The final version of this paper was published in Proceedings of Second Biennial Conference of the South African Society for Engineering Education, June 2013, Cape Town, South Africa, 39-47. Any citations should refer to that version.

Observations and Conclusions of Dynamics Students' Mathematical Fluency

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The course Dynamics I in mechanical engineering is a challenging course for many reasons, one of them being its mathematical demands. A collaboration between the first author (a mathematics lecturer and mathematics education researcher) and the second author (a mechanical engineer and the Dynamics I lecturer) sought to answer the question "What specific and identifiable mathematical difficulties are experienced by the Dynamics I students?" The observational results of this, in essence, ethnographic case study suggest that there are two levels of mathematical challenge, namely specific symbolic and computational difficulties as well as the need for well-developed problem-solving processes. We discuss our observations and provide pedagogic advice for lecturers of mathematics to help ease the transition to Dynamics I.

Introduction

The introductory course in engineering dynamics, Dynamics I, has traditionally been amongst the most challenging course encountered by 2nd year engineering students. While there are a wide variety of reasons for this, the difficulties that students experience, from a mathematical perspective, appear to resolve into two degrees of mathematical fluency: symbolic fluency, *i.e.* the ability to rapidly transition from one notation (symbols, style, operators, etc.) style to another, and problem-solving, *i.e.* the behaviour and processes employed when the solution strategy is not immediately obvious. Both of these levels of engagement suffer from a disconnect between the Dynamics I content and that of the mathematics courses completed by the students. In particular, problem-solving, in the strict sense, is typically not addressed in the conventional mathematics courses and the symbolic language used in the two subjects differs in small but significant ways.

"Problem-solving" refers to the processes and behaviours employed when attempting to solve a problem as opposed to an exercise. An exercise is a mathematical question which the student has a good idea of how to solve. Typically, an exercise in mathematics would include a mathematical expression as well as some indication (either symbolic or verbal) of what result is expected. Furthermore, the solution process will have been well-covered in the course material and is either unique or limited to a small set of techniques – this solution "package" is often called a schema (Schoenfeld, 1985). The focus of an exercise is to evaluate whether the student has mastered the consistent hierarchical system presented in the course. By contrast, a problem is a question which the student does not initially know how to approach. A problem typically needs to be manipulated, represented differently (typically by a diagram) and generally experimented with before the student has a clear idea of how to proceed to a solution. In other words, the student is required to interpret and transform the problem into a form that is amenable to solution techniques presented in the mathematics courses. Almost every mathematical question encountered in first- and second-year engineering mathematics is an exercise. This is not surprising since the undergraduate mathematics curricula are based on the traditional and widespread model that proficiency (mastery of exercises) is required before insight (related to mastery of problems) can be developed. However, our contention is that the relative absence of problems is not ideal. It must be noted that some in the mathematics education community argue convincingly for mathematics to be taught from a problem-solving point of view from early childhood (Romberg, 1994).

At a finer grained cognitive level than problem-solving, the symbolic demands of Dynamics I confront the students, veterans of three semesters of calculus, with some unexpected challenges. While it is true that the use of mathematical symbols transcends verbal language and is globally understood, it is also true that different fields which use mathematics have different requirements, emphases, and symbolic choices. For example, mathematicians tend to value the system of notation that is most concise, precise and generally applicable, while being less concerned as to whether the symbols have any physical meaning. By contrast, engineers tend to adopt notations that may sacrifice generality in favour of physical meaning, which often leads to a proliferation of symbols not encountered in mathematics texts. The symbolic usage in the mathematics courses completed by the students considered in this study is underpinned by the textbooks used as well as the habits of the lecturers involved, who are often pure mathematicians or have been educated within a pure mathematics context. It is the transition in symbolic usage from this context to that of Dynamics I which is the central focus of our study. A pertinent example of this shift is found in calculus notation, where the overdot notation is rarely seen in a mathematics course yet is widely used in certain applied mathematics courses or in engineering.

In this paper we shall address several of the symbolic stumbling blocks apparently presented by Dynamics I in some detail. We shall extend our discussion to include aspects of mathematical problem-solving and the role it has to play in Dynamics. We shall conclude with suggestions for lecturers of both subjects for smoothing the transition for students.

Rationale and Context

The subject of Engineering Dynamics is encountered by mechanical engineering students in their second year of registration for the standard degree. Dynamics contains much work that is mathematical in nature, both in a "bigger picture" sense of problem formulation and in the more symbolic sense of algebra and calculus. As part of our collaboration, the first author (a lecturer of mathematics) and the second author (the Dynamics lecturer) agreed that the first author would sit in on the Dynamics course and observe the challenges encountered during the course. In essence, this project is an ethnographic case study (Case and Light, 2011) with the first author as participant observer and the second author as key informant. Our aim is to aggregate our respective observations and work to address student difficulties. In time, our expectation (hope?) is that our distinct academic backgrounds, roles in the lectures and relationship to the students, will allow us to filter out the bias in our respective observations and elucidate the essence of the difficulties the students face. An inherent advantage of our collaboration is that the sum of our experiences allows us to be thoroughly familiar with the content covered in first year, including engineering mathematics, physics and engineering statics. However, for this paper we shall be approaching all observations from the point of view of first-year mathematics as required in Dynamics I.

In this paper we primarily address one of the clusters of observations made during a semester of Dynamics I related to the symbolic requirements of the course. However, we have found that this issue cannot be viewed in isolation and will, therefore, include some limited discussion of problem-solving and the distinct approaches taken in the mathematics and dynamics courses as a result of a fundamental disconnect in the underlying discourse of the fields.

Observations of symbolic challenges

At the outset of this study, the attention of the first author, *i.e.* the participant observer, was drawn to subtle, yet distinct, differences between mathematics and dynamics in the use of common symbols and notations. Through the interrogation of these variations, which at first seem to be sporadic and arbitrary, we discerned what appears to be a consistent underlying pattern which may provide insight into mathematical fluency difficulties.

Symbolic difficulties displayed by the students include struggling with a shift in the frequently seen variables as well as a new form of calculus notation, a distressing inclination to not recognise the chain rule and being slow to pick up on the existence and utility of certain expressions and identities. In the forthcoming sections we will describe the most common symbolic fluency difficulties displayed by the students in Dynamics I.

Greater variety of variables

The most superficially apparent difference between a mathematics and dynamics text is the greater variety of variables that appear in the dynamics expressions. Typically, this is motivated by the requirement to represent specific physical quantities with uniquely defined variables. By contrast, mathematics teaching and mathematics textbooks tend to use x and y widely. The reason for this choice is that, in an abstract calculus context, the variables used in, say, differentiating an expression, are immaterial. The mathematical concepts behind the notation are what is important, rather than the physical meaning of the variables which are being manipulated.

For example, the two expressions below have the same meaning from a mathematician's point of view.

$$y = k\frac{dx}{dt} \quad \Rightarrow \quad \frac{dy}{dt} = k\frac{d^2x}{dt^2}$$ [Note: y and x are abstract variables]

$$v_\theta = r\frac{d\theta}{dt} \quad \Rightarrow \quad a_\theta = r\frac{d^2\theta}{dt^2}$$ [Note: a, v and θ have physical meaning]

Hence, it is possible, from the mathematical standpoint, to say, “since the variables we are using to demonstrate this calculus concept/process are immaterial, let’s use the standard *x* and *y*.”

While an ‘*x* and *y*’ standard is entirely appropriate if the sole aim is to contrast various mathematical operations, it would appear to also present a danger. The students can get so used to seeing ‘*x* and *y*’ that if one gives them an expression in ‘*r* and *θ’* they struggle to fluently apply the rules of calculus or, as we saw more often, even recognise the application of calculus when demonstrated. It is evident to us that this is not an issue of fundamental understanding, but rather one of a lack of exposure.

Differing calculus notation

A similar obstacle to fluency, as that regarding symbols, was found to occur with calculus notation. In first-year mathematics the following notation is used almost exclusively for derivatives,

$$\frac{dy}{dx} \text{ and } f'(x).$$

In first-year mathematics, if *t* is the independent variable (and it rarely is) then *t* is simply substituted for *x* in the above. Some diversity is encountered regarding the symbol used for the independent variable in text book sections on “optimisation” and “related rates”, which are typically included within a chapter on applications of differentiation. However, it must be emphasized that those sections cover, at most, four days in the first-year syllabus and, incidentally, are regarded by students as relatively difficult. In addition, the notation D_x is seen very occasionally.

In contrast, dynamics texts such as Meriam and Kraige (2008) frequently use the “overdot” notation (Cajori, 1923), for example,

$$a_r = \ddot{r} - r\dot{\theta}^2.$$

The overdot notation is encountered occasionally in applied mathematics settings (Fowler, 1997), but is almost never seen in standard first-year science or engineering mathematics courses. Certainly, the textbooks used for undergraduate engineering mathematics at our institution for the last several years do not mention the overdot notation at all.

The reason for these distinct approaches is similar to that noticed regarding symbolic usage. Mathematics has a preference for an operator notation that is abstract and general, with good reason, while dynamics places an emphasis on the physical meaning of the operated variable and can, therefore, accommodate a notation that is concrete and less general.

Given the above, we find it interesting to note the historical origins of the two notations in question. The "overdot" notation was introduced by Newton (circa 1665), who used it to represent velocities as 'fluxions' of position (Cajori, 1923). In other words, Newton, being a polymath, always appears to have had the physical of the derivative in mind. By contrast, the standard notation for the derivative, as used in modern mathematics texts, was introduced by Leibniz (circa 1675), Newton's great rival as the 'father of Calculus', who appears to have been more focused on the needs of the pure mathematician.

In retrospect it would appear that, while meandering through their history, two subtly difference notation systems have come to be entrenched in the two closely related fields of mathematics and engineering dynamics. A consequence of this is that when moving from mathematics to dynamics, the student experiences not only a change of topic but also a change of notation and, moreover, a shift in emphasis regarding the notation. We have observed that students find this 'shifting' to be non-trivial, and the process takes a long time during which the students experience much frustration.

Lack of recognition of the chain rule

Speaking as the first author, I admit to being shocked by how often in class a student raised his or her hand to ask a question about some algebraic work to which the accurate answer was "it's the chain rule". As a lecturer of first- and second-year mathematics, which is 70% calculus, it was simultaneously alarming and illuminating to see how little depth there was to the students' understanding of the well-known chain rule. The following shows one Dynamics use of the rule:

$$\frac{d\hat{e}_t}{d\beta} = \hat{e}_n \quad \text{therefore} \quad \frac{d\hat{e}_t}{dt} = \frac{d\beta}{dt}\frac{d\hat{e}_t}{d\beta} = \dot{\beta}\hat{e}_n$$

From our observations in the course, many students struggle to follow that step, despite it being an instance of a rule they know extremely well in other contexts, using different symbols. For example, ask the students to calculate

$$\frac{d}{dt}[\sin(t^2)]$$

and they will do it quickly, confidently and accurately.

We note that in single variable calculus, covered in first year, almost every exercise involving the chain rule is of the latter form; the students are required to differentiate one variable with respect to another (almost always y and x). The chain rule is initially introduced using expressions such as

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

However, thereafter, the students rarely see the chain rule expressed using derivative notation. In second-year multivariate calculus, a problem such as

$$\text{If } z = f(u,v),\ u = xy \text{ and } v = \frac{y}{x}\text{, find } \frac{d^2 z}{dx^2}$$

is poorly performed by the students and is regarded as extremely confusing. We conclude that the chain rule is shallowly understood by students throughout their mathematics courses and that this lack of depth is masked by the narrow expectations made by their assessment activities.

Novel ubiquitous symbolic expressions

Certain expressions are used frequently in Dynamics, usually as one small part of a larger and more complicated whole. An example is

$$vdv = ads$$

derived through simple velocity and acceleration identities and use of the chain rule. This expression should not be arduous for the students to derive or use. The students have dealt with more intricate expressions in their mathematics courses. A crucial difference may be that in the mathematics courses students are predominantly required to answer questions using a given expression, while a dynamics student is required to discern when the expression is required. This is closely related to the fluency related issue of problem-solving which we discuss in the next section of this paper. Nevertheless, we have noted that students take a while to get used to using such expressions, at a time when they need to be familiar with the expressions immediately.

Problem-Solving

Throughout the study presented in this paper, a recurring refrain we encountered amongst the student was a sense of being lost with regard to how to approach the problems in dynamics. It became clear to us that this was not merely an issue of difficulties experienced in understanding the theory presented in the course. In general, and upon dealing with the fluency issues discussed previously, the students did not experience major difficulties in following the example problems presented in class. Furthermore, good average marks were attained in the weekly tutorial tests. We feel this indicates that the students have a basic level of proficiency that is adequate for the course. However, when faced with a collection of essentially the same questions in a test setting, the students struggle to solve the problems. It is here that we noticed the pervasive lack of problem-solving skills.

Upon our interaction with the students, we noticed some interesting characteristics of the difficulties they experience. For example, the difficulties seem to have to do with not knowing where to start than not knowing how to finish. One memorable comment by a student was, “How do you expect me to solve a problem when I don’t know which chapter it comes from?”

Another instructive exercise in which the 2nd author often engages during tutorials, is to assemble a group of student who are all struggling with the same problem. We assemble at a chalk board and I take the part of the scribe. I emphasize to the students that I am only going to write down what comes out of their mouths. In other words, I studiously avoid providing any information, be it theory or assumption, pertinent to the problem. However, I do ask questions such as, “have you considered all the information?”, “have you considered all the theory?”, “is the sketch complete?”. Invariably we arrive at a solution without too much trouble. At this point I take great care to emphasize to the students that the exercise shows that they do not lack the knowledge, but need to practise the skill of problem-solving.

The word “problem” can take on many different meanings. In this paper we shall be using the word to mean questions of a type where the problem solver cannot immediately draw on a taught solution schema (Stanic and Kilpatrick, 1989). In the mathematics education literature one finds definitions such as

> A task that is difficult for the individual who is trying to solve it. Moreover, that difficulty should be an intellectual impasse rather than a computational

> one….To state things more formally, if one has ready access to a solution schema for a mathematical task, that task is an exercise and not a problem. (Schoenfeld, 1985, p. 74)

although often the term is used more loosely to mean any typical word problem (Kilpatrick, 1983; Bell and Bell, 1985). When one aims to solve a problem, one needs to apply well-chosen heuristic strategies, or "rules of thumb" (Pólya, 1945). In abstract mathematical problems these might include: draw a diagram, solve for a subset of the domain, solve a related simpler problem, and so on. A successful problem-solving attempt would use those heuristic strategies to get a handle on the problem, as it were, to get started on a solution process sufficiently to understand what the problem is about and how to carry the solution process to completion.

In Dynamics I, heuristic strategies can also be deployed such as: draw a diagram, categorise as kinetics vs. kinematics or energy vs. momentum. In this sense, problem-solving in both the mathematics and dynamics context involves 'educated guessing', i.e. assumptions. However, there is a subtle, yet profound, difference in the nature of the assumptions used. The assumptions employed in mathematics, such as the assumed form of a trial function or a trigonometric substitution, are all self-validating. By this we mean that the validity of an assumption is confirmed by the fact that a solution is obtained. The confirmation is internal and intrinsic with no external justification required. By contrast, the assumptions used to solve dynamics problems are not self-validating. That is, the mere fact that it leads to a solution does not confirm the validity of a given assumption. In fact, two alternative assumptions can lead to two possible solutions (this duality tends not to occur in mathematical problems). Rather, the assumption must be motivated by physical reasoning, that is, their validity is external to the mathematical solution of the problem.

The distinction drawn above suggests that a pure mathematics course can never be expected to provide an environment that can teach problem-solving in the manner required by dynamics. In our view, this is a task better suited to courses such as first year physics or statics. However, we contend that subtle changes to the questions posed in mathematics course can lead to an improvement of problem-solving fluency and its transfer to other subjects.

It is a matter of much discussion in the mathematics education literature that problem-solving – distinct from exercise solving – is rarely encountered in a traditional undergraduate mathematics course. The reasons for this are multiple, an example being that there is a huge amount of exercise solving which *must* be covered in a short time for the students to pick up very necessary skills, as well as the fact that teaching problem-solving is notoriously difficult (Schoenfeld, 1985).

In contrast, much of Dynamics I can be classified as requiring well developed problem-solving behaviour throughout the course. Once the students have learned to categorise and successfully solve many problems, certain types of problems can become routine and be regarded as exercises. Getting to that stage requires hours of problem-solving, however, and certain of the particularly challenging types of problems are probably never regarded as routine exercises by the majority of the students.

Problem-solving, in both the mathematics and Dynamics I context, requires an element of 'hit and miss'. When faced with a new problem, the first solution strategy we choose may not work, but in the process we learn about the problem and are (hopefully) able to make a better second choice. This is particularly evident in the environment of divergent design problems where there is no 'right' answer. This notion is completely foreign to standard mathematical texts where the sure and steady logical and consistent progression from one result to another is prized above all, and for good reason.

Solomon (2006), in the context of epistemic fluency, discusses the difference in the approach to mathematical proof in research and teaching environments. She points out that "...research mathematician handles proof procedures which include...intuition, trial, error, speculation, conjecture, proof,..." while "...undergraduate teaching stresses instead a much simpler and very different model of definition, theorem, proof,...". Furthermore, the reason for this is that "...whereas mathematical practice uses proof to justify and verify, mathematics teaching uses it to explain." While a simplified approach for the sake of teaching may initially be justified, Solomon goes on to point out that "...first year mathematics students see mathematics learning simply as a rote learning task...." with the consequence that "...The exclusion of students from the knowledge construction process results in undergraduates who 'exhibit a lack of concern for meaning,..." (*ibid.,* p. 377).

Our view has certain similarities to that of Solomon. In particular, one consequence of the teaching approach used in mathematics is that the original motivation for a given assumption is never discussed. In other words, the underlying problem-solving heuristics that pervade the field of mathematics are rarely evident to the student. We feel that some subtle changes to the presentation of first year mathematics courses, such as discussing some of the 'tell tale' signs that a particular assumption will work, could go some way toward laying a foundation for problem-solving that successive course can build on. A useful result would be to create an awareness that judicious assumptions don't magically appear, as it may seem in the text books, and cannot be learnt by rote, but are the fruit of heuristics, built upon the experience gained through many hours of practice.

In short, first- and second-year mathematics more than prepares the student for the computational demands of Dynamics I, but poorly prepares the student for the problem-solving demands of the course, resulting a steep learning curve for most (if not all) of the students.

Toward an essential distinction between Mathematics and Dynamics

The symbolic, notational and terminological difficulties we have reported, as experienced by students of Dynamics, are, arguably, superficial indicators of much deeper underlying differences between the two subjects. Our brief discussion of problem-solving is one attempt to uncover more fundamental differences, however we contend that the differences are deeper than "simply" problem-solving. Lengthier discussion is beyond the scope of this paper, however we suggest further avenues for investigation.

The object of mathematical proofs has no clear parallel in Dynamics. Proofs in mathematics frequently begin with an assumption, such as "Assume D is a simply connected region in $\mathbf{R}^2$" which, although possibly a hard won starting point for the person who originated the proof, is given without explanation or history to the student. In contrast, derivations (tellingly not proofs) in Dynamics might begin with an assumption which has firm physical underpinnings, such as "Assume negligible air resistance" or "assume a constant friction coefficient". The contrasts between the abstractions necessary for and valued by mathematics and the concrete, applied thought necessary for Dynamics can be elaborated on, however we shall leave that as a topic for another paper.

Discussion

There is much about Dynamics which is challenging for the students to master, but it should not be the calculus requirements. The students registered for Dynamics I have completed three semesters of demanding mathematics, concluding with Vector Calculus. Second derivatives and the chain rule are about as demanding as the symbolic calculus requirements get in Dynamics I. Yet we personally observed, on several occasions, that the students struggle to recognise calculus when the lecturer was demonstrating problems in the classroom. The problem is one of familiarity and recognition, not of calculus knowledge.

From the context of one of us being a regular lecturer of first-year mathematics we suggest that one fault is with the way first-year mathematics is presented. Our recommendations to lecturers of first-year mathematics to engineers are to:

- use alternate variables to x and y as often as possible. However, this is often hampered by the standard textbooks, where the bulk of homework problems are located.
- make frequent use the overdot notation for time derivatives, which is not usually encountered in mathematics textbooks. We have a large collection of widely used first-year maths textbooks (Stewart, 2010; Swokowsky, 1988; Thomas, 2006, among others) and we could find the overdot notation in none of them.
- imbue the variables with meaning, such as using velocity, wind speed, impedance or viscosity, instead of the exclusive use of abstract symbols.
- Increase the proportion of chain rule exercises in first-year calculus which delve at the conceptual underpinnings of the rule or at least require symbolic manipulation more sophisticated than mere carrying out of the rule on multi-level single variable expressions.
- motivate the use of a particular assumption in a proof or solution, or, at least, overtly discuss the "tell tale" features that you look out for as part of your heuristic strategy.

The issue of providing a good grounding in problem-solving in the mathematics courses is harder to address. First-year mathematics, with its packed and jumbled syllabus of differential and integral calculus, vector geometry, complex numbers, basic linear algebra and differential equations, as well as second-year mathematics, with its semester of vector calculus preceding Dynamics, require new topics with new algorithms, methods and processes to be covered every day. Teaching and learning problem-solving requires time and freedom from the pressures of learning new content daily. With colleagues, we (Craig, 2011, 2012) have attempted various teaching and learning initiatives over the years, aimed at improving mathematical problem-solving, but they frequently either do not work very well or cannot be sustained, usually both. Perhaps it must simply be accepted that students are required to swiftly develop sound problem-solving behaviours in Dynamics from very little background, but we prefer to believe that problem-solving, or at least a precursor, can be incorporated in mathematics and shall continue to try to do so.

Conclusion

Mathematical difficulties experienced by students of Engineering Dynamics include symbolic, notational and terminological shifts from their first- and second-year mathematics courses. Several of these shifts are calculus-specific and all can be addressed on both sides of the divide by lecturers of Mathematics and Dynamics who are aware of the mismatches.

A further struggle related to mathematics is the shift to a problem-solving way of working and thinking which is at odds with the largely exercise focus of undergraduate mathematics, despite any individual mathematics lecturer’s attempts to emphasise problem-solving. Problem-solving is difficult and time consuming to teach well and becomes almost impossible to teach when a course syllabus is as exercise heavy as first- and second-year mathematics are.

Finally, we contend that the mathematical disconnect between the two types of courses runs far deeper than the, arguably, superficial points discussed in this paper. The objects of study differ between the disciplines, as do the epistemological nature of assumptions and conceptual underpinnings. In the short term we shall focus on the superficial and easily addressed issues of symbols and notation and leave as an object of further study the conflicting epistemologies of Dynamics and mathematics.

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