

DYNAMICS OF CLASSICAL STRINGS
IN RINDLER SPACE

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Abstract

The fundamental degrees of freedom in string theory are extended objects. Solving their equations of motion can be difficult unless they are considered in very constrained situations. We investigate the dynamics of gravitational D-brane radiation. Results of others are reviewed which show that in the static case the string profile of Newtonian and relativistic strings are the same. We show that for slow moving strings the relativistic solution agrees with the classical one.

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Declaration

I know the meaning of plagiarism and declare that all of the work in the thesis, save for that which is properly acknowledged, is my own. In particular I would like to point out that Chapters 5 and 6 are based on work done in collaboration with Jeff Murugan and Jean Philippe Uzan.

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Chapter 1

Introduction

Gravitational radiation [1] was first predicted by Einstein in 1916 and has been an active research topic ever since. In general relativity gravitational radiation manifests in the form of waves that distorts spacetime.

More recently string theory and the AdS/CFT correspondence [2] has made a different flavour of gravitational radiation popular. The equations of motion of string theory allows for two different kinds of strings. The first kind forms closed loops, free to move unconstrained in space, and the second are open strings that have end points which, for energy considerations, has to be attached to objects called D-branes. There exists situations where the endpoints of an open string can meet to form a closed string and in doing so “escape” from the D-brane. Closed strings are closely related to gravitons and since they are free to carry energy away from the brane, this process is interpreted as gravitational radiation.

The AdS/CFT correspondence proposes that there exists a duality between string theory and certain gauge theories. More interestingly, the correspondence is a strong weak duality, meaning that there is a one-to-one mapping between the weak coupling limit of one theory to the high coupling limit of its dual. This opens up opportunities to use perturbation methods on one side of the correspondence to study the dual theory at high coupling. Many of these models feature a D-brane in some gravitational potential and strings ending on it would be interacting with this potential too. The dynamics of D-branes and its strings near such gravitational potentials have been the subject of intense research.

In Brane-world models [3] gravitational radiation has been proposed as a mechanism to explain the hierarchy problem. The idea is that our observable universe is a brane embedded in a higher dimensional space called the bulk. Three fundamental forces electromagnetism, the weak and strong nuclear forces are described by Yang-Mills theories that are confined to the brane. The higher dimensional space has no bearing on them. Gravity, on the other hand, is a force that can propagate in the higher dimensional space. Because gravity is a theory in the bulk, there exists decay channels where energy can escape the brane and the end result is that gravity is much weaker than the other fundamental forces on the brane.

A second scenario where gravitational radiation has become interesting is in AdS/QCD correspondence. In these models quarks bound in a meson are represented by the endpoints of an open string. The thermodynamics of the gauge theory on the brane correspond to the thermodynamics of a black hole in the bulk [4]. The interaction between the D-brane and the black hole is not trivial and the brane can fall into the black hole [5]. This happens when the mass, and therefore the Hawking temperature, of the black hole becomes large enough. The Hawking temperature of the brane is related to the temperature of the mesons on the brane and the collapse of the D-brane is associated with a quark gluon plasma forming in the gauge theory.

Gravitational radiation still occur when the D-brane does not collapse into the black hole. In the high temperature Sakai-Sugimoto model there exists conditions where strings attached to D-branes will peel off from the D-brane, carrying energy with them [6]. In the gauge theory dual living on the D-brane world volume, this phenomenon will show up as energy loss.

In both these situations the D-brane is at a constant distance from the black hole. For it to stay in the same place it must be accelerating to counteract the gravitational force acting on it. Strings are not charged under the force that keep the brane in place and are therefore in free fall under the gravitational force, while the endpoints of the string are still stuck to the brane.

It is hoped that a solution of the string equations of motion will be useful in building a toy model of gravitational radiation. Unfortunately the string equations of motion are too complicated to solve in the AdS background, however they have been solved for a string undergoing a uniform acceleration. Rindler space – a parametrisation of Minkowski space from the point of an observer ex-

perceiving a uniform acceleration – seems like a natural starting point for such a solution.

1.1 Motivation and Outline

The AdS/CFT correspondence have found application in modeling a number of phenomena including the quark-gluon plasma and cosmological models of the early universe.

In these models gravitational radiation of D-branes plays a central role. The string theory picture of gravitational radiation is that of open strings attached to the brane peeling off to form closed strings. This can happen in two different ways: the endpoints of the string may meet and join to form a closed string that is no longer attached to the brane, or the string could self intersect to form a closed string with a smaller open string still attached to the brane.

Our aim is to explore a toy model for gravitational radiation in terms of a relativistic string attached to a brane that is undergoing a constant acceleration. Rindler space provides a natural embedding for such a system. Others have shown¹ that the solution to the static problem is the well known catenary – the profile of a suspended Newtonian string under the influence of gravity. Before we attempt to solve the full dynamical problem in Rindler space, we turn our attention to its classical counterpart in hopes that the latter may provide some insight to the solution of the former.

The rest of this thesis is outlined as follows. Chapter 2 gives an overview of bosonic string theory and introduces fundamental concepts such as the string action, equations of motion and symmetries. D-branes are also introduced and some of their properties discussed. In Chapter 3 introduces the AdS/CFT correspondence and discusses gravitational radiation in this context. Chapter 4 gives reviews static strings under constant acceleration, where the string equations of motion are embedded in Rindler space and the solution is the well know catenary. In Chapter 5 we study the classical catenary, but where the end points of the string is allowed to move in one direction. Chapter 6 proposes some future work related to dynamical strings in Rindler space and makes some concluding remarks.

¹We provide references and a detailed review in a later chapter.

Chapter 2

Relativistic String Theory

One dimensional relativistic strings, as opposed to point particles or fields, are the fundamental objects of string theory. They are extended objects and are therefore not made from smaller constituents in the way that classical strings are made from atoms. A brief overview of the dynamics of relativistic strings and their implications is given in this section. Starting from the action, we derive the equations of motion, take a look at the symmetries they preserve and study the motion of string endpoints.

A number reviews and introductory textbooks have been written on the material covered in this chapter. In particular we highlight the key ideas and arguments discussed in [7, 8, 9, 10, 11, 12].

2.1 The Dynamics of 1-Dimensional Extended Objects

We will consider a relativistic string in a $(d + 1)$ -dimensional space-time with metric $g_{\mu\nu}$. In bosonic and super string theory d is fixed at 25 and 10, respectively to preserve Lorentz invariance at all energy scales. However in many situations a suitable choice of coordinates can be made so that string dynamics is restricted to a lower dimensional subspace.

As a string moves in space-time it traces out a two dimensional worldsheet. The worldsheet can be parametrised by two coordinates ξ^α , with $\alpha = 1, 2$, that can be chosen independent of its dynamics. The freedom to choose a coordi-

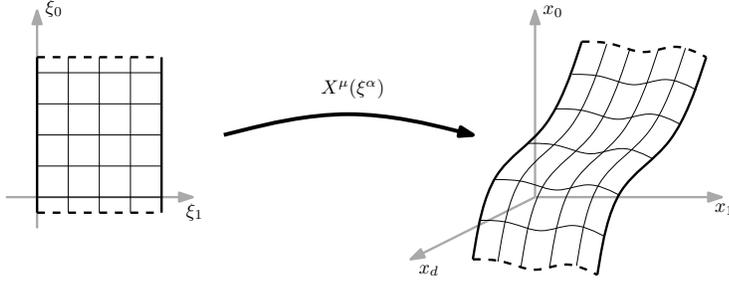


FIGURE 2.1: The string world sheet and coordinates.

nate system on the world sheet leads to a symmetry called reparametrisation invariance. In spacetime, the world sheet is described by the string coordinates $X^\mu(\xi^\alpha)$ which is d functions of the world sheet coordinates ξ^α . These functions map a point on the world sheet, a ξ^α coordinate, to a point in spacetime, a X^μ coordinate (see Figure 2.1). It is often convenient to choose $\xi^0 = \tau$ as a time like parameter and $\xi^1 = \sigma$ as a space like parameter. Further more, derivatives of the string coordinates are abbreviated by $\dot{X} = \partial X / \partial \tau$ and $X' = \partial X / \partial \sigma$.

The embedding spacetime metric, $g_{\mu\nu}$, induces a metric on the string world-sheet $\gamma_{\alpha\beta}$ which is given by

$$\gamma_{\alpha\beta} = g_{\mu\nu} \frac{dX^\mu}{d\xi^\alpha} \frac{dX^\nu}{d\xi^\beta} . \quad (2.1)$$

The induced metric describes worldsheet geometry in the same way that the metric $g_{\mu\nu}$ describes the spacetime geometry. This allows us to calculate quantities such as the worldsheet's surface area and distances between points. For our purposes, the surface area is the most use full quantity and is given by $\sqrt{-\det \gamma_{ab}}$.

The dynamics and symmetries of a string is encoded in an action. We require that this action be Lorentz and reparametrisation invariant. The simplest such action is proportional to the area of the world sheet, called the Nambu-Goto action[7], and is given by

$$S_{NG} = -T_0 \int d^2\xi \sqrt{-\gamma} \quad (2.2)$$

$$= \int d^2\xi \mathcal{L}(\dot{X}^\mu, X'^\mu; X^\mu) , \quad (2.3)$$

where $\gamma = \det \gamma_{\alpha\beta}$. The constant of proportionality, T_0 , is the interpreted as the string tension, since it carries the dimension of force. The expression in the

integral is identified as the Lagrangian, \mathcal{L} .

The equations of motion are obtained by varying the above action with respect to the string coordinates are

$$\frac{\partial \mathcal{P}_\mu^\tau}{\partial \tau} + \frac{\partial \mathcal{P}_\mu^\sigma}{\partial \sigma} = \frac{\partial \mathcal{L}}{\partial g_{\nu\rho}} \frac{\partial g_{\nu\rho}}{\partial X^\mu} . \quad (2.4)$$

We have used the definitions of the world sheet momenta $\mathcal{P}_\mu^\tau = \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu}$ and $\mathcal{P}_\mu^\sigma = \frac{\partial \mathcal{L}}{\partial X'^\mu}$. The term on the right hand side is zero if the spacetime metric $g_{\mu\nu}$ is not a function of the string coordinates, which is the case for Minkowski spacetime.

A second form of the equations of motion can be found from the Polyakov action[9]

$$S_P = -T_0 \int d^2\xi \sqrt{-\gamma} \gamma^{\alpha\beta} \gamma_{\alpha\beta} . \quad (2.5)$$

Here $\gamma^{\alpha\beta}$ is defined by $\gamma^{\alpha\beta} \gamma_{\beta\epsilon} = \delta_\epsilon^\alpha$. The Polyakov action is quadratic in X^μ which lends itself much better to string quantisation.

Varying this action with respect to the worldsheet metric gives the equations of motion

$$\frac{1}{\sqrt{-\gamma}} \partial_\alpha (\sqrt{-\gamma} \gamma^{\alpha\beta} \partial_\beta X^\mu) + \Gamma_{\nu\rho}^\mu \gamma^{\alpha\beta} \partial_\alpha X^\nu \partial_\beta X^\rho = 0 . \quad (2.6)$$

Here the second term is due to the curvature of the background spacetime, which is encoded in the Christoffel symbols $\Gamma_{\nu\rho}^\mu$, defined as

$$\Gamma_{\nu\rho}^\mu = g^{\mu\lambda} (\partial_\nu g_{\lambda\rho} + \partial_\rho g_{\lambda\nu} - \partial_\lambda g_{\nu\rho}) . \quad (2.7)$$

This second form of the string equations of motion is a generalisation of the geodesic equation of a point particle to higher dimensional objects[].

Boundary Conditions

A choice has to be made regarding how the endpoints of open strings are treated when the equations of motion were derived from the Nambu-Goto action[7]. By requiring that the variation of the action must be zero, a boundary term must also vanish. The boundary term in question is at the boundary of the string world sheet and can be made to disappear by imposing one of two boundary conditions.

The first condition requires

$$\mathcal{P}_\mu^\sigma = 0 , \quad (2.8)$$

and corresponds to free endpoints. It can be shown that the endpoints will always move at light speed and that the energy of the string is conserved.

The second condition, called Dirichlet boundary conditions, requires that

$$\frac{\partial X^\mu}{\partial \tau} = 0, \quad \mu \neq 0, \quad (2.9)$$

Dirichlet boundary conditions leads to much richer physics. This condition fixes the string endpoints at a spacial point and means that the energy at the string end point is not conserved, which seems concerning. However, such boundary conditions appear in classical strings too, where they are imposed to fix the string end point to some object. Even though energy is not conserved in the classical string, the energy of the system, with the string and the object that it is fixed to, remains conserved. The question is, what plays the role of such an object in string theory?

For Dirichlet boundary conditions to be consistent in string theory, higher dimensional objects must also exist in the theory. They must be dynamical object, to facilitate the transfer of energy between themselves and strings. These objects are called Dp -branes, where p refers to the number of spacial dimensions of the object. The properties of D-branes are studied through their interactions with quantised open strings. Some of there properties will be discussed in the following sections.

World Sheet Symmetries

The Nambu-Goto and Polyakov actions preserve three symmetries. Lorentz and reparametrization invariance was built onto the action, but there is also an additional symmetry, conformal symmetry on the world sheet, that the string action was not expected to preserve. This section gives a brief description of each of these symmetries.

Spacetime Poincare symmetry means that the action remains invariant under the global transformation

$$X^\mu \rightarrow \Lambda^\mu{}_\nu X^\nu + A^\mu, \quad (2.10)$$

with $\Lambda^\mu{}_\nu$ a Lorentz transformation and A^μ a constant translation. If string theory is to be a viable theory of everything, it must be Poincare invariant at large length scales. The requirement that superstring theory lives in a $d = 10$

dimensional spacetime follows from the requirement that Poincare invariance holds at all length scales.

World sheet reparametrisation invariance leaves the action unchanged under the simultaneous transformation

$$X^\mu \rightarrow X^\mu + \zeta^\alpha \partial_\alpha X^\mu \quad (2.11)$$

$$\gamma^{ab} \rightarrow \gamma^{ab} + \zeta^c \partial_c \gamma^{ab} - \partial_c \zeta^a \gamma^{cb} - \partial_c \zeta^b \gamma^{ac} . \quad (2.12)$$

with $\zeta^a(\xi^b)$ a two parameter set. Reparametrisation invariance means that the action, and hence the dynamics, is the same for any choice of the world sheet coordinates. This symmetry allows us to choose a reparametrisation in which the equations of motion can be written in their simplest form. This is the symmetry exploited the most in the study of string dynamics and string quantisation.

Conformal symmetry means that the action is invariant under scale transformations of the world sheet metric

$$\gamma_{ab} \rightarrow e^{2\omega} \gamma_{ab} . \quad (2.13)$$

This extra symmetry arises due to the fact that there is no natural energy, mass or length scale in either forms of the action. It comes as a pleasant surprise, since the action was not manipulated to have this symmetry, but one would expect a theory of every thing to be conformally symmetric.

As a result of conformal symmetry, the trace of the stress tensor on the world sheet is zero. In fact, gauge choices for conformal and reparametrisation invariance can be imposed by requiring that all components of the stress tensor be zero. The resulting equations are called Virasoro constraints.

Fundamental String Charge

Up to now we have discussed string by looking at the Polyakov and Nambu-Goto forms of the action. A lot about strings can be learn from the dynamical part of its action but they do not describe string interactions with other degrees of freedom in string theory.

To consider interactions additional terms must be added to the string action. The new terms must involve the string coordinates somehow and must also be dimensionless scalars. In Maxwell electrodynamics the tangent vector to the

world line of a point charge couples to the electric field strength with a single Lorentz index. Since strings are two dimensional they have tangent planes and should couple to an analogous field strength that carries two Lorentz indices.

Quantisation of the closed string reveals a field with two Lorentz indices, the Kalb-Rammond field $B^{\mu\nu}$. The following interaction term can be added to the string action to describe the coupling

$$S_{\text{KR}} = - \int d\tau d\sigma \frac{\partial X^\mu}{\partial \tau} \frac{\partial X^\nu}{\partial \sigma} B_{\mu\nu} . \quad (2.14)$$

S_{KR} must be dimensionless so the Kalb-Rammond field carries units M^2 .

The interaction between the string world sheet and the Kalb-Rammond field implies that strings carries a new kind of charge - called string charge. There is also a current on the world sheet given by

$$j^{\mu\nu} = \frac{1}{\kappa^2} H_{\mu\nu\rho} \quad (2.15)$$

$$H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu} , \quad (2.16)$$

where H is the field strength of the Kalb-Ramond field and κ is a dimensionfull constant to ensure that the field strength term in the action remains dimensionless.

String reparametrisation invariance is broken by adding S_{KR} to the string action. That is because the current flows tangentially to the string and introduces a preferred direction on the string. Strings are said to be oriented in the direction of increasing σ .

The current j also violates energy conservation for open strings. The intuitive understanding is that a current flowing along the string, must flow from somewhere and into something. This broken symmetry is also apparent when considering the gauge transformation

$$\delta B_{\mu\nu}(x) = \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu . \quad (2.17)$$

The Maxwell field is gauge invariant under such a transformation, but in the case of the Kalb-Ramond field gauge invariance is broken at the string endpoints. This disparity is another hint that string endpoints need dynamical degrees of freedom on which to end and can be resolved by introducing D-branes into the theory.

2.2 D-Branes

D-Branes are needed in string theory to allow open strings to have Dirichlet boundary conditions. D-Branes also play an important role in charge conservation. A current flowing along the string came as a result of the coupling between the string world sheet and the Kalb-Rammond field introduced in the previous chapter. D-Branes offer a mechanism that prevents charge from building up at one end of the string.

D-Branes, or more formally Dp -branes, are extended objects in p spatial dimensions. D-branes are defined as the objects on which open strings end. Therefore, not all extended objects in string theory are D-branes. A string is a 1-brane, but not a D1-brane. They are dynamical objects and carry their own set of charges, some of which will be explored in this section. Nothing is assumed about D-brane dynamics; D-branes are studied through their interactions with open and closed strings.

Consider a Dp -brane embedded in a $d+1$ dimensional spacetime. Naturally, we must have $p < d$. Endpoints of open strings attached to the brane will be constrained to move only in the p dimensional subspace in which the brane is extended. Say the brane is extended in the spacetime dimensions x^m with $m = 0, \dots, p$ and located at some set of constant values in the remaining spatial directions $x^n = C^n$, where $n = p+1, \dots, d$. The restrictions on the movement of the string endpoints are described by Neumann boundary conditions for the directions labelled by m and Dirichlet boundary conditions for the directions labelled by n . The situation can be summarised as follows

$$X'^m(\tau, \sigma)|_{\text{endpoints}} = 0, \quad m = 0, 1, \dots, p. \quad (2.18)$$

$$X^n(\tau, \sigma)|_{\text{endpoints}} = C^n, \quad n = p+1, p+2, \dots, d. \quad (2.19)$$

When quantising open strings one can consider different brane configurations and in particular different ways of attaching how strings are attached to the branes. The case of both endpoints attached to the same brane reveals several interesting properties of D-branes. Firstly, the creation and annihilation operators for string fluctuations can be grouped into two categories - operators that create excitations orthogonal and parallel to the brane's world volume. This is a direct consequence of the boundary conditions (2.18) and (2.19). The excitations of the parallel operators transform as Lorentz vectors on the D-brane's

world volume and therefore, D-branes have a Maxwell field on their world volume. The excitations of the orthogonal operators transform as Lorentz scalars on the world volume and are interpreted as D-brane excitations.

If there is a Maxwell field living on the world volume of D-branes, there must also be objects that are charged under this field. Furthermore, they must be point-like and confined to the world volume of the brane. String endpoints are likely candidates to carry this charge. This identification also solves the problem with energy conservation that arose due to the coupling between the string world volume and the Kalb-Rammond field.

The coupling between the Kalb-Ramond field and the string current (2.14) on the brane world can be written in the following form

$$S_{\text{KR}} = - \int d^D x B_{\mu\nu} j^{\mu\nu}(x) . \quad (2.20)$$

As mentioned in the previous section, this term is not gauge invariant at the string endpoints under the gauge transformation (2.17). The boundary terms that do not disappear under the transformation can be made to vanish by adding an additional coupling term to the action and requiring that the gauge transformation between the Kalb-Ramond and Maxwell field be related. The addition to the action is

$$S_{\text{M}} = \int d\tau A_m(X) \left. \frac{dX^m}{d\tau} \right|_{\sigma=\pi} - \int d\tau A_m(X) \left. \frac{dX^m}{d\tau} \right|_{\sigma=0} , \quad (2.21)$$

where A is the Maxwell field that lives on the world volume of the Dp -brane, and the new gauge transformation is the simultaneous variation

$$\delta B_{\mu\nu} = \delta_\mu \Lambda_\nu - \delta_\nu \Lambda_\mu \quad (2.22)$$

$$\delta A_m = -\Lambda_m . \quad (2.23)$$

The requirement that these variations be simultaneous implies that the physical degrees of freedom on the brane world volume is the quantity

$$\mathcal{F}_{mn} = F_{mn} + B_{mn} . \quad (2.24)$$

D-brane stability

D-branes have been introduced to solve problems with charge conservation at string endpoints. They are required in order to allow for open strings in

string theory, but D-branes must also be stable objects in the theory. In other words there is some conservation law needed to prevent D-branes states from decaying into string states.

Such a mechanism would require D-branes to carry a charge that is not carried by strings. A coupling between the world volume of a Dp -brane and mass-less antisymmetric tensor field with $(p+1)$ indices would imply that there is a suitable conservation law to prevent D-brane's from decaying.

Type IIA and type IIB superstring theories have such tensor fields in their Ramond-Ramond sectors. Type IIA superstring theory has suitable fields to allow for stable D0 and D2-branes and Type IIB superstring theory can allow for stable D1 and D3-branes.

Chapter 3

The AdS/CFT Correspondence

3.1 Introduction

String theory was first conceived as a theory of strong interactions. Later QCD, a gauge theory with gauge group $SU(3)$, gave another description of strong interactions and it was hoped that there was some relation between string theory and QCD.

In an attempt to consolidate these theories, t'Hooft [13] realised that $SU(N)$ gauge theories admit a $1/N$ expansion. The expansion is made in the limit where $N \rightarrow \infty$, the gauge coupling $g_{YM} \rightarrow 0$, but where the combination

$$\lambda = g_{YM}^2 N , \tag{3.1}$$

remains fixed. The link between the gauge theory and string theory, is that in t'Hooft's expansion, Feynman diagrams of the gauge theory become ribbon diagrams that have exactly the same topology of interacting string world sheets of quantised strings. This was the first indication that gauge theories may be dual to string theories, but it was not clear which particular theories would be dual to each other.

It was later proposed [2, 14, 15] that string theories on an AdS backgrounds are equivalent to a super conformal gauge theories. Relationships between pa-

parameters of two dual theories have been established as

$$\frac{R^2}{\alpha'} = \sqrt{\lambda} \quad \text{and} \quad g_s = \frac{\lambda}{N} . \quad (3.2)$$

Here R is the radius of the AdS background and g_s the string coupling.

This is the celebrated AdS/CFT correspondence and to date a number of examples of this correspondence have been found. That is to say specific string theories are known to be dual to certain gauge theories.

One significant feature of the AdS/CFT correspondence is that is a strong/weak coupling duality. From (3.2) one can see that when the perturbation expansion in the gauge theory is valid (when $\lambda \ll 1$), the curvature of the AdS space is small and one cannot work with the classical gravity description on the string theory side. On the other hand, when the classical gravity description is valid, the perturbation expansion fails on the gauge theory side.

The strong/weakly coupled nature of the duality makes it difficult to test it directly by matching the results of calculations done in both theories. Fortunately there are certain objects that are protected by supersymmetry, and can be calculated exactly at all coupling strengths.

3.2 An example of the correspondence

We now turn our attention to the first known, and most studied, example of the AdS/CFT correspondence. In [2] it was argued that Type IIB Superstring Theory on $AdS_5 \times S^5$ is dual to $\mathcal{N} = 4$ Super Yang-Mills gauge theory, with gauge group $SU(N)$, on a $\mathbb{R} \times S^3$ background.

Type IIB String Theory

The close string sector of Type IIB string theory contains left and right moving perturbations with the same chirality (as opposed to Type IIA string theory where left and right movers have opposite chirality). The Ramond-Ramond sector has two gauge fields $A_{\mu\nu}$ and $A_{\mu\nu\rho\sigma}$ that couple electrically to the world volume of D1 and D3-brane, respectively [7]. These gauge fields are necessary to allow stable D-branes in the theory, and D-branes are necessary to allow open strings. Charge and energy conservation prevent D-branes to decay.

Stable D-Branes proved to be a good place to test the AdS/CFT correspondence, because their dual operators in the gauge theory are 1/2 BPS states.

These states are protected by supersymmetry and results computed at weak coupling also holds at strong coupling.

The dual gauge theory

The action (on $\mathbb{R} \times S^3$) of $\mathcal{N} = 4$ SYM is given by

$$S = \frac{N}{4\pi\lambda} \int dt \int_{S^3} \frac{d\Omega_3}{2\pi^2} \left(\frac{1}{2} (D\phi^i)(D\phi^i) + \frac{1}{4} ([\phi^i, \phi^j])^2 - \frac{1}{2} \phi^i \phi^j \right), \quad (3.3)$$

where $\lambda = g_{YM}^2 N$ is the t'Hooft coupling and $i, j = 1..6$ labels the Higgs fields. The gauge and fermionic fields are not shown. The mass term comes from the conformal coupling to the metric of S^3 .

Fields transform in the adjoint representation of the gauge group (SU(N)) and must therefore be matrices. The six real scalar fields are combined into three complex scalar fields:

$$X = \phi^1 + i\phi^2, \quad Y = \phi^3 + i\phi^4 \quad \text{and} \quad Z = \phi^5 + i\phi^6. \quad (3.4)$$

The free field propagators for the complex fields are given by

$$\langle Z_{ij}^\dagger Z_{kl} \rangle = \frac{4\pi\lambda}{N} \delta_{ij} \delta_{jk} \quad (3.5)$$

Operators in the 1/2 BPS sector of the gauge theory are made from a single complex scalar field. Usually Z from (3.4) is chosen but any one can be used. Operators are written as a product of traces, with one or more Z in each trace. The number of Z 's in an operator corresponds to its \mathcal{R} charge, J .

It was argued [16] that the low energy states of the 1/2 BPS sector forms a closed subspace. These low energy states are protected by supersymmetry and will not turn into higher energy states through interactions and high energy states will not turn into low energy states.

In order to do any sensible calculations in the 1/2 BPS sector, one must know the correlation functions of these operators. Therefore a suitable basis is needed and one that diagonalise the correlators will simplify calculations considerably. Initially, single trace operators was thought to be a good basis,

$$\mathcal{O}_n \propto \text{Tr}(Z^n).$$

However, even though this basis diagonalises the two point functions, it does not diagonalise three point functions. A new basis was proposed in terms of

Schur polynomials [17]. A Schur polynomial is defined as

$$\begin{aligned}\chi_R(Z) &= \frac{1}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma) Z_{i_{\sigma(1)}}^{i_1} Z_{i_{\sigma(2)}}^{i_2} \cdots Z_{i_{\sigma(n)}}^{i_n} \\ &= \frac{1}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma) \text{Tr}(\sigma Z^{\otimes n}) ,\end{aligned}\tag{3.6}$$

where $\chi_R(\sigma)$ is the character of $\sigma \in S_n$. The label R is a representation of the symmetric group S_n and is usually depicted by a Young diagram with n boxes. These operators carry conformal dimension and \mathcal{R} charge n .

The two and three point functions in terms of Schur polynomials are given by

$$\langle \chi_R(Z) \chi_S(Z) \rangle = \delta_{RS} f_R \tag{3.7}$$

$$\langle \chi_R(Z) \chi_S(Z) \chi_T(Z) \rangle = g(R, S, T) f_T \tag{3.8}$$

where $g(R, S, T)$ is the Littlewood-Richardson coefficient. The constant f_R is the product of weights of the Young diagram corresponding to the representation R . The weight for a box in row i and column j is $N - i + j$, so f_R can be calculated by

$$f_R = \prod_{i=1}^I \prod_{j=1}^{J_i} (N - i + j) , \tag{3.9}$$

where I is the number of rows in the diagram and J_i the number of boxes in each row.

The AdS/CFT Dictionary

The AdS/CFT Dictionary (see [18] for a recent review) relates operators in the 1/2 BPS sector of the gauge theory to states in the string theory. The symmetries in the gauge theory match the isometries of the $AdS_5 \times S^5$ background. Therefore conserved quantities that arise due to these symmetries are matched when an operator is matched with a state. Two sets of matching quantities are the scaling dimension and \mathcal{R} of operators with the energies and angular momentum of states in the string theory, respectively.

This line of reasoning has been quite fruitful and for different values of \mathcal{R} -charge the following relations between operators and states have been identified. Operators with $J \sim \mathcal{O}(1)$ are dual to point gravitons. For $J \sim \mathcal{O}(\sqrt{N})$ the dual object is a string [19]. For $J \sim \mathcal{O}(N)$ the dual object is a giant graviton [20].

For $J \sim \mathcal{O}(N^2)$ the dual object's size is divergent. This is interpreted as new backreacted backgrounds.

The underlying principle is that there is a non zero five form field present in the string theory that couples to the world volume of D3 branes. In fact, the coupling between the five form flux and the world volume is related to the R -charge of the brane's dual operator. Moreover the radius of the brane is given by

$$R = \sqrt{\frac{J}{N}} R_{\text{AdS}} \quad \text{with} \quad R_{\text{AdS}}^2 = \sqrt{g_{YM}^2 N \alpha'} , \quad (3.10)$$

in other words the brane will expand as J increases. As J reaches the different values mentioned above, the state in the string theory changes. As $J \sim \mathcal{O}(N^2)$ however, the back reaction between the brane and the background spacetime causes geometry of the background to change entirely.

3.3 Giant gravitons

Giant gravitons [20, 21, 22] are stable, spherically symmetric D3-branes of IIB supergravity. Their stability is achieved by balancing the tension of the D-brane with the five form flux on the brane. Giants extend on the sphere or in the AdS part of the background spacetime. Since the sphere is a compact space, a giant cannot grow larger than the radius of the S^5 , where giants in the AdS_5 part can grow to any size. This places some constraints on how the dual operators can be constructed and the distinction between a giant in AdS_5 and S^5 must arise naturally in the gauge theory and must pose the same limits on the operator's 'size'.

It should also be noted that the size of a giant graviton is proportional to its angular momentum, and hence also its energy.

These D-brane configurations are called giant gravitons because they preserve the same symmetries and share the same energy spectrum of point gravitons [21]. This is a very important result, because point gravitons preserve supersymmetry and as such the operators dual to gravitons (point like and giants) will be in the 1/2 BPS sector of the dual gauge theory. This provides an opportunity to study the interactions of giants through their dual operators.

As for any D-brane, excitations of giants are interpreted as open strings attached to them. Excitation modes of giants have been studied [23] and sur-

prisingly it was found that the spectrum does not depend on any geometrical quantity of giants.

Operators dual to giants

So operators dual to giant gravitons have to be in the 1/2 BPS sector of $\mathcal{N} = 4$ SYM theory, since they share the same energy and quantum states of point like gravitons. Operators dual to giants [17, 24] are Schur polynomials with $n \sim \mathcal{O}(N)$.

In this identification also matches some fundamental properties of giants with that of the symmetric group S_n . For example, it was mentioned that giants can expand in the S^5 part of $AdS_5 \times S^5$. This places an upper bound on the size of the giant, and hence also on its angular momentum. The angular momentum maps to the \mathcal{R} charge of the dual operator, so one should expect a similar bound on some of the dual operators. The completely antisymmetric representation of S_n have exactly such a cut of on its \mathcal{R} charge and one would therefore naturally identify a Schur, with the antisymmetric representation, with a giant expanding in the compact space.

Operators dual to excited giants are restricted Schur polynomials [25]. They are near BPS states which allows us to study them using perturbation theory. A restricted Schur is defined by

$$\chi_{R,R_1}^{(k)} = \frac{1}{(n-k)!} \sum_{\sigma \in S_n} \text{Tr}_{R_1}(\Gamma_R(\sigma)) \text{Tr}(\sigma Z^{\otimes n-k} W^{(1)} \dots W^{(k)}) , \quad (3.11)$$

$$\begin{aligned} & \text{Tr}(\sigma Z^{\otimes n-k} W^{(1)} \dots W^{(k)}) \\ &= Z_{i_{\sigma(1)}}^{i_1} \dots Z_{i_{\sigma(n-k)}}^{i_{n-k}} \left(W^{(1)} \right)_{i_{\sigma(n-k+1)}}^{i_{n-k+1}} \dots \left(W^{(k)} \right)_{i_{\sigma(n)}}^{i_n} . \end{aligned} \quad (3.12)$$

Again R is an irreducible representation of S_n denoted by a Young diagram with n boxes and $\Gamma_R(\sigma)$ is the matrix representation of σ in R . R_1 is a representation of S_{n-k} and is denoted by a Young diagram with $n-k$ boxes.

Here Γ_R is a matrix in R , but the Tr_{R_1} is only taken over the indices that belong to the R_1 subgroup. This is achieved as follows. First consider the subgroup $H = S_{n-k} \otimes S_k \subset S_n$. In general it is possible to find a similarity transformation S , that block diagonalises all elements in the subgroup H . Under this restriction (by using considering $\sigma \in H$) R can be reduced to $R_1 \otimes R_2$, for

some representation R_2 of S_k , and we can write

$$\Gamma_R(\sigma) = \begin{bmatrix} \Gamma_{R_1}(\sigma) & 0 \\ 0 & \Gamma_{R_2}(\sigma) \end{bmatrix} \quad \forall \sigma \in H . \quad (3.13)$$

Now the trace is calculated as the trace over the block corresponding to the R_1 representation, i.e.

$$\mathrm{Tr}_{R_1} \Gamma_R(\sigma) = \mathrm{Tr} \Gamma_{R_1}(\sigma) . \quad (3.14)$$

Notice that the sum in (3.11) is over group elements in S_n . The definition for Tr_{R_1} mentioned above is not suitable for $\sigma \notin S_{n-k} \otimes S_k$, because in this case the representation of σ is not block diagonal. It is however possible to generalise (3.14) for all elements in S_n , by projecting $\Gamma(\sigma)$ onto H . Suppose P_{R_1} represents such a projection, then $P_{R_1} \Gamma_R(\sigma)$ will be a matrix like (3.13), but with $\Gamma_{R_2} = 0$, for all $\sigma \in S_n$. One can then define the trace Tr_{R_1} as

$$\mathrm{Tr}_{R_1} \Gamma_R(\sigma) = \mathrm{Tr} P_{R_1} \Gamma_R(\sigma) . \quad (3.15)$$

These operators capture the constraints placed on how strings are allowed to be attached to giants [25]. Since the world volume of a giant is a compact space and the total charge on it must sum to zero, Gauss' law implies that the must be an equal number of strings ending one the giant as ones leaving it.

3.4 Gravitational Radiation

In the context of string theory, gravitational radiation is understood as excited D-branes that undergoes a transition from a higher excited state to a lower excited state [26, 27]. The energy difference is emitted as closed strings. Into the bulk.

The qualitative argument [27] is that the giant is charged under the four form flux, accelerating the brane relative to the background spacetime. Strings, however, are not charged under the four form flux and in particular strings that are attached to the brane will only be dragged along. From the brane's perspective the strings will be in geodesic free fall. The apparent force acting on the string will cause it to stretch and bring the endpoints together.

This situation is understood very well in the dual gauge theory [26]. The operator dual to an excited giant, of momentum p , expanding in the S^5 is given

by restricted Schur polynomial

$$\chi_{1^{p+1}, 1^p}(Z, W), \quad (3.16)$$

where 1^p is the totally antisymmetric representation of S_p , or a Young diagram of p boxes with only one row. The open string word W is Y^J , where J is the mass of the string. According to the AdS/CFT dictionary p and J must be $\mathcal{O}(N)$ and $\mathcal{O}(\sqrt{N})$, respectively.

The operator dual to the ground state giant and a closed string is the product of a Schur polynomial with a single trace operator

$$\text{Tr}(Y^J)\chi_{1^p}(Z). \quad (3.17)$$

Amplitude of the excited giant decaying to its ground state by emitting a closed string is given by

$$\mathcal{A} = \frac{1}{\sqrt{A}} \langle \text{Tr}(Y^{\dagger J}) \chi_{1^p}^\dagger \chi_{1^{p+1}, 1^p} \rangle \quad (3.18)$$

where the normalisation, A , is given by

$$A = \langle \text{Tr}(Y^{\dagger J})\chi_{1^p}^\dagger \text{Tr}(Y^J)\chi_{1^p} \rangle \langle \chi_{1^{p+1}, 1^p}^\dagger \chi_{1^{p+1}, 1^p} \rangle. \quad (3.19)$$

The two point functions are easy to calculate using machinery accosted with Schur polynomials [26] and is found to be

$$\mathcal{A} = \sqrt{\frac{J(N-p)}{pN}} \left(1 + \mathcal{O}\left(\frac{J^4}{N^2}\right) \right). \quad (3.20)$$

One can also calculate the amplitude of pieces of the open string radiating off the brane. This situation can be realised if the string self intersects before the endpoints reach each other. In this situation the final state is given by the product of a ‘smaller’ single trace operator, dual to a closed string with mass J_1 , and a restricted Schur, dual to a giant with momentum p and attached string with mass J_2 ,

$$\text{Tr}(Y^{J_1})\chi_{1^{p+1}, 1^p}(Z, Y^{J_2}), \quad (3.21)$$

with $J = J_1 + J_2$. This amplitude is

$$\mathcal{A} = \frac{\sqrt{J_1}(J_2 + 1)}{N} \left(1 + \mathcal{O}\left(\frac{J^4}{N^2}\right) \right) \quad (3.22)$$

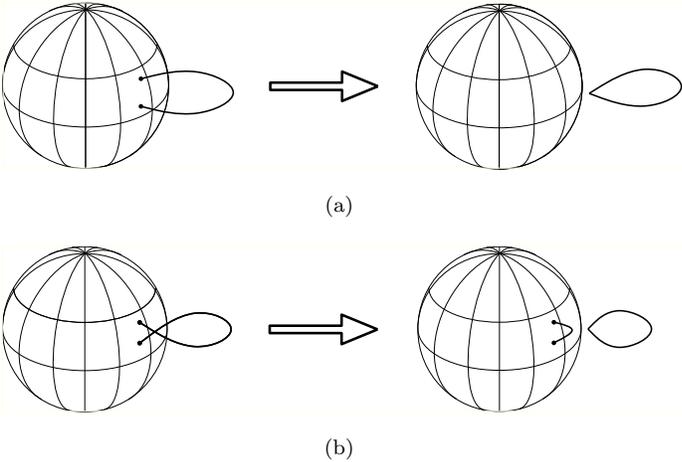


FIGURE 3.1: A depiction of the two modes of gravitational radiation from an excited giant graviton. In one case (a) the entire string peels off and for the other (b) only a part of the string breaks off, leaving a shorter string attached to the brane.

Chapter 4

Static Strings in Rindler Space

Accelerated frames have become an important tool in studying strings dynamics near black holes. Rindler space provides a natural background for studying relativistic objects under constant acceleration. In particular strings have been studied in this setting [28, 29, 30].

In this chapter we review the solutions of relativistic static string under a constant acceleration [31].

4.1 Rindler Space

Rindler space[32] is a parametrisation of a wedge in Minkowski space, as seen from an observer under the influence of a constant acceleration. Consider a flat spacetime with line element $ds^2 = -dT^2 + dw^2$. Suppose we have a point particle moving in this spacetime with a constant acceleration acting on the particle in the positive w direction. If $x^\mu(t)$ is the world line of the particle, with t the proper time of the accelerated observer, the vector tangent to the world line is given by $u^\mu = dx^\mu/dt$ and the acceleration by $\gamma^\mu = du^\mu/dt$. The tangent and acceleration is defined such that $u^\mu u_\mu = -1$ and $\gamma^\mu \gamma_\mu = a^2$. Since the w axis was chosen to aligned with the acceleration to give

$$-u_0^2 + u_1^2 = -1 \quad \text{and} \quad -\gamma_0^2 + \gamma_1^2 = a^2 . \quad (4.1)$$

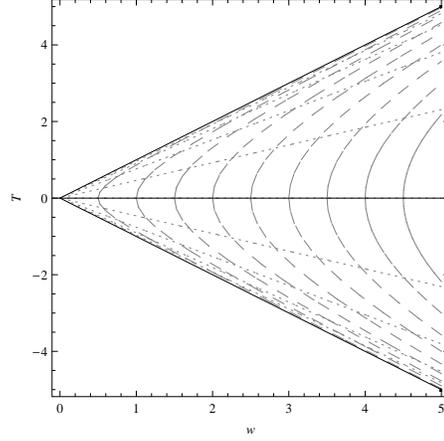


FIGURE 4.1: The Rindler Wedge. Constant x coordinates are shown as dashed lines and constant t coordinates as dotted lines.

The first equation suggests that we can parametrise the world line using $u_0 = \cosh f(t)$ and $u_a = \sinh f(t)$, and the second fixes $f = at$. The particles world line in terms of its acceleration and proper time is given by

$$w = \frac{1}{a} \sinh(at) \quad \text{and} \quad T = \frac{1}{a} \cosh(at) . \quad (4.2)$$

We use this parametrisation if a particle's world line to define a parametrisation of Minkowski space. Rindler space just that parametrisation and is defined by the change of coordinates

$$w = x \sinh(at) \quad \text{and} \quad T = x \cosh(at) . \quad (4.3)$$

In Rindler space, point particles subject to a constant acceleration have a fixed x coordinate. The relationship between a particles x position and its acceleration is $x = 1/a$.

The line element in the new coordinates is

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -a^2 x^2 dt^2 + dx^2 . \quad (4.4)$$

This is the metric of a Rindler spacetime, which describe a wedge of Minkowski spacetime called the Rindler Wedge. See Figure 4.1.

The near-horizon limit of AdS-Schwarzschild

Black holes in anti-de Sitter space was considered in [33] and later in the context of the AdS/CFT correspondence in [4].

The AdS_n -Schwarzschild metric is given by

$$ds^2 = -V(r)dt^2 + (V(r))^{-1}dr^2 + r^2d\Omega_{n-2}^2 \quad (4.5)$$

$$V(r) = 1 + \frac{r^2}{b^2} - \frac{\omega_n M}{r^{n-2}} . \quad (4.6)$$

Here M is the mass of the black hole... The event horizon is at r_+ , with r_+ being the largest solution of $V(r) = 0$.

4.2 Strings in Rindler Space

Strings in a Rindler background have been studied in a variety of scenarios. These include strings near black holes [34] and even meson decay [35]. The common thread is that Rindler space provides a simple background where accelerated objects can be studied in flat spacetime. However, all the references considers only rigid strings. In other words, the string is parametrised using $X^0 = t$ and $X^i = X^i(\sigma)$ for $i = 1, 2, \dots, d$, which does not allow the string to move relative to itself.

Although we are interested in describing strings peeling off D-branes, for which this description will be insufficient, some useful lessons can be learned about how strings behave in this background by studying the static case.

Lets consider a semi classical open string. Since open strings have to end somewhere and for the time being we are only interested in static strings, the string end points are assumed to be attached to two D0-branes. The D-branes are charged under the RR field and one can imagine a small constant field strength to provide a constant acceleration to the branes. The string, however is not charged under the RR field and are only be dragged along with the branes, but will feel no direct acceleration due to the RR field. The main concern of this section is the shape of the string profile as it is dragged by the branes.

In this set up, the string is embedded in a Rindler Space background with one extra transverse direction. The line element is

$$ds^2 = g_{\mu\nu}dX^\mu dX^\nu = -a^2x^2dt^2 + dx^2 + dy^2 . \quad (4.7)$$

Boundary conditions are provided by the D-branes, that are held at a constant separation d in the y direction. The branes are accelerating the x direction, since that is the spacial coordinate that appear in the first term of (4.7).

The string world sheet can be parametrised by two world sheet coordinates $\xi^a = (\tau, \sigma)$. There are two natural ways to choose the parametrisation for σ . The first by simply setting $y(\sigma) = \sigma$ and the second by making σ a parameter along the length of the string.

In the static gauge, the string coordinates in the embedding space is

$$t = \tau, \quad x = x(\sigma) \quad \text{and} \quad y = y(\sigma) . \quad (4.8)$$

Some common factors that appear in the equations of motions can be simplified. By using the shorthand notation $a \cdot b = g_{\mu\nu} a^\mu b^\nu$, these are

$$X' \cdot \dot{X} = 0 \quad (4.9)$$

$$(\dot{X})^2 = -a^2 x^2 \quad (4.10)$$

$$(X')^2 = (x')^2 + (y')^2 \quad (4.11)$$

The momenta along the string world sheet are

$$\mathcal{P}_\mu^\tau = -T_0 \frac{(x')^2 + (y')^2}{ay\sqrt{(x')^2 + (y')^2}} \dot{X}^\mu \quad (4.12)$$

$$\mathcal{P}_\mu^\sigma = -T_0 \frac{ax}{\sqrt{(x')^2 + (y')^2}} X'^\mu \quad (4.13)$$

The only non-zero term on the left hand side of (2.4) is when $\nu = \rho = 0$:

$$\frac{\partial \mathcal{L}}{\partial g_{00}} = -\frac{T_0}{2} \frac{\sqrt{(x')^2 + (y')^2}}{ax} \quad (4.14)$$

Finally, the equations of motions are

$$\frac{d}{d\tau} \left(\frac{\sqrt{(x')^2 + (y')^2}}{ax} \right) = 0 \quad (4.15)$$

$$\frac{d}{d\sigma} \left(\frac{xx'}{\sqrt{(x')^2 + (y')^2}} \right) = -\sqrt{(x')^2 + (y')^2} \quad (4.16)$$

$$\frac{d}{d\sigma} \left(\frac{xy'}{\sqrt{(x')^2 + (y')^2}} \right) = 0 \quad (4.17)$$

The condition that the spatial world sheet coordinate must parametrise the length of the string is enforced with $(x')^2 + (y')^2 = 1$. Now the equations of motion reduces to same set of equations as in the classical case.

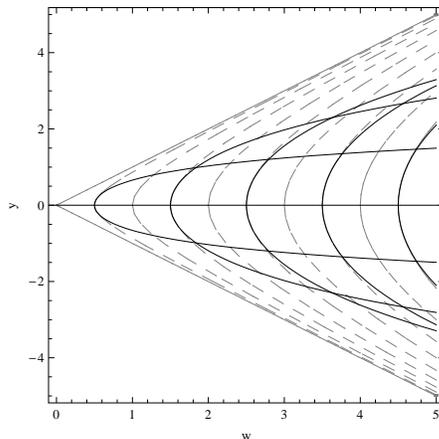


FIGURE 4.2: Static string solutions in Rindler space.

The fact that semi-classical strings in Rindler Space have the same profile as classical strings should be of some reassurance that, in this particular set up, our intuition of classical string also apply for relativistic strings. Some of the string solutions are plotted in Figure 4.2.

Energy Analysis

From Figure 4.2, one can see that for any two points $(x_0, -d/2)$ and $(x_0, d/2)$ there are two possible string solutions. In addition to these two solutions, a third solution is also possible [31]. This additional solution is given by two strings connecting $(x_0, -d/2)$ and $(x_0, d/2)$ to the origin in a straight line.

Physical strings can only be described by one solution, so we need to find out which of these solutions are stable.

The energy of the catenary solution can be calculated from the Lagrangian and the $X = f(\sigma)$, $Y = \sigma$ parametrisation. The string's energy is given by $E_{\text{cat}} = -\mathcal{L}$, since it is static. The integral over σ is changed to an integral over f so that the total energy is a function of h , the y component of the endpoint.

$$E_{\text{cat}} = aT \int_0^{x_0} d\sigma f(\sigma) \sqrt{(\partial_\sigma f(\sigma))^2 + 1} \quad (4.18)$$

$$= 4aTL^2 \int_1^{\bar{u}} du \frac{u^2}{\sqrt{u^2 - 1}}, \quad (4.19)$$

where we also made the substitution $u = f/L$ and $\bar{u} = h/L$. The energy for the

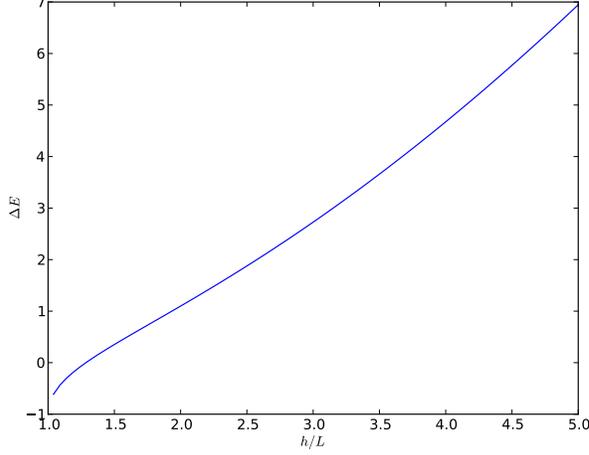


FIGURE 4.3: The energy difference between the catenary and a pair of straight lines between the origin and the endpoints (a, h) .

straight line solutions is given by

$$E_{\text{st}} = aT \int d\sigma f^2(\sigma) \quad (4.20)$$

$$= 2aTL^2 \int_0^{\bar{u}} du 2u \quad (4.21)$$

$$= 4aTL^2 \left(\int_1^{\bar{u}} du u + \frac{1}{2} \right) . \quad (4.22)$$

The energy difference between the two configurations is

$$\Delta E = E_{\text{cat}} - E_{\text{st}} \quad (4.23)$$

$$= 4aTL^2 \left[\int_1^{\bar{u}} du \left(\frac{u^2}{\sqrt{u^2 - 1}} - u \right) - \frac{1}{2} \right] \quad (4.24)$$

$$= 2aTL^2 \left[\bar{u} \sqrt{\bar{u}^2 - 1} + \ln \left(\bar{u} + \sqrt{\bar{u}^2 - 1} \right) - \bar{u} \right] . \quad (4.25)$$

The catenary solution will be favoured when $\Delta E < 0$. Figure 4.3 shows ΔE as a function of h/L , in which it is clear that the catenary solution with $h \sim L$ is the string configuration with the lowest energy.

Energy of the configuration

Now that we know which solution correspond to stable string configurations it is necessary to verify that the total energy of a string decreases as the end points move closer. The total energy is now calculated by integrating over σ in (4.18), which gives

$$E_{\text{cat}} = \kappa T L \int_{-a}^a d\sigma \cosh^2(\sigma/L) \quad (4.26)$$

$$= \frac{\kappa T}{2} \sinh(2a/L) . \quad (4.27)$$

Clearly string configurations with a small separation between the end points will be favoured to ones with larger separations.

Chapter 5

Newtonian Strings

The main results in the previous chapter is that static strings in Rindler space take the shape of a catenary, the same as shape made by a classical string in a uniform gravitational potential. Ultimately we are interested in the dynamics of strings in Rindler space, but before we jump into that, we will revisit the classical catenary. There are two reasons for this, firstly there is a strong correlation between the static configurations of relativistic and classical strings and secondly, classical strings are much easier to deal with mathematically and in studying them we hope to gain a few clues as how to deal with the relativistic system.

The content of this chapter was taken from [36].

5.1 The Static Catenary

The static catenary has proven to be much richer problem than than what it first appears to be and continues to be the subject of active research. These topics include the elastic, centrally loaded chains [37] and a number of studies of chains with varying mass density (s) [38]. A uniform ideal chain hanging under its own weight from two fixed points at a fixed height assumes the shape of a catenary. Calculating the shape function of the chain has become a routine exercise in variational calculus [39, 40].

Consider a string with constant length $2\ell_0$ and linear mass density μ . Further suppose that the string is attached by its end-points at the same height and

hanging in a gravitational field. One example of such a system would be a segment of a telephone line attached to two telephone poles. The question is: what will the string profile ($x = X(\xi), y = Y(\xi)$) be that minimises the gravitational potential energy of the string? A very elegant solution is given by the variational principle.

The action and Lagrangian for this system is given by

$$S = \int d\xi L \quad \text{and} \quad L = g\mu Y \sqrt{(X'(\xi))^2 + (Y'(\xi))^2}, \quad (5.1)$$

respectively. Here ξ is a spacial parameter along the string that can be chosen in a convenient way.

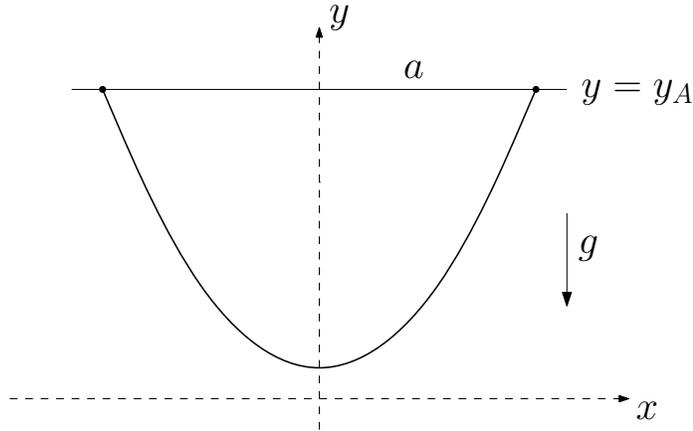


FIGURE 5.1: A string of length $2\ell_0$ and linear mass μ is hanging in a gravitational field g .

Choosing $\xi = x$

For this choice of the length parameter the action becomes

$$L = g\mu \int dx Y \sqrt{1 + (Y'(x))^2}. \quad (5.2)$$

Because there is no explicit dependence on x , the Euler-Lagrange equation implies that $L - Y'\partial L/\partial Y'$ is constant. Therefore Y must satisfy

$$g\mu Y = k_x \sqrt{1 + Y'^2}, \quad (5.3)$$

where k_x is a constant related to the string length. This equation can be integrated to give

$$Y(x) = k \cosh\left(\frac{x}{k}\right) + C \quad \text{with} \quad k = \frac{k_x}{g\mu}. \quad (5.4)$$

This is the famous catenary solution. The constant C is fixed by boundary conditions at the string end points $Y(\pm a) = y_A$, yields $C = y_a - k \cosh(a/k)$. The string length is simply given by

$$2\ell_0 = \int_{-a}^a dx \sqrt{1 + Y'^2} = 2k \sinh\left(\frac{a}{k}\right). \quad (5.5)$$

Therefore the full solution is

$$Y(x) = k \left[\cosh\left(\frac{x}{k}\right) - \cosh\left(\frac{a}{k}\right) \right] + y_A, \quad (5.6)$$

with the constraint

$$\ell_0 = k \sinh\left(\frac{a}{k}\right). \quad (5.7)$$

Here k is indeed given by a complicated expression of the two parameters ℓ_0 and a , but the solution is nevertheless unique and completely determined. In particular, we can compute the angle between the tangent to the string and the x -axis as

$$\tan \alpha = \frac{dY}{dx} = \sinh\left(\frac{x}{k}\right). \quad (5.8)$$

This angle is a useful quantity to describe the tension in the string and when dynamic strings are considered.

Choosing $\xi = s$

An alternative to the first choice, is for ξ to parametrise the length of the string. This choice is expressed by $ds^2 = dX^2 + dY^2$ and results in an additional constraint equation

$$(X')^2 + (Y')^2 = 1.$$

A Lagrange multiplier, λ , is added to (5.1) to impose this constraint. In order to obtain the system of differential equations that describe the string profile, we now need to extremise

$$S = g\mu \int dt Y \sqrt{(X')^2 + (Y')^2} + \lambda \left(\int dt \sqrt{(X')^2 + (Y')^2} - 1 \right). \quad (5.9)$$

The Euler-Lagrange equations for λ , X and Y , respectively, gives

$$(X')^2 + (Y')^2 = 1 , \quad (5.10)$$

$$X'(g\mu + \lambda) = kg\mu , \quad (5.11)$$

$$\frac{d}{ds} \left[\frac{Y'(g\mu Y + \lambda)}{\sqrt{(X')^2 + (Y')^2}} \right] = g\mu . \quad (5.12)$$

By making the substitution $\tilde{Y} = Y + \lambda/g\mu$ and combing the first two equations, we have

$$\frac{k^2}{\tilde{Y}^2} + \tilde{Y}'^2 = 1 ,$$

which has the general solution

$$\tilde{Y}(s) = \pm \sqrt{(s - s_0)^2 + k^2} .$$

We can choose s_0 to that $s = 0$ coincides with the middle of the string and choose the domain of the parameter $s \in [-\ell_0, \ell_0]$. X can be obtained by solving the separable differential equation $X'\tilde{Y} = k$. The full solution of the system of equations is given by

$$X(s) = k \sinh^{-1} \left(\frac{s}{k} \right) \quad (5.13)$$

$$Y(s) = \sqrt{s^2 + k^2} - \sqrt{\ell_0^2 + k^2} + y_A . \quad (5.14)$$

The parameter k is fixed by the boundary conditions $-X(-\ell_0) = X(\ell_0) = a$, so that

$$a = k \sinh^{-1} \left(\frac{\ell_0}{k} \right) \quad \text{or} \quad \ell_0 = k \sinh \left(\frac{a}{k} \right) , \quad (5.15)$$

which is the same as the $\xi = x$ choice. The only other parameter is λ , which is fixed by the boundary condition for Y

$$\frac{\lambda}{g\mu} = \sqrt{\ell_0^2 + k^2} + y_A . \quad (5.16)$$

This is solution is the same as the one obtained for the choice $\xi = x$, which is easily checked by eliminating s from equations (5.13) and (5.14).

From (5.10) we can define a function $\alpha(s)$ that satisfies

$$X'(s) = \cos \alpha(s) \quad \text{and} \quad Y'(s) = \sin \alpha(s) . \quad (5.17)$$

The physical interpretation for this function is the angle between vector tangent to the string, $\mathbf{t} = (X'(s), Y'(s))$ and the x -axis. Using the solutions of X and

Y from (5.13) and (5.14), we have

$$\tan \alpha(s) = \frac{s}{k} \quad (5.18)$$

$$\cos \alpha(s) = \frac{s/k}{\sqrt{1 + (s/k)^2}} \quad (5.19)$$

$$\sin \alpha(s) = \frac{1}{\sqrt{1 + (s/k)^2}}. \quad (5.20)$$

Tension along the string

From the $\xi = s$ solution, one can derive the tension along the string. Since the tension is an internal force, it cannot be calculated in the Lagrangian formalism. It is still possible to find an expression for the tension using the Newtonian approach.

Even though the tension is not of great interest in the static case, identifying it will help identifying the tension for dynamic case, which is not so simple to obtain. Further more, the tension is one of the important points of difference between classical and relativistic strings.

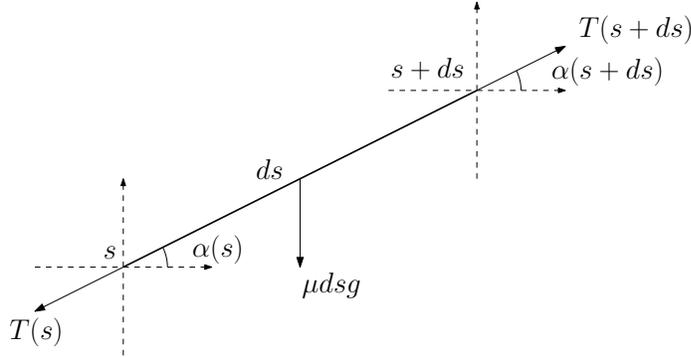


FIGURE 5.2: A string segment with all the forces acting on it.

The tension is a vector tangential to the string and can be written as $\mathbf{T}(s) = (T_x(s), T_y(s)) = T(s)\mathbf{t}(s)$. The string segment is in equilibrium so the x components are equal and the difference in the y components must balance the weight of the segment (see Figure 5.2). This gives

$$T_x(s + ds) - T_x(s) = 0 \quad \text{and} \quad T_y(s + ds) - T_y(s) = g\mu. \quad (5.21)$$

If the string segment is taken to be infinitesimal, the left hand side of these

expressions can be written in terms of derivatives with respect to s to give

$$T'_x = \frac{d}{ds}(T \cos \alpha) = 0 \quad \text{and} \quad T'_y = \frac{d}{ds}(T \sin \alpha) = g\mu . \quad (5.22)$$

From the second expression, we have $T_y(s) = T(s) \sin \alpha(s) = sg\mu$. There is no constant term because $T_y(0) = 0$. Using (5.19), we can write the tension as

$$T(s) = kg\mu\sqrt{1 + (s/k)^2} . \quad (5.23)$$

Using this definition of the tension (5.11) and (5.12) can be written as

$$\frac{d}{ds} [X'(s)T(s)] = 0 \quad (5.24)$$

$$\frac{d}{ds} [\tilde{Y}'(s)T(s)] - g\mu = 0 . \quad (5.25)$$

Energy of the configuration

The energy of the catenary can be calculated and written as a function of the two parameters (ℓ_0, a) of the problem

$$E(a, \ell_0) = g\mu \int_{-\ell_0}^{\ell_0} ds Y(s) \quad (5.26)$$

$$= 2g\mu\ell_0 \left(y_A - \sqrt{\ell_0^2 + k^2} \right) + \int_{-\ell_0}^{\ell_0} ds \sqrt{s^2 + k^2} \quad (5.27)$$

$$= 2g\mu\ell_0 \left(y_A - \sqrt{\ell_0^2 + k^2} \right) + \frac{1}{2} \left[s\sqrt{k^2 + s^2} + k^2 \ln(s + \sqrt{k^2 + s^2}) \right]_0^{\ell_0} . \quad (5.28)$$

We can use (5.15) to write $k^2 \ln(s + \sqrt{k^2 + s^2}) = k^2 \ln k + a/k$. Then, if we let the mass of the string be $m = 2\mu\ell_0$ the final expression for the energy is

$$E(a, \ell_0) = mgy_A - \frac{1}{2}mg\ell_0\sqrt{1 + \frac{k^2}{\ell_0^2}} + \frac{1}{2}\frac{m}{\ell_0}gka . \quad (5.29)$$

In the limit $a \rightarrow \ell_0$, the string is in a straight line along $y = y_A$ and, using (5.15), we see that $k \rightarrow \infty$ so that $E \rightarrow gmy_A$, as expected. In the limit $a \rightarrow 0$, the string hangs vertically and (5.15) implies that $k \rightarrow 0$ so that $E \rightarrow gm(y_A\ell_0/2)$, which is also expected since the position of the barycentre of the string is at $y_A\ell_0/2$. Figure 5.3 gives the dependence of E with a .

This analysis shows that string configurations with a small separation between endpoints are energetically favoured to configurations with large a separation. Subsequently we would expect the endpoints to move together, if they are allowed to do so.

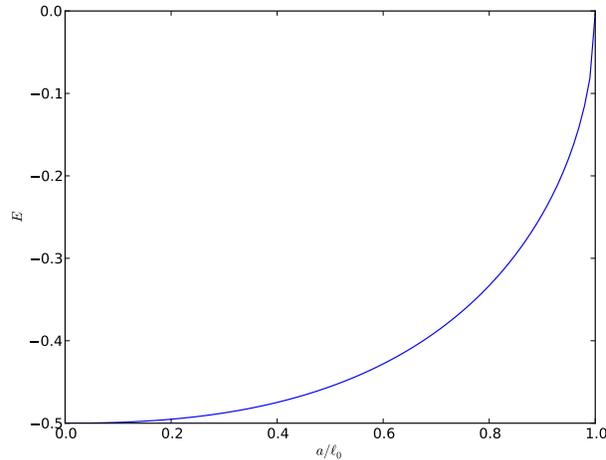


FIGURE 5.3: The variable part of E , in units of $gm\ell_0$, as a function of a . Note that k is an implicit function of a , which is why the graph is not linear.

5.2 The Dynamical Catenary

We now turn our attention to a string dangling in a uniform gravitational field that is allowed to move in time.

There are two situations that are of particular interest. Firstly, the endpoints are restricted to move only in the x direction. This situation is similar to a chain attached to a horizontal rod, see Figure 5.4. Secondly, the endpoints are fixed, but waves are allowed to propagate along the string. This is similar to standing waves in a rubber band with the difference being that the string is not pulled ‘tight’.

In the first case, the string will try to minimise its energy, so we expect the end-points will move together as we discussed in the previous section.

To model this problem the string coordinates become function of time in addition to a spacial parameter. The $\xi = s$ choice is used which will allow solutions that form loops and cusps, unlike the $\xi = x$ parameter choice. The

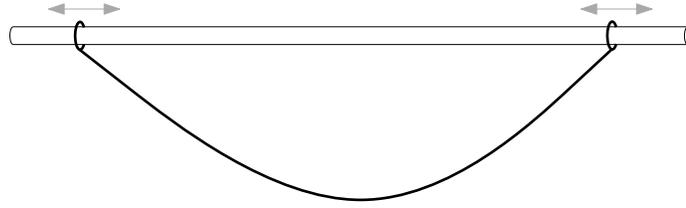


FIGURE 5.4: The system that we are interested in is like a rope suspended from a horizontal rod. The end points are allowed to move horizontally along the rod, but not transverse to it.

Lagrangian includes a momentum term, and now reads

$$\begin{aligned} \mathcal{L} = \mu \int ds dt \sqrt{(X')^2 + (Y')^2} \left[\frac{1}{2}(\dot{X}^2 + \dot{Y}^2) + gY \right] \\ + \lambda \left(\int ds dt \sqrt{(X')^2 + (Y')^2} - 1 \right) , \end{aligned} \quad (5.30)$$

where \dot{X} and \dot{Y} are derivatives with respect to time.

Equations of motion

The equations of motion can be obtained by the Euler-Lagrange equations. For X , Y and λ , these are

$$\ddot{X} = \frac{d}{ds} \left[X' \left(gY + \frac{1}{2}\dot{X} + \frac{1}{2}\dot{Y} + \frac{\lambda}{\mu} \right) \right] \quad (5.31)$$

$$\ddot{Y} = \frac{d}{ds} \left[Y' \left(gY + \frac{1}{2}\dot{X} + \frac{1}{2}\dot{Y} + \frac{\lambda}{\mu} \right) \right] - g \quad (5.32)$$

$$(X')^2 + (Y')^2 = 1 . \quad (5.33)$$

If we identify the string tension as

$$T(s, t) = \mu \left(gY + \frac{1}{2}\dot{X} + \frac{1}{2}\dot{Y} + \frac{\lambda}{\mu} \right) , \quad (5.34)$$

the equations of motion simplify to their static counterparts (5.26) and (5.28), after time derivatives are set to zero.

Moving end-points

We are interested in a configuration where the end-points are allowed to move in x direction, but not in the y direction. This constraint is imposed by choosing

the following boundary conditions for Y

$$Y(\pm\ell_0, t) = y_A \quad \text{and} \quad \dot{Y}(\pm\ell_0, t) = \ddot{Y}(\pm\ell_0, t) = 0 . \quad (5.35)$$

There is a discontinuity at the string end-points which leads to the following relationships between the forces acting on the segments at the end-points

$$\mu ds \ddot{X}(\ell_0, t) = -T_x(\ell_0, t) \quad (5.36)$$

$$\mu ds \ddot{Y}(\ell_0, t) = -T_y(\ell_0, t) - g\mu ds . \quad (5.37)$$

Using the boundary conditions, the second equation become $T_y(\ell_0, t) = -g\mu ds$ and the ratio T_x/T_y gives the following boundary condition for X

$$\ddot{X}(\ell_0, t) = -g \cot \alpha(\ell, t) . \quad (5.38)$$

Using the static configuration as the initial condition at $t = 0$, the final system is

$$\ddot{X}(s, t) = \frac{d}{ds} \left[\frac{T}{\mu} X' \right] \quad (5.39)$$

$$\ddot{Y}(s, t) = \frac{d}{ds} \left[\frac{T}{\mu} Y' \right] - g \quad (5.40)$$

$$(X')^2 + (Y')^2 = 1 \quad (5.41)$$

$$\ddot{X}(\pm\ell_0, t) = \mp g \cot \alpha(\pm\ell, t) \quad (5.42)$$

$$X(s, 0) = X_{\text{stat}}(s) \quad (5.43)$$

$$Y(s, 0) = Y_{\text{stat}}(s) \quad (5.44)$$

Adiabatic Solution

We can find an approximate solution to the system by assuming that the motion of a string is a sequence of static configurations. This assumption is reasonable for slow moving end-points.

Under this assumption $\alpha(\ell_0, t) = \alpha_{\text{stat}}(t) = k(t)/\ell_0$. Substituting this into the boundary condition for X (5.38) and the static solution (5.13), we find the closed system for the end-points

$$\ddot{X}(\ell_0, t) = -\frac{gk(t)}{\ell_0} \quad \text{and} \quad \sinh \frac{X(\ell_0, t)}{k(t)} = \frac{\ell_0}{k(t)} . \quad (5.45)$$

This system can be solved numerically, however it is useful to introduce dimensionless quantities $\alpha(t) = X(\ell_0, t)/\ell_0$, $K(t) = k(t)/\ell_0$ and $\tau = t/t_c$ with

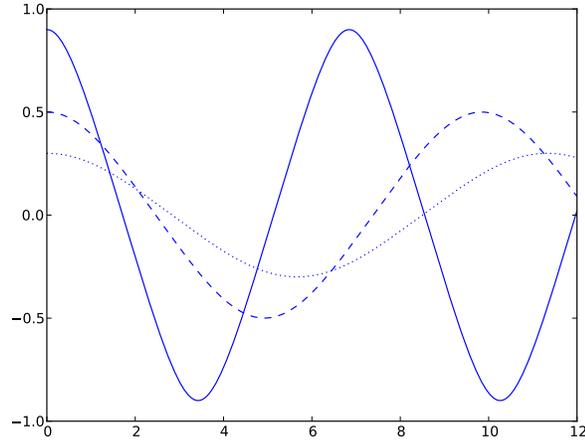


FIGURE 5.5: Numerical solutions for the adiabatic approximation.

$t_c^2 = l_0/g$, in which the system reads

$$\ddot{\alpha}(t) = -K(\tau) \quad \text{and} \quad K(\tau) \sinh \frac{\alpha(\tau)}{K(\tau)} = 1, \quad (5.46)$$

with dot being derivatives with respect to τ .

The numerical solution is plotted in Figure 5.5. Notice that the period changes with the initial amplitude, contrary to a harmonic oscillator.

Numeric Solution

The system (5.39-5.44) can be solved numerically. However, to facilitate the numerical integration, it will prove more convenient to rewrite it as a system of first order equations of the form,

$$\dot{X}(s, t) = u(s, t), \quad \dot{u}(s, t) = \frac{d}{ds} F[X(s, t), X'(s, t)]. \quad (5.47)$$

Differential equations of this form are of the general class that may be solved by either the Lax-Friedrichs finite difference [41] or Smoothed Particle Hydrodynamics [42] numerical integration schemes. To understand the exact behaviour of the dynamical catenary (and check the range of validity of the assumption of adiabaticity), we have implemented both of them and checked that they give the same results.

A surface plot of a solution is depicted in Figure 5.6(a) and the motion of the endpoints for different configurations is shown in Figure 5.6(b) for the case of a constant length string. The first plot shows that the adiabatic solution is valid at most for the first oscillations since after this, cusps developed and cannot be treated by the adiabatic approximation.

Waves

We can also consider waves propagating along a static catenary. For small deviations on the static solution, the string profile can be written as

$$\begin{aligned} \mathbf{X}(s, t) &= \{X(s, t), Y(s, t), Z(s, t)\} \\ &= \mathbf{X}_{\text{stat}}(s) + \mathbf{u}(s, t) \\ &= \{X_{\text{stat}}(s), Y_{\text{stat}}(s), 0\} + \{u_x(s, t), u_y(s, t), u_z(s, t)\} . \end{aligned} \quad (5.48)$$

Firstly we ignore perturbations transverse to the plane of the string by setting $u_z = 0$. The condition in (5.33) and the definition of \mathbf{u} requires that that

$$\mathbf{t} \cdot \mathbf{u}' = X'_{\text{stat}} u'_x + Y'_{\text{stat}} u'_y = 0 . \quad (5.49)$$

If we further restrict our attention to perturbations perpendicular to the string, we must have $\mathbf{u} = f(s, t)\mathbf{N}(s)$. With the normal vector chosen as $\mathbf{N}(s) = (-Y'_{\text{stat}}, X'_{\text{stat}}, 0)$ and $f(s, t) = X'_{\text{stat}} u_y - Y'_{\text{stat}} u_x$.

Expanding the equations of motion (5.31) and (5.32) using (5.48), one can extract the following equation of motion for f

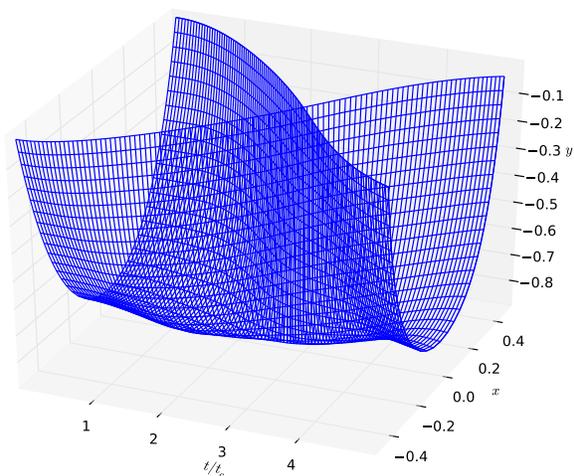
$$\ddot{f} - \frac{T_{\text{stat}}}{\mu} f'' = \frac{T_{\text{stat}}'}{\mu} f' . \quad (5.50)$$

$T_{\text{stat}}(s)$ is the tension along the string (the same as for the static case) and is given in (5.23). The equation above describes damped waves with sound speed

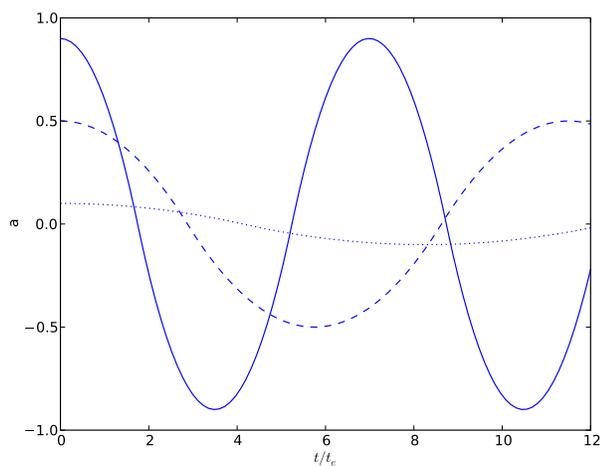
$$c^2(s) = kg \sqrt{1 + \left(\frac{s}{k}\right)^2} . \quad (5.51)$$

The boundary conditions are $f(\pm\ell_0, t) = \dot{f}(\pm\ell_0, t) = f'(\pm\ell_0, t) = 0$ since the endpoints remain fixed.

In order to include perturbations transverse to the plane of the static string, one first needs to extend the action to include a Z coordinate and derive an



(a)



(b)

FIGURE 5.6: (a) Numerical solution of the full string dynamics with $a(0) = 0.5$. (b) The position of an endpoint for $a(0) = 0.9$ (solid), 0.5 (dashed) and 0.1 (dotted). We see that the adiabatic solution is a good qualitative description before the first crossing of the end-points. Thereafter, the development of cusps causes the adiabatic approximation to break down.

equation of motion for it. The extended action is

$$S = \int ds dt \mu \sqrt{X'^2 + Y'^2 + Z'^2} \left[\frac{1}{2} (\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2) + gY \right] + \lambda \left(\int ds \sqrt{X'^2 + Y'^2 + Z'^2} - 1 \right). \quad (5.52)$$

The variation with respect to Z at first order in u_z is (since it has no zeroth order)

$$\ddot{u} - \frac{T_{\text{stat}}}{\mu} u''_z = \frac{T_{\text{stat}}}{\mu} u'_z. \quad (5.53)$$

This is the same equation as for waves in the plane of the catenary.

Figure 5.7(a) depicts the evolution of f with time assuming an initial profile with $u_x(s, t) = A \sin(\pi s)$ and $u_y(s, t) = A \sin(1.5\pi s)$. A is chosen so that the perturbed string's length is the same as the static string. Using the relation $\mathbf{u}(s, t) = f(s, t)\mathbf{N}(s)$, we can calculate the perturbations at later time steps. Figure 5.7(b) shows the string profile at different times.

Plateau-Rayleigh Instabilities

Plateau-Rayleigh instabilities [43] are wave like perturbations that occur in water jets. They can commonly be seen in a small stream of water from a running tap, where a uniform stream develops small ripples and eventually breaks into a number of individual droplets.

It can be shown [44] that a fluid jet, shot downward from a circular orifice, with radius a , and falling under the influence of gravity, will have radius given by

$$\frac{R(z)}{a} = \left[1 + \frac{2}{F_r} \frac{z}{a} + \frac{2}{W_e} \left(1 - \frac{a}{r} \right) \right]^{-1/4}. \quad (5.54)$$

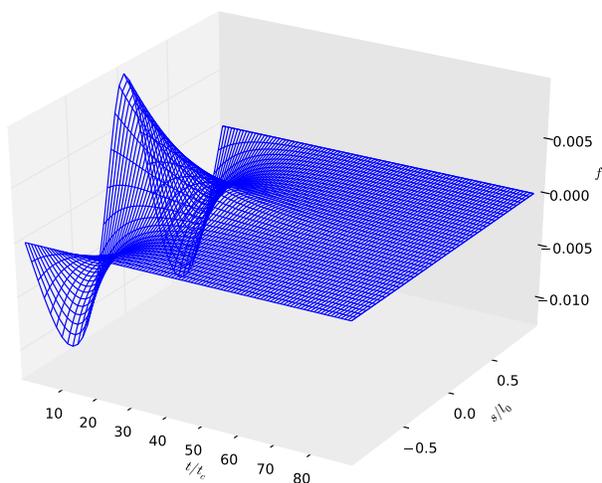
Here F_r and W_e are dimensionless constants given by

$$F_r = \frac{U_0^2}{ga} \quad \text{and} \quad W_e = \frac{\rho U_0^2 a}{\sigma}, \quad (5.55)$$

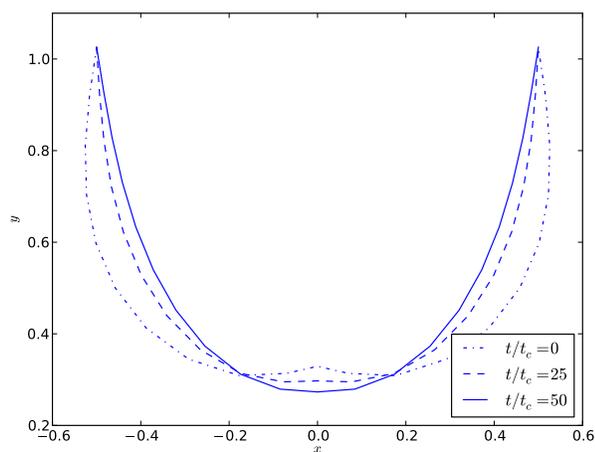
respectively. The other constants are jet velocity at the orifice U_0 , the fluid density σ , gravitational acceleration g and the surface tension σ .

The shape of a fluid jet given by (5.54) is not stable. Suppose the surface of the fluid jet is perturbed by

$$\tilde{R}(z) = R(z) + r(z) \exp(\omega t + ikz). \quad (5.56)$$



(a)



(b)

FIGURE 5.7: (a) A numerical solution of $f(t, s)$ with $u_x(0, s) = A \sin(\pi s)$ and $u_y(0, s) = A \sin(1.5\pi s)$. A is fixed by the string length. The perturbations decay like an over damped harmonic oscillator. This can be seen in the figure as the bumps die out monotonically. (b) The string profile at different times using the same initial configuration as in (a). The string coordinates here were calculated using $X = X_{\text{stat}} - fY'$ and $Y = Y_{\text{stat}} + fX'$.

It is possible to linearise the governing equations, the Navier-Stokes equations, and find a differential equation for that perturbation amplitude $r(z)$. By analysing the solution for $r(z)$ one can show that the wave like perturbation grows until it becomes larger than the equilibrium radius $R(z)$. When this happens the fluid jet breaks up into droplets.

Waves in strings as Plateau-Rayleigh Instabilities

We would like to investigate whether Plateau-Rayleigh Instabilities can be understood as waves propagating on a static string, as in the previous section. The goal is to study these instabilities in relativistic strings in the hope that they will give a viable model for gravitational radiation in D-brane decay. We look at the classical case as a test run before we tackle the relativistic problem.

We make the assumption that u_y and u_x are of the form of Plateau-Rayleigh Instabilities

$$u_x(s, t) = x(s)e^{\kappa s + i\omega t} \quad \text{and} \quad u_y(s, t) = y(s)e^{\kappa s + i\omega t} \quad (5.57)$$

and then solve for the amplitudes $x(s)$ and $y(s)$. After substituting (5.57) into (5.50) one obtains a second order ODE in both x and y . The length constraint (5.49) can be used to eliminate $x''(s)$ and $x'(s)$ and one obtains an ODE of the form

$$P(\sigma)y''(\sigma) + Q(\sigma)y'(\sigma) + R_y(\sigma)y(\sigma) + R_x(\sigma)x(\sigma) = 0 \quad (5.58)$$

where the coefficient functions are given by

$$P(\sigma) = (-1 + \sigma^2)(1 + \sigma^2)^2 \quad (5.59)$$

$$Q(\sigma) = 2(1 + \sigma^2)(\sigma(2 + \sigma^2) + K(-1 + \sigma^4)) \quad (5.60)$$

$$R_y(\sigma) = 1 + K^2(-1 + \sigma^2)(1 + \sigma^2)^2 + 2K\sigma(2 + 3\sigma^2 + \sigma^4) - \sqrt{1 + \sigma^2\Omega^2} - \sigma^2(1 + \sqrt{1 + \sigma^2\Omega^2}) \quad (5.61)$$

$$R_x(\sigma) = \sigma(2 - (1 + \sigma^2)^{3/2}\Omega^2) . \quad (5.62)$$

In obtaining these results we have already used non-dimensional variables $\sigma = s/k$, $\Omega = \sqrt{k/g\omega}$ and $K = k\kappa$.

Numerical solution

We introduce a dummy variable $z(\sigma) = y'(\sigma)$ and then the system of first order equations from (5.49) and (5.58) is

$$x'(\sigma) = -Kx(\sigma) - \sigma(z(\sigma) + Ky(\sigma)) \quad (5.63)$$

$$y'(\sigma) = z(\sigma) \quad (5.64)$$

$$z'(\sigma) = -\frac{1}{P(\sigma)}(Q(\sigma)z(\sigma) + R_y y(\sigma) + R_x x(\sigma)) . \quad (5.65)$$

The original equation (5.58) have vanishing boundary conditions for x , x' , y and y' on $\sigma \in [-1, 1]$. Systems like these give trivial numerical solutions. To avoid this, we solve an initial value problem on $\sigma \in [0, 1]$. Initial conditions are then deduced at the centre point from the the fact that y and x is symmetric and anti-symmetric, respectively. So $x(0) = 0$ and $y'(0) = 0$ while $y(0) = -0.1$ (this choice is arbitrary but will only change the amplitude of y by a multiplicative constant).

To ensure that the boundary conditions are met, K is specified and a root finding routine is used to find Ω which yields $y(1) = x(1) = 0$.

When calculating the coordinates for the string profile a suitable static solution must first be found. The separation of the string endpoints are specified by $X(1) = a$ and the string length by ℓ_0 . The parameter k is then calculated so that it satisfies

$$k \sinh(a/k) = \ell_0 .$$

The string length was always set to $\ell_0 = 1$. The string profile is then given by $(X(s) + x(s), Y(s) + y(s))$, where X and Y are the static string solution, given in (5.13) and (5.14).

In Figure 5.8 string profiles are shown for different oscillation modes. The values for a where chosen to show how a string self intersects. Initially a large value for a is used, which is then made smaller until $X(\sigma) + x(\sigma) < 0$ on the interval $\sigma \in [0, 1]$. If a suitable value for a was not found when $a < 10^{-6}$, a was set to zero (as is the case for the lowest mode).

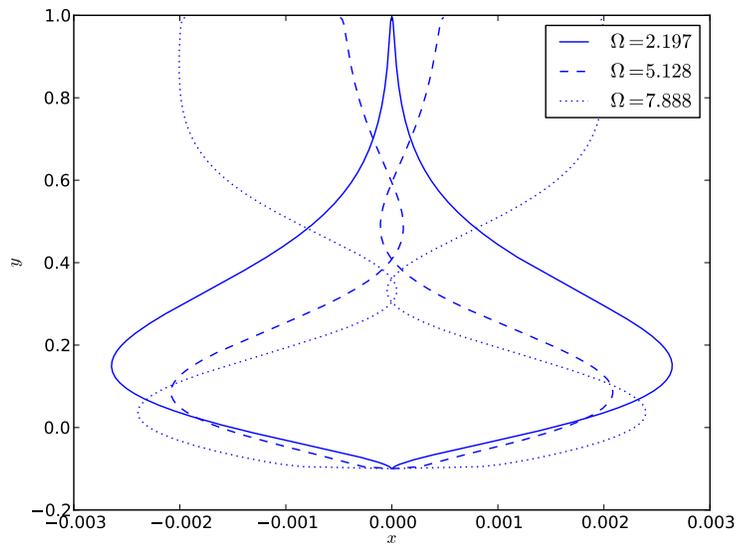


FIGURE 5.8: Two strings with different values for Ω and $K = 10$. For the lowest mode of Plateau-Rayleigh instability the string endpoints meet while for higher modes the string self intersects before the endpoints meet. The values for $X(1) = a$ were chosen such that $X(\sigma) + x(\sigma) < 0$ on the interval $\sigma \in [0, 1]$ or $a = 0$ if the first condition was not met.

Chapter 6

Discussion and conclusion

In this chapter discusses some possibilities for future work and gives a brief summary of this thesis.

6.1 Dynamic Strings in Rindler Space

Again we consider relativistic strings in Rindler Space, now allow the string coordinates to be functions of time. Recall that the background metric is given by

$$ds^2 = g_{\mu\nu}dX^\mu dX^\nu = -a^2x^2dt^2 + dx^2 + dy^2 . \quad (6.1)$$

The equations of motion for a general string was derived in Chapter 4 and are given by

$$\frac{1}{\sqrt{-\gamma}}\partial_\alpha (\sqrt{-\gamma}\gamma^{\alpha\beta}\partial_\beta X^\mu) + \Gamma_{\nu\rho}^\mu\gamma^{\alpha\beta}\partial_\alpha X^\nu\partial_\beta X^\rho = 0 . \quad (6.2)$$

The equations of motion can be simplified a lot by imposing gauge choices on the string coordinates. The intrinsic coordinates are chosen as $\xi_0 = t$ and $\xi_1 = s$, that is the temporal gauge for the first coordinate and the second coordinate as some parameter along the length of the string. The temporal gauge does not allow us to choose s is such a way that $\gamma_{\alpha\beta}$ is conformal to $\eta_{\alpha\beta}$. However s can still be chosen so that the string coordinates satisfy both

$$g_{\mu\nu}X'^\mu(s, t)X'^\nu(s, t) = 1 \quad (6.3)$$

and

$$g_{\mu\nu}X'^\mu(s, t)\dot{X}^\nu(s, t) = 0 . \quad (6.4)$$

Concerning the boundary conditions, we assume that the string hangs on a brane that seats at a constant distance from the Rindler horizon so that it undergoes a constant acceleration. Note however that the string does not lie at a constant distance from the horizon so that the local acceleration (or gravitational potential) will vary along the string, which is a major difference compared to the Newtonian case.

The equations of motion are

$$\ddot{X} + f(X, Y)\dot{X} = -\sqrt{-\gamma}(X'\sqrt{-\gamma})' \quad (6.5)$$

$$\ddot{Y} + f(X, Y)\dot{Y} = -\sqrt{-\gamma}(Y'\sqrt{-\gamma})' + a^2Y \quad (6.6)$$

where f is given by

$$f(X, Y) = -\frac{d}{dt} \ln \sqrt{-\gamma} .$$

The constraints that impose our gauge choices are given by

$$(X'(s, t))^2 + (Y'(s, t))^2 = 1 \quad (6.7)$$

$$\dot{X}X' + \dot{Y}Y' = 0 . \quad (6.8)$$

Adiabatic Approximation

These equations are different from the Newtonian version in particular because the gravitational potential is not constant along the string since its y coordinate is not constant. However, thanks to the fact that the static solution is the same as in the Newtonian case, we can use the adiabatic solution to investigate the motion of the end-points. Evaluating (6.5) at the end points, one finds

$$\ddot{X}_A + f\dot{X}_A + \sqrt{-\gamma}(X'\sqrt{-\gamma})' \Big|_{s=\ell_0} . \quad (6.9)$$

The boundary conditions for Y and (6.6) implies that $\sqrt{-\gamma}(Y'\sqrt{-\gamma})' \Big|_{s=\ell_0} = a^2y_A$. As for the classical case, we define α as the angle of a tangent to the string through $\tan \alpha(s, t) = Y'/X'$. Then we can write

$$\sqrt{-\gamma}(X'\sqrt{-\gamma})' \Big|_{s=\ell_0} = (a^2y_A) \cot \alpha_0(t) = g \cot \alpha_0(t) ,$$

which simplifies the friction coefficient to

$$f(\ell_0, t) = \frac{\dot{X}_A \ddot{X}_A}{1 - \dot{X}_A^2} . \quad (6.10)$$

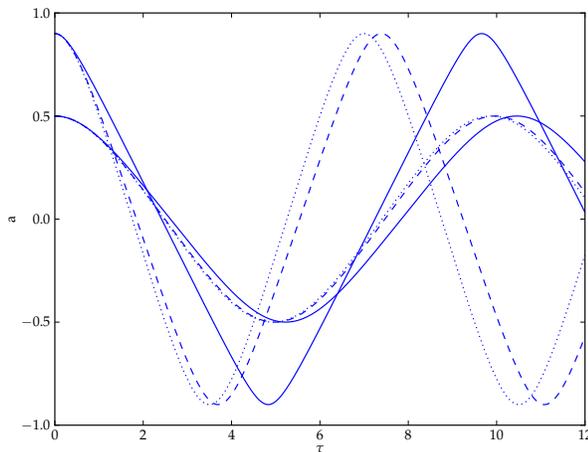


FIGURE 6.1: Numerical solutions for the adiabatic approximation of a relativistic string. Solutions for $a(0) = 0.9$ and $a(0) = 0.5$ are shown with $\ell_0/y_A = 0.3, 1.0, 3.0$ shown in dotted, dashed and solid lines, respectively.

The equation of motion for the end point is therefore

$$\frac{\ddot{X}_A(t)}{1 - \dot{X}_A^2} = -g \cot \alpha(\ell_0, t) . \quad (6.11)$$

We see that what appeared as a friction term recombines in order to give an equation which is the pure relativistic invariant of (5.45).

The adiabatic approximation assumes that $\cot \alpha_0(\ell_0, t) = \cot \alpha_{\text{stat}}(t) = k(t)/\ell_0$ (since the static solution is identical to the Newtonian case) so that the final system takes the form

$$\frac{\ddot{a}(\tau)}{1 - \frac{\ell_0}{y_A} \dot{a}^2} = -K(\tau) \quad (6.12)$$

$$K(\tau) \sinh \frac{a(t)}{K(\tau)} = 1 , \quad (6.13)$$

in the same dimensionless units as the static case and dots denote derivatives with respect to τ . We thus expect differences from the Newtonian analysis when $\ell_0/y_A \ll 1$, that is when the brane is seating close enough from the Rindler horizon.

We can integrate this equation in the same way as the Newtonian case and the solution is depicted on Figure 6.1. We deduce the typical collapse time of the

string in function of the initial separation of the end-points and the distance of the D-brane from the Rindler horizon. By comparing to Figure 5.5, we conclude that the order of magnitude obtained from the Newtonian case is usually a good estimate. Indeed, when $\ell_0/yA \ll 1$, the Newtonian description starts to fail.

6.2 Conclusion

We have revisited the problem of a Newtonian string suspended by its endpoints in a uniform gravitational field. The solution of the string profile is the catenary, a well known problem in undergraduate physics. The catenary also turns out to be the solution of a relativistic string in Rindler space where the end points are held at a fixed distance from the Rindler horizon.

The mechanism for D-brane radiation in string theory is through open strings attached to the brane peeling off to form closed strings. This can happen in two different ways: the endpoints of the string may meet and join to form a closed string that is no longer attached to the brane, or the string could self intersect to form a closed string with a smaller open string still attached to the brane.

Motivated by the similarity between the solutions of Newtonian and Rindler strings, we have extended the analysis of static Newtonian strings by studying its dynamic counterpart – where the endpoints are free to move in the line connecting them. It was our hope that the dynamic Newtonian string would shed some light on dynamic relativistic strings in Rindler space, which in turn could be used as a toy model for gravitational radiation of D-branes. An open string attached to an uniformly accelerating D-brane is equivalent to an open string attached to D-brane in a gravitational field.

The endpoints of slowly moving strings were also investigated for the Newtonian and Rindler string and we found that the time it takes for the endpoints to meet are remarkably similar.

We have studied the evolution the Newtonian string numerically and found the catenary solution forms cusps or self intersections. If similar features in the solution of relativistic strings are present it may provide a better understanding of the additional mode in which strings can radiate from D-branes.

A full solution of the dynamic relativistic string in Rindler space may provide a useful model for the dynamics of gravitational radiation from accelerated branes, but this is left for future work. Such a toy model may be useful to answer

more fundamental questions in other models where gravitational radiation play an important role. Some of these questions include related to investigate the hierarchy problem in.

This work can be extended in a number of ways. The dynamic problem in Rindler space still needs be solved numerically. A method for doing so could then be used to further study the evolution and stability of various string configurations, which in turn could point to new ways of testing the AdS/CFT correspondence.

A numeric solution of the string equation of motion in Rindler space might lead to new insights into solving the string equations in the AdS-Schwarzschild background.

A toy model in Rindler space could be used to explore energy loss in AdS/QGP models and the effects of gravitational radiation in brane world models, which might provide additional ways of testing the AdS/CFT correspondence.

Appendix A

A word on Numerical Methods

The numerical methods presented here were developed to solve partial differential equations of the form

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x} F\left(u, \frac{\partial u}{\partial x}\right), \quad (\text{A.1})$$

where $u = u(x, t)$. This form of differential equations is commonly referred to as conservation equations.

In the case where the PDE in question is second order in time, as encountered in this thesis, an analogous system of first order equations can be found by defining an auxiliary variable U as the first derivative of u with respect to time. This system will read

$$\frac{\partial u}{\partial t} = U \quad (\text{A.2})$$

$$\frac{\partial U}{\partial t} = -\frac{\partial F}{\partial x}. \quad (\text{A.3})$$

The second method can be solved with the methods discussed here. The first can be solved with any number of numeric methods for solving ordinary differential equations.

Finite Difference Methods

Finite difference methods replaces the smooth domain of dependant variables with a discrete grid. Derivatives at grid points are approximated by the average slope between two adjacent points.

The Lax-Friedrich method [41] is the simplest of these schemes. The numerical solution is given by a set of grids $u_j^n = u(n, j)$, so that n is the time index and j the spacial index. Time derivatives are approximated by forward Euler difference and spacial derivatives by central Euler difference,

$$\left. \frac{\partial u}{\partial t} \right|_{n,j} \approx \frac{u_j^{n+1} - u_j^n}{\Delta t}, \quad (\text{A.4})$$

$$\left. \frac{\partial u}{\partial x} \right|_{n,j} \approx \Delta u_j^n = \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}. \quad (\text{A.5})$$

After substituting these approximations for the derivatives in (A.1), one can find an expression for unknown grid location (at time step $n + 1$) in terms of known grid locations (at time step n). The update equation for u is given by

$$u_j^{n+1} = \frac{1}{2} (u_{j+1}^n + u_{j-1}^n) - \frac{\Delta t}{2\Delta s} [F_{j+1}^n - F_{j-1}^n], \quad (\text{A.6})$$

where we introduced

$$F_j^n = F(u_{j+1}^n, \Delta u_{j+1}^n). \quad (\text{A.7})$$

The Lax-Wendorff method [41] is a refinement on the Lax-Friedrich method, where the independent variable is calculated at half step points to achieve a higher accuracy at integral grid points.

Intermediate grid points are calculated using

$$u_{j+1/2}^{n+1/2} = \frac{1}{2} (u_{j+1}^n + u_{j-1}^n) - \frac{\Delta t}{\Delta s} [F_{j+1}^n - F_{j-1}^n]. \quad (\text{A.8})$$

It is now possible to evaluate F at the half grid points, and then calculate the u at the next integral grid point using

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta s} [F_{j+1/2}^{n+1/2} - F_{j-1/2}^{n+1/2}]. \quad (\text{A.9})$$

Smoothed Particle Hydrodynamics

Smoothed Particle Hydrodynamics (SPH) [45, 46, 42]. In order to solve the system numerically, two successive approximations are made.

The smoothed approximation

The first is called the smoothed approximation. This is based on the fact that any smooth function can be written as an integral over its domain as

$$f(x) = \int_{\Omega} dx' f(x') \delta(x - x') . \quad (\text{A.10})$$

For a function $W(x - x', h)$, called the kernel, that is sufficiently ‘delta like’, f can be approximated by

$$f(x) \approx \langle f(x) \rangle = \int_{\Omega} dx' f(x') W(x - x', h) . \quad (\text{A.11})$$

The angle bracket notation is used to specify that the smoothed approximation was made. The parameter h , the smoothing length, defined the area of influence of the kernel.

The kernel should satisfy the following conditions to be considered sufficiently ‘delta’ like. The kernel should be normalised

$$\int_{\Omega} W(x - x', h) dx' = 1 .$$

In the limit where the smoothing length goes to zero, the Dirac delta function should be retained

$$\lim_{h \rightarrow 0} W(x - x', h) = \delta(x - x') .$$

The kernel should also be compact on a finite domain

$$W(x - x', h) = 0 \quad \text{when} \quad |x - x'| > \kappa h ,$$

where κ is some constant.

One can also find an approximation for the derivative of f by using Stokes’ Theorem and the compact support of W

$$\begin{aligned} \left\langle \frac{df}{dx} \right\rangle &= \int_{\Omega} dx' \frac{d}{dx'} f(x') W(x - x', h) \\ &= \int_S dx' f(x') W(x - x', h) - \int_{\Omega} dx' f(x') \frac{d}{dx'} W(x - x', h) \end{aligned} \quad (\text{A.12})$$

$$= \int_{\Omega} dx' f(x') W'(x - x', h) , \quad (\text{A.13})$$

where S is the boundary of the domain Ω and W' is the derivative of the kernel with respect to its first argument.

The particle approximation

The particle approximation is made when the integral is written as a sum

$$f(x_j) \approx [f(x)] = \sum_i f(x_i)W(x_j - x_i, h)\Delta x_i \quad (\text{A.14})$$

The derivative of f is approximated using by

$$\left[\frac{df}{dx} \right] = \sum_i f(x_i)W'(x_j - x_i, h)\Delta x_i \quad (\text{A.15})$$

Conservative equations

Conservative PDE's are of the general form

$$\dot{u} = -\frac{\partial}{\partial x}F(u) , \quad (\text{A.16})$$

with $u = u(t, x)$. Now consider the following Taylor expansion around Δt

$$u(t + \Delta t, x) = u(t, x) + \Delta t \dot{u}(t, x) + \mathcal{O}((\Delta t)^2) \quad (\text{A.17})$$

$$= u(t, x) - \Delta t F'(u(t, x)) + \mathcal{O}((\Delta t)^2) . \quad (\text{A.18})$$

Now the result from (A.14) can be used to approximate F' to obtain the following scheme for approximating time derivatives

$$u_j^{n+1} = u_j^n + \Delta t \sum_i F'(u_i^n)W'(x_j - x_i, h)\Delta x_i . \quad (\text{A.19})$$

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