Heavy Neutrino Matter at Finite Temperature

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Abstract

We study, for a system of massive fermions which interact only gravitationally, the phase transition that is associated with gravitational collapse. It is shown that by cooling a non-degenerate gas of massive neutrinos below some critical temperature, a condensed phase emerges, consisting of quasi-degenerate supermassive neutrino stars. These compact dark objects could play an important role in structure formation in this universe, as they might in fact provide the seeds for galactic nuclei and quasi-stellar objects.

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Chapter 1

Introduction

Systems of massive fermions which interact only gravitationally will undergo gravitational collapse under certain conditions. This collapse will be accompanied by a phase transition which occurs only for the attractive gravitational interaction of neutral particles obeying Fermi-Dirac statistics [1, 2, 3, 4, 5]. The phase transition neither takes place in the case of charged fermions [6], nor uncharged particles obeying Bose-Einstein or Maxwell-Boltzmann statistics. This result remains basically correct if the fermions have an additional weak interaction, as for example neutrinos do. We focus here on the heaviest neutrino ν_τ, although this is not essential for most of this thesis.

Systems of massive neutrinos that interact only gravitationally also have interesting and important consequences for the early universe. The ground state of a gravitationally condensed neutrino cloud, with mass below the black hole mass limit, is a cold neutrino star, in which the gravitational attraction balances the degeneracy pressure of the neutrinos [7, 8, 9]. Degenerate stars consisting of neutrinos in the mass range of m_ν = 10 to 25 keV are particularly interesting. They can explain some of the features observed around supermassive compact dark objects, which exist at the centres of quite a few galaxies [10, 11, 12, 13, 14, 15, 16], including the Milky Way [17, 18] and quasi-stellar objects [19, 20, 21]. In fact, the difference between a supermassive black hole and a neutrino star of the same mass near the black hole limit, a few Schwarzschild radii away from the object, is small.

The existence of a quasi-stable neutrino in the mass range between 10 and 25 keV is neither ruled out by particle physics or nuclear physics experiments, nor by astrophysical observations [9, 22, 23]. However, such a neutrino would lead to an early matter dominated phase in the early universe, which may have occurred some time after nucleosynthesis and prior to recombination. Thus, if the Standard Model of Cosmology is correct, the universe would reach the microwave background temperature much too early to accommodate the oldest stars in globular clusters, cosmochronology and the Hubble expansion age.

It is conceivable, however, that in the presence of these heavy neutrinos, the early universe might have evolved quite differently than described in the Standard Model of Cosmology. Neutrino stars might have been formed in local condensation processes during a gravitational phase transition. The latent heat produced by such a phase transition, apart from reheating the gaseous phase, might have reheated the radiation background as well. Annihilation of the heavy neutrinos into light neutrinos via the Z^0 in the interior of these neutrino stars [9, 22, 23] might have taken place. Both these processes will decrease the contribution of the heavy neutrinos to the critical density today and therefore increase the age of the universe [24]. Thus a quasi-stable neutrino in the mass range between 10 and 25 keV does not seem excluded by astrophysical observations [9].

The purpose of this thesis is to study the phase transition, that occurs during gravitational collapse of neutrino matter, and the subsequent formation of these neutrino stars. This scenario will be studied without the presence of a radiation background.

Chapter 2

The Thomas-Fermi and Lané-Emden Equations

The Fermi-Thomas statistical model of the atom has been used by several investigators for approximate calculations of potential fields and charge densities in metals as a function of lattice spacing. The method has also served as a starting point for the study of the behaviour of matter under extremely high pressures as found, for example in stars.
In its original form, the theory makes several simplifying assumptions: the effect of exchange forces is not taken into account, and the temperature of electrons and nuclei is taken as zero degrees absolute, $T = 0$. With these simplifications, a set of universal potential functions may be found, applicable to all atomic numbers, Z, by a simple change in scale of linear dimensions.

- **Feynman, Metropolis and Teller** [6].

In this chapter we derive the Thomas-Fermi equation for a degenerate Fermi gas of electrons around a nucleus at temperature $T = 0$. We then use the same techniques to derive the Lané-Emden equation for degenerate neutrino matter at

$T = 0$ in a gravitational field. In both cases we deal with interacting degenerate fermion gases. A discussion follows in which the similarities and differences between the two equations are compared with each other and discussed.

2.1 The Thomas-Fermi Equation for a Degenerate Electron Gas around a Nucleus

The study of a degenerate Fermi gas and its properties at zero temperature is of fundamental significance in statistical physics and plays a central role in this thesis. Therefore, let us now consider the well studied case of an electron gas in an ion. We neglect finite temperature and exchange effects. In such a gas the electrons will be distributed among the various quantum states so that the total energy of the gas has its least possible value. Since no more than one electron can be in one quantum state, the electrons occupy all states with energies from a smallest value to some largest value, which of course depends on the number of electrons in the gas [25].

With $g_e = 2$ the spin degeneracy factor of the electron, the number of electrons with momentum between p and $p + dp$, moving in a large volume V, is

$$dN_e = g_e \cdot \frac{4\pi p^2 dpV}{(2\pi\hbar)^3} . \tag{2.1}$$

The electrons occupy all states with momenta from zero to a limit $p = p_F$, called the radius of the Fermi sphere in momentum space. The total number of electrons in these states is [25]

$$N_e = \frac{g_e V}{2\pi^2\hbar^3} \int_0^{p_F} p^2 dp = \frac{g_e V p_F^3}{6\pi^2\hbar^3} . \tag{2.2}$$

Hence the Fermi momentum is given by

$$p_F = \left(\frac{6\pi^2}{g_e}\right)^{1/3} \left(\frac{N_e}{V}\right)^{1/3} \hbar . \tag{2.3}$$

We now assume that we are dealing with the non-relativistic case, i.e. the Fermi velocity is much smaller than the velocity of light, c.

$$v_F = \frac{p_F}{m_e} = \frac{\hbar}{m_e}\left(\frac{6\pi^2}{g_e}\right)^{1/3}\left(\frac{N_e}{V}\right)^{1/3} \ll c\,. \tag{2.4}$$

The total energy of the electron gas is obtained by multiplying the number of states, (eq. (2.1)) with $p^2/2m_e$, where m_e is the electron mass, and integrating it over all momenta, yielding

$$\begin{aligned} E &= \frac{g_e 4\pi V}{2m_e(2\pi\hbar)^3}\int_0^{p_F} p^4 dp \\ &= \frac{g_e V p_F^5}{20 m_e \pi^2 \hbar^3}\,. \end{aligned} \tag{2.5}$$

Now we substitute p_F from eq. (2.3) into this equation and find the expression for the total energy

$$E = \frac{3V\hbar^2}{10m_e}\left(\frac{6\pi^2}{g_e}\right)^{2/3}\left(\frac{N_e}{V}\right)^{5/3}\,. \tag{2.6}$$

In order to obtain the other thermodynamical properties like pressure, we use a result from [25], which is derived in Appendix A and which gives the pressure P in terms of the energy and volume, i.e.

$$PV = \frac{2}{3}E\,. \tag{2.7}$$

From eqs. (2.6) and (2.7) we find the degeneracy pressure

$$P = \frac{\hbar^2}{5m_e}\left(\frac{6\pi^2}{g_e}\right)^{2/3}\left(\frac{N_e}{V}\right)^{5/3}\,. \tag{2.8}$$

At this point it is helpful to write eq. (2.2) and eq. (2.6) in terms of *densities.* The electron number density, $n_e = N_e/V$, and energy density, $\varepsilon = E/V$, are then respectively

$$n_e = \frac{g_e p_F^3}{6\pi^2\hbar^3} \text{ and } \varepsilon = \frac{3\hbar^2}{10m_e}\left(\frac{6\pi^2}{g_e}\right)^{2/3} n_e^{5/3}\,. \tag{2.9}$$

We now introduce $\Phi(r)$ as the scalar potential. The potential energy of an electron then is

$$V(r) = -e\Phi(r) \,. \tag{2.10}$$

The local Fermi momentum $p_F(r)$ is therefore implicitly given by

$$\frac{p_F^2(r)}{2m_e} = e[\Phi(r) - \Phi(r_0)] \,, \tag{2.11}$$

where $\Phi(r_0)$ is the total energy, since the kinetic energy vanishes at the classical turning point r_0.

The Fermi momentum and electron number density are then respectively written as

$$p_F(r) = \sqrt{2m_e e[\Phi(r) - \Phi(r_0)]} \tag{2.12}$$

and

$$\begin{aligned} n_e(r) &= \frac{g_e p_F^3(r)}{6\pi^2\hbar^3} \\ &= \frac{g_e}{6\pi^2\hbar^3}\left[2m_e e[\Phi(r) - \Phi(r_0)]\right]^{3/2} \,. \end{aligned} \tag{2.13}$$

If we approximate the nucleus as a point source, we can use Poisson's equation for $r \neq 0$ with the electron density only as the source term. It will be valid everywhere except at the origin, i.e.

$$\Delta\Phi(r) = 4\pi e n_e(r) \,. \tag{2.14}$$

We now investigate the case of spherical symmetry, yielding

$$\frac{d^2\Phi(r)}{dr^2} + \frac{2}{r}\frac{d\Phi(r)}{dr} = \frac{4\pi e g_e}{6\pi^2\hbar^3}\left[2em_e[\Phi(r) - \Phi(r_0)]\right]^{3/2} \,, \tag{2.15}$$

and after making the substitution $\Phi(r) - \Phi(r_0) = u/r$ one finds

$$\Delta\frac{u(r)}{r} = \frac{1}{r}\frac{d^2u(r)}{dr^2} = \frac{2eg_e}{3\pi\hbar^3}\left(2em_e\frac{u(r)}{r}\right)^{3/2} \,. \tag{2.16}$$

The Thomas-Fermi differential equation for an electron gas around a point nucleus is thus

$$\frac{d^2u(r)}{dr^2} = \frac{2eg_e}{3\pi\hbar^3}(2em_e)^{3/2}\frac{u(r)^{3/2}}{\sqrt{r}} \,. \tag{2.17}$$

The Fermi velocity, electron number density and degeneracy pressure can now also be written in terms of u/r

$$v_F(r) = \left(\frac{e}{m_e}\right)^{1/2}\left(\frac{2u(r)}{r}\right)^{1/2} \tag{2.18}$$

$$n_e(r) = \frac{g_e}{6\pi^2\hbar^3}(m_e e)^{3/2}\left(\frac{2u(r)}{r}\right)^{3/2} \tag{2.19}$$

$$P_e(r) = \frac{(m_e e)^{5/2}}{5m_e\hbar^3}\frac{g_e}{6\pi^2}\left(\frac{2u(r)}{r}\right)^{5/2} \tag{2.20}$$

In order to specify the solutions of eq. (2.17) completely, we need boundary conditions. The first boundary condition follows from our choice of $\Phi(r_0)$ and from $\Phi(r) - \Phi(r_0) = u/r$. At the classical turning point $r = r_0$ the electron number density vanishes. The second condition is that for $r \to 0$ the potential becomes Coulomb like. Our boundary conditions are therefore

$$u(r_0) = 0 \text{ since } n_e(r_0) = 0, \text{ and} \tag{2.21}$$

$$u(0) = Ze \text{ since } \Phi(r) \to \tfrac{Ze}{r} \text{ for } r \to 0. \tag{2.22}$$

We may now introduce the dimensionless units defined as

$$x = \frac{r}{a} \text{ and } v = \frac{u}{e}\,, \tag{2.23}$$

yielding the Thomas-Fermi equation for an ion

$$\frac{d^2v}{dx^2} = \frac{v^{3/2}}{\sqrt{x}} \tag{2.24}$$

with the boundary conditions

$$v(0) = Z \text{ and } v(x_0) = 0 \text{ for } x_0 = \frac{r_0}{a}\,. \tag{2.25}$$

In these equations the scale a is given by

$$a = a_0\left(\frac{3\pi^2}{32g_e^2}\right)^{1/3} = 0.74370\ \text{Å}\left(\frac{0.511\ \text{MeV}}{m_e c^2}\right)(g_e)^{-2/3}\,, \tag{2.26}$$

where $a_0 = \hbar^2/(m_e c^2)$ is the Bohr radius.

Eq. (2.24) plays an important role in the context of this thesis. The Lané-Emden equation for degenerate neutrinos around a baryonic star, will be exactly the same, except for the sign.

2.2 The Lané-Emden Equation for a Degenerate Neutrino Halo around a Baryonic Star

Let us now consider a degenerate cloud of neutrinos around a baryonic star [7, 8, 9]. As in the Thomas-Fermi model, we ignore exchange and finite temperature effects. We also assume that the neutrino cloud is in the state of lowest possible energy and that the Fermi velocity is much smaller than the speed of light.
The local Fermi energy is now again set equal to the local gravitational binding energy (compare eq. (2.11)), i.e.

$$\frac{p_F^2(r)}{2m_\nu} = m_\nu[\Phi(r) - \Phi(r_0)] \,. \tag{2.27}$$

We obtain the local Fermi momentum as

$$p_F(r) = 2m_\nu[m_\nu\Phi(r) - m_\nu\Phi(r_0)]^{1/2} \tag{2.28}$$

and therefore the number density can be expressed as

$$n_\nu(r) = \frac{g_\nu}{6\pi^2\hbar^3}[2m_\nu(m_\nu\Phi(r) - m_\nu\Phi(r_0))]^{3/2} \,. \tag{2.29}$$

Here the spin degeneracy factor of the neutrino g_ν, includes both spin and antiparticle degrees of freedom. Thus g_ν is either 2 (for Majorana neutrinos and antineutrinos, where the neutrino and antineutrino are the same particle) or 4 (for Dirac neutrinos and antineutrinos, where the neutrino and antineutrino are two distinct particles). Of course if there are only particles or antiparticles present in the halo these numbers must be divided by two. Please note that throughout this thesis we assume that $g_\nu = 2$, for simplicity.
If the baryonic star can be approximated by a point source, the gravitational potential will obey Poisson's equation for $r \neq 0$ with only the neutrino density as the source term, i.e.

$$\Delta\Phi(r) = -4\pi G m_\nu n_\nu(r) \,. \tag{2.30}$$

If we restrict ourselves to spherical symmetry, we find similar to the Thomas-Fermi case,

$$\frac{d^2\Phi(r)}{dr^2} + \frac{2}{r}\frac{d\Phi(r)}{dr} = \frac{-4\pi G m_\nu}{6\pi^2\hbar^3}[2m_\nu(m_\nu\Phi(r) - m_\nu\Phi(r_0))]^{3/2} , \tag{2.31}$$

and after making the substitution $\Phi(r) - \Phi(r_0) = u/r$ one finds

$$\Delta\frac{u(r)}{r} = \frac{1}{r}\frac{d^2u(r)}{r} = -\frac{G m_\nu^4 g_\nu 4\sqrt{2}}{3\pi\hbar^3}\left(\frac{u(r)}{r}\right)^{3/2} , \tag{2.32}$$

and we arrive at the Lané-Emden equation with polytrope index $n = 3/2$, i.e.

$$\frac{d^2u(r)}{dr^2} = -\frac{G m_\nu^4 g_\nu 4\sqrt{2}}{3\pi\hbar^3}\frac{u(r)^{3/2}}{\sqrt{r}} . \tag{2.33}$$

The relevant boundary conditions are

$$u(0) = GM_B \text{ and } u(r_0) = 0 , \tag{2.34}$$

where r_0 is the radius of the neutrino star and M_B the mass of the pointlike baryonic star at the centre of the neutrino halo.

Of course we may put this differential equation into dimensionless form as well, using

$$v = \frac{u}{GM_\odot} \text{ and } x = \frac{r}{R_0} . \tag{2.35}$$

This finally leads to the dimensionless Lané-Emden equation for degenerate neutrinos in a neutrino star or a halo around a baryonic star

$$\frac{d^2v(x)}{dx^2} = -\frac{v(x)^{3/2}}{\sqrt{x}} , \tag{2.36}$$

where the scale R_0 is defined as

$$R_0 = \left(\frac{3\pi\hbar^3}{4\sqrt{2} m_\nu^4 g_\nu G^{3/2} M_\odot^{1/2}}\right)^{2/3} . \tag{2.37}$$

The boundary conditions now become

$$v(0) = \frac{M_B}{M_\odot} \text{ and } v(x_0) = 0 \text{ with } x_0 = \frac{r_0}{R_0} . \tag{2.38}$$

2.3 Remarks on the Thomas-Fermi and Lané-Emden equations

The Thomas-Fermi and Lané-Emden equations differ merely by a sign which is due to the fact that in the Thomas-Fermi case, the *repulsive* electrostatic interaction of the electrons is at work, while in the Lané-Emden case, we have the gravitational *attraction* between the neutrinos.

Another difference may be the value of the degeneracy factors. In the case of the electron g_e is always two, but in the case of the neutrino, g_ν could be either two if the neutrino is a Majorana particle, or four if it is a Dirac particle, provided of course that both particles *and* antiparticles are present in the halo. The similarity between the two differential equations is even more remarkable considering the fact that these two equations are roughly 26 orders of magnitude apart in size: In the case of electrons around a nucleus, the characteristic scale

$$a = \frac{\hbar^2}{m_e e^2}\left(\frac{3\pi^2}{32 g_e^2}\right)^{1/3} = 0.74370\ \text{Å}\left(\frac{0.511\ \text{MeV}}{m_e c^2}\right)(g_e)^{-2/3} \tag{2.39}$$

is of the order of the Bohr radius, while in the case of neutrinos around a baryonic star the characteristic distance

$$R_0 = \left(\frac{3\pi\hbar^3}{4\sqrt{2} m_\nu^4 g_\nu G^{3/2} M_\odot^{1/2}}\right)^{2/3} = 2.1376\ \text{lyr}\left(\frac{17.2\ \text{keV}}{m_\nu c^2}\right)^{8/3}(g_\nu)^{-2/3} \tag{2.40}$$

is of the order of a light year or two.

The scale R_0 strongly depends on the neutrino mass like $m_\nu^{-8/3}$ while the scale a depends on the electron mass as m_e^{-1}. This is due to the fact that in both cases the mass is responsible for degeneracy pressure, but in neutrino stars m_ν is also the source of gravity, while in the ion case this role is taken over by the charge e. Both equations are differential equations of second order. Because of the sign difference the Thomas-Fermi differential equation is a concave, while the solution of the Lané-Emden equation is convex. In both cases the solution will therefore depend on two integration constants, of which one is trivial due to the homology theorem [7]. Indeed, if $v(x)$ is a solution of

$$\frac{d^2v}{dx^2} = \pm\frac{v^{3/2}}{\sqrt{x}}$$

then

$$\tilde{v}(x) = A^3 v(Ax)$$

is also a solution, where A is any positive real number.

Proof:

$$\tilde{v}(x) = A^3 v(Ax)$$

The second derivative of $\tilde{v}(x)$ with respect to x is given by

$$\tilde{v}''(x) = A^5 v''(Ax) \ .$$

Hence

$$\frac{\tilde{v}(x)^{3/2}}{\sqrt{x}} = \frac{A^{9/2}[v(Ax)]^{3/2}}{\sqrt{Ax}}\sqrt{A} = \frac{A^5[v(Ax)]^{3/2}}{\sqrt{Ax}} \tag{2.41}$$

and therefore it follows that

$$\tilde{v}''(x) \pm \frac{\tilde{v}(x)^{3/2}}{\sqrt{x}} = A^5 \underbrace{\left(v''(Ax) \pm \frac{[v(Ax)]^{3/2}}{\sqrt{Ax}}\right)}_{=0 \text{ by definition}}$$

and thus

$$\tilde{v}''(x) \pm \frac{\tilde{v}(x)^{3/2}}{\sqrt{x}} = 0 \ .$$

Both of these equations also have a scaling property. Let us consider the Lané-Emden equation first. If $v(x)$ extends from $x = 0$ to $x = x_0$ then $\tilde{v}(x)$ extends from $x = 0$ to $x = x_0/A$. The masses and the radius of the neutrino gas and baryonic source scale as [7, 9]

$$\tilde{M}_\nu = M_\odot \int_0^{x_0/A} [A^3 v(Ax)]^{3/2} x^{1/2} dx$$

$$\begin{aligned} &= A^3 M_\odot \int_0^{x_0} [v(x)]^{3/2} x^{1/2} dx \\ &= A^3 M_\nu \ , \end{aligned} \tag{2.42}$$

$$\tilde{M}_B = M_\odot \tilde{v}(0) = A^3 M_\odot v(0) = A^3 M_B , \tag{2.43}$$

with

$$\tilde{r}_0 = r_0 / A \ . \tag{2.44}$$

The consequence of this result is that $M_\nu r_0^3$ and M_B/M_ν are independent of A. We now show that the Thomas-Fermi equation also has a scaling property. In this case Zr_0^3, Nr_0^3 and N/Z will all be independent of A.

$$\begin{aligned} \tilde{N} &= \int_0^{x_0/A} \left[A^3 v(Ax)\right]^{3/2} x^{1/2} dx \\ &= A^3 \int_0^{x_0} [v(x)]^{3/2} x^{1/2} dx \\ &= A^3 N \end{aligned} \tag{2.45}$$

and

$$\tilde{Z} = \tilde{v}(0) = A^3 v(0) = A^3 Z \ . \tag{2.46}$$

For electrons orbiting a nucleus we can write the number of electrons N as

$$N = \int_0^{x_0} [v(x)]^{3/2} \sqrt{x} dx \tag{2.47}$$

and the screened charge of the nucleus will be

$$Z - N = -x_0 v'(x_0) \ , \tag{2.48}$$

with Z being the atomic number.

For a cold neutrino star, the equivalent analysis yields

$$M = M_\odot \int_0^{x_0} [v(x)]^{3/2} \sqrt{x} dx = -x_0 v'(x_0) M_\odot \ . \tag{2.49}$$

Therefore the total mass is related to its derivative of $v(x)$ at x_0 by [7, 9]

$$M = M_B + M_\nu = -x_0 v'(x_0) M_\odot \,. \tag{2.50}$$

2.4 Results for the Lané-Emden Equation

We already know that $M_\nu r_0^3$, $M_B r_0^3$ and M_B/M_ν are constant with respect to the homology transformation. It is thus sufficient to study one example of the differential equation (2.36), for each ratio of M_B/M_ν.

For a pure neutrino star without any baryonic matter at the centre, we have $M_B = 0$ and $v(0) = 0$. Choosing $v'(0) = 1$ and solving the differential equation, we obtain the E-type solution [7, 9, 26]. In this case the second zero of $v(x)$ is at $x_0 = 3.65375$ and the slope at this position is given by $v'(x_0) = -0.742813$, which means that $M = M_\nu = -x_0 v'(x_0) M_\odot = 2.71405 M_\odot$.

If we now integrate back to the origin, starting with the same value $v(x_0) = 0$ and a different slope $-x_0 v'(x_0) < 2.71405$, we arrive at $v(0) > 0$, an M-type solution which corresponds to $M_B > 0$. However, this still yields a finite M_ν.

For $-x_0 v'(x_0) > 2.71405$, $v(x)$ will have at least one zero in the interval $0 < x < x_0$, which obviously represents a gravitationally unstable and thus unphysical F-type solution [7, 26] that must be discarded. These scenarios are plotted in fig. (2.1).

Now, due to the homology theorem, the radii and masses of the neutrino stars or halos obey the relation

$$M_\nu r_0^3 = C_\nu(\mu) \,, \tag{2.51}$$

where $C_\nu(\mu)$ depends merely on the ratio $\mu = M_\nu/M_B$ and can be calculated solving eq. (2.36). From eqs. (2.38) and (2.49) we can find C_ν in the absence of any baryonic matter ($M_B = 0$), yielding

$$\begin{aligned} C_\nu(\mu) &= -x_0 v'(x_0) M_\odot (x_0 R_0)^3 \\ &= x_0^4 v'(x_0) R_0^3 M_\odot \end{aligned}$$

$$
\begin{aligned}
&= x_0^4 v'(x_0) M_\odot \left(\frac{3\pi\hbar^3}{4\sqrt{2} m_\nu^4 g_\nu G^{3/2} M_\odot^{1/2}} \right)^2 && (2.52) \\
&= \frac{9\pi^2 x_0^4 v'(x_0) \hbar^6}{32 g_\nu^2 m_\nu^8 G^3} \\
C_\nu(\mu) &= 5.4446 \cdot 10^{80} \ \mathrm{kg.m}^3 \left(\frac{17.2 \ \mathrm{keV}}{m_\nu c^2} \right)^8 \left(\frac{2}{g_\nu} \right)^2 \\
&= 323.25 \ M_\odot \ (\mathrm{lyr})^3 \left(\frac{17.2 \ \mathrm{keV}}{m_\nu c^2} \right)^8 \left(\frac{2}{g_\nu} \right)^2 && (2.53)
\end{aligned}
$$

2.5 Black Hole Mass Limit

A neutrino star will become a black hole when the escape velocity from the surface reaches the speed of light, which yields the Schwarzschild radius

$$
r_s = \frac{2GM_s}{c^2} \ . \tag{2.54}
$$

From eq. (2.53) we obtain for this black hole limit

$$
M_s r_s^3 = \frac{9\pi^2 x_0^4 \mid v'(x_0) \mid \hbar^6}{32 g_\nu^2 m_\nu^8 G^3} \tag{2.55}
$$

or

$$
\begin{aligned}
M_s &= \frac{\sqrt{3}\pi x_0 \mid v'(x_0) \mid^{1/2} \hbar^{3/2} c^{3/2}}{4\sqrt{g_\nu} m_\nu^2 G^{3/2}} \\
&= 1.015 \cdot 10^{10} \ M_\odot \left(\frac{17.2 \ \mathrm{keV}}{m_\nu c^2} \right)^2 \left(\frac{2}{g_\nu} \right)^{1/2} . && (2.56)
\end{aligned}
$$

This corresponds to a Schwarzschild radius of

$$
r_s = \frac{2GM_s}{c^2} = 1.16 \left(\frac{17.2 \ \mathrm{keV}}{m_\nu c^2} \right)^2 g^{-1/2} \ \mathrm{ld} \ . \tag{2.57}
$$

Of course, this result is not completely correct, as we have used the Newtonian approximation with $v_F \ll c$ to obtain it. Nevertheless it shows where the Newtonian approximation fails, as can be proven in the correct general relativistic treatment using the Tolman-Oppenheimer-Volkoff equation [29, 30].

The equations derived in Chapter 2.2 apply to massive neutrino concentrations. We assumed that neutrino matter in such stars is in a state of lowest energy and therefore a degenerate Fermi-gas. Degeneracy pressure will prevent gravitational collapse of a neutrino star, as long as the actual radius of the star is greater than the Schwarzschild radius. This is fulfilled for masses smaller than $1.015 \cdot 10^{10} M_\odot$ (eq. (2.56)) corresponding to a Schwarzschild radius of 1.16 ld (eq. (2.57)). This is for the static non-relativistic case. General relativistic corrections will reduce the limit, while the inclusion of rotation will increase it again. There might be other small corrections due to exchange and finite temperature effects as well as baryonic impurities. One can thus conclude that stars consisting of 10 to 25 keV/c^2 neutrinos can exist up to masses of a few times $10^9 M_\odot$, which is about the upper limit for the masses of the "conjectured" black holes (see Chapter 2.6) at the centre of galaxies.

2.6 Conclusion

There is increasing evidence for the existence of supermassive compact dark objects in the centre of galaxies, for example in two of our neighbouring galaxies, M31 [9, 12, 13, 14] and M32 [10, 11, 13] as well as NGC 4594 [15] and NGC 3115 [16]. All these compact dark objects have masses in the range of $10^{6.5}$ to $10^{9.5} M_\odot$. There is also further evidence which suggest that active galactic nuclei and quasi-stellar objects are powered by compact energy sources in the mass range of 10^8 to $10^9 M_\odot$. We can estimate the sizes of the light emitting regions of quasi-stellar objects from the time variability of the the energy output as being of the order of a light day. However, the closest of all these compact objects seems is at the centre of our own galaxy, the Milky Way [17, 18], with a mass of about $2.5 \cdot 10^6 M_\odot$. It is usually identified with by the radio point source SgrA* [27, 28] or the centroid of the complex infrared source IRS 16 [31]. This compact dark object must have a radius of 30 light days or less as Genzel et al. [32, 33] have determined from the motion of stars close to SgrA*.

There is no known state of baryonic matter that seems to fit the requirements of a mass between $10^{6.5}$ and $10^{9.5} M_{\odot}$ and a size of less than a few tens to one light day. The current explanation of these supermassive compact dark objects, is that they are black holes, since *all* other conventional explanations fail and there are no other forms of matter that could be arranged into such massive and compact objects.

However, it is important to mention that all the dynamical evidence is also consistent with the hypothesis that these compact dark objects are in fact degenerate neutrino stars with neutrino masses in the range of 10 to 25 keV/c^2. Indeed on the one hand, a neutrino star of $M = 3 \cdot 10^9 M_{\odot}$ will behave almost as a supermassive black hole, since the escape velocity from the surface of the neutrino star is a sizable fraction of the velocity of light. On the other hand a neutrino star of $M = 2.5 \cdot 10^6 M_{\odot}$, as in the centre of our galaxy, would have a radius of about 20 to 30 ld and an escape velocity of about 600 km/s, which is as far as you can possibly get from a black hole scenario.

Thus for a $2.5 \cdot 10^6 M_{\odot}$ neutrino star the potential will be much shallower, and the free fall energy of accreting matter onto this object much smaller than in the black hole scenario. Therefore, replacing the $2.5 \cdot 10^6 M_{\odot}$ black hole by a neutrino star of the same mass, could resolve the longstanding puzzle, known as the blackness problem, that the X-ray luminosity of the centre of our galaxy is far too small to be consistent with a supermassive black hole. It would also explain the radio spectrum of SgrA* much better than the black hole scenario. As heavy neutrinos will decay into light neutrinos through standard weak and electromagnetic interaction by emission of 5 to 12.5 keV photons, neutrino stars could perhaps be observable.

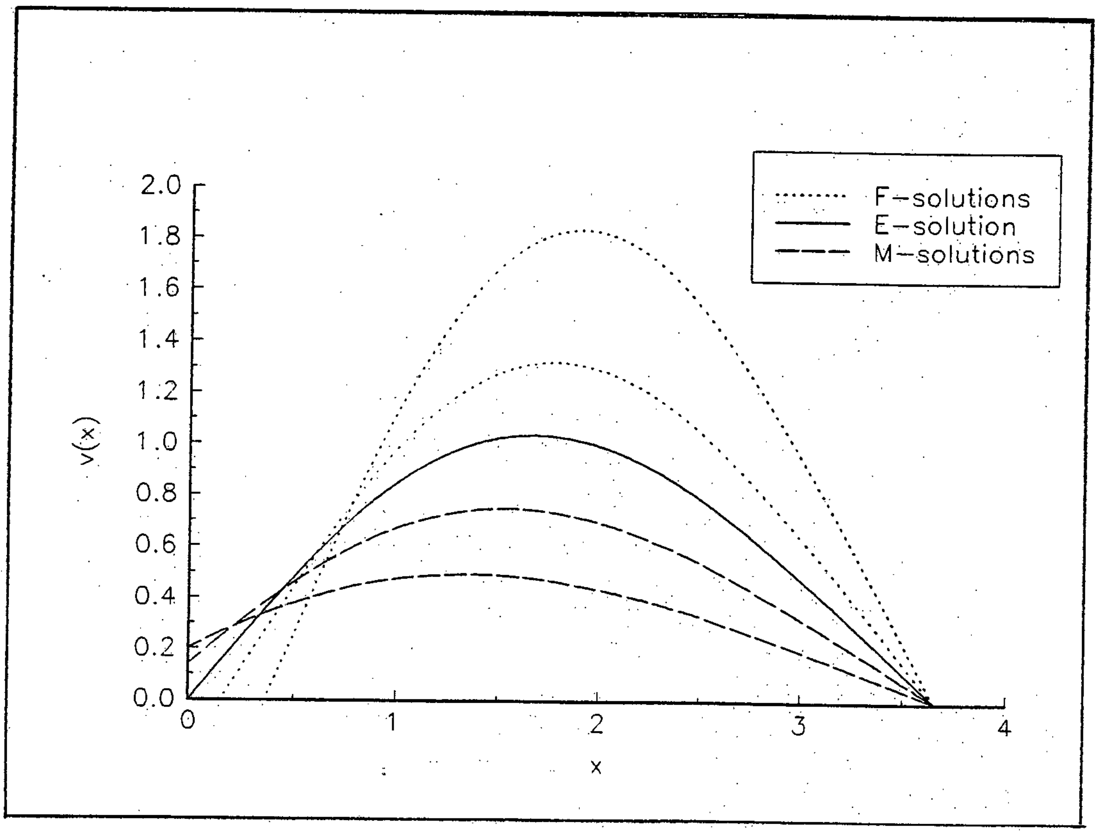

Figure 2.1: Typical solutions of the Lané-Emden differential equation. Pure neutrino star (E-solution), and pointlike baryonic star with neutrino halo (M-Solutions). The F-Solutions are unphysical.

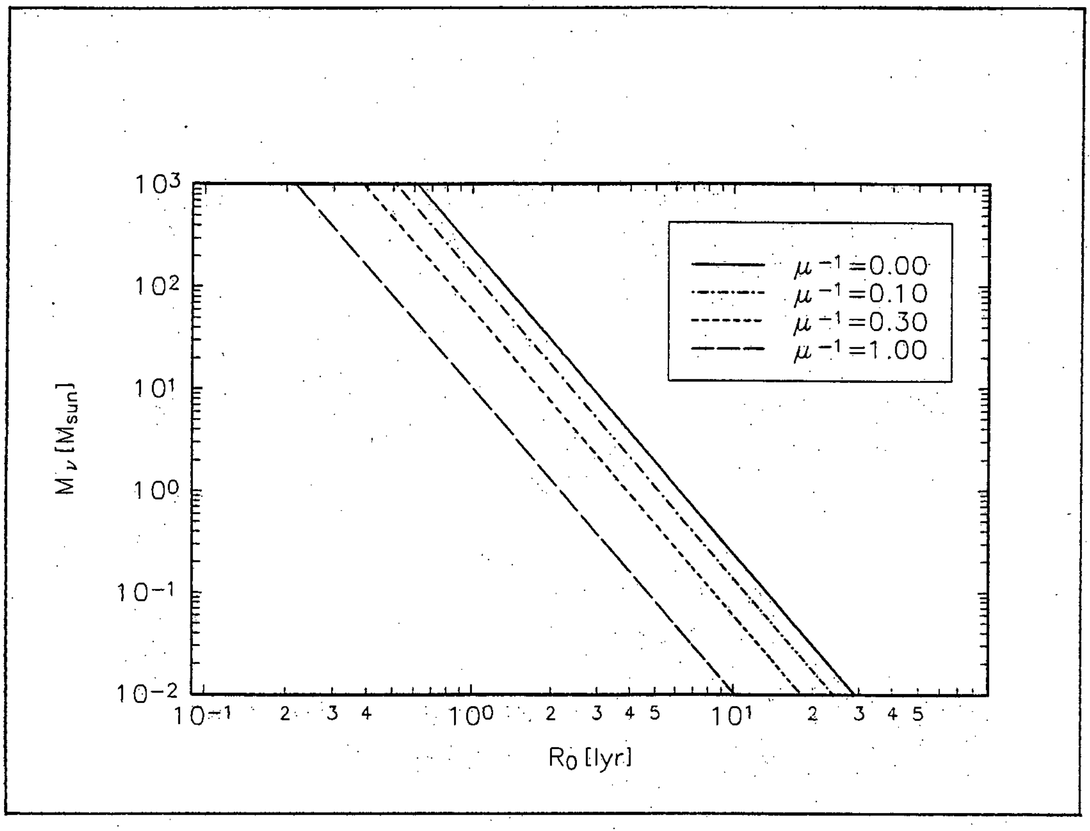

Figure 2.2: Masses and radii of degenerate neutrino halos around pointlike baryonic stars for various neutrino to baryon mass ratios. In this figure, $\mu = M_\nu/M_B$, $g_\nu = 2$ and $m_\nu = 17.2$ keV/c^2.

Chapter 3

Heavy Neutrino Matter at Finite Temperature

Having presented in the first two chapters, degenerate electron and neutrino matter at zero temperature, we now focus on the specific topic of this thesis: *heavy neutrino matter at finite temperature.* We will describe a neutrino star in terms of the Fermi-Dirac distribution, in which neutrinos are no longer degenerate but have a finite temperature T. We also assume that the neutrino gas is enclosed in a spherical cavity.

3.1 The Lané-Emden Equation for Heavy Neutrino Matter at Finite Temperature

The number density of neutrinos with momentum between p and $p + dp$ is given by

$$dn_\nu = \frac{g_\nu p^2 dp}{2\pi^2\hbar^3(e^{(E-\mu)/kT} + 1)} , \tag{3.1}$$

where usually $E = p^2/2m$ is the kinetic energy and μ the chemical potential [25]. Here, we are concerned with neutrinos moving in a gravitational potential $V(r)$

and therefore we have

$$E = \frac{p^2}{2m} + V(r) \; . \tag{3.2}$$

Integrating over all momenta we arrive at the expression for the (total) neutrino number density

$$n_\nu(r) = \frac{g_\nu}{2\pi^2\hbar^3} \int\limits_0^\infty \frac{p^2 dp}{1 + e^{\left(\frac{p^2}{2m_\nu} + V(r) - \mu\right)/kT}} \; . \tag{3.3}$$

We now make in the expression for the neutrino number density, eq. (3.3), the substitutions

$$\xi = \frac{p^2}{2m_\nu kT} \tag{3.4}$$

and

$$\eta = \frac{\mu - V(r)}{kT} \; , \tag{3.5}$$

and find the respective differentials

$$d\xi = \frac{pdp}{m_\nu kT}$$

or

$$dp = \frac{m_\nu kT d\xi}{(2\xi m_\nu kT)^{1/2}} \; . \tag{3.6}$$

η will be zero at r_0, where r_0 is the classical turning point defined through

$$\mu - V(r_0) = 0 \; . \tag{3.7}$$

We may rewrite eq. (3.3) as

$$\begin{aligned} n_\nu(r) &= \frac{g_\nu}{2\pi^2\hbar^3} \int\limits_0^\infty \underbrace{2\xi m_\nu kT}_{p^2} \underbrace{\frac{m_\nu kT d\xi}{(2\xi m_\nu kT)^{1/2}}}_{dp} \frac{1}{1 + e^{\xi - \eta}} \\ &= \frac{g_\nu}{4\pi^2\hbar^3} (2m_\nu kT)^{3/2} \underbrace{\int\limits_0^\infty \frac{\sqrt{\xi} d\xi}{1 + e^{\xi - \eta}}}_{I_{\frac{1}{2}}(\eta)} \\ &= \frac{g_\nu}{4\pi^2\hbar^3} (2m_\nu kT)^{3/2} I_{\frac{1}{2}}(\eta) \; . \end{aligned} \tag{3.8}$$

Here we have defined

$$I_n(\eta) = \int_0^\infty \frac{\xi^n d\xi}{1+e^{\xi-\eta}} . \tag{3.9}$$

By using the expressions of eq. (3.5) once again, the number density, eq. (3.8), can be rewritten as

$$n_\nu(r) = \frac{g_\nu}{4\pi^2\hbar^3}(2m_\nu kT)^{3/2} I_{\frac{1}{2}}\left(\frac{\mu - V(r)}{kT}\right) . \tag{3.10}$$

Now that we have our expression for number density in a convenient form, we introduce u as the "reduced" potential

$$\eta = \frac{\mu - V(r)}{kT} = \frac{m_\nu}{kT}\frac{u}{r} \tag{3.11}$$

or

$$\frac{u}{r} = \frac{\mu - V(r)}{m_\nu} = \frac{kT}{m_\nu}\eta . \tag{3.12}$$

If the baryonic star can be approximated by a point source, we can use Poisson's equation as in the case of the degenerate neutrino star. From this approximation we arrive at the boundary condition of

$$u(0) = GM_B. \tag{3.13}$$

If there is no baryonic matter present, $u(0) = 0$.

Poisson's equation will then be valid as long as $r \neq 0$. The neutrino density is then used as the source term

$$\Delta\frac{u(r)}{r} = -4\pi G m_\nu n_\nu = \frac{1}{r}\frac{d^2u(r)}{dr^2} . \tag{3.14}$$

By substituting the expression for neutrino number density, eq. (3.10), we find

$$\frac{1}{r}\frac{d^2u}{dr^2} = -4\pi G m_\nu \frac{g_\nu}{4\pi^2\hbar^3}(2m_\nu kT)^{3/2} I_{\frac{1}{2}}\left(\frac{m_\nu}{kT}\frac{u}{r}\right) , \tag{3.15}$$

and thus arrive at the non-linear Lané-Emden differential equation [7, 26],

$$\frac{d^2u(r)}{dr^2} = -\frac{G m_\nu g_\nu}{\pi\hbar^3}(2m_\nu kT)^{3/2} r I_{\frac{1}{2}}\left(\frac{m_\nu}{kT}\frac{u(r)}{r}\right) . \tag{3.16}$$

It is noted that one can obtain the same differential equation, eq. (3.16), with a more conventional approach from the assumption of hydrostatic equilibrium in the gravitational field of the neutrinos [7, 38]. However, this will not be discussed in this thesis.

Here again we introduce a dimensionless form of the differential equation (3.16). This can be derived by using the same substitutions as in Chapter 2, eq. (2.35), namely

$$\begin{aligned} v &= \frac{u}{GM_\odot} \\ x &= \frac{r}{R_0} \, . \end{aligned} \tag{3.17}$$

By combining these expressions with eq. (3.11) we find

$$\eta = \frac{m_\nu}{kT}\frac{u(r)}{r} = \underbrace{\frac{m_\nu}{kT}\frac{GM_\odot}{R_0}}_{\beta}\frac{v(x)}{x} = \beta\frac{v(x)}{x} \, . \tag{3.18}$$

We introduce the inverse temperature $\beta = T_0/T$ with the temperature scale T_0 given as

$$T_0 = \frac{m_\nu GM_\odot}{kR_0} \, . \tag{3.19}$$

Our differential equation, eq. (3.16) now becomes the dimensionless Lané-Emden equation for heavy neutrino matter at finite temperature

$$\frac{d^2v(x)}{dx^2} = -\frac{3}{2}\beta^{-3/2}xI_{\frac{1}{2}}\left(\beta\frac{v(x)}{x}\right) \, . \tag{3.20}$$

From eqs. (3.13) and (3.17), our boundary conditions can also be expressed in terms of v, i.e.

$$v(0) = \frac{M_B}{M_\odot} \, , \tag{3.21}$$

where we have assumed that there is a compact seed at the centre.

As a check on our finite temperature equation, we can try to recover the Lané-Emden equation for degenerate neutrino stars, as derived in Chapter 2. In fact, in the limit of $T \to 0$ the integral $I_n(\eta)$ can be approximated by

$$I_{\frac{1}{2}}(\eta) = \int_0^\infty \frac{\sqrt{\xi}d\xi}{1+e^{\xi-\eta}} \approx \int_0^\eta \xi^{1/2}d\xi = \frac{\eta^{3/2}}{3/2} = \frac{2}{3}\eta^{3/2} . \tag{3.22}$$

Thus by starting with the finite temperature result and using the approximation

$$\begin{aligned} \frac{d^2v(x)}{dx^2} &= -\frac{3}{2}\beta^{-3/2}xI_{\frac{1}{2}}\left(\beta\frac{v(x)}{x}\right) \\ &= -\frac{3}{2}\beta^{-3/2}x\left[\frac{2}{3}\left(\beta\frac{v(x)}{x}\right)^{3/2}\right] \\ &= -\frac{v(x)^{3/2}}{\sqrt{x}} , \end{aligned} \tag{3.23}$$

we arrive at the Lané-Emden equation, eq. (2.36).

3.2 Remarks on the Finite Temperature Lané-Emden equation

Similar to the Lané-Emden equation for the cold case, eq. (2.36), it can be shown that this equation also has a scaling property: if $v(x)$ is a solution of eq. (3.20) at a temperature T and a cavity radius R, then $\tilde{v}(x) = A^3v(Ax)$ with $(A > 0)$ is also a solution at the temperatures $\tilde{T} = A^4T$ and $\tilde{R} = R/A$ or in mathematical terms:

If

$$\frac{d^2v}{dx^2} = -\frac{3}{2}\beta^{-3/2}xI_{\frac{1}{2}}\left(\beta\frac{v}{x}\right) \tag{3.24}$$

then

$$\frac{d^2\tilde{v}}{dx^2} = -\frac{3}{2}\tilde{\beta}^{-3/2}xI_{\frac{1}{2}}\left(\tilde{\beta}\frac{v}{x}\right) , \tag{3.25}$$

where $\tilde{v}(x) = A^3v(Ax)$ and $\tilde{\beta} = \beta A^{-4}$.

Proof:

On the left hand side we have from eq. (2.41)

$$\frac{d^2\tilde{v}}{dx^2} = A^5 \frac{d^2v(Ax)}{d(Ax)^2} .$$

and by manipulating the right hand side

$$\begin{aligned}
& -\frac{3}{2}\tilde{\beta}^{-3/2} x I_{\frac{1}{2}}\left(\tilde{\beta}\frac{\tilde{v}}{x}\right) \\
= & -\frac{3}{2}\tilde{\beta}^{-3/2} A A^{-1} x I_{\frac{1}{2}}\left(\tilde{\beta}\frac{\tilde{v}}{x}\right) \\
= & -\frac{3}{2}(\beta A^{-4})^{-3/2} A A^{-1} x I_{\frac{1}{2}}\left(\beta A^{-4} A^4 \frac{v(Ax)}{Ax}\right) \\
= & -\frac{3}{2}\beta^{-3/2} A^6 A^{-1} x I_{\frac{1}{2}}\left(\beta\frac{v(Ax)}{Ax}\right) \\
= & -\frac{3}{2}\beta^{-3/2} Ax I_{\frac{1}{2}}\left(\beta\frac{v(Ax)}{Ax}\right) A^5 \\
= & A^5 \frac{d^2v(Ax)}{d(Ax)^2}
\end{aligned} \tag{3.26}$$

we arrive at the same result.

In the next few chapters we will derive other important thermodynamical properties, i.e. energy, free energy and entropy. To be able to write these quantities in a simple form, we need the neutrino number density and pressure in other convenient forms. This is derived first.

Section 3.8 provides a summary of the results that have been derived thus far and which will be derived in the next few sections.

3.3 Further Manipulation of the Neutrino Number Density n_ν

From eq. (3.18) we find

$$kT = \frac{m_\nu G M_\odot}{\beta R_0} . \tag{3.27}$$

Substituting this expression in eq. (3.10), the neutrino number density may be written as

$$\begin{aligned} n_\nu(x) &= \frac{g_\nu}{4\pi^2\hbar^3}\left(2m_\nu \frac{m_\nu G M_\odot}{\beta R_0}\right)^{3/2} I_{\frac{1}{2}}\left(\beta\frac{v}{x}\right) \\ n_\nu(x) &= \frac{g_\nu m_\nu^3}{\sqrt{2}\pi^2\hbar^3}\left(\frac{GM_\odot}{\beta R_0}\right)^{3/2} I_{\frac{1}{2}}\left(\beta\frac{v}{x}\right) . \\ &= \frac{g_\nu m_\nu^3}{\sqrt{2}\pi^2\hbar^3}\left(\frac{GM_\odot}{R_0}\right)^{3/2} \beta^{-3/2} I_{\frac{1}{2}}\left(\beta\frac{v}{x}\right) \\ &= \frac{g_\nu m_\nu^3}{\sqrt{2}\pi^2\hbar^3}\frac{G^{3/2}M_\odot^{3/2}R_0^{3/2}}{R_0^3}\beta^{-3/2} I_{\frac{1}{2}}\left(\beta\frac{v}{x}\right) . \end{aligned} \tag{3.28}$$

By raising the power of eq. (2.37) to 3/2 on both sides we find

$$R_0^{3/2} = \left(\frac{3\pi\hbar^3}{4\sqrt{2}m_\nu^4 g_\nu G^{3/2} M_\odot^{1/2}}\right) \tag{3.29}$$

and using this result, the number density becomes

$$n_\nu(x) = \frac{g_\nu m_\nu^3}{\sqrt{2}\pi^2\hbar^3}\frac{G^{3/2}M_\odot^{3/2}}{R_0^3}\left(\frac{3\pi\hbar^3}{4\sqrt{2}m_\nu^4 g_\nu G^{3/2} M_\odot^{1/2}}\right)\beta^{-3/2} I_{\frac{1}{2}}\left(\beta\frac{v}{x}\right) , \tag{3.30}$$

which simplifies to

$$n_\nu(x) = \frac{M_\odot}{m_\nu R_0^3}\frac{3}{8\pi}\beta^{-3/2} I_{\frac{1}{2}}\left(\beta\frac{v}{x}\right) . \tag{3.31}$$

3.4 Derivation of the Neutrino Pressure P

From eqs. (3.11) and (3.12) the potential can be expressed as

$$V(r) = \mu - m_\nu\frac{u(r)}{r} = \mu - m_\nu\frac{GM_\odot}{R_0}\frac{v(x)}{x} . \tag{3.32}$$

The kinetic energy density is

$$\varepsilon_{kin}(r) = \frac{g_\nu}{2\pi^2\hbar^3 2m_\nu}\int_0^\infty \frac{p^4 dp}{1+e^{(\frac{p^2}{2m_\nu}+V(r)-\mu)/kT}} . \tag{3.33}$$

For a interacting non-relativistic gas we use the same result as in Chapter 2, which is derived in Appendix A, namely

$$PV = \frac{2}{3}E_{\text{kin}} . \tag{3.34}$$

Thus in terms of energy density

$$\begin{aligned} P_\nu(r) &= \frac{2}{3}\varepsilon_{\rm kin}(r) \\ &= \frac{g_\nu}{6\pi^2\hbar^3 m_\nu}\int_0^\infty \frac{p^4 dp}{1+e^{(\frac{p^2}{2m_\nu}+V(r)-\mu)/kT}} \ . \end{aligned} \tag{3.35}$$

By using the same substitutions as in the case of the number density, eq. (3.5) the pressure may be written as

$$P_\nu(r) = \frac{g_\nu}{6\pi^2\hbar^3 2m_\nu}(2m_\nu kT)^{5/2}\underbrace{\int_0^\infty \frac{\xi^{3/2}}{1+e^{\xi-\eta}}}_{I_{\frac{3}{2}}(\eta)} \ . \tag{3.36}$$

Now by using eq. (3.18) and substituting kT, (eq. (3.27)), we find the expression for the pressure to be

$$\begin{aligned} P_\nu(x) &= \frac{\sqrt{2}g_\nu m_\nu^4}{3\pi^2\hbar^3}\left(\frac{GM_\odot}{\beta R_0}\right)^{5/2} I_{\frac{3}{2}}\left(\beta\frac{v}{x}\right) &(3.37)\\ &= \frac{\sqrt{2}g_\nu m_\nu^4}{3\pi^2\hbar^3}\left(\frac{GM_\odot}{R_0}\right)^{5/2}\beta^{-5/2} I_{\frac{3}{2}}\left(\beta\frac{v}{x}\right) \ . &(3.38) \end{aligned}$$

We now introduce kT_0 (from eq. (3.19)) into the equation and find

$$P_\nu(x) = \frac{\sqrt{2}g_\nu m_\nu^3}{3\pi^2\hbar^3}\frac{G^{3/2}M_\odot^{3/2}kT_0}{R_0^{3/2}}\beta^{-5/2} I_{\frac{3}{2}}\left(\beta\frac{v}{x}\right) \ . \tag{3.39}$$

Finally, if we multiply by $R_0^{3/2}/R_0^{3/2}$ and then replace $R_0^{3/2}$ (eq. (3.29)) in the numerator, the pressure is found to be

$$\begin{aligned} P_\nu(x) &= \frac{\sqrt{2}g_\nu m_\nu^3}{3\pi^2\hbar^3}\frac{G^{3/2}M_\odot^{3/2}kT_0 R_0^{3/2}}{R_0^3}\beta^{-5/2} I_{\frac{3}{2}}\left(\beta\frac{v}{x}\right) \\ &= \frac{\sqrt{2}g_\nu m_\nu^3}{3\pi^2\hbar^3}\frac{G^{3/2}M_\odot^{3/2}kT_0}{R_0^3}\left(\frac{3\pi\hbar^3}{4\sqrt{2}m_\nu^4 g_\nu G^{3/2}M_\odot^{1/2}}\right)\beta^{-5/2} I_{\frac{3}{2}}\left(\beta\frac{v}{x}\right) \end{aligned}$$

which simplifies to the final expression

$$P_\nu(x) = \frac{M_\odot kT_0}{m_\nu R_0^3}\frac{1}{4\pi}\beta^{-5/2} I_{\frac{3}{2}}\left(\beta\frac{v}{x}\right) \ . \tag{3.40}$$

3.5 The Free Energy F

3.5.1 Derivation of the Free Energy

From [1, 2, 4] the free energy functional is

$$F = -W + \mu N_\nu - g_\nu kT \int d^3r \frac{d^3p}{(2\pi\hbar)^3} \ln\left[1 + e^{\left(-\frac{p^2}{2m_\nu kT} - \frac{V'(r)}{kT} + \frac{\mu}{kT}\right)}\right] \tag{3.41}$$

where

$$\begin{aligned} W &= -\frac{1}{2} G m_\nu^2 \int d^3r d^3r' \frac{n_\nu(r) n_\nu(r')}{|\mathbf{r} - \mathbf{r}'|} \\ &= \frac{1}{2} \int d^3r n_\nu V'(r) \end{aligned} \tag{3.42}$$

and

$$\begin{aligned} V'(r) &= -G m_\nu^2 \int \frac{n_\nu(r')}{|\mathbf{r} - \mathbf{r}'|} d^3r' \\ &= V(r) + \frac{m_\nu G M_B}{r} \\ &= \left(\mu - m_\nu \frac{u(r)}{r}\right) + m_\nu \frac{u(0)}{r} . \end{aligned} \tag{3.43}$$

By substituting the latter expression into the first, W becomes

$$\begin{aligned} W &= \frac{1}{2} \int d^3r n_\nu(r) \left[\mu - m_\nu \frac{u(r)}{r} + m_\nu \frac{u(0)}{r}\right] \\ &= \frac{1}{2} \int d^3r \mu n_\nu(r) - \frac{1}{2} \int d^3r n_\nu(r) \left[m_\nu \frac{u(r)}{r} - m_\nu \frac{u(0)}{r}\right] \end{aligned} \tag{3.44}$$

and by using this in the free energy functional

$$\begin{aligned} F &= \frac{1}{2} \mu N_\nu + \underbrace{\frac{1}{2} \int d^3r n_\nu(r) \left[m_\nu \frac{u(r)}{r} - m_\nu \frac{u(0)}{r}\right]}_{\mathcal{K}_2} \\ &\quad \underbrace{- g_\nu kT \int d^3r \frac{d^3p}{(2\pi\hbar)^3} \ln\left[1 + e^{\left(-\frac{p^2}{2m_\nu kT} - \frac{V'(r)}{kT} + \frac{\mu}{kT}\right)}\right]}_{\mathcal{K}_3} . \end{aligned} \tag{3.45}$$

To simplify matters, we evaluate the terms separately and add them up again afterwards.

The Integral $\mathcal{K}_2$

First let us look at the integral $\mathcal{K}_2$. By introducing kT into the equation and substituting eq. (3.11) and the number density (eq. (3.31)), we arrive at

$$\begin{aligned}\mathcal{K}_2 &= \frac{1}{2}\int d^3r n_\nu(r) m_\nu \left[\frac{u(r)}{r} - \frac{u(0)}{r}\right] \\ &= \frac{1}{2}kT\int d^3r n_\nu(r)\frac{m_\nu}{kT}\left[\frac{u(r)}{r} - \frac{u(0)}{r}\right] \\ &= \frac{1}{2}\frac{g_\nu kT}{4\pi^2\hbar^3}(2m_\nu kT)^{3/2}\int d^3r I_{\frac{1}{2}}\left(\frac{m_\nu}{kT}\frac{u}{r}\right)\frac{m_\nu}{kT}\left[\frac{u(r)}{r} - \frac{u(0)}{r}\right] .\end{aligned} \tag{3.46}$$

We want to write the integral in terms of our dimensionless variables, v and x. This is done by using eqs. (2.35) and (3.18):

$$\mathcal{K}_2 = \frac{1}{2}\frac{g_\nu kT}{4\pi^2\hbar^3}(2m_\nu kT)^{3/2}\beta R_0^3\int d^3x I_{\frac{1}{2}}\left(\beta\frac{v}{x}\right)\left[\frac{v(x)}{x} - \frac{v(0)}{x}\right] . \tag{3.47}$$

We now write the equation in terms of the neutrino number density, using eq. (3.31)

$$\mathcal{K}_2 = \frac{1}{2}\frac{g_\nu kT}{4\pi^2\hbar^3}(2m_\nu kT)^{3/2}\beta\left(\frac{8m_\nu R_0^3\pi\beta^{3/2}}{3M_\odot}\right)\int d^3x n_\nu(x)\left[\frac{v(x)}{x} - \frac{v(0)}{x}\right] . \tag{3.48}$$

By substituting for β (eq. (3.18)), multiplying by T_0/T_0, and then substituting T_0 in the denominator (eq. (3.19)) we can simplify the latter equation to

$$\mathcal{K}_2 = \frac{1}{2}kT_0R_0^3\int d^3x n_\nu(x)\left[\frac{v(x)}{x} - \frac{v(0)}{x}\right] . \tag{3.49}$$

The integral $\mathcal{K}_3$

The third term in the Free Energy functional is given by

$$\begin{aligned}\mathcal{K}_3 &= -g_\nu kT\int d^3r\frac{d^3p}{(2\pi\hbar)^3}\ln\left[1 + e^{\left(-\frac{p^2}{2m_\nu kT} - \frac{V'(r)}{kT} + \frac{\mu}{kT}\right)}\right] \\ &= -g_\nu kT\int d^3r\frac{4\pi p^2 dp}{(2\pi\hbar)^3}\ln\left[1 + e^{\left(-\frac{p^2}{2m_\nu kT} - \frac{V'(r)}{kT} + \frac{\mu}{kT}\right)}\right] .\end{aligned} \tag{3.50}$$

By making use of partial integration

$$\int f'g = fg - \int fg' , \tag{3.51}$$

where we have conveniently chosen

$$\frac{df}{dp} = p^2 \tag{3.52}$$

and

$$g = \ln\left[1 + e^{\left(-\frac{p^2}{2m_\nu kT} - \frac{V'(r)}{kT} + \frac{\mu}{kT}\right)}\right] \tag{3.53}$$

we can substitute the following in eq. (3.51):

$$\begin{aligned} f &= \frac{1}{3}p^3 && (3.54) \\ \frac{\partial g}{\partial p} &= \frac{-\frac{p}{m_\nu kT}}{\left[1 + e^{\left(\frac{p^2}{2m_\nu kT} + \frac{V'(r)}{kT} - \frac{\mu}{kT}\right)}\right]} . && (3.55) \end{aligned}$$

Thus eq. (3.50) becomes

$$\begin{aligned} \mathcal{K}_3 &= -g_\nu kT \int d^3r \frac{4\pi p^3}{3(2\pi\hbar)^3} \ln\left[1 + e^{\left(-\frac{p^2}{2m_\nu kT} - \frac{V'(r)}{kT} + \frac{\mu}{kT}\right)}\right]\Bigg|_0^\infty \\ &\quad + g_\nu kT \int d^3r \int \frac{4\pi p^3 dp}{3(2\pi\hbar)^3} \frac{-\frac{p}{m_\nu kT}}{\left[1 + e^{\left(\frac{p^2}{2m_\nu kT} + \frac{V'(r)}{kT} - \frac{\mu}{kT}\right)}\right]} . \end{aligned} \tag{3.56}$$

The first part of this equation equals zero since $\ln(1+x) \approx x$ with $x \ll 1$ and $\lim_{p\to\infty} p^3 e^{-p^2} = \lim_{p\to\infty} \frac{p^3}{e^{p^2}} = 0$. Thus the first term of eq. (3.56) drops away, and we are left with

$$\begin{aligned} \mathcal{K}_3 &= -g_\nu kT \int d^3r \frac{2 \cdot 4\pi p^2 dp}{3(2\pi\hbar)^3} \frac{\frac{p^2}{2m_\nu kT}}{\left[1 + e^{\left(\frac{p^2}{2m_\nu kT} + \frac{V'(r)}{kT} - \frac{\mu}{kT}\right)}\right]} \\ &= -\frac{2}{3} g_\nu kT \int d^3r \frac{4\pi p^2 dp}{(2\pi\hbar)^3} \frac{p^2}{2m_\nu kT} \frac{1}{\left[1 + e^{\left(\frac{p^2}{2m_\nu kT} + \frac{V'(r)}{kT} - \frac{\mu}{kT}\right)}\right]} . \end{aligned} \tag{3.57}$$

Now by making use of the substitution $x = p^2/(2m_\nu kT)$ and substituting the expression for the potential, the above equation becomes

$$\mathcal{K}_3 = -\frac{2}{3} g_\nu kT \frac{4\pi}{(2\pi\hbar)^3} \int d^3r \underbrace{(2m_\nu kTx)}_{p^2} \underbrace{(2m_\nu kT)^{1/2}\frac{1}{2}x^{-1/2}dx}_{dp} \frac{x}{\left[1 + e^{\left(x - \frac{m_\nu}{kT}\frac{u}{r}\right)}\right]}$$

$$\mathcal{K}_3 = -\frac{1}{2}\frac{2}{3}\frac{g_\nu kT}{(2\pi\hbar)^3}4\pi(2m_\nu kT)^{3/2}\int d^3r \underbrace{\int \frac{x^{3/2}dx}{\left[1+e^{\left(x-\frac{m_\nu}{kT}\frac{u}{r}\right)}\right]}}_{I_{\frac{3}{2}}\left(\frac{m_\nu}{kT}\frac{u(r)}{r}\right)} . \quad (3.58)$$

First we write this in terms of our dimensionless units v and x, (eqs. (2.35) and (3.18)) and then in terms of $P_\nu(x)$ using eq. (3.40)

$$\begin{aligned}\mathcal{K}_3 &= -\frac{g_\nu kT}{6\pi^2\hbar^3}(2m_\nu kT)^{3/2}R_0^3\int d^3x I_{\frac{3}{2}}\left(\beta\frac{v}{x}\right) \\ &= -\frac{g_\nu kT}{6\pi^2\hbar^3}(2m_\nu kT)^{3/2}\left(\frac{4\pi m_\nu R_0^3}{kM_\odot T_0}\right)\beta^{5/2}R_0^3\int d^3x P_\nu(x) . \end{aligned} \quad (3.59)$$

By substituting T_0 (eq. (3.19)) and β (eq. (3.18))

$$\begin{aligned}\mathcal{K}_3 &= -\frac{g_\nu kT}{6\pi^2\hbar^3}(2m_\nu kT)^{3/2}\frac{4\pi m_\nu R_0^3}{kM_\odot}\left(\frac{kR_0}{m_\nu GM_\odot}\right)\left(\frac{m_\nu GM_\odot}{kTR_0}\right)^{5/2}R_0^3\int d^3x P_\nu(x) \\ &= -\left(\frac{4\sqrt{2}m_\nu^4 G^{3/2}M_\odot^{1/2}}{3\pi\hbar^3}\right)R^{3/2}R_0^3\int d^3x P_\nu(x) . \end{aligned} \quad (3.60)$$

By applying eq. (3.29) again this simplifies to

$$\begin{aligned}\mathcal{K}_3 &= -\frac{R_0^{3/2}}{R_0^{3/2}}R_0^3\int d^3x P_\nu(x) \\ &= -R_0^3\int d^3x P_\nu(x) . \end{aligned} \quad (3.61)$$

If we now look at eq. (3.45) and substitute the results for the integral $\mathcal{K}_2$ (eq. (3.49)) and $\mathcal{K}_3$ (eq. (3.61)) we retrieve our final expression for the free energy functional, namely

$$F = \frac{1}{2}\mu N_\nu + \frac{1}{2}kT_0R_0^3\int d^3x n_\nu\left[\frac{v(x)}{x} - \frac{v(0)}{x}\right] - R_0^3\int d^3x P_\nu(x) . \quad (3.62)$$

3.5.2 Extrema of the Free Energy

The free energy functional, $F[n]$, will have an extremum at $n = n_0(r)$ if $n_0(r)$ is a solution to the Thomas-Fermi equation. In other words we now prove that

$$\left.\frac{\partial F[n]}{\partial n_\nu(r)}\right|_{n=n_0(r)} = 0 . \quad (3.63)$$

Proof:

From eq. (3.41) we already know the expression for the free energy:

$$\begin{aligned} F[n] &= -g_\nu kT \int \frac{d^3r d^3p}{(2\pi\hbar)^3} \ln\left[1 + e^{\left(-\frac{p^2}{2mkT} - \frac{V'[n(r)]}{kT} + \frac{\mu[n(r)]}{kT}\right)}\right] \\ &\quad -W[n] + \mu[n]N_\nu \ , \end{aligned} \tag{3.64}$$

where $W[n]$ is given by

$$W[n] = \frac{1}{2}\int d^3r n(r) V'[n] \tag{3.65}$$

and $V'[n]$ by

$$V'[n(r)] = -Gm_\nu^2 \int \frac{n(r')}{|\boldsymbol{r}-\boldsymbol{r}'|} d^3r' \ . \tag{3.66}$$

Therefore by finding the derivative of $W[n]$

$$\begin{aligned} \frac{\partial W[n]}{\partial n_\nu(r)} &= -\frac{1}{2}Gm_\nu^2 \int d^3r d^3r' \frac{\partial(n(r)n(r'))}{\partial n_\nu(r)} \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} \\ &= -\frac{1}{2}Gm_\nu^2 \int d^3r d^3r' 2n(r) \frac{\partial n(r')}{\partial n_\nu(r)} \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} \\ &= \int d^3r n(r) \frac{\partial V'[n(r)]}{\partial n_\nu(r)} \ , \end{aligned} \tag{3.67}$$

we can write the derivate of the free energy functional as

$$\begin{aligned} \frac{\partial F[n]}{\partial n} &= -g_\nu kT \int d^3r \frac{d^3p}{(2\pi\hbar)^3} \frac{1}{\left[1 + e^{\left(\frac{p^2}{2mkT} + \frac{V'[n]}{kT} - \frac{\mu[n]}{kT}\right)}\right]} \left(\frac{\partial \mu}{\partial n} - \frac{\partial V'}{\partial n}\right) \frac{1}{kT} \\ &\quad - \int d^3r n(r) \frac{\partial V'[n(r)]}{\partial n_\nu(r)} + \frac{\partial \mu[n]}{\partial n_\nu(r)} N_\nu \\ &= \int -d^3r \underbrace{\left[n(r) - g \int \frac{d^3p}{(2\pi\hbar)^3} \frac{1}{\left[1 + e^{\left(\frac{p^2}{2mkT} + \frac{V'[n]}{kT} - \frac{\mu[n]}{kT}\right)}\right]} \right]}_{=0 \text{ for } n = n_0} \frac{\partial V'[n]}{\partial n_\nu(r)} \\ &\quad + \underbrace{\left(N_\nu - g \int d^3r \frac{d^3p}{(2\pi\hbar)^3} \frac{1}{\left[1 + e^{\left(\frac{p^2}{2mkT} + \frac{V'[n]}{kT} - \frac{\mu[n]}{kT}\right)}\right]} \right)}_{=0 \text{ for } n = n_0} \frac{\partial \mu[n]}{\partial n_\nu(r)} \end{aligned}$$

$$= \quad 0 \text{ for } n_\nu(r) = n_0(r) \, . \tag{3.68}$$

It is important to know that only those solutions that minimize the free energy are physical.

3.6 Derivation of the Energy E

The energy is given by

$$E = \frac{\partial}{\partial\beta}(\beta F)\Big|_{N_\nu, V_0} \, , \tag{3.69}$$

where β now depicts $1/kT$. If we now substitute our original expression for the free energy, as given in eq. (3.41) the energy can be written as

$$\begin{aligned} E &= -W + \mu N_\nu - \beta\frac{\partial W}{\partial\beta} + \beta\frac{\partial\mu}{\partial\beta}N_\nu \\ &\quad -g\frac{\partial}{\partial\beta}\int d^3r\frac{d^3p}{(2\pi\hbar)^3}\ln\left[1 + e^{\left(-\frac{p^2}{2m_\nu} - V'(r) + \mu\right)\beta}\right] \, . \end{aligned} \tag{3.70}$$

Since

$$\begin{aligned} &\frac{\partial}{\partial\beta}\left[\ln\left[1 + e^{\left(-\frac{p^2}{2m_\nu} - V'(r) + \mu\right)\beta}\right]\right] \\ &= \frac{\left[-\frac{p^2}{2m_\nu} - V'(r) + \mu\right] + \beta\frac{\partial}{\partial\beta}\left[-\frac{p^2}{2m_\nu} - V'(r) + \mu\right]}{\left[1 + e^{\left(\frac{p^2}{2m_\nu} + V'(r) - \mu\right)\beta}\right]} \end{aligned} \tag{3.71}$$

and $\frac{\partial}{\partial\beta}\left(\frac{p^2}{2m_\nu}\right) = 0$ the energy is expressed as

$$\begin{aligned} E &= -W + \mu N_\nu - \beta\frac{\partial W}{\partial\beta} + \beta\frac{\partial\mu}{\partial\beta}N_\nu \\ &\quad +g_\nu\int d^3r\frac{d^3p}{(2\pi\hbar)^3}\frac{p^2}{2m_\nu}\frac{1}{\left[1 + e^{\left(\frac{p^2}{2m_\nu} + V'(r) - \mu\right)\beta}\right]} \\ &\quad +g_\nu\int d^3r\frac{d^3p}{(2\pi\hbar)^3}V'(r)\frac{1}{\left[1 + e^{\left(\frac{p^2}{2m_\nu} + V'(r) - \mu\right)\beta}\right]} \end{aligned}$$

$$-g_\nu \int d^3r \frac{d^3p}{(2\pi\hbar)^3} \mu \frac{1}{\left[1+e^{\left(\frac{p^2}{2m_\nu}+V'(r)-\mu\right)\beta}\right]}$$
$$+g_\nu \int d^3r \frac{d^3p}{(2\pi\hbar)^3} \beta \frac{\partial V'}{\partial \beta} \frac{1}{\left[1+e^{\left(\frac{p^2}{2m_\nu}+V'(r)-\mu\right)\beta}\right]}$$
$$-g_\nu \int d^3r \frac{d^3p}{(2\pi\hbar)^3} \beta \frac{\partial \mu}{\partial \beta} \frac{1}{\left[1+e^{\left(\frac{p^2}{2m_\nu}+V'(r)-\mu\right)\beta}\right]} . \quad (3.72)$$

Now we can simplify matters by writing this expression in terms of the neutrino number density. This is done by using the next two equations

$$n_\nu(r) = g_\nu \int \frac{d^3p}{(2\pi\hbar)^3} \frac{1}{\left[1+e^{\left(\frac{p^2}{2m_\nu}+V'(r)-\mu\right)\beta}\right]} \quad (3.73)$$

and

$$N_\nu(r) = g_\nu \int d^3r \int \frac{d^3p}{(2\pi\hbar)^3} \frac{1}{\left[1+e^{\left(\frac{p^2}{2m_\nu}+V'(r)-\mu\right)\beta}\right]} . \quad (3.74)$$

The energy then simplifies to

$$\begin{aligned} E &= -W + \mu N_\nu - \beta \frac{\partial W}{\partial \beta} + \beta \frac{\partial \mu}{\partial \beta} N_\nu \\ &+ g_\nu \int d^3r \frac{d^3p}{(2\pi\hbar)^3} \frac{p^2}{2m_\nu} \frac{1}{\left[1+e^{\left(\frac{p^2}{2m_\nu}+V'(r)-\mu\right)\beta}\right]} \\ &+ \int d^3r V'(r) n_\nu(r) - \mu N_\nu \\ &+ \int d^3r \beta \frac{\partial V'}{\partial \beta} n_\nu(r) - \beta \frac{\partial \mu}{\partial \beta} N_\nu . \end{aligned} \quad (3.75)$$

We can write the sixth term in this expression in terms of W, using eq. (3.42).

$$\begin{aligned} E &= W + g_\nu \int d^3r \frac{d^3p}{(2\pi\hbar)^3} \frac{p^2}{2m_\nu} \frac{1}{\left[1+e^{\left(\frac{p^2}{2m_\nu}+V'(r)-\mu\right)\beta}\right]} \\ &- \beta \frac{\partial W}{\partial \beta} + \int d^3r \beta \frac{\partial V'}{\partial \beta} n_\nu(r) . \end{aligned} \quad (3.76)$$

It is now shown that that last two terms cancel

$$\begin{aligned} W &= \frac{1}{2}\int d^3r d^3r' \left(\frac{-Gm_\nu^2}{|\boldsymbol{r}-\boldsymbol{r}'|}\right) n_\nu(r)n(r') & (3.77) \\ \frac{\partial W}{\partial \beta} &= \frac{1}{2}\int d^3r d^3r' \left(\frac{-Gm_\nu^2}{|\boldsymbol{r}-\boldsymbol{r}'|}\right)\left[\frac{\partial n(r')}{\partial\beta}n_\nu(r) + \frac{\partial n_\nu(r)}{\partial\beta}n(r')\right] \\ &= \frac{1}{2}\int d^3r n_\nu(r) 2\frac{\partial V'(r)}{\partial\beta} & (3.78) \\ \beta\frac{\partial W}{\partial\beta} &= \int d^3r \beta\frac{\partial V'}{\partial\beta}n_\nu(r)\ . & (3.79) \end{aligned}$$

After this cancellation of terms, we are left with

$$E = W + g_\nu \int d^3r \frac{d^3p}{(2\pi\hbar)^3}\frac{p^2}{2m_\nu}\frac{1}{\left[1+e^{\left(\frac{p^2}{2m_\nu}+V'(r)-\mu\right)\beta}\right]}\ . \quad (3.80)$$

By using our expression for W (eq. (3.42)) we write the energy as

$$\begin{aligned} E &= \frac{1}{2}\int d^3r V'(r) n_\nu(r) \\ &\quad + g_\nu kT \int d^3r \frac{d^3p}{(2\pi\hbar)^3}\frac{p^2}{2m_\nu kT}\frac{1}{\left[1+e^{\left(\frac{p^2}{2m_\nu kT}+\frac{V'(r)}{kT}-\frac{\mu}{kT}\right)}\right]}\ . & (3.81) \end{aligned}$$

We use eq. (3.43) to substitute for $V'(r)$, and find

$$\begin{aligned} E &= \frac{1}{2}\int d^3r \mu n_\nu(r) - \frac{1}{2}\int d^3r n_\nu(r)\left[m_\nu\frac{u(r)}{r} - m_\nu\frac{u(0)}{r}\right] \\ &\quad + g_\nu kT \int d^3r \frac{d^3p}{(2\pi\hbar)^3}\frac{p^2}{2m_\nu kT}\frac{1}{\left[1+e^{\left(\frac{p^2}{2m_\nu kT}+\frac{V'(r)}{kT}-\frac{\mu}{kT}\right)}\right]} \\ &= \frac{1}{2}\mu N_\nu \underbrace{-\frac{1}{2}\int d^3r n_\nu(r)\left[m_\nu\frac{u(r)}{r} - m_\nu\frac{u(0)}{r}\right]}_{\mathcal{K}_2} \\ &\quad \underbrace{+ g_\nu kT \int d^3r \frac{d^3p}{(2\pi\hbar)^3}\frac{p^2}{2m_\nu kT}\frac{1}{\left[1+e^{\left(\frac{p^2}{2m_\nu kT}+\frac{V'(r)}{kT}-\frac{\mu}{kT}\right)}\right]}}_{\mathcal{K}_3}\ . & (3.82) \end{aligned}$$

It is now simply a matter of comparing this last expression with the one we had for the free energy before we simplified the terms. The difference to the integral $\mathcal{K}_2$ we had for the free energy is only the minus sign. If we compare the integral $\mathcal{K}_3$ with eq. (3.57) the difference is a factor of $-\frac{3}{2}$. We thus know how the simplified terms will differ from those derived for the free energy. The energy is then simply written as

$$E = \frac{1}{2}\mu N_\nu - \frac{1}{2}kT_0R_0^3 \int d^3x n_\nu \left[\frac{v(x)}{x} - \frac{v(0)}{x}\right] + \frac{3}{2}R_0^3 \int d^3x P_\nu(x) \ . \tag{3.83}$$

3.7 The Entropy S

From [39] we find the expression for the Helmholtz free energy to be

$$F = E - TS \tag{3.84}$$

and the entropy is then given by

$$S = \frac{1}{T}(E - F) \ . \tag{3.85}$$

3.8 Summary

The Lané-Emden equation for a neutrino star at finite temperature, that needs to be solved is

$$\frac{d^2v}{dx^2} = -\frac{3}{2}\beta^{-3/2}xI_{\frac{1}{2}}\left(\beta\frac{v}{x}\right) \ . \tag{3.86}$$

All the other important thermodynamical quantities such as number density, pressure, free energy, energy and entropy can be expressed in terms of v/x

$$n_\nu(x) = \frac{M_\odot}{m_\nu R_0^3}\frac{3}{8\pi}\beta^{-3/2}I_{\frac{1}{2}}\left(\beta\frac{v}{x}\right) \tag{3.87}$$

$$P_\nu(x) = \frac{M_\odot kT_0}{m_\nu R_0^3}\frac{1}{4\pi}\beta^{-5/2}I_{\frac{3}{2}}\left(\beta\frac{v}{x}\right) = \frac{2}{3}\varepsilon_{\text{kin}} \tag{3.88}$$

$$F = \frac{1}{2}\mu N_\nu + \frac{1}{2}kT_0R_0^3 \int d^3x n_\nu \left[\frac{v(x)-v(0)}{x}\right] - R_0^3 \int d^3x P_\nu(x) \quad (3.89)$$

$$E = \frac{1}{2}\mu N_\nu - \frac{1}{2}kT_0R_0^3 \int d^3x n_\nu \left[\frac{v(x)-v(0)}{x}\right] + \frac{3}{2}R_0^3 \int d^3x P_\nu(x) \quad (3.90)$$

$$= \frac{1}{2}\mu N_\nu - \frac{1}{2}kT_0R_0^3 \int d^3x n_\nu \left[\frac{v(x)-v(0)}{x}\right] + R_0^3 \int d^3x \varepsilon_{\rm kin} \quad (3.91)$$

$$S = \frac{1}{T}(E-F) . \quad (3.92)$$

When solving this case, the chemical potential varies with density so that the number of neutrinos $N_\nu = M_\nu/m_\nu$ is kept fixed [1].

3.9 Results

The differential equation (eq. (3.20)) requires boundary conditions. We define R as the radius of the spherical cavity in which the neutrinos are enclosed, corresponding to $x_1 = \frac{R}{R_0}$ [1]. We also require that the total neutrino mass be M_ν, with a possible pointlike mass M_B at the origin. $v(x)$ is then related to its derivative at $x = x_1$ by

$$v'(x_1) = \frac{1}{x_1}\left(v(x_1) - \frac{M_B + M_\nu}{M_\odot}\right) , \quad (3.93)$$

which is in turn related to the chemical potential by $\mu = kT_0v'(x_1)$ since

$$V(r) = \mu - m_\nu \frac{u(r)}{r} \quad (3.94)$$

$$\left.\frac{dV}{dr}\right|_{r=R} = \frac{GmM}{R^2} \quad (3.95)$$

$$= \frac{mu}{R^2} - \left.\frac{m}{R}\frac{du}{dr}\right|_{r=R} \quad (3.96)$$

$$\frac{M}{x_1} = \frac{M_\odot}{x_1}v - \left. M_\odot \frac{dV}{dx_1}\right|_{x=x_1} , \quad (3.97)$$

where $M = M_B + M_\nu$. $v(x)$ at $x = 0$ is related to the point mass at the origin by $v(0) = M_B/M_\odot$ as shown earlier.

We have done a numerical study on a system of self-gravitating massive neutrinos where the total mass of the neutrinos was chosen to be $10M_{\odot}$. The cavity radius was also arbitrarily chosen to be $100R_0$. The neutrino mass m_ν was set equal to 17.2 keV, although the graphs are represented in such a way, that a change in neutrino mass would not make a difference, since it changes the quantities in such a way that there is no net effect.

In the temperature interval, $T = (0.0397 - 0.3109)T_0$, three distinct solutions were found of which only two solutions are physical, namely those two for which the free energy assumes a minimum. We refer to the solution for temperatures higher than those mentioned in the above interval as "gas", and those solutions which exist at colder temperatures and eventually becomes a degenerate Fermi gas at $T = 0$ as "condensate" [1].

In the figures we plot various thermodynamic quantities per neutrino, as a function of neutrino temperature. The phase transition takes place where the free energy of the gas and condensate become equal. This transition temperature is $T_t = 0.19441T_0$. (The transition temperature is shown as the dotted line on all the relevant graphs.) The top curve in the free energy plot is the unphysical solution as any system will try to minimize the free energy. Both the energy and entropy have a discontinuity at T_t.

From Fig. (3.5) we see that because of the scaling laws in the cold case, we have a straight line. Because the existence of scaling laws in the finite temperature case, but different from the ones in the cold case, the graphs have the same shape for different temperatures. The system just shifts parallel to the cold case line.

3.10 Conclusion

At the critical temperature T_t, the two states of the neutrino matter, gas and condensate, exist together. Although there is no discontinuity in the free energy, it shows in the entropy and energy. If the conditions change, one or the other state

will be advantaged. If the temperature is above T_t the gaseous phase is favoured and below the critical temperature, the condensate phase is more likely. This happens irrespective of background radiation. Radiation pressure may only speed up the process. This is because of the fact that only the solutions which minimizes the free energy are valid solutions. This has been discussed in Section 3.5.2.
When the first order phase transition occurs latent heat is released. This is evident from Fig. (3.2). Thus, the condensate formation is accompanied by a release of a considerable amount of energy that will reheat the environment. When the universe expands and cools, a phase transition will occur in the neutrino stars, as soon as a certain temperature is reached.
As mentioned earlier, the properties of the graphs have been chosen in such a way that the neutrino mass will have no effect on them. A different neutrino mass (for example 5 keV or 45 keV) will have an effect on the critical temperature and cavity radius. For a neutrino mass lighter than 17.2 keV the cavity radius will increase and the critical temperature will decrease and a heavier neutrino will have the opposite effect.

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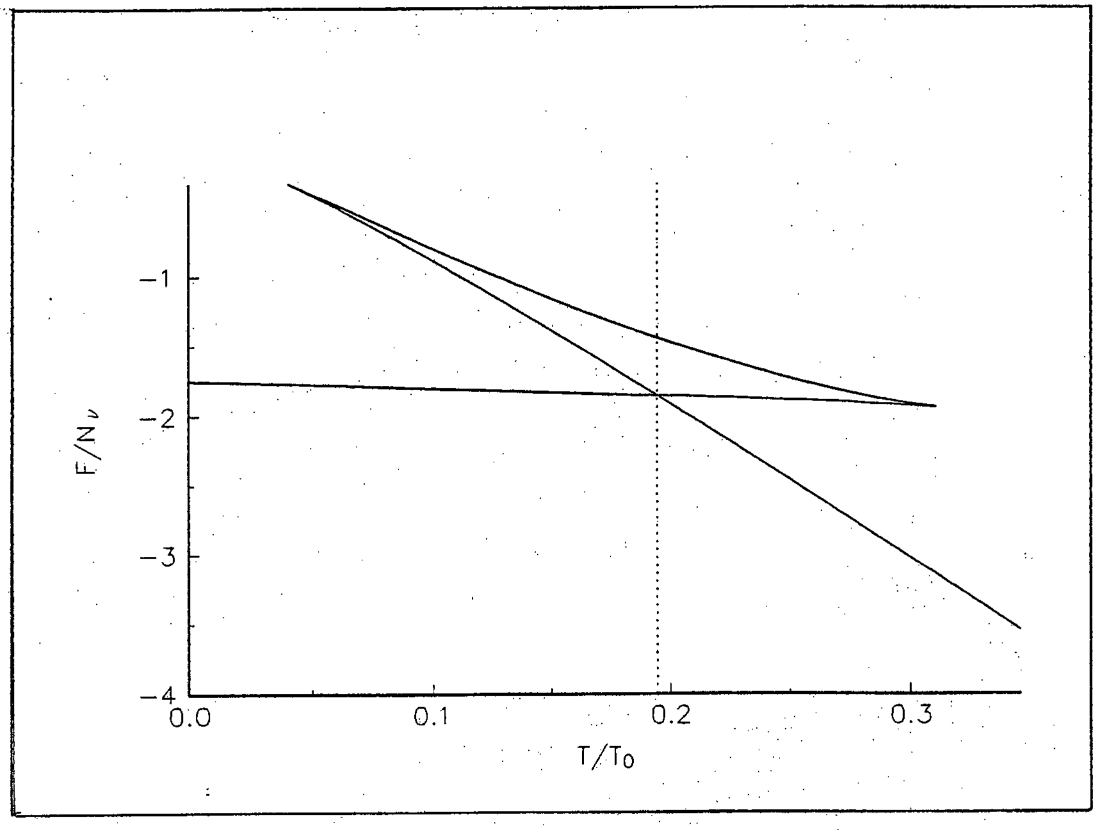

Figure 3.1: The free energy, F, per neutrino as a function of temperature. Free energy in units of kT_0 and temperature in units of T_0. The top curve in the plot is unphysical.

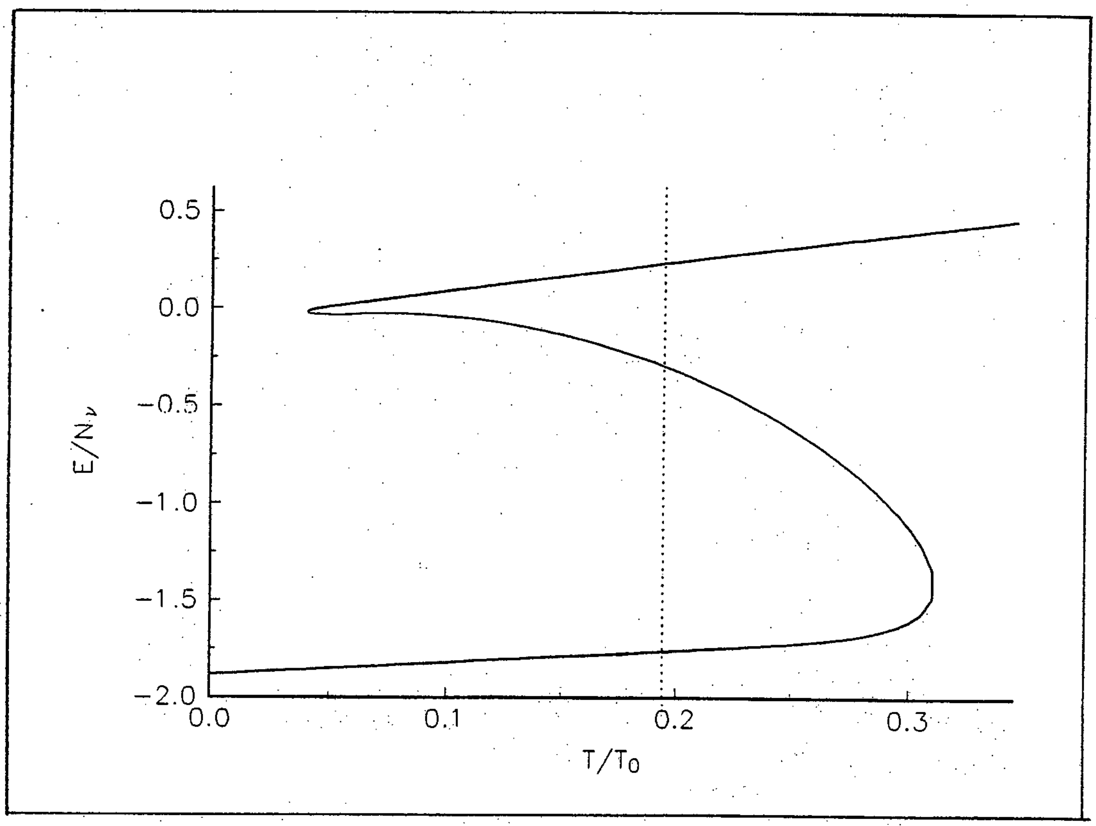

Figure 3.2: The energy, E, per neutrino as a function of temperature. Temperature in units of T_0 and energy in units of kT_0. The energy has a discontinuity at T_t. This transition temperature is indicated by the dotted line.

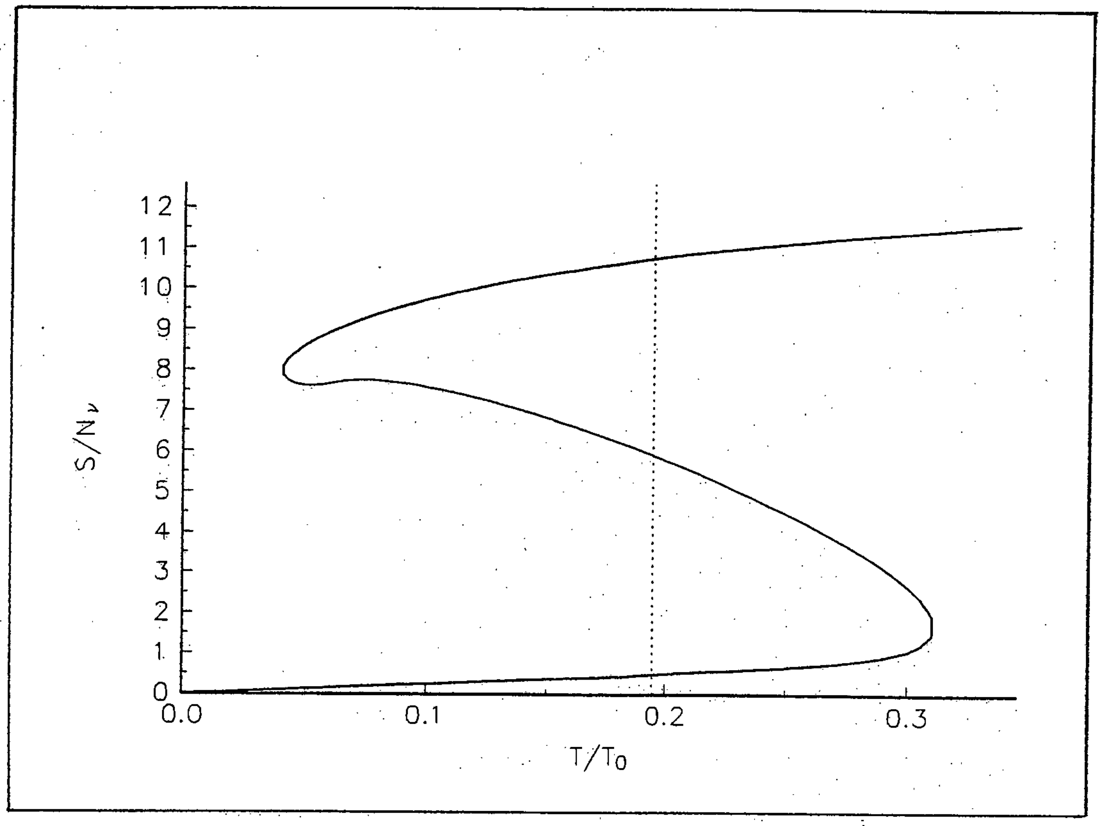

Figure 3.3: The entropy, S, per neutrino as a function of temperature. The temperature in units of T_0. The entropy has a discontinuity at T_t. This transition temperature is indicated by the dotted line.

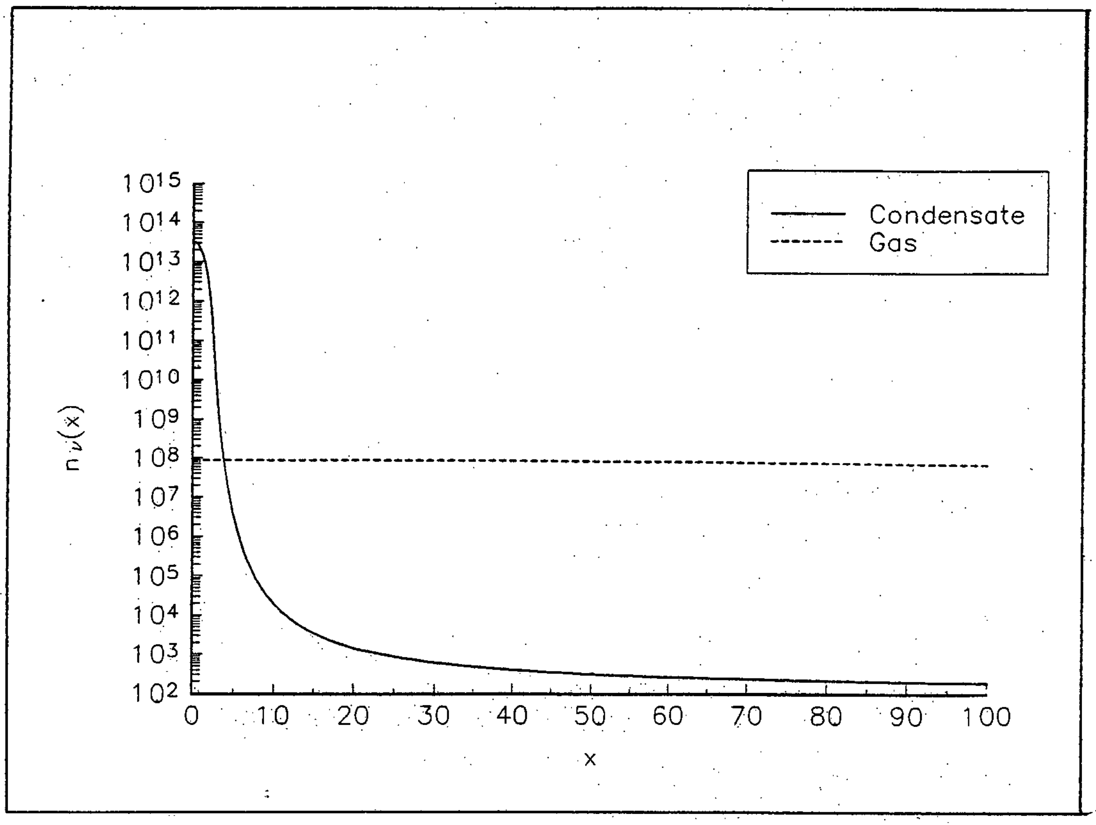

Figure 3.4: A graph of the number densities of the two solutions at T_t which minimizes the free energy. This clearly shows the phase transition, since one represents the condensate phase and the other one the gaseous phase.

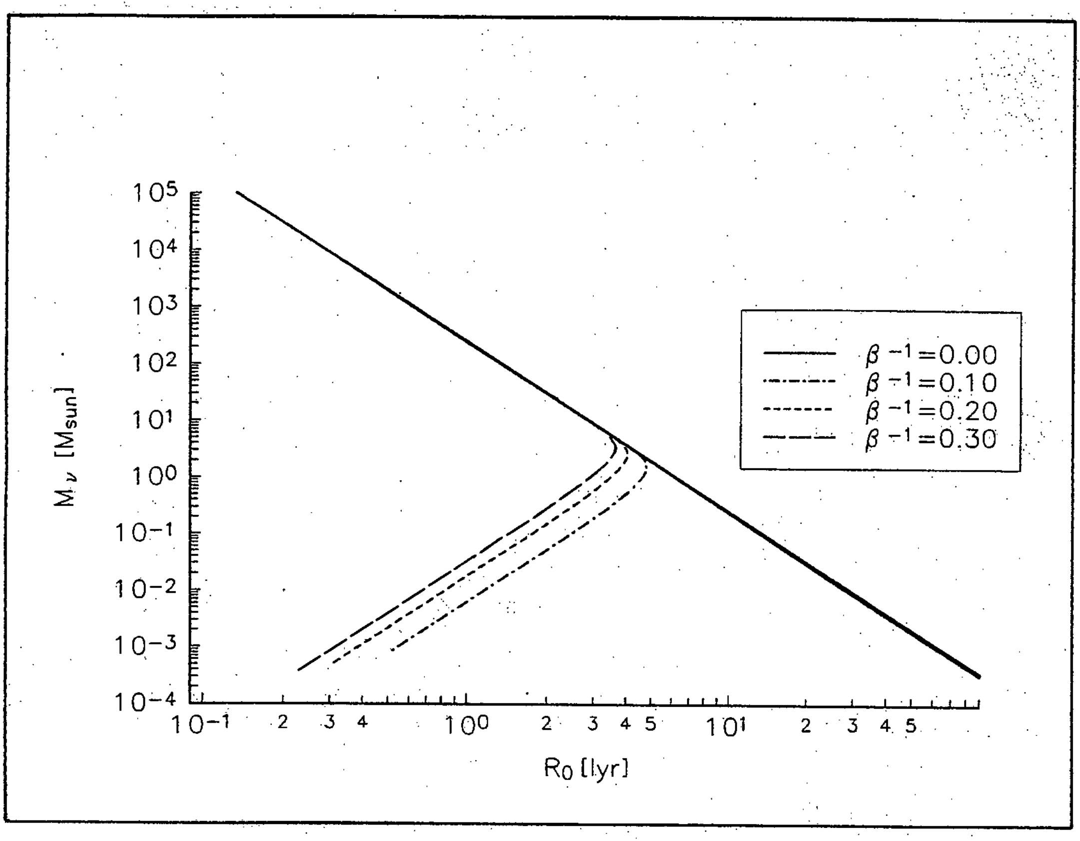

Figure 3.5: Masses and radii of neutrino stars at different finite temperatures, with respect to the pure cold degenerate neutrino star solution (solid line). In this figure $\mu = M_\nu/M_\odot$ and $\beta = T_0/T$. In both cases there are no baryonic matter present ($M_B = 0$).

Appendix A

The Thermodynamical Potential, Ω

In this appendix we derive (following [25]) the results quoted in Chapter 2 and Chapter 3.

Let us consider a gas consisting of elementary particles. The energy of an elementary particle is just the translational kinetic energy. We therefore have

$$\varepsilon = (p_x^2 + p_y^2 + p_z^2)/2m \ . \tag{A.1}$$

One must remember, however, that the state of the particle still depends on the orientation of the spin. Hence, the number of particles in a volume element $dp_x dp_y dp_z dV$ in phase space is then found by multiplying the Fermi-distribution

$$\bar{n} = \frac{1}{e^{(\varepsilon-\mu)/kT} + 1} \tag{A.2}$$

by

$$gd\tau = gdp_x dp_y dp_z dV/(2\pi\hbar)^3 \tag{A.3}$$

where $g = 2s + 1$ (s being the spin of the particle), giving

$$dN = \frac{gd\tau}{e^{(\varepsilon-\mu)/kT} + 1} \ . \tag{A.4}$$

Integrating over the volume V, we find the distribution for the absolute magnitude of the momentum to be

$$dN_p = \frac{gVp^2dp}{2\pi^2\hbar^3(e^{(\varepsilon-\mu)/kT}+1)}, \tag{A.5}$$

where $\varepsilon = p^2/2m$, or the energy distribution

$$dN_\varepsilon = \frac{gVm^{3/2}}{\sqrt{2}\pi^2\hbar^3}\frac{\sqrt{\varepsilon}d\varepsilon}{e^{(\varepsilon-\mu)/kT}+1}. \tag{A.6}$$

Integrating with respect to ε, we obtain the total number of particles in the gas

$$N = \frac{gVm^{3/2}}{\sqrt{2}\pi^2\hbar^3}\int_0^\infty \frac{\sqrt{\varepsilon}d\varepsilon}{e^{(\varepsilon-\mu)/kT}+1}. \tag{A.7}$$

If we introduce a new variable of integration, $z = \varepsilon/kT$ the above equation can be written as

$$\frac{N}{V} = \frac{g(mkT)^{3/2}}{\sqrt{2}\pi^2\hbar^3}\int_0^\infty \frac{\sqrt{z}dz}{e^{z-\mu/kT}+1}. \tag{A.8}$$

This formula implicitly determines the chemical potential μ of the gas as a function of its temperature T and density N/V.

The thermodynamical potential Ω of the gas is obtained by summation of Ω over all quantum states

$$\Omega = -\frac{gVkTm^{3/2}}{\sqrt{2}\pi^2\hbar^3}\int_0^\infty \sqrt{\varepsilon}\log\left(1+e^{(\mu-\varepsilon)/kT}\right)d\varepsilon. \tag{A.9}$$

Integration by parts yields

$$\Omega = -\frac{2}{3}\frac{gVm^{3/2}}{\sqrt{2}\pi^2\hbar^3}\int_0^\infty \frac{\varepsilon^{3/2}d\varepsilon}{e^{(\varepsilon-\mu)/kT}+1}. \tag{A.10}$$

This expression is, apart from the factor $-2/3$, the total energy of the gas:

$$E = \frac{gVm^{3/2}}{\sqrt{2}\pi^2\hbar^3}\int_0^\infty \frac{\varepsilon^{3/2}d\varepsilon}{e^{(\varepsilon-\mu)/kT}+1}. \tag{A.11}$$

Now, since $\Omega = -PV$ [25], we have

$$PV = \frac{2}{3}E. \tag{A.12}$$

Appendix B

Calculations

B.1 The integral $I_n(\eta)$

We have encountered these integrals in Chapter 2 and Chapter 3. They were solved numerically, since no analytical solution exists. The integration range, $[0, \infty)$, were divided into three parts which were solved separately and then added up as the next equation demonstrates

$$I_n(\eta) = \int_0^\infty \frac{\xi^n d\xi}{1 + e^{\xi - \eta}} \tag{B.1}$$

$$= \int_0^a \frac{\xi^n d\xi}{1 + e^{\xi - \eta}} + \int_a^b \frac{\xi^n d\xi}{1 + e^{\xi - \eta}} + \int_b^\infty \frac{\xi^n d\xi}{1 + e^{\xi - \eta}} \tag{B.2}$$

where $n = \frac{1}{2}$ (e.g. eq. (3.88)) or $n = \frac{3}{2}$ (e.g. eq. (3.89)). The integral limits (i.e. a and b) was chosen such as to obtain the best accuracy.

The first two intervals, with finite limits, were solved using the weights and abscissas of the Gauss-Legendre N-point quadrature formula [40]. N was varied in order to obtain the best results.

The third region was calculated with Laguerre integration (using 15 points) as this is not a finite interval [41].

Our results have been compared to those listed in [42]. In all cases the results agreed to the maximum amount of significant figures listed.

B.2 The Lané-Emden Equation

Both non-linear equations, eqs. (2.36) and (3.20), have been solved using fifth order adaptive stepsize control for Runge-Kutta [40]. The pressure and number density (needed to evaluate the free energy and energy) have been calculated by introducing them as extra differential equations into the adaptive stepsize control routine.

Brent's method [40] was used to calculate the roots of functions in order to determine the exact solutions to the differential equations.

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