



UNIVERSITY OF CAPE TOWN

DEPARTMENT OF MATHEMATICS

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Aspects of Compactness in Convex Spaces

by

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A thesis prepared under the supervision  
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Preface.

This thesis has its origin in a ten minute paper presented at the British Mathematical Colloquium in London in 1966. At this reading, a certain class of spaces was introduced by Mr. D. H. Fremlin, who furthermore, cited interesting compactness properties pertaining to these spaces.

This work shall be concerned, foremostly, with the provision of proofs for the statements made by Mr. D. H. Fremlin in 1966, and with demonstrations of how these purely topological results have applications to linear topological spaces, with the emphasis placed on those which are locally convex.

Our investigations of the 'Fremlin' spaces, and the search for known spaces which satisfy the 'Fremlin' conditions has led to a concentration on the separable and Fréchet spaces. These are discussed in chapter 11.

On reading this thesis, it will become apparent that an underlying theme is that of weak compactness. We have therefore taken the liberty, of concluding with a chapter which gives various characterisations of this.

Each chapter will introduce its own subject matter.

New results and different proofs of existing theorems have been obtained.

These are to be found in the following:

1.1.8, 1.1.9, 1.1.10, 1.2.3, 1.2.4, 1.2.6, 2.1.1, 2.1.2, 2.1.3,  
2.2.3, 2.2.4, 2.2.5, 2.2.6, 2.2.8 and 2.2.10.

We shall state clearly which of these have been adapted from previous research articles.

It is my pleasure to thank sincerely Dr. J. H. Webb for his assistance with this thesis, for his many helpful and interesting suggestions, and for first posing to me the problem of theorem 1.1.8, the investigation of which has been both stimulating and enjoyable.

Errata:

- 1) p 30, L 8 : Replace 'on' by 'in'.
- 2) p 38, L 1 : Delete 'then'.
- 3) p 43, L 3 : Insert 'in'.

Notation.

Let  $(E, F)$  be a dual pair of vector spaces. The following shall denote the most frequently used topologies on  $E$ :

$\sigma(E, F)$  - the weak topology, or the topology of uniform convergence on the finite subsets of  $F$ ;

$\mu(E, F)$  - the Mackey topology, or the topology of uniform convergence on the absolutely convex  $\sigma(F, E)$ -compact subsets of  $F$ ;

$\beta(E, F)$  - the strong topology, or the topology of uniform convergence on the  $\sigma(F, E)$ -bounded subsets of  $F$ .

If  $(E, \tau)$  is a linear topological space, then

$E^*$  = algebraic dual of  $E$ .

$E'$  =  $(E, \tau)'$  = topological dual of  $E$ .

$E^+$  =  $\{ f \in E^* \mid f(x_n) \rightarrow 0 \text{ whenever } x_n \rightarrow 0 \text{ in } E. \}$

Further, if  $F \subset E$  then,

$\tau|_F$  = the restriction of  $\tau$  to  $F$ , that is, the relative topology of  $E$  on  $F$ .

The scalar field shall be the field of real or complex numbers.

Finally, we shall use the abbreviation 'a convex space', to mean 'a locally convex, linear topological, Hausdorff ( $T_2$ ) space'.

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## Chapter 1.

### 1. Fremlin Spaces.

Our attention in this chapter is to be focused mainly, on a certain kind of topological space, which we have called a Fremlin space. As far as we know, it was Mr. D. H. Fremlin, who first publicly stated, without proofs, the results of theorem 1.1.8.

To the best of our knowledge, the proofs of these results are as yet unpublished, and hence those which we shall give here, are entirely independent of those which must have been obtained by him.

We wish however, to draw attention to a reference, [12], which appeared in the Proceedings of the Cambridge Philosophical Society, Volume 63, 1967, from which possibly, further information is to be gleaned.

Definition 1.1.1. A subset  $A$  of a linear topological space  $E$  is said to be total if and only if  $\overline{\text{lin } A} = E$ .

It is well-known that this definition is equivalent to saying: A subset  $A$  of a linear topological space  $E$  is total if and only if every continuous linear functional vanishing on  $A$  is identically zero.

Definition 1.1.2. A subset  $A$  of a topological space  $E$  is said to be relatively compact if and only if its closure in  $E$  is compact.

Definition 1.1.3. A subset  $A$  of a topological space  $E$  is said to be (relatively) sequentially compact if and only if every sequence in  $A$  contains a subsequence converging to a point in  $A$ , ( $E$ ).

Definition 1.1.4. A subset  $A$  of a topological space  $E$  is said to be (relatively) countably compact if and only if every countable subset of  $A$  has a cluster point in  $A$ , ( $E$ ).

Note that, while the closure of a relatively compact set is compact, the closure of a relatively sequentially compact set need not be sequentially compact, nor need the closure of a relatively countably compact set be countably compact.

Definition 1.1.5. A topological space  $E$  is said to be a KC space if every compact subset  $A$  of  $E$  is closed. See [33].

Definition 1.1.6. A Fremlin space  $E$  is a regular KC space satisfying the following conditions:

- (Fi). Every relatively countably compact (RCC) subset  $A$  of  $E$  is relatively compact (RC).

(Fii) If  $A \subset E$ , and  $A$  is relatively compact, and if  $x \in \bar{A}$  then there exists a sequence  $(x_n)$  in  $A$  such that  $x_n \rightarrow x$ .

By regular we shall understand that a closed subset  $A$  of  $E$  and a point  $b \in X - A$  can be separated by means of two disjoint open sets.

Remarks.

(i).  $T_2 \Rightarrow KC \Rightarrow T_1$ .

(ii). Every Fremlin space is necessarily a  $T_2$  space.

(iii). Every subspace of a Fremlin space is a Fremlin space.

(iv). In a Fremlin space every relatively compact set is sequential if and only if it is closed. Thus, every compact set is sequential. (A topological space  $E$  is said to be sequential if every subset  $A$  of  $E$  which contains the limit points of all its convergent sequences is closed in  $E$ ).

Proposition 1.1.7. Every convergent sequence in a  $KC$  space has a unique limit point.

Proof. Since each singleton is compact, every  $KC$  space is  $T_1$ . Suppose  $X$  is a  $T_1$  space in which there is a sequence



converging to two distinct points  $a$  and  $b$ . Suppose that  $(x_n)$  is such a sequence. Now  $\{b\}$  is closed and  $X - \{b\}$  is an open neighbourhood of  $a$ .

$\therefore \exists m: n \geq m \Rightarrow x_n \in X - \{b\}$ .

Then  $K = \{a, x_m, x_{m+1}, \dots\}$  is a compact subset of  $X$  which is not closed since  $b \in \bar{K}$  and  $b \notin K$ . So,  $X$  is not a KC space.

Theorem 1.1.8. Let  $X$  be a Fremlin space. Then  $X$  has these properties:

- (i). Every relatively countably compact (RCC) subset  $A$  of  $X$  is relatively sequentially compact (RSC).
- (ii). In  $X$ , the (relatively) compact, (relatively) countably compact and (relatively) sequentially compact sets, all coincide.
- (iii). If  $Y$  is a  $T_2$ , regular topological space,  $X$ , a Fremlin space, and  $f: Y \rightarrow X$  is a one-to-one, continuous mapping from  $Y$  into  $X$ , then  $Y$  is a Fremlin space.

Proof. (i) Let  $A \subset X$ , and let  $A$  be RCC. Let  $(x_n) \subset A$ . Then  $\exists x_0 \in X$  with  $x_0 \in \overline{\{x_n\}_n}$ . Now,  $\{x_n\}_n \subset \bar{A}$  which is compact by (Fi). Thus by (Fii),  $\exists$  a subsequence  $(x_{n_k})$  of  $(x_n)$  such that  $x_{n_k} \rightarrow x_0$ . Hence,  $A$  is (RSC).

(ii). Since every RSC set is RCC, (Fi) together with (i) above, yield that the relatively compact, relatively sequentially compact and relatively countably compact subsets of  $X$  are identical. We shall now establish that every SC subset of  $X$  is compact.

Suppose  $A \subset X$ , and  $A$  is SC. Then  $A$ , being RSC is relatively compact. Let  $x \in \bar{A}$ . Then there exists a sequence  $(x_n) \subset A$  with  $x_n \rightarrow x$ , (by Fii). By the sequential compactness of  $A$ , there exists a subsequence  $(x_{n_k})$  of  $(x_n)$  with  $x_{n_k} \rightarrow y \in A$ . But  $x_{n_k} \rightarrow x$ . Hence by prop. 1.1.7,  $x = y \in A$ . Therefore,  $A$  is closed and relatively compact, and thus compact. We now have for  $A \subset X$ ,

$A$  sequentially compact  $\Rightarrow A$  compact  $\Rightarrow A$  countably compact  
It remains to show that if  $A$  is CC then  $A$  is SC.

Suppose that  $A$  is CC, and that  $(x_n)$  is a sequence in  $A$  having  $x_0$  as cluster point in  $A$ . By (Fi)  $A$  is RC, hence so is  $\{x_n\}$ . By (Fii) there exists a subsequence  $(x_{n_k})$  of  $(x_n)$  such that  $x_{n_k} \rightarrow x_0 \in A$ .  $A$  is therefore sequentially compact.

This then, completes the proof that in a Fremlin space, the notions of compactness, countable compactness and sequential compactness coincide.

(iii). Let  $Y$  be a regular,  $T_2$ , topological space,  $X$  a Fremlin space and  $f: Y \rightarrow X$ , a one-to-one, continuous

mapping of  $Y$  into  $X$ . We wish to establish that  $Y$  is a Fremlin space. We shall show:

1. If  $A \subset Y$  and  $A$  is RCC then  $f(\bar{A}) = \overline{f(A)}$ .
2. If  $A \subset Y$  and  $A$  is RCC,  $x \in \bar{A}$  then  $\exists (x_n) \subset A$  such that  $x_n \rightarrow x$ .
3. If  $A \subset Y$  and  $A$  is RCC then  $f: \bar{A} \rightarrow \overline{f(A)}$  is a homeomorphism.

Then for  $Y$ , 1. and 3. and the fact that  $X$  is a Fremlin space yield property (Fi), while 2. gives property (Fii).

1. Let  $A \subset Y$ , and let  $A$  be RCC. By the continuity of  $f$ ,  $f(\bar{A}) \subset \overline{f(A)}$  and  $f(A)$  is RCC. We must prove that  $\overline{f(A)} \subset f(\bar{A})$ .

Since  $X$  is a Fremlin space,  $f(A)$  is relatively compact. i.e.  $\overline{f(A)}$  is compact. Let  $y \in \overline{f(A)}$ . By (Fii) there exists a sequence  $f(x_n) \subset f(A)$  such that  $(x_n) \subset A$  and  $f(x_n) \rightarrow y$ . Then  $y \in \overline{f\{x_n\}}$ . As  $A$  is RCC, there exists  $x_0 \in \overline{\{x_n\}} \subset \bar{A}$ .  $\therefore f(x_0) \in f(\bar{A})$  and  $f(x_0) \in \overline{f\{x_n\}} \subset \overline{f\{x_n\}} \subset \overline{f(A)} \dots\dots\dots(a)$ . Hence  $\overline{f\{x_n\}}$  is compact in the Fremlin space  $X$ . By (Fii) there exists a subsequence  $f(x_{n_k})$  of  $f(x_n)$  with  $f(x_{n_k}) \rightarrow f(x_0)$ . By (a)  $f(x_{n_k}) \rightarrow y$ . So  $y = f(x_0) \in f(\bar{A})$ . Thus  $\overline{f(A)} \subset f(\bar{A})$  and so  $f(\bar{A}) = \overline{f(A)}$ .

2. By a similar argument to that which we have just used, and bearing in mind that  $f$  is one-to-one, it is not difficult to see that if  $A$  is RCC,  $(x_n) \subset A$  and  $f(x_n) \rightarrow f(x)$  then,

$x$  is a unique cluster point of  $(x_n)$  and  $x_n \rightarrow x$ . .....(c).  
 This is so since every subsequence of  $(x_n)$  has  $x$  as its  
 unique cluster point, (by repetition of our previous argument).  
 Also, if  $x_n \not\rightarrow x$ , then there exists a neighbourhood  $V$  of  $x$   
 and a subsequence  $(x_{n_k})$  of  $(x_n)$  outside of  $V$ . This would  
 then contradict  $x$  being a cluster point of  $(x_{n_k})$ .

Suppose that  $A$  is RCC and  $x \in \bar{A}$ . Then  $f(x) \in \overline{f(A)}$   
 which is compact, (by Fi). By (Fii) there exists a sequence  
 $f(x_n) \in f(A)$  such that  $f(x_n) \rightarrow f(x)$  and  $(x_n) \subset A$ . By (c),  
 $x_n \rightarrow x$ ,  $x$  being the only cluster point of  $(x_n)$ . So,  $x \in \bar{A}$   
 implies that there exists a sequence  $(x_n) \subset A$  with  $x_n \rightarrow x$ .  
 In particular, this is true if  $A$  is relatively compact.  
 This establishes 2. and also (Fii) for  $Y$ .

3. Let  $A$  be RCC and consider  $f: \bar{A} \rightarrow f(\bar{A})$ . By 1.  $f$  is  
 onto.  $f$  is also one-to-one and continuous. We are required  
 to prove that  $f$  is a closed map, and hence a homeomorphism.

Let  $B$  be closed in  $\bar{A}$ . Let  $f(b) \in f(B) \subset \overline{f(A)} = f(\bar{A})$ .  
 By (Fii)  $\exists f(b_n) \in f(B)$ ,  $(b_n) \subset B$ , with  $f(b_n) \rightarrow f(b)$ .

Suppose that  $b \notin \overline{\{b_n\}}$ . Then by the regularity of  $Y$ ,  
 $\exists$  an open  $W$  with  $b \in W \subset \bar{W}$  and  $W \cap \overline{\{b_n\}} = \emptyset$ . .....(d).  
 For each  $n$ ,  $b_n \in \bar{A}$ . By 1. for each  $n$ ,  $\exists (a_{mn})_m \subset A$   
 such that  $a_{mn} \xrightarrow{m} b_n$ . Also, since  $X - \bar{W}$  is an open neighbour-  
 hood of  $b_n$ , for each  $n$  we may choose the  $a_{mn}$ 's to lie  
 outside of  $W$  for all  $m, n$ .

Now,  $f(a_{mn}) \rightarrow f(b_n)$  as  $m \rightarrow \infty$ , and  $f(b_n) \rightarrow f(b)$ . Hence,  $f(b) \in \overline{f\{a_{mn}\}}$ . But  $\{a_{mn}\}$  is RCC for each  $n$ . Thus, applying 1. we have that  $\overline{f\{a_{mn}\}} = \overline{f\{a_{mn}\}}$ . Therefore,  $b \in \overline{\{a_{mn}\}}$ . This is however impossible, since  $\overline{\{a_{mn}\}} \subset X - W$  and  $\overline{X - W} = X - W$ . So,  $b \notin W$ . This contradicts (d).  
 $\therefore b \in \overline{\{b_n\}} \subset \overline{B} = B$  and  $f(b) \in f(B)$ , that is,  $f(B)$  equals  $\overline{f(B)}$  and  $f: \overline{A} \rightarrow \overline{f(A)}$  is a homeomorphism.

Finally, since  $f$  is a homeomorphism of  $\overline{A}$  onto  $\overline{f(A)}$  and  $\overline{f(A)} = \overline{f(A)}$  which is compact, it follows that  $\overline{A}$  is compact and (Fii) is satisfied in  $Y$ .

Note. In proving (i) and (ii) the regularity of  $X$  was not used and the results remain true for a KC space satisfying (Fi) and (Fii). In establishing (iii) the regularity of  $Y$  was only needed in proving that  $f|_{\overline{A}}$  was a homeomorphism. Also, it was not necessary that  $X$  be a regular space, again, a KC space with properties (Fi) and (Fii) would have sufficed.

Theorem 1.1.8 gives rise to a very powerful corollary, namely,

Corollary 1.1.9. A Fremlin space remains a Fremlin space under any finer regular topology.

Proof. If  $(X, \tau)$  is a Fremlin space with  $\iota \geq \tau$  and  $(X, \iota)$  is a regular space, then  $(X, \iota)$  is  $T_2$  and the identity map

$i: (X, \mathcal{l}) \rightarrow (X, \tau)$  is one-to-one and continuous. The result follows from theorem 1.1.8 (iii).

Corollary 1.1.10. If  $(X, \tau)$  is a regular,  $T_2$ , topological space, and  $(X, \mathcal{l})$  is a Fremlin space with  $\mathcal{l} \leq \tau$ , then:  $(X, \tau)$  and  $(X, \mathcal{l})$  have the same convergent sequences if and only if they have the same compact sets.

Proof. Sufficiency: By corollary 1.1.9,  $(X, \tau)$  is a Fremlin space. Now the identity map  $i: (X, \tau) \rightarrow (X, \mathcal{l})$  is one-to-one and continuous, hence, every  $\tau$ -compact set is  $\mathcal{l}$ -compact and every  $\tau$ -convergent sequence is  $\mathcal{l}$ -convergent. Let  $(x_n) \subset X$  and suppose that  $x_n \rightarrow x$  with respect to  $\mathcal{l}$ . Then  $A = \{x_n, x\}$  is  $\mathcal{l}$ -compact and so by hypothesis, also  $\tau$ -compact. Since  $(X, \tau)$  is a Fremlin space,  $(x_n)$  has a  $\tau$ -cluster point. In fact,  $x$  is the only cluster point of  $(x_n)$  and  $(x_n)$  converges to  $x$  with respect to  $\tau$ . (See theorem 1.1.8, 2 .....(c)).

Necessity: Let  $A$  be  $\mathcal{l}$ -compact. Then by theorem 1.1.8 (ii)  $A$  is  $\mathcal{l}$ -sequentially compact. By hypothesis  $A$  is then  $\tau$ -sequentially compact. Since  $(X, \tau)$  is a Fremlin space,  $A$  is  $\tau$ -compact.

Remarks.

(i). The property of being a Fremlin space is a topological one.

Then the set  $\{ \rho_\mu \mid \mu \in \lambda^+ \}$  of semi-norms, generates a convex topology  $\mid \sigma \mid(\lambda, \lambda^+)$  on  $\lambda$ , called the 'normal' or Köthe topology. This topology is finer than  $\sigma(\lambda, \lambda^+)$  but weaker than  $\mu(\lambda, \lambda^+)$ . (See [22; 30; 2]).

When  $\lambda \supset \varphi$ ,  $\sigma(\lambda, \varphi)$  is metrisable, since  $(\lambda, \sigma(\lambda, \varphi))$  is a subspace of  $(\omega, \sigma(\omega, \varphi))$ , which is metrisable, being the topological product of countably many lines. Hence,  $(\lambda, \sigma(\lambda, \varphi))$  is a Fremlin space. Thus, if  $\tau$  is any regular linear topology on  $\lambda$  for which the co-ordinate projection maps are continuous, then  $\tau \geq \sigma(\lambda, \varphi)$ . By corollary 1.1.9  $(\lambda, \tau)$  is then a Fremlin space. In particular,  $(\lambda, \sigma(\lambda, \lambda^+))$  and  $(\lambda, \mid \sigma \mid(\lambda, \lambda^+))$  are Fremlin spaces. It follows that:

$$A \text{ } \sigma(\lambda, \lambda^+)\text{-compact} \Leftrightarrow A \text{ } \sigma(\lambda, \lambda^+)\text{-countably compact}$$

$$\Leftrightarrow A \text{ } \sigma(\lambda, \lambda^+)\text{-sequentially compact for } A \subset \lambda$$

Similarly for  $(\lambda, \mid \sigma \mid(\lambda, \lambda^+))$ .

We note that the topology  $\tau$  need not be locally convex, nor indeed even connected with the dual pair  $(\lambda, \lambda^+)$ .

The above observations now supply the proof of theorem 7, as stated by D. J. H. Garling [13; pp. 1010], which we shall now prove. For further interesting results on sequence spaces we refer the reader to [22; 30] and [23]. Theorem 7 of D. J. H. Garling reads as follows:

Theorem. If  $\lambda$  is a sequence space containing  $\phi$ , and  $\tau$  is a regular vector topology on  $\lambda$  for which the projection maps are co-ordinatewise continuous, then if  $A \subset \lambda$ , the following are equivalent:

- (i).  $A$  is RCC.
- (ii).  $A$  is RSC.
- (iii).  $A$  is relatively compact.
- (iv). If  $A$  is co-ordinatewise bounded and  $(x^n) \subset A$ , and  $x_i^n \rightarrow x_i^o$  for each  $i$ , then  $x^n \rightarrow x^o$  with respect to  $\tau$ , where  $x^o = (x_i^o)$ .

Proof. By our previous remarks, since  $(\lambda, \tau)$  is a Fremlin space, the equivalence of (i), (ii) and (iii) is immediate.

We shall show that (iv)  $\Leftrightarrow$  (ii).

(iv)  $\Rightarrow$  (ii). Let  $A$  be co-ordinatewise bounded, and let  $(x^n) \subset A$  with  $x_i^n \rightarrow x_i^o$  for each  $i$ . Then  $(x_i^n)$  is a bounded sequence in the scalars for each  $i$ . Therefore, by a diagonal process we can select a subsequence  $(x^{nn})$  of  $(x^n)$  such that  $x_i^{nn} \rightarrow x_i^n$  for each  $i$ . But then  $x^{nn} \rightarrow x^o$  with respect to  $\tau$ . Therefore  $A$  is RSC.

(ii)  $\Rightarrow$  (iv). Suppose that  $A$  is RSC. Then  $A$  is bounded. Let  $(x^n) \subset A$  with  $x_i^n \rightarrow x_i^o$  for each  $i$ . Then we know that there exists a subsequence  $(x^{n_k})$  of  $(x^n)$  such that  $x^{n_k} \rightarrow y$ .



Hence  $x_i^{n_k} \rightarrow y_i$  for each  $i$ . Also,  $x_i^{n_k} \rightarrow x_i^0$  for each  $i$ . Therefore  $x^0 = y$ . In the same way every subsequence of  $(x^n)$  contains a subsubsequence converging to  $x^0$ . Thus,  $x^n \rightarrow x^0$  with respect to  $\tau$ , and the proof is complete.

Example. Consider  $\ell^1 = \{ (x_i) \in \omega \mid \sum |x_i| < \infty \} \supset \varphi$ . Then  $(\ell^1)^+ = \ell^\infty = (\ell^1, \| \cdot \|)$  where the norm on  $\ell^1$  is defined by:

$$\| x \| = \sum |x_i| < \infty \quad x \in \ell^1 .$$

Then by corollary 1.1.10 every  $\sigma(\ell^1, \ell^\infty)$ -compact set is compact. This is so since  $(\ell^1, \sigma(\ell^1, \ell^\infty))$  and  $(\ell^1, \| \cdot \|)$  have the same convergent sequences. See [22; 22; 4] and J. B. Conway [5], who recently gave a different proof of this result, by making use of the Baire Category theorem.

2. Let  $E$  be a convex separable space, with a countable dense subset  $A$ . Then  $\sigma(E', A) \leq \sigma(E', E)$ , and since  $A$  is dense in  $E$ ,  $\sigma(E', A)$  is  $T_2$ . As  $A$  is countable,  $\sigma(E', A)$  is metrisable. Thus by corollary 1.1.9,  $(E', \sigma(E', E))$  is a Fremlin space.

## 2. The Eberlein-Šmulian Theorem.

In studying the Fremlin space, we were, quite unexpectedly led to the consideration of the Eberlein-Šmulian theorem:

If  $X$  is a Banach space, dual  $X'$  and  $A \subset X$ , then

- (i).  $A$  is  $\sigma(X, X')$ -RCC.
- (ii).  $A$  is  $\sigma(X, X')$ -relatively compact.
- (iii).  $A$  is  $\sigma(X, X')$ -RSC are all equivalent.

There are in existence several proofs of this theorem and its generalisation to Fréchet spaces. These we do not propose to give here. See for example [4], [25], and [32].

We should like however, to comment only on one of these - an elementary proof given in 1967, for the case of a Banach space, by R. Whitley. We shall obtain this theorem, for the case of a Fréchet space, by employing a method analogous to one which was used by him.

Lemma 1.2.1. If  $(E, \tau)$  is a convex space, then:

$(E, \tau)$  is separable if and only if there exists a countable, total subset  $A$  in  $E$ .

Proof. The necessity is obvious. Sufficiency: If  $A$  is a countable, total subset in  $E$ , take  $B$  as the set of all rational, finite linear combinations of members in  $A$ . Then  $B$  is countable and total in  $E$  since  $\text{lin } A \subset \bar{B}$ . (This is so since scalar multiplication in  $E$  is continuous ).

Lemma 1.2.2. If  $(E, \tau)$  is a separable, convex, metrisable space then  $(E', \sigma(E', E))$  is separable.

Proof. See [22; 21; 3(5)].

Remark. If  $(E, \tau)$  is a separable, convex, metrisable space, then  $(E, \sigma(E, E'))$  is a Fremlin space. This follows from lemma 1.2.2 and application 2 of section 1.

Eberlein's Theorem. If  $(E, \tau)$  is a convex space such that  $(E, \mu(E, E'))$  is complete, then the  $\sigma(E, E')$ -RCC subsets of  $E$  are  $\sigma(E, E')$ -relatively compact. (We refer to [27; ch. 6; 1.4])

Theorem 1.2.3. Let  $(E, \tau)$  be a Fréchet space, dual  $E'$ . Then for every  $A \subset E$  the following are equivalent:

- (i).  $A$  is  $\sigma(E, E')$ -RCC.
- (ii).  $A$  is  $\sigma(E, E')$ -relatively compact.
- (iii).  $A$  is  $\sigma(E, E')$ -RSC.

Proof. (i)  $\Rightarrow$  (ii). This is a direct consequence of Eberlein's theorem, which we have stated above, and which shall be discussed in chapter 111.

(iii)  $\Rightarrow$  (i). This is implied by the definition of a RSC and a RCC set.

(ii)  $\Rightarrow$  (iii). Let us assume that  $A \subset E$  and that  $A$  is  $\sigma(E, E')$ -relatively compact. Let  $(x_n)$  be a sequence in  $A$ . Consider  $S = \overline{\text{lin}\{x_n\}}$  (closure in  $E$ ). Then  $\{x_n\}$  is a countable, total, subset of  $S$ , with respect to the relative topology on  $S$ . Hence, by lemma 1.2.1,  $S$  is a separable Fréchet space, and by lemma 1.2.2  $(E', \sigma(E', S))$  is separable. Thus in  $S$ , the  $\sigma(S, E')$ -compact sets are  $\sigma(S, E')$ -metrizable. Also,  $\text{lin}\{x_n\}$  is convex, so  $S$  is  $\sigma(E, E')$ -closed. Now,  $S \cap \bar{A}$ , (closure in  $\sigma(E, E')$ ), is a subset of  $\bar{A}$ . Therefore,  $\bar{A} \cap S$  is  $\sigma(E, E')$ -compact. So,  $A \cap S$  is  $\sigma(S, E')$ -relatively compact. In a metric space, the relatively compact, relatively sequentially compact and relatively countably compact sets all coincide. Hence,  $A \cap S$  is  $\sigma(S, E')$ -RSC. Consequently, there exists a subsequence  $(x_{n_k})$  of  $(x_n)$  such that  $x_{n_k} \rightarrow x \in S$ , with respect to  $\sigma(S, E')$ . But  $\sigma(E, E')|_S = \sigma(S, E')$ . It follows that  $x_{n_k} \rightarrow x$  with respect to  $\sigma(E, E')$ , and that  $A$  is  $\sigma(E, E')$ -relatively sequentially compact.

The above proof is analogous to a lemma of R. Whitley, [32].

Let  $S$  be a subspace of a complete convex space  $E$ . If  $A \subset S$ , we shall say that  $S$  satisfies property (A) if and only if the following are equivalent:

- (i).  $A$  is  $\sigma(S, E')$ -RCC.
- (ii).  $A$  is  $\sigma(S, E')$ -relatively compact.
- (iii).  $A$  is  $\sigma(S, E')$ -RSC.

Proposition 1.2.4. Let  $(E, \tau)$  be a complete convex space, with dual  $E'$ . Then, (A) holds for  $E$  if and only if (A) holds for each closed, separable subspace  $S$  of  $E$ .

Proof. Let  $S$  be a closed, separable subspace of  $E$ .

We must show that  $S$  satisfies property (A).

(i)  $\Rightarrow$  (ii). Let  $A \subset S$ , and assume that  $A$  is  $\sigma(S, E')$ -RCC. Then  $A$  is  $\sigma(E, E')$ -RCC. By hypothesis  $A$  is  $\sigma(E, E')$ -relatively compact, that is,  $\bar{A}$  (closure in  $\sigma(E, E')$ ) is  $\sigma(E, E')$ -compact.

$$\text{Now, } \bar{A} \text{ (closure in } \sigma(S, E') \text{)} = S \cap \bar{A} \text{ (closure in } \sigma(E, E') \text{)},$$

and  $S \cap \bar{A}$  is contained in  $\bar{A}$  (closure in  $\sigma(E, E')$ ).

Since  $S$  is  $\tau$ -closed and convex,  $S$  is  $\sigma(E, E')$ -closed.

Hence  $\bar{A}$  (closure in  $\sigma(S, E')$ ) is  $\sigma(E, E')$ -closed. Thus,  $\bar{A}$  is  $\sigma(E, E')$ -compact. As  $\sigma(S, E') = \sigma(E, E')|_S$ , we have that  $A$  is  $\sigma(S, E')$ -relatively compact.

(ii)  $\Rightarrow$  (iii). Suppose that  $A \subset S$  and that  $A$  is  $\sigma(S, E')$ -relatively compact. Then  $\bar{A}$  (closure in  $\sigma(S, E')$ ) is  $\sigma(E, E')$ -compact. By property (A) in  $E$ ,  $\bar{A}$  is  $\sigma(E, E')$ -RSC.

who obtained that if:

$A$  is a weakly-RCC subset of a Banach space  $X$ , then  $A$  is weakly-RSC, weakly-relatively compact and weakly-sequentially dense, where the latter means, that if  $x \in \bar{A}$ , then there exists a sequence  $(x_n) \subset A$  such that  $x_n \rightarrow x$  with respect to the weak topology on  $X$ .

We thus observe that if  $X$  is a Banach space with dual  $X'$ , then  $(X, \sigma(X, X'))$  is a Fremlin space. Further, theorem 1.1.8 yields that  $X$  is reflexive if and only if every bounded subset of  $X$  is  $\sigma(X, X')$ -relatively compact, (respectively  $\sigma(X, X')$ -RSC, respectively  $\sigma(X, X')$ -RCC). Moreover, suppose that  $X$  is a reflexive Banach space. Then  $(X', \beta(X', X))$  is a Banach space with  $\beta(X', X)$  equal to  $\mu(X', X)$ . Thus  $(X', \sigma(X', X''))$  is a Fremlin space. But since  $X = X''$ ,  $(X', \sigma(X', X))$  is a Fremlin space.

Note. In general,  $X$  a Banach space does not imply that  $(X', \sigma(X', X))$  is a Fremlin space.

Example. Let  $X = \ell^\infty$  with the supremum norm. Then  $\ell' \subset (\ell^\infty)'$ . Consider  $\{e_n\}_n \subset \ell'$  where  $e_n = 1$  at the  $n^{\text{th}}$  co-ordinate and  $e_n = 0$  at all other co-ordinates. Now  $\{e_n\}$  is  $\sigma(\ell', \ell^\infty)$ -bounded, hence  $\sigma((\ell^\infty)', \ell^\infty)$ -bounded.

Since  $\ell^\infty$  is barrelled, the  $\sigma((\ell^\infty)', \ell^\infty)$ -bounded sets are equicontinuous, and thus  $\{e_n\}$  is  $\sigma((\ell^\infty)', \ell^\infty)$ -relatively compact. However,  $\{e_n\}$  is not  $\sigma((\ell^\infty)', \ell^\infty)$ -RSC:

Suppose that  $e_{n_k} \rightarrow x \in (\ell^\infty)'$  with respect to  $\sigma((\ell^\infty)', \ell^\infty)$ , for some subsequence  $(e_{n_k})$  of  $(e_n)$ . Then  $e_{n_k}^i \rightarrow x^i$  for each  $i$ . But then  $x^i = 0$  for each  $i$ . Therefore,  $e_{n_k} \rightarrow 0$  with respect to  $\sigma((\ell^\infty)', \ell^\infty)$ . This is impossible since  $e = (1, 1, 1, \dots) \in \ell^\infty$  and  $\langle e_{n_k}, e \rangle = 1$ . So,  $\langle e_{n_k}, e \rangle \not\rightarrow 0$ , and  $e_{n_k} \not\rightarrow 0$  in terms of  $\sigma((\ell^\infty)', \ell^\infty)$ . This shows that we have in  $(\ell^\infty)'$  a  $\sigma((\ell^\infty)', \ell^\infty)$ -relatively compact set which is not  $\sigma((\ell^\infty)', \ell^\infty)$ -RSC. Consequently, by theorem 1.1.8 (ii),  $((\ell^\infty)', \sigma((\ell^\infty)', \ell^\infty))$  is not a Fremlin space.

The observations we have made above, together with H. B. Cohen's paper, gave rise to two natural questions:

(i). If  $(E, \tau)$  is a Fréchet space, with dual  $E'$ , is  $(E, \sigma(E, E'))$  a Fremlin space?

(ii). If  $(E, \tau)$  is a reflexive Fréchet space, can we say that  $(E', \sigma(E', E))$  is a Fremlin space?

Certainly, for  $(E, \tau)$  separable  $(E', \sigma(E', E))$  is a Fremlin space. Until now, we have been unable to settle problem (ii) and it therefore remains for us, an open question. The answer to (i) is to be found in the following theorem.

Lemma 1.2.5. Let  $G$  be the family of all continuous functions on a compact space  $S$  to a compact metric space  $Z$ . Assume that  $G$  has the topology of pointwise convergence, and that  $F$  is a subfamily of  $G$  such that each sequence in  $F$  has a cluster point in  $G$ . Then each  $f \in \overline{F}$  is a cluster point of a sequence in  $F$ , and each cluster point of a sequence in  $F$  is the limit of a subsequence.

Proof. See [19; ch. 2; 8.20.]

Theorem 1.2.6. If  $(E, \tau)$  is a Fréchet space with dual  $E'$ , then  $(E, \sigma(E, E'))$  is a Fremlin space.

Proof. We wish to show that  $(E, \sigma(E, E'))$  satisfies the conditions (Fi) and (Fii) of a Fremlin space.

(Fi). Since  $(E, \tau)$  is complete and  $\tau = \mu(E, E')$ , by Eberlein's theorem the  $\sigma(E, E')$ -RCC subsets of  $E$  are  $\sigma(E, E')$ -relatively compact.

(Fii). We shall prove that if  $E$  is any metrisable, convex space, then  $(E, \sigma(E, E'))$  satisfies (Fii). The proof shall be given in full. An outline of it is to be found in [19; ch. 5; 17L.]

Let  $(U_n)_n$  be a decreasing sequence of basic neighbourhoods of 0 in  $E$ . Suppose that  $A \subset E$  and that  $A$  is  $\sigma(E, E')$ -relatively compact. Then  $E' = \bigcup_n U_n^0$ , and



each  $U_n^0$  is  $\sigma(E', E)$ -compact. Each  $x \in A$  can be considered as a continuous linear functional on each  $(U_n^0, \sigma(U_n, E))$  into the scalars, (metric space). Let  $x \in \bar{A}$  ( $\sigma(E, E')$ -closure). By lemma 1.2.5  $x$  is a  $\sigma(E, E')$  cluster point of a sequence in  $A$  and is the limit of a subsequence of this sequence. Hence for each  $n$ , there exists a sequence  $(x_j^n) \subset A$  converging pointwisely ( $\sigma(E, U_n^0)$ ), to  $x$  on  $U_n^0$ . Enumerate the points of all these sequences to obtain a sequence  $(x_k) \subset A$ . Let  $H = \overline{\text{lin } \{x_k\}}$ . Then  $H$  is a separable Fréchet space. Thus by lemma 1.2.2  $(E', \sigma(E', H))$  is separable. Suppose that  $D = \{f_n\}_n$  is a  $\sigma(E', H)$  dense, countable subset of  $E'$ . Now  $x \in \overline{\{x_k\}}$  (closure in  $\sigma(E, E')$ ), therefore  $f(x) \in \overline{f\{x_k\}} \subset \overline{f\{x_k\}}$  which is bounded for all  $f \in D$ . Hence there exists a subsequence  $(x_{1k})$  of  $(x_k)$  with  $\langle x_{1k}, f_1 \rangle \rightarrow \langle x, f_1 \rangle$ . Also,  $\{\langle x_{1k}, f_2 \rangle\}$  is bounded, so there exists a convergent subsequence  $\{\langle x_{2k}, f_2 \rangle\}$  of  $\{\langle x_{1k}, f_2 \rangle\}$  such that  $\langle x_{2k}, f_2 \rangle \rightarrow \langle x, f_2 \rangle$ . Continuing thus, by the diagonal process we can obtain a subsequence  $(x_{k_i})$  of  $(x_k)$  so that  $\langle x_{k_i}, f \rangle \rightarrow \langle x, f \rangle$  for all  $f \in D$ . Since  $(x_{k_i} - x) \in H$  and  $D$  is  $\sigma(E', H)$  dense in  $E'$ , given any  $g \in E'$ , there exists  $f \in D$  such that  $f$  belongs to  $g + \frac{1}{n} \{x_{k_n} - x\}^0$ . Then  $|\langle x_{k_n} - x, f - g \rangle| \leq \frac{1}{n}$ . Let  $n \rightarrow \infty$ , then  $\langle x_{k_n}, g \rangle \rightarrow \langle x, g \rangle$ . Thus we have

that  $\langle x_{k_n}, g \rangle \rightarrow \langle x, g \rangle$  for all  $g \in E'$ . Now  $\{x_{k_n}\}$  has at least one  $\sigma(E, E')$  cluster point. Let  $y \in \overline{\{x_{k_n}\}}$  (closure in  $\sigma(E, E')$ ) and  $g \in E'$ . For each  $j$ , there exists a subsequence  $(x_{k_{n_j}})$  of  $(x_{k_n})$  such that  $x_{k_{n_j}} \in y + \frac{1}{j} \{g\}^0$ . So,  $|\langle x_{k_{n_j}}, g \rangle - \langle y, g \rangle| \leq \frac{1}{j}$ . Therefore, for every  $g \in E'$ , there exists a subsequence  $(x_{k_{n_j}})$  of  $(x_{k_n})$  with  $\langle x_{k_{n_j}}, g \rangle \rightarrow \langle y, g \rangle$ . But  $\langle x_{k_{n_j}}, g \rangle \rightarrow \langle x, g \rangle$ , so for all  $g \in E'$   $\langle x, g \rangle$  equals  $\langle y, g \rangle$ . Thus  $x = y$  and  $\overline{\{x_{k_n}\}}$  (closure in  $\sigma(E, E')$ ) has a unique cluster point. Since  $\overline{\{x_{k_n}\}}$  is  $\sigma(E, E')$ -compact, it follows that  $x_{k_n} \rightarrow x$  with respect to  $\sigma(E, E')$ . Hence (Fii) is satisfied and  $(E, \sigma(E, E'))$  is a Fremlin space.

### Remarks.

(i). If  $(E, \tau)$  is a Fréchet space, then since  $(E, \sigma(E, E'))$  is a Fremlin space, by theorem 1.1.8 (ii) we have that the  $\sigma(E, E')$ -relatively compact,  $\sigma(E, E')$ -RSC and the  $\sigma(E, E')$ -RCC subsets of  $E$  all coincide - thus obtaining the Eberlein-Šmulian Theorem for the case of a Fréchet space, in an essentially different manner to that of theorem 1.2.3.

(ii). The generalisation of H. B. Cohen's result (which has previously been mentioned) to a Fréchet space is now an immediate consequence of theorem 1.2.6. In conclusion, we remark that the proof of theorem 1.2.6 is not analogous to that of H. B. Cohen, given for a Banach space, since this proof relies heavily on the use of the norm.

## Chapter 11.

In our previous chapter, we saw that every separable convex space  $E$ , gives rise to a Fremlin space, namely,  $(E', \sigma(E', E))$  in its dual. In particular, when  $E$  is a separable, metrisable, convex space then both  $(E, \sigma(E, E'))$  and  $(E', \sigma(E', E))$  are Fremlin spaces.

In section 1, we shall characterise separable (LF) and separable, metrisable, convex spaces. We then proceed to section 2 in which a publication by C. L. DeVito [7], will be enlarged upon and discussed. This publication is concerned with Eberlein's Theorem and conditions under which the  $\sigma(E, E')$ -relatively countably compact subsets of a convex space  $E$  are  $\sigma(E, E')$ -relatively compact. We note in passing, that the equivalence of the  $\sigma(E, E')$ -relatively compact and  $\sigma(E, E')$ -relatively countably compact subsets of a convex space  $E$ , is the first condition (F<sub>1</sub>), which must be satisfied whenever  $E$  with its weak topology is to be a Fremlin space.

### 1. Separable (F) and (LF) Spaces.

If  $(E, \tau)$  is a separable convex space, with dual  $E'$ , then the  $\sigma(E', E)$ -compact subsets of  $E'$  are  $\sigma(E', E)$ -metrisable.

The converse statement is, in general, false:

Example. Consider  $E = \prod \mathbb{R}$ , the topological product of uncountably many lines.  $E' = \sum \mathbb{R}$ . The bounded sets in  $E'$  are finite dimensional, and hence  $\sigma(E', E)$ -metrisable. Yet  $E$  is a non-separable, non-metrisable, reflexive convex space.

We shall now see that the situation is somewhat different when  $E$  is a metrisable convex space.

Theorem 2.1.1. If  $(E, \tau)$  is a metrisable convex space, in the dual of which the equicontinuous sets are  $\sigma(E', E)$ -metrisable then  $E$  is separable.

Proof. Let  $(U_n)_n$  be a sequence of basic neighbourhoods of 0 in  $E$ . By hypothesis each  $U_n^0$  is  $\sigma(E', E)$ -metrisable. Thus for each  $n$ , there exists a sequence  $(A_j^n)_j$  of finite subsets of  $E$  such that  $(U_n^0 \cap (A_j^n)^0)_j$  is a local base at 0 for  $\sigma(E', E)|_{U_n^0}$ . Let  $B = \cup A_j^n$  where the union is taken over  $n$  and  $j$ . Then  $B$  is countable and total in  $E$ ; for suppose that  $x \in E'$ ,  $x \neq 0$ . Then  $x \in U_n^0$  for some  $n$ . Since  $U_n^0$  is  $\sigma(E', E)$ -metrisable, there exists an  $A_j^n$  with  $x \notin (A_j^n)^0 \cap U_n^0$ . Then  $x \notin (A_j^n)^0$ . Therefore, we can find  $y \in (A_j^n)$  with  $|\langle y, x \rangle| > 1$ . And so, there

exists  $\forall \epsilon \in B$  with  $|\langle y, x \rangle| > 1$ . Hence  $B$  is total in  $E$ , and by lemma 1.2.1  $E$  is separable.

Corollary 2.1.2. Let  $E$  be a metrisable convex space, with dual  $E'$ . If the equicontinuous subsets of  $E'$  are  $\sigma(E', E)$ -metrisable then both  $(E, \sigma(E, E'))$  and  $(E', \sigma(E', E))$  are Fremlin spaces.

Proof. By theorem 2.1.1  $E$  is separable, thus  $(E', \sigma(E', E))$  is separable, (lemma 1.2.2). It follows now that both  $E$  and  $E'$  with their weak topologies are Fremlin spaces.

Remarks.

(i). The proof of theorem 2.1.1 has been adapted from an article by I. Amemiya [1], where it is shown that if  $E$  is a (F) space such that in  $E'$  every bounded set is  $\beta(E', E)$ -metrisable, then  $E$  contains a bounded, total set.

(ii). An alternative proof of theorem 2.1.1 can be obtained by embedding  $E$  isomorphically into  $\prod_n C(U_n^0)$  - the topological product of countably many separable Banach spaces,  $C(U_n^0)$  being the space of all continuous maps from  $U_n^0$  with the topology  $\sigma(E', E)|_{U_n^0}$  on  $U_n^0$ , into the scalar field, where  $(U_n)$  is a sequence of basic neighbourhoods of 0 in  $E$ , and  $C(U_n^0)$  has the uniform norm topology. This proof is

however more complicated. Refer to [29] and [30; th. 7].

(iii). It is clear that a Banach space  $E$  in which the unit ball is separable is itself separable. In the case of a non-complete, metrisable convex space  $E$ , J. Dieudonné [8], has shown that the separability of the bounded sets does not give the separability of  $E$ . Y. Komura [21; 3], has constructed an example of a Fréchet space which is non-separable but has all its bounded sets separable. It is interesting to note that in both examples use was made of the continuum hypothesis.

Theorem 2.1.3. Let  $(E, \tau)$  be an (LF) space, with defining sequence  $(E_n, \tau_n)$ . Then the following are equivalent:

1.  $(E, \tau)$  is separable.
2.  $(E, \sigma(E, E'))$  is separable.
3. The  $\sigma(E', E)$ -compact subsets of  $E'$  are  $\sigma(E', E)$ -metrisable.
4. Each  $(E_n, \tau_n)$  is separable.

Proof.  $1 \Rightarrow 2 \Rightarrow 3$  is straightforward.

$3 \Rightarrow 4$ . Assume that the  $\sigma(E', E)$ -compact subsets of  $E'$  are  $\sigma(E', E)$ -metrisable. Let  $\mathcal{M}$  be the class of all the absolutely convex  $\sigma(E', E)$ -compact subsets of  $E'$ . Then  $\mathcal{M}$  generates the Mackey topology on  $E$ , which is in fact, equal to  $\tau$ . Let us fix  $n$ . Since  $(E_n, \tau_n)$  is a Fréchet space, and

$\tau_n = \tau|_{E_n}$  there is a countable subfamily  $(M_n)$  of  $\mathfrak{m}$  such that for every  $V$ , a  $\tau_n$ -neighbourhood of  $0$  in  $E_n$  we can find a  $M_k$  with  $M_k^0$  (polar in  $E_n$ ) contained in  $V$ . Furthermore, for each  $k$ ,  $M_k$  is  $\sigma(E', E)$ -compact. Also,  $\sigma(E', E) \geq \sigma(E'_n, E_n)|_{E'}$  and  $\sigma(E'_n, E_n)|_{E'}$  is  $T_2$ . Hence,  $\sigma(E', E) = \sigma(E'_n, E_n)|_{E'}$  on each  $M_k$ . By hypothesis every  $M_n$  is  $\sigma(E', E)$ -metrizable. Thus every  $M_n$  is  $\sigma(E'_n, E_n)$ -metrizable. By theorem 2.1.1,  $(E_n, \tau_n)$  is separable.

All this is true for each  $n$ , and the proof is now complete.

4  $\Rightarrow$  1. We shall show that in general, the strict inductive limit of separable spaces is separable. Assume that  $(E, \tau)$  is the strict inductive limit of a sequence of separable spaces  $(E_n, \tau_n)$ . For each  $n$ , let  $A_n$  be a countable, dense subset of  $E_n$ . Let  $A = \bigcup_n A_n$ . Then  $A$  is a countable, dense subset of  $(E, \tau)$ :

Let  $x \in E$  and let  $V$  be an open neighbourhood of  $x$ .

Then  $x \in E_n$  for some  $n$ . Since  $V \cap E_n$  is an open neighbourhood of  $x$  in  $E_n$ , and  $A_n$  is dense in  $E_n$ ,  $V \cap E_n \cap A_n \neq \emptyset$ . Therefore,  $V \cap A \neq \emptyset$  and  $(E, \tau)$  is separable.

#### Remarks.

(i). If  $(E, \tau)$  is a (LF) space with defining sequence  $(E_n, \tau_n)$ , and each  $E_n$  is separable, then  $(E, \tau)$  is complete



and separable and  $(E', \sigma(E', E))$  is a Fremlin space. (The completeness of  $E$  is a property of the strict inductive limit of complete convex spaces, (see 27; ch. 7; 1.3).

(ii). If  $(E, \tau)$  is the strict inductive limit of convex Fremlin spaces  $(E_n, \tau_n)$  and each  $E_n$  is closed in  $E_{n+1}$  then  $(E, \tau)$  is a Fremlin space. This is a consequence of the fact that the  $\tau$ -RCC subsets of  $E$  are bounded, and hence contained on some  $E_n$ . (Refer to [27; ch. 7; 1.4.]). Thus, since  $\tau|_{E_n} = \tau_n$  it is not difficult to show that property (Fi) of a Fremlin space is satisfied in  $(E, \tau)$ . Property (Fii) is similarly obtained.

(iii). Every (LF) space is a Fremlin space. Recall that theorem 1.2.6 gave that every (F) space with its weak topology is a Fremlin space. We may ask whether this is true for a (LF) space with its weak topology. Certainly, (Fi) and (Fii) hold for the convex sets. We leave this question unanswered.

## 2. Notes on Eberlein's Theorem.

Recently, a publication by C. L. DeVito [7], appeared, in which Eberlein's theorem was obtained for a certain class of convex spaces, those having property S, where property S was defined as follows:

Definition 2.2.1. Let  $(E, \tau)$  be a convex space with bidual  $E''$ . Then  $E$  has property S if and only if for every  $\omega \in E''$  such that  $\omega(f_n) \rightarrow 0$  whenever  $f_n \rightarrow 0$  with respect to  $\sigma(E', E)$  and  $(f_n)_n$  is  $\beta(E', E)$ -bounded, we have  $\omega \in E$ .

We state below the main results of this work.

1. If  $E$  is a convex space with property S, then every  $\sigma(E, E')$ -closed, countably compact subset of  $E$  is  $\sigma(E, E')$ -compact.
2. If  $E$  is a convex space and  $\gamma(E, E')$  is the strong topology on  $E''$ , restricted to  $E$ , and furthermore,  $(E, \gamma(E, E'))$  is separable, then  $E$  has property S if and only if  $E$  is a closed linear subspace of  $(E'', \beta(E'', E'))$ .
3. If  $(E, \gamma(E, E'))$  is both complete and separable then  $E$  has property S.

Definition 2.2.2. We shall say that a convex space  $(E, \tau)$  has property S\* if and only if for every  $\omega \in E''$  such that  $\omega(f_n) \rightarrow 0$  whenever  $f_n \rightarrow 0$  with respect to  $\sigma(E', E)$ , then  $\omega \in E$ .

Remarks.

(i). If  $E = [E', \sigma(E', E)]^+$  then clearly  $E$  has property  $S^*$ .

(ii). Property  $S \Leftrightarrow$  Property  $S^*$ .

(iii). If  $E$  is a barrelled or complete convex space then property  $S \Leftrightarrow$  property  $S^*$ , since under these conditions, the  $\beta(E', E)$ -bounded and the  $\sigma(E', E)$ -bounded subsets of  $E'$  coincide.

In this section, we intend first of all, to demonstrate that DeVito's result 1 above, is true for a convex space having property  $S^*$ . Having done this, we shall proceed to discuss, generally, convex spaces with this property.

Theorem 2.2.3. If  $E$  is a convex space with property  $S^*$ , then every  $\sigma(E, E')$ -closed, countably compact subset of  $E$  is  $\sigma(E, E')$ -compact. (We shall in future write  $E$  for  $(E, \tau)$ ).

Proof. Let  $M$  be a  $\sigma(E, E')$ -closed, countably compact subset of  $E$ . Let  $\omega \in \bar{M}$  (closure in  $\sigma(E'', E')$ ). Then, since  $M$  is bounded,  $M^{\circ}$  (polar in  $E'$ ) is a  $\beta(E', E)$ -neighbourhood of  $0$ . Hence,  $M^{\circ\circ}$  (bipolar in  $E''$ ) is a  $\sigma(E'', E')$ -compact subset of  $E''$ . We must show that  $\omega \in E$ . Let  $f_n \rightarrow 0$  in terms of  $\sigma(E', E)$ , where  $(f_n) \subset E'$ . Then

$\{ f_1, \dots, f_n \}^{\circ}$  (polar in  $E''$ ) is a  $\sigma(E'', E')$ -neighbourhood of 0, for each  $n$ . Therefore, for each  $k$ , there exists  $x_k \in M$  with  $x_k \in \omega + \frac{1}{k} \{ f_1, \dots, f_k \}^{\circ}$ . Then  $\| \langle x_k - \omega, f_n \rangle \| \leq \frac{1}{k} \quad | \leq n \leq k$ . Since  $M$  is  $\sigma(E, E')$ -countably compact,  $(x_k)_k$  has a  $\sigma(E, E')$ -cluster point  $x_0 \in M$ . Thus for each fixed  $n$ ,  $\langle x_k, f_n \rangle \rightarrow \langle \omega, f_n \rangle$ , so  $\langle x_0, f_n \rangle = \langle \omega, f_n \rangle$  for each  $n$ . Now  $f_n \rightarrow 0$  with respect to  $\sigma(E', E)$ , and  $x_0 \in E$ . Hence  $\langle x_0, f_n \rangle \rightarrow 0$ , and so  $\omega(f_n) \rightarrow 0$ . Since  $E$  has property  $S^*$ , it follows that  $\omega \in E$ . Thus  $M$  is  $\sigma(E, E')$ -compact.

Let  $(E, \tau)$  be a convex space. Let  $(E')^b$  equal  $(E', \sigma(E', E))^b$  = family of bounded linear functionals on  $(E', \sigma(E', E))$ . Then  $(E, \beta(E, E'))$  is a topological subspace of  $( (E')^b, \beta( (E')^b, E' ) )$ , since the  $\sigma(E', E)$ -bounded subsets of  $E'$  are  $\sigma(E', (E')^b)$ -bounded and vice versa.

Proposition 2.2.4. Let  $(E, \tau)$  be a convex space such that  $(E, \beta(E, E'))$  is separable. Then  $E$  has property  $S^*$  if and only if  $E$  is a closed linear subspace of  $(E'', \beta( (E')^b, E' )|_{E''}) = F$ .

Proof. Necessity: Let  $\omega \in \bar{E}$  (closure in  $F$ ). Then

$\omega \in \overline{E} \cap E''$  (closure in  $((E')^b, \beta((E')^b, E'))$ ). Let  $(f_n) \subset E'$  such that  $(f_n)$  is  $\sigma(E', E)$ -convergent to 0. Then  $(f_n)$  is  $\sigma(E', E)$ -bounded and hence  $\sigma(E', (E')^b)$ -bounded. Thus given  $k \in \mathbb{N}$  there exists  $x_k \in E$  with  $x_k \in \omega + \frac{1}{k} \{f_n\}^0$  (polar in  $E''$ ). Therefore,

$$| \langle x_k - \omega, f_n \rangle | \leq \frac{1}{k} \quad \forall n.$$

$$\therefore | \langle \omega, f_n \rangle | \leq | \langle x_k, f_n \rangle | + \frac{1}{k} \quad \forall n.$$

Given  $\varepsilon > 0$ . Choose  $k$  such that  $\frac{1}{k} < \frac{1}{2}\varepsilon$ .

Since  $\langle x_k, f_n \rangle \rightarrow 0$ ,  $\exists N: n > N \Rightarrow | \langle x_k, f_n \rangle | < \frac{1}{2}\varepsilon$ .

Then  $n > N \Rightarrow | \langle \omega, f_n \rangle | < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon$ .

Therefore,  $\omega(f_n) \rightarrow 0$ . By property  $S^*$ ,  $\omega \in E$ .

Sufficiency: Let  $\omega \in E''$  such that  $\omega(f_n) \rightarrow 0$  whenever  $f_n \rightarrow 0$  in terms of  $\sigma(E', E)$ ,  $(f_n) \subset E'$ . .....(a). Since  $(E, \beta(E, E'))$  is separable, the  $\sigma(E', E)$ -bounded subsets of  $E'$  are  $\sigma(E', E)$ -metrisable. By (a)  $\omega$  is  $\sigma(E', E)$ -sequentially continuous, and hence  $\sigma(E', E)$ -continuous on the  $\sigma(E', E)$ -bounded subsets of  $E'$ . Thus,  $\omega \in E'' \cap (\overline{E, \beta(E, E')})$  where  $(\overline{E, \beta(E, E')})$  is the completion of  $(E, \beta(E, E'))$ . Since  $E$  is closed in  $F$ , it follows that  $\omega \in E$ , and the proposition is proved.

Corollary 2.2.5. If  $(E, \tau)$  is a convex space, such that  $(E, \beta(E, E'))$  is separable and complete, then  $E$  has property  $S^*$ .

Corollary 2.2.5 is an immediate consequence of the previous proposition. Observe that the approach of the proof of the above proposition, makes use of the fact that  $\beta(E, E')$  is a  $(E, E')$  polar topology. We inquire now, as to whether  $\beta(E, E')$  in corollary 2.2.5, can be replaced by any  $(E, E')$  polar topology on  $E$ . We answer in the affirmative.

Theorem 2.2.6. Let  $(E, \tau)$  be a convex space, with dual  $E'$ . Let  $\tau^\alpha$  be any  $(E, E')$  polar topology on  $E$ . Then if  $(E, \tau^\alpha)$  is separable and complete,  $E$  has property  $S^*$  and thus the  $\sigma(E, E')$ -closed countably compact subsets of  $E$  are  $\sigma(E, E')$ -compact.

Proof. Let  $(\widetilde{E}, \widetilde{\tau}^\alpha)$  be the completion of  $(E, \tau^\alpha)$ . By Grothendieck's completeness theorem we have that  $(\widetilde{E}, \widetilde{\tau}^\alpha)$  is the set of all  $\omega \in F^*$  such that  $\omega$  is  $\sigma(F, E)$ -continuous on the equicontinuous subsets of  $F$ , where  $F = (E, \tau^\alpha)'$ . Since  $\tau^\alpha$  is a  $(E, E')$  polar topology, the equicontinuous subsets for  $\tau^\alpha$  are subsets of  $E'$ . By the separability of  $(E, \tau^\alpha)$  and by its completeness,  $E$  is the same as  $(\widetilde{E}, \widetilde{\tau}^\alpha)$ , and  $(\widetilde{E}, \widetilde{\tau}^\alpha)$  is the set of all  $\omega \in (E')^*$  such that  $\omega$  is  $\sigma(E', E)$  continuous on the equicontinuous subsets for  $\tau^\alpha$ , this subset being equivalent to that of all  $\omega \in (E')^*$  for which  $\omega$  is  $\sigma(E', E)$  sequentially continuous on the equi-

continuous subsets for  $\tau^\alpha$ . Thus if  $\omega \in E''$  such that  $\omega(f_n) \rightarrow 0$  whenever  $f_n \rightarrow 0$  with respect to  $\sigma(E', E)$ , then  $\omega \in (\widetilde{E, \tau^\alpha}) = E$ . Therefore  $(E, \tau)$  has property  $S^*$  and by theorem 2.2.3, the  $\sigma(E, E')$ -closed, countably compact subsets of  $E$  are  $\sigma(E, E')$ -compact.

### Notes.

1. In general we have avoided the use of Eberlein's theorem in the above proofs. However, if  $\sigma(E, E') \leq \tau^\alpha \leq \mu(E, E')$  then  $(E, \tau^\alpha)$  complete implies that  $(E, \mu(E, E'))$  is complete, and Eberlein's theorem would give that the  $\sigma(E, E')$ -closed, countably compact subsets of  $E$  are  $\sigma(E, E')$ -compact.
2. Note that, while the separability of a convex space is preserved under a weaker topology, the reverse situation holds when dealing with completeness. For if  $E$  is a vector space and  $\mathcal{U}$  and  $\tau$  are  $T_2$  vector topologies on  $E$  such that  $\mathcal{U} \leq \tau$  and  $E$  has a  $\tau$ -base at 0 of  $\mathcal{U}$ -closed sets, then completeness with respect to  $\mathcal{U}$  implies completeness with respect to  $\tau$ .

Definition 2.2.7. A convex space  $(E, \tau)$  is said to be sequentially barrelled if and only if every  $\sigma(E', E)$ -null sequence in  $E'$  is an equicontinuous set for  $\tau$ .

Proposition 2.2.8. Let  $(E, \tau)$  be a sequentially barrelled convex space. Let  $\tau^\alpha \geq \tau$ , where  $\tau^\alpha$  is a  $(E, E')$  polar topology on  $E$  such that  $(E, \tau^\alpha)$  is separable. Then if  $E$  has property  $S^*$  it follows that  $E'' \cap \widetilde{(E, \tau^\alpha)} = E$ , where  $E''$  is the bidual of  $(E, \tau)$  and  $\widetilde{(E, \tau^\alpha)}$  is the completion of  $(E, \tau^\alpha)$ .

Proof. Since  $(E, \tau^\alpha)$  is separable, the  $\tau^\alpha$ -equicontinuous sets are  $\sigma(E', E)$ -metrisable. Hence,  $\widetilde{(E, \tau^\alpha)}$  is that subset of  $(E')^*$  consisting of all  $\omega$  which are  $\sigma(E', E)$  sequentially continuous on the equicontinuous sets for  $\tau^\alpha$ . Let  $\omega \in \widetilde{(E, \tau^\alpha)} \cap E''$ . Since  $(E, \tau)$  is sequentially barrelled, any sequence  $(f_n) \subset E'$  which is  $\sigma(E', E)$ -convergent to 0, is  $\tau$ -equicontinuous. Thus  $(f_n)$  is  $\tau^\alpha$ -equicontinuous. Therefore,  $\omega(f_n) \rightarrow 0$  since  $\omega \in \widetilde{(E, \tau^\alpha)}$ . By property  $S^*$ ,  $\omega \in E$ .

Further information on sequentially barrelled spaces can be found in [31; 4].

Remarks.

(i). If  $(E, \tau)$  is a sequentially barrelled convex space and  $\tau^\alpha$  is a  $(E, E')$  polar topology on  $E$ , such that  $\tau^\alpha \geq \tau$ , and  $\widetilde{(E, \tau^\alpha)}$  is



a subset of  $E''$  (the bidual of  $(E, \tau)$ ). then  $E$  has property  $S^*$  if and only if  $(E, \tau^\alpha)$  is complete.

(ii). If  $(E, \tau)$  is a sequentially barrelled convex space, and  $\tau^\alpha$  is a  $(E, E')$  polar topology on  $E$ , such that  $\tau^\alpha \geq \tau$ , and  $(E, \tau^\alpha)$  is separable, then if  $E$  coincides with  $[E', \sigma(E', E)]^+$  then  $(E, \tau^\alpha)$  is complete.

(iii). If  $E$  is a normed space, then clearly  $\tilde{E} \subset E''$ , where  $\tilde{E}$  is the completion of  $E$ . Moreover, if  $E$  is a convex space which is quasi-barrelled, then if  $(E', \beta(E', E))$  is bornological, it follows that  $\tilde{E} \subset E''$ . This was remarked by R. Nielsen [24]. A. Grothendieck [15; cor. 3, th. 1], has shown that when  $E$  is a metrisable convex space,  $\tilde{E} \subset E''$ . Note that this is not a consequence of R. Nielsen's observation above, since there does exist a metrisable convex space such that  $(E', \beta(E', E))$  is not bornological. (See [21; 4]).

The result of Grothendieck mentioned above, can be extended to a wider class of spaces than the class of metrisable convex spaces. We defer the proof of this statement until later, when we shall show that an extension to T. Husain's  $(S)$ -spaces, [16; ch. 6], can be obtained.

(iv). Proposition 2.2.8 together with our above remark on  $(S)$ -spaces, yields that a separable, barrelled,  $(S)$ -space

with property  $S^*$  is complete. In particular, a separable barrelled, metrisable convex space with property  $S^*$  will be complete. By virtue of theorem 2.1.1, this may be re-phrased as follows:

If  $E$  is a barrelled, metrisable convex space, with property  $S^*$ , such that in the dual of  $E$ , the equicontinuous sets are  $\sigma(E', E)$ -metrisable, then  $E$  is complete.

(v). We cannot improve (iv) to read:

A separable barrelled space, with property  $S^*$  is complete.

I. Amemiya and Y. Komura, [2; 4], have constructed an example of a separable, non-complete, Montel space. Such a space, by its reflexivity, will satisfy property  $S^*$ .

Let  $(E, \tau)$  be a convex space, with dual  $E'$ . We shall denote by  $\tau^f$  the finest topology on  $E'$ , coinciding with  $\sigma(E', E)$  on the  $\tau$ -equicontinuous subsets of  $E'$ . Let  $\tau^0$  be the topology on  $E'$  generated by the  $\tau$ -precompact subsets of  $E$ . (Note that the topology  $\tau^f$ , need not be locally convex, nor even linear.)

Definition 2.2.9. A convex space,  $(E, \tau)$ , is said to be a (S)-space if and only if  $\tau^0 = \tau^f$ . In such a case,  $\tau^f$  is necessarily locally convex.

When  $(E, \tau)$  is a metrisable convex space, the Banach-Dieudonné Theorem gives  $\tau^{\circ} = \tau^f$ . Hence, every metrisable convex space is a (S)-space. The converse is however, false. We refer the reader to [16; ch. 6].

Theorem 2.2.10. If  $(E, \tau)$  is a (S)-space, then  $\tilde{E} \subset E''$ , where  $\tilde{E}$  is the completion of  $(E, \tau)$ .

Proof.  $E$  consists of all  $f \in (E')^*$  which are  $\sigma(E', E)$  continuous on the equicontinuous subsets of  $E'$ . Now,  $\tau^{\circ} = \tau^f$ . Hence, if  $M \subset E'$ , then  $M$  is  $\tau^f$  ( $\tau^{\circ}$ ) closed if and only if  $M \cap A$  is  $\sigma(E', E)$ -closed in  $E'$ , for every  $\sigma(E', E)$ -closed equicontinuous set  $A$  in  $E'$ . Let  $f \in \tilde{E}$ , and let  $A$  be a  $\sigma(E', E)$ -closed equicontinuous subset of  $E'$ . Then  $f^{-1}(0) \cap A$  is  $\sigma(E', E)$ -closed in  $A$ , thus also in  $E'$ . Therefore  $f^{-1}(0)$  is  $\tau^{\circ}$ -closed. Hence  $f \in (E', \tau^{\circ})'$  and  $(E', \tau^{\circ})' \subset (E', \beta(E', E)) = E''$ . Thus  $\tilde{E} \subset E''$ .

Note that if  $(E, \tau)$  is a complete (S)-space, then the  $\tau$ -precompact subsets of  $E$  are  $\sigma(E, E')$ -relatively compact. So,  $\sigma(E', E) \leq \tau^{\circ} \leq \mu(E', E)$ , that is,  $(E', \tau^{\circ})' = E$ . It follows now, that if  $M \subset E'$  and  $M$  is convex, then  $M$  is  $\sigma(E', E)$ -closed ( $\tau^{\circ}$ -closed) if and only if  $M \cap U^{\circ}$  is  $\sigma(E', E)$ -closed in  $E'$  for each  $U$ , a  $\tau$ -neighbourhood of  $0$ .

are  $(\omega_0, \varphi)$ -compact. But the completion of  $(\omega_0, \sigma(\omega_0, \varphi))$  is  $\omega$ , thus  $(\omega_0, \sigma(\omega_0, \varphi))$  is non-complete. Furthermore,  $(\omega_0, \sigma(\omega_0, \varphi))$  is barrelled:

Let  $K$  be  $\sigma(\varphi, \omega_0)$ -bounded. Then  $K$  is  $|\sigma|(\varphi, \omega_0)$ -bounded and hence  $|\sigma|(\varphi, \omega)$ -bounded. (See [31; pp. 358]). Since  $(\omega, \sigma(\omega, \varphi))$  is semi-reflexive,  $K$  is  $\sigma(\varphi, \omega)$ -relatively compact. Thus  $K$  is  $\sigma(\varphi, \omega_0)$ -relatively compact and  $(\omega_0, \sigma(\omega_0, \varphi))$  is barrelled.

Chapter 111.

The appearance of W. Eberlein's celebrated theorem on weak compactness in Banach spaces, in 1947, led to an enlivened and renewed interest this topic. The weakly compact sets in  $(F)$  and  $(LF)$  spaces were characterised by J. Dieudonné and L. Schwartz, [9; prop. 17, 18], in 1949 - 1950. Some time later, A. Grothendieck, [14; 4.2], inspired by Eberlein's result, treated the case of a complete convex space with the Mackey topology. A concise description of the progress made in this field until 1955, can be found in H. S. Collin's publication, [4], in which the concept of pseudo-completeness is used, in order to connect the various weak compactness properties in convex spaces.

A further important contribution to the theory of weak compactness, was made by R. C. James, [18], when he answered in the affirmative a conjecture raised by V. Klee, [20], in 1962, namely,

A closed convex subset  $C$  of a Banach space  $E$  is weakly compact if and only if every continuous linear functional on  $E$  attains its supremum on  $C$  at some point of  $C$ .

James showed that this result could be extended to a complete convex space  $E$ . Employing basically the same

techniques as those of James, and leaning on Eberlein's theorem, a slightly simpler proof of this result was given by J. D. Pryce [26], in 1966. His final result states:

If  $E$  is a real convex space and  $A$  is a weakly closed subset of  $E$ , such that the closed convex hull of  $A$  is complete with respect to the Mackey topology, then  $A$  is weakly compact if and only if every continuous linear functional on  $E$  attains its supremum on  $A$ , at some point of  $A$ .

In this chapter, we shall give, firstly, a simple proof of Krein's theorem, as was recently obtained by I. Tweddle [28], and which relies on the results of R. C. James and J. D. Pryce, which we have mentioned above. Secondly, we shall give a proof of the theorem now known as Eberlein's Compactness theorem, and at the same time give further characterisations of weak compactness in convex spaces, which we have found interesting. In this connection see [17].

### 1. Krein's Theorem.

If  $E$  is a complex convex space, then  $E$  may be considered as a real space  $E_R$ , and if  $E'_R$  is the real dual, then  $\sigma(E, E')$  and  $\mu(E, E')$  coincide with  $\sigma(E_R, E'_R)$  and  $\mu(E_R, E'_R)$  respectively.

Theorem 3.1.1. Let  $E$  be a real convex space, with dual  $E'$ . Let  $A$  be a  $\sigma(E, E')$ -compact subset of  $E$ . Then  $Y = \overline{\Delta(A)}$  (closed convex hull of  $A$ ), is  $\sigma(E, E')$ -compact if and only if  $Y$  is  $\mu(E, E')$ -complete.

Proof. The necessity is immediate, since  $Y$  is  $\sigma(E, E')$ -complete and  $\mu(E, E')$  has a local base at  $0$  of  $\sigma(E, E')$ -closed sets. Let us assume that  $Y$  is  $\mu(E, E')$ -complete. Let  $f \in E'$ . Let  $\alpha = \inf_A \langle x, f \rangle$ . Let  $\beta = \sup_A \langle x, f \rangle$ . Since  $A$  is  $\sigma(E, E')$ -compact, there exists  $x_0 \in A$ , such that  $\langle x_0, f \rangle = \beta$ . By the continuity of  $f$ , we have that  $\sup \{ \langle y, f \rangle \mid y \in \Delta(A) \} = \sup \{ \langle y, f \rangle \mid y \in \overline{\Delta(A)} \}$ . Moreover, since  $Y$  is convex, and  $\sup_{\Delta(A)} \langle y, f \rangle = \sup_A \langle y, f \rangle$   $f(Y) = [\alpha, \beta]$ . Thus,  $\sup_Y \langle y, f \rangle = \beta = \langle x_0, f \rangle$ .

So  $f$  attains its supremum on  $Y$  at some point of  $Y$ . Since  $Y$  is  $\sigma(E, E')$ -closed and  $\mu(E, E')$ -complete, on application of the result of Pryce, we obtain that  $Y$  is  $\sigma(E, E')$ -compact.

Remark. If  $E$  is a quasi-complete convex space, then if  $A$  is  $\sigma(E, E')$ -compact it follows that  $\overline{\Delta(A)}$  is  $\mu(E, E')$ -complete.

## 2. Eberlein's Compactness Theorem.

Lemma 3.2.1. If  $E$  is a real convex space, and  $K$  is a closed convex subset of  $E$ . then if  $x \notin K$ , there exists  $f \in E'$  with  $\langle x, f \rangle > \sup_K \langle y, f \rangle$ .

Proof. See [6; sect. 6, th. 5].

Theorem 3.2.2. Let  $(E, \tau)$  be a real convex space, and let  $A$  be a bounded,  $\sigma(E, E')$ -closed subset of  $E$ , such that  $\overline{\Delta(A)}$  is complete. Then the following are equivalent:

1.  $A$  is  $\sigma(E, E')$ -compact.
2.  $A$  is  $\sigma(E, E')$ -countably compact.
3. For each sequence  $(x_n) \subset A$ , there exists  $x \in A$  such that for all  $f \in E'$ ,
 
$$\underline{\lim}_n f(x_n) \leq f(x) \leq \overline{\lim}_n f(x_n).$$
4. If  $(K_n)$  is a nested sequence of closed convex sets, and  $A \cap K_n$  is non-empty for each  $n$ , then  $\bigcap_n (A \cap K_n)$  is non-empty.
5. For each sequence  $(x_k) \subset A$ , and for each equicontinuous sequence  $(f_n) \subset E'$ , we have that
 
$$\lim_n \lim_k f_n(x_k) = \lim_k \lim_n f_n(x_k)$$
 whenever all these limits exist.



Proof.  $1 \Rightarrow 2$  is immediate.

$2 \Rightarrow 3$ . Since  $A$  is  $\sigma(E, E')$ -countably compact, every sequence  $(x_n) \subset A$  has a  $\sigma(E, E')$ -cluster point  $x_0 \in A$ . Then for any  $f \in E'$ , we have that  $\lim_n f(x_n) \leq f(x_0)$  and  $f(x_0) \leq \overline{\lim_n f(x_n)}$ .

$3 \Rightarrow 4$ . Let  $(K_n)$  be a nested sequence of closed convex subsets of  $E$ , such that  $A \cap K_n = \emptyset$  for each  $n$ . For each  $n$ , select one  $x_n \in A \cap K_n$ . We thus obtain a sequence  $(x_n) \subset A$ . By hypothesis we can find  $x \in A$ , which satisfies the inequality of 3. But then  $x \in \bigcap_n K_n$ . For if not, there exists  $n$  with  $x \notin K_n$ . By lemma 3.2.1, there exists  $f \in E'$  such that  $f(x) > \sup_{K_n} f(y)$ . But then  $\overline{\lim_n f(x_n)} < f(x)$ . This contradicts 3. Hence  $x \in \bigcap_n (A \cap K_n)$ .

$4 \Rightarrow 5$ . Let  $(x_n)$  be a sequence in  $A$ , and let  $(f_n)$  be an equicontinuous sequence in  $E'$ . Suppose that  $\lim_n \lim_k f_n(x_k)$  and  $\lim_k \lim_n f_n(x_k)$  both exist. For each  $n$ , form the set  $K_n = \overline{\Delta(x_{n+1}, \dots)}$ . Then  $(K_n)$  is a nested sequence of closed convex subsets of  $E$ . Further,  $A \cap K_n$  is non-empty for each  $n$ . By hypothesis, there exists  $x \in E$  such that  $x \in \bigcap_n K_n$ . Then for every  $n$ ,  $\lim_k f_n(x_k) = f_n(x)$ . Thus,  $\lim_n \lim_k f_n(x_k) = \lim_n f_n(x)$ . Let  $L = \lim_k \lim_n f_n(x_k)$ . Given  $\varepsilon > 0$ : Choose  $K$  such that  $|L - \lim_n f_n(x_k)| < \varepsilon$ .

for all  $k > K$ . Then  $|\lim_n f_n(x) - L| \leq \varepsilon$ , for all  $k > K$ .  
 $\therefore \lim_k \lim_n f_n(x_k) = L = \lim_n \lim_k f_n(x_k) = \lim_n f_n(x)$ .

5  $\Rightarrow$  1. Without loss of generality, we may assume that  $E$  is complete. Let  $F = E'$ , and embed  $E$  in  $(E')^*$ . Since  $((E')^*, \sigma((E')^*, E'))$  is complete,  $\bar{A}$ , closure in  $\sigma((E')^*, E')$ , is  $\sigma((E')^*, E')$ -compact. Also,  $\sigma(E, E')$  is the restriction of  $\sigma((E')^*, E')$  to  $E$ , hence it remains to prove that  $\bar{A} \subset E$ . Now,  $\tilde{E} = E$  (completion of  $E$ ).

$E$  consists of all  $f \in (E')^*$  which are  $\sigma(E', E)$ -continuous on each  $U^0$ , where  $U$  is a neighbourhood of 0 in  $E$ . Let  $\varphi \in \bar{A}$ , closure in  $\sigma((E')^*, E')$ . Let  $B \subset E'$ , where  $B$  is  $\sigma(E', E)$ -closed and equicontinuous. Then  $B$  is  $\sigma(E', E)$ -compact. We must show that  $\varphi$  is  $\sigma(E', E)$ -continuous on  $B$ . Suppose that this is not so. Then there exists  $\varepsilon > 0$ , such that for every  $\sigma(E', E)$ -neighbourhood  $V'$  of 0 in  $E'$ ,  $\varphi \notin \varepsilon(V' \cap B)^0$ . Take  $V' = E'$ , then  $\varphi \notin \varepsilon B^0$ . Thus, there exists  $x'_1 \in B$  with  $|\langle \varphi, x'_1 \rangle| > \varepsilon$ . Since  $\varphi \in \bar{A}$ , there exists  $x_1 \in A$  with  $|\langle \varphi - x_1, x'_1 \rangle| < \varepsilon/3$ . Take  $V^2 = \varepsilon/3 \{x_1\}^0$ , then  $[\{\varepsilon/3 x_1\}^0 \cap B]^0$ . Continue as before, to obtain a sequence  $(x_n)$  in  $A$  and a sequence  $(x'_n)$  in  $B$ , with:

1.  $|\langle \varphi - x_m, x'_n \rangle| < \varepsilon/3$  .....  $n \leq m$ .
2.  $|\langle \varphi, x'_n \rangle| > \varepsilon$
3.  $|\langle x_m, x'_n \rangle| \leq \varepsilon/3$  .....  $m < n$ .

Now,  $\lim_n \langle x_m, x'_n \rangle = 0$ . This is true for each  $m$ , by 3. Hence,  $\lim_m \lim_n \langle x_m, x'_n \rangle$ , which by hypothesis exists, must equal zero. But  $|\langle \varphi, x'_n \rangle - \langle x_m, x'_n \rangle| < \varepsilon/3$ , for all  $n \leq m$ , by 1. Thus,  $\lim_m \langle x_m, x'_n \rangle = \langle \varphi, x'_n \rangle$  for all  $n$ . By 2 it follows that  $\lim_n \lim_m \langle x_m, x'_n \rangle$  equals  $\lim_n \langle \varphi, x'_n \rangle$  (which exists), and  $\lim_n \langle \varphi, x'_n \rangle \geq \varepsilon$ . We so obtain a contradiction.

The proof given above, is essentially that of Eberlein.

Theorem 3.2.3. Let  $(E, \tau)$  be a convex space, and let  $A$  be a  $\sigma(E, E')$ -closed, bounded subset of  $E$ , such that  $\overline{\Delta(A)}$  is complete. Then the following are equivalent to each other, and also to each of the statements of theorem 2.2.2.

6. Each  $\sigma(E, E')$ -continuous functional (not necessarily linear) on  $A$  is bounded.
7. Each continuous linear functional on  $E$  attains its supremum on  $A$ .
8.  $\overline{\Delta(A)}$  is  $\sigma(E, E')$ -compact.
9. If  $(x_n)$  is a sequence in  $A$ , and  $g \in E'$  such that  $\lim_n g(x_n)$  exists, then there exists  $x \in A$  with  $\lim_n g(x_n) = g(x)$ .
10. If  $K$  is a closed, convex subset of  $E$ , and  $A \cap K = \emptyset$ , then  $0 \notin \overline{A - K}$ .

Proof.  $1 \Rightarrow 6$ . Suppose that  $g$  is an unbounded,  $\sigma(E, E')$ -continuous functional on  $A$ . Then  $\{g^{-1}(-n, n)\}_n$  is a  $\sigma(E, E')$ -open covering of  $A$  with no finite subcover, contradicting the weak compactness of  $A$ .

$6 \Rightarrow 7$ . Let  $f$  be a continuous linear functional on  $E$ . Then  $f$  is bounded on  $A$ . Suppose that  $f$  does not attain its supremum on  $A$ . Define  $f^*$  on  $A$  by:

$$f^*(x) = [\sup \{f(y) \mid y \in A\} - f(x)]^{-1} \quad \forall x \in A.$$

Then  $f^*$  is a  $\sigma(E, E')$ -continuous functional which is unbounded on  $A$ . This is a contradiction to 6.

$7 \Rightarrow 8$ . Let  $f \in E'$  such that  $f$  attains its supremum on  $A$ . Now since  $\Delta(A)$  is convex and contains  $A$ ,  $f$  will attain its supremum on  $\Delta(A)$  at some point of  $\Delta(A)$ . But  $\overline{\Delta(A)}$  is complete and  $\sigma(E, E')$ -closed, and thus by applying the result of R. C. James (stated previously), we obtain that  $\overline{\Delta(A)}$  is  $\sigma(E, E')$ -compact.

$8 \Rightarrow 1$ . This is immediate since  $\overline{\Delta(A)}$  is  $\sigma(E, E')$ -compact and  $A$  is  $\sigma(E, E')$ -closed in  $\overline{\Delta(A)}$ .

$1 \Rightarrow 9$ . Suppose that  $A$  is  $\sigma(E, E')$ -compact. Let  $(x_n) \subset A$  and let  $g$  be a  $\sigma(E, E')$ -continuous functional such that  $\lim_n g(x_n)$  exists. Set  $L = \lim_n g(x_n)$ . Define for each  $n$ ,  $U_n = \{y \mid |L - g(y)| > 1/n\}$ . Then there exists  $x \in A$

with  $g(x) = L$ . If this was not so,  $\{U_n\}_n$  would form a  $\sigma(E, E')$ -open covering of  $A$  with no finite subcover, contradicting the  $\sigma(E, E')$ -compactness of  $A$ . Since  $(E, \sigma(E, E'))' = E' = (E, \tau)'$ , this result is true if  $g$  is a continuous linear functional on  $E$ .

$9 \Rightarrow 7 \Rightarrow 1$  is immediate.

It remains to prove that  $10 \Rightarrow 1$  and  $1 \Rightarrow 10$ .

$10 \Rightarrow 7 (\Rightarrow 1)$ . Suppose that  $f \in E'$  and  $f$  does not attain its supremum on  $A$ . Let  $\sup_A f(x) = m$ . Then if we let  $K = \{y \mid f(y) = m\}$ , then  $K$  is a closed convex subset of  $E$ , with  $K \cap A = \emptyset$ , and  $0 \in \overline{A - K}$ .

$1 \Rightarrow 10$ . Let  $K$  be a closed convex subset of  $E$ , with  $A \cap K = \emptyset$ . For each continuous linear functional  $f$  and number  $\alpha$  such that  $\sup \{f(y) \mid y \in K\} < \alpha$ , define  $W_f^\alpha = \{y \mid f(y) > \alpha\}$ . Then for each pair  $(f, \alpha)$ ,  $W_f^\alpha$  is  $\sigma(E, E')$ -open. Given  $x \in A$ , since  $A \cap K = \emptyset$ , and  $K$  is closed and convex, there exists  $f \in E'$  with  $\sup_K f(y) < f(x)$ . The family  $\{W_f^\alpha\}_{f, \alpha}$  form a  $\sigma(E, E')$ -open covering of  $A$ . As  $A$  is  $\sigma(E, E')$ -compact, we can select a finite subcover  $W_{f_1}^{\alpha_1} \dots W_{f_n}^{\alpha_n}$ . Now  $\sup_K f_i(y) < \alpha_i$  for  $i = 1 \dots n$ . Choose  $\varepsilon > 0$ , with  $\alpha_i - \sup_K f_i(y) > \varepsilon$

for  $i = 1, \dots, n$ . Let  $U = \{x \mid |f_i(x)| < \varepsilon, 1 \leq i \leq n.\}$

Then  $U$  is a neighbourhood of  $0$  which does not meet  $A - K$ :

For if there exists  $z = x - y \in A - K$ , and  $z \in U$ , then  $x \in W_{f_i}^{\alpha_i}$  for some  $i$ . Hence,  $f_i(x) > \alpha_i > \sup_K f(y) + \varepsilon \dots (i)$ .

Since  $z = x - y \in U$ ,  $|f_i(x) - f_i(y)| < \varepsilon$ . Therefore,  $f_i(x) < \sup_K f_i(y) + \varepsilon$ , contradicting (i). This completes the proof.

### Remarks.

(i). For further characterisations of weak compactness on the bounded weakly closed subsets of a complete convex space (real field), we refer the reader to R. C. James [17].

R. J. Fleming has noted that these characterisations lead to a variety of characterisations of semi-reflexivity in a quasi-complete convex space. See [11; section 4].

(ii). In the case of  $E$  a complex convex space we obtain that  $1 \Leftrightarrow 2 \Leftrightarrow 4 \Leftrightarrow 5 \Leftrightarrow 7$ . The proofs of these equivalences can be found in any well-known textbook. See, for example [22; pp. 319] and [19; ch. 5, 17.12].

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