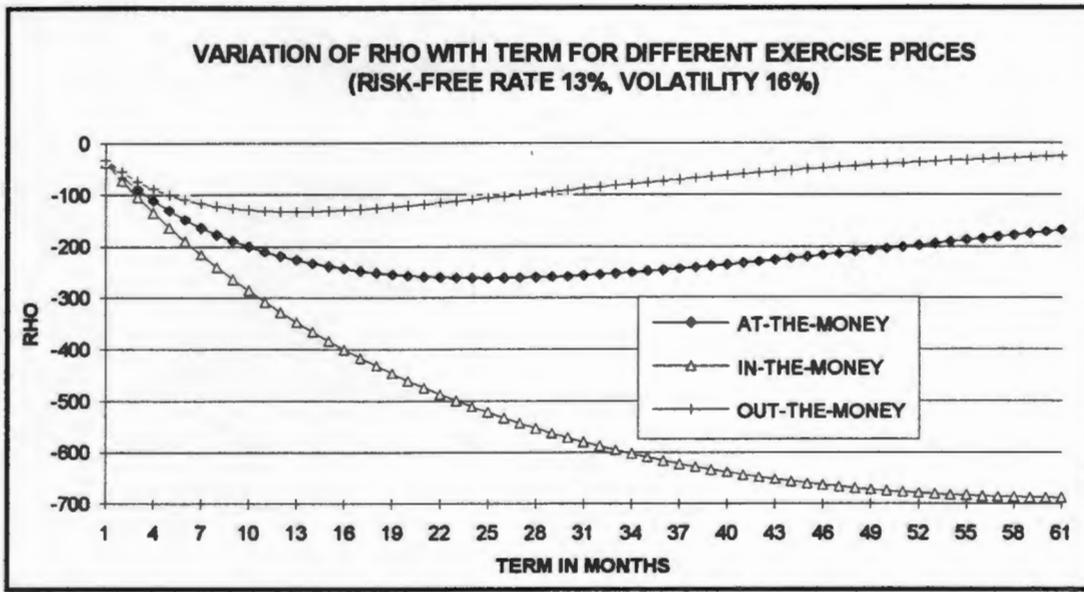
This trend in rho is explained as follows:

The closer the exercise price of the option is to the mean of the stock price distribution at maturity, the greater the impact of a change in the risk-free rate will be, since a greater amount of probability weight is being pushed out of the positive payoff region of the option, by the effect of the drift rate of the underlying asset, as described above. Conversely the further the positive payoff region of the option is into the lower tail of the distribution the smaller the impact of a point change in interest rates will be.

The increasing and decreasing trend in rho can then be explained in terms of the two conflicting effects of volatility and drift, similar to that discussed in section 4.3. For short durations, an increase in term increases the spread of the distribution since volatility is proportional to the square root of time. This increases the extent of the probability weight below the exercise price. The impact of a change in the interest rate is thus, in line with the argument of the paragraph above, greater in these years. For further increased terms the drift effect overshadows the volatility spread effect. This pushes the positive payoff region of the option into the tail of the distribution, and so in line with the above argument, reduces the impact of a change in interest rates.

The impact on rho of being in- or out-the-money is shown in Figure 5.2. For in-the-money options, the exercise price is set equal to the stock price increased by 5% p.a. and for out-the-money options, the exercise price is somewhat arbitrarily set equal to the stock price decreased by 5% p.a. (volatility 16%, risk-free rate 13%).

Figure 5.2



The graph shows, by comparison with at-the-money put options, that, for out-the-money puts, the absolute size of rho is smaller throughout and the turning point from increasing trend to decreasing occurs earlier. For in-the-money puts, the absolute size of rho is larger throughout and the turning point occurs later. This shows, that as in the case of volatility, when pricing a prospective guarantee of 5% p.a., the sensitivity of the option price to the interest rate is very high and remains high at 5 years and beyond.

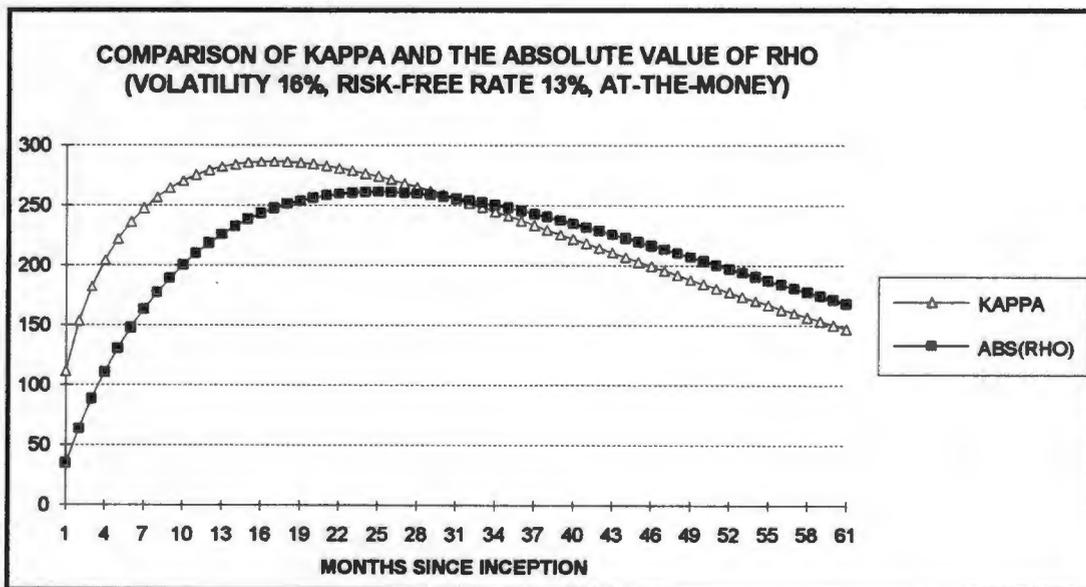
Table 5.1 overpage quantifies the impact of a 1% and 2% error in estimating interest rates in pricing a prospective guarantee of 5% p.a. using the Black-Scholes valuation. The figures calculated are for a five year put option with a volatility of 16% on an asset with price R1000. The Black-Scholes price at a risk-free rate of 13% is R18.5. Biases have been calculated as a percentage of this price.

**Table 5.1**

	RFR	BLACK-SCHOLES PRICE (in Rands)	BIAS
<b>1%</b>	12%	24.92	34.74%
<b>ERROR</b>	14%	13.53	-26.84%
<b>2%</b>	11%	33.11	79.02%
<b>ERROR</b>	15%	11.61	47.26%

A comparison of these figures with those in Table 4.1 shows that the biases due to mis-estimation of the risk-free rate are greater than those of volatility mis-estimation (where the error in estimation is the same size in both - 1% or 2%). Gastinaeu's comment in the opening paragraph of this section is thus borne out. Figure 5.3 below provides a comparison of kappa and rho over term for an at-the-money option (volatility 16%, risk-free rate 13%). Rho has been plotted on the same scale as kappa by taking its absolute values.

**Figure 5.3**



The graph shows that at short durations  $\kappa$  is greater than  $\rho$ , but that at some point they cross over, and for longer durations  $\rho$  is greater. The cross-over point will be dictated by size of the volatility, the drift rate and the extent to which the option is in/out-the-money. One can conclude that for the levels of volatility and risk-free rates considered here, short term options will be more sensitive to a given percentage error in volatility than in interest rates, while for long term options the interest rates are more important.

## **5.4 IMPACT OF STOCHASTICITY**

The impact of moving from a constant to a stochastic interest rate is considered firstly in terms of the impact on the hedging argument and then in terms of the impact on the option price.

### **5.4.1 HEDGING**

A stochastic interest rate does not present the same problems here as did stochastic volatility in Chapter 4, since traded securities exist which enable the interest rate risk to be hedged, namely bonds of appropriate term, and hence it is possible to create a hedged portfolio.

The hedge portfolio, assuming a constant volatility, would consist of stock, options and bonds, where the bonds have a maturity equal to the expiration date of the option. Merton (1973) sets up such a hedged portfolio for valuation of stock options with stochastic interest rates. His hedge portfolio is a self-financing strategy which involves borrowing bonds (and writing call options) and using the proceeds to purchase stocks. Since the portfolio requires no investment, the expected return on the portfolio is zero. Merton's partial differential equations are then set up from the equation for this zero return portfolio. The Black-Scholes derivation differs in that a bond of any maturity can be used, since the yield curve is assumed to be flat.

Furthermore, in order to derive the price of the bond for a given stochastic interest rate process, it is necessary to know the market price of risk on the bond (the algebra of this is detailed in 5.4.2 below). Some of the models discussed in section 5.5 have been designed to overcome this problem. This thesis is concerned only with the process followed by the interest rate in a risk-neutral world and it is not necessary for the purpose of pricing the stock options to determine the price of the bond. No assumption is thus required here about the market price of risk. To value the stock option one needs only to generate the path followed by the risk-free rate, which is used to determine the drift of the stock and the discount factor to use.

#### 5.4.2 PRICING

The only models to date for pricing stock options which incorporate stochasticity of interest rates and provide analytical solutions appear to be those by Merton (1973) and Rabinovitch (1989). Merton (1973) was the first to develop a stock option pricing model which allows for stochastic interest rates. While the model was developed for a specific form of the interest rate process, its closed form solution provides a useful tool and is used here to give insight into how stochasticity of interest rates in general affects option prices.

In his model Merton assumes that the diffusion processes for the stock price  $S$ , and the bond price  $P$  are:

$$\frac{dS}{S} = \alpha \cdot dt + \sigma \cdot dz, \text{ and}$$

$$\frac{dP(\tau)}{P(\tau)} = \mu(\tau) \cdot dt + \delta(\tau) dq(\tau)$$

where,

$\alpha$  = the instantaneous drift rate of the stock price  $S$ ,

$\sigma$  = the instantaneous standard deviation of the stock price returns,

$dz$  is a Weiner process,

$P(\tau)$  is the price of a bond with term to maturity  $\tau$ ,

$\mu(\tau)$  = the instantaneous expected return on a bond with term to maturity  $\tau$ ,

$\delta(\tau)$  = the instantaneous variance of a bond with term to maturity  $\tau$ ,

$dq$  is Weiner process for a bond with term to maturity  $\tau$ .

By making the transformation  $x = S/X \cdot P(\tau)$  in the Black-Scholes equation, Merton derives the value of a call as:

$$S \cdot N(D_1) - X \cdot P(\tau) \cdot N(D_2)$$

where,

$$D_1 = [\ln(S/X \cdot P(\tau)) + V^2 \tau / 2] / (V \sqrt{\tau}),$$

$$D_2 = D_1 - V \sqrt{\tau},$$

$$V^2 = \sigma^2 + \delta^2 - 2\rho\sigma\delta, \text{ and}$$

$\rho = dz \cdot dq(\tau)$  is the instantaneous correlation coefficient between returns on stock and on bond.

This option price is of the same form as the Black-Scholes equation but with a different risk-free rate,  $r$  (described in detail below), and a different volatility,  $V^2$ . In particular, the volatility of the underlying asset has been increased by the volatility of the bond. Recall from section 2.3 that the underlying asset is assumed in the risk-neutral valuation to drift upward at the risk-free rate. In effect  $\alpha$  in the diffusion process of  $S$  above is replaced by  $r$ . Having a stochastic interest rate thus introduces additional variance into the risk-neutral path followed by  $S$  through the drift term - as opposed to stochastic volatility which introduced additional volatility through the variance term,  $\sigma$ , of  $S$ .

Rabinovitch (1989) used Merton's approach in the derivation of the option price, but adopted a different process for the interest rate. This approach assumes that the short term interest rate follows a mean-reverting Ornstein-Uhlenbeck process of the form,

$$dr = q(m - r)dt + vdz$$

where,

$r$  = short term interest rate, i.e., the yield to maturity (annualised) on a bond that pays one

$R$  and in the next instant of time,

$q(m - r)$  = is the instantaneous expected change in the short term interest rate

$v^2$  = the instantaneous variance of the short term interest rate,

$m$  = the interest rate long term mean to which  $r$  reverts at a speed proportional to  $q$ ,

$\rho$  = the correlation between unanticipated changes in the short term rate and returns on the stock, and

$dz$  is a Weiner process.

For this Ornstein-Uhlenbeck process, the price of the bond is as follows, Vasicek (1977):

$$P(\tau) = A \cdot \exp(-rB) \quad \dots(1)$$

where,

$$B = B(\tau) = (1 - \exp[-q\tau]) / q,$$

$$A = A(\tau) = \exp[k(B - \tau) - (vB/2)^2 / q],$$

$$k = m + v\lambda / q - (v/q)^2 / 2,$$

$$\lambda = (\gamma - \tau) / \delta,$$

where,

$\gamma$  is the instantaneous expected return of the bond,

$\delta^2$  is the instantaneous return variance of the bond,

$\lambda$  is the market price of risk,

and from Ito' Lemma (Rabinovitch, 1989) one has that,

$$\delta(\tau) = vB(\tau).$$

Applying Merton's option valuation approach, Rabinovitch derives the same formula for the call value but with the term  $V^2 r$  replaced with,

$$T = \sigma^2 r + (r - 2B + (1 - \exp[-2qr]) / 2q)(v/q)^2 - 2pq(r - B)v/q \quad \dots(3)$$

### 5.4.2.1 Magnitude of Price Bias

The impact of including a stochastic interest rate, using Merton's model, and Rabinovitch's assumed interest rate process, is considered in terms of three effects. In each case Rabinovitch's model is used to establish the pricing biases that exist relative to Black-Scholes. The prices are in all cases calculated for *put* options (Rabinovitch calculates only call option prices). The three effects considered are:

- i) Interest rate effect,
- ii) Volatility effect, and
- iii) Mean reversion effect:

#### *i) Interest Rate Effect*

In Black-Scholes the bond price,  $P(\tau) = e^{-\tau r}$ , whereas in the Merton model  $P(\tau)$  allows for a stochastic  $r$ . For a given market bond price, the instantaneous rate of interest for the models will thus differ. For the Black-Scholes model the instantaneous rate is

$r = -\ln P(\tau) / \tau$ , while for the Rabinovitch model, applying equation (1), the instantaneous rate is  $r = -(\ln P(\tau) / A) / B$ . According to Rabinovitch (1989) the impact of this is negligible.

The impact on price of a change in the volatility of the stochastic interest rate,  $v$ , is shown in Table 5.2. The following assumptions have been used:

The option is at - the - money

Stock price = R50

Exercise price = R50

Term = 1 year

$\sigma = 0.16$

$r = 0.10$

The impact of mean reversion has been reduced by setting ,  
 $q = 0.01$ , i.e. the speed of adjustment to the long term mean is very low , and  
 $m = 0.1$ , i.e. the long term mean is set equal to the instantaneous rate of interest  
The effect of correlation is also ignored by setting  $\rho = 0$   
The market price of risk is assumed to be a constant  $\lambda = 0.2$ .

**Table 5.2**

$v$	<b>R</b>	<b>B</b>	<b>Rand bias = R - B</b>	<b>Percentage Bias = ( R -B) / B</b>
0.02	1.21	1.20	0.01	0.53%
0.06	1.22	1.16	0.06	4.84%
0.1	1.28	1.12	0.15	13.54%
0.14	1.38	1.09	0.29	26.54%
0.18	1.54	1.07	0.47	43.53%

*R denotes the price as calculated by Rabinovitch's model, and B the corresponding price according to the Black-Scholes model. The bias in the last column has been calculated without rounding errors and does not correspond exactly with the rounded figures presented in the previous three columns.*

The decrease in B observed is due to the increase in the Black-Scholes risk-free rate as the stochasticity of the short term interest rate is increased. This results from the fact that an increase in  $v$  reduces the price of the bond  $P(\tau)$  as dictated by equation (1) and consequently the Black-Scholes constant risk-free rate which equates to this bond price is increased. This higher Black-Scholes interest rate results in roughly a 1 cent reduction in the Black-Scholes price for every 1% increase in  $v$ . A large part of the biases observed are due to the increases in R - these increases are attributed to the volatility effect described in ii) below. The interest rate effect is thus not very substantial, although Rabinovitch's proposal that the effect is *negligible* is probably an understatement.

*ii) Volatility Effect*

Merton's model is, apart from the interest rate effect, equivalent to a Black-Scholes

model with an effective volatility of,

$$V^2 = \sigma^2 + \delta^2 - 2\rho\sigma\delta.$$

Rabinovitch's model (1989) incorporates the effect of mean reversion into the term for volatility,  $T$ , as reflected in the equation (3). The impact of the change in volatility will thus depend on the size and sign of  $\rho$  and  $\delta$ . Rabinovitch (1989) compares the prices generated by his model with that of Black-Scholes and finds that the biases are large when  $|\rho|$  and  $\nu$  and hence  $\delta$  are large (since a large  $\nu$  is equivalent to a large  $\delta$ ).

When  $\rho = +1$ , that is, the returns on the underlying stock and the default free bond behave identically, then the variance  $V^2$  is at its lowest. At the other extreme, when  $\rho = -1$ , the variance  $V^2$  is at its highest. In these scenarios the effect of stochastic interest is greatest. In Table 5.3 below pricing biases relative to the Black-Scholes price have been calculated for different levels of  $\rho$ . The assumptions are the same as for Table 5.2 with  $\nu = 0.1$ .

The table shows that, as predicted, the impact of volatility is greatest when the correlation is at the extremes, -1 and +1. Even when correlation is zero, R is 13.54% higher than Black-Scholes. This is due to the increase in the effective volatility,  $V^2$ , by the addition of the term  $\delta^2$ . As  $\nu$  is increased,  $\delta^2$  will increase through the dynamics of equation (2). For this one year term the increase in  $\delta$  is almost linear in  $\nu$  and results in significant increases in the biases above when  $\nu$  is increased.

**Table 5.3**

$\rho$	R	B	R - B	(R - B)/B
-1	1.94	1.12	0.82	72.96%
-1/2	1.62	1.12	0.50	44.36%
0	1.28	1.12	0.15	13.54%
1/2	0.9	1.12	-0.22	-19.93%
1	0.49	1.12	-0.63	-56.33%

### iii) Mean reversion effect

The difference between the current short term rate and the long term mean,  $r-m$ , is accounted for in the Rabinovitch model, in the equation for  $T$ , equation (3). The impact of mean reversion is considered here by investigating the impact on R and B of a change in the speed of adjustment of the instantaneous interest rate to its long term mean. Table 5.4 below shows the extent of biases for different values of  $q$  (assumptions as for Table 5.2 with  $\nu = 0.1$  and  $\rho = -1$ ):

**Table 5.4**

$q$	R	B	R - B	(R - B)/B
0.01	1.94	1.12	0.82	72.96%
1	1.74	1.15	0.59	51.7%
2	1.61	1.16	0.45	38.85%

We see from this table that the impact of mean reversion is to reduce the extent of the bias caused by stochasticity of interest rate. The greater the speed of adjustment of the instantaneous rate to its long term mean the smaller the pricing bias. This effect is the same as was observed for volatility (see 4.4.3.3 and 4.4.3.5), and may once again be attributed to a reduction in the spread of the distribution, caused by mean reversion (the distribution here being that of the interest rate).

#### 5.4.2.2 Extension of Term Effect

One is also able to investigate the effect of an extension in term using Rabinovitch's pricing model. Table 5.5 overpage shows the how the biases change with an increased term of the option. The assumptions are as for Table 5.2 with  $\nu = 0.1$ .

**Table 5.5**

TERM	R	B	R - B	(R - B) / B
1 year	1.28	1.12	0.15	13.54%
2 years	1.45	0.84	0.61	73.28%
5 years	3.50	0.37	3.13	837.28%

The table shows that the impact increases significantly with term. The increases are largely due to the effect of the increased volatility in Rabinovitch's model. This affects the size of the bias in two ways. Firstly, for longer term options, a change in volatility has a greater impact on price, which was discussed in 4.3. Secondly, for longer term options the size of  $V^2$  is greater. When  $\rho = 0$ , the increase in the effective volatility is determined by the size of  $\delta^2$ , and hence the impact of term on the effective volatility can be determined by investigating the impact of term on  $\delta^2$ . This is illustrated in Table 5.6 for two values of  $q$  (assumptions as for Table 5.2).

**Table 5.6**

TERM	SIZE OF $\delta^2$	
	q=1	q=1/2
1 year	6.3%	7.9%
2 years	8.6%	8%
5 years	9.9%	18.4%
10 years	10%	19.9%

As mentioned before,  $\delta^2$  refers to the volatility of the bond with maturity equal to the expiration date of the option. The increase in  $\delta^2$  with term merely reflects the increase in volatility of the bond with increased duration. The increase is greater the lower the value of  $q$ . It thus appears that mean reversion reduces the volatility of the longer term bonds. In the case of  $q = 1/2$  the volatility effect for a 5 and 10 year option is extremely high. The stochastic interest rate has in these circumstances caused

more than an effective doubling of the underlying asset's assumed volatility. The price biases corresponding to these figures are extremely high.

As with volatility it is to be expected that the effect of mean reversion is increased with an extension of term. Table 5.7 below shows how the biases between R and B vary with term for two values of  $q$ . Assumptions as for Table 5.2 with  $\nu = 0.1$ .

**Table 5.7**

TERM	(R - B) / B	
	q = 0.01	q = 1
1 year	13.54%	6.86%
2 years	73.28%	20.83%
5 years	837.28%	69.14%

The table shows that the *extent* of the bias and *rate of increase* in bias with term is reduced with an increase in the "speed adjustment factor",  $q$ , but that the net effect of an extension of term is still a significant increase in the bias. The reduction in bias due to mean reversion is greater for the longer terms. For a 5 year option the bias is reduced from 837% to 69% whereas for a one year term the reduction is only from 13.5% to 7%. One could thus conclude that mean reversion becomes more significant as the term is extended. Failure to account for mean reversion of the interest rate would thus result in significant overstatement of the option price.

## 5.5 THEORY AND EVIDENCE OF INTEREST RATES

Some of the more important models that have been used to characterize interest rate behaviour are listed in Table 5.8 overpage:

**Table 5.8**

<b>Author</b>	<b>Model Specification</b>
Merton (1973)	$dr = \theta.dt + \sigma.dz$
Vasicek (1977)	$dr = k(\theta - r).dt + \sigma.dz$
Dothan (1978)	$dr = \sigma.r.dz$
Courtadon (1982)	$dr = k(\theta - r).dt + \sigma.r.dz$
Cox, Ingersoll, Ross (1985b)	$dr = k(\theta - r).dt + \sigma.\sqrt{r}.dz$

These are all one-factor models in that only one diffusion process is used to describe the interest rate dynamics. Two factor models have been proposed by, amongst others, Brennan and Schwartz (1979), Schaefer and Schwartz (1984), and Longstaff and Schwartz (1992). A three factor model has recently been proposed by Chen (1994).

As can be seen from the model specifications in the above table, the models differ mostly in respect to allowance for mean reversion and allowance for volatility which depends on the current level of interest rates. In Merton and Vasicek's models volatility does not depend on the current level of interest rates. Rabinovitch (see section 5.4) used Vasicek's model.

The models listed in the above table all adopt the same method to derive the model. In all of them a plausible model for the short term interest rate was proposed and from this the current yield curve is deduced. The parameters of the model are chosen so that it reflects market data as closely as possible. A more recent approach to interest rate modeling involves taking the current term structure of interest rates as given, and developing an arbitrage-free yield curve model that is perfectly consistent with this current term structure. Within these arbitrage-free yield curve models three different approaches have been used to date. Ho and Lee (1986) constructed a binomial tree for bond prices which provides an exact fit to the current term structure of interest rates. Their model however, results in interest rates of all maturities having the same

volatility, which is not plausible. An alternative approach was adopted by Black, Derman and Toy (1990) who use a binomial tree to construct a model for short term interest rates that fits the current structure as well as the current volatilities of all bond yields. This approach to modeling short term rates so as to fit the current term structure has also been used by Hull and White (1990), Black and Karasinski (1991), and Hull and White (1993). Finally, Heath, Jarrow and Morton (1992) have developed a model for the process followed by instantaneous forward rates which fits the current term structure.

There is no general agreement on which set of assumptions is best and little work appears to have been done on comparing the models. A recent paper Chan, Karolyi, Longstaff and Sanders (1992) compares eight of these one factor models by investigating (for the USA) which fits short term Treasury bill yield data best. Their results show that one of the most crucial elements of modeling interest rates is the modeling of volatility of the short term interest rate. They find that the best fit models are those that allow the volatility of interest rate changes to be highly dependent on the current level of the interest rate.

## 5.6 CONCLUSION

The following important features of the interest rate for the purposes of long term option pricing have emerged in this chapter.

1. The sensitivity of the option price to the interest rate increases with increased term of the option. For prospective nominal guarantees such as 5% per annum the sensitivity is high.
2. For long term options a mis-estimation of the level of interest rates will lead to a greater error in price than a similar mis-estimation of the level of volatility.
3. Stochasticity of interest rates introduces additional volatility into the path of the price of the underlying asset in the risk-neutral valuation.

4. The extent of the increase in volatility depends crucially on  $\rho$ , the correlation between the bond and stock price changes and on the size of  $v$ , the variance of short term interest rate. The larger the size of  $v$  and  $|\rho|$  the greater the extent of this increase in volatility.
5. The magnitude of the increase in volatility due to stochasticity of interest rates increases with the term of the option.
6. Mean reversion of the stochastic interest rate reduces significantly the extent of biases and this mitigating effect increases with term.

It is crucial thus that any model for pricing long term options should incorporate the following main features of interest rate behaviour:

1. variance of short term interest rate,
2. correlation between bond and stock price changes, and
3. mean reversion of the short term interest rate.

Vasicek's interest rate model (1977) is convenient in that it lends itself to Merton's (1973) closed form solution - as used by Rabinovitch (1989). This model is not recommended here however, for a number of reasons. Firstly, it leads to long run possibilities of negative interest rates. While this is not likely in the short term and can hence be ignored in pricing short term options, it may be problematic for the longer term. Secondly, the evidence of Chan, Karolyi, Longstaff and Sanders (1992) indicates that the volatility of interest rate changes should depend on the current level of interest rates and this is not accounted for in Vasicek's model. Finally, the convenience of Merton's closed form solution is not of benefit to this thesis since stochastic volatility still needs to be accounted for and this necessitates a non-analytical solution.

In line with the second of these reasons, this paper recommends the use of models of the form of Cox, Ingersoll, and Ross (1985b):

$$dr = k(\theta - r).dt + v.\sqrt{r}.dz_2$$

or Courtadon (1982):

$$dr = k(\theta - r).dt + v.r.dz_2$$

where,

the process followed by the stock price  $S$  is as defined in 5.4.2, i.e.

$$\frac{dS}{S} = \alpha.dt + \sigma.dz_1$$

$r$  = short term interest rate,

$\theta$  = the interest rate long term mean to which  $r$  reverts,

$k$  = the speed of adjustment of short term interest rate to its long term mean,

$v$  = the instantaneous standard deviation of the short term interest rate,

$dz_1$  and  $dz_2$  are Weiner processes with,

$$dz_1.dz_2 = \rho,$$

$\rho$  = the instantaneous correlation between returns on stock and bond.

In both cases the volatility of  $r$  depends on the current level of  $r$ . These interest rate models render Merton's option price model inappropriate since the variance of the bond price is then,  $\delta^2 = (\sqrt{r}vB(\tau))^2$  for the Cox, Ingersoll and Ross's model, or  $\delta^2 = (rvB(\tau))^2$ , for Courtadon's model. The bond price is thus less volatile the lower the short term interest rate. Merton's derivation is based on the assumption that  $\delta$  is non-stochastic and independent of the level of  $P$  - which is violated in both the models. Cox, Ingersoll and Ross (1985b) have also shown that their interest rate model leads to a steady state distribution for the future interest rate. The distribution of the interest rate at maturity of the bond (and hence maturity of the option) for long durations approaches a gamma distribution.

Whatever model is adopted it is essential that the parameters for the interest rate process be consistent with the bond price data and fit the term structure of interest rates. If this is not the case then the interest rate process has not been accurately defined and can result in significant biases in the option price. Further research is required into the term structure of interest rates in the South African market.

## CHAPTER 6

### CONCLUSION

This thesis has shown that it is not sound to use the Black-Scholes model for the purposes of costing long term maturity guarantees. The Black-Scholes model is widely accepted in the literature for costing such guarantees and has been applied without questioning of the underlying assumptions. The Black-Scholes assumptions have been shown in this thesis to lead to significant biases, and these biases increase with increased term to expiry. The major assumptions underlying the Black-Scholes model have been investigated separately and the findings are summarized below.

With respect to the distribution of underlying stock price, evidence indicates that the assumption of a log-normal distribution of returns is not appropriate for short term returns, as the distribution of these exhibit leptokurtosis. Daily returns, furthermore, do not appear to be stable - there being a reduction in the "fatness" of the tails for longer term returns, and greater success has been achieved in fitting a student-t distribution. In light of this, it is suggested that a log-normal distribution of the returns conditional on the distribution of the volatility be attempted. If this results in a student-t distribution for short term returns then it is not unreasonable to use a log-normal distribution for long term returns.

The investigations of volatility have shown that, when pricing guarantees of a nominal rate of return per annum, volatility is more important for longer term options than short term options. It has also been shown that the stochasticity of volatility leads to significant biases, and that the direction of these biases depends on the correlation between volatility and stock returns. Increased term of the option was shown to be associated with increased biases, but mean reversion of volatility reduces the extent of this increased bias. Evidence of correlation between volatility and returns is not conclusive, but there is strong evidence for mean reversion of volatility both

internationally and locally. It is recommended that an Ornstein-Uhlenbeck process be adopted to model volatility. This process incorporates mean reversion behaviour and allows for correlation between volatility and stock returns.

The investigations into the interest rate assumption show that, as for volatility, the sensitivity of the option price to changes in the risk-free rate of interest is greater for longer term options, and particularly so for options which secure a guaranteed nominal rate of return per annum. For short term options, prices are more sensitive to volatility, but for longer term options it was shown that prices are more sensitive to changes in interest rate than volatility. It was demonstrated that stochasticity of interest rates introduces additional volatility into the risk-neutral distribution of the returns on the underlying asset, and that the extent of this increase in volatility depends on the extent of the variance of the short term interest rate and the size of the correlation between bond and stock price changes. As in the case of volatility, it was shown that stochasticity of interest rate leads to significant biases and that these biases increase with the term of the option. Mean reversion of the interest rate was shown to reduce the extent of this increase.

The scope of this thesis prevented further investigation of empirical evidence of interest rates in the South African market, nevertheless international interest rate models have been presented. Two such models are recommended for use in long term option pricing. These models incorporate the major features of interest rate behaviour which affect the option price, namely, mean reversion, correlation between stock and bond price changes, and a variance of the interest rate which depends on the current level of the short term interest rate. Further work needs to be done to test and parametrize these models using South African data.

Based on the recommended adjustments to the Black-Scholes model, the net effect on the option price relative to Black-Scholes will depend on the major features of volatility and interest rates identified above. The *direction* of the bias will depend on the sign of the correlation between volatility and stock returns and on the sign of the

correlation between bond returns and stock returns. For both volatility and bond returns, a zero or negative correlation with stock returns will cause the resultant price to be higher than Black-Scholes. However, a positive correlation with stock returns, for either volatility or bond returns, will reduce the resultant price relative to Black-Scholes. The *magnitude* of the bias will depend on size of the respective correlations as well as on the extent of mean reversion in volatility and the short term interest rate. For both volatility and interest, a stronger mean reversion reduces the extent of the bias. The variance of the short term interest rate will also have a significant influence on the magnitude of the bias in that a higher variance increases the resultant price relative to Black-Scholes.

The option needs to be valued by modeling the three stochastic processes simultaneously, namely, the stock price process, the volatility of the stock return process, and the interest rate process. Monte Carlo simulation is recommended and the reader is referred to the paper by Boyle (1977) for an exposition of the method of simulation. This technique, which is based on the risk-neutral valuation approach as outlined in 2.4, would, by way of a brief summary, involve the following. The paths of each of the three stochastic variables would be simulated using appropriate correlations between their Weiner processes. The terminal payoff of the option would be calculated for each simulation run from the sample paths generated. The terminal payoff is then discounted at the average value of the stochastic interest rate, and the option price is taken as the arithmetic average of the discounted terminal values. It must be remembered that the sample paths simulated for the underlying variables must correspond to the stochastic process that the variable follows in a risk-neutral world, in particular, for stock prices the simulation will be based on the asset having an underlying drift equal to the risk-free rate. The Monte Carlo approach for valuing options has been used by, Johnson and Shanno (1987), Hull and White (1987), and Scott (1987), and has the advantage that it is flexible and can be modified to accommodate any stochastic process for the underlying variables.

The specification of two additional stochastic processes, means that additional parameters need to be estimated. Whether the increased accuracy from the more detailed model specification justifies the effort required for additional parameter estimation is an empirical question. The stochastic processes suggested need to be parametrized on the relevant local data and the resulting option values compared with those of Black-Scholes to establish how significant the price change is. It is possible that for a given parametrization, the resultant option prices are not significantly different from the Black-Scholes prices due to counteractive effects of the new stochastic features introduced. In this case the Black-Scholes price will be approximately correct only by coincidence, since the assumptions underlying Black-Scholes have been shown to deviate significantly from empirical evidence of the long term behaviour of the variables.

Thus it must be concluded that the assumptions underlying the Black-Scholes model are not empirically sound for long term option pricing, and that the combined effect of all the deviations from the assumptions could lead to significant biases. It is recommended that the long term options be valued by a risk-neutral valuation of the option payoff produced by the three simultaneous stochastic equations suggested, and this be done using Monte Carlo simulation. This approach has been used by Nomura Bank for long term foreign exchange options (Dolbear and Tsivandis, 1990) and they have concluded that the extra effort of estimating the additional parameters has been of net benefit to the bank, and that the model "allows a more pragmatic modeling of the real world" (p. 53). It is hoped that this thesis will stimulate more research into the underlying assumptions of long term option pricing models and that a parametrization and pricing exercise for the South African market will be performed to establish the extent of the deviation of the long term option value from the Black-Scholes price.

## APPENDIX A

### DERIVATION OF NORMAL PROBABILITY PLOT

Let,

$$U = \ln(S_t / S_{t-1}),$$

$S_t$  = stock price at time  $t$ ,

$U_i$  = ordered values of  $U$ ,  $i = 1$  to  $N$ ,

$N$  = sample size = 1809.

Let  $f$  be the fractile, where  $f$  is taken as

$$(3 \times i - 1) / (3 \times N + 1) .$$

This formula for calculating  $f$  has been shown to give reasonable estimates (Gumbel, 1954) and was used in the Fama (1965) study.

Let,

$$z_i = (U_i - \mu) / \sigma$$

where,

$\mu$  is the mean of the  $U_i$ 's, and

$\sigma$  is the standard deviation of the  $U_i$ 's.

Now, if  $U_i$  are normally distributed then  $z_i$  will be a standard unit normal variable.

Since  $z$  is just a linear transformation of  $U$ , the graph of  $z$  against  $U$  should be a straight line. For the purposes of the probability plot  $z_i$  are not calculated from the data but are rather derived from the cumulative normal distribution function. If  $U_i$  are in fact normal then the relationship should be a straight line (subject to departures due to sampling error).

## APPENDIX B

### METHOD OF CALCULATION OF ALPHA, $\alpha$

Let,

$f$  denote the fractile,

$x_f$  be the  $f$  sample fractile,

$c$  = dispersion parameter.

For some large  $f$  calculate,

$$\hat{z}_f = (\hat{x}_f - \hat{x}_{1-f}) / 2\hat{c},$$

where,  $c$  is estimated as

$$\hat{c} = 0.5 \times (\hat{x}_{0.72} - \hat{x}_{0.28}) / 0.827,$$

(Finite sample properties of  $\hat{c}$  are given in Fama and Roll (1968))

and the value of  $f$  was taken as,

$$(3 \times i - 1) / (3 \times N + 1)$$

as used in 3.3.2.2.

Given a symmetric stable variable with exponent  $\alpha$ , and scale  $c$ ,  $z_f$  is an estimator of the  $f$  fractile of the standardized symmetric stable distribution with exponent  $\alpha$ . The estimate of  $\alpha$  is obtained from tables of standardized symmetric stable distribution fractiles (Fama and Roll, 1968) as that value of  $\alpha$  whose  $f$  fractile matches  $\hat{z}_f$  most closely. The values for  $\alpha$  have been quoted to two decimal places - these values have been obtained by interpolating between the one decimal values quoted in the above mentioned tables.

## APPENDIX C

### CHOICE OF $f$ VALUE FOR ESTIMATION OF ALPHA

The choice of the fractile value can have a significant impact on the accuracy of results. Fama and Roll (1971) have shown that a higher  $f$  value has benefit in that differences between fractiles for different values of  $\alpha$  increases with  $f$ . However, the sampling dispersion of fractile estimates is inversely proportional to the density of the underlying distribution which decreases as we choose fractiles further into the tails. This means that the estimate of the fractile will be subject to greater sampling error for higher values of  $f$ .

There is some value of  $f$  which gives an optimal balance between the above two effects. Fama and Roll (1971) used Monte Carlo sampling distributions of  $\alpha$  for different values of  $f$  between (and inclusive of) 0.93 and 0.99. They found that when  $\alpha \geq 1.9$  the best estimator in terms of low bias and standard deviation is  $\hat{\alpha}_{0.99}$ . This requires large samples to get reliable estimates of these extreme tails. However, if one has no knowledge that the true value of  $\alpha$  is close to 2 and do not have large sample sizes then  $\hat{\alpha}_{0.99}$  is a relatively poor estimator and is dominated by both  $\hat{\alpha}_{0.95}$  and  $\hat{\alpha}_{0.97}$ .

## GLOSSARY

**ALSI** - The JSE All-Share-Index.

**American Option** - An option that may be exercised at any time up to and including the expiration date of the option.

**At-the-money** - An option for which the current price of the underlying asset is equal to the exercise price of the option.

**Delta** - The first order partial derivative of the option price with respect to the price of the underlying asset. This is used as a measure of the exposure that the option gives to the underlying asset.

**Delta neutral portfolio** - A portfolio which is managed so as to maintain its delta as close to zero as possible. Such a portfolio is hedged in the sense that small changes in the price of the underlying asset will not result in any loss to it.

**European Option** - An option that may be exercised only on the expiration date of the option.

**Fair value of future** - The value that the future would have in an efficient market. This is calculated as the value of the underlying asset plus the net carry cost on the asset.

**Gamma** - The second order partial derivative of the option price with respect to the price of the underlying asset. It is the derivative of delta with respect to the price of the underlying asset.

**Gamma neutral portfolio** - A portfolio which is managed so as to maintain its Gamma as close to zero as possible.

**In-the-money** - An option that has intrinsic value. A call option is in-the-money if the underlying asset price exceeds the exercise price, and a put option is in-the-money if the reverse holds.

**Intrinsic value** - The value of the option deriving from the difference between the exercise price and the underlying asset price. In the case of a call option it is the underlying asset price less the exercise price. In the case of a put option it is the exercise price less the underlying asset price. Where this leads to a negative amount the option is said to have no intrinsic value.

**JSE** - The Johannesburg Stock Exchange

**Long position** - Adopting the strategy of buying an asset

**Martingale Measure** - A stochastic process lying in a probability space which satisfies a convergent conditional expectation rule, algebraically:

A stochastic process  $\{Z_n, n \geq 1\}$  is said to be a Martingale process if,

$$E[|Z_n|] < \infty \text{ for all } n$$

and,

$$E[Z_{n+1} | Z_1, Z_2, \dots, Z_n] = Z_n$$

**Net carry cost** - The storage cost plus interest that would have to be paid to finance the purchase of the asset underlying the future contract less the income that would be earned on the underlying asset. For a non-dividend paying asset the net carry cost would equal the risk-free rate of interest.

**Out-the-money** - An option that has no intrinsic value. A call option is out-the-money if the exercise price exceeds the underlying asset price, and a put option is out-the-money if the reverse holds.

**SAFEX** - The South African Futures Exchange.

**Short position** - Adopting the strategy of selling an asset which one does not hold.

**Spot price** - The current price ruling in the market.

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