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Hyperconvex hulls in categories of quasi-metric spaces

by

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Abstract

Isbell showed that every metric space has an injective hull, that is, every metric space has a “minimal” hyperconvex metric superspace. Dress then showed that the hyperconvex hull is a tight extension. In analogy to Isbell’s theory Kemajou et al. proved that each T_0 -quasi-metric space X has a q -hyperconvex hull Q_X , which is joincompact if X is joincompact. They called a T_0 -quasi-metric space q -hyperconvex if and only if it is injective in the category of T_0 -quasi-metric spaces and non-expansive maps. Agyingi et al. generalized results due to Dress on tight extensions of metric spaces to the category of T_0 -quasi-metric spaces and non-expansive maps.

In this dissertation, we shall study tight extensions (called uq -tight extensions in the following) in the categories of T_0 -quasi-metric spaces and T_0 -ultra-quasi-metric spaces. We show in particular that most of the results stay the same as we move from T_0 -quasi-metric spaces to T_0 -ultra-quasi-metric spaces. We shall show that these extensions are maximal among the uq -tight extensions of the space in question.

In the second part of the dissertation we shall study the q -hyperconvex hull by viewing it as a space of minimal function pairs. We will also consider supseparability of the space of minimal function pairs. Furthermore we study a special subcollection of bicomplete supseparable quasi-metric spaces: bicomplete supseparable ultra-quasi-metric spaces. We will show the existence and uniqueness (up

to isometry) of a Urysohn Γ -ultra-quasi-metric space, for an arbitrary countable set Γ of non-negative real numbers including 0.

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Dedication

I dedicate this dissertation to my parents Isaiah Angono Agyingi and Ndongwo Esther Mbanwi, to my brothers Cedric Agyingi Agyingi and Blaise Nche Agyingi, to my sisters Claudia Edeh Atah and Cludith Asissia Agyingi and to my wife Ijang Anwi Agyingi, for their patience and understanding.

Declaration

I, COLLINS AMBURO AGYINGI

hereby declare that this thesis is my own unaided work which is being submitted for the degree of Doctor of Philosophy at the University of Cape Town. It has not been submitted for any degree or examination in any other university.

SIGNED:.....

DATE:.....

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Chapter 0

Introduction

The concept of hyperconvexity of a metric space was introduced in 1956 by Aron-szajn and Panitchpakdi [4], where they proved that a metric space is hyperconvex if and only if it is injective. They showed that a hyperconvex metric space is a non-expansive retract of any metric space in which it is isometrically embedded. Since then several research articles by Isbell, Dress, Khamsi, Espinola, Kirk, Smyth, Tsaur etc. on hyperconvex metric spaces have appeared in the literature.

Isbell [15] proved that every metric space has a hyperconvex hull T_X which is compact if X is compact. Recall that a metric space is said to be hyperconvex (see for example [21, p. 78]) if and only if it is injective in the category of metric spaces and non-expansive maps. Dress [11] later gave an independent, but equivalent approach to Isbell's theory that is based on the concept of a tight extension.

Kemajou et al. proved in [20] that every T_0 -quasi-metric space X has an injective hull, Q_X , in the category of T_0 -quasi-metric spaces and non-expansive mappings (which they called q -hyperconvex hull or the di-injective hull or better still Isbell-hull), which is joincompact if X is joincompact. Analogously Otafudu et al. [23] presented a similar construction in the category of T_0 -ultra-quasi-metric spaces

and non-expansive maps. It should be noted that comparable studies in the area of ultrametric spaces have been conducted before by Bayod and Martínez-Maurica (compare [5]). Otafudu et al. showed that a T_0 -ultra-quasi-metric space is ultra-quasi-metrically injective if and only if it is q -spherically complete. They presented an explicit construction of the ultra-quasi-metrically injective hull of a T_0 -ultra-quasi-metric space. They also showed that the ultra-quasi-metrically injective hull of a totally bounded T_0 -ultra-quasi-metric space is joincompact.

The following natural questions motivated part of the research in Chapter 5 (particularly Section 5.2): Do there exist universal spaces for the class of all supseparable ultra-quasi-metric spaces? Are there any such spaces with the Urysohn property?

The collection of bicomplete supseparable ultra-quasi-metric spaces is a subcollection of bicomplete supseparable quasi-metric spaces. We recall here that the existence up to isometry of a q -universal bicomplete supseparable quasi-metric space had been shown recently by Künzi and Sanchis [25]. This was done by modifying a construction due to Katětov for metric spaces. Section 5.2 of this dissertation is focused on bicomplete supseparable ultra-quasi-metric spaces. In this section we consider a simplified variant of our original question stated two paragraphs before.

0.0.1 Definition. A T_0 -ultra-quasi-metric space (X, d) with $\Gamma = \{d(a, b) : a, b \in X\}$ is called **Γ -Urysohn** if for any finite ultra-quasi-metric space A with $\{d(x, y) : x, y \in A\} \subseteq \Gamma$, and any subspace $B \subseteq A$, every isometric embedding $f : B \hookrightarrow X$ can be extended to an isometric embedding $g : A \hookrightarrow X$.

We shall show that any supseparable ultra-quasi-metric space (X, d) realizes only a countable set of distances, i.e. the set $\Gamma_X = \{d(x, y) : x, y \in X\}$ is countable. Thus to continue our study it is natural to consider a fixed countable set $\Gamma \subseteq \mathbb{R}_+$ of potential values of the ultra-quasi-metric. We shall call an ultra-quasi-metric space X with $\Gamma_X \subseteq \Gamma$ a Γ -ultra-quasi-metric space. We now have the following

natural question: For what sets Γ does there exist an ultra-quasi-universal Γ -ultra-quasi-metric space? Similar studies for ultrametric spaces had been done by Shao in [35].

A brief outline of the thesis

This dissertation is organized as described below.

Chapter 1. In the first chapter we give a brief overview of certain well-known basic concepts from the theory of quasi-pseudometric and ultra-quasi-pseudometric spaces. Some interesting examples of T_0 -quasi-metric spaces (Example 1.1.2 and Example 1.1.3) and T_0 -ultra-quasi-metric spaces (Example 1.2.1) are presented.

The main part of this dissertation is contained in Chapters 2, 3, 4, and 5.

Chapter 2. In this chapter we recall the concept of tight extensions in the category of T_0 -quasi-metric spaces and non-expansive maps from [1]. It is known that the “Isbell-hull” Q_X of a T_0 -quasi-metric space (X, d) is a tight extension of X . Moreover it was also shown that this extension Q_X is maximal among the tight extensions of X (see Proposition 2.2.4).

Chapter 3. This chapter generalizes the concept of a tight extensions to the category of T_0 -ultra-quasi-metric spaces and non-expansive maps. One of our main result is Proposition 3.3.1. We provide two proofs to this proposition: namely a constructive and non-constructive proof (by Zorn’s Lemma). In particular it is noted that the symmetry of the metric is not needed in the development of the theory. We show also that every T_0 -ultra-quasi-metric space X has a uq -tight extension uQ_X in which X can be isometrically embedded. Moreover we show that uQ_X is maximal among the uq -tight extensions of X .

Chapter 4. In this chapter, we will view the q -hyperconvex hull Q_X of a T_0 -quasi-

metric space X as a space of minimal function pairs on X . We will concentrate on supseparability of this space of minimal function pairs on an arbitrary T_0 -quasi-metric space. We have results for some specific classes of T_0 -quasi-metric spaces. It is known from [20] that every joincompact T_0 -quasi-metric space X has a joincompact q -hyperconvex hull Q_X , thus their spaces of minimal function pairs are supseparable. Also, since every T_0 -quasi-metric space can be embedded in its q -hyperconvex hull, we have that a non-supseparable T_0 -quasi-metric space has a non-supseparable q -hyperconvex hull.

Chapter 5. In this chapter we study supseparable ultra-quasi-metric spaces. In Section 5.1 we modify a construction due to Katětov for a T_0 -ultra-quasi-metric space. We prove the uniqueness of Urysohn Γ -ultra-quasi-metric spaces, for an arbitrary countable set Γ of non-negative real numbers including 0.

Chapter 6. In this last chapter we conclude our investigations by reflecting on the main results of the dissertation and highlight some connections of this current work with old work found in the literature. Furthermore we make mention of some open problems which can constitute some topics for further research. The study of q -hyperconvex T_0 -quasi-metric spaces and ultra-quasi-metrically injective T_0 -ultra-quasi-metric spaces leads to some open problems. Indeed one could investigate whether the theory of q -hyperconvexity can be applied to asymmetric normed spaces with all of its related areas.

As was shown by Rao [32] (and also by Dress [11]) the concept of tight extension for metric spaces is equivalent to the injective hull of metric spaces. It follows from that result that the injective hull of a Banach space in the category of Banach spaces agrees with its injective hull in the category of metric spaces; so its injective metric hull can be given the structure of a Banach space. Very recently Agyingi et al. [1] studied tight extensions for T_0 -quasi-metric spaces. They obtained results similar to those in the classical case. One can guess that a similar study can be done for asymmetric normed spaces.

Chapter 1

Preliminaries

We start the next section by recalling some basic concepts from the theory of quasi-pseudometric spaces and ultra-quasi-pseudometric spaces. For further readings and recent results in the area of asymmetric topology, the reader is advised to consult [23, 24, 34].

1.1 Quasi-pseudometric spaces

1.1.1 Definition. (Compare [20, Page 3]) Let X be a nonempty set and $d : X \times X \rightarrow [0, \infty)$ be a mapping into the set $[0, \infty)$ of non-negative reals. Then d is a **quasi-pseudometric** on X if

- (a) $d(x, x) = 0$ for all $x \in X$, and
- (b) $d(x, z) \leq d(x, y) + d(y, z)$ whenever $x, y, z \in X$.

The pair (X, d) is said to be a **quasi-pseudometric space**.

We say that d is a T_0 -quasi-metric (or a di-metric) provided that it satisfies the additional condition that for any $x, y \in X$, $d(x, y) = 0 = d(y, x)$ implies that $x = y$. The set X together with a T_0 -quasi-metric (di-metric) is called a T_0 -quasi-metric space (di-space). Note that if d is a quasi-pseudometric on X , then $d^{-1} : X \times X \rightarrow [0, \infty)$ defined by $d^{-1}(x, y) = d(y, x)$ whenever $x, y \in X$ is also a quasi-pseudometric on X , called the **conjugate quasi-pseudometric** of d . As usual, a quasi-pseudometric d on X such that $d = d^{-1}$ is called a **pseudometric** on X . For example $d^s = d \vee d^{-1}$ is a pseudometric on X . If d is a T_0 -quasi-metric, then d^s is a **metric** on X .

Let (X, d) be a quasi-pseudometric space and for each $x \in X$, $\epsilon \in [0, \infty)$, let $C_d(x, \epsilon) = \{y \in X : d(x, y) \leq \epsilon\}$ be the $\tau(d^{-1})$ -closed ball (compare [7, Proposition 1.5(1)]) of radius ϵ centered at x . We shall represent the $\tau(d)$ -open ball of radius ϵ centered at x by $B_d(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\}$.

A map $f : (X, d_X) \rightarrow (Y, d_Y)$ between two quasi-pseudometric spaces (X, d_X) and (Y, d_Y) is called **non-expansive** provided that $d_Y(f(x), f(y)) \leq d_X(x, y)$ whenever $x, y \in X$.

A map $f : (X, d_X) \rightarrow (Y, d_Y)$ between two quasi-pseudometric spaces (X, d_X) and (Y, d_Y) is called an **isometry** provided that $d_Y(f(x), f(y)) = d_X(x, y)$ whenever $x, y \in X$. Note that each isometric map with a T_0 -quasi-metric domain is a one-to-one map. Two quasi-pseudometric spaces (X, d_X) and (Y, d_Y) will be said to be isometric provided that there exists a bijective isometry between them.

We next define an asymmetric norm which we shall use in the construction of Example 1.1.2.

1.1.2 Definition. (compare [7, Section 1.1]) Let X be a non-empty real vector space and $p : X \rightarrow [0, \infty)$ be a mapping into the set $[0, \infty)$ of non-negative reals. Then p is called an **asymmetric norm** on X if for all $x, y \in X$ and $\alpha \geq 0$:

$$(a) \quad p(x) = p(-x) = 0 \Rightarrow x = 0,$$

$$(b) p(\alpha x) = \alpha p(x),$$

$$(c) p(x + y) \leq p(x) + p(y).$$

We call the pair (X, p) an asymmetric normed space.

Sometimes p will be allowed to take value ∞ , in which case we shall call it an extended asymmetric norm. We define the conjugate p^t of p as $p^t(x) = p(-x)$, $x \in X$. Its not difficult to see that $p^s = \max\{p, p^t\}$ is a norm on X . We shall use the symbol $\|\cdot\|$ (compare [22, Ch. IX, Section 5]) to denote an asymmetric norm.

The following (Example 1.1.1) is an example of an asymmetric norm on \mathbb{R} .

1.1.1 Example. (compare [7, Example 1.2]) Define on \mathbb{R} the map $u : \mathbb{R} \rightarrow [0, \infty)$ by $\alpha \mapsto u(\alpha) = \alpha^+ := \max\{\alpha, 0\}$. Then its not hard to see that u is an asymmetric norm on \mathbb{R} . The conjugate u^t of u is defined by $u^t(\alpha) = \alpha^- := \max\{-\alpha, 0\}$ and $u^s = \max\{u, u^t\} = |\alpha|$ is a norm on X .

1.1.2 Example. (The general quasi-metric “segment I_{ab} ”) (compare [1, Remark 2]) Let $X = [0, 1]$. Choose $a, b \in [0, \infty)$ such that $a + b \neq 0$. Set $d_{ab}(x, y) = (x - y)a$ if $x > y$ and $d_{ab}(x, y) = (y - x)b$ if $y \geq x$. Then $([0, 1], d_{ab})$ is a T_0 -quasi-metric space as it is readily checked, by considering the various cases for the underlying asymmetric norm n_{ab} on \mathbb{R} defined by $n_{ab}(x) = xa$ if $x > 0$ and $n_{ab}(x) = -xb$ if $x \leq 0$.

1.1.3 Example. Given two nonnegative real numbers a and b we shall write $a \dot{-} b$ for $\max\{a - b, 0\}$, which in a more lattice theoretic terminology we shall also denote by $(a - b) \vee 0$. It should be noted that $u(x, y) = x \dot{-} y$ with $x, y \in [0, \infty)$ defines the standard T_0 -quasi-metric on $[0, \infty)$. Thus $([0, \infty), u)$ is a T_0 -quasi-metric space.

1.2 Ultra-quasi-pseudometric spaces

We mention that the ultra-quasi-pseudometric spaces should not be confused with quasi-ultra-metric spaces as they are discussed in the theory of dissimilarities (check for instance [9]).

1.2.1 Definition. (Compare [23, page 2]) Let X be a set and $d : X \times X \rightarrow [0, \infty)$ be a function mapping into the set $[0, \infty)$ of non-negative reals. Then d is an **ultra-quasi-pseudometric** on X if

(a) $d(x, x) = 0$ for all $x \in X$, and

(b) $d(x, z) \leq \max\{d(x, y), d(y, z)\}$ whenever $x, y, z \in X$.

We remark here that the conjugate d^{-1} of d where $d^{-1}(x, y) = d(y, x)$ whenever $x, y \in X$ is also an ultra-quasi-pseudometric on X .

If d also satisfies the condition:

(c) For any $x, y \in X$, $d(x, y) = 0 = d(y, x)$ implies that $x = y$, then d is called a T_0 -ultra-quasi-metric on X . Notice that $d^s = \sup\{d, d^{-1}\} = d \vee d^{-1}$ is an ultrametric on X .

In the literature, T_0 -ultra-quasi-metric spaces are also known as non Archimedean quasi-metric spaces and the set of open balls $\{\{y \in X : d(x, y) < \epsilon\} : x \in X, \epsilon > 0\}$ yields a base for the topology $\tau(d)$ induced by d on X .

1.2.1 Example. (compare [23, Example 2]) Let $X = [0, \infty)$. Define $n(x, y) = x$ if $x, y \in X$ and $x > y$, and $n(x, y) = 0$ if $x, y \in X$ and $x \leq y$. Then (X, n) is a T_0 -ultra-quasi-metric space: we show the strong triangle inequality $n(x, z) \leq \max\{n(x, y), n(y, z)\}$ whenever $x, y, z \in X$ since the other conditions are obvious. For $n(x, y) = x$, the result is trivial, since then $n(x, z) \leq n(x, y)$. Similarly the case that $n(x, y) = 0$ and $n(y, z) = y$ is obvious, since then $x \leq y$ and $n(x, z) \leq n(y, z)$. In the remaining case that $n(x, y) = 0 = n(y, z)$, we have

by transitivity of \leq that $x \leq z$, and thus $n(x, z) = 0$. It is obvious that n satisfies the T_0 -condition.

Notice also that for $x, y \in [0, \infty)$, we have $n^s(x, y) = \max\{x, y\}$ if $x \neq y$ and $n^s(x, y) = 0$ if $x = y$. The ultra-metric n^s is complete on $[0, \infty)$ since n and n^{-1} are bicomplete on $[0, \infty)$. Recall that a T_0 -ultra-quasi-metric space (X, d) is said to be **bicomplete** if the ultrametric space (X, d^s) is complete. Furthermore (X, d) will be said to be **supseparable** if the ultrametric space (X, d^s) is separable.

Furthermore 0 is the only non-isolated point of $\tau(n^s)$. Indeed $A = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ is a compact subspace of $([0, \infty), n^s)$.

In some cases we will replace $[0, \infty)$ with $[0, \infty]$ and in this case we shall speak of an extended ultra-quasi-pseudometric.

1.2.1 Lemma. (Compare [6, Proposition 2.1]) Let $\alpha, \beta, \gamma \in [0, \infty)$. Then the following are equivalent:

- (a) $n(\alpha, \beta) \leq \gamma$,
- (b) $\alpha \leq \max\{\beta, \gamma\}$.

Proof.

(a) \Rightarrow (b)

To reach a contradiction, suppose that $\alpha > \max\{\beta, \gamma\}$. Since $\alpha > \beta$, we have $n(\alpha, \beta) = \alpha \leq \gamma$ by part (a) and the way n was defined. Thus we have that $\alpha \leq \max\{\beta, \gamma\} < \alpha$ and this is a contradiction.

(b) \Rightarrow (a)

Suppose on the contrary that $n(\alpha, \beta) > \gamma$. Then $n(\alpha, \beta) = \alpha$ and $\alpha > \beta$ and hence $\alpha > \gamma$ which implies that $\alpha > \max\{\beta, \gamma\}$. We have by (b) that $\alpha \leq \max\{\beta, \gamma\}$ which is a contradiction.

□

The following corollaries are immediate. Their proofs rely on Lemma 1.2.1.

1.2.1 Corollary. (see [23]) Let (X, d) be an ultra-quasi-pseudometric space. Consider a map $f : X \rightarrow [0, \infty)$ and let $x, y \in X$. Then the following are equivalent:

(a) $n(f(x), f(y)) \leq d(x, y)$,

(b) $f(x) \leq \max\{f(y), d(x, y)\}$.

1.2.2 Corollary. (see [23]) Let (X, d) be an ultra-quasi-pseudometric space. Then

(a) $f : (X, d) \rightarrow ([0, \infty), n)$ is a non-expansive map if and only if $f_2(x) \leq \max\{f_1(y), d(x, y)\}$ whenever $x, y \in X$,

(b) $f : (X, d) \rightarrow ([0, \infty), n^{-1})$ is a non-expansive map if and only if $f_1(x) \leq \max\{f_2(y), d(y, x)\}$ whenever $x, y \in X$.

Chapter 2

Tight extensions of quasi-metric spaces

In this chapter we present a summary of the construction of a tight extension of a T_0 -quasi-metric space as it was developed in [1] and then establish some propositions (Proposition 2.2.1 and Proposition 2.2.4) that helps us to better understand the maximality of the tight extension Q_X (among the tight extensions) of the T_0 -quasi-metric space X .

2.1 q -hyperconvex hulls of T_0 -quasi-metric spaces

Let us recall that the concept of hyperconvex metric spaces was introduced in [4] and was investigated later by many authors (see for instance [11, 12, 15, 16, 17, 21]).

In this section, we shall recall some results from the theory of q -hyperconvex hulls of T_0 -quasi-metric spaces due to [20].

Let (X, d) be a T_0 -quasi-metric space. We shall say that a function pair $f = (f_1, f_2)$ on (X, d) where $f_i : X \rightarrow [0, \infty)$ ($i = 1, 2$) is ample if for all $x, y \in X$, $d(x, y) \leq f_2(x) + f_1(y)$.

Let P_X denote the set of all ample function pairs on (X, d) . (We may also write $P_{(X, d)}$ in cases where d is not obvious.) For each $f, g \in P_X$, define

$$D(f, g) = \sup\{f_1(x) \dot{-} g_1(x) : x \in X\} \vee \sup\{g_2(x) \dot{-} f_2(x) : x \in X\}.$$

Then D is an extended T_0 -quasi-metric on P_X .

We shall say that a function pair f is minimal on (X, d) (among the ample function pairs on (X, d)) if it is an ample function pair and if $g = (g_1, g_2)$ is an ample function pair on (X, d) and for each $x \in X$, $g_1(x) \leq f_1(x)$ and $g_2(x) \leq f_2(x)$, then $f = g$. Minimal ample function pairs are also called extremal function pairs. It is well known that Zorn's lemma implies that below each ample function pair there is a minimal ample function pair (for a constructive proof, see Proposition 2.2.1 below). By Q_X we shall denote the set of all minimal ample function pairs on (X, d) equipped with the restriction of D to $Q_X \times Q_X$, which we still denote by D . Recall that D is actually a T_0 -quasi-metric on $Q_X \times Q_X$ by [20, Remark 6].

Recall from [1, Lemma 3.4] that $f \in Q_X$ if and only if

$$f_1(x) = \sup\{d(y, x) \dot{-} f_2(y) : y \in X\} \tag{2.1}$$

and

$$f_2(x) = \sup\{d(x, y) \dot{-} f_1(y) : y \in X\} \tag{2.2}$$

whenever $x \in X$.

It is known that if f is an extremal function pair on (X, d) , then $f_1(x) \dot{-} f_1(y) \leq d(y, x)$ and $f_2(x) \dot{-} f_2(y) \leq d(x, y)$.

2.1.1 Lemma. Let (X, d) be a T_0 -quasi-metric space. For each $x \in X$, define $f_x(y) = (d(x, y), d(y, x))$ whenever $y \in X$. Then f_x is ample.

The map $x \mapsto f_x$ whenever $x \in X$ defines an isometric embedding of (X, d) into (Q_X, D) (see [20, Lemma 1]).

Recall that (Q_X, D) is said to be the q -hyperconvex hull of (X, d) . A T_0 -quasi-metric space X is said to be q -hyperconvex if $f \in Q_X$ implies that there exists an $x \in X$ such that $f = f_x$ (compare [20, Corollary 4]). See also [20, Definition 2] for an intrinsic characterization of q -hyperconvexity. Note that $D(f, f_x) = f_1(x)$ and $D(f_x, f) = f_2(x)$ whenever $x \in X$ and $f \in Q_X$ (check for instance [20, Lemma 8]).

2.1.1 Remark. For the usual construction of the (metric) hyperconvex hull T_X of a metric space (X, d) , the reader is advised to consult [11, 15].

Indeed T_X consists of functions $f : X \rightarrow [0, \infty)$ which are extremal. Recall that f is said to be extremal if it is ample (i.e., $d(x, y) \leq f(x) + f(y)$ whenever $x, y \in X$) and f is minimal (among the ample functions) with respect to the pointwise order on these functions. If we define $D(f, g) = \sup_{x \in X} |f(x) - g(x)|$ whenever $f, g \in T_X$, then D defines a metric on T_X and the pair (T_X, D) is the hyperconvex hull of (X, d) .

2.2 T_0 -quasi-metric tight extensions

In this section we shall generalize some results about tight extensions from [11] to the quasi-metric setting.

2.2.1 Proposition. (Compare [11, Section 1.9]) Let (X, d) be a T_0 -quasi-metric space. There exists a retraction map $p : P_X \rightarrow Q_X$, i.e., a map that satisfies the following conditions:

- (a) $D(p(f), p(g)) \leq D(f, g)$ whenever $f, g \in P_X$.
- (b) $p(f) \leq f$ whenever $f \in P_X$.

(In particular $p(f) = f$ whenever $f \in Q_X$, since each $f \in Q_X$ is minimal.)

Proof.

Given $f \in P_X$, set $f_1^*(y) = \sup\{d(x', y) \dot{-} f_2(x') : x' \in X\}$ whenever $y \in X$ and $f_2^*(x) = \sup\{d(x, y') \dot{-} f_1(y') : y' \in X\}$ whenever $x \in X$. We have the following claims:

Claim 1: $f^* \leq f$.

For any $x, y' \in X$, we have $d(x, y') \leq f_2(x) + f_1(y')$ and thus $d(x, y') \dot{-} f_1(y') \leq f_2(x)$. Therefore $f_2^*(x) = \sup\{d(x, y') \dot{-} f_1(y') : y' \in X\} \leq f_2(x)$. In a similar manner, one can show that $f_1^*(y) \leq f_1(y)$ whenever $y \in X$. Thus the claim holds.

Claim 2: $d(x, y) \leq f_2^*(x) + f_1(y)$ and $d(x, y) \leq f_2(x) + f_1^*(y)$ whenever $x, y \in X$. Let $x, y \in X$. Then it is clear that

$$\sup\{d(x, y') \dot{-} f_1(y') : y' \in X\} + f_1(y) \geq d(x, y).$$

In a similar way we get $d(x, y) \leq f_2(x) + \sup\{d(x', y) \dot{-} f_2(x') : x' \in X\}$. We have therefore that $d(x, y) \leq f_2^*(x) + f_1(y)$ and $d(x, y) \leq f_2(x) + f_1^*(y)$ whenever $x, y \in X$.

Define the map $q : P_X \rightarrow P_X$ by

$$f \mapsto q(f) = \left(\frac{1}{2}(f_1 + f_1^*), \frac{1}{2}(f_2 + f_2^*) \right)$$

whenever $f \in P_X$.

Claim 3: $q(f)$ is ample and $q(f) \leq f$.

Indeed

$$\begin{aligned}
(q(f))_2(x) + (q(f))_1(y) &= \frac{1}{2}(f_2(x) + f_2^*(x)) + \frac{1}{2}(f_1(x) + f_1^*(x)) \\
&= \frac{1}{2}(f_1(y) + f_2^*(x)) + \frac{1}{2}(f_1^*(y) + f_2(x)) \\
&\geq \frac{1}{2}d(x, y) + \frac{1}{2}d(x, y) = d(x, y)
\end{aligned}$$

whenever $x, y \in X$.

This shows that $q(f)$ is ample. Obviously $q(f) \leq f$, since we have that $f^* \leq f$.

Claim 4: $D(q(f), q(g)) \leq D(f, g)$ whenever $f, g \in P_X$:

Let $f, g \in P_X$ and $x \in X$. Then we have that

$$\begin{aligned}
f_1^*(x) &= \sup\{d(y, x) \dot{-} f_2(y) : y \in X\} \\
&\leq \sup\{d(y, x) \dot{-} g_2(y) + g_2(y) \dot{-} f_2(y) : y \in X\} \\
&\leq \sup\{d(y, x) \dot{-} g_2(y) : y \in X\} + \sup\{g_2(y) \dot{-} f_2(y) : y \in X\} \\
&\leq g_1^*(x) + D(f, g).
\end{aligned}$$

Therefore

$$\begin{aligned}
\sup_{x \in X}\{(q(f))_1(x) \dot{-} (q(g))_1(x)\} &\leq \frac{1}{2} \sup_{x \in X}\{f_1(x) \dot{-} g_1(x)\} + \frac{1}{2} \sup_{x \in X}\{f_1^*(x) \dot{-} g_1^*(x)\} \\
&\leq \frac{1}{2}D(f, g) + \frac{1}{2}D(f, g) = D(f, g).
\end{aligned}$$

Similarly one can show that

$$\sup_{x \in X}\{(q(g))_2(x) \dot{-} (q(f))_2(x)\} \leq D(f, g)$$

whenever $f, g \in P_X$. Hence $D(q(f), q(g)) \leq D(f, g)$ as required.

For the remainder of the proof, given $f \in P_X$, we obtain a minimal ample function pair below f as the pointwise limit of the sequence $(g^n(f))_{n \in \mathbb{N}}$ where g^n is the n^{th} -iteration of q :

let us fix $f \in P_X$. Then by a simple computation, it is clear that for $n \in \mathbb{N}$ the n^{th} -iteration q^n of q yields a monotonically decreasing sequence $(q^n(f))$ that is bounded below by the 0-pair.

Hence the map $p(f) := \lim_{n \rightarrow \infty} q^n(f)$ exists, where we take the pointwise limit pair with respect to the usual topology $\tau(u^s)$ on $[0, \infty)$.

It should be noted that for each $n \in \mathbb{N}$, $q^n(f)$ belongs to P_X and q^n satisfies conditions (a) and (b), too. Thus $p(f) \in P_X$ and p satisfies condition (b).

For each $n \in \mathbb{N}$, $f, g \in P_X$ and $x \in X$ we have

$$\max\{(q^n(f))_1(x) \dot{-} (q^n(g))_1(x), (q^n(g))_2(x) \dot{-} (q^n(f))_2(x)\} \leq D(f, g);$$

thus $D(p(f), p(g)) \leq D(f, g)$ and p also satisfies (a).

To complete the proof, we show that $p(f) \in Q_X$ whenever $f \in P_X$.

Let $f \in P_X$. For every $n \in \mathbb{N}$ we have that $p(f) \leq q^n(f)$ and hence $q^n(f)^* \leq p(f)^*$ by the way we defined the $*$ -operation.

Thus $0 \leq p(f) - p(f)^* \leq q^n(f) - q^n(f)^* = 2(q^n(f) - q^{n+1}(f))$ (compare [26, Proof of Proposition 3.1]), since

$$q^{n+1}(f) = \frac{q^n(f) + q^n(f)^*}{2}.$$

This yields in particular that $p(f) = p(f)^*$.

Let now $g \leq h$, that is $g_1 \leq h_1$ and $g_2 \leq h_2$, where $h := p(f)$ and let g be an ample function pair. Then for each $x \in X$,

$$h_2(x) = \sup_{y \in X} \{d(x, y) \dot{-} h_1(y)\} \leq \sup_{y \in X} \{d(x, y) \dot{-} g_1(y)\} \leq g_2(x)$$

by ampleness of g . So we have $g_2 = h_2$. Similarly one can show that $g_1 = h_1$ and therefore the pair h is minimal ample.

□

2.2.1 Remark. Let us remark that Proposition 2.2.1 can be established by the use of Zorn's Lemma (compare [11, Section 1.9]).

Proof.

Let \mathcal{P} be the set of all maps $p : P_X \rightarrow P_X$ satisfying the conditions (1) $p(f) \leq f$ and (2) $D(p(f), p(g)) \leq D(f, g)$ whenever $f, g \in P_X$. Then $\mathcal{P} \neq \emptyset$ since the identity map belongs to \mathcal{P} .

Define \preceq on \mathcal{P} as follows:

$p \preceq q$ if and only if $[p(f) \leq q(f) \text{ and } D(p(f), p(g)) \leq D(q(f), q(g))]$ whenever $f, g \in P_X$. Then by a routine check, one can see that \preceq is a partial order on \mathcal{P} and hence (\mathcal{P}, \preceq) is a partially ordered set.

Let $\emptyset \neq \mathcal{K}$ be a chain in (\mathcal{P}, \preceq) . Define a map $t : P_X \rightarrow P_X$ by

$$t(f)(x) := \left(\inf_{k \in \mathcal{K}} (k(f))_1(x), \inf_{k \in \mathcal{K}} (k(f))_2(x) \right)$$

whenever $x \in X$, where the infima are taken pointwise in $[0, \infty)$. Then $t(f) \in \mathcal{P}$ since $k(f) \in \mathcal{P}$. Indeed see immediately that

$$t(f) \leq k(f) \leq f,$$

for every $f \in P_X$. Thus $t(f) \leq f$ and part (a) holds.

$D(f, g) \geq D(k(f), k(g))$, since $k \in \mathcal{P}$

$$\begin{aligned} &= \sup_{x \in X} \{(k(f))_1(x) \dot{-} (k(g))_1(x)\} \vee \sup_{x \in X} \{(k(g))_2(x) \dot{-} (k(f))_2(x)\} \\ &\geq \inf_{k \in \mathcal{K}} \left(\sup_{x \in X} \{(k(f))_1(x) \dot{-} (k(g))_1(x)\} \vee \sup_{x \in X} \{(k(g))_2(x) \dot{-} (k(f))_2(x)\} \right) \\ &= \sup_{x \in X} \inf_{k \in \mathcal{K}} \{(k(f))_1(x) \dot{-} (k(g))_1(x)\} \vee \sup_{x \in X} \inf_{k \in \mathcal{K}} \{(k(g))_2(x) \dot{-} (k(f))_2(x)\} \\ &\geq \sup_{x \in X} \{ \inf_{k \in \mathcal{K}} (k(f))_1(x) \dot{-} \inf_{k \in \mathcal{K}} (k(g))_1(x) \} \vee \sup_{x \in X} \{ \inf_{k \in \mathcal{K}} (k(g))_2(x) \dot{-} \inf_{k \in \mathcal{K}} (k(f))_2(x) \} \\ &= D(t(f), t(g)). \end{aligned}$$

Thus part (b) is satisfied. The fact that t is a map from P_X to P_X , we conclude that $t \in \mathcal{P}$ and is a lower bound (by the way it was constructed) of the chain \mathcal{K} in (\mathcal{P}, \preceq) . Hence by Zorn's Lemma \mathcal{P} has a minimal element, say m .

To complete the proof of Remark 2.2.1, we need only show that $m(f) \in Q_X$ whenever $f \in P_X$. To do this we shall need the following result.

2.2.1 Lemma. (compare [11, Section 1.3]) Let (X, d) be a T_0 -quasi-metric space and let $f \in P_X$. For each $x \in X$ set $(p_x(f))_1(z) = f_1(z)$ if $z \in X \setminus \{x\}$ and

$$(p_x(f))_1(x) = \sup\{d(y, x) \dot{-} f_2(y) : y \in X\}$$

and $(p_x(f))_2(z) = f_2(z)$ if $z \in X \setminus \{x\}$ and

$$(p_x(f))_2(x) = \sup\{d(x, y) \dot{-} f_1(y) : y \in X\}.$$

Then for each $x \in X$, $p_x \in \mathcal{P}$.

Proof.

We show first that $p_x(f) \in P_X$. We explore the following cases:

Case 1: $z = x$ and $y = x$. Then the result follows since $d(x, x) = 0$.

Case 2: $z \neq x$ and $y \neq x$. Then $(p_x(f))_1(z) = f_1(z)$ and $(p_x(f))_2(z) = f_2(z)$ so that

$$(p_x(f))_2(z) + (p_x(f))_1(y) = f_2(z) + f_1(y) \geq d(z, y).$$

Case 3: $z = x$ and $y \neq x$. In this case $(p_x(f))_1(y) = f_1(y)$ and $(p_x(f))_2(z) = \sup\{d(z, y) \dot{-} f_1(y) : y \in X\}$ so that

$$(p_x(f))_2(z) + (p_x(f))_1(y) = \sup\{d(z, y) \dot{-} f_1(y) : y \in X\} + f_1(y) \geq d(z, y).$$

The case $z \neq x$ and $y = x$ is similar to Case 3 and hence we leave it to the interested reader to verify.

Thus $p_x(f)$ is ample. $p_x(f)$ also satisfies $p_x(f) \leq f$. Indeed if $z = x$ the result is obvious. Suppose now that $z \neq x$. Then we have that

$$(p_x(f))_1(x) = \sup\{d(y, x) \dot{-} f_2(y) : y \in X\}$$

and

$$(p_x(f))_2(x) = \sup\{d(x, y) \dot{-} f_1(y) : y \in X\}.$$

Suppose on the contrary that $(p_x(f))_1(x) \not\leq f_1(x)$ for every $x \in X$. Then we must have that $(p_x(f))_1(x) > f_1(x)$ for every $x \in X$. This implies that $\sup\{d(y, x) \dot{-} f_2(y) : y \in X\} > f_1(x)$ for every $x \in X$, i.e., $d(y, x) \dot{-} f_2(y) > f_1(x)$ for every $x, y \in X$. Hence we have that $d(y, x) > f_2(y) + f_1(x)$. This however contradicts $f \in P_X$. Thus we must have that $(p_x(f))_1(x) \leq f_1(x)$ for every $x \in X$.

Similarly one can show that $(p_x(f))_2(x) \leq f_2(x)$ for every $x \in X$. Hence $p_x(f) \leq f$.

Finally it is not difficult to see that $D(p_x(f), p_x(g)) \leq D(f, g)$ and this completes the proof.

□

We now finish the proof of Remark 2.2.1. For each $x \in X$ we have obviously that $p_x \circ m \in \mathcal{P}$ and $p_x \circ m \preceq m$. Hence by minimality of m , $p_x \circ m = m$ whenever $x \in X$.

It therefore follows that for each $x \in X$, $p_x(m(f)) = m(f)$ whenever $f \in P_X$. Thus by the definition of the elements of Q_X we conclude that $m(f) \in Q_X$ whenever $f \in P_X$ (compare [25, Remark 2]).

□

2.2.2 Remark. (compare [11, Section 1.10]) Using some p (as in Proposition 2.2.1) we can define for any $f \in Q_X$ a map $\Psi : [0, 1] \times Q_X \rightarrow Q_X$ as follows:

$$(t, g) \mapsto p_t(g) = p(tf + (1 - t)g)$$

from p_0 , the identity on Q_X , to p_1 , the constant map $Q_X \rightarrow \{f\} \subseteq Q_X$. We used the fact that $tf + (1-t)g \in P_X$.

Moreover note that for any $0 \leq t \leq s \leq 1$ one has

$$\begin{aligned}
D(p_s(g), p_t(g)) &= D(p(sf + (1-s)g), p(tf + (1-t)g)) \\
&\leq D(sf + (1-s)g, tf + (1-t)g) \text{ by condition (a) of Proposition 2.2.1} \\
&= \sup_{x \in X} \{(s-t)f_1(x) \dot{-} (s-t)g_1(x)\} \vee \sup_{x \in X} \{(s-t)g_2(x) \dot{-} (s-t)f_2(x)\} \\
&= (s-t)D(f, g).
\end{aligned}$$

Furthermore

$$\begin{aligned}
D(f, g) &\leq D(f, p_s(g)) + D(p_s(g), p_t(g)) + D(p_t(g), g) \\
&= D(p_1(g), p_s(g)) + D(p_s(g), p_t(g)) + D(p_t(g), p_0(g)) \\
&= (1-s)D(f, g) + D(p_s(g), p_t(g)) + tD(f, g).
\end{aligned}$$

Therefore

$$D(p_s(g), p_t(g)) \geq (s-t)D(f, g)$$

and thus

$$D(p_s(g), p_t(g)) = (s-t)D(f, g)$$

whenever $g \in Q_X$ and $0 \leq t \leq s \leq 1$.

If $0 \leq s \leq t \leq 1$ and $g \in Q_X$, then a similar computation gives

$$D(p_s(g), p_t(g)) = D^{-1}(p_t(g), p_s(g)) = (t-s)D^{-1}(f, g) = (t-s)D(g, f).$$

If we set $a = D(f, g)$ and $b = D(g, f)$, then we see that the map $([0, 1], d_{ab}) \rightarrow (Q_X, D)$ defined by $s \mapsto p_s(g)$ yields an isometric map connecting g to f , where d_{ab} is as defined in Example 1.1.2.

If we equip the interval $[0, 1]$ of the real numbers with its usual topology $\tau(u^s)$ (as usual, u^s also denotes the restriction of the metric u^s to $[0, 1]$) and Q_X with the

topology $\tau(D)$, then we see that the map Ψ is continuous, that is, $\Psi : [0, 1] \times Q_X \rightarrow Q_X$ is a homotopy and Q_X is contractible in the classical sense.

Indeed suppose that the sequence $(s_n)_{n \in \mathbb{N}}$ converges to s in $([0, 1], \tau(u^s))$ and the sequence $(g_n)_{n \in \mathbb{N}}$ converges to $g \in (Q_X, \tau(D))$, that is $D(g, g_n) \rightarrow 0$ as $n \rightarrow \infty$. Then for each $n \in \mathbb{N}$, by the triangle inequality we have that

$$D(p_s(g), p_{s_n}(g_n)) \leq D(p_s(g), p_{s_n}(g)) + D(p_{s_n}(g), p_{s_n}(g_n)).$$

Therefore we see that for each $n \in \mathbb{N}$

$$D(p_s(g), p_{s_n}(g)) = (s - s_n)D(f, g)$$

if $s \geq s_n$ and

$$D(p_s(g), p_{s_n}(g)) = (s_n - s)D(f, g)$$

if $s_n \geq s$, according to our previous calculations above.

Moreover for each $n \in \mathbb{N}$ we get that

$$D(p_{s_n}(g), p_{s_n}(g_n)) = (1 - s_n)D(g, g_n)$$

by the definition of D .

Hence by our assumptions we conclude that

$$D(p_s(g), p_{s_n}(g)) \rightarrow 0$$

and

$$D(p_{s_n}(g), p_{s_n}(g_n)) \rightarrow 0$$

as $n \rightarrow \infty$. Therefore $D(p_s(g), p_{s_n}(g_n)) \rightarrow 0$ as $n \rightarrow \infty$, and hence Ψ is indeed continuous.

2.2.2 Proposition. (compare [1, Proposition 3]) Let (Y, d) be a T_0 -quasi-metric space and let (X, d) be a nonempty subspace of (Y, d) . Then there exists an isometric embedding $\tau : Q_X \rightarrow Q_Y$ such that $\tau(f)|_X = f$ whenever $f \in Q_X$.

Proof.

Let us fix $x_0 \in X$ and choose a retraction $p : P_Y \rightarrow Q_Y$ satisfying Proposition 2.2.1. Furthermore let $s : Q_X \rightarrow P_Y$ be defined as $s(f) = f'$ where $f'_1(y) = f_1(y)$ whenever $y \in X$, and $f'_1(y) = f_1(x_0) + d(x_0, y)$ whenever $y \in Y \setminus X$. Similarly $f'_2(y) = f_2(y)$ whenever $y \in X$, and $f'_2(y) = f_2(x_0) + d(y, x_0)$ whenever $y \in Y \setminus X$. We shall prove that $f' \in P_Y$ by considering the following four cases:

Suppose $x \in X$ and $y \in X$; then

$$f'_2(x) + f'_1(y) = f_2(x) + f_1(y) \geq d(x, y).$$

Suppose now that $x \in Y \setminus X$ and $y \in Y \setminus X$; then

$$f'_2(x) + f'_1(y) = d(x, x_0) + f_2(x_0) + f_1(x_0) + d(x_0, y) \geq d(x, x_0) + d(x_0, y) \geq d(x, y).$$

Suppose that $x \in X$ and $y \in Y \setminus X$; then

$$f'_2(x) + f'_1(y) = f_2(x) + f_1(x_0) + d(x_0, y) \geq d(x, x_0) + d(x_0, y) \geq d(x, y).$$

Suppose that $x \in Y \setminus X$ and $y \in X$; then

$$f'_2(x) + f'_1(y) = d(x, x_0) + f_2(x_0) + f_1(y) \geq d(x, x_0) + d(x_0, y) \geq d(x, y).$$

Thus $f' \in P_Y$.

Define $\tau : p \circ s : Q_X \rightarrow Q_Y$. Notice that $\tau(f)|_X = p(f')|_X = f$ whenever $f \in Q_X$, since $p(f') \leq f'$, therefore we have that $p(f')|_X \leq f'|_X = f$, and f minimal on X .

Furthermore for any $f, g \in Q_X$,

$$\begin{aligned} D(f, g) &= D(\tau(f)|_X, \tau(g)|_X) \\ &\leq D(\tau(f), \tau(g)) \\ &= D(p(f'), p(g')) \\ &\leq D(f', g') \\ &= D(f, g) \end{aligned}$$

where the last equality follows from the definition of f' and g' .

□

Let us now state precisely what we mean by tight extension of a T_0 -quasi-metric space.

2.2.1 Definition. (compare [20, Remark 7]) Let X be a subspace of a T_0 -quasi-metric space (Y, d) . Then Y is called a **tight extension** of X if for any quasi-pseudometric ρ on Y that satisfies $\rho \leq d$ and $\rho = d$ on $X \times X$ we have that $\rho = d$.

Recall that it was shown in [20, Remark 7] that for any T_0 -quasi-metric space (X, d) , the isometric embedding $\varphi : X \rightarrow Q_X$ is tight, that is, Q_X is a tight extension of $\varphi(X)$.

2.2.3 Remark. (compare [11, Section 1.12]) For any T_0 -quasi-metric tight extension Y_1 of X , any T_0 -quasi-metric extension (Y_2, d) of X and any non-expansive map $\phi : Y_1 \rightarrow Y_2$ satisfying $\phi(x) = x$ whenever $x \in X$, ϕ is necessarily an isometric map.

Proof.

Otherwise the quasi-pseudometric $\rho : Y_1 \times Y_1 \rightarrow [0, \infty)$ defined by $\rho(x, y) = d(\phi(x), \phi(y))$ would contradict the tightness of the extension Y_1 of X .

□

2.2.3 Proposition. (compare [11, Section 1.13]) Let (Y, d) be a T_0 -quasi-metric tight extension of X . Then the restriction map defined by $f \mapsto f|_X$ whenever $f \in Q_Y$ is a bijective isometric map $Q_Y \rightarrow Q_X$.

Proof.

Choose a retraction map $p : P_X \rightarrow Q_X$ satisfying the conditions of Proposition 2.2.1 and let $\phi : Q_Y \rightarrow Q_X$ defined as $f \mapsto p(f|_X)$ denote the composition of p

with the restriction map. Then ϕ is a non-expansive map and Q_X and Q_Y are T_0 -quasi-metric extensions of X . By Remark 2.2.3 ϕ must be an isometric map, since Q_Y is a tight extension of X , because Q_Y is a tight extension of Y and Y is a tight extension of X .

We therefore have by Proposition 2.2.2 that there exists an isometric embedding $\tau : Q_X \rightarrow Q_Y$ satisfying $\tau(f)|_X = f$ for every $f \in Q_X$. Then we have

$$\phi(\tau(f)) = p(\tau(f)|_X) = p(f) = f$$

for all $f \in Q_X$ and thus ϕ is necessarily surjective. But a surjective isometric map on a T_0 -quasi-metric domain is necessarily bijective. So $\tau : Q_X \rightarrow Q_Y$ has to be the inverse map of ϕ and thus for every $f \in Q_Y$ we have the formula

$$f|_X = \tau(\phi(f))|_X = \phi(f) \in Q_X,$$

that is, the restriction map

$$Q_Y \rightarrow Q_X : f \mapsto f|_X$$

maps Q_Y already onto Q_X without having to be composed with the restriction map p . Hence we see that for any T_0 -quasi-metric tight extension Y of X the restriction map $Q_Y \rightarrow Q_X : f \mapsto f|_X$ yields a bijective isometric map between Q_Y and Q_X .

□

2.2.4 Proposition. (compare [14, Theorem 2]) Let X be a subspace of the T_0 -quasi-metric space (Y, d) . Then the following conditions are equivalent:

- (a) Y is a tight extension of X .
- (b) $d(y_1, y_2) = \sup\{(d(x_1, x_2) - d(x_1, y_1) - d(y_2, x_2)) \vee 0 : x_1, x_2 \in X\}$ whenever $y_1, y_2 \in Y$.
- (c) $f_y|_X(x) = (d(y, x), d(x, y))$ with $x \in X$ is minimal on X whenever $y \in Y$ and the map $\phi : (Y, d) \rightarrow (Q_X, D) : y \mapsto f_y|_X$ is an isometric embedding.

Proof.

(a) \implies (b):

Let Y be a T_0 -quasi-metric tight extension of X . By Proposition 2.2.3 the map $Q_Y \rightarrow Q_X$ defined by $f \mapsto f|_X$ defines a bijective isometric map between Q_Y and Q_X . Hence the extension Y of X , as a subspace of Q_Y , fulfils condition (b) of Proposition 2.2.4, since the extension Q_X of X satisfies it by [20, Remark 7].

(b) \implies (c):

For any $x_1, x_2 \in X$ and $y_1 \in Y$ we have that

$$d(x_1, x_2) \leq d(x_1, y_1) + d(y_1, x_2).$$

Therefore for any $x_1, x_2 \in X$ and $y_1, y_2 \in Y$ we see that

$$d(x_1, x_2) - d(x_1, y_1) - d(y_2, x_2) \leq d(y_1, x_2) - d(y_2, x_2).$$

Consequently for any $y_1, y_2 \in Y$ we have by (b) that

$$\begin{aligned} d(y_1, y_2) &= \sup\{(d(x_1, x_2) - d(x_1, y_1) - d(y_2, x_2)) \vee 0 : x_1, x_2 \in X\} \\ &\leq \sup\{(d(y_1, x_2) - d(y_2, x_2)) : x_2 \in X\} \\ &\leq d(y_1, y_2). \end{aligned}$$

Similarly

$$d(x_1, x_2) \leq d(x_1, y_2) + d(y_2, x_2)$$

whenever $x_1, x_2 \in X$ and $y_2 \in Y$.

It follows that for each $x_1, x_2 \in X$ and $y_1, y_2 \in Y$ we have that

$$d(x_1, x_2) - d(y_2, x_2) - d(x_1, y_1) \leq d(x_1, y_2) - d(x_1, y_1).$$

Thus for any $y_1, y_2 \in Y$ we get by (b) that

$$\begin{aligned} d(y_1, y_2) &= \sup\{(d(x_1, x_2) - d(y_2, x_2) - d(x_1, y_1)) \vee 0 : x_1, x_2 \in X\} \\ &\leq \sup\{(d(x_1, y_2) - d(x_1, y_1)) : x_1 \in X\} \\ &\leq d(y_1, y_2). \end{aligned}$$

Hence we conclude that $d(y_1, y_2) = D(f_{y_1}|_X, f_{y_2}|_X)$ whenever $y_1, y_2 \in Y$.

As we have just shown above, for any $y_1, y_2 \in Y$ we have that

$$d(y_1, y_2) = \sup\{d(x_1, y_2) \dot{-} d(x_1, y_1) : x_1 \in X\}$$

and

$$d(y_1, y_2) = \sup\{d(y_1, x_2) \dot{-} d(y_2, x_2) : x_2 \in X\}.$$

Substituting $x_2 \in X$ for y_2 and $x_1 \in X$ for y_1 , respectively, we obtain the two equations

$$(f_{y_1})_1(x_2) = d(y_1, x_2) = \sup\{d(x_1, x_2) \dot{-} d(x_1, y_1) : x_1 \in X\}$$

whenever $y_1 \in Y$ and $x_2 \in X$, and

$$(f_{y_2})_2(x_1) = d(x_1, y_2) = \sup\{d(x_1, x_2) \dot{-} d(y_2, x_2) : x_2 \in X\}$$

whenever $y_2 \in Y$ and $x_1 \in X$.

By [25, Remark 2] the restriction $f_y|_X$ is minimal on X whenever $y \in Y$.

(c) \implies (a):

Let $q : Y \times Y \rightarrow [0, \infty)$ be a quasi-pseudometric on Y such that $q \leq d$ and $q|_{X \times X} = d|_{X \times X}$.

Then by (c) and since $f_y|_X$ is minimal whenever $y \in X$ we have

$$\begin{aligned} d(y_1, y_2) &= D(f_{y_1}|_X, f_{y_2}|_X) \\ &= \sup\{d(y_1, x) \dot{-} d(y_2, x) : x \in X\} \\ &= \sup\{d(x, y_2) \dot{-} d(x, y_1) : x \in X\} \end{aligned}$$

whenever $y_1, y_2 \in Y$ by [20, Lemma 7].

Substituting

$$d(x_1, y_2) = \sup\{d(x_1, x_2) \dot{-} d(y_2, x_2) : x_2 \in X\}$$

into the formula

$$d(y_1, y_2) = \sup\{d(x_1, y_2) - d(x_1, y_1) : x_1 \in X\},$$

we obtain

$$\begin{aligned} d(y_1, y_2) &= \sup_{x_1 \in X} \sup_{x_2 \in X} \sup\{(d(x_1, x_2) - d(x_1, y_1) - d(y_2, x_2)) \vee 0\} \\ &\leq \sup_{x_1, x_2 \in X} \{(q(x_1, x_2) - q(x_1, y_1) - q(y_2, x_2)) \vee 0\} \\ &\leq q(y_1, y_2) \end{aligned}$$

whenever $y_1, y_2 \in Y$ by our assumption. Therefore (a) is satisfied. □

2.2.4 Remark. (compare [11, Section 1.14]) Let (Y, d) be a T_0 -quasi-metric tight extension of X . Elaborating further on Proposition 2.2.4 we see that there exists only one isometric embedding $\phi : Y \rightarrow Q_X$ satisfying $\phi(x) = f_x$ whenever $x \in X$, since for such an embedding $\phi : Y \rightarrow Q_X$, $y \in Y$ and $x \in X$, we have

$$\begin{aligned} (f_y)_2|_X(x) &= d(x, y) \\ &= D(\phi(x), \phi(y)) \\ &= D(f_x, \phi(y)) \\ &= (\phi(y))_2(x); \end{aligned}$$

therefore $(f_y)_2|_X = (\phi(y))_2$. In a similar way, we can show that $(f_y)_1|_X = (\phi(y))_1$ whenever $y \in Y$.

In particular we see that the tight extension Y of X can be understood as a subspace of the extension Q_X of X . Thus we have that Q_X is maximal among the T_0 -quasi-metric tight extensions of X .

We next discuss an asymmetric version of a result due to Herrlich.

In a quasi-pseudometric space (X, d_X) we say that the (double) family

$$\mathcal{C} = (C_{d_X}(x_i, r_i), C_{d_X^{-1}}(y_i, s_i))_{i \in I}$$

of such balls meets potentially provided that there exists a T_0 -quasi-metric extension (Y, d_Y) of (X, d_X) such that

$$\bigcap_{i \in I} (C_{d_Y}(x_i, r_i) \cap C_{d_Y^{-1}}(y_i, s_i)) \neq \emptyset.$$

2.2.5 Proposition. (compare [14, Proposition]) If $\mathcal{C} = (C_{d_X}(x_i, r_i), C_{d_X^{-1}}(y_i, s_i))_{i \in I}$ is a (double) family of balls in a T_0 -quasi-metric space (X, d_X) , then the following conditions are equivalent:

- (1) \mathcal{C} meets potentially in X .
- (2) For any $i, j \in I$, $C_{d_X}(x_i, r_i)$ meets with any $C_{d_X^{-1}}(x_j, s_j)$ potentially in X .
- (3) $d_X(x_i, x_j) \leq r_i + s_j$ whenever $i, j \in I$.
- (4) There exists a minimal (ample) function pair t on X with $t_2(x_i) \leq r_i$ and $t_1(x_i) \leq s_i$ whenever $i \in I$.

Proof.

The implications (1) \implies (2) \implies (3) are obvious. For $I = \emptyset$ the implication (3) \implies (4) is also obvious, otherwise define a function pair f on $Y = \{x_i : i \in I\}$ by setting for each $y \in Y$, $f_1(y) = \inf\{s_i : x_i = y\}$ and $f_2(y) = \inf\{r_i : x_i = y\}$. Choose $y_0 \in Y$.

Set $g_1(x) = f_1(x)$ if $x \in Y$, and $g_1(x) = f_1(y_0) + d_X(y_0, x)$ if $x \in X \setminus Y$.

Furthermore set $g_2(x) = f_2(x)$ if $x \in Y$, and $g_2(x) = f_2(y_0) + d_X(x, y_0)$ if $x \in X \setminus Y$.

Then $g_1(x_i) \leq s_i$ and $g_2(x_i) \leq r_i$ whenever $i \in I$. Furthermore we have that $d_X(x, x') \leq g_2(x) + g_1(x')$ whenever $x, x' \in X$. Thus g as defined above is an ample function pair on X and by Zorn's Lemma there is a minimal ample pair t on X such that $t \leq g$.

(4) \implies (1):

Let t be a minimal ample pair on X with $t_1(x_i) \leq s_i$ and $t_2(x_i) \leq r_i$ whenever $i \in I$. If $t = f_x$ for some $x \in X$, then

$$x \in \bigcap_{i \in I} (C_{d_X}(x_i, r_i) \cap C_{d_X^{-1}}(x_i, s_i)).$$

Thus the family \mathcal{C} meets in X .

Otherwise extend X to a space Y by adding one point y_0 to X and by defining a T_0 -quasi-metric d_Y on Y extending d_X and satisfying $d_Y(x, y_0) = t_2(x)$ and $d_Y(y_0, x) = t_1(x)$ whenever $x \in X$. It is readily checked that d_Y is a T_0 -quasi-metric on Y (compare [20, end of proof of Theorem 1]). Then

$$y_0 \in \bigcap_{i \in I} (C_{d_Y}(x_i, r_i) \cap C_{d_Y^{-1}}(x_i, s_i))$$

and we are done.

□

Chapter 3

uq-tight extensions of T_0 -ultra-quasi-metric spaces

3.1 Introduction

In [1] a concept of extension (called “tight extension”, check [11]) that is appropriate in the category of T_0 -quasi-metric spaces and non-expansive maps was studied. In particular such an extension was constructed and it was shown that this extension is maximal.

In this chapter we shall show how the studies of [1] can be modified in order to obtain a theory that is appropriate for T_0 -ultra-quasi-metric spaces. Eventhough our studies follow essentially the articles [11, 1], we found it imperative to work out every detail of this theory in this chapter.

We will show in this chapter that every T_0 -ultra-quasi-metric space X has a *uq*-tight extension which is maximal amongst the *uq*-tight extensions of X . This agrees with the result we have for T_0 -quasi-metric spaces (check [1]).

3.2 Hulls of T_0 -ultra-quasi-metric spaces

We shall recall some results from the theory of uq -hyperconvex hulls of T_0 -ultra-quasi-metric spaces due to [23].

3.2.1 Definition. (Compare [23, Definition 1, p.4]) Let (X, d) be a T_0 -ultra-quasi-metric space. We shall say that a function pair $f = (f_1, f_2)$ on (X, d) where $f_i : X \rightarrow [0, \infty)$ ($i = 1, 2$) is **ultra-ample** if for all $x, y \in X$, we have $d(x, y) \leq \max\{f_2(x), f_1(y)\}$.

Let us denote by uP_X the set of all ultra-ample function pairs on a T_0 -ultra-quasi-metric space (X, d) . For each $f, g \in uP_X$, define

$$N(f, g) = \max \left\{ \sup_{x \in X} n(f_1(x), g_1(x)), \sup_{x \in X} n(g_2(x), f_2(x)) \right\},$$

where the T_0 -ultra-quasi-metric n^1 is as defined in Example 1.2.1. Then N is an extended T_0 -ultra-quasi-metric on uP_X .

We say that a function pair f is uq -minimal among the ultra-ample function pairs on (X, d) if it is an ultra-ample function pair and if $g = (g_1, g_2)$ is an ultra-ample function pair on (X, d) and for each $x \in X$ $g_1(x) \leq f_1(x)$ and $g_2(x) \leq f_2(x)$, implies $f = g$. We shall also call a uq -minimal ultra-ample function pair a uq -extremal (ultra-ample) function pair. By uQ_X we shall denote the set of all uq -extremal function pairs on (X, d) equipped with the restriction of N to $uQ_X \times uQ_X$, which we still denote by N . Of course N is actually a T_0 -ultra-quasi-metric on uQ_X (compare [23, Corollary 5]). We shall call (uQ_X, N) the ultra-quasi-metrically injective hull of (X, d) .

We recall also by [23, Lemma 5] that if $f, g \in uQ_X$ of a T_0 -ultra-quasi-metric space (X, d) , then

$$N(f, g) = \sup\{n(f_1(x), g_1(x)) : x \in X\} = \sup\{n(g_2(x), f_2(x)) : x \in X\}.$$

¹Ofcourse we can use T_0 -ultra-quasi-metric n here since the function pairs take values in $[0, \infty)$.

Recall that $f \in uP_X$ belongs to uQ_X if and only if

$$f_1(x) = \sup\{d(y, x) : y \in X \text{ and } d(y, x) > f_2(y)\} \quad (3.1)$$

and

$$f_2(x) = \sup\{d(x, y) : y \in X \text{ and } d(x, y) > f_1(y)\} \quad (3.2)$$

whenever $x \in X$ (compare [25, Remark 2]).

It is known (see [23, Corollary 3]) that if f is a uq -extremal ultra-ample function pair on (X, d) , then $f_1(x) \leq \max\{f_1(y), d(y, x)\}$ and $f_2(x) \leq \max\{f_2(y), d(x, y)\}$.

3.2.1 Lemma. (Compare [23, Lemma 2]) Let (X, d) be a T_0 -ultra-quasi-metric space. For each $x \in X$, define $f_x(y) = (d(x, y), d(y, x))$ whenever $y \in X$. Then f_x is ultra-ample.

The map $x \mapsto f_x$ whenever $x \in X$ defines an isometric embedding of (X, d) into (uQ_X, N) .

3.2.1 Proposition. (Compare [11, Section 1.3]) Let (X, d) be a T_0 -ultra-quasi-metric space. Then uQ_X consists of all function pairs which are “ uq -minimal” in uP_X .

Proof.

To prove the above proposition, we need to prove that there is no $g \in uP_X$ with $g < f$ but $g \neq f$. This is so since on the one hand, $g \leq f \in uQ_X$ and $g \in uP_X$ implies

$$\begin{aligned} f_1(x) &= \sup\{d(y, x) : y \in X \text{ and } d(y, x) > f_2(y) \geq g_2(y)\} \\ &= \sup\{d(y, x) : y \in X \text{ and } d(y, x) \geq g_2(y)\} \\ &\leq g_1(x). \end{aligned}$$

Thus

$$f_1(x) \leq g_1(x). \quad (3.3)$$

Similarly we have

$$\begin{aligned}
f_2(x) &= \sup\{d(x, y) : y \in X \text{ and } d(x, y) > f_1(y) \geq g_1(y)\} \\
&= \sup\{d(x, y) : y \in X \text{ and } d(x, y) \geq g_1(y)\} \\
&\leq g_2(x)
\end{aligned}$$

implies

$$f_2(x) \leq g_2(x). \quad (3.4)$$

Using (3.3) and (3.4) and the condition that $g \leq f$, we have thus shown that $f = g$.

On the other hand, for some $x \in X$ and $f \in uP_X$, we have that $f_1(x) > \sup\{d(y, x) : y \in X \text{ and } d(y, x) > f_2(y)\}$ or $f_2(x) > \sup\{d(x, y) : y \in X \text{ and } d(x, y) > f_1(y)\}$.

For each $x \in X$ and $f \in uP_X$ set $(p_x(f))_1(z) = f_1(z)$ if $z \in X \setminus \{x\}$ and $(p_x(f))_1(x) = \sup\{d(y, x) : y \in X \text{ and } d(y, x) > f_2(y)\}$.

Similarly set $(p_x(f))_2(z) = f_2(z)$ if $z \in X \setminus \{x\}$ and $(p_x(f))_2(x) = \sup\{d(x, y) : y \in X \text{ and } d(x, y) > f_1(y)\}$.

To show that $p_x(f)$ is ultra-ample, we shall consider the following cases:

Case 1: If $z = x$ and $y = x$, then the result holds since $d(x, x) = 0$.

Case 2: If $z \neq x$ and $y \neq x$, then $(p_x(f))_1(z) = f_1(z)$ and $(p_x(f))_2(z) = f_2(z)$ so that

$$\max\{(p_x(f))_2(z), (p_x(f))_1(y)\} = \max\{f_2(z), f_1(y)\} \geq d(z, y).$$

Case 3: $z = x$ and $y \neq x$. In this case $(p_x(f))_1(y) = f_1(y)$ and $(p_x(f))_2(z) = \sup\{d(z, y) : y \in X \text{ and } d(z, y) > f_1(y)\}$ so that

$$(p_x(f))_2(z) \vee ((p_x(f))_1(y)) = (\sup\{d(z, y) : y \in X \text{ and } d(z, y) > f_1(y)\}) \vee f_1(y) \geq d(z, y).$$

Thus $p_x(f) = ((p_x(f))_1, (p_x(f))_2)$ is ultra-ample and also satisfies $p_x(f) \leq f$ by the way it was constructed.

Notice that $(p_x(f))_1 \neq f_1$ since $(p_x(f))(x)$ is either 0 (whereas $f_1(x) > \sup\{d(y, x) : y \in X \text{ and } d(y, x) > f_2(y)\}$ implies that $f_1(x) > d(x, x) = 0$ and we have that $f_1(x) > 0$ since $f_2(x) \geq 0$) or $((p_x(f))_1)(x) = \sup\{d(y, x) : y \in X \text{ and } d(y, x) > f_2(y)\} \leq f_1(x)$.

In a similar fashion, we can show that $(p_x(f))_2 \neq f_2$. One can thus conclude now that $p_x(f) \neq f$.

Thus by defining $g = ((p_x(f))_1, (p_x(f))_2)$, we can conclude using Zorn's lemma that for any $f \in uP_X$, $g \leq f$.

It is an easy exercise to show that $N(p_x(f), p_x(g)) \leq N(f, g)$.

□

3.2.2 Lemma. (Compare [23, Theorem 1]) Let (X, d) be a T_0 -ultra-quasi-metric space and $f \in uQ_X$ and $a \in X$. Then $N(f, f_a) = f_1(a)$ and $N(f_a, f) = f_2(a)$.

3.2.1 Corollary. (compare [23, Corollary 5]) Let (X, n) be a T_0 -ultra-quasi-metric space. Then N is indeed a T_0 -ultra-quasi-metric on uQ_X .

3.3 T_0 -ultra-quasi-metric tight extensions

We will in this section study the variants of some crucial results about tight extensions of T_0 -quasi-metric spaces from [1] to uq -tight extensions of T_0 -ultra-quasi-metric spaces.

3.3.1 Lemma. (compare [23, Lemma 8]) Let (X, d) be a T_0 -ultra-quasi-metric space. Then for any $f, g \in uQ_X$, we have that

$$N(f, g) = \sup\{d(x_1, x_2) : x_1, x_2 \in X, d(x_1, x_2) > f_2(x_1) \text{ and } d(x_1, x_2) > g_1(x_2)\}.$$

3.3.1 Proposition. (Compare [1, Proposition 1]) Let (X, d) be a T_0 -ultra-quasi-metric space. There exists a retraction map $p : uP_X \rightarrow uQ_X$, i.e., a map that satisfies the following conditions

(a) $N(p(f), p(g)) \leq N(f, g)$ whenever $f, g \in uP_X$.

(b) $p(f) \leq f$ whenever $f \in uP_X$.

(In particular we have that $p(f) = f$ whenever $f \in uQ_X$.)

Proof.

Given a pair $f \in uP_X$, we set

$$f_1^*(y) = \sup\{d(x', y) : x' \in X \text{ and } d(x', y) > f_2(x')\}$$

whenever $y \in X$ and

$$f_2^*(x) = \sup\{d(x, y') : y' \in X \text{ and } d(x, y') > f_1(y')\}$$

whenever $x \in X$.

Claim 1: $f^* \leq f$.

Note that for any $x, y' \in X$, we have $d(x, y') \leq \max\{f_2(x), f_1(y')\}$ (since $f \in uP_X$) and thus if $d(x, y') > f_1(y')$, we have $d(x, y') \leq f_2(x)$ so that by taking supremum over $y' \in X$, we get that $f_2^*(x) \leq f_2(x)$. In a similar manner, we can show that $f_1^*(y) \leq f_1(y)$ whenever $y \in X$. Thus $f^* \leq f$ as required.

Claim 2: $d(x, y) \leq \max\{f_2^*(x), f_1(y)\}$ and $d(y, x) \leq \max\{f_1^*(x), f_2(y)\}$ whenever $x, y \in X$.

$$\begin{aligned} \max\{f_2^*(x), f_1(y)\} &= \max\{\sup\{d(x, y) : y \in X \text{ and } d(x, y) > f_1(y)\}, f_1(y)\} \\ &\geq \max\{d(x, y), f_1(y)\} \quad \forall y \in X \\ &= d(x, y) \end{aligned}$$

implying that

$$d(x, y) \leq \max\{f_2^*(x), f_1(y)\}. \quad (3.5)$$

Also

$$\begin{aligned} \max\{f_1^*(x), f_2(y)\} &= \max\{\sup\{d(y, x) : y \in X \text{ and } d(y, x) > f_2(y)\}, f_2(y)\} \\ &\geq \max\{d(y, x), f_2(y)\} \quad \forall y \in X \\ &= d(y, x) \end{aligned}$$

implying that

$$d(y, x) \leq \max\{f_1^*(x), f_2(y)\}. \quad (3.6)$$

By Equation (3.6) and Equation (3.5), the claim follows.

Let us now define the map $q : uP_X \rightarrow uP_X$ by $f \mapsto q(f) = (\max\{f_1, f_1^*\}, \max\{f_2, f_2^*\})$ whenever $f \in uP_X$. Then the following claim holds.

Claim 3: $q(f)$ is indeed ultra-ample and $q(f) \leq f$.

Indeed

$$\begin{aligned} \max\{(q(f))_2(x), (q(f))_1(y)\} &= \max\{\max\{f_1(y), f_1^*(y)\}, \max\{f_2(x), f_2^*(x)\}\} \\ &= \max\{\max\{f_2(x), f_1^*(y)\}, \max\{f_1(y), f_2^*(x)\}\} \\ &\geq \max\{d(x, y), d(x, y)\} \\ &= d(x, y) \text{ whenever } x, y \in X. \end{aligned}$$

This shows that $q(f)$ is ultra-ample.

We now show that $q(f) \leq f$. Indeed

$$\begin{aligned} f_1^* \leq f_1 &\Rightarrow (q(f))_1 = \max\{f_1, f_1^*\} \leq \max\{f_1, f_1\} = f_1, \text{ i.e. } (q(f))_1 \leq f_1, \\ f_2^* \leq f_2 &\Rightarrow (q(f))_2 = \max\{f_2, f_2^*\} \leq \max\{f_2, f_2\} = f_2, \text{ i.e. } (q(f))_2 \leq f_2. \end{aligned}$$

We get immediately from the last two lines that $((q(f))_1, (q(f))_2) \leq (f_1, f_2)$, i.e. $q(f) \leq f$.

Claim 4: $N(q(f), q(g)) \leq N(f, g)$ whenever $f, g \in uP_X$.

Let $f, g \in uP_X$ and $x \in X$. Then

$$\begin{aligned}
f_1^*(x) &= \sup\{d(y, x) : y \in X \text{ and } d(y, x) > f_2(y)\} \\
&= \sup\{\max\{d(y, x), n(g_2(y), f_2(y))\} : y \in X \text{ and } d(y, x) > \max\{f_2(y), g_2(y)\}\} \\
&= \max\{\sup\{d(y, x) : y \in X \text{ and } d(y, x) > g_2(y)\}, \sup\{n(g_2(y), f_2(y)) : y \in X\}\} \\
&\leq \max\{g_1^*(x), N(f, g)\}.
\end{aligned}$$

Thus $n(f_1^*(x), g_1^*(x)) \leq N(f, g)$.

In a similar way, we can show that $n(g_2^*(x), f_2^*(x)) \leq N(f, g)$.

By taking the maximum of the above two expressions, we get that:

$$N(f^*, g^*) \leq N(f, g) \tag{3.7}$$

for all $f, g \in uP_X$.

Observe first that

$$\begin{aligned}
n((q(f))_1(x), (q(g))_1(x)) &= n(\max\{f_1(x), f_1^*(x)\}, \max\{g_1(x), g_1^*(x)\}) \\
&= n(\max\{f_1(x), g_1(x)\}, \max\{f_1^*(x), g_1^*(x)\}) \\
&= \max\{n(f_1(x), g_1(x)), n(f_1^*(x), g_1^*(x))\},
\end{aligned}$$

so that

$$\begin{aligned}
\sup_{x \in X} \{n((q(f))_1(x), (q(g))_1(x))\} &= \sup_{x \in X} \{\max\{n(f_1(x), g_1(x)), n(f_1^*(x), g_1^*(x))\}\} \\
&= \max\{\sup_{x \in X} \{n(f_1(x), g_1(x))\}, \sup_{x \in X} \{n(f_1^*(x), g_1^*(x))\}\} \\
&\leq \max\{N(f, g), N(f^*, g^*)\} \\
&\leq \max\{N(f, g), N(f, g)\} \\
&= N(f, g).
\end{aligned}$$

This shows that

$$\sup\{n((q(f))_1(x), (q(g))_1(x)) : x \in X\} \leq N(f, g). \quad (3.8)$$

In a similar way, we get that

$$\sup\{n((q(g))_2(x), (q(f))_2(x)) : x \in X\} \leq N(f, g). \quad (3.9)$$

By Equations (3.8) and (3.9), we get immediately that

$$N(q(f), q(g)) \leq N(f, g)$$

whenever $f, g \in uP_X$.

To complete the proof, given $f \in uP_X$ we obtain a uq -minimal ample function pair below f as the point-wise limit of the sequence $(q^n(f))_{n \in \mathbb{N}}$ where q^n is the n^{th} -iteration of q . Indeed:

let $f \in uP_X$ be fixed so that $q(f) \in uP_X$ and also $f^* \leq f$. Then

$$f_1^* \leq f_1 \Rightarrow f_1^* \leq \max\{f_1, f_1^*\} = (q(f))_1.$$

The last inequality and the fact that $q(f) \in uP_X$ implies that

$$f_1^* \leq (q(f))_1^* \leq (q(f))_1.$$

By iterating the map q , we get for every $f \in uP_X$ a monotonically decreasing sequence of functions

$$q(f) \geq q^2(f) \geq q^3(f) \geq \dots$$

in uP_X .

Thus we can define the map

$$p = \lim_{k \rightarrow \infty} q^k : uP_X \rightarrow uP_X$$

$$f \mapsto \lim_{k \rightarrow \infty} q^k(f) = \left(\lim_{k \rightarrow \infty} (q^k(f))_1, \lim_{k \rightarrow \infty} (q^k(f))_2 \right) = ((p(f))_1, (p(f))_2)$$

The above point-wise limit pair is taken with respect to the usual topology $\tau(n^s)$ on $[0, \infty)$ and the limit exists because (q^k) is monotonically decreasing and is bounded below by the 0-pair.

Note that obviously for each $k \in \mathbb{N}$, $q^k(f) \in uP_X$ and q^k satisfies the conditions (a) and (b), too. Therefore $p(f) \in uP_X$ and p also satisfies condition (b).

For each $k \in \mathbb{N}$, $f, g \in uP_X$ and $x \in X$ we have

$$\left[\sup_{x \in X} n((q^k(f))_1(x), (q^k(g))_1(x)) \right] \vee \left[\sup_{x \in X} n((q^k(g))_2(x), (q^k(f))_2(x)) \right] \leq N(f, g);$$

thus $N(p(f), p(g)) \leq N(f, g)$ and p satisfies condition (b), too.

We finally show that $p(f) \in uQ_X$ whenever $f \in uP_X$.

Let $f \in uP_X$. Then we have that

$$n((q(f))_1(x), (q(f))_1^*(x)) \leq n((q(f))_1(x), (q(f))_1(x)) = 0$$

which in turn implies that

$$n((q^k(f))_1(x), (q^k(f))_1^*(x)) \leq n((q^k(f))_1(x), (q^k(f))_1(x)) = 0$$

and we get that $n((q^k(f))_1(x), (q^k(f))_1^*(x)) = 0$.

In a similar manner, we get $n((q^k(f))_1^*(x), (q^k(f))_1(x)) = 0$.

By appealing to the T_0 condition and taking limits as $k \rightarrow \infty$, we get that $p(f) = (p(f))^*$ or, equivalently $p(f) \in \nu_q(X)$.

Moreover the function pair $h := p(f)$ is uq -extremal among the ultra-ample function pairs.

Indeed let $g \leq h$ and let g be an ultra-ample function pair. Then for each $x \in X$,

$$\begin{aligned} h_2(x) &= \sup\{d(x, y) : y \in X \text{ and } d(x, y) > h_1(y) \geq g_1(y)\} \\ &\leq \sup\{d(x, y) : y \in X \text{ and } d(x, y) > g_1(y)\} \\ &\leq g_2(x), \text{ by ultra-ampleness of } g. \end{aligned}$$

So $g_2 = h_2$. Similarly $g_1 = h_1$ and therefore the pair h is a uq -extremal ultra-ample.

□

3.3.1 Remark. It can be shown that Proposition 3.3.1 can be established by the use of Zorn's Lemma (compare [11, Section 1.9]).

Indeed, let (X, d) be a T_0 -ultra-quasi-metric space and let \mathcal{P} be the set of all maps from uP_X to uP_X satisfying conditions (a) and (b) in Proposition 3.3.1.

Order \mathcal{P} by

$$p \preceq q \Leftrightarrow p(f) \leq q(f) \text{ and } N(p(f), p(g)) \leq N(q(f), q(g))$$

for all $f, g \in uP_X$ and $p, q \in \mathcal{P}$. Then $\mathcal{P} \neq \emptyset$ since the identity map belongs to \mathcal{P} .

We have to check now that \preceq is actually a partial order.

Reflexivity is obvious since every map is equal to itself.

Let now $p, q \in \mathcal{P}$ such that $p \preceq q$ and $q \preceq p$.

$p \preceq q \Rightarrow (p(f))_1 \leq (q(f))_1, (p(f))_2 \leq (q(f))_2$ and $N(p(f), p(g)) \leq N(q(f), q(g))$
 $q \preceq p \Rightarrow (q(f))_1 \leq (p(f))_1, (q(f))_2 \leq (p(f))_2$ and $N(q(f), q(g)) \leq N(p(f), p(g))$
 $(p(f))_1 \leq (q(f))_1$ and $(q(f))_1 \leq (p(f))_1$ implies that $(p(f))_1 = (q(f))_1$. In a similar manner, we have that $(p(f))_2 = (q(f))_2$ so that we can conclude that $p = q$.

Also $N(p(f), p(g)) \leq N(q(f), q(g))$ and $N(q(f), q(g)) \leq N(p(f), p(g))$ imply that $p = q$. This shows that \preceq is anti-symmetric.

Suppose now that $p, q, s \in \mathcal{P}$ such that $p \preceq q$ and $q \preceq s$.

$p \preceq q \Rightarrow (p(f))_1 \leq (q(f))_1, (p(f))_2 \leq (q(f))_2$ and $N(p(f), p(g)) \leq N(q(f), q(g))$
 $q \preceq s \Rightarrow (q(f))_1 \leq (s(f))_1, (q(f))_2 \leq (s(f))_2$ and $N(q(f), q(g)) \leq N(s(f), s(g))$

$(p(f))_1 \leq (q(f))_1$ and $(q(f))_1 \leq (s(f))_1$ implies that $(p(f))_1 \leq (s(f))_1$ by transitivity of $[0, \infty)$ as a subset of \mathbb{R} with the usual ordering \leq . Similarly, we can show that $(p(f))_2 \leq (s(f))_2$.

Also $N(p(f), p(g)) \leq N(q(f), q(g))$ and $N(q(f), q(g)) \leq N(s(f), s(g))$ imply that $N(p(f), p(g)) \leq N(s(f), s(g))$. Thus $p \preceq s$. This proves that \preceq is transitive. Therefore (\mathcal{P}, \preceq) is a partially ordered set.

To complete the proof, we have to show that every chain in \mathcal{P} has a lower bound.

Let $\emptyset \neq \mathcal{K} \subseteq \mathcal{P}$ be a chain and define $s : uP_X \rightarrow uP_X$ by

$$s(f)(x) := \left(\inf_{k \in \mathcal{K}} (k(f))_1(x), \inf_{k \in \mathcal{K}} (k(f))_2(x) \right)$$

whenever $x \in X$. Since $k(f) \in \mathcal{P}$, we have that $s(f) \in \mathcal{P}$.

Indeed observe that $s(f) \leq k(f) \leq f$, $\forall f \in uP_X$. Thus $s(f) \leq f$ and condition (a) is satisfied.

To check condition (b), we check that $N(s(f), s(g)) \leq N(k(f), k(g)) \leq N(f, g)$.

Indeed

$$\begin{aligned} N(f, g) &\geq N(k(f), k(g)), \text{ since } k \in \mathcal{P} \\ &\geq \sup_{x \in X} \{n((k(f))_1(x), (k(g))_1(x))\} \vee \sup_{x \in X} \{n((k(g))_2(x), (k(f))_2(x))\} \\ &\geq \sup_{x \in X} \inf_{k \in \mathcal{K}} \{n((k(f))_1(x), (k(g))_1(x))\} \vee \sup_{x \in X} \inf_{k \in \mathcal{K}} \{n((k(g))_2(x), (k(f))_2(x))\} \\ &\geq \sup_{x \in X} n \left(\inf_{k \in \mathcal{K}} (k(f))_1(x), \inf_{k \in \mathcal{K}} (k(g))_1(x) \right) \vee \sup_{x \in X} n \left(\inf_{k \in \mathcal{K}} (k(g))_2(x), \inf_{k \in \mathcal{K}} (k(f))_2(x) \right) \\ &= N(s(f), s(g)) \end{aligned}$$

where the last inequality above is established as follows: We consider only the first coordinate; the case of the second coordinate is established analogously.

Thus we have that condition (b) is satisfied and since s is a map from uP_X to uP_X , we conclude that $s \in \mathcal{P}$ and s is a lower bound of the chain \mathcal{K} by construction.

We therefore appeal to Zorn's lemma to conclude that \mathcal{P} has a minimal element, say m , with respect to the partial order \preceq .

We show that $m(f) \in uQ_X$ whenever $f \in uP_X$.

For each $x \in X$, we have that $p_x \circ m \in \mathcal{P}$ and $p_x \circ m \preceq m$ (where p_x is as defined in the proof of Proposition 3.2.1). Thus by minimality of m , we have $p_x \circ m = m$. It therefore follows that for each $x \in X$, $p_x(m(f)) = m(f)$ whenever $f \in uP_X$. Thus by the way elements in uQ_X are defined, we conclude that $m(f) \in uQ_X$ whenever $f \in uP_X$ (compare [25, Remark 2]).

3.3.2 Proposition. (compare [1, Proposition 3]) Let (Y, d) be a T_0 -ultra-quasi-metric space and $\emptyset \neq X$ be a subspace of (Y, d) . Then there exists an isometric embedding $\tau : uQ_X \rightarrow uQ_Y$ such that $\tau(f)|_X = f$ whenever $f \in uQ_X$.

Proof.

Let us fix $x_0 \in X$ and choose a retraction $p : uP_Y \rightarrow uQ_Y$ satisfying the conditions of Proposition 2.2.1. Also let $s : uQ_X \rightarrow P_Y$ be defined as $s(f) = f'$ where $f'_1(y) = f_1(y)$ whenever $y \in X$, and $f'_1(y) = \max\{f_1(x_0), d(x_0, y)\}$ whenever $y \in Y \setminus X$. The coordinate f'_2 of the pair f' is defined similarly.

We shall consider the following cases to prove that f' belongs to uP_Y :

Case 1: $x \in X$ and $y \in X$.

Then $\max\{f'_2(x), f'_1(y)\} = \max\{f_2(x), f_1(y)\} \geq d(x, y)$.

Case 2: $x \in Y \setminus X$ and $y \in Y \setminus X$.

Then $\max\{f'_2(x), f'_1(y)\} = \max\{f_2(x_0), f_1(x_0), d(x, x_0), d(x_0, y)\} \geq \max\{d(x, x_0), d(x_0, y)\} \geq d(x, y)$.

Case 3: $x \in X$ and $y \in Y \setminus X$.

Then $\max\{f'_2(x), f'_1(y)\} = \max\{f_2(x), f_1(x_0), d(x_0, y)\} \geq \max\{d(x, x_0), d(x_0, y)\} \geq d(x, y)$.

Case 4: $x \in Y \setminus X$ and $y \in X$.

Then $\max\{f'_2(x), f'_1(y)\} = \max\{f_2(x_0), f_1(y), d(x, x_0)\} \geq \max\{d(x, x_0), d(x_0, y)\} \geq d(x, y)$.

Thus $f' \in uP_Y$.

Define the map $\tau = p \circ s$. Then $\tau(f)|_X = p(f')|_X = f$ whenever $f \in uQ_X$ since $p(f') \leq f'$, thus $p(f')|_X \leq f'|_X = f$, and f is minimal on X .

Moreover for any $f, g \in uQ_X$, we have

$$\begin{aligned} N(f, g) &= N(\tau(f)|_X, \tau(g)|_X) \\ &\leq N(\tau(f), \tau(g)) \\ &= N(p(f'), p(g')) \\ &\leq N(f', g') \\ &= N(f, g). \end{aligned}$$

The last equality follows from the definition of f' and g' .

□

3.3.1 Definition. (compare [20, Remark 7]) Let X be a subspace of a T_0 -ultra-quasi-metric space (Y, d_Y) . Then (Y, d_Y) is called a ***uq-tight extension*** of X if for any ultra-quasi-pseudometric ρ on Y that satisfies $\rho \leq d_Y$ and agrees with d_Y on $X \times X$, we have that $\rho = d_Y$.

3.3.2 Remark. For any T_0 -ultra-quasi-metric *uq-tight* extension Y_1 of X , any T_0 -ultra-quasi-metric extension (Y_2, d) of X and any non-expansive map $\varphi : Y_1 \rightarrow Y_2$ satisfying $\varphi(x) = x$ whenever $x \in X$ must necessarily be an isometric map.

Indeed if that is not the case then the ultra-quasi-pseudometric $\rho : Y_1 \times Y_1 \rightarrow [0, \infty)$ defined by $(x, y) \mapsto \rho(x, y) = d(\varphi(x), \varphi(y))$ would contradict the *uq-tightness* of the extension Y_1 of X .

It was shown in [23, Theorem 1] that the map $e_X : (X, d) \rightarrow (uQ_X, N)$ from

a T_0 -ultra-quasi-metric space (X, d) to its ultra-quasi-metrically injective hull (uQ_X, N) defined by $e_X(a) = f_a$ whenever $a \in X$ is an isometric embedding. We shall proceed now with the help of Lemma 3.3.1 to show that the embedding is *uq-tight*, that is, uQ_X is a *uq-tight* extension of $e_X(X)$.

3.3.3 Proposition. Let (X, d) be a T_0 -ultra-quasi-metric space and $e_X : X \rightarrow uQ_X$ be as defined above. Then uQ_X is a *uq-tight* extension of $e_X(X)$.

Proof.

Let ρ be an ultra-quasi-pseudometric on uQ_X such that $\rho \leq N$ and $\rho(f_x, f_y) = N(f_x, f_y)$ whenever $x, y \in X$. By Lemma 3.3.1 and the fact that $\rho \leq N$, for any $f, g \in uQ_X$, we have

$$\begin{aligned} N(f, g) &= \sup_{x_1, x_2 \in X} \{N(f_{x_1}, f_{x_2}) : N(f_{x_1}, f_{x_2}) > N(f_{x_1}, f), N(g, f_{x_2})\} \\ &\leq \sup_{x_1, x_2 \in X} \{\rho(f_{x_1}, f_{x_2}) : \rho(f_{x_1}, f_{x_2}) > \rho(f_{x_1}, f), \rho(g, f_{x_2})\} \\ &\leq \rho(f, g), \text{ since } \rho(f_{x_1}, f_{x_2}) \leq \max\{\rho(f_{x_1}, f), \rho(f, g), \rho(g, f_{x_2})\}. \end{aligned}$$

Thus $\rho = N$.

□

3.3.4 Proposition. (compare [11, Section 1.13]) Let (Y, d) be a T_0 -ultra-quasi-metric *uq-tight* extension of X . Then the restriction map defined by $f \mapsto f|_X$ whenever $f \in uQ_Y$ is a bijective isometric map $uQ_Y \rightarrow uQ_X$.

Proof.

Let us choose a retraction map $p : uP_X \rightarrow uQ_X$ that satisfies the conditions of Proposition 2.2.1 and let $\varphi : uQ_Y \rightarrow uQ_X : f \mapsto p(f)|_X$ denote the composition of the retraction map p with the restriction map. It is easy to check that φ is non-expansive. Thus by Lemma 3.3.2, φ must be an isometry, because uQ_Y is a *uq-tight* extension of X (this is so since uQ_Y is a *uq-tight* extension of Y and Y is a *uq-tight* extension of X).

We can find an isometric embedding $\tau : uQ_X \rightarrow uQ_Y$ such that $\tau(f)|_X = f$ for every $f \in uQ_X$ (compare Proposition 3.3.2). We therefore have

$$\varphi(\tau(f)) = p(\tau(f)|_X) = p(f) = f \text{ for every } f \in uQ_X.$$

This implies that φ is surjective. Injectivity of φ is clear since uQ_X is a T_0 -ultra-quasi-metric space (compare Corollary 3.2.1). Thus φ is bijective. In this case, the inverse of φ has to be the inverse of τ and hence for any $f \in uQ_Y$, we have $f|_X = \tau(\varphi(f))|_X = \varphi(f) \in uQ_X$, that is the map

$$uQ_Y \rightarrow uP_X : f \mapsto f|_X$$

maps uQ_Y onto uQ_X , without it being composed with p . Hence for any T_0 -ultra-quasi-metric uq -tight extension Y of X , the map

$$uQ_Y \rightarrow uQ_X : f \mapsto f|_X$$

is a bijective isometry between uQ_X and uQ_Y .

3.3.1 Theorem. (compare [1, Proposition 5]) Let X be a subspace of the T_0 -ultra-quasi-metric space (Y, d) . Then the following are equivalent:

- (a) Y is a T_0 -ultra-quasi-metric uq -tight extension of X .
- (b) $d(y_1, y_2) = \sup\{d(x_1, x_2) : x_1, x_2 \in X, d(x_1, x_2) > d(x_1, y_1), d(x_1, x_2) > d(y_2, x_2)\}$ whenever $y_1, y_2 \in Y$.
- (c) $f_y|_X(x) = (d(y, x), d(x, y))$, $x \in X$ is ultra-minimal on X whenever $y \in Y$ and the map $(Y, d) \rightarrow (uQ_X, N)$ defined by $y \mapsto f_y|_X$ is an isometric embedding.

Proof.

(a) \Rightarrow (b)

Let Y be a T_0 -ultra-quasi-metric uq -tight extension of X . By Proposition 3.3.4, the restriction map $uQ_Y \rightarrow uQ_X$ is a bijective isometry between uQ_Y and uQ_X .

Thus the extension $Y \subseteq uQ_Y$ satisfies condition (b), since uQ_X satisfies it by [23, Lemma 8].

(b) \Rightarrow (c)

Let $x_1, x_2 \in X$ and $y_1 \in Y$. Then we have that $d(x_1, x_2) \leq \max\{d(x_1, y_1), d(y_1, x_2)\}$.

Thus by condition (b) we have that $d(x_1, x_2) \leq d(y_1, x_2)$. Also

$$d(x_1, x_2) \leq n(d(y_1, x_2), d(y_2, x_2)) \leq d(y_1, y_2).$$

Consequently for $y_1, y_2 \in Y$ we have by (b) that

$$\begin{aligned} d(y_1, y_2) &= \sup\{d(x_1, x_2) : x_1, x_2 \in X, d(x_1, x_2) > d(x_1, y_1), d(x_1, x_2) > d(y_2, x_2)\} \\ &\leq \sup\{d(y_1, x_2) : x_2 \in X, d(y_1, x_2) > d(y_2, x_2)\} \\ &\leq d(y_1, y_2). \end{aligned}$$

Similarly we have that $d(x_1, x_2) \leq \max\{d(x_1, y_2), d(y_2, x_2)\}$ whenever $x_1, x_2 \in X$ and $y_2 \in Y$ so that by condition (b) we get $d(x_1, x_2) \leq d(x_1, y_2)$. It therefore follows that for each $x_1, x_2 \in X$ and $y_1, y_2 \in Y$,

$$d(x_1, x_2) \leq n(d(x_1, y_2), d(x_1, y_1)) \leq d(y_1, y_2).$$

Thus for $y_1, y_2 \in Y$ we see by (b) that

$$\begin{aligned} d(y_1, y_2) &= \sup\{d(x_1, x_2) : x_1, x_2 \in X, d(x_1, x_2) > d(x_1, y_1), d(x_1, x_2) > d(y_2, x_2)\} \\ &\leq \sup\{d(x_1, y_2) : x_1 \in X, d(x_1, y_2) > d(x_1, y_1)\} \\ &\leq d(y_1, y_2). \end{aligned}$$

Thus we conclude that $d(y_1, y_2) = N(f_{y_1}|_X, f_{y_2}|_X)$.

As we have above, for any $y_1, y_2 \in Y$ we have that

$$d(y_1, y_2) = \sup\{d(y_1, x_2) : x_2 \in X, d(y_1, x_2) > d(y_2, x_2)\}$$

and

$$d(y_1, y_2) = \sup\{d(x_1, y_2) : x_1 \in X, d(x_1, y_2) > d(x_1, y_1)\}.$$

Observe that if we substitute $x_1 \in X$ for y_1 and $x_2 \in X$ for y_2 , respectively, we obtain the following equations

$$(f_{y_1})_1(x_2) = d(y_1, x_2) = \sup\{d(x_1, x_2) : x_1 \in X \text{ and } d(x_1, x_2) > d(x_1, y_1)\}$$

whenever $y_1 \in Y$ and $x_2 \in X$

and

$$(f_{y_2})_2(x_1) = d(x_1, y_2) = \sup\{d(x_1, x_2) : x_2 \in X \text{ and } d(x_1, x_2) > d(y_2, x_2)\}$$

whenever $y_2 \in Y$ and $x_1 \in X$. We have therefore that the restriction $f_y|_X$ is ultra-minimal on X whenever $y \in Y$.

(c) \Rightarrow (a)

Let ρ be an ultra-quasipseudometric on Y such that $\rho(y_1, y_2) \leq d(y_1, y_2)$ whenever $y_1, y_2 \in Y$ and $\rho(x_1, x_2) = d(x_1, x_2)$ whenever $x_1, x_2 \in X$. Then according to part (c) and the fact that $f_y|_X$ is ultra-minimal whenever $y \in X$, we have

$$\begin{aligned} d(y_1, y_2) &= N(f_{y_1}|_X, f_{y_2}|_X) \\ &= \sup\{d(y_1, x) : x \in X, d(y_1, x) > d(x_1, y_1), d(y_1, x) > d(y_2, x_2)\}. \end{aligned}$$

By substituting

$$d(x_1, y_2) = \sup\{d(x_1, x_2) : x_2 \in X \text{ and } d(x_1, x_2) > d(y_2, x_2)\}$$

into the formula

$$d(y_1, y_2) = \sup\{d(x_1, y_2) : x_1 \in X \text{ and } d(x_1, y_2) > d(x_1, y_1)\}$$

we obtain

$$\begin{aligned} d(y_1, y_2) &= \sup\{d(x_1, y_2) : x_1 \in X \text{ and } d(x_1, y_2) > d(x_1, y_1)\} \\ &= \sup\{d(x_1, x_2) : x_1, x_2 \in X \text{ and } d(x_1, y_2) > d(x_1, y_1), d(x_1, x_2) > d(y_2, x_2)\} \\ &\leq \sup\{\rho(x_1, x_2) : x_1, x_2 \in X \text{ and } \rho(x_1, y_2) > \rho(x_1, y_1), \rho(x_1, x_2) > \rho(y_2, x_2)\} \end{aligned}$$

whenever $y_1, y_2 \in Y$. Thus (a) follows. □

3.3.3 Remark. We see from Theorem 3.3.1 that there is only one isometric embedding $\varphi : Y \rightarrow uQ_X$ satisfying $\varphi(x) = f_x$ whenever $x \in X$, since for such an embedding we have

$$(f_y)_2|_X(x) = d(x, y) = N(\varphi(x), \varphi(y)) = N(f_x, \varphi(y)) = (\varphi(y))_2(x);$$

therefore $(f_y)_2|_X = (\varphi(y))_2$. Similarly, one can show that $(f_y)_1|_X = (\varphi(y))_1$ whenever $y \in Y$.

Thus we see that the uq -tight extension Y of X can be understood as a subspace of the extension uQ_X of X . Hence uQ_X is maximal among the T_0 -ultra-quasi-metric uq -tight extensions of X .

3.3.4 Remark. Bayod proved in [5] that if X is a compact ultrametric space, there are no proper ultrametrically tight extensions of X . They also proved that for a compact ultrametric space, ultrametric injectivity and spherical completeness are the same. Recall that an ultrametric space is said to be spherically complete if every collection of closed balls with the binary intersection property has a nonempty intersection.

We end this chapter with the following example.

3.3.1 Example. Let $X = \{0, 1\}$ be equipped with the discrete metric d defined by $d(x, x) = 0$ whenever $x \in X$ and $d(x, y) = 1$ whenever $x \neq y$. We have by [20, Proposition 5] that the (metric) hyperconvex hull T_X of the metric space (X, d) is isometric to the metric subspace of function pairs (f_1, f_2) satisfying $f_1 = f_2$.

The Isbell-hull Q_X consists of exactly the function pairs defined by : for each $(u, v) \in [0, 1] \times [0, 1]$, define the function pair $(u, v) = ((u, v)_1, (u, v)_2)$ as follows: $(u, v)_1(0) = u$, $(u, v)_1(1) = v$, $(u, v)_2(0) = 1 - v$, $(u, v)_2(1) = 1 - u$. Moreover for

each $(u, v), (u', v') \in [0, 1] \times [0, 1]$, one can define

$$D((u, v), (u', v')) = (u \dot{-} u') \vee (v \dot{-} v').$$

This shows that the Isbell-hull Q_X of (X, d) can be identified with $([0, 1] \times [0, 1], D)$. It is thus clear that Q_X is larger than T_X .

The ultra-quasi-metrically injective hull uQ_X consists of the four pairs $((f_1(0), f_1(1)), (f_2(0), f_2(1)))$ determined as follows:

$$((0, 1), (0, 1)), ((1, 1), (0, 0)), ((0, 0), (1, 1)), ((1, 0), (1, 0))$$

(compare [23, Example 3]).

If we identify the points (f_1, f_2) according to $(f_1(0), f_1(1)) = (u, v)$ with $u, v \in X$, we get $N((u, v), (u', v')) = 1$ if $(u, u') = (1, 0)$ or $(v, v') = (1, 0)$ and $N((u, v), (u', v')) = 0$ otherwise.

Thus in summary, for the above metric space (X, d) , Q_X is the unit square with the max metric, T_X is the diagonal (this is the metric hyperconvex hull, isometric to the unit interval), uQ_X is the space $\{(0, 0), (0, 1), (1, 0), (1, 1)\}$ and the ultrametric hull consists of the points $(0, 1)$ and $(1, 0)$.

Chapter 4

The space of minimal and Katětov function pairs

Extremal function pairs were introduced by Kemajou et al. in [20]. Sanchis et al. in [25] introduced the notion of Katětov function pairs that is appropriate in the category of T_0 -quasi-metric spaces and non-expansive maps. They used these Katětov function pairs for the construction of a universal space in the category of T_0 -quasi-metric spaces and non-expansive maps. This was done by modifying a construction due to Katětov (compare [18]). For the definition of Katětov function pair, see Definition 5.1.1. It is known that extremal function pairs are also Katětov as we shall see in Proposition 4.1.1.

In this chapter we shall try to understand several aspects of q -hyperconvex hulls of T_0 -quasi-metric spaces. In particular we wish to study supseparability of Q_X for a T_0 -quasi-metric space X . For instance it is known that if X is a joincompact T_0 -quasi-metric space, then Q_X is joincompact. Thus in this way Q_X will be supseparable. It is also known that for any T_0 -quasi-metric space X , there exists an isometric embedding $X \hookrightarrow Q_X$.

During our investigations we shall introduce two new classes of function pairs namely the minimum and maximum function pairs. This will be possible with the help of Inequalities 4.1 and 4.2 used in Definition 5.1.1. We will determine sufficient conditions under which Q_X will be supseparable.

We wish to remark here that the treatment of the material in this chapter is motivated by [10] which deals with the metric case.

4.1 Katětov function pairs

4.1.1 Definition. Let (X, d) be a T_0 -quasi-metric space. We shall say that a function pair on (X, d) is **Katětov** if it satisfies the following inequalities:

$$d(x, y) \leq f_2(x) + f_1(y) \text{ whenever } x, y \in X, \quad (4.1)$$

$$f_1(x) \dot{-} f_1(y) \leq d(y, x) \text{ and } f_2(x) \dot{-} f_2(y) \leq d(x, y) \text{ whenever } x, y \in X. \quad (4.2)$$

We shall denote the set of all Katětov function pairs on (X, d) by $Q(X, d)$ (or simply $Q(X)$ when there is no confusion on d).

For each $f, g \in Q(X)$ we set

$$D(f, g) = \sup_{x \in X} \{f_1(x) \dot{-} g_1(x)\} \vee \sup_{x \in X} \{g_2(x) \dot{-} f_2(x)\}.$$

Then D is an extended T_0 -quasi-metric on $Q(X)$.

Notice from Definition 5.1.1 that Inequality (4.1) simply says that f is ample. It is clear that all extremal function pairs are Katětov. We state this as a proposition.

4.1.1 Proposition. Let (X, d) be a T_0 -quasi-metric space. Then $Q_X \subseteq Q(X)$.

Proof.

Suppose $f \in Q_X$. Then it is clear from the definition of Q_X that f is ample. It remains only to show that f satisfies inequality (4.2), i.e., we show that for every $x, y \in X$, $f_1(x) \dot{-} f_1(y) \leq d(y, x)$ and $f_2(x) \dot{-} f_2(y) \leq d(x, y)$.

Suppose on the contrary that for some $x, y \in X$, we had $f_2(x) \dot{-} f_2(y) > d(x, y)$. Then for all $y_0 \in X$, $f \in Q_X$ implies that $d(y, y_0) \leq f_2(y) + f_1(y_0)$. We therefore have that

$$\begin{aligned} f_2(x) + f_1(y) &\geq f_2(x) + d(y, y_0) \dot{-} f_2(y) \\ &= d(x, y) + d(y, y_0) + [(f_2(x) \dot{-} f_2(y)) - d(x, y)] \\ &\geq d(x, y_0) + [(f_2(x) \dot{-} f_2(y)) - d(x, y)] \end{aligned}$$

Thus we have that $f_2(x) > \sup\{d(x, y_0) \dot{-} f_1(y_0) : y_0 \in X\}$ which contradicts Definition 5.1.1. Thus we must have that $f_2(x) \dot{-} f_2(y) \leq d(x, y)$.

Similarly one can show that $f_1(x) \dot{-} f_1(y) \leq d(y, x)$. Thus we have that $f \in Q(X)$.

□

Next we introduce some notions which are needed in the sequel. Recall from [1, Lemma 3.4] that $f \in Q_X$ if and only if

$$f_1(x) = \sup\{d(y, x) \dot{-} f_2(y) : y \in X\} \tag{4.3}$$

and

$$f_2(x) = \sup\{d(x, y) \dot{-} f_1(y) : y \in X\} \tag{4.4}$$

whenever $x \in X$. If there exists a finite subset Y of X such that Equations 4.4 and 4.3 holds, then we obtain the concept of a finitely witnessed function pair (see Definition 4.1.2 below) and in this case the supremum is replaced by the maximum.

4.1.2 Definition. (Compare [10, Definition 2.2]) Let (X, d) be a T_0 -quasi-metric space. A function pair $f \in Q_X$ is said to be **finitely witnessed** if there exists a

finite set $Y \subseteq X$ such that for every $x \in X$,

$$f_1(x) = \max\{d(y, x) + f_2(y) : y \in Y\}$$

and

$$f_2(x) = \max\{d(x, y) + f_1(y) : y \in Y\}.$$

The set Y is called a **witness set** of the pair f and we say that f is n -witnessed if the cardinality of the set Y is n . We shall denote by Q_X^ω the set of all finitely witnessed function pairs in $Q(X)$ and Q_X^n the set of all finitely n -witnessed function pairs in $Q(X)$.

Using Inequalities 4.2 one can similarly define the concept of finitely supported function pair as in Definition 4.1.3 below.

4.1.3 Definition. (Compare [10, Definition 2.1]) Let (X, d) be a T_0 -quasi-metric space. A function pair $f \in Q_X$ is said to be **finitely supported** if there exists a finite set $Y \subseteq X$ such that for every $x \in X$,

$$f_1(x) = \min\{d(y, x) + f_1(y) : y \in Y\}$$

and

$$f_2(x) = \min\{d(x, y) + f_2(y) : y \in Y\}.$$

The set Y is called a **support** of the pair f and we say that f is n -supported if the cardinality of the set Y is n . We shall denote by $Q(X, \omega)$ the set of all finitely supported function pairs in $Q(X)$ and $Q(X, n)$ the set of all finitely n -supported function pairs in $Q(X)$.

Using these concepts of finitely supported and finitely witnessed function pairs, we now introduce two new classes of function pairs which are generalizations of the finitely supported and finitely witnessed function pairs.

4.1.4 Definition. Let (X, d) be a T_0 -quasi-metric space and $F \subseteq X$ be finite. Let $\alpha_i : F \rightarrow [0, \infty)$, $i = 1, 2$ be function pairs. We define the **minimum function**

pair on (F, α) as

$$(m_{F,\alpha})_1(x) = \min\{d(y, x) + \alpha_1(y) : y \in F\}$$

and

$$(m_{F,\alpha})_2(x) = \min\{d(x, y) + \alpha_2(y) : y \in F\}.$$

4.1.5 Definition. Let (X, d) be a T_0 -quasi-metric space and $F \subseteq X$ be finite. Let $\alpha_i : F \rightarrow [0, \infty)$, $i = 1, 2$ be a function pair. We shall define the **maximum function pair** on (F, α) as

$$(M_{F,\alpha})_1(x) = \max\{d(y, x) + \alpha_2(y) : y \in F\}$$

and

$$(M_{F,\alpha})_2(x) = \max\{d(x, y) + \alpha_1(y) : y \in F\}.$$

We shall denote the set of all minimum function pairs by

$$m(X, \omega) := \bigcup_n m(X, n)$$

and the set of all maximum pairs by

$$M(X, \omega) := \bigcup_n M(X, n)$$

where $m(X, n)$ and $M(X, n)$ are defined by

$$\begin{aligned} m(X, n) &:= \{m_{F,\alpha} : F \subseteq X, |F| = n, \alpha : F \rightarrow [0, \infty)\}, \\ M(X, n) &:= \{M_{F,\alpha} : F \subseteq X, |F| = n, \alpha : F \rightarrow [0, \infty)\}. \end{aligned}$$

We see from the above definitions that

$$Q(X, \omega) \subseteq m(X, \omega) \text{ and } Q_X^\omega \subseteq M(X, \omega).$$

This is clear since

$$Q(X, n) \subseteq m(X, n) \text{ and } Q_X^n \subseteq M(X, n).$$

It becomes natural to ask the following questions:

- Under what conditions is a minimum pair (maximum pair) Katětov?
- Under what conditions is a minimum pair (maximum pair) extremal?
- If a minimum pair (maximum pair) is Katětov, is it automatically finitely supported (finitely witnessed)?

To answer the above questions, we shall state and prove a series of propositions and lemmas.

4.1.2 Proposition. Let (X, d) be a T_0 -quasi-metric space, $F \subseteq X$ finite and $\alpha : F \rightarrow [0, \infty)$.

- (1) $m_{F,\alpha} \in Q(X)$ if and only if for every $y, y_0 \in F$, $d(y, y_0) \leq \alpha_2(y) + \alpha_1(y_0)$;
- (2) $M_{F,\alpha} \in Q(X)$ if and only if for every $x, x_0 \in X$, there exists $y, y_0 \in F$ such that $\alpha_2(y) + \alpha_1(y_0) \leq (d(y, x) + d(x_0, y_0)) \dot{-} d(x, x_0)$;
- (3) $m_{F,\alpha} \in Q_X$ if and only if $m_{F,\alpha} \in Q(X)$ and $m_{F,\alpha} = M_{F,\alpha}$;
- (4) $M_{F,\alpha} \in Q_X$ if and only if $M_{F,\alpha} \in Q(X)$ and for every $y, y_0 \in F$ $d(y, y_0) \leq \alpha_2(y) + \alpha_1(y_0)$.

4.1.3 Proposition. Let (X, d) be a T_0 -quasi-metric space, $F \subseteq X$ finite and $\alpha : F \rightarrow [0, \infty)$.

- (1) $m_{F,\alpha} \in Q(X)$ implies $m_{F,\alpha} \in Q(X, |F|)$ and F is a support of $m_{F,\alpha}$;
- (2) $M_{F,\alpha} \in Q_X$ implies $M_{F,\alpha} \in Q_X^{|F|}$ and F is a witness set of $M_{F,\alpha}$.

Thus for any n , $Q(X, n) = Q(X) \cap m(X, n)$ and $Q_X^n = Q_X \cap M(X, n)$, so

$$Q(X, \omega) = Q(X) \cap m(X, \omega)$$

and

$$Q_X^\omega = Q_X \cap M(X, \omega).$$

We will now state and prove the following lemmas which shall account for the proofs of Propositions 4.1.2 and 4.1.3.

4.1.1 Lemma. Let (X, d) be a T_0 -quasi-metric space, $F \subseteq X$ finite and $\alpha : F \rightarrow [0, \infty)$. For every $x, x_0 \in X$,

$$(a) (m_{F,\alpha})_1(x) \dot{-} (m_{F,\alpha})_1(x_0) \leq d(x_0, x) \text{ and } (m_{F,\alpha})_2(x) \dot{-} (m_{F,\alpha})_2(x_0) \leq d(x, x_0);$$

$$(b) (M_{F,\alpha})_1(x) \dot{-} (M_{F,\alpha})_1(x_0) \leq d(x_0, x) \text{ and } (M_{F,\alpha})_2(x) \dot{-} (M_{F,\alpha})_2(x_0) \leq d(x, x_0).$$

Proof.

We shall prove only part (a) since the proof of part (b) is similar.

Let $x, x_0 \in X$ fixed and let $y_0, y \in F$ be such that

$$(m_{F,\alpha})_1(x) = d(y, x) + \alpha_1(y) \text{ and } (m_{F,\alpha})_1(x_0) = d(y_0, x_0) + \alpha_1(y_0)$$

and

$$(m_{F,\alpha})_2(x) = d(x, y) + \alpha_2(y) \text{ and } (m_{F,\alpha})_2(x_0) = d(x_0, y_0) + \alpha_2(y_0).$$

We have that

$$\begin{aligned} d(y, x) + \alpha_1(y) &= (m_{F,\alpha})_1(x) \\ &= \min\{d(y_0, x) + \alpha_1(y_0) : y_0 \in F\} \\ &\leq d(y_0, x) + \alpha_1(y_0) \end{aligned}$$

so that

$$\begin{aligned} (m_{F,\alpha})_1(x) \dot{-} (m_{F,\alpha})_1(x_0) &= (d(y, x) + \alpha_1(y)) \dot{-} (d(y_0, x_0) + \alpha_1(y_0)) \\ &\leq (d(y_0, x) + \alpha_1(y_0)) \dot{-} (d(y_0, x_0) + \alpha_1(y_0)) \\ &\leq d(x_0, x). \end{aligned}$$

Similarly, we have that

$$\begin{aligned}
d(x, y) + \alpha_2(y) &= (m_{F,\alpha})_2(x) \\
&= \min\{d(x, y_0) + \alpha_2(y_0) : y_0 \in F\} \\
&\leq d(x, y_0) + \alpha_2(y_0)
\end{aligned}$$

so that

$$\begin{aligned}
(m_{F,\alpha})_2(x) - (m_{F,\alpha})_2(x_0) &= (d(x, y) + \alpha_2(y)) - (d(x_0, y_0) + \alpha_2(y_0)) \\
&\leq (d(x, y_0) + \alpha_2(y_0)) - (d(x_0, y_0) + \alpha_2(y_0)) \\
&\leq d(x, x_0).
\end{aligned}$$

□

4.1.2 Lemma. Let (X, d) be a T_0 -quasi-metric space, $F \subseteq X$ finite and $\alpha : F \rightarrow [0, \infty)$. Then the minimum function pair $m_{F,\alpha}$ is ample if and only if the function pair α is ample.

Proof.

We have to show that for every $x, x_0 \in X$,

$$d(x, x_0) \leq (m_{F,\alpha})_2(x) + (m_{F,\alpha})_1(x_0)$$

if and only if for every $y, y_0 \in F$,

$$d(y, y_0) \leq \alpha_2(y) + \alpha_1(y_0).$$

(\implies)

Suppose that for all $x, x_0 \in X$, $d(x, x_0) \leq (m_{F,\alpha})_2(x) + (m_{F,\alpha})_1(x_0)$, then we must have in particular that $d(y, y_0) \leq (m_{F,\alpha})_2(y) + (m_{F,\alpha})_1(y_0)$ for all $y, y_0 \in F$. Thus

we have that

$$\begin{aligned}
(m_{F,\alpha})_1(y_0) &= \min\{d(y, y_0) + \alpha_1(y) : y \in F\} \\
&\leq d(y_0, y_0) + \alpha_1(y_0) \\
&= \alpha_1(y_0).
\end{aligned}$$

Similarly we have

$$\begin{aligned}
(m_{F,\alpha})_2(y) &= \min\{d(y, y_0) + \alpha_2(y_0) : y_0 \in F\} \\
&\leq d(y, y) + \alpha_2(y) \\
&= \alpha_2(y).
\end{aligned}$$

We have therefore that for all $y, y_0 \in F$,

$$d(y, y_0) \leq (m_{F,\alpha})_2(y) + (m_{F,\alpha})_1(y_0) \leq \alpha_2(y) + \alpha_1(y_0)$$

and the result holds.

(\Leftarrow)

Suppose now that $d(y, y_0) \leq \alpha_2(y) + \alpha_1(y_0)$ for all $y, y_0 \in F$. Fix $x, x_0 \in X$. By the definition of $m_{F,\alpha}$, we may fix $y, y_0 \in F$ such that

$$(m_{F,\alpha})_1(x_0) = d(y_0, x_0) + \alpha_1(y_0) \text{ and } (m_{F,\alpha})_2(x) = d(x, y) + \alpha_2(y).$$

Then

$$\begin{aligned}
(m_{F,\alpha})_2(x) + (m_{F,\alpha})_1(x_0) &= d(x, y) + \alpha_2(y) + d(y_0, x_0) + \alpha_1(y_0) \\
&\geq d(x, y) + d(y_0, x_0) + d(y, y_0) \\
&\geq d(x, x_0).
\end{aligned}$$

□

It turns out that for $M_{F,\alpha}$ to satisfy Inequality 4.1, the condition on the function pair α is different. We shall see this in the next lemma.

4.1.3 Lemma. Let (X, d) be a T_0 -quasi-metric space, $F \subseteq X$ finite and $\alpha : F \rightarrow [0, \infty)$. For every $x, x_0 \in X$,

$$d(x, x_0) \leq (M_{F,\alpha})_2(x) + (M_{F,\alpha})_1(x_0)$$

if and only if for every $x, x_0 \in X$, there exists $y, y_0 \in F$ such that

$$\alpha_2(y) + \alpha_1(y_0) \leq (d(y, x) + d(x_0, y_0)) \dot{-} d(x, x_0).$$

Proof.

$$\forall x, x_0 \in X, (M_{F,\alpha})_2(x) + (M_{F,\alpha})_1(x_0) \geq d(x, x_0)$$

\Updownarrow

$$\forall x, x_0 \in X, \exists y, y_0 \in F : (d(y, x) \dot{-} \alpha_2(y)) + (d(x_0, y_0) \dot{-} \alpha_1(y_0)) \geq d(x, x_0)$$

\Updownarrow

$$\forall x, x_0 \in X, \exists y, y_0 \in F : (d(y, x) + d(x_0, y_0)) \dot{-} d(x, x_0) \geq \alpha_2(y) + \alpha_1(y_0).$$

□

4.1.1 Remark. Notice that Lemmas 4.1.1, 4.1.2 and 4.1.3 state exactly when minimum function pairs and maximum function pairs are Katětov, thus proving parts (1) and (2) of Proposition 4.1.2. In the next lemma we shall show conditions under which a maximum function pair is extremal, and in that case the defining set F is a witness set, thus proving part (4) in Proposition 4.1.2 and part (2) of Proposition 4.1.3.

4.1.4 Lemma. Let (X, d) be a T_0 -quasi-metric space, $F \subseteq X$ finite and $\alpha : F \rightarrow [0, \infty)$. Then $M_{F,\alpha}$ is extremal if and only if $M_{F,\alpha}$ is Katětov and α is ample on F .

Moreover, if $M_{F,\alpha} \in Q(X)$, then F is a witness set of $M_{F,\alpha}$.

Proof.

(\implies)

Assume that $M_{F,\alpha}$ is extremal. We shall prove this direction by the contrapositive, i.e., if $M_{F,\alpha} \notin Q(X)$ then $M_{F,\alpha} \notin Q_X$ since $Q_X \subseteq Q(X)$.

Suppose now that $M_{F,\alpha}$ is not Katětov and that there exists $y, y_0 \in F$ such that $\alpha_2(y) + \alpha_1(y_0) < d(y, y_0)$.

Claim: $M_{F,\alpha}$ is not extremal at y .

Let $x \in X$. Then by the definition of the maximum function pair, we have that

$$(M_{F,\alpha})_2(x) \geq d(x, y) \dot{-} \alpha_1(y)$$

so that

$$(M_{F,\alpha})_2(x) + (M_{F,\alpha})_1(y) \geq d(x, y) \dot{-} \alpha_1(y) + (M_{F,\alpha})_1(y).$$

We also know from the definition of the maximum pair that

$$(M_{F,\alpha})_1(y) \geq d(x, y) \dot{-} \alpha_2(y) > \alpha_1(y_0).$$

Thus $(M_{F,\alpha})_1(y) \dot{-} \alpha_1(y_0) > 0$.

This proves that $M_{F,\alpha}$ is not minimal at y and hence $M_{F,\alpha} \notin Q_X$.

(\impliedby)

To show that $M_{F,\alpha}$ is extremal, we shall now show that for all $x \in X$,

$$(M_{F,\alpha})_2(x) = \max\{d(x, y) \dot{-} (M_{F,\alpha})_1(y) : y \in F\}.$$

The above equation together with the fact that $M_{F,\alpha} \in Q(X)$ implies that $M_{F,\alpha} \in Q_X$ and F is a witness set of $M_{F,\alpha}$. A similar argument can be used to show that

$$(M_{F,\alpha})_1(x) = \max\{d(y, x) \dot{-} (M_{F,\alpha})_2(y) : y \in F\}.$$

$M_{F,\alpha} \in Q(X)$ implies that for every $x \in X$,

$$(M_{F,\alpha})_2(x) \geq \max\{d(x, y) \dot{-} (M_{F,\alpha})_1(y) : y \in F\}.$$

Indeed if $M_{F,\alpha} \in Q(X)$, then $M_{F,\alpha}$ is ample, i.e., $(M_{F,\alpha})_2(x) + (M_{F,\alpha})_1(y) \geq d(x, y)$, and we have from the above inequality that $(M_{F,\alpha})_2(x) \geq d(x, y) \dot{-} (M_{F,\alpha})_1(y)$. Thus the conclusion follows by taking the maximum over all $y \in F$ since F is finite.

By our assumption, we have $d(y, y_0) \leq \alpha_2(y) + \alpha_1(y_0)$ for all $y, y_0 \in F$ implies that $(M_{F,\alpha})_1(y) \leq \alpha_1(y)$ for all $y \in F$. Hence for all $x \in X$,

$$(M_{F,\alpha})_2(x) = \max\{d(x, y) \dot{-} \alpha_1(y) : y \in F\} \leq \max\{d(x, y) \dot{-} (M_{F,\alpha})_1(y) : y \in F\}.$$

□

To complete the proofs of Proposition 4.1.2 and 4.1.3, we shall need the following lemma.

4.1.5 Lemma. (Compare [2, Lemma 2]) Let (X, d) be a joincompact T_0 -quasi-metric space and $f \in Q_X$. Given $x \in X$ such that $f_2(x) > 0$ there is a $y \in X$ such that $d(x, y) = f_2(x) + f_1(y)$.

Similarly for each $x \in X$ such that $f_1(x) > 0$ there is a $y_0 \in X$ such that $d(y_0, x) = f_2(y_0) + f_1(x)$.

4.1.1 Corollary. (Compare [2, Corollary 1]) Let (X, d) be a T_0 -quasi-metric space, $f \in Q_X$, $\epsilon > 0$ and $x \in X$. Suppose that $f_2(x) > 0$. Then there is a $y \in X$ such that $d(x, y) + \epsilon > f_2(x) + f_1(y)$.

Similarly suppose that $f_1(x) > 0$ then there is a $y_0 \in X$ such that $d(y_0, x) + \epsilon > f_2(y_0) + f_1(x)$.

4.1.6 Definition. (Compare [10, Definition 2.11]) Let (X, d) be a T_0 -quasi-metric space $f \in Q_X$ and $x \in X$ such that $f_2(x) > 0$. We say that a sequence $(y_n)_n$ in

X is a **witness for the extremality** of f at x with respect to f_2 if

$$\lim_{n \rightarrow \infty} [f_2(x) + f_1(y_n) - d(x, y_n)] = 0$$

where the limit is taken with respect to the $\tau(u^s)$ -topology on $[0, \infty)$.

Dually, for $f_1(x) > 0$ we say that a sequence $(y_n)_n$ in X is a **witness for the extremality** of f at x with respect to f_1 if

$$\lim_{n \rightarrow \infty} [f_2(y_n) + f_1(x) - d(y_n, x)] = 0.$$

We shall say that a point $y \in X$ is a witness for the extremality of f at x if the constant sequence $(y)_n$ witnesses the extremality of f at x . It is clear that if y witnesses the extremality of f at x with respect to f_2 then $f_2(x) + f_1(y) = d(x, y)$. Similarly if y witnesses the extremality of f at x with respect to f_1 then $f_2(y) + f_1(x) = d(y, x)$.

We can now complete the proof of Proposition 4.1.2 since part (3) is still to be proved.

Proof of Proposition 4.1.2(3)

\Leftarrow

If $m_{F,\alpha} \in Q(X)$, then we have by part (1) that for all $y, y_0 \in F$, $d(y, y_0) \leq \alpha_2(y) + \alpha_1(y_0)$. The above inequality together with part (4) implies that $M_{F,\alpha} \in Q_X$ and hence $m_{F,\alpha} \in Q_X$ as required.

\Rightarrow

Let $x \in X$ be fixed. Since $m_{F,\alpha} \in Q(X)$, we have

$$\begin{aligned} (m_{F,\alpha})_2(x) &\geq \max\{d(x, y) - (m_{F,\alpha})_1(y) : y \in F\} \\ &\geq \max\{d(x, y) - \alpha_2(y) : y \in F\} \\ &= (M_{F,\alpha})_2(x). \end{aligned}$$

Similarly it can be shown that $(m_{F,\alpha})_1(x) \geq (M_{F,\alpha})_1(x)$. Thus we have $m_{F,\alpha} \geq M_{F,\alpha}$.

Since $m_{F,\alpha}$ is extremal, we can find some sequence $(y_n)_n$ in X that witnesses its extremality. Since F is finite, by the definition of the minimum function pair, we may assume that for $y \in F$ fixed,

$$\begin{aligned}(m_{F,\alpha})_1(y_n) &= d(y, y_n) + \alpha_1(y) \\ (m_{F,\alpha})_2(y_n) &= d(y_n, y) + \alpha_2(y)\end{aligned}$$

for all n . Let $z \in F$ be fixed such that

$$\begin{aligned}(m_{F,\alpha})_1(x) &= d(z, x) + \alpha_1(z) > 0 \\ (m_{F,\alpha})_2(x) &= d(x, z) + \alpha_2(z) > 0,\end{aligned}$$

then

$$\begin{aligned}d(x, y_n) &\leq d(x, z) + d(z, y) + d(y, y_n) \\ &\leq d(x, z) + (m_{F,\alpha})_2(z) + (m_{F,\alpha})_1(y) + d(y, y_n) \\ &\leq d(x, z) + \alpha_2(z) + \alpha_1(y) + d(y, y_n) \\ &= (m_{F,\alpha})_2(x) + (m_{F,\alpha})_1(y_n).\end{aligned}$$

Thus

$$d(x, y_n) \leq (m_{F,\alpha})_2(x) + (m_{F,\alpha})_1(y_n).$$

Since $(m_{F,\alpha})_2(x) + (m_{F,\alpha})_1(y_n) - d(x, y_n) \rightarrow 0$ (according to Definition 4.1.6), we must have that $d(x, y_0) = d(x, z) + \alpha_2(z) + \alpha_1(y) + d(y, y_0)$ and $\alpha_2(z) + \alpha_1(y) = d(z, y)$ for some $y_0 \in F$ such that y_n converges to y_0 . Thus

$$\begin{aligned}(m_{F,\alpha})_2(x) &= d(x, y_0) + (m_{F,\alpha})_1(y_0) \\ &= d(x, y_0) + d(y_0, y) \dot{-} (m_{F,\alpha})_1(y) \\ &= d(x, y_0) + d(y_0, y) \dot{-} (d(y_0, y) + \alpha_1(y_0)) \\ &= d(x, y_0) \dot{-} \alpha_1(y_0) \\ &\leq (M_{F,\alpha})_2(x).\end{aligned}$$

Similarly we have

$$\begin{aligned}
d(y_n, x) &\leq d(y_n, y) + d(y, z) + d(z, x) \\
&\leq d(y_n, y) + (m_{F,\alpha})_2(y) + (m_{F,\alpha})_1(z) + d(z, x) \\
&\leq d(y_n, y) + \alpha_2(y) + \alpha_1(z) + d(z, x) \\
&= (m_{F,\alpha})_2(y_n) + (m_{F,\alpha})_1(x).
\end{aligned}$$

Hence by considering the fact that $(m_{F,\alpha})_2(y_n) + (m_{F,\alpha})_1(x) - d(y_n, x) \rightarrow 0$ one can show that $(m_{F,\alpha})_1(x) \leq (M_{F,\alpha})_1(x)$ and hence we conclude that $(m_{F,\alpha})(x) \leq (M_{F,\alpha})(x)$. We have therefore proved that $m_{F,\alpha} = M_{F,\alpha}$.

□

Proof of Proposition 4.1.3.

(1)

Let $G = \{y \in F : m_{F,\alpha}(y) = \alpha(y)\}$ and α restricted to G gives α_0 . Consider m_{G,α_0} . Then $m_{G,\alpha_0}(y_0) = \alpha_0(y_0)$ for all $y_0 \in G$. It follows from the definition of minimum function pairs that for all $x \in X$,

$$\begin{aligned}
(m_{G,\alpha_0})_1(x) &= \min\{d(y, x) + (m_{G,\alpha_0})_1(y) : y \in G\} \\
(m_{G,\alpha_0})_2(x) &= \min\{d(x, y) + (m_{G,\alpha_0})_2(y) : y \in G\}.
\end{aligned}$$

Since F is a support of $m_{F,\alpha}$ and $G \subseteq F$ we have that G is also a support of $m_{F,\alpha}$. It suffices to show only that $m_{F,\alpha} = m_{G,\alpha_0}$.

Let $x \in X$ be given. If $(m_{F,\alpha})_1(x) = d(y, x) + \alpha_1(y)$ for some $y \in G$, then $(m_{F,\alpha})_1(x) = (m_{G,\alpha_0})_1(x)$. In a similar manner if $(m_{F,\alpha})_2(x) = d(x, y) + \alpha_2(y)$ for some $y \in G$, then $(m_{F,\alpha})_2(x) = (m_{G,\alpha_0})_2(x)$. Hence $m_{F,\alpha} = m_{G,\alpha_0}$ as required.

Suppose now that for some $y \in F \setminus G$, we have

$$\begin{aligned}(m_{F,\alpha})_1(x) &= d(y, x) + \alpha_1(y) \\ (m_{F,\alpha})_2(x) &= d(x, y) + \alpha_2(y).\end{aligned}$$

Then by the way G was defined, we have that $m_{F,\alpha}(y) \neq \alpha(y)$ for $y \in F \setminus G$. Thus there exists some $y_0 \in F$ such that

$$\begin{aligned}(m_{F,\alpha})_1(y) &= d(y_0, y) + \alpha_1(y_0) < \alpha_1(y) \\ (m_{F,\alpha})_2(y) &= d(y, y_0) + \alpha_2(y_0) < \alpha_2(y).\end{aligned}$$

Thus

$$\begin{aligned}(m_{F,\alpha})_2(x) &= \min\{d(x, y_0) + \alpha_2(y_0) : y_0 \in F\} \\ &\leq d(x, y_0) + \alpha_2(y_0) \\ &< d(x, y_0) + (\alpha_2(y) - d(y, y_0)) \\ &\leq d(x, y) + \alpha_2(y).\end{aligned}$$

Thus we have that $(m_{F,\alpha})_2(x) < d(x, y) + \alpha_2(y)$. Similarly one can show that $(m_{F,\alpha})_1(x) < d(y, x) + \alpha_1(y)$ which is a contradiction. We must therefore have that $m_{F,\alpha} = m_{G,\alpha_0}$ as required.

(2)

This follows immediately from Lemma 4.1.4.

(1) and (2) of Proposition 4.1.3 imply that

$$m(X, n) \cap Q(X) \subseteq Q(X, n)$$

and

$$M(X, n) \cap Q_X \subseteq Q_X^n$$

for all n . The result now follows since the reverse inclusion is immediate from the definitions.

□

In the next section (Section 4.2), we shall discuss some facts about preservation of supseparability of the space of extremal function pairs which we shall need in the sequel.

4.2 Supseparability of the space of minimal function pairs

The first property we shall prove is an extension property of the extremal function pairs.

4.2.1 Lemma. (compare [10, Lemma 3.1]) Let (X, d) be a T_0 -quasi-metric space. Any extremal function pair on a subspace of X extends to an extremal function pair on the whole of X .

Proof.

Let $Y \subseteq X$ be any subspace and let $f \in Q(Y)$. We shall need the following general facts from the literature.

1. A Katětov function pair on a subspace of X can be extended to a Katětov function pair on the whole of X (check for instance [25, Lemma 6]).
2. Zorn's lemma implies that below any ample function pair there is an extremal function pair.

Taking the above facts into consideration, we can extend f to some $f' \in Q(X)$ (by the first fact) and then take some extremal function pair f'' below f' (by the second fact). To complete the proof, it suffices to show that f'' is an extension of f .

Notice (by the choice of f' and f'') that

$$f = f'|_Y \geq f''|_Y.$$

On the other hand since f'' is extremal, it cannot be strictly less than f at any point $y \in Y$ and still be ample for all $x, y \in X$. So $f = f''$ as required.

□

4.2.1 Proposition. (compare [1, Proposition 3]) Let (Y, d) be a T_0 -quasi-metric space and let X be a (nonempty) subspace of (Y, d) . Then there exists an isometric embedding $\tau : Q_X \rightarrow Q_Y$ such that $\tau(f)|_X = f$ whenever $f \in Q_X$.

4.2.1 Corollary. Let (X, d) be a T_0 -quasi-metric space and $Y \subseteq X$ be any sup-dense subspace of X . Then Q_X and Q_Y are isometric.

Corollary 4.2.1 says in other words that supseparability of a space X is preserved if we are restricted to a subspace of X . A natural question one can ask is the following: what happens if we move up to a superspace of X ? We shall see that if we take an arbitrary joincompact space K and we consider $X' = X \cup K$, then $Q_{X'}$ is supseparable. This leads us to the following proposition.

4.2.2 Proposition. Let (X, d) be a T_0 -quasi-metric space, K be a joincompact space and $X' = X \cup K$. If Q_X is supseparable then $Q_{X'}$ is also supseparable, where in this case the T_0 -quasi-metric on X' is any T_0 -quasi-metric on X that extends that on X .

The breakdown of the proof will be as follows:

1. We show first that if we take a point $x_0 \notin X$, then $Q_{X \cup \{x_0\}}$ will be supseparable whenever Q_X is supseparable by Lemma 4.2.1.
2. We next take a finite set $A = \{x_0, x_1, \dots, x_n\}$ for $x_0, x_1, \dots, x_n \notin X$. Then $Q_{X \cup A}$ will be supseparable whenever Q_X is supseparable.

3. Finally we take an arbitrary joincompact space K and show that $Q_{X'}$ is supseparable whenever Q_X is supseparable.

4.2.2 Lemma. Let X be a T_0 -quasi-metric space and $x_0 \notin X$. If Q_X is supseparable then so is $Q_{X \cup \{x_0\}}$ where the T_0 -quasi-metric on $X \cup \{x_0\}$ is any T_0 -quasi-metric that extends the T_0 -quasi-metric on X .

Proof.

Let d be a T_0 -quasi-metric on $X \cup \{x_0\}$ that extends the T_0 -quasi-metric on X . By supseparability of Q_X , we can fix $\{f^i : i < \omega\}$ a countable supdense subset of Q_X . Let $r \in [0, \infty)$ and $i \in \mathbb{N}$. Define $f^{i,r}(x_0) = r$ and for every element $x \in X$,

$$f_1^{i,r}(x) = \max\{d(x, x_0) - r, f_2^i(x)\}$$

and

$$f_2^{i,r}(x) = \max\{d(x_0, x) - r, f_1^i(x)\}.$$

Then we have that $\{f^{i,r} : i \in \mathbb{N}, r \in [0, \infty)\}$ is supseparable since

$$\{f^{i,q} : i \in \mathbb{N}, q \in \mathbb{Q}\} \subseteq \{f^{i,r} : i \in \mathbb{N}, r \in [0, \infty)\}$$

and $\{f^{i,q} : i \in \mathbb{N}, q \in \mathbb{Q}\}$ is countable. To show that $Q_{X \cup \{x_0\}}$ is supseparable, it suffices to show that $Q_{X \cup \{x_0\}}$ is contained in the $\tau(D^s)$ -closure of $\{f^{i,r} : i \in \mathbb{N}, r \in [0, \infty)\}$.

Claim: For $f \in Q_{X \cup \{x_0\}}$ and $\epsilon > 0$, there exist some $i \in \mathbb{N}$ and $r \in [0, \infty)$ such that $D^s(f, f^{i,r}) \leq \epsilon$.

Proof of Claim.

Consider $f|_X$. Then f is Katětov (it may not necessarily be extremal on X). Thus we may fix $f' \in Q_X$ such that $f'(x) \leq f(x)$ for every $x \in X$. Let us choose f^i such that $D^s(f', f^i) \leq \epsilon$ and $r = f(x_0)$. We shall now show that $D^s(f, f^{i,r}) \leq \epsilon$. Let $x \in X$. From the definition of $f^{i,r}$, we have the following cases:

Case 1: $f^{i,r} = f^i$.

In this case if $f'(x) = f(x)$ then we are done, since we will have

$$\sup\{f_1^i(x) \dot{-} f_1'(x) : x \in X\} \vee \sup\{f_2^i(x) \dot{-} f_2'(x) : x \in X\} \leq \epsilon.$$

On the other hand if $f'(x) \neq f(x)$, then we claim that $f_2(x) = d(x, x_0) \dot{-} r$ and $f_1(x) = d(x_0, x) \dot{-} r$. This will suffice since by the definition of $f^{i,r}$ and the assumption that $f^{i,r} = f^i$, we get that

$$f_2(x) = d(x, x_0) \dot{-} f_1(x_0) \leq f_2^i(x) = f_2^{i,r}(x).$$

Moreover our choice of f' and f^i gives $f'(x) \leq f(x)$ and $f(x) \geq f^i(x) + \epsilon$. Thus we have that

$$f_2'(x) + \epsilon \geq f_2^i(x) = f_2^{i,r}(x) \geq f_2'(x)$$

and hence $f_2^{i,r}(x)$ is within ϵ of $f_2(x)$.

If $f_2(x) \neq d(x, x_0) \dot{-} f_1(x_0)$, then $f \in Q_{X \cup \{x_0\}}$ implies that there exists $y \in X$ such that

$$f_2(x) + f_1(y) < d(x, y) + (f_2(x) \dot{-} f_2'(x)) \text{ (compare Corollary 4.1.1).}$$

In that way $f_1'(y) \leq f_1(y)$ gives

$$\begin{aligned} d(x, y) + (f_2(x) \dot{-} f_2'(x)) &> f_2(x) + f_1(y) \\ d(x, y) &> f_1(y) + f_2'(x) \\ &> f_1'(y) + f_2'(x), \end{aligned}$$

but this contradicts $f' \in Q_X$. Thus we must have that $f_2(x) = d(x, x_0) \dot{-} f_1(x_0)$ as required. In a similar manner, we can show that $f_1(x) = d(x_0, x) \dot{-} f_2(x_0)$.

Case 2: $f_1^{i,r}(x) = d(x_0, x) \dot{-} r$ and $f_2^{i,r}(x) = d(x, x_0) \dot{-} r$.

This means that $f_1^{i,r}(x) = d(x_0, x) \dot{-} r \geq f_2^i(x)$. Since f is Katětov and by our choice of f^i we get that

$$f_2(x) \dot{-} \epsilon \leq f_2^i(x) \leq f_2^{i,r}(x) = d(x, x_0) \dot{-} f_1(x_0) \leq f_2(x).$$

Thus $f_2(x) - \epsilon \leq f_2^{i,r}(x) \leq f_2(x)$ and this shows that $f_2^{i,r}(x)$ is within some ϵ of $f_2(x)$. Similarly it can be shown that $f_1^{i,r}(x)$ is within some ϵ of $f_1(x)$. Hence $f^{i,r}(x)$ is within some ϵ of $f(x)$. Thus we have that $D(f^{i,r}, f) \leq \epsilon$. The proof that $D(f, f^{i,r}) \leq \epsilon$ is similar. Hence $D^s(f, f^{i,r}) \leq \epsilon$ as required.

□

4.2.3 Lemma. For a T_0 -quasi-metric space X and $A = \{x_0, x_1, \dots, x_n\}$ where $x_0, x_1, \dots, x_n \notin X$, if Q_X is supseparable then $Q_{X \cup A}$ is supseparable where the T_0 -quasi-metric on $X \cup A$ extends that on X .

Proof.

By the fact that an extremal pair on X extends to an extremal pair on $X \cup A$ (compare Lemma 4.2.1) we have that if Q_X is non-supseparable, then $Q_{X \cup A}$ will be non-supseparable. Thus by applying Lemma 4.2.2 recursively $(n + 1)$ times for Q_X supseparable, we get the supseparability of $Q_{X \cup A}$.

□

Proof of Proposition 4.2.2.

Let d be any T_0 -quasi-metric on $X \cup K$ that extends the T_0 -quasi-metric on X and the T_0 -quasi-metric on K and assume that Q_X is supseparable. For each n , let $F_n \subseteq K$ be a finite subset of K such that for each $x \in K$ there exists $y \in F_n$ such that $d^s(x, y) < 2^{-n}$, i.e., F_n is 2^{-n} supdense in K . It is clear from Lemma 4.2.3 that $Q_{X \cup F_n}$ is supseparable. Thus we can fix for each n a countable supdense subspace $A_n \subseteq Q_{X \cup F_n}$. By Lemma 4.2.1, each function pair $g \in A_n$ extends to an extremal function pair over $X \cup K$. Fix $B_n \subseteq Q_{X \cup K}$ such that B_n is countable and for each function pair $g' \in A_n$, there is some function pair $g \in B_n$ such that g extends g' .

Claim: $\bigcup_n B_n$ is supdense in $Q_{X \cup K}$.

Let $f \in Q_{X \cup K}$ and $\epsilon > 0$ be given. Let n be such that $\frac{\epsilon}{6} > 2^{-n}$. Consider $f|_{X \cup F_n}$. Then f is Katětov. Thus we may fix $f' \in Q_{X \cup F_n}$ such that f' is below f . By our choice of A_n and B_n , we may take $g' \in A_n$ such that $D^s(g', f') \leq \frac{\epsilon}{6}$ and $g \in B_n$ such that g' extends to g . We will now show that $D^s(f, g) \leq \epsilon$.

First we show that $f' \geq f \dot{-} \frac{\epsilon}{3}$.

Assume on the contrary that for some $x \in X \cup F_n$, $f'_2(x) < f_2(x) \dot{-} \frac{\epsilon}{3}$. Since f is extremal, we fix $y \in X \cup K$ such that $f_2(x) + f_1(y) < d(x, y) + f_2(x) \dot{-} (f'_2(x) + \frac{\epsilon}{3})$.

By our choice of F_n , let us fix $y_0 \in X \cup F_n$ such that $d^s(y, y_0) < \frac{\epsilon}{6}$. Since f is Katětov, we have $f_1(y_0) \dot{-} f_1(y) \leq d(y, y_0)$. Thus

$$\begin{aligned} f_2(x) + f_1(y_0) &\leq f_2(x) + d(y, y_0) + f_1(y) \\ &< d(x, y) + f_2(x) \dot{-} \left(f'_2(x) + \frac{\epsilon}{3}\right) + \frac{\epsilon}{6} \\ &\leq d(x, y_0) + d(y_0, y) + f_2(x) \dot{-} f'_2(x) \dot{-} \frac{\epsilon}{6} \\ &< d(x, y_0) + f_2(x) \dot{-} f'_2(x) \\ f_1(y_0) &< d(x, y_0) \dot{-} f'_2(x). \end{aligned}$$

On the other hand, $f' \leq f$, so that $f'_1(y_0) \leq f_1(y_0) < d(x, y_0) \dot{-} f'_2(x)$. Thus $f'_2(x) + f'_1(y_0) < d(x, y_0)$ which is a contradiction to f' extremal. Hence we must have that $f' \geq f \dot{-} \frac{\epsilon}{3}$ as required.

Let us now finish the proof by showing that $D^s(f, g) \leq \epsilon$.

Fix $x \in X \cup K$. By the way F_n was picked, we can fix $y \in X \cup F_n$ such that $d^s(x, y) \leq \frac{\epsilon}{6}$. Since g' extends to g and $y \in X \cup F_n$, we have $g'(y) = g(y)$. Thus

$$\begin{aligned}
g_1(x) \dot{-} f_1(x) &= g_1(x) \dot{-} g_1(y) + g_1(y) \dot{-} g'_1(y) + g'_1(y) \dot{-} f'_1(y) + f'_1(y) \dot{-} f_1(y) + f_1(y) \dot{-} f_1(x) \\
&\leq d(y, x) + 0 + D(g', f') + |f'_1(y) - f_1(y)| + d(x, y) \\
&\leq d^s(x, y) + D^s(f', g') + \frac{\epsilon}{3} + d^s(x, y) \\
&\leq \frac{\epsilon}{6} + \frac{\epsilon}{6} + \frac{\epsilon}{3} + \frac{\epsilon}{6} \\
&\leq \epsilon.
\end{aligned}$$

Hence $D(f, g) \leq \epsilon$. By doing a similar computation, one can prove that $D(g, f) \leq \epsilon$. Hence we have that $D^s(f, g) \leq \epsilon$.

□

Let us remark that the definition of extremal function pairs depends heavily on the quasi-metric on the space. Thus if we modify the quasi-metric on the space, we would have modified the q -hyperconvex hull. We have the following definition.

4.2.1 Definition. (compare [10, Definition 3.6]) Let d_1 and d_2 be two T_0 -quasi-metrics on the same set, say X . We define the **q -uniform quasi-metric** between d_1 and d_2 by

$$\rho(d_1, d_2) = \sup\{d_1(x, y) \dot{-} d_2(x, y) : x, y \in X\}^1.$$

We say that a sequence of T_0 -quasi-metrics $(d_n)_n$ on the same space X , converge **q -uniformly** to a T_0 -quasi-metric d on X if

$$\lim_{n \rightarrow \infty} \rho^s(d, d_n) = 0.$$

4.2.4 Lemma. (compare [10, Lemma 3.8]) If d_1 and d_2 are two T_0 -quasi-metrics on the same set, say X , and $\rho^s(d_1, d_2) \leq \epsilon$ for $\epsilon > 0$, then for every $f \in Q_{(X, d_1)}$ there exists some $g \in Q_{(X, d_2)}$ such that $D^s(f, g) \leq \epsilon'$ for some $\epsilon' > 0$.

Proof.

¹We remark that ρ is an extended T_0 -quasi-metric.

Let $f \in Q_{(X,d_1)}$ be given. Define g' by $g'(x) = f(x) + \frac{\epsilon}{2}$ for all $x \in X$. Then g' satisfies

$$\begin{aligned} g'_2(x) + g'_1(y) &= f_2(x) + \frac{\epsilon}{2} + f_1(y) + \frac{\epsilon}{2} \\ &\geq d_1(x, y) + \epsilon \\ &\geq d_2(x, y) \end{aligned}$$

for every $x, y \in X$. Thus we have that there is a $g \in Q_{(X,d_2)}$ such that g is below g' . Let now $x \in X$ be given. By our choice of g and the definition of g' , we have that $g'(x) \leq f(x) + \frac{\epsilon}{2}$. Since f is extremal, we have that for all $\delta > 0$ there exists some $y \in X$ such that $f_2(x) + f_1(y) \leq d_1(x, y) + \delta$ (compare Corollary 4.1.1). Thus

$$\begin{aligned} g_2(x) + g_1(y) &\geq d_2(x, y) \quad \text{since } g \text{ is extremal,} \\ g_2(x) + g'_1(y) &\geq d_1(x, y) - \epsilon \quad \text{since } g \leq g' \text{ and } \rho^s(d_1, d_2) \leq \epsilon, \forall x, y \in X, \\ g_2(x) + f_1(y) + \frac{\epsilon}{2} &\geq d_1(x, y) - \epsilon, \\ g_2(x) &\geq (d_1(x, y) - f_1(y)) - \frac{3\epsilon}{2}, \\ g_2(x) &\geq (f_2(x) - \delta) - \frac{3\epsilon}{2}. \end{aligned}$$

Since δ was chosen arbitrarily small, we have that $g_2(x) \geq f_2(x) - \frac{3\epsilon}{2}$. It therefore follows that $f_2(x) - g_2(x) \leq \frac{3\epsilon}{2}$ so that

$$\sup\{f_2(x) - g_2(x) : x \in X\} \leq \frac{3\epsilon}{2}.$$

Choose $\epsilon' = \frac{3\epsilon}{2} > 0$. Similarly one can show that

$$\sup\{g_1(x) - f_1(x) : x \in X\} \leq \epsilon'.$$

Hence we have that $D(g, f) \leq \epsilon'$. To show that $D(f, g) \leq \epsilon'$, we follow the same procedure as above. Therefore $D^s(f, g) \leq \epsilon'$ as required.

□

4.2.3 Proposition. Let $(d_i)_i$ be a sequence of T_0 -quasi-metrics on a set X such that d_i converges q -uniformly to d for some T_0 -quasi-metric d on X . If each $Q_{(X,d_i)}$ is supseparable, so is $Q_{(X,d)}$.

Proof.

By supseparability of $Q_{(X,d_i)}$, we may fix for each $i \in \omega$ a countable set $\{f^{i,n} : n \in \omega\}$ which is supdense in $Q_{(X,d_i)}$. Then $Q_{(X,d)}$ is contained in the $\tau(D^s)$ -closure of $\{f^{i,n} : i, n \in \omega\}$ in the space of all real-valued function pairs on X by Lemma 4.2.4. Hence we have that $Q_{(X,d)}$ is supseparable.

□

Chapter 5

The Urysohn

T_0 -ultra-quasi-metric space

5.1 The Katětov construction modified for a T_0 -ultra-quasi-metric space

5.1.1 Definition. Let (X, d) be a T_0 -ultra-quasi-metric space. We shall say that a function pair on (X, d) is **Katětov** if it satisfies the following inequalities:

(a) f is ultra-ample, i.e., $d(x, y) \leq \max\{f_2(x), f_1(y)\}$ whenever $x, y \in X$,

(b) f is non-expansive, i.e., $f_1(x) \leq \max\{f_1(y), d(y, x)\}$ and

$f_2(x) \leq \max\{f_2(y), d(x, y)\}$ whenever $x, y \in X$.

5.1.1 Remark. Note that we should call these function pairs ultra-Katětov to avoid any confusion with possible Katětov function pairs on (X, d) . Nevertheless since we do not speak about the latter pairs in this section, we avoid this complication of terminology and hope that this will not lead to confusion.

We shall denote the set of all Katětov function pairs on (X, d) by $Q(X, d)$ (or

simply $Q(X)$ when there is no confusion on d). Elements in $Q(X, d)$ will be called Katětov function pairs (or simply Katětov pairs) on (X, d) or ultra-quasi-metric admissible.

5.1.2 Remark. We do not require that $f \in Q(X, d)$ be uq -minimal with respect to the pointwise order on function pairs. Hence in general our function pairs f do not satisfy

$$f_1(x) = \sup\{d(y, x) : y \in X \text{ and } d(y, x) > f_2(y)\}$$

and

$$f_2(x) = \sup\{d(x, y) : y \in X \text{ and } d(x, y) > f_1(y)\}$$

whenever $x \in X$, otherwise we will have that f is uq -minimal.

Indeed, let $g_1 \leq f_1$ and $g_2 \leq f_2$, where f satisfies the above inequalities and g is ultra-ample. Then for $x \in X$

$$\begin{aligned} f_2(x) &\leq \sup\{d(x, y) : y \in X \text{ and } d(x, y) > f_1(y) \geq g_1(y)\} \\ &= \sup\{d(x, y) : y \in X \text{ and } d(x, y) > g_1(y)\} \\ &= g_2(x) \text{ by ultra-ampleness of } g. \end{aligned}$$

This proves that $g_2 = f_2$. Similarly one can show that $g_1 = f_1$ and therefore f is uq -minimal.

5.1.2 Definition. (Compare [25, page 713]) Let $f, g \in Q(X, d)$. Then

$$D(f, g) = \sup\{n(f_1(x), g_1(x)) : x \in X\} \vee \sup\{n(g_2(x), f_2(x)) : x \in X\}.$$

Then D is an extended T_0 -**ultra-quasi-metric** on $Q(X, d)$. We shall call $(Q(X, d), D)$ the Katětov pairspace on (X, d) .

5.1.1 Lemma. (Compare [23, Lemma 2]) Let (X, d) be a T_0 -ultra-quasi-metric space. For each $a \in X$, $f_a(x) := (d(a, x), d(x, a))$ belongs to $Q(X, d)$ whenever $x \in X$.

Proof.

Let $a \in X$ and $x, y \in X$. It is clear by Lemma ?? that f_a is ultra-ample. It remains only to show that f_a is non-expansive.

$$\begin{aligned}(f_a)_1(x) &= d(a, x) \leq \max\{d(a, y), d(y, x)\} = \max\{(f_a)_1(y), d(y, x)\} \\ (f_a)_2(x) &= d(x, a) \leq \max\{d(y, a), d(x, y)\} = \max\{(f_a)_2(y), d(x, y)\}.\end{aligned}$$

Thus by Corollary 1.2.2 we have that f_a is non-expansive.

□

5.1.2 Lemma. (Compare [23, Theorem 1]) Let (X, d) be a T_0 -ultra-quasi-metric space. For each $a, b \in X$, we have that the map

$$e_X : (X, d) \rightarrow (Q(X, d), D) : a \mapsto e_X(a) = f_a$$

whenever $a \in X$, is an isometry. Also, e_X is injective.

Proof.

We prove injectivity first.

Suppose that for any $a \in X$, we have that $f_a(a) = f_a(b)$, i.e., $(f_a)_1(a) = (f_a)_1(b)$ and $(f_a)_2(a) = (f_a)_2(b)$.

$$\begin{aligned}(f_a)_1(a) = (f_a)_1(b) &\Rightarrow 0 = d(a, a) = d(a, b) \\ (f_a)_2(a) = (f_a)_2(b) &\Rightarrow 0 = d(a, a) = d(b, a).\end{aligned}$$

$d(a, b) = 0 = d(b, a) \implies a = b$ by the T_0 property. Thus f_a is injective.

If $a = b$ then $f_a = f_b$ and we have trivially that $D(f_a, f_b) = 0 = d(a, b)$. Without loss of generality, suppose $f_a > f_b$. Then we have that

$$n((f_a)_1(x), (f_b)_1(x)) = (f_a)_1(x) \leq \max\{(f_a)_1(b), (f_b)_1(x)\}$$

and by taking $b = x$, we get that $n((f_a)_1(x), (f_b)_1(x)) = d(a, b)$ and hence

$$\sup\{n((f_a)_1(x), (f_b)_1(x)) : x \in X\} = d(a, b).$$

Similarly we can show that

$$\sup\{n((f_b)_2(x), (f_a)_2(x)) : x \in X\} = d(a, b).$$

Therefore $D(f_a, f_b) = d(a, b)$.

□

5.1.3 Lemma. Let (X, d) be a T_0 -ultra-quasi-metric space, $f \in Q(X, d)$ and $a \in X$. Then $D(f, f_a) = f_1(a)$ and $D(f_a, f) = f_2(a)$.

Proof.

By taking $x = a$ we have that

$$f_1(a) \leq n(f_1(a), d(a, a)) \leq \sup\{n(f_1(x), d(a, x)) : x \in X\}.$$

Also by taking $x = a$ we have that $n(f_1(a), d(a, a)) \leq f_1(a)$. Thus

$$\sup\{n(f_1(x), d(a, x)) : x \in X\} \leq f_1(a)$$

and hence the equality

$$f_1(a) = \sup\{n(f_1(x), d(a, x)) : x \in X\}.$$

We that $d(x, a) \leq \max\{f_2(x), f_1(a)\}$ (by ultra-ampleness of f) implies $n(d(x, a), f_2(x)) \leq f_1(a)$. Thus by taking supremum over $x \in X$, we get

$$\sup\{n(d(x, a), f_2(x)) : x \in X\} \leq f_1(a).$$

Equality holds in the statement. Suppose it does not. This means that

$$\sup\{n(d(x, a), f_2(x)) : x \in X\} < f_1(a)$$

which implies that $d(x, a) < \max\{f_2(x), f_1(a)\}$ and this contradicts the ultra-ampleness of f . Thus we have that

$$\sup\{n(d(x, a), f_2(x)) : x \in X\} = f_1(a).$$

We have finally that

$$f_1(a) = \sup\{n(d(x, a), f_2(x)) : x \in X\} \vee \sup\{n(d(a, x), f_1(x)) : x \in X\} = F(f, f_a).$$

In a similar way, we can show that $f_2(a) = D(f_a, f)$.

□

5.1.1 Corollary. (Compare [25, Corollary 1]) For any $f, g \in Q(X, d)$ and $a \in X$, we have

(a) $D(f, g) \leq \max\{f_1(a), g_2(a)\}$.

(b) D is a bicomplete T_0 -ultra-quasi-metric on $Q(X, d)$.

Proof.

(a) Let $a \in X$. Then $f_a \in Q(X, d)$ by Lemma 5.1.1. Thus

$$\begin{aligned} D(f, g) &\leq \max\{D(f, f_a), D(f_a, g)\} \text{ by the strong triangle inequality} \\ &= \max\{f_1(a), g_2(a)\} \text{ by Lemma 5.1.3} \end{aligned}$$

for every $f, g \in Q(X, d)$.

(b) We need to prove that $(Q(X, d), D)$ is a bicomplete space. Let $((f_k)_1, (f_k)_2)_{k \in \mathbb{N}}$ be a Cauchy sequence with respect to D^s in $Q(X, d)$. Since $((f_k)_1(x))_{k \in \mathbb{N}}$ and $((f_k)_2(x))_{k \in \mathbb{N}}$ are Cauchy sequences in $([0, \infty), n^s)$ for each $x \in X$, we have by completeness of $([0, \infty), n^s)$ (see [23, Example 2]) that $((f_k)_1(x))_{k \in \mathbb{N}}$ converges to some $f_1(x) \in [0, \infty)$ (i.e., for every $\epsilon_1 > 0$, there is an $l_1 \in \mathbb{N}$ such that for all $k > l_1$, we have that $n^s((f_k)_1(x), f_1(x)) < \epsilon_1$) and $((f_k)_2(x))_{k \in \mathbb{N}}$ converges to some $f_2(x) \in [0, \infty)$ (i.e., for every $\epsilon_2 > 0$, there is an $l_2 \in \mathbb{N}$ such that for all $k > l_2$, we have that $n^s((f_k)_2(x), f_2(x)) < \epsilon_2$). To complete the proof, we must show that $((f_k)_1, (f_k)_2)_{k \in \mathbb{N}}$ converges to (f_1, f_2) and $(f_1, f_2) \in Q(X, d)$.

Choose $l = \max\{l_1, l_2\}$, then $k > l$. Also choose $\epsilon = \max\{\epsilon_1, \epsilon_2\}$. Then we have

that

$$\begin{aligned} D^s(f_k, f) &= \sup\{n^s((f_k)_1(x), f_1(x)) : x \in X\} \vee \sup\{n^s(f_2(x), (f_k)_1(x)) : x \in X\} \\ &< \max\{\epsilon_1, \epsilon_2\} = \epsilon. \end{aligned}$$

Hence $D^s(f_k, f) < \epsilon$. This shows that f_k converges to f .

Since $(f_k)_{k \in \mathbb{N}} \in Q(X, d)$, it means that each f_k is ultra-ample (i.e., $d(x, y) \leq \max\{(f_k)_2(x), (f_k)_1(y)\}$) and non-expansive (i.e., $n((f_k)_1(x), (f_k)_1(y)) \leq d(y, x)$ and $n((f_k)_2(x), (f_k)_2(y)) \leq d(x, y)$). Thus we have that

$$n((f_k)_2(x), (f_k)_2(y)) \leq d(x, y) \leq \max\{(f_k)_2(x), (f_k)_1(y)\}$$

and

$$n((f_k)_1(x), (f_k)_1(y)) \leq d(y, x) \leq \max\{(f_k)_2(y), (f_k)_1(x)\}.$$

Since $(f_k)_{k \in \mathbb{N}}$ converges to f , we get that

$$n(f_2(x), f_2(y)) \leq d(x, y) \leq \max\{f_2(x), f_1(y)\}$$

and

$$n(f_1(x), f_1(y)) \leq d(y, x) \leq \max\{f_2(y), f_1(x)\}.$$

We have thus shown that f is ultra-ample and non-expansive. Hence $f \in Q(X, d)$.

□

5.1.4 Lemma. (Compare [25, Lemma 4]) Let (X, d) be a T_0 -ultra-quasi-metric space. For any $f \in Q(X, d)$ and $a \in X$, the following conditions are equivalent:

(a) $f_2(a) = 0$.

(b) $d(a, x) \leq f_1(x)$ and $f_2(x) \leq d(x, a)$ whenever $x \in X$.

(c) $D(f_a, f) = 0$.

Proof.

(a) \Rightarrow (b)

Suppose that (a) holds. By ultra-ampleness of f , we have $d(a, x) \leq \max\{f_2(a), f_1(x)\} = f_1(x)$ whenever $x \in X$. Thus $d(a, x) \leq f_1(x)$. Since f is non-expansive, we have that $n(f_2(x), f_2(a)) \leq d(x, a)$, i.e., $n(f_2(x), 0) \leq d(x, a)$. Thus $f_2(x) \leq d(x, a)$ whenever $x \in X$.

(b) \Rightarrow (c)

Suppose (b) holds. Then

$$\begin{aligned} D(f_a, f) &= \sup\{n(d(a, x), f_1(x)) : x \in X\} \vee \sup\{n(d(x, a), f_2(x)) : x \in X\} \\ &= \max\{0, 0\} \\ &= 0. \end{aligned}$$

(b) \Rightarrow (c)

Suppose now that (c) holds. Then $0 = D(f_a, f) = f_2(a)$.

□

Analogously, one can prove the following lemma.

5.1.5 Lemma. (Compare [25, Lemma 5]) Let (X, d) be a T_0 -ultra-quasi-metric space. For any $f \in Q(X, d)$ and $a \in X$, the following conditions are equivalent:

(a) $f_1(a) = 0$.

(b) $d(x, a) \leq f_2(x)$ and $f_1(x) \leq d(a, x)$ whenever $x \in X$.

(c) $D(f, f_a) = 0$.

5.1.2 Corollary. (Compare [25, Corollary 2]) Given any T_0 -ultra-quasi-metric space (X, d) , any $f \in Q(X, d)$ and any $a \in X$, we have that $f_1(a) = 0 = f_2(a)$ if and only if $f = f_a$.

Proof.

Suppose that $f_1(a) = 0 = f_2(a)$.

$f_1(a) = 0$ implies (by Lemma 5.1.5(b)) that $d(x, a) \leq f_2(x)$ whenever $x \in X$.

$f_2(a) = 0$ implies (by Lemma 5.1.4(b)) that $f_2(x) \leq d(x, a)$ whenever $x \in X$.

Thus we have $f_2(x) \leq d(x, a) \leq f_2(x)$, i.e., $f_2(x) = d(x, a) = (f_a)_2(x)$ whenever $x \in X$. Therefore $f_2 = (f_a)_2$.

$f_1(a) = 0$ implies (by Lemma 5.1.5(b)) that $f_1(x) \leq d(a, x)$ whenever $x \in X$.

$f_2(a) = 0$ implies (by Lemma 5.1.4(b)) that $d(a, x) \leq f_1(x)$ whenever $x \in X$.

Thus we have $f_1(x) \leq d(a, x) \leq f_1(x)$, i.e., $f_1(x) = d(a, x) = (f_a)_1(x)$ whenever $x \in X$. Therefore $f_1 = (f_a)_1$. We therefore have that

$$f = (f_1, f_2) = ((f_a)_1, (f_a)_2) = f_a.$$

Suppose now that $f = f_a$. Then $D(f, f_a) = 0 = D(f_a, f)$.

$D(f, f_a) = 0$ implies by Lemma 5.1.5 that $f_1(a) = 0$.

$D(f_a, f) = 0$ implies by Lemma 5.1.4 that $f_2(a) = 0$.

Thus $f_1(a) = 0 = f_2(a)$.

□

5.1.1 Proposition. (Compare [20, Proposition 4]) Let (X, d) be a T_0 -quasi-metric space. Then $(f_1, f_2) \in Q(X, d)$ implies that $(f_2, f_1) \in Q(X, d^{-1})$ and the map

$$\begin{aligned} \varphi : (Q(X, d), D) &\rightarrow (Q(X, d^{-1}), D^{-1}) \\ &: (f_1, f_2) \mapsto \varphi(f_1, f_2) = (f_2, f_1) \end{aligned}$$

is an isometry.

Proof.

Suppose $(f_1, f_2) \in Q(X, d)$. Let $x, y \in X$. Then

$$d^{-1}(y, x) = d(x, y) \leq \max\{f_2(x), f_1(y)\}.$$

Also

$$d^{-1}(x, y) = d(y, x) \leq \max\{f_2(y), f_1(x)\}.$$

Hence (f_2, f_1) is ultra-ample.

Moreover $n(f_1(x), f_1(y)) \leq d(y, x) = d^{-1}(x, y)$, which implies that $n(f_1(x), f_1(y)) \leq d^{-1}(x, y)$. Similarly we have that $n(f_2(x), f_2(y)) \leq d(x, y) = d^{-1}(y, x)$ implies that $n(f_2(x), f_2(y)) \leq d^{-1}(y, x)$. This proves nonexpansivity of (f_2, f_1) . Hence we have that $(f_2, f_1) \in Q(X, d^{-1})$.

$$\begin{aligned} D^{-1}(\varphi(f), \varphi(g)) &= D^{-1}((f_2, f_1), (g_2, g_1)) \\ &= D((g_2, g_1), (f_2, f_1)) \\ &= D((f_1, f_2), (g_1, g_2)) \end{aligned}$$

whenever $f, g \in Q(X, d)$. Thus φ is an isometry.

5.1.3 Remark. Let (X, d) be a T_0 -ultra-quasi-metric space and $f \in Q(X, d)$ be such that $f \neq f_x$ whenever $x \in X$. We can obtain a T_0 -ultra-quasi-metric one-point extension $X^+ = X \cup \{f\}$ of X by extending d to X^+ as follows: $d(f, x) = f_1(x)$ and $d(x, f) = f_2(x)$ whenever $x \in X$, and $d(f, f) = 0$. One can show that d is a T_0 -ultra-quasi-metric on X^+ .

Thus if (X^+, d) is a T_0 -ultra-quasi-metric one-point extension, where $X^+ = X \cup \{\omega\}$, we can set $f_1(x) = d(\omega, x)$ and $f_2(x) = d(x, \omega)$ whenever $x \in X$ in order to have $f \in Q(X, d)$ such that $f \neq f_x$ whenever $x \in X$.

5.1.6 Lemma. (Compare [18, Fact 1.4]) Let (Y, d) be an ultra-quasi-pseudometric space and $X \subseteq Y$. Then we can interpret $(Q(X, d), D)$ as a subspace of $(Q(Y, d), D)$. Indeed, for any pair $f \in Q(X, d)$, we define an extension f_Y of f to Y as follows

$$(f_Y)_1(y) = \inf_{z \in X} \max\{f_1(z), d(z, y)\}$$

and

$$(f_Y)_2(y) = \inf_{z \in X} \max\{f_2(z), d(y, z)\}$$

whenever $y \in Y$. (We shall say that the pair f_Y is controlled by the subspace X or that X is a support of the pair f_Y .)

Proof.

Let $f \in Q(X, d)$. Since f is non-expansive on X , we have that $n(f_1(x), f_1(y)) \leq d(y, x)$ and $n(f_2(x), f_2(y)) \leq d(x, y)$ whenever $x, y \in X$.

$n(f_1(x), f_1(y)) \leq d(y, x) \implies f_1(x) \leq \max\{f_1(y), d(y, x)\}$. By taking $x = y$, the equality $f_1(y) = (f_Y)_1(y)$ follows since $d(y, y) = 0$.

Similarly one can show that $f_2(y) = (f_Y)_2(y)$. Thus f_Y extends f to Y .

Let $x, y \in Y$. Then for each $z \in X$, we have $d(x, z) \leq \max\{d(x, y), d(y, z)\}$, so that

$$\begin{aligned} \max\{d(x, z), f_2(z)\} &\leq \max\{\max\{d(x, y), d(y, z)\}, f_2(z)\} \\ &= \max\{\max\{d(y, z), f_2(z)\}, d(x, y)\}. \end{aligned}$$

By taking the infimum over $z \in X$, we get

$$\begin{aligned} \inf_{z \in X} \max\{d(x, z), f_2(z)\} &\leq \inf_{z \in X} \max\{\max\{d(y, z), f_2(z)\}, d(x, y)\} \\ &= \max\left\{ \inf_{z \in X} \max\{d(y, z), f_2(z)\}, d(x, y) \right\}. \end{aligned}$$

We therefore have that $(f_Y)_2(x) \leq \max\{(f_Y)_2(y), d(x, y)\}$ which implies that

$$n((f_Y)_2(x), (f_Y)_2(y)) \leq d(x, y).$$

In the same way we can show that

$$n((f_Y)_1(x), (f_Y)_1(y)) \leq d(y, x)$$

so as to conclude that f_Y is non-expansive on Y .

Let now $x, y \in Y$ and $\epsilon > 0$.

$(f_Y)_1(y) = \inf_{z \in X} \max\{f_1(z), d(z, y)\}$ implies that there exists an $a_1 \in X$ such that $\max\{f_1(a_1), d(a_1, y)\} - \epsilon \leq (f_Y)_1(y)$.

$(f_Y)_2(y) = \inf_{z \in X} \max\{f_2(z), d(y, z)\}$ implies that there exists an $a_2 \in X$ such that $\max\{f_2(a_2), d(y, a_2)\} - \epsilon \leq (f_Y)_2(y)$.

$$\begin{aligned} d(y, z) &\leq \max\{d(y, a_2), d(a_2, a_1), d(a_1, z)\} \\ &\leq \max\{d(y, a_2), \max\{f_2(a_2), f_1(a_1)\}, d(a_1, z)\} \\ &= \max\{\max\{d(y, a_2), f_2(a_2)\}, \max\{f_1(a_1), d(a_1, z)\}\} \\ &\leq \max\{(f_Y)_1(z) + \epsilon, (f_Y)_2(y) + \epsilon\}. \end{aligned}$$

Since ϵ was chosen arbitrarily, we have that $d(y, z) \leq \max\{(f_Y)_1(z), (f_Y)_2(y)\}$. It therefore follows that f_Y is ultra-ample on Y .

Finally we have to show that $\varphi : (Q(X, d), D) \rightarrow (Q(Y, d), D)$ defined by $\varphi(f) = f_Y$ is an isometric map.

Let $f, g \in Q(X, d)$ be such that $f \neq g$. Then

$$\begin{aligned} (f_Y)_1(y) &= f_1(y) \text{ and } (f_Y)_2(y) = f_2(y) \\ (g_Y)_1(y) &= g_1(y) \text{ and } (g_Y)_2(y) = g_2(y). \end{aligned}$$

From the above expressions we see that $n((f_Y)_1(y), (g_Y)_1(y)) \geq n(f_1(y), g_1(y))$ so that

$$\sup_{y \in Y} n((f_Y)_1(y), (g_Y)_1(y)) \geq \sup_{y \in Y} n(f_1(y), g_1(y)).$$

Similarly

$$\sup_{y \in Y} n((g_Y)_2(y), (f_Y)_2(y)) \geq \sup_{y \in Y} n(g_2(y), f_2(y)).$$

By taking the maximum we get $D(f_Y, g_Y) \geq D(f, g)$.

By the definition of f_Y and g_Y , we have that $(f_Y)_1(y) \leq \max\{f_1(t), d(t, y)\}$ and

$(g_Y)_1(y) \leq \max\{g_1(t), d(t, y)\}$ for $t \in X$. Thus we have

$$n((f_Y)_1(y), (g_Y)_1(y)) \leq n(\max\{f_1(t), d(t, y)\}, \max\{g_1(t), d(t, y)\}).$$

Claim: $\max\{f_1(t), d(t, y)\} = f_1(t)$ and $\max\{g_1(t), d(t, y)\} = g_1(t)$.

Proof of the Claim

Case 1: $\max\{f_1(t), d(t, y)\} = d(t, y)$ and $\max\{g_1(t), d(t, y)\} = d(t, y)$. Then

$$n(\max\{f_1(t), d(t, y)\}, \max\{g_1(t), d(t, y)\}) = n(d(t, y), d(t, y)) = 0.$$

Also

$$n(\max\{g_1(t), d(t, y)\}, \max\{f_1(t), d(t, y)\}) = n(d(t, y), d(t, y)) = 0.$$

Then by the T_0 -condition we have that $f_1 = g_1$ which is not possible.

Case 2: $\max\{f_1(t), d(t, y)\} = f_1(t)$ and $\max\{g_1(t), d(t, y)\} = d(t, y)$. Then

$$n(\max\{f_1(t), d(t, y)\}, \max\{g_1(t), d(t, y)\}) = n(f_1(t), d(t, y)) = f_1(t) = n(f_1(t), g_1(t)).$$

Case 3: $\max\{f_1(t), d(t, y)\} = d(t, y)$ and $\max\{g_1(t), d(t, y)\} = g_1(t)$. Then

$$n(\max\{f_1(t), d(t, y)\}, \max\{g_1(t), d(t, y)\}) = n(d(t, y), g_1(t)) = 0 = n(f_1(t), g_1(t)).$$

Case 4 is straightforward.

Thus

$$n((f_Y)_1(y), (g_Y)_1(y)) \leq n(f_1(t), g_1(t))$$

and by taking supremum on both sides, we get $D(f_Y, g_Y) \leq D(f, g)$ and hence $D(f_Y, g_Y) = D(f, g)$.

□

5.1.7 *Lemma.* (Compare [18, Fact 1.6]) Let (X, d) be a T_0 -ultra-quasi-metric space and $\varphi : X \rightarrow X$ be an isometry. Then there exists a unique isometry from $(Q(X, d), D)$ to $(Q(X, d), D)$ which extends φ .

Proof.

Define $\phi : (Q(X, d), D) \rightarrow (Q(X, d), D)$ by $\phi(f) = f \circ \varphi^{-1}$. Let $x \in X$. Then for each $y \in Y$,

$$\begin{aligned}
 \phi(f_x)(y) &= (f_x \circ \varphi^{-1})(y) \\
 &= (((f_x)_1 \circ \varphi^{-1})(y), ((f_x)_2 \circ \varphi^{-1})(y)) \\
 &= (d(x, \varphi^{-1}(y)), d(\varphi^{-1}(y), x)) \\
 &= (d(\varphi(x), y), d(y, \varphi(x))) \\
 &= f_{\varphi(x)}(y).
 \end{aligned}$$

Thus $\phi(f_x) = f_{\varphi(x)}$ and $\phi|_{f_x} = \varphi$ for every $x \in X$.

Let $g \in Q(X, d)$. Then we have that $g \circ \varphi \in Q(X, d)$ and $g = (g \circ \varphi) \circ \varphi^{-1}$ which shows that ϕ is surjective.

$$\begin{aligned}
D(\phi(f), \phi(g)) &= D(f \circ \varphi^{-1}, g \circ \varphi^{-1}) \\
&= \max \left\{ \sup_{x \in X} n(f_1 \circ \varphi^{-1}(x), g_1 \circ \varphi^{-1}(x)), \sup_{x \in X} n(g_2 \circ \varphi^{-1}(x), f_2 \circ \varphi^{-1}(x)) \right\} \\
&= \max \left\{ \sup_{x \in X} n(f_1(\varphi^{-1}(x)), g_1(\varphi^{-1}(x))), \sup_{x \in X} n(g_2(\varphi^{-1}(x)), f_2(\varphi^{-1}(x))) \right\} \\
&= \max \left\{ \sup_{x \in X} n(d(f, \varphi^{-1}(x)), d(g, \varphi^{-1}(x))), \sup_{x \in X} n(d(\varphi^{-1}(x), g), d(\varphi^{-1}(x), f)) \right\} \\
&= \max \left\{ \sup_{x \in X} n(d(\varphi(f), x), d(\varphi(g), x)), \sup_{x \in X} n(d(x, \varphi(g)), d(x, \varphi(f))) \right\} \\
&= \max \left\{ \sup_{x \in X} n(d(f, x), d(g, x)), \sup_{x \in X} n(d(x, g), d(x, f)) \right\} \\
&= \max \left\{ \sup_{x \in X} n(f_1(x), g_1(x)), \sup_{x \in X} n(g_2(x), f_2(x)) \right\} \\
&= D(f, g)
\end{aligned}$$

whenever $f, g \in Q(X, d)$. Thus ϕ is an isometry.

Notice by Lemma 5.1.2 that since $Q(X, d)$ is a T_0 -space and ϕ is an isometric map, ϕ is injective. We therefore conclude that ϕ is an isometry.

It remains to show only that ϕ is unique.

Suppose $\rho : (Q(X, d), D) \rightarrow (Q(X, d), D)$ is an isometry extending φ . Then by Lemma 5.1.3, for each $a \in X$, we have

$$(\rho(f))_1(\varphi(a)) = D(\rho(f), \rho(f_a)) = D(f, f_a) = f_1(a).$$

Hence $(\rho(f))_1 \circ \varphi = f_1$ and thus $(\rho(f))_1 = f_1 \circ \varphi^{-1}$. Similarly, one can show that $(\rho(f))_2 = f_2 \circ \varphi^{-1}$. Thus $\rho(f) = f \circ \varphi^{-1}$.

□

5.2 Urysohn ultra-quasi-metric spaces

5.2.1 Ultra-quasi-admissible sets of distances

In this subsection, we introduce the notion of ultra-quasi-admissible sets and give some characterizations. Comparable studies in the category of ultrametric spaces had been conducted by Shao [35].

5.2.1 Proposition. Let (X, d) be an ultra-quasi-metric space. Then every triangle in $(X, d^s)^1$ is isosceles.

Proof.

Let $x, y, z \in X$. If $d^s(x, y) = d^s(x, z) = d^s(y, z)$ then we have nothing to show. Otherwise, without loss of generality suppose $d^s(x, y) > d^s(y, z)$. Then we have

$$d(x, z) \leq \max\{d(x, y), d(y, z)\} \leq \max\{d^s(x, y), d^s(y, z)\} = d^s(x, y).$$

Thus we have $d(x, z) \leq d^s(x, y)$. Dually we have that

$$d(z, x) \leq \max\{d(y, x), d(z, y)\} \leq \max\{d^s(x, y), d^s(y, z)\} = d^s(x, y).$$

Thus we have $d(z, x) \leq d^s(x, y)$. Thus we conclude that $d^s(x, z) \leq d^s(x, y)$. To finish the proof, we show that $d^s(x, y) \leq d^s(x, z)$ so as to conclude that $d^s(x, z) = d^s(x, y)$. Notice that we have

$$d(x, y) \leq d^s(x, y) \leq \max\{d^s(x, z), d^s(z, y)\} = d^s(x, z),$$

so that $d^s(x, z) \geq d(x, y)$. Dually we have

$$d(y, x) \leq d^s(y, x) \leq \max\{d^s(x, z), d^s(z, y)\} = d^s(x, z),$$

so that $d^s(x, z) \geq d(y, x)$ and hence we have that $d^s(x, z) \geq d^s(y, x)$. Therefore $d^s(x, z) = d^s(x, y)$ and this finishes the proof.

□

¹Of course d^s is an ultrametric (check for instance [23])

5.2.2 Proposition. If (X, d) is a supseparable T_0 -ultra-quasi-metric space, then $\{d(x, y) : x, y \in X\}$ is countable.

Proof.

Let D be a countable supdense subset of X , say $D = \{x_i\}_{i=1}^{\infty}$. Then $\{d(x_i, x_j) : x_i, x_j \in D\}$ is countable. We will show that for every $x, y \in X$, there exist $x_i, x_j \in D$ such that $d(x_i, x_j) = d(x, y)$. If $d(x, y) = 0$, then the result holds since we can set $d(x, y) = d(x_i, x_i)$ for some $i \in I$. Thus assume that $d(x, y) > 0$. Let $r_1, r_2 < d(x, y)$. Consider $B_d(x, r_1)$ and $B_d(y, r_2)$. Since D is a countable supdense set in X , there exist $x_i \in B_d(x, r_1) \cap D$. Therefore by using the strong triangle inequality twice and our assumption on r_1 , we obtain

$$d(x, y) \leq \max\{d(x, x_i), d(x_i, y)\} = d(x_i, y) \leq \max\{d(x_i, x), d(x, y)\} = d(x, y).$$

Thus $d(x, y) = d(x_i, y)$. Dually by countable supdensity of D in X , there exists $x_j \in B_d(x_i, r_2) \cap D$ such that $d(x_i, y) = d(x_i, x_j)$ (note that we replace for instance the variable y by x_j in this way). Hence we have that $d(x, y) = d(x_i, x_j)$.

□

5.2.1 Example. Let (X, \leq) be a partially ordered set. Define an ultra-quasi-metric d on X as follows: $d(x, y) = 1$ if $x > y$ and $d(x, y) = 0$ otherwise, so that (X, d) is an ultra-quasi-metric space. Then we have that $\Gamma = \{0, 1\}$ is countable.

5.2.1 Definition. A T_0 -ultra-quasi-metric space X is said to be **ultra-quasi-universal** for a family U of T_0 -ultra-quasi-metric spaces if any space from U is isometrically embeddable in X .

5.2.1 Corollary. There does not exist an ultra-quasi-universal supseparable T_0 -ultra-quasi-metric space.

Proof.

Suppose (X, d_X) is an ultra-quasi-universal supseparable T_0 -ultra-quasi-metric space, then $A = \{d_X(x, y) : x, y \in X\}$ is countable. Let $\alpha \in \mathbb{R} \setminus A$, and let

(Y, d_Y) be with $y_1, y_2 \in Y$ such that $d_Y(y_1, y_2) = \alpha$. Then we cannot embed (Y, d_Y) in (X, d_X) .

□

5.2.2 Definition. A countable set $\Gamma \subseteq \mathbb{R}$ is said to be **ultra-quasi-admissible** if there exists a T_0 -ultra-quasi-metric space (X, d) such that $\Gamma = \{d(x, y) : x, y \in X\}$.

$\Gamma \subseteq \mathbb{R}$ will be said to be ultra-quasi-admissible in the sense of joincompactness if there exists a joincompact T_0 -ultra-quasi-metric space (X, d) such that $\Gamma = \{d(x, y) : x, y \in X\}$.

In the following we shall characterize the ultra-quasi-admissible sets.

5.2.3 Proposition. A countable set $\Gamma \subseteq \mathbb{R}$ is said to be ultra-quasi-admissible in the sense of joincompactness if and only if $\Gamma = \{a_n\}_{n \geq 1} \cup \{0\}$ with $a_{n+1} < a_n$ and $a_n \rightarrow 0$.

Proof.

(\implies)

Let (X, d) be a joincompact Γ -ultra-quasi-metric space such that $\Gamma = \{d(x, y) : x, y \in X\}$. Without any loss of generality assume that $\Gamma = \{a_n\}_{n \geq 1} \cup \{0\}$.

Claim: if there exists an infinite subsequence $\{a_{n_k}\}$ of $\{a_n\}$ with $\{a_{n_k}\} \rightarrow a$, where $a > 0$, then Γ is not ultra-quasi-admissible in the sense of joincompact.

Proof of Claim.

Let us reenumerate the sequence $\{a_{n_k}\} = \{b_i\}_{i \geq 1}$. Then for each positive integer n , pick $x_n, y_n \in X$ such that $d^s(x_n, y_n) = a_n$. Then clearly $\{x_n\}$ and $\{y_n\}$ are infinite sequences in X . By joincompactness of X , let $\{x_{n_k}\}$ be a convergent subsequence of $\{x_n\}$, say $x_{n_k} \rightarrow x_\infty$ with respect to the topology $\tau(d^s)$. Also $\{y_{n_k}\}$ is an infinite sequence in X so that by joincompactness of X there is a

convergent subsequence of $\{y_{n_{k_j}}\}$ of $\{y_{n_k}\}$, say $y_{n_{k_j}} \rightarrow y_\infty$ with respect to the topology $\tau(d^s)$. Then we see that $x_{n_{k_j}} \rightarrow x_\infty$.

Indeed since

$$\begin{aligned} d(x_{n_{k_j}}, x_\infty) &\leq \max\{d(x_{n_{k_j}}, x_{n_k}), d(x_{n_k}, x_\infty)\} \\ &\leq \max\{d^s(x_{n_{k_j}}, x_{n_k}), d^s(x_{n_k}, x_\infty)\} \end{aligned}$$

and

$$\begin{aligned} d(x_\infty, x_{n_{k_j}}) &\leq \max\{d(x_{n_k}, x_{n_{k_j}}), d(x_\infty, x_{n_k})\} \\ &\leq \max\{d^s(x_{n_k}, x_{n_{k_j}}), d^s(x_\infty, x_{n_k})\} \end{aligned}$$

we have that

$$d^s(x_{n_{k_j}}, x_\infty) \leq \max\{d^s(x_{n_{k_j}}, x_{n_k}), d^s(x_{n_k}, x_\infty)\}$$

which implies that $x_{n_{k_j}} \rightarrow x_\infty$ with respect to the topology $\tau(d^s)$ since $x_{n_{k_j}} \rightarrow x_{n_k}$ and $x_{n_k} \rightarrow x_\infty$ with respect to the topology $\tau(d^s)$.

Moreover we have that $d^s(x_\infty, y_\infty) = a$ since $\{a_{n_k}\} \rightarrow a$.

Let now $y_m \in \{y_{n_{k_j}}\}$, with $d^s(y_\infty, y_m) < \frac{a}{2^m}$. If we consider the triangle formed by the points x_∞, y_∞, y_m , then since X is a T_0 -ultra-quasi-metric space, we have by Proposition 5.2.1 that $d^s(x_\infty, y_\infty) = d(x_\infty, y_m) = a$. Since $x_{n_{k_j}} \rightarrow x_\infty$, we have that $d^s(x_m, x_\infty)$ is arbitrarily small, but $d^s(x_m, y_m) = a_m$ and $d^s(y_m, x_\infty) = a$ and this means that the triangle formed by x_∞, x_m, y_m is not isosceles, which contradicts X being a T_0 -ultra-quasi-metric space.

End of Proof of Claim.

It is clear that by our claim above that Γ does not contain any decreasing sequence which converges to a positive number. It is also clear that Γ does not contain any infinite increasing sequence.

Indeed suppose Γ contains an infinite increasing sequence $\{a_k\}_{k \geq 1}$. Then since X is joincompact, the sequence $\{a_k\}_{k \geq 1}$ is bounded above. Let $a = \sup_{k \geq 1} a_k$, then $a > 0$ and there exists a subsequence $\{a_{k_j}\}$ of $\{a_k\}_{k \geq 1}$ such that $a_{k_j} \rightarrow a$. Hence $\Gamma = \{a_n\}_{n \geq 1} \cup \{0\}$, with $a_{n+1} < a_n$ and $a_n \rightarrow 0$.

(\Leftarrow)

We will now prove the converse of Proposition 5.2.3. Suppose that $\Gamma = \{a_n\}_{n \geq 1} \cup \{0\}$, with $a_{n+1} < a_n$ and $a_n \rightarrow 0$. Let $X = \Gamma$, and define a T_0 -ultra-quasi-metric n on X by $n(x, y) = x$ if $x, y \in X$ and $x > y$, and $n(x, y) = 0$ if $x, y \in X$ and $x \leq y$ (check [23, Example 1]). Then we have that n^s is an ultrametric on X defined by $n^s(x, y) = \max\{x, y\}$ if $x \neq y$ and $n^s(x, y) = 0$ if $x = y$. Furthermore the ultrametric n^s is complete (check [23, Example 2]).

Since $a_n \rightarrow 0$ with respect to the topology $\tau(n^s)$, we have that (X, n^s) is a joincompact T_0 -ultra-quasi-metric space.

□

5.2.1 Remark. If we have $\{b_n\} \subseteq \Gamma$ with $b_n \rightarrow a$, where $a > 0$, then Γ is not ultra-quasi-admissible in the sense of joincompactness.

In the next proposition we shall characterize ultra-quasi-admissible sets without joincompactness.

5.2.4 Proposition. (Compare [35, Proposition 2.15]) Let (X, d) a T_0 -ultra-quasi-metric space without joincompactness. A countable $\Gamma \subseteq \mathbb{R}$ is ultra-quasi-admissible if and only if Γ contains 0.

Proof.

(\Rightarrow)

This direction follows immediately from Proposition 5.2.3.

(\Leftarrow)

Suppose $\Gamma = \{a_n\}_{n \geq 1} \cup \{0\}$ and let $X = \Gamma$. By defining a T_0 -ultra-quasi-metric n as in the proof of Proposition 5.2.3, we have that (X, n) is a T_0 -ultra-quasi-metric space.

□

5.2.2 Countable Γ -ultra-quasi-Urysohn space

In this subsection we shall consider only Γ that is ultra-quasi-admissible without joincompactness with $0 \in \overline{\Gamma^*}$, where by Γ^* we mean $\Gamma \setminus \{0\}$ and $\overline{\Gamma}$ is the $\tau(D^s)$ -closure of Γ .

The question we shall answer in this subsection is the following: For what set Γ does there exist an ultra-quasi-universal Γ -ultra-quasi-metric space? It turns out that the answer is affirmative for any $\{r_i\}_{i \geq 0} \subseteq \mathbb{R}_+$ with $r_0 = 0$. Thus we state first the main theorem which we shall prove in the sequel.

5.2.1 Theorem. For any countable $\Gamma \subseteq \mathbb{R}_+$ with Γ containing 0, there exists a unique corresponding ultra-quasi-universal bicomplete supseparable Γ -ultra-quasi-metric space which we shall denote by \mathbb{U}_Γ^{uq} .

It should be noted that such a Γ can even be a supdense subset of \mathbb{R} like the set of all rational numbers. To prove Theorem 5.2.1 we shall use an ultra-tree construction. Note that the ultra-tree that we shall use is different from the trees that is commonly commonly talked about (check for instance [11] for the notion of trees in metric spaces and [29] for the notion of trees in quasi-metric spaces). The ultra-tree that we shall construct will have the property that if you pick an arbitrary node u in the tree, then the collection of all nodes that extends from u should give a linear order while the tree that people normally use has the property that if you take an arbitrary node v in the tree, then the collection of all nodes that extends from v gives a well order.

Let S be a set and $n \in \omega$. We denote the set of all finite sequences of length n in S by S^n . Denote $S_\omega = \bigcup_{n \in \omega} S^n$. By an ultra-tree T on the set S we shall mean a set of finite sequences from S , closed under subsequences; i.e., if $u \in T$ and $v \subseteq u$ then $v \in T$. In general we have that $T \subseteq S_\omega$ and the empty sequence is always a member of a nonempty ultra-tree. A node of an ultra-tree is just a sequence in that ultra-tree. Let $S^\omega := \{g \mid g : \omega \rightarrow S\}$.

5.2.3 Definition. Let ω^Γ denote the set of all functions $g : [a, \infty) \cap \Gamma \rightarrow \omega$ where $a \in \Gamma$ such that the set $b \in \Gamma \cap [a, \infty) : g(b) = 0$ is finite. If $u \in \omega^\Gamma$ and $b \in \text{dom}(g)$, where $\text{dom}(g)$ denotes the domain of the function g , then we denote by $g|_b$ the function $g|_{([b, \infty) \cap \Gamma)}$, which is also an element of ω^Γ . For $f, g \in \omega^\Gamma$, we shall say that f is an initial segment of g , or g extends f , and denote by $f \subseteq g$ or $g \supseteq f$, if there is $b \in \text{dom}(g)$ such that $f = g|_b$. We call $(\omega^\Gamma, \subseteq)$ the full Γ -ultra-tree.

Observe that for any $f \in \omega^\Gamma$, the set of initial segments of f is linearly ordered by \subseteq .

5.2.4 Definition. For every $f \in \omega^\Gamma$, we define the **level of f** by $\text{lev}(f) = \inf \text{dom}(f) = \min \text{dom}(f)$. We conclude therefore from this notation that $f \subseteq g$ if and only if $f = g|_{\text{lev}(f)}$. We now present the following formal definition of a Γ -ultra-tree.

5.2.5 Definition. A subset T of ω^Γ is called a **Γ -ultra-tree** if it is closed under taking initial segments, i.e., if $f \subseteq g$ and $g \in T$ then $f \in T$. A Γ -ultra-tree T is **pruned** if for every $f \in T$ and $a \in \Gamma$ with $a < \text{lev}(f)$, there is $g \in T$ with $f \subseteq g$.

5.2.6 Definition. Let T be a Γ -ultra-tree. A **branch of T** is a function $f \in \omega^\Gamma$ such that for all $a \in \Gamma$, $f|_a \in T$, where by $f|_a$ we mean $f|_{([a, \infty) \cap \Gamma)}$. If we have $f = g|_a$ we will write $f \subseteq g$ and say that f is an initial segment of g . We denote the set of all branches of T by $[T]$. See immediately that $[T] \subseteq \omega^{\Gamma*}$.

5.2.2 Remark. Notice that for every $f \in [T]$ the set $\{a \in \Gamma : f(a) = 0\}$ is either finite or a decreasing sequence converging to 0.

5.2.5 Proposition. If $f \neq g \in [T]$, then the set $\{a \in \Gamma : f(a) \neq g(a)\}$ has a maximum.

Proof.

Suppose that is not the case. Then we can find an infinite increasing sequence $\{a_n\}_{n \geq 1}$ in Γ with $f(a_l) = g(a_l)$, for every $l \in \mathbb{N}$. Fix $l \in \mathbb{N}$ and consider $u = f|_{a_l}$ and $v = g|_{a_l}$. Then we have $u(a_m)$ will be different from $v(a_m)$ for every $m \geq l$. Thus at least one of the following sets $\{\alpha \in \Gamma : v(\alpha) = 0\}$ and $\{\beta \in \Gamma : u(\beta) = 0\}$ is infinite, which is a contradiction (check Remark 5.2.2).

□

Proposition 5.2.5 allows us to define a Γ -ultra-quasi-metric on $[T]$ for any Γ -ultra-tree T .

5.2.7 Definition. Let T be a Γ -ultra-tree. Define an ultra-quasi-metric on $[T]$ by

$$D(f, g) = \begin{cases} 0 & \text{if } f \leq g \\ a_n & \text{least level } n \text{ such that } f(n) > g(n). \end{cases}$$

Take $\Gamma = \{a_n\}_{n \geq 1} \cup \{0\}$ with $a_{n+1} < a_n$ and $a_n \rightarrow 0$.

We shall denote by X_T the space $([T], D)$. Also, we denote by X_Γ the space $([\omega^\Gamma], D)$. It is not difficult to check that D is a Γ -ultra-quasi-metric. The definition of D as it stands is independent of the specific Γ -ultra-tree T : notice that all these ultra-quasi-metrics are simply the restrictions to $[T]$ of the corresponding ultra-quasi-metrics defined for $[\omega_\Gamma]$. Indeed X_T has the subspace topology inherited from X_Γ . From the D defined above, one can show easily that D^s given by

$$D^s(f, g) = \begin{cases} 0 & \text{if } f = g \\ \max\{a \in \Gamma : f(a) \neq g(a)\} & \text{otherwise} \end{cases}$$

is a Γ -ultra-metric on $[T]$.

5.2.1 *Notation.* For any $u \in \omega_\Gamma$, we define

$$N_u = \{f \in [\omega_\Gamma] : u \subseteq f\}.$$

Clearly for any $f \in N_u$, we have that

$$N_u = \{g \in [\omega_\Gamma] : D^s(f, g) < lev(u)\}$$

and in this way we see that the collection of all N_u for u in ω_Γ forms a countable base for X_Γ . Thus X_Γ is second countable. Since X_T is a topological subspace of X_Γ , we have that all X_T are also second countable.

5.2.6 *Proposition.* If T is an infinite Γ -ultra-tree, then X_T is a supseparable bi-complete Γ -ultra-quasi-metric space.

Proof.

We will show first that X_T is bicomplete. Let $\{f_n\}_{n \geq 1}$ be a $\tau(D^s)$ -Cauchy sequence in X_T . We shall show that $f_n \rightarrow f$ with respect to the topology $\tau(D^s)$ for some $f \in X_T$.

Let $\epsilon > 0$ be fixed. Then there exists $N \in \mathbb{N}$ such that for all $m, n > N$, we have $D^s(f_m, f_n) < \epsilon$ (this is possible since $\{f_n\}_{n \geq 1}$ is D^s -Cauchy). Thus we have that $f_n|_\epsilon = f_m|_\epsilon$. Define f up to the ϵ^{th} level by letting $f|_\epsilon = u_\epsilon = f_m|_\epsilon$.

We continue the construction of f by extending u_ϵ : since $\{f_n\}_{n \geq 1}$ is D^s -Cauchy, we can find an element $a_1 \in \Gamma \cap (\frac{\epsilon}{2}, \epsilon)$ and there exists a natural number $N_1 > N$ such that for every $i, j \geq N_1$, we have $D^s(f_i, f_j) < a_1$. Define now f up to the a_1^{th} level by letting $f|_{a_1} = u_{a_1} = f_i|_{a_1}$. Then clearly $u_\epsilon \subseteq u_{a_1}$.

We continue this construction inductively until we have $f|_{a_n} = u_{a_n} = f_t|_{a_n}$, for some $a_n \in \Gamma \cap (\frac{\epsilon}{n+1}, \frac{\epsilon}{n})$ and for some integer t .

Again since $\{f_n\}_{n \geq 1}$ is D^s -Cauchy, there exists $a_{n+1} \in \Gamma \cap (\frac{\epsilon}{n+2}, \frac{\epsilon}{n+1})$ and there is a natural number $N_{n+1} > N_n$ such that for every $s, t \geq N_{n+1}$, we have $D^s(f_s, f_t) < a_{n+1}$. Continue defining f by letting $f|_{a_{n+1}} = u_{a_{n+1}} = f_s|_{a_{n+1}}$.

By continuing this process we will get f such that for any $a > 0$, there exists some integer M and for all $k \geq M$ we have that $D^s(f, f_k) < a$, i.e., $f_k \rightarrow f$. Of course f is an element of X_T , since for any $0 \neq a \in \Gamma$, if we write $u_a = f|_a$, then $\{b \in \Gamma : u_a(b) \neq 0\}$ is finite, by the way we constructed f . This proves bicompleteness of X_T .

We will now show that X_T is supseparable. Note first that $N_u = \{f \in [\omega_\Gamma] : u \subseteq f\}$ is $\tau(D^s)$ -open: since for any $\epsilon \leq \text{lev}(u)$ and any $f \supseteq u$, we have that the ball $B_{D^s}(f, \epsilon) = \{g : D^s(f, g) < \epsilon\} \subseteq N_u$ is open. Secondly by the definition of ω_Γ , notice that $\{N_u : u \in T\}$ is countable. Thus X_T has a countable base and hence is supseparable.

□

5.2.8 Definition. (Compare [35, Definition 2.33]) A T_0 -ultra-quasi-metric space (X, d) is called **ultra-homogeneous** if for every $x, y \in X$, there is an isometry φ on X such that $\varphi(x) = y$.

We call a T_0 -ultra-quasi-metric space (X, d) with $\Gamma_X = \{d(x, y) : x, y \in X\} \subseteq \Gamma$ (for a fixed countable set $\Gamma \subseteq \mathbb{R}_+$ of potential values of d) a Γ -ultra-quasi-metric space.

5.2.9 Definition. (Compare [35, Definition 2.34]) A Γ -ultra-quasi-metric space is **Γ -ultra-quasi-homogeneous** if any partial isometry between finite subsets of it can be extended to an isometry of the whole space.

5.2.10 Definition. (Compare [35, Definition 2.37]) A Γ -ultra-quasi-metric space (X, d_X) is called **Γ -ultra-quasi-Urysohn** if for any finite ultra-quasi-metric space A with $\{d(x, y) : x, y \in A\} \subseteq \Gamma$, and any subspace $B \subseteq A$, every isometric embedding from B onto X can be extended to an isometric embedding of A onto X .

5.2.11 Definition. (Compare [35, Definition 2.40]) A Γ -ultra-quasi-metric space U is said to be **Γ -ultra-quasi-universal** for a family \mathcal{X} of Γ -ultra-quasi-metric spaces if any space from \mathcal{X} can be isometrically embedded in U .

5.2.7 Proposition. A supseparable bicomplete Γ -ultra-quasi-metric space X is Γ -ultra-quasi-Urysohn if and only if X is Γ -ultra-quasi-universal and ultra-quasi-homogeneous.

Proof.

We have the following claim.

Claim: If $\varphi : (X, d_X) \rightarrow (Y, d_Y)$ is an isometry, and $(X', d_{X'})$ and $(Y', d_{Y'})$ are the bicompletions of (X, d_X) and (Y, d_Y) , respectively, then there exist a unique isometry $\varphi' : X' \rightarrow Y'$ which extends φ .

Proof of Claim

It is easy to check that for each d_X^s -Cauchy sequence $\{x_i\}$ in X , if we define the map $\bar{\varphi}$ by $\bar{\varphi}(\{x_i\}) = \{\varphi(x_i)\}$ then $\bar{\varphi}$ uniquely extends φ .

End of proof of Claim

Recall that Proposition 5.2.2 shows that if X is a supseparable ultra-quasi-metric space then $\Gamma = \{d(x_1, x_2) : x_1, x_2 \in X\}$ is determined by any countable supdense subset S of X , i.e., $\{d(s_1, s_2) : s_1, s_2 \in S\} = \Gamma = \{d(x_1, x_2) : x_1, x_2 \in X\}$.

(\implies)

Let us show first that if X is a supseparable bicomplete Γ -ultra-quasi-Urysohn space, then X is ultra-quasi-homogeneous.

Let $S = \{s_1, s_2, s_3, \dots, s_n, \dots\}$ be a countable supdense subset of X . Let $\varphi : \Lambda \rightarrow \Omega$ be an isometry from Λ into Ω with Λ and Ω finite subsets of X . Define sequences of sets $\Lambda_n, \Omega_n \subseteq X$ and isometries $\varphi_n : \Lambda_n \rightarrow \Omega_n$ by induction on n .

Put $\Lambda_0 = \Lambda, \Omega_0 = \Omega$, and $\varphi_0 = \varphi$. We study the following cases for $n \geq 1$.

Case 1: n is even.

Let $s_i \in S$ with i the least such that $s_i \notin \Omega_{n-1} = \varphi_{n-1}(\Lambda_{n-1})$. Let $\Omega_n =$

$\Omega_{n-1} \cup \{s_i\}$. Clearly, Ω_n is a finite subset of X , and $\{d(a_1, a_2) : a_1, a_2 \in \Lambda_n\} \subseteq \Gamma$. Note that $\varphi_{n-1} : \Lambda_{n-1} \rightarrow \Omega_{n-1}$ is an isometry, hence $\varphi_{n-1}^{-1} : \Omega_{n-1} \rightarrow \Lambda_{n-1}$ is also an isometry. Since X is Γ -ultra-quasi-Urysohn and $\varphi_{n-1}^{-1} : \Omega_{n-1} \rightarrow (\Lambda_{n-1} \subseteq)X$ is an isometric embedding, we can find an isometric embedding $\nu_{n-1} : \Omega_n \rightarrow X$ which extends φ_{n-1}^{-1} . Let $\Lambda_n = \nu_{n-1}(\Omega_n)$ then $\nu_{n-1} : \Omega_n \rightarrow \Lambda_n$ is an isometry. Hence we must have that $\nu_{n-1}^{-1} : \Lambda_n \rightarrow \Omega_n$ is an isometry. Put $\varphi_n = \nu_{n-1}^{-1}$.

Case 2: n is odd.

Let $s_j \in S$ with j the least such that $s_j \notin \Lambda_{n-1}$. Let $\Lambda_n = \Lambda_{n-1} \cup \{s_j\}$. Clearly, Λ_n is a finite subset of X , and $\{d(a_1, a_2) : a_1, a_2 \in \Lambda_n\} \subseteq \Gamma$. Since X is Γ -ultra-quasi-Urysohn and $\varphi_{n-1}^{-1} : \Lambda_{n-1} \rightarrow (\Omega_{n-1} \subseteq)X$ is an isometric embedding, we can find an isometric embedding $\phi_{n-1} : \Lambda_n \rightarrow X$ which extends φ_{n-1}^{-1} . Let $\Omega_n = \varphi_n(\Lambda_n)$ and $\varphi_n = \phi_{n-1}$. This finishes the inductive construction of Λ_n, Ω_n and φ_n .

Let now

$$\Lambda_\omega = \bigcup_{n \in \omega} \Lambda_n, \Omega_\omega = \bigcup_{n \in \omega} \Omega_n, \text{ and } \varphi_\omega = \bigcup_{n \in \omega} \varphi_n.$$

Then φ_ω is an isometry from Λ_ω into Ω_ω . Note by the construction of Λ_ω and Ω_ω that $S \subseteq \Lambda_\omega$ and $S \subseteq \Omega_\omega$. Hence Λ_ω and Ω_ω are supdense in X . We therefore have by the above claim that φ_ω can be extended uniquely to $\bar{\varphi} : X \rightarrow X$ which is an isometry from X onto itself. Of course $\bar{\varphi}$ extends $\varphi_0 = \varphi$.

We will now show that if X is a supseparable bicomplete Γ -ultra-quasi-Urysohn space, then X is Γ -ultra-quasi-universal. It is enough to show that if (X, d_X) is Γ -ultra-quasi-Urysohn then for any supseparable Γ -ultra-quasi-metric space (Y, d_Y) there is an isometric embedding from (Y, d_Y) into (X, d_X) .

Let $Y_s = \{y_0, y_1, y_2, \dots, y_n, \dots\}$ be a supdense subset of Y . Let $\Lambda_n = \{y_0, y_1, y_2, \dots, y_n\}$. Then there exists an isometric embedding from Λ_0 into X . Let us call it φ_0 . Since X is Γ -ultra-quasi-Urysohn, we can obtain the following chain:

$$\varphi_0 \subseteq \varphi_1 \subseteq \varphi_2 \subseteq \dots \subseteq \varphi_n \subseteq \dots$$

such that $\varphi_i : \Lambda_i \rightarrow X$ is an isometric embedding from Λ_i into X for every i . Now let

$$\varphi_\omega = \bigcup_{n \in \omega} \varphi_n,$$

then $\varphi_\omega : Y_s \rightarrow X$ is also an isometric embedding. By our claim, there is an isometric embedding from Y into X which we call φ .

(\Leftarrow)

Let X be a supseparable bicomplete Γ -ultra-quasi-metric space which is Γ -ultra-quasi-universal and ultra-quasi-homogeneous. Let $\Omega \subseteq \Lambda$ be both finite with $\varphi : \Omega \rightarrow X$ be an isometric embedding and $\{d_\Lambda(\alpha_1, \alpha_2) : \alpha_1, \alpha_2 \in \Lambda\} \subseteq \Gamma$. Since X is Γ -ultra-quasi-universal, there is an isometric embedding $\Psi : \Lambda \rightarrow X$. Let $\Lambda' = \Psi(\Lambda) \subseteq X$, $\Omega' = \Psi(\Omega)$ and $\Omega'' = \varphi(\Omega)$. Consider $\sigma = \varphi \circ \Psi^{-1}, \sigma : \Omega \rightarrow \Omega$ is an isometry from Ω onto itself. Since X is ultra-quasi-homogeneous, there is an isometry $\bar{\sigma} : X \rightarrow X$ of X onto itself which extends σ . Let $\Lambda'' = \bar{\sigma}(\Lambda')$, then $\Omega'' \subseteq \Lambda''$. Let $\varphi' = \bar{\sigma} \circ \Psi$, then φ' is an isometric embedding of Λ into X which extends φ .

□

Let us denote the supseparable bicomplete Γ -ultra-quasi-Urysohn space as \mathbb{U}_Γ^{uq} . In what follows we shall show the existence and uniqueness of \mathbb{U}_Γ^{uq} up to isometry.

5.2.8 Proposition. For every countable subset $\Gamma \subseteq \mathbb{R}$, the supseparable bicomplete Γ -ultra-quasi-Urysohn space is unique. This means, if X and Y are both supseparable bicomplete Γ -ultra-quasi-Urysohn spaces then there exists an isometry from X onto Y .

Proof.

Suppose X and Y are supseparable bicomplete Γ -ultra-quasi-Urysohn spaces. Let S_X and S_Y be countable supdense subsets of X and Y respectively. Without loss of generality, we may denote $S_X = \{x_0, x_1, x_2, \dots, x_n, \dots\}$ and $S_Y =$

$\{y_0, y_1, y_2, \dots, y_n, \dots\}$. Note that $\{d(x, x') : x, x' \in X\} = \Gamma = \{d(a, b) : a, b \in S_X\}$ and $\{d(y, y') : y, y' \in Y\} = \Gamma = \{d(e, f) : e, f \in S_Y\}$.

Let $\Lambda = \{x_0, x_1\}$ and pick $y, y' \in Y$ such that $d(x_0, x_1) = d(y, y')$. Let $\Omega = \{y, y'\}$. Define $\varphi : \Lambda \rightarrow \Omega$ such that $\varphi(x_0) = y$ and $\varphi(x_1) = y'$. Therefore φ is an isometry.

Define sequences of sets $\Lambda_n, \Omega_n \subseteq X$ and isometries $\varphi_n : \Lambda_n \hookrightarrow \Omega_n$ by induction on n .

Put $\Lambda_0 = \Lambda, \Omega_0 = \Omega$, and $\varphi_0 = \varphi$. We study the following cases for $n \geq 1$.

Case 1: n is even.

Let $y_i \in S_Y$ with i the least such that $y_i \notin \Omega_{n-1} = \varphi_{n-1}(\Lambda_{n-1})$. Let $\Omega_n = \Omega_{n-1} \cup \{y_i\}$. Clearly, Ω_n is a finite subset of X , and $\{d(\alpha_1, \alpha_2) : \alpha_1, \alpha_2 \in \Omega_n\} \subseteq \Gamma$. Note that $\varphi_{n-1} : \Lambda_{n-1} \rightarrow \Omega_{n-1}$ is an isometry, hence $\varphi_{n-1}^{-1} : \Omega_{n-1} \rightarrow \Lambda_{n-1}$ is also an isometry. Since X is Γ -ultra-quasi-Urysohn and $\varphi_{n-1}^{-1} : \Omega_{n-1} \hookrightarrow (\Lambda_{n-1} \subseteq) X$ is an isometric embedding, we can find an isometric embedding $\nu_{n-1} : \Omega_n \rightarrow X$ which extends φ_{n-1}^{-1} . Let $\Lambda_n = \nu_{n-1}(\Omega_n)$ then $\nu_{n-1} : \Omega_n \rightarrow \Lambda_n$ is an isometry. Hence we must have that $\nu_{n-1}^{-1} : \Lambda_n \hookrightarrow \Omega_n$ is an isometry. Put $\varphi_n = \nu_{n-1}^{-1}$.

Case 2: n is odd.

Let $x_j \in S_X$ with j the least such that $x_j \notin \Lambda_{n-1}$. Let $\Lambda_n = \Lambda_{n-1} \cup \{x_j\}$. Clearly, Λ_n is a finite subset of X , and $\{d(\alpha_1, \alpha_2) : \alpha_1, \alpha_2 \in \Lambda_n\} \subseteq \Gamma$. Since Y is Γ -ultra-quasi-Urysohn and $\varphi_{n-1} : \Lambda_{n-1} \rightarrow (\Omega_{n-1} \subseteq) Y$ is an isometric embedding, we can find an isometric embedding $\phi_{n-1} : \Lambda_n \rightarrow Y$ which extends φ_{n-1} . Let $\Omega_n = \varphi_n(\Lambda_n)$ and $\varphi_n = \phi_{n-1}$. This finishes the construction (by induction) of Λ_n, Ω_n and φ_n .

Let now

$$\Lambda_\omega = \bigcup_{n \in \omega} \Lambda_n, \Omega_\omega = \bigcup_{n \in \omega} \Omega_n, \text{ and } \varphi_\omega = \bigcup_{n \in \omega} \varphi_n.$$

Then φ_ω is an isometry from Λ_ω onto Ω_ω . Note by the construction of Λ_ω and Ω_ω that $S_X \subseteq \Lambda_\omega$ and $S_Y \subseteq \Omega_\omega$. Hence Λ_ω and Ω_ω are supdense in X . We therefore

have by the above claim in Proposition 5.2.7 that φ_ω can be extended uniquely to $\bar{\varphi} : X \rightarrow Y$ which is an isometry from X onto Y .

□

5.2.2 Theorem. For every countable $\Gamma \subseteq \mathbb{R}$, $\mathbb{U}_\Gamma^{uq} = [\omega_\Gamma]$ is a bicomplete supseparable Γ -ultra-quasi-Urysohn space.

Proof.

Let $\Lambda = \{x_1, x_2, \dots, x_n, a\}$ and $\Omega = \{x_1, x_2, \dots, x_n\}$ where a is distinct from x_i , $i = 1, 2, \dots, n$. Let d_Λ be a T_0 -ultra-quasi-metric on Λ such that $\{d_\Lambda(x, y) : x, y \in \Lambda\} \subseteq \Gamma$. Let $\varphi : \Omega \rightarrow [\omega_\Gamma]$ such that $f_i = \varphi(x_i)$. Denote $X_\Gamma = ([\omega_\Gamma], D)$. We are going to show there exists $g \in \mathbb{U}_\Gamma^{uq}$ such that $D^s(f_i, g) = d_\Lambda^s$ for any $i \in \{1, 2, \dots, n\}$. Assume without any loss of generality that for all $i < j$, $d_\Lambda^s(x_j, a) = c_j \geq c_i = d_\Lambda^s(x_i, a)$. Let $g(x) = f_n(x)$ for every $x \in (c_n, \infty) \cap \Gamma$. Without loss of generality let us suppose that $l = \min\{i : c_i = c_n\}$. Define $g(c_n)$ to be an arbitrary natural number in $\mathbb{N} \setminus \{f_i(c_n)\}_{i=1}^n$. Finally put $g(x) = 0$ for every $x \in (-\infty, c_n) \cap \Gamma$. By this construction of g , it is clear that $g \in \mathbb{U}_\Gamma^{uq}$, since $\{g(x) \neq 0 : x \in \Gamma\}$ is finite.

To complete the proof, we show that $D^s(f_i, g) = c_i$ for all $i \in \{1, 2, \dots, n\}$. Note first that $D^s(f_i, f_n) = d_\Lambda^s(x_i, x_n)$. Indeed $D^s(f_i, f_n) = D^s(\varphi(x_i), \varphi(x_n)) = d_\Lambda^s(x_i, x_n)$ since φ is an isometry (check Proposition 5.2.8). Consider the triangle formed by a, x_i and x_n . For $i < l \leq n$, we have that $d_\Lambda^s(x_n, a) = c_n < c_l = d_\Lambda^s(x_i, a)$. By Proposition 5.2.1, we have $d_\Lambda^s(x_i, x_n) = c_i$. Hence $D^s(f_i, f_n) = c_i$. We shall now proceed to show that $D^s(f_i, g) = c_i$ for all $i < l$.

By the construction of g , we have that $g(c_n) \neq f_j(c_n)$ for all $j \in \{l, l+1, \dots, n\}$. Thus we must get that $D^s(f_j, g) \geq c_n = c_j$ for all $j \in \{l, l+1, \dots, n\}$. To conclude we need only argue that $D^s(f_j, g) \leq c_j$.

Well, we know that $D^s(f_n, g) = c_n$ and notice also that

$$D^s(f_n, f_j) = d_\Lambda^s(x_n, x_j) \leq \max\{d_\Lambda^s(x_n, a), d_\Lambda^s(x_j, a)\} = c_n,$$

since $c_n = c_j$. Thus we have that

$$D^s(f_n, g) \leq \max\{D^s(f_j, f_n), D^s(f_n, g)\} = c_n = c_j.$$

This shows that $D^s(f_j, g) \leq c_j$ and hence we have the equality $D^s(f_j, g) = c_j$ for all $j \in \{l, l+1, \dots, n\}$. Therefore $D^s(f_i, g) = c_i$ for every $i \in \{1, 2, \dots, n\}$.

□

5.2.3 Remark. Proposition 5.2.8 and Theorem 5.2.2 shows that for each countable $\Gamma \subseteq \mathbb{R}$, there exists a unique bicomplete supseparable Γ -ultra-quasi-Urysohn space which we denote by \mathbb{U}_Γ^{uq} .

For $\Gamma \subseteq \mathbb{R}$ countable and including 0, we have the following:

5.2.9 Proposition. If a supseparable bicomplete Γ -ultra-quasi-metric space X has the Γ -ultra-quasi-Urysohn property, then every countable supdense subspace S of X is Γ -ultra-quasi-Urysohn.

Proof.

Let $\Lambda = \{x_1, x_2, \dots, x_n, a\}$ and $\Omega = \{x_1, x_2, \dots, x_n\}$ where a is distinct from x_i , $i = 1, 2, \dots, n$. Let d_Λ be a T_0 -ultra-quasi-metric on Λ such that $\{d_\Lambda(x, y) : x, y \in \Lambda\} \subseteq \Gamma$. Let $\varphi : \Omega \rightarrow S$ be an isometric embedding. Note that φ is also an isometric embedding from Ω into X . Since X is Γ -ultra-quasi-Urysohn space, there exists an isometric embedding $\psi : \Lambda \rightarrow X$ which extends φ , say $\psi(a) = z$. Since S is supdense in X , for arbitrary small $\epsilon > 0$, there exists $y \in S$ such that $z \in B_{d^s}(y, \epsilon)$. Therefore $d^s(y, \psi(x_i)) = d^s(z, \psi(x_i))$ for each $i \in \{1, 2, \dots, n\}$ by Proposition 5.2.1. Define $\gamma : \Lambda \rightarrow S$ as follows: let $\gamma(x_i) = \psi(x_i)$, for each $i \in \{1, 2, \dots, n\}$; and let $\gamma(a) = y$. Hence γ is an isometric embedding from Λ into S which extends φ . Therefore S is Γ -ultra-quasi-Urysohn by Definition 5.2.10.

□

5.2.2 Corollary. If X is a Γ -ultra-quasi-Urysohn space, then every supdense subset of X is Γ -ultra-quasi-Urysohn.

5.2.10 Proposition. For every $\Gamma \subseteq \mathbb{R}$ countable, there exists a countable Γ -ultra-quasi-Urysohn space.

Proof.

This follows immediately from Theore 5.2.2 and Proposition 5.2.9.

□

5.2.11 Proposition. The Γ -ultra-quasi-Urysohn space (for Γ countable) in Proposition 5.2.9 is ultra-quasi-homogeneous.

Proof.

The proof is similar to that in Proposition 5.2.7.

□

5.2.12 Proposition. The Γ -ultra-quasi-Urysohn space (for Γ countable) in Proposition 5.2.9 is unique.

Proof.

Let S_1 and S_2 be two supdense subsets of a Γ -ultra-quasi-Urysohn space (X, d) enumerated as follows: $S_1 = \{x_0, x_1, \dots, x_n, \dots\}$ and $S_2 = \{y_0, y_1, \dots, y_n, \dots\}$. By Proposition 5.2.9, S_1 and S_2 are Γ -ultra-quasi-Urysohn. Note that $\{d(x, x') : x, x' \in S_1\} = \Gamma = \{d(y, y') : y, y' \in S_2\}$, as X is a Γ -ultra-quasi-metric space. We will show that there exists an isometry from S_1 onto S_2 .

Let $\Lambda = \{x_0, x_1\}$ and pick $y_s, y_t \in S_2$ such that $d(x_0, x_1) = d(y_s, y_t)$. Let $\Omega = \{y_s, y_t\}$. Define $\varphi : \Lambda \rightarrow \Omega$ such that $\varphi(x_0) = y_s$ and $\varphi(x_1) = y_t$. Therefore φ is an isometry.

Define sequences of sets $\Lambda_n \subseteq S_1$, $\Omega_n \subseteq S_2$ and isometries $\varphi_n : \Lambda_n \rightarrow \Omega_n$ by induction on n .

Put $\Lambda_0 = \Lambda$, $\Omega_0 = \Omega$, and $\varphi_0 = \varphi$. We study the following cases for $n > 1$.

Case 1: n is even.

Let $y_j \in S_2$ with j the least such that $y_j \notin \Omega_{n-1} = \varphi_{n-1}(\Lambda_{n-1})$. Let $\Omega_n = \Omega_{n-1} \cup \{y_j\}$. Clearly, Ω_n is a finite subset of X , and $\{d(\alpha_1, \alpha_2) : \alpha_1, \alpha_2 \in \Lambda_n\} \subseteq \Gamma$. Note that $\varphi_{n-1} : \Lambda_{n-1} \rightarrow \Omega_{n-1}$ is an isometry, hence $\varphi_{n-1}^{-1} : \Omega_{n-1} \rightarrow \Lambda_{n-1}$ is also an isometry. Since S_1 is Γ -ultra-quasi-Urysohn and $\varphi_{n-1}^{-1} : \Omega_{n-1} \rightarrow (\Lambda_{n-1} \subseteq) S_1$ is an isometric embedding, we can find an isometric embedding $\nu_n : \Omega_n \hookrightarrow X$ which extends φ_{n-1}^{-1} . Let $\Lambda_n = \nu_n(\Omega_n)$ then $\nu_n : \Omega_n \rightarrow \Lambda_n$ is an isometry. Hence we must have that $\nu_n^{-1} : \Lambda_n \rightarrow \Omega_n$ is an isometry. Put $\varphi_n = \nu_n$.

Case 2: n is odd.

Let $x_i \in S_1$ with i the least such that $x_i \notin \Lambda_{n-1}$. Let $\Lambda_n = \Lambda_{n-1} \cup \{x_i\}$. Clearly, Λ_n is a finite subset of X , and $\{d(\alpha_1, \alpha_2) : \alpha_1, \alpha_2 \in \Lambda_n\} \subseteq \Gamma$. Since S_2 is Γ -ultra-quasi-Urysohn and $\varphi_{n-1} : \Lambda_{n-1} \rightarrow (\Omega_{n-1} \subseteq) S_2$ is an isometric embedding, we can find an isometric embedding $\psi_n : \Lambda_n \rightarrow S_2$ which extends φ_{n-1} . Let $\Omega_n = \psi_n(\Lambda_n)$ and $\varphi_n = \psi_n$. Then φ_n is an isometry from Λ_n onto Ω_n . This finishes the inductive construction of Λ_n, Ω_n and φ_n .

Let

$$\Lambda_\omega = \bigcup_{n \in \omega} \Lambda_n, \Omega_\omega = \bigcup_{n \in \omega} \Omega_n, \text{ and } \varphi_\omega = \bigcup_{n \in \omega} \varphi_n.$$

Then φ_ω is an isometry from Λ_ω onto Ω_ω . Note that by the construction of Λ_ω and Ω_ω we have that $S_1 = \Lambda_\omega$ and $S_2 = \Omega_\omega$. So φ_ω is an isometry from S_1 to S_2 .

□

We have thus shown that for every supseparable bicomplete Γ -ultra-quasi-Urysohn space X , the countable supdense subspace $S \subseteq X$ is unique up to isometry.

Chapter 6

Conclusion and open problems

In this last chapter, we draw the conclusions of our investigations and underline some open problems found throughout the work that can constitute topics of further research.

The dissertation achieved the task of first establishing some new results from the theory of tight extensions that were developed in [1]. Also we established some results concerning supseparability of the space of minimal function pairs of a T_0 -quasi-metric space X .

Below we give a summary of the work which we studied in each chapter of the dissertation and then suggest two important areas of future research which are related to q -hyperconvex hulls of quasi-metric spaces.

6.1 Summary of the achieved work

In Chapter 1, we present some preliminaries from the theory of quasi-pseudometric spaces and ultra-quasi-pseudometric spaces that will ease the reading of this work.

In Chapter 2, we present some results about tight extensions of T_0 -quasi-metric spaces that was studied in [1]. We also review and present some results about hyperconvex hulls of T_0 -quasi-metric spaces according to [20].

In Chapter 3, we present some new results about tight (which we call *uq-tight*) extensions of T_0 -ultra-quasi-metric spaces. We show that for a T_0 -ultra-quasi-metric space X , the *uq-tight* extension uQ_X of X is maximal, among the *uq-tight* extensions of X (see Remark 3.3.1). An example describing the Isbell-hull Q_X (according to [20]) and the ultra-quasi-metrically injective hull uQ_X (according to [23]) for a discrete metric space (X, d) is presented (see Example 3.3.1).

In Chapter 4, Section 4.1, we introduce the concept of Katětov function pairs, their properties, and some examples. We also introduce some examples of function pairs called minimum and maximum function pairs and we give conditions under which such pairs will be Katětov (see Remark 4.1.1).

In Section 4.2 we give some facts relating to when supseparability of the space of extremal function pairs is preserved.

In Chapter 5, we concentrate on ultra-quasi-pseudometric spaces. In Section 5.1, we modify a construction due to Katětov ([18]) to the category of T_0 -ultra-quasi-metric spaces. In Section 5.2 we study Γ -Urysohn-ultra-quasi-metric spaces for some countable set $\Gamma \subseteq \mathbb{R}_+$ and we prove uniqueness of such spaces.

The following problems are related to our studies in Chapter 4.

6.1.1 Problem. Are the spaces of extremal function pairs over subspaces of finite dimensional real biBanach spaces supseparable?

6.1.2 Problem. For which T_0 -ultra-quasi-metric spaces X , is uQ_X supseparable?

6.1.3 Problem. Suppose that (X, d) is a T_0 -quasi-metric space. Consider the (metric) Isbell hull $\epsilon_m(X)$ of (X, d^s) and the q -hyperconvex hull Q_X of (X, d) . If $\epsilon_m(X)$ is separable, is Q_X supseparable? If Q_X is supseparable, is $\epsilon_m(X)$ separable?

A simpler question which could first be studied is the following:

6.1.4 Problem. If (X, m) is a metric space and $\epsilon_m(X)$ is separable, is Q_X supseparable, too?

Note that in this case $\epsilon_m(X)$ is a subspace of Q_X , so supseparability of Q_X implies that $\epsilon_m(X)$ is separable.

The following problems are related to our studies in Chapter 5.

6.1.5 Problem. How do we characterize the isometry groups of supseparable T_0 -ultra-quasi-metric spaces?

6.2 Two possible areas for future work

The theory of tight extensions of quasi-metric spaces may have some interesting applications in other structures of mathematics. For instance results from the theory of tight extensions of T_0 -quasi-metric spaces were recently applied to some investigations about endpoints in T_0 -quasi-metric spaces (compare [2, 3]). In the following we point out two areas of future work related to the theories of q -hyperconvex hulls and tight extensions.

6.2.1 The q -hyperconvex hull of an asymmetric normed space

In [8] Cohen constructed the injective envelope of Banach spaces and showed that it is unique (up to isometry).

Rao [32] showed that the injective hull of a Banach space X in the category of metric spaces and contractions coincides with the injective hull of X in the category of (real) normed spaces and contractions.

6.2.1 Problem. Let (X, p) be an asymmetric normed space. Does there exist a q -hyperconvex hull of (X, p) in the category of asymmetric normed spaces and contractions? Furthermore does the Isbell hull Q_X coincide with the q -hyperconvex hull of (X, p) ?

It should be noted that Problem 6.2.1 had already been stated by Otafudu [28] but the problem still remains open till date. Nevertheless, very recently Otafudu made a first attempt towards solving the problem (check [30]).

The next problem about tight extensions of asymmetric normed spaces is related to Problem 6.2.1 and we state it next.

6.2.2 Tight extensions of asymmetrically normed spaces

In [1], Agyingi et al. studied tight extensions in the category of T_0 -quasi-metric spaces and non-expansive maps. They showed that for a T_0 -quasi-metric space (X, d) , the q -hyperconvex hull Q_X is a maximal (among the) tight extension of X .

Tight extensions of normed spaces had been studied by Rao [33] in the context of Banach spaces. He showed that bound extensions as defined by Kaufman [19] and tight extensions as discussed by Dress [11] are the same.

6.2.2 Problem. Let (X, p) be an asymmetric normed space. Is there a reasonable way to define tight extension in the category of asymmetric normed spaces and contractions? If yes, is there a maximal one?

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